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The Magnetic Weak Gravity Conjecture, the Quantum Gravity cutoff and simple Type IIA compactifications

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Abstract

The magnetic weak gravity conjecture (MWGC) imposes constraints on the maximum value of the cutoff Λ associated with a gravitational Effective Field Theory (EFT). In its original formulation, the MWGC only included the gravitational and U(1) gauge interaction. One of the main goals of this thesis is to study the MWGC in the presence of scalar fields. We start by studying a general bosonic action describing a source interacting with three kinds of fields: scalar, gravitational, and U(1) gauge field. We continue looking for from the equations of motion and from them we obtain the so-called no-force condition, in the presence of these three interactions. We review the theory behind magnetic monopoles and from the aforementioned condition, by using the fact that the mass of a monopole can be estimated as $\frac{\Lambda}{e^2}$, where Λ can be identified as the EFT cutoff and e the electric coupling, we finally extract the generalization of the MWGC in the presence of scalar fields. Furthermore, we explore the interplay between the Distance Conjecture and the extension of the no-force condition. In the second part of the thesis, we review the theory behind compactifications, paving the way for a focused study of Type IIA compactification on a toroidal orbifold. We extract the spectrum of four-dimensional particles and strings arising from Dp-branes wrapping the corresponding cycles. The spectrum strictly depends on the Kähler moduli t^A. We apply the extension of the MWGC for both particles and strings in various large volume limits, and after a review and computation of the species scale, we compare these three cutoffs. We show that the smaller cutoff, which means the first one to affect our theory, is the one associated with particles arising from D6-branes wrapping 6-cycles; which corresponds to the heaviest magnetic monopole. In this context, we finally introduce and test the so-called Distant Axionic String Conjecture, which relates the mass m_* of the lightest tower near an infinite distance limit to the tension $\mathcal{T}$ of an axionic string that dynamically drives the moduli towards that limit. We examine the conjectured relation $m_*^2 \sim \mathcal{T}^w$ with $w = 1, 2, 3$. The conjecture states that w takes the integer values 1, 2 and 3. Our results show that these integer exponents are only recovered along certain paths but seem not to be a completely general property of all geodesics towards infinite distance limits.

Contents

Introduction

One of the greatest challenges of modern theoretical physics is the quantum physical description of the four fundamental interactions. We have managed to do this in the case of electromagnetic, strong, and weak interactions [1], but we find it extremely challenging to accommodate the principles of quantum physics within the gravitational interaction. Our current description is based on Albert Einstein's General Relativity [2]. One of the leading approaches attempting to unify all the fundamental interactions into a single framework, the String Theory, see e.g. [3]. In spite of its tremendous theoretical success, connecting String Theory with the observable universe in a concrete manner is known to be a challenge. In particular, String Theory is known to generally be consistent in more than four dimensions, and in order to connect it to the real world, some spatial dimensions usually need to be compactified. This results in a vast number of possible geometries for these compact dimensions, yielding an incredibly rich vacuum structure for the theory. In fact, for each different vacuum, we have an associated Effective Field Theory (EFT), which is a low-energy description of a particular solution, and altogether these constitute the so-called Landscape [4]. However, it is worth noting that as has been more recently understood, not all existing QFTs can be completed into Quantum Gravity (QG) in the ultraviolet (UV), which indicates their inconsistency as gravitational theories. It is important to remark that this inconsistency arises only when coupling such QFT to gravity, meaning it is not present at the level of the QFT itself. These inconsistent low-energy EFTs theories constitute what is known as the Swampland [5].

The second complication is related to the fact that direct observation of quantum gravitational effects is naively thought to only appear at extremely high-energy scales, of the order of the Planck scale ($\sim 10^18$ GeV), extremely higher than those currently available in high-energy particle accelerators. In this context, a complementary approach to the one obtained from specific top-down constructions is the one motivate by the Swampland Program (for recent reviews on the topic see [6, 7]). This new way of proceed offers a complementary new perspective in building such a connection: instead of starting from String Theory, compactifying it on some particular manifold, and then trying to match the EFT with the observable universe, this approach moves in a new direction. Inspired by the general lessons drawn from String Theory as a quantum theory of gravity, it tries to starts directly from the EFT side, constraining the space of such EFTs that can be consistently coupled to QG (a more detailed description is presented in chapter 1).

This thesis mainly aims to understand and study some of the building blocks of the Swampland Program, namely the Magnetic Weak Gravity Conjecture (MWGC) [8], the Species Scale [9–12], and the Distance Conjecture [13], together with the closely related Distant Axionic String Conjecture [14]. This work is mainly divided into two parts. In the first part, we will start by reviewing some relevant aspects of the current status of the Swampland program, analyze and motivate the aforementioned conjectures, and extend the MWGC in the presence of scalar fields. In the second part, we will test these results and the DASC in a particular string theory setup. Our results show some discrepancy with a particular aspect of the DASC, related to the integer coefficients relating the masses of towers and the tension of EFT strings. We will explain how these integral coefficients are recovered along particular geodesic trajectories but are not a general property of all infinite distance geodesic in the particular toroidal compactification under consideration.

The structure of this thesis is the following. The initial chapters of the thesis focus on the study of the key Swampland Conjectures and the extension of the MWGC in the presence of scalar fields. In Chapter 1, we begin by reviewing the WGC, with a particular focus on the magnetic part which imposes a bound on the cutoff of the EFT that we are studying from QG considerations. We then explore the various implications, introduce supporting arguments, and motivate the general interest in studying the MWGC. Specifically, our main goal is to extend the original formulation, which considers only the gravitational interaction and a U(1) gauge interaction, to account for the presence of scalar fields. This extension has already been studied in the case of the electric part of the WGC and then it is natural to try to understand how the magnetic version is affected by it, we do this in chapter 3. To do this, we introduce and review in detail the so-called repulsive force condition. Starting from this condition and using the scaling of the mass of a monopole, estimated from the energy stored in its own electromagnetic field $M_{\text{mon}} \sim \frac{\Lambda}{e^2}$ (where Λ is identified as the EFT cut off and e^2 is the electric gauge coupling), we obtain an upper bound for the cutoff, similar to the original MWGC. To achieve this goal, we first need to understand the interaction potentials by computing the propagators of various mediators and the vertex strengths. Once we have the potentials, we review the theory behind magnetic monopoles, focusing on the Dirac quantization condition and the relationship between mass and cutoff. With all these building blocks in place, we then proceed to construct the repulsive force condition and extract the final formula.

After obtaining the final extension, we apply it to a specific string theory setup and compare it with another important cutoff in the Swampland Program: the species scale.

This constitutes the second goal of the thesis, which we will pursue in the second part, beginning in chapter 5. Our specific setup is Type IIA superstring theory compactified on a toroidal orbifold down to a four-dimensional $\mathcal{N} = 2$ EFT. Before performing explicit calculations, we review the theory behind Calabi-Yau compactifications, understanding how the reduced action looks, which are the various multiplets, and what states constitute the BPS spectrum of particles and strings arising from Dp-branes and NS5-branes wrapping supersymmetric internal cycles. These classes of towers are used to compute the cutoff associated with the MWGC and the species scale, Λ_{s}, which is also introduced in some detail in the initial part of the thesis. Our analysis involves a systematic analysis of various large volume limits, which correspond to sending different Kähler moduli (on which the masses of these towers depend) to infinity along different geodesic trajectories. As we will see, the smallest cutoff responsible for the breakdown of our EFT comes from towers of particles generated by D6-branes wrapping internal 6-cycles, in all the analyzed cases.

The final part of the thesis, in particular chapter 6, is devoted to the Distant Axionic String Conjecture [15]. It consists of two parts: the first states that every infinite distance limit can be reached as an RG flow endpoint of an axionic string (this will be extensively explained and motivated in Chapter 6, and the second connects the mass scale of the lightest tower of particles along any possible infinite distance geodesic trajectory in Kähler moduli space to the tension of the axionic string responsible for this limit: $m_*^2 \simeq \mathcal{T}^w$. The interesting aspect for us is the second part, which also states that the exponent w can only take the integer values 1, 2, or 3. By taking various infinite distance limits and following geodesic paths, we find that in our particular setup, the exponents are not always the ones stated in the conjecture. In fact, as we will see, in the original paper they were working in the so-called strict asymptotic regime which does not capture all geodesics in our setup and this is at the core of the discrepancy that we find. It is worth noting that by selecting the particular ones which are associated to the aforementioned regime, indeed, the exponent takes the values stated in the conjecture.

Part I

The Swampland Program

Chapter 1

The relevant conjectures

The idea behind the Swampland Program, originally introduced in [5], is based on the importance of a gravitational theory's self-consistency and its power can be understood in String Theory. As we have seen in the introduction, the different EFTs can live in the Landscape but also in the Swampland then it is important at this point, to have a clear set of gravitational constraints and criteria to discern when an EFT lives in one or in the other place: this is the final goal of the Swampland Program. The construction of these constraints is based on the analysis of the QG landscape that we know (i.e. some parts of the ST landscape) by extracting common patterns amongst the different consistent EFTs and also in general bottom-up (usually more heuristic) arguments, like the ones involving BHs. A lot of the criteria that we use today, have not yet been proved from a microscopic perspective and this is why they are stated as conjectures. This because in order to prove something one needs a well-defined, rigorous framework, and we don't have such a thing for QG. However, they can be attempted to be proven when such a framework is well-defined, like in e.g. some particular ST setups or in AdS/CFT. In this section we present a review of three milestones of the Swampland Program: the No Global Symmetry Conjecture, the Distance Conjecture and the Weak Gravity Conjecture. In the purpose of this review I mostly took the material from [6], but see also [5] and [7] for a complementary point of view.

Let us start by analyzing the so called No Global Symmetries Conjecture (NGSC) [16,17]. It is not the main focus of the thesis, but it is extremely important to motivate the following part of the chapter; moreover the NGSC is the oldest, and better estabilished QG conjecture. It tells us that a theory with a finite number of states and which is coupled to gravity, can have no exact global symmetries. There are a lot of arguments supporting this conjecture and I will introduce one which uses black holes. Consider a four-dimensional Schwarzschild black hole, described by the following solution:

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})dr^2 + r^2 d\Omega^2, \tag{I.1.1}$$

where we have used spherical coordinates. The No Hair Theoreme [18] tells us that the metric in eq. I.1.9 is unique for uncharged stationary BHs. This means that if the BH is formed by throwing inside matter charged under a global $U(1)$ symmetry, this is not reflected into properties of the BH's horizon. And from a semi-classical perspective this is extremely dangerous. In fact, the BH will lose mass via Hawking radiation but it will not lose charge. This means that, in this theory, is impossible to know what the global charge of the BH, from the outside, is. This uncertainty can be associated to an infinite entropy violating the Bekenstein-Hawking entropy which is proportional to M^2. This result shows the problematic existence of global symmetries in Quantum Gravity. But it is worth noting that this violation is strictly connected to the spectrum of particles, in fact it can happen that the number of them with a mass below the cut off is still not too large and that would not give any problem.

Let us move the analysis on the first important conjecture for this work, the Weak Gravity Conjecture (WGC) [19]. This conjecture is composed by two parts: the electric WGC and the magnetic WGC. The idea behind the electric version is that whenever a U(1) gauge theory is coupled to gravity, there must exist at least one charged state which is self-repulsive. Which means that the repulsion experienced from the gauge interaction has to be stronger than the attraction coming from the gravitational one.

In the end the comparison between these two forces, taking two identical particles, reduces to the analysis of the relation between the gauge charge q and mass m. In particular, as the aforementioned conjecture states, we expect q/m (in Planck units) to be grater than some constant factor c that is extracted from the charge-to-mass ratio of large extremal black holes:

$$\frac{m}{q} \leq \lim_{Q\to\infty} \frac{M_{ext}(Q)}{Q} = c \tag{I.1.2}$$

The principal argument for this conjecture comes from the idea that (sub)- extremal black holes must be able to decay without becoming superextremal. Consider a massive charged extremal black hole which satisfies the extremality condition $M_{ext}(Q) = cQ$, with c the constant in eq. I.1.2. If the black hole is not BPS, there is no particular reason for it to be stable, therefore let us assume that it decays by emitting a particle of charge q and mass m. As a result of that, there is a new black hole with mass $M - m$ and charge $Q - q$ which must satisfies the extremality inequality $M - m \geq c(Q - q)$ and another state (a particle) which instead satisfies the following inequality $m \leq cq$, which is nothing but the Weak Gravity Conjecture.

Let us now focus on the WGC in the context of supersymmetric ST, where BPS states are present [20, 21]. The BPS bound [3]:

$$m \geq |Z(q)|\,, \tag{I.1.3}$$

tells that the mass has to be bigger or equal to the central charge. Comparing this with the EWGC, they seems to be in contrasts with each other, but what the conjecture is telling us is that must exist *some* states which satisfies the constraint while the BPS bound applies to *all* the states. This imply that some BPS states, the ones which satisfy the EWGC, must exactly saturate the bound in eq. I.1.3 and in this sense the conjecture can be understood as the statement that BPS states *must* exist in supersymmetric theories. It is worth noting that these states, not only satisfy the WGC but they exactly saturate it.

This understood, it easy to see how the EWGC is strictly related to the stability of these objects. Consider a state with charge q and mass m which decays into n lighter objects with charge and mass (q_i, m_i), requiring the usual conservation of both energy and charge we get:

$$\begin{aligned} &q_1 + ... + q_n = q, \\ &m_1 + ...m_n \leq m. \end{aligned} \tag{I.1.4}$$

Now, it is easy to find:

$$\frac{|q|}{m} \leq \frac{|\sum_i q_i|}{\sum_i m_i} \leq \frac{\sum_i |q_i|}{\sum_i m_i} \leq \max_i \frac{|q_i|}{m_i}. \tag{I.1.5}$$

Thus, there is always a particle in the decay (therefore more stable) which satisfies the EWGC better than the parent one. In fact, the BPS states which saturate the WGC are stable.

There is also another way in which eq. I.1.2 arise, namely by using the self-repulsive condition; strictly connected to the problem of the species as illustrated in the next part of the chapter. This condition clearly states that the sum of the forces acting on two identical particles has to be grater or equal than zero (repulsive force). Consider Einstein-Maxwell theory and the particle with the largest charge-to-mass ratio in the theory. If we imagine to have two particles of this type at distance r, the forces acting on them have the following forms:

$$\begin{aligned} F_{\text{gravity}} &= \frac{m^2}{8\pi M_p^2 r^2}, \\ F_{\text{gauge}} &= \frac{(gq)^2}{4\pi r^2}. \end{aligned} \tag{I.1.6}$$

By imposing that the self repulsion has to be grater than the self attraction, namely $F_{\text{gauge}} \geq F_{\text{gravitational}}$, we exactly get the EWGC. Then, let us suppose to violate the conjecture, what happens is that the two particles form a bound state. This state has a charge of $2q$ but a mass smaller than $2m$ (because of the gravitational potential); we just get a state with the charge-to-mass ratio bigger than the one at the beginning. Moreover, due to the conservation of energy and charge, the bound state cannot decay and therefore would be stable. But now, simply varying the number of constituent in it, I would get a huge number of different species N_s and this paves the way, as we will see, for possible problems. Then, it is clear how the non-violation of the WGC is crucial to avoid them.

The WGC not only contains an electric part but also exists the Magnetic version of the Weak Gravity Conjecture (MWGC).
This part of the conjecture states that the cutoff scale Λ of the effective field theory (introduced for the electric one) is bounded from above by the gauge coupling, in the following way:

$$\Lambda \lesssim g M_p. \tag{I.1.7}$$

What is written in eq. I.1.7 is strictly connected and motivated by the No Global Symmetry Conjecture. Consider a theory weakly coupled to gravity and a $U(1)$ gauge symmetry, with coupling g; by sending $g \to 0$ the differences with a global $U(1)$ symmetry are no longer present and we expect, due to the No Global Symmetry Conjecture, the theory to break down. The magnetic WGC exactly fulfill this goal; in fact as can be seen from eq. I.1.7, by sending $g \to 0$, $\Lambda \to 0$ making the theory useless.
Now that we have seen both the parts which constitute the WGC, let us try to connect them. This part is extremely important for the purpose of the thesis, since I will go through the same steps (in chapter 3) to obtain an extension of eq. I.1.7. Consider the no-force condition:

$$\frac{(gq)^2}{4\pi r^2} - \frac{m^2}{8\pi M_p^2 r^2} = 0. \tag{I.1.8}$$

Imagine that we apply this to a magnetic monopole with mass m (in chapter 2 there is a detailed review of them). It is known from the theory that the mass of these objects goes like $m \sim \frac{\Lambda}{e^2}$. At this point, by substituting the previous expression in eq. I.1.8 and solving for Λ, we exactly get the Magnetic Weak Gravity Conjecture in eq. I.1.7.

Let us now move to the other relevant conjecture in the thesis, the Distance Conjecture (DC) [13].
The different Effective Field Theories which arise from String Theory, can be explored moving in the so called moduli space $\mathcal{M}$. Moving in this space means varying the vev of some scalar fields ϕ^i, that do not have a potential. It can be interesting understand what happens when we move at the asymptotic regions of such a space; in fact, as one expects, in these limits some global symmetries are restored and this goes against the

No Global Symmetries Conjecture, one of the most important and tested conjectures in the Swampland Program. Consider a theory coupled to gravity, with a moduli space space $\mathcal{M}$ parametrized by some fields ϕ^i with no potential and with a metric g_{ij} (the one from the kinetic terms). The first statement of the conjecture is the following one: starting from a point $P \in \mathcal{M}$, there always exists a point $Q \in \mathcal{M}$ such that the geodesic distance, $d(P, Q)$, is infinite. The second statement tells instead that exists a tower of particles, with mass scale M, such that (in Planck units):

$$M(Q) \sim M(P)e^{-\alpha d(P,Q)}, \tag{I.1.9}$$

with α positive constant. In order to understand better the subtleties related to the conjecture let us consider a very general setup. Take the following action:

$$S = \int d^dx\sqrt{-g}\left[\frac{R}{2} - g_{ij}(\phi^i)\partial\phi^i\partial\phi^j. + ...\right] \tag{I.1.10}$$

From the kinetic term we get the metric g_{ij} on the moduli space and the index i is related to the dimension of the space. The case in which $g_{ij} = \delta_{ij}$ represent a flat moduli space. This understood, the geodesic distance between two points P and Q, in the moduli space, is:

$$d(P, Q) = \int_\gamma \left(g_{ij}\frac{\partial\phi^i}{\partial s}\frac{\partial\phi^j}{\partial s}\right)^{\frac{1}{2}} ds \tag{I.1.11}$$

with γ the shortest geodesic between the two point and ds the line element along the geodesic.

The first subtlety come from the first statement: given a point P exists a point Q at infinite distance in the moduli space. It seems not to be always true, in fact taking a periodic scalar we could have a moduli space of this type $\mathcal{M} = \mathcal{S}^1$. Then, what the conjecture tells us is that these kind of moduli are necessary included in a bigger moduli space. Regarding the second statement instead we cannot expect the exponential behaviour to be a general property of the moduli space but it is related to the behavior expected at the asymptotic regions of the moduli space.

The classical example, which shows how the Distance Conjecture is satisfied in String Theory, is the KK circle compactification to d-dimensions. Consider the following action:

$$S \supset M_p^{d-2}\int d^dx\sqrt{-h}\left(\frac{R}{2} - \frac{1}{2}\frac{d-1}{d-2}\frac{(\partial r)^2}{r^2}\right), \tag{I.1.12}$$

which is the Einstein-Hilbert term associated to a modulus r. The proper field distance, using the canonically normalized field, take this form:

$$\Delta R = \sqrt{\frac{d-1}{d-2}}\log r. \tag{I.1.13}$$

The infinite distance limits in moduli space can be reached at $r \to 0$ and $r \to \infty$. Taking the latter limit, the KK tower becomes light in the following way:

$$m_{\mathrm{KK}} = \frac{q}{r^{\frac{d-1}{d-2}}} = q e^{-\sqrt{\frac{d-1}{d-2}}\Delta R}, \tag{I.1.14}$$

exactly following what is predicted by the Distance Conjecture. Furthermore, taking the opposite limit $r \to 0$, we observe that the towers which are becoming light are the ones associated to the winding modes of the string. They follow the same exact behaviour of the KK tower in eq. I.1.16. It is worth noting that this is something intrinsic from string theory (i.e. extended objects), since without strings there is no winding tower and thus no tower as $r \to 0$ (therefore in field theory there is no need for infinite towers, but as soon as you couple it to gravity -via ST- the towers arise). Since we are in a supersymmetric setup (no potential) we are able to exactly compute the exponential rate in terms of the dimensions of the EFT:

$$\alpha = \sqrt{\frac{d-1}{d-2}}. \tag{I.1.15}$$

It is worth noting that the DC is strictly related to the WGC. In particular, the No Global Symmetry Conjecture, which was used to support the Weak Gravity Conjecture, can also be used to support the Distance Conjecture and directly connect them. We have seen that the global symmetries seem to be restored at infinite distance in moduli space. In [22] the opposite was proposed (every time that we are in the asymptotic regions of the moduli space, we restore a global symmetry), but not proven in general. And if we assume this to be true, we expect that as we reach infinite distance points in the moduli space $g \to 0$, and also $m \to 0$ due to the EWGC; but then, this tower becomes a candidate for the one satisfying the DC which is clearly another way to protect a theory from restoring a global symmetry.

Let us finally introduce one of the most important scale in the Program, the Species Scale Λ_{s} [9–12]. It is extremely relevant for the aforementioned conjectures since it acts as the upper bound for the UV cutoff in gravitational EFTs. Consider a d-dimensional EFT, coupled to gravity, with a Planck mass M_p^d and with N_{s} species under the cutoff. Within any weakly coupled regime exists a bound on the maximum value of the cutoff given by:

$$\Lambda < \Lambda_{\mathrm{s}} \equiv \frac{M_{P,d}}{N_{\mathrm{s}}^{\frac{1}{d-2}}}. \tag{I.1.16}$$

In order to understand better the conjecture, let us consider a D-dimensional theory (UV completed theory) with a Planck mass $M_{p,D}$ and the EFT arising from compactifying it along a periodic and compact direction which dimensionless angular coordinate satisfies

$X_d \sim X_d+1$; it is a d-dimensional theory ($d = D-1$) with a Planck mass $M_{p,d}$. Reducing the D-dimensional metric:

$$ds^2 = g_{\mu\nu}dX^\mu dX^\nu + (2\pi R)^2(dX^d)^2 \tag{I.1.17}$$

we get the following relation between the Planck masses:

$$(M_{p,d})^{d-2} = (M_{p,D})^{D-2}2\pi R \tag{I.1.18}$$

which tells us that in d-dimensional Planck units $M_{p,D} \sim R^{\frac{1}{2-D}}$. Using that, in this frame, the KK scale goes like $m_{\mathrm{KK}} \sim \frac{1}{R}$, the number of states present before the quantum gravity cutoff $M_{p,D}$ are equal to:

$$N_s \sim \frac{M_{p,D}}{m_{\mathrm{KK}}} \sim R^{\frac{d-2}{D-2}}. \tag{I.1.19}$$

From which we extract that the quantum gravity cutoff in terms of N_{s}:

$$M_{p,D} \sim \frac{1}{N_{\mathrm{s}}^{\frac{1}{d-2}}}. \tag{I.1.20}$$

And it represents the scale at which the gravity becomes strongly coupled. But then, for a large number of species ($R \gg 1$), can happen that $M_{p,D} < M_{p,d}$ and gravity becomes strongly coupled at a lower scale than one would expect. In this explanation KK modes were used, but it holds in general (i.e. with string oscillator towers).

As mentioned in the previous discussions regarding the conjectures, a large number of species could be, in particular cases, problematic. Let us understand what are the issues that could arise. In order to do that, it is important to introduce the so called Bousso Bound [23, 24], in flat space time it restricts the entropy S inside a sphere of radius R in the following way:

$$S \leq \frac{A(R)M_P^{2-D}}{4}. \tag{I.1.21}$$

Strictly related to this bound there is the so called Bekenstein bound [25–27], which constraints the entropy S in the following way:

$$S \leq 2\pi ER, \tag{I.1.22}$$

where E is the total energy inside a sphere of radius R. It is worth noting that eq. I.1.22 does not include M_p (gravity). Consider the presence of a number N_s of species in our theory and let us couple eq. I.1.22 to gravity. In this way we can directly compare the Bousso bound in eq. I.1.21 and the Bekenstein bound in eq. I.1.22; and we require the mass inside the sphere not to collapse to a black hole $2E \leq M_P^{d-2}R^{d-3}$. Following the analysis in [24], we can associate, at the different species, a temperature T such that

$E \approx N_s R^3 T^4$ and $S \approx N_s R^3 T^3$. Using the constraint for the gravitational stability and equation I.1.21, we get:

$$N_s \lesssim M_P^2 R^2. \tag{I.1.23}$$

From this relation it is clear how a large number of species could violate the bound. It is worth noting, again, that if the number of particles with mass below the cut off is not too large, the problem is not present.

Now that the stage is set, let us understand the goals of the thesis. As we have seen in this section, the no-force condition is strictly related to the WGC. In fact, by imposing that self-repulsion must win over the self-attraction, we get the aforementioned conjecture. Moreover, starting from the electric WGC, we have seen how relating it to a magnetic monopole, we get the so called magnetic WGC. In the previous discussion we only considered the presence of two interactions, the gravitational one and the gauge one. Then, the question that automatically arise is: how does the WGC changes in the presence of other types of interactions? The first goal of the thesis is exactly this one: understand how the conjecture (more specifically the magnetic part) changes under the introduction of a scalar-mediated interaction. We have also seen how in QG different types of cutoffs exist: the WGC one and the species scale. Then, it might be interesting to understand the interplay between these cutoffs, with a particular attention on which one of the two is smaller (the WGC cutoff is extracted by the extended case with scalars). Finally, due to the connection that we have seen exists between the DC and the WGC, could be interesting to understand the interplay and the possible implications that one have on the other.

The next chapter (2) aims to set the stage to build the aforementioned extension; in chapter 3 we explicitly compute it and we analyze the relations between the DC and the WGC. Finally, in the second part of the thesis (II), through a direct test in String Theory, we compare the behaviour of the two different cutoffs.

Chapter 2

Building the tools for the extension

In the previous section we understood what are the interesting conjectures for this work, what motivates them and the direction that the thesis aims to take. Now it is time to go deep in the topic, trying to analyze the different building blocks which will help us constituting the extension, with a particular attention on the magnetic monopoles. And only after that, the ground will be ready for the introduction of scalar fields in the WGC. To carry on the following analysis I mainly took the material from [28]. Let us start by introducing a very general action which is constituted by a part associated to a massive boson in slowly varying background fields and a part associated to these fields:

$$S = -\int M(\phi) ds - Q\int A + \int d^4x\sqrt{-g}(\frac{1}{2k_D^2}R - \\ - \frac{1}{2}G_{\phi_i\phi_j}\partial_\mu\phi^i\partial^\mu\phi^j - \frac{1}{4e(\phi)^2}F_{\mu\nu}F^{\mu\nu}). \tag{I.2.1}$$

As can be seen, the boson interacts with a gauge field described by the field strength $F_{\mu\nu}$, interacts with the gravitational field through the Einstein-Hilbert term and finally interacts with complex scalar fields ϕ^i (interaction that will emerge through the mass dependence $M(\phi)$ and the gauge coupling $e(\phi)$). The first step is to extract the strength of the various interactions; this is important in order to understand what are the terms that we must include in the no-force condition. It can be done through an analysis of the propagators and potentials of interaction. Let us start with the propagators.

2.1 Propagators

The first propagator that we analyze is the scalar one. In general, the propagator is defined as the inverse of the kinetic term, and we will use this definition for the derivation in the three cases. In particular, for the scalar case we will go through all the steps, but for the other cases the explicit computations can be found in A. The term that we need to invert arise from the following part of the Lagrangian:

$$\mathcal{L} = \frac{1}{2} G_{\phi_i\phi_j} \partial_\mu \phi^i \partial^\mu \phi^j = \frac{1}{2} G_{\phi_i\phi_j} \phi^i \partial_\mu \partial^\mu \phi^j. \tag{I.2.2}$$

This means that the equation to solve is:

$$G_{\phi_i\phi_j} \partial_\mu \partial^\mu G(x-y) = \delta^3(x-y). \tag{I.2.3}$$

The general way to get $G(x-y)$ is, at first, move in the Fourier space and solve the equation, obtaining:

$$\tilde{G}(\vec{k}) = -\frac{G^{\phi_i\phi_j}}{\vec{k}^2}, \tag{I.2.4}$$

where $\vec{k}^2$ is the modulus square of the scalar's momentum. And then take the inverse Fourier transform, finally getting the propagator:

$$< \phi^i \phi^j >= -i\tilde{G}(\vec{k}). \tag{I.2.5}$$

Now let us move to the second propagator, the gauge field propagator. The complete computations can be found in A. In this case the kinetic term that has to be inverted, as can be seen from I.2.1, is the following one:

$$-\frac{1}{e^2}(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) D_{\nu\lambda}(\vec{x}-\vec{y}) = \delta^\mu_\nu \delta^3(\vec{x}-\vec{y}). \tag{I.2.6}$$

Then going in the Fourier space, using a particular ansatz and then taking the inverse Fourier transformation, we get the following gauge propagator:

$$< A_\mu A_\nu >= -\frac{ie^2}{\vec{k}^2} g_{\mu\nu}, \tag{I.2.7}$$

where $\vec{k}^2$ is the modulus square of the gauge field's momentum.

Finally, let us move to the last propagator, the graviton's one. As the previous case the full computations can be found in A. This is the most complex to obtain. In fact, starting from the Einstein-Hilbert term in I.2.1, we need at first to expand the metric around the flat one. Then, expanding also the action we extract the part containing the quadratic contribution of the graviton II.A.8. And finally, using the fact that this action enjoys the so called BRST symmetry we obtain the propagator:

$$\begin{aligned} < h_{\mu\nu} h_{\rho\sigma} > &= \frac{-iP_{\mu\nu,\rho\sigma}}{\vec{k}^2}, \\ P_{\mu\nu,\alpha\beta} &= (\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\nu\alpha}\eta_{\mu\beta}) - \eta_{\mu\nu}\eta_{\alpha\beta}. \end{aligned} \tag{I.2.8}$$

and again, $\vec{k}^2$ represent the square of the graviton's momentum. Second necessary step to construct the no-force condition is to compute the different potentials of interaction. In our case, the particle experiences three different interactions.

2.2 Interaction potentials

The potentials are extremely important in understanding whether or not we must include an interaction in the no-force condition for the boson: they give us the strength of the vertex. In order to extract them, we must find the equations of motion and solve them. The solutions are directly connected to the vertexes. To do that, we at first linearize the action I.2.1 around a background $g_{\mu\nu} = \eta_{\mu\nu}$ and $\phi = \phi_0$, then we compute the equations of motion in the usual way. In these computations we always consider the particle to be static in the origin, namely $x^i = 0$. Let us start with the scalar contribution. As for the propagators, we will go through all the computations for the scalar case, while the other two cases are explained in details in B. This contribution is extremely interesting and, as we will see, is composed by two parts:

- **Mass-term contribution**: The first term in the action which give us a contribution is the mass one. In fact, after having linearized it around the aforementioned background, we have:

$$-\int M(\phi)ds = -\int M(\phi_0)ds - \int \partial_\phi M|_{\phi_0}\delta\phi ds. \tag{I.2.9}$$

From I.2.9 and from the scalar kinetic term in I.2.1, the following equation of motion emerge:

$$G_{\phi\phi}\partial^2\phi(\vec{x}) = M'(\phi_0)\delta^3(\vec{x}), \tag{I.2.10}$$

taking the Fourier transform of ϕ and substituting inside I.2.10, we get:

$$\tilde{\phi}(\vec{k}) = -\frac{M'(\phi_0)G^{\phi\phi}}{\vec{k}^2}. \tag{I.2.11}$$

Applying the inverse Fourier transform (using spherical coordinates), we get the following integral:

$$\phi(\vec{x}) = -\frac{M'(\phi_0)G^{\phi\phi}}{(2\pi)^3}\int_0^\infty \frac{k^2dk}{k^2}\int_0^\pi sin\theta d\theta\int_0^{2\pi} d\phi e^{ikrcos\theta}, \tag{I.2.12}$$

from which the potential emerges:

$$V_{\text{scalar}} = -\frac{G^{\phi\phi}M'^2(\phi_0)}{4\pi r}. \tag{I.2.13}$$

In d-dimensions, using the fact that the potential goes like r^{3-d} and that:

$$\begin{aligned}\partial_i\partial^i\frac{1}{r^{d-3}} &= -(d-3)V_{d-2}\delta^{(d-1)}(x_i),\\ V_{d-2} &= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})},\end{aligned} \tag{I.2.14}$$

I.2.13 can be generalized in this way:

$$V^1_{\text{scalar}} = -\frac{M'^2(\phi_0) G^{\phi\phi}}{(d-3)V_{d-2} r^{d-3}}, \tag{I.2.15}$$

where V_{d-2} is the volume of a $(d-2)$-dimensional sphere.

- **Gauge coupling contribution**: In this case the part interesting for us is the gauge field kinetic term; after having expanded the gauge coupling around ϕ_0 in the following way:

$$\frac{1}{e^2(\phi)} = \frac{1}{e^2(\phi_0)} + \frac{2e'(\phi_0)}{e^3(\phi_0)}\delta\phi, \tag{I.2.16}$$

  this contribution emerges:

$$G_{\phi\phi}\partial^2\phi(\vec{x}) = \frac{e'(\phi_0)}{2e^3(\phi_0)} F_{\mu\nu}F^{\mu\nu}. \tag{I.2.17}$$

  This interaction term does not involve the boson but it is describing the gauge field self-interaction mediated by a scalar. Then, it does not directly affect the particle self-interaction at three level but enters in the loop corrections. And the strength of the vertex scalar-gauge field (identified by χ) is the following one:

$$\chi \sim \frac{e'(\phi_0)}{2e^3(\phi_0)}. \tag{I.2.18}$$

  Since we are interesting in the three-level no-force condition we will not take into account this contribution.

Now let us move to the gauge interaction, where a detailed computation can be found again in appendix B. The equations of motion for the gauge field is the following one:

$$\frac{1}{e^2}\partial_\nu F^{\mu\nu} = Q\delta^3(\vec{x}). \tag{I.2.19}$$

We use the fact that $A = \Phi dt$, where $F = dA$; then, going through the same steps seen for the scalar case, we get the following potential of interaction (in d-dimensions):

$$V_{\text{gauge}} = \frac{e^2 Q^2}{(d-3)V_{d-2} r^{d-3}}. \tag{I.2.20}$$

What remains to do, is to compute the gravitational potential. The equations of motion for the graviton, in the Lorentz gauge $\partial_\mu \bar{h}^{\mu\nu} = 0$, are the following ones:

$$-\frac{1}{2}\partial^2 \bar{h}^{\mu\nu} = k_D^2 M(\phi_0)\delta^3(\vec{x})\delta^{\mu 0}\delta^{\nu 0}, \tag{I.2.21}$$

where $\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$. Then using the same procedure as before (detailed calculation can be found in B), we get (in d-dimensions):

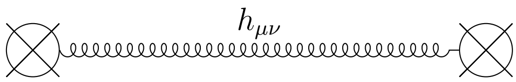
$$V_{\text{gravitational}} = -\frac{k_D^2 M^2(\phi_0)}{(d-3)V_{d-2}r^{d-3}}. \tag{I.2.22}$$

Now that all the building blocks are present let us see whether or not all of them will contribute in the construction of the no-force condition. We have to keep into account that we are considering magnetic monopoles; and in order to do that we anticipate an important result that we will get in the next section, which is the following one $e \sim \frac{1}{g}$, where g is the magnetic coupling. And it is worth noting that in this case, the gauge field propagator is proportional to $g^2 \sim e^{-2}$.

- **Graviton exchange**
  The contribution of this interaction (defined by $\mathcal{A}_{\text{gravitational}}$), considering that the strength of the vertex is the one in the right hand side of eq. II.B.7 and that the propagator of the graviton is the one in I.2.8, is the following one:

$$\mathcal{A}_{\text{gravitational}} = \frac{k_D^2 M^2}{\vec{k}^2} \tag{I.2.23}$$

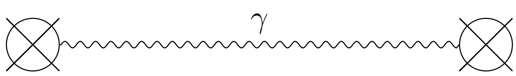
  And the following one is the Feynman diagram which describes the interaction:



- **Gauge field exchange**
  This contribution (defined by $\mathcal{A}_{\text{gauge}}$), considering that the strength of the vertex is the one in the right hand side of eq. II.B.1 and that the propagator of the gauge field is the one in I.2.7, is the following one:

$$\mathcal{A}_{\text{gauge}} = \frac{Q^2}{e^2\vec{k}^2} \tag{I.2.24}$$

  And the following one is the Feynman diagram which describes the interaction:

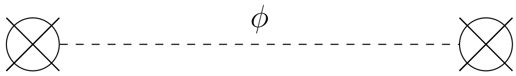


- **Scalar field exchange**
  This contribution (defined by $\mathcal{A}_{\text{scalar}}$), considering that the strength of the vertex

is the one in the right hand side of eq. I.2.10 and that the propagator of the scalar is the one in I.2.5, is the following one:

$$\mathcal{A}_{\text{scalar}} = \frac{G^{\phi\phi}M'^2}{\vec{k}^2} \tag{I.2.25}$$

And the following one is the Feynman diagram which describes the interaction:



Now, all the needed terms are present and we are ready to proceed in the computation of the no-force condition at three-level; we will do it in chapter 3 generalizing eq. I.1.8. But first, we need to know better the objects involved, the magnetic monopoles.

2.3 The magnetic monopoles

The large amount of the topics treated in this discussion come from [29].
A magnetic monopole is an object characterised by the emission of a magnetic field of this form:

$$\mathbf{B} = \frac{g\hat{\mathbf{r}}}{4\pi r^2}, \tag{I.2.26}$$

where g is the magnetic charge. Analyzing the Maxwell equations it seems that such an object cannot exists, in fact:

$$\nabla \cdot \mathbf{B} = 0. \tag{I.2.27}$$

And since the magnetic field is defined in terms of the gauge potential $\mathbf{A}$, such that $\mathbf{B} = \nabla \times \mathbf{A}$, I.2.26 cannot be different from zero. Moreover, $\mathbf{A}$ is necessary since every time that we want to describe the quantum physics of a particle which move in magnetic fields we need to introduce it. But Dirac discovered something that totally changed our view about the existence of these objects: in fact, there exists an ambiguity in how we define the gauge potential. Let us analyze how this fact introduce a new perspective regarding magnetic monopoles. Suppose to have a particle, of electric charge e, which moves in a background field generated by a magnetic monopole; and imagine that the particle moves along a closed path C. When the particle returns to its original position, the wave function ψ , which describe the particle, will get a phase:

$$\begin{aligned} &\psi \to e^{ie\omega/\hbar}\psi, \\ &\alpha = \int_C \mathbf{A}d\mathbf{x} = \int_S \mathbf{B}d\mathbf{s}, \end{aligned} \tag{I.2.28}$$

where S is the surface of the region included inside the path C. Now, imagine S cover a solid angle Ω, using $\mathbf{B}$=g, we get:

$$\alpha = \frac{g\Omega}{4\pi}. \tag{I.2.29}$$

But there is an ambiguity in this computation, in fact, we could also integrate on S', the other region that we can consider to be included in C (the one which together with S cover the 4π solid angle around the magnetic monopole), in this case we would get:

$$\alpha = \frac{(4\pi - \Omega)g}{4\pi}. \tag{I.2.30}$$

The phase shift that we get is an observable, and then, we need that the final wave function is the same in the two cases: $e^{ie\alpha/\hbar} = e^{ie\alpha'/\hbar}$. This imply that:

$$eg = 2\pi\hbar n \qquad n \in \mathbf{Z}. \tag{I.2.31}$$

And this is the famous Dirac quantisation condition. This important result tells us that the existence of the magnetic monopoles imply the quantisation of the magnetic charge. But this important result, by itself, does not solve explicitly the question about the existence of these objects. Let us see how to motivate it. Our goal is to find a configuration which lies at the origin and which give arise to $\mathbf{B}$ in I.2.26. First thing to clarify is that we cannot require the gauge field to be well-defined at the origin. This allows us to write down a gauge field on $R^3\backslash\{0\}$ instead of R^3: the non trivial topology on the origin is fundamental for the final result. Consider the following potential (in spherical coordinates):

$$A_\phi^N = \frac{g^2}{4\pi r}\frac{1-\cos\theta}{\sin\theta}, \tag{I.2.32}$$

substituting it in $\mathbf{B} = \nabla \times \mathbf{A}$, we get the magnetic field in I.2.26, which is what we were looking for. But how is it possible? It is possible because $\mathbf{A}^N$ is not only singular at the origin, but it is singular along a line (for $\theta = \pi$) which extends from the origin to infinity. Then, the connection is not well-defined along the south pole line but is fine elsewhere. But we might as well proceed in a different way by defining a new type of connection, of this form:

$$A_\phi^S = -\frac{g}{4\pi r}\frac{1+\cos\theta}{\sin\theta}, \tag{I.2.33}$$

and this give arise to the same magnetic field as before, the one in I.2.26. Now the potential is singular for $\theta = 0$. And this is exactly the aforementioned ambiguity of the gauge connection. At this point, in order to have such a $\mathbf{B}$, we can proceed in the following way: we associate $\mathbf{A}^N$ at the northern hemisphere while $\mathbf{A}^S$ at the southern hemisphere. This can be done since the two gauge potentials are the same, up to a gauge transformation:

$$\begin{aligned} A_\phi^N &= A_\phi^S + \frac{1}{r\sin\theta}\partial_\phi\omega, \\ \omega &= \frac{g\phi}{2\pi}. \end{aligned} \tag{I.2.34}$$

This understood, let us introduce the last block concerning magnetic monopoles: their mass. In order to compute it, we use the fact that it should be at least equal to the energy stored in the electromagnetic field, and we neglect the gravitational backreaction. Then, this is the known result:

$$M_{\text{mon}} = \int_V d^3x \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2), \tag{I.2.35}$$

where V is the volume of a sphere with radius R, which include almost the total amount of energy stored in the electromagnetic field. Then, in our case, considering the magnetic filed in I.2.26 and the absence of the electric field, $\mathbf{E} = 0$, we get:

$$M_{\text{mon}} = \frac{1}{2} \int_R^\infty r^2 dr \int_0^\pi d\theta sin\theta \int_0^{2\pi} d\phi \frac{g^2}{16\pi^2 r^4} = \frac{g^2}{8\pi R}. \tag{I.2.36}$$

And this mass, being related to the total energy stored in the electromagnetic field, tells us what is the cutoff of our theory: $M_{mon} \sim \frac{\Lambda}{e^2}$. This is another important relation that we will use to achieve the goals of the thesis and that we have already used to get I.1.8 (magnetic WGC). We finally have all the ingredients to compute the extension of the WGC, that we will find and analyze in the following chapter 3.

Chapter 3

Extending the WGC and the relation with the DC

In the previous sections we understood what are the goals of the thesis, in particular the one to extend the WGC with a scalar mediated interaction. In order to do that, we computed the propagators of the various mediators and the different vertexes strength. This was necessary to understand how to properly extend the no-force condition. Now that we have all the ingredients we can extend it. At first, we will generalize eq. I.1.8 and then using the monopole mass dependence on the cutoff, we can finally obtain the extended magnetic WGC. Therefore, using the forces associated to the potentials in eq. I.2.20 (where e^2 is substituted by g^2 since we are considering magnetic monopoles), I.2.15 and I.2.22, we can finally generalize the eq. I.1.8 in the following way:

$$\frac{g^2(\phi_0)Q^2}{(d-3)V_{d-2}r^{d-3}} - \frac{G^{\phi\phi}M'^2_{\text{mon}}(\phi_0)}{(d-3)V_{d-2}r^{d-3}} - \frac{k_D^2 M^2_{\text{mon}}(\phi_0)}{(d-3)V_{d-2}r^{d-3}} = 0. \tag{I.3.1}$$

And this equation simply reduces to:

$$g^2(\phi_0)Q^2 - G^{\phi\phi}M'^2_{\text{mon}}(\phi_0) - k_D^2 M^2_{\text{mon}}(\phi_0) = 0. \tag{I.3.2}$$

What remains to do, like we have seen in 1, is to write the mass in term of the cutoff of the theory, $M_{mon} = \frac{\Lambda_{\text{WGC}}}{e^2}$, and then solve for Λ_{WGC}:

$$\Lambda^2_{\text{WGC}} = \frac{e^2(\phi_0)Q^2}{k_D^2} - \frac{e^4(\phi_0)G^{\phi\phi}M'^2_{\text{mon}}}{k_D^2}. \tag{I.3.3}$$

What we just got is nothing but the extension of the magnetic WGC in the presence of scalar fields. Of course, it changes the value of the cutoff with respect to the one in I.1.8; it is interesting, as already remarked, understand how the new cutoff interplay with the species scale (analysis that we will do in the the second part of the thesis II).

Let us now move to the other interesting thing: understand how the extension I.3.2 and the DC influence each other and what are the useful information that arise from this connection. We have seen how the distance conjecture introduces an exponential behaviour in the masses of the towers, in the asymptotic regions of the moduli space. And this can be seen as a protection toward restoring global symmetries (as explained in chapter 1). This conjecture, as already remarked, give us information only in these particular regions of the moduli space and does not hold in general in it. In this thesis, for our analysis, we will take into account (in the second part II) the so called large volume limits, and this is why we are extremely interested in the effects that the DC could have: in these limits the DC holds. There are two ways in which we can carry this analysis:

- To assume the DC and see what are the implications, by using eq. I.3.2.
- To not assume the DC and see whether or not we can make it arise in some way.

Let us start with the first way of proceeding. Consider a local patch at infinite distance in the moduli space. We are working with the canonical variables $\tilde{t}^i$, which are the ones associated to the moduli t^i, which parametrize the aforementioned space. For simplicity let us phrase the moduli dependence in the following way $t_1 = t_2 = t_3 = t$ which implies, $\tilde{t}_1 = \tilde{t}_2 = \tilde{t}_3 = \tilde{t}$. Using the canonical variable, the expression of the no-force condition I.3.2, is the following one:

$$g^2(\tilde{t}) - \delta^{ij} M_i'(\tilde{t}) M_j'(\tilde{t}) - k_D^2 M^2(\tilde{t}) = 0. \tag{I.3.4}$$

Now, we use the fact that $M(\tilde{t})$ is associated to a KK tower. Then the mass, as the DC suggests, behaves in the following way:

$$M(\tilde{t}) \sim e^{-\alpha \Delta \tilde{t}}, \tag{I.3.5}$$

where α is a constant factor. Given M, it is trivial to derive M'; which, as one would expect, has the same exponential behaviour:

$$M' \sim -\alpha M. \tag{I.3.6}$$

Now that we have computed the last two terms in I.3.4, we see how both of them follow an exponential behaviour. The fact that we have an equality in I.3.4, that must be exactly satisfied in the region under analysis, forces the gauge term to have the same exponential behaviour. And this is the first interesting result that we get. But we can do more. Let us extract the moduli dependence of $g(\tilde{t})$, in the following way:

$$g(\tilde{t}) = \hat{g} e^{-\alpha \Delta \tilde{t}}. \tag{I.3.7}$$

Now, simply substituting eq. I.3.5, I.3.6, and I.3.7 in eq. I.3.4 and solving for the gauge coupling, we get:

$$\hat{g}^2 = \alpha^2 - 1. \tag{I.3.8}$$

And this is the second and last interesting information that we get by using this approach. It imposes a bound on the gauge term which strictly depends on the value of the DC coefficent α.

Let us now move to the second approach, the one which does not assumes the DC as a starting point. In this discussion plays again an important role the fact that we have an equality, more specifically I.3.4, that must be saturated everywhere in the region under analysis (at infinite distance limit in the moduli space). In fact, as we will see, it will shed light on particular characteristics of the exponential behaviour. Without assuming anything, the mass term could have different behaviours, so let us study them:

- **Polynomial/sub-exponential behaviour of M**:
  in this first case, since we have a polynomial mass, the first derivative of it will be polynomial as well; what changes between the two terms is the degree, which in fact is smaller for M'. Due to this fact, the scalar contribution (proportional to M') can be neglected. What remain in the end are the gauge term and the mass term. Due to the saturation of the equality I.3.4, the gauge term is forced to be again polynomial, exactly like the mass M.

- **Super-exponential behaviour of M**:
  in this second case, since the mass is super-exponential, the first derivative of it will be again super-exponential; but differently from the previous case, is now the M' term which dominates. But at this point is the mass term that can be neglected. What remain are the gauge term and the scalar term. Again, due to the saturation of eq. I.3.4, the gauge term is forced to be super-exponential, in this case like M'.

- **Exponential behaviour of M**:
  this last case, is particularly interesting. Since the mass has an exponential behaviour, the first derivative, as we have seen in I.3.6, will be exponential as well; but now the peculiarity is that differently from the previous cases, neither the mass term nor the scalar term dominates on the other and then both cannot be neglected. In the end, we remain with all the three terms. And, due to the saturation of I.3.4, the gauge term is forced, from M and M', to be exponential too.

After this analysis, we can state that the only case in which all the three terms in I.3.4 contribute in the same way, and no one of them is negligible, is the case in which the mass has an exponential behaviour.

Let us summarize what we have done so far in this first part of the thesis, which ends here. We achieved two of the goals of the thesis: the generalization of the WGC, in the presence of scalar fields, and we tried to connect it with the DC. Now let us move to the second part which will make an extensive use of I.3.3. In fact, it aims to apply it in particular String Theory setups with particular attention at the comparison with the other important cutoff in the Swampland Program, the species scale.

Part II

Test in String Theory

In this part of the thesis, as already remarked, there will be an extensive use of the magnetic WGC I.3.3. This law tells us the value of the cutoff (Λ_{WGC}) of an effective field theory, in which we consider the following mediators: a graviton, a gauge field and a scalar. But in the Swampland Program is present another very important cutoff: the species scale Λ_s; it tells us the scale at which gravity becomes strongly coupled. The question that automatically arise from this fact is: what is the cutoff that I have to take into account? This is the question that we want to answer in the following chapters. In order to have explicit results to consider, we are going to analyze a specific setup in String Theory: the EFT arising from Type IIA superstring compactified on a Calabi-Yau threefold (in particular the toroidal orbifold $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2'$). The complete analysis will go through different steps. At first, in chapter 4, there will be a review on Type IIA compactified on toroidal orbifold, with particular attention at the towers of particles and strings arising from the compactification of wrapped branes. And then, in chapter 5 there will the computation of both Λ_{WGC} and Λ_s for the aforementioned towers, with a final comparison. Finally, in chapter 6, there will be a test of the so called Distant Axionic String Conjecture [15]. In order to do that, we will need the mass scale of the different towers of particles and the tension of the strings, things that will be computed in chapter 4.

Chapter 4

Type IIA compactified on a toroidal orbifold

Let us start to analyze the theory behind the compactification of type IIA superstring on a Calabi-Yau threefold, mainly following [30–32]. In the discussion we will not turn on fluxes since we are interested in studying the case in which we can move freely in moduli space, without a scalar potential, for complete reviews on that see [33, 34]. At first, it is important to understand the properties of the manifold on which we are compactifying. We know that String Theory is formulated in ten dimensions which we can decompose in this way: $\mathcal{M}_{10} = \mathbb{M}^{3,1} \times Y$. Y is the compact manifold on which we are going to compactify. Since we decompose the ten-dimensional space in this way, we have that the Lorentz group decomposes as well, $S = (1, 9) \to SO(1, 3) \times SO(6)$. In our case, we demand Y to preserve the minimal amount of supersymmetry and this forces us to pick a manifold with structure group $SU(3)$. This kind of manifolds admit a globally defined spinor η with the characteristic to be covariantly constant; this force the manifold Y to have an $SU(3)$ holonomy [35, 36]. The spaces, with such characteristics, are called Calabi-Yau manifolds and are complex Kähler manifolds which are also Ricci flat [37]. Using η one can define a covariantly constant two-form J and a three-form Ω, which satisfy the following conditions:

$$J \wedge J \wedge J \propto \Omega \wedge \bar{\Omega}, \quad J \wedge \Omega = 0. \tag{II.4.1}$$

In our particular case the manifold is the so-called toroidal orbifold $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2'$; the objects that we are going to introduce, describe Calabi-Yau manifolds from a general perspective, then to understand exactly how these objects look like for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2'$ you can see section 4.1.

Now let us move the the compactification. When we will perform a KK reduction on a background, the massless four-dimensional fields arise as the zero modes of the following

Laplacian [20, 21]:

$$\Delta_6 \Phi(x, y) = 0. \tag{II.4.2}$$

These zero modes are in one-to-one correspondence with the harmonic forms on Y; then, the dimensions of the various cohomologies are related to the multiplicity of these fields. The only non-vanishing cohomologies of a Calabi-Yau are the following one:

$$\begin{aligned} H^{even} &= H^{(0,0)} \oplus H^{(1,1)} \oplus H^{(2,2)} \oplus H^{(3,3)}, \\ H^{odd} &= H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)} \oplus H^{(0,3)}, \end{aligned} \tag{II.4.3}$$

with dimension identified by $h^{(p,q)}$: $h^{(0,0)} = h^{(3,3)} = h^{(3,0)} = h^{(0,3)} = 1$, $h^{(1,1)} = h^{(2,2)}$ and $h^{(2,1)} = h^{(1,2)}$. In particular since we are considering a toroidal orbifold we have $h^{(1,1)} = 51$ and $h^{(2,1)} = 3$. Let us see what are the different basis for the cohomology groups. We have ω^A and $\tilde{\omega}^A$ respectively for the dual spaces $H^{(1,1)}$ and $H^{(2,2)}$. Then, $(\alpha_{\hat{K}}, \beta^{\hat{L}})$ is the symplectic basis on $H^{(3)}$. Where the only non vanishing intersection numbers are the following ones:

$$\int_Y \omega_A \wedge \tilde{\omega}^B = \delta_A^B, \quad \int_Y \alpha_{\hat{K}} \wedge \alpha^{\hat{L}} = \delta_{\hat{A}}^{\hat{B}}. \tag{II.4.4}$$

And finally the harmonic three-forms are written in term of the harmonic volume vol(Y). As remarked, these harmonic forms are strictly related to the four-dimensional massless fields which arise from compactifying. Regarding this let us introduce the massless modes which arise from the deformation of the Calabi-Yau metric. We know, by definition, that the Calabi-Yau is a compact Kähler manifold of vanishing first Chern class. An interesting question could be: what are the allowed variations of the metric, $g_{\mu\nu} + \delta g_{\mu\nu}$, such that the Calabi-Yau condition is not disturbed? Which means:

$$R_{\mu\nu}(g) = 0 \rightarrow R_{\mu\nu}(g + \delta g) = 0. \tag{II.4.5}$$

When we look for these variations is important to exclude the ones which are coordinate transformations. In fact, given g a Ricci-flat metric, if we vary it with a diffeomorphism it will remain Ricci-flat as well. Then, we fix the diffeomorphism invariance by requiring $\nabla^\mu \delta g_{\mu\nu} = 0$. Now, what we do is to expand to first order the equation II.4.5, using the relation $R(g) = 0$, and to take the trace obtaining $\nabla^\rho \nabla_\rho \left(g^{\mu\nu} \delta g_{\mu\nu}\right) = 0$. Which in the end, brings us to:

$$\Delta_L \delta g_{\mu\nu} \equiv \nabla^\rho \nabla_\rho \delta g_{\mu\nu} + 2 R_{\mu\ \nu}^{\ \rho\ \sigma} \delta g_{\rho\sigma} = 0, \tag{II.4.6}$$

where Δ_L is the so called Lichnerowicz operator. Since we are working with a Calabi-Yau manifold we can study the condition on $\delta g_{i\bar{j}}$ and δg_{ij} in a separate way. Let us see:

- $\delta g_{i\bar{j}}$: in this case, the condition II.4.5 is equivalent to $(\Delta \delta g)_{i\bar{j}} = 0$. And we can consider $\delta g_{i\bar{j}}$ as the components of a (1,1)-form. Then expanding $\delta g_{i\bar{j}}$ in a basis of

(1,1)-forms, denoted by $\omega^A, A = 1, ..., h^{(1,1)}$, we get:

$$\delta g_{i\bar{j}} = \sum_{A=1}^{h^{(1,1)}} t^A \omega^A_{i\bar{j}},\ v^A \in \mathbb{R}, \tag{II.4.7}$$

where t^A represent the so-called Kähler moduli. They are called in this way since are deformations of the Kähler form:

$$J = t^A \omega_A, \tag{II.4.8}$$

and we can understand why it holds from eq. II.4.32.

- $\delta g_{\bar{i}\bar{j}}$: in this second case, the condition II.4.5 reduces to:

$$\Delta_{\bar{\partial}} \delta g^i = 0, \tag{II.4.9}$$

where:

$$\begin{aligned} &\Delta_{\bar{\partial}} = \left(\bar{\partial} + \bar{\partial}^*\right)^2 = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}, \quad \bar{\partial}^* = - * \partial *, \\ &\delta g^i = \delta g^i_{\bar{j}} d\bar{z}^{\bar{j}}, \quad \delta g^i_{\bar{j}} = g^{i\bar{k}} \delta g_{\bar{k}\bar{j}}, \end{aligned} \tag{II.4.10}$$

is nothing but a (0,1)-form with values in $T^{(1,0)}$, which is the holomorfic tangent bundle. This object consist of all the tangent vectors on the Calabi-Yau which only depend on the real part of the coordinates, and it is called holomorfic since the transition functions are holomorfic. The associated cohomology group is $H^1(Y, T^{(1,0)})$. The correct interpretation of these deformations is the following one: the new metric in order to be a Kähler metric, can be rewritten in a way where only the mixed components are non-zero, but to do that we need to apply a non-holomorfic transformation: it induces a new complex structure. Therefore, these are deformations of the complex structure. Using Ω, the holomorfic three-form, we define an isomorphism between $H^1(Y, T^{(1,0)})$ and $H^{(2,1)}(Y)$; this allows us to expand the deformations in a basis $\omega^{\hat{K}}_{ij\bar{l}}$, $\hat{K} = 1, ..., h^{(2,1)}$ of harmonic (2,1)-forms:

$$\Omega_{ijk} \delta g^k_{\bar{l}} = \sum_{A=1}^{h^{(2,1)}} \tilde{t}^A \omega^{\hat{K}}_{ij\bar{l}}, \tag{II.4.11}$$

where $\tilde{t}^A$ are the so-called complex structure moduli.

Together these moduli, t^A and $\tilde{t}^A$, span the so called moduli space, that locally can be decomposed in the following way:

$$\mathcal{M}^{cs} \times \mathcal{M}^{ks}. \tag{II.4.12}$$

Let us analyze a bit $\mathcal{M}^{cs}$. The associated metric is the following one [38]:

$$G_{K\bar{L}} = -\frac{\int_Y \chi_K \wedge \bar{\chi}_L}{\int_Y \Omega \wedge \bar{\Omega}}, \tag{II.4.13}$$

where χ_K is defined as:

$$\chi_K(z,\bar{z}) = \partial_{z^K}\Omega(z) + \Omega(z)\partial_{z^K}K^{cs}. \tag{II.4.14}$$

Now, one can shows that this space is a Kähler manifold, since exists a function $K(z,\bar{z})$, such that:

$$\begin{aligned} G_{K\bar{L}} &= \partial_{z^K}\partial_{\bar{z}^L}K^{cs}, \\ K^{cs} &= -\ln\left[i\int_Y \Omega\wedge\bar{\Omega}\right] = -\ln i\left[\bar{Z}^{\hat{K}}\mathcal{F}_{\hat{K}} - Z^{\hat{K}}\bar{\mathcal{F}}_{\hat{K}}\right] = 4D, \end{aligned} \tag{II.4.15}$$

where D is the four dimensional dilaton (defined in II.4.25) and the holomorphic periods $Z^{\hat{K}}$ and $\mathcal{F}_{\hat{K}}$ can be expressed as:

$$Z^{\hat{K}}(z) = \int_Y \Omega(z)\wedge\beta^{\hat{K}}, \quad \mathcal{F}_{\hat{K}}(z) = \int_Y \Omega(z)\wedge\alpha_{\hat{K}}. \tag{II.4.16}$$

Moreover, M^{cs} is a special Kähler manifold (more information about it can be found in appendix C) since $\mathcal{F}_{\hat{K}}$ is the first derivative, with respect to $Z^{\hat{K}}$, of the prepotential $\mathcal{F} = \frac{1}{2}Z^{\hat{K}}\mathcal{F}_{\hat{K}}$.
Let us now analyze M^{ks}. The metric of this manifold is given by:

$$G_{AB} = \frac{3}{2\mathcal{K}}\int_Y \omega_A\wedge *\omega_B = -\frac{3}{2}\left(\frac{\mathcal{K}_{AB}}{\mathcal{K}} - \frac{3}{2}\frac{\mathcal{K}_A\mathcal{K}_B}{\mathcal{K}^2}\right) = \partial_{t^A}\partial_{\bar{t}^B}K^{ks}, \tag{II.4.17}$$

where $*$ is the six-dimensional Hodge star on Y and the intersection numbers can be written in this way:

$$\begin{aligned} \mathcal{K}_{ABC} &= \int_Y \omega_A\wedge\omega_B\wedge\omega_C, \quad \mathcal{K}_{AB} = \int_Y \omega_A\wedge\omega_B\wedge J = \mathcal{K}_{ABC}t^C, \\ \mathcal{K}_A &= \int_Y \omega_A\wedge J\wedge J = \mathcal{K}_{ABC}t^Bt^C, \quad \mathcal{K} = \int_Y J\wedge J\wedge J = \mathcal{K}_{ABC}t^At^Bt^C, \end{aligned} \tag{II.4.18}$$

where $\mathcal{K}_{ABC}$ is the so-called triple intersection number. Instead, the Kähler potential K^{ks} is expressed in the following way:

$$K^{ks} = -\ln\left(\frac{4}{3}\mathcal{K}_{ABC}t^At^Bt^C\right). \tag{II.4.19}$$

Again, $\mathcal{M}^{ks}$ is a special Kähler manifold since K^{ks} can be derived from the holomorfic prepotential $f(t) = -\frac{1}{6}\mathcal{K}_{ABC}t^A t^B t^C$.

Now that we have a clear understanding of the compact manifold, with particular attention at the fields (moduli) that will emerge after the compactification, it is now time to analyze the type IIA superstring action. In the Einstein frame, it takes the form:

$$\begin{aligned} S^{(10)}_{IIA} = \int & -\frac{1}{2}\hat{R} * \mathbf{1} - \frac{1}{4}d\hat{\phi} \wedge *d\hat{\phi} - \frac{1}{4}e^{-\phi}\hat{H}_3 \wedge *\hat{H}_3 - \frac{1}{2}e^{\frac{3}{2}\hat{\phi}}\hat{F}_2 \wedge *\hat{F}_2 \\ & - \frac{1}{2}e^{\frac{1}{2}\hat{\phi}}\hat{F}_4 \wedge *\hat{F}_4 + \mathcal{L}', \end{aligned} \tag{II.4.20}$$

where

$$\mathcal{L}' = -\frac{1}{2}\left[\hat{B}_2 \wedge d\hat{C}_3 \wedge d\hat{C}_3 - \left(\hat{B}_2\right)^2 \wedge d\hat{C}_3 \wedge d\hat{A}_1\right], \tag{II.4.21}$$

and the field strength are defined in the following way:

$$\hat{H}_3 = d\hat{B}_2,\ \hat{F}_2 = d\hat{A}_1,\ \hat{F}_4 = d\hat{C}_3 - \hat{A}_1 \wedge \hat{H}_3. \tag{II.4.22}$$

We have that the dilaton $\hat{\phi}$, the ten-dimensional metric $\hat{g}$ and the two-form $\hat{B}_2$ are the massless fields in the NS sector, while the one-form $\hat{A}_1$ and the three-form $\hat{C}_3$ are the ones in the RR sector. When we compactify it on a Calabi-Yau threefold Y, we get a four-dimensional theory with $\mathcal{N} = 2$. Then, the zero-modes of Y, which are in one-to-one correspondence with the harmonic forms, are collected into massless $\mathcal{N} = 2$ multiplets. And their multiplicity is exactly counted by the dimension of the cohomologies $H^{(1,1)}$ and $H^{(1,2)}$. Then, let us understand more about these multiplets, in particular about the modes which characterized them.

We have already seen the ones which comes from the Calabi-Yau metric, then let us analyze what emerge from the other fields in eq. II.4.20. We can expand them, as always, in terms of the harmonic forms on Y:

$$\begin{aligned} &\hat{A}_1 = A^0(x), \quad \hat{B}_2 = B_2(x) + b^A(x)\omega_A, \quad A = 1, ..., h^{(1,1)}, \\ &\hat{C}_3 = A^A(x) \wedge \omega_A + \xi^{\hat{K}}(x)\alpha_{\hat{K}} - \tilde{\xi}_{\hat{K}}(x)\beta^{\hat{K}}, \quad \hat{K} = 0, ..., h^{(2,1)}. \end{aligned} \tag{II.4.23}$$

Where $b^A, \xi^{\hat{K}}$ and $\tilde{\xi}_{\hat{K}}$ are scalars, A^0, A^A are one-forms and B_2 is a two-form. Since the Calabi-Yau has no harmonic one-forms the expansion of $\hat{A}_1$ only contains A^0. Now, we have all the ingredients of the resulting EFT, let us see what are the different multiplets:

- there is 1 **gravity multiplet** assembled in this way $(g_{\mu\nu}, A^0)$.
- there are $h^{(1,1)}$**vector multiplets** assembled in this way (A^A, t^A, b^A).

- there are $h^{(2,1)}$ **hypermultiplets** assembled in this way $(\tilde{t}^A, \xi^{\hat{K}}, \tilde{\xi}_{\hat{K}})$.
- there is one **tensor multiplet** assembled in this way $(B_2, \phi, \xi^0, \tilde{\xi}_0)$.

But what about the four-dimensional effective action? In order to do that we have to redefine the variables. Let us start with the so-called complexification of the Kähler cone. What we have to do is to mix the two real scalars t^A and b^A, in the following way:

$$T^A = t^A + ib^A, \tag{II.4.24}$$

where now, the moduli T^A are complex, and they will have an important role in this part of the thesis. Using this variable we can also introduce the four dimensional dilaton D. It can be done using $\mathcal{K}$, seen in II.4.18:

$$e^D = e^{\phi} \left(\mathcal{K}/6\right)^{-\frac{1}{2}}, \tag{II.4.25}$$

where t^A and $\mathcal{K}/6$ are evaluated in the string frame. Using this frame, the Einstein-Hilbert term looks like $\int \frac{1}{2} e^{-2\hat{\phi}} R * \mathbf{1}$ and $J = t^A \omega_A$ is related to internal part of the metric. Since in the Einstein frame it looks like $\int R * \mathbf{1}$, we conclude that $J = e^{\phi/2} J_E$. Now, we are ready to obtain the four-dimensional effective action, more details can be found in C. Using the expansions II.4.23 inside II.4.22 and then II.4.20, reducing the Ricci scalar and performing a Weyl rescaling to get the standard Einstein-Hilbert term, we get [39–41]:

$$\begin{aligned} S^{(4)}_{IIA} = \int & -\frac{1}{2} R * \mathbf{1} + \frac{1}{2} \mathfrak{Im} \mathcal{N}_{\hat{A}\hat{B}} F^{\hat{A}} \wedge * F^{\hat{B}} + \frac{1}{2} \mathfrak{Re} \mathcal{N}_{\hat{A}\hat{B}} F^{\hat{A}} \wedge F^{\hat{B}} \\ & - G_{AB} dT^A \wedge * d\bar{T}^B - h_{uv} d\tilde{q}^u \wedge * d\tilde{q}^v, \end{aligned} \tag{II.4.26}$$

where $F^{\hat{A}} = dA^{\hat{A}}$. Now let us analyze the contributions related to the various multiplets. The coupling of the vector ones are encoded in G_{AB}, which only depends on T^A, and $\mathcal{N}_{\hat{A}\hat{B}}$; the explicit from of this last matrix can be seen in eq. II.C.9.
Regarding the hypermultiplets, the coupling are encoded in h_{uv}. It is a quaternionic metric and the term looks like [42]:

$$\begin{aligned} h_{uv} d\tilde{q}^u d\tilde{q}^v = & (dD)^2 + G_{K\bar{L}} dz^K d\bar{z}^L + \frac{1}{4} e^{4D} \left(da - \left(\tilde{\xi}_{\hat{K}} d\xi^{\hat{K}} - \xi^{\hat{K}} \tilde{\xi}_{\hat{K}} \right) \right)^2 \\ & - \frac{1}{2} e^{2D} \left(\mathfrak{Im} \mathcal{M} \right)^{-1 \hat{K}\hat{L}} \left(d\tilde{\xi}_{\hat{K}} - \mathcal{M}_{\hat{K}\hat{N}} d\xi^{\hat{N}} \right) \left(d\tilde{\xi}_{\hat{L}} - \bar{\mathcal{M}}_{\hat{L}\hat{M}} d\xi^{\hat{M}} \right), \end{aligned} \tag{II.4.27}$$

where $G_{K\bar{L}}$ is given in eq. II.4.13. Instead, the complex matrix $\mathcal{M}_{\hat{K}\hat{L}}$ depends on the fields z^K, and the definition is the following one:

$$\begin{aligned} & \int \alpha_{\hat{K}} \wedge \alpha_{\hat{L}} = - \left(\mathfrak{Im} \mathcal{M} + \left(\mathfrak{Re} \mathcal{M} \right) \left(\mathfrak{Im} \mathcal{M} \right)^{-1} \left(\mathfrak{Re} \mathcal{M} \right) \right)_{\hat{K}\hat{L}}, \\ & \int \beta^{\hat{K}} \wedge * \beta^{\hat{L}} = - \left(\mathfrak{Im} \mathcal{M} \right)^{-1 \hat{K}\hat{L}}, \\ & \int \alpha_{\hat{K}} \wedge * \beta^{\hat{L}} = - \left(\left(\mathfrak{Re} \mathcal{M} \right) \left(\mathfrak{Im} \mathcal{M} \right)^{-1} \right)^{\hat{L}}_{\hat{K}}. \end{aligned} \tag{II.4.28}$$

As already seen, the moduli space can be rewritten as:

$$\mathcal{M}^{SK} \times \mathcal{M}^{Q}, \tag{II.4.29}$$

where the first one in the decomposition is nothing but $\mathcal{M}^{ks}$, and is spanned by the scalars in the vector multiplets (or the complexified deformations of the Calabi-Yau) while $\mathcal{M}^{Q}$ is the quaternionic manifold and is spanned by the scalar in the hypermultiplets.

After this introduction, we now have a clear idea about the ten-dimensional Type IIA superstring, the six-dimensional compact manifold (Calabi-Yau) and the resulting four-dimensional EFT. We have seen what are the different ingredients which characterised them, how they are connected amongst each other and how they emerge. Now let us briefly described what is the explicit from of the objects introduced in chapter 4 in the particular case of the toroidal orbifold.

4.1 The $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold

T^6 is the product of $\otimes_{j=1}^{3} T_j^2$. The sub-torus has a square lattice described by R_x^i and R_y^i and we can write the area and the complex structure in the following way $A_i = R_x^i R_y^i$ and $\tau_i = R_y^i/R_x^i$. The metric takes the following form:

$$G = \mathrm{diag}\left(\frac{A_1}{\tau_1}, A_1\tau_1, \frac{A_2}{\tau_2}, A_2\tau_2, \frac{A_3}{\tau_3}, A_3\tau_3\right). \tag{II.4.30}$$

We define the following complex coordinates $z^i = R_x^i x^i + i R_y^i y^i$. Then, the orbifold action are:

$$(z^1, z^2, z^3) \rightarrow \begin{cases} (-z^1, -z^2, z^3) & \mathbb{Z}_2 \\ (z^1, -z^2, -z^3) & \mathbb{Z}_2' \end{cases} \tag{II.4.31}$$

The Kähler from is $J = i\sum_{k=1}^{3} G_{k\bar{k}} dz^k \wedge dz^{\bar{k}}$. And extracting $G_{k\bar{k}} = \frac{1}{2}$ from II.4.30, we get $J = \sum_{k=1}^{3} A_k dx^k \wedge dy^k$. As we can see it is invariant under the actions II.4.31 and the manifold has a volume $\mathcal{V} = \frac{1}{4}A_1A_2A_3$. We define $t^k = A_k/2^{\frac{2}{3}}$, such that:

$$J = t^k\omega_k = t^1\omega_1 + t^2\omega_2 + t^3\omega_3, \quad \omega_k = 2^{\frac{2}{3}} dx^k \wedge dy^k. \tag{II.4.32}$$

We have seen that the only non-vanishing triple intersection number is $\mathcal{K}_{123} = 1$. The basis of dual four-forms is $\tilde{\omega}^i$, with e.g. $\tilde{\omega}^1 = 2^{\frac{4}{3}} dx^1 \wedge dy^2 dx^3 \wedge dy^3$. Instead, the holomorphic three-form Ω takes the following form:

$$\Omega = (dx^1 + i\tau^1 dy^1) \wedge (dx^2 + i\tau^2 dy^2) \wedge (dx^3 + i\tau^3 dy^3), \tag{II.4.33}$$

which again is invariant under the action II.4.31. Finally, the basis of three-forms has to be chosen in such a way that satisfies eq. II.4.4, for example:

$$\alpha_0 = 2dx^1 \wedge dx^2 \wedge dx^3, \quad \beta^0 = 2dy^3 \wedge dy^2 \wedge dy^1. \tag{II.4.34}$$

The Hodge dual can be easily extracted since the metric is diagonal, $*\alpha_0 = \tau_1\tau_2\tau_3\beta^0$. The Kähler potential in this particular case, takes the following form:

$$K^{ks} = -\ln(8\mathcal{V}) \tag{II.4.35}$$

where $\frac{1}{6}\mathcal{K} \equiv \mathcal{V}$ represents the volume of the toroidal orbifold and in the last equality we used the fact that $\mathcal{K}_{123} = 1$. The explict expression of the Kähler form is the one in II.4.32 and thank to it, we can compute the volume of the various supersymmetric p-cycles. In order to get it we use the fact that the Kähler form and the holomorfic three-form are calibrations and the volume of a supersymmetric cycle can be computed by integrating these forms along any cycle in the same homology class. In particular, since we are considering even cycles, the volumes can be computed by integrating the Kähler form elevated at some powers, on a cycle in the same homology class. Let us analyze case by case:

- **Volume of two − cycles**:

$$\mathcal{V}_2^i = \int_{\gamma_2^i} J = \int_{\mathcal{M}} J \wedge \tilde{\omega}^i = \int_{\mathcal{M}} t^j \omega_j \wedge \tilde{\omega}^i = t^i, \tag{II.4.36}$$

  where in the second equality we used the Poincaré duality and in the last, the equation II.4.4. We can generalize the result introducing the B-filed, which means $J \to J + iB$, we get:

$$\mathcal{V}_2^i = T^i. \tag{II.4.37}$$

  which means $\mathcal{V}_2^1 = T_1$, $\mathcal{V}_2^2 = T_2$ and $\mathcal{V}_2^3 = T_3$.

- **Volume of four − cycles**:

$$\mathcal{V}_{4,i} = \frac{1}{2}\int_{\gamma_4^i} J \wedge J = \frac{1}{2}\int_{\mathcal{M}} J \wedge J \wedge \omega_i = \frac{1}{2}\int_{\mathcal{M}} t^j t^k \omega_j \wedge \omega_k \wedge \omega_i, \tag{II.4.38}$$

  where in the second equality we used again Poincaré duality and in the last one eq. II.4.18. Promoting again $J \to J + iB$:

$$\mathcal{V}_{4,i} = \frac{1}{2}\sum_{jk} \mathcal{K}_{ijk} T^j T^k, \tag{II.4.39}$$

  which means $\mathcal{V}_{4,1} = T_1T_2$, $\mathcal{V}_{4,2} = T_2T_3$ and $\mathcal{V}_{4,3} = T_1T_3$.

- **Volume of six – cycles**:

$$\mathcal{V}_6 = \frac{1}{3!}\int_{\mathcal{M}} J \wedge J \wedge J = \int_{\mathcal{M}} t^i t^j t^k \omega_i \wedge \omega_j \wedge \omega_k = \frac{1}{6}\sum_{ijk} \mathcal{K}_{ijk} t^i t^j t^k, \qquad \text{(II.4.40)}$$

where in the last equality we used again eq. II.4.18. Then, introducing the B-field $J \to J + iB$:

$$\mathcal{V}_6 = \frac{1}{6}\sum_{ijk} \mathcal{K}_{ijk} T^i T^j T^k, \qquad \text{(II.4.41)}$$

which means $\mathcal{V}_6 = T_1 T_2 T_3$.

It is time to switch the analysis toward other objects which always arise after the compactification. As we will see, they emerge from Dp-branes which wrap supersymmetric internal cycles (cycles belonging to the Calabi-Yau). The towers of BPS states that we are going to study are the ones composed by particles and strings, since this is what we need to fulfill the next goals of the thesis: the computation of the two cutoffs, Λ_{WGC} and Λ_s, and their comparison.

4.2 Towers of massless particles

As just introduced, these towers of states (deeply analyzed in [43]) arise from the compactification of Dp-branes wrapped around internal cycles. From the title of the section is clear how we are only interested in the massless ones, and this is related to the fact that are those towers which contribute in the final computation of the species scale Λ_s. In particular we will focus on the ones which become massless in the asymptotic regions of the moduli space, which correspond to take (in different combination) the limit $t^A \to \infty$ for the Kähler moduli. Then, let us understand where these towers come from explicitly and what are their properties. As we know, particles are object with a one-dimensional world volume, while the Dp-branes are objects with a $(p+1)$-dimensional world volume. Then, we expect particles to arise from Dp-branes wrapping the so-called p-cycles (where p identifies the dimension). In our case, the Dp-branes which make it possible are the following ones: D0, D2, D4 and D6; which respectively have to wrap 0-cycles, 2-cycles, 4-cycles and 6-cycles.

To compute the mass of these towers, we need to introduce the so-called DBI action; it is associated and describes unmagnetized Dp-branes. In the ten-dimensional string frame takes this form [3]:

$$S_{\text{DBI}} = -\mu_p \int_{W_{p+1}} d^{p+1}\xi e^{-\phi}\sqrt{-det\left(P\left[g_{\mu\nu} + B_{\mu\nu}\right]_{mn}\right)}, \qquad \text{(II.4.42)}$$

where $\mu_p = 2\pi/l_s^{p+1}$ is the brane tension from a ten-dimensional perspective, W_{p+1} is the $(p+1)$-dimensional world volume of the Dp-brane and $P\left[g_{\mu\nu} + B_{\mu\nu}\right]$ is the pullback on the world volume of the tensor $(g_{\mu\nu} + B_{\mu\nu})$. In the following discussion we will not take into account the B-field, and then the axions b^A. In fact, in the large volume limit (the one that we analyze in this thesis) the contribution of these objects is irrelevant. What we have to do now, is to dimensional reduce the action II.4.42, like we have done for the one in eq. II.4.20. The resulting action will exactly give us the mass of the different towers. The explicit reduction computations can be found in appendix C. Consider W_{p+1} to be the product of γ_p (internal p-cycle) and the world volume of the particle, and that $\mu_p = 2\pi/l_s^{p+1}$, if we integrate over the internal space, we obtain:

$$S_{\text{DBI}} = -\frac{2\pi\mathcal{V}_p}{g_s l_s}\int d\xi\sqrt{-g^{(1)}}, \tag{II.4.43}$$

where $\mathcal{V}_p$ is the volume of the p-cycles and $g^{(1)}$ the determinant of the one-dimensional metric. Then, in terms of the Planck scale, the mass takes the following form:

$$M_{\text{particle}}(\gamma_p) = \frac{2\pi\mathcal{V}_p}{g_s}M_s = \sqrt{\pi}M_P\frac{\mathcal{V}_p}{\sqrt{\mathcal{V}}}, \tag{II.4.44}$$

where to get the second equality we have used the relation II.C.22 (explained in details in appendix C). At this point, having computed all the ingredients needed in eq. II.4.44, we simply substitute them inside it(the volume of the various p-cycles can be found in the previous section, namely 4.1). Let us analyze case by case:

- **D0 – brane**: in this case $\mathcal{V}_0 = 1$, and using the fact that $\frac{1}{\sqrt{\mathcal{V}}} = \sqrt{8}e^{K^{ks}/2}$ (from eq. II.4.35), we get $M_0 = \sqrt{8\pi}M_P e^{K^{ks}/2}$.
- **D2 – brane**: using eq. II.4.44 and eq. II.4.37, the mass takes this form:

$$M_2 = \sqrt{8\pi}M_P e^{K^{ks}/2}T^i \tag{II.4.45}$$

- **D4 – brane**: using eq. II.4.44 and eq. II.4.39, the mass takes this form:

$$M_4 = \sqrt{8\pi}M_P e^{K^{ks}/2}\sum_{jk}\mathcal{K}_{ijk}T^jT^k. \tag{II.4.46}$$

- **D6 – brane**: using eq. II.4.44 and eq. II.4.41, the mass takes the following form::

$$M_6 = \sqrt{8\pi}M_P e^{K^{ks}/2}\sum_{ijk}\mathcal{K}_{ijk}T^iT^jT^k. \tag{II.4.47}$$

Now that we have the general form for the masses, let us find the explicit expressions in our case. First of all, since we are interested in infinite distance regions in the moduli space ($t^A \to \infty$) we neglect the B-field; then, using the fact the we are working with a toroidal orbifold ($\mathcal{K}_{ABC} = 1$), we get:

$$(M_0, M_{2_i}, M_{4_i}, M_6) = \sqrt{\pi} M_P\left(\frac{1}{\sqrt{t^1 t^2 t^3}}, \sqrt{\frac{t^i}{t^j t^k}}, \sqrt{\frac{t^j t^k}{t_i}}, \sqrt{t^1 t^2 t^3}\right). \tag{II.4.48}$$

It is clear from this point that taking various large volume limits forces some masses to go to zero; these are the towers, as already said, in which we are interested. We will go into details in chapter 5. Now, let us start the analysis of the string spectrum.

4.3 Towers of tensionless strings

In the case of strings, the game that we are going to play is the same. We expect some Dp-branes to wrap $(p-1)$-cycles (the world volume of a string is two-dimensional) and then after the compactification, to get the towers of strings. Indeed this is what happens, and the branes which are now interesting for us are the so-called NS5-branes. In principle, we could get strings also from D4-branes wrapping 3-cycles, but the final mass does not depends on the moduli t^A, then it is not interesting for our purpose. The starting point is the same as form particles, namely the action II.4.42, with an extra factor $e^{-\phi} = g_s$. In this case, we can decompose the world volume W_{p+1} of the Dp-brane, in a $(p-1)$-cycle and the world volume of the string. Then, we are going to integrate over the $(p-1)$-cycle, in order to get the action associated to the string. The way in which we can do this, is well explained in the appendix C; even if the case showed in the appendix is related to the particles, it is trivial to generalize the procedure and get the final result for strings. After the integration over the internal cycle, the action II.4.42 becomes:

$$S_{\text{DBI}} = -\frac{2\pi\mathcal{V}_4}{g_s^2 l_s^2}\int d^2\xi\sqrt{-g^{(2)}}, \tag{II.4.49}$$

which means that the string tension is the following one:

$$T_{NS5}(\gamma_4) = \frac{2\pi\mathcal{V}_4}{g_s^2} M_s^2 = \frac{M_P^2\mathcal{V}_4}{2\pi\mathcal{V}}, \tag{II.4.50}$$

where in the last equality we used the relation II.C.22. We can use, inside II.4.50, the usual relation between the volume and the Kähler potential, $\frac{1}{\sqrt{\mathcal{V}}} = \sqrt{8}e^{K^{ks}/2}$, obtaining:

$$T_{NS5} = 8M_P^2 e^{K^{ks}}\mathcal{V}_4. \tag{II.4.51}$$

Finally, plugging inside it the explicit expression for $\mathcal{V}_4$ in eq. II.4.39, we get:

$$T^i_{NS5} = M_P^2\left(\frac{2}{t^i}\right). \tag{II.4.52}$$

We are at the end of this chapter, in which we finally got the spectrum of particles and strings which arise from the compactification of the aforementioned branes. Now that we have the explicit values, we can finally go through the calculations needed to obtain at first, Λ_{WGC}, and then Λ_s. We will do this, and the comparison as well, in the next chapter.

Chapter 5

WGC cutoff vs Species cutoff

We have all the ingredients to finally achieve the second goal of the thesis. Since in the Swampland Program we have two different cutoffs, it would be interesting understand what is the right one to take into account. With the *right one*, I mean the one which is smaller; because this is the first one which will lead our theory to the breakdown. We perfectly know how to calculate the first cutoff, Λ_{WGC}; it is explicitly given by the equation I.3.3. It is worth noting that to compute it, we will not use directly the equation I.3.3, but a generalization of it. In fact, we have seen that from the compactification of the various Dp-branes, different towers of particles and strings arise. To make the discussion as general as possible, we are not only interested in the single tower which becomes massless/tensionless in a particular limit, but we are also interested in bound states of them, which do the same. Then, we need to extend I.3.3 in a way that could also take into account these objects. There is also something else to add. From the DBI action seen in various parts of the previous chapter, we know that it depends on the volume of the p-cycles and these objects not only depends on the Kähler moduli t^A but also on the axions b^A. Then when we generalize the scalar term we have to take into account also the scalar interaction mediated by the field b^A. In order to do that, we take the DBI action, we plug inside the various volumes of the p-cycles (we compute them in the previous chapter 4) and than we take the variation of the action with respect to b^A; what we get is the strenght of the vertex, that we must evaluate for a background $b^A = 0$. Following these steps we see that this contribution is exactly zero (result that can be seen in eq. II.C.24), for all the different volumes, implying that we do not have to consider the aforementioned interaction in the generalization of eq. I.3.3, which takes this form:

$$\Lambda_{\text{WGC}}^2 = \frac{q_a E^{ab} q_b}{k_D^2} - \frac{\left(q_a E^{ab} q_b\right)^2 G^{\phi_i \phi_j} M_i' M_j'}{k_D^2}, \tag{II.5.1}$$

where E^{ab} is the gauge kinetic matrix (an extension of $(\mathfrak{Im}\mathcal{N})^{-1}$ in II.C.9, which include both the electric and the magnetic contributions), q_a represent the charges of the towers and M' is the first derivative first derivative of the mass only with respect to the Kähler moduli t^A; in few lines we will see explicitly the expressions for these terms. It is worth noting that when we are going to apply II.5.1 we have to pay attention at the charges that we are turning on. In fact, since the M' term is directly associated to the monopole we have to set the charge of the monopole itself different from zero; but the gauge term, and in eq. I.3.3 it can be seen clearly, is representing the coupling dual to the one of the monopole, therefore we have to turn on the charge associated to its electric dual. Let us start with the case of particles to see how it works. The mass terms for these towers are the ones in eq. II.4.48, the generalization for bound states is the following one:

$$\begin{aligned} M_{\text{particles}} = \pi^{1/2} M_p \Big(& \frac{n_0^2}{t_1 t_2 t_3} + \frac{n_{21}^2 t_1}{t_2 t_3} + \frac{n_{22}^2 t_2}{t_1 t_3} + \frac{n_{23}^2 t_3}{t_1 t_2} + \frac{n_{41}^2 t_3 t_2}{t_1} \\ & + \frac{n_{42}^2 t_1 t_3}{t_2} + \frac{n_{43}^2 t_1 t_2}{t_3} + n_6^2 t_1 t_2 t_3 \Big)^{1/2}, \end{aligned} \tag{II.5.2}$$

where with n_i we identify the integer charges of the various towers. In fact, the term q_a has the following form:

$$q_a = (n_0, n_{21}, n_{22}, n_{23}, n_{41}, n_{42}, n_{43}, n_6). \tag{II.5.3}$$

Regarding $M'_{\text{particles}}$, it is not difficult to extract it from II.5.2 and the final form of the scalar term which appears in eq. II.5.1 can be found in eq. II.D.5. Now, let us move to the metric $G_{\phi_i \phi_j}$. In our case the scalars which define the metric are the moduli t^A, therefore the right way to write it, is $G_{\phi_i \phi_j} \equiv G_{t_i t_j}$. It is associated to a Kähler manifold $\mathcal{M}^{ks}$ which is described by the Kähler potential K^{ks} in eq. II.4.35. Then, by using the formula II.4.17, we get the following metric:

$$G_{t_i t_j} = \frac{1}{4} \begin{pmatrix} \frac{1}{t_1^2} & 0 & 0 \\ 0 & \frac{1}{t_2^2} & 0 \\ 0 & 0 & \frac{1}{t_3^2} \end{pmatrix}. \tag{II.5.4}$$

Let us finally analyze the gauge kinetic matrix E^{ab}. As already remarked, it is nothing but an extension of $(\mathfrak{Im}\mathcal{N})^{-1}$ (its explicit from is the one in eq. II.D.4) since it includes the contribution of both the electric objects and the magnetic duals. To get the expression for it, we will obviously use eq. II.D.4 and the fact that the electric and magnetic couplings are connected by the Dirac quantization condition, that we use with this particular normalization $eg = 4\pi$; the final result can be seen in eq. II.D.5.

Now, let us move the analysis to the terms for the strings. The starting equation is always II.5.1 but some of the terms will be obviously different from the previous case. And not only the terms, but is different also the approach that we will use to get the final cutoff. But why? We have seen that Λ_{WGC} enters the game because we are considering the presence of monopoles. In the case of the strings, in four dimensions, the magnetic counterpart is given by the axions. The problem is that we do not have the explicit form of the different terms for these objects, but only for strings. But here is the solution. By analyzing the equation I.3.3 and its generalization II.5.1, it is clear that in order to exactly saturate the equation in the entire region under analysis, we need that $\Lambda_{\text{WGC}} \sim e \sim \left(q_a E^{ab} q_b\right)^{1/2}$. Then, the problem reduces to compute the gauge kinetic matrix in the case of strings. But we have all the ingredients to do that. The Kähler metric $G^{t_i t_j}$ is exactly the same; the mass term, which is given by the square root of the tension T, is the following one (it comes from eq. II.4.52):

$$M_{\text{strings}} = \pi^{1/2} M_p \sqrt{\frac{2n_1}{t_1} + \frac{2n_2}{t_2} + \frac{2n_3}{t_3}}, \tag{II.5.5}$$

where now the dependence is on the square root of the charges. Finally, the M' term can be easily extracted from eq. II.5.5, and the full expression of the scalar interaction contribution can be found in eq. II.D.6. At this point, in order to get the gauge kinetic matrix, we have simply to solve for E^{ab}_{string} the generalization of the no-force condition I.3.2, which is this one:

$$q_a E^{ab}_{\text{string}} q_b - G^{t_i t_j} M'_i M'_j - k_D^2 M^2 = 0, \tag{II.5.6}$$

getting:

$$E^{ab}_{\text{string}} = 4\pi \begin{pmatrix} \frac{1}{t_1} & 0 & 0 \\ 0 & \frac{1}{t_2} & 0 \\ 0 & 0 & \frac{1}{t_3} \end{pmatrix}. \tag{II.5.7}$$

We finally have all the building blocks, both for particles and strings, to compute Λ_{WGC}. (We will do it explicitly after the following discussion on the species scale).

But what about Λ_s? The procedure to compute this object is algorithmic and these are the steps [44, 45]:

- 1) We have to take the tower with the lightest mass scale in a particular (large volume) limit and apply, at first, two laws:

$$\Lambda_{\text{s}}^{p_1} = N_{\text{s}}^{(1)} M^{p_1}_{\text{tower},1}, \tag{II.5.8}$$

$$\Lambda_{\text{s}} = \frac{M_P}{N_s}, \tag{II.5.9}$$

  where with the first one we extract $N_s(\Lambda_s)$ and then plugging it into the second one we finally get $\Lambda_s^{(1)}$; the exponent p_1 identify the number of towers with the same mass gap, and the index (1) simply that this is the first tower that we are considering.

- 2) Compare $\Lambda_s^{(1)}$ with the mass scale of the second lightest tower, in the limit. In particular we are interested in the maximum excitation number of the second tower falling below the first cutoff, it can be computed via:

$$N_2 = \left(\frac{\Lambda_s^{(1)}}{M_{\text{tower},2}}\right)^{p_2}. \tag{II.5.10}$$

  If we get $N_2 \leq 1$ then the first tower saturates the number of species and the species scale is equal to $\Lambda_s^{(1)}$.

- 3) If $N_2 \gg 1$ than we need to proceed in the following way. We need to define an effective tower with $M_{tower,(2)}^{p_1+p_2} = M_{\text{tower},1}^{p_1} M_{\text{tower},(2)}^{p_2}$. Once we have this new mass scale, we plug it inside II.5.8, getting $N_s(\Lambda_s)$, and then we substitute it in II.5.9, finally getting Λ_s.

- 4) We take the third lightest tower, we iterate 2) and if necessary 3) and 4).

Now that the stage is set, it is time to go through the explicit computations. As already mentioned, we are interested in the so called large volume limits, which corresponds to move at the asymptotic regions of the moduli space. These limits are strictly related to the behaviour of the moduli t^A (which from now, for convenience, are written with subscripts), and they are the following ones:

- $t_1 = t_2 = t_3 = t \to \infty$.
- $t_1 \to \infty, \quad t_2 = t_3 = \alpha = O(1)$.
- $t_1 = t_2^\alpha \to \infty, \quad t_3 = \alpha = O(1)$.

Let us finally move to the computation of Λ_{WGC} (for both particles and strings), Λ_s and their comparison in the various limits.

5.1 $t_1 = t_2 = t_3 = t \to \infty$ case

In this particular case the spectrum of the particles in eq. II.4.48 takes the following form:

$$(M_0, M_2, M_4, M_6) = \sqrt{\pi} M_P \left(\frac{1}{t^{3/2}}, \frac{1}{t^{1/2}}, t^{1/2}, t^{3/2}\right). \tag{II.5.11}$$

From the previous expression it is easy to see that the mass associated to n_{21}, n_{22} and n_{23} are equal and the same holds for n_{41}, n_{42} and n_{43}. Therefore, the electric and magnetic duality, in this particular case, work in this way: $M_0 \leftrightarrow M_6$ and $M_2 \leftrightarrow M_4$. Since the masses of the particles coming from D4-branes and D6-branes go to infinity in the large volume limit, these are the magnetic counterparts, the monopoles. Then, we will use these towers to compute Λ_{WGC}. Therefore what we need to do is the following: take the equation II.5.1, substitute the various ingredients (that we construct in chapter 4), where we turn on only $n_{41} = 1$ and $n_6 = 1$ in the scalar term (singularly and together, to take into account bound states) and respectively n_{21} and n_0 in the gauge term (the electric duals). Working in this way, we get the following cutoffs:

$$\begin{aligned} \Lambda^{(4)}_{\text{WGC}} &= \frac{1}{t^{1/2}}, \\ \Lambda^{(6)}_{\text{WGC}} &= \frac{1}{t^{3/2}}, \end{aligned} \tag{II.5.12}$$

and it is clear that $\Lambda^{(6)}_{\text{WGC}}$ is the smaller one, and of the same order of the lightest tower in the theory (the one coming from D0-branes, which not by chance is the one dual to the D6-branes tower).

Let us now compute the ones associated to the strings. We have seen how in this case what is interesting for us, is the gauge interaction term, since the cutoff is strictly connected to it. Then using the equation II.D.6, where we turn on only one n_i (since each term depends on the moduli in the same way $\sim t^{-1/2}$, it does not make any difference), we obtain this cutoff:

$$\Lambda^{\text{s}}_{\text{WGC}} = \frac{1}{t^{1/2}}. \tag{II.5.13}$$

What remains to do, is to compute the species scale. We have an algorithm which allows us to do that. Then, starting with the lightest tower of particles (in this case $\sim \frac{1}{t^{3/2}}$) and applying it, we see that also the second tower must be included, the one which goes like $\sim t^{-\frac{1}{2}}$, and we get:

$$\Lambda_{\text{s}} = \frac{1}{t^{1/2}}. \tag{II.5.14}$$

We are finally ready for the comparison, in this particular limit. What emerge is the following:

$$\Lambda_{\text{s}} \sim \Lambda^{\text{s}}_{\text{WGC}} \sim \Lambda^{(4)}_{\text{WGC}} \gg \Lambda^{(6)}_{\text{WGC}}. \tag{II.5.15}$$

This result is telling us that the true cutoff, the first one to affect our theory is the WGC cutoff which come from particles of D6 type and it is of the same order of the lightest tower in the theory. Since these are the heaviest objects in the theory, and the magnetic dual of the D0 type particles (which are the lightest), we can easily understand why. It is worth noting that the species scale is equal to the cutoff that we get from D4 type particles; it seems to be an interesting result since Λ_s, which is associated to the lightest towers in the theory is equal to the cutoff introduced by the magnetic dual (D4 type particles) of the heaviest tower which enters the computations of the species scale (D2 type particles). But as we will see this this is an artifact coming from the equality between the species scale and the mass of the D2 type particles ($\sim t^{-1/2}$). Finally, the fact that the true cutoff is smaller than Λ_s is telling us that when the theory breaks down, gravity is still weakly coupled.

5.2 $t_1 \to \infty, \quad t_2 = t_3 = \alpha = O(1)$ case

In this second case the spectrum takes the following form:

- $M_0 = \sqrt{\pi} M_P \frac{1}{\alpha\sqrt{t_1}}$,
- $M_2 = \sqrt{\pi} M_P (\frac{\sqrt{t_1}}{\alpha}, \sqrt{\frac{1}{t_1}}, \sqrt{\frac{1}{t_1}})$,
- $M_4 = \sqrt{\pi} M_P (\alpha\sqrt{\frac{1}{t_1}}, \sqrt{t_1}, \sqrt{t_1})$,
- $M_6 = \sqrt{\pi} M_P \alpha \sqrt{t_1}$,

and, as can be seen, now the electromagnetic duality works in the following way: $M_0 \longleftrightarrow M_6$, $M_4^{(1)} \longleftrightarrow M_2^{(1)}$, $M_2^{(2)} \longleftrightarrow M_4^{(2)}$, $M_2^{(3)} \longleftrightarrow M_4^{(3)}$. Therefore in this particular case, in order to compute Λ_{WGC}, we take eq. II.5.1 and we turn on the following charges (not only singularly, but also in different combination to take into account the various bound states): n_6, n_{21}, n_{42}, n_{43} (in the scalar term) and respectively n_0, n_{41}, n_{22}, n_{23} (in the gauge term). Again, the smallest cutoff is the one related to the heaviest monopole in the theory, the D6 type particles, and behaves in the following way:

$$\Lambda_{\text{WGC}}^{(6)} \sim \frac{1}{t_1^{1/2}}, \tag{II.5.16}$$

which like before is of the same order of the lightest tower in the theory. Let us analyze the cutoff which comes from strings. We know that $\Lambda_{\text{WGC}} \sim e$, and in this limit we get:

$$\Lambda_{\text{WGC}}^{\text{s}} = \frac{1}{t_1^{1/2}}. \tag{II.5.17}$$

The last object that we need to analyze is Λ_{s}, and using the usual algorithm (starting with the D0 type tower), we obtain:

$$\Lambda_{\text{s}} = \frac{1}{t_1^{1/3}}. \tag{II.5.18}$$

Finally, we can proceed with the comparison, obtaining:

$$\Lambda_{\text{s}} \gg \Lambda_{\text{WGC}}^{\text{s}} \sim \Lambda_{\text{WGC}}^{(6)}, \tag{II.5.19}$$

and this result is telling us that, again, the cutoff that we need to consider, is the WGC cutoff for D6 type particles. It is worth noting that in this limit, the cutoff introduced by the strings follow the same behaviour of $\Lambda_{\text{WGC}}^{(6)}$. Finally, since the cutoff is not Λ_{s}, at the theory break down, gravity remains weakly coupled.

5.3 $t_1 = t_2^\gamma, \to \infty, \gamma \geq 1, \quad t_3 = \alpha = O(1)$ case

In this last case the particles mass spectrum take the following from:

- $M_0 = \sqrt{\frac{\pi}{\alpha}} M_P \frac{1}{\sqrt{t_2^{\gamma+1}}}$
- $M_2 = \sqrt{\pi} M_P (\sqrt{\frac{1}{\alpha}}, \sqrt{\frac{1}{\alpha}}, \sqrt{\frac{\alpha}{t_2^{\gamma+1}}})$
- $M_4 = \sqrt{\pi} M_P (\sqrt{\alpha}, \sqrt{\alpha}, \sqrt{\frac{t_2^{\gamma+1}}{\alpha}})$
- $M_6 = \sqrt{\pi} M_P \sqrt{t_2^{\gamma+1} \alpha}$.

Analyzing the spectrum, it is clear that the electromagnetic duality is satisfied by the following couples: $M_0 \longleftrightarrow M_6$, $M_2^{(1)} \longleftrightarrow M_4^{(1)}$, $M_2^{(2)} \longleftrightarrow M_4^{(2)}$ and $M_2^{(3)} \longleftrightarrow M_4^{(3)}$. The way to proceed is the same as before. We apply eq. II.5.1 by turning on n_6, n_{41}, n_{42}, n_{43} (in the scalar term) and respectively n_0, n_{21}, n_{22}, n_{23} (in the gauge term). What emerge is that, the smallest cutoff, is the one associated to D6 type monopole and behaves in the following way:

$$\Lambda_{\text{WGC}}^{(6)} \sim \frac{1}{t_2^{\frac{\gamma+1}{2}}}. \tag{II.5.20}$$

Let us move to the strings case. Using the usual method and equation II.D.6, we see that the couplings which go to zero are these two $e \sim t_1^{-1}, t_2^{-1}$, then in this particular limit the result is the following one:

$$\Lambda_{\text{WGC}} \sim \sqrt{\frac{n_2 + n_1 t_2^{1-\gamma}}{t_2}}. \tag{II.5.21}$$

Now, two different scenarios are in front of us:

- $n_1 = 0$. In this case, the cutoff takes this form:

$$\Lambda_{\text{WGC}}^{\text{s}} = \frac{1}{t_2^{1/2}}. \tag{II.5.22}$$

- $n_1 \neq 0$. In this case, the cutoff takes this form:

$$\Lambda_{\text{WGC}}^{\text{s}} = \frac{1}{t_2^{\frac{\gamma+1}{4}}}. \tag{II.5.23}$$

Then, looking at eq. II.5.22 and eq. II.5.23 it is clear that the second case introduces a cutoff which is always smaller than the first one, except for $\gamma = 1$; this is why we consider (in the following part of the section) $\Lambda^{\mathrm{s}}_{\mathrm{WGC}}$ equal to the one in eq. II.5.23. Let us move to the species scale. The lightest towers are the ones with M_0 and M_{23}, and since they behave in the same way ($\sim t_2^{-\frac{\gamma+1}{2}}$), the coefficent p is exactly 2. Then applying the algorithm, we get:

$$\Lambda_{\mathrm{s}} \sim \frac{1}{t_2^{\frac{\gamma+1}{4}}}. \tag{II.5.24}$$

Finally, we have all the building blocks for the final comparison. In this limit, this is what emerges:

$$\Lambda^{\mathrm{s}}_{\mathrm{WGC}} \sim \Lambda_{\mathrm{s}} \gg \Lambda^{(6)}_{\mathrm{WGC}}, \tag{II.5.25}$$

and can be seen, that again the right cutoff is the WGC cutoff associated to D6 monopoles (heaviest in the theory). Since the species scale is still bigger than $\Lambda^{(6)}_{\mathrm{WGC}}$, at the break down gravity is weakly coupled.

We finally achieved the other important goal of the thesis, the extraction of the EFT cutoff in a particular setup. At first, we understood how to compute the various cutoffs in the Swampland Program, then we explicitly applied these calculation in Type IIA compactified on a toroidal orbifold and finally we compare the results. And we can summarize what we got, in the following way: the cutoff of the EFT, in the various large volume limits, is given by the WGC cutoff which comes from towers of particles, in particular the ones which arise from D6-branes wrapping 6-cycles.

Now let us move the analysis toward the last topic treated in this thesis. The Distant Axionic String Conjecture (DASC). In chapter 6 we will start with a review of this conjecture, necessary to motivate our interest toward it; and in the final part, there will be a direct test using the data and results obtained in the thesis, in order to understand whether or not they support the aforementioned conjecture.

Chapter 6

Test of the Distant Axionic String Conjecture

The material used to carry on the discussion in this section mainly comes from [14, 15]. Before writing down what the conjecture states let us understand a little bit about the background. The objects at which the conjecture refers are the fundamental strings, namely the ones with the tension satisfying:

$$\Lambda^2 < \mathcal{T}_{\rm str} < M_P^2, \tag{II.6.1}$$

where Λ is the EFT cutoff scale. This objects, as can be seen from the definition, must be included in the theory as localised operators in the theory, since cannot be resolved from the EFT perspective. The standard four-dimensional $\mathcal{N} = 1$ bosonic effective action, describing a set of chiral multiplets $\{\phi^a\}$, is:

$$S = \int \left(\frac{M_P^2}{2} R * 1 - M_P^2 K_{\alpha\bar{\beta}} d\phi^\alpha \wedge * d\bar{\phi}^{\bar{\beta}} - V * 1 \right), \tag{II.6.2}$$

where R is the Ricci scalar, $K_{\alpha\bar{\beta}}$ the Kähler metric (defined as in eq. II.4.17), and V the scalar potential. In particular regimes, we can employ a dual formulation. For example, when we have periodic directions in the moduli space which are promoted to approximate axionic shift symmetries. Imagine that we have a subset of fields $\{t^i\} \subset \{\phi^a\} = \{t^i, \chi^k\}$ which are periodic:

$$\mathfrak{Re} t^i \simeq \mathfrak{Re} t^i + e^i, \quad e^i \in \mathbb{Z}, \tag{II.6.3}$$

then an approximate continuous isometry, $\mathfrak{Re} t^i \to \mathfrak{Re} t^i + \lambda e^i$, $\lambda \in \mathbb{R}$, is present for the field space metric. In this particular case, we can dualise the chiral fields t^i to a linear multiplet which contains a dual saxion ℓ_i and a two-form potential $\mathcal{B}_{2i}$. Introducing the saxion $s^i \equiv \mathfrak{Im} t^i$, we define the dual variables in this way:

$$\ell_i = -\frac{1}{2}\frac{\partial K}{\partial s^i}, \quad \mathcal{H}_{3i} = d\mathcal{B}_{2i} = -M_P^2 \mathcal{G}_{ij} *_4 d\mathfrak{Re} t^j, \tag{II.6.4}$$

and

$$\mathcal{G}_{ij} = \frac{1}{2}\frac{\partial^2 K}{\partial s^i \partial s^j}. \tag{II.6.5}$$

The new kinetic term associated to the dual variables, takes the form:

$$-\frac{1}{2}\int \mathcal{G}^{ij}\left(M_P^2 d\ell_i \wedge * d\ell_j + \frac{1}{M_P^2}\mathcal{H}_{3i}\wedge *\mathcal{H}_{3i}\right), \tag{II.6.6}$$

and we can add to it the action of a string which couples, with charge e^i, to the two-form field:

$$S_{str} = -M_P^2 \int_{\mathcal{S}} \left|e^i \ell_i\right| \sqrt{-h} + e^i \int_{\mathcal{S}} \mathcal{B}_{2i}. \tag{II.6.7}$$

From the last equation it is easy to extract the tension and the charge of the string:

$$\mathcal{T}_{\mathbf{e}} = M_P^2 \left|e^i \ell_i\right|, \quad Q_{\mathbf{e}} = M_P \sqrt{\mathcal{G}_{ij} e^i e^j}. \tag{II.6.8}$$

Now, that the stage is set let us start going into the details. Imagine to have a single axionic chiral field such that $t \simeq t+1$, and with the following Kähler potential:

$$K = -n \log \mathfrak{Im} t. \tag{II.6.9}$$

Applying the definition of the dual variables in eq. II.6.4 we get the following dual saxion:

$$\ell = \frac{n}{2s}, \tag{II.6.10}$$

and using eq. II.6.8, this string tension:

$$\mathcal{T}(\ell) = M_p^2 e \ell. \tag{II.6.11}$$

This localised source produce a flow of the scalar field, which in a neighbourhood of the string takes this form [46]:

$$t(z) = t_0 + \frac{e}{2\pi i}\log\frac{z}{z_0}, \tag{II.6.12}$$

where z parametrize the transverse string directions ($z=0$ identifies the string) and t_0 gives the value of t at z_0. We can set $z = re^{i\theta}$ and this forces $a = a_0 + \frac{e\theta}{2\pi}$, which tells us that the axion undergoes a monodromy $a \to a + e$ around $z = 0$. While for the saxion we have the following flows (along the radial direction):

$$s(r) = s_0 - \frac{e}{2\pi}\log\frac{r}{r_0}, \qquad \ell(r) = \frac{n}{2s_0 - \frac{e}{\pi}\log\frac{r}{r_0}}. \tag{II.6.13}$$

If we plug eq. II.6.13 inside the tension II.6.11, since the string is located at $r = 0$, one would get a vanishing tension, violating eq. II.6.1. But considering that a Wilsonian EFT (therefore also the string tension), is associated with a given cutoff Λ, we can introduce a minimal distance $r_\Lambda = \Lambda^{-1}$, and thanks to what emerge in [47–49], we can write down the following formula for the cutoff scale:

$$\frac{\mathcal{T}(\Lambda)}{M_P^2} = e\ell(r_\Lambda) = \frac{ne}{2s_0 + \frac{e}{\pi}\log(\Lambda r_0)}. \tag{II.6.14}$$

Then, the RG-flow differential equation takes the following form:

$$\Lambda\frac{d}{d\Lambda}\left(\frac{\mathcal{T}}{M_P^2}\right) = -\frac{1}{n\pi}\left(\frac{\mathcal{T}}{M_P^2}\right)^2, \tag{II.6.15}$$

and integrating it, we get:

$$\frac{\mathcal{T}(\Lambda')}{M_P^2} = \frac{1}{\frac{M_P^2}{\mathcal{T}(\Lambda)} + \frac{1}{n\pi}\log\frac{\Lambda}{\Lambda'}}. \tag{II.6.16}$$

What we could also do, is to choose r_0 in such a way that $r_0 = r_\Lambda \equiv \Lambda^{-1}$, and write eq. II.6.13 in the following way:

$$s(r) = \frac{neM_P^2}{2\mathcal{T}(\Lambda)} - \frac{e}{2\pi}\log(\Lambda r), \tag{II.6.17}$$

and if Λ changes according to eq. II.6.15, this expression depends on the cutoff. This description of the string backreaction in terms of the RG-flow of the string tension has different consequences. First of all, moving from r_Λ to bigger distances makes $s(r)$ decrease. Since in quantum gravity we can only have approximate axionic symmetries, which means $s \gg 1$, the flow II.6.17 drives the theory at the break down ($s \simeq 0$). Instead, if we increase Λ, the string tension decreases and the saxionic vev $s(r_\Lambda)$ increases; this ensure that the EFT description in terms of the localised string is self-consistent (large $s(r_\Lambda)$).

Now, let us return to the case of multiple t^i. Suppose to have the following saxion domain $s^i > 0$ and $\ell_i > 0$. Since $\mathcal{T}_{\mathbf{e}} > 0$, we can assume that, at least, one $e^i > 0$. then, moving toward increasing values of r, $s^i \to 0$ and we go in the interior of $\mathcal{M}$ and instanton effects bring us in the non perturbative regime. If we instead move toward $r \to 0$, with $e^i \geq 0$, $\forall i$, we get:

$$s^i \to s^i_\infty = e^i \cdot \infty. \tag{II.6.18}$$

When this is mapped into a trajectory of the moduli space of vacua, it drives the scalars t^i toward the boundary of $\mathcal{M}$. And if the charge $\mathbf{e}$ is such that $e^i m_i \geq 0$, where m_i are the instanton charges, the corrections $e^{2\pi i m_i t^i}$ will die off in this limit and we remain in $\mathcal{M}_{parturbative}$ (approaching the string core). The fact that these corrections disappear, restore an exact axionic symmetry approaching the core, namely $a^i \to a^i + e^i$. For this reason we call these string as fundamental axionic strings. But what we know from quantum gravity arguments, [16,17], is that we can only restore global symmetries at the boundary of the moduli space: than, the consistent EFTs should map string locations to points s^i_∞, which are at infinite distance in the moduli space, and backreaction of the type II.6.13 to infinite distance path in $\mathcal{M}$.

Now, we have set the stage to understand the first part of the DASC: all the infinite distance limits of a four-dimensional EFT can be realised as RG flow endpoint of a fundamental axionic string [15]. As a support of this conjecture, was shown in different papers, [43, 50, 51], that each infinite distance limit is characterised by a monodromy, and its realization from a four-dimensional perspective is exactly $t^i \to t^i + e^i$. It is worth noting that the non trivial statement is the fact that at every infinite distance limit there is a continuous shift symmetry being restored.

This understood, we can move to the second part of the conjecture, called Cutoff asymptotics, which is the one directly tested in this thesis. Let us understand what it states. We have that the EFT strings are asymptotically tensionless, therefore we expect the EFT to break down along the infinite distance trajectories. Then, whenever the leading tower, with mass m_*, is not given by the winding modes of the string, there exists a direct link between this scale and the tension of the string describing this particular limit:

$$\frac{m_*^2}{M_P^2} \simeq A \left(\frac{\mathcal{T}}{M_P^2}\right)^w \quad \text{for some positive integer } w = 1, 2, ... \tag{II.6.19}$$

with A depending on the non-flowing chiral fields.
This conjecture was extensively tested in the paper and in the various examples was shown that the exponent w only assumes integer values, as stated by the conjecture. This is what we are trying to verify in the following work, but there are few subtleties to remark. In [15], they did not use the field dependent tension $\mathcal{T}_{\mathbf{e}}$ but the probe tension: the one computed at the cutoff. This, as can be seen from the previous formulae, cancels the effect of the backreaction which means that is the same type of tension introduced in 4 and 5. Moreover, the infinite distance trajectories described in the paper are geodesic path and the same holds for the ones that we will take into account in this description. A direct proof of that is present in appendix E. We are now ready to start the analysis. It is important to say that the string tension that we are going to use is the one in II.4.52 and the mass scale m_* is the one associated to lightest tower of particles in the limit (obtained from II.4.48):

- $t_1 = t_2 = t_3 = t \to \infty$: in this case

$$(\frac{1}{t})^\omega \sim (\frac{1}{t^{3/2}})^2, \tag{II.6.20}$$

  from which we get $\omega = 3$.

- $t_1 \to \infty$: in this case
$$\left(\frac{1}{t_1}\right)^w \sim \left(\frac{1}{t_1^{1/2}}\right)^2, \tag{II.6.21}$$
from which we get $w = 1$.
- $t_1 = t_2^\alpha \to \infty$ in this case
$$\left(\frac{1}{t_2^\alpha}\right)^w \sim \left(\frac{1}{t_2^{\frac{\alpha+1}{2}}}\right)^2, \tag{II.6.22}$$
from which we get $w = \frac{\alpha+1}{\alpha}$.
- $t_1 \gg t_2 \gg t_3$ with $t_1 = t_3^\alpha, t_2 = t_3^\beta$ and $\alpha \gg \beta \gg 1$ in this case
$$\left(\frac{1}{t_3^\alpha}\right)^w \sim \left(\frac{1}{t_3^{\frac{\alpha+\beta+1}{2}}}\right)^2, \tag{II.6.23}$$
from which we get $w = \frac{\alpha+\beta+1}{\alpha}$.
- $t_1 \gg t_2 \sim t_3$ with $t_1 = t_2^\alpha$ in this case
$$\left(\frac{1}{t_2^\alpha}\right)^w \sim \left(\frac{1}{t_2^{\frac{\alpha+2}{2}}}\right)^2, \tag{II.6.24}$$
from which we get $w = \frac{\alpha+2}{\alpha}$.
- $t_1 \sim t_2 \gg t_3$ with $t_1 = t_3^\alpha$ in this case
$$\left(\frac{1}{t_3^\alpha}\right)^w \sim \left(\frac{1}{t_3^{\frac{2\alpha+1}{2}}}\right)^2 \tag{II.6.25}$$
from which we get $w = \frac{2\alpha+1}{\alpha}$.

What emerges from the analysis, at first glimpse, seems not to support what stated in the conjecture. In fact, except from the first two cases, the other cases have a w which depends on particular combination of α and β. But it is worth nothing that in the limit in which $\alpha \to \infty$ or $\alpha = 1$, the conjecture is satisfied. This coincidence can be perfectly explained. The Kähler potential used in the paper, takes the form in eq. II.6.9 where n is an integer, and this the so called *strict asymptotic regime*. Instead, looking at the Kähler potential that we are using, the one in eq. II.4.35, it is clear that we are not in this regime. Roughly speaking this means that we are considering a larger number of geodesics reaching the particular infinite distance limit. But in the moment in which we take the limits $\alpha = 1$ or $\alpha \to \infty$ we get exactly integer values of w, and in fact, not by chance, it exactly corresponds to moving in the regime adopted in the original paper. And this shows that this work supports what stated in the conjecture.

Conclusions

Let us summarize the findings of this thesis work. We began by introducing one of the biggest challenges of modern theoretical physics: quantum gravity. Starting from String theory, which constitutes the main framework aiming to unify quantum physics and gravitation, and we presented some of the main challenges to connect it with observations. In this context, we introduced the main ideas behind the Swampland Program and some of its new perspective on solving this challenging problem.

We started by reviewing some of the conjectures and the objects of interest in this work. The first one was the Weak gravity Conjecture [8], which we have seen is composed by two different parts, the electric and the magnetic one. We focused our attention on the MWGC, which imposes bounds on the gravitational EFT cutoff, and after having introduced why it makes sense to extend it in the case of scalar fields, we constructed all the building blocks needed to achieve the goal. After that, we shifted our attention towards the comparison of the WGC cut off with another fundamental UV scale quantity in the Swampland Program [5]: the species scale Λ_{s} [9, 10]; that we also introduced and described outlining the steps to compute it asymptotically, mainly following [44]. Then, after discussing the magnetic monopoles, the connection between the self-repulsive condition and the MWGC, and the interaction potentials, we extracted the final law which gives us Λ_{WGC}, see eq. II.5.1.

After that, we started to set the stage for the comparison of the two aforementioned cutoffs. To do that, we started by reviewing Type IIA compactification on a general Calabi–Yau, and introduced a particular toroidal orbifold, $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2'$, to test things explicitly. We focused out attention on how the effective action is reduced, the resulting multiplets, and explored the towers of particles and strings emerging from Dp-branes and NS5-branes wrapping supersymmetric p-cycles. We also explained and computed how these towers depend on the Kähler moduli and using the extension of the MWGC and the algorithm for the species scales we calculated them in various large volume limits:

- $t_1 = t_2 = t_3 = t \to \infty$.
- $t_1 \to \infty \quad t_2 = t_3 = \alpha = O(1)$.

- $t_1 = t_2 \to \infty \quad t_3 = \alpha = O(1)$.

Our results showed that in all the cases, the smallest cutoff, which directly influences the breakdown of our EFT, is the WGC cutoff associated with D6-branes wrapping 6-cycles, which correspondingly gives rise to the heaviest monopole in the theory. Moreover, since it is smaller than the species scale, we observed that at the breakdown, gravity remains weakly coupled and it indicates that the EFT could be extended by including some states in the tower.

We then analyzed the Distant Axionic String Conjecture [14,15], which states that infinite distance limits can be associated with the RG flow endpoint of an axionic fundamental string and relates the mass scale m_* of the lightest tower in each particular limit to the tension $\mathcal{T}$ of the axionic string responsible for this limit: $m_*^2 \simeq \mathcal{T}^w$, with $w = 1, 2, 3$. We particularly focused on the second part and studied it in the previously introduced toroidal orbifold. Our findings showed that the coefficient w not only took the values stated in the conjecture but, in some cases, could be expressed in terms of combinations involving α, the exponent of the Kähler moduli t_A. While this initially seems to contrast with the original formulation, it is perfectly explainable and consistent. The original paper considered a Kähler potential depending on a single integer exponent n: $K = -n \log \mathfrak{Im} t$; known as the strict asymptotic regime, which was not the regime we considered in our work. By transitioning our setup to this regime, we would expect our results to align with those presented in the original paper. Indeed, by taking the limits $\alpha = 1$ or $\alpha \to \infty$, corresponding to move in this regime, we precisely recover $w = 1, 2, 3$.

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Appendix A

Gauge and graviton propagator

Let us start with the gauge propagator. We have the following Lagrangian:

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{e^2}A_\mu(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu, \tag{II.A.1}$$

and then we need to invert the following kinetic term:

$$-\frac{1}{2e^2}(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)D_{\nu\lambda}(\vec{x}-\vec{y}) = \delta^\mu_\nu\delta^3(\vec{x}-\vec{y}) \tag{II.A.2}$$

Taking the Fourier transform of $D_{\nu\lambda}(\vec{x}-\vec{y})$, we get:

$$-\frac{1}{e^2}(-k^2 g^{\mu\nu} - k^\mu k^\nu)\tilde{D}_{\nu\lambda} = \delta^\mu_\nu. \tag{II.A.3}$$

Now, we use the following ansatz for $\tilde{D}_{\nu\lambda}$:

$$\tilde{D}_{\nu\lambda} = A g_{\nu\lambda} + B k_\nu k_\mu \tag{II.A.4}$$

. Finally, substituting the ansatz we get the propagator:

$$< A_\mu A_\nu >= -\frac{ie^2}{k^2}g_{\mu\nu} \tag{II.A.5}$$

Let us now move the other case, the graviton propagator. We have the following action:

$$S_{EH}[g_{\mu\nu}] = \frac{1}{2k^2}\int d^4x\sqrt{-g}R(g). \tag{II.A.6}$$

We can expand the metric around the flat metric, $g_{\mu\nu} = \eta_{\mu\nu} + k_D h_{\mu\nu}$, and then put it inside the action II.A.6, getting:

$$S_{EH}[g_{\mu\nu} = S_2[h_{\mu\nu}] + \sum_{n=3}^{\infty} S_n[h_{\mu\nu}]. \tag{II.A.7}$$

To extract the propagator we need $S_2[h_{\mu\nu}]$, which is:

$$S_2[h_{\mu\nu}] = \frac{1}{4}\int d^4x(h^{\mu\nu}\partial^2 h_{\mu\nu} - \frac{1}{2}h\partial^2 h + 2(\partial^\nu h_\nu\mu - \frac{1}{2}\partial_\mu h)), \tag{II.A.8}$$

and this action has a gauge symmetry:

$$\delta h_{\mu\nu} = \partial_\mu \chi_\nu + \partial_\nu \chi_\mu. \tag{II.A.9}$$

After promoting it to a BRST symmetry (introducing the ghosts c^μ), we need to introduce the gauge fermion:

$$\psi = \bar{c}_\mu(f^\mu + \alpha B^\mu), \tag{II.A.10}$$

where

$$f^\mu = -(\partial_\mu h^{\nu\mu} - \frac{1}{2}\partial^\mu h). \tag{II.A.11}$$

Then we have:

$$s\psi = -\frac{1}{4\alpha^2}f^2 + \bar{c}_\mu \partial^2 c^\mu. \tag{II.A.12}$$

Inserting it in the quadratic action, with $\alpha = \frac{1}{2}$ (Feynman gauge), we get:

$$S_2[h_{\mu\nu}] = \int d^4x(\frac{1}{4}h^{\mu\nu}\partial^2 h_{\mu\nu} - \frac{1}{8}h\partial^2 h + \bar{c_\mu}\partial^2 c^\mu). \tag{II.A.13}$$

The lagrangian which is interesting for us is the following one:

$$\mathcal{L}_h = \frac{1}{2}h_{\mu\nu}\tilde{P}^{\mu\nu,\alpha\beta}\partial^2 h_{\alpha\beta}, \tag{II.A.14}$$

where

$$\tilde{P}^{\mu\nu,\alpha\beta} = \frac{1}{4}(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\nu\alpha}\eta^{\mu\beta}) - \frac{1}{4}\eta^{\mu\nu}\eta^p ha\beta. \tag{II.A.15}$$

From here we can get the inverse:

$$P_{\mu\nu,\alpha\beta} = (\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\nu\alpha}\eta_{\mu\beta}) - \eta_{\mu\nu}\eta_{\alpha\beta}, \tag{II.A.16}$$

such that

$$\tilde{P}^{\mu\nu,\alpha\beta}P_{\alpha\beta,\rho\sigma} = \frac{1}{2}(\delta^\mu_\rho \delta^\nu_\sigma + \delta^\mu_\sigma \delta^\nu_\rho). \tag{II.A.17}$$

And finally we have the propagator:

$$< h_{\mu\nu}h_{\rho\sigma} >= \frac{-iP_{\mu\nu,\rho\sigma}}{k^2}. \tag{II.A.18}$$

Appendix B

Gauge and gravitational potential

It is important to remark that in the computations we start from the action I.2.1, and we linearized it around a background $g_{\mu\nu}$ and $\phi = \phi_0$. Furthermore, we consider the particle to be static at the origin, $x^i = 0$. Let us start with the gauge field potential. We have the following equations of motion:

$$\frac{1}{2e^2}\partial_\nu F^{\mu\nu} = Q\delta^3(\vec{x}). \tag{II.B.1}$$

Using the fact that $A = \Phi dt$ ($F = dA$), we can rewrite the previous equation as:

$$-\partial^2\Phi(\vec{x}) = e^2 Q\delta^3(\vec{x}). \tag{II.B.2}$$

Taking the Fourier transform of ϕ and substituting it inside eq. II.B.2, we get:

$$\tilde{\Phi}(\vec{k}) = \frac{e^2 Q}{k^2}. \tag{II.B.3}$$

Performing the inverse Fourier transform on it (using spherical coordinates), we end up with:

$$\Phi(\vec{x}) = \frac{e^2 Q}{(2\pi)^3}\int_0^\infty \frac{k^2 dk}{k^2}\int_0^\pi sin\theta d\theta \int_0^{2\pi} d\phi e^{ikrcos\theta}, \tag{II.B.4}$$

and then with the following potential:

$$V_{\text{gauge}} = \frac{e^2 Q^2}{4\pi r}. \tag{II.B.5}$$

We can generalize it in d-dimensions using eq. I.2.14:

$$V_{\text{gauge}} = \frac{e^2 Q^2}{(d-3)V_{d-2}r^{d-3}}. \tag{II.B.6}$$

Let us now move to the gravitational potential of interaction. The equations of motion, in the Lorentz gauge $\partial_\mu \bar{h}^{\mu\nu} = 0$, are:

$$-\frac{1}{2k_D^2}\partial^2 \bar{h}^{\mu\nu} = M(\phi_0)\delta^3(\vec{x})\delta^{\mu 0}\delta^{\nu 0}, \tag{II.B.7}$$

where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$. Taking the Fourier transform of $\bar{h}^{tt}$ and substituting inside eq. II.B.7, we get:

$$\tilde{\bar{h}}^{tt}(\vec{k}) = -\frac{2k_D^2 M(\phi_0)}{\vec{k}^2}. \tag{II.B.8}$$

Then, performing the inverse Fourier transform on eq. II.B.8 (using spherical coordinates) we end up with:

$$\bar{h}^{tt}(\vec{x}) = \frac{2k_D^2 M(\phi_0)}{(2\pi)^3}\int_0^\infty \frac{k^2 dk}{k^2}\int_0^\pi sin\theta d\theta \int_0^{2\pi} d\phi e^{ikrcos\theta}, \tag{II.B.9}$$

from which we obtain $\bar{h}^{tt}$. Knowing that the potential is equal to $-\frac{1}{4}\bar{h}^{tt}$, we get:

$$V_{\text{gravitational}} = -\frac{k_D^2 M^2(\phi_0)}{4\pi r}. \tag{II.B.10}$$

We can generalize it in d dimensions using eq. I.2.14:

$$V_{\text{gravitational}} = -\frac{k_D^2 M^2(\phi_0)}{(d-3)V_{d-2}r^{d-3}}. \tag{II.B.11}$$

Appendix C

Understanding the $\mathcal{N} = 2$ superstring reduction

A special Kähler manifold $\mathcal{M}$ is characterised, as we have seen, by the presence of the three-form Ω. We can define also the covariant derivatives of this object that we named χ_K in eq. II.4.13. These objects can be expanded in the following way:

$$\Omega = Z^{\hat{K}}\alpha_{\hat{K}} - \mathcal{F}_{\hat{K}}\beta^{\hat{K}}, \quad \chi_K = \chi_K^{\hat{L}}\alpha_{\hat{L}} - \chi_{\hat{L}K}\beta^{\hat{L}}. \tag{II.C.1}$$

The functions $Z^{\hat{K}}(z)$ and $\mathcal{F}_{\hat{K}}(z)$ (both holomorfic) are the so-called periods of Ω, while $\chi_K^{\hat{L}}(z,\bar{z})$ and $\chi_{\hat{L}K}(z,\bar{z})$ are the periods of ξ_K. For every special Kähler manifold there exists a matrix $\mathcal{M}_{\hat{K}\hat{L}}(z,\bar{z})$ that is defined as:

$$\mathcal{M}_{\hat{K}\hat{L}} = \left(\bar{\chi}_{\hat{K}\hat{M}}\mathcal{F}_{\hat{K}}\right)\left(\bar{\chi}_{\hat{M}}^{\hat{L}} Z^{\hat{L}}\right)^{-1}. \tag{II.C.2}$$

Moreover, from eq. II.C.2 can be extracted:

$$\mathcal{F}_{\hat{K}} = \mathcal{M}_{\hat{K}\hat{L}} Z^{\hat{L}}, \quad \chi_{\hat{L}M} = \bar{\mathcal{M}}_{\hat{L}\hat{M}}\chi_K^{\hat{M}}. \tag{II.C.3}$$

If we take the Jacobian $\partial_{z^L}\left(Z^K/Z_0\right)$ to be invertible, $\mathcal{F}_{\hat{K}}$ can be written as the derivative of a holomorfic prepotential $\mathcal{F}$ with respect to $Z^{\hat{K}}$:

$$\mathcal{F} = \frac{1}{2}Z^{\hat{K}}\mathcal{F}_{\hat{K}}, \quad \mathcal{F}_{\hat{K}} = \partial_{Z^{\hat{K}}}\mathcal{F}, \quad \mathcal{F}_{\hat{K}\hat{L}} = \partial_{Z^{\hat{K}}}\mathcal{F}_{\hat{L}}, \quad \mathcal{F}_{\hat{L}} = Z^{\hat{K}}\mathcal{F}_{\hat{K}\hat{L}}. \tag{II.C.4}$$

The complex matrix $\mathcal{M}_{\hat{K}\hat{L}}$ in eq. II.C.2, can be rewritten as:

$$\mathcal{M}_{\hat{K}\hat{L}} = \bar{\mathcal{F}}_{\hat{K}\hat{L}} + 2i\frac{(\mathfrak{Im}\mathcal{F})_{\hat{K}\hat{M}}\, Z^{\hat{M}}\, (\mathfrak{Im}\mathcal{F})_{\hat{L}\hat{N}}\, Z^{\hat{N}}}{Z^{\hat{N}}\, (\mathfrak{Im}\mathcal{F})_{\hat{N}\hat{M}}\, Z^{\hat{M}}}. \tag{II.C.5}$$

It is worth noting that Ω is defined up to complex rescalings by a holomorfic function $\Omega e^{-h(z)}$. This symmetry makes the periods Z_0 unphysical, such that we can fix it with the so called Kähler gauge: $Z_0 = 1$. The remaining $h^{(2,1)}$ periods can be identified with the complex structure deformations z^K by setting $z^K = Z^K/Z_0$. The homogeneity of $\mathcal{F}$ makes possible to define a prepotential $f(z)$ (holomorfic) which satisfies:

$$\mathcal{F}(Z) = \left(Z^0\right)^2 f(z). \tag{II.C.6}$$

. Then the Kähler potential, which is expressed as in eq. II.4.14, takes the following form:

$$K = -\ln i \left|Z^0\right|^2 \left[2\left(f - \bar{f}\right) - \left(\partial_K f + \partial_{\bar{K}} \bar{f}\right)\left(z^K - \bar{z}^K\right)\right]. \tag{II.C.7}$$

An example of this past discussion is the special Kähler manifold $\mathcal{M}^{\mathcal{SK}}$ introduced in chapter 4. The deformations t^A, introduced in eq. II.4.23 are the moduli which span this manifold. The Kähler potential of the metric G_{AB} can be written as in eq. II.C.7 with a prepotential:

$$f(t) = -\frac{1}{6}\mathcal{K}_{ABC} T^A T^B T^C. \tag{II.C.8}$$

Finally, substituting eq. II.C.8 in eq. II.C.6 and then plugging it in eq. II.C.5, we get the expression for $\mathcal{N}_{\hat{A}\hat{B}}(t, \bar{t})$:

$$\begin{aligned}
\mathfrak{Re}\mathcal{N} &= \begin{pmatrix} -\frac{1}{3}\mathcal{K}_{ABC} b^A b^B b_C & \frac{1}{2}\mathcal{K}_{ABC} b^B b^C \\ \frac{1}{2}\mathcal{K}_{ABC} b^B b^C & -K_{ABC} b^C \end{pmatrix}, \\
\mathfrak{Im}\mathcal{N} &= -\frac{\mathcal{K}}{4\pi}\begin{pmatrix} 1 + 4G_{AB} b^A b^B & -4G_{AB} b^B \\ -4G_{AB} b^B & 4G_{AB} \end{pmatrix}, \\
\left(\mathfrak{Im}\mathcal{N}\right)^{-1} &= -\frac{4\pi}{\mathcal{K}}\begin{pmatrix} 1 & b^A \\ b^A & \frac{1}{4}G^{AB} + b^A b^B \end{pmatrix}.
\end{aligned} \tag{II.C.9}$$

where G_{AB} is the one defined in eq. II.4.16.

Let us now move to the analysis of the DBI-action reduction. The action that we are considering is the one in eq. II.4.42. In order to understand how to properly reduce it, let us analyze the pullback (remember that we neglect the effect of the B-field). It is representing the metric of the $(p+1)$-dimensional world volume, seen by a $(p+1)$-dimensional observer, embedded in background described by $g_{\mu\nu}$. Without lose of generality we can assume $P\left[g_{\mu\nu}\right]_{mn}$ to be block diagonal, such that it can be written in this form:

$$g_{mn} = g_{00} dx^0 dx^0 + g_{rs} dx^r dx^s, \tag{II.C.10}$$

where g_{00} is the $(0,0)$ component of g_{mn} and then describes a one-dimensional time direction, while g_{rs} is what remains of g_{mn} and describes a p-dimensional spatial part. By taking the determinant it is clear that what we get is:

$$g = g^{(1)} \times g', \tag{II.C.11}$$

where g is the determinant of g_{00} and g' the ones of g_{rs}. At this point, by using the fact that W_{p+1} can be decomposed as a p-cycle and the world-volume of a particle, we get:

$$S_{\mathrm{DBI}} = -\frac{2\pi}{l_s^{p+1} e^{\phi}} \int d\xi \sqrt{-g^{(1)}} \int_{W_p} d^p\xi \sqrt{-g'}. \tag{II.C.12}$$

The second integral is nothing but $\mathcal{V}_p l_s^p$ (with $\mathcal{V}_p$ volume of the p-cycle), and using the relation $e^{\phi} = g_s$, where g_s is the string coupling, we exactly get the equation II.4.43. Then same discussion procedure can be applied to get the DBI action for strings in eq. II.4.49

Let us finally focus on the relation between M_p and M_s used in the second equality of eq. II.4.44. To get the final result we start with the ten-dimensional Einstein-Hilbert action in the string frame:

$$S^S = \frac{1}{2k_S^2} \int d^{10}x \sqrt{-G_{10}^S} e^{-2\phi} R_{10}^S, \tag{II.C.13}$$

where the index S identify the frame in which we are working (E for the Einstein one). This action is related to the one in the Einstein frame by a conformal transformation, this one:

$$G_{MN}^E = e^{-\frac{\phi-\phi_0}{2}} G_{MN}^S, \tag{II.C.14}$$

and using it in eq. II.C.13, of course we get:

$$S_{10}^E = \frac{1}{2k_E^2} \int d^{10}x \sqrt{-G_{10}^E} R_{10}^E. \tag{II.C.15}$$

Now, we need to compactify the theory on the internal manifold (six-dimensional); in order to do that we need a general ansatz for the metric, this one:

$$ds_{10}^2 = H^{-1/2}(y) e^{2\omega(x)} g_{\mu\nu} dx^\mu dx^\nu + H^{1/2}(y) \mathcal{V}^{1/3} g_{mn} dy^m dy^n, \tag{II.C.16}$$

in the Einstein frame. Using eq. II.C.16, the Ricci tensor takes the following form:

$$R_{10}^E = H^{1/2}(y) e^{-2\omega(x)} R_4 + H^{-1/2}(y) \mathcal{V}^{-1/3} R_6 + ..., \tag{II.C.17}$$

where we only consider the terms necessary to obtain the final relation.
Plugging eq. II.C.17 inside the action II.C.15, we obtain:

$$S_4^E \supset \frac{1}{2k_E^2} \int d^4x \sqrt{-g_{(4)}^E} e^{2\omega(x)} (\mathcal{V} \int d^6y \sqrt{-g_{(6)}^E} H(y)) R_4^E. \tag{II.C.18}$$

Now we require the canonical form for eq. II.C.18, this implies that:

$$e^{2\omega(x)} = \frac{e^{2\omega_0} l_s^6}{\int d^6y \sqrt{-g^E_{(6)}} H(y)}, \tag{II.C.19}$$

and we plug it inside eq. II.C.18:

$$S_4^E \supset \frac{e^{2\omega_0} l_s^6}{2k_E^2} \int d^4x \sqrt{-g^E_{(4)}} R_4^E = \frac{M_p^2}{2} \int d^4x \sqrt{-g^E_{(4)}} R_4^E. \tag{II.C.20}$$

We are at the end, in fact knowing that $M_s = 1/l_s$, $e^{2\omega_0} = \mathcal{V}$ and using the following relations (known from String Theory):

$$\begin{aligned} 2k_S^2 &= (2\pi)^7 \alpha'^4 \\ 2k_E^2 &= 2e^{2\phi_0} k_S^2 = \frac{g_s^2 l_s^8}{(2\pi)}, \end{aligned} \tag{II.C.21}$$

we can equate the two pre-factors in eq. II.C.20, getting:

$$M_s^2 = \frac{g_s^2 M_P^2}{4\pi \mathcal{V}}. \tag{II.C.22}$$

This is exactly what we were looking for.

Let us finally see why the scalar interaction does not include the contribution of the axions b^A. We will proof this result for the case of particles arising from wrapping D2-branes around two cycles, but it holds is general. We take the DBI action in eq. II.4.43 and utilizing eq. II.C.22 and eq. II.4.37, we get:

$$S_{\text{DBI}} = \frac{\pi^{1/2} M_P (t_A^2 + b_A^2)^{1/2}}{(t_1 t_2 t_3)^{1/2}} \int \sqrt{g^{(1)}}. \tag{II.C.23}$$

In order to get the strength of the vertex we, at first, take the variation of the action with respect to b_A and than we evaluate it for a background $b_A = 0$. What emerges is the following:

$$\frac{\delta S}{\delta b_A} = \left(\frac{\pi^{1/2} M_P b_A}{(t_1 t_2 t_3)^{1/2} (t_A^2 + b_A^2)^{1/2}} \int \sqrt{g^{(1)}} \right)_{b_A = 0} = 0 \tag{II.C.24}$$

And this result means that the strength of the vertex is zero and we must not include it in the generalization of eq. I.3.3.

Appendix D

Full expressions for the computation of $\Lambda_{\mathbf{WGC}}$

We start with the terms in the case of particles. At first let us see how the inverse metric looks like:

$$G^{t_i t_j} = 4\begin{pmatrix} t_1^2 & 0 & 0 \\ 0 & t_2^2 & 0 \\ 0 & 0 & t_3^2 \end{pmatrix}. \tag{II.D.1}$$

Then, the scalar contribution can be computed from the mass term in eq. II.5.2 and the inverse metric in eq. II.D.1:

$$\begin{aligned} G^{t_i t_j} M_i' M_j' = \frac{\pi}{K}[& \frac{3n_0^4}{(t_1t_2t_3)^2} + \frac{3n_{21}^4 t_1^2}{(t_2t_3)^2} + \frac{3n_{22}^4 t_2^2}{(t_1t_3)^2} + \frac{3n_{23}^4 t_3^2}{(t_1t_2)^2} + \frac{3n_{41}^4 (t_2t_3)^2}{t_1^2} + \\ & + \frac{3n_{42}^4 (t_1t_3)^2}{t_2^2} + \frac{3n_{43}^4 (t_1t_2)^2}{t_3^2} + 3n_6^4 (t_1t_2t_3)^2 + \frac{2n_0^2 n_{21}^2}{(t_2t_3)^2} + \\ & + \frac{6n_0^2 n_{22}^2}{(t_1t_3)^2} + \frac{2n_0^2 n_{23}^2}{(t_1t_2)^2} - \frac{2n_0^2 n_{41}^2}{t_1^2} - \frac{2n_0^2 n_{42}^2}{t_2^2} - 2n_0^2 n_{43}^2 t_3^2 - \\ & - 2n_0^2 n_6^2 + \frac{2n_{21}^2 n_{22}^2}{t_3^2} - \frac{2n_{21}^2 n_{23}^2}{t_2^2} - 6n_{21}^2 n_{41}^2 + \frac{2n_{21}^2 n_{42}^2 t_1^2}{t_2^2} + \\ & + \frac{2n_{21}^2 n_{43}^2 t_1^2}{t_3^2} - 2n_{21}^2 n_6^2 t_1^2 + \frac{2n_{22}^2 n_{23}^2}{t_1^2} - \frac{2n_{22}^2 n_{41}^2 t_2^2}{t_1^2} - \\ & - 2n_{22}^2 n_{42}^2 - \frac{2n_{22}^2 n_{43}^2 t_2^2}{t_3^2} - 6n_{22}^2 n_6^2 t_2^2 + \frac{2n_{23}^2 n_{41}^2 t_3^2}{t_1^2} + \\ & + \frac{2n_{23}^2 n_{42}^2 t_3^2}{t_2^2} - 6n_{23}^2 n_{43}^2 - 2n_{23}^2 n_6^2 t_3^2 - 2n_{41}^2 n_{42}^2 t_3^2 - \\ & - 2n_{41}^2 n_{43}^2 t_2^2 + 2n_{41}^2 n_6^2 (t_1t_3)^2 - 2n_{42}^2 n_{43}^2 t_1^2 + 2n_{42}^2 n_6^2 (t_1t_3)^2 + \\ & + 2n_{43}^2 n_6^2 (t_1t_2)^2], \end{aligned} \tag{II.D.2}$$

where K takes the following form:

$$\begin{aligned} K = & \frac{n_0^2}{t_1 t_2 t_3} + \frac{n_{21}^2 t_1}{t_2 t_3} + \frac{n_{22}^2 t_2}{t_1 t_3} + \frac{n_{23}^2 t_3}{t_1 t_2} + \frac{n_{41}^2 t_3 t_2}{t_1} + \\ & + \frac{n_{42}^2 t_1 t_3}{t_2} + \frac{n_{43}^2 t_1 t_2}{t_3} + n_6^2 t_1 t_2 t_3. \end{aligned} \tag{II.D.3}$$

Now let us extract the gauge interaction term of the generalized no-force condition in eq. II.5.1. In order to do that, it is important to get the explicit expression of $(\mathfrak{Im}\mathcal{N})^{-1}$ in eq. II.C.9. Which, using $b^A = 0$ and $G^{AB} \equiv G^{t_i t_j}$, is the following one:

$$(\mathfrak{Im}\mathcal{N})^{-1} = \frac{4\pi}{t_1 t_2 t_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t_1^2 & 0 & 0 \\ 0 & 0 & t_2^2 & 0 \\ 0 & 0 & 0 & t_3^2 \end{pmatrix}. \tag{II.D.4}$$

Finally, utilizing the Dirac quantization condition $eg = 4\pi$, we get the extension of it. In fact using eq. II.5.3 and the extension of II.D.4, we obtain:

$$\begin{aligned} q_a E^{ab} q_b = & \frac{4\pi n_0^2}{t_1 t_2 t_3} + \frac{4\pi n_{21}^2 t_1}{t_2 t_3} + \frac{4\pi n_{22}^2 t_2}{t_1 t_3} + \frac{4\pi n_{23}^2 t_3}{t_1 t_2} + \frac{4\pi n_{41}^2 t_2 t_3}{t_1} + \\ & + \frac{4\pi n_{42}^2 t_1 t_3}{t_2} + \frac{4\pi n_{43}^2 t_1 t_2}{t_3} + 4\pi n_6^2 t_1 t_2 t_3. \end{aligned} \tag{II.D.5}$$

Let us now move to the terms for strings. We have seen that in the end the only interesting term for us, is the gauge one. By taking the expression for E^{ab}_{string} in eq. II.5.7 and $q_a = (\sqrt{n_1}, \sqrt{n_2}, \sqrt{n_3})$, the gauge interaction term looks like:

$$q_a E^{ab}_{\text{string}} q_b = \frac{4\pi n_1}{t_1} + \frac{4\pi n_2}{t_2} + \frac{4\pi n_3}{t_3}. \tag{II.D.6}$$

Appendix E

Geodesic trajectories

In this appendix we want to make sure that the various infinite distance limits taken in chapter 6, truly can be reached via geodesic trajectories. The geodesic equation takes the following form:

$$\frac{dx^\mu}{d^2s^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds} fracdx^\beta ds = 0, \tag{II.E.1}$$

where s is a parameter and the Christoffel symbols are defined as:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\sigma}\left(g_{\alpha\sigma,\beta} + g_{\sigma\beta,\alpha} - g_{\alpha\beta,\sigma}\right), \tag{II.E.2}$$

where $g_{\mu\nu}$ is the space-time metric. Using the expression II.5.4 for the metric, the non-vanishing Christoffel symbols are the following ones:

$$\Gamma^0_{00} = -\frac{1}{t_1}, \quad \Gamma^1_{11} = -\frac{1}{t_2}, \quad \Gamma^2_{22} = -\frac{1}{t_3}. \tag{II.E.3}$$

In the various limits, we always have some fields t^i which varies and which are some powers of the other (i.e. $t_1 = t_2^\alpha$ etc.) while the others remain constant. The ones which does not vary, will automatically satisfy eq. II.E.1, while for the others we have to solve solve the geodesic equation. Let us take the particular case of $t_1 = t_2^\alpha$ (the result can be trivially generalized to the other cases). We take eq. II.E.1, we plug it inside eq. II.E.3, and we get:

$$\begin{aligned} \frac{d^2t_1}{ds^2} - \frac{1}{t_1}\left(\frac{dt_1}{ds}\right)^2 = 0, \\ \frac{d^2t_2}{ds^2} - \frac{1}{t_2}\left(\frac{dt_2}{ds}\right)^2 = 0. \end{aligned} \tag{II.E.4}$$

The general solution of the first differential equation is $t_1 \sim e^{\gamma s}$, with γ some constant of integration. Since $t_1 = t_2^\alpha$, we can write $t_2 \sim e^{\frac{\gamma}{\alpha}s}$, which satisfies the second equation.

We just proved that this is a geodesic trajectory. Taking into account that in all the other cases the relation between the moduli is always the one that we just analyzed, we conclude that all the other trajectories are geodesic.

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