Alma Mater Studiorum - Università di Bologna

Department of Physics and Astronomy
Bachelor in Astronomy

GRAVITATIONAL ENERGY IN ASTROPHYSICS

Bachelor's Thesis

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Abstract

Gravitational potential energy is the energy an object possesses due to its position in a gravitational field. In astrophysics, this energy plays a fundamental role in various processes, from star formation to regulating the maximum luminosity an accreting object, such as a black hole, can emit before radiation pressure counteracts gravity and slows further accretion. It also governs the dynamics of stars within galaxies. In this thesis, we explore the key aspects of gravitational energy in astrophysical phenomena. **Chapter 1** focuses on the theoretical foundation, covering Newton's laws of motion and the law of universal gravitation. **Chapter 2** applies these principles to astrophysical contexts, as discussed earlier.

Sommario

L'energia potenziale gravitazionale è l'energia che un oggetto possiede a causa della sua posizione in un campo gravitazionale. In astrofisica, questa energia gioca un ruolo fondamentale in vari processi, dalla formazione stellare alla regolazione della massima luminosità che un oggetto in accrescimento, come un buco nero, può emettere prima che la pressione di radiazione contrasti la gravità e rallenti ulteriormente l'accrescimento. Governa anche la dinamica delle stelle all'interno delle galassie. In questa tesi, esploreremo i principali aspetti dell'energia gravitazionale nei fenomeni astrofisici. **Il Capitolo 1** si concentra sulle fondamenta teoriche, trattando le leggi del moto di Newton e la legge di gravitazione universale. **Il Capitolo 2** applica questi principi a contesti astrofisici, come discusso in precedenza.

Contents

1 Theoretical background

1.1 Historical background on gravity

Isaac Newton (1642–1727), arguably the greatest of any scientific mind in history, was born on Christmas Day in the year of Galileo's death. At age 18, Newton enrolled at Cambridge University and subsequently obtained his bachelor's degree. In the two years following the completion of his formal studies, and while living at home in Woolsthorpe, in rural England, away from the immediate dangers of the Plague, Newton engaged in what was likely the most productive period of scientific work ever carried out by one individual. During that interval, he made significant discoveries and theoretical advances in understanding motion, astronomy, optics, and mathematics. Although his work was not published immediately, the *Philosophiae Naturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy), now simply known as the *Principia* , finally appeared in 1687 and contained much of his work on mechanics, gravitation, and the calculus. The publication of the *Principia* came about largely as a result of the urging of Edmond Halley, who paid for its printing. Another book, Optiks , appeared separately in 1704 and contained Newton's ideas about the nature of light and some of his early experiments in optics. Although many of his ideas concerning the particle nature of light were later shown to be in error, much of Newton's other work is still used extensively today.
Today, classical mechanics is described by Newton's three laws of motion, along with his **universal law of gravity** . Outside of the realms of atomic dimensions, velocities approaching the speed of light, or extreme gravitational forces, Newtonian physics has proved very successful in explaining the results of observations and experiments.
We will introduce the Newtonian mechanics starting by the *Newton's laws of motion* and then pass to the *Newtonian gravity.*

1.2 Newton's Laws of motion

Newton's laws of motion, in the rather obscure language of *Principia*, takes the following form:

1. *The law of inertia.* An object at rest will remain at rest and an object in motion will remain in motion in a straight line at a constant speed unless acted upon by an external force. To establish whether an object is actually moving, a reference frame must be established. Non inertial reference frames are accelerated with respect to inertial frames. The first law may be restated in terms of the momentum of an object, $\mathbf{p} = m\mathbf{v}$, where m and $\mathbf{v}$ are mass and velocity, respectively. Thus Newton's first law may be expressed as *"the momentum of an object remains constant unless it experiences an external force".*

   The second law is actually a definition of the concept of force:

2. The *net* force (the sum of all forces) acting on an object is proportional to the object's mass and its resultant acceleration. If an object is experiencing n

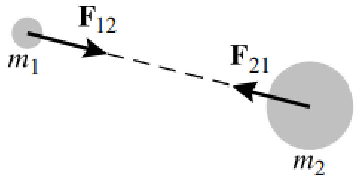


Figure 1: Newton's third law.

forces, then the net force is given by

$$\mathbf{F_{net}} = \sum_{i=1}^{n} \mathbf{F_i} = m\mathbf{a}. \tag{1.1}$$

However, assuming that the mass is constant and using the definition $\mathbf{a} \equiv \frac{d\mathbf{v}}{dt}$, Newton's second law may be expressed as

$$\mathbf{F_{net}} = m\frac{d\mathbf{v}}{dt} = \frac{d(m\mathbf{v})}{dt} = \frac{d\mathbf{p}}{dt}; \tag{1.2}$$

the net force on an object is equal to the time rate of change of its momentum, $\mathbf{p}$. $\mathbf{F_{net}} = \frac{d\mathbf{p}}{dt}$ actually represents the most general statement of the second law, allowing for a time variation in the mass of the object.

3. For every action there is an equal and opposite reaction. In this law, *action* and *reaction* are to be interpreted as forces acting on different objects. Consider the force exerted *on* one object (object 1) *by* a second object (object 2), $\mathbf{F_{12}}$. Newton's third law states that the force exerted on object 2 by object 1, $\mathbf{F_{21}}$, must necessarily be of the same magnitude but in opposite direction (see Fig. 1). Mathematically the third law can be represented as

$$\mathbf{F_{12}} = -\mathbf{F_{21}} \tag{1.3}$$

1.3 Newtonian gravity

Classical gravity, which is invariably the dominant force in celestial dynamic systems, was first correctly described in Newton's *Principia*. According to Newton, any two point objects exert a **gravitational force** of attraction to each other. This force is directed along the straight line joining the two objects, is directly proportional to the product of their masses, and is inversely proportional to the square of the distance between them. Consider two point objects of mass m_1 and m_2 that are located at position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, respectively. The gravitational force $\boldsymbol{f}_{12}$ that mass m_2 exerts on mass m_1 is written

$$\boldsymbol{f}_{12} = G m_1 m_2 \frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3} \tag{1.4}$$

The gravitational force $\boldsymbol{f}_{21}$ that mass m_1 exerts on mass m_2 is equal and opposite: $\boldsymbol{f}_{21}$ = -$\boldsymbol{f}_{12}$. Here the constant of proportionality G, is called the *universal gravitational constant* and takes the value:

$$G = 6.673 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}.$$

1.4 Potential theory

Much of the mass of a galaxy resides in stars. Considering that a typical galaxy contains $\approx 10^{11}$ stars, the task of compute the gravitational potential of such a large quantity of stars is not practicable if we simply add the point-mass potentials of all the stars together. For most purposes it is sufficient to model the potential by smoothing the mass density in stars on a scale that is small compared to the size of the galaxy, but large compared to the mean distance between stars.
Gravitational potential energy or just **gravitational energy** can be derived from Newton's laws and its universal law of gravity. Our goal is to calculate the force $\mathbf{F(x)}$ on a particle of mass m_s at a position $\mathbf{x}$ that is generated by the gravitational attraction of a distribution of mass $\rho(\mathbf{x}')$. Considering the small mass contributions caused by the small volumes $\delta^3\mathbf{x}'$ in **x'**:

$$\delta\mathbf{F}(\mathbf{x}) = Gm_s\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\,\delta m(\mathbf{x}') = Gm_s\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\,\rho(\mathbf{x}')d^3\mathbf{x}' \tag{1.5}$$

to the overall force from each small element of volume $d^3\mathbf{x}'$ located at $\mathbf{x}'$.
Thus

$$\mathbf{F}(\mathbf{x}) = m_s\mathbf{g}(\mathbf{x}) \quad \text{where} \quad \mathbf{g}(\mathbf{x}) \equiv G\int \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\,\rho(\mathbf{x}')d^3\mathbf{x}' \tag{1.6}$$

is the **gravitational field**, the force per unit mass.
If we define the **gravitational potential** $\boldsymbol{\Phi}(\mathbf{x})$ by

$$\boldsymbol{\Phi}(\mathbf{x}) \equiv -G\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}'-\mathbf{x}|}\,d^3\mathbf{x}' \tag{1.7}$$

And notice that:

$$\boldsymbol{\nabla}_{\mathbf{x}}\left(\frac{1}{|\mathbf{x}'-\mathbf{x}|}\right) = \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3} \tag{1.8}$$

We can write $\mathbf{g}$ as:

$$\mathbf{g}(\mathbf{x}) = \boldsymbol{\nabla}_{\mathbf{x}}\int \frac{G\rho(\mathbf{x}')}{|\mathbf{x}'-\mathbf{x}|}\,d^3\mathbf{x}' = -\boldsymbol{\nabla}\boldsymbol{\Phi} \tag{1.9}$$

The potential is useful because is a scalar field that is easier to visualize than the vector gravitational field but contains the same information.
If we take the divergence of (1.6), we find

$$\boldsymbol{\nabla}\cdot\mathbf{g}(\mathbf{x}) = G\int \nabla_{\mathbf{x}}\left(\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\right)\rho(\mathbf{x}')d^3\mathbf{x}' \tag{1.10}$$

We have that

$$\nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) = 0 \quad \text{for} \quad \mathbf{x} \neq \mathbf{x}' \tag{1.11}$$

Therefore, any contribution to the integral of equation (1.7) must come from the point **x'** $= \mathbf{x}$ and we may restrict the volume of integration to a small sphere of radius h centered on this point. Since for sufficiently small h, the density will be almost constant through this volume, we can take out $\rho(\mathbf{x})$ out of the integral. The remaining terms of the integrand, after a few steps and applying the divergence theorem[1] can be arranged as follows:

$$\boldsymbol{\nabla} \cdot \mathbf{g}(\mathbf{x}) = -G\rho(\mathbf{x}) \int_{|\mathbf{x}'-\mathbf{x}|=h} \frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^3} \cdot d^2\mathbf{S}' \tag{1.12}$$

Now on the sphere $|\mathbf{x}' - \mathbf{x}| = h$ we have $d^2\mathbf{S}' = (\mathbf{x}' - \mathbf{x})hd^2\Omega$, where $d^2\Omega$ is a small element of solid angle. Hence equation (1.9) becomes

$$\nabla \cdot \mathbf{g}(\mathbf{x}) = -G\rho(x) \int d^2\Omega = -4\pi G\rho(x). \tag{1.13}$$

If we substitute equation (1.6) for $\nabla \cdot \mathbf{g}$, we obtain **Poisson's equation** relating the potential Φ to the density ρ;

$$\nabla^2\boldsymbol{\Phi} = 4\pi G\rho. \tag{1.14}$$

Since $\mathbf{g}$ is determined from the gradient of a potential, the gravitational field is conservative, that means, the work done against gravitational forces in moving two stars from infinity to a given configuration is independent of the path along which they are moved, and is defined to be the potential energy of the configuration. Similarly, the work done against gravitational forces in assembling an arbitrary continuous distribution of mass $\rho(x)$ in independent of the details of how the mass distribution was assembled, and is defined to be equal to the **potential energy** of the mass distribution. An expression for the potential energy can be obtained by the following argument. Suppose that some of the mass is already in place so that the density and potential are $\rho(x)$ and $\Phi(x)$. If we now bring in an additional small mass δm from infinity to position $\mathbf{x}$, the work done is $\delta m\Phi(x)$. Thus, if we add a small increment of density $\delta\rho(x)$, the change in potential energy is

$$\delta W = \int \delta\rho(x)\boldsymbol{\Phi}(\mathbf{x})d^3\mathbf{x} \tag{1.15}$$

According to Poisson's equation the resulting change in potential $\delta\boldsymbol{\Phi}$ satisfies $\nabla^2(\delta\boldsymbol{\Phi}) = 4\pi G(\delta\rho)$, so

$$\delta W = \frac{1}{4\pi G} \int \boldsymbol{\Phi}\nabla^2(\delta\rho)d^3\mathbf{x}. \tag{1.16}$$

Using the divergence theorem, we may write this as

$$\delta W = \frac{1}{4\pi G} \int \boldsymbol{\Phi}\nabla(\delta\boldsymbol{\Phi}) \cdot d^2\mathbf{S} - \frac{1}{4\pi G} \int \nabla\boldsymbol{\Phi} \cdot \nabla(\delta\boldsymbol{\Phi})d^3\mathbf{x}, \tag{1.17}$$

[1]The divergence theorem states that $\int_V \nabla \cdot \mathbf{F}\, dV = \oint_{\partial V} \mathbf{F} \cdot d\mathbf{S}$

where the surface integral vanishes because $\mathbf{\Phi} \propto r^{-1}$, $|\nabla\delta\mathbf{\Phi}| \propto r^{-2}$ as $r \to \infty$, so the integrand $\propto r^{-3}$ while the total surface area $\propto r^2$. But $\nabla\mathbf{\Phi}\cdot\nabla(\delta\mathbf{\Phi}) = \frac{1}{2}\delta(\nabla\mathbf{\Phi}\cdot\nabla\mathbf{\Phi}) = \frac{1}{2}\delta|(\nabla\mathbf{\Phi})|^2$. Hence

$$\delta W = -\frac{1}{8\pi G}\delta\left(\int |\nabla\mathbf{\Phi}|^2 d^3\mathbf{x}\right). \tag{1.18}$$

If we now sum up all of the contributions δW, we have a simple expression for the potential energy,

$$W = -\frac{1}{8\pi G}\int |\nabla\mathbf{\Phi}|^2. \tag{1.19}$$

To obtain an alternative expression for W, we again apply the divergence theorem and replace $\nabla^2\Phi$ by $4\pi G\rho$ to obtain

$$W = \frac{1}{2}\int \rho(x)\mathbf{\Phi}(\mathbf{x})d^3\mathbf{x}. \tag{1.20}$$

1.5 Spherical systems

Newton's theorems

Newton proved two results that enable us to calculate the gravitational potential of any spherically symmetric distribution of matter easily:

- **Newton's first theorem** *A body that is inside a spherical shell of matter experiences no net gravitational force from that shell.*
- **Newton's second theorem** *The gravitational force on a body that lies outside a spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its center.*

An important corollary of Newton's first theorem is that the gravitational potential inside an empty spherical shell is constant because $\nabla\mathbf{\Phi} = -\mathbf{g} = 0$. Thus we may evaluate the potential $\mathbf{\Phi}(\mathbf{r})$ inside the shell by calculating the integral expression (1.4) for $\mathbf{r}$ located at any interior point. The most convenient place for $\mathbf{r}$ is the center of the shell, for then all points on the shell are at the same distance R, and one immediately has

$$\Phi = -\frac{GM}{R} \tag{1.21}$$

From Newton's first and second theorems, it follows that the gravitational attraction of a spherical density distribution $\rho(r')$ on a unit mass at radius r is entirely determined by the mass interior to r:

$$\mathbf{F}(\mathbf{r}) = -\frac{GM(r)}{r^2}\hat{\mathbf{e}}_\mathbf{r}, \tag{1.22}$$

where

$$M(r) = 4\pi\int_0^r \rho(r')r'^2 dr'. \tag{1.23}$$

An important property of a spherical matter distribution is its **circular speed** $v_c(r)$, defined to be the speed of a particle of negligible mass (a **test particle**) in a circular orbit radius r. We may readily evaluate v_c by equating the gravitational attraction $|\mathbf{F}|$ from equation (1.19) to the centripetal acceleration v_c^2/r:

$$v_c^2 = r|\mathbf{F}| = r\frac{d\Phi}{dr} = \frac{GM(r)}{r}. \tag{1.24}$$

Another important quantity is the **escape speed** v_e defined by[2]

$$v_e(r) \equiv \sqrt{2|\Phi(r)|}. \tag{1.25}$$

A star at r can escape from the gravitational field represented by Φ only if it has a speed at least great as $v_e(r)$, for only then does its (positive) kinetic energy $\frac{1}{2}v^2$ exceed the absolute value of its (negative) potential energy Φ. The escape at r depends on the mass both inside and outside r.

1.6 Virial theorem for a self-gravitating system

What is a self-gravitating system? is a system in which each part of the system moves under the influence of the gravitational field generated by all the other parts of the system. If the system consist in more that two objects the equations of motion cannot in general be solved analytically. Given some initial values the, the orbits[3] can, of course be found by numerical integration, but this does not tell us anything about the general properties of all possible orbits. The only integration constant available for an arbitrary system are the total momentum, angular momentum and energy. In addition to these, it is possible to derive certain statistical results like the *virial theorem*[4]. It concerns time averages only, but does not say anything about the actual state of the system at some specified moment.
Suppose we have a system of n points masses m_i with a radius vector $\mathbf{r_i}$ and velocities $\dot{\mathbf{r}}_\mathbf{i}$. We define a quantity A (the "virial" of the system) as follows:

$$A = \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_\mathbf{i} \cdot \mathbf{r_i}. \tag{1.26}$$

The time derivative of this is

$$\dot{A} = \sum_{i=1}^{n} (m_i \dot{\mathbf{r}}_\mathbf{i} \cdot \dot{\mathbf{r}}_\mathbf{i} + m_i \ddot{\mathbf{r}}_\mathbf{i} \cdot \mathbf{r_i}). \tag{1.27}$$

The first term equals twice the kinetic energy of the ith particle, and the second term contains a factor $m_i \ddot{\mathbf{r}}_\mathbf{i}$ which according to Newton's laws, equals the force applied to the ith particle. Thus we have

$$\dot{A} = 2K + \sum_{i=1}^{n} \mathbf{F_i} \cdot \mathbf{r_i}, \tag{1.28}$$

[2]This result is correct only if the potential $\Phi(r) \to 0$ as $r \to \infty$ we have assumed this so far for systems with very extended mass distributions other zero points may be necessary

[3]Due to the limited length of this work we could not introduce the orbits inside a system

[4]Proof of the virial theorem taken from the text book "Fundamental Astronomy"

where K is the total kinetic energy of the system. If $\langle x \rangle$ denotes the time average x in the time interval $[0, \tau]$, we have

$$\langle \dot{A} \rangle = \frac{1}{\tau} \int_0^\tau \dot{A} dt = \langle 2K \rangle + \left\langle \sum_{i=1}^{n} \mathbf{F_i} \cdot \mathbf{r_i} \right\rangle. \tag{1.29}$$

If the system remains bound i.e. none of the particles escapes, all $\mathbf{r_i}$'s as well as all velocities will remain bound. In such case, A does not grow without limit, and the integral of the previous equation remains finite. When the time interval becomes longer ($\tau \to \infty$), $\langle \dot{A} \rangle$ approaches zero, and we get

$$\langle 2K \rangle + \left\langle \sum_{i=1}^{n} \mathbf{F_i} \cdot \mathbf{r_i} \right\rangle = 0. \tag{1.30}$$

This is the general form of the *virial theorem.* If the forces are due to mutual gravitation only, they have the expressions

$$\mathbf{F_i} = -Gm_i \sum_{i=1, j\neq i}^{n} m_j \frac{\mathbf{r_i} - \mathbf{r_j}}{r_{ij}^3}, \tag{1.31}$$

where $r_{ij} = |\mathbf{r_i} - \mathbf{r_j}|$. The latter term in the virial theorem is now

$$\sum_{i=1}^{n} \mathbf{F_i} \cdot \mathbf{r_i} = -G \sum_{i=1}^{n} \sum_{i=1, j\neq i}^{n} m_i m_j \frac{\mathbf{r_i} - \mathbf{r_j}}{r_{ij}^3} \cdot \mathbf{r_i} = -G \sum_{i=1}^{n} \sum_{j=i+1}^{n} m_i m_j \frac{\mathbf{r_i} - \mathbf{r_j}}{r_{ij}^3} \cdot (\mathbf{r_i} - \mathbf{r_j}), \tag{1.32}$$

where the latter form is obtained by rearranging the double sum, combining the terms

$$m_i m_j \frac{\mathbf{r_i} - \mathbf{r_j}}{r_{ij}^3} \cdot \mathbf{r_i} \tag{1.33}$$

and

$$m_j m_i \frac{\mathbf{r_j} - \mathbf{r_i}}{r_{ji}^3} \cdot \mathbf{r_j} = m_i m_j \frac{\mathbf{r_i} - \mathbf{r_j}}{r_{ij}^3} \cdot (-\mathbf{r_j}). \tag{1.34}$$

Since $(\mathbf{r_i} - \mathbf{r_j}) \cdot (\mathbf{r_i} - \mathbf{r_j}) = r_{ij}^2$ the sum reduces

$$-G \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{m_i m_j}{r_{ij}} = W \tag{1.35}$$

Where W is the **potential energy** of the system. Thus the virial theorem becomes simply[5]

$$\boxed{\langle K \rangle = -\frac{1}{2} \langle W \rangle} \tag{1.36}$$

The following image illustrates how the virial theorem in its simplest form allows us to derive fundamental physical properties of self-gravitating stellar systems, such as their negative specific heat and their slow collapse as a consequence of "gravitational evaporation" of stars, finally leading to the so-called *gravothermal catastrophe*, one

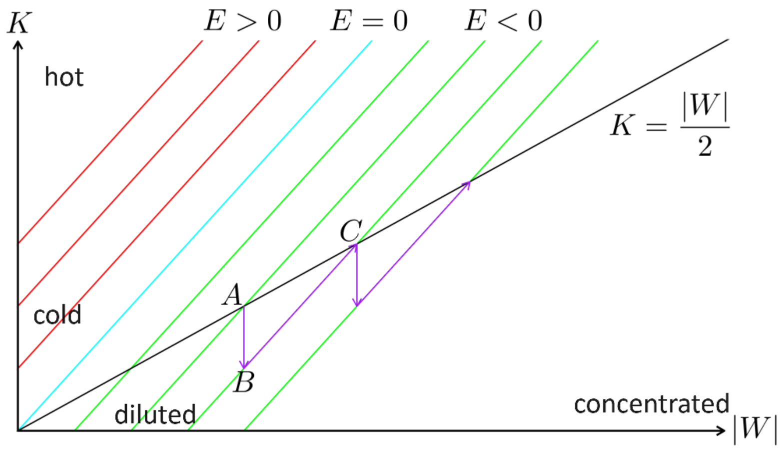


Figure 2: Virial plane for self-gravitating systems. The straight line VT, in black, represents, the equilibrium configuration of the system. The parallel lines in colors represents the conservation of the total energy. We can see from the plot that only $E \leqslant 0$ can be at equilibrium, because they are the only ones that intercept the line VT. This graphic is a personal adaptation by professor Luca Ciotti (University of Bologna), plotted by Claudia Crastolla (2022)

of the most important concepts for understanding the dynamical evolution of globular clusters.
No virialized states exist for positive values of E. Self-gravitating systems are virialized if and only if its representative point is placed on the *virial theorem line* $K = \frac{|W|}{2}$ like in Figure 2. On the same plane its drawn a family of straight lines representing energy conservation $K = |W| + E$. If a system evolves at a constant E, then its representative point can only move on the line fixed by the initial conditions. Obviously, the line $K = |W|$ represents systems of zero total energy, parallel line above represent unbound systems, while parallel lines below represent systems with total negative energy. The first important consequence that can be obtained by inspection of Figure 2 is that only systems with total negative energy have a chance of being virialized, as only the constant energy lines of this family have intersections with the VT line. In practice, suppose we have a stellar system with given initial conditions, so that we can determine the initial position of its representative point in the virial plane. If the point is on the VT line, then the system is in equilibrium and, if the equilibrium is stable, the point will remain there forever. If the total energy is positive and energy losses are forbidden, the point will necessarily move on its energy conservation line in the equation $K = |W| + E$ without having the possibility of virializing, as no intersections can occur with the VT line. Finally, if the total initial energy is negative, one possibility –compatible with energy conservation – is that the point moves and finally stops (after the phase of violent relaxation) on the VT line at the position consistent with its negative total energy. In Figure 2 we can see the violet path indicates a displacement of a auto-gravitating system through different states of equilibrium. We can see that arising the temperature (decrease in gravitational energy), the total energy of the system decreases, for this reason

[5] The scalar virial theorem

self-gravitating systems are to be said that have **negative specific heat capacity**.

2 Gravitational energy in astrophysics and its applications

2.1 Globular clusters and gravothermal catastrophe

A globular cluster is a system of stars gravitationally bound. As we know, in a globular cluster there is a phenomena called *gravitational evaporation* in which the lighter stars that have $v > v_e$ "evaporate" from the system, this phenomena comes from a series of interactions at high impact parameter with other stars in which for every collision the star gains a little quantity of kinetic energy T until it overcomes the energy of the gravitational potential Φ. In this case, the stars that escape through gravitational evaporation have a total energy ($E = T - \Phi$) that is very close to zero ($T \sim \Phi$), when this happens, the systems starts losing stars, it contracts a little bit and stars move faster. When the density of the central regions is ~ 700 times higher than the average density, the central region starts to separate from the outer regions. If we look at the graphic in Figure 2 and suppose we start with a virialized system at point A and assume that a fraction of its kinetic energy is lost because for some reason is ejected carrying away its kinetic energy (such in case of gravitational evaporation) so that T is reduced and $|W|$ remains almost unaffected. Graphically this is represented by the arrow from point A to point B, in practice we cooled the system at a fixed W. The system is now out of the equilibrium and its point cannot stay at point B if the system is virialized the only place possible in the virial plane is at point C corresponding to a more concentrated configuration and a higher "temperature". In other words we can say that the system cooled, but after virialization the self-gravity finally heated it to a greater temperature than the initial one. What we have just descrived is the the physical basis of the phenomenon of *gravothermal catastrophe* occurring in globular clusters.The progressive contraction of their inner regions associated with the continuous ejection of low-mass stars produce the tendency toward equipartition[6].

2.2 The formation of protostars and Jeans criterion

We will briefly study the birth of stars. The initial state is, roughly speaking, a gas cloud that begins to collapse due to its own gravity. If the mass of the cloud is high enough, its **potential energy** exceeds the kinetic energy and the cloud collapses. From the *virial theorem* we can deduce that the potential energy must be at least twice the kinetic energy. This provides a criterion for the critical mass necessary for the cloud to collapse. This criterion was first suggested by *Sir James Jeans* in 1902.

[6]Means that each component of the system have the same kinetic energy and different potential energy

Jeans Criterion

Despite many successes, important questions remain concerning how stars change during their lifetimes. One area where the picture is far from complete is in the earliest stage of evolution, the formation of prenuclear-burning objects known as protostars from interstellar molecular clouds. If globules and cores in molecular clouds are the sites of star formation, what conditions must exist for collapse to occur? Sir James Jeans (1877–1946) first investigated this problem in 1902 by considering the effects of small deviations from hydrostatic equilibrium. Although several simplifying assumptions are made in the analysis, such as neglecting effects due to rotation, turbulence, and galactic magnetic fields, it provides important insights into the development of protostars. The virial theorem,

$$2K + W = 0, \tag{2.1}$$

describes the condition of equilibrium for a stable, gravitationally bound system. It can be seen that the virial theorem arises naturally in the discussion of orbital motion, and can be invoked to estimate the amount of gravitational energy contained within a star. The virial theorem may also be used to estimate the conditions necessary for protostar collapse. If twice the total internal kinetic energy of a molecular cloud ($2K$) exceeds the absolute value of the gravitational potential energy ($|W|$), the force due to the gas pressure will dominate the force of gravity and the cloud will expand. On the other hand, if the internal kinetic energy is too low, the cloud will collapse. The boundary between these two cases describes the critical condition for stability when rotation, turbulence, and magnetic fields are neglected. Assuming a spherical cloud of constant density, the **gravitational potential energy** is approximately

$$W \sim -\frac{3}{5}\frac{GM_c^2}{R_c^2} \tag{2.2}$$

Where M_c and R_c are the mass and the radius of the cloud respectively. We may also estimate the cloud's internal kinetic energy, given by

$$K = \frac{3}{2}NkT \tag{2.3}$$

where N is the number of particles. But N is just

$$N = \frac{M_c}{\mu m_H} \tag{2.4}$$

where μ is the mean molecular weight. Now, by the virial theorem, the condition for collapse ($2K < |W|$) becomes

$$\frac{3M_c kT}{\mu m_H} < \frac{3}{5}\frac{GM_c^2}{R_c} \tag{2.5}$$

The radius may be replaced by using the initial mass density of the cloud, ρ_0, assumed here to be constant throughout the cloud

$$R_c = \left(\frac{3M_c}{4\pi\rho_0}\right)^{1/3} \tag{2.6}$$

After substitution into Eq.(2.5), we may solve for the minimum mass necessary to initiate the spontaneous collapse of the cloud. This condition is know as the **Jeans criterion**:

$$M_c > M_j \tag{2.7}$$

where

$$\boxed{M_j \backsimeq \left(\frac{5kT}{G\mu m_H}\right)^{3/2} \left(\frac{3}{4\pi\rho_0}\right)^{1/2}} \tag{2.8}$$

is called the **Jeans mass**. A cloud will collapse only if its mass is larger than the Jeans mass.

2.3 Accretion as a source of energy

For the nineteenth century physicists, gravity was the only conceivable source of energy in celestial bodies, but gravity was inadequate to power the Sun for its known lifetime. In contrast, at the beginning of the twenty-first century it is to gravity that we look to power the most luminous objects in the Universe, for which the nuclear sources of the stars are wholly inadequate. The extraction of gravitational potential energy from material which accretes on to a gravitating body is now known to be the principal source of power in several types of close binary systems, and is widely believed to provide the power supply in active galactic nuclei and quasars. This increasing recognition of the importance of accretion has accompanied the dramatic expansion of observational techniques in astronomy, in particular the exploitation of the full range of the electromagnetic spectrum from the radio to X-rays and γ-rays. At the same time, the existence of compact objects has been placed beyond doubt by the discovery of the pulsars, and black holes have been given a sound theoretical status. Thus, the new role for gravity arises because accretion on to compact objects is a natural and powerful mechanism for producing high-energy radiation.

Black holes and Eddington luminosity

Black holes (BH) represent the ultimate degree of compactness to which a stellar configuration can evolve. Already at the end of the 18th century Laplace showed that a sufficiently massive body would prevent the escape of light from its surface. According to classical mechanics, the escape velocity from a body of radius R and mass M is

$$v_e = \sqrt{\frac{2GM}{R}} \tag{2.9}$$

This is greater than the speed of light, if the radius is smaller than the critical radius

$$R_S{}^7 = \frac{2GM}{c^2} \tag{2.10}$$

The properties of black holes have to be studied on the basis of the general theory of relativity, which is beyond the scope of this work. A black hole is black because light undergoes a gravitational redshift. The photon, being redshifted infinitely has energy and $\lambda \to 0$.

[7] Schwarzschild radius

Eddington luminosity

The Eddington luminosity is the maximum luminosity that a star, or in this case, a black hole, can achieve where the outward force of radiation pressure is balanced by the inward force of gravity. This concept is named after the British astrophysicist Sir Arthur Eddington. We can make some assumptions to make the calculation of this luminosity:

- Spherically symmetric accretion
- Optically thin regime
- $L_{BH}(\nu) = L_{BH} \cdot f(\nu)$[8]

Let's consider the flux, i.e the number of ergs per unit frequency passing through the surface per second.

$$S(\nu) = \frac{L_{BH}(\nu)}{4\pi r^2} \tag{2.11}$$

And consider the number of photons:

$$\frac{L_{BH}(\nu)}{4\pi r^2 \cdot h\nu} \tag{2.12}$$

We know that the energy of a photon is $E_{\text{photon}} = pc$, where $p = \frac{h\nu}{c}$. The momentum carried by photons emitted from the central regions, per unit area per second, is proportional to the energy flux and can be expressed as the following radiation pressure:

$$P_{\text{photon}} = \frac{L_{\text{BH}}(\nu)}{4\pi r^2 c} \tag{2.13}$$

where $L_{\text{BH}}(\nu)$ is the luminosity of the black hole at frequency ν, and r is the radial distance from the source.
A "wind" of electromagnetic impulse. But we also have matter that is falling into the BH, let's suppose is pure ionized hydrogen, so heavy protons and light electrons. What does this EM wind do? it transfers a bit of impulse to protons and electrons. The impulse hits the electrons only,[9] we have for the cross section of the electron:

$$\sigma_T = 6.65 \cdot 10^{-25} cm^2 \tag{2.14}$$

Let's suppose that our EM "wind" is going to hit the electrons, how much of impulse is transmitted? Our gas has an electron number n_e, i.e the number of electrons per unit volume at a distance r from the center. The answer to the question is

$$\frac{L_{BH}(\nu) n_e \sigma_T}{4\pi r^2 c} \tag{2.15}$$

And the total impulse per unit time is given from the sum of all impulses transferred at all frequencies

$$\frac{L_{BH} f(\nu) n_e(r) \sigma_T}{4\pi r^2 c} \tag{2.16}$$

[8]Where $\int_0^\infty L_{BH}(\nu) d\nu = L_{BH}$ and $\int_0^\infty f(\nu) d\nu = 1$ it's the SED

[9]because $\sigma_T \propto m^{-2}$ and we know that m_p is ~ 2000 times m_e

that integrated at all ν gives us

$$F_{rad} = \frac{L_{BH} n_e(r) \sigma_T C_k}{4\pi r^2 c} \tag{2.17}$$

This equation gives us the quantity of impulse per unit time that is deposited in the element of volume. The C_K factor is the cross section of Klein-Nishina[10], that is when the cross section depends on ν. Eddington's idea was that when the force with which the wind presses equals the force of gravity, maybe the matter stops falling. We have a deceleration, exceeding Eddington luminosity does not mean that accretion stops, the effect is not instantaneous.
The force of gravity with which the BH pulls is

$$F_{grav} = \frac{-GM_{BH}(n_e m_e + n_p m_p)}{r^2} \tag{2.18}$$

gravity pulls electrons and protons, instead the pressure is applied on electrons only. Having neutral H we have $n_e \sim n_p$. But as $m_p \sim 2000$ times m_e we will consider gravity that the volume undergoes is applied only to n_p. Finally we have the total force:

$$F_{tot} = \frac{1}{r^2}\left(-GM_{BH} n_e m_p + \frac{L_{BH} n_e \sigma_T}{4\pi c}\right) \tag{2.19}$$

We rewrite the F_{tot} as

$$F_{tot} = \frac{-GM_{BH} n_e(r) m_p}{r^2}\left[1 - \frac{L_{BH} n_e \sigma_T}{4\pi r^2 c} \frac{r^2}{GM_{BH} n_e m_p}\right] \tag{2.20}$$

We are in optically thin regime. Now, we take the bracket and we put it equal to zero, in doing so we have the critic luminosity, so when [...] is zero we have

$$\boxed{L_{edd} = \frac{4\pi c G m_p M_{BH}}{\sigma_T}} \tag{2.21}$$

Exceeding the Eddington luminosity means that the gravity field is inverted at every point. Being super-Eddington for some time **is not** forbidden[11], is deceleration that commands so the velocity slows down until is inverted outwards. We can see that $L_{edd} \propto M_{BH}$, so a massive black hole can accrete large amounts of matter and emit significant radiation until it exhausts its surrounding material.

2.4 Rotation curve in spiral galaxies and dark matter

Rotation curve of a galaxy is a property of the gravitational field. If we have a system with cylindrical symmetry we will have a rotation curve. Let's consider a

[10] If we consider QED we discover that the cross section is not independent of ν, so if we have high energy photons we have C_K, in this case we consider $C_K \sim 1$

[11] Is forbidden being *always* super-Eddington

spiral galaxy in equilibrium and a test mass located at a point r from its center. We remember equation for circular velocity

$$v_c = \sqrt{\frac{GM}{r}} = \sqrt{|\Phi|} \tag{2.22}$$

We see that this equation depends on the potential generated from the mass distribution, the rotation curve vary depending on the specific area we are analyzing. If we consider a spiral galaxy, we have different regimes to consider with their own circular velocities generated:

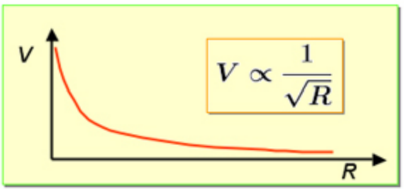
- Point-like mass M: the simplest case we have the potential:

$$\Phi(r) = \frac{GM}{r} \tag{2.23}$$

  and its angular velocity:

$$v_c = \sqrt{\frac{GM}{r}} \tag{2.24}$$

  We can see that $v_c \propto \frac{1}{\sqrt{r}}$ and this kind of rotation curve is said to be **Keplerian**.



Keplerian Rotation

Figure 3

- Homogeneous sphere with constant density $\rho(r)$, let's consider Newton's theorems seen in Chapter 1 and we imagine the sphere made of shells, the gravitational field from all the shells is given by:

$$\mathbf{g} = -\frac{GM(r)}{r^2} \quad , \quad M(r) = 4\pi \int_0^r t^2 \rho(t) dt \tag{2.25}$$

  We have that the rotation curve of such a distribution is:

$$v = \sqrt{\frac{GM(r)}{r}} \tag{2.26}$$

  We can see that is obviously **not** Keplerian. Because it does not only depends from the distance from the center but also from how the mass distribution is spread.

We have that in a spiral galaxy there is the *bulge* that we can approximate to a homogeneous sphere as in equation (2.24). For the thin disk, things get complicated because the rotation curve for a disk mass distribution is not easy to analyze because inside a disk, the gravitational field at a distance r from the center not only depends from the mass that is *inside* r, but also from the mass that is *outside* r. We can consider a surface density $\sigma(r)$ that has a constant rotation curve. For distances far away from the central regions we can consider the mass distribution as a point-like mass and the curve is Keplerian. But, the rotation curves observed does not present the Keplerian behavior, instead they present a flat pattern as seen in Figure 4 represented by the green line. Most of the stars in a typical spiral galaxy are in the

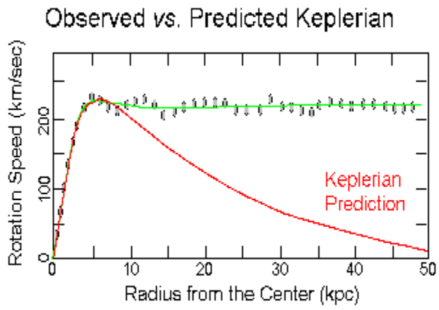


Figure 4: Credits: Prof. Richard Pogge from Ohio State University

inner 10 kpc or so. If stars provided all of its mass we would expect the following:

- The rotation speed should rise to a maximum speed in the inner parts.
- The rotation speed would then fall steadily with radius outside a radius of $\sim$ 10 kpc (i.e., Keplerian), since most of the mass in stars is inside of our orbit beyond that distance.

But, the observations show that the rotation speed stays roughly constant (a "flat rotation curve") at large radii. Higher speeds at large radii mean you need to have more mass at large radii than is observed in the stars and gas alone.
Astronomers using doppler effect manage to measure the rotation speed to observe the rotation curve of the galaxy and verify if that matches the empiric one. In particular it was Vera Rubin that studied the empiric rotation curves and observed that they raise and then they remain flat. This gives us a hint of the presence of some kind of matter that we are unable to observe, in fact Vera Rubin hypothesizes that the galaxy is surrounded by a halo of dark matter (this theory was previously hypothesized) with a density profile of $\sim R^{-2}$. This model however cannot work properly because if we consider the disk of a spiral galaxy viewed face-on (let's suppose with no spiral arms) the brightness profile is described by an **exponential** and its rotation curve rises and then **remains flat**. In the case of a disk mass distribution, it is not possible to apply Newton's theorems. We can observe however inside the disk there is no need of dark matter because it follows the theoretical rotation curve (flat). This occurs until $\sim 90\%$ of the disk, after which the theoretical rotation curve becomes Keplerian because when we are far away from the object we can consider the gravitational field as it was produced from a point-like object. Observing galaxies in HI, the part of the stars is microscopic with respect to the rest of the galaxy, indeed there is a big amount of neutral hydrogen. Radio astronomers discovered that the rotation curve of the clouds of HI **remains flat**, and therefore there must be dark matter.

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