Alma Mater Studiorum · Università di Bologna

Scuola di Scienze
Dipartimento di Fisica e Astronomia
Corso di Laurea in Fisica

Weighted projective space and Kaluza Klein theory

Relatore:
Prof. Michele Cicoli

Correlatore:
Prof.ssa Antonella Grassi

Presentata da:
Andrea Silingardi

Anno Accademico 2023/2024

Abstract

Nel corso di questo elaborato tratteremo la teoria di Kaluza Klein, essa unifica gravità ed elettromagnetismo sfruttando le proprietà geometriche dello spazio. Nello specifico si basa sull'assumere l'universo come localmente composto da cinque dimensioni di cui quattro spaziali e una temporale ponendo la quarta dimensione spaziale come cerchio sull'ordinario spazio di Minkowski quadridimensionale ($M_4 \times S^1$). Questo unito alle condizioni di periodicità e invarianza necessarie nella quinta dimensione permette diverse considerazioni fisiche. Prima di affrontare il caso gravitazionale ci concentreremo sul caso scalare e vettoriale andando a studiare come campi appunto scalari e vettoriali nelle cinque dimensioni, così strutturate, producano diversi oggetti matematici nelle sole quattro dello spazio di Minkowski.[Que15]

Il caso gravitazionale prevederà l'aggiunta della descrizione metrica dello spazio-tempo cinque-dimensionale e noteremo come questo porti ad oservare, nello spazio quadri-mensionale, non solo un campo gravitazionale ma anche uno elettromagnetico e uno scalare ottenendo così l'unificazione voluta. [BL87] [Chu22]

Per finire tratteremo gli spazi proiettivi pesati e le loro varietà: enti matematici necessari ad una trattazione più moderna delle idee proposte da Kaluza e Klein. In questa parte daremo alcune definizione, lemmi e teoremi fondamentali per una prima comprensione dell'argomento, per poi passare ad un esempio svolto.

Introduction

The first example of a theory unifying gravitation and electromagnetism was developed by Kaluza in 1921 and then quantized by Klein in 1926.

The theory is developed starting from the simplest case in which a single extra spatial dimension is compactified to a circle, and an electromagnetic-like field structure arises in four dimensions from the higher-dimensional metric.

Although Kaluza-Klein theory makes incorrect predictions about the masses of elementary particles, it is of great importance because it establishes a framework that has become one of the foundational pillars of modern physics, leading to advanced theories such as Yang-Mills theory and string theory.

In modern theory the extra dimensions of the model are described by complex manifolds; obviously, a one-dimensional sphere is not enough to encompass all the possibilities of generality. Therefore, weighted projective varieties are introduced, starting from fundamental definitions to a worked example via lemmas and theorems.

Contents

Chapter 1

Kaluza Klein Theory

5 dimensional Kaluza Klein theory (Kaluza 1921, Klein 1926) unifies electromagnetism with gravitation by starting from a theory of Einstein gravity in five dimensions. Thus, the initial theory has five-dimensional general coordinate invariance. However, it is assumed that one of the spatial dimensions compactifies so as to have the geometry of a circle S^1 of very small radius. Then, there is a residual four-dimensional general coordinate invariance.

Before addressing the gravitational case, we first discuss simpler scenarios, examining scalar and vector cases. For this chapter we use mostly what is in [BL87] [Que15]

1.1 Scalar in 5 dimensions

If we consider a massless 5D scalar $\varphi(x^M)$, $M = 1, 2, 3, 4, 5$ and put $x^5 = y$ spanning a circle of radius r with $y \equiv y + 2\pi r$. Our spacetime is now $M_4 \times S^1$. We can consider the Fourier expansion

$$\varphi(x^M) = \varphi(x^\mu, y) = \sum_{n=-\infty}^{\infty} \varphi_n(x^\mu) \exp\left(\frac{iny}{r}\right),$$

because of the periodicity in y. Notice that the Fourier coefficients are functions of the 4D coordinates and therefore are, infinitely many, 4D scalar. Consider an action of the form

$$\mathcal{S}_{5D} = \int \mathrm{d}^5 x \partial^M \varphi \partial_M \varphi,$$

from which we can derive the equation of motion, using Eulero-Lagrange equation for quantum field

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \partial_M \frac{\partial \mathcal{L}}{\partial(\partial_M \varphi)}$$

we get

$$\begin{aligned}\partial^M \partial_M \varphi = 0 &\Longrightarrow \sum_{n=-\infty}^{\infty} \left(\partial^\mu \partial_\mu - \frac{n^2}{r^2} \right) \varphi_n(x^\mu) \exp\left(\frac{iny}{r} \right) = 0 \\ &\Longrightarrow \left(\partial^\mu \partial_\mu - \frac{n^2}{r^2} \right) \varphi_n(x^\mu) = 0 \\ &\Longrightarrow \left(\partial^\mu \partial_\mu - m_n^2 \right) \varphi_n(x^\mu) = 0\end{aligned}$$

where we used the Fourier expansion. These are infinitely many Klein-Gordon equations for massive 4D fields with mass mass $m_n^2 = \frac{n^2}{r^2}$. It is easy to see why only the zero mode ($n = 0$) is massless. We can visualize the states as an infinite tower of massive states. This is called Kaluza Klein tower and the massive states are called Kaluza Klein or momentum states, since they come from the momentum in the extra dimension:

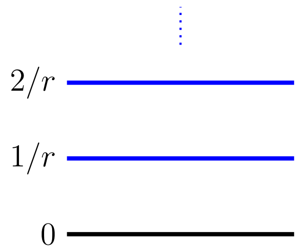

Figure 1.1: The Kaluza Klein tower of massive states. Masses $m_n = \frac{|n|}{r^2}$ grow linearly with the fifth dimension's wave number $n \in Z$.

To obtain the 4D action for all this particle we plug the Fourier expansion in the 5D action and notice we can separate the integral int the fifth dimension because no quantities depend on it

$$\begin{aligned}\mathcal{S}_{5D} &= \int \mathrm{d}^4x \int \mathrm{d}y \partial^M \varphi \partial_M \varphi = 2\pi r \int \mathrm{d}^4 x \sum_{n=-\infty}^{\infty} \left(\partial^\mu \varphi_n(x^\mu) \partial_\mu \varphi_n(x^\mu)^* - \frac{n^2}{r^2} |\varphi_n|^2 \right) \\ &= 2\pi r \int \mathrm{d}^4 x \left(\partial^\mu \varphi_n(x^\mu) \partial_\mu \varphi_n(x^\mu)^* + \ldots \right) = \mathcal{S}_{4D} + \ldots\end{aligned}$$

The 5D action reduces to one 4D action for a massless scalar field plus an infinite sum of massive scalar actions in 4D. We can focus only on the zero mode (as Kaluza did), then $\varphi_n(x^M) = \varphi_n(x^\mu)$. We speak in this case of dimensional reduction, it is equivalent to truncating all the massive fields. In this case we are only interested in energies smaller than $\frac{1}{r}$. More generally, if we keep all the massive modes we talk about compactification, we consider the extra dimension compact and we account for it including all the Fourier modes.

1.2 Vector in 5 dimensions

Let us now consider the case of an abelian vector field $A_M(x^M)$ in 5D, which is similar to the electromagnetic field in 4D. We can decompose the 5D massless vector field A_M as:

$$A_M = \begin{cases} A_\mu \quad \text{(4D vector field)} \\ A_4 =: \rho \quad \text{(4D scalar field)} \end{cases}$$

Each component of the 5D field can be expanded in a discrete Fourier series over the compactified dimension:

- For the vector field A_μ:

$$A_\mu(x^\mu, y) = \sum_{n=-\infty}^{\infty} A_\mu^n(x^\mu) \exp\left(\frac{iny}{r}\right),$$

where $A_\mu^n(x^\mu)$ are the Fourier modes, and r is the radius of the compactified dimension.

- For the scalar field ρ (from A_4):

$$\rho(x^\mu, y) = \sum_{n=-\infty}^{\infty} \rho^n(x^\mu) \exp\left(\frac{iny}{r}\right).$$

Thus, after dimensional reduction, we have an infinite tower of massive modes corresponding to the Fourier coefficients A_μ^n and ρ^n, as we have seen for the 5D scalar.

As the 5D action for an abelian gauge field we consider:

$$\mathcal{S}_{5D} = \int \mathrm{d}^5x \, \frac{1}{g_{5D}^2} F_{MN}F^{MN},$$

where $F_{MN} = \partial_M A_N - \partial_N A_M$ is the field strength tensor in 5D, and A_M is the gauge field.

The equations of motion are thus:

$$\partial_M \partial^M A_N - \partial_N(\partial_M A^M) = 0.$$

In the Lorenz gauge $\partial_M A^M = 0$, this reduces to:

$$\partial_M \partial^M A_N = 0.$$

The 5D action reduces to the 4D effective action:

$$\mathcal{S}_{5D} \to \mathcal{S}_{4D} = \int \mathrm{d}^4x \left(\frac{2\pi r}{g_{5D}^2} F^{(0)}_{\mu\nu} F^{(0)\mu\nu} + \frac{2\pi r}{g_{5D}^2} \partial_\mu \rho_0 \partial^\mu \rho_0 + \dots \right),$$

where $F^{(0)}_{\mu\nu}$ is the field strength of the 4D gauge field. Therefore we end up with a 4D theory of a gauge particle (massless), a massless scalar and infinite towers of massive vector and scalar fields.

The 4D and 5D gauge couplings (coefficients of $F_{\mu\nu}F^{\mu\nu}$ and $F_{MN}F^{MN}$) are related by:

$$\frac{1}{g_4^2} = \frac{2\pi r}{g_{5D}^2}.$$

Notice that $2\pi r$ is the volume of the conpactified extra dimension S^1.

1.3 Gravitation in 5 dimensions: Kaluza Klein theory

We can now consider the graviton of the Kaluza Klein Theory. The metric G_{MN} can be expressed as:

$$G_{MN} = \begin{cases} G_{\mu\nu} & \text{(graviton)} \\ G_{\mu 4} & \text{(vector)} \\ G_{44} & \text{(scalar)} \end{cases}$$

where $\mu, \nu = 0, 1, 2, 3$.

One possible solution is the 5D Minkowski metric: $G_{MN} = \eta_{MN} = (+, -, -, -, -)$, as we can expect another one is a 4D Minkowski spacetime M_4 times a circle S^1, it is the metric proposed by Kluza and is of the $M_4 \times S^1$ type:

$$ds^2 = W(y)\eta_{\mu\nu}dx^\mu dx^\nu - dy^2,$$

Where $W(y)$ is a warp factor that is allowed by the symmetries of the background, and y is restricted to the interval $[0, 2\pi r]$, as we have already seen for the other cases. For simplicity, we will set the warp factor to a constant.

Consider the physical excitation to the background metric:

$$G_{MN} = \phi^{-1/3} \begin{pmatrix} g_{\mu\nu} - \kappa^2 \phi A_\mu A_\nu & -\kappa\phi A_\mu \\ -\kappa\phi A_\nu & -\phi \end{pmatrix}$$

where κ is a constant to be fixed. As for the other cases we take the discrete Fourier expansion over the compactified dimension:

$$g_{\mu\nu} = \sum_{n=-\infty}^{\infty} g^n_{\mu\nu} e^{\frac{iny}{r}}, \quad A_\mu = \sum_{n=-\infty}^{\infty} A^n_\mu e^{\frac{iny}{r}}, \quad \phi = \sum_{n=-\infty}^{\infty} \phi^n e^{\frac{iny}{r}}$$

we can plug the Fourier expansion in the metric and write it as:

$$G_{MN} = \underbrace{\phi^{(0)-1/3} \begin{pmatrix} g^{(0)}_{\mu\nu} - \kappa^2 \phi^{(0)} A^{(0)}_\mu A^{(0)}_\nu & -\kappa\phi^{(0)} A^{(0)}_\mu \\ -\kappa\phi^{(0)} A^{(0)}_\nu & -\phi^{(0)} \end{pmatrix}}_{\text{Kaluza Klein ansatz}} + \text{infinite tower of massive modes.}$$

Consider a 5D Einstein-Hilbert action proportional to the simplest curvature invariant, ${}^{(5)}R$, the Ricci curvature scalar in 5D. Notice is the same form as the action for 4D general relativity.

$$\mathcal{S} = -M_*^3 \int \mathrm{d}^5 x \sqrt{|G|}\, {}^{(5)}R.$$

Where M_* is the fundamental mass scale of the high-dimensional theory and $G = \det(G_{MN}) = -\phi^{\frac{2}{3}} g$ with $g = \det(g_{\mu\nu})$ determinant of the 4D metric. Next, we plug the zero mode part into the 5D Einstein-Hilbert action reducing it to a 4D action, as we have done for the vector field:

$$\mathcal{S}_{4D} = -\int d^4x \sqrt{|g|} \left(M^2_{pl} R + \frac{1}{4} \phi^{(0)} F^{(0)}_{\mu\nu} F^{(0)\mu\nu} + \frac{M^2_{pl}}{6} \frac{\partial_\mu \phi^{(0)} \partial^\mu \phi^{(0)}}{(\phi^{(0)})^2} + \dots \right),$$

where in order to absorb the constant in the Maxwell term we have set $\kappa^{-1} = M_{pl} = \sqrt{\frac{hc}{G}}$ the 4D Planck scale, and put $M^2_{pl} = M^3_*\, 2\pi r$. Notice we can adjust M_* and r to get the right Planck mass, but nothing else constrain M_* and r to a fixed value.

We have obtained a unified theory of gravity, electromagnetism, and scalar fields!

$$\begin{aligned} \mathcal{S}_{GR} &= -\int d^4x \sqrt{|g|} M^2_{pl} R, \\ \mathcal{S}_{EM} &= -\int d^4x \sqrt{|g|} \frac{1}{4} \phi^{(0)} F^{(0)}_{\mu\nu} F^{(0)\mu\nu}, \\ \mathcal{S}_{SC} &= -\int d^4x \sqrt{|g|} \frac{M^2_{pl}}{6} \frac{\partial_\mu \phi^{(0)} \partial^\mu \phi^{(0)}}{(\phi^{(0)})^2}. \end{aligned}$$

Notice the EM Lagrangian is multiplied by the scalar field ϕ. However, if ϕ is slowly varying then we can approximate it as a constant and absorb it as a constant multiple of the entire Lagrangian. The action literally reduces to

$$\mathcal{S}_{4D} = \mathcal{S}_{EM} + \mathcal{S}_{GR}$$

1.3.1 A purely gravitational theory in 5 dimensions

It is possible to show that the results we obtained in the previous section can be derived from a purely gravitational Theory in 5D

Consider the physical excitation to the background metric:

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} + \kappa^2\phi^2 A_\mu A_\nu & \kappa\phi^2 A_\mu \\ \kappa\phi^2 A_\nu & \phi^2 \end{pmatrix}$$

Index juggling is performed with the 4D metric $g_{\mu\nu}$ and $G = \phi^2 g$.

To be able to project physics down into the base 4D space-time, we impose the cylinder condition:

$$\partial_5 G_{MN} = 0 \tag{1.3.1}$$

This means that physical quantities should not change while moving along the fifth dimension. It is possible to show that for Condition 1.3.1 rotating locally around the S^1 components only changes the gauge field A_μ, particularly the 4D space-time metric is left invariant.[Chu22]

Consider a 5D Einstein-Hilbert action like the one in the last section:

$$\mathcal{S} = M_*^3 \int \mathrm{d}^5x \sqrt{|G|}\, {}^{(5)}R.$$

The expression for ${}^{(5)}R$ follows formally from the form of the metric and the Christoffel symbols, we postulate it is in the from:

$${}^{(5)}R = R - \frac{\kappa^2}{4}\phi^2 F_{\mu\nu}F^{\mu\nu} - 2\frac{\partial_\mu\partial^\mu\phi}{\phi}$$

where R is the 4D Ricci curvature scalar.

Notice the term:

$$L_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

has the same form of the electromagnetic Lagrangian; this is a hint of the separation we have seen before. Therefore the action becomes:

$$\mathcal{S}_{5D} = M_*^3 \int \mathrm{d}^5x\, \phi\sqrt{|g|}\left(R - \frac{\kappa^2}{4}\phi^2 F_{\mu\nu}F^{\mu\nu} - 2\frac{\partial_\mu\partial^\mu\phi}{\phi}\right)$$

By the cylinder condition 1.3.1, none of these quantities depends on x^5, so we can integrate out x^5. Suppose that C is the volume of the fifth dimension or equivalently the circumference of the compactified dimension. Then the Kaluza Klein action becomes:

$$\mathcal{S}_{5D} = M_*^3 \int \mathrm{d}^4x\, C\phi\sqrt{|g|}\left(R - \frac{\kappa^2}{4}\phi^2 F_{\mu\nu}F^{\mu\nu} - 2\frac{\partial_\mu\partial^\mu\phi}{\phi}\right)$$

Since the constant κ is arbitrary, set: $\kappa^{-1} = M_{pl} = \sqrt{\frac{hc}{G}}$, at last:

$$\mathcal{S}_{5D} = \int \mathrm{d}^4x\,\sqrt{|g|}\phi\, R M_{pl}^2 - \int \mathrm{d}^4x\,\sqrt{|g|}\frac{1}{4}\phi^3 F_{\mu\nu}F^{\mu\nu} - 2M_{pl}^2 \int \mathrm{d}^4x\,\sqrt{|g|}\frac{\partial_\mu\partial^\mu\phi}{\phi}$$

which can be interpreted as:

$$\mathcal{S}_{4D} = \mathcal{S}_{GR} + \mathcal{S}_{EM} + \mathcal{S}_{SF}$$

with the same notation as before. Notice that, as before we have to play with the scalar field to get the desired action.

Chapter 2

Weighted projective space

Weighted projective spaces appear to be generalizations of the usual projective space (herein referred to as straight projective space), especially when we show that we can simply embed weighted projective space into a large enough standard projective space. But it turns out that, often, weighted projective spaces are more manageable than embedding them in standard projective space. Some projective varieties can be more easily described using weighted projective spaces.

Every algebraic geometric topic can be described in a geometric or algebraic way; in this chapter, we will focus on the geometric description of weighted projective space. The algebraic version uses some language from scheme theory that we will not discuss in this thesis. See [Rei02] and [Hos16] for more. We discuss this topic because modern theories utilize complex manifolds and particularly weighted projective space to describe the extra dimensions in their model. For example Calabi-Yau manifolds can be described with this formalism

2.1 Construction of Weighted projective space

Definizione 2.1.1. (Weighted projective space) A weighted projective space (wps) is the quotient:

$$\mathbb{P}(a_0, \dots, a_n) = (\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m^{(a)}, \tag{2.1.1}$$

where $(a_0, \dots, a_n)$ with $a_i \in \mathbb{N}$ is called a weight and $\mathbb{G}_m^{(a)}$, that is the multiplicative group of a field $(k^\times)$ with reference to a, act on $(\mathbb{A}^{n+1} \setminus 0)$ by:

$$\lambda \cdot (x_0, \dots, x_n) = (\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n) \quad \forall \lambda \in \mathbb{G}_m^{(a)}.$$

We write point in $\mathbb{P}(a_0, \ldots, a_n)$ as $|x_0 : \ldots : x_n|_a$, omitting the subscript a if it is clear that we are working in $\mathbb{P}(a) = \mathbb{P}(a_0, \ldots, a_n)$.

It is obvious that setting $a = (1, \ldots, 1)$ gives us the straight projective space $\mathbb{P}^n = \mathbb{P}(1, \ldots, 1) = (\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m^{(a)}$ where the action of a is:

$$\lambda \cdot (x_0, \ldots, x_n) = (\lambda x_0, \ldots, \lambda x_n) \quad \forall \lambda \in \mathbb{G}_m^{(a)}.$$

We denote the coordinates by $[x_0 : \ldots : x_n]$ to distinguish it from the point $|x_0 : \ldots : x_n|$

2.2 Coordinate patches

In straight projective space we can define a standard decomposition give by:

$$U_i = \{|x_0 : \ldots : x_n| \in \mathbb{P}^n \mid x_i \neq 0\},$$

$$H_i = \{|x_0 : \ldots : x_n| \in \mathbb{P}^n \mid x_i = 0\},$$

$$\mathbb{P}^n = H_i \cup U_i \cong \mathbb{P}^{n-1} \cup \mathbb{A}^n, \tag{2.2.1}$$

where $0 \leq i \leq n$ and U_i are called *affine patches* or *coordinate patches*. These patches are useful as they provide a full covering of the projective space. So given some projective variety, we can see how it intersects with the affine patches U_i and study these using all our familiarity with affine space.

It is logical to define patches in wps in the same way, but unfortunately we have a slight issue, the U_i are not isomorphic to $\mathbb{A}^n$, but instead some quotient of An by a finite group.

Definizione 2.2.1. (Quotient of affine space by a cyclic group) Define an action of μ^{a_i} (cyclic group of order n) on An, called the action of type $\frac{1}{a_i}(a_0, \ldots, \widehat{a_i}, \ldots, a_n)$ by

$$\omega_{a_i} \cdot (x_0, \ldots, \widehat{x_i}, \ldots, x_n) = (\omega_{a_i}^{a_0} x_0, \ldots, \widehat{\omega_{a_i}^{a_i} x_i}, \ldots, \omega_{a_i}^{a_n} x_n),$$

this induces an action on $k[x_0, \ldots, \widehat{x_i}, \ldots, x_n]$ given by $\omega_{a_i} \cdot x_j = \omega_{a_i}^{a_i} x_j$ and thus gives rise to the affine quotient variety

$$\mathbb{A}^n/\mu^{a_i} = \mathrm{mSpec}(k[x_0, \ldots, \widehat{x_i}, \ldots, x_n]^{a_i})$$

as well as the map $\pi_i = (\iota_i)_\# : \mathbb{A}^n \to \mathbb{A}^n/\mu^{a_i}$ corresponding to the inclusion

$$\iota_i : k[x_0, \ldots, \widehat{x_i}, \ldots, x_n]^{\mu^{a_i}} \hookrightarrow k[x_0, \ldots, \widehat{x_i}, \ldots, x_n].$$

Lemma 2.2.2. *(Affine patches in wps) with* $U_i = \{|x_0 : \ldots : x_n| \in \mathbb{P}(a_0, \ldots, a_n) \,|\, x_i \neq 0\}$ *we have*

$$U_i = \mathbb{A}^n/\mu^{a_i}$$

where we mean isomorphic in the usual sense: there exists an algebraic morphism given by a polynomial map with polynomial inverse. We often write $\mathbb{A}_i = \mathbb{A}^n/\mu^{a_i}$

Although the U_i is a nice affine space, it is a quotient one and so can be quite tricky at times. Much easier is the idea of looking at the covering space of the affine patches.

Definizione 2.2.3. (Quotient and covering affine patches) Given some subset $X \subseteq \mathbb{P}(a_0, \ldots, a_n)$ we define the following:

$$X_i = X \cap U_i \subseteq A_i$$
$$\overline{X_i} = {\pi_i}^{-1} X \cap U_i \subseteq A^n,$$

we call X_i *quotient affine patches* and $\overline{X_i}$ *covering affine patches* of X

2.3 The problem of polynomial

Before we can define the notion of weighted projective varieties, we need to define a homogeneous polynomial. We also need to discuss evaluating a polynomial at a point and seeing whether or not a point is a zero of a polynomial. It turns out that the first is not well defined while the second is (as long as our polynomial is weighted-homogeneous).

We will start by covering these topics and then proceed to define a variety in a weighted projective space.

Definizione 2.3.1. (weighted polynomial ring) Define the *polynomial ring in n+1 variables with weight* $a = (a_0, \ldots, a_n)$ as

$$k_a[x_0, \ldots, x_n] \,\text{with}\, \text{wt} x_i = a_i.$$

we think x_i a degree a_i monomial, for example,

$$\deg({x_i}^{c_i}) = a_i c_i$$

it follows

$$\deg\left(\prod_{i=0}^{n} {x_i}^{c_i}\right) = \sum_{i=0}^{n} a_i c_i$$

Notice that $\deg \lambda = 0$ for any $\lambda \in k$. For a general polynomial $f \in k_a[x_0, \ldots, x_n]$ we define the degree $\deg f$ as the maximum of all the degrees of the monomials in f.

Definizione 2.3.2. (weighted-homogeneous polynomial)
Let $f \in k[x_0, \ldots, x_n]$ where $\mathrm{wt} x_i = a_i$ for some weight a = (a0, . . . , an). We say that f is a-weighted-homogeneous of degree d if each monomial in f is of weighted degree d: $\exists c_i \in k$ and some $m \in \mathbb{N}$ such that

$$f = \sum_{j=1}^{m} c_i \left(\prod_{i=0}^{n} x_i^{d(j)} \right)$$

and far all $j \in \{0, \ldots, m\}$

$$\sum_{i=0}^{n} a_i d_i^{(j)} = d$$

We write $k_a[x_0, \ldots, x_n]_d \subset k_a[x_0, \ldots, x_n]$ as the additive group of all weighted-homogeneous polynomials of degree d.

It can now be demonstrated why evaluating a weighted-homogeneous polynomial f at a point $p \in \mathbb{P}(a)$ doesn't make sense in general. From definition 1.3.2 we can see that:

$$f(\lambda^{a_0} x_0, \ldots, \lambda^{a_n} x_n) = \lambda^d f(x_0, \ldots, x_n), \tag{2.3.1}$$

with $\lambda \in \mathbb{G}_m^{(a)}$.

Let's take $p \in \mathbb{P}(a_0, \ldots, a_n)$ then $p = |p_0 : \cdots : p_n|$ but by definition we also have $p = |\lambda^{a_0} p_0 : \cdots : \lambda^{a_n} p_n|$ for any $\lambda \in \mathbb{G}_m^{(a)}$. Assume $\lambda \neq 1$, then using Equation 2.3.1

$$f(\lambda^{a_0} x_0, \ldots, \lambda^{a_n} x_n) = f(x_0, \ldots, x_n) \quad \text{if and only if} \quad f(x_0, \ldots, x_n) = 0,$$

looking at the points $p \in \mathbb{P}(a)$ at which f vanishes does make sense. It is well defined to write that $f(p) = 0$ for some $f \in k_a[x_0, \ldots, x_n]$ and $p \in \mathbb{P}(a)$.

With weighted-homogeneous polynomial we can define

Definizione 2.3.3. (Weighted-homogeneous ideal)
We say that an ideal $I \subset k_a[x_0, \ldots, x_n]$ is a-weighted-homogeneous if it is generated by a-weighted-homogeneous elements (of not necessary the same degree).

An equivalent definition can be: I is weighted-homogeneous if and only if every element $f \in I$ can be written as

$$f = \sum_{i=0}^{\deg f} f_i,$$

for a unique choice of $fi \in K_a[x_0, \ldots, x_n] \cap I$. This definition need to be proved, a good proof can be find in [Hos16] Lemma (3.0.7).

Lemma 2.3.4. *(Prime ideal)*
A weighted-homogeneous ideal $I \subset k_a[x_0, \ldots, x_n]$ is prime if and only if, whenever $fg \in I for f, g \in k_a[x_0, \ldots, x_n]$ with f, g both homogeneous, either $f in I$ or $g \in I$. That is, when considering primality of the ideal, it is enough to check the usual definition on only the homogeneous. elements of the ideal.

2.4 Weighted projective varieties

Aided by the definitions from the previous section, we can now define a weighted projective variety in a similar manner to how we would for standard projective space. [RR88].

Definizione 2.4.1. (Weighted projective varieties and their ideals)
let $I \subset k[x_0, \ldots, x_n]$ be a weighted homogeneous ideal. Define the weighed projective variety associated to I by

$$\mathbb{V}(I) = \{p \in \mathbb{P}(a_0, \ldots, a_n) | f(p) = 0 \text{ far all } f \in I\}.$$

Let $V \subseteq \mathbb{P}(a_0, \ldots, a_n)$. Define the ideal associated to V by

$$\mathbb{I}(V) = \{f \in k_a[x_0, \ldots, x_n] | f(p) = 0 \quad \text{for all } p \in V \text{ and } f \text{ is a-weighted homogeneous}\}.$$

A subset $V \subseteq \mathbb{P}(a)$ is a weighted projective variety if it is of the form $\mathbb{V}(I)$ for $I \subset k_a[x_0, \ldots, x_n]$ some weighted homogeneous ideal.

It follows naturally that if $V \subseteq W$ V is a subvariety of W. V weighted projective varety is irreducible if it has no non-trivial decomposition into subvarietes:
$V = V_i \cup V_j$, $V_i, V_j \neq \varnothing, V$. We'll write $\mathbb{V}\mathbb{I}$ to mean $\mathbb{V} \circ \mathbb{I}$

We now list some properties of weighted-homogeneous ideal that will be needed to define a Zariski like topology on our weighted projective varieties.

Lemma 2.4.2. *Let $I, J \subset k_a[x_0, \ldots, x_n]$ be weighted-homogeneous ideals. Then*

(i) $\mathbb{V}(I) \cap \mathbb{V}(J) = \mathbb{V}(IJ)$

(ii) $\mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(I + J)$

(iii) $\varnothing = \mathbb{V}(k_a[x_0, \ldots, x_n])$

(iv) $\mathbb{P}(a_0, \ldots, a_n) = \mathbb{V}(\{0\})$.

Lemma 2.4.3. *An arbitrary sum of weighted homogeneous ideals is a weighted homogeneous ideal*

$$I = \sum_{\alpha \in \mathcal{A}} I_\alpha = \left\{ \sum_{\beta \in \mathcal{B}} f_\beta \in I_\beta \text{ and} \mathcal{B} \subset \mathcal{A} \text{is finite} \right\}.$$

Corollario 2.4.4. *An arbitrary intersection of weighted projective varieties is a weighted projective varietie:*

$$\bigcap_{I \in \mathcal{I}} \mathbb{V}(I) = \mathbb{V}\left(\sum_{I \in \mathcal{I}} I \right) = \mathbb{V}(J)$$

where $\sum_{I \in \mathcal{I}} = J \subset k_a[x_0, \ldots, x_n]$.

We can finally define a topology of varieties in a weighted projecive space

Definizione 2.4.5. (Zariski topology) The Zariski topology on $\mathbb{P}(a_0, \ldots, a_n)$ is given by defining the closed sets of $\mathbb{P}(a_0, \ldots, a_n)$ to be those of the form $\mathbb{V}(I)$ for some weighted-homogeneous ideal $I \subset k_a[x_0, \ldots, x_n]$, that is, the weighted projective varieties.

One final thing to note before moving on is how we can use the construction of weighted projective space to understand weighted projective varieties. The way that we define $f(p) = 0$ for some a-weighted-homogeneous f and point $p \in \mathbb{P}(a)$ is really by requiring that $f(\hat{p}) = 0$, where $\hat{p} \in \mathbb{A}^{n+1} \setminus \{0\}$ is a representative of p. We use the requirement of f being a-weighted-homogeneous to ensure that this definition is well-defined under a change of representatives.

So we can think of $\mathbb{V}(I)$ as a quotient of the affine ‘cone’:

$$\mathbb{V}(I) = \frac{\mathbb{V}_{\mathit{aff}}(I) \setminus \{0\}}{\mathbb{G}_m} \subseteq \frac{\mathbb{A}^{n+1} \setminus \{0\}}{\mathbb{G}_m} \tag{2.4.1}$$

where $\mathbb{V}_{\mathit{aff}}(I) = \{x \in \mathbb{A}^{n+1} \mid f(x) = 0 \text{ for all } f \in I\}$ and we consider $I \in k[x_0, \ldots, x_n]$ as an ideal in the usual polynomial ring (i.e. with all weights equal to 1)

Definizione 2.4.6. Given $X = \mathbb{V}(I)$ for some weighted-homogeneous ideal $I \in k_a[x_0, \ldots, x_n]$, we write $\hat{X}$ to mean $\mathbb{V}_{\text{aff}}(I)$, so that Eq. 2.4.1 can be written as

$$X = \frac{\hat{X} \cap (\mathbb{A}^{n+1} \setminus \{0\})}{\mathbb{G}_m}.$$

Notice that we don’t simply write $\hat{X} \setminus \{0\}$ in Definition 2.4.6 since we don’t know a priori that $0 \in \hat{X}$.

2.5 The weighted projective Nullstellensatz

Lemma 2.5.1. *Let* $\subset k_a[x_0,\dots,x_n]$ *be a weighted-homogeneous ideal and let* $V, W \subset \mathbb{P}(a_0,\dots,a_n)$. *then*

(i) $\mathbb{I}(V) \subseteq k_a[x_0,\dots,x_n]$ *is a radical weighted-homogeneous ideal*

(ii) if $I \subseteq J$ *then* $\mathbb{V}(J) \subseteq \mathbb{V}(I)$

(iii) if $V \subseteq W$ *then* $\mathbb{I}(W) \subseteq \mathbb{I}(V)$

(iv) $I \subseteq \mathbb{IV}(I)$

(v) $\mathbb{V}(I) = \mathbb{VIV}(I)$

Definizione 2.5.2. (Relevant ideals) An ideal $I \subset k_a[x_0,\dots,x_n]$ is relevant if:

(i) $I \subset (x_0,\dots,x_n)$ (irrelevant ideal)

(ii) $\mathbb{V}(I) \neq \varnothing$.

Notice that if I is weighted-homogeneous then it is always strictly contained inside the irrelevant ideal.

Lemma 2.5.3. *(Equivalent definition of relevant) Let* $I \subset k_a[x_0,\dots,x_n]$ *be a weighted homofigeneous ideal. then the following are equivalent:*

(i) I *is relevant*

(ii) I *is is strictly contained inside* $k_a[x_0,\dots,x_n]$ *and is not equal to the irrelevant ideal*

(iii) $(x_0,\dots,x_n) \nsubseteq \mathrm{rad}(I)$

Definizione 2.5.4. (Maximal weighted-homogeneous ideals) Let $I, J \subset k_a[x_0,\dots,x_n]$ ideal, I is said to be a maximal weighted-homogeneous ideal if $I \subsetneq J$, then $J = (x_0,\dots,x_n)$ (irrelevant). That is if it is relevant and maximal amongst relevant weighted-homogeneous ideals.

With all we have defined till now, we can state the weighted projective Nullstellensatz.

Teorema 2.5.5. *(Weighted projective Nullstellensatz) Let* $I \subset k_a[x_0,\dots,x_n]$ *be a weighted-homogeneous relevant ideal. Then*

$$\mathbb{IV}(I) = \mathrm{rad}(I)$$

Proof. Appendix A.1 □

Corollario 2.5.6. *(Applied weighted projective Nullstellensatz) The maps* $\mathbb{V}$ *and* $\mathbb{I}$ *give us an inclusion reversing bijection between weighted projective varieties and radical weighted-homogeneous relevant ideals:*

$$\underbrace{\begin{Bmatrix} radicalw.h.relevantideals \\ I \subset k_a[x_0,\ldots,x_n] \end{Bmatrix}}_{\mathbb{I}\mathbb{V}\ act\ as\ the\ identity} \begin{matrix} \xrightarrow{\mathbb{V}} \\ \xleftarrow{\mathbb{I}} \end{matrix} \underbrace{\begin{Bmatrix} radicalw.h.relevantideals \\ I \subset k_a[x_0,\ldots,x_n] \end{Bmatrix}}_{\mathbb{V}\mathbb{I}\ act\ as\ the\ identity}$$

$$I \subseteq J \Rightarrow \mathbb{V}(J) \subseteq \mathbb{V}(I)$$

$$\mathbb{I}(Y) \subseteq \mathbb{I}(X) \Rightarrow X \subseteq Y$$

Further, under this bijection, prime weighted-homogeneous ideals correspond to irreducible varieties, and maximal weighted-homogeneous ideals to points.

2.6 Coordinate ring

Definizione 2.6.1. (Weighted-homogeneous coordinate rings) Let $X = \mathbb{V}(I)$ be a non-empty weighted projective variety. Then define the weighted-homogeneous coordinate ring of X to be

$$S(X) = \frac{k_a[x_0,\ldots,x_n]}{\mathbb{I}(X)}.$$

If we write $\hat{X} = \mathbb{V}_aff(I)$ to mean the affine version of the correspondence $\mathbb{V}$, $\mathbb{I}_aff$ for $\mathbb{I}$ and A(Y) for the coordinate ring of an affine variety Y then

$$S(X) = \frac{k_a[x_0,\ldots,x_n]}{\mathbb{I}(X)} = \frac{k_a[x_0,\ldots,x_n]}{\mathbb{I}_aff(\hat{X})} = A(\hat{X})$$

2.7 Worked example

This section presents a simple example to understand how a weighted projective space works.

Consider $\mathbb{P}(1,1,2)$, it is defined by:

$$\mathbb{P}(1,1,2) = \mathbb{A}^3 \setminus 0)/\mathbb{G}_m^{(a)}$$

The points in $\mathbb{P}(1,1,2)$ are invariant under scaling with respect to the weighting. For example

$$|0:3:2| = 2 \cdot |0:3:2| = |0:6:8|$$

To understand the space as a whole, we need to define a map

$$\varphi : [x_0 : x_1 : x_2] \mapsto [x_0^2 : x_0x_1 : x_1^2 : x_2].$$

We claim that this map has its image in $\mathbb{P}^3$. Firstly at least one of the monomials will be non-zero, then we need to check if the image is invariant under scaling, we find out it is:

$$\lambda \cdot [x_0^2 : x_0x_1 : x_1^2 : x_2] = [\lambda x_0^2 : \lambda x_0x_1 : \lambda x_1^2 : \lambda x_2] = [x_0^2 : x_0x_1 : x_1^2 : x_2].$$

We have not defined an isomorphism between weighted projective space, but if we can find an inverse map that is a polynomial in each coordinate (even though we have not defined an isomorphsm for weighted projective space) we can think of $\mathbb{P}(1,1,2)$ as isomorphic to the image of ϕ in $\mathbb{P}^3$. We see ϕ as an embedding of $\mathbb{P}(1,1,2)$ in $\mathbb{P}^3$. To construct our inverse map we take some point $[y_0, y_1, y_2, y_3]$ in the image. Unfortunately, even though k is algebraically closed, we cannot take $|y_0^{\frac{1}{2}} : y_1^{\frac{1}{2}} : y3|$ as our inverse map, since this is not a polynomial in each coordinate. However, we notice that

$$y_0 = x_0^2, \quad y_1 = x_0x_1, \quad y_2 = x_1^2, \quad y_3 = x_2$$

for some $|x_0 : x_1 : x_2| \in \mathbb{P}(1,1,2)$, and so

$$|x_0 : x_1 : x_2| = x_0 \cdot |x_0 : x_1 : x_2| = |x_0^2 : x_0x_1 : x_0^2x_2| = |y_0 : y_1 : y_0y_3|,$$

where $y_0, y_1, y_3 \neq 0$ and

$$|x_1 : x_0 : x_2| = x_1 \cdot |x_0 : x_1 : x_2| = |x_0x_1 : x_1^2 : x_2| = |y_1 : y_2 : y_2y_3|,$$

elsewhere.

Thus, we identify two mutually inverse polynomial maps, which we, for now, consider as an isomorphism:

$$\begin{aligned} \varphi : \quad & \mathbb{P}(1,1,2) \to X \subset \mathbb{P}^3 \\ & |x_0 : x_1 : x_2| \mapsto |x_0^2 : x_0x_1 : x_1^2 : x_2| \end{aligned}$$

$$\begin{aligned} \varphi^{-1} : \quad & X \to \mathbb{P}(1,1,2) \\ |y_0 : y_1 : y_2 : y_3| \mapsto & \begin{cases} |y_0 : y_1 : y_0y_3| & \text{if } y_0, y_1, y_3 \neq 0; \\ |y_1 : y_2 : y_2y_3| & \text{otherwise.} \end{cases} \end{aligned}$$

understanding $\mathbb{P}(1,1,2)$ becomes a matter of understanding the set $X \subset \mathbb{P}^3$.

Another approach can be looking for the affine patches, $\mathbb{P}(1,1,2)$ can be covered by 3 patches. We notice that in U_0

$$|x_0, x_1, x_2| = \left|1, \frac{x_1}{x_0}, \frac{x_2}{x_0^2}\right| \quad \text{because}\, x_0 \neq 0$$

that is isomorphic to $\mathbb{A}^2$, the same can be done for $U_1 = \{x_1 \neq 0\}$ giving us the 2 affine patches:

$$U_0 \cong \mathbb{A}^2 \quad \text{with coordinates} \quad \left(\frac{x_1}{x_0}, \frac{x_2}{x_0^2}\right)$$
$$U_1 \cong \mathbb{A}^2 \quad \text{with coordinates} \quad \left(\frac{x_0}{x_1}, \frac{x_2}{x_1^2}\right)$$

for the last patches we need to consider $U_2 = \mathbb{A}^2/\mu^2$ that we write as the action $\frac{1}{2}(1,1)$ on $\mathbb{A}^2$, to have a better understanding of U_2 let's take into account it's embedding φ in $\mathbb{P}^3$, we have

$$|x_0 : x_1 : x_2| \mapsto |x_0^2 : x_0x_1 : x_1^2 : x_2|$$

and because $\mathbb{P}^3$ is a straight projective space in $X \subset \mathbb{P}^3$ it is true that

$$|x_0^2 : x_0x_1 : x_1^2 : x_2| = \left|\frac{x_0^2}{x_2} : \frac{x_0x_1}{x_2} : \frac{x_1^2}{x_2} : 1\right|$$

but that is isomorphic to $Y \subset \mathbb{A}^3$ with coordinates

$$\left(\frac{x_0^2}{x_2}, \frac{x_0x_1}{x_2}, \frac{x_1^2}{x_2}\right)$$

notice that they are not independent if we write (u, v, w) as coordinates in Y it is obvious that $uw = v^2$ so $Y = V(uw = v^2)$ and

$$U_2 \cong V(uw = v^2)$$

Conclusion

In this thesis, we have explored the fundamental principles of 5-dimensional Kaluza-Klein theory, which elegantly unifies electromagnetism and gravitation. By compactifying one of the spatial dimensions to a circle S^1, we demonstrated how the theory retains four-dimensional general coordinate invariance, paving the way for a deeper understanding of fundamental forces in a higher-dimensional context.

Furthermore, we extended our discussion to the mathematical structures that underpin modern version of these theories, particularly focusing on weighted projective spaces and varieties. This exploration highlighted the importance of algebraic geometry in providing a rigorous framework for studying the implications of Kaluza-Klein theory.

The insights gained from analyzing weighted projective spaces not only enhance our comprehension of geometric constructs in theoretical physics but also offer potential pathways for further research. This thesis bridges high-dimensional theories and algebraic geometry, laying the groundwork for exploring theories related to the unification of fundamental interactions.

This work highlights the importance of Kaluza-Klein theory as a pioneering attempt in the quest for a unified framework in modern physics and suggests that further exploration in both geometric and physical realms may reveal deeper connections between these two fields.

Appendix A

Proof

A.1 Nullstellensatz proof

The proof of the weighted projective Nullstellensatz is a technical one, we use the approach explained in [Hos16]

Write $X = \mathbb{V}(I)$, then we have to prove that

$$\mathbb{IV}(I) = \mathbb{I}(X) = \mathbb{I}_{aff}(\hat{X}) = rad(I)$$

Since $I_{aff}(\hat{X})$ is all about affine quantities (remember $\hat{X} = \mathbb{V}_{\mathit{aff}}(I) = \{x \in \mathbb{A}^{n+1} \mid f(x) = 0 \text{ for all } f \in I\}$) we know from the affine Nullstellensatz that the last equality is true. We work now to show the second one.

$$\mathbb{I}(X) = \mathbb{I}_{aff}(\hat{X})$$

Since I is relevant, $X = \mathbb{V}(I) \neq \emptyset$. We also know that $X = \mathbb{VI}(X)$, thus $\mathbb{VI}(X) \neq \emptyset$. Hence $\mathbb{I}(X)$ is also relevant. So there are no constant polynomials in $\mathbb{I}(X)$, otherwise $\mathbb{I}(X)$ would be the whole of $k_a[x_0, \ldots, x_n]$. Hence if $f \in \mathbb{I}(X)$, then $f(0) = 0$, because $f(0)$ is a polynomial with no constant term.

Also, if $f \in \mathbb{I}(X)$, then $f(x) = 0$ for all $x \in X$, i.e. $f(\hat{x}) = 0$ for all representatives $\hat{x} \in \hat{X} \setminus \{0\}$, as already stated we don't know a priori if $0 \in \hat{X}$. So $f \in \mathbb{I}_{\mathit{aff}}(\hat{X} \setminus \{0\})$. But since $f(0) = 0$ as well, $f \in \mathbb{I}_{\text{aff}}(\hat{X})$, hence

$$\mathbb{I}(X) \subseteq \mathbb{I}_{\text{aff}}(\hat{X}).$$

For the other inclusion let begin by proving that $\mathbb{I}_{\mathit{aff}}(\hat{X})$ is weighted-homogeneous.

To do so we will show that if $I \subset k_a[x_0, \dots, x_n]$ is a weighted-homogeneous ideal, then $rad(I) \subset k_a[x_0, \dots, x_n]$ is also a weighted-homogeneous ideal, then well use the affine Nullstellensatz once again (remember that $\mathbb{I}_{aff}(\hat{X})$ are all affine quantities) to go back.

To do so let $g \in rad(I)$, so that $g^k \in I$ for some $k \in \mathbb{N}$. Write $d = \deg g$ and let

$$g_i = g \cap k_a[x_0, \dots, x_n]_i \quad \textit{for} \quad 0 \le i \le d$$

notice that for $i > d$ this intersection will be empty. Furthermore by Definition 2.3.3 g_i are uniquely determined by g and so it is enough to show that $g_i \in rad(I)$ for all $0 \le i \le d$, to prove that g is a weighted-homogeneous polynomial.

We look first at g_d. Because $g_d^k = g^k \cap k_a[x_0, \dots, x_n]_d$ (since it is the only term of high enough degree) and I is weighted-homogeneous, we must have that $g_d^k \in I$, and so $g_d \in rad(I)$. But then $(g - g_d) \in rad(I)$ is a polynomial of strictly smaller degree with homogeneous components $g_0, \dots, g_{d-1}$, thus $(g - g_d)^{k'} \in I$ for some $k' \in \mathbb{N}$, so we repeat the above process with g_{d-1} to show that $g_{d-1}^{k'} \in I$, and thus $g_{d-1} \in rad(I)$. After repeating this finitely many times (since the total degree strictly decreases each time), we have that $g_i \in rad(I)$ for all $0 \le i \le d$. That prove $rad(I)$ is weighted-homogeneous.

Now we just need to remember the affine Nullstellensatz, since we have called$\hat{X} = \mathbb{V}_{aff}(I)$:

$$\mathbb{I}_{aff}(\hat{X}) = \mathbb{I}_{aff}\mathbb{V}_{aff}(I) = rad(I),$$

is weighted-homogeneous.

Now let f be a generator of $\mathbb{I}_{aff}(\hat{X})$, f is weighted-homogeneous (as we have just demonstrated). Also $f(\hat{x}) = 0$ for all $\hat{x} \in \hat{X}$, and so in particular $f(\hat{x}) = 0$ for all $\hat{x} \in \hat{X} \setminus \{0\}$.

Combining these two facts, we see that $f \in \mathbb{I}(X)$. Then, since all the generators of $I_{\text{aff}}(\hat{X})$ are in $I(X)$, we have

$$I_{\text{aff}}(\hat{X}) \subseteq I(X).$$

This concludes our proof.

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