Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE

Corso di Laurea in Matematica

Coxeter systems and the combinatorial invariance conjecture

Tesi di Laurea in Algebra

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Introduzione

I gruppi di Coxeter vennero definiti da H. S. Coxeter in un articolo pubblicato nel 1936 (si veda [2]) come gruppi astratti che generalizzano i gruppi di riflessioni finiti. Come questi ultimi, essi ammettono una presentazione in termini di riflessioni, ma non tutti i gruppi di Coxeter sono finiti e non sempre possono essere descritti in termini di riflessioni in uno spazio euclideo. I gruppi di Coxeter finiti vennero classificati, sempre da H. S. Coxeter (si veda [3]), nel 1935 e risultano essere tutti e soli i gruppi di riflessioni finiti.

La definizione di questi gruppi è sufficientemente generale da incapsulare molti gruppi che si incontrano naturalmente nello studio di altre strutture matematiche. Ad esempio, i gruppi di simmetria di politopi regolari sono gruppi di Coxeter, lo sono anche i gruppi di Weyl delle algebre di Lie semplici, oltreché i gruppi di Weyl delle algebre di Kac-Moody, che tipicamente hanno dimensione infinita. Dedicheremo il primo capitolo di questa tesi allo studio delle proprietà principali dei gruppi di Coxeter, come la Proprietà di Scambio Forte, e lo concluderemo dimostrando la classifiazione dei gruppi di Coxeter finiti. Di particolare importanza per i capitoli successivi sarà l'introduzione di un ordinamento parziale su questi gruppi detto *ordinamento di Bruhat*.

Nel secondo capitolo, parleremo dell'algebra di Hecke di un gruppo di Coxeter, che è costruita a partire dallo $\mathbb{Z}[q, q^{-1}]$-modulo libero su un gruppo di Coxeter, su cui successivamente si definisce un prodotto che, in qualche modo, rispetta la struttura interna del gruppo. A partire da questa algebra, in un lavoro pubbicato nel 1979 (si veda [12]), David Kazhdan e George Lusztig definirono due classi di polinomi indicizzate da coppie di elementi del gruppo di Coxeter, gli *R-polinomi* e i *polinomi di Kazhdan-Lusztig*, questi ultimi vennero usati per costruire rappresentazioni dell'algebra di Hecke di un gruppo di Coxeter. I polinomi di Kazhdan-Lusztig hanno successivamente trovato numerose applicazioni inaspettate in vari ambiti della matematica, come la teoria delle rappresentazioni dei gruppi algebrici semisemplici, la teoria dei moduli di Verma, e la geometria delle varietà di Schubert. Ad esempio, proprio in [12], Kazhdan e Lusztig congetturarono che i valori dei polinomi di Kazhdan-Lusztig in 1 fossero connessi con delle quantità importanti della teoria dei moduli di Verma. Queste congetture furono dimostrate in maniera indipendente da A. Beilinson e J. Bernstein in [1] e da J. L. Brylinski e M. Kashiwara in [9] nel 1981. Inoltre, sempre grazie a un lavoro di Kazhdan e Lusztig (si veda [13]), si trovarono delle interpretazioni dei coefficienti dei polinomi di Kazhdan-Lusztig in termini di invarianti topologici delle varietà di Schubert, degli oggetti geometrici indicizzati da elementi del gruppo simmetrico (che è un gruppo di riflessioni finito, e quindi un gruppo di Coxeter).

Tutti questi collegamenti con altre parti della matematica rendono il calcolo di questi polinomi un problema di grande interesse. Purtroppo, il calcolo di quest'ultimi risulta essere complesso. Una delle più importanti congetture aperte su questi oggetti, la cui risoluzione semplificherebbe notevolmente il problema del loro calcolo, è la *congettura di invarianza combinatorica*, che afferma che questi polinomi dipendono solo dalla struttura di insieme parzialmente ordinato del gruppo. Più precisamente, la congettura asserisce che se u, v sono due elementi di un gruppo di Coxeter ordinato con l'ordinamento di Bruhat, il polinomio di Kazhdan-Lusztig $P_{u,v}$ dipende solo dalla struttura di poset dell'intervallo $[u, v]$. La congettura rimane aperta in generale, ma è stata dimostrata in alcuni casi particolari. Ad esempio, in [8] F. Brenti ha dimostrato la congettura per gli intervalli detti *short edge* e F. Brenti, F. Caselli e M. Marietti hanno dimsotrato la congettura per *lower intervals* in [7], cioè intervalli della forma $[1, w]$. Nel terzo capitolo di questa tesi, forniremo una dimostrazione della congettura di invarianza combinatorica per lower intervals seguendo il lavoro di Brenti, Caselli e Marietti appena citato.

Contents

Chapter 1

Coxeter groups

In the present chapter, we will introduce the main object of study: Coxeter groups. The principal motivations for the definition of this special class of groups are finite reflection groups and so called affine Weyl groups, which contain affine reflections as well as reflections that fix the origin of some Euclidean space. After having dealt with the basic properties and definitions, we will study the central results about such groups: the *Strong Exchange Condition* and the *Deletion Condition*. Moreover, we will introduce and study a special partial order on Coxeter groups called the *Bruhat order*, this is what makes Coxeter systems interesting combinatorical objects. At the end on the chapter, we will also state the classification of finite Coxeter groups. For the most part, we will follow [11] and [5].

1.1 Coxeter groups and Coxeter systems

Definition. A *Coxeter system* is a pair (W, S) where W is a group and $S \subseteq W$ is a generating subset of W. To be a Coxeter system, we require that W has a presentation of the following form:

$$W = \langle S \mid (ss')^{m(s,s')} = 1 \; \forall s, s' \in S \text{ such that } m(s,s') \neq \infty \rangle.$$

The numbers $m(s, s')$ are positive integers or infinity, respecting the following conditions:

$$\begin{aligned} &\forall s \in S \qquad m(s,s) = 1 \\ &\forall s \neq s' \qquad m(s,s') = m(s',s) \geqslant 2 \end{aligned}$$

If no relation occurs between two generators, we set $m(s, s') = \infty$ by convention. We define the *rank* of the Coxeter system (W, S) to be $|S|$. From now on we will assume $|S| < +\infty$. When the presentation is clear, we will simply refer to W as a Coxeter group.

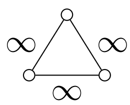
Remark 1.1.1. Formally, W is constructed by taking the free group over S, which we will denote F_S. Then we let N be the normal subgroup generated by all the elements $(ss')^{m(s,s')}$, and take W to be the quotient F_S/N.

Remark 1.1.2. It is not immediately clear from the definition that all the elements of S are of order two, nor is it clear that the integers $m(s,s')$ are precisely the order of the elements ss'. This will turn out ot be the case.

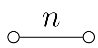
To define a Coxeter system it is sufficient to specify the generating set S, and to give a symmetric matrix M whose entries are parametrized by pairs of elements of S and take values in $\mathbb{Z}\cup\{\infty\}$. These will need to respect the relations of the first definition. Another way to represent in a compact way all the relations between generators is to draw a graph with labelled vertices. The set S will be the vertex set and we will join two vertices s and s' when $m(s,s') \geqslant 2$. The labels on the edges will just be the integers $m(s,s')$. In order to prevent clutter on these graphs, the label is omitted when $m(s,s') = 3$. We call this a *Coxeter graph*.

Example 1. The group $\mathbb{Z}_2^n$ is a Coxeter group. Just take $S := \{1,\ldots,n\}$ and the graph with n isolated vertices as Coxeter graph.

Example 2. The *Universal Coxeter group* of order n is the free group on n generators. In the case $n = 3$ it has Coxeter graph

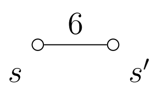


Example 3. The dihedral group D_n is a Coxeter group with Coxeter graph

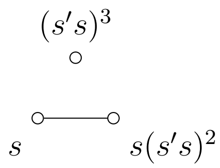


Example 4. The symmetric group $\mathfrak{S}_n$ is generated by the transpositions $(1,2),\ldots,(n-1,n)$. The product $(i,i+1)(j,j+1)$ has order 2 if $|i-j| \neq 1$, and 3 when $|i-j| = 1$. It turns out that by choosing $S = \{(1,2),\ldots,(n-1,n)\}$, $(\mathfrak{S}_n, S)$ is a Coxeter system whose Coxeter graph is:

Remark 1.1.3. It is important to notice that two different Coxeter systems can give rise to the same Coxeter group. For example the group D_6 is the Coxeter group of the Coxeter system with graph



But we can present D_6 in a different way. If S is the generating set of the previous presentation, the set $S' := \{s, (s's)^2, s(s's)^2\}$ also generates D_6 and leads to a Coxeter system with Coxeter graph



Proposition 1.1.1. There is a unique surjective homomorphism $\varepsilon : W \to \{-1, 1\}$ sending each element of S to -1. As a consequence every element in S has order 2.

Proof. The propositions follows by observing that there is an homomorphism

$$\begin{aligned} F_S &\to \{-1, 1\} \\ s &\mapsto -1 \end{aligned}$$

and that all elements of the form $(ss')^{m(s,s')}$ are in the kernel. Therefore this map factors to a morphism $W \to \{-1, 1\}$ sending each generator to -1. $\square$

Definition. A Coxeter system (W, S) is said to be *irreducible* if its Coxeter graph is connected. A Coxeter system which is not irreducible is said to be *reducible*

Since every element in S has order 2, we can write each $w \in W$ as $w = s_1 s_2 \dots s_r$ for some $s_i \in S$.

Definition. Let (W, S) be a Coxeter system, we define the *lenght function* $\ell : W \to \mathbb{N}$ as $\ell(w) := \min\{r \in \mathbb{N} \ : \ \exists s_1, \dots, s_r \in S \ s.t. \ w = s_1 s_2 \dots s_r\}$. By convention $\ell(1) = 0$.

Here are some of the main properties of the length function:

(L1) $\ell(w) = \ell(w^{-1})$;

(L2) $\ell(w) = 1 \iff w \in S$;

(L3) $\ell(ww') \leqslant \ell(w) + \ell(w')$;

(L4) $\ell(ww') \geqslant \ell(w) - \ell(w')$;

(L5) If $s \in S$ and $w \in W$ we have $\ell(w) - 1 \leqslant \ell(ws) \leqslant \ell(w) + 1$.

Proposition 1.1.2. The homomorphism ε defined in Proposition 1.1.1 can be expressed as $\varepsilon(w) = (-1)^{\ell(w)}$. As a consequence of this and property (L5), we have that for all $s \in S$ and all $w \in W$ it holds that $\ell(ws) = \ell(w) \pm 1$.

Proof. Let w be an element of the group W. By definition of the length function, w can be written in the form $w = s_1 s_2 \dots s_{\ell(w)}$ with each $s_i \in S$, since ε sends each element in S to -1, the first part of the proposition is clear. The fact that $\varepsilon(sw) = -\varepsilon(w)$ implies, using the first part of the proposition, that $\ell(ws) \neq \ell(w)$. This and the property (L5) prove the second part of the proposition. □

1.2 The geometric representation of a Coxeter group

The length function introduced in the previous section is of vital importance because many of the following results will be proved by induction on $\ell(w)$. Because of this, we need to study in more depth the precise relation between $\ell(sw)$ and $\ell(w)$. In order to do this we need to construct a representation of W. In most of the examples given, the group W admits a representation as a group of orthogonal reflections in some Euclidean space V. Unfortunately, this cannot be done in general. What can be done in the general case is to allow the bilinear form on the vector space to be degenerate, and replace orthogonal reflections with endomorphisms which fix an hyperplane pointwise and send a vector to its negative. Let V be a vector space with a basis $\{\alpha_s \mid s \in S\}$ in bijection with S. Generalizing what is seen in the case of dihedral groups, we define a symmetric bilinear form B on V by

$$B(\alpha_s, \alpha_{s'}) = -\cos\left(\frac{\pi}{m(s, s')}\right)$$

If $m(s, s') = \infty$, this is interpreted to be -1.

Remark 1.2.1. Observe that each element α_s is non-isotropic. Therefore, if H_s is the orthogonal space of $\mathbb{R}\alpha_s$, we have $V = \mathbb{R}\alpha_s \oplus H_s$.

Proposition 1.2.1. There is a unique homomorphism $\sigma : W \to \mathrm{End}(V)$ sending $s \in S$ to

$$\sigma_s(v) = v - 2B(\alpha_s, v)\alpha_s.$$

This is called the *geometric representation of* W. Moreover, the form B is $\sigma(W)$-invariant.

Proof. Firstly, we check that each σ_s preserves the form B. Let $s \in S$ and $v, w \in V$, then:

$$\begin{aligned} B(\sigma_s(v), \sigma_s(w)) &= B(v - 2B(v, \alpha_s)\alpha_s, w - 2B(w; \alpha_s)\alpha_s) \\ &= B(v, w) - 2B(v, \alpha_s)B(w, \alpha_s) - 2B(v, \alpha_s)(B(\alpha_s, w) - 2B(w, \alpha_s)) \\ &= B(v, w) \cancel{-4B(v, \alpha_s)B(w, \alpha_s)} \cancel{+4B(v, \alpha_s)B(w, \alpha_s)} \\ &= B(v, w). \end{aligned}$$

and we are done.

To show the existence and uniqueness of σ we need to prove the following statement:

$$\forall s, s' \in S \ \ (\sigma_s \sigma_{s'})^{m(s,s')} = \mathrm{id}_V.$$

We first fix $s, s' \in S$, set $m := m(s, s')$ and define $V_{s,s'} := \mathbb{R}\alpha_s \oplus \mathbb{R}\alpha_{s'}$. We observe that $V_{s,s'}$ is left stable by σ_s and $\sigma_{s'}$, therefore their restrictions are endomorphisms of $V_{s,s'}$ that, with a little abuse of notation, we will denote again by σ_s and $\sigma_{s'}$. Furthermore, we observe that the restriction of B to $V_{s,s'}$ is positive semidefinite, and it is degenerate precisely when $m = \infty$, this is made clear by the following calculation: let $v = a\alpha_s + b\alpha_{s'}$, with $a, b \in \mathbb{R}$, then

$$B(v, v) = a^2 - 2\,ab\cos(\pi/m) + b^2 = (a - b\cos(\pi/m))^2 + b^2 \sin^2(\pi/m) \geqslant 0.$$

Moreover this quantity is always strictly positive if $m < \infty$. We distinguish two cases:

1. $m < \infty$. From what we said above, the form B restricted to $V_{s,s'}$ is positive definite. Therefore V splits as the direct sum of $V_{s,s'}$ and its orthogonal complement, which is fixed by σ_s and $\sigma_{s'}$ pointwise. Thus, we only need to calculate the order of $\sigma_s\sigma_{s'}$ on $V_{s,s'}$. By choosing the basis $\{\alpha_s, \alpha_{s'}\}$ we obtain an isomorphism with the Euclidean plane $(\mathbb{R}^2, \langle\cdot,\cdot\rangle_{\mathrm{st}})$. The lines spanned by α_s and $\alpha_{s'}$ form an angle of π/m, therefore we observe that $\sigma_s\sigma_{s'}$ is just a rotation of $2\pi/m$, hence it has order m.

2. $m = \infty$. We have $B(\alpha_s, \alpha_{s'}) = -1$. We define $v^* = \alpha_s + \alpha_{s'}$, this vector is fixed by both σ_s and $\sigma_{s'}$. Moreover, $\sigma_s\sigma_{s'}(\alpha_s) = 3\alpha_s + 2\alpha_{s'} = 2v^* + \alpha_s$. Inductively we get $(\sigma_s\sigma_{s'})^k = 2k\,v^* + \alpha_s$, which is never equal to α_s. Therefore $\sigma_s\sigma_{s'}$ has infinite order on $V_{s,s'}$ and therefore as a map on V.

This concludes the proof by defining the map first on the free group on S, and then letting the map descend to the quotient. □

Corollary 1.2.1. For all $s, s' \in S$, $m(s, s')$ is precisely the order of ss'.

Proof. In the proof we have showed that the order of $\sigma_s\sigma_{s'}$ is equal to $m(s, s')$, thus the order of ss' cannot be smaller than $m(s, s')$. □

The geometric representation of a Coxeter group turns out to be always faithful, but this will be proved only in the next section and requires more work.

1.3 Root systems

In this section we obtain the main result needed to better understand the behaviour of the length function. In order to do this, we need to study how W acts on V via the geometric representation.

Definition. The *root system* Φ of (W, S) is the collection of vectors

$$\Phi = \{\, w(\alpha_s) \mid w \in W,\ s \in S \,\}.$$

Its elements are called *roots*.

Remark 1.3.1. All roots are unit vectors since the geometric action of W preserves the form B. Furthermore, if α is a root so is $-\alpha$, since $s(\alpha_s) = -\alpha_s$.

Let α be a root, then we can write

$$\alpha = \sum_{s \in S} c_s \alpha_s.$$

We call α *positive* and write $\alpha > 0$ if for all $s \in S$ we have $c_s \geqslant 0$, while we call the root *negative* if for all $s \in S$ we have $c_s \leqslant 0$. We call Φ^+ the set of all positive roots and Φ^- the set of all negative roots.
In order to prove the next theorem, we need to introduce a special class of subgroups of W.

Definition. Let (W, S) be a Coxeter system. Let $I \subseteq S$, then we define W_I as the subgroup of W generated by the elements of I. A subgroup obtained in this way is called a *parabolic subgroup*.

Remark 1.3.2. Let (W, S) be a reducible Coxeter system, let $\Gamma_1, \dots \Gamma_n$ be the connencted components of the Coxeter graph of (W, S). Then if I_i is the subset of S consisting of the vertices of Γ_i, W splits in the direct sum $W = W_{I_1} \times \cdots \times W_{I_n}$.

Remark 1.3.3. Every $w \in W_I$ can be written as product of elements in I, therefore we can define a length function ℓ_I that tells us the minimum length of any such expression. Of course, it is true that if $w \in W_I$, then $\ell(w) \leqslant \ell_I(w)$. It will be proved in the next section that these two functions take the same values.

The following theorem is a key element in the proofs of much of what follows in the next sections. It gives us a way to use the geometric representation to study the length function, and is also essential in the proof of the main properties of parabolic subgroups, stated in Theorem 1.5.1.

Theorem 1.3.1. Let $w \in W$ and $s \in S$. If $\ell(ws) > \ell(w)$ then $w(\alpha_s) > 0$. Similarly if $\ell(ws) < \ell(w)$ then $w(\alpha_s) < 0$.

Proof. First of all, we observe that the second statement follows from the first one. If $\ell(ws) < \ell(w)$, then $\ell((ws)s) = \ell(w) > \ell(ws)$. By the first statement $ws(\alpha_s) = -w(\alpha_s) > 0$, so $w(\alpha_s) < 0$. We prove the first statement by induction on $\ell(w)$. If $\ell(w) = 0$, then $w = 1$ and the claim is clear. If $\ell(w) > 0$, let $w = s_1 s_2 \dots s_r$ be a reduced word and let us denote $s' := s_r$, then s' is such that $\ell(ws') < \ell(w)$. This shows that $s \neq s'$, therefore we can define $I := \{s, s'\}$ so that W_I is dihedral (finite or infinite). We can now define the following set

$$A := \{v \in W \mid v^{-1}w \in W_I \text{ and } \ell(v) + \ell_I(v^{-1}w) = \ell(w)\}.$$

This set is non-empty since $w \in A$. We take $v \in A$ with minumum length among the elements of A and denote $v_I := v^{-1}w$. By definition $w = vv_I$ and $\ell(w) = \ell(v) + \ell_I(v_I)$. The strategy is to use the induction hypothesis on the pair v, s, and then use the relation $w = vv_I$ to prove the theorem for w. In order to do so, we first need to show the following inequalities:

$$\text{(a)}\ \ \ell(v) < \ell(w), \qquad \text{(b)}\ \ \ell(vs) > \ell(v).$$

We recall that s' is such that $\ell(ws') = \ell(w) - 1$, therefore $(ws')^{-1}w = s' \in W_I$ and $\ell(w) = \ell(ws') + 1 = \ell(ws') + \ell(s')$, this shows that $ws' \in A$. Because of how we chose v, it holds that $\ell(v) \leqslant \ell(ws') = \ell(w) - 1$ and we have showed (a). The proof of (b) is by contradiction, let us suppose that $\ell(vs) < \ell(v)$, then we would have

$$\begin{aligned}
\ell(w) &\underset{(L3)}{\leqslant} \ell(vs) + \ell((sv^{-1})w) \\
&\leqslant \ell(vs) + \ell_I(sv^{-1}w) && [\text{using } sv^{-1}w \in W_I \text{ and } \ell \leqslant \ell_I] \\
&= (\ell(v) - 1) + \ell_I(sv_I) \\
&\underset{(L5)}{\leqslant} \ell(v) - 1 + \ell_I(v_I) + 1 \\
&= \ell(v) + \ell_I(v_I) \\
&= \ell(w).
\end{aligned}$$

Thus, all these quantities are equal, giving us $\ell(w) = \ell(vs) + \ell((sv^{-1})w)$. Since $sv^{-1}w \in W_I$ as observed in the second line of the calculation, we can conclude that $vs \in A$, which is absurd since we assumed $\ell(vs) < \ell(v)$ and $v \in A$ of minimal length. Consequently $\ell(vs) > \ell(v)$. Following the exact same steps made explicit for s, we also obtain $\ell(vs') > \ell(v)$. By induction we can conclude that $v(\alpha_s), v(\alpha_{s'}) > 0$.

The next step is to show that also $w(\alpha_s)$ and $w(\alpha_{s'})$ are positive. Since $w = vv_I$, it will be enough to show that v_I sends α_s to a linear combination of α_s and $\alpha_{s'}$ with non-negative coefficients. Initially, we state that $\ell_I(v_I s) \geqslant \ell_I(v_I)$, if this were not the case, we would get

$$\begin{aligned} \ell(ws) = \; & \ell(vv^{-1}ws) \leqslant \ell(v) + \ell(v^{-1}ws) \\ & = \ell(v) + \ell(v_I s) \leqslant \ell(v) + \ell_I(v_I s) \\ & < \ell(v) + \ell_I(v_I) = \ell(w). \end{aligned}$$

Which is a contradiction since $\ell(ws) > \ell(w)$. We observe that since W_I is dihedral, all reduced expressions are alternating products of s and s', and therefore all reduced expressions of v_I end in s. Let $m := m(s, s')$, two cases are now possible:

1. $m = \infty$. In this situation we have that $B(\alpha_s, \alpha_{s'}) = -1$, and iteratively we get

$$\begin{aligned} s'(\alpha_s) &= \alpha_s + 2\alpha_{s'} \\ ss'(\alpha_s) &= 2\alpha_{s'} + 3\alpha_s \\ s'ss'(\alpha_s) &= 3\alpha_s + 4\alpha_{s'} \\ &\vdots \end{aligned}$$

2. $m < \infty$. In this situation W_I is just the finite dihedral group D_m, and we can think of s, s' as reflections over two axis that form an angle of π/m. Clearly m is the maximum of ℓ_I, but the only element of length m is the reflection $\underbrace{\dots ss'}_{m} = \underbrace{\dots s's}_{m}$. Therefore v_I has a reduced expression $v_I = \underbrace{\dots s's}_{<m}$. We observe that ss' is a rotation through an angle of $2\pi/m$. We remark that in a fixed reduced expression for v_I we have at most $m/2$ such reflection. If $m = 2k+1$ is odd and $\ell(v_I) = 2k$, then $v_I(\alpha_s) = \alpha_{s'}$. Otherwise the rotation part of v_I moves α_s towards $\alpha_{s'}$ through an angle of at most $\pi - 2\pi/m$, still within the positive cone defined by α_s and $\alpha_{s'}$. If the reduced expression for v_I starts with s', then the resulting vector is sent again in the positive cone since the angle between the reflecting line and α_s is $\pi/2 - \pi/m$.

□

Remark 1.3.4. Clearly, the two statements of the theorem put together give us the converse implications. By this we mean that both statements of the theorem are an "if and only if".

Corollary 1.3.1. The geometric representation σ is faithful.

Proof. Suppose $w \in \mathrm{Ker}(\sigma)$ and $w \neq 1$. Since w is not the identity element, it has positive length and it is possible find $s \in S$ such that $\ell(ws) < \ell(w)$, then Theorem 1.3.1 says that $w(\alpha_s) < 0$, but since w acts as the identity, we have $w(\alpha_s) = \alpha_s > 0$, finding a contradiction. $\square$

Corollary 1.3.2. If $\alpha \in \Phi$, and $\alpha = \sum_{s \in S} c_s \alpha_s$, then one of the following holds:

(1) $c_s \geqslant 0 \ \ \forall s \in S$;

(2) $c_s \leqslant 0 \ \ \forall s \in S$.

Which means that $\Phi = \Phi^+ \amalg \Phi^-$.

Proof. By definition of a root, there exists $w \in W$ and $s \in S$ such that $\alpha = w(\alpha_s)$, but we recall that as a consequence of Proposition 1.1.1, $\ell(ws)$ and $\ell(w)$ are never equal. Therefore either $\ell(ws) > \ell(w)$ or $\ell(ws) < \ell(w)$. By Theorem 1.3.1 either $\alpha > 0$ or $\alpha < 0$. $\square$

1.4 The geometric interpretation of the length function

The goal of the next section is to better understand how W permutes the roots. We will obtain a result that connects the behaviour of the length function with the geometric representation. This result will be used to prove the so called *Exchange Condition* and *Deletion Condition*, which are the main combinatorial properties of words in a Coxeter group. As expected, Theorem 1.3.1 will be a key ingredient in the proof of the main proposition.

Proposition 1.4.1.

1. If $s \in S$, the only positive root sent to a negative root by s is α_s.
2. For all $w \in W$, $\ell(w)$ is the number of positive root sent to negative roots by w.

Proof.

1. Let $\alpha \in \Phi^+ \setminus \{\alpha_s\}$. Since α is a unit vector it cannot be a multiple of α_s, and therefore it can be written in the form

$$\alpha = \sum_{t \in S} c_t \alpha_t,$$

where at least one $c_t > 0$ for some $t \neq s$. By definition, the action of s on a vector only changes the α_s component, hence the coefficient c_t is left unchanged, and this proves the root must be positive and also shows that it is different from α_s. We thus have $s(\Phi^+ \setminus \{\alpha_s\}) \subseteq \Phi^+ \setminus \{\alpha_s\}$. Applying s to both sides of the inclusion gives us the opposite one, and we have proved 1.

2. Let us define

$$\Pi(w) := \Phi^+ \cap w^{-1}(-\Phi^+); \qquad n(w) := |\Pi(w)|.$$

In words, $n(w)$ is the number of positive root sent to negative root by w. Note that $n(w)$ may be infinite. We proceed by induction on $\ell(w)$. The first part of the proposition tells us that if $s \in S$ we have $n(w) = 1$. We observe that, because of part 1 of the proposition, if $w(\alpha_s) > 0$ then $\Pi(ws) = s(\Pi(w)) \amalg \{\alpha_s\}$, whereas if $w(\alpha_s) < 0$ we get $\Pi(ws) = s(\Pi(w) \setminus \{\alpha_s\})$. We omit the details but the result follows using the main properties of the image set of a function. Theorem 1.3.1 tells us $\ell(ws) = \ell(w) + 1$ exactly when $w(\alpha_s) > 0$, and $\ell(ws) = \ell(w) - 1$ exactly when $w(\alpha_s) < 0$, using the induction hypothesis we are done.

□

1.5 Parabolic subgroups

By using the results we have just obtained about the geometric representation and using Theorem 1.3.1, we can get a lot of information about the subgroup structure of W. Before anything else, we recall that if (W, S) is a Coxeter system and $I \subseteq S$, we defined W_I to be the subgroup generated by the elements of I.

Theorem 1.5.1. Let (W, S) be a Coxeter system with Coxeter graph Γ, let $I \subseteq S$, then the following hold:

1. The pair (W_I, I) is a Coxeter system whose Coxeter graph is the subgraph of Γ induced by the set of vertices I.

2. If $w \in W_I$ and $w = s_1 s_2 \dots s_r$ is a reduced expression, then every element of the expression is in I. Therefore $\ell = \ell_I$ on W_I.

3. There is a lattice isomorphism between the subsets of S and parabolic subgroups of W, the isomoprhism is given by

$$I \mapsto W_I.$$

4. No subset of S generates W.

Proof. 1. Let Γ be the Coxeter graph of W, the set I and the subgraph Γ_I generated by I (considered as a subset of vertices) defines abstractly a Coxeter system $(\overline{W}_I, I)$. Let $\bar{\sigma}$ be the geometric representation of $(\overline{W}_I, I)$ and let $V_I := \mathrm{Span}\{\ \alpha_s \mid s \in I\}$. Since the action each element $s \in I$ is determined purely by the values $m(s, s')$, we can state that if $s \in I$, then $\bar{\sigma}(s) = \sigma(s)_{|V_I}$. Lastly, we observe that there is a natural surjective homomorphism

$$\overline{W}_I \to W_I$$
$$s_1 s_2 \dots s_r \mapsto s_1 s_2 \dots s_r.$$

The observations made above let us claim that the following diagram commutes

$$\begin{tikzcd} \overline{W}_I \arrow[r, two heads] \arrow[dr, hook, "\bar{\sigma}"'] & W_I \arrow[d] \\ & \mathrm{End}(V_I) \end{tikzcd}$$

Since $\bar{\sigma}$ is injective as proved in Corollary 1.3.1, the upper arrow is an isomorphism.

2. The proof is by induction on $\ell(w)$, the case $w = 1$ is clear. Suppose $w \neq 1$, let $w = s_1 s_2 \dots s_r$ be a reduced expression and denote $s := s_r$. By Theorem 1.3.1, we have $w(\alpha_s) < 0$. From the fact that $w \in W_I$, we can say that w can be written as $w = t_1 t_2 \dots t_q$ $(t_i \in I)$. Thus

$$w(\alpha_s) = \alpha_s + \sum_{i=1}^{q} c_i \alpha_{t_i}$$

for some $c_i \in \mathbb{R}$. Since $w(\alpha_s) < 0$, there must be an index i such that $s = t_i$, obtaining $s \in I$. By the fact that $ws = s_1 \dots s_{r-1}$ is reduced, we conclude inductively.

3. Suppose $I, J \subseteq S$. If $W_I \subseteq W_J$, then $I = W_I \cap S \subseteq W_J \cap S = J$ as a consequence of 2. Trivially, $W_{I \cup J}$ is the subgroup generated by W_I and W_J. Thanks to 2 we have the relation $W_{I \cap J} = W_I \cap W_J$. This gives us the lattice isomorphism.

4. This is a direct consequence of 3.

$\square$

We end this section by presenting an important result about parabolic subgroups. For further details, the reader may consult [11, §5.12]. This result will be used in Chapter 3. Firstly, we define for every $J \subseteq S$ the sets:

$$W^I := \{w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in I\}, \quad {}^I W := \{w \in W \mid \ell(sw) > \ell(w) \text{ for all } s \in I\}.$$

Proposition 1.5.1. Let $I \in J$. Then:

(i) Every $w \in W$ has a unique factorization $w = w^I w_I$ with $w^I \in W^I$, $w_I \in W_I$ and $\ell(w) = \ell(w_I) + \ell(w^I)$.

(ii) Every $w \in W$ has a unique factorization $w = {}_I w\, {}^I w$, where ${}_I w \in W_I$, ${}^I w \in {}^I W$ and $\ell(w) = \ell({}_I w) + \ell({}^I w)$.

1.6 The Exchange and Deletion conditions

In this section we prove the most important properties of Coxeter groups, not only do these result shed light on the combinatorial behaviour of expressions in a Coxeter group, but also characterize Coxeter groups completely. We omit the proof of this result here, the reader may consult [11, Chapter 1] or [5, Chapter 1]. The first step is to generalize which elements act as reflections. Let $\alpha = w(\alpha_s)$ $(s \in S, w \in W)$ be a root. Then if $v \in V$ we get:

$$\begin{aligned} wsw^{-1}(v) &= w[w^{-1}(v) - 2B(w^{-1}(v), \alpha_s)\alpha_s] \\ &= v - 2B(w^{-1}, \alpha_s)w(\alpha_s) \\ &= v - 2B(v, w(\alpha_s))w(\alpha_s) \\ &= v - 2B(v, \alpha)\alpha. \end{aligned}$$

Therefore the action of an element wsw^{-1} does not depend on the elements s, w themselves, but only on the root $w(\alpha_s)$. Because of this, we can denote this map s_α. This endomorphism fixes the orthogonal complement of $\mathbb{R}\alpha$ and sends α to $-\alpha$. Therefore, we can define the set of all such reflections as

$$T := \bigcup_{w \in W} wSw^{-1}.$$

Lemma 1.6.1. The map $\Phi^+ \to T$ sending $\alpha_s \mapsto s_\alpha$, where s_α is the map defined above, is a bijection.

Proof. The map is surjective by definition. If $s_\alpha = s_\beta$, then

$$s_\alpha(\beta) = \beta - 2B(\beta, \alpha)\alpha = -\beta = s_\beta(\beta) \implies \beta = B(\beta, \alpha)\alpha.$$

Since both are unit vectors and positive roots, it must be that $\alpha = \beta$. □

Lemma 1.6.2. If α, β are roots and if there exists $w \in W$ such that $\beta = w(\alpha)$, then $ws_\alpha w^{-1} = s_\beta$.

Proof. Let $v \in V$:

$$\begin{aligned} ws_\alpha w^{-1}(v) &= w[w^{-1}(v) - 2B(w^{-1}(v), \alpha)\alpha] \\ &= v - 2B(w^{-1}(v), \alpha)w(\alpha) \\ &= v - 2B(v, w(\alpha))w(\alpha) \\ &= v - 2B(v, \beta)\beta = s_\beta(v). \end{aligned}$$

□

There is an analogue of Theorem 1.3.1 that holds for arbitrary reflections.

Proposition 1.6.1. Let $w \in W$, $\alpha \in \Pi$, then $\ell(ws_\alpha) > \ell(w)$ if and only if $w(\alpha) > 0$.

Proof. We prove the statement by induction on $\ell(w)$, if $\ell(w) = 0$ then $w = 1$ and the statement is evident. We can assume $\ell(w) > 0$, therefore it is possible to find $s \in S$ such that $\ell(sw) < \ell(w)$, and

$$\ell((sw)s_\alpha) = \ell(s(ws_\alpha)) \underset{(L5)}{\geqslant} \ell(ws_\alpha) - 1 > \ell(w) - 1 = \ell(sw).$$

Because of the choice of s, we can use the inductive hypothesis and get $sw(\alpha) > 0$. If it were true that $w(\alpha) < 0$, since the only negative root sent to a positive one by s is $-\alpha_s$, thus we have $w(\alpha) = -\alpha_s$ and $sw(\alpha) = \alpha_s$. Then by the lemma we just proved, we would get $(sw)s_\alpha(sw)^{-1} = s$, and obtain $ws_\alpha = sw$, which is absurd since $\ell(ws_\alpha) = \ell(w) + 1$ while $\ell(sw) = \ell(w) - 1$. □

It is possible to generalize the notion of parabolic subgroups to arbitrary reflections. A *reflection subgroup* of W is any subgroup of the form $W' := \langle A \rangle$ with $A \subseteq T$. It turns out that these subgroups are Coxeter groups. Let $w \in W$ and define $N(w) := \{t \in T \mid \ell(wt) < \ell(w)\}$, then by setting $S' := \{t \in T \mid N(t) \cap W' = \{t\}\}$ it is true that (W', S') is a Coxeter system. We call elements of S' *canonical generators* of W' and say that W' is *dihedral* if $|S'| = 2$. This result is presented, for example, in [11, Theorem 8.2].

Theorem 1.6.1 (Strong exchange condition)**.** Let $w = s_1 s_2 \dots s_r$ and let $t \in T$ be such that $\ell(wt) < \ell(w)$. Then there exists $i \in \{1, \dots, r\}$ such that $wt = s_1 \dots \widehat{s_i} \dots s_r$. Moreover, if $w = s_1 s_2 \dots s_r$ is a reduced expression, then the index i is unique.

Proof. Let $\alpha \in \Phi^+$ be such that $t = s_\alpha$. Since $\ell(wt) < \ell(w)$, the last proposition tells us that $w(\alpha) < 0$. But α is positive, so there must be an index i such that $s_{i+1} \dots s_r(\alpha) > 0$ but $s_i \dots s_r(\alpha) < 0$. Using what we have proved in the last section, we get $s_{i+1} \dots s_r(\alpha) = \alpha_{s_i}$. Using the previous lemma we obtain:

$$s_i = (s_{i+1} \dots s_r) s_\alpha (s_{i+1} \dots s_r)^{-1}.$$

Therefore $ws_\alpha = wt = s_1 \dots \widehat{s_i} \dots s_r$. If $w = s_1 s_2 \dots s_r$ is a reduced expression, then if there were indices $i < j$ such that

$$wt = s_1 \dots \widehat{s_i} \dots s_j \dots s_r = s_1 \dots s_i \dots \widehat{s_j} \dots s_r.,$$

by cancelling and manipolating the expression, we would then be able to find $w = s_1 \dots \widehat{s_i} \dots \widehat{s_j} \dots s_r$ and the expression $w = s_1 s_2 \dots s_r$ would not be reduced. □

Corollary 1.6.1 (Deletion Condition)**.**
(1) Let $w = s_1 s_2 \dots s_r$. If the expression is not reduced, then there are indices $i < j$ such that $w = s_1 \dots \widehat{s_i} \dots \widehat{s_j} \dots s_r$.

(2) It is possible to obtain a reduced expression by an arbitrary one omitting an even number of factors.

Proof. The second part of the statement is a direct consequence of the first one. The hypothesis entails that there is an index j such that $\ell(s_1 \dots s_{j-1} s_j) < \ell(s_1 \dots s_{j-1})$, using the Exchange Condition we get $s_1 \dots s_j = s_1 \dots \widehat{s_i} \dots s_{j-1}$, thus we can write $w = s_1 \dots \widehat{s_i} \dots \widehat{s_j} \dots s_r$. □

1.7 The Bruhat order

In this section we define a special partial order on a Coxeter groups W and study its properties. The order is defined in such a way that the order relation reflect the behaviour of the length function. One of the main reasons Coxeter groups are interesting combinatorial objects is because of this particular poset structure. It is important to say that the Bruhat order also arises in other branches of mathematics, such as the study of Schubert varieties in algebraic geometry.

Definition. Let (W, S) be a Coxeter system and T its set of reflections. Let $u, w \in W$ and $t \in T$, we write $u \xrightarrow{t} w$ if $w = vt$ and $\ell(u) < \ell(w)$. We may write $u \to w$ to say that $u \xrightarrow{t} w$ holds for some $t \in T$.
We define $w' < w$ if there are elements $w_1, \ldots, w_m$ such that $w' = w_0 \to \ldots \to w_m = w$. We call this the *Bruhat order* of the group W. The *Bruhat graph* of (W, S) is the graph whose vertices are the elemets of W, and there is an edge between $u, w \in W$ if $u \to w$.

Remark 1.7.1. In the previous definition, if we restricted ourselves to the case $t \in S$, we would have obtain an order called the *weak ordering*.

Remark 1.7.2. If $u \to w$, by definition $\ell(w) > \ell(u)$, but we do not know what the length difference is, since we are not working with just the generators anymore and property (L5) does not hold anymore. We will prove more information about this length difference.

Remark 1.7.3. We omit the proof, but defining $u \to tw$ if $w = tu$ and $\ell(w) > \ell(u)$, defines the same partial order.

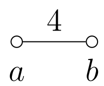
Lemma 1.7.1. (1) $u < w$ implies $\ell(u) < \ell(w)$;

(2) $\forall u \in W \quad \forall t \in T \;\; u < ut \iff \ell(u) < \ell(ut)$;

(3) $1 \in W$ is the smallest element.

Proof. All of these are direct consequences of the definition. □

Example 5. Let us consider the group D_4 with Coxeter graph



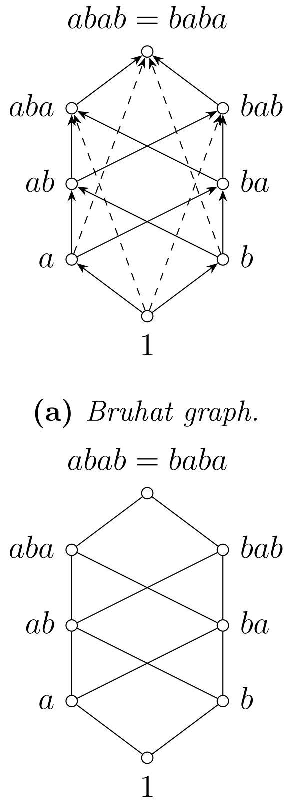
Then the set of reflections is $T = \{a,\ b,\ aba,\ bab\}$ and D_4 has the Bruhat graph depicted in Figure 1.1 . Some edges are dashed to make the picture clearer.

Lemma 1.7.2. Let $u, w \in W$ be distinct, $w = s_1 s_2 \ldots s_r$ be a reduced expression and let u be such that u has a reduced expression which is a subword of $s_1 s_2 \ldots s_r$. Then there is an element $v \in W$ such that

(1) $u < v$;

(2) $\ell(v) = \ell(u) + 1$;

(3) v has a reduced expression that is a subword of $s_1 s_2 \ldots s_r$.



(b) *Bruhat ordering.*

Figure 1.1: *The poset structure of D_4.*

Proof. Out of all the reduced expression for u which are subwords of rw, choose one in which the last omitted index i_k is minimal. Then consider the reflection $t = s_r s_{r-1} \dots s_{i_k} \dots s_{r-1} s_r$. So we obtain the following:

$$u = s_1 \dots \widehat{s_{i_1}} \dots \widehat{s_{i_k}} \dots s_r, \qquad ut = s_1 \dots \widehat{s_{i_1}} \dots \widehat{s_{i_{r-1}}} \dots s_{i_k} \dots s_r.$$

Thus $\ell(ut) < \ell(u) + 1$. Suppose $ut < u$, then the Strong Exchange Condition gives us a reduced expression for ut by omitting a reflection. By equating the terms and cancelling we get that either

$$t = s_r s_{r-1} \dots s_p \dots s_{r-1} s_r, \qquad \text{for } p > i_k$$

or

$$t = s_r \dots \widehat{s_{i_k}} \dots \widehat{s_{i_d}} \dots s_q \dots \widehat{s_{i_d}} \dots \widehat{s_{i_k}} \dots s_r, \qquad \text{for some } r < i_k,\ r \neq i_j.$$

In the first case:

$$\begin{aligned} w &= wt^2 \\ &= (s_1 s_2 \dots s_r)(s_r \dots s_{i_k} \dots s_r)(s_r \dots s_p \dots s_r) \\ &= s_1 \dots \widehat{s_{i_k}} \dots \widehat{s_p} \dots s_r. \end{aligned}$$

This is impossible since $w = s_1 s_2 \dots s_r$ was assumed to be reduced. In the second case:

$$\begin{aligned} u &= ut^2 \\ &= (s_1 \dots \widehat{s_{i_1}} \dots \widehat{s_{i_k}} \dots s_r)(s_r \dots \widehat{s_{i_k}} \dots s_q \dots \widehat{s_{i_k}} \dots s_r)(s_q \dots s_{i_k} \dots s_q) \\ &= s_1 \dots \widehat{s_{i_1}} \dots \widehat{s_q} \dots s_{i_k} \dots s_r. \end{aligned}$$

Which is impossible since i_k was assumed to be minimal. We now define $v := ut$ and we are done. $\square$

Theorem 1.7.1 (Subword property)**.** Let $w \in W$ and $w = s_1 s_2 \dots s_r$ be a reduced expression. Then $w' \leqslant w$ if and only if there are indices $1 \leqslant i_1 < \dots < i_k \leqslant r$ such that $w' = s_{i_1} s_{i_2} \dots s_{i_k}$.

Proof. By definition we have $w' = w_0 \to \dots \to w_m = w$, the statement is proved by induction on m. If $m = 0$ the statement is trivial. If $w' \to w$, by definition there is $t \in T$ such that $w = w't$ and $\ell(w) > \ell(w')$. Thus, we can use the Strong Exchange Condition and obtain $w' = s_1 \dots \widehat{s_i} \dots s_r$. Since $w_{m-1} \to w$, the element w_{m-1} is obtained as a subword of $w = s_1 s_2 \dots s_r$, which is necessarily reduced. By the fact that $w' = w_0 \to \dots \to w_{m-1}$, inductively we can conclude that w' can be obtained as a subword of the reduced expression obtained for w_{m-1} and we are done. For the converse implication, the proof is by induction on $\ell(w) - \ell(u)$, the case $\ell(w) - \ell(u) = 0$ being clear. Let $u = s_{i_1} s_{i_2} \dots s_{i_k}$ be a subword of $s_1 s_2 \dots s_r$, then by the previous lemma we have an element $v \in W$ that has a reduced expression which is a subword of $s_1 s_2 \dots s_r$, it is strictly greater that u and $\ell(w) - \ell(v) = \ell(w) - \ell(u) - 1$. By using the induction hypothesis, we can conclude that $u < v \leqslant w$. $\square$

Theorem 1.7.2 (Chain property)**.** If $u < w$, there exists a chain $u = x_0 < x_1 < \dots < x_k = w$ such that $\ell(x_i) = \ell(u) + i$ for $1 \leqslant i \leqslant k$.

Proof. We prove the result by induction on $\ell(w) - \ell(u)$, with the case $\ell(w) - \ell(u) = 0$ being trivial. Let $w = s_1 s_2 \dots s_r$ be a reduced expression for w. If $u < w$, by the Subword Property there are indices $1 \leqslant i_1 < \dots < i_k \leqslant r$ such that $u = s_{i_1} \dots s_{i_k}$ and the expression is reduced. Using the last lemma, we have an element $v > u$ with all the properties proved in the lemma. Therefore by induction we have a chain

$$v = x_1 < \cdots < x_k = w$$

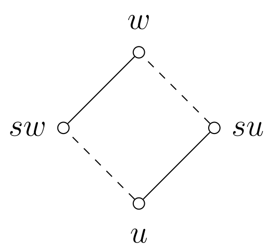
since $\ell(v) = \ell(u) + 1$, therefore the chain

$$u = x_0 < x_1 = v < \cdots < x_k = w$$

has the desired properties. □

Notation. We write $u \lhd w$ if $u < w$ and $\ell(w) = \ell(u) + 1$.

Proposition 1.7.1 (Lifting Property)**.** Let $u < w$, let $s \in S$ be such that $\ell(sw) < \ell(w)$ but $\ell(su) > \ell(u)$. Then $u \leqslant sw$ and $su \leqslant w$.



Proof. We denote $\alpha \prec \beta$ the relation given by "being a subword of". Let $sw = s_1 s_2 \dots s_r$ be a reduced expression, then write $w = s s_1 s_2 \dots s_r$ (note that this expression is reduced). Therefore we have a reduced subword

$$u = s_{i_1} \dots s_{i_k} \prec s s_1 s_2 \dots s_r.$$

Since we have assumed $su > u$, s and s_{i_1} must be distinct. Thus we conclude that $s_{i1} \dots s_{i_k} \prec s_1 s_2 \dots s_r$, meaning $u \leqslant sw$ and $s s_{i_1} \dots s_{i_k} \prec s s_1 s_2 \dots s_r$. Thus, by the subword property $su \leqslant w$. □

1.8 Fundamental domain for σ

In this section we further study the geometric action of a Coxeter group, these results are a key step in the classification of all finite Coxeter groups which we will study in the next section. Firstly, we recall that if $\rho : G \to \mathrm{End}(V)$ is a finite dimensional representation of a group G, then its *dual action* $\rho^* : G \to \mathrm{End}(V^*)$ is defined as $\rho(g)(\varphi) := \varphi \circ \rho(g^{-1})(v)$. From now on we will omit ρ and ρ^* from the notation in order to make the exposition clearer, as done in the past sections when using the geometric representation. The dual action has the following property:

$$\forall \varphi \in V^* \quad \forall g \in G \quad \forall v \in V \qquad g(\varphi)(g(v)) = \varphi(v).$$

Let (W, S) be a Coxeter system, $\sigma : W \to V$ be its geometric representation and σ^* its dual representation. We define the hyperplanes $Z_s := \{\, \varphi \in V^* \mid \varphi(\alpha_s) = 0 \,\}$, and the relative half-spaces $A_s := \{\, \varphi \in V^* \mid \varphi(\alpha_s) > 0 \,\}$, $A'_s := \{\, \varphi \in V^* \mid \varphi(\alpha_s) < 0 \,\}$. We can then define

$$C := \bigcap_{s \in S} A_s.$$

Remark 1.8.1. The action of s fixes Z_s pointwise since the action of s only changes the α_s coordinate, which is sent to zero by any element of Z_s.

We can fix a basis consisting of the vectors α_s, obtaining an identification of V and V^* with $\mathbb{R}^n$ (with $n = |S|$). The identification of V^* with $\mathbb{R}^n$ is given by the dual basis of the basis chosen for V. Clearly Z_s is a closed set, while A_s, A'_s and C are open. Moreover, all elements of W act as continuous functions both on V and V^*.

Remark 1.8.2. Note that if $s \in S$ it is true that $\overline{A_s} = A_s \cup Z_s$, and $D := \overline{C} = \bigcap_{s \in S} \overline{A_s}$, then D turns out to be a convex cone.

We define the following partition of D: if $I \subseteq S$, we call

$$C_I := \left(\bigcap_{s \in I} Z_s \right) \cap \left(\bigcap_{s \notin I} A_s \right).$$

We have observed that s fixes Z_s pointwise, therefore W_I fixes C_I pointwise. Conversely, if $s \in S$ fixes a point $\varphi \in C_I$, then we have:

$$\varphi(\alpha_s) = s(\varphi)(s(\alpha_s)) = -\varphi(\alpha_s) \implies \varphi \in Z_s.$$

Since C_I intersects Z_s only if $s \in I$, it must be that $s \in I$. This argument is not enough to show that W_I is precisely the stabilizer of C_I, this will turn out to be true and we will prove this in one of the next results. We define

$$U := \bigcup_{w \in W} w(D).$$

Since D is a cone, so is U, which is called the *Tits cone*. We will now proceed and study the action of W on the Tits cone.

Lemma 1.8.1. Let $s \in S$ and $w \in W$. Then $\ell(sw) > \ell(w)$ if and only if $w(C) \subseteq A_s$, whereas $\ell(sw) < \ell(w)$ if and only if $w(C) \subseteq A'_s$.

Proof. If $\ell(sw) > \ell(w)$ then $\ell(w^{-1}s) > \ell(w)$ and by Theorem 1.3.1 we get $w^{-1} > 0$. If $\varphi \in C$, then $w(\varphi)(\alpha_s) = \varphi(w^{-1})(\alpha_s) > 0$, which is equivalent to $w^{-1} > 0$ by how C is defined. So $w(C) \in A_s$ if and only if $\ell(sw) > \ell(w)$. □

Theorem 1.8.1. (1) Let $w \in W$ and $I, J \subseteq S$. If $w(C_I) \cap C_J \neq \emptyset$, then $I = J$ and $w \in W_I$, so $w(C_I) = C_I$. In particular W_I is the stabilizer of each point of C_I, and the family $\mathcal{C} = \{\, w(C_I) \mid I \subseteq S,\ w \in W\}$ is a partition of U.

(2) D is a fundamental domain of the action of W on U, meaning the orbit of each point of U has exactly one element in D.

(3) The Tits cone U is convex. Furthermore, every closed segment in U meets finitely many elements of the partition $\mathcal{C}$.

Proof. (1) We proceed by induction on $\ell(w)$, the case $\ell(w) = 0$ being trivial. If $\ell(w) > 0$, there exists $s \in S$ such that $\ell(sw) < \ell(w)$, writing $w = s(sw)$ and applying the previous lemma to this situation we get $w(C) \subseteq s(A_s) = A'_s$. Since the action of each element of W is continuous, we get $w(D) \subseteq \overline{A'_s}$. Since $D \subseteq \overline{A_s}$, we have $D \cap w(D) \subseteq Z_s$. This means that s fixes pointwise the set $w(C_I) \cap C_J \neq \emptyset$. We can therefore deduce the following:

- The reflection s fixes an element in C_J, applying the reasoning followed after the definition of C_I, we can deduce that $s \in J$. Thus s fixes C_J pointwise.
- $C_J \cap sw(C_I) = s(C_J \cap w(C_I))$ is nonempty.

Since $\ell(sw) < \ell(w)$, we can use the induction hypothesis and get $I = J$ and $sw \in W_I$, but since $s \in J = I$, we have $w \in W_I$. The rest of the claim is a direct consequence of what we have just showed.

(2) We only need to prove the uniqueness of the element in D in each orbit. Let $\varphi, \psi \in D$ be in the same orbit, meaning there is an element $w \in W$ such that $w(\varphi) = \psi$. There are $I, J \subseteq S$ such that $\varphi \in C_I$ and $\psi \in C_J$, thus $w(C_I) \cap C_J \neq \emptyset$, by part (1) we obtain $I = J$ and $w \in W_I$ and therefore $\varphi = \psi$.

(3) It is sufficient to prove that if $\varphi, \psi \in U$, the segment $[\varphi, \psi]$ can be covered by finitely many sets in the family $\mathcal{C}$. Firstly, we observe that the action of an element w permutes the elements of the family $\mathcal{C}$, meaning that the claim holds for a segment $[\varphi, \psi]$ if and only if it also holds for $[w(\varphi), w(\psi)]$. This means that we can assume $\varphi \in D$ and $\psi \in w(D)$. We proceed by induction

on $\ell(w)$, the case $w = 1$ being part (1). Let $\ell(w) > 0$, since D is a convex cone, the intersection of the segment $[\varphi, \psi]$ with D is a closed segment $[\varphi, \eta]$ which by part (1) is covered by finitely many elements of the family $\mathcal{C}$. It remains to show that we can cover $[\varphi, \eta]$. We can assume $\psi \notin D$ since we have already covered this case. This means that for some $s \in I$ and $s \notin I$, we have $\psi \in A'_s \cap \overline{A_{s'}}$. Let us assume that $\eta \in A_s$ for all $s \in I$, then this would have to be true for all points in a neighbourhood of η, thus the whole neighbourhood would lie in D which is a contradiction. This means that for some $s \in I$ we have $\eta \in Z_s$. Furthermore, because of the fact that $\psi \in A'_s$ it must be that $w(D) \subseteq \overline{A'_s}$ thus $w(C) \subseteq A'_s$. Using the previous lemma we obtain $\ell(sw) < \ell(w)$ and using the induction hypothesis to the pair $\eta \in D$ and $s(\psi) \in sw(D)$ we can cover $[\eta, s(\psi)]$, taking the image set via s lets us conclude the proof.

□

1.9 Classification of finite Coxeter groups

In this section we state the classification of finite Coxeter groups, which turn out to be all the finite reflection groups. These in turn are classified, and we will state the classification by giving the list of all the possible Coxeter graphs of these groups, without giving the detailed proof. The interest reader can look at [11, Chapter 2]. Firstly, we need to better study the properties of the geometric representation $\sigma : W \to \mathrm{End}(V)$. As before, we fix a basis for V, and therefore we identify V and V^* with $\mathbb{R}^n$. Moreover, we identify $\mathrm{End}(V)$ and $\mathrm{End}(V^*)$ with $\mathrm{GL}(n, \mathbb{R})$.

Proposition 1.9.1. With the topologies definied above, $\sigma(W)$ is a discrete subgroup of $\mathrm{End}(v)$.

Proof. We keep the notation of the previous section for all the important subsets of V^*. Firstly, we observe that for all $\varphi \in V^*$, the valuation map $v_\varphi : \mathrm{End}(V^*) \to V^*$ is continuous. This tells us that the set $C_0 := v_\varphi^{-1}(C)$ is an open set containing the identity. By choosing $\varphi \in C$, part (1) of Theorem 1.8.1 lets us conclude that $\sigma^*(W) \cap C_0 = \{1\}$. In turn, an arbitrary element $g = \sigma^*(w)$ has an open neighbourhood gC_0 intersecting $\sigma^*(W)$ in $\{g\}$. This means that $\sigma^*(W)$ is a discrete subset of $\mathrm{GL}(V^*)$. By trasport of structure we obtain the desired claim. □

Lemma 1.9.1. Every discrete subgroup of a compact Hausdorff topological group is closed and therefore finite.

Proof. Omitted. □

Corollary 1.9.1. If the form B induced by the Coxeter system (W, S) is positive definite, then W is finite.

Proof. If B is positive definite, we get an identification of V and V^* with $\mathbb{R}^n$ endowed with the Euclidean scalar product. Because of how the geometric action is defined, $\sigma(W)$ contains only isometries. Since the group of isometries $O(n, \mathbb{R})$ is compact. By the previous theorem $\sigma(W)$ is discrete and we can therefore conclude using the lemma. □

Proposition 1.9.2. Let (W, S) be an irreducible Coxeter system, then the following hold:

(1) Every subrepresentation of the geometric representation of W is a subset of $V^\perp = \mathrm{rad}(B) = \bigcap_{s\in S} H_s$.

(2) If B is degenerate, meaning $\mathrm{rad}(B) \neq \emptyset$, then V is not completely reducible.

(3) If B is non-degenerate then the geometric representation is irreducible.

(4) The only W-module homomorphisms are multiples of the identity.

Proof. (1) Let V' be a proper submodule of V. Let us assume that V' does not contain any roots. The maps σ_s is diagonizable with eigenvalues 1 and -1. Since the eigenspace of eigenvalue -1 is generated by α_s, which we assumed to be outside V', σ_s fixes V' pointwise. This is true for all $s \in S$, meaning $V' \subseteq \bigcap_{s\in S} H_s = V^\perp$. If α_s is in V' for some $s \in S$, then we can choose an element $t \in S$ adjancent to s in the Coxeter graph of (W, S), therefore $\sigma_t(\alpha_s) = \alpha_s + c\alpha_t$, but V' is W-invariant, forcing α_t to be in V'. Since (W, S) is connected, we can iterate the argument and say that $\alpha_s \in V'$ for all $s \in S$, thus $V = V'$.

(2) Using part (1) of the proposition, any submodule of V is inside the radical, hence it cannot have a direct complement that is also a submodule.

(3) This is a direct consequence of part (1).

(4) Let $f \in \mathrm{End}(V)$ commute with all elements of $\sigma(W)$. If s is in S, then f commutes with σ_s and therefore the subspace $\mathbb{R}\alpha_s$ is fixed by f and $f(\alpha_s) = c\alpha_s$ for some $c \in \mathbb{R}$. Let g be the linear map defined by $g(v) = f(v) - cv$, since f commutes with all elements in $\sigma(W)$, the map g is a W-module homomorphism and therefore $\mathrm{Ker}(g)$ is a subrepresentation that contains α_s. By part (1) of the proposition $\mathrm{Ker}(g) = V$.

□

In order to obtain the main result, the next fact about group representations is needed. We state the result without proof, the interested reader can consult [11]. Statement (b) of the lemma is also known as *Maschke's theorem.*

Lemma 1.9.2. Let $\rho : G \to \mathrm{End}(E)$ be a group representation, with E a finite dimensional vector space over $\mathbb{R}$.

(a) If G is finite, then there exists a positive definite G-invariant bilinear form on E.

(b) If G is finite, then ρ is completely reducible.

(c) Suppose the only endomorphisms of E commuting with $\rho(G)$ are the scalars. If β and β' are non-degenerate symmetric bilinear forms on E, both G-invariant, then β' is a scalar multiple of β.

Theorem 1.9.1. The following conditions on a Coxeter group W are equivalent:

(1) W is finite.

(2) The bilinear form B is positive definite.

(3) W is a finite reflection group, meaning a group isomorphic to a finite group of orthogonal transformations in Euclidean space generated by reflections.

Proof. Without loss of generality we assume that (W, S) is irreducible.

(a) $\Longrightarrow$ (b) As a consequence of Maschke's theorem, the geometric representation is completely reducible. Thus, using the previous proposition we get the following results: the form B must be non-degenerate, σ is irreducible and the only W-module homomorphisms are multiples of the identity. Moreover, by using part (c) of the lemma stated above, we can conclude that all non-degenerate symmetric bilinear forms that are W-invariant are multiples of B. By part (1) of the lemma one of these forms is positive definite, let us call it B'. Then $B' = cB$ for some $c \in \mathbb{R}^*$. By evaluating at α_s we obtain

$$c = c \cdot 1 = cB(\alpha_s, \alpha_s) = B'(\alpha_s, \alpha_s) > 0.$$

Thus B is positive definite.

(b) $\Longrightarrow$ (c) This is Corollary 1.9.1.

(c) $\Longrightarrow$ (a) This statement is trivial.

∎

To conclude the chapter, we list all of the Coxeter graphs which give rise to a positive definite bilinear form B. The result will not be proven here. As advised at the beginning of the section, we suggest the interest reader to consult [11, Chapter 2].

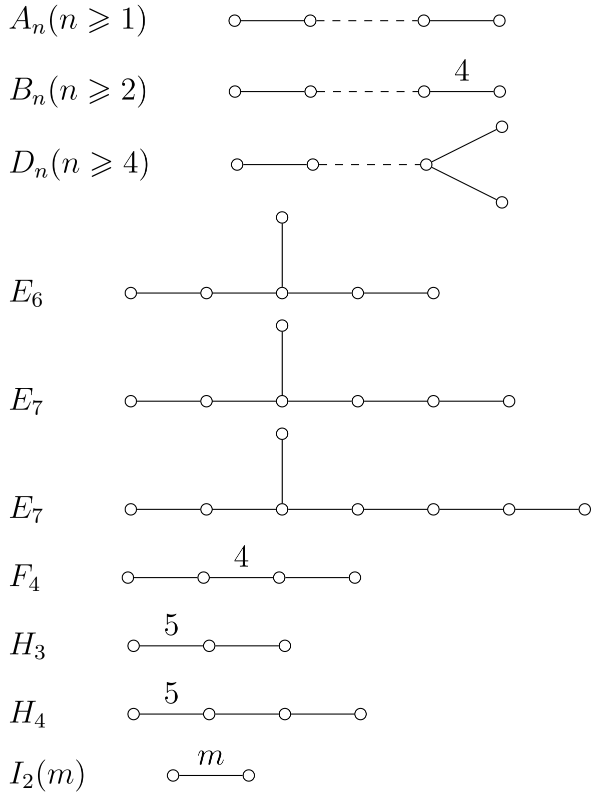


Figure 1.2: *Coxeter graphs of finite Coxeter groups.*

Chapter 2

Hecke algebras and Kazhdan–Lusztig polynomials

In this chapter we start by giving a definition of the Hecke algebra of a Coxeter group. This object is constructed by giving an algebra structure to the free $\mathbb{Z}[q, q^{-1}]$-module on W, whose product is defined on the free basis in a way that reflects the behaviour of the length function of the Coxeter group. Afterwards, we go on defining the R-polynomials and the Kazhdan-Lusztig polynomials. These polynomials were defined by David Kazhdan and George Lusztig in [12] in order to construct representations of the associated Hecke algebra. Roughly speaking, the R-polynomials are related to how the inverse of an element in the free basis is written with respect to the same basis. These are used to define the Kazhdan–Lusztig polynomials. The Hecke algebra of a Coxeter group has another natural basis other than the free basis indexed by W. The Kazhdan–Lusztig polynomials essentially are the coefficients of the basis change matrix between these two bases.

The importance of Kazhdan–Lusztig polynomials goes further than the construction of representations of the Hecke algebra. Since they first appeared in [12], unanticipated applications of these polynomials have been found: they appear in the representation theory of semisimple algebraic groups, the theory of Verma modules and in the study of Schubert varieties in algebraic geometry. The work that connects Schubert varieties and Kazhdan–Lusztig polynomials was done by Kazhdan and Lusztig in [13].

2.1 Hecke algebras

We begin this chapter with the construction of the Hecke algebra of a Coxeter group. Let (W, S) be a Coxeter system, we call $\mathcal{H}$ the free $\mathbb{Z}[q, q^{-1}]$-module on W. Let $\{ T_w \mid$

$w \in W\}$ be the free basis indexed by elements of W.

Theorem 2.1.1. There is a unique associative algebra structure on the free module $\mathcal{H}$, having T_1 as identity element, such that the product operation behaves as follows:

$$\begin{cases} (1)\ T_sT_w = T_{sw} & \text{if } \ell(sw) > \ell(w) \\ (2)\ T_sT_w = (q-1)T_w + qT_{sw} & \text{if } \ell(sw) < \ell(w). \end{cases}$$

Proof. The uniqueness part is clear, since once the product is defined on a basis, the product between any to elements is uniquely determined since we are asking for a product that is bilinear (we are asking for the distributive property). We simply write a reduced expression $w = s_1s_2\dots s_r$, and use the first relation to obtain $T_w = T_{s_1}\dots T_{s_r}$. We give only the idea of the existence of the structure and omit the details. The strategy is to use the already existing associative algebra structure on $\mathrm{End}(\mathcal{H})$. Observe that if $\mathcal{H}$ did in fact have such a structure, the left multiplication maps on $\mathcal{H}$ would generate a subalgebra of $\mathrm{End}(\mathcal{H})$ isomorphic to $\mathcal{H}$ as an algebra. Therefore, we just need to find the right subalgebra of $\mathrm{End}(\mathcal{H})$. We proceed by defining the following operators. We define λ_s $(s \in S)$ (which correspond to the left multiplication operators) by

$$\begin{aligned} \lambda_s(T_w) &= T_{sw} && \text{if } \ell(sw) > \ell(w), \\ \lambda_s(T_w) &= (q-1)T_w + qT_{sw} && \text{if } \ell(sw) < \ell(w). \end{aligned}$$

And we define the operators corresponding to right multiplication as follows $(t \in S)$

$$\begin{aligned} \rho_s(T_w) &= T_{wt} && \text{if } \ell(wt) > \ell(w), \\ \rho_s(T_w) &= (q-1)T_w + qT_{wt} && \text{if } \ell(wt) < \ell(w). \end{aligned}$$

The proof consists in showing that every λ_s commutes with every ρ_t. Then define $\mathcal{L}$ to be the subalgebra of $\mathrm{End}(\mathcal{H})$ generated by the maps λ_s. To finish, we define the map $\varphi : \mathcal{L} \to \mathcal{H}$ to be the valuation map at T_1. The map turns out to be bijective, and the fact that the operators commute is used to show this. We have therefore transferred an algebra structure on $\mathcal{H}$, and the last step of the proof is to show that the relations required actually hold. $\square$

Remark 2.1.1. The relations that appear in the theorem are actually equivalent to the following

$$\begin{cases} (3)\ T_sT_w = T_{sw} & \text{if } \ell(sw) > \ell(w) \\ (4)\ T_s^2 = (q-1)T_s + qT_1. \end{cases}$$

These two relations clearly follow from the previous one, (1) is unchanged and (4) is just a special case of the one that appears in the theorem. To prove the converse, we have to show (2). If $w \in S$ we have (4). Assume $\ell(sw) < \ell(w)$, then $\ell(s(sw)) > \ell(sw)$. Thus relation (3) gives us $T_s T_{sw} = T_w$. Using the relation (4) we get

$$T_s T_w = T_s^2 T_{sw} \underset{(4)}{=} ((q-1)T_s + qT_1)T_{sw} = (q-1)T_w + qT_{sw}.$$

Notation. To avoid excessive use of parentheses, making the notation cumbersome, we introduce the notation $\varepsilon_w = (-1)^{\ell(w)}$ and $q_w = q^{\ell(w)}$.

2.2 R-polynomials and the ι involution

In this section we give the main theorem-definition of R-polynomials, these polynomials occur when studying the invertibility of the elements of the free basis of $\mathcal{H}$, whose coordinates are essentially given by the R-polynomials. Moreover, we state and prove the main properties of such objects.

Lemma 2.2.1. Suppose $s \in S$ and $w \in W$ were such that $sw < w$. Suppose $x \in W$ was such that $x < w$. Then the following hold:

(a) If $sx < x$, then $sx < sw$.

(b) If $sx > x$, then $sx \leqslant w$ and $x \leqslant sw$.

In either case, we obtain $sx \leqslant w$.

Proof. Both statements are, essentialy, a consequence of the Subword Property of the Bruhat ordering. Using the Exchange Condition, the fact that $sw < w$ garantees us that w has a reduced expression $w = s_1 s_2 \dots s_r$ ending in s, meaning $s_1 = s$. Let $sx < x$, the Subword Property tells us that x has a reduced expression that is a subword of $s_1 s_2 \dots s_r$. The fact that $sx < x$ forces the last term of this expression to be s. Thus $sx \prec s_2 \dots s_r = sw$, and we conclude using the Subword Property again, noting that $sx \neq sw$ since $x < w$. In case (b) the same reasoning lets us conclude that x has a reduced expression $x = ss_2 \dots \widehat{s_i} \dots \widehat{s_j} \dots s_r$, while in case (a) there must have been one element omitted, this is not the case here since this term could be s alone. We are done by applying the Subword Property. $\square$

Theorem 2.2.1. For all $x, w \in W$, there exists polynomials $R_{x,w}(q) \in \mathbb{Z}[q]$ of degree $\ell(w) - \ell(x)$ such that

$$(T_{w^{-1}})^{-1} = \varepsilon_w q_w^{-1} \sum_{x \leqslant w} \varepsilon_x R_{x,w}(q) T_x.$$

Furthermore, for all $w \in W$, we have that $R_{w,w}(q) = 1$.

Proof. We proceed by induction on w. The proof will also provide an algorithm for the computation of these polynomials. If $\ell(w) = 1$, then $w \in S$. By a small manipulation of relation (4), we get

$$T_s^{-1} = q^{-1}T_s - (1 - q^{-1})T_1 = -q^{-1}(-T_s + (q-1)T_1).$$

And we are done by setting $R_{1,s}(q) = q - 1$. For convenience, in what follows we set $R_{x,w} = 0$ when $x \not\leqslant w$. Let w be of positive length, then we can find $s \in S$ and $v \in W$ such that $w = sv$ and $\ell(v) < \ell(w)$. We observe that this means that $\varepsilon_w = -\varepsilon_v$ and $q_w = q_v q$. Using the induction hypothesis we get:

$$\begin{aligned}
(T_{w_{-1}})^{-1} &= (T_{v^{-1}}T_s)^{-1} \\
&= (T_s)^{-1}(T_{v^{-1}})^{-1} \\
&= q^{-1}(T_s - (q-1)T_1)(\varepsilon_v q_v^{-1} \sum_{y \leqslant v} \varepsilon_y R_{y,v} T_y) \\
&= \varepsilon_w q_w^{-1} \left[(q-1) \sum_{y \leqslant v} \varepsilon_y R_{y,v} T_y - \sum_{y \leqslant v} \varepsilon_y R_{y,v} T_s T_y \right]. \qquad (5)
\end{aligned}$$

The second sum that appears in the last right term involves two types of possible terms. If $sy > y$, we get $\varepsilon_y R_{y,v} T_{sy}$. But if $sy < y$, we get a term of the following form:

$$(q-1)\varepsilon_y R_{y,v} T_y + q\varepsilon_y R_{y,v} T_{s,y}.$$

The first term of this is equal and opposite of a term in the first sum of (5). Therefore we can divide all the summing terms in (5) in three categories:

$$\begin{aligned}
&y \leqslant v,\ y < sy, \qquad && (q-1)\varepsilon_y R_{y,v} T_y; \qquad && (6)\\
&y \leqslant v,\ y < sy, \qquad && -\varepsilon_y R_{y,v} T_{sy}; \qquad && (7)\\
&y \leqslant v,\ y > sy, \qquad && q\varepsilon_y R_{y,v} T_{sy}. \qquad && (8)
\end{aligned}$$

In each case we have $y < w$, and using the previous lemma we also have $sy \leqslant w$. Notice that, thinking about elements as subexpression, that every $x \leqslant w$ occurs either as $y \leqslant v$ or as sy with $y \leqslant v$. So the only thing left to do is check that the coefficient in (5) satisfies the required properties.

Let us consider the case $x \leqslant w$ and $x > sx$. In this situation T_x can appear only in case (8), with $x = sy$ (with $y \leqslant v$), and our coefficient is $-\varepsilon_y R_{y,v} = \varepsilon_x R_{sx,sw}$, whose degree is $\ell(sw) - \ell(sx) = \ell(w) - \ell(x)$. In the boundary case $x = w$, we have $y = v$ and $R_{v,v} = 1$. We finish by defining $R_{x,w} := R_{sx,sw}$.

We go on taking into exam the case $x < w$ and $x < sx$. We consider two possible situations:

1. If $sx < v$, T_x occurs both in a term of type (6), with $x = y$, and in a term of type (8) with $x = sy$, so that $y = sx \leqslant v$. Adding the two terms together, we get a coefficient equal to
$$(q-1)\varepsilon_x R_{x,v} - q\varepsilon_{sx} R_{sx,v}.$$
We can thus define $R_{x,w} := (q-1)R_{x,sw} + qR_{sx,sw}$, this has the right degree since by induction $\deg(qR_{sx,v}) = \ell(v) - \ell(sx) + 1 = (\ell(w)-1) + (\ell(x)+1) + 1 = \ell(w) - \ell(x) - 1$ while $\deg((q-1)R_{x,v}) = \ell(v) - \ell(x) + 1 = (\ell(w)-1) - \ell(x) + 1 = \ell(w) - \ell(x)$, thus no cancellation can occur in the leading term and the polynomial $R_{x,w}$ has the right degree.

2. If $sx \not\leqslant v$, then T_x occurs in a term of type (6) with coefficient $\varepsilon_x(q-1)R_{x,c}$. Using the convention $R_{sx,v} = 0$ introduced earlier, we can conclude exactly as we did in 1.

□

We now introduce an involution $\iota : \mathcal{H} \to \mathcal{H}$, defined as follows
$$\sum_{w \in W} p_w(q) T_w \overset{\iota}{\mapsto} \sum_{w \in W} p_w(q^{-1})(T_{w^{-1}})^{-1}.$$
The fact that $\iota^2(T_s) = T_s$ is a consequence of relation (4), which we used to prove the inversion formula for T_s, explicitly:
$$\begin{aligned}\iota^2(T_s) &= \iota(q^{-1}T_s - (1-q^{-1})T_1) = q \cdot \iota(T_s) - (1-q)T_1 \\ &= q(q^{-1}T_s - (1-q^{-1})T_1) - (1-q)T_1 = T_s - (q-1)T_1 + (q-1)T_1 \\ &= T_s.\end{aligned}$$
Therefore, it is enough to show that ι is a ring homomorphism to show that it is an involution, remembering that if $w = s_1 s_2 \dots s_r$ is a reduced expression then $T_w = T_{s_1} \dots T_{s_r}$. To begin, we show that of $s \in S$ and $w \in W$, then $\iota(T_s T_w) = \iota(T_s)\iota(T_w)$. If $\ell(sw) > \ell(w)$, then
$$\iota(T_s T_w) = \iota(T_{sw}) = (T_{w^{-1}s})^{-1} = (T_{w^{-1}} T_s) = T_s^{-1}(T_{w^{-1}})^{-1} = \iota(T_s)\iota(T_w).$$
If $\ell(sw) < \ell(w)$, we define $v = (sw)^{-1}$, so that $w^{-1} = vs$, then we can calculate
$$\iota(T_s T_w) = \iota(qT_{sw} + (q-1)T_w) = q^{-1}T_v^{-1} + (q^{-1}-1)(T_{w^{-1}})^{-1}.$$
We remark that the following facts hold:

1. $q^{-1} - 1 = -q^{-1}(q-1)$;

2. $(T_{w^{-1}})^{-1} = (T_{vs})^{-1} = T_s^{-1}T_v^{-1}$;

3. $T_s^{-1} = q^{-1}(T_s - (q-1)T_1)$.

Substituting in the previous expression we obtain:

$$\begin{aligned}
\iota(T_sT_w) &= q^{-1}T_v^{-1} - q^{-1}(q-1)(T_s^{-1}T_v^{-1}) \\
&= q^{-1}T_v^{-1} - q^{-1}(q-1)[(T_s - (q-1)T_1)q^{-1}T_v^{-1}] \\
&= [q^{-1} + q^{-2}(q-1)^2]T_v^{-1} - q^{-2}(q-1)T_sT_v^{-1} \\
&= q^{-2}(q^2 - q + 1)T_v^{-1} - (q-1)q^{-2}T_sT_v^{-1}.
\end{aligned}$$

On the other hand we calculate directly $\iota(T_s)\iota(T_w)$, using relation (4) and the already used expression for T_s^{-1} and obtain:

$$\begin{aligned}
\iota(T_s)\iota(T_w) &= T_s^{-1}(T_w^{-1})^{-1} = (T_s)^{-2}T_v^{-1} \\
&= [q^{-1}(T_s - (q-1)T_1)]^2T_v^{-1} \\
&= q^{-2}[T_s^2 - 2(q-1)T_s + (q-1)^2T_1]T_v^{-1} \\
&= q^{-2}[(q-1)T_s + qT_1 - 2(q-1)T_s + (q^2 - 2q + 1)T_1]T_v^{-1} \\
&= q^{-2}[(q^2 - q + 1)T_1 - (q-1)T_s]T_v^{-1}.
\end{aligned}$$

And these two quantities are equal as desired. We can prove that $\iota(T_{w'}T_w) = \iota(T_{w'})\iota(T_w)$ using induction on $\ell(w')$ (the case $\ell(w') = 1$ being already dealt with). Let $s \in S$ be such that $\ell(w's) < \ell(w')$. Then using induction we get:

$$\begin{aligned}
\iota(T_{w'}T_w) &= \iota(T_{w's}T_sT_w) \\
&= \iota(T_{w's})\iota(T_sT_w) \\
&= \iota(T_{w's})\iota(T_s)\iota(T_w) \\
&= \iota(T_{w's}T_s)\iota(T_w) \\
&= \iota(T'_w)\iota(T_w).
\end{aligned}$$

And we are done.

The involution we have just defined will be crucial later on, when defining Kazhdan-Lusztig polynomials. It is used to define another natural basis of the Hecke algebra, whose change of base matrix is given by these important polynomials. To end this section, we prove some of the main results about the behaviour of the R-polynomials with this involution.

To continue we prove some of the main properties of these polynomials, both for the sake of completeness and to get used to working with these objects. In what follows, in order to avoid cumbersome notation, we introduce the following notation: if $p \in \mathbb{Z}[q]$ is a polynomial, we write $\overline{p}$ for $p(q^{-1})$.

Proposition 2.2.1. If $x, w \in W$, then the following hold:

(a) $\bar{R}_{x,w} = \varepsilon_x \varepsilon_w q_x q_w^{-1} R_{x,w}$;

(b) $(T_{w^{-1}})^{-1} = \sum_{x \leqslant w} q_x^{-1} \bar{R}_{x,w} T_x$;

(c) $\sum_{x \leqslant y \leqslant w} \varepsilon_x \varepsilon_y R_{x,y} R_{y,w} = \delta_{x,w}$.

Proof. (a) To prove this result we will use much of the inductive definition of R-polynomials made explicit in the proof of existence of these polynomials. We will therefore divide our proof into cases. Let $s \in S$ be such that $sw < w$, we divide our argument into cases. If $x < w$, $sx < x$ and $sw < w$, by how the R-polynomials were constructed we have $R_{x,w} = R_{sx,sw}$; thus by induction we get

$$\bar{R}_{sx,sw} = \varepsilon_{sx} \varepsilon_{sw}\, q_{sx} q_{sw}^{-1}\, R_{sx,sw} = (-\varepsilon_x)(-\varepsilon_w)\, q_x q^{-1} q_w^{-1} q\, R_{x,w} = \varepsilon_x \varepsilon_w\, q_x q_w^{-1}\, R_{x,w}.$$

If $x < w$, $x < sx$ and $sw < w$, the existence proof of the previous theorem tells us that $R_{x,w} = (q-1)R_{x,sw} + qR_{sx,sw}$. Applying ι and using induction we get:

$$\begin{aligned}
\bar{R}_{x,w} &= -q^{-1}(q-1)\bar{R}_{x,sw} + q^{-1}\bar{R}_{sx,sw} \\
&= q^{-1}(q-1)\ \varepsilon_x \varepsilon_{sw}\, q_x q_{sw}^{-1}\, R_{x,sw} + q^{-1} \varepsilon_{sx} \varepsilon_{sw}\ q_{sx} q_{sw}^{-1}\, R_{sx,sw} \\
&= -q^{-1}(q-1)\ \varepsilon_x(-\varepsilon_w)\ q_x q_w^{-1} q\ R_{x,sw} + q^{-1}\ (-\varepsilon_x)(-\varepsilon_w)\ (q_x q)(q_w^{-1} q)\ R_{sx,sw} \\
&= \varepsilon_x \varepsilon_w\ q_x q_w^{-1}\ ((q-1)R_{x,sw} + qR_{sx,sw}) \\
&= \varepsilon_x \varepsilon_w q_x q_w^{-1} R_{x,w}.
\end{aligned}$$

As required.

(b) By the definition of the R-polynomials we have

$$(T_{w^{-1}})^{-1} = \sum_{x \leqslant w} \varepsilon_w \varepsilon_x\ q_w^{-1} R_{x,w} T_x \underset{(a)}{=} \sum_{x \leqslant w} q_x^{-1} \bar{R}_{x,w} T_x.$$

(c) Applying ι to the equation written in part (b) and substituting the expression that characterizes R-polynomials we get:

$$T_w = \sum_{y \leqslant w} q_y R_{y,w} (T_{y^{-1}})^{-1} = \sum_{y \leqslant w} q_y R_{y,w}\ \varepsilon_y q_y^{-1} \sum_{x \leqslant y} \varepsilon_x R_{x,y} T_x.$$

Comparing the coefficients of each term we get the desired result.

$\square$

2.3 Kazhdan-Lusztig polynomials

In this section we construct the Kazhdan-Lusztig polynomials. These polynomials are much less predictable in their behaviour than R-polynomials (for example their degree is not easily predictable) and understanding their properties is still a topic of active research. The first definition of these polynomials was given by David Kazhdan and George Lusztig in [12]. In the same paper, they conjectured a relationship between values of these polynomials at 1 and certain numbers important for the study of certain algebraic objects called *Verma modules.*

The first step in the definition is the construction of another basis for $\mathcal{H}$, whose elements are left invariant by ι. First of all, we enlarge the ring $\mathbb{Z}[q, q^{-1}]$ and work with $\mathcal{H}$ considered as a $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-module. The previous results and calculations are left unchanged by this operation. We observe that, for example, the following elements are fixed by ι:

$$C_s := q^{-1/2}(T_s - qT_1).$$

This will be our first step in the definition of the desired basis.

Theorem 2.3.1. For each $w \in W$ there is a unique element $C_w \in \mathcal{H}$ satisfying the following:

(1) $\iota(C_w) = C_w$;

(2) There exist a unique set of polynomials $P_{x,w} \in \mathbb{Z}[q]$ called *Kazhdan-Lusztig polynomials* that satisty the following :

  (a) $P_{w,w} = 1 \quad \forall w \in W$;

  (b) $\deg(P_{x,w}) \leqslant \frac{1}{2}(\ell(w) - \ell(x) - 1)$ when $x < w$;

  (c) $C_w = \varepsilon_w q_w^{1/2} \sum_{x \leqslant w} \varepsilon_x q_x^{-1} \bar{P}_{x,w} T_x$.

Proof. We begin with proving uniqueness. For notational ease, we define $a(x, w) = \varepsilon_w \varepsilon_x q_w^{1/2} q_x^{-1}$. We fix $w \in W$ and proceed by induction on $\ell(w) - \ell(x)$, and start by requiring that $P_{w,w} = 1$. We start by observing that this equality holds:

$$\begin{aligned} C_w = \iota(C_w) &= \sum_{y \leqslant w} a(y, w) P_{x,w} (T_{y^{-1}})^{-1} \\ &= \sum_{y \leqslant w} a(y, w) P_{x,w} \varepsilon_y q_w^{-1} \sum_{x \leqslant y} \varepsilon_x R_{x,y}(q) T_x \\ &= \varepsilon_w q_w^{-1/2} \sum_{x \leqslant y \leqslant w} \varepsilon_x R_{x,y} P_{y,w} T_x. \end{aligned}$$

By equating this quantity to the original expression given in (c) for C_w, and by equating the coefficient of T_x in both expressions we get that these implications are true:

$$\begin{aligned}
&\varepsilon_w q_w^{1/2}\varepsilon_x q_x^{-1}\bar{P}_{x,w} = \varepsilon_w q_w^{-1/2}\sum_{x\leqslant y\leqslant w}\varepsilon_x R_{x,y}P_{y,w}\\
\Longrightarrow\ & q_w^{1/2}q_x^{-1/2}\bar{P}_{x,w} = q_w^{-1/2}q_x^{1/2}\sum_{x\leqslant y\leqslant w}R_{x,y}P_{y,w}\\
\Longrightarrow\ & q_w^{1/2}q_x^{-1/2}\bar{P}_{x,w} - q_w^{-1/2}q_x^{1/2}P_{x,w} = q_w^{-1/2}q_x^{1/2}\sum_{x< y\leqslant w}R_{x,y}P_{y,w}. \qquad (6)
\end{aligned}$$

We can now assume that all $P_{y,w}$ with $x < y \leqslant w$ are uniquely determined by induction. Since $x < w$, we can assume that the degree assumption stated in (b) holds, meaning that the first term on the left side is a polynomial in $q^{1/2}$ that has no constant term, because of the multiplication with $q_w^{1/2}q_x^{-1/2}$. The second term, on the other hand, is a polynomial in $q^{-1/2}$ without constant term (for the same reason). Thus, there is no cancellation between the terms and the relation we just found forces the choice of all the coefficients.

We continue by proving the existence of the basis C_w and of the polynomials. We start by defining a relation. We write $x \prec w$ (not to be confused with the "being a subword of" relation, which is a relation on expressions and not on elements of the group) when $P_{x,w}$ has degree $1/2(\ell(w) - \ell(x) - 1)$, the maximum possible one. This is possible only if $\varepsilon_w = -\varepsilon_x$. When $x \prec w$ we define $\mu(x, w)$ to be the leading coefficient of $P_{x,w}$. We proceed by induction on $\ell(w)$. If $s \in S$, then we define $C_s := q^{-1/2}(T_s - qT_1)$. Find $s \in S$ such that $\ell(sw) < \ell(w)$ and define $v = sw$, thus C_v is assumed to be known. Note that $a(x, w) = -q^{1/2}a(x, v)$. We define

$$C_w := C_sC_v - \sum_{\substack{z\prec v\\ sz<z}}\mu(z, v)C_z.$$

The fact that this element is invariant under ι is a direct consequence of the induction hypothesis. By induction, C_z is combination of T_x with $x \leqslant z < w$, the same holds for C_v, by how C_s is defined, it is clear that C_w is a linear combination of T_x with $x \leqslant w$. Therefore by convention, we set $P_{x,w} = 0$ if $x \nleqslant w$.

We now have to analyse the coefficients of each T_x, define the polynomials in each case and prove the degree inequality. Firstly, we study the coefficients appearing in the product C_sC_v, the ones appearing in the summation will be considered later.

If $x = w$, the only place in which T_w can occur is in the product C_sC_v, its coefficient is $q^{-1/2}a(v, v)\bar{P}_{v,v} = q^{-1/2}q_v^{1/2}q_v^{-1} = q_w^{-1/2}$, as required.

Let $x < w$ then T_x can occur either in C_v already, and it stays in the expression when multiplied by T_1, or in the product T_sC_v, where a product T_sT_{sx} appears if $sx \leqslant v$.

If $x < sx$, then $T_sT_{sx} = qT_x + (q-1)T_{sx}$, and $q^{-1/2}T_sC_v$ involves T_x with coefficient (remember $q^{-1/2}$ from the definition of C_s):

$$q^{-1/2}q\, a(sx,v)\bar{P}_{sx,v} = q^{-1/2}(-1^{-1})a(x,v)\bar{P}_{sx,v} = q^{-1}a(x,w)\bar{P}_{sx,v}.$$

While the other term in the product C_sC_v, which is $-q^{-1/2}T_1T_v$ involves T_x with coefficient:

$$q^{-1/2}a(x,v)\bar{P}_{x,v} = a(x,v)\bar{P}_{x,v}.$$

Thus, the combined coefficient of T_x in the product C_sC_v is

$$q^{-1}a(x,w)\bar{P}_{x,v} + a(x,w)\bar{P}_{x,v}.$$

Suppose now $sx < x$, so $T_sT_{sx} = T_x$ and $T_sT_x = qT_{sx} + (q-1)T_x$. Then, reasoning as done before, in the term $q^{-1/2}T_sC_v$ we get coefficients of T_x equal to

$$q^{-1/2}a(sx,v)\bar{P}_{sx,v} = q^{-1/2}(-q)a(x,v)\bar{P}_{sx,v} = a(x,w)\bar{P}_{sx,v},$$

$$(q-1)q^{-1/2}\, a(x,v)\bar{P}_{x,v}.$$

On the other hand, the term $-q^{-1/2}T_1C_v$ involves T_x with coefficient

$$-q^{1/2}a(x,v)\bar{P}_{x,v} = a(x,w)\bar{P}_{x,v}.$$

Combining these, the coefficient of T_x is equal to

$$a(x,w)\bar{P}_{sx,v} + q^{-1}a(x,w)\bar{P}_{x,v}.$$

To conclude, we have to check the coefficients that appear in the sum $-\sum_z \mu(z,v)C_z$. Recalling the fact that since all z in the sum are such that $z \prec v = sw$, then necessarily $\varepsilon_z\varepsilon_w = 1$, and we can start by making explicit the terms in the sum:

$$-\sum_{\substack{z\prec v\\ sz<z}} \mu(z,v)a(x,v)\bar{P}_{x,z} = -\sum_{\substack{z\prec v\\ sz<z}} \mu(z,v)q_z^{1/2}q_w^{-1/2}a(x,w)\bar{P}_{x,z}.$$

By defining c to be 0 if $x < sx$ and 1 if $sx < x$, we can state these results together as

$$P_{x,w} := q^{1-c}P_{sx,v} + q^cP_{x,v} - \sum_{\substack{z\prec v\\ sz<z}} \mu(z,v)q_z^{-1/2}q_w^{1/2}P_{x,z}.$$

We now have to prove the upper bound on the degree of these polynomials. We omit the explicit calculation in the cases when $c = 0$ or $sx > x$. In the only left case, the calculation is less direct, since the term $qP_{x,v}$ could have degree $1/2(\ell(w) - \ell(x))$. But in this case, by definition, $x \prec v$ and since $sx < x$, there is a term in the sum correspondig to $z = x$ which is equal to $-\mu(x,v)q_x^{-1/2}q_w^{1/2}P_{x,x} = -\mu(x,v)q^{1/2(\ell(w)-\ell(x))}$, which is exactly the term we wanted to be cancelled. Moreover we observe that this is the only case in which $P_{x,x}$ appears in the sum, thus we can inductively apply the bound in all the other situations.

This concludes the proof. □

Remark 2.3.1. It is important to note that equation (6) can be written as

$$q_u^{-1}q_w\bar{P}_{x,w} = \sum_{x\leqslant y\leqslant w} R_{x,y}P_{y,w}.$$

This gives us a way to construct inductively Kazhdan-Lusztig polynomials if we already know all R-polynomials. This will be crucial in the next chapter, as we will prove an important result about R-polynomials and, because of this remark, we will be able to extend the claim to Kazhdan-Lusztig polynomials.

Chapter 3

Special matchings and the combinatorial invariance conjecture

Having defined the R-polynomials and the Kazhdan-Lusztig polynomials in Chapter 2, the present chapter is dedicated in exposing a partial result of one of the main open problems regarding these polynomials, the so called *combinatorial invariance conjecture.* Because of the surprising connections with algebra and geometry that these polynomials have, it has become of interest understanding how to ease the hard task of carrying out their calculation explicitly. To state the conjecture we introduce the following notations: if $(P, \leqslant)$ is a partially ordered set and $x, y \in P$, we denote $[x, y]$ the set $\{z \in P : x \leqslant z \leqslant y\}$ and call this an *interval* of P. In what follows, we assume that every Coxeter group and its subsets are partially ordered by the Bruhat order. A poset isomorphism is an order preserving bijection whose inverse is order preserving.

The conjecture states the following:

Conjecture (Combinatorial invariance conjecture)**.** Suppose that $(W, S), (W', S')$ are two Coxeter systems and that the elements $u, v \in W$ and $u', v' \in W'$ are such that $[u, v]$ and $[u', v']$ are isomorphic as posets. Then $P_{u,v} = P_{u',v'}$.

This conjecture has been proved in several special cases. For example it has been verified for small rank finite Coxeter groups, for so called *short edge intevals* (see [8]) and for *lower intervals* (as in [7]). An interval $[u, v]$ in a Coxeter group is a short edge interval if all edges $y \to y'$ in the Bruhat graph restricted to $[u, v]$ are such that $\ell(y') - \ell(y) = 1$. Lower intervals are intervals in the Bruhat order of the type $[1, w]$. In this chapter, we will define and study the main tools used for the proof of the conjecture for lower intervals, following [7].

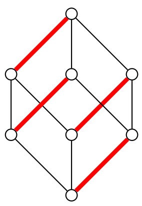
3.1 Special matchings

Firstly, we establish some notations and terminology for partially ordered sets. Let $(P, \leqslant)$ be a partially ordered set (poset for short) and $x, y \in P$. We say that x and y are comparable if either $x \leqslant y$ or $y \leqslant x$ and incomparable otherwise. If $x \leqslant y$ and $|[x, y]| = 2$ we say that y covers x and write $x \lhd y$. If $z \in [x, y]$, we say that z is an atom (respectively coatom) of $[x, y]$ if $x \lhd z$ (respectively $z \lhd y$). A poset P has a minimum (respectively a maximum) if there is an element $\hat{0}$ (respectevely $\hat{1}$) such that for all $x \in P$ $\hat{0} \leqslant x$ (respectively $x \leqslant \hat{1}$). A graded poset is a poset P that has a minimum and has a rank function on it. A rank function is a function $\rho : P \to \mathbb{N}$ such that $\rho(\hat{0}) = 0$ and whenever $x \lhd y$, we have $\rho(y) = \rho(x) + 1$.

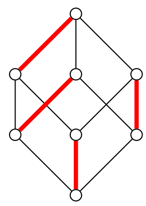
Remark 3.1.1. Let (W, S) be a Coxeter system, then from the results we got in Chapter 1, we can conclude that W with the Bruhat ordering is a graded poset whose rank function is the length function ℓ.

If $G = (V, E)$ is a graph, a matching of G is an involution $M : V \to V$ such that for all vertices $v \in V$, we have $\{v, M(v)\} \in E$. The Hasse diagram of P is the graph $H(P) = (P, E)$ having defined $E = \{\{x, y\} \mid x, y \in P \text{ and either } x \lhd y \text{ or } y \lhd x\}$. Now we have the necessary terminology to define a special matching

Definition. Let P be a poset, a matching M of the Hasse diagram of P is said to be a *special matching* if for all $x, y \in P$ such that $x \lhd y$ and $M(x) \neq y$, we have $M(x) \leqslant M(y)$.



(a) *A special matching*



(b) *Not a special matching*

Figure 3.1: *Examples.*

Remark 3.1.2. By the definition of a special matching we can see that if $x, y \in P$ are such that $x \lhd y$ and $x \lhd M(x)$, then $M(x) \lhd M(y)$. This will be extensively used in what follows.

Let v be an element of a Coxeter group W and let s be such that $\ell(vs) < \ell(v)$ (respectively $\ell(sv) < \ell(v)$), we can define the matching ρ_s (respectively λ_s) of $[1, v]$ by $\rho_s(u) = us$ (respectively $\lambda_s(u) = su$). A matching of this form is called a *multiplication matching.* The lifting property of the Bruhat ordering tells us that these matchings are special matchings of $[1, v]$.

3.2 Some additional facts about the Bruhat order

We will gather here some preliminary results about the Bruhat order that were not proved in previous chapters. We will prove just the most important result that will be used repeatedly in future proofs.

Lemma 3.2.1. Let (W, S) be a Coxeter system and and let $t_1, \ldots t_{2n} \in T$ $(n \in \mathbb{N})$ be such that $t_1t_2 = t_3t_4 = \cdots = t_{2n-1}t_{2n} \neq 1$. Then $W' = \langle\{t_1, \ldots, t_{2n}\}\rangle$ is a dihedral reflection subgroup.

Proof. See [4, Lemma 3.1]. □

Theorem 3.2.1. Suppose that (W, S) is a Coxeter system and that $a, b \in W$ are such that either $|\{w \in W \mid w \lhd a,\ w \lhd b\}| \geqslant 3$ or $|\{w \in W \mid a \lhd w,\ b \lhd w\}| \geqslant 3$. Then $a = b$.

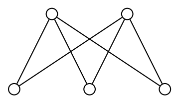


Figure 3.2: *This configuration is forbidden in a Coxeter group by Theorem 3.2.1.*

Proof. We prove the first case, the second is proved using the same argument. Suppose that $a \neq b$ and let $x, y, z \in \{w \in W \mid w \lhd a,\ w \lhd b\}$. By definition of the Bruhat order, there are $t_1, \ldots, t_6 \in T$ such that $at_1 = x$, $at_3 = y$, $at_5 = z$, $bt_2 = x$, $bt_4 = y$, $bt_5 = z$. Then the following implication is true:

$$at_1t_2 = xt_2 = at_3t_4 = at_5t_6 = b \implies t_1t_2 = t_3t_4 = t_5t_6 = a^{-1}b \neq 1.$$

Using the above lemma, we can conclude that $W' := \langle\{t_1, \ldots, t_6\}\rangle$ is a dihedral reflection subgroup, and clearly a, b, x, y, z are in the coset aW'. Using Theorem 1.4 of [4], we know that the subgraph of the Bruhat graph of W generated by the set of vertices aW' is isomorphic to the Bruhat graph of W'. But this is a contradiction since W' is dihedral, and x, y, z are incomparable. Thus $a = b$. □

This result motivates the following definition.

Definition. Let P be a graded poset, we say that P *avoids* $K_{3,2}$ if there does not exist distict elements $a_1, a_2, a_3, b_1, b_2 \in P$ such that $a_i \lhd b_j$ for all $i = 1, 2, 3$, $j = 1, 2$ or $b_j \lhd a_i$ for all $i = 1, 2, 3$, $j = 1, 2$.

So the theorem we just proved says that a Coxeter group with the Bruhat order always avoids $K_{3,2}$. We also give here the definition of a *dihedral interval.*

Definition. Let P be a poset and let $u, v \in P$, we call the interval $[u, v]$ *dihedral* if it is isomorphic to a finite Coxeter system of rank $r \leqslant 2$ ordered with the Bruhat ordering.

Corollary 3.2.1. Let (W, S) be a Coxeter system, and $u, v \in W$. If $[u, v]$ has two coatoms, then it is dihedral.

Proof. It is true that if $x, y \in W$ are such that $y \leqslant x$ and $\ell(x) - \ell(y) = 2$ then $[y, x]$ is a Boolean poset of rank 2 (see [5, Lemma 2.7.3]). Using this fact and Theorem 3.2.1 it is possible to prove by induction that for all $i \in \{1, \dots, \ell(v) - \ell(u) - 1\}$ it holds that $|\{w \in [u, v] \mid \ell(v) - \ell(u) = i\}| = 2$. □

We now introduce the following notation, if $I \subseteq S$ and $w \in W$, we denote $W_I \cap [1, w]$ as $W_I(w)$. It is known that there exists a unique maximal element in $W_I(w)$, denoted $w[I]$, such that $W_I(w) = [1, w[I]]$. For a proof of this result, see [10, Lemma 7].

3.3 Pairs of special matchings

In this section we move away from the setting of a Coxeter group and prove some general results about special matchings. Firstly, we remark that since a matching is an function, two matching can be composed. Given two matchings M, N of a poset P, we can study the orbits of the group $\langle M, N\rangle \subseteq \mathrm{Sym}(P)$. If $x \in P$, we denote the orbit of x by $\langle M, N\rangle(x)$.

Lemma 3.3.1. If P is a finite poset, M, N are two special matchings of P and $u \in P$, then $\langle M, N\rangle(u)$ is a dihedral interval.

Proof. Since P is finite, every orbit has to be finite. Thus, there exist an element x such that $M(x), N(x) \lhd x$. If $M(x) = N(x)$ we have that $\langle M, N\rangle(x) = \{x, M(x)\}$ and we are done. Otherwise, using Remark 3.1.2, we can see that the relation below hold:

$$NM(x) \lhd M(x), \qquad NM(x) \lhd N(x), \qquad MN(x) \lhd N(x), \qquad MN(x) \lhd M(x).$$

If $MN(x) = NM(x)$, then $\langle M, N\rangle(u) = \{x, M(x), N(x), MN(x)\}$ and we are done. Otherwise we iterate the previous argument, the iteration must terminate since the orbit is finite. □

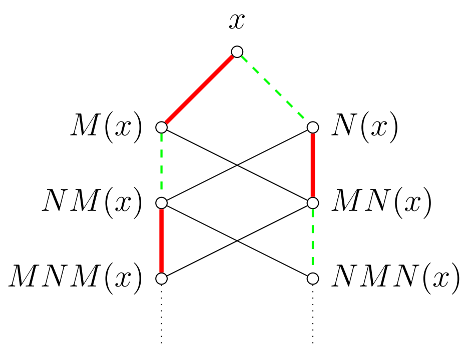


Figure 3.3: *Proof of Lemma 3.3.1*

Proposition 3.3.1. Let P be a finite graded poset that avoids $K_{3,2}$, and let $v \in P$, and M, N be two special matchings of P such that $M(v) \neq N(v)$. If $v' \in P$ is distinct from $M(v)$ and $N(v)$, and it is such that one of the two following conditions hold:

(i) $M(v) \lhd v$, $N(v) \lhd v$ and $v' \lhd v$,

(ii) $v \lhd M(v)$, $v \lhd N(v)$ and $v \lhd v'$;

then $|\langle M, N\rangle(v)| = |\langle M, N\rangle(v')|$.

Proof. We write the proof with hypothesis (i), the case (ii) is similar. Let $|\langle M, N\rangle(v)| = 2n$ and $|\langle M, N\rangle(v')| = 2m$. If v' was in the same orbit of v, because of their length difference it would have to be either $M(v)$ or $N(v)$, which we supposed to be false. Thus, the two orbits have to be disjoint, so we have no matchings between $\langle M, N\rangle(v)$ and $\langle M, N\rangle(v')$. Starting from the hypotheses and using repeatedly Remark 3.1.2, we get the following relations, which hold for all $k \leqslant n$:

$$\underbrace{MNM\ldots}_{k}(v') \lhd \underbrace{MNM\ldots}_{k}(v), \qquad \underbrace{MNM\ldots}_{k}(v') \lhd \underbrace{NMN\ldots}_{k-1}(v'),$$
$$\underbrace{NMN\ldots}_{k}(v') \lhd \underbrace{NMN\ldots}_{k}(v), \qquad \underbrace{NMN\ldots}_{k}(v') \lhd \underbrace{MNM\ldots}_{k-1}(v').$$

So it must be that $m \geqslant n$. If $m \neq n$, the we would get that $\underbrace{MNM\ldots}_{n}(v') \neq \underbrace{NMN\ldots}_{n}(v')$ but $\underbrace{MNM\ldots}_{n}(v) = \underbrace{NMN\ldots}_{n}$, and this contradicts the fact that P avoids $K_{3,2}$. The following figure, drawn in the case $n = 3$ should make the argument clear. $\square$

Lemma 3.3.2. Let P be a graded poset, M a special matching of P, and $u, v \in P$ be such that $M(v) \lhd v$ and $M(u) \lhd u$. Then M restricts to a special matching of $[u, v]$.

Proof. Omitted. $\square$

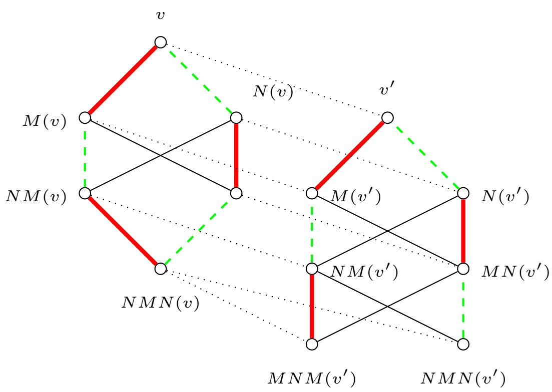


Figure 3.4: *Proof of Proposition 3.3.1 in the case* $n = 3$.

We now return to the setting of a Coxeter group W ordered with the Bruhat order. We consider the poset P to be an interval of the form $[1, v]$ with $v \in W$. To simplify the terminology we will refer to a special matching of $[1, v]$ as a special matching of v.

Lemma 3.3.3. Let $u, v \in W$, $u \leqslant v$ and let M, N be two special matchings of v. If $|\langle M, N\rangle(u)| = 2m > 2$, then there exists an element $u' \in W$ and a dihedral interval $I \subseteq W$ such that $1, N(1), M(1) \in I$, $\langle M, N\rangle(u') \subseteq I$. In particular, if $M(1) \neq N(1)$ then $W_{\{M(1),N(1)\}}$ contains an orbit of cardinality $2m$.

Proof. We have shown that every orbit of $\langle M, N\rangle$ has the form shown in figure 3.3, therefore we can assume that $M(u), N(u) \lhd u$. To prove the result we will find a sequence $u = u_1 \rhd u_2 \rhd \cdots \rhd u_k$ with the following properties holding for all $i = 1, \ldots, k$:

$$M(u_i), N(u_i) \lhd u_i, \qquad |\langle M, N\rangle(u_i)| = 2m, \qquad [1, u_k] \text{ is dihedral.}$$

In fact, if $[1, u]$ only has the coatoms $M(u)$ and $N(u)$, by Corollary 3.2.1 we are done. Otherwise we choose u_2 two be one of the coatoms different from $M(u)$ and $N(u)$, by Proposition 3.3.1 we have that $|\langle M, N\rangle(u_2)| = 2m$ and $M(u_2), N(u_2) \lhd u_2$ (see the figure above). If $M(u_2)$ and $N(u_2)$ are the only coatoms of u_2 we are done, otherwise we iterate the argument. □

3.4 Algebraic properties of special matchings

We gather here some results about special matchings of lower intervals. All the results here are preliminaries for the proof of the main result.

Lemma 3.4.1. Let $u, v \in W$, $u \leqslant v$ and M a special matching of w. If $u \notin \bigcup_{t \in S} W_{\{t, M(1)\}}$ and $u \lhd M(u)$, then

$$|\{x \in [1, u] \mid x \lhd u \text{ and } x \lhd M(x)\}| \geqslant 2.$$

Proof. By Lemma 3.3.2, if we have an element $v \in W$ satisfying $M(v) \lhd v$, then M restricts to a special matching of v, and in particular it must be that $M(1) \leqslant v$. Thus, if $M(1) \nleqslant u$, then $x \lhd M(x)$ must hold for all $x \in [1, u]$ and we are done. If $M(1) \leqslant u$, then our hypotheses tell us that u is not in any parabolic subgroup of W, thus $[1, u]$ cannot be dihedral, hence $[1, M(u)]$ has at least two distinct coatoms, say x_1 and x_2. Using Remark 3.1.2, we conclude that $M(x_i) \lhd x_i$ and $M(x_i) \lhd u$ for $i = 1, 2$, and we are done. □

The following lemma is used in the proof of many of the results of the next sections.

Lemma 3.4.2. Let $u, w \in W$, $u \leqslant w$, M a special matching of w and $s := M(1)$. If for all $x \in \bigcup_{t \in S} W_{\{s,t\}}(u)$ we have that $M(x) = xs$, then $M(u) = us$.

Proof. The proof is made by induction on $\ell(u)$. The case $u = 0$ is clear. If $M(u) \lhd u$ by induction $u = MM(u) = M(u)s$, multiplying both side by s to the right gives us $M(u) = us$. Therefore, we can assume $u \lhd M(u)$, and similarly we obtain that $u \lhd us$. Clearly, we can assume that $u \notin \bigcup_{t \in S} W_{\{s,t\}}$, otherwise the claim would be immediate. Thus, using the previous lemma, there are two distinct elements u_1, u_2 such that $u_i \lhd u$, $M(u_i) \rhd u_i$. Using induction, we can assume that $M(u_i) = u_i s$. So we have that $us \rhd u, M(u_1), M(u_2)$, but using Remark 3.1.2, we can conclude that the same holds for $M(u)$. Using Theorem 3.2.1, we obtain the desired claim. □

The next two results tell us respectively how special matchings behave with respect to parabolic subgroups, and what condition they must satisfy in order to be different from a multiplication matching. The following "invariance" property will be used in what follows.

Proposition 3.4.1. If $w \in W$ and M is a special matching of w, then, for all $I \subseteq S$ such that $M(1) \in I$, M stabilizes $W_I(w)$.

Proof. Let $u \in W_I(w)$, we proceed by induction on $\ell(u)$. The case $\ell(u) = 0$ is trivial. We recall that there exists a unique element $w[I]$ such that $W_I(w) = [1, w[I]]$. If $M(u) \lhd u$, then $1 \leqslant M(u) \lhd u \leqslant w[I]$, and we are done. If $u \lhd M(u)$, let $x \lhd M(u)$, $x \neq u$. Then $M(x) \lhd u$ and by induction $x \in W_I(w)$. Hence, all the coatoms of $[1, M(u)]$ are in $W_I(w)$, so $M(u) \in W_I(w)$. □

Corollary 3.4.1. If M, N are two special matchings of w, and $M = N$ on $\bigcup_{t\in S} W_{\{s,t\}}(u)$ (with $s = M(1)$), then $M(u) = N(u)$.

Lemma 3.4.3. Let $w \in W$, M a special matching of w, $s := M(1)$ and $r, t \in S$. If $M(t) = ts \neq st$ and $M(r) = sr \neq rs$, then $rst \not\leqslant w$. Moreover, if $rt \neq tr$, then $rt \not\leqslant w$.

Proof. Suppose that $rt \leqslant w$. Using Remark 3.1.2 we obtain that $rt \lhd M(rt)$, $ts \lhd M(rt)$ and $sr \lhd M(rt)$. If $rt \neq tr$ then there are no such elements and this proves the second claim. If $rt = tr$, then it must be that $M(rt) = tsr$. If we had that $rst \leqslant w$ then $M(rst)$ would cover both tsr and rst and there are no such elements. □

3.5 Coxeter systems of rank 3

In this section we will study the behaviour of special matchings in Coxeter groups of rank 3. Most importantly, we state results that give information about the action of special matchings on lower intervals. These results will be fundamental for the study of the general case, since they will be applyed to rank 3 parabolic subgroups of general Coxeter groups.

Firstly, we will fix some notation that will be used through all this section. The Coxeter system (W, S) will always be of rank 3 and we will name r, s, t the elements of S. We let $w \in W$ be fixed but arbitrary, M a special matching of w and we assume that $s = M(1)$. For $x, y \in S$ we denote by $\dots xyx$ (respectively $xyx\dots$) be a word given by an alternating product of x and y that ends (respectively begins) with x. All expressions considered for elements of a Coxeter groups are assumed to be reduced unless specified otherwise.

Lemma 3.5.1. If $rs, st \leqslant w$, $rs \neq sr$, $st \neq ts$, $M(t) = ts$ and $M(r) = rs$, then we have that $M(st) = sts$ and $M(sr) = srs$.

Proof. By symmetry of the hypotheses made, it is enough to show that $M(st) = sts$. Using Remark 3.1.2 we have that $st, ts \lhd M(st)$, thus it must be that $M(st) \in \{sts, tst\}$. Using the same argument we can also conclude that $M(sr) \in \{srs, rsr\}$. By contradiction, suppose it were true that $M(st) = tst$. If $str \leqslant w$, then we could apply M and by Remark 3.1.2 we could conclude that $tst \lhd M(str)$ and $M(sr) \lhd M(str)$. But there cannot exist elements that cover both tst and $M(sr)$ (because of the Subword Property), thus $srt \not\leqslant w$. By an analogous argument we conclude that $srt \not\leqslant w$. Considering a reduced expression for w, what we have just proved tells us that tst and either srs or rsr are subwords of this expression. This forces either str or srt to be a subword, contradicting the fact that $str \not\leqslant w$ and $srt \not\leqslant w$. □

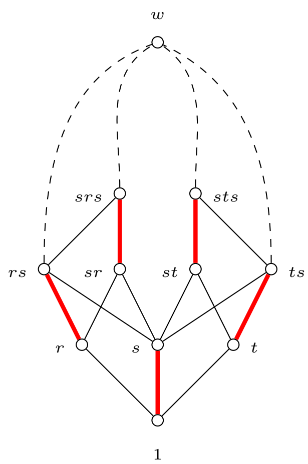


Figure 3.5: *Proof of the Lemma 3.5.1.*

Lemma 3.5.2. Suppose $M(t) = ts$ and $M(r) = rs$, but $M \neq \rho_s$ on $W_{\{s,t\}}(w)$. If x_0 is a minimal element of $W_{\{s,t\}}(w)$ such that $M(x_0) \neq x_0 s$, then

$$\{u \leqslant w \mid x_0 \lhd u,\ u \notin W_{\{s,t\}}\} \subseteq \begin{cases} \{x_0 r, r x_0\} \text{ if } sr = rs, \\ \{r x_0\} \text{ if } sr \neq rs. \end{cases}$$

Proof. Because of the minimality of x_0, we have that $\ell(x_0 s) > \ell(x_0)$, otherwise we would have that $M(x_0 s) = x_0 ss = x_0$, thus $M(x_0) = x_0 s$, contrary to our assumption. With a similar argument we can conclude that $x_0 \lhd M(x_0)$. Let $x_0 = \underbrace{\alpha\beta\alpha \ldots tst}_{k}$ be a reduced expression, with $\{\alpha, \beta\} = \{s, t\}$. Since we have assumed $M \neq \rho_s$ on $W_{\{s,t\}}(w)$, it must be that $st \leqslant w$ and $st \neq ts$, otherwise $W_{\{s,t\}}(w)$ would be too small in all cases to allow a matching to be different from ρ_s. Let u be such that $u \leqslant w$, $x_0 \lhd u$, $u \notin W_{\{s,t\}}$ and assume $u \notin \{x_0 r, r x_0\}$ if $sr = rs$ and $u \neq r x_0$ if $sr \neq rs$. Thus, by the Subword Property, u is obtained by inserting an r in the unique reduced expression of x_0.

We define $y := \alpha u$, by what we have just remarked $y \lhd u$, therefore all the elements in $W_{\{s,t\}}(y)$ are strictly smaller that x_0 (this is evident thinking about the Subword Property). Moreover, the elements in $W_{\{s,r\}}(y)$ are all smaller than srs if $sr \neq rs$ or smaller than sr if $sr = rs$. Hence, using Lemma 3.4.2 and Lemma 3.5.2, we conclude that $M(y) = ys$. Because of the fact that x_0 and y are both covered by u, using the properties of special matching we obtain $u \lhd M(u)$, $M(u) \rhd M(x_0) = \underbrace{\beta\alpha\beta \ldots tst}_{k+1} \neq \underbrace{\alpha\beta\alpha \ldots sts}_{k+1}$ and $M(y) \lhd M(u)$. From these conditions, it is not hard to see that $M(u) = yst$, which is a contradiction since, as one can verify, $yst \ngtr u$. $\square$

From now on, a set of three kinds of hypotheses will be frequently used, for the sake of brevity, we will list them here.

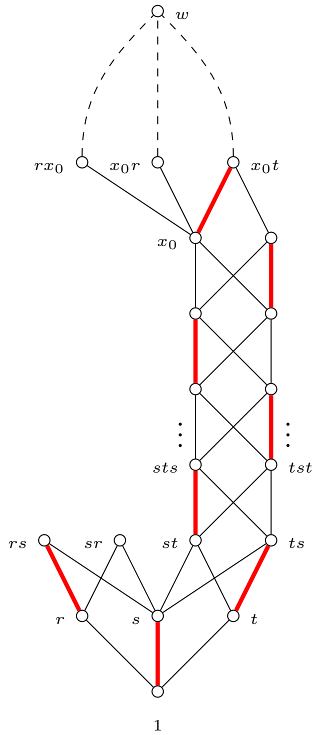


Figure 3.6: *Proof of Lemma 3.5.2.*

(1) $M(t) = ts \neq st$, $M(r) = rs \neq sr$ and $M \neq \rho_s$ on $W_{\{s,t\}}(w)$.

(2) $M(t) = ts \neq st$, $M(r) = rs = sr$ and $M \neq \rho_s$ on $W_{\{s,t\}}(w)$.

(3) $M(t) = ts \neq st$, $M(r) = sr \neq rs$.

In cases (1) and (2), we let x_0 be the unique minimal element of $W_{\{s,t\}}(w)$ such that $M(x_0) \neq x_0 s$ and $\alpha\beta\alpha \ldots tst$ be its unique reduced expression. As observed in the proof of Lemma 3.5.2, $x_0 \lhd x_0 s$.

Proposition 3.5.1. Under hypothesss (1) any element $u \leqslant w$ has a reduced expression of the form $(\ldots r\beta r)\eta(\alpha\beta\alpha\ldots)$, where $\eta \in \{1, \beta\}$.

Under hypothesis (2) any element $u \leqslant w$ has a reduced expression of the form $(\ldots r\beta r)\eta(\alpha\beta\alpha\ldots)\delta$, where $\eta \in \{1, \beta\}$ and $\delta \in \{1, r\}$.

Under hypothesis (3) any element $u \leqslant w$ has a reduced expression of the form $(\ldots tst)\varepsilon(rsr\ldots)$, where $\varepsilon \in \{1, s\}$.

Proof. The proof is omitted here. The interested reader may consult [7, §6]. □

Definition. Let $w \in W$, we say that w is *dihedral* if the interval $[1, w]$ is a dihedral inteval.

Theorem 3.5.1. If (W, S) is a Coxeter system of rank 3, $w \in W$, M a special matching of w and $s := M(e)$, then there exists $x \in S \setminus \{s\}$ such that either $M = \lambda_s$ or $M = \rho_s$ on $W_{\{s,x\}}(w)$.

Proof. We may assume that w is not dihedral, that M is not a multiplication matching and, as a consequence of Proposition 3.4.1, we have that neither of $W_{\{r,s\}}(w)$ and $W_{\{t,s\}}(w)$ have cardinality equal to 4. In particular, $rs \neq sr$ and $ts \neq st$.

Before going on with the proof, we remark that the result is true for a special matching M of w if and only if it is true for the special matching $\tilde{M}$ of w^{-1} defined as $\tilde{M}(x) := (M(x^{-1}))^{-1}$ for $x \leqslant w^{-1}$. If $M(r) = rs$ and $M(t) = ts$ then, using Lemma 3.4.2, we conclude that $M \neq \rho_s$ on $W_{\{s,t\}}(w) \cup W_{\{s,r\}}(w)$ so M satisfies hypothesis (1) (possibly renaming the canonical generators). If $M(r) = sr$ and $M(t) = st$ then $\tilde{M}$ satisfies hypotheses (1). If $M(r) = sr$ and $M(t) = ts$, then M satisfies hypotheses (3). If $M(r) = rs$ and $M(t) = st$ then $\tilde{M}$ satisfies (3). Thus, we only need to consider two cases.

If M is in case (1) we have $\beta = s$, otherwise, as a consequence of the last proposition, $W_{\{r,s\}}(w) = \{1, s, r, rs\}$, which is impossible since $W_{\{r,s\}}(w)$ cannot have this cardinality. By contradicion, assume that $M \neq \rho_s$ on $W_{\{r,s\}}(w)$, let us denote $y_0 \in W_{\{r,s\}}(w)$ a minimal element such that $M(y_0) \neq y_0 s$. Since w is not dihedral, any of its reduced expressions must include the letter t, thus as a consequence of the last proposition $y_0 t \leqslant w$. This, and Lemma 3.5.2 implies $y_0 t = t y_0$, which is a contradiction since $ts \neq st$.

If M is in case (3) we will prove that either $M = \rho_s$ on $W_{\{t,s\}}(w)$ or $M = \lambda_s$ on $W_{\{r,s\}}(w)$. We will proceed by induction on $\ell(w)$. By the last proposition we can write a reduced expression $w = (\underbrace{\ldots tst}_{k})\varepsilon(\underbrace{rsr\ldots}_{h})$ with $\varepsilon \in \{1, s\}$. Since $W_{\{r,s\}}(w)$ and $W_{\{t,s\}}(w)$ cannot have cardinality equal to 4, we have that $h, k \geqslant 2$. Let w_1 and w_2 be two coatoms of $[1, w]$ obtained by omitting respectively the first and last letter of this reduced expression for w. Since w is matched with only one element, either w_1 or w_2 is matched with an element of lower length. Without loss of generality, we assume it to be w_1. By Lemma 3.3.2, M restricts to a special matching of $[1, w_1]$. By induction, either $M = \rho_s$ on $W_{\{t,s\}}(w_1)$ or $M = \lambda_s$ on $W_{\{r,s\}}(w_1)$. In the second case k is odd and we are done since $W_{\{r,s\}}(w_1) = W_{\{r,s\}}(w)$. If $M = \rho_s$ on $W_{\{t,s\}}(w_1)$, then $W_{\{t,s\}}(w) \setminus W_{\{t,s\}}(w_1) = \{\underbrace{\ldots tst}_{k}, \underbrace{\ldots sts}_{k+1}\}$ and since M stabilizes $W_{\{t,s\}}(w)$ by Proposition 3.4.1, it must be that $M(\underbrace{\ldots tst}_{k}) = \underbrace{\ldots sts}_{k+1}$ and hence $M = \rho_s$ on $W_{\{t,s\}}(w)$. □

The next two proposition tell us how special matchings act and behave on a lower interval under the sets of hypotheses (1), (2) and (3). In order to make the claims clearer to read, we introduce the following sets:

$$D_R(w) := \{s \in S \mid \ell(ws) < \ell(w)\}, \qquad D_L(w) := \{s \in S \mid \ell(sw) < \ell(w)\}.$$

Proposition 3.5.2. Under hypotheses (1) if $u \leqslant w$, $u = (\dots r\beta r)\eta(\alpha\beta\alpha\dots)$ where $\eta \in \{1,\beta\}$ and $\beta \notin D_R(\dots r\beta r)$, then $M(u) = (\dots r\beta r)M(\eta\alpha\beta\alpha\dots)$.

Under hypotheses (2) if $u \leqslant w$, $u = (\dots r\beta r)\eta(\alpha\beta\alpha\dots)\delta$ where $\eta \in \{1,\beta\}$, $\delta \in \{1,r\}$ and $\beta \notin D_R(\dots r\beta r)$, then $M(u) = (\dots r\beta r)M(\eta\alpha\beta\alpha\dots)\delta$.

Under hypotheses (3) if $u \leqslant w$, $u = (\dots tst)\varepsilon(rsr\dots)$ where $\varepsilon \in \{1,s\}$ and $s \notin D_L(rsr\dots)$, then $M(u) = M(\dots tst)\varepsilon(rsr\dots)$.

Proof. See [7, Proposition 6.5]. □

Proposition 3.5.3. Under hypotheses (1) write $w = (\underbrace{\dots r\beta r}_{h})\eta(\alpha\beta\alpha\dots)$, with $\eta \in \{1,r\}$ and $\beta \notin D_R(\dots r\beta r)$. If $h \geqslant 2$ and $\beta \in D_L(w)$, then $M\lambda_\beta = \lambda_\beta M$.

Under hypotheses (2) write $w = (\underbrace{\dots r\beta r}_{h})\eta(\alpha\beta\alpha\dots)\delta$, with $\eta \in \{1,\beta\}$, $\delta \in \{1,r\}$ and $\beta \notin D_R(\dots r\beta r)$. If $h \geqslant 2$ and $\beta \in D_L(w)$, then $M\lambda_\beta = \lambda_\beta M$.

Under hypotheses (3) write $w = (\dots tst)\varepsilon(\underbrace{rsr\dots}_{h})$, with $\varepsilon \in \{1,s\}$ and $s \notin D_L(rsr\dots)$. If $h \geqslant 2$ and $s \in D_R(w)$, then $M\rho_s = \rho_s M$.

Proof. See [7, Proposition 6.6]. □

3.6 The combinatorial invariance conjecture for lower intervals

As we have remarked at the end of Chapter 2, if we know the R-polynomials, we can inductively construct the Kazhdan-Lusztig polynomials. As a consequence of this, the R-polynomials only depend on the poset structure of intervals if and only if the Kazhdan-Lusztig polynomials only depend on the poset structure of intervals. In this chapter, we will show the proof of one of the main theorems in [7], which tells us how to build the R-polynomials of a lower interval using special matchings. Since this procedure only uses the poset structure of the intervals, it proves the combinatorial invariance conjecture in this special case.

Before proving this result, we also prove a result which describes all special matchings of any element of a Coxeter group. From now on (W,S) is an arbitrary Coxeter system.

Lemma 3.6.1. If $w \in W$, M is a special matching of w and $s := M(1)$, then there exists at most one $x \in S$ such that $M \neq \lambda_s$ and $M \neq \rho_s$ on $W_{\{s,x\}}(w)$.

Proof. If there were two such elements r and t. Using Proposition 3.4.1, M would restrict to a special matching of $[1, w[\{s,t,r\}]]$, contradicting Theorem 3.5.1. $\square$

The next result is functional only to the proof of the next proposition.

Lemma 3.6.2. Let $w \in W$, M a special matching of w and $s = M(1)$. Let $t, r \in S$ be such that $M(t) = ts \neq st$ and $M(r) = sr \neq r$ and let $k_1, \ldots, k_p \in S \setminus \{s\}$ ($p \in \mathbb{N} \setminus \{0\}$) be such that for all $j = 1, \ldots, p$ we have $k_j s = s k_j$. If $rk_1 \ldots k_p t \leqslant w$ and $\ell(rk_1 \ldots k_p t) = p+2$, then there exist $h_1, \ldots, h_p \in S$ and $i \in \mathbb{N}$ such that $rk_1 \ldots k_p t = h_1 \ldots h_i t r h_{i+1} \ldots h_p$.

Proof. See [7, Lemma 7.2]. $\square$

We now define some important subsets of W for the next results. Let $w \in W$, M a special matching of w and $s := M(1)$, we let:

$$S' := S \cap [1, w], \qquad J := \{r \in S' \mid M(r) = sr\}, \qquad J' := \{r \in J \mid rs \neq sr\}.$$

Thus $S' \setminus J' := \{r \in S \mid M(r) = rs\}$. We recall here that for every subset $I \subseteq S$, we can factorize uniquely every $w \in W$ as $w = w_I w^I$ with $w_I \in W_I$ and $w^I \in W^I$. There is an analogous unique factorization $w = {}_I w \; {}^I w$ with ${}_I w \in W_I$ and ${}^I w \in {}^I W$. These results can be found in [11].

Proposition 3.6.1. If $u \leqslant w$, then $u^J \in W_{S \setminus J'}$.

Proof. Consider a reduced expression for u^J. By contradiction, suppose that at least one of the letters in this expression is in J', consider the one that is further left, let us denote this as r. Now, consider the first letter after r that is not in J, call it t. Because of Lemma 3.4.3, $rst \not\leqslant u^J$ so there is no s between r and t, and by the same lemma there can be only letters that commute with s. Using iteratively Lemma 3.6.2, we find a reduced expression for u^J that ends with a letter in J which is a contradiction. $\square$

Proposition 3.6.2. Let $t \in S$ be such that M is not a multiplication matching on $W_{\{s,t\}}(w)$ and $x_0 = \alpha\beta\alpha\ldots$ be a minimal element in $W_{\{s,t\}}(w)$ such that $M(x_0) \neq x_0 s$. If $M(t) = ts$, then $\alpha \not\leqslant (u^J)^{\{s,t\}}$ for all $u \leqslant w$.

Proof. It is sufficient to prove the claim in the case $u = w$, since it turns out, and is not hard to see, that if $u \leqslant w$ then $(u^J)^{\{s,t\}} \leqslant (w^J)^{\{s,t\}}$. As previously proved in Section 3.5, $s \notin D_R(x_0)$, thus we can write $x_0 = \alpha\beta\alpha\ldots tst$ and $x_0 = x_0^J \leqslant w^J$. Consider a reduced expression for w^J and a subword of this expression of the form $\alpha\beta\alpha\ldots tst$, choosing the leftmost α and the rightmost t. Now consider the first letter different from s and t, let us call it r. Then, using Lemma 3.5.2, it is possible to see that either this letter can be "pushed" to the left of the first α, or it is located to the right of t. Thus, we may assume that the first such letter r appears after the last t. Using again Lemma 3.5.2, it turns out that all the letters after the last t are in J. Hence, w^J has a reduced expression in which all the letters after the first α are either s or t and this implies our claim. □

Lemma 3.6.3. If $t \in S$ is such that $M(t) \in ts$ but $M \neq \rho_s$ on $W_{\{t,s\}}(w)$, and $u \leqslant w$, then

$$(u^J)^{\{s,t\}}(\underbrace{\ldots tst}_{k}) \in W^J$$

for all $1 < k < m(s,t)$.

Proof. Let $r \in J$, remembering that $W^J := \{w \in W \mid \ell(ws) > \ell(w) \text{ for all} s \in J\}$, we wish to show that $\ell((u^J)^{\{s,t\}} \ldots tstr) > \ell((u^J)^{\{s,t\}} \ldots tst)$. If $r = s$ or $r \in J'$, then Proposition 3.6.1 lets us conclude. Thus, we can assume $r \in J \setminus (J' \cup \{s\})$. Recalling Theorem 1.3.1, we will show that $(u^J)^{\{s,t\}}(\ldots tst)(\alpha_r)$ is a positive root. Since $r \in J \setminus (J' \cup \{s\})$, r must be different from both s and t, and (by definition of J') $rs = sr$. We consider the two possibilities: $rt = tr$ or $rt \neq tr$. If $rt = tr$ then $B(r,s) = B(r,t) = 0$, thus $(u^J)^{\{s,t\}}(\ldots tst)(\alpha_r) = (u^J)^{\{s,t\}}(\alpha_r)$ is positive since $r \in J$. If $rt \neq tr$, then recalling the definition of the geometric representation, one obtains after a quick induction that, for all $1 < k < m(s,t)$,

$$\underbrace{\ldots tst}_{k}(\alpha_r) = \alpha + b\alpha_s + c\alpha_t$$

for some $b, c \in \mathbb{R}_{>0}$. As a consequence of Proposition 3.6.2, either $s \nleqslant (u^J)^{\{s,t\}}$ or $t \nleqslant (u^J)^{\{s,t\}}$. If $s \nleqslant (u^J)^{\{s,t\}}$, then the coefficient of α_s in $(u^J)^{\{s,t\}}(\alpha_r + b\alpha_s + c\alpha_t)$ is b, since a root can either be positive or negative, the root has to be positive, as desired. if $t \nleqslant (u^J)^{\{s,t\}}$, the proof is similar. □

The next result will describe the action of any special matching on any lower interval. Note that it is possible to factorize any $u \in W$ as

$$u = u^J u_J = (u^J)^{\{s,t\}} (u^J)_{\{s,t\}\ \{s\}} (u_J)^{\ \{s\}} (u_J).$$

Theorem 3.6.1. Let (W, S) be a Coxeter system, $w \in W$, M a special matching of w and $s = M(1)$.

(i) If there exists a (necessarily unique) $t \in S$ such that $M(t) = ts$ but $M \neq \rho_s$ on $W_{\{s,t\}}(w)$, then

$$M(u) = (u^J)^{\{s,t\}} M\left((u^J)_{\{s,t\}\ \{s\}} (u_J) \right)^{\ \{s\}} (u_J)$$

for all $u \leqslant w$.

(ii) If M is a multiplication matching on $W_{\{x,s\}}(w)$ for all $x \in S$, then

$$M(u) = u^J s u_J$$

for all $u \leqslant w$.

Proof. (i) We proceed by induction on $\ell(u)$ the result being clear if $\ell(u) = 0$. Note that, by Proposition 3.4.1, $M((u^J)_{\{s,t\}}\ {}_{\{s\}}(u_J)) \in W_{\{s,t\}}(w)$ and so, if we set

$$v := (u^J)^{\{s,t\}} M\left((u^J)_{\{s,t\}}\ {}_{\{s\}}(u_J)\right)\ {}^{\{s\}}(u_J)$$

then, by Lemma 3.6.3, $(v^J)_{\{s,t\}}\ {}_{\{s\}}(v_J) = M\left((u^J)_{\{s,t\}}\ {}_{\{s\}}(u_J)\right)$.

If $v := M(u) \lhd u$ then by induction $u = M(v) = (v^J)^{\{s,t\}} M\left((v^J)_{\{s,t\}}\ {}_{\{s\}}(v_J)\right)^{\{s\}}\ (v_J)$ and so by what we just remarked we obtain

$$\begin{aligned}
(u^J)^{\{s,t\}} &= (v^J)^{\{s,t\}}, \\
(u^J)_{\{s,t\}}\ {}_{\{s\}}(u_J) &= M\left((v^J)_{\{s,t\}}\ {}_{\{s\}}(v_J)\right), \\
{}^{\{s\}}(u_J) &= {}^{\{s\}}(v_J).
\end{aligned}$$

Hence,

$$M(u) = (v_J)^{\{s,t\}}(v^J)_{\{s,t\}}\ {}_{\{s\}}(v_J)\ {}^{\{s\}}v_J = (u^J)^{\{s,t\}}\ M\left((u^J)_{\{s,t\}}\ {}_{\{s\}}(u_J)\right)\ {}^{\{s\}}(u_J)$$

as desired. We may therefore assume that $u \lhd M(u)$. Similarly we may assume that $M\left((u^J)_{\{s,t\}}\ {}_{\{s\}}(u_J)\right) \rhd (u^J)_{\{s,t\}}\ {}_{\{s\}}(u_J)$.

If $u = (u^J)^{\{s,t\}}$ then, by Proposition 3.6.2, either $s \not\leqslant u$ or $t \not\leqslant u$. Therefore, if $a \in \bigcup_{x \in S} W_{\{x,s\}}(u)$, then either $a \in \{s,t\}$ or, by Proposition 3.6.1, $a \in W_{\{r,s\}}(u)$ for some $r \in S \setminus J'$, with $r \neq t$. Hence, by Lemma 3.6.1, $M(a) = as$ so $M(u) = us$ by Lemma 3.4.2 and the result holds in this case. Similarly, the result holds if $u = {}^{\{s\}}(u_J)$, while it is trivial if $u = (u^J)_{\{s,t\}}\ {}_{\{s\}}(u_J)$.

Now consider the following three definitions:

1. If $(u^J)^{\{s,t\}} \neq 1$ let $x_1 \in D_L((u^J)^{\{s,t\}})$ and $u_1 := x_1 u$.
2. If $(u^J)_{\{s,t\}}\ {}_{\{s\}}(u_J) \neq 1$ let $v \lhd (u^J)_{\{s,t\}}\ {}_{\{s\}}(u_J)$ be such that $v \lhd M(v)$ and let $u_2 := (u^J)^{\{s,t\}} v^{\{s\}}(u_J)$.
3. If ${}^{\{s\}}(u_J) \neq 1$ let $x_3 \in D_R({}^{\{s\}}(u_J)$ and $u_3 := u x_3$.

By our last remark we may assume that there exist $i, j \in \{1,2,3\}$ with $i \neq j$, such that u_i and u_j can be defined as above. Applying our induction hypothesis to u_i and u_j we have that $u_i \lhd M(u_i)$, $u_j \lhd M(u_j)$, and $(u^J)^{\{s,t\}} M\left((u^J)_{\{s,t\}}\ {}_{\{s\}}(u_J)\right)\ {}^{\{s\}}(u_J)$ covers $M(u_i)$ and $M(u_j)$. On the other hand, by definition of a special matching, $M(u) \rhd M(u_i), M(u_j)$. Since $(u^J)^{\{s,t\}} M\left((u^J)_{\{s,t\}}\ {}_{\{s\}}(u_J)\right)\ {}^{\{s\}}(u_J) \rhd u$ and $M(u) \rhd u$ we conclude using Theorem 3.2.1 that

$$M(u) = (u^J)^{\{s,t\}} M((u^J)_{\{s,t\}}\ {}_{\{s\}}(u_J))\ \ {}^{\{s\}}(u_J),$$

as desired.

(ii) This is similar and simpler than case (i) and is left to the reader.

□

The next theorem gives us the main link between special matchings and Kazhdan-Lusztig polynomials.

Theorem 3.6.2. If (W, S) is a Coxeter system, $w \in W \setminus \{1\}$ is not dihedral, and M is a special matching of w, then there exist a multiplication matching N of w such that $NM(u) = MN(u)$ for all $u \leqslant w$, and $N(w) \neq M(w)$.

Proof. Note first that the result is true for a special matching M if and only if it is true for the special matching $\tilde{M}$ defined as in Theorem 3.5.1. Hence we may assume that M is in one of the cases of the last theorem.

Suppose that M is in case (i). Then, by Lemma 3.6.1, $M = \rho_s$ on $W_{\{s,y\}}(w)$ for all $y \in S \setminus J'$, with $y \neq t$ and $M = \lambda_s$ on $W_{\{s,y\}}(w)$ for all $s \in J'$.

If $(w^J)^{\{s,t\}} \neq 1$ let $x \in D_L((w^J)^{\{s,t\}})$. If $x \notin \{s,t\}$ then $M = \rho_s$ on $W_{\{s,t\}}(w)$ so $M\lambda_x = \lambda_x M$ on $W_{\{s,x\}}(w)$ and we are done by Lemma 3.3.3. If $x \in \{s,t\}$ then, by Proposition 3.6.2, $x = \beta$ and there exists $r \in S$ with $r < (w^J)^{\{s,t\}}$ such that $\beta r \neq r\beta$. Furthermore, by Proposition 3.6.1, $r \in S \setminus J'$ so $M(r) = rs$. Let $K := \{s,t,r\}$, then by Proposition 3.4.1 M and λ_β restrict to special matchings of $[1, w[K]] = W_K(w)$ and M satisfies either hypotheses (1) or (2) (those of Section 3.5). Therefore, by Proposition 3.5.3, $M\lambda_\beta = \lambda_\beta M$ on $[1, w[K]]$ and hence on $W_{\{s,t\}}(w)$ and the claim follows by Lemma 3.3.3. Note that $M(w) \neq \lambda_x(w)$ by what we have proved in the last theorem.

If $(w^J)^{\{s,t\}} = 1$ then necessarily ${}^{\{s\}}(w_J) \neq 1$ (otherwise w would be dihedral) and we proceed in a similar way considering a right descend x of ${}^{\{s\}}(u_J)$. In this case M will satisfy hypotheses (3) in Section 3.5 and one concludes that $M\rho_x = \rho_x M$. If M is in case (ii) the proof is similar and simpler and is left to the reader. □

We remark that the above result does not hold if w is dihedral.

The following result will be the conclusion of this work, and it shows that R-polynomials, and thus Kazhdan-Lusztig polynomials as a consequence of Remark 2.3.1, can be computed from the poset structure of a lower interval.

Theorem 3.6.3. If (W, S) is a Coxeter system, $w \in W$ and M is a special matching of w, then

$$R_{u,w}(q) = q^c R_{M(u),M(w)}(q) + (q^c - 1) R_{u,M(w)}(q)$$

for all $u \leqslant w$, where $c := 1$ if $M(u) \rhd u$ and $c := 0$ otherwise.

Proof. We proceed by induction on $\ell(w)$, the result being clearly true if $\ell(w) \leqslant 2$. So let $\ell(w) \geqslant 3$. If w is dihedral then the result is easy to check, so suppose that w is not dihedral. Then, by Theorem 3.6.2, there exists a multiplication matching N of w such that $NM(u) = MN(u)$ for all $u \leqslant w$, and $N(w) \neq M(w)$.

Fix $u \leqslant w$. There are four cases to distinguish. We consider only two of them, the other two being exactly similar. Since $M(w) \neq N(w)$, we have that $M(w) \rhd NM(w) = MN(w) \lhd N(w)$ so M restricts to a special matching of $[1, N(w)]$.

1. $N(u) \rhd u$, $M(u) \lhd u$.

   Then, since $MN(u) = NM(w)$, $M(u) \lhd MN(u) \lhd N(u)$. Therefore, by the properties of R-polynomials and our induction hypothesis,

$$\begin{aligned} R_{u,w} &= qR_{N(u),N(w)} + (q-1)R_{u,N(w)} \\ &= qR_{MN(u),MN(w)} + (q-1)R_{M(u),MN(w)} \\ &= qR_{NM(u),NM(w)} + (q-1)R_{M(u),NM(w)} \\ &= R_{M(u),M(w)}, \end{aligned}$$

   as desired

2. $N(u) \rhd u$, $M(u) \rhd u$.

   If $M(u) \neq N(u)$ then $MN(u) \rhd N(u)$ and $MN(u) \rhd M(u)$ so, by the properties of R-polynomials and our induction hypothesis we have:

$$\begin{aligned} R_{u,w} &= qR_{N(u),N(w)} + (q-1)R_{u,N(w)} \\ &= q(qR_{MN(u),MN(w)} + (q-1)R_{N(u),MN(w)}) \\ &+ (q-1)(qR_{M(u),MN(w)} + (q-1)R_{u,MN(w)}) \\ &= q^2 R_{NM(u),NM(w)} + q(q-1)R_{N(u),NM(w)} \\ &+ q(q-1)R_{M(u),NM(w)} + (q-1)^2 R_{u,NM(w)} \\ &= qR_{M(u),M(w)} + (q-1)R_{u,M(w)}, \end{aligned}$$

as desired. If $M(u) = N(u)$ then we have similarly that

$$\begin{aligned}
R_{u,w} &= qR_{N(u),N(w)} + (q-1)R_{u,N(w)} \\
&= qR_{MN(u),MN(w)} + (q-1)(qR_{M(u),MN(w)} + (q-1)R_{u,MN(w)}) \\
&= qR_{M(u),M(w)} + (q-1)R_{u,M(w)}
\end{aligned}$$

and the result again follows. $\square$

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Ringraziamenti