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Color structures in quantum chromodynamics

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Abstract

This thesis project aims to calculate important color factors of processes mediated by the strong interaction, which are visually represented by Feynman diagrams. These factors are relevant in the context of probability calculation of given processes. To achieve this goal, it is essential to understand the properties of the strong interaction and to develop the appropriate tools to describe it. This includes the mathematical framework of quantum field theory, specifically non-abelian gauge theories, and the algebra of the color group $SU(3)$. Color Feynman rules for quantum chromodynamics are derived and then applied to calculate color factors of different processes, ultimately providing a qualitative argument for the color-singlet nature of quarks bound states.

Questo progetto di tesi ha come obiettivo il calcolo di importanti fattori di colore relativi a processi mediati dall'interazione forte, visualmente rappresentati da diagrammi di Feynman. Per raggiungere questo obiettivo, è necessario comprendere le proprietà di questa interazione fondamentale e sviluppare i metodi adeguati per descriverla. Questi includono la struttura matematica delle teorie di campo quantistiche, in particolare le teorie di gauge non abeliane, e l'algebra del gruppo di colore $SU(3)$. Le regole di colore di Feynman per la cromodinamica quantistica vengono derivate e successivamente utilizzate per calcolare fattori di colore di altri processi, fornendo supporto qualitativo alla spiegazione del fenomeno della neutralità di colore degli stati legati di quark.

Contents

Chapter 1

Introduction

Physics is the natural science that seeks to explain natural phenomena. Throughout history, humanity's understanding of the universe has continuously deepened and become more sophisticated. Despite the significant progress that has been made, it is astonishing that a human being can conceive of scenarios such as an object falling into a black hole or a particle colliding with a nucleus, and, through the application of mathematical and physical laws, not only describe that phenomenon but also gain a deeper understanding of it and possibly obtain experimental evidences.

One of the main areas of science that has been proliferating in the past century is the study of quantum mechanics, particularly its modified version to include special relativity, known as Quantum Field Theory (QFT). QFT is a powerful mathematical framework that describes three out of the four fundamental interactions: the electromagnetic force, the weak force, and the strong force. It notably lacks a description of the gravitational force, which is not described by QFTs at higher energies. The three QFTs that describe these interactions form the larger framework of the Standard Model (SM), which accounts for almost all phenomena occurring at the quantum level or at high energy scales.

Although these three interactions are all described by QFTs, they exhibit different characteristics. For example, depending on the energy scale of a given process, one interaction may be more dominant than the others. Specifically, the strong interaction dominates at the nuclear and subnuclear levels, or generally speaking, at high energy scales accessible to current experiments. This implies that a deep understanding of such interaction is necessary to describe phenomena at these scales, where other interactions also play a role.

One of the most intriguing aspects of the strong force is the concept of *color*, which is analogous to the electric charge in the context of the electromagnetic interaction. Particles that have color can interact via the strong force. However, unlike the electric charge, color is not a simple scalar value, it is represented

by the $SU(3)$ group, which is the symmetry group at the hearth of Quantum Chromodynamics (QCD), the QFT describing the strong force. The introduction of color will be further explained later in the thesis and linked to the physical properties of the strong force.

As all theoretical models, QFT is required to accurately describe physical phenomena. This is achieved by developing theoretical ideas that can be experimentally tested. Usually, in experiments involving particles collisions or accelerations, observables are linked to the probability of a given process. Depending on the interaction responsible for the process, different factors contribute to this probability. Specifically, if one considers processes mediated by the strong interaction, then color is a fundamental factor. It is possible to calculate the color structure of a certain process represented by a Feynman diagram, that is a critical component in determining the process's probability.

This thesis addresses the reasons that led to the development of QFTs and explains the methods used to construct them, providing examples along the way. The primary objective is to calculate important color factors for processes mediated by the strong interaction and demonstrate how they can be used to explain certain physical phenomena related to the strong force.

The structure of the thesis is as follows: the second chapter provides an overview of classical field theory, explaining the concept of field and the Lagrangian formalism before taking into account quantum mechanics and thus culminate into quantum field theory. The third chapter discusses the importance of gauge transformations in electromagnetism and quantum mechanics, these are then combined with local $U(1)$ transformations to build a scalar QFT for the electromagnetic interaction. The fourth chapter focuses on Lie groups and $SU(N)$ algebra, developing the mathematical tools necessary for later sections. In the fifth chapter, an outline of the strong force is presented, including the significance of color, before deriving the mathematical framework needed to explain this interaction through *non-abelian gauge theories*. Finally, in the sixth chapter, the algebra developed in chapter 4 is used to calculate color factors of several QCD processes represented by Feynman diagrams.

Chapter 2

Classical and quantum field theory

In this chapter, I will be giving an overview of classical field theory and the Lagrangian formalism, before diving into QFT, as it is necessary to recall basic concepts and ideas that will become helpful later on. What follows is inspired by [1], [2], [3] and [4].

2.1 Classical field theory

A field ϕ is a quantity that is defined in every point in space and time, hence $\phi(\vec{x}, t)$. The nature of a field can mainly be of four different types depending on the quantity or the spin (s) of the particle it describes, in the case of QFT. There are real and complex fields that can be scalar fields ($s = 0$), spinor fields (fermions fields for instance, $s = \frac{1}{2}$), vector fields ($s = 1$), or tensor fields. The difference between standard mechanics and classical field theory is that the latter is just a mechanical system with a continuous set of degrees of freedom. That is because instead of having a finite number of generalized coordinated $q_a(t)$ indexed by a label a, in a field theory, different points in space and time are considered individuals degrees of freedom, hence they are labels. The dynamics of a field is governed by a Lagrangian or an Hamiltonian that are written as integrals over all space of Hamiltonian and Lagrangian densities:

$$H = \int d^3x \, \mathcal{H} \qquad\qquad L = \int d^3x \, \mathcal{L}. \tag{2.1}$$

However, the weird shaped letters are usually referred to as Hamiltonian and Lagrangian, losing the word "density". The Hamiltonian of a system is defined as a functional of fields and their first derivative, $\mathcal{H}[\phi, \pi]$, while the Lagrangian is

defined as the Legendre transform of the Hamiltonian. Formally:

$$\mathcal{L}[\phi, \dot{\phi}] = \pi[\phi, \dot{\phi}]\dot{\phi} - \mathcal{H}[\phi, \pi[\phi, \dot{\phi}]], \tag{2.2}$$

where $\dot{\phi} = \frac{\partial \phi}{\partial t}$ and $\pi[\phi, \dot{\phi}]$ is the conjugate momentum of the field, defined by one of Hamilton's equations $\frac{\partial \mathcal{H}[\phi,\pi]}{\partial \pi} = \dot{\phi}$. The inverse transform gives us the Hamiltonian density:

$$\mathcal{H}[\phi, \pi] = \pi\dot{\phi}[\phi, \pi] - \mathcal{L}[\phi, \dot{\phi}[\phi, \pi]]. \tag{2.3}$$

For example, let us consider the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial_\mu \phi) - \mathcal{V}[\phi] = \frac{1}{2}(\dot{\phi})^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \mathcal{V}[\phi], \tag{2.4}$$

where $\mathcal{V}[\phi]$ is the potential (density). Then one can notice that $\pi = \dot{\phi}$ and therefore:

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \mathcal{V}[\phi]. \tag{2.5}$$

This shows that the Hamiltonian (density) is the sum of kinetic and potential energy, while the Lagrangian (density) is the difference of the two. Therefore, the Hamiltonian corresponds to a quantity that is conserved, the total energy, while the Lagrangian does not. However, when dealing with field theories and especially quantum field theories, Lorentz invariance is fundamental to achieve a theory that accurately describes natural phenomena, since it must be compatible with special relativity. The total energy of a system is the temporal component of a 4-vector, namely the 4-momentum vector $P^\mu = (\frac{E}{c}, \vec{p})$, and the Hamiltonian density often coincides with the 00 component of the so called energy-momentum tensor $T^{\mu\nu}$, whose four components are the conserved currents arising from translational invariance of a Lagrangian. Therefore they are not manifestly Lorentz invariant. This is the reason why Lagrangians are used in field theories instead of Hamiltonians. Furthermore, Lagrangians depend only on the fields and their first derivatives, because higher order derivative terms have led to absurd theories that do not describe the physical world we live in. In addition, an explicit dependence on the coordinates x^μ will result in a theory that is not translational invariant, that is absurd since space is homogeneous. Moreover, in field theory one usually refers to kinetic energy as kinetic terms and potential energy as interactions. Kinetic terms are bilinear, meaning they contain 2 fields, such as the first term of eq. (2.4), while interaction terms have three or more fields, such as $\lambda\phi^3$, and these can be grouped into a single Lagrangian describing the interactions between the fields, $\mathcal{L}_{int}$.

2.1.1 Euler-Lagrange equations

The dynamic of a Lagrangian system is determined by using the principle of least action. The action of a field is defined as the integral over time of the Lagrangian:

$$S = \int dt\, L = \int_\Omega d^4x\, \mathcal{L}(\phi, \partial_\mu \phi). \tag{2.6}$$

The principle of least action states that the true configuration of the field is the one that makes the action an extreme, that is the action is stationary with respect to a small arbitrary variation of the fields $\delta\phi$, meaning $\frac{\delta S}{\delta \phi} = 0$. If we consider a small arbitrary variation of the fields, $\phi \to \phi + \delta\phi$, where this small variation vanishes at the boundary of the hyper-volume Ω, $\partial\Omega$, which means that the fields vanish at spacial and temporal infinity, then the variation of the action is:

$$\delta S = \int_\Omega d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi)\right] = \tag{2.7}$$

$$= \int_\Omega d^4x \left\{\left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right]\delta\phi + \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right]\right\}, \tag{2.8}$$

where we have used the fact that $\delta(\partial_\mu\phi) = \partial_\mu(\phi + \delta\phi) - \partial_\mu\phi = \partial_\mu\delta\phi$ thus allowing us to write the derivative of a product and then perform integration by parts. Now, the last term is a total derivative, and therefore, by using Gauss's theorem in 4 dimensions we can write it as

$$\int_{\partial\Omega} d\sigma_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right],$$

where $d\sigma_\mu$ is the infinitesimal element of the oriented hyper-surface enclosing the volume Ω. However, the variation of the fields vanish on the boundary of the volume Ω. Therefore, this last term can be discarded and we are left with the following:

$$\delta S = \int_\Omega d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right]\delta\phi. \tag{2.9}$$

Now applying the least action principle suggests that the integral is zero, but since the variation $\delta\phi$ is arbitrary it must hold for every possible variation, therefore, we are left with Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0. \tag{2.10}$$

These provide the equations of motion for a field described by a given Lagrangian. The simplest example can be discussed considering the Lagrangian in eq. (2.4). By applying Euler-Lagrange equations we have:

$$\partial_\mu \partial^\mu \phi = \Box\phi = -\frac{d\mathcal{V}}{d\phi}. \tag{2.11}$$

If the potential has the form $\mathcal{V}[\phi] = \frac{1}{2}m^2\phi^2$, then the equations of motion become:

$$(\Box + m^2)\phi = 0, \tag{2.12}$$

This is the Klein-Gordon equation, that describes the dynamic of a free scalar field.

An important step in the derivation of Euler-Lagrange equations involves discarding a term containing a total derivative. This is actually a general property of Lagrangian systems that is worth pointing out. Indeed, the equations of motion arising from a given Lagrangian are invariant under the addition to the latter of a total derivative of a field-depending function. Consider a Lagrangian $\mathcal{L}$ and the corresponding action S. Then if:

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \partial_\mu \Lambda^\mu(\phi), \tag{2.13}$$

where $\Lambda^\mu(\phi)$ is a field, the action changes as follows:

$$S \to S' = \int_\Omega d^4x[\mathcal{L} + \partial_\mu \Lambda^\mu(\phi)] = S + \int_\Omega d^4x\, \partial_\mu \Lambda^\mu(\phi). \tag{2.14}$$

Then by applying Gauss's theorem in 4 dimensions:

$$S' = S + \int_{\partial\Omega} d\sigma_\mu \Lambda^\mu(\phi). \tag{2.15}$$

Since the variation of the field vanishes on the hyper surface $\partial\Omega$, then:

$$\delta \int_{\partial\Omega} d\sigma_\mu \Lambda^\mu(\phi) = 0, \tag{2.16}$$

and therefore:

$$\delta S = 0 \implies \delta S' = 0, \tag{2.17}$$

which implies that the equations of motion obtained from the principle of least action on S are the same as those coming form S'.

2.1.2 Noether's theorem

One of the most important theorems in mechanics and field theory is Noether's theorem, which states that for every continuous symmetry of a system of a given number of fields there exists one or more conserved currents or integrals of motion. A conserved current $J^\mu(x)$ satisfies the following continuity equation:

$$\partial_\mu J^\mu = 0 \to \frac{1}{c}\frac{\partial J^0}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0. \tag{2.18}$$

However, this also implies that a certain quantity is conserved. Indeed, if we consider the quantity Q defined as:

$$Q = \int d^3x \, J^0, \tag{2.19}$$

given that eq. (2.18) holds, we have:

$$\partial_t Q = \int d^3x \, \partial_t J^0 = -\int d^3x \, \vec{\nabla} \cdot \vec{J} = 0, \tag{2.20}$$

where in the last step we have used Gauss's theorem and assumed that $\vec{J}$ vanishes at spacial infinity, that is reasonable since by assumption there is nothing leaving the boundary. Therefore, the quantity Q is constant and thus conserved. For example, this is the total charge if the current comes from a Lagrangian containing complex fields.

To prove Noether's theorem it is necessary to work infinitesimally, hence the reason why the symmetry must be continuous. Noether's theorem does not work for discrete symmetries.

Proof. First, let us define what we mean by continuous symmetry. A transformation of the fields $\delta\phi(x) = X(\phi)$ is a symmetry if the Lagrangian changes by a total derivative:

$$\delta\mathcal{L} = \partial_\mu F^\mu(\phi), \tag{2.21}$$

for some function $F^\mu(\phi)$. Consider now a small transformation of the fields, then the Lagrangian transforms as follows:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) = \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right). \tag{2.22}$$

If the equations of motion are satisfied, that is the fields are *on-shell*, the term between square brackets is zero. Therefore, we have:

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right). \tag{2.23}$$

However, if the transformation of the fields is a continuous symmetry for the Lagrangian, eq. (2.21) holds, hence:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - F^\mu(\phi)\right) = 0 \implies \partial_\mu J^\mu = 0, \tag{2.24}$$

where J^μ is the conserved current, called Noether current. □

This result can be generalized for an arbitrary number of fields:

$$J^\mu = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \delta\phi_n. \tag{2.25}$$

For example, consider the following Lagrangian for a complex field ϕ:

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi \phi^*. \tag{2.26}$$

The equations of motion can be easily found to be:

$$(\Box + m^2)\phi = 0 \qquad\qquad (\Box + m^2)\phi^* = 0. \tag{2.27}$$

It can be noticed that there is a continuous symmetry since the Lagrangian is invariant ($\delta\mathcal{L} = 0$) under the following transformation of the fields:

$$\phi \to \phi e^{ia} \qquad\qquad \phi^* \to \phi^* e^{-ia}, \tag{2.28}$$

where a is a real number. This is just a phase redefinition of the fields, precisely a global $U(1)$ transformation, since a is not a function of space and time. The infinitesimal variation of the fields is:

$$\delta\phi = ia\phi \qquad\qquad \delta\phi^* = -ia\phi^*, \tag{2.29}$$

Therefore, the conserved current is:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \delta\phi^* = ia(\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi). \tag{2.30}$$

This satisfies the continuity equation when the fields are on-shell because:

$$\partial_\mu J^\mu = ia(\phi \Box \phi^* - \phi^* \Box \phi) = -ia(\phi m^2 \phi^* - \phi^* m^2 \phi) = 0, \tag{2.31}$$

and the corresponding conserved quantity is the total charge Q.

2.2 Quantum field theory

In this section, I shall explain some of the reasons why QFTs have been developed and how they describe some of the main phenomena. The focus will be on the fundamental principles of these theories rather than the technical details, which are beyond the scope of this thesis.

First of all, the concept of field was introduced in classical mechanics to remove the "spooky action at a distance" that characterizes Newton's and Coulomb's law.

Their theories were improved by Maxwell and Einstein by introducing the concept of the electromagnetic and gravitational fields. This also underlies an important aspects of physical laws, that is *locality* [2]. The laws of physics are local, meaning that physical objects are influenced by their immediate surroundings, not by distant bodies instantaneously. Given that special relativity dictates that the speed of light in the vacuum is the maximum speed at which information can travel, this results in a causal structure of space time. Therefore, the principle of locality ensures the agreement of any physical laws with special relativity. In addition, special relativity is by nature incompatible with standard quantum mechanics. This is due to the fact that the combination of the two implies that particle number is not conserved. It has been experimentally proven that there exist natural processes that create and annihilate particles. Therefore, it is reasonable to conclude that conservation of particle number is not a fundamental principle of physics. However, standard quantum mechanics does not provide the tools required to deal with systems whose particle number is not conserved. If one tries to write a relativistic Schrodinger equation, the consequences are negative probabilities and non-sense energy levels. Hence, a new mathematical framework is needed: that is QFT.

Another important argument in support of the need for a QFT is that all particles of the same type are the same. A proton produced in the core of a star and a proton produced at the Large Hadron Collider (LHC) in Geneva are the same in terms of mass, charge and spin. This may seem normal but these two protons have been produced in different ways and yet they have the same physical properties. QFT provides an explanation to this phenomena and many others. The basic procedure of QFT is the quantization of a classical field, as the name suggests. In ordinary quantum mechanics, classical degrees of freedom are promoted to operator's acting on an infinite dimensional Hilbert space. In QFT the same is true, but since we have fields, there are an infinite numbers of degrees of freedom.

To summarize how a QFT works, consider the Klein-Gordon equation (2.12), then by taking the Fourier transform of the field $\phi(\vec{x}, t)$, we have [3]:

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \phi(\vec{p}, t). \tag{2.32}$$

Since the box operator has a temporal and spacial component, substituting this expression in the Klein-Gordon equation gives us the following:

$$(\partial_t^2 + (\vec{p}^{\,2} + m^2))\phi(\vec{p}, t) = 0. \tag{2.33}$$

This is the equation of a harmonic oscillator that vibrates at a frequency $\omega_{\vec{p}} = +\sqrt{\vec{p}^{\,2} + m^2}$. Therefore, the free scalar field described by the Klein-Gordon equation has a general solution consisting of a linear superposition of independent harmonic oscillators vibrating at different frequency linked to a different value of $\vec{p}$.

To quantize the field it is necessary to quantize this infinite collection of harmonic oscillators, that is known from standard quantum mechanics. The main idea is that when the oscillators are all in their non trivial ground state, the field is said to be in the *vacuum* configuration and there are no particles. Excitations from the ground state are interpreted as particles of mass m and momentum $\vec{p}$ and it can also be seen that in this case the particle has no internal angular momentum. Therefore, the quantization of a free scalar field gives rise to a spin 0 particle. This can be done for spinor and vector fields, giving particles with spin $\frac{1}{2}$ such as fermions like quarks, and spin 1, such as mediators of interactions like gluons or protons. Hence, in QFT every particle is associated to a field that permeates the universe, for example the electronic field corresponds to electrons. Excitations of these fields result in the particles we observe. This provides an explanation to one of the problems stated before. All particle of the same type are identical in terms of mass, spin and charge because they are excitations of the same field, that is the same across the entire universe.

An important factor when discussing QFTs and how to build them is symmetry, specifically gauge symmetry. In section 2.1.2, I have outlined how Noether's theorem connects conserved currents and quantities to global continuous symmetries. However, the symmetry under gauge transformation that was first noticed in Maxwell's equations of electromagnetism is a different type of symmetry, meaning that it is local instead of global. This leads to a very different interpretation of the symmetry and its consequences. This gauge symmetry will be the subject of the next chapter.

Chapter 3

Gauge invariance and local symmetry

In this chapter, I shall introduce the concept of gauge symmetry in standard electromagnetism [1], explore its consequences when taking into account quantum mechanics [5] and how it can be used as a principle to derive a scalar version of the QFT used to describe the electromagnetic interaction, that is scalar Quantum Electrodynamics or simply scalar QED [4].

3.1 Gauge invariance in electromagnetism

Maxwell's equations of electromagnetism are usually written in terms of the electric and magnetic fields, $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$. In Heaviside-Lorentz units, these are known to take the following form [6]:

$$\vec{\nabla}\cdot\vec{E}(\vec{x},t)=\rho(\vec{x},t) \qquad \vec{\nabla}\times\vec{E}(\vec{x},t)+\frac{1}{c}\frac{\partial\vec{B}(\vec{x},t)}{\partial t}=0 \tag{3.1}$$

$$\vec{\nabla}\cdot\vec{B}(\vec{x},t)=0 \qquad \vec{\nabla}\times\vec{B}(\vec{x},t)-\frac{1}{c}\frac{\partial\vec{E}(\vec{x},t)}{\partial t}=\frac{1}{c}\vec{J}(\vec{x},t), \tag{3.2}$$

where $\vec{J}(\vec{x},t)$ and $\rho(\vec{x},t)$ are the current density and charge density, respectively. However, by looking at the homogeneous equations one can see that the magnetic field can be written as the curl of another vector field, since the divergence of a curl is always zero. Hence the equation still holds. Precisely:

$$\vec{B}(\vec{x},t)=\vec{\nabla}\times\vec{A}(\vec{x},t), \tag{3.3}$$

where $\vec{A}$ is called vector potential. Substituting this new expression of the magnetic field into the other homogeneous equation, we find that (the dependence of space

and time is implied):

$$\vec{\nabla} \times \left(\vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right) = 0. \tag{3.4}$$

Therefore, the field between brackets is irrotational and it can be written as the gradient of a scalar field, since the curl of a gradient is always zero. Hence:

$$\vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\phi \implies \vec{E} = -\vec{\nabla}\phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}. \tag{3.5}$$

Thus, the electric and magnetic fields can be written in terms of two other fields $\vec{A}(\vec{x}, t)$ and $\phi(\vec{x}, t)$, that are called vector and scalar potential, respectively. In this way, one can rewrite Maxwell's equations using these potentials, that by definition automatically satisfy the homogeneous equations. The in-homogeneous ones become:

$$\vec{\nabla}^2 \phi + \frac{1}{c} \frac{\partial (\vec{\nabla} \cdot \vec{A})}{\partial t} = -\rho \qquad \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}^2 \vec{A} + \vec{\nabla} \left[\vec{\nabla} \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} \right] = \frac{1}{c} \vec{J}. \tag{3.6}$$

It can be noticed that the expressions of the electric and magnetic fields in terms of the potentials are invariant under a transformation of these last. Specifically, if the scalar and vector potential transform as follows:

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\chi \qquad\qquad \phi \to \phi' = \phi - \frac{1}{c} \frac{\partial \chi}{\partial t}, \tag{3.7}$$

where $\chi = \chi(\vec{x}, t)$ is an arbitrary scalar function of space and time, the transformed electric and magnetic fields are invariant. This is called a *gauge transformation* of the potentials and Maxwell's equations are invariant under such transformation. Using the covariant formalism of special relativity, which is the natural language of QFT, all of this can be written by considering the electromagnetic field strength $F^{\mu\nu}$, that is a $(2, 0)$-rank tensor that replaces the electric and magnetic fields in Maxwell's equations. These can be written as follows:

$$\partial_\mu F^{\mu\nu} = \frac{1}{c} J^\nu \qquad\qquad \partial_\mu (F^{\mu\nu})^* = 0, \tag{3.8}$$

where $(F^{\mu\nu})^* = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$ is the dual tensor and $J^\nu = (c\rho, \vec{J})$ is the 4-current. The homogeneous equations have a solution provided by Poincare's lemma, that is with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, where A^μ is a 4-vector and it can be proven that its temporal component is the scalar potential ϕ, while its spacial components are the three components of the vector potential $\vec{A}$. Hence, this 4-vector is called the four-potential, $A^\mu = (\phi, \vec{A})$. Since the gauge transformation acts on the potentials,

in this new formalism it becomes a single transformation that acts on the four-potential, that is:

$$A^\mu \to A'^\mu = A^\mu - \partial^\mu \chi, \tag{3.9}$$

where $\chi = \chi(\vec{x}, t)$ is the arbitrary function of space and time. Since the field strength $F^{\mu\nu}$ contains the electric and magnetic fields, that are invariant under a gauge transformation, it is reasonable to expect that $F^{\mu\nu}$ is also invariant under such transformation. Indeed:

$$\begin{aligned} F'^{\mu\nu} &= \partial^\mu A'^\nu - \partial^\nu A'^\mu = \\ &= \partial^\mu (A^\nu - \partial^\nu \chi) - \partial^\nu (A^\mu - \partial^\mu \chi) = \\ &= \partial^\mu A^\nu - \partial^\mu \partial^\nu \chi - \partial^\nu A^\mu + \partial^\nu \partial^\mu \chi = \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}, \end{aligned}$$

where it was used the fact that the 4-gradients commute, so $\partial^\mu \partial^\nu = \partial^\nu \partial^\mu$. This gauge invariance may seem like a simple mathematical transformation, but it has several physical implications that reflect the massless nature of the photon and thus properties of electromagnetic waves. However, delving into this is beyond the scope of the thesis. Instead, I will focus on how this gauge invariance impacts the quantum world.

3.2 Gauge invariance in quantum mechanics

The invariance under gauge transformation of Maxwell's equations has a direct implication when considering the Schrodinger equation for a charged particle in an electromagnetic field. The classic Hamiltonian for such particle is given by:

$$H = \frac{1}{2m}\left[\vec{p} - \frac{q}{c}\vec{A}\right]^2 + q\phi, \tag{3.10}$$

where $\vec{A}$ is the vector potential, ϕ is the scalar potential and q is the charge of the particle. In quantum mechanics, this becomes the Hamiltonian operator acting on the wave function according to the Schrodinger equation, with the substitution of $\vec{p}$ with the corresponding momentum operator $-i\hbar\vec{\nabla}$. The Schrodinger equation thus becomes the following:

$$\frac{1}{2m}\left[-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right]^2 \Psi + q\phi\Psi = i\hbar\frac{\partial\Psi}{\partial t}. \tag{3.11}$$

This equation can be written in an easier way recalling the structure of the Schrodinger equation for a free particle, introducing two operators that basically generalize the space and time derivatives (using $c = 1$):

$$\hat{D} = -\hbar\vec{\nabla} + iq\vec{A} \qquad\qquad \hat{D}^0 = \hbar\frac{\partial}{\partial t} + iq\phi.$$

The equation then becomes:

$$\frac{1}{2m}(i\hat{D})^2\Psi = i\hat{D}^0\Psi. \tag{3.12}$$

It can be observed that under a gauge transformation of the potentials eq. (3.11) is not invariant. This is theoretical problem because from electromagnetism it is known that Maxwell's equations are invariant under such transformation and it has important consequences regarding the physical degrees of freedom of the theory. Hence, it is reasonable to expect that it should also hold at the quantum level and, therefore, that the quantum theory of a charged particle in an electromagnetic field should have this invariance or symmetry. The only way for the above equation to be invariant under a gauge transformation of the potentials is that while the potentials are subject to this transformation, the wave function must transform in some way to maintain the equation unchanged. It can be shown that the wave function transformation is a local phase redefinition, precisely a $U(1)$ local transformation, that is:

$$\Psi \to \Psi' = e^{iq\chi(\vec{x},t)}\Psi. \tag{3.13}$$

Given that the function χ is a function of space and time, the derivative operators act on it, thus leading to the invariance of the Schrodinger equation as requested. The process just shown can also be reversed, meaning we firstly consider the Schrodinger equation for a free particle, and secondly we perform a local $U(1)$ transformation. By requiring the invariance of the equation, we need to include two fields that have special transformations: these will be the known gauge transformation of the vector and scalar potential. Furthermore, the introduction of these fields will result in a theory that doesn't describe a free particle, but instead a particle interacting with these fields, which in the case shown above is the electromagnetic field. This procedure can be generalized to what is called a gauge principle, and it is a key component of QFT. Since these theories are based on gauge transformations, they are called *gauge theories*. The principle can be summarized as follows: first, it is necessary to analyze the theory for a free particle; if this theory has a global symmetry we can try to promote that symmetry to a local one, introducing a certain number of gauge fields with special transformations. This will eventually result in the theory of a particle interacting with such gauge fields.

In the next paragraph, I shall demonstrate how this can be used to derive the scalar QED Lagrangian from a local symmetry without using Maxwell's equations.

3.3 Scalar QED Lagrangian

The Lagrangian describing the electromagnetic interaction requires a QFT for charged fermions, since the basic electromagnetic processes involve electrons and photons. However, electrons are fermions, spin $\frac{1}{2}$ particles, and are mathematically described by spinors, that have a specific algebra which is somewhat complicated. To avoid dealing with this, I show how a simplified version of QED can be built: QED with spin 0 particles, known as scalar QED or sQED for short. This theory actually describes particles that can be observed, such as pions.

Let us consider a complex scalar field ϕ described by the complex Klein-Gordon Lagrangian in eq. (2.26). I have already obtained the equation of motion for both degrees of freedom and also noticed how there is a global $U(1)$ symmetry that leads to the conservation of electric charge. It can immediately be said that since these fields are their respective complex conjugate, this kind of theory describes particle anti-particle pairs, with same mass and opposite charge. Since they are charged, it is reasonable to assume that there will be an electromagnetic interaction among them. Thus, it is necessary to introduce it somehow. This can be done by applying the gauge principle outlined in the previous section. The first step is verified, a $U(1)$ global symmetry is present. Let us try and promote this to a local $U(1)$ symmetry. This leads to the following transformation of the fields:

$$\phi \to \phi' = \phi e^{ig\alpha(x)} \qquad\qquad \phi^* \to \phi'^* = \phi^* e^{-ig\alpha(x)}, \tag{3.14}$$

where $\alpha(x) = \alpha(\vec{x}, t)$ is a function of space and time and g is just a constant for now. Under this transformation, the mass term in the Lagrangian is invariant, since there are no derivatives. However, the kinetic terms does change since the derivatives transform as follows:

$$\partial_\mu \phi \to \partial_\mu \phi' = \partial_\mu (\phi e^{ig\alpha(x)}) = e^{ig\alpha(x)} \partial_\mu \phi + ig(\partial_\mu \alpha) e^{ig\alpha(x)} \phi \tag{3.15}$$

$$\partial^\mu \phi^* \to \partial^\mu \phi'^* = \partial^\mu (\phi^* e^{-ig\alpha(x)}) = e^{-ig\alpha(x)} \partial^\mu \phi^* - ig(\partial^\mu \alpha) e^{-ig\alpha(x)} \phi^*, \tag{3.16}$$

Therefore, the kinetic term becomes:

$$(\partial_\mu \phi')(\partial^\mu \phi'^*) = (\partial_\mu \phi)(\partial^\mu \phi^*) + ig(\phi \partial_\mu \phi^* - \phi^* \partial_\mu \phi)\partial^\mu \alpha + g^2 \partial_\mu \alpha \partial^\mu \alpha \phi \phi^*. \tag{3.17}$$

This was expected since derivatives, by definition, involve the difference between functions valued at two different points in space, and these transform differently under a local transformation. It is then obvious that something needs to be modified in order to have a $U(1)$ local symmetry, specifically in the derivative term. This motivates the introduction of a new field $W(x, y)$, called a Wilson line, that

is a scalar bi-local field, meaning it depends on two points of space and time. The important thing is that under a local transformation it transform as follows:

$$W(x,y) \to W(x,y)' = e^{i\alpha(x)} W(x,y) e^{-i\alpha(y)}, \tag{3.18}$$

so that:

$$\begin{aligned} W(x,y)\phi(y) - \phi(x) \to W(x,y)'\phi'(y) - \phi'(x) = & \qquad (3.19)\\ = e^{i\alpha(x)} W(x,y) e^{-i\alpha(y)} e^{i\alpha(y)}\phi(y) - e^{i\alpha(x)}\phi(x) = & \qquad (3.20)\\ = e^{i\alpha(x)}[W(x,y)\phi(y) - \phi(x)]. & \qquad (3.21) \end{aligned}$$

In this way, the difference in square brackets is independent of the chosen local phase convention. Thus, the Wilson line compensates the difference in local transformations of fields at different points in space. In addition, if the coordinate y is close to x, that is $y^\mu = x^\mu + \delta x^\mu$, if we divide by δx^μ and then let $\delta x^\mu \to 0$, it becomes a derivative:

$$D^\mu \phi(x) = \lim_{\delta x^\mu \to 0} \frac{1}{\delta x^\mu}[W(x, x+\delta x)\phi(x+\delta x) - \phi(x)], \tag{3.22}$$

and since eq. (3.19) holds, under a local $U(1)$ transformation it transform as follows:

$$D^\mu \phi(x) \to (D^\mu \phi(x))' \to e^{i\alpha(x)} D^\mu \phi(x). \tag{3.23}$$

This property allows us to rewrite the kinetic term in the Lagrangian using this new derivative operator, called a *covariant gauge derivative*. The Lagrangian thus becomes:

$$\mathcal{L}_0 = (D_\mu \phi)(D^\mu \phi)^* - m^2 \phi \phi^*, \tag{3.24}$$

which under a $U(1)$ local transformation turns into:

$$\begin{aligned} \mathcal{L}_0' = (D_\mu \phi)'(D^\mu \phi)'^* - m^2 \phi' \phi'^* = & \qquad (3.25)\\ = e^{i\alpha(x)}(D_\mu \phi) e^{-i\alpha(x)} (D^\mu \phi)^* - m^2 \phi \phi^* = & \qquad (3.26)\\ = (D_\mu \phi)(D^\mu \phi)^* - m^2 \phi \phi^* = \mathcal{L}_0, & \qquad (3.27) \end{aligned}$$

and it is therefore invariant. It is also required that $W(x,x) = 1$ since there is nothing to compensate in this case. Since a $U(1)$ transformation is basically a multiplication by an imaginary exponential, the Wilson line can be written as:

$$W(x,y) = e^{i\phi(x,y)}, \tag{3.28}$$

where $\phi(x,x) = 0$ and such that $\frac{\partial \phi(x,y)}{\partial y^\mu}\Big|_{x=y} = gA^\mu$, where A^μ is a vector field and g is the coupling constant for such vector field. If δx^μ is small enough, the Wilson line can be expanded as:

$$W(x, x+\delta x) = 1 - ig\delta x^\mu A_\mu(x) + \mathcal{O}(\delta x^2) \tag{3.29}$$

Then, it follows from eq. (3.18) that under a local $U(1)$ transformation the field transforms as:

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{1}{g}\partial_\mu \alpha(x), \tag{3.30}$$

which leads to the following expression of the covariant derivative through its definition (3.22):

$$D^\mu\phi(x) = \lim_{\delta x^\mu \to 0} \frac{1}{\delta x^\mu}[(1 - ig\delta x^\mu A_\mu(x))\phi(x + \delta x) - \phi(x)] \tag{3.31}$$

$$D^\mu\phi(x) = \lim_{\delta x^\mu \to 0} \frac{1}{\delta x^\mu}[\phi(x + \delta x) - \phi(x)] - igA_\mu(x)\phi(x) \tag{3.32}$$

$$\implies D_\mu\phi = \partial_\mu\phi - igA_\mu\phi. \tag{3.33}$$

In this way, the vector field A_μ is introduced as a connection, allowing us to compare fields at different points in space, despite their different phase. This is called a *gauge field*. It is worth noticing that since the action of the covariant derivative on the field has a nice transformation, so will have $D_\mu D_\nu\phi(x)$. Therefore, we can compute the action of the commutator of D_μ and D_ν on the field as follows:

$$[D_\mu, D_\nu]\phi = [\partial_\mu - igA_\mu, \partial_\nu - igA_\nu]\phi = \tag{3.34}$$

$$= [\partial_\mu, \partial_\nu]\phi - g^2[A_\mu, A_\nu]\phi - ig([\partial_\mu, A_\nu] + [A_\mu, \partial_\nu])\phi = \tag{3.35}$$

$$= -ig(\partial_\mu(A_\nu\phi) - A_\nu\partial_\mu\phi + A_\mu\partial_\nu\phi - \partial_\nu(A_\mu\phi)) = \tag{3.36}$$

$$= -ig(\partial_\mu A_\nu - \partial_\nu A_\mu)\phi = -igF_{\mu\nu}\phi, \tag{3.37}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength and it is just a function, not an operator. Therefore, the field strength for QED ($g = e$) can be defined as :

$$F_{\mu\nu} = \frac{i}{e}[D_\mu, D_\nu]. \tag{3.38}$$

This leads to a geometric interpretation: the electromagnetic field strength is the difference between applying $D_\mu D_\nu$ (which compares the values of the fields separated in the ν direction and then in the μ direction) and the reverse order. Equivalently, it represents the result of comparing the fields around an infinitesimal closed loop in the $\mu - \nu$ plane. Going back to the Lagrangian where we have substituted the standard derivatives with covariant derivatives, there is now the new gauge field. Therefore, we need to include a kinetic term describing its dynamics, which should also be invariant under local $U(1)$ transformation. Indeed, we can use the field strength to build such term, since the combination $F^{\mu\nu}F_{\mu\nu}$ is invariant. Including this new term in the Lagrangian gives us:

$$\mathcal{L} = (D_\mu\phi)(D^\mu\phi)^* - m^2\phi\phi^* - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{3.39}$$

where the constant $\frac{1}{4}$ is introduced for normalization. This can be written in another form, recalling the expression of the covariant derivatives:

$$\mathcal{L} = (D_\mu \phi)(D^\mu \phi)^* - m^2 \phi \phi^* - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \tag{3.40}$$

$$= (\partial_\mu - ieA_\mu)\phi(\partial^\mu + ieA^\mu)\phi^* - m^2 \phi \phi^* - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \tag{3.41}$$

$$= (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi \phi^* - ieA^\mu(\phi \partial_\mu \phi^* - \phi^* \partial_\mu \phi) + e^2 A_\mu A^\mu \phi \phi^* - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \tag{3.42}$$

$$= \mathcal{L}_0 + \mathcal{L}_{int} + \mathcal{L}_M, \tag{3.43}$$

where we have divided the Lagrangian in three terms: $\mathcal{L}_0$ describes the free scalar fields, $\mathcal{L}_{int} = -A^\mu J_\mu + e^2 A_\mu A^\mu \phi \phi^*$, with $J_\mu = ie(\phi \partial_\mu \phi^* - \phi^* \partial_\mu \phi)$ the Noether current associated with $U(1)$ global symmetry. This part describes the interactions of the fields ϕ and ϕ^* with the gauge field. Lastly, $\mathcal{L}_M$ is Maxwell's Lagrangian for the vector field A_μ, describing the dynamics of this gauge field. By carrying out the quantization procedure of these fields, it can be seen that the theory describes a particle and anti-particle pair, coupled to a spin 1 particle, that is the gauge field. The existence of antiparticles is automatically implied by this coupling.

To summarize, starting from a Lagrangian with a global $U(1)$ symmetry, the requirement of a $U(1)$ local symmetry led to the introduction of a new vector field with a particular transformation law, that introduced interactions between particles and the field. Finally, it worth noticing that, in this case, the symmetry group was $U(1)$, an abelian group, and for this reason this type of theories are called *abelian gauge theories*. However, the general case may involve a non-abelian symmetry group, and these gauge theories are thus generalized to what are called *non-abelian gauge theories* or *Yang-Mills theories* that are heavily based on groups properties and algebra, which will be the subject of the next chapter.

Chapter 4

Lie groups

A very important concept when discussing QFT is the concept of groups, and more specifically Lie groups. In this chapter, I shall review some of the basic notions about groups [1], outline Lie groups and their properties, focusing on representations and useful results of algebra [4] that will be useful later on when computing color factors of Feynman diagrams in chapter 6.

4.1 General properties

First, recall that a group is a set that has a internal composition law denoted by the symbol $\cdot$, that is often called product or multiplication. It has four fundamental properties:

1. *Closure*: $\forall g_1, g_2 \in G,\ g_1 \cdot g_2 \in G$;
2. *Associativity*: $\forall g_1, g_2, g_3 \in G,\ g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$;
3. *Existence of neutral element*: $\forall g \in G, \exists \mathbf{I}\ s.t.\ g \cdot \mathbf{I} = \mathbf{I} \cdot g = g$;
4. *Existence of inverse element*: $\forall g \in G, \exists g^{-1}\ s.t.\ g \cdot g^{-1} = g^{-1} \cdot g = \mathbf{I}$.

In addition, if $\forall g_1, g_2 \in G,\ g_1 \cdot g_2 = g_2 \cdot g_1$, the group is said *abelian*, if not it is said *non-abelian*.

If one looks at an infinitesimal group transformation, the corresponding operators are called the *generators* of the group. They are called generators because all the other elements of the group can be written in terms of these generators, specifically for any group G, an element g can be written as $g = \exp\left(ic_i^g \lambda_i\right)$, where the c_i^g are real numbers and the λ_i are the group generators.

Lie groups are a class of groups that have an infinite number of elements, but a finite number of generators. The number of generators is given by the number

of independent parameters of a general group transformation. The elements of the group are also differentiable manifolds. The commutation relations between the group generators form the so called *Lie algebra*, which is important to understanding QFTs. The SM is based on three important Lie groups that are the symmetry groups at the hearth of the three QFTs describing the three fundamental interactions, these are $U(1)$, $SU(2)$, and $SU(3)$. The important fact about Lie groups is that any group element connected to the identity can be written as:

$$U = \exp(i\alpha^a T^a) \cdot \mathbf{I}, \tag{4.1}$$

where α^a are numbers parametrizing the group elements (i.e. the three components of the velocity and the three Euler's angle for the Lorentz group) and T^a are the group generators. The Lie algebra is defined by the commutation relations of the generators:

$$[T^a, T^b] = if^{abc}T^c, \tag{4.2}$$

where f^{abc} are the *structure constants* of the group and these are independent of the representation. It follows then that a Lie group is abelian if $f^{abc} = 0$, non-abelian otherwise. For example, the structure constants for $SU(2)$ are $f^{abc} = \epsilon^{abc}$, hence it is not an abelian group. From now on I will focus on the $SU(N)$ group in general, that is usually represented as the group of $N \times N$ special hermitian matrices, special meaning with determinant equals to 1. The number of independent parameters is $N^2 - 1$, hence there are $N^2 - 1$ generators for $SU(N)$. This is because initially there are N^2 complex parameters in an $N \times N$ hermitian matrix, and so $2N^2$ real parameters. However, the unitary condition, $U^\dagger U = \mathbf{I}$, imposes N^2 real constrains while the special condition imposes one additional constrain, hence $2N^2 - N^2 - 1 = N^2 - 1$. Furthermore, the special condition implies that the generators are traceless, that is:

$$tr(T^a) = 0 \ \forall a = 1, 2, ..., N^2 - 1. \tag{4.3}$$

4.2 Representations

A *representation* is a particular embedding of the elements of the group in operators acting on a vector space. The dimension of this vector space is said to be the dimension of the representation, which can be infinite or finite dimensional. The two most important representations of a group are the fundamental and adjoint representation.

The fundamental representation is the smallest non trivial representation of the Lie algebra. For $SU(N)$ this is the set of $N \times N$ hermitian matrices with

determinant equal to 1. Consider a set of N fields denoted by ϕ_i that transform in this representation, under an infinitesimal group transformation they become:

$$\phi_i \to \phi_i' = \phi_i + i\alpha^a (T^a_{fund})_{ij}\phi_j, \tag{4.4}$$

where α^a are real numbers. This is obtained from the Taylor expansion of the general expression of an element of the group. Similarly, the complex conjugate fields transform in the anti-fundamental representation, where $T^a_{anti-fund} = -(T^a_{fund})^*$, hence:

$$\phi_i^* \to \phi_i'^* = \phi_i^* + i\alpha^a (T^a_{anti-fund})_{ij}\phi_j^* = \phi_i^* - i\alpha^a \phi_j^* (T^a_{fund})_{ij}, \tag{4.5}$$

where we used the fact that T^a_{fund} is hermitian for $SU(N)$. The algebra can be found by expanding a basis of group elements near $\mathbf{I}$. In this way, the generators also can be found. For $SU(2)$, the generators in the fundamental representation are the Pauli matrices σ^a divided by 2, for the normalization convention:

$$T^a = \tau^a \equiv \frac{\sigma^a}{2}. \tag{4.6}$$

They satisfy the well known commutation relation $[\frac{\sigma^a}{2}, \frac{\sigma^b}{2}] = i\epsilon^{abc}\frac{\sigma^c}{2}$. For $SU(3)$, there are 8 generators, that in this representation are known as the Gell-Mann matrices divided by 2: $T^a = \frac{\lambda^a}{2}$. In physics, usually the structure constants are normalized by imposing the following normalization condition:

$$\sum_{c,d} f^{acd} f^{bcd} = N\delta^{ab}. \tag{4.7}$$

This implies that the $SU(N)$ generators in the fundamental representation are normalized so that:

$$tr(T^a T^b) = \frac{1}{2}\delta^{ab}. \tag{4.8}$$

One can observe that this relation implies that the product of two generators is well defined, but in reality in a Lie algebra only the commutator of them is defined.

The adjoint representation consists instead of operators that act directly on the vector space spanned by the generators themselves, basically the Lie algebra. Since for $SU(N)$ there are $N^2 - 1$ generators, this representation has dimension $N^2 - 1$ The matrices representing the adjoint generators are given by $(T^a_{adj})^{bc} = -if^{abc}$. For $SU(2)$, these are 3×3 traceless hermitian matrices, since the structure constants are given by the anti-symmetric 3×3 tensor ϵ^{abc}. Similarly, for $SU(3)$ these will be 8×8 matrices. The commutation relations still hold, as this is independent of the representation. The physical relevance of this representation is related to the transformation of gauge fields.

Another important notion to define is Casimir operators or simply *Casimir*. These are operators that commute with all other elements of the group. AS a result, the representation can be characterized in a basis-independent way. For instance, in $SU(2)$ it is known that the operator $\hat{J}^2 = \sum_a T^a T^a$ is a Casimir operator with eigenvalue $j(j+1)$ where j is denoted as *spin*. In general, the quadratic Casimir $C_2(R)$ is defined by

$$\sum_a T^a T^a = C_2(R)\mathbf{I}. \tag{4.9}$$

This will always be proportional to the unit operator because of Schur's lemma which states that a group element commuting with all other elements in any irreducible representation must be proportional to $\mathbf{I}$. Since in a Lie group the elements can be obtained from the generators, it is sufficient to show that the quadratic Casimir commutes with these. Indeed, implying the sum over a using Einstein's convention, we have:

$$[T_R^a T_R^a, T_R^b] = T_R^a [T_R^a, T_R^b] + [T_R^a, T_R^b] T_R^a = i f^{abc} (T_R^a T_R^c + T_R^c T_R^a) = 0,$$

where Leibniz's rule was used for the first step and the antisymmetry of the structure constants in the last one. It is convenient to define an inner product on the generators in order to evaluate the quadratic Casimir. The generators can be chosen so that:

$$tr(T_R^a T_R^b) = T(R)\delta^{ab}, \tag{4.10}$$

where $T(R)$ (or sometimes $C(R)$) is denoted as the index of the representation R. The convention set for the fundamental representation, eq. (4.8) implies that $T(Fund) \implies T_F = \frac{1}{2}$. While in the adjoint representation $T(adj) \implies T_A = N$. To determine the quadratic Casimir we can set $a = b$ in the above equation, obtaining the definition of $C_2(R)$:

$$tr(T_R^a T_R^a) = T(R)\delta^{aa} = C_2(R)\mathbf{I} \tag{4.11}$$

This can be written as:

$$d(R)C_2(R) = T(R)d(G), \tag{4.12}$$

where $d(R)$ is the dimension of the representation and $d(G)$ is the dimension of the group. The dimension of $SU(N)$ is N^2-1, while in the fundamental representation we have $d(F) = N$ and in the adjoint representation $d(A) = N^2 - 1$. Therefore:

$$C_2(F) = C_F = \frac{N^2-1}{2N} \qquad\qquad C_2(A) = C_A = N, \tag{4.13}$$

Then, for $SU(2)$ we have $C_F = \frac{3}{4}$ and $C_A = 2$, while for $SU(3)$ we have $C_F = \frac{4}{3}$ and $C_A = 3$. Furthermore, in any representation one can write:

$$tr([T_R^a, T_R^b]T_R^c) = if^{abd}tr(T_R^d T_R^c) = if^{abd}T(R)\delta^{dc} = if^{abc}T(R). \tag{4.14}$$

It follows that the structure constants can be always written in terms of the commutator of generators or products of them:

$$f^{abc} = -\frac{i}{T(R)}tr([T_R^a, T_R^b]T_R^c). \tag{4.15}$$

This property has important implication in the context of gluon scattering. Another important identity that follows from the properties of the generators is the Fierz identity:

$$T_{ij}^a T_{kl}^a = \frac{1}{2}(\delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl}), \tag{4.16}$$

where the indices i, j, k and l take value from 1,2,.. N. This identity is derived in appendix A and it will be fundamental when calculating color factors of Feynman diagrams. Relations (4.7), (4.10), (4.13), and (4.16) are constantly used when doing calculations in QCD, especially regarding color factors in transition amplitudes. This is the end goal of the thesis and will be discussed in the next chapters.

Chapter 5

Non-abelian gauge theories

In this chapter, I aim at explaining some of the basic properties of the strong interaction and how it is mathematically described by a specific type of QFT based on a gauge transformation of a non-abelian group and, therefore, called non-abelian gauge theory.

5.1 Quantum chromodynamics

In the following section, I shall provide an overview of the basics facts that characterize the strong interaction and that eventually lead to the formulation of the QFT that describes it, Quantum Chromodynamics or simply QCD. These notions are taken from [7] and [8].

The strong interaction, as the name suggests, is really strong. Among the four fundamental forces of nature, it is ranked as the strongest one at the nuclear and sub-nuclear level, or generally speaking at the high energy scales reached by current experiments. Particles that interact via the strong force are called *hadrons*, which are divided in two categories: *baryons* and *mesons*, with half-integer spin and integer spin. There are mainly two types of particles in QCD: fermions called *quarks*, and massless spin 1 gauge bosons called *gluons*, that are the mediators of the interaction. It is useful to compare these characteristics with those of QED. In this analogy, gluons play the same role of photons. However, gluons have a charge in the context of the strong force, meaning they can directly interact with each other through the strong interaction, unlike photons that are known to be neutral in terms of electric charge and thus there is no possible direct self interaction among them. Only at the quantum level photons can interact indirectly via charged virtual fermions that only exists for a brief time interval. Throughout the years, a total of six quarks have been discovered: they are fermions with

fractional charge and whose masses are some of the only free parameters of the SM. These six different types of quarks are usually denoted as *flavours*, and they are up (u),down (d), charm (c), strange (s), top (t), and bottom (b). Masses and charges of these quarks are listed in the table below.

	u	d	c	s	t	b
Mass (MeV)	2.15	4.70	1270	93.5	163000	4180
Charge (e)	2/3	-1/3	2/3	-1/3	2/3	-1/3

Table 5.1: Masses and charges of quarks [4].

One of the key aspects of QCD is *color*, that is the charge of QCD, meaning that particles that have a color charge can interact via the strong interaction. Again, comparing between QCD and QED, this is analogous to the electric charge, as it is known that particles with electric charge different from zero can interact via the electromagnetic force. However, this new charge characterizing the strong interaction is quite peculiar.

Color was introduced because some observations of the hadrons spectrum, that consisted of the various particles observed to be interacting via the strong force, seemed to be inconsistent with Pauli exclusion principle. It was known that baryons were composed of three quarks, but some of them were observed to have three quarks of the same flavour, and with total spin $\frac{3}{2}$. For example, the baryons $\Delta^{++} = uuu$ and $\Omega^{-} = sss$. Assuming that the baryons are in a state where the total orbital angular momentum vanishes, such value of the total spin implied that the three quarks have their spin aligned and therefore the total wave function of the baryon is symmetric under the exchange of two of the three equal quarks. Since quarks are fermions, this violated Pauli exclusion principle which states that the total wave function of a system composed of two or more fermions must be anti-symmetric under the exchange of any two fermions. To fix this problem, a new quantum number, called *color*, was introduced. In this way, if the three quarks in the hadron have the same flavour but different color, then the total wave function can be anti-symmetric as requested by the spin-statistics theorem. However, when researchers tried to find experimental evidence of the color of a hadron, no results were obtained, that is no hadron had a specific color. This lead to the *confinement law*, stating that all free particles are color singlet, or equivalently, colored particles must be bound into color-less states. Since in $SU(2)$ there is no way of making a singlet with three particles, which is like trying to make a spin 0 state when adding three spin 1/2 particles, and since some hadrons were made up of three quarks, the simplest hypothesis is to consider the $SU(3)$ group. This has been experimentally validated and thus color is mathematically described by the symmetry group $SU(3)$. Therefore, in the fundamental representation of $SU(3)$, we can identify a

basis of 3 color states that are conventionally called red (R), green (G), and blue (B). Since quarks are electrically charged, there are also anti-quarks, that have the respective anti-color: anti-red ($\overline{R}$), anti-blue ($\overline{B}$), and anti-green ($\overline{G}$). Color singlets can be formed by building invariants under the action of the color group $SU(3)$. These have the following form:

$$\Psi_b = \sum_{i,j,k \in R,G,B} \epsilon_{ijk} q_i q'_j q''_k \qquad\qquad \Psi_m = \sum_{j \in R,G,B} q_j \bar{q}'_j \,, \tag{5.1}$$

where q_i identifies a quark with i, j, and k color indexes and flavour $q \in \{u, d, c, s, t, b\}$. These two configurations represent baryons and mesons and, since quarks have spin 1/2, we conclude that baryons have half-integer spin while mesons have integer spin.

Since color is responsible for the property of confinement, which is one of the main features and characteristics of the strong interaction, understanding color is the key to comprehend the strong interaction. Therefore, one can derive QCD through a local $SU(3)$ color symmetry, much like scalar QED was derived from a $U(1)$ local symmetry in section 3.3. However, while $U(1)$ is an abelian group, $SU(3)$ is non-abelian, therefore we need to develop the adequate tools to describe such symmetry, which shall be the discussion of the next chapter.

5.2 Yang-Mills theories

Yang-Mills theories are a generalization of QED, meaning they are theories that describe massless spin-1 particles that have the property of interacting among themselves. This is the main characteristic of the strong interaction that results in weird phenomena, such as asymptotic freedom. These theories are based on a generalization of the gauge symmetry that was used to derive scalar QED in section 3.3, that is called non-abelian gauge symmetry, and for this reason these theories are also known as non-abelian gauge theories [4].

Consider two complex fields ϕ_1 and ϕ_2. The kinetic Lagrangian describing the fields is then:

$$\mathcal{L}_{kin} = (\partial_\mu \phi_1^*)(\partial^\mu \phi_1) + (\partial_\mu \phi_2^*)(\partial^\mu \phi_2) = (\partial_\mu \vec{\phi})^\dagger (\partial^\mu \vec{\phi}), \tag{5.2}$$

where $\vec{\phi} = (\phi_1, \phi_2)^T$, and it is invariant under a global $SU(2)$ transformation that can be expressed as $\vec{\phi} \to \vec{\phi}' = U\vec{\phi}$, where U is a special 2×2 unitary matrix. Recalling section 4.1, any element can be written as eq. (4.1), where in this case the T^a are the $SU(2)$ generators, that are the Pauli matrices σ^a divided by

2, $T^a = \tau^a = \frac{\sigma^a}{2}$. Considering an infinitesimal group transformation, the fields transform as:

$$\vec{\phi} \to \vec{\phi'} = \vec{\phi} + i\alpha^a T^a \vec{\phi}, \tag{5.3}$$

where α^a are real numbers. The application of the gauge principle involves promoting the global $SU(2)$ symmetry to a local one, that can be done by promoting the α^a from real numbers to real functions of space and time, $\alpha^a(x)$, where $x = (\vec{x}, t)$ denotes spacial and time coordinates. This clearly has implications on the Lagrangian, since it contains derivatives that are now acting on $\alpha^a(x)$. Hence:

$$\partial^\mu \phi_1 \to \partial^\mu \phi_1' = \partial^\mu \phi_1 (1 + i\alpha^a(x)\tau^a) + i\partial^\mu \alpha^a(x)\tau^a \phi_1 \tag{5.4}$$

$$\partial_\mu \phi_1^* \to \partial_\mu \phi_1^{*'} = \partial_\mu \phi_1^* (1 - i\alpha^a(x)\tau^a) - i\partial_\mu \alpha^a(x)\tau^a \phi_1^*. \tag{5.5}$$

Therefore, the Lagrangian is clearly not invariant under such transformation. To solve this problem, we can proceed as in section 3.3, by substituting the ordinary derivative with a covariant derivative defined as follows:

$$D_\mu \vec{\phi} = \partial_\mu \vec{\phi} - igA_\mu^a \tau^a \vec{\phi}, \tag{5.6}$$

where g is a constant representing the strength of the force described and A_μ^a is a set of three gauge fields, since $a = 1, 2, 3$ in $SU(2)$, while with $U(1)$ there was only a vector field. By recalling the definition of covariant derivatives, eq. (3.22), it can be seen that under a local $SU(2)$ transformation each one of the three gauge fields transform as follows:

$$A_\mu^a(x) \to A_\mu^a(x)' = A_\mu^a(x) + \frac{1}{g}\partial_\mu \alpha^a(x) - f^{abc}\alpha^b(x)A_\mu^c(x), \tag{5.7}$$

where $f^{abc} = \epsilon^{abc}$ are the structure constants for $SU(2)$. The non-abelian nature of $SU(2)$ is highlighted in this transformation law, which differs from eq.(3.30) for the presence of the term containing the structure constants. Again, as previously done when addressing the local $U(1)$ symmetry, it is now necessary to include a term describing the dynamic of these three gauge fields in the Lagrangian, and it can be proven that the only gauge-invariant kinetic term for these fields is:

$$\mathcal{L}_{YM} = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} = -\frac{1}{4}\sum_a (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c)^2. \tag{5.8}$$

This kinetic term, also known as Yang-Mills Lagrangian, is crucial to understand the physics of Yang-Mills theories, because it induces renormalizable interactions among the three gauge fields of $SU(2)$, which are the mediators of the weak interactions. This has very important consequences, for example virtual gauge bosons can interact among themselves producing a vacuum polarization effect that behaves

in the opposite way to that of QED. That is, while in QED the coupling constant becomes logarithmically weaker at larger distances, here it becomes stronger at larger distances or low energies. This means that at high energies or shorter distances, the strong force is weak. This property is known as *asymptotic freedom* and it is one of the peculiarities of the strong interaction. However, at the high energy scales reached by current experiments the strong force is typically ten times stronger than the other interactions. The kinetic term representing the dynamics of the three gauge fields can also be derived from a geometrical point of view, that shall be the discussion of the next paragraph.

5.3 Geometrical derivation of Y-M Lagrangian

In section 3.3, the Wilson line was introduced to allows us to compare fields valued at different points in space, without worrying about their possible different phase convention. In that case, the required symmetry was a local $U(1)$, however the general case involves more than a simple phase rotation and it will be explored in this section [4].

The end goal in the non-abelian case remains unchanged: we want to compare fields valued at different points in space. Hence the procedure is the same, except the definition of the Wilson line will change since the transformation on the fields changes too. For $SU(N)$, a local transformation is defined as

$$\vec{\phi} \to \vec{\phi}' = \phi e^{i\alpha^a(x)T^a}, \tag{5.9}$$

where T^a are the group generators, $\alpha^a(x)$ are real functions of space and time, and $a = 1, ..., N^2 - 1$. Therefore, the Wilson line is required to transform as:

$$W(x,y) \to W(x,y)' = e^{i\alpha^a(x)T^a} W(x,y) e^{-i\alpha^a(y)T^{a\dagger}} = \tag{5.10}$$

$$= e^{i\alpha^a(x)T^a} W(x,y) e^{-i\alpha^a(y)T^a}, \tag{5.11}$$

where it was used the fact that $T^{a\dagger} = T^a$ for $SU(N)$. In the non-abelian case, it is convenient to represent the gauge field as a Lie algebra-valued field, that is instead of assigning a real number for each point of space and time as for a standard field, here it is assigned an element of the Lie algebra associated with a given Lie group, in this case $SU(N)$. Explicitly, the gauge fields are written as:

$$\vec{A}_\mu \equiv A^a_\mu(x) T^a, \tag{5.12}$$

where $A^a_\mu(x)$ are the components of the fields valued on the Lie algebra. The infinitesimal expansion of the Wilson line can be found as in the abelian case by

writing $W(x,y) = e^{i\phi^a(x,y)T^a}$, where $\left.\frac{\partial\phi^a(x,y)}{\partial y^\mu}\right|_{x=y} = gA^a_\mu$, with A^a_μ a set of N^2-1 gauge fields and g a constant. Then, if y^μ is sufficiently near to x^μ, that is $y^\mu = x^\mu + \delta x^\mu$, we can expand the Wilson line as follows:

$$W(x^\mu, x^\mu + \delta x^\mu) = \mathbf{I} - ig\vec{A}_\mu \delta x^\mu. \tag{5.13}$$

In order to derive the transformation law of the gauge fields, it is convenient to consider the transformation of the covariant derivative, that must be the same of the original fields, instead of expanding the transformation of the Wilson line, as was done in section 3.3. Following this, it is known that the fields under a $SU(N)$ local transformation change as follow:

$$\vec{\psi}(x) \to \vec{\psi}'(x) = U(x)\vec{\psi}(x), \tag{5.14}$$

where $U(x) = e^{i\alpha^a(x)T^a} \in SU(N)$ is the group element for the transformation at point x. Then, the covariant derivative must transform as the fields:

$$D_\mu\vec{\psi}(x) \to D'_\mu\vec{\psi}(x)' = U(x)D_\mu\vec{\psi}(x). \tag{5.15}$$

Therefore, by using the definition of the covariant derivative we have:

$$(\partial_\mu - ig\vec{A}'_\mu)U(x)\vec{\psi}(x) = U(x)(\partial_\mu - ig\vec{A}_\mu)\vec{\psi}(x) \tag{5.16}$$

$$\partial_\mu(U\vec{\psi}) - ig\vec{A}'_\mu U\vec{\psi} = U\partial_\mu\vec{\psi} - igU\vec{A}_\mu\vec{\psi} \tag{5.17}$$

$$\implies \partial_\mu U - ig\vec{A}'_\mu U = -igU\vec{A}_\mu, \tag{5.18}$$

and thus we have:

$$\vec{A}'_\mu = U\vec{A}_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}, \tag{5.19}$$

which the infinitesimal version, where $U(x) \approx \mathbf{I} + i\alpha^a(x)T^a$, in terms of components becomes:

$$A^a_\mu(x) \to A'^a_\mu(x) = (\mathbf{I} + i\alpha^a(x)T^a)A^b_\mu T^b(\mathbf{I} - i\alpha^a(x)T^a) - \tag{5.20}$$

$$- \frac{i}{g}(i\partial_\mu(\alpha^a(x))T^a)(\mathbf{I} + i\alpha^b(x)T^b) = \tag{5.21}$$

$$= A^b_\mu T^b + i\alpha^a A^b_\mu(T^aT^b - T^bT^a) + \frac{1}{g}\partial_\mu(\alpha^a)T^a + \tag{5.22}$$

$$+ \mathcal{O}(\alpha^2(x)) = \tag{5.23}$$

$$= A^b_\mu T^b + \frac{1}{g}\partial_\mu(\alpha^a)T^a - f^{abc}\alpha^a(x)A^b_\mu T^c \tag{5.24}$$

$$\implies A'^a_\mu = A^a_\mu(x) + \frac{1}{g}\partial_\mu\alpha^a(x) - f^{abc}\alpha^b(x)A^c_\mu(x). \tag{5.25}$$

This is precisely eq. (5.7). Again, the difference between this transformation of the gauge fields and eq. (3.30) is the last term, that contains the structure constants of $SU(N)$, which differ from zero due to the non-abelian nature of the group. Finally, we can also calculate the action of the commutator of covariant derivatives on the field as follows:

$$[D_\mu, D_\nu]\vec{\psi}(x) = [\partial_\mu - ig\vec{A}_\mu, \partial_\nu - ig\vec{A}_\nu]\vec{\psi} = \tag{5.26}$$
$$= [\partial_\mu, \partial_\nu]\vec{\psi} - g^2[\vec{A}_\mu, \vec{A}_\nu]\vec{\psi} - ig([\vec{A}_\mu, \partial_\nu] + [\partial_\mu, \vec{A}_\nu])\vec{\psi} = \tag{5.27}$$
$$= (-ig(\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu) - g^2[\vec{A}_\mu, \vec{A}_\nu])\vec{\psi}(x). \tag{5.28}$$

Therefore, as in the abelian case, this is just a function, since there are no derivatives acting on the field. Furthermore, the natural field strength for the non abelian case can be written as:

$$\vec{F}_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = (\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu) - ig[\vec{A}_\mu, \vec{A}_\nu], \tag{5.29}$$

which in terms of components becomes $\vec{F}_{\mu\nu} = F^a_{\mu\nu}T^a$, where $F^a_{\mu\nu}$ is the following:

$$F^a_{\mu\nu}T^a = (\partial_\mu A^b_\nu T^b - \partial_\nu A^b_\mu T^b) - ig(A^b_\mu T^b A^c_\nu T^c - A^c_\nu T^c A^b_\mu T^b) = \tag{5.30}$$
$$= (\partial_\mu A^b_\nu - \partial_\nu A^b_\mu)T^b - igA^b_\mu A^c_\nu(T^bT^c - T^cT^b) = \tag{5.31}$$
$$= (\partial_\mu A^b_\nu - \partial_\nu A^b_\mu)T^b + gA^b_\mu A^c_\nu f^{bcd}T^d \tag{5.32}$$
$$\implies F^a_{\mu\nu} = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + gf^{abc}A^b_\mu A^c_\nu, \tag{5.33}$$

where we have used the commutation relation (4.2), renamed the contracted indices and used the antisymmetry of f^{abc}. The square of this term is precisely the kinetic term introduced in eq. (5.8). Once again, I would like to highlight the difference from the natural field strength in the abelian case is the last term, that contains the structure constants. Interestingly, under a global or local $SU(N)$ transformation, the field strength transforms covariantly as it contains gauge fields, that is:

$$\vec{F}'_{\mu\nu} = U(x)\vec{F}_{\mu\nu}U^{-1}(x). \tag{5.34}$$

This ensures that quantities derived from $\mathcal{L}_{YM}$ are gauge invariant, and thus represent physical degrees of freedom. That said, by considering the infinitesimal representation of the group elements, $U(x) = e^{i\alpha^a(x)T^a} \approx \mathbf{I} + i\alpha^a(x)T^a$, the components of the field strength transform as follows:

$$F^a_{\mu\nu}T^a \to F'^a_{\mu\nu}T^a = (\mathbf{I} + i\alpha^a(x)T^a)F^b_{\mu\nu}T^b(\mathbf{I} - i\alpha^a(x)T^a) = \tag{5.35}$$
$$= F^b_{\mu\nu}T^b + i\alpha^a(x)(T^aT^b - T^bT^a)F^b_{\mu\nu} + \mathcal{O}(\alpha^2(x)) = \tag{5.36}$$
$$= F^b_{\mu\nu}T^b - f^{abc}\alpha^a(x)F^b_{\mu\nu}T^c \tag{5.37}$$
$$\implies F'^a_{\mu\nu} = F^a_{\mu\nu} - f^{abc}\alpha^b(x)F^c_{\mu\nu} \tag{5.38}$$

which is the same for both α global or $\alpha(x)$ local because there are no derivatives involved. Since this transformation doesn't involve the generators, it implies that the kinetic term depends only on the fields $F^a_{\mu\nu}$ that transform in the adjoint representation, while initially $\vec{F}_{\mu\nu}$ transformed in the fundamental representation.

Chapter 6

Color factors calculation

In this chapter, I want to demonstrate how the algebra developed in chapter 4 can be used to calculate important color factors of some Feynman diagrams representing basic QCD processes, leading to the *color Feynman rules* for QCD. For this reason, it is necessary to briefly introduce the concept of Feynman diagrams and how they are connected to probability through a transition amplitude.

6.1 Feynman diagrams and probability

Feynman diagrams are a visual representation of complex mathematical formula describing physical processes between particles interacting in space time. Each process has a certain probability of happening and, from quantum mechanics, it is known that probabilities are linked to the square modulus of an amplitude, a matrix element. In QFT, the same holds true, and the amplitude in question, usually denoted as $\mathcal{M}$, is proportional to the square of an element of the S *matrix* [4], that encodes the various ways given initial states turn into given final states. Mathematically:

$$prob \propto |\mathcal{M}|^2 \propto |\langle i|S|f\rangle|^2,$$

where i and f denote the initial and final states.

Each of the terms in the expansion of the S matrix can be represented as a Feynman diagram. The rules needed to derive the formula behind the diagrams are called *Feynman rules*, and they can be obtained in various contexts using different approaches. The transition amplitude $\mathcal{M}$ is a scalar function that contains all dynamical information about the process [9]. It usually depends on the momenta of the interacting particles, but it also contains color factors if the particles interact via the strong interaction. The last point is what I want to focus on, that is calculating the color factor of some Feynman diagrams representing QCD processes.

First, I shall remind some basic Feynman diagrams notation and properties [10].

Particles are represented by lines. Due to crossing symmetry, particles in diagrams going from point A to point B are equivalent to anti-particles going from B to A. Fermions are represented by straight lines, photons by wiggly lines, and gluons by curly lines or two straight lines with arrows in opposite direction. Angles are basically useless, meaning a loop can be represented by a circle or a square without changing its physical meaning. Unless specified otherwise, diagrams will be drawn with time going from left to right. See Fig. 6.1 for reference.

6.2 Color Feynman rules for QCD

In order to calculate color factors, it is necessary to represent color properties of quark and gluons via Feynman diagrams. A useful way of doing that is to think about quarks as objects having 3 possible states, one for each color, while gluons carry all possible combination of color anti-color pair, without the neutral one that would be formed by the sum over all equal color anti-color pairs [10]. This leads to the double line formalism, where a fermion-gluon vertex has the following visual representation:

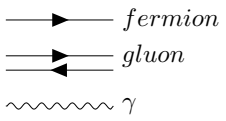

Figure 6.1: Representation of different particles in Feynman diagrams.

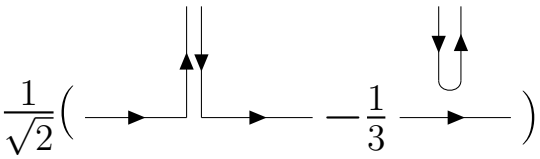

Figure 6.2: Fermion-gluon vertex representation.

Mathematically, this corresponds to a generator of the $SU(3)$ group, T^a. The physical interpretation is that the first term represents the interaction of the quark with the gluon, hence I shall refer to it as *interaction term*. Instead, in the second term there is no interaction as the fermion line is not altered by the gluon, thus it will be referred to as *non-interaction term*. This means that in the first term the color of the quark changed from initial to final state, while in the second term color is unchanged.

As explained in section 5.2, gluons are charged and therefore they can interact among themselves. Thus, 3 or 4-gluon vertexes must be accounted for. In the

double line formalism, the 3-gluon vertex is represented by the following:

Figure 6.3: 3-gluon vertex (f^{abc}). Image taken from [11].

This corresponds to the structure constants of $SU(3)$, since these interactions exist due to the non-abelian nature of the group. Fig. 6.2 and Fig. 6.3 constitute the fundamental blocks of the color Feynman rules for QCD, where the $\frac{1}{\sqrt{2}}$ factor present in both figures is to ensure the right normalization constant set to $\frac{1}{2}$ for T_F as in eq.(4.10). These rules can be combined with $SU(3)$ color algebra and then applied to calculate color factors for several diagrams.

In these representations, contraction over color indices is implied by connecting the respective color or anti-color line. A close loop gives a factor $N = 3$ since it is the trace of the unit matrix in $SU(3)$ [11]. With this knowledge, one is able to compute the color factors of almost all QCD processes. As a first example, consider the quark self-energy diagram shown below:

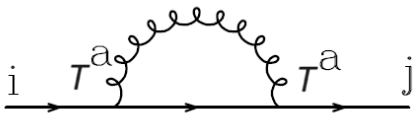

Figure 6.4: Quark self-energy Feynman diagram. A virtual gluon is emitted and re-absorbed by the quark that changes color. Image taken from [11].

For reason that will be clear in a moment, it is useful to add a color index for the fermion line in the middle, say k. Computing the color factor using section 4.2 algebra, involves considering that each fermion-gluon vertex is a generator T^a. Thus, since the diagram above has two of these vertexes, we have the product of two generators, $T^a_{jk}T^b_{ki}$, where the indices are written backwards by convention. However, since the virtual gluon is re-absorbed by the quark, we need to add a δ^{ab} factor. Hence, there is an implicit sum over a:

$$T^a_{jk}\delta^{ab}T^b_{ki} = T^a_{jk}T^a_{ki} = \sum_a (T^aT^a)_{ji}. \tag{6.1}$$

This is the left hand side of the Fierz identity, eq. (4.16), with $i \to j$, $j \to k$, $l \to i$. Therefore:

$$T^a_{jk} T^a_{ki} = \frac{1}{2}(\delta_{ji}\delta_{kk} - \frac{1}{N}\delta_{jk}\delta_{ki}) = \tag{6.2}$$

$$= \frac{1}{2}\delta_{ji}(N - \frac{1}{N}) = \tag{6.3}$$

$$= \delta_{ji}(\frac{N^2 - 1}{2N}). \tag{6.4}$$

However, by eq.(4.9) we also have:

$$\sum_a (T^a T^a)_{ji} = C_F \delta_{ji}. \tag{6.5}$$

Thus, by comparing these two equations, the value of C_F in the fundamental representation can be found to be:

$$C_F = \frac{N^2 - 1}{2N} = \frac{4}{3} \tag{6.6}$$

This can also be obtained using visual representations of the fermion-gluon vertex. Since there are two such vertices, the product of two Fig. 6.2 representations is needed. The visual calculation proceeds as follows:

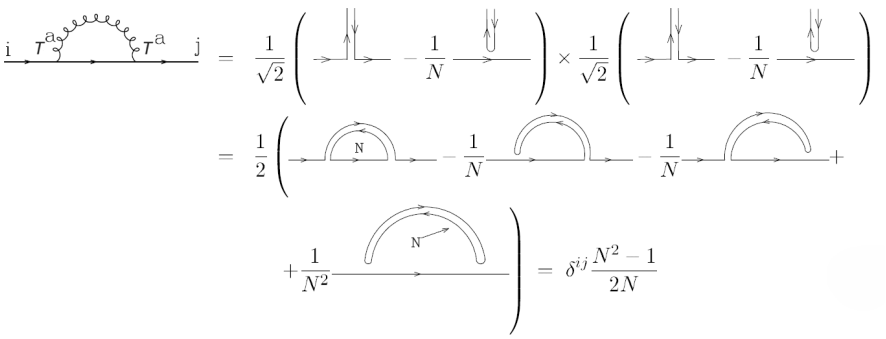

Figure 6.5: Color factor calculation of a quark self-energy Feynman diagram using visual representation of fermion-gluon vertexes. Image taken from [11].

The product of the two quark-gluon interaction terms gives the first diagram in the second line. It basically is a loop, thus giving a factor N, and a fermion line giving δ^{ij}. The second term in the second line is the product of the interaction term for the T_{jk} vertex, and the non-interaction term for the T_{ki}. The third term is the opposite. These are essentially fermion lines, therefore, each one of these

gives a factor δ^{ij}. Finally, the fourth term is the product of both non-interaction terms, that produces a loop giving a factor N and a fermion line giving δ^{ij}. Thus, the calculation proceeds as follows:

$$\frac{1}{2}(N\delta^{ij} - \frac{2}{N}\delta^{ij} + \frac{1}{N}\delta^{ij}) = \delta^{ij}\frac{N^2-1}{2N}. \tag{6.7}$$

Hence, using this double line formalism the same result is obtained.

As another example, consider the diagram below:

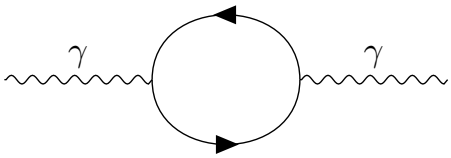

Figure 6.6: Feynman diagram representing a photon creating a quark anti-quark ($q\bar{q}$) pair that subsequently annihilates into another photon.

The color factor of this diagram can easily be found by observing that the photon is color free, meaning it is transparent with respect to the strong interaction. Therefore, the diagram reduces to a simple loop, that gives a factor $N = 3$.

An interesting variation of the diagram shown above is the following:

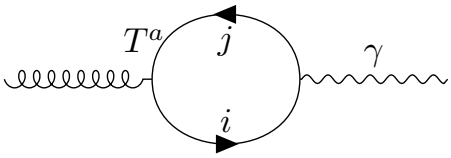

Figure 6.7: Feynman diagram representing a gluon creating a $q\bar{q}$ pair that subsequently annihilates into another photon.

As shown in the previous example, the photon can be neglected because it is color free. Therefore, the diagram becomes a gluon creating a $q\bar{q}$ pair that closes into a loop. Since a color line is connected, there will be a δ_{ij} factor:

$$T^a_{ji}\delta_{ij} = T^a_{ii} = 0, \tag{6.8}$$

using the fact that $SU(N)$ generators are traceless. The color factor of this diagram is 0, therefore, the probability that this process occurs is also zero. This reflects the fact that color must be conserved because initially, the gluon has a certain color. Since the color line of the two quarks is connected, it implies that they are a

color anti-color pair. However, this would result in the gluon being a color singlet, which is absurd.

The same result can be obtained using the visual representation, since there is basically a single quark-gluon vertex. Hence:

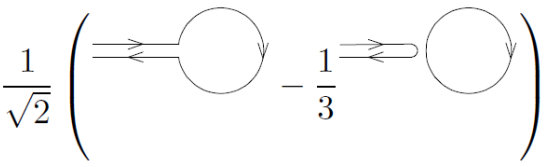

Figure 6.8: Visual representation of the color factor calculation of the Feynman diagram shown in Fig. 6.7. Image taken from [10].

This is basically the same as Fig. 6.2, except that it is sideways since the gluon creating the $q\bar{q}$ pair is coming from left to right, and the quark lines are closed since the quarks make a loop in the original diagram. The loop in the second term gives a $N = 3$ factor that cancels out the $\frac{1}{3}$. Therefore, we are left with the gluon line that is equivalent to the first term. Hence, they cancel out, giving a total color factor of 0, as expected.

Finally, to complete the color Feynman rules for QCD, let us consider the following diagram:

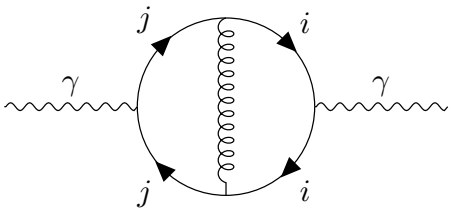

Figure 6.9: Feynman diagram representing a photon creating a $q\bar{q}$ pair that subsequently exchanges a gluon and then annihilates into another photon.

Again, the photons are irrelevant in computing color factors. Thus, we are left with the central loop, but now there is an exchanged gluon. This creates two fermion-gluon vertexes. Therefore, the calculation is very similar to that shown in Fig. 6.5, with the difference that in this case the quark lines are connected. This produces the following visual calculation:

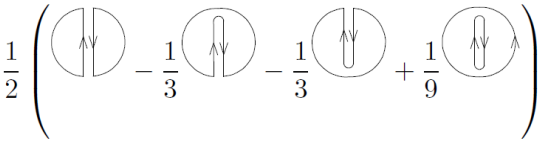

Figure 6.10: Visual calculation of the color factor of Fig.6.9 Feynman diagram. Image taken from [10].

The first and fourth term have 2 closed loops, thus each one of them results in a N^2 factor. The first one represents both quarks interacting with the gluon, while the fourth term represents both of them not interacting. Instead, the second term represents the top quark interacting and the bottom quark not interacting, while the third term is the opposite. These both give a factor N as they are closed loops. Thus, the color factor is:

$$\frac{1}{2}(N^2 - \frac{2}{3}N + \frac{N^2}{9}) = \frac{1}{2}(9 - 2 + 1) = 4. \tag{6.9}$$

The same result can be obtain using the group algebra, recalling that each fermion-gluon vertex is a group generator. Here we have two vertexes with the same gluon, thus:

$$T^a_{ij}T^b_{ji}\delta^{ab} = T^a_{ij}T^a_{ji} = \frac{1}{2}(\delta_{ii}\delta_{jj} - \frac{1}{N}\delta_{ij}\delta_{ji}) = \tag{6.10}$$

$$= \frac{1}{2}(N^2 - 1) = 4, \tag{6.11}$$

where Fierz identity was used, with $k \to j$, and $l \to i$.

6.3 Physical relevance of color factors

In order to show the physical relevance of these calculations, it is interesting to evaluate the color factor for the one-loop interaction vertex with a photon and with a gluon. These diagrams will have a different color factor that leads to a very interesting conclusion and provides a qualitative argument in support of the physical phenomenon that involves quark bound states to be color-singlets.

First, consider the photon interaction vertex. It is basically a photon creating a $q\bar{q}$ pair, that subsequently exchanges a gluon. This is represented by the following diagram:

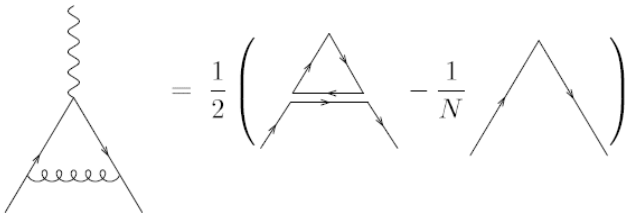

Figure 6.11: Visual calculation of the color factor of the one-loop interaction vertex with photon. Time going downward. Image taken from [11].

The process to obtain the right hand side of the above figure is the following: the photon is negligible as in diagrams shown before, therefore, we are left with a fermion line exchanging a gluon. However, since angles are irrelevant, one can imagine to move the quark lines so that they make a 180° angle. The new diagram resembles that of Fig. 6.4. Indeed, there are 2 fermion-gluon vertexes, thus, the calculation is the same as in Fig. 6.5, except that the lines have a different orientation, hence the triangles. That said, the first term contains a closed triangle, that is a loop, which gives a factor N, leaving a fermion line with a δ_{ij} factor. The same is true for the second term, except the loop. Therefore:

$$\frac{1}{2}(N\delta_{ij} - \frac{1}{N}\delta_{ij}) = \frac{N^2-1}{2N}\delta_{ij} = C_F\delta_{ij}. \tag{6.12}$$

This can also be obtained from the product of two generators using Fierz identity exactly as in Fig. 6.4. Notice how the color factor is positive.

A variation of this diagram is the one-loop interaction vertex with a gluon, shown below:

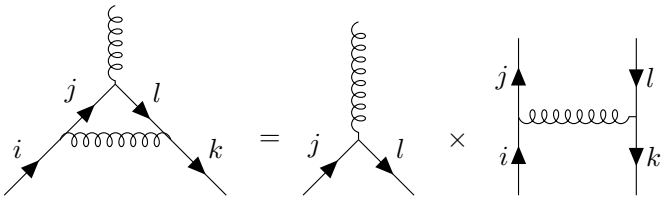

Figure 6.12: Feynman diagram representing a one-loop interaction vertex with a gluon decomposed into the product of two other diagrams. Time direction is downward.

The substitution of the photon with a gluon creates an additional fermion-gluon vertex that needs to be accounted for and it cannot be neglected as the photon was. The calculation can be broken into a product of two factors or diagrams as outlined in Fig. 6.12. The first one can be derived by neglecting the gluon exchanged in

the middle, leaving a diagram representing a gluon creating a $q\bar{q}$ pair. Instead, the second diagram can be obtained by neglecting the vertical gluon, thus leaving the quark lines not connected. The second term is basically a $q\bar{q}$ scattering, $q\bar{q} \to q\bar{q}$, that can be represented by the following diagram:

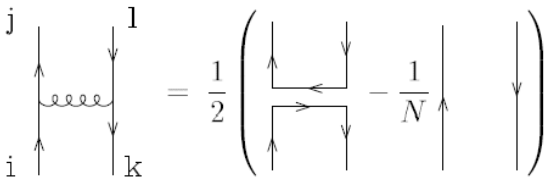

Figure 6.13: Feynman diagram representing a $q\bar{q}$ scattering on the left hand side and corresponding visual calculation on the right hand side. Time direction is downward. Image taken from [11].

The diagram has 2 fermion-gluon vertexes, therefore ,it can be broken down into the product of two diagrams similar to those in Fig. 6.2, with the lines following the original directions. This product gives four terms representing the possible combinations of which quark is interacting and which is not. The ones where only one of the two quarks is interacting are equals, but with opposite sign and therefore cancel out. Thus, the remaining terms are the one where both quarks interact, that is the first diagram on the right hand side of the above figure, and the one where both are not interacting, that is the term multiplied by $\frac{1}{N}$. Algebraically, this corresponds exactly to the following Fierz identity:

$$T^a_{ij} T^a_{lk} = \frac{1}{2}(\delta_{ik}\delta_{lj} - \frac{1}{N}\delta_{ij}\delta_{lk}), \tag{6.13}$$

since two fermion-gluon vertexes correspond to the product of two generators.

Then, going back to Fig. 6.12, the product of the two diagrams visually becomes:

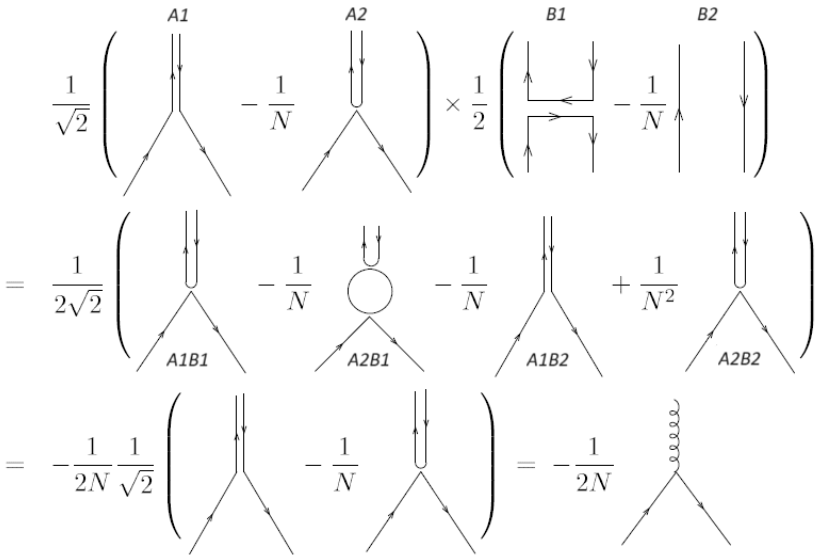

Figure 6.14: Visual calculation of the color factor of the Fig. 6.12 Feynman diagram. Original image without labels taken from [11].

The diagram containing *A1* and *A2* in the first line is a simple fermion-gluon vertex, where the lines follow the orientation of the initial diagram. The term containing *B1* and *B2*, instead, comes from Fig. 6.13, as claimed before. The product of *A2* with *B1* produces the *A2B1* term in the second line. Connecting the lines results in a loop, thus giving a factor N that cancels out the $\frac{1}{N}$, leaving basically a diagram equivalent to *A1B1*. Therefore, these two cancel out, leaving only the third and fourth term in the second line. It can be factored out a $-\frac{1}{2N}$, leading to the fourth line, where it can be noticed that the two diagrams left are the interacting term and non-interacting term for a simple fermion-gluon vertex, hence the result.

This can also be derived using the $SU(3)$ color algebra. Keeping in mind that the original diagram can be broken down into the product of two diagrams, as shown in Fig. 6.12, it is necessary to multiply their corresponding algebraic factors. This is a Fierz identity for the $q\bar{q}$ scattering, as there are two fermion-gluon vertexes, and a single generator for the other fermion-gluon vertex. Thus:

$$T^a_{jl} \cdot \frac{1}{2}(\delta_{ik}\delta_{lj} - \frac{1}{N}\delta_{ij}\delta_{lk}) = \frac{1}{2}(T^a_{jj}\delta_{ik} - \frac{1}{N}T^a_{ik}) = -\frac{1}{2N}T^a_{ik}, \tag{6.14}$$

where it was used $T^a_{jj} = 0$. Since a generator corresponds to a fermion-gluon vertex by Fig. 6.2, the result is obtained.

It is worth noticing that the color factor here has a negative sign. Indeed, in the one loop interaction vertex with a photon, the gluon exchange between quarks resulted in a positive factor, C_F to be precise. This reflects an attraction between the quarks, that initially were in a color singlet state, since they were created by a photon, that is a color singlet. Meanwhile, in the one loop interaction vertex with a gluon, the gluon exchange between quarks led to a negative factor, $-\frac{1}{2N}$. This represents a repulsion between the $q\bar{q}$ pair, that initially were in a color-octet state, since they were created by the gluon, which is not a color singlet. Because of this, it can be concluded that the force between a $q\bar{q}$ pair is attractive if the pair is a color singlet, and repulsive if it is in a color-octet state. This is qualitative evidence as for the reason why no bound $q\bar{q}$ pair observed is in a color-octet state, they simply repel [11].

To summarize, the calculation of color factors for QCD processes has been demonstrated, leading to the derivation of the color Feynman rules for QCD. These rules were then applied to more interesting diagrams, providing a qualitative argument in support of an important physical phenomenon, which is the color-singlet nature of quarks bound states, else known as confinement.

Chapter 7

Conclusion

In conclusion, this thesis set out to calculate important color factors of QCD processes visually represented by Feynman diagrams. These color factors appear in transition amplitudes and, consequently, are essential for calculating probabilities in quantum processes. Achieving this, required a deep understanding of the strong interaction and its main properties that spurred the formulation of the adequate tools required to describe the pertinent physical phenomena. Namely, the mathematical framework of QFT.

It was explained why these new theories were necessary, showing how they combine quantum mechanics with classical field theory. Furthermore, the key role of gauge transformations was highlighted, beginning from their relevance in the context of standard electromagnetism and extending into the quantum mechanical world. The combination of such transformations with local $U(1)$ transformations led to abelian gauge theories, with scalar QED provided as an example. Instead, when local $SU(N)$ transformations were considered, this combination led to the development of non-abelian gauge theories or Yang-Mills theories, which introduced the concept of direct self interactions among gauge bosons as a result of the non-abelian nature of the group.

Lastly, using the results of $SU(3)$ color algebra, I was able to calculate color factors of some basic QCD processes and derive the color Feynman rules for QCD, that were later applied to more complex diagrams. This provided qualitative arguments supporting the color-singlet nature of quarks bound states, showing a significant physical application of these calculations.

This thesis serves as an introduction to these theories and the mathematical framework used to describe fundamental interactions. Much more could be said, such as the relevance of color factors in processes like gluon scattering, but doing so would require a deeper mathematical understanding of QFTs that I have yet to acquire.

In summary, this thesis provided an overview of the methods used to describe

fundamental interactions, specifically QFT with a focus on QCD and the calculation of color factors for processes represented by Feynman diagrams.

Appendix A

Fierz identity for $SU(N)$

Here, following the work in [12], a derivation of the Fierz identity in the general $SU(N)$ case is provided. That is:

$$T^a_{ij}T^a_{kl} = \frac{1}{2}(\delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl}), \tag{A.1}$$

where the indices i, j, k and l take values from 1,2...N.

Proof. Any $N \times N$ complex matrix M can be expanded in a basis of $SU(N)$ elements consisting of the unit matrix $\mathbf{I}$ and the remaining $N^2 - 1$ generators T^a, where $a = 1, 2, ..., N^2 - 1$. Hence:

$$M = \alpha_0\mathbf{I} + \alpha_a T^a, \tag{A.2}$$

where α_0 and α_a are coefficients that can be determined as follows. By tracing the above equation, we have:

$$tr(M) = \alpha_0 tr(\mathbf{I}) + \alpha_a tr(T^a).$$

Using the fact that $tr(\mathbf{I}) = N$ and that $SU(N)$ generators are traceless, $tr(T^a) = 0$, because of the special condition, it follows that:

$$\alpha_0 = \frac{1}{N}tr(M). \tag{A.3}$$

To determine α_a instead, we can multiply eq. (A.2) by a generator T^b. Tracing the obtained equation then one finds:

$$tr(MT^b) = \alpha_0 tr(T^b) + \alpha_a tr(T^aT^b). \tag{A.4}$$

Using the normalization condition of the generators, eq. (4.8), it follows that:

$$\alpha_a = 2tr(MT^a). \tag{A.5}$$

Therefore, rewriting eq. (A.2) with the new expressions for the coefficients it becomes:

$$M = \frac{1}{N} tr(M)\mathbf{I} + 2tr(MT^a)T^a, \tag{A.6}$$

which can be written in terms of the matrix elements:

$$M_{ij} = \frac{1}{N} M_{kk}\delta_{ij} + 2M_{lk}T^a_{kl}T^a_{ij}. \tag{A.7}$$

This can be written as:

$$\delta_{il}\delta_{jk}M_{lk} = \frac{1}{N}\delta_{ij}\delta_{kl}M_{lk} + 2M_{lk}T^a_{kl}T^a_{ij} = (\frac{1}{N}\delta_{ij}\delta_{kl} + 2T^a_{kl}T^a_{ij})M_{lk}, \tag{A.8}$$

and it follows that:

$$[T^a_{kl}T^a_{ij} - \frac{1}{2}(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl})]M_{lk} = 0. \tag{A.9}$$

Since M_{lk} are the elements of an arbitrary $N \times N$ complex matrix, this equation must hold for every M_{lk}. Therefore, the expression between square brackets must vanish, thus leading to the Fierz identity as requested. □

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