Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE
Corso di Laurea in Matematica

AN INTRODUCTION TO THE POLYTOPE ALGEBRA

With a view toward the *g*-Theorem

Tesi di Laurea in Geometria Discreta

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Anno Accademico 2023-2024

Introduction

Polytopes are the higher dimensional generalization of polygons in the plane and polyhedra in three-dimensional space. Their study is rooted in the classical work of Euclid and since then it has grown into an active area of research of modern mathematics.

Nowadays the theory of polytopes can be considered part of combinatorics and discrete geometry, in fact the main questions concern either the combinatorial properties of a polytope or its metric and geometric features. Far from being a stand alone branch of mathematics, a lot of recent progress in the theory stemmed from the developing relationship between commutative algebra (and algebraic geometry) and combinatorics. Pioneering the dialogue between the two distinct areas is the proof of Richard Stanley of the Upper Bound Theorem for f-vectors of simplicial spheres [Sta75]. He showed how the combinatorial properties of a triangulation Δ of the sphere can be related to algebraic properties of an appropriate ring $K[\Delta]$, he then used commutative algebra techniques to complete the proof of the theorem. Stanley's proof relies on a previous result of G. Reisner, stating that under some topological conditions on Δ (satisfied when the simplicial complex is a triangulation of a sphere), $K[\Delta]$ is a Cohen-Macaulay ring.

Going back to polytopes, in 1971 Peter McMullen conjectured a complete characterization of the f-vectors of simplicial polytopes [McM71]. He proposed certain combinatorial conditions on a sequence of integers $g = (g_{-1}, g_0, g_1, ..., g_d)$, then he conjectured that they were necessary and sufficient for the existence of a simplicial polytope P with g as its g-vector. A decade later, the proof of the conjecture was established, marking what is arguably the single most important result in the modern theory of polytopes.

Billera and Lee proved the sufficiency of the conditions via an ingenious construction [BL81]: from the combinatorial properties of the sequence g, they used a particular order on monomials to choose a collection of facets from the cyclic polytope such that their boundary had the desired combinatorics. From there they constructed a simplicial polytope having the same face numbers. (See [Bil14] for a simple presentation).

In the same year, Stanley proved the necessity of the conditions on the g-vector by using techniques from algebraic geometry [Sta80]. In particular, he applied the Hard Lefschetz Theorem to the cohomology ring of the toric variety associated to a simplicial polytope with rational coordinates.

The cohomology of such a toric variety forms a graded algebra that is zero in odd degrees, is generated in degree 2 and has as even Betti numbers the h-numbers of P. The Hard Lefschetz Theorem gives a class ω its cohomology ring in degree 2 such that multiplication by ω induces an isomorphism between the component in degree k with the component in degree $2d-k$, for all k up to half the dimension. Therefore, the quotient of the cohomology ring by the ideal $\langle\omega\rangle$ is a graded algebra with Hilbert series (in even degrees) the g-vector of P; due to a result of Macaulay, this implies the numerical conditions McMullen conjectured.

McMullen, perhaps dissatisfied with Stanley's "not that polytopal" proof, demonstrated the necessity of the conditions on the g-vector with more elementary methods [McM89; McM93]. In a way, his approach mirrors that of Stanley: he translated and proved established results on the cohomology of toric varieties within a more combinatorial framework he developed: the polytope algebra.

In this thesis, we focus on the work of McMullen on the polytope algebra, with particular emphasis on the key results he used in the combinatorial proof of the g-Theorem. In the first chapter, we give a concise introduction to the study of convex polytopes: after the first definitions and results, we briefly talk about the face lattice and polarity, and then we prove the equations of Dehn and Sommerville for the h-vector of a simple polytope. We introduce the cyclic polytope, and in the end we give the precise statement for both the Upper Bound Theorem for convex polytopes and the g-Theorem.

The second chapter is dedicated to the study of the polytope algebra Π. We show that, in all but a single trivial aspect, it is a graded commutative algebra over $\mathbb{Q}$. The key element in the proof is the ingenious definition of the *logarithm* of a polytope, which encodes the polytope in a particularly well-behaved way from an algebraic point of view. Then we find a family of group homomorphisms that separates Π, these correspond to taking volumes of lower dimensional faces in different directions. We find some *syzygies* between them, that will later used in the definition of weights on a polytope.

Lastly, in Chapter 3, we study the subalgebra $\Pi(P)$ associated to a simple polytope P: we show that this algebra is generated by polytopes strongly isomorphic to P, in particular by the open cone of all their logarithms. Then we introduce the notion of a weight on a polytope, and use it to compute the Hilbert series of $\Pi(P)$, showing that this equals the h-polynomial of P. We end the thesis by showing how the Upper Bound Theorem for polytopes can be deduced as a corollary of our results on $\Pi(P)$, echoing Stanley's approach for the more general statement about triangulated spheres.

Contents

Chapter 1

Polytopes

In this first chapter we recall the main definitions and results about convex polytopes. The goal is not to give a complete overview of the theory, but rather to introduce the vocabulary we need for the later chapters and to explore some elementary examples.
In this first chapter of the exposition, our main reference is [Zie94], the reader is referred there to see proofs and many details we omitted.

Remark. We shall generally consider V to be a real finite-dimensional vector space of dimension d. Although most of what we say is applicable to vector spaces over any ordered field $\mathbb{F}$, this chapter will not explore these generalizations. In the following chapters, we will make only occasionally reference to polytopes in a vector space not defined over the reals, mostly dealing with some differences when restricting to a d-vector space over the rationals.

1.1 Preliminaries

Let V denote a real vector space of dimension d, equipped with the euclidean topology.

Definition 1.1. Let S be a subset of V, we respectively say that S is:

- *convex* if for every pair $x, y \in S$ and $\lambda \in [0, 1]$ we have $\lambda x + (1 - \lambda)y \in S$;
- a *cone* if it is convex and if for $x \in S$ and $t \geq 0$ also tx is in S;
- a *hyperplane* if there exists $\psi \in V^*$ non zero and $a \in \mathbb{R}$ such that $S = \{x \in V \mid \psi(x) = a\}$;
- a *halfspace* if there exists ψ and a as above such that $S = \{x \in V \mid \psi(x) \leq a\}$;
- an *affine subspace* if it is an intersection of hyperplanes;
- a *polyhedron* if it is a finite intersection of halfspaces.

Definition 1.2. Let S be a subset of V, we call *convex hull* of S, and write $\mathrm{conv}(S)$, the intersection of all convex sets containing S.
Similarly the *affine hull* of S $\mathrm{aff}(S)$ is the intersection of all affine subspaces containing S.
With simple computations one can prove that

$$\mathrm{conv}(S) = \Big\{\sum_{i=1}^{n} \lambda_i s_i \mid \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \geq 0,\ s_i \in S\Big\},$$
$$\mathrm{aff}(S) = \Big\{\sum_{i=1}^{n} \lambda_i s_i \mid \sum_{i=1}^{n} \lambda_i = 1,\ s_i \in S\Big\}.$$

Definition 1.3. We say that two subsets $S, R \subset V$ are *parallel* to each other if the affine hull of either of one can be translated into the affine hull of the other.

Definition 1.4. The *dimension* of an affine subspace is the dimension of its parallel vector subspace (passing through the origin).

Definition 1.5. A *polytope* in V is the convex hull of a finite set S. The dimension of a polytope P is the dimension of its affine hull, if that dimension is k we say that P is a k-polytope (we set $\dim(\emptyset) := -1$ by definition).

Example 1.6. Here are some elementary examples of polytopes we are going to encouter in the next sections.

- Let $V = \mathbb{R}^3$, denoting by $e_i = (\delta_{1i}, \delta_{2i}, \delta_{3i})$ the standard basis vectors, consider $S = \{e_1, e_2, e_3\}$. The convex hull of S is triangle, the vectors in S form the set of its vertices.

- Generalizing the previous example, the *standard simplex* of dimension $d-1$ is the polytope $\Delta_{d-1} = \mathrm{conv}(\{e_1, ..., e_d\})$ lying in $\mathbb{R}^d$. Again, observe that the standard simplex is not full dimensional, in fact it lies in the affine hyperplane where the sum of the coordinates equals 1.

- The *cube* C_d of dimension d is the convex hull of the 2^d vectors in $\mathbb{R}^d$ with all coordinates $+1$ or -1. Note that we can obtain the cube also as a bounded polyhedron by intersecting the halfspaces $\{v \in \mathbb{R}^d \mid v_i \leq 1\}$ and $\{v \in \mathbb{R}^d \mid v_i \geq -1\}$, for $i = 1, ..., d$.

- The *crosspolytope* of dimension d is the convex hull of the vectors $+e_1, -e_1, +e_2, -e_2, ..., -e_d$.
  In 3-dimensional euclidean space this is the well-known regular *octahedron*.
  Again, observe that the crosspolytope coincides with the set $\{v \in \mathbb{R}^d \mid \sum |v_i| \leq 1\}$ that can easily be obtained as an intersection of halfspaces.

Theorem 1. *A subset P of V is a polytope if and only if it is a bounded polyhedron.*

The proof is definitely longer than one may expect, therefore we refer to [Zie94] for it.

Corollary 1.7. *The image under an affine map of a polytope is a polytope.*
The intersection of a polytope with an affine subspace is a polytope.

Proof. A hyperplane is the intersection of two halfspaces, so the intersection of a polytope P with an affine subspace is the intersection between a bounded polyhedron with finitely many halfspaces.
Since it holds for linear maps and for translations, for an affine map we have $\Phi(tx + (1-t)y) = t\Phi(x) + (1-t)\Phi(y)$, thus if $P = \text{conv}(S)$, $\Phi(P) = \text{conv}(\Phi(S))$. □

Remark. In the process of defining a polytope P as the convex hull of a finite set, we may have some redundant data. A set S satisfying $P = \text{conv}(S)$ minimal with respect to inclusion is said a set of *vertices* of P.
Similarly, in describing P as a bounded polyhedron we may have more inequalities than needed.

Definition 1.8. Let ψ be an element of V^* and b be a real number, we say that the inequality $\psi \leq b$ is *valid* for a polytope P if P is contained in its set of solutions.

Definition 1.9 (Face)**.** Let P be a polytope in V, a *face* F of P is a set of the form $P \cap \{\psi = b\}$, where $\psi \leq b$ is a valid inequality for P. If b is minimal such that $\psi \leq b$ is valid for P, then the face $P \cap \{\psi = b\}$ is precisely P_ψ (see in the next page).

By considering the inequalities $\{0 \leq 1\}$ and $\{0 \leq 0\}$ we always have both the empty face $\emptyset$ and the whole polytope P as faces of P. Furthermore, the faces of P are obtained by intersecting it with hyperplanes, so from Theorem 1 each face of a polytope is still a polytope.

Proposition 1.10. *The vertices of a polytope are its zero dimensional faces.*

Proposition 1.11. *If F and G are faces of a polytope P, then $F \cap G$ is also a face of P. The faces of F are precisely those of P that lie in F.*

As a consequence we have that P has only finitely many faces of each dimension.

Example 1.12. The faces of Δ_d are straightforward to compute: for each non empty subset of the vertices $I \subseteq \{e_1, ..., e_{d+1}\}$ of size $k+1$ consider the functional $\psi_I(x) = \sum_{i \in I} x_i$; the face identified by the valid inequality $\{\psi_I \leq 1\}$ is a k-simplex and I is its set of vertices.
The number of faces of dimension k of a simplex is $\binom{d+1}{k+1}$, as any choice of $k+1$ vertices defines a unique k-face.

Example 1.13. The cube C_d has $2d$ faces of dimension d, obtained by the inequalities $\{\pm v_i \leq 1\}$, and of course it has 2^d vertices. A face of dimension $d-k$ can be obtained as follows: for any choice of a set of indices $I = \{i_1, ..., i_k\} \subseteq \{1, ..., d\}$ and vector $\delta \in \{1, -1\}^k$, consider the functional $\psi(v) = \delta_1 v_{i_1} + ... + \delta_k v_{i_k}$. This is maximized by those vectors of C_d where the coordinates indexed by elements in I satisfy $v_{i_j} = \delta_j$, and the other $d-k$ coordinates are free to vary in $[-1, 1]$. We deduce that C_d has $2^k\binom{d}{k}$ faces of dimension $d-k$.

Definition 1.14. If P is a non empty polytope, its *supporting function* $h(P, \cdot)\colon V^* \longrightarrow \mathbb{R}$ is defined by:

$$h(P, \psi) := \max_{x \in P} \psi(x).$$

Since polytopes are compact subsets of V the definition is well posed. We write

$$P_\psi := \{x \in P \mid \psi(x) = h(P, \psi)\}$$

to indicate the face of P in direction ψ. We call the map $P \mapsto P_\psi$ the *face map.*

Remark. If V is equipped with a positive definite symmetric bilinear form, we can define $h(P, u) := \max\langle x, u\rangle$, where $u \in V$ and x ranges over P.
In that case we call the hyperplane $\{x \in V \mid \langle x, u\rangle = h(P, u)\}$ the *supporting hyperplane* of P with outer normal vector u, the intersection between this hyperplane and P is P_u.
A *k-frame* is and ordered k-tuple of orthonormal vectors $U = (u_1, ..., u_k)$. If U is a k-frame, we define recursively

$$P_U := (P_{u_1})_{(u_2,...,u_k)}.$$

Definition 1.15. If P and Q are polytopes, we can use them to build new examples of polytopes in a controlled way. In particular we can construct:

- the *pyramid* over P: first embed V in $V \oplus \mathbb{R}$ and then we define $\text{pyr}(P) := \text{conv}(P \cup (0, 1))$;
- the *Minkowski sum* $P + Q$ as $\{x + y \in V \mid x \in P, y \in Q\}$;
- the *product* $P \times Q$ as $\{(x, y) \in V \oplus W \mid x \in P, y \in Q\}$; this time P and Q are not required to lie in the same vector space.

In the next sections we especially make use of the Minkowski sum, therefore, we state some facts about it that we may need to use. We redirect to [Grü03, §15] for a complete overview.

Proposition 1.16. *If Q and R are polytopes, respectively with vertices $q_1, ..., q_n$ and $r_1, ..., r_m$, then:*

$$Q + R = \text{conv}(\{q_i + r_j \mid i = 1, ..., n, j = 1, ..., m\});$$
$$h(Q + \lambda R, \cdot) = h(Q, \cdot) + \lambda h(R, \cdot) \ \textit{for}\ \lambda \geq 0;$$
$$(Q + R)_\psi = Q_\psi + R_\psi \ \ \textit{for each non zero vector}\ \psi.$$

1.2 The face lattice

The faces of a polytope, naturally ordered by inclusion, form a bounded poset (the maximal and minimal element are respectively P itself and the empty face). Before stating the precise result, we need to introduce some necessary terminology for partially ordered sets.

Definition 1.17. Let $(X, \leq)$ be a partially ordered set, let $x \leq y$ be two elements of X, then the *interval* between x and y is defined as

$$[x, y] := \{z \in X \mid x \leq z \leq y\}.$$

If $(X, \leq)$ and $(Y, \preceq)$ are two partially ordered sets, an isomorphism between them is a bijection $\Phi\colon X \to Y$ such that $a \leq b$ if and only if $\Phi(a) \preceq \Phi(b)$.
An interval in $(X, \leq)$ is said to be *boolean* if it is isomorphic to a poset of the form $(\mathcal{P}(I), \subseteq)$ for some finite set I.
A *chain* in $(X, \leq)$ is a subset of X totally ordered with the induced order relation, the *length* of a chain is the number of its elements minus 1.
We say that $(X, \leq)$ is bounded if it has a unique maximal element $\hat{1}$ and a unique minimal element $\hat{0}$. A bounded poset is *graded* if each maximal chain has the same length. On a graded poset we define a *rank function* $\rho\colon X \to \mathbb{Z}$ that associates to an element $x \in X$ the length of any maximal chain in $[0, x]$.
A *lattice* is a bounded poset such that for each pair of elements $x, y \in X$ there is a unique least upper bound $x \vee y$ called the *join* of x and y, and a unique greatest lower bound $x \wedge y$ called the *meet* of x and y.

Theorem 2. *Let P be a convex polytope, then:*

- *the faces of P form a graded lattice $\mathcal{L}(P)$, called its* face lattice*;*
- *each interval $[F, G]$ is the face lattice of a polytope of dimension $\rho(G) - \rho(F) - 1$;*
- *the face lattice $\mathcal{L}(P)$ is atomic, i.e. each element is the join of elements of rank 0;*
- *the opposite poset $\mathcal{L}(P)^{op}$ is also the face lattice of a polytope;*
- *for each k-face F and $k+2$-face G that contains F, there are precisely two $k+1$-faces H, H' containing F and contained in G.*

Definition 1.18. We say that two polytopes P and Q are *combinatorially isomorphic* if their face lattices are isomorphic.
A *combinatorial dual* of P is any polytope with face lattice isomorphic to $\mathcal{L}(P)^{op}$.

From the combinatorial point of view, the face lattice of a polytope encodes all the interesting information, for example: the incidence relations of the faces, the "shape" of each face, the number of faces of each dimension, and others. We could restrict ourselves to the study of such posets, but considering polytopes as rigid metric objects in a real vector space V gives us more tools to study their combinatorics, and therefore this is the approach we will adopt.
For example, throughout the thesis we will often distinguish between a square, a rectangle or a trapezoid; we will generally consider linear and piecewise linear maps and sometimes orthogonal projections and reflections (in the second chapter a scalar product on V is introduced).

The face lattice $\mathcal{L}(P)$ of a general polytope can be quite intricate and complex. However, there are classes of polytopes which face lattices are well-behaved, in the sense that little data is needed to specify the whole poset.

Definition 1.19. A polytope P is said *simplicial* if all its facets are simplices.
A polytope of dimension d is said *simple* if all its vertices lie in precisely d facets.

Remark. When defining a polytope as the convex hull of a finite set S, if the points in S are chosen with sufficient generality no $d+1$ of them are contained in a single hyperplane. Consequently, the polytope $P = \text{conv}(S)$ is simplicial.

Similarly, if we construct a d-polytope P as a bounded polyhedron, and the inequalities are chosen generally enough, the hyperplanes affinely spanned by the facets are in general position. Thus, k of them intersect in a linear subspace of dimension $d-k$, a vertex lies in d facets, making P a simple polytope.

A rephrasing we will often use of these observations, is that the only combinatorial type that is stable under small perturbations of the vertices is that of a simplicial polytope, and the only one stable under perturbations of the defining inequalities is that of a simple polytope.

Remark. Since the faces of a simplex are also simplices, the face lattice of a simplicial polytope is characterized by the property that the interval $[\emptyset, F]$ is a boolean lattice for each proper face F.

Proposition 1.20. *Let P be a polytope and v a vertex of P identified by the valid inequality $\{\psi \leq b\}$, let $c < b$ be such that $c > \psi(v')$ for each other vertex v' of P. Then*

$$\mathcal{L}(P \cap \{\psi = c\}) \cong [v, P].$$

With a slight abuse of notation, the polytope $P \cap \{\psi = c\}$ is called the *vertex figure* of P at v, even though only its combinatorial type is uniquely determined by P and v.

Remark. The vertex figures of a simple polytope P are $d-1$-simplices, since they are $d-1$-polytopes with only d facets (these correspond to the facets of P containing v). We deduce that a polytope is simple if and only if for each proper face F the interval $[F, P]$ is boolean.

The notions of simple and simplicial polytope are dual to each other in a precise way: a combinatorial dual of a simple polytope is simplicial and vice versa.
Theorem 2 asserts the existence of combinatorial duals, if P is a full dimensional polytope and has the origin of V in its interior, there is a standard way of constructing a dual.

Definition 1.21. If S is a subset of V, its *polar subset* is defined as:

$$S^\dagger := \{\varphi \in V^* \mid \sup_{x \in S} \varphi(x) \leq 1\}.$$

Theorem 3. *[Zie94, §2.3] Let P be a full dimensional polytope in V with the origin in its interior. Then $P^\dagger$ is also a convex polytope of dimension d. Further more*

$$\mathcal{L}(P^\dagger) \cong \mathcal{L}(P)^{op}.$$

Example 1.22. Consider the standard cube $C_d = [-1, 1]^d$ inside $\mathbb{R}^d$, its polar polytope is

$$C_d^\dagger = \{\varphi \in (\mathbb{R}^d)^* \mid \max_{x \in C_d} \varphi(x) \leq 1\} = \{\sum_i^d a_i e_i^* \in (\mathbb{R}^d)^* \mid \sum_i |a_i| \leq 1\},$$

which is the regular crosspolytope in the dual vector space. The cube is simple and the crosspolytope is simplicial, the vertex figures of one correspond to the facets of the latter.

Classifying polytopes up to combinatorial equivalence is a daring project. On that direction one may look for a weaker invariant: given a polytope P a very naive and intuitive invariant is the number of k-faces $f_k(P)$.
We can collect all those integers in the *f-polynomial* $f(P, t) := \sum f_k(P)t^k$, its coefficients form the *f-vector* $f(P) = (f_0(P), f_1(P), ..., f_{d-1}(P), f_d(P))$. In spite of their simple definition, not much can be said about f-vectors of arbitrary polytopes: already the problem of characterizing f-vectors of 4-polytopes is open [Zie94, Ex. 8.29].

Exercise. Characterizing f-vectors of 2-polytopes is straightforward. Assuming the Euler formula $v - e + f = 2$, prove that in dimension 3 a triple of positive integers (v, e, f) corresponds to the number of vertices, edges and faces of a 3-polytope if and only if

$$\begin{cases} v - e + f = 2 \\ 2v \leq f + 4 \\ 2f \leq v + 4. \end{cases}$$

Definition 1.23. Let P be a simple polytope, its *h-vector* is defined by the identity

$$h(P,t) = f(P,t-1)$$
$$\sum_{k=0}^{d} h_k(P)t^k = \sum_{k=0}^{d} f_k(P)(t-1)^k.$$

Notice that the two vectors carry the same amount of information since we can recover the f-vector by $f(P,x) = h(P,x+1)$.

Remark. The h-vector is typically defined for simplicial polytopes. We have chosen to define it for simple polytopes instead since as Chapter 3 primarily focuses on them. It is important to note that due to polarity, the study of the combinatorics of simple polytopes is equivalent to that of simplicial polytopes, therefore every statement about the first can be appropriately translated into a statement about the latter.

For non simple polytopes Definition 1.23 makes perfectly sense, but while the f-vector always has an intuitive meaning, that of the h-vector is not very clear in general. When P is simple, on the other hand, its h-vector does have a nice combinatorial interpretation, which makes very transparent relations between face numbers otherwise much harder to spot in the f-vector.

Theorem 4 (Dehn-Sommerville)**.** *Let P be a simple d-polytope, then $h_k = h_{d-k}$.*

Proof. The idea is to consider the halfspace $H_t = \{x \in V \mid \varphi(x) \leq t\}$ for some general enough functional $\varphi \in V^*$, and count the faces of P that are fully contained in it as t goes to infinity.

Since P is simple, through each vertex v pass only d edges, the subsets of these edges are in bijection with the faces of P containing v. When the halfspace H_t passes through v we count only k of these edges (just those "pointing into H_t"). In doing so we add a single k-face together with all its faces: in total we count $\binom{k}{r}$ new r-faces for each $r \leq k$. In this case we say that v is a k-vertex with respect to φ, or, interchangeably, a vertex of type k.

The total increment on the f-polynomial is $\sum_{r=0}^{k} \binom{k}{r} t^r$, the increment of the h-polynomial is just x^k: h_k counts the number of k-vertices. We deduce that the number of k-vertices is independent on φ.
By repeating the same procedure with $-\varphi$ each k-vertex turns into a $d-k$-vertex and vice versa, proving the thesis.

The generality conditions on φ are easily satisfied, in fact if $p_1, ..., p_n$ are the vertices of P, those amount to asking that all the finitely many vectors of the form $p_i - p_j$ do not lie in the hyperplane corresponding to the kernel of φ. □

Example 1.24. We can use this new interpretation of the h-vector to compute it for some polytopes without having to evaluate and simplify long sums.

If P is a simple polytope, also the prism $P \times [0,1]$ is simple; to each vertex v of type k of P we can find an appropriate functional on $V \oplus \mathbb{R}$ so that the vertices $(v,0)$ and $(v,1)$ are respectively of type k and $k+1$. We deduce that:

$$h(P \times [0,1], t) = (1+t)h(P,t).$$

Generalizing the argument, if v is a k-vertex of $P \subseteq V$ and w is an r-vertex of $Q \subseteq W$, then one can find an appropriate functional[1] on $V \oplus W$ such that (v,w) is a $k+r$-vertex of $P \times Q$, yielding the formula:

$$h(P \times Q, t) = h(P,t)h(Q,t).$$

If we now decided to "slice off" a vertex from a simple polytope P (a process similar to that of Proposition 1.20) obtaining a polytope P', we can chose φ so that the new d vertices $v_1, ..., v_d$ are the first to be counted. If v_1 is the 0-vertex of P' (v was the 0 vertex of P) the other $d-1$ form the vertices of a $d-2$ simplex. Since they are all connected to v_1, the new h-vector is given by:

$$h(P',t) = h(P,t) + t \cdot h(\Delta_{d-2}, t) = h(P,t) + h(\Delta_{d-1}, t) - 1.$$

Going up one dimension, we can cut a whole edge e connecting two vertices v and v' of P. This is achieved by finding $\psi \in V^*$ such that $\psi(v) > 0$, $\psi(v') > 0$ and $\psi(w) < 0$ for each other vertex of P, such a ψ exists since the edge connecting v and v' does not lie in the convex hull of all the other vertices (for more details look at the Farkas Lemmas [Zie94, §1.4]). Being P simple, the "edge figure" at e, that is the intersection between P and the hyperplane $\{\psi = 0\}$, is combinatorially isomorphic to the product $\Delta_{d-2} \times [0,1]$. Via an argument similar to the one above we see that first we removed a 0-vertex and a 1-vertex, and then added the vertices of $\Delta_{d-2} \times [0,1]$, obtaining:

$$h(P',t) = h(P,t) + (1+t)h(\Delta_{d-2}, t) - (1+t)$$

Corollary 1.25. *The h-polynomials of the simplex and of the cube are:*

$$h(\Delta_d, t) = \sum_{k=0}^{d} t^k;$$
$$h(C_d, t) = \sum_{k=0}^{d} \binom{d}{k} t^k.$$

[1] If $\varphi \in V^*$ and $\psi \in W^*$ are the functionals that make v of type k and w of type r, it suffices to consider $\varphi + N\psi \in (V \oplus W)^*$ for a constant $N \gg 1$.

1.3 The cyclic polytope

Now we follow an important and interesting construction that can be found in [Zie94]: we define the *cyclic polytope* and study some aspects of its combinatorics.
Consider the moment curve

$$\begin{aligned} \nu\colon \mathbb{R} &\longrightarrow \mathbb{R}^d \\ t &\longmapsto (t, t^2, ..., t^d). \end{aligned}$$

If $t_1, ..., t_n$ are distinct real numbers, $n > d$, we denote by $C_d(t_1, ..., t_n)$ the polytope obtained as the convex hull of the points $\nu(t_1), ..., \nu(t_n)$; the polytopes we obtain in this way are called *cyclic polytopes*.

Theorem 5 ([Zie94]). *Let $n > d$ be an integer and $t_1 < t_2 < ... < t_n$ real numbers. The cyclic polytope $C_d(t_1, ..., t_n)$ is a simplicial d-polytope and its combinatorial type only depends on d and n. More precisely a set of indices $I \subseteq \{1, ..., n\}$ of size d determines the vertices of a facet if and only if for each pair i, j not in I*

$$|\{k \in I \mid i < k < j\}| \equiv 0 \pmod 2.$$

Proof. First we observe that $C_d(t_1, ..., t_n)$ is simplicial: if $s_1, ..., s_{d+1} \in \{t_1, ..., t_n\}$, the vectors $\nu(s_1), ..., \nu(s_{d+1})$ cannot lie in any common hyperplane as they are affinely independent, in fact we have[2]:

$$\det\begin{pmatrix} 1 & 1 & \dots & 1 \\ \nu(s_1) & \nu(s_2) & \dots & \nu(s_{d+1}) \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & \dots & 1 \\ s_1 & s_2 & \dots & s_{d+1} \\ \vdots & & \ddots & \\ s_1^d & s_2^d & \dots & s_{d+1}^d \end{pmatrix} = \prod_{i<j}(s_j - s_i) \neq 0.$$

Now let $S = \{s_1, ..., s_d\} \subseteq \{t_1, ..., t_n\}$, consider the linear functional of V^*

$$\psi_S(v) = \det\begin{pmatrix} 1 & 1 & \dots & 1 \\ v & \nu(s_1) & \dots & \nu(s_d) \end{pmatrix}.$$

This is a non zero functional since $\psi_S(\nu(t_i)) \neq 0$ for $t_i \notin S$; it vanishes on $\nu(s_1), ..., \nu(s_d)$ so its kernel is the hyperplane passing through those points. We deduce that $\nu(s_1), ..., \nu(s_d)$ are the vertices of a facet if and only if $C_d(t_1, ..., t_n)$ is contained in one of the two closed halfspaces bounded by $\{\psi_S = 0\}$, or equivalently if $\psi_S(\nu(t_i))$ has the same sign for all $t_i \notin S$. Observe that $\psi_S \circ \nu(t)$ is a polynomial in t of degree d which vanishes on $s_1, ..., s_d$, so it has no multiple roots: each time it vanishes it changes sign. We deduce that for i, j not in S we have $sgn\ \psi_S \circ \nu(t_i) = sgn\ \psi_S \circ \nu(t_j)$ if and only if there is an even number of elements in S between i and j. □

[2] Recall that $v_1, ..., v_k$ are affinely independent if and only if the columns of $\begin{pmatrix} 1 & \dots & 1 \\ v_1 & \dots & v_k \end{pmatrix}$ are linearly independent.

Remark. In the proof we have only described the "facet-vertex" incidences, these are sufficient to determine the face lattice since we can easily find the vertices of all faces by intersecting different facets. So the combinatorial type of $C_d(t_1, ..., t_n)$ does not depend on the particular choices of the t_i, we will denote its combinatorial type by $C_d(n)$ and call it *the* cyclic polytope of dimension d with n vertices.

Proposition 1.26. *Each subset I of $\{1, ..., n\}$ of size at most $\lfloor d/2 \rfloor$ identifies the vertices of a face of $C_d(n)$.*

Proof. Let $C_d(n) = C_d(t_1, ..., t_n)$, $S = \{s_1, ..., s_k\} \subseteq \{t_1, ..., t_n\}$ with $2k \le n$, let $\epsilon > 0$ be small enough so that for each $j = 1, ..., k$ there is no t_i contained in the interval $]s_j, s_j + \epsilon[$, and let $M_{2k+1}, ..., M_d \gg t_n$. Consider the functional

$$\psi(v) = \det \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ v & \nu(s_1) & \nu(s_1+\epsilon) & \dots & \nu(s_k+\epsilon) & \nu(M_{2k+1}) & \dots & \nu(M_d) \end{pmatrix}.$$

Again $\psi(\nu(t))$ is a polynomial of degree d with simple roots $s_1, s_1+\epsilon, ..., M_d$. By construction $\psi \circ \nu$ has the same sign on $\{t_1, ..., t_n\} \setminus S$ since each pair t_i, t_j not in S is separated by an even number of simple zeros. Without loss of generality we can assume the sign to be negative, so $C_d(t_1, ..., t_n) \subseteq \{\psi \le 0\}$ and the points $\nu(s_1), ..., \nu(s_k)$ are the vertices of the face corresponding to the hyperplane $\{\psi = 0\}$. □

While studying polytopes one may ask how to maximize the number $f_k(P)$. In other words: fixing the dimension d and the number of vertices n, what is the biggest possible value of $f_k(P)$ between all d-polytopes with n vertices? Can a single polytope simultaneously maximize $f_k(P)$ for all k?

Since each k-face has at least $k+1$ vertices, we have the obvious upper bound $f_k(P) \le \binom{n}{k+1}$, and in the previous proposition we showed $f_k(C_d(n)) = \binom{n}{k+1}$ for all $k = 0, ..., \lfloor d/2 \rfloor$[3]. When trying to answer the questions above, it is sufficient to restrict the attention to simplicial polytopes:

Lemma 1.27. *For each polytope Q with n vertices there exists a simplicial polytope P also with n vertices, such that $f_k(Q) \le f_k(P)$ for each k.*

The idea is that one can "wobble" slightly the vertices of Q and only increase the number of faces.
Since $C_d(n)$ is simplicial, a combinatorial dual $C_d(n)^\dagger$ will be a simple polytope, the values of $f_0(C_d(n)), ..., f_{\lfloor d/2 \rfloor}(C_d(n))$ form the last half of its f-vector (the first half for $C_d(n)$) and with the Dehn-Sommerville relations we can compute the entire f-vector.

[3] In dimension 2 and 3 this simply means that there are n vertices, however in dimension at least 4 one obtains some highly counter intuitive results by considering the cases $k > 0$.

Summarizing, we said that in order to maximize $f_k(P)$ we can restrict the attention to simplicial polytopes, then we showed in Proposition 1.26 that $C_d(n)$ maximizes f_k for all $k \leq \lfloor d/2 \rfloor$, and lastly we observed that these values completely determine the f-vector of $C_d(n)$.
Unfortunately it is false that the linear combinations of the numbers $f_0, ..., f_{\lfloor d/2 \rfloor}$ yielding the other half of the f-vector of a simplicial polytope have non-negative coefficients. Therefore maximizing those numbers does not obviously imply we maximize the whole f-vector.

In 1957 Motzkin made the following conjecture, that has been proved by McMullen in 1970 and since then has been known as the Upper Bound Theorem (for convex polytopes).

Theorem 6 (Upper Bound Theorem). *Let P be a d-polytope with n vertices. Then for each k, P has at most as many k-faces as the cyclic polytope $C_d(n)$:*

$$f_k(P) \leq f_k(C_d(n)).$$

We postpone the proof of the theorem since it comes as a corollary of our results in Chapter 3. We end the section by computing the h-vector of $C_d(n)^\dagger$, we follow [Zie94, Ex.

8.20]. The result of the computations will used in the proof of the Upper Bound Theorem for polytopes.

Lemma 1.28. *Let $n > d$ be an integer, then for each $k = 0, ..., \lfloor d/2 \rfloor$*

$$h_k(C_d(n)^\dagger) = \binom{n-d-1+k}{k}.$$

Proof. By polarity we have $f_k(C_d(n)^\dagger) = f_{d-k-1}(C_d(n))$: if $d-k-1 \leq \lfloor d/2 \rfloor$, meaning $k \geq \lceil d/2 \rceil - 1$, we have $f_k(C_d(n)^\dagger) = \binom{n}{d-k}$.
We know the second half of the f-vector of $C_d(n)^\dagger$ so we are able to directly compute the the second half of its h-vector, this is sufficient thanks to the Dehn-Sommerville equations. We will show that for $k \geq \lceil d/2 \rceil - 1$ we have $h_k(C_d(n)^\dagger) = \binom{n-k-1}{d-k}$, this can be checked to be equivalent to the thesis. For a general polytope P we have:

$$\begin{aligned} h(P,t) &= \sum_{i=0}^{d} f_i(P)(t-1)^i = \sum_{i=0}^{d} f_i(P) \sum_{k=0}^{i} t^k (-1)^{i-k} \binom{i}{k} \\ &= \sum_{k=0}^{d} t^k \cdot \sum_{i=k}^{d} (-1)^{i-k} \binom{i}{k} f_i(P); \end{aligned}$$

therefore if $P = C_d(n)^\dagger$ and $k \geq \lceil d/2 \rceil - 1$:

$$h_k(C) = \sum_{i=k}^{d} (-1)^{i-k} \binom{i}{k} \binom{n}{d-i}$$

$$= \sum_{j=0}^{d-k} (-1)^{d-j-k} \binom{d-j}{k} \binom{n}{j}.$$

We now prove that for all $k = 0, ..., d$, this last sum equals $\binom{n-k-1}{d-k}$.
If $k = d$ both terms trivially equal 1.
If $k = 0$ we proceed by induction on d:

$$\sum_{j=0}^{d} (-1)^{d-j} \binom{d-j}{0} \binom{n}{j} = \binom{n}{d} - \sum_{j=0}^{d-1} (-1)^{(d-1)-j} \binom{(d-1)-j}{0} \binom{n}{j}$$
$$= \binom{n}{d} - \binom{n-1}{d-1} = \binom{n-1}{d}.$$

Finally, if $0 < k < d$ we can trace back to a case in dimension $d-1$ and apply the inductive hypothesis:

$$\binom{n-k-1}{d-k} = \binom{n-k}{d-k} - \binom{n-k-1}{d-k-1} = \binom{n-(k-1)-1}{(d-1)-(k-1)} - \binom{n-k-1}{(d-1)-k}$$
$$= \sum_{j=0}^{d-k} (-1)^{d-j-k} \binom{(d-1)-j}{k-1} \binom{n}{j} + \sum_{j=0}^{d-k-1} (-1)^{d-j-k} \binom{(d-1)-j}{k} \binom{n}{j}$$
$$= \sum_{j=0}^{d-k-1} (-1)^{d-j-k} \binom{d-j}{k} \binom{n}{j} + \binom{n}{d-k}$$
$$= \sum_{j=0}^{d-k} (-1)^{d-j-k} \binom{d-j}{k} \binom{n}{j}.$$

Completing the computations. □

1.4 The g-Theorem

We intend to state the g-Theorem: a characterization of the f-vectors of simple polytopes. First we need to give some preliminary notions.

Lemma 1.29. *For each pair of integers $a, i > 0$ there exist unique integers $a_i > a_{i-1} > ... > a_k \geq k > 0$ such that*

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + ... + \binom{a_k}{k}.$$

Proof. Consider a reverse lexicographic order on the set of subsets of $\mathbb{N}$ of size i, that is $\{b_i, b_{i-1}..., b_1\} < \{c_i, c_{i-1}, ..., c_1\}$[4] if either $b_i < c_i$, or $b_i = c_i$ and $\{b_{i-1}, ..., b_1\} <$

[4] We are assuming that the elements of the sets are already listed in descending order $b_i > b_{i-1} > ...$

$\{c_{i-1}, ..., c_1\}$ in the reverse lexicographic order for sets of size $i-1$. For example the first sets of size 3 with this order are (omitting commas and parenthesis)

$$012 < 013 < 023 < 123 < 014 < 024 < ...$$

Now consider the set in position $a+1$ in this ordering, it will be of the form $\{a_i+1, ..., a_1+1\}$ for some integers $a_i, ..., a_1$; the subsets strictly smaller in the order can be partitioned into those with all elements smaller than $a_i + 1$, those containing $a_i + 1$ but with all other elements smaller that $a_{i-1} + 1$, and so on, therefore we have

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + ... + \binom{a_1}{1};$$

with the convention $\binom{k}{j} = 0$ if $k < j$.

The coefficients $a_i, ..., a_1$ are unique since any other set $\{a'_i + 1, ..., a'_1 + 1\}$ will come in a different position in the reverse lexicographic order, therefore corresponding to a different integer a'. □

We call the sum in the previous lemma the *i-canonical representation* of a. We define also the *i-partial power* of a as the sum

$$a^{\langle i \rangle} = \binom{a_i+1}{i+1} + \binom{a_{i-1}+1}{(i-1)+1} + ... + \binom{a_k+1}{k+1}.$$

Definition 1.30. A sequence of integers $(h_0, h_1, h_2, ...)$ is called an *M-sequence* if it exists a graded commutative algebra $A = \bigoplus_{i\geq 0} A_i$ over a field $\mathbb{F}$ generated in degree 1 such that $\dim_{\mathbb{F}} A_i = h_i$.

The following lemma is due to Macaulay, it gives concrete numerical properties characterizing M-sequences.

Lemma 1.31. *A sequence of non-negative integers $h = (h_0, h_1, h_2, ...)$ is an M-sequence if and only if $h_0 = 1$ and $h_{k+1} \leq h_k^{\langle k \rangle}$ for each $k \geq 1$.*[5]

See [BH98, Theorem 4.2.10] for a detailed proof and a complete account of the topic.

Recall that the h-vector of a simple polytope is always a palindromic sequence, by looking at some examples we computed in the previous sections, one can observe another kind of pattern: the h-vector seems to be a *unimodal* sequence.

[5] Originally, the definitions were reversed: M-sequences have been first defined as those sequences of integers satisfying the numerical conditions of the lemma, and later Macaulay showed the equivalence with the Hilbert series of graded algebras.

Definition 1.32. The *g-vector* $(g_0(P), ..., g_{d+1}(P))$ of a simple polytope P is defined by the polynomial relation

$$g(P, t) = (1 - t)h(P, t).$$

So $g_0 = h_0 = 1$, $g_{d+1} = -h_d = -1$ and $g_k = h_k - h_{k-1}$. We can recover the h-vector from the g-vector by observing that $h_k = \sum_{i \leq k} g_i$. The Dehn-Sommerville equations for the h-vector now take this the form:

$$g_k = -g_{d-k+1}.$$

Having established all the preliminaries, we can now state the g-Theorem.

Theorem 7. *(g-Theorem) For a sequence of integers $g = (g_0, g_1, ..., g_{d+1})$ there exists a simple polytope having g as its g-vector if and only if*

- $g_k = -g_{d-k+1}$ *for all k,*
- *the sequence $(g_0, g_1, ..., g_{\lfloor d/2 \rfloor})$ is an M-sequence.*

As mentioned in the introduction, half of the theorem was proven by Billera and Lee [BL81], that established the sufficiency of McMullen's conditions through a clever combinatorial-geometric construction. They presented a constructive way to obtain a simplicial polytope with any prescribed M-sequence as its g-vector. The second half was proved by Stanley [Sta80], by using the Hard Lefschetz Theorem to the cohomology of the projective toric variety X associated to a polytope equivalent to P with rational coordinates. He derived a graded algebra generated in degree 1, with the Hilbert function being the g-vector of P, thereby proving that the g-vector of a simplicial polytope is an M-sequence.

In the next chapters we follow the construction of McMullen of the graded algebra $\Pi(P)$ associated to a simple polytope P. This algebra will turn out to have the h-vector of P as its Hilbert function. In fact, this will be the content the very last theorem of this thesis. McMullen later proved an analogue of the Hard Lefschetz Theorem for the algebra $\Pi(P)$, providing a proof of the g-Theorem entirely withing the realm of convexity and polytope theory.

The proof of Stanley, relies on the the fact that the vertices of a simplicial polytope can be slightly perturbed to have rational coordinates, a condition that may seem artificial from perspective of a combinatorialist. This innocent detail prevents the proof from being generalized to non-simplicial polytopes, as in this case there is no toric variety in general to associate to the polytope (non-simplicial polytopes may not be realizable with rational coordinates [Zie94, §6.5]). Conversely, McMullen's proof does not depend on this fact. Therefore, even though his proof, as originally proposed, only applies to simple polytopes, it offers hope for finding more general results applicable to arbitrary polytopes.

Chapter 2

The Polytope Algebra

Recall that V is a real vector space of dimension d, we denote by $\mathcal{P} = \mathcal{P}(V)$ the set of all polytopes in V (not necessarily full dimensional). From now on we will suppose a symmetric positive definite bilinear form $\langle \cdot, \cdot \rangle$ is defined on V.
We are following the first parts of [McM89].

Definition 2.1. Let A be an abelian group, a function $\phi\colon \mathcal{P} \longrightarrow A$ is called a *valuation* if $\phi(P \cap Q) + \phi(P \cup Q) = \phi(P) + \phi(Q)$ whenever P and Q in $\mathcal{P}$ are such that also $P \cup Q$ is in $\mathcal{P}$, we also required that $\phi(\emptyset) = 0$.

On the one hand the definition is intuitive: valuations are functions on polytopes that behave additively if we break the polytope into pieces; on the other hand it is not very practical to work with. Instead of seeing valuations as functions between a set and an abelian group satisfying some relations, we prefer to see them as morphisms between appropriate abelian groups.

Definition 2.2. We denote by $\Pi = \Pi(V)$ the abelian group with generators the classes of polytopes $[P]$ for P varying in $\mathcal{P}$, and relations between them generated by the "translation invariance": for each t in V, $[P] = [P + t]$; and the "valuation property": whenever P and Q are such that $P \cup Q$ is in $\mathcal{P}$, $[P \cup Q] + [P \cap Q] = [P] + [Q]$; moreover we impose $[\emptyset] = 0$. For the moment Π is just an abelian group, nonetheless we refer to it as the *polytope algebra* of V.

From the properties of groups defined by generators and relations we see that:

Proposition 2.3. *Let A be an abelian group, a function $\phi\colon \mathcal{P}(V) \longrightarrow A$ is a translation invariant valuation if and only if it induces a group homomorphism $\phi\colon \Pi(V) \longrightarrow A$.*

Example 2.4. Some key examples of valuations on polytopes are the following.

- The Lebesgue measure

$$Leb_n\colon \mathcal{P}(\mathbb{R}^n) \longrightarrow \mathbb{R};$$

that sends a polytope to its volume.

- The Euler characteristic
$$\chi\colon \mathcal{P} \longrightarrow \mathbb{Z};$$
simply sending each non empty polytope to 1.

- We can combine the Lebesgue measure with a face map and a translation to get a more elaborate example: first identify $\mathbb{R}^{n-1}$ with the hyperplane $\mathbb{R}^{n-1} \times \{0\} \subseteq \mathbb{R}^n$ orthogonal to e_n, then set
$$\begin{aligned}\phi\colon \mathcal{P}(\mathbb{R}^n) &\longrightarrow \mathbb{R}\\ P &\mapsto Leb_{n-1}(P_{e_n});\end{aligned}$$
we are supposing that P has been translated in order for P_{e_n} to lie in $\mathbb{R}^{n-1} \times \{0\}$.

- We can make the assignment $P \mapsto h(P,\cdot)$ from $\mathcal{P}$ to the vector space of piecewise linear functionals on V^*. One can check that this also is a valuation on $\mathcal{P}$.

On the classes of polytopes we can define the operation $[P]\cdot[Q] = [P+Q]$ induced by the Minkowsky sum, and extended on Π by linearity.

Lemma 2.5. *If $P, Q, R \in \mathcal{P}$ are such that $P \cup Q \in \mathcal{P}$, then $(P+R)\cap(Q+R) = (P\cap Q)+R$, and $(P+R)\cup(Q+R) = (P\cup Q)+R$.*

Proof. Out of the four, the only non straightforward inclusion is $(P+R)\cap(Q+R) \subseteq (P\cap Q)+R$. If $p \in P$, $q \in Q$ and $r, r' \in R$ satisfy $p+r = q+r'$, for every $t \in [0,1]$
$$p+r = t(p+r)+(1-t)(q+r') = (tp+(1-t)q)+(tr+(1-t)r'),$$
and since $P \cup Q$ and R are convex there is a particular $\tilde{t}$ in $[0,1]$ giving a point $\tilde{x} \in P\cap Q$ and $\tilde{r} \in R$ such that $\tilde{x}+\tilde{r} = p+r$. □

Proposition 2.6. *The group Π with product induced by the Minkowski sum is a commutative ring with unit.*

Proof. First we verify that the product does not depend on the representative chosen.
Being the Minkowski sum commutative we have $(P+t)+Q = (P+Q)+t$. To check the valuation property let P and Q be such that $P\cup Q \in \mathcal{P}$, then
$$([P]+[Q])\cdot[R] = [P+R]+[Q+R] = [(P+R)\cap(Q+R)]+[(P+R)\cup(Q+R)] =$$
$$= [(P\cap Q)+R]+[(P\cup Q)+R] = ([P\cap Q]+[P\cup Q])\cdot[R],$$
where in the third equality we used the previous lemma.
The distributive law holds since we defined the product only on the classes of polytopes

and extended by linearity; the Minkowski sum is associative so also the induced product is. $[\emptyset + P] = [\emptyset]$ for each P but we have imposed $[\emptyset] = 0$ from the beginning; the class of a point $[*]$ is the unit. $\square$

The condition $P \cup Q \in \mathcal{P}$ is not always easy to verify. Additionally, when breaking a polytope into pieces one would like to consider *partially open polytopes* (i.e. polytopes lacking some faces of certain dimensions) rather than having to account for the corresponding faces with a minus sign.

Let $\mathcal{U}$ denote the set $\{A \setminus B \mid A = \bigcup_i P_i,\ B = \bigcup_j Q_j,\ P_i, Q_j \in \mathcal{P}\}$ of differences between finite unions of polytopes. A function $\phi\colon \mathcal{U} \to A$ with values in an abelian group A is a valuation if for each $X, Y \in \mathcal{U}$ it satisfies $\phi(X) + \phi(Y) = \phi(X \cup Y) + \phi(X \cap Y)$.

Lemma 2.7. *Valuations on $\mathcal{P}$ admit a unique extension to valuations on $\mathcal{U}$.*

Proof. Let $\widetilde{\phi}\colon \mathcal{U} \longrightarrow A$ be a valuation, denote by ϕ its restriction to $\mathcal{P}$, we first want to show that $\widetilde{\phi}$ is determined by ϕ. Being $\widetilde{\phi}$ a valuation, for each $P, Q \in \mathcal{P}$ we have

$$\widetilde{\phi}(P \cup Q) = \widetilde{\phi}(P) + \widetilde{\phi}(Q) - \widetilde{\phi}(P \cap Q) = \phi(P) + \phi(Q) - \phi(P \cap Q),$$

and if $A \setminus B \in \mathcal{U}$ is a difference between unions of polytopes, in particular $B = \bigcup_i P_i$, we have $\widetilde{\phi}(A \setminus B) = \widetilde{\phi}(A) - \widetilde{\phi}(\bigcup_i (A \cap P_i))$. Thus $\widetilde{\phi}$ is fully determined by its restriction ϕ to $\mathcal{P}$.

On the other hand for every valuation ϕ on $\mathcal{P}$ the function $\widetilde{\phi}$ defined by the computations above is also a valuation, this can be checked via some tedious calculations. $\square$

This means that if $X \in \mathcal{U}$ we can extend our notation and consider the class $[X]$ as an element of Π, intending the appropriate set of sums and differences of classes of polytopes.

Remark. Even though we will now talk about classes of elements of $\mathcal{U}$, there is no reason to expect them to behave like classes of polytopes. For example if $X = [a, b[$ is a partially open real segment, one might expect $[X] \cdot [X] = [X + X] = [\,[2a, 2b[\,]$, however $X = [a, b] \setminus \{b\}$, and playing with the relations in Π one gets

$$[X]^2 = ([\,[0, b-a]\,] - 1)^2 = [\,[0, 2b-2a]\,] - [\,[0, b-a]\,] - [\,[0, b-a]\,] + 1 = 0.$$

Lemma 2.8. *If $P_1, P_2, ..., P_r$ are partially open polytopes that form a partition of a polytope P and ϕ is a valuation,*

$$\phi(P) = \sum_{i=1}^{r} \phi(P_i).$$

Lemma 2.9. *Each polytope admits a triangulation.*

Proof. If P is a segment there is nothing to do. Proceeding by induction on the dimension of P, we can chose a vertex v_0 of P, and a triangulation of each facet of P not containing v_0. For each simplex in those triangulations take the cone with vertex v_0, what we get is a triangulation of P. $\square$

Corollary 2.10. Π *is generated by the classes of simplices.*

Theorem 8. *Let* V, W *be finite dimensional* $\mathbb{R}$*-vector spaces. Affine maps* $T\colon V \longrightarrow W$ *induce ring homomorphisms* $T\colon \Pi(V) \longrightarrow \Pi(W)$ *that commute with dilatations.*

Proof. We can assume T to be linear since $T' = T - T(0)$ induces the same function on Π. To check that the function $T[P] := [TP]$ is well defined we first observe that $T[P+t] = [TP + Tt] = [TP]$. If $P \cup Q \in \mathcal{P}$ we only get

$$T([P] + [Q]) = [TP] + [TQ] = [TP \cap TQ] + [TP \cup TQ],$$

to end we need to show that $T[P \cap Q] = [TP \cap TQ]$.
If $x \in TP \cap TQ$ we have $p \in P$ and $q \in Q$ such that $Tp = Tq = x$, so the span of $q - p$ is contained in $\ker T$. Being $P \cup Q$ convex there is y in the intersection between the segment $[p, q]$ and $P \cap Q$, it follows that $x = Ty \in T(P \cap Q)$.
From linearity of T follows that $T[\lambda P] = \lambda T[P]$ and $T[P + Q] = [TP + TQ]$. □

Corollary 2.11. *The dilatation by a factor* $\lambda \in \mathbb{R}$ *on* V *induces a ring endomorphism* $\Delta(\lambda)$ *on* Π*.*

Remark. Multiplication by scalars in Π does not correspond to a dilatation: consider for example s the class of a closed real segment, then $\Delta(2)s = 2s - 2$. Furthermore, $[\frac{1}{2}P]$ is simply the class of a polytope, while it is not even clear if writing $\frac{1}{2}[P]$ makes sense (it would correspond to an element $x \in \Pi$ such that $x + x = [P]$). Moreover, if we just look at the classes polytopes, dilatations appear to be more related to the product in Π than the sum: $[nP] = [P + P + ... + P] = [P]^n$.

Theorem 9. *Let* U *be a* k*-frame, the map* $P \mapsto [P]_U := [P_U]$ *induces an ring endomorphism* $x \mapsto x_U$ *that commutes with non-negative dilatations.*

Proof. Consider the composition $P \mapsto P_U \mapsto [P_U]$, this is a translation invariant valuation on $\mathcal{P}$: clearly $(P + t)_u = P_u + t$; secondly, if $P \cup Q$ is in $\mathcal{P}$ we have two options:

- if the supporting hyperplane to $P \cup Q$ corresponding to u meets both P and Q we see that
$$(P \cup Q)_u = P_u \cup Q_u \text{ and } (P \cap Q)_u = P_u \cup Q_u;$$
- if on the other hand it meets only one of them, say P, we see that
$$(P \cup Q)_u = P_u \text{ and } (P \cap Q)_u = Q_u.$$

In both cases we have:
$$[P \cap Q]_u + [P \cup Q]_u = [P]_u + [Q]_u.$$

Finally recalling from chapter 1 that $(P + Q)_u = P_u + Q_u$, and noting that $(\lambda P)_u = \lambda P_u$ for non-negative λ we have the thesis. □

Remark. In general we cannot allow negative dilatations since $(\Delta(\lambda)x)_u = \Delta(\lambda)(x_{-u})$ for $\lambda < 0$.

Our goal in this chapter is to establish the main algebraic properties of Π, in particular, there is the following structure theorem:

Theorem 10. *Π is almost a graded commutative $\mathbb{R}$-algebra generated in degree 1. More precisely:*

- *as abelian groups we have*
$$\Pi = \bigoplus_{r=0}^{d} \Pi_r;$$
- *multiplication satisfies*
$$\Pi_r \cdot \Pi_s = \Pi_{r+s};$$
- *$\Pi_0 \cong \mathbb{Z}$, and for $r \geq 1$ Π_r is a real vector space, in particular $\Pi_d \cong \mathbb{R}$;*
- *if $x, y \in \bigoplus_{r=1}^{d} \Pi_r, \lambda \in \mathbb{R}$ then $(\lambda x)y = x(\lambda y)$;*
- *the vector spaces Π_r are the eigenspaces of the non-negative dilatations $\Delta(\lambda)$, in particular if $\lambda \geq 0$ and $x \in \Pi_r$ then*
$$\Delta(\lambda)x = \lambda^r x.$$

  (With the convention $0^0 := 1$).

Remark. In [McM89], the theorem is proved over an arbitrary ordered field. Here, we will limit our proof to showing that Π is a rational graded algebra, effectively proving the entire theorem but with scalars restricted to $\mathbb{Q}$. At the end of the chapter we include remarks concerning multiplication by a real scalar. This is needed because in Chapter 3, we will rely on the full statement of Theorem 10.

2.1 Rational structure

Let Π_0 the subgroup of Π generated by $[*] = 1$, we see that $\Pi_0 \cong \mathbb{Z}$ since the map $\chi \colon \Pi \longrightarrow \mathbb{Z}$ induced by the Euler characteristic maps $[*]$ to 1. Denoting by Z_1 the subgroup generated by elements of the form $([P] - 1)$ for $P \in \mathcal{P} \setminus \{\emptyset\}$, we have

Proposition 2.12. *Z_1 is an ideal of Π and as abelian groups we have*
$$\Pi = \Pi_0 \oplus Z_1.$$

Proof. If $x \in \Pi$, we can write $x = \sum_i \alpha_i[P_i]$ with α_i integers. Then $x = \sum_i \alpha_i + \sum_i \alpha_i([P_i] - 1)$ so the sum is indeed Π. Finally the intersection is empty since Z_1 is contained in the kernel of $\Delta(0)$ while $\Delta(0)1 = 1$, it follows also that Z_1 coincides with the kernel of $\Delta(0)$ and thus it is an ideal. □

Our next goal is to prove that Z_1 is a nilpotent ideal, in order to do this, we need to establish some technical results.
Consider $\{a_1, ..., a_k\}$ a set of linearly independent vectors in V, fix $a_0 \in V$. Denote by

$$T(a_1, ..., a_k) := \text{conv}(a_0, a_0 + a_1, ..., a_0 + ... + a_k),$$
$$s(a_1, ..., a_k) := [T(a_1, ..., a_k)] - [T(a_1, ..., a_{k-1})],$$

respectively a simplex and the class of the partially open simplex obtained by removing from T the facet opposite to the vertex $a_0 + ... + a_k$. $(s(\emptyset) := 1)$.

Remark. Since $[T(a_1, ..., a_k)] = \sum_{i=o}^{k} s(a_1, ..., a_i)$, we deduce that Π is generated by the various $s(a_1, ..., a_k)$, those with $k \geq 1$ generate Z_1.

Lemma 2.13. *Let $\lambda, \mu \geq 0$, then*

$$\Delta(\lambda + \mu)s(a_1, ..., a_k) = \sum_{i=0}^{k} (\Delta(\lambda)s(a_1, ..., a_i))(\Delta(\mu)s(a_{i+1}, ..., a_k)).$$

Proof. If we translate the class of $s(a_1, ..., a_i)$ by the vector $\mu a_1 + ... + \mu a_i$ we see that the i-th addendum of the sum corresponds to the class of the partially open polytope $\{\sum_j \xi_j a_j \mid 0 < \xi_k \leq ... \leq \xi_{i+1} \leq \mu < \xi_i \leq ... \leq \xi_1 \leq \lambda + \mu\}$.
These sets are disjoint for different i and their union is the set $\{\sum_j \xi_j a_j \mid 0 < \xi_k \leq ... \leq \xi_1 \leq \lambda + \mu\}$, which class in Π is $\Delta(\lambda + \mu)s(a_1, ..., a_k)$. □

An induction argument yields the following technical result:

Lemma 2.14. *Let $n \geq 0, k \geq 1$ be integers, then*

$$\Delta(n)s(a_1, ..., a_k) = \sum_{i=1}^{k} \binom{n}{k} z_i;$$

$$z_i = \sum_{\substack{J \subseteq [k] \\ |J| = i}} \prod_{r=1}^{i} s(a_{j_{(r-1)}+1}, ..., a_{j_{(r)}});$$

in particular we have

$$z_k = s(a_1)s(a_2)...s(a_k).$$

Remark. The formula for the z_i is quite intimidating, nevertheless it shows that they do not depend from n and that they all lie in Z_1.

Lemma 2.15. *For each $x \in \Pi$ there exist unique $y_0 \in \Pi_0$ and $y_1, ..., y_d \in Z_1$ such that for each integer $n \geq 1$*

$$\Delta(n)x = \sum_{i=0}^{d} \binom{n}{i} y_i.$$

Proof. The previous lemma yields the existence, for the uniqueness part we see that the square matrix with entries $\binom{i}{j}$ $i, j = 0, ..., d$ is triangular with all diagonal elements equal to 1, and thus invertible, so the y_i can be calculated by the dilates of x. □

Remark. Since the matrix with entries the binomial coefficients remains invertible regardless if its size, in the proof of the lemma we actually proved a slightly stronger result: the only possible way of extending the sum in the statement is by setting $y_i = 0$ for $i > d$.

Proposition 2.16. *Let P be a non empty polytope, then for $k > d$*

$$([P] - 1)^k = 0.$$

Proof.

$$\Delta(n)[P] = [P]^n = (1 + ([P] - 1))^n = \sum_{k=0}^{n} \binom{n}{k} ([P] - 1)^k.$$

The thesis follows from the previous remark. □

We are now almost in position to define on Z_1 the structure of $\mathbb{Q}$-vector space, we will achieve so by showing that for $m \in \mathbb{Z}$ and $y \in Z_1$, equations of the form $mz = y$ admit a unique solution in Z_1. The first step is to realize we have a filtration: denoting Z_r the subgroup generated by r-powers of elements of Z_1, we get:

$$Z_1 \supset Z_2 \supset ... \supset Z_d \supset Z_{d+1} = 0.$$

Remark. $\Delta(\lambda)Z_r \subseteq Z_r$ since $\Delta(\lambda)([P] - 1)^r = ([\lambda P] - 1)^r$. Also recall that for $\lambda \neq 0$, $\Delta(\lambda)$ is invertible with inverse $\Delta(\lambda^{-1})$.

Lemma 2.17. *For $x \in Z_r$ and n natural, we have $\Delta(n)x - n^r x \in Z_{r+1}$.*

Proof. It suffices to prove so for the generators. From the proof of the last proposition we know that $\Delta(n)[P] - 1 = \sum_{k \geq 1} \binom{n}{k} ([P] - 1)^k$. Taking the r-th power on both sides and subtracting $n^r([P] - 1)^r$ we get a sum in Z_{r+1}. □

Lemma 2.18. *The subgroup Z_1 is torsion free.*

Proof. Let $x \in Z_1$ be a torsion element, let $n \in \mathbb{Z}$ be a non zero integer such that $nx = 0$. If $x \in Z_r$, then

$$\Delta(n)x = \Delta(n)x - n^r x \in Z_{r+1}.$$

We deduce that $x \in Z_{d+1}$ and so $x = 0$. □

Lemma 2.19. *The subgroup Z_1 is divisible.*

Proof. Let $y \in Z_1$ and m be a non zero integer, if $y \in Z_d$, then $\Delta(m)y - m^d y = 0$, so

$$y = m \cdot \Delta(1/m)m^{d-1}y.$$

Now suppose Z_{r+1} is divisible. If $y \in Z_r$, then $\Delta(m)y - m^r y = x \in Z_{r+1}$ so $x = mz$ for some z, then

$$y = m \cdot \Delta(1/m)(z + m^{r-1}y).$$

□

Remark. We have established that Z_1 is a nilpotent ideal and a divisible torsion-free abelian group (so a $\mathbb{Q}$-vector space). Therefore, the following definitions make sense since the multiplication by a rational is well-defined and the sums that appear are actually finite.

Definition 2.20. For $z \in Z_1$ we define the *logarithm* and the *exponential* by the familiar power series:

$$\log(1+z) := \sum_{k>0} \frac{(-1)^{k+1}}{k} z^k;$$
$$\exp(z) := \sum_{k\geq 0} \frac{z^k}{k!}.$$

In particular, for a non empty polytope P we can define $\log\, P := \log[P] = \log(1+([P]-1))$.

Proposition 2.21. *The usual properties of* log *and* exp *continue to hold, in particular if $x, y \in \Pi$ are such that $x-1, y-1 \in Z_1$ (equivalently $\Delta(0)x = \Delta(0)y = 1$), then $\exp\circ\log(x) = x$ and $\log(xy) = \log(x) + \log(y)$.*

Notice also that if n, m are positive integers,

$$\Delta(n)\log[P] = \log[nP] = \log[P]^n = n \cdot \log[P].$$

We deduce that $\Delta(1/m)\log[P]$ is the solution of the equation $mz = \log[P]$: the logarithms of polytopes are eigenvectors for non-negative rational dilatations.

Let $\mathfrak{p} := \log\, P$ and denote by Π_r the subgroup of Z_1 generated by elements of the form $\mathfrak{p}^r$. We are now in position to prove most of Theorem 10 with scalars restricted to being rational.

Theorem 11. *Π is almost a graded $\mathbb{Q}$-algebra in the sense of Theorem 10, precisely we have*

$$\Pi = \bigoplus_{k=0}^{d} \Pi_k$$

as abelian groups, and $x \in \Pi_k$ if and only if $\Delta(\lambda)x = \lambda^k x$ for each $\lambda \geq 0$ rational (with the convention $0^0 := 1$).

Proof. First observe that $\mathfrak{p}^{d+1} = 0$, therefore $\Pi_r = 0$ for $r > d$.
Being $\Delta(n)$ a ring endomorphism, if $x \in \Pi_r$ and $\lambda \in \mathbb{Q}_{\geq 0}$ we see that $\Delta(\lambda)x = \lambda^r x$. By considering the sum

$$\Delta(\lambda)[P] = \exp(\log[\lambda P]) = \exp(\lambda \mathfrak{p}) = \sum_{r=0}^{d} \lambda^r \frac{\mathfrak{p}^r}{r!}$$

for $\lambda = 1$, we see that Π is generated by the Π_r. Their sum is direct since $\Pi = \Pi_0 \oplus Z_1$, and for $r > 0$ the Π_r are vector subspaces of Z_1 contained in pairwise distinct eigenspaces for dilatations, so they themselves have trivial intersection.
Being $Z_1 = \bigoplus_{r>0} \Pi_r$ we deduce that Π_r is exactly the eigenspace of $\Delta(\lambda)$ of eigenvalue λ^r.
If $x \in \Pi_r$, $y \in \Pi_s$ and $\lambda \in \mathbb{Q}_{\geq 0}$, we have $\Delta(\lambda)(xy) = \Delta(\lambda)x \cdot \Delta(\lambda)y = \lambda^r x \cdot \lambda^s y = \lambda^{r+s} xy \in \Pi_{r+s}$. And since $\mathfrak{p}^r \cdot \mathfrak{p}^s = \mathfrak{p}^{r+s}$ we deduce $\Pi_r \cdot \Pi_s = \Pi_{r+s}$. □

Definition 2.22. Let k be a non-negative integer, a valuation $\phi\colon \mathcal{P} \longrightarrow A$ is said to be *homogeneous* of degree k if for each integer $n \geq 0$ it satisfies $\phi(nP) = n^k \phi(P)$.

Corollary 2.23. *Each translation invariant valuation ϕ on $\mathcal{P}$ admits a unique decomposition $\phi = \sum_{k=0}^{d} \phi_k$, with ϕ_k a translation invariant valuation homogeneous of degree k.*

Proof. It suffices to consider the restrictions of ϕ to the k-th graded component Π_k. □

2.2 Volume

We wish to give a more precise description of Π_d. To do so, it suffices to look at the d-components of the semi-open simplices $s(a_1, ..., a_k)$ introduced in the previous section. We notice that the coefficient of n^d in the expansion of Lemma 2.14 just appears for $k = d$, and it is

$$s(a_1, ..., a_d)_d = \frac{1}{d!} s(a_1)s(a_2)...s(a_d).$$

Observe that $s(a_1)s(a_2)...s(a_d)$ corresponds to the class of the partially open "d-parallelogram"

$$\Big\{ \sum_{i=1}^{d} \xi_i a_i \mid 0 < \xi_i \leq 1 \Big\}.$$

Of course the order of the a_i is irrelevant, thanks to the translation invariance we have $s(a_i) = s(-a_i)$, and a simple picture drawing will convince the reader that for $i \neq j$ and $\lambda \in \mathbb{R}$

$$s(a_i + \lambda a_j)s(a_j) = s(a_i)s(a_j).$$

From linear algebra we know that if we chose $\{v_1, ..., v_d\}$ a basis of V, the above operations are sufficient to get $s(a_1)s(a_2)...s(a_d) = s(\mu v_1)s(v_2)...s(v_d)$, where μ is the absolute value

of the determinant of the tuple $(a_1, a_2, ..., a_d)$ relative to the basis chosen.
Since $s((\nu+\mu)v_1) = s(\nu v_1) + s(\mu v_1)$, the volume map

$$s(a_1)s(a_2)...s(a_d) \overset{\text{vol}}{\longmapsto} |det(a_1, ..., a_d)|,$$

is a well defined isomorphism of abelian groups between Π_d and $\mathbb{R}$.

Corollary 2.24. *Let $\Phi\colon \mathcal{P} \longrightarrow \mathbb{R}$ be a non-negative translation invariant valuation, homogeneous of degree d, then Φ is a positive multiple of volume.*

We recall that we say that two subsets S, R of V are parallel to each other if the affine hull of either of the two translates into the affine hull of the other.

Remark. For a linear subspace $L \subseteq V$ of dimension k we have the subalgebra $\Pi(L)$ with its own volume, by choosing a basis of L we have k-volume for polytopes parallel to L, that we will denote by vol_L. In order to choose volume for each subspace with continuity, we can pick a full dimensional polytope P that contains the origin in its interior (for example $\text{conv}(v_1, ..., v_d, -\sum_i v_i)$ for some basis $v_1, ..., v_d$ of V) and then scale vol_L so that the L-volume of $P \cap L$ is 1.
Since k-volume on a subspace is uniquely determined up to scaling, if we want to compare volume between different k-subspaces L and M, there is a unique non-negative scalar $\theta(L, M)$ such that for any polytope P parallel to L

$$\theta(L, M)\text{vol}_L(P) = \text{vol}_M(\pi_M P),$$

where π_M is the orthogonal projection onto M.

Remark. We are working over a real vector space V with a scalar product, so it may feel not natural to consider different volumes in each subspace and then use some scalars to compare them. For a linear subspace L, one can scale vol_L so that the intersection of L with the unit sphere in V has the expected volume, then compare different volumes as usual, playing with the scalar product and measuring some angles. This approach is valid and will be used in Chapter 3. However, it is important to note that this method is not applicable if the vector space does not have a norm. Additionally, if one is working over an ordered field other that $\mathbb{R}$ (for example the rationals) a "uniform" scaling of L-volumes may not exist.

2.3 Separation

We saw the isomorphism between $\mathbb{R}$ and Π_d induced by volume, a rephrasing of this is that we can think of vol as a functional from Π to $\mathbb{R}$ that separates Π_d (meaning $x, y \in \Pi_d$ have the same volume only if $x = y$).
We wish to do a similar construction using volume on lower dimensional subspaces.

Definition 2.25. If U is a frame, we can associate to it the *frame functional* f_U defined by

$$f_U(x) = \mathrm{vol}_{U^\perp} x_U.$$

If U is a k-frame, we say that f_U is a frame functional of type $d-k$.
Note that $f_\emptyset$ is just volume and if U is any d-frame, f_U coincides with $\Delta(0)$.

Theorem 12. *Frame functionals separate* Π.

It is sufficient to show that if $x \in \Pi$ is such that $f_U(x) = 0$ for each frame U, then $x = 0$.

Remark. If u is a non zero vector, then x_u lies in $\Pi(H_u)$ (where H_u is the hyperplane orthogonal to u) and for each frame U in the orthogonal of u, $f_U(x_u) = f_{(u,U)}(x) = 0$, thus, imagining an inductive argument in the proof of Theorem 12, we have $x_u = 0$.

We need to set some notations: fix a hyperplane H passing through the origin, a vector w spanning $L = H^\perp$ and a segment E in L, $e := \log E = [E] - 1$. Let Λ be the subgroup of Π generated by elements of the form $y \cdot \Delta(\lambda)e$ for $\lambda \in \mathbb{R}$ and $y \in \Pi(H)$.

Lemma 2.26. *Let H be a hyperplane and L a complementary line, $e \in \Pi_1(L)$ and $y \in Z_1(H)$, then $\Delta(\lambda)y \cdot e = y \cdot \Delta(\lambda)e$ for each $\lambda > 0$.*

We will not provide a proof for the lemma, that can be found in [McM89, §9].

Lemma 2.27. *If x is such that $f_U(x) = 0$ for each frame U, then $x \in \Lambda$.*

Proof. We want to describe the quotient map $\rho\colon \Pi \twoheadrightarrow \Pi/\Lambda$. Let π_H be the orthogonal projection on H and u be a vector not in H; if $Q \in \mathcal{P}(H_u)$, we can suppose it has been translated into the upper halfspace bounded by H (we call "upper one" the one containing w). The class of $\overline{Q} = \mathrm{conv}(Q \cup \pi_H Q)$ is determined by Q up to an element in Λ, thus $\rho([\overline{Q}]) = \rho([Q])$. For a polytope P we define it upper and lower boundaries as

$$P_+ = \{v \in P \mid \forall\mu > 0; v + \mu w \notin P\},$$
$$P_- = \{v \in P \mid \forall\mu > 0; v - \mu w \notin P\}.$$

These are elements in $\mathcal{U}$, so there is a unique way of defining the classes $[\overline{P}_+]$ and $[\overline{P_-}]$ (we again suppose P lies in the upper halfspace). Since

$$[P] = [\overline{P}_+] - [\overline{P}_-] + [P_-],$$

we have a decomposition $x = \overline{x}_+ - \overline{x}_- + x_-$ for all $x \in \Pi$. The condition $x_u = 0$ for all u implies $x_- = x_+ = 0$, since both are just sums of x_{u_i} for some vectors u_i. Finally, under the projection on Π/Λ we have $\rho(\overline{x}_-) = \rho(x_-) = 0 = \rho(\overline{x}_+)$, completing the proof of the lemma. □

Proof of Theorem 12. Every x in the intersection of the kernels of all frame functionals has the form $x = \lambda e + ey$ with $y \in Z_1(H)$; if we choose a $d-1$-frame U in H, we see that $0 = f_U(x) = \lambda \mathrm{vol}_L(e)$, so $\lambda = 0$.
On the other hand, for each linear subspace M of H we can choose a scaling of M-volume such that $\mathrm{vol}_M(P) = \mathrm{vol}_{M+L}(P+E)$ for each polytope $P \in \mathcal{P}(M)$. Hence, for a frame U in H we get $0 = f_U(\lambda e + ey) = 0 + f'_U(y)$ (where f'_U indicates the induced frame functional on $\Pi(H)$). By the inductive assumption that Theorem 12 holds in $\Pi(H)$, we conclude that $y = 0$. □

Corollary 2.28. *For each $k = 0, 1, ..., d$, the frame functionals of type k separate Π_k*

Proof. A frame functional of type k is a homogeneous valuation of degree k, therefore it vanishes on Π_r for $r \neq k$. □

We now want to find some non trivial linear relations between frame functionals. For example, we already observed that for any U, U' d-frames $f_U = f_{U'}$. Another kind of relation arises from the analogue of Minkowski's theorem on facet areas of polytopes.
If U is a k-frame and is v a vector in $U^\perp$, we denote by L_v the span of (U, v).

Theorem 13. *For each frame U and vector $v \in U^\perp$*

$$\sum_{w \in U^\perp} sgn\langle v, w\rangle \theta(L_w, L_v) f_{(U,w)} = 0.$$

Proof. It suffices to check the relation for a polytope $P \in \mathcal{P}(U^\perp)$. The sum actually becomes a finite sum since all vectors w in the sum not orthogonal to a facet of P account for a zero addendum. For the remaining vectors we are just summing the areas of the facets of P orthogonal to them after a projection on L_v (accounting for the orientation with a sign). We can divide the facets in upper and lower ones by looking at $\mathrm{sgn}\langle v, u\rangle$ for the u that identifies them, we get:

$$\begin{aligned}
\sum_{w \in U^\perp} \mathrm{sgn}\langle v, w\rangle \theta(L_w, L_v) f_{(U,w)}(P) &= \sum_{\text{upper}} \mathrm{vol}_{L_v}(\pi_{L_v} F) - \sum_{\text{lower}} \mathrm{vol}_{L_v}(\pi_{L_v} F) \\
&= \mathrm{vol}_{L_v}(\pi_{L_v}(\sum_{\text{upper}} F)) - \mathrm{vol}_{L_v}(\pi_{L_v}(\sum_{\text{lower}} F)) \\
&= 0
\end{aligned}$$

The last equality follows from observing that the sums of the upper facets and of the lower facets both project on $\pi_{L_v} P$. □

We end the chapter with some remarks concerning results of [McM89] that will be used in the subsequent chapter but were not proven.

First of all, we will make use of Theorem 10 in its full statement even though we did not completed the proof with the scalars in $\mathbb{R}$. A note is needed for the definition of the multiplication by real scalars.
The idea is to first define the scalar multiplication in Π_1 using dilatations, specifically, for $x \in \Pi_1$ and $\lambda \in \mathbb{R}$ we set $\lambda \cdot x = \Delta(\lambda)x$ if $\lambda \geq 0$, and $\lambda \cdot x = -\Delta(-\lambda)x$ if $\lambda < 0$. This definition is then extended on all graded components by setting $\lambda \cdot x_1...x_k = (\lambda x_1)x_2...x_k$ for monomials, and then extended by linearity.
Verifying that everything is well-defined requires a discrete amount of work, for which we refer to [McM89].

Recall that affine maps and frames induce morphisms of rings as stated in Theorem 8 and Theorem 9. We already showed these morphisms commute with non negative dilatations, so it will follow immediately from Theorem 10 that the two maps are indeed morphism of $\mathbb{R}$-algebras.

Lemma 2.29. *The map $T\colon \Pi(V) \longrightarrow \Pi(W)$ induced by an affine map $T\colon V \longrightarrow W$ is a morphism of $\mathbb{R}$-algebras.*
The endomorphism of rings induced by the valuation $P \mapsto [P]_U$ is an endomorphism of $\mathbb{R}$-algebras.

Chapter 3

Simple Polytopes

In the previous chapter, we introduced the polytope algebra Π: a graded infinite dimensional algebra over $\mathbb{R}$ associated to a real vector space V. Now we turn our attention to a particular subalgebra of Π associated to a fixed polytope P; we intend to study some properties of this algebra and to show how these are related to both the combinatorics and the geometry of P. To ensure the full algebra properties (but blurring some of the geometric meaning of the objects) we will replace Π_0 with $\Pi_0 \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 3.1. If P is a polytope in V, $\Pi(P)$ is the subalgebra of $\Pi(V)$ generated by the classes of Minkowski summands of P.
Similarly to Π, using the logarithm we get the grading $\Pi(P) = \bigoplus_{k=0}^{d} \Pi_k(P)$.

Remark. If Q is a polytope in V and λ is a non-negative scalar, the class in Π of the polytope λQ coincides with $\exp(\lambda \mathfrak{q})$, which is a polynomial in the operations of the algebra evaluated at $[Q]$. Therefore in $\Pi(P)$ we find the classes of all the non-negative dilates of the Minkowski summands of P.
Motivated by this, we say that Q is a *weak Minkowski summand* of P if it is a Minkowski summand of a non-negative dilate of P.

Our main goal in this final chapter is to compute the Hilbert series of $\Pi(P)$ for the case where P is a simple polytope, and show that it coincides with the h-polynomial of P.
We briefly outline the structure of the chapter. In the first section we give a more practical description of $\Pi_1(P)$, in particular, we explicitly construct an isomorphism with a quotient of $\mathbb{R}^n$ (where n indicates the number of facets of P), that allows us to set some sort of coordinates on $\Pi_1(P)$. Next, we introduce the concept of a weight on P; using this technical tool together with the new description of the elements of $\Pi_1(P)$, we are able to prove that the spaces $\Pi_k(P)$ and $\Pi_{d-k}(P)$ are dual to each other. Lastly, we use weights to compute the Hilbert series of $\Pi(P)$. We end the chapter by computing the algebra $\Pi(P)$ for some examples of polytopes and by proving the Upper Bound Theorem for polytopes as a corollary to the theory developed.

3.1 Strong isomorphism

In order to better understand the algebra $\Pi(P)$, first of all we need to improve our understanding of the Minkowski sum and the Minkowski summands of a polytope. We have the definition but we have not provided any criterion to concretely check the property.

Definition 3.2. Let P and Q be two polytopes in V, we write $Q \preceq P$ if $\dim Q_u \leq \dim P_u$ for each non zero vector u. We say that two polytopes P and Q are *strongly isomorphic*, and we write $P \cong Q$, if they are equivalent with respect to the equivalence relation induced by $\preceq$. We denote by $\mathcal{P}(V, P)$ the strong isomorphism class of P in V and by $\mathcal{P}(V, P)/T$ its quotient by the equivalence relation of being a translate of one another.

Lemma 3.3. *Strong isomorphism implies combinatorial isomorphism.*

Proof. We want to "build up" the face lattice of Q from that of P, we give a sketch of how that goes. The claim is that the map sending a k-face F of P of the form P_u to the k-face Q_u is a combinatorial isomorphism. It clear that is a bijection between the sets of k-faces, in particular the vertices, what is less obvious is that it preserves the inclusion relations. For each i let u_i a fixed vector that "identifies" the vertices p_i and q_i. If p_i, p_j are connected by the edge $P_{u_{ij}}$, we have that $Q_{u_{ij}}$ is an edge of Q; the faces of $P_{u_{ij}}$ are $p_i = P_{u_{ij}+u_i}$ and $p_j = P_{u_{ij}+u_j}$, so $Q_{u_{ij}+u_i} = q_i$ and $Q_{u_{ij}+u_j} = q_j$ are the faces of $Q_{u_{ij}}$.

With a similar argument we deal with higher dimensional faces: if $F = P_u$ is a k-face of P with vertices $p_1, ..., p_m$, these are identified by the vectors $u + u_1, ..., u + u_m$, so Q_u is the k-face of Q with vertices $Q_{u+u_1} = q_1, ..., Q_{u+u_m} = q_m$. □

Lemma 3.4. *Let P and Q be two polytopes in V, then Q is a weak Minkowski summand of P if and only if $Q \preceq P$.*

Proof. We give just a sketch of the proof, and refer to [Grü03, §15] for a more detailed one. If Q is a summand of λP, then Q_u is a summand of λP_u, so $\dim Q_u \leq \dim \lambda P_u \leq \dim P_u$ since $\lambda \geq 0$. Now suppose $Q \preceq P$, suppose also P is full dimensional (otherwise restrict the attention to $\operatorname{aff}(P)$), we need to construct a polytope R such that $Q + R = \lambda P$ for some $\lambda \geq 0$. Write $P = \operatorname{conv}(p_1, ..., p_n)$ as the convex hull of its vertices, respectively identified by the vectors $u_1, ..., u_n$, calling $q_i := Q_{u_i}$, we deduce that $\{q_1, ..., q_n\}$ is the set of vertices of Q (potentially with repetitions). If a polytope R as we want it was to exists, certainly $\lambda P_u = Q_u + R_u$, thus, following the earlier notation for the vertices in direction u_i, we must have $r_i = \lambda p_i - q_i$; set $r_i = \lambda p_i - q_i$ and call $R = \operatorname{conv}(r_1, ..., r_n)$. Recalling the properties of the Minkowski sum of Chapter 1, we have:

$$Q + R = \operatorname{conv}(\{q_i + r_j \mid i, j = 1, ...n\}) \supseteq \operatorname{conv}(\{q_i + r_i \mid i = 1, ..., n\}) = \lambda P.$$

If R is such that r_i is the vertex in direction u_i[1], we have that $\langle q_k + r_j, u_i\rangle \leq \langle q_i + r_i, u_i\rangle$ for each i, and since the u_i positively span V we deduce $q_k + r_j \in \text{conv}(\{q_i + r_i \mid i = 1, ...n\})$, yielding the missing inclusion.

To complete the proof we need to choose λ large enough so that all r_i are the vertices respectively in direction u_i, equivalently, we need for each i to have:

$$0 < \langle r_i - r_j, u_i\rangle = \langle \lambda(p_i - p_j) - (q_i - q_j), u_i\rangle = \lambda\langle p_i - p_j, u_i\rangle - \langle q_i - q_j, u_i\rangle.$$

Such a scalar λ exists since $\langle p_i - p_j, u_i\rangle$ is positive. □

Corollary 3.5. *Two polytopes P and Q are weak Minkowski summand of each other if and only if they are strongly isomorphic.*

Polytopes strongly isomorphic to each other are characterized by the property of having the respective facets parallel to each other (and being of the same dimension); they can be obtained from one another by translating the hyperplanes spanned by the facets. We can keep track of how much we translate one of the hyperplanes with a scalar in $\mathbb{R}$, thus, if P is a full dimensional polytope with n facets, its strong isomorphism $\mathcal{P}(V, P)$ class can be parameterized by a subset of $\mathbb{R}^n$ with the euclidean topology.

More precisely, let $U = (u_1, ..., u_n)$ be an ordered tuple of non zero vectors, having no pair of vectors positive multiples to each other and that positively spans V (for example the outer normal vectors at the facets of a full dimensional polytope). We denote by $\mathcal{P}(U)$ the family of polytopes obtained by intersecting halfspaces with outer normal vectors in U. If Q is such a polytope, there exist $\eta_1, ..., \eta_n \in \mathbb{R}$, called *support parameters* of Q, such that:

$$Q = \{x \in V \mid \langle x, u_i\rangle \leq \eta_i, i = 1, ..., n\}.$$

The full dimensional polytopes in $\mathcal{P}(U)$ having exactly n facets form a strong isomorphism equivalence class, moreover, their support parameters are uniquely determined[2]. Recalling that the combinatorial type of a simple polytope is stable under small perturbations of the facet-defining hyperplanes, we deduce that the subset of $\mathbb{R}^n$ identified by a simple polytope P via its strong isomorphism class is open in the euclidean topology.

Lemma 3.6. *We can parameterize $\mathcal{P}(V, P)/T$ with an open subset of $\mathbb{R}^{n-d}$.*

Proof. Since the vectors u_i span V, the linear map $V \hookrightarrow \mathbb{R}^n$ given by $t \mapsto (\langle t, u_1\rangle, ..., \langle t, u_n\rangle)$ is injective; in particular, its image $T = \{(\langle t, u_1\rangle, ..., \langle t, u_n\rangle) \in \mathbb{R}^n \mid t \in V\}$ is a d-dimensional subspace. A translation of $Q \in \mathcal{P}(U)$ by the vector $t \in V$ corresponds to an increment

[1] For the way we defined R it may very well not be the case.

[2] On the other hand, if a polytope $R \in \mathcal{P}(U)$ is lower dimensional or has less than n facets, it may happen that one of the halfspaces is not necessary in the description of R, therefore the corresponding support parameter is free to vary in one direction.

of the support parameters by the vector $(\langle t, u_1\rangle, ..., \langle t, u_n\rangle) \in T$. Since the quotient map $\mathbb{R}^n \longrightarrow \mathbb{R}^n/T \cong \mathbb{R}^{n-d}$ is linear and surjective, it is open; therefore the image of the open set that parameterizes $\mathcal{P}(V, P)$ is open too. □

Definition 3.7. Denote by $\mathcal{K}(P)$ the subset of $\Pi_1(P)$ consisting of the logarithms of polytopes in $\mathcal{P}(V, P)$. Since $\lambda \log R + \log Q = \log(\lambda R + Q)$, $\mathcal{K}(P)$ forms a cone (without the origin) in $\Pi_1(P)$.

Moreover, with a similar argument to that in [McM89, §8], one can show that there is a natural isomorphism of semigroups between $\mathcal{P}(V, P)/T$ with the Minkowski sum[3] and $\mathcal{K}(P)$. This should not come as a surprise since we already know that the sum of two logarithms is the logarithm of the Minkowski sum, and that the support parameters behave similarly. Under this isomorphism, the logarithm $\mathfrak{q}$ of a polytope is identified with the support parameters $(\eta_1, ..., \eta_n)$ up to a vector in the subspace we previously called T (we refer to these coordinates also as *generalized* support parameters).
This isomorphism, toghether with the previous lemma, imply:

Theorem 14. *Let P be a simple polytope of dimension d, then $\Pi_1(P) \cong \mathbb{R}^{n-d}$.*

Lemma 3.8. *The classes of the polytopes in any neighbourhood N of P in its strong isomorphism class generate $\Pi(P)$.*

Proof. Let Q be a summand of P, the function $\lambda \mapsto P + \lambda Q$ if read in coordinates is a well defined continuous map from $\mathbb{R}$ to $\mathbb{R}^n$, so there exists $\lambda > 0$ small enough such that $P' = \lambda Q + P$ is in N, in that case we have $Q = \frac{1}{\lambda}\exp(\mathfrak{p}' - \mathfrak{p})$.

□

Recalling that if Q is a summand of P then Q_u is a summand of P_u, if $F = P_U$ is a face of P, the *face map* $x \mapsto x_U$ is a well defined morphism of algebras from $\Pi(P)$ to $\Pi(F)$.

Lemma 3.9. *Let P be a simple polytope, and $F = P_U$ a k-face of P.*
Then the face map $x \mapsto x_U$ is surjective, and if U is a $d-k$-frame, the restriction of the face map from $\Pi_r(P)$ to $\Pi_r(F)$ is well defined and also surjective, for each $r = 0, ..., k$.

Proof. Since P is simple, there is a neighborhood of F in its strong isomorphism class that can be obtained as "U-faces" of polytopes in the strong isomorphism class of P, making the face map surjective.
Calling $\phi\colon \Pi(P) \longrightarrow \Pi_r(F)$ the composition of the face map with the projection onto $\Pi_r(F)$, we see that $\phi(\Delta(n)x) = n^r\phi(x)$. It induces a translation invariant valuation homogeneous of degree r, so is non zero just on $\Pi_r(P)$. It follows that $\Pi_r(P)$ surjects via the face map onto $\Pi_r(F)$ for each r. □

[3] If we look at the support parameters, this is the sum induced from $\mathbb{R}^n$

In the remaining part of the section we delve into an interesting linear algebra construction, that was used in [McM93] during the proof of the g-Theorem; its purpose was to keep track of changes in the combinatorics of P while its facets were being translated. These results do not play any role in the next sections, so at a first reading they should be skipped.

Definition 3.10 (Linear Transform)**.** As in the definition of $\mathcal{P}(U)$, let $U = (u_1, ..., u_n)$ be an ordered tuple of non zero vectors, with no pair of vectors positive scalar multiple to each other and that positively spans V. A *linear transform* of U is a tuple $\overline{U} = (\overline{u}_1, ..., \overline{u}_n)$ of vectors in $\mathbb{R}^{n-d}$, satisfying:

$$\sum_{i=1}^{n} u_i \otimes_{\mathbb{R}} \overline{u}_i = 0.$$

Remark. More explicitly, a linear transform of U can be obtained by choosing a basis $\{\alpha_1, ..., \alpha_{n-d}\}$ of the space of linear dependencies of U (i.e. the kernel of the map $e_i \mapsto u_i$ from $\mathbb{R}^n$ to V), and then putting $\overline{u}_i = (\alpha_{1,i}, ..., \alpha_{n-d,i})$. A particular linear transform depends on a choice of a basis, so is defined up to linear equivalence. Observe that the $\overline{u}_i$ span $\mathbb{R}^{n-d}$ since the α_i are linearly independent (the column rank of a rectangular matrix equals the row rank).

Definition 3.11. Let Q be in $\mathcal{P}(U)$ with support parameters $(\eta_1, ..., \eta_n)$, let $\overline{U}$ be a linear transform of U, we call the vector $q = \sum \eta_i \overline{u}_i$ a *representative* of Q (relative to $\overline{U}$). From a previous remark, if Q has n facets and is full dimensional it has a unique representative, if not, it may have multiple representatives.

Lemma 3.12. *Let Q be a polytope in $\mathcal{P}(U)$ with support parameters $(\eta_1, ..., \eta_n)$, let $\overline{U}$ be a linear transform of U. Then the vector $q = \sum_i \eta_i \overline{u}_i$ is the representative of precisely the translates of Q.*

Proof. We want to understand the kernel of the surjective map $\pi\colon \mathbb{R}^n \longrightarrow \mathbb{R}^{n-d}$ sending $e_i \mapsto \overline{u}_i$; in particular, we claim that the kernel of π is precisely the d-dimensional subspace $T = \{(\langle t, u_1\rangle, ..., \langle t, u_n\rangle) \in \mathbb{R}^n \mid t \in V\}$, that we previously observed to correspond to the translations of the polytopes in V.

Consider the map $B_t\colon \mathbb{R}^n \times \mathbb{R}^{n-d} \longrightarrow \mathbb{R}^{n-d}$ defined by $B_t(v, w) = \langle t, v\rangle w$; it is bilinear, so it factors through the linear map $\widetilde{B}_t\colon \mathbb{R}^n \otimes \mathbb{R}^{d-n} \longrightarrow \mathbb{R}^{n-d}$. Via direct computation we have:

$$\widetilde{B}_t\Big(\sum_i u_i \otimes \overline{u}_i\Big) = \sum_i \langle t, u_i\rangle \overline{u_i} = \pi(\langle t, u_1\rangle, ..., \langle t, u_n\rangle),$$

so the property of the linear transform implies that T is in the kernel of π, by counting dimensions we conclude that the two coincide. □

Theorem 15. *Let P be a full dimensional simple polytope with n facets, let $U = (u_1, ..., u_n)$ be a tuple of non zero outer normal vectors to the facets of P. Then every linear transform $\overline{U}$ of U defines an isomorphism between $\Pi_1(P)$ and $\mathbb{R}^{n-d}$.*

Proof. The map π considered in the proof of Lemma 3.12 is linear and surjective, thus open. The set of the representatives of the polytopes in the strong isomorphism class of P is the image under π of the open cone $\mathcal{P}(V, P)$, therefore it is an open cone too.
Combining the previous results we have a natural isomorphism of semigroups $\mathcal{K}(P) \cong \mathcal{P}(V, P)/T$ and another one induced by π between $\mathcal{P}(V, P)/T$ and its image. Since $\mathbb{R}^{n-d}$ is the group generated by this last semigroup and $\Pi_1(P)$ the group generated by $\mathcal{K}(P)$, we have the thesis. □

Remark. If we fix a linear transform $\overline{U}$, with the isomorphisms we constructed we can identify a polytope Q in $\mathcal{P}(V, P)/T$ with its logarithm $\mathfrak{q}$ in $\mathcal{K}(P)$, with its representative q in $\pi(\mathcal{P}(V, P))$ and with its vector of generalized support parameters $[(\eta_1, ..., \eta_n)]$ in an open cone in $\mathbb{R}^n/T$.

The following theorem shows how to reconstruct the facial structure of Q from its representative q. We will not prove the theorem, the reader interested on a proof is referred to [McM73].

Definition 3.13. Let Q be in $\mathcal{P}(U)$, we say that a subset $S \subseteq U$ is *facial* for Q if the vectors in S identify facets of Q, and if those are precisely the facets containing a face of Q.
A subset $\overline{S} \subseteq \overline{U}$ is said *cofacial* for Q if q lies in the relative interior of the cone $cone(\overline{S})$.

Theorem 16. *Let Q be a polytope in $\mathcal{P}(U)$, then $S \subseteq U$ is facial for Q if and only if $\overline{S} \subseteq \overline{U} = \{\overline{u} \mid u \notin S\}$ is cofacial for Q.*

Remark. Through its representative q, a polytope Q in $\mathcal{P}(U)$ identifies its *type cone*, obtained by intersecting all the relative interiors of the various $cone(\overline{S})$ that contain q. For the representative r of R to lie in the type cone of Q it is just a rephrasing of saying that R is strongly isomorphic to Q, we deduce that the type cone of Q is again $\mathcal{K}(Q)$.

3.2 Weights

In this section we define weights and use them, together with the results of the previous section, to prove that $\Pi_k(P)$ and $\Pi_{d-k}(P)$ are dual to each other. Specifically, we employ weights to exhibit a perfect paring $\Pi_k(P) \times \Pi_{d-k}(P) \longrightarrow \mathbb{R}$.

Definition 3.14. If P is a d-polytope, a k-weight on P is a function $\omega\colon \mathcal{F}_k(P) \longrightarrow \mathbb{R}$ from the k-faces of P to $\mathbb{R}$ satisfying the Minkowski relations of Theorem 13. This means that for each $k+1$-face G, if L is a subspace of $\mathrm{aff}(G)$ of dimension k and v is a vector in $\mathrm{aff}(G)$ orthogonal to L, we require:

$$\sum_{F\in\mathcal{F}_k(G)} \mathrm{sgn}\langle v,w\rangle\theta(L,F)\omega(F)=0,$$

where w is any vector in $\mathrm{aff}(G)$ such that $F=G_w$.
The vector space of k-weights on P is denoted by $\Omega_k(P)$, this is a linear subspace of $\mathbb{R}^{f_k(P)}$, the real vector space with coordinates indexed by k-faces of P.

The definition is more complicated than it should be (recall also the remark of Section 2.2): since we are working over the real numbers we can choose a scaling factor of k-volume for each subspace such that the intersection with the unit sphere in V has the expected volume (depending on k). This way for each pair of affine subspaces L and M of codimension 1, the scaling factors $\theta(L,M)$ is just the cosine of the angle between them as usual (accounting with orientation). This cannot be done over $\mathbb{Q}$, so in this case the scaling factors $\theta(L,F)$ are needed.
Therefore, the sum in Definition 3.14 takes the following form:

$$\sum_{F\in\mathcal{F}_k(G)} \omega(F)\cdot\mu_{F,G}=0;$$

where $\mu_{F,G}$ is the vector in $\mathrm{aff}(G)$ normal to F of norm 1 pointing outwards.

Example 3.15. Let T be the triangle $\mathrm{conv}(0,e_1,e_2)$ in $\mathbb{R}^2$. We call F_0,F_1 and F_2 the edges opposite respectively to the vertices 0, e_1 and e_2. As we said, since we are in a real vector space we consider as scaling factors for 1-volume the "usual ones". A 1-weight on T is a vector $\omega=(\omega(F_0),\omega(F_1),\omega(F_2))$ in $\mathbb{R}^3$ satisfying the Minkowski relations: $\omega(F_0)=\sqrt{2}\cdot\omega(F_1)$ and $\omega(F_0)=\sqrt{2}\cdot\omega(F_2)$; therefore the vector space $\Omega_1(T)$ can be identified with the 1-dimensional subspace of $\mathbb{R}^3$ spanned by $(\sqrt{2},1,1)$.

Now let T' be the triangle $\mathrm{conv}(0,e_1,e_2)$ in $\mathbb{Q}^2$, call F_0, F_1 and F_2 its edges as before. First of all we need to define 1-volume in each of the three 1-dimensional vector spaces parallel to the three edges: to do so, we specify a basis of each of them: in order $\{e_2-e_1\}$, $\{e_2\}$ and $\{e_1\}$. Secondly, we compute the three scaling factors, recall that they are determined by the condition $\theta(L,M)\mathrm{vol}_L(P)=\mathrm{vol}_M(\pi_M P)$ for each polytope P in L. By looking at the volume of the projection of the three edges F_0, F_1 and F_2 onto each others affine hulls, we obtain: $\theta(F_0,F_1)=1$, $\theta(F_0,F_2)=1$ and $\theta(F_1,F_2)=0$. With elementary computations one can solve the system arising from the Minkowski relations, finding that the weight space $\Omega_1(T')$ is identified with the span of $(1,1,1)$ in $\mathbb{Q}^3$.

Remark. In both of the examples we observe an apparent coincidence: the 1-weights with positive coordinates are in natural bijection with the cone $\mathcal{K}$ of the polytope considered.

It seems to be that for each positive weight ω, the value $\omega(F)$ on a face is the volume of the face corresponding to F of a polytope $Q \cong P$. Consider $\Omega_1(T)$ as an example: for each positive weight $\omega \in \Omega_1(T)$ consider the triangle $\widetilde{T} = \omega(F_1) \cdot T$, a scaled of T. The values ω takes on the edges of T are the lengths of the corresponding edges of $\widetilde{T}$.

Lemma 3.16. *There is a natural inclusion* $\Pi_k(P) \hookrightarrow \Omega_k(P)$.

Proof. If U is a $d-k$-frame and $F = P_U$ is a k-face, the inclusion is given by the map sending $x \in \Pi_k(P)$ to the weight $\omega_x \in \Omega_k(P)$, defined on each k-face F by $\omega_x(F) = f_U(x)$. To check that the definition does not depend on the choice of U, we can assume x to be the k-component of the class of a polytope Q strongly isomorphic to P, since these form a set of generators of $\Pi_k(P)$. In this case $f_U([Q]_k) = f_U([Q]) = \text{vol}_F(F')$, where F' is the face of Q corresponding to F and vol_F indicates k-volume for polytopes parallel to F. The map is linear, Theorem 13 assures us it is well defined, and Theorem 12 implies the map is injective. □

Remark. If $F = P_U$ is any r-face of P and $k \leq r$, we have the restriction map $\Omega_k(P) \longrightarrow \Omega_k(F)$ sending $\omega \mapsto \omega|_F$, the weight relations follow from those of ω.

Lemma 3.17. *Let U be a $d-r$-frame and $F = P_U$ an r-face, for each $k = 0, ..., r$ the diagram*

$$\begin{array}{ccc} \Pi_k(P) & \xrightarrow{x_U} & \Pi_k(F) \\ \downarrow & & \downarrow \\ \Omega_k(P) & \xrightarrow{\omega|_F} & \Omega_k(F) \end{array}$$

is commutative.

Proof. It is sufficient to check the lemma for the k-components $[Q]_k$ of the classes of polytopes strongly isomorphic to P. If G is a k-face of F, both Q and Q_U have the same face k-face G', their corresponding weights evaluated on G both equal the k-volume of G'. □

Remark. A methodical way to compute the volume of a d-polytope Q is to translate it so that the origin is in its interior, each facet F now defines a pyramid pointed at the origin. The volume of P is then the sum of the volumes of these pyramids. The volume of such a pyramid is (up to a constant) the $d-1$-volume of the facet F multiplied by the corresponding support parameter.

We can define on $\Omega(P) = \bigoplus_k \Omega_k(P)$ a structure of $\Pi(P)$-module using the previous simple observation. Since $\Pi(P)$ is generated in degree 1, it is sufficient to define a linear action of $\Pi_1(P)$ on $\Omega(P)$, in particular, we define a hybrid multiplication $\Pi_1(P) \times \Omega_k(P) \longrightarrow \Omega_{k+1}(P)$.

Let $y \in \Pi_1(P)$ correspond to the vector $(\eta_1, ..., \eta_n)$ of generalized support parameters, let $F_1, ..., F_n$ be the facets of P. If $\omega \in \Omega_{d-1}(P)$, we define[4] the d-weight $y \cdot \omega$ by:

$$y\omega(P) := \sum_{i=1}^{n} \eta_i \, \omega(F_i).$$

The definition does not depend on the particular choice of $(\eta_1, ..., \eta_n)$ since $\sum_i \langle t, u_i \rangle \omega(F_i) = \langle t, \sum_i \omega(F_i) u_i \rangle = 0$; furthermore, on d-weights there are no Minkowski relations to be satisfied.
Now we define the action of $\Pi_1(P)$ over $\Omega_k(P)$, we do so in the only possible way so that the restriction map $\Omega_k(P) \longrightarrow \Omega_k(F)$ is a morphism of $\Pi(P)$-modules.
It is tautological that a weight on P is determined by its value on the faces of P, this means that the local information is all that is needed to specify it: if U is a $d-k-1$-frame, and $G = P_U$ is a $k+1$-face of P, we define the $k+1$-weight $y \cdot \omega$ at the face G by:

$$y\omega(G) := y_U \omega\big|_G(G).$$

Explicitly we have that $y\omega(G) = \sum_{F \in \mathcal{F}_k(G)} \eta_{F,G} \, \omega(F)$, where F varies across all k faces of G and $\eta_{F,G}$ is the generalized support parameter of y_U at the facet[5] F, thought inside aff(G). The Minkowski relations continue to hold, as shown by the computations in [McM93]. Instead of repeating them here, we prefer to show an example.

Example 3.18. Let $P = \text{conv}(-2e_1, 2e_1, e_2+e_1, e_2-e_1)$ be a trapezoid in $\mathbb{R}^2$, we denote by F_1 the upper edge, and then proceed with F_2, F_3 and F_4 in anticlockwise order. The triangle $S = conv(-2e_1, 2e_1, 2e_2)$ is a weak Minkowski summand of P (Lemma 3.4), so its logarithm $\mathfrak{s}$ is an element of $\Pi_1(P)$, it corresponds to the vector of generalized support parameters $(2, 0, 0, 2\sqrt{2})$ (we translated S so that its lower left vertex is in the origin). Consider the 1-weight on P $\omega = (\omega(F_1), \omega(F_2), \omega(F_3), \omega(F_4)) = (0, 1, \sqrt{2}, 1)$, then $\mathfrak{s}\omega(P) = 2\sqrt{2}$.

Now consider the 0-weight ω' with value constant -1 on each vertex (up to scalar, these are the only 0-weights on a polytope), to compute the values of the 1-weight $\mathfrak{s}\omega'$ we need to determine the new support parameters. For example, the facets of $F_1 = P_{e_2}$ are the intersection of F_1 respectively with F_4 and F_2, they are just the two vertices $e_2 + e_1$ and $e_2 - e_1$ (we order them in anticlockwise order). In this case S_{e_2} is just a point, so its support parameters (in aff(F_1)) are just $(0, 0)$, therefore $\mathfrak{s}\omega'(F_1) = 0$.
Now consider $F_2 = P_{e_2 - e_1}$, denote by v_1 and v_2 the unit normal vectors in aff(F_2) respectively at the two facets $e_2 - e_1$ and $-2e_1$, this time $S_{e_2-e_1}$ is a segment, with support parameters $\eta_{v_1, F_2} = 2\sqrt{2}$ relative to v_1 and $\eta_{v_2, F_2} = 0$ relative to v_2, therefore $\mathfrak{s}\omega'(F_2) = -2\sqrt{2}$.

[4] We will only deal with the case of V a vector space over $\mathbb{R}$, with "uniform" choices of volume as we described previously. The general case of a vector space over an ordered field involves more scaling factors to make sure volumes in different subspaces interact in the appropriate way.

[5] F is a facet of G.

Via similar computations one should get $\mathfrak{s}\omega'(F_3) = -4$ and $\mathfrak{s}\omega'(F_4) = -2\sqrt{2}$. It is easy to see that $\mathfrak{s}\omega'$ is indeed a 1-weight since $-\mathfrak{s}\omega'$ is the image of the logarithm of S under the inclusion $\Pi_1(P) \hookrightarrow \Omega_1(P)$.

Proposition 3.19. [6] *Let $0 \leq k < d$, if $\omega \in \Omega_k(P)$ is a non zero weight, it exists $y \in \Pi_1(P)$ such that $y\omega$ is a non zero $k+1$-weight.*

Proof. If ω is a non zero $d-1$-weight, let F be a facet of P such that $\omega(F) \neq 0$, without loss of generality we can suppose that F is the facet relative to the vector u_1. It is sufficient to choose y as the vector of generalized support parameters $(1, 0, ..., 0)$, by direct calculations we have $y\omega(P) = \omega(F) \neq 0$.

If ω is a non zero k-weight, let F be a k-face of P on which $\omega(F) \neq 0$, let $G = P_u$ be a facet of P containing F. Then $\omega|_G$ is a non zero k-weight on G, which is a lower dimensional polytope. By inductive hypothesis on the dimension there is $z \in \Pi_1(G)$ such that $z\omega|_G$ is non zero in $\Omega_{k+1}(G)$. Since P is simple, the face map $\Pi_1(P) \to \Pi_1(G)$ is surjective, so $z = x_u$ for some $x \in \Pi_1(P)$. It follows that $(x\omega)|_G = x|_G\omega|_G \neq 0$, therefore also the $k+1$ weight $x\omega$ is non zero. □

Theorem 17. *For P a simple polytope, the embedding $\Pi_k(P) \hookrightarrow \Omega_k(P)$ is an isomorphism of vector spaces.*

Proof. For a fixed $x \in \Pi_{d-k}(P)$, the map $\omega \mapsto x\omega(P)$ is linear from $\Omega_k(P)$ to $\mathbb{R}$. It follows that we have a linear map from $\Pi_{d-k}(P)$ to $\Omega_k(P)^*$, the previous proposition implies this last map is surjective. A simple dimension counting shows

$$\dim \Pi_k(P) \leq \dim \Omega_k(P) \leq \dim \Pi_{d-k}(P) \leq \dim \Omega_{d-k}(P) \leq \dim \Pi_k(P).$$

□

Remark. For P an arbitrary polytope, the theorem is false. For example, if P is simplicial of dimension d, its 2-faces are triangles so it has no non trivial summands [Grü03, §15]: $\Pi_k(P)$ are all of dimension 1. On the other hand, regardless of the combinatorics of P being full dimensional with n facets implies $\dim \Omega_{d-1}(P) = n - d$.

From the previous results, it straightforwardly follows that:

Corollary 3.20. *The bilinear map*

$$\begin{aligned} \Pi_k(P) \times \Pi_{d-k}(P) &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto x\omega_y(P) \end{aligned}$$

is a perfect paring.

[6] Compare this proposition with [McM89, Theorem 11].

3.3 Hilbert series

We now have the necessary tools to compute the Hilbert series of $\Pi(P)$. In particular, we are going show that it corresponds to the h-polynomial of P. In light of this fact, the Dehn-Sommerville equations for $h(P,t)$ are just a shadow of the duality between the spaces $\Pi_k(P)$ and $\Pi_{d-k}(P)$.

Theorem 18. *Let P be a simple d-polytope with n facets, then* dim $\Pi_k(P) = h_k(P)$.

We start in the same set up of the proof of the Dehn-Sommerville equations (Theorem 4). Recall that φ is a functional in V^* and we are moving the hyperplane $H_t = \{x \in V \mid \varphi(x) \leq t\}$ by gradually increasing the value of t; φ is generic with respect to the vertices of P, meaning $\varphi(v) \neq \varphi(v')$ for each pair of distinct vertices v, v'. Recall that a vertex v is said of type r relative to φ if for $t = \varphi(v)$, precisely r of the edges passing through v are contained in H_t.
We will show that the dimension of $\Omega_k(P)$ coincides with the number of vertices of type k relative to φ.
Since it will be heavily used in the proof, we also recall that two subsets of V are said parallel if the affine hull of either of the two can be translated into the affine hull of the other.

Proof. The plan is to gradually define a k-weight $\omega \in \Omega_k(P)$ as we move the halfspace $H_t = \{x \in V \mid \varphi(x) \leq t\}$ through P, paying attention to what happens we add new vertices; there are 3 possibilities we need to consider: the vertex we are adding can be either of type $r < k$, of type k or of type $r > k$.
In the first option we do not have any new k-face, so there is nothing to say about ω. If we instead add a vertex of type k, there is a single new k-face F and no new $k+1$-faces, therefore we can freely assign the value of ω at F as there are no Minkowski relations to be satisfied. Lastly, for vertices of type $r > k$ we claim that the weight relations on the new $k+1$-faces completely determine the value of ω on the k-faces we have added.

If G is one of the new $k+1$-faces, together with G we add its $k+1$ k-faces $F_1, ..., F_{k+1}$ containing the new vertex v. The $k+1$ vectors in aff(G) normal to $F_1, ..., F_{k+1}$ are linearly independent, the Minkowski relations on G imply that the values ω takes on $F_1, ..., F_{k+1}$ are uniquely determined by the values of ω we previously chose on the other k-faces of G. Therefore, we do not have any freedom on the values of ω on the new k-faces when adding a vertex of type $r > k$: each new $k+1$-face determines the value of ω on the new k-faces it contains. What is not clear, is whether or not the value of ω on such k-faces is independent from the particular $k+1$-face we compute it from.

If $r = k + m$, each k-face F lies in precisely m distinct new $k+1$-faces, from Theorem 2, each pair of them identifies a unique $k+2$-face containing F, so checking the result for

$r = k + 2$ is sufficient. If we restricting the attention to the affine hull of the $k + 2$-face we have chosen, this face becomes a full dimensional polytope, therefore we can suppose we are assigning a $d - 2$ weight ω on a simple d-polytope P, that we have already assigned its value on all the faces that do not contain a fixed vertex v and the weight relations are all satisfied when necessary.

Let F be a $d-2$-face of P containing v, F contains precisely $d-2$ of the edges through v, the two facets G_1, G_2 containing F respectively contain the remaining two edges l_1 and l_2. Let L be the affine plane spanned by the two edges l_1 and l_2, then choose an orientation of L; this gives a notion of "up" and "down" in all the facets of P not parallel to L. In fact, by considering the intersection between $L + t$, a general translate of L intersecting P, and P itself, we get an oriented polygon with the edges corresponding to some facets of P, these edges are oriented segments in the affine hulls of the respective facets.

Let G be a facet of P not parallel to L, proceding similarly to the proof of Theorem 13, we divide its $d - 2$-faces into *upper*, *lower* and *vertical*, according to the direction of their outer normal vectors in $\text{aff}(G)$ relative to the oriented segment $G \cap (L + t)$. A face is an upper face if its outer normal vectors point "up", similarly, we say that it is a lower face if its outer normal vectors point "down"; the face is said vertical if it is parallel to the oriented segment $G \cap (L + t)$. Observe that G_1 and G_2 are the only facets at v not parallel to L and F is an upper face in one facet and a lower face in the other one.

Now if we project orthogonally on the $d - 2$-dimensional linear space $L^\perp$, the weight relations precisely say that that for each facet G:

$$\sum_{F' \text{ lower}} \theta(F', L^\perp)\omega(F') = \sum_{F' \text{ upper}} \theta(F', L^\perp)\omega(F').$$

By summing over all the facets of P we observe that most things cancel out: if a $d-2$-face intersects non trivially a translate of L, it accounts for a zero addendum (it does not project on a full dimensional polytope in $L^\perp$). Even though ω might take different values at those faces containing v if computed as upper or lower faces, we can still cancel out those that not containing v, as each $d-2$-face F' not parallel to L appears once as an upper face and once as a lower face.

We observed in the previous paragraph that all those faces but F are not full dimensional if projected on $L^\perp$, therefore do not influence the sum. Every other term canceled out leaving just $\omega_{\text{lower}}(F) = \omega_{\text{upper}}(F)$, we deduce the assignment $\omega(F)$ does not depend on the $k + 1$-face we compute it from, completing the proof of the theorem. □

Remark. Throughout the proof we showed that to a generic functional $\varphi \in V^*$ we can associate a basis of $\Omega_k(P)$ indexed by the set of k-vertices of P: the weight ω_v takes the value 1 on the k-face F relative to the k-vertex v and 0 on all the other k-faces associated to k-vertices.

Definition 3.21. To each $\varphi \in V^*$ general enough, the corresponding basis of $\Pi_k(P)$ is called the *section basis* relative to φ.

We now intend to compute a few examples of such algebras to get a feel of how they look like, and see in practice the properties we observed so far.

Example 3.22. Let us first consider $P = \Delta_d$ the standard d-simplex (we think of it as being full dimensional in the affine hyperplane in $\mathbb{R}^{d+1}$ where coordinates add to 1). The computations are straightforward: we know that $\Pi(\Delta_d)$ is generated in degree 1 and from Theorem 14 the dimension of $\Pi_1(\Delta_d) = (d+1) - d = 1$; the dimension of $\Pi(\Delta_d) = \sum h_k = f_0(\Delta_d) = d + 1$. We have

$$\Pi(\Delta_d) \cong \frac{\mathbb{R}[x]}{(x^{d+1})}.$$

Example 3.23. Now let $P = C_d$ be the standard d-cube C_d, for a better notation consider the d-cube $[0,1]^d$, which is strongly isomorphic to C_d and therefore identifies the same subalgebra of Π. The algebra $\Pi(C_d)$ is generated by the d-dimensional $\Pi_1(C_d)$, with basis the logarithms $\mathfrak{q}_1, ..., \mathfrak{q}_d$ of the edges $Q_1 = [0, e_1], ..., Q_d = [0, e_d]$, they are 1-dimensional polytopes so $\mathfrak{q}_i^2 = 0$. We deduce there is a surjective algebra morphism

$$\frac{\mathbb{R}[x_1, ..., x_d]}{(x_1^2, ..., x_d^2)} \longrightarrow \Pi(C_d)$$

sending $x_i \mapsto \mathfrak{q}_i$, both algebras have dimension 2^d, therefore the map is an isomorphism of algebras. Under this isomorphism the square-free monomial $x_{i_1}...x_{i_k}$ corresponds to $\mathfrak{q}_{i_1}...\mathfrak{q}_{i_k} = k!\,[Q]_k$, the k-component (up to a constant) of the cube $Q = [0, e_{i_1}] \times [0, e_{i_2}] \times ... \times [0, e_{i_k}]$.

Example 3.24. Let $v_1, ..., v_d$ be linearly independent vectors in $\mathbb{R}^d$. Consider the non degenerate d-parallelogram $P = \operatorname{conv}\left(\{\sum_{i \in I} v_i \mid I \subseteq [d]\}\right)$. The linear map $T\colon \mathbb{R}^d \to \mathbb{R}^d$ sending $v_i \mapsto e_i$ induces an automorphism of $\Pi(\mathbb{R}^d)$ that restricts to an isomorphism between $\Pi(P)$ with $\Pi(C_d)$, that we just computed.

Example 3.25. Recalling that the h-vector counts the number of k-vertices relative to a general linear functional, we deduce that for a prism $P \times [0,1]$ we have $h(P \times [0,1], t) = (1+t)h(P,t)$ (this gives an elegant way of computing $h(C_d, t)$). Inspired by this simple relation we intend to compute the algebra $\Pi(P \times [0,1])$ in terms of $\Pi(P)$.

If $\varphi \in V^*$ gives us the section basis $\{\omega_1, ..., \omega_r\}$ for $\Pi_k(P)$, by "tilting up" φ enough we can get $\psi \in (V \oplus \mathbb{R})^*$ such that for each vertex v of P of type k the vertices $(v, 0)$ and $(v, 1)$ are respectively of type k and $k+1$. The corresponding section basis we get is $\{\omega_1, ..., \omega_r, \omega_1', ..., \omega_r'\}$, where ω' indicates the section weight associated to the vertex "over" that of ω.

Calling $\mathfrak{s}$ the logarithm of the class of the segment $[0,1]$ in $\Pi(V \oplus \mathbb{R})$, with little computations one observes that $\mathfrak{s}\omega_i = \omega_i'$, therefore having

$$\Pi(P \times [0,1]) \cong \Pi(P) \otimes_{\mathbb{R}} \frac{\mathbb{R}[t]}{(t^2)}.$$

Example 3.26. Let Q be the pentagon in $\mathbb{R}^2$ with vertices given by the columns of the matrix $\begin{pmatrix} 0 & 0 & 2 & 2 & 3 \\ -1 & 1 & -1 & 1 & 0 \end{pmatrix}$, and order its facets (edges) $F_1, ..., F_5$ starting from the bottom one and proceeding in anticlockwise order. Consider the functional of $(\mathbb{R}^2)^*$ corresponding to the row vector $u = (2,1)$, the corresponding section basis of $\Pi_1(Q)$ is given by the weights

$$\begin{aligned} \omega_A &= (0, \sqrt{2}/2, \sqrt{2}/2, 0, 1), \\ \omega_B &= (1, -\sqrt{2}/2, \sqrt{2}/2, 0, 0), \\ \omega_C &= (0, \sqrt{2}/2, -\sqrt{2}/2, 1, 0). \end{aligned}$$

Consider now the change of coordinates:

$$\begin{aligned} \omega_1 &= \omega_A + \omega_B = (1, 0, \sqrt{2}, 0, 1); \\ \omega_2 &= \omega_A + \omega_C = (0, \sqrt{2}, 0, 1, 1); \\ \omega_3 &= \omega_B + \omega_C = (1, 0, 0, 1, 0). \end{aligned}$$

Calling T_1 and T_2 respectively the logarithms of the two triangles with vertices respectively $\{0, e_1, e_2\}$ and $\{0, e_1, -e_2\}$ and S the segment between 0 and e_1, denoting then by $\mathfrak{t}_1, \mathfrak{t}_2$ and $\mathfrak{s}$ their logarithms, we observe that $\omega_1 = \omega_{\mathfrak{t}_1}$, $\omega_2 = \omega_{\mathfrak{t}_2}$ and $\omega_3 = \omega_{\mathfrak{s}}$. We deduce that $\{\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{s}\}$ is a basis of $\Pi_1(Q)$. To compute whole algebra $\Pi(Q)$, we can now look at volumes of the various Minkowski sums of the elements of the basis. We recall that the sum of the logarithms is the logarithm of the Minkowski sum of the polytopes, and that $\Pi_2(Q)$ can be identified with $\mathbb{R}$ via the map induced by volume. With easy computations of the volumes of some polygons, we get:

$$\begin{aligned} &\mathfrak{s}^2 = 0; \\ &\mathfrak{t}_1^2 - \mathfrak{t}_2^2 = 0; \\ &2\mathfrak{t}_2^2 + \mathfrak{t}_1^2 = (\mathfrak{t}_2 + \mathfrak{s})^2 = \mathfrak{t}_2^2 + 2\mathfrak{s}\mathfrak{t}_2; \\ &2\mathfrak{t}_1^2 + \mathfrak{t}_2^2 = (\mathfrak{t}_1 + \mathfrak{s})^2 = \mathfrak{t}_1^2 + 2\mathfrak{s}\mathfrak{t}_1. \end{aligned}$$

If we call J the ideal of $\mathbb{R}[x,y,z]$ generated by the polynomials $z^2, x^2 - y^2, x^2 - xz, y^2 - yz$, the relations above are sufficient to get the isomorphism

$$\Pi(Q) \cong \frac{\mathbb{R}[x,y,z]}{J}.$$

Recall from Chapter 1 the following theorem.

Theorem 19 (Upper Bound Theorem). *If P is a d-polytope with n vertices, for each integer k it has at most as many k-faces as the cyclic polytope $C_d(n)$:*

$$f_k(P) \leq f_k(C_d(n)).$$

Remark. We now see how the Upper Bound Theorem for polytopes follows smoothly from the results of this chapter. It is worth noting that this is not the most direct or fastest approach; for example, in [Zie94, §8] is presented the original proof of McMullen, that uses little more that the definition of h-vectors. Nonetheless, the following approach in intriguing as it exemplifies a connection between algebra and combinatorics. One might describe the following as a simplified version of Stanley's proof of the Upper Bound Theorem for simplicial spheres [Sta75].

Proof. Follows immediately from Theorem 18 that the h-vector of a simple polytope is an M-sequence, from their characterization of Lemma 1.31 we have some upper bounds on the possible h-vectors of simple polytopes: $h_{k+1}(P) \leq h_k^{\langle k \rangle}(P)$ for each $k \geq 1$. We want to end up with a simplicial polytope with n vertices so we start with a simple polytope with n facets, in this case we know that $h_1(P) = n - d = \binom{n-d}{1}$, therefore the bound on $h_2(P)$ is

$$h_2(P) \leq h_1(P)^{\langle 1 \rangle} = \binom{n-d+1}{2}$$

Similarly we get

$$h_3(P) \leq h_2(P)^{\langle 2 \rangle} \leq \left(\binom{n-d+1}{2} + \binom{0}{1} \right)^{\langle 2 \rangle} = \binom{n-d+2}{3},$$

and with an induction argument we have

$$h_k(P) \leq \binom{n-d-1+k}{k}.$$

These bounds are interesting only up to $k = \lfloor d/2 \rfloor$ since the h-vector of a simple polytope is a palindromic sequence. From our computations in Lemma 1.28, we see that a dual of the cyclic polytope $C_d(n)$ reaches the bound on all the coordinates of the h-vector, therefore for any simple d-polytope P with n facets and any integer $k \leq d$, $h_k(P) \leq h_k(C_d(n)^\dagger)$.
In conclusion, observe that the numbers $f_k(P)$ are obtained as linear combinations with positive coefficients of the numbers $h_k(P)$, since $f(P,t) = h(P,t+1)$, therefore an upper bound on the h-vector immediately yields an upper bound on the f-vector. By polarity we deduce that for convex d-polytopes with n vertices, the cyclic polytope with n vertices maximizes all the components of the f-vector . □

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