Alma Mater Studiorum · University of Bologna

School of Science
Department of Physics and Astronomy
Master Degree in Physics

Integrable Structures in Resurgent Quantum Mechanics and Wall-crossing

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Academic Year 2022/2023

Abstract

This Thesis is centered on the analysis of structures typical of integrable models that are found also in the context of the Schrödinger equation. Among the many relations that can be derived, particular focus is devoted to the study of the Thermodynamic Bethe Ansatz (TBA) equations, and the related Y-systems, for the Borel resummed version of the quantum periods associated to a given choice of potential function. These quantum periods are defined as loop integrals of formal solutions to the Riccati equation, which get promoted to actual functions through the resummation procedure. They are part of the monodromy data of the equation, and can be used to find exact quantisation conditions that allow to solve for the energy spectrum. Once the type of potential has been selected, the form of the TBA equations for the resummed quantum periods depends on the value chosen for the parameters entering in the potential; the parameter space may be in fact organised in regions, called chambers, differing for the type of TBA. Building on the results found in [1], we have developed an algorithmic procedure that allows to find the TBA equations for the resummed quantum periods associated to a generic polynomial potential, and so in a generic chamber of the related parameter space, only through algebraic manipulations of the Y-system of a minimal chamber, where the form of the Y-system is always known. To further investigate the realm of applicability of this procedure, we then studied its possible implementation for the case of the modified Mathieu equation, a Schrödinger-like equation with a periodic potential, whose relevance lies in a connection with the deformation of the 4d $\mathcal{N} = 2$ supersymmetric $SU(2)$ pure gauge theory.

Contents

Introduction

The study on integrable systems, at first in the classical setting, dates back to the very beginning of Classical Mechanics, in its Newtonian formulation [2]. The appeal of integrable models as systems that can be exactly solved has since gathered more and more interest from the physics community, which developed strategies and techniques to tackle the problem of extracting from these theories information on their observables, and on other relevant physical quantities characterizing the models. As is common in the field, many of these techniques have then been found to be be applicable in the most disparate areas of physics, even in contexts which at first were seemingly unrelated with the integrability domain. Among these cases, the one that is central to this thesis work is a connection between integrable models and the study of the Schrödinger equation, stepping thus into the realm of Quantum Mechanics (QM). The presence of integrable structures hidden in particular quantum mechanical settings may be already found in the foundational works of A. Voros (most importantly [3], but also e.g. [4–6]), and Y. Sibuya [7], but a properly formalized connection, soon named ODE-IM correspondence (from the name of the two worlds it brings together, namely ordinary differential equations (ODEs), like the Schrödinger one, and integrable models (IMs)), was put forward by P. Dorey and R. Tateo in [8, 9], and completed at the same time by V.V. Bazhanov, S.L. Lukyanov and A.B.Zamolodchikov in [10]. In this series of papers, the authors showed how to use asymptotic solutions of a simple class of Schrödinger equations (the ones having as potential a monomial + the angular momentum term) to build objects which satisfy functional relations completely analogous of those encountered in the analysis of integrable models. These functional relations, which are called Q system, T system, TQ system and Y system, from the name of the quantities that they involve, allow, once found in the QM setting, to determine in an exact way relevant quantities for the Schrödinger problem, among all the energy spectrum. It is worth mentioning that the type of analysis pioneered by Dorey, Tateo, Bazhanov, Lukyanov and Zamolodchikov does not limit to the simple models studied in the first papers [8–10], and has since then been generalised to more complicate type of potentials and even different type of ODEs [11, 12].

The route used in [8–10] however, is not the only one that allows to find integrable structures in a given form of Schrödinger equation. If we focus on the Y-system and the

associated TBA equations, similar relations can in fact be obtained by exploiting another approach to the analysis of the Schrödinger problem, called *resurgent approach* to Quantum Mechanics, and based on an exact version of the WKB approximation method. This type of procedure is indeed the one used by Voros in its aforementioned works [3–6], and has been developed thanks to the efforts of many contributors, among them E. Delabaere, H. Dillinger and F. Pham (see e.g. [13]), J. Ecalle [14], T. Kawai and Y. Takei [15, 16], and H.J. Silverstone [17]. This exact WKB method have then been used in a wide variety of contexts related to the Schrödinger equation, and has gained recently quite some popularity.

In particular, in 2019, K. Ito, M. Mariño and H. Shu used this type of analysis (presenting also a re-derivation with a more "traditional" ODE-IM oriented perspective) to find a set of TBA equations governing the Borel resummed WKB periods associated to a Schrödinger equation with a generic type of polynomial potential with complex coefficients [1], extending the original works about the ODE-IM correspondence. In the paper, the authors first obtained the set of TBA equations for a simplified class of polynomial potentials, and then used a process that combined the analytic continuation of the TBA equations and a redefinition of the related pseudoenergies, to extend the results to the general case. This analytic continuation process is usually called *wall-crossing* of the TBA equations; one can in fact see that there exist boundaries in the space of parameters associated to the potential functions, which separate regions with different type of TBA equations, and when one performs an analytic continuation of the equations between these regions, he should be ready to modify them as the wall is encountered.

The first goal of the present thesis was to deepen the understanding of the process used by Ito and collaborators to derive the TBA equations in the various regions (more technically, *chambers*) of the space of parameters associated to a polynomial potential of any degree, and in particular study its consequences from the point of view of the related Y-system. The main outcome of this analysis was the development of an algorithmic procedure involving only Y functions and Y-systems, that allows one to obtain the TBA equations in a generic chamber (i.e. for a generic choice of polynomial potential) in a fast and consistent way. The main advantage of working with Y-systems, instead that with the TBA equations, lies in their algebraic, rather than integral, structure, which strongly facilitates their manipulation. Most part of this section of the thesis work has been done in collaboration with the colleague S. Franzoni, who has published its version of the results in [18]. Here, we try to give an alternative type of presentation with respect to the one described in [18], working from the start with a more Y-system oriented perspective, and deepening the motivations that lie behind some of the choices that guide the procedure. To build the intuition on how the procedure has been derived, we use as exemplification model a quartic potential, and then present the general version of the algorithm.

As a second type of application of this kind of analysis, the attention has been then turned onto the modified Mathieu equation, mainly due to its link with a deformation

of the 4d $\mathcal{N} = 2$ supersymmetric $SU(2)$ pure gauge theory [19], whose undeformed version is the well-known Seiberg-Witten (SW) theory [20]. The modified Mathieu equation, with an appropriate identification of its parameters, can in fact be considered as a Schrödinger-type equation with a periodic potential[1], making it also a relevant candidate to investigate the applicability of the procedure developed in the case of the polynomial potential to a wider class of potential functions. What's more, thanks to the connection with SW theory, the resummed quantum periods that usually enter into the TBA equations associated to a given choice of Schrödinger equation, take on in this case also an additional role, as they can be put [22] in 1-1 correspondence with the BPS states of the gauge theory. Driven by these motivations, we performed a study of the analytic continuation of the TBA equations from the strong to the weak coupling regions (in the SW language) of the parameter space, and evaluated the feasibility of generating the weak coupling TBA equations from the strong coupling ones, in a similar fashion of what previously done for the case of a polynomial potential. By looking at the Stokes graphs associated to the modified Mathieu equation in the two regions, we have indeed found a possible identification between the quantum periods and the spectrum of BPS states in the two regions, but we faced serious complications in the process of generating the new TBA equations from the old ones. Ultimately, the reason behind these complications shall be identified with a different type of wall-crossing with respect to the polynomial case, which do not allow for a direct application to the present case of the method developed in the latter one. To overcome this problem, we experimented with new strategies, which we discuss.
In order to summarize the whole content of this thesis work, we now briefly describe how the material is organized into the different chapters

- *Chapter 1* contains an overview of all the techniques developed in the context of integrable models which later will play a role in the study of the Schrödinger equation. The perspective used here is mainly that of 2D integrable quantum field theories, for which there exist a wide array of methods of analysis. We briefly present the S-matrix approach to integrable theories, which then allows us to derive the Thermodynamic Bethe Ansatz and the Y-system. We show how the TBA may be used to obtain the spectrum of the field theory, by using the mirror argument for the ground state, and then a process of analytic continuation of the TBA equations to reach the excited states (following [23]). This procedure will also give a flavour of the similar continuation process we will then use in Chapter 4 and 5. Finally, we present also the so-called BLZ construction for conformal field theories (as prime examples of integrable theories) [24–27], that allows us to introduce in the field theory context the notion of Q,T operators and their functional relations, and allow for a different way of deriving the Y-system and the TBA equations

[1]For the Generalized Mathieu equation, of which the modified Mathieu equation is just a special case, an ODE-IM analysis has been performed in [21]

which is more closely related to that used in the works [8–10] on the ODE-IM correspondence

- *Chapter 2* is devoted to the presentation of the ODE-IM correspondence, in its original formulation [8–10]. The "integrability side" of the correspondence has been already explained in the first chapter, so the focus of this part is primarily oriented towards the Schrödinger equation, as prototypical example of ODE (and also as the type of ODE concerning this thesis work). Here we show how to build the analog of the Q,T and Y functions appearing in integrable models, together with their functional relations, and make explicit the identification between the quantities belonging to the two different sides of the correspondence. We initially apply this type of analysis to the model that has been historically studied as first, namely that of a potential of monomial type + the angular momentum term, and then repeat the construction for the case of a generic polynomial potential, since this is the setting in which the Y-system algorithm presented in Chapter 4 has been derived

- *Chapter 3* contains a self-contained introduction to the last main piece of theoretical background needed to understand the original part of this thesis work, namely that of the exact WKB method. An overview of the main concepts involved is presented, together with some derivations, and a particular attention is posed towards the understanding of the Stokes phenomenon for the resummed series, which lies at the heart of the reason why TBA-like equations emerge in this context. We give then an explicit derivation, from the point of view of exact WKB theory, of the TBA equations associated to a Schrödinger equation with polynomial potential, in the particularly chosen region of the parameters space which we will than use as base point for the analytic continuation process (called *minimal chamber*). To conclude, we compare this derivation of the TBA equations with that coming from the ODE-IM approach of Chapter 2, making thus contact between the two worlds.

- In *Chapter 4* we pick up from the minimal chamber's TBA equations derived at the end of the previous chapter, and perform the study of analytic continuation of the TBA equations (along the lines of [1]) associated to a Schrödinger equation with polynomial potential. Using as explicit example a quartic potential, we show how this process can be translated in terms of Y functions and Y-systems, explaining the origin of the algorithmic procedure we developed, which we recall allows to readily find the TBA equations in any chamber of the parameters space through only algebraic manipulations of Y functions. The chapter concludes with a general formulation of the algorithm, valid for all polynomial potentials having distinct turning points.

- *Chapter 5* is dedicated to a study of the resummed quantum periods, and the

related TBA equations, in the case of the modified Mathieu equation, with special care to its connection with the deformed Seiberg-Witten theory. After a very brief excursus on the different approaches to this type of analysis that have been performed in the recent literature, we tackle the problem of first writing the TBA equations in the strong coupling region of the parameter space, and then analytically continue them onto the weak coupling one. We discuss the feasibility of applying to this case the algorithm we developed for a Schrödinger equation with polynomial potential, and comment on problems and possible solutions.

Chapter 1

Exact techniques for the analysis of 2D integrable quantum field theories

A classical system with a finite number n of degrees of freedom is said to be *integrable* (in the Liouville sense) if there exist n functions $F_i(\vec{p}, \vec{q})$, $i = 1, ..., n$ defined on its phase space (where $\vec{p}, \vec{q}$ are respectively the vectors of the generalised momenta and position) that are

(i) *independent*, i.e. the subset of phase space defined by fixing each F_i to a certain value is n dimensional

(ii) *conserved in time*, that is $\frac{dF}{dt} = \{H, F_i\} = 0$, where H is the Hamiltonian of the system and $\{\cdot, \cdot\}$ is the Poisson bracket

(iii) *in involution*, which means pairwise commuting under the Poisson bracket operation

An integrable system has the peculiar feature that its dynamics can be solved through a canonical transformation of phase space coordinates to the so called action-angle variables, in terms of which the motion is linear in time. This notion of integrability can be generalised also to classical fields, where the phase space is parametrised by the configurations of the field $\phi(x^\mu)$ and its conjugate momentum field $\Pi(x^\mu)$, and the F_i are local functionals of ϕ, Π, where for locality we mean that they are spatial integral of densities that depend only on $\phi(x^\mu), \Pi(x^\mu)$, and a finite number of their derivatives, all evaluated at the same space-time point x^μ. In this case, being the number of degrees of freedom of a field infinite, to have integrability there must be an infinite family of local integral of motions F_i.
Quantizing the field theory, the Poisson brackets get promoted to commutators, and the functionals acting on the phase space become operators acting on a Fock space. With these adaptations, integrability can be defined also for a quantum field theory (QFT):

> *We define integrable a QFT that has an infinite number of local conserved charges Q_i in involution (i.e. pairwise commuting)*

If a quantum field theory is integrable, it has the very remarkable property to be exactly solvable (that is, solvable not by means of a perturbative series), which implies that one is able to compute exactly:

- the mass spectrum of its excitations
- its S-matrix and correlation functions
- the thermodynamics of the associated statistical model

Actually, in $D > 2$ dimensions, one can show that any QFT which contains a parity conjugate pair of conserved charges of spin > 1 must be trivial, i.e. a free theory (for an heuristic explanation see [28], chapter 17.2.4). Relevant integrable quantum field theories are thus present only in (1+1) dimensions, which is the setting we will focus on in this chapter. Perhaps the most widely studied class of 2D integrable quantum field theories are the conformal field theories, for which an infinite tower of conserved charges can be built by considering descendant fields in the Verma module of the identity operator (among which the stress-energy tensor is present). Being conformal, these theories are massless. Massive integrable theories may be obtained for example by perturbing a conformal field theory via one of its relevant primary fields, but such a perturbation is not guaranteed to preserve integrability.
As the name suggests, this first introductive chapter presents different techniques that have been developed to study and solve in an exact way integrable QFTs in 2D. Our interest in presenting these results comes from a remarkable connection that was found, after the theory of 2D conformal theories was well established, between integrable models and a particular class of Schrödinger equations. Some of the fundamental set of equations that we will derive in this section, as for example the so called thermodynamic Bethe ansatz equation, will then appear also in the context of one dimensional Schrödinger problems, but with a different meaning and implications.

1.1 S-matrix theory in 2D and constraints from integrability

We start our presentation on 2D integrable QFTs by describing the peculiarities of the S-matrix theory when working in only (1+1) dimensions, and how integrability of a field theory limits the possibilities of the scattering phenomena. In particular, we will present the general form of the two particle S-matrix for a diagonal theory.
Following [29], we will work using light cone coordinates in Minkowski space-time, so

that the energy-momentum vector (p^0, p^1) will be given as $(p, \bar{p}) = (p^0 + p^1, p^0 - p^1)$, and the mass-shell condition becomes $p^\mu p_\mu = p\bar{p} = m^2$. We will also attach a letter $a, b, c, ...$ to the subscript of quantities in order to distinguish different particles in a theory. We could parametrise $(p_a, \bar{p}_a)$ as $(m_a a, m_a a^{-1})$, $a \in \mathbb{R}$, to explicitly fulfill the mass-shell condition, where $a > 0$ for particles and $a < 0$ for antiparticles (since the energy $p_a^0 = (p_a + \bar{p}_a)/2 = \frac{m_a a}{2}(1 + a^{-2})$ has the same sign of a), but a more convenient parametrisation is done by using the rapidity variable $\theta_a = \log(a)$, so that

$$p_a = m_a e^{\theta_a} \qquad \bar{p}_a = m_a e^{-\theta_a} \tag{1.1}$$

or, in terms of energy and momentum

$$p_a^0 = (p_a + \bar{p}_a)/2 = m_a \cosh\theta_a \qquad p_a^1 = (p_a - \bar{p}_a)/2 = m_a \sinh\theta_a \tag{1.2}$$

Differently from a, we regard θ_a as a complex variable. For particles, θ_a ranges on all the real line, while antiparticles are shifted on the line $\mathrm{Im}\,\theta_a = \pi$ due to the definition of the complex logarithm ($\theta_a = \ln(a) = \ln|a| + i\pi$, for $a < 0$). Exploiting the fact that we have only one spatial dimension, we can represent asymptotic states as

- an *in state* is characterised by the absence of further interactions at $t \to -\infty$, and we can write it as (we will usually omit the bracket notation)

$$A_{a_1}(\theta_1)A_{a_2}(\theta_2)\cdots A_{a_n}(\theta_n), \qquad \text{with } \theta_1 > \theta_2 > \cdots > \theta_n \tag{1.3}$$

  where $A_{a_i}(\theta_i)$ describes a particle of type a_i travelling with rapidity θ_i, and we have assumed an in state containing n particles.

- an *out state* is characterised by the absence of further interactions at $t \to +\infty$, and we similarly represent it as

$$A_{b_1}(\theta_1)A_{b_2}(\theta_2)\cdots A_{b_n}(\theta_n), \qquad \text{with } \theta_1 < \theta_2 < \cdots < \theta_n \tag{1.4}$$

  for an out state containing n particles

Ordering of symbols is very important in this notation, and we can consider the $A_{a_i}(\theta_i)$ as non-commutative objects. The S-matrix provides the mapping between the in state basis and the out state one. For example, for a 2-particle in state

$$A_{a_1}(\theta_1)A_{a_2}(\theta_2) = \sum_{n=2}^{\infty} \sum_{\{b_1\cdots b_n\}} \sum_{\theta_1' < \cdots < \theta_n'} S_{a_1 a_2}^{b_1\cdots b_n}(\theta_1, \theta_2; \theta_1' \cdots \theta_n') A_{b_1}(\theta_1') \cdots A_{b_n}(\theta_n') \tag{1.5}$$

where the first sum runs over the number of particles in the out state, the second one over their type, and the last one represents a number of integrals over the rapidities θ_i',

with constraints imposed on the θ_i' by conservation of left- and right- lightcone momenta.

The above discussion was generic to any quantum field theory in (1+1) dimensions. Let's now investigate the consequence of the existence of local conserved charges in the theory. In an integrable (relativistic) QFT, we can always arrange the conserved charges Q_i so that they are labelled by a spin index $s \in \mathbb{Z}$, whose value describes the transformation properties of the operator Q_i under boosts [30]. The integrable theory can then be characterised by an infinite set of spin values $\{s_i\}$ (that obviously will not cover all the numbers in $\mathbb{Z}$), each with its associated conserved charge Q_{s_i}. The action a charge Q_s of spin s on a 1-particle asymptotic state $A_a(\theta_a)$ is given by

$$Q_s A_a(\theta_a) = q_a^{(s)} e^{s\theta_a} A_a(\theta_a) \tag{1.6}$$

where $q_a^{(s)}$ is a factor depending on the particle type a and on the charge Q_s chosen. For example, if we consider as charge the left component P in lightcone coordinates of the energy momentum operator, which has spin 1, then $q_a^{(s)} = m_a$, so that $PA_a(\theta_a) = p_a A_a(\theta_a)$ as expected.
Since we have restricted ourselves to considering charges that are integrals of a local density, their action on a multiparticle state is just additive

$$Q_s\, A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n) = \left(q_{a_1}^{(s)} e^{s\theta_{a_1}} + \cdots + q_{a_n}^{(s)} e^{s\theta_{a_n}}\right) A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n) \tag{1.7}$$

In presence of a conserved charge Q_s, the S-matrix must have nonzero elements only between states that share the same total value of Q_s. For the matrix element $S_{a_1\cdots a_n}^{b_1\cdots b_m}$ describing a $n \to m$ scattering, this traduces in the constraint

$$q_{a_1}^{(s)} e^{s\theta_{a_1}} + \cdots + q_{a_n}^{(s)} e^{s\theta_{a_n}} = q_{b_1}^{(s)} e^{s\theta'_{b_1}} + \cdots + q_{b_m}^{(s)} e^{s\theta'_{b_m}} \tag{1.8}$$

In an integrable theory, we will have an infinite tower of these relations, one for each spin value s associated to a conserved charge of the theory. These conditions do not force the set of outgoing particles to be the same of that of the initial ones, but still severely constrain the form of the S-matrix. In particular, they imply:

- no particle production (so same number of particles in in and out states)
- equality of the set of initial and final momenta
- factorisability of any $n \to n$ S-matrix element into $2 \to 2$ matrix elements

This last point motivates us to focus only onto the two-particle S-matrix, whose action is encoded in

$$A_i(\theta_1)A_j(\theta_2) = S_{ij}^{kl}(\theta_1 - \theta_2)A_l(\theta_2)A_k(\theta_1), \qquad \theta_1 > \theta_2 \tag{1.9}$$

where a sum over the indices k, l is implied, and the fact that S_{ij}^{kl} depends not separately on θ_1, θ_2 but only on their difference is due to invariance of the theory under boosts, that shift all the θ_i by a constant. The two particle S-matrix is further constrained by energy-momentum conservation, which implies $m_i = m_k$ and $m_j = m_l$, charge, parity, and time-reversal symmetry, crossing symmetry and unitarity. In what follows we will also restrict to the simplified setting where all the asymptotic states can be distinguished by their values of the conserved charges of the theory (mass included), so that the the in and out state will contain the same number of particles for each type. In this case the two particle S-matrix $S_{ij}(\theta)$ (which now has only two indices) is called *diagonal*, and has a simple analytic structure in the complex θ plane. Unitarity and crossing symmetry imply $S_{ij}(\theta + 2\pi i) = S_{ij}(\theta)$, and so we can capture the full behaviour of the scattering process by considering only the region $-i\pi < \mathrm{Im}\,\theta < i\pi$. In particular, simple poles of the S-matrix displaced along the imaginary axis inside the so called "physical strip" $0 < \mathrm{Im}\,\theta < i\pi$, are usually associated to virtual bound states of the theory (that is, excitations contained into the spectrum of the theory, eventually decaying at $t \to \infty$ into asymptotic states). The location $\theta_k = iu_{ij}^k$ of each of these poles is linked to the mass squared of the related bound state as [1]

$$m_k^2 = m_i^2 + m_j^2 + 2m_i m_j \cosh(\theta_k) = m_i^2 + m_j^2 + 2m_i m_j \cos(u_{ij}^k) \tag{1.10}$$

The most remarkable consequence of all the constraints imposed on the S-matrix is however that we are able to write an explicit form for S_{ij} that holds for a generic theory. The only additional requirement we have to assume, which is an expected feature of a local QFT, is for scattering amplitudes to be polynomially bounded on the momenta. Any two particle S-matrix of a diagonal integrable QFT can then be presented as

$$S_{ab}(\theta) = \prod_{\alpha \in A_{ab}} s_\alpha(\theta), \quad \text{where } s_\alpha(\theta) = \frac{\sinh(\theta + i\alpha\pi)/2}{\sinh(\theta - i\alpha\pi)/2} \tag{1.11}$$

where $\alpha \in \mathbb{R}$, $-1 \leq \alpha \leq 1$, and A_{ab} is a family of values for α that describes the location of the simple poles of $S_{ab}(\theta)$, and so are linked to the masses of virtual bound states that can be created by the scattering of a particle of type a with one of type b. It is immediate to see in fact that each factor s_α entering in $S_{ab}(\theta)$, which is an instance of a so called *CDD factor*, has a simple pole at $\theta = i\alpha\pi$.

[1] This relation can be understood by interpreting the pole of S_{ij} as coming from a propagator contained into it, representing the propagation of the bound state. Expressing the S-matrix not in terms of θ, but of the Mandelstam variable s, the propagator is just $\sim i/(s - m_k^2)$, and so the location of the pole in the s plane coincides with m_k^2. The equation (1.10) is then just a consequence of the relation between s and θ

$$s = m_i^2 + m_j^2 + 2m_i m_j \cosh(\theta)$$

1.2 Thermodynamics and the TBA

The Thermodynamic Bethe Ansatz (TBA) is a technique that allows one to obtain the thermodynamics of a integrable scattering theory from its S-matrix. Building on previous results found by C. Yang and C.P. Yang for non-relativistic models [31], the field theory version of the TBA has been first proposed by Al. B. Zamolodchikov in [32] for two simple purely elastic (i.e. diagonal) theories, and later generalised to a vast class of diagonal and non diagonal integrable theories. In our discussion, we will focus on the diagonal case, to present just the essential features of the TBA.
In this method, the calculation is reduced to the solution first of a set of nonliner integral equations for one-particle excitation energies and then of another set of nonlinear integral equations for the rapidity distributions of the species of particles in the theory. The derivation of these equations consists of two parts:

1. first, we use the assumption that the asymptotic wave function for our system of particles has a very simple form. Putting the system in a circle with periodic or antiperiodic boundary conditions, and exploiting the fact that different configurations of the asymptotic wavefunction are related through the S-matrix, one obtains then quantisation conditions for the set $\{\theta_i\}$ of the rapidities of the particles in the asymptotic states.

2. The second part is just statistical mechanics. One performs the thermodynamic limit (TD) on the system and uses the minimisation of the free energy to find the dominant macroscopic configurations (i.e. the rapidity distributions) consistent with the constraint of having fixed the (spatial) density of each particle type. This procedure leads to the nonlinear integral equations mentioned before

Let's now discuss the two steps in detail, using as main reference [33, 34].

Asymptotic wave function and quantisation of rapidities at finite size

Consider a QFT on a circle of circumference L (with L large) that has purely elastic scattering. Let there be n different species of asymptotic particles in the theory, labelled by an index $a = 1, ..., n$, with masses m_a. We define the correlation length ξ of the theory as the inverse of the smallest of these masses. Since the theory is diagonal, we can specify the S-matrix via the scattering phases, defined as

$$S_{ab}(\theta) = e^{i\delta_{ab}(\theta)}, \quad \text{i.e. } \delta_{ab}(\theta) = -i \ln S_{ab}(\theta) \tag{1.12}$$

where we choose for the complex logarithm its first determination. As already discussed in the previous section, the Hilbert space of a theory with diagonal scattering is rather simple: given any N-particle state with N_a particles of each type the integrability of the theory ensures that time evolution preserves both the identity of the particles and the set

of their momenta. What's more, if we choose to focus on those regions of configuration space $(x_1, ..., x_N)$ of the system where particles are strongly separated (so that no virtual processes are involved), which means $|x_i - x_j| \gg \xi,\ i, j = 1, ..., N\ ,\ i \neq j$, then it makes sense to associate to an N-particle state of the system a wavefunction $\psi(x_1, ..., x_N)$. We will call these regions *free regions*, and distinguish them only by the ordering of particles $Q(1, ..., N)$ in space, where $Q(1, ..., N)$ indicates a permutation of the indices of the particles that reflects their order in space, i.e. $x_{Q_1} < x_{Q_2} < ... < x_{Q_N}$ (Q_n is the n-th number in the permutation Q). Given a set of rapidities $\{\theta_i\}$ for an N-particle state, to build the corresponding wavefunction ψ we exploit the fact that in each free region ψ must be proportional to a superposition of plane waves, one for each particle i, with the appropriate momentum $p_i = m_i \sinh(\theta_i)$, and a proportionality coefficient A that depends on the free region considered (so on the related ordering Q):

$$\psi(x_1, ..., x_N) = e^{i \sum_i^N p_i x_i} \sum_{Q \in S_N} A(Q) \Theta(x_Q) \tag{1.13}$$

where S_N stands for the set of all possible permutations of $\{1, ..., N\}$ and the symbol $\Theta(x_Q)$ is defined to be

$$\Theta(x_Q) = \begin{cases} 1, & \text{if } x_{Q_1} < x_{Q_2} < ... < x_{Q_N} \\ 0, & \text{otherwise} \end{cases} \tag{1.14}$$

We will refer to this form of wavefunction as *asymptotic wavefunction.*
The S-matrix provides the matching condition for ψ across each free region, and so allows to determine the coefficients $A(Q)$ up to an irrelevant overall factor. For example, if two permutations $Q = (..., i, j, ...)$ and $Q' = (..., j, i, ...)$ differ only for the exchange of two numbers i, j, we have

$$A(Q') = S_{ij}(\theta_i - \theta_j) A(Q) \tag{1.15}$$

Boundary conditions at $x = 0$ and $x = L$ induce then a quantisation on the possible sets $\{\theta_i\}$ of particle rapidities. In particular, if we choose periodic (antiperiodic) boundary conditions for bosons (fermions)[2], we have the quantisation conditions

$$\psi(..., x_i = L, ...) = (-1)^{F_i} \psi(..., x_i = 0, ...), \qquad i = 1, ..., N \tag{1.16}$$

with $F_i = 0$ $(= 1)$ if the i^{th} particle is a boson (fermion). Using our expression (1.13) for ψ, this leads to

$$A(i, Q_2, ..., Q_N) = (-1)^{F_i} e^{i p_i L} A(Q_2, ..., Q_N, i) \tag{1.17}$$

[2]This prescription is important to have a proper correspondence between the generating functional of the theory in the Euclidean path integral formulation and its partition function in the statistical mechanics approach

for every $Q \in S_N$ such that $Q_1 = i$. We have such a constraint for every particle $i = 1, ..., N$. Using now the fact that the two A coefficients in this formula can be related by the S-matrix via a repeated use of (1.15), we obtain

$$\begin{aligned} A(i, Q_2, ..., Q_N) &= (-1)^{F_i} e^{ip_i L} \left(\prod_{j \neq i} S_{ij}(\theta_i - \theta_j) \right) A(i, Q_2, ..., Q_N) \\ \Rightarrow \quad e^{ip_i L} &\left(\prod_{j \neq i} S_{ij}(\theta_i - \theta_j) \right) = (-1)^{F_i}, \quad i = 1, ..., N \end{aligned} \tag{1.18}$$

Taking the logarithm, we can rewrite this set of relations in terms of the phase shifts, and expressing the momenta through their rapidities (this is allowed since particles in the free regions are considered on-shell), one arrives to the so called *coordinate Bethe ansatz equations*:

$$L m_i \sinh \theta_i + \sum_{j \neq i} \delta_{ij}(\theta_i - \theta_j) = 2\pi n_i, \quad i = 1, ..., N \tag{1.19}$$

where the quantum numbers n_i we have introduced are integers or half-integers depending if the i^{th} particle is a boson or a fermion. From now on, we choose to focus on the latter case.

The coordinate Bethe ansatz equations are N coupled (via the phase shifts) transcendental equations which determine the allowed sets $\{\theta_i\}$ of quantised rapidities for a chosen N-particle asymptotic state. The admissible rapidity sets are specified via the sets of quantum numbers $\{n_i\}$, and for several models one can show that the solution is unique for every set $\{n_i\}$ of pairwise distinct numbers, and the totality of the resulting Bethe states $\psi_{\{\theta_i\}}$ form a basis for the interacting Hamiltonian. We will assume this as an hypothesis. The energy and momentum associated with these states will have the simple form:

$$E(\{\theta_i\}) = \sum_{i=1}^{N} m_i \cosh \theta_i, \quad p(\{\theta_i\}) = \sum_{i=1}^{N} m_i \sinh \theta_i \tag{1.20}$$

Before moving on the next step in the derivation of the TBA equations, it is worthwhile to present an alternative form of equations (1.19)

$$p_i = \frac{2\pi n_i}{L} - \frac{1}{L} \sum_{j \neq i} \delta_{ij}(\theta_i - \theta_j), \quad i = 1, ..., N \tag{1.21}$$

We see that if all interactions are trivial, i.e. the theory is a free theory, the Bethe equations coincide with the usual quantisation conditions $p_i = 2\pi n_i / L$ for a free particle in a box. The effect of interactions, encoded in the phase shifts, is then to introduce a variation in the sets of admissible values for the momenta, and so accordingly shift the energy levels of the multiparticle system via (1.20).

Derivation of the thermodynamics

What we have done up to now was a microscopic description of the system. To obtain from it the thermodynamics, we have first to implement the TD limit, and then look at the macroscopic variables, such as (1.20), at thermal equilibrium. As we will see in a moment, the TD limit will help simplify the difficult transcendental equations (1.19) and lead to a series of integral equations.
In the TD limit $L \to \infty$ and $N_a \to \infty$, with the densities N_a/L staying finite and fixed for each specie of particles. This implies:

- The Bethe states are now formed by an infinite number of particles, and so the corresponding rapidity sets $\{\theta_i\}$ have an infinite number of elements. What's more, since (from the coordinate Bethe ansatz equations) the difference between two consecutive rapidities in a Bethe state behaves like $|\theta_i - \theta_{i-1}| \sim 1/mL$, the "occupied" rapidities in a state condense, and can so be described by a continuous distribution $\rho^{(r)}(\theta)$ (r here stands for roots, i.e. roots of Bethe eqs.). By definition, this particle density $\rho^{(r)}(\theta)$ is such that the number of particles in a state with rapidity in the interval $[\theta, \theta + \Delta\theta]$ is given by $\rho^{(r)}(\theta)\Delta\theta L$. In what follows, it is convenient to separate in each set $\{\theta_i\}$ the subsets $\{\theta_{a,i}\}$ relative to each single particle type a, and call the corresponding subset of quantum numbers $\{n_{a,i}\}$. In the TD limit, we can then associate a density of roots $\rho_a^{(r)}(\theta)$ to each particle type.

- Considering now the coordinate Bethe ansatz equations (1.19), in the TD limit we can rewrite them using the newly introduced root densities as

$$\frac{m_a}{2\pi}\sinh\theta_{a,i} + \sum_b \int \frac{d\theta}{2\pi}\delta_{ab}(\theta_{a,i} - \theta)\rho_b^{(r)}(\theta) = \frac{n_{a,i}}{L} \tag{1.22}$$

  We give a name to the function that is on the left hand-side of this equation, calling it *counting function*

$$J_a(\theta) \equiv \frac{m_a}{2\pi}\sinh\theta + \frac{1}{2\pi}\sum_b \left(\delta_{ab} * \rho_b^{(r)}\right)(\theta) \tag{1.23}$$

  where we have used the convolution symbol to simplify the form of the expression. In terms of the counting function, the coordinate Bethe ansatz equations (in the TD limit) are just

$$J_a(\theta_{a,i}) = \frac{n_{a,i}}{L} \tag{1.24}$$

  with one of equation for each particle i.
  Being an infinite number of equations, the Bethe ansatz equations in the TD limit are not very useful as they stand, but they can be recast in a more manageable form. Assume for the moment that the counting functions $J_a(\theta)$ are monotonically

increasing; we will prove this assumption at the end of the analysis. Choosing the labelling of the $\theta_{a,i}$ values in a state such that $\theta_{a,i} < \theta_{a,i+1}$, the monotonicity of the J_a implies that also the sequence of $\{n_{a,i}\}$ is monotonically increasing with i (recall (1.24)), and in particular is strictly increasing since fermionic particles in a state can't have the same value of θ.
As already stressed, of all possible integer numbers, only a subset of them $\{n_{a,i}\}$ will solve the Bethe equations. If we associate a "hole" density of states $\rho_a^{(h)}(\theta)$ related to the non realised $\{n_{a,i}\}$, we will have that given any number $\bar{n}_a \in \mathbb{Z}$, it will correspond to the total number of realised + not realised values of θ in a state up to the value $\bar{\theta}$ associated to it, i.e.

$$LJ_a(\bar{\theta}) = \bar{n}_a = L \int_{-\infty}^{\bar{\theta}} d\theta \, \rho_a^{(r)}(\theta) + \rho_a^{(h)}(\theta) \tag{1.25}$$

Restoring instead of J_a its defining expression (1.23), and deriving the above equation one gets

$$\rho_a(\theta) = \frac{m_a}{2\pi} \cosh\theta + \frac{1}{2\pi} \sum_b \left(\phi_{ab} * \rho_b^{(r)}\right)(\theta) \tag{1.26}$$

where we have called $\rho_a(\theta) \equiv \rho_a^{(r)}(\theta) + \rho_a^{(h)}(\theta)$ the density of collectively roots and holes, and $\phi_{ab}(\theta) \equiv \frac{d}{d\theta}\delta_{ab}(\theta) = -i\frac{d}{d\theta} \ln S_{ab}(\theta)$.

At this stage, we have managed to convert the infinite number of coordinate Bethe ansatz equations to the finite (one for each type of particle) number of equations (1.26) in terms of the densities. However, in order to do so, we had to introduce also the density of holes, which makes the set of equations (1.26) not sufficient to determine both the root and the holes densities. If we are interested in the thermodynamics of the system, what is left to do is to impose the minimisation of free energy.
The total entropy per unit length $s[\rho_a^{(r)}, \rho_a^{(h)}]$ associated to a certain density of roots and holes (for all particle types) can be calculated by noticing that, even if we have defined $\rho_a(\theta)$ so that $\rho_a(\theta)L\Delta\theta$ corresponds to the total number of roots and holes of a species contained in the interval $[\theta, \theta + \Delta\theta]$, we can reshuffle the roots and the holes in that interval without changing $\rho_a(\theta)$, and so the macroscopic variables of the system (this is the typical situation encountered in statistical mechanics where many microscopic states $\{\theta_{a,i}\}$ correspond to the same macroscopic description $\rho_a^{(r)}(\theta)$). Recalling that for fermionic particles the number of ways we can reshuffle is

$$\Omega_a = \frac{(n_a^{(r)} + n_a^{(h)})!}{n_a^{(r)}!\, n_a^{(h)}!}$$

we get

$$
\begin{aligned}
s[\rho_a^{(r)},\rho_a^{(h)}] &= \sum_a s_a[\rho_a^{(r)},\rho_a^{(h)}] = \sum_a \int_{-\infty}^{+\infty} d\theta \ln \frac{(\rho_a^{(r)}+\rho_a^{(h)})!}{\rho_a^{(r)}!\ \rho_a^{(h)}!} = \\
&= \sum_a \int_{-\infty}^{+\infty} d\theta \left[(\rho_a^{(r)}+\rho_a^{(h)}) \ln (\rho_a^{(r)}+\rho_a^{(h)}) - \rho_a^{(r)} \ln \rho_a^{(r)} - \rho_a^{(h)} \ln \rho_a^{(h)}\right]
\end{aligned}
\tag{1.27}
$$

where in the last step we have used the Stirling approximation for the logarithm. The energy per unit length is just the continuous version of (1.20), i.e.

$$
h[\rho_a^{(r)}] = \sum_a \int_{-\infty}^{+\infty} d\theta \rho_a^{(r)} m_a \cosh\theta \tag{1.28}
$$

When minimising the free energy per unit length $f = h - Ts$ (here T is the temperature) we have as usual to take account of the constraints imposed on the system, adding them to f as Lagrange multipliers. In this case, the constraints come from equation (1.26), and the functional to minimize is $\hat{f} \equiv f - \sum_a \lambda_a \Phi_a$, with

$$
\Phi_a[\rho_a^{(r)},\rho_a^{(h)}] = \int_{-\infty}^{+\infty} d\theta \left[(\rho_a^{(r)}+\rho_a^{(h)}) - \frac{m_a}{2\pi}\cosh\theta - \frac{1}{2\pi}\sum_b \left(\phi_{ab} * \rho_b^{(r)}\right)(\theta)\right] \tag{1.29}
$$

Deriving $\hat{f}$ separately with respect to $\rho_a^{(r)}$ and $\rho_a^{(h)}$ and setting the two derivatives to zero, one obtains

$$
\begin{cases}
\frac{\partial \hat{f}}{\partial \rho_a^{(r)}} = \int_{-\infty}^{+\infty} d\theta \ \ m_a \cosh\theta - T \ln \frac{\rho_a^{(r)}+\rho_a^{(h)}}{\rho_a^{(r)}} - \lambda_a + \frac{1}{2\pi}\sum_b (\phi_{ab} * \lambda_b)(\theta) = 0 \\
\frac{\partial \hat{f}}{\partial \rho_a^{(h)}} = \int_{-\infty}^{+\infty} d\theta \ \ - T \ln \frac{\rho_a^{(r)}+\rho_a^{(h)}}{\rho_a^{(h)}} - \lambda_a = 0
\end{cases}
\tag{1.30}
$$

Finally, using the second of the two equations to solve for the λ_a, and putting the resulting expression into the first, one obtains the celebrated *thermodynamic Bethe ansatz (TBA) equations*:

$$
\epsilon_a(\theta) = m_a R \cosh\theta - \frac{1}{2\pi}\sum_b (\phi_{ab} * L_b)(\theta) \tag{1.31}
$$

where we have called $R \equiv T^{-1}$, and introduced the *pseudoenergies* $\epsilon_a(\theta)$ as

$$
\epsilon_a(\theta) \equiv -\ln \rho_a^{(r)}/\rho_a^{(h)} \tag{1.32}
$$

together with the related functions

$$
L_a(\theta) \equiv \ln\left(1+e^{-\epsilon_a(\theta)}\right) = \ln \frac{\rho_a^{(r)}+\rho_a^{(h)}}{\rho_a^{(h)}} \tag{1.33}
$$

The TBA equations are a finite (one for each particle type a) set of nonlinear integral equations for the pseudoenergies. To determine the thermodynamics of the system one first solves the TBA equations to find the pseudoenergies, and then via equations (1.32, 1.26) is able to reconstruct the density of both roots and holes, in term of which any thermodynamic state function can be expressed. We list below just some of the expressions for the most relevant thermodynamic quantities, as functions of $\rho_a^{(r)}, \rho_a^{(h)}$ or the $\epsilon_a(\theta)$:

- ENERGY: refer back to equation (1.28)
- ENTROPY: one can use the expression (1.27), which can be also transformed into the equivalent form

$$s[\rho_a^{(r)}, \rho_a^{(h)}] = \sum_a \int_{-\infty}^{\infty} d\theta \;\; R\rho_a^{(r)} m_a \cosh\theta + \frac{m_a}{2\pi} \cosh\theta L(\theta) \tag{1.34}$$

- FREE ENERGY: from the equations for the energy and the entropy, one can extract

$$f[\rho_a^{(r)}, \rho_a^{(h)}] = -\frac{1}{2\pi R} \sum_a \int_{-\infty}^{+\infty} d\theta \;\; m_a \cosh\theta L(\theta) \tag{1.35}$$

- PARTITION FUNCTION:

$$Z(L,R) = e^{-\frac{1}{T}Lf} = e^{\sum_a \int_{-\infty}^{+\infty} \frac{d\theta}{2\pi}\, m_a \cosh\theta L(\theta)} \tag{1.36}$$

  One can also show (see [28], section 20.5) that this expression is equivalent to

$$Z(L,R) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi} \, \langle \theta_n \cdots \theta_1 | \theta_1 \cdots \theta_n \rangle \, \prod_{i=1}^{n} e^{-\epsilon(\theta_i)} \tag{1.37}$$

  where the scalar product is computed using the rules of free fermionic systems. This form for the partition function clearly shows how a diagonal scattering theory in the TD limit can be interpreted as a gas of free quasi-particles, with effective energy not given by the usual dispersion relation of particles $m_a \cosh\theta$, but equal to $e_a(\theta) \equiv \epsilon(\theta)/R$. Similarly, the effective momentum of these quasi-particles is given by $2\pi J_a(\theta)$ (recall eq. (1.24)), and includes the usual term $m_a \sinh\theta$ plus a correction due to the presence of interactions.

To conclude our discussion on the TBA equations, we make two final comments:

(i) we have left to prove the statement made earlier that the counting functions $J_a(\theta)$ are monotonically increasing. The TBA equations imply that the pseudoenergies are real functions of θ for any value of $r \equiv R/\xi$. This then means that $\rho_a, \rho_a^{(r)} > 0$, and since $J_a(\theta) = \frac{d}{d\theta}\rho_a(\theta)$ the monotonicity of the $J_a(\theta)$ follows

(ii) the TBA equations clearly show a conceptual difference between the energy levels of a free theory and those of an interacting theory with diagonal scattering. For free theories the levels are simply determined by the quantisation of the single particle states, and are the same whatever is the number of particles in the system and the distributions of occupation of the levels. In interacting theories, instead, the levels are determined in a self-consistent way with the statistical distribution of the particles themselves; this is the meaning of the non-linear TBA equations for the pseudoenergies.

1.3 Y-system

There is a class of two dimensional integrable QFTs, defined in terms of their S-matrix, for which the TBA equations can be recast in a more "universal" form [35, 36]: these are the so-called *ADE theories*, that is, diagonal scattering theories whose S-matrix is related to a certain simply-laced affine Lie algebra of the series A_n, D_n, E_6, E_7, E_8. These theories can be identified as perturbations of certain conformal field theories which preserve integrability (for a precise identification of the related CFTs see [33]). The ADE theories have a number of particles r equal to rank of the related Lie algebra, and to each of these particles may be associated a node on the Dynkin diagram of the algebra. With this correspondence in mind, one can show that the conserved charges associated to a certain spin value s in the field theory can be arranged into eigenvectors of the incidence matrix I_{ab} of the Dynkin diagram, with a spin dependent eigenvalue

$$\sum_b I_{ab} q_b^{(s)} = 2\cos\left(\frac{\pi s}{h}\right) q_a^{(s)} \tag{1.38}$$

being h the Coxeter number of the Lie algebra.
This relation, specialised for $s = 1$, tells us in particular that the masses of the particles in the theory are proportional to the Perron-Frobenius eigenvector of I_{ab}, namely the eigenvector corresponding to its highest eigenvalue (that for a matrix with non-negative entries like I_{ab} is unique). Being diagonal theories, ADE theories have a two particle S-matrix of the form (1.11), with poles corresponding to bound states located along the imaginary θ axis at integer multiples of π/h. Using the explicit form of the S-matrix, one can show then the following relation, peculiar to ADE theories (for a rigorous derivation refer to [36])

$$\mathcal{S}_{ab}\left(\theta + \frac{i\pi}{h}\right)\mathcal{S}_{ab}\left(\theta - \frac{i\pi}{h}\right) = \prod_c \mathcal{S}_{ac}(\theta)^{I_{bc}} e^{-2\pi i I_{ab}\Theta(\theta)} \tag{1.39}$$

where $\Theta(\theta)$ is the step function

$$\Theta(\theta) = \begin{cases} 0, & \text{if } \theta < 0 \\ \frac{1}{2}, & \text{if } \theta = 0 \\ 1, & \text{if } \theta > 0 \end{cases} \tag{1.40}$$

that implies the presence of a corrective term into equation (1.39) only when $\theta = 0$. Expressing the relation in terms of the functions $\phi_{ab}(\theta) = -i\frac{d}{d\theta}\ln S_{ab}(\theta)$ that enter into the standard TBA equations, one has

$$\phi_{ab}\left(\theta + \frac{i\pi}{h}\right) + \phi_{ab}\left(\theta - \frac{i\pi}{h}\right) = \sum_c I_{bc}\phi_{ac}(\theta) - 2\pi\delta(\theta)I_{ab} \tag{1.41}$$

which Fourier transformed is (we call k the momentum corresponding to θ)

$$2\cosh\left(\frac{k\pi}{h}\right)\tilde{\phi}_{ab}(k) = \sum_c I_{bc}\tilde{\phi}_{ac}(k) - 2\pi I_{ab} \tag{1.42}$$

where we used the following convention for the Fourier transform

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} d\theta\, e^{ik\theta} f(\theta) \tag{1.43}$$

The stage for obtaining the universal form of the TBA equations has now been set up. What is left to do then is to first Fourier transform the standard TBA equations (1.31) to obtain

$$\tilde{\epsilon}_a(k) = \tilde{\nu}_a(k) - \frac{1}{2\pi}\sum_b \tilde{L}_b(k)\tilde{\phi}_{ab}(k) \tag{1.44}$$

where we have introduced the common notation $\nu_a(\theta) \equiv m_a R\cosh\theta$. We then multiply to both sides of this equation the matrix element $\delta_{ac} - \tilde{R}(k)I_{ac}$, where $\tilde{R}(k) \equiv \frac{1}{2\cosh k\pi/h}$, sum over repeated indices, and use the relation (1.42) to get (after a relabelling of the indices)

$$\tilde{\epsilon}_a(k) = \tilde{\nu}_a(k) - \tilde{R}(k)\sum_b I_{ab}\left(\tilde{\nu}_b(k) - \tilde{\epsilon}_b(k) - \tilde{L}_b(k)\right) \tag{1.45}$$

which transformed back to the θ space reads

$$\begin{aligned} \epsilon_a(\theta) &= \nu_a(\theta) - \frac{1}{2\pi}\phi_h * \sum_b I_{ab}\left(\nu_b(\theta) - \epsilon_b(\theta) - L_b(\theta)\right) = \\ &= \nu_a(\theta) - \frac{1}{2\pi}\phi_h * \sum_b I_{ab}\left(\nu_b(\theta) - \ln(1 + e^{\epsilon(\theta)})\right) \end{aligned} \tag{1.46}$$

in terms of the universal kernel (it's the antitransform of $\tilde{R}(k)$, up to a 2π factor)

$$\phi_h \equiv \frac{h}{2\cosh h\theta/2} \tag{1.47}$$

that depends only on the Coxeter number of the Lie algebra. The equations (1.46), equivalent to the usual TBA ones, constitute the so-called *universal form* of the TBA equations (for the ADE theories). The relevance of this form for the TBA equations lies in the fact that they depend only on features of the Dynkin diagram (and thus the Lie algebra) associated to the theory, bypassing the S-matrix description. Another important consequence of having a unique kernel for all the terms that enter in the TBA equations, as first observed in [35], is that we can obtain from the TBA a set of *algebraic*, rather than integral, equations for the pseudoenergies, known as *Y-system*. To obtain them one performs a smooth shift of the universal TBA equations in the direction $\theta \to \theta + i\pi/h$ and then add them to the equations shifted to $\theta \to \theta - i\pi/h$. Taking account of the residues from the poles of the kernel, one finds

$$\epsilon_a(\theta + i\pi/h) + \epsilon_a(\theta - i\pi/h) = \nu_a(\theta + i\pi/h) + \nu_a(\theta - i\pi/h) - \sum_b I_{ab}\left(\nu_b(\theta) - \ln(1 + e^{\epsilon(\theta)})\right) \tag{1.48}$$

This expression can be greatly simplified by restoring the explicit form for the $\nu_a(\theta)$

$$\begin{aligned} \nu_a(\theta + i\pi/h) + \nu_a(\theta - i\pi/h) &= m_a R\left(\cosh\left(\theta + i\pi/h\right) + \cosh\left(\theta - i\pi/h\right)\right) = \\ &= 2m_a R\cosh\left(\theta\right)\cosh\left(\pi/h\right) = \\ &= R\cosh\left(\theta\right)\sum_{b=1}^{r} I_{ab} m_b = \sum_{b=1}^{r} I_{ab}\nu_b \end{aligned} \tag{1.49}$$

where in the penultimate step we used the relation (1.38) specialised in the case $s = 1$ (where the charges are the masses m_a). Looking back at the equations (1.48) we see that the term just calculated gets cancelled by a part of the contribution coming from the residues of the kernels, leaving with

$$\epsilon_a(\theta + i\pi/h) + \epsilon_a(\theta - i\pi/h) = \sum_b I_{ab}\ln(1 + e^{\epsilon(\theta)}) \tag{1.50}$$

which, rewritten in terms of the so-called *Y-functions*, $Y(\theta) \equiv e^{\epsilon(\theta)}$, becomes the aforementioned Y-system

$$Y_a(\theta + i\pi/h)Y_a(\theta - i\pi/h) = \prod_b \left[1 + Y(\theta)\right]^{I_{ab}} \tag{1.51}$$

The Y-system equations have been derived here through the original route, the one proposed in [35], that involves obtaining them through the universal form of the TBA,

but as it has become clear by further investigations on integrable QFTs, these are far more general than the original TBA equations. They can be extracted directly from the integrable structure of the theory, as we will see in section 1.5, and the same Y-system can describe several different theories. Y-systems and TBA equations have been derived in a wider class of theories than diagonal ones, and also appeared in other areas of physics.
We conclude by mentioning that looking at the Y-system, one can extract the periodicity of the Y-functions, which for ADE theories reads [36]

$$Y_a(\theta + Pi\pi) = Y_{\bar{a}}(\theta)\,, \qquad P = \frac{h+2}{h} \tag{1.52}$$

where $\bar{a}$ represents the antiparticle of a. A remarkable property is that the periods are directly related to the conformal dimension Δ of the perturbing field that makes the related CFT flow into the ADE theory along the RG flow:

$$\Delta = 1 - \frac{1}{P} \tag{1.53}$$

1.4 Using the TBA to find the spectrum

The ground state

In section 1.2 we presented the TBA equations as a technique to find the thermodynamics of a (1+1) dimensional integrable scattering theory. In that case we considered a spatial dimension compactified on a circle of length L, and made no assumption on the other direction, that in the relativistic perspective may be related to time. Assume now that also this other direction is compactified to a circle (so that the overall 2D space is a torus), of length R. Thanks to the correspondence between a (1+1) QFT in its Euclidean formulation and its thermodynamics at finite temperature, the length R of the compactified "time" direction can be identified with the inverse temperature of the quantum statistical model. This is why we have chosen the same letter R here as the one used in chapter for T^{-1}, i.e. the inverse temperature. One can go also further, and use the equivalence of time and space directions in the Euclidean formulation of a QFT, to relate the theory quantised using the L direction as "space" (and the R one as "time") with the same theory but quantised using the opposite perspective (L as time and R as space). This symmetry goes under the name of *mirror symmetry*, and implies the equivalence of the following two expressions for the partition function of the model

$$Z(R,L) = \mathrm{Tr}\, e^{-L\mathcal{H}_R} \quad \text{and} \quad Z(R,L) = \mathrm{Tr}\, e^{-R\mathcal{H}_L} \tag{1.54}$$

where $\mathcal{H}_R, \mathcal{H}_L$ are the Hamiltonians of the system quantised respectively using R or L as spatial direction, which can be obtained from the stress-energy tensor of the theory

$T_{\mu\nu}$ as

$$\mathcal{H}_R = \frac{1}{2\pi}\int T_{yy}dx \qquad \mathcal{H}_L = \frac{1}{2\pi}\int T_{xx}dy \tag{1.55}$$

having called x and y the coordinates along the R and L axes. In the following, we will denote the quantization scheme in which the role of the time direction is played by the L axis as the *L-channel*, while the other one as the *R-channel*.
This mirror symmetry, as was already noticed by Al.B. Zamolodchikov in the seminal paper [32], allows to relate the thermodynamical results obtained in the R-channel quantisation scheme through the TBA (those presented in section 1.2) to the ground state energy of the theory in the L-channel quantisation approach. To see why precisely the ground state is singled out, we must recall that in the derivations of the TBA equations one performs the TD limit, which implies sending $L \to \infty$. In the R-channel quantisation scheme (used in section 1.2), the TD limit is needed to recover the thermodynamics of the theory, and in this limit the usual relation between the free energy per unit length $f(R)$ and the partition function holds

$$Z(R, L) = e^{-LRf(R)} \tag{1.56}$$

Considering the L-channel instead, sending $L \to \infty$ into the first of the two expressions in (1.54) reduces the relevant terms in the trace to only that corresponding to the ground state (supposedly unique), leaving with

$$Z(R, L) = e^{-LE_0(R)} \tag{1.57}$$

Comparing the two formulas in the $L \to \infty$ limit, one finds

$$E_0(R) = Rf(R) \tag{1.58}$$

that is a relation between the Casimir energy $E_0(R)$ of the ground state of a (1+1) dimensional QFT at finite volume and its free energy $f(R)$ (per unit length) at infinite volume but at finite temperature $T = R^{-1}$, for which we have an expression (look back at formula (1.35)) in terms of the TBA pseudoenergies. The TBA equations presented in section 1.2 can then be interpreted to describe not only the thermodynamics of a scattering QFT, but also the evolution of its Casimir energy (when considered on a cylinder) along its Renormalisation Group (RG) flow.

Relation to the central charge

Following the reasoning above, one can also think to use the TBA equations to deduce information about the theory that sits at the UV critical point of the RG flow related to a given diagonal scattering theory (i.e. the CFT which, once perturbed, leads along

the RG flow to the scattering theory considered). The Casimir energy is in fact simply expressed in terms of the so called scaling function of the theory $\tilde{c}(r)$ (here $r \equiv R/\xi$)

$$E_0(R) = -\frac{\pi \tilde{c}(r)}{6R} \tag{1.59}$$

which in turn, at the UV critical point, is closely linked to the central charge c of the CFT associated to that critical point

$$\lim_{r\to 0} \tilde{c}(r) = c - 12(\Delta_{min} + \bar{\Delta}_{min}) \tag{1.60}$$

where $\Delta_{min}, \bar{\Delta}_{min}$ are the left and right lowest conformal dimensions of the CFT. This quantity, when $\Delta_{min} = \bar{\Delta}_{min}$, is usually called *effective central charge* and indicated as c_{eff}.
Using the relation (1.35), we can also directly express the effective central charge of the UV CFT in terms of the pseudoenergies. For the scaling function we have (here we use the reduced masses $\hat{m}_a \equiv m_a \xi$, to rewrite the formulas not in terms of R but on the dimensionless factor r)

$$\tilde{c}(r) = \frac{3}{\pi^2} r \sum_a \hat{m}_a \int_{-\infty}^{+\infty} d\theta \cosh(\theta) L_a(\theta) \tag{1.61}$$

which in the $r \to 0$ limit can be presented as, after some manipulations (for a detailed computation refer for example to [34])

$$\tilde{c}(0) = \frac{6}{\pi^2} \sum_a \mathcal{L}\left(\frac{1}{1+e^{\epsilon_a}}\right) \tag{1.62}$$

where the symbol $\mathcal{L}(x)$ stands for the Roger's Dilogarithm function

$$\mathcal{L}(x) \equiv -\frac{1}{2}\int_0^x \mathrm{d}t \left(\frac{\ln(1-t)}{t} + \frac{\ln(t)}{1-t}\right) \tag{1.63}$$

that is related to the usual Dilogarithm function $\mathrm{Li}_2(x)$ as

$$\mathcal{L}\left(\frac{1}{1+x}\right) = \mathrm{Li}_2(x) + \frac{1}{2}\ln(1-x)\ln(x) \tag{1.64}$$

and the ϵ_a are a constant version of the pseudonergies that solve the following coupled transcendental equations

$$\epsilon_a = \sum_a N_{ab} \ln\left(1 + e^{-\epsilon_b}\right) \tag{1.65}$$

with

$$N_{ab} \equiv -\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}\theta \phi_{ab}(\theta) = -\frac{1}{2\pi}\left[\delta_{ab}(+\infty) - \delta_{ab}(-\infty)\right] \tag{1.66}$$

These constant pseudonergies enter the picture through the observation that, as $r \to 0$, the pseudonergies $\epsilon_a(\theta)$ that solve the usual TBA equations (1.31) develop a plateau in the region $-\ln(2/r) \ll \theta \ll \ln(2/r)$. In this region, that increases as r decreases, we can consider $\epsilon_a(\theta) = \epsilon_a$ as constants, and the TBA equations then become (1.65) (in these equations the driving term $m_a R \cosh\theta = \hat{m}_a r \cosh\theta$ is not present since it vanishes in the $r \to 0$ limit). This connection between the TBA of a scattering theory and the central charge of the related CFT has been proven to be very useful in the past to test conjectured form of S-matrices associated to integrable perturbations of a given CFT. Borrowing the words of Klassen and Melzer [34], we may say that if the proposed S-matrix leads (via the TBA) to a value of the effective central charge that is the same of the CFT whose perturbation is supposed to give rise to the scattering theory in question, this provides strong evidence that the conjectured S-matrix is correct[3].

As a concluding remark of this small excursus on the relation between the TBA and the effective central charge we stress that, in the case of ADE theories (presented in section 1.3), we can express the matrix N_{ab} that appears in the constant TBA equations directly in terms of the incidence matrix I_{ab} of the Dynkin diagram of the Lie algebra associated to the theory. Looking at the definition of N_{ab}, we can in fact see that it coincides (up to a prefactor) to the Fourier transform $\tilde{\phi}_{ab}(k)$ of $\phi_{ab}(\theta)$ evaluated at $k = 0$, and we know we have, in case of ADE theories, the equation (1.42) that relates $\tilde{\phi}_{ab}(k)$ to I_{ab}. To obtained the desired formula, we first rewrite (1.42) in the following equivalent form, more convenient for the present purpose

$$\tilde{\phi}_{ab}(k) = -2\pi \left[I \left(2\cos\left(\frac{\pi k}{h}\right) - I \right)^{-1} \right]_{ab} \tag{1.67}$$

and we then get

$$N = -\frac{1}{2\pi}\tilde{\phi}(0) = I(2-I)^{-1} \tag{1.68}$$

which allows to rephrase the TBA equations (1.65) for the constants ϵ_a as

$$y_a^2 = \prod_b (1+y_b)^{I_{ab}} \tag{1.69}$$

where we have introduced the symbols $y_a = e^{\epsilon_a}$. As for the universal TBA equations (1.46) or the Y-system (1.51), this form has the advantage of being directly linked to the Lie algebra underlying the ADE theory, without passing through the S-matrix formulation. Indeed, looking back at (1.51), we see that this equations could have been obtained directly from the Y-system by searching for stationary solutions $Y_a(\theta) = const. \equiv y_a$.

[3]Actually [34], one can also separate into c_{eff} the central charge c from the scaling dimension Δ_{min} only using the S-matrix data, since Δ_{min} can be extracted directly from the small-r expansion of $\tilde{c}(r)$.

The excited states

From the discussion presented above on the mirror symmetry, and in particular from the fact that the TD limit in the L-channel quantisation perspective projects the partition function onto the lowest energy state, one is induced to think that the possibility of using the TBA equations to find the energy levels of a QFT is limited to the ground state (and at most some few states that become degenerate with it in large volumes). Nevertheless, building on results already obtained in the quantum mechanical setting (among all the historical work of Bender and Wu on the quantum anharmonic oscillator [37]), in a remarkable paper [23] of 1996, P.Dorey and R.Tateo suggested a way to sidestep the problem through analytic continuation of the TBA equations. The argument is quite generic, however, in the paper [23] the authors worked in the simplified context of the scaling Lee-Yang model (SLYM), a perturbation of the non-unitary minimal model $\mathcal{M}_{2,5}$ by its unique relevant operator φ. This model has a single neutral particle in its asymptotic states, with S-matrix

$$S(\theta) = \frac{\sinh(\theta) + i\sinh(\pi/3)}{\sinh(\theta) - i\sinh(\pi/3)} \tag{1.70}$$

and the single TBA equation associated to the particle reads

$$\epsilon(\theta) = r\cosh\theta - \int_{-\infty}^{+\infty} \frac{d\theta'}{2\pi}\phi(\theta-\theta')L(\theta') \tag{1.71}$$

where as usual $\phi = -i\frac{d}{d\theta}\ln S(\theta)$ and $L(\theta) = \log\left(1+e^{-\epsilon(\theta)}\right)$, and we have explicitly written the convolution integral for the purpose of the analysis that will follow. To obtain the $\epsilon_a(\theta)$ functions associated to the excited states, the idea is to perform analytic continuation of the TBA equations along a closed path of the complex plane related to a parameter entering into the equations. If, at a certain point P along this path, looking at the θ' plane, a pole θ'_p of the function $\phi(\theta-\theta')L(\theta')$ contained into the convolution integral reaches the real θ' line along which the integration is performed, to analytically continue the solution beyond the point P one has to distort the integration contour so that θ'_p is avoided, or equivalently, pick the residue of the integrand function $\phi(\theta-\theta')L(\theta')$ evaluated at $\theta' = \theta'_p$ (this mechanism of analytic continuation is described in detail in [38], chapter 2). This residue coming from the convolution term will then change the form of the TBA equations, modifying in particular its asymptotics.
In the cases analysed in the article [23], the authors considered analytic continuation in the r parameter, and found some closed paths in the r complex plane along which (complex conjugate) pairs of poles $\{\theta_0, \bar{\theta}_0\}$ of the $L(\theta')$ function present in the convolution moved across the integration contour $\mathbb{R}_{\theta'}$, causing new terms to appear into the TBA equation. In the simplest case, the modified TBA equation read

$$\epsilon(\theta) = r\cosh\theta + \log\frac{S(\theta-\theta_0)}{S(\theta-\bar{\theta}_0)} - \phi * L(\theta) \tag{1.72}$$

Also the integral equation (1.35) that through the mirror symmetry determines the ground state energy must be changed accordingly (here m is the mass of the only particle contained into the theory)

$$E(r) = -im(\sinh\theta_0 - \sinh\bar{\theta}_0) - \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\theta \;\; m\cosh\theta L(\theta) \tag{1.73}$$

and this modification can be interpreted to describe a change of meaning for the formula, from representing the energy of the ground state to describing that of a single particle excited state. Although, looking at these equations, the point θ_0 may seem a parameter to be determined elsewhere, perhaps through the study of the analytic continuation of the original TBA equation into the modified one, this is not the case. Self-consistency of the analytic continuation process implies that the θ_0 point must coincide with the position of the singularity of $L(\theta)$ that caused the modification of the TBA equation. A pole for $L(\theta) = \log\left(1 + e^{-\epsilon(\theta)}\right)$ implies $\epsilon(\theta) = (2n+1)\pi i$, for a certain integer n, and so θ_0 can be found by imposing this condition into the equation (1.72) , to get

$$2n\pi i = r\cosh\theta_0 - \log S(2i\,\mathrm{Im}(\theta_0)) - \phi * L(\theta_0) \tag{1.74}$$

In their paper [23], Dorey and Tateo explicitly found the result also for two particle excited states, and conjectured a form for a generic n-particle state of the scaling Lee-Yang model. However, the line of reasoning followed by the authors to find the excited states, briefly presented in this section, is straightforward enough to be used also for other field theories, and opens to various generalisations.

In our presentation we discussed the consequences of the process of analytic continuation focusing on the TBA equations, but one could ask what happens to the Y-system associated to the scaling Lee-Yang model, which reads

$$Y(\theta + i\pi/3)Y(\theta - i\pi/3) = 1 + Y(\theta) \tag{1.75}$$

As one could expect, being the Y-system a set of algebraic, rather than integral equations, they do not change in the analytic continuation process. This can be seen also by re-deriving the Y-system directly from the modified TBA equations (1.72), by the usual process of smooth shifting the equations in $\theta \to \theta \pm i\pi/3$. It is then manifest how the Y-system contains the information on both the ground state *and* the excited states; what changes in these different cases are the asymptotics of the Y functions, which ultimately have to be imposed to recover back a set of TBA equations from the Y-system . This is another reason for the more general nature of the Y-system when compared to the TBA equations.

1.5 T,Q operators and the BLZ construction

When dealing with integrable models, finding an explicit set of integrals of motions in involution equal in number to that of the degrees of freedom of the system is usually not an easy task, especially in the case of field theories, where the degrees of freedom are infinite. In the classical setting, a powerful approach to tackle the problem is the one of *Lax pairs*. A Lax pair is a pair of two matrices L, M depending on the dynamical variables of the system (and possibly also on a spectral parameter), which satisfies the condition

$$\frac{dL}{dt} = [M, L] \tag{1.76}$$

Once a Lax pair is found, building integrals of motion is immediate. For any value $n \in \mathbb{N}$, we have in fact that the quantities

$$I_n = \operatorname{Tr} L^n \tag{1.77}$$

are conserved in time. In particular, if L is diagonalizable, this implies the conservation of all the d eigenvalues of L. The existence of this set of integrals of motion (IMs) does not directly imply the integrability of the system (in the Liouville sense), since we have no guarantee that the IMs are in involution under the Poisson bracket. One can show however (refer to [2] for a detailed presentation on the subject of integrability), that the involution property of the eigenvalues of L is equivalent to the existence of a quantity $r = \sum_{\alpha,\beta} a_\alpha \otimes a_\beta$, called *r-matrix*, where a_α, a_β are $d \times d$ matrices which we suppose to have constant entries, such that the following Jacobi-like identity is satisfied

$$[r_{12}, r_{13}] + [r_{12}, r_{12}] + [r_{23}, r_{13}] = 0 \tag{1.78}$$

where we have used the notation $r_{12} = \sum_{\alpha,\beta} a_\alpha \otimes a_\beta \otimes \mathbb{1}$, $r_{13} = \sum_{\alpha,\beta} a_\alpha \otimes \mathbb{1} \otimes a_\beta$ and similarly for the other choices of indices. When r is antisymmetric, that is $r_{12} = -r_{21}$, this equation is called *classical Yang-Baxter equation*.

Considering the quantum case, and in particular the standard example of integrable quantum field theories, i.e. CFTs, the integrability structure of these theories, such as the integrals of motions, or the existence of an r-matrix, remains somewhat hidden in the standard presentation of the theories, which involves studying modules over the Virasoro algebra. The S-matrix approach, from which the thermodynamic Bethe ansatz is derived, is a step closer to the integrable description of conformal field theories, but there is another way to tackle the problem, developed in the mid-nineties by V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov in the series of works [24–27], which mirrors methods and results developed in the classical setting to unveil the more profound integrable structure of CFTs (for a review on the subject see [39]; we will use this as main reference in the forthcoming exposition). This approach, called BLZ approach from the name of

the authors, has then been extended also to massive integrable deformations of conformal field theories. In our presentation of this method, that will be minimal to avoid a long detour on the subject, we will limit to the case of CFTs. Our interest in discussing the method is that the central objects appearing in this type of analysis, namely the Q,T and Y functions, together with the functional relations among them, are found also in the context of differential equations, and represent the "integrability side" of the so-called ODE-IM correspondence, that will be discussed in the next chapter.

The KdV equation and the classical version of the method

The starting point is the connection with the classical case, in order to build intuition and try then to reproduce the results in the quantum theory. It is known that the CFTs with $c < 1$ have a classical counterpart in the Korteweg–de Vries (KdV) equation, an equation historically introduced as an approximation of hydrodynamical equations, to describe unidimensional long waves in shallow water. Considering in fact the "classical limit" $c \to -\infty$, and performing the following substitutions

$$T(w) \to -\frac{c}{6}U(w) \qquad [\,,\,] \to \frac{6\pi}{ic}\{\,,\,\} \tag{1.79}$$

where $\{\,,\,\}$ stands for the Poisson brackets, the Virasoro algebra of the CFT reduces to the Poisson algebra

$$\{U(w), U\,(w')\} = 2\,(U(w) + U\,(w'))\,\delta'\,(w - w') + \delta'''\,(w - w') \tag{1.80}$$

that describes the second Hamiltonian structure of the KdV equation, provided the Hamiltonian is chosen amongst the classical integrals of motion I^{cl}_{2n-1}. Different choices for the Hamiltonian lead to different equation of motion for the field U, which are collectively called *KdV hierarchy*. All of them can be shown to be equivalent to a description of isospectral deformations of the following Lax differential operator, depending on a spectral parameter λ:

$$L(w \mid \lambda) \equiv \partial_w^2 + U(w) - \lambda^2 \tag{1.81}$$

The different expressions for the equation of motion are recovered, considering as time variable the one t_{2n-1} related to a chosen I^{cl}_{2n-1}, by a different form for the second operator $M(w)$ in the Lax pair

$$\frac{d}{dt_{2n-1}}L(w \mid \lambda) = [M_{2n-1}(w), L(w \mid \lambda)] \Longleftrightarrow \text{KdV equation associated to } I^{\mathrm{cl}}_{2n-1} \text{ is satisfied.} \tag{1.82}$$

Associated to a Lax operator, there exists a differential equation, usually called *auxiliary equation*, which in this case takes the form

$$L(w \mid \lambda)\psi(w \mid \lambda) = \psi''(w) - \left(\lambda^2 - U(w)\right)\psi(w) = 0 \tag{1.83}$$

This is a second order differential equation and, as such, possesses two linearly independent solutions $\psi_1(w \mid \lambda)$ and $\psi_2(w \mid \lambda)$. We are interested in analyzing the monodromy problem of these solutions, encoded in the *monodromy matrix* $\mathbf{M}(\lambda)$:

$$(\psi_1(w \mid \lambda), \psi_2(w \mid \lambda))\, \mathbf{M}(\lambda) = (\psi_1(w + 2\pi \mid \lambda), \psi_2(w + 2\pi \mid \lambda)) \tag{1.84}$$

The trace $\mathbf{T}(\lambda) \equiv \operatorname{tr} \mathbf{M}(\lambda)$ of the monodromy matrix, which is called T-function, is a central object in integrable systems, since it can be shown to contain into its asymptotic λ expansion the local classical integrals of motion

$$\frac{1}{2\pi} \ln [\mathbf{T}(\lambda)] \simeq \lambda - \sum_{n=1}^{\infty} c_n I^{cl}_{2n-1} \lambda^{1-2n} \tag{1.85}$$

The construction just presented can be extended to a more general setting by first factorising $L(w \mid \lambda)$ as

$$L(w \mid \lambda) = (\partial_w + \phi'(w))(\partial_w - \phi'(w)) - \lambda^2 \tag{1.86}$$

where $\phi(w)$ is the *Miura transform* of $U(w)$:

$$U(w) = -\phi'(w)^2 + \phi''(w) \tag{1.87}$$

Considering the QFT on a cylinder of circumference 2π, we have $T(w + 2\pi) = T(w)$ for the stress-energy tensor, and so similarly for $U(w)$. This implies for the Miura field the quasi-periodicity

$$\phi(w + 2\pi) = \phi(w) + 2\pi i p \tag{1.88}$$

The Miura transform reduces the second order differential equation (1.83) into a system of two first order differential equations, presented in matrix form as (here the σ^i are the Pauli matrices)

$$\left(\partial_w - \phi'(w)\sigma^3 - \lambda\sigma^1\right) \Psi(w \mid \lambda) = 0 \tag{1.89}$$

This form can be generalised by considering the abstract Lax operator

$$\mathcal{L}(w \mid \lambda) \equiv \partial_w - \phi'(w) H - \lambda(E + F) \tag{1.90}$$

in terms of the generators H, E, F of the $sl(2)$ Lie algebra. The commutation relations of H, E, F

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = 2H \tag{1.91}$$

imply that the commutation properties of $\mathcal{L}$ are the same of those of L, and on top of this $\mathcal{L}$ reduces to L when the 2-dimensional representation of $sl(2)$ is chosen. Having introduced the Lie algebra however, opens the possibility to consider also different matrix

representations of $sl(2)$. Calling π_j, with $j \in \frac{1}{2}\mathbb{N}$ the representation of $sl(2)$, such that $\pi_j[H] = \mathrm{diag}(2j, 2j-2, \ldots, -2j+2, -2j)$, we can in fact consider the matrix equation

$$\pi_j[\mathcal{L}(w \mid \lambda)]\Psi_j(w \mid \lambda) = 0 \tag{1.92}$$

where Ψ_j is a $(2j+1)$ dimensional vector. The solution of this equation can be presented in terms of a *path-ordered exponential*

$$\Psi_j(w \mid \lambda) = \pi_j \left\{ e^{\phi(w)H} \mathcal{P} \exp\left[\lambda \int_0^w dw' \left(e^{-2\phi(w')} E + e^{2\phi(w')} F \right) \right] \right\} \Psi_j^0 \tag{1.93}$$

where Ψ_j^0 is an arbitrary constant vector describing the integration constants. The path ordering is just an ordering for decreasing argument (from left to right)

$$\mathcal{P}\left[a\left(w_1\right) a\left(w_2\right)\right] = \begin{cases} a\left(w_1\right) a\left(w_2\right) & w_1 > w_2 \\ a\left(w_2\right) a\left(w_1\right) & w_1 < w_2 \end{cases} \tag{1.94}$$

and its action on an exponential is defined through the series expansion

$$\mathcal{P} \exp\left[\int_0^w a\left(w'\right) dw' \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^w \cdots \int_0^w \mathcal{P}\left[a\left(w_1'\right) \cdots a\left(w_n'\right)\right] dw_1' \cdots dw_n' \tag{1.95}$$

As for the case (1.84), given a representation π_j we can then consider the monodromy matrix associated to a solution Ψ_j, which can be written in the explicit form

$$\mathbf{M}_j(\lambda) = \pi_j \left\{ e^{2\pi i p H} \mathcal{P} \exp\left[\lambda \int_0^{2\pi} dw \left(e^{-2\phi(w)} E + e^{2\phi(w)} F \right) \right] \right\} \tag{1.96}$$

Giving a name to the trace functions

$$\mathbf{T}_j(\lambda) = \mathrm{tr}\, \mathbf{M}_j(\lambda) \tag{1.97}$$

where $\mathbf{T}_{\frac{1}{2}}(\lambda) = \mathbf{T}(\lambda)$, one can show that these quantities are in involution under Poisson brackets

$$\left\{ \mathbf{T}_j(\lambda), \mathbf{T}_{j'}\left(\lambda'\right) \right\} = 0 \tag{1.98}$$

and are expected to contain the classical IMs in their asymptotic expansion.

The quantum case: T and Y operators

Having analyzed the "classical limit", the goal is now to reproduce similar results in the quantum setting. The quantum analog of $U(w)$ is the stress-energy tensor $T(w)$ (recall

(1.79)), and the Miura transform (1.87) is the $c \to -\infty$ counterpart of the so-called *Feigin-Fuchs free field representation*

$$-\beta^2 T(w) =: \left(\varphi'(w)\right)^2 : + \left(1-\beta^2\right)\varphi''(w) + \frac{\beta^2}{24} \tag{1.99}$$

where $\varphi(w)$ is a free field (whose classical counterpart is $\phi(w)$),

$$\varphi(w) = \mathrm{i}Q + \mathrm{i}Pw + \sum_{n\neq 0} \frac{a_{-n}}{n} e^{\mathrm{i}nw} \tag{1.100}$$

the parameter β is related to the central charge

$$\beta \equiv \sqrt{\frac{1-c}{24}} - \sqrt{\frac{25-c}{24}} \tag{1.101}$$

and the normal ordering prescription $: \cdot :$ implies placing the a_n operators in increasing value of their index n from left to right. The operators Q, P and a_n, $n \neq 0$, generate an Heisenberg algebra:

$$[Q, P] = \frac{\mathrm{i}}{2}\beta^2, \quad [a_n, a_m] = \frac{n}{2}\beta^2 \delta_{n+m,0} \tag{1.102}$$

Using this representation, one can give an equivalent description of the chiral Hilbert space[4] $\mathcal{H}_{chiral}$ associated to a CFT, in terms of modules not over the Virasoro algebra, but the Heisenberg one. Calling "Fock space" $\mathcal{F}_p$ the highest weight module over the Heisenberg algebra having as highest weight vector the vector $|p\rangle$ defined by the equations

$$P|p\rangle = p|p\rangle, \quad a_n|p\rangle = 0, \ \ \forall n > 0. \tag{1.103}$$

we have that $\mathcal{F}_p$ is isomorphic to the Verma module $\mathcal{V}_\Delta$ with dimension

$$\Delta = \left(\frac{p}{\beta}\right)^2 + \frac{c-1}{24} \tag{1.104}$$

and we can thus describe the chiral Hilbert space as

$$\mathcal{H}_{chiral} = \bigoplus_n \mathcal{F}_{p_n}, \tag{1.105}$$

where the sum runs over the values of p corresponding to the dimensions of the Virasoro modules present in the CFT.

[4]Recall that thanks to the splitting of the stress-energy tensor in a holomorphic and anti-holomorphic part, the physical Hilbert space of a CFT can be embedded into the tensor product $\mathcal{H}_{chiral} \otimes \bar{\mathcal{H}}_{chiral}$

Continuing with the classical to quantum mapping, to build the quantum version of the monodromy matrices (1.96) we consider a quantum counterpart of the Lie algebra $sl(2)$, namely the quantum enveloping algebra $\mathcal{U}_q(sl(2))$, whose generators E, F, H satisfy now the commutation relations

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = \frac{q^H - q^{-H}}{q - q^{-1}} \tag{1.106}$$

where $q \equiv e^{\mathrm{i}\pi\beta^2}$. Denoting as before π_j the $(2j+1)$-dimensional representation of this algebra, the *quantum monodromy matrices* are then defined as

$$\mathbf{M}_j(\lambda) \equiv \pi_j \left\{ e^{2\pi\mathrm{i}PH} \mathcal{P} \exp\left[\lambda \int_0^{2\pi} dw \left(V_-(w) q^{\frac{H}{2}} E + V_+(w) q^{-\frac{H}{2}} F \right) \right] \right\} \tag{1.107}$$

where $V_\pm(w)$ are just the vertex operators

$$V_\pm(w) \equiv : e^{\pm 2\varphi(w)} := \exp\left[\pm 2 \sum_{n=1}^{\infty} \frac{a_{-n}}{n} e^{\mathrm{i}nw} \right] e^{\pm 2\mathrm{i}(Q+Pw)} \exp\left[\mp 2 \sum_{n=1}^{\infty} \frac{a_n}{n} e^{-\mathrm{i}nw} \right] \tag{1.108}$$

Recalling that $\varphi(w)$ is the quantum counterpart of $\phi(w)$, we see that indeed this expression completely mirrors the classical one (1.96), a part from minor differences like the presence of the factors $q^{\frac{H}{2}}, q^{-\frac{H}{2}}$ in front of respectively E and F, coming from the fact that the correct construction of these $\mathbf{M}_j(\lambda)$ should start from the quantum *affine* enveloping algebra $\mathcal{U}_q(\widehat{sl(2)})$ rather than $\mathcal{U}_q(sl(2))$ (for an explanation see [39]). Notice that the definition (1.107) has to be intended as a power series in λ, which converges for any λ only for $-\infty < c < -2$. To extend this range, one needs to use a proper regularisation prescription in the definition.
As in the classical case, we also introduce the *quantum transfer matrices* as

$$\mathbf{T}_j(\lambda) \equiv \mathrm{tr}_{\pi_j} \mathbf{M}_j(\lambda) \tag{1.109}$$

By using their definition in terms of the $\mathbf{M}_j(\lambda)$, one can show[5] that these quantum transfer matrices $\mathbf{T}_j(\lambda)$ commute among themselves

$$[\mathbf{T}_j(\lambda), \mathbf{T}_{j'}(\lambda')] = 0 \tag{1.110}$$

and also with the operator P, so that they act invariantly on each Fock space $\mathcal{F}_p$. Looking at the asymptotic expansion around $\lambda^2 \to -\infty$ of $\mathbf{T}(\lambda) \equiv \mathbf{T}_{\frac{1}{2}}(\lambda)$, one can find all the

[5]To be precise, in the conventional BLZ approach, to prove the commutation properties of the $\mathbf{T}_j(\lambda)$, one introduces another set of operators $\mathbf{L}_j(\lambda)$, also built upon the quantum monodromy matrices as is the case for $\mathbf{T}_j(\lambda)$, and closely related to them. The usefulness of these $\mathbf{L}_j(\lambda)$ lies in the existence of a matrix relation between them and the trigonometric R-matrix of $\mathcal{U}_q(sl(2))$ acting on $\pi_j \otimes \pi_{j'}$ (i.e. the trigonometric matrix that solves the Yang-Baxter relation acting on $\pi_j \otimes \pi_{j'}$), called *RLL relation*, upon which the relations for the $\mathbf{T}_j(\lambda)$ can be derived.

local (quantum) integrals of motion I_{2n-1} of the CFT that can be built considering densities T_{2n} of the stress-energy tensor $T(w)$:

$$I_{2n-1} \equiv \int_0^{2\pi} \frac{dw}{2\pi} T_{2n}(w) \tag{1.111}$$

These densities are in general regularised polynomials of $T(w)$ and its derivatives (refer to [24, 39] for a proper explanation).
More interesting for the following analysis is however the expansion of the $\mathbf{T}_j(\lambda)$ around $\lambda^2 = 0$, which directly comes from the definition of the related quantum monodromy matrices (1.107)

$$\mathbf{T}_j(\lambda) = \frac{\sin(2(2j+1)\pi P)}{\sin(2\pi P)} + \sum_{n=1}^{\infty} \lambda^{2n} \mathbf{G}_{2n}^{(j)} \tag{1.112}$$

where the $\mathbf{G}_{2n}^{(j)}$ can be regarded as *nonlocal* integrals of motion of the CFT. The remarkable fact of these expansions is that they are deeply interrelated, being the non-local integrals of motion $\mathbf{G}_{2n}^{(j)}$ with $j > \frac{1}{2}$ all expressible in terms of the basic ones $\mathbf{G}_{2n}^{\left(\frac{1}{2}\right)} \equiv \mathbf{G}_{2n}$, as for example

$$\mathbf{G}_2^{(j)} = A_j\left(2\pi P, \pi\beta^2\right) \mathbf{G}_2, \tag{1.113}$$

$$\mathbf{G}_4^{(j)} = A_j\left(2\pi P, 2\pi\beta^2\right) \mathbf{G}_4 + B_j\left(2\pi P, \pi\beta^2\right) \mathbf{G}_2, \tag{1.114}$$

$$\dots \tag{1.115}$$

where $A_j(a,b), B_j(a,b)$ are particular functions that depend on j (for their explicit form see [39]). This interdependence of the different $\mathbf{T}_j(\lambda)$ matrices suggests the existence of a functional equation among them, which is indeed the case. The $\mathbf{T}_j(\lambda)$ satisfy in fact the following fusion relation

$$\mathbf{T}_j\left(q^{\frac{1}{2}}\lambda\right) \mathbf{T}_j\left(q^{-\frac{1}{2}}\lambda\right) = 1 + \mathbf{T}_{j+\frac{1}{2}}(\lambda)\mathbf{T}_{j-\frac{1}{2}}(\lambda) \tag{1.116}$$

which is universally known as *T-system* or *Hirota bilinear equations*, and allows one to express all the $\mathbf{T}_j(\lambda)$ in terms of the fundamental one $\mathbf{T}(\lambda)$. Another profound consequence of the existence of these equations is that the Y-system, presented in section 1.3 for ADE theories, can be recovered directly through this formalism, without passing through the S-matrix one. To obtain it one defines the Y-operators as

$$\mathbf{Y}_j(\theta) \equiv \mathbf{T}_{j-\frac{1}{2}}(\lambda)\mathbf{T}_{j+\frac{1}{2}}(\lambda) \tag{1.117}$$

with the convention $\mathbf{T}_0(\lambda) = 1$ and $\mathbf{T}_{-\frac{1}{2}}(\lambda) = 0$, having introduced the variable θ such that

$$\lambda^{1+\xi} = e^{\theta}, \qquad \text{where } \xi \equiv \frac{\beta^2}{1-\beta^2} \tag{1.118}$$

It is then immediate use the T-system (1.116) to find for the $\mathbf{Y}_j(\theta)$ the relations

$$\mathbf{Y}_j\left(\theta + i\pi\xi/2\right)\mathbf{Y}_j\left(\theta - i\pi\xi/2\right) = \left(1 + \mathbf{Y}_{j+\frac{1}{2}}(\theta)\right)\left(1 + \mathbf{Y}_{j-\frac{1}{2}}(\theta)\right) \tag{1.119}$$

This is an infinite set (one for each representation π_j) of equations that indeed closely resembles the Y-system seen in section 1.3. The fact that here the equations are infinite in number is however a big discrepancy with the Y-system of the ADE case, and this is somewhat disturbing, since one expects a sort of consistency between the BLZ approach and the S-matrix one (from which the Y-system in section 1.3 was derived), when applied to the same theory.
The solution to this discrepancy lies in a phenomenon called *truncation* of the T and Y-systems. When q is a N-th root of unity the $(N+1)$-dimensional representation $\pi_{\frac{N}{2}}$ of $\mathcal{U}_q(sl(2))$ becomes reducible, while all the representations with $\frac{1}{2} \leq j < \frac{N}{2}$ remain irreducible. This implies that the infinite tower of relations for the $\mathbf{T}_j(\theta)$ and $\mathbf{Y}_j(\theta)$ closes on themselves by considering only a finite number of them. Noticing that for q to be a root of unity implies that β^2 is a rational number $\beta^2 = m'/m$, this singles out among the values of the central charge those precisely corresponding to the minimal models $\mathcal{M}_{m,m'}$

$$c = 13 - 6(\beta^{-2} + \beta^2) = 1 - 6\frac{(m-m')^2}{mm'} \tag{1.120}$$

The minimal models represent the CFTs with $c < -2$ having a known ADE classification; we thus find a truncation of the functional equations in the BLZ approach exactly for those theories from which we expect it to happen. In this cases, also the precise form (1.51) of the Y-sytem, with the right hand side depending on the incidence matrix of a Dynkin diagram, can be reproduced, provided possibly a slight modification of the definition of the $\mathbf{Y}_j(\theta)$.
A last remark we want to make on the derivation of the Y-system via the BLZ construction concerns the possibility to obtain from it a set of TBA equations. In order to do so, one considers the Y-system equations (1.119) for the $\mathbf{Y}_j$ operators common eigenvalues

$$\mathbf{Y}_j(\theta)\left|\ell\right\rangle = Y_j^{(\ell)}(\theta)\left|\ell\right\rangle \tag{1.121}$$

where ℓ labels the eigenstate under consideration. Defining then as usual the pseudoenergies as $\epsilon_j^{(\ell)}(\theta) \equiv \log Y_j^{(\ell)}(\theta)$, one can then, computing the asymptotic behaviour of the $\epsilon_j^{(\ell)}(\theta)$ through the series expansions for $\mathbf{T}_j(\theta)$, together with the knowledge of their analytic structure, deduce a set of TBA equations in the form

$$\epsilon_j^{(\ell)}(\theta) = \mathcal{Z}^{(\ell)}(\theta) - \sum_k \int_{-\infty}^{+\infty} d\theta' \phi_{jk}\left(\theta - \theta'\right) \log\left[1 + e^{-\epsilon_k^{(\ell)}(\theta')}\right] \tag{1.122}$$

where the driving term $\mathcal{Z}^{(\ell)}(\theta)$ encodes the the asymptotic behaviour of the $\epsilon_j^{(\ell)}(\theta)$, and the kernel $\phi_{jk}\left(\theta - \theta'\right)$ the algebraic structure of the Y-system. Different values of ℓ will

correspond to a set of TBA equations with different driving terms; recalling the form of the TBA equations obtained in section 1.4 for the excited states, these distinct forms of the driving terms can be interpreted as describing different excited states of the theory. Diversely from the S-matrix approach to arrive at the excited states TBA, in the BLZ one there is no need to perform analytic continuation, since one directly obtains the Y-system, together with the control over the form of the possible asymptotics for the pseudoenergies, from which the various excited states TBA can be derived (for a more in depth analysis and some explicit examples refer to [27]).

The quantum case: Q operators

We conclude our presentation on the BLZ method by precising that all the above construction, involving the T and Y operators, can be obtained starting from similar, but more fundamental objects, called Q-operators. Like the T operators, they are defined as traces of some particular monodromy matrix built out of vertex operators and the generators of an algebra. The difference lies exactly in the choice of the algebra. In this case one considers the *q-oscillator algebra* osc_q, defined trough the commutation relations of its generators $\{\mathcal{H}, \mathcal{E}_+, \mathcal{E}_-\}$ as

$$[\mathcal{H}, \mathcal{E}_\pm] = \pm 2\mathcal{E}_\pm, \qquad q\mathcal{E}_+\mathcal{E}_- - q^{-1}\mathcal{E}_-\mathcal{E}_+ = \frac{1}{q-q^{-1}} \tag{1.123}$$

Choosing any representation ρ of osc_q such that the following object exists for $\mathrm{Im}\, p > 0$

$$Z(p) = \mathrm{tr}_\rho \left[e^{2\pi i p \mathcal{H}} \right] \tag{1.124}$$

we first define the $\mathbf{A}_\pm$ operators in analogy with the $\mathbf{T}_j$

$$\mathbf{A}_\pm(\lambda) \equiv \frac{1}{Z(\pm P)} \mathrm{tr}_\rho \left\{ e^{\pm 2i\pi P\mathcal{H}} \mathcal{P} \exp\left[\lambda \int_0^{2\pi} du \left(V_-(u) q^{\pm\frac{\mathcal{H}}{2}} \mathcal{E}_\pm + V_+(u) q^{\mp\frac{\mathcal{H}}{2}} \mathcal{E}_\mp \right) \right] \right\} \tag{1.125}$$

and then the Baxter $\mathbf{Q}_\pm$ operators as a simple rescaling of them

$$\mathbf{Q}_\pm(\lambda) = \lambda^{\pm 2P/\beta^2} \mathbf{A}_\pm(\lambda) \tag{1.126}$$

The expansion of the logarithm of $\mathbf{A}_+$ (and so also that of $\mathbf{Q}_+$) contains a set of nonlocal integrals of motion $\mathbf{H}_{2n}$, alternative to the $\mathbf{G}_{2n}$ already seen (thus algebraically related to them)

$$\ln \mathbf{A}_+(\lambda) = -\sum_{n=1}^{\infty} y^{2n} \mathbf{H}_{2n} \tag{1.127}$$

The relevance of the $\mathbf{Q}_\pm$ operators lies however in a list of remarkable properties which they satisfy:

1. they commute among themselves and with the operators $\mathbf{T}_j$

$$[\mathbf{Q}_\pm(\lambda), \mathbf{Q}_\pm(\lambda')] = [\mathbf{Q}_\pm(\lambda), \mathbf{T}_j(\lambda')] = 0 \tag{1.128}$$

2. they satisfy the *Baxter TQ relation*

$$\mathbf{T}(\lambda)\mathbf{Q}_\pm(\lambda) = \mathbf{Q}_\pm(q\lambda) + \mathbf{Q}_\pm\left(q^{-1}\lambda\right) \tag{1.129}$$

3. they satisfy the *quantum wronskian relation* (or *QQ relation*)

$$2i\sin(2\pi P) = \mathbf{Q}_+\left(q^{\frac{1}{2}}\lambda\right)\mathbf{Q}_-\left(q^{-\frac{1}{2}}\lambda\right) - \mathbf{Q}_+\left(q^{-\frac{1}{2}}\lambda\right)\mathbf{Q}_-\left(q^{\frac{1}{2}}\lambda\right) \tag{1.130}$$

which is actually just the particular case for $j = 0$ of the more general relation

$$2i\sin(2\pi P)\mathbf{T}_j(\lambda) = \mathbf{Q}_+\left(q^{j+\frac{1}{2}}\lambda\right)\mathbf{Q}_-\left(q^{-j-\frac{1}{2}}\lambda\right) - \mathbf{Q}_+\left(q^{-j-\frac{1}{2}}\lambda\right)\mathbf{Q}_-\left(q^{j+\frac{1}{2}}\lambda\right) \tag{1.131}$$

This last property clearly shows how the $\mathbf{T}_j$ operators can be built starting from the $\mathbf{Q}_\pm$ ones. In this perspective the $\mathbf{Q}_\pm$ are the most fundamental objects in the construction; one can start by defining the $\mathbf{Q}_\pm$, then from them obtain the $\mathbf{T}_j$ using (1.131), and from those the $\mathbf{Y}_j$ via (1.117). The Baxter TQ relation, the T-system and the related Y-system are all just a consequence of the quantum wronskian relation.

Chapter 2

The ODE-IM correspondence

This chapter is about a correspondence that brings together two worlds that are central in modern mathematical physics: on one side, there are integrable models (IMs), and in particular integrable quantum field theories in (1+1) dimensions; on the other there is the spectral analysis of ordinary differential equations (ODEs). For this reason, this link has taken the name of *ODE-IM correspondence*.
The correspondence has been first properly formalised in the foundational works of P.Dorey and R.Tateo [8, 9] and V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov [10], which noticed that a series of functional equations that were emerging in the context of the Schrödinger equation (especially in the works of Voros [4–6] and Sibuya [7]) had a striking similarity with those characterising integrable systems, and understood that these equations were just a hint of a full integrable structure present in a wide class of ODEs.
The "IM side" of the correspondence concerns all the material we already presented in chapter 1 (where we focused on 2D integrable QFTs, but similar structures can be found also in lattice models). In this chapter we will instead show the other side of the correspondence, namely how mathematical objects and relations typical of integrable models can be found in the study of ODEs in the complex domain. Of the many approaches to the analysis of integrable QFTs, it is the BLZ "functional" approach that allows to establish a more direct link with the theory of differential equations. We will see how objects like Q,T and Y operators can be built starting from solutions of an ODE, and recover functional relations closely mirroring those seen in section 1.5.[1]

The first works [8–10] about the ODE-IM correspondence considered (on the ODE

[1]From the presentation of the BLZ construction given in section 1.5, that was strongly based on mirroring results known for the classical KdV model, the presence of integrable structures in particular cases of the Schrödinger equation could perhaps have already been guessed. The Lax differential operator (1.81) matches in fact with the Schrödinger one, where U plays the role of the potential and λ of the energy.

side) the Schrödinger equation with a potential of the monomial form $V(z) = z^{r+1}$, $r \in \mathbb{R}, r > -2$ and possibly also the angular momentum term $l(l+1)/z^2$, but a similar correspondence has been then found also for more complicated potentials, and even extended to a wider class of ODEs (see e.g. [40, 41]). In the present chapter, our presentation on the ODE-IM correspondence will be centered on two selected models, both concerning the Schrödinger equation, but with a different form of the potential. First, we will discuss the historical model that involves a potential of monomial type + the angular momentum term (the one studied in [8–10]), and then generalise these result to a generic polynomial potential, since this will be the setting in which the main part of the original works of this thesis (explained in chapter 4) has been developed. In this second case we will leave aside the angular momentum term, to keep the discussion simpler[2].

2.1 The original version

The following analysis can be regarded as a synthesis and a reworking of the original papers ([8–10]) about the ODE-IM correspondence, in light of more recent revisions on the subject (in particular the review [43]) and some contamination coming from other areas of physics where a connection with integrability has played an important role, among all the *AdS/CFT* correspondence (see e.g.[44]). The ODE we first consider is the 1D stationary Schrödinger equation with a monomial and the angular momentum term as potential, having set $\hbar = 1$

$$\left(-\frac{d^2}{dz^2} + z^{r+1} + \frac{l(l+1)}{z^2} - E\right)\psi(z) = 0 \tag{2.1}$$

where we take r to be positive integer for simplicity, and exclude the case $r = 1$ that needs some special care[3]. The presence of the angular momentum term introduces a branch point for $\psi(z)$ at $z = 0$, so the solutions of the differential equation should be considered to be defined on a Riemann surface that is a cover of the punctured complex plane. The equation (2.1) has a solution $y = y(z, E, l)$ that is an entire function of (z, E) (where z is taken on the Riemann surface) and uniquely characterised by its asymptotic expansion [9]

$$y \sim z^{-\frac{r+1}{4}} \exp\left(-\frac{2}{r+3} z^{\frac{r+3}{2}}\right), \quad \text{for } |z| \to +\infty, \quad |\arg z| < \frac{3\pi}{r+3} \tag{2.2}$$

If we consider the so-called *Stokes sectors*

$$\mathcal{S}_k = \left\{ z \in \mathbb{C} \mid \left| \arg(z) - \frac{2k\pi}{r+3} \right| < \frac{\pi}{r+3} \right\}, \quad k \in \mathbb{Z} \tag{2.3}$$

[2] An analysis that includes also the angular momentum term can be found in [42]

[3] We note however that his case corresponds to the three dimensional harmonic oscillator, for which the wavefunctions are known exactly

which cover the complex plane (multiple times, as $r+3$ of them cover it once), from the asymptotic expansion (2.2) we see that the solution $y(z,E,l)$ tends to zero as $|z| \to +\infty$ in $\mathcal{S}_0$ and to infinity in the adjacent sectors $\mathcal{S}_{\pm 1}$. More technically, one says that this solution is *subdominant* in $\mathcal{S}_0$ and *dominant* in $\mathcal{S}_{\pm 1}$. We will call *anti-Stokes* lines the boundaries of the Stokes sectors. To generate other solutions of the differential equation (2.1) we can take advantage of a symmetry of the equation which is broken by the solution $y(z,E,l)$. We consider in particular the symmetry (sometimes referred to as *Symanzik rotation*)

$$\Omega : (z,E,l) \to (\omega^{-1}z, \omega^2 E, l), \quad \text{with } \omega = e^{\frac{2\pi i}{r+3}} \tag{2.4}$$

and define from $y(z,E,l)$ other solutions as

$$y_k(z,E,l) \equiv \omega^{k/2}\, \Omega^k\, y(z,E,l) = \omega^{k/2}\, y(\omega^{-k}z, \omega^{2k}E, l) \tag{2.5}$$

where the prefactor $\omega^{k/2}$ is included for later convenience. Each solution y_k is dominant in $\mathcal{S}_k$ and subdominant in $\mathcal{S}_{k\pm 1}$, and thus any couple $\{y_{k-1}, y_k\}$ provides a basis of solutions to the second order differential equation (2.1).
Another basis of solutions can be found by expanding $\psi(z)$ around $z=0$, where the wavefunction has two possible behaviours, either $\sim z^{l+1}$ or $\sim z^{-l}$. We fix one of the two solutions ψ^+ by requiring the first type of behaviour (the decaying one, if $l>0$)

$$\psi^+(z,E,l) \sim z^{l+1} + O(z^{l+3}) \tag{2.6}$$

while the second independent solution can be obtained applying to ψ^+ another symmetry of equation (2.1)

$$\Lambda : (z,E,l) \to (z,E,-1-l) \tag{2.7}$$

so that

$$\psi^-(z,E,l) \equiv \Lambda\, \psi^+(z,E,l) = \psi^+(z,E,-1-l) \tag{2.8}$$

This procedure to obtain a second independent solution from ψ^+ fails for some values of l, the most obvious being $l=-\frac{1}{2}$, where $\psi^-(z,E,l) = \psi^+(z,E,l)$. In these cases the second solution should be replaced with another one, having a logarithmic component. This situation is described in [9], where the authors show how the modification of this basis influences the functional relations that appear in the theory. Here we will exclude these peculiar cases to keep the discussion contained.
The action of the Ω symmetry on the basis $\psi^\pm$ is equivalent to a simple rescaling

$$\Omega^k\, \psi^\pm(z,E,l) = \psi^\pm(\omega^{-k}z, \omega^{2k}E, l) = \omega^{\mp k(l+\frac{1}{2})-\frac{k}{2}}\, \psi^\pm(z,E,l) \tag{2.9}$$

a property that will be useful in the following analysis.

The solutions y_k and the pair $\psi^\pm$ are all the building blocks required to extract the integrable structures concealed into the differential equation (2.1). In the following we

will show how to use them in order to construct objects that satisfy relations completely analog to those satisfied by the common eigenvalues of the T,Q, and Y operators seen in section 1.5. To be more precise, if we introduce the functions[4]

$$T_j(\lambda, p) = \langle p| \, \mathbf{T}_j(\lambda) \, |p\rangle \,, \qquad j = \frac{1}{2},\, 1,\, \frac{3}{2}, ... \tag{2.10}$$

and

$$Q_\pm(\lambda, p) = \langle p| \, \lambda^{\mp P/\beta^2} \mathbf{Q}_\pm(\lambda) \, |p\rangle = \langle p| \, \mathbf{A}_\pm(\lambda) \, |p\rangle \tag{2.11}$$

where $|p\rangle$ are the highest weight vectors of the Heisenberg algebra seen in section 1.5, what we want to reproduce are the following relations (from now on we will usually leave understood the p dependence of the T,Q functions):

- the TQ relation for the fundamental $T \equiv T_{\frac{1}{2}}$ and the $Q_\pm$ functions, coming from (1.129)
$$T(\lambda)Q_\pm(\lambda) = q^{\mp 2p/\beta^2} Q_\pm(q^{-1}\lambda) + q^{\pm 2p/\beta^2} Q_\pm(q\lambda) \tag{2.12}$$
where we recall that $q = e^{i\pi\beta^2}$

- the fusion relation for the T_j functions, coming from (1.116)
$$T_j(q^{\frac{1}{2}}\lambda) T_j(q^{-\frac{1}{2}}\lambda) = 1 + T_{j+\frac{1}{2}}(\lambda) T_{j-\frac{1}{2}}(\lambda) \tag{2.13}$$

- the expression that gives the T_j functions in terms of the $Q_\pm$ functions, coming from (1.131)
$$\begin{aligned} 2i \sin(2\pi p) T_j(\lambda) = {} & q^{(4j+2)p/\beta^2} Q_+(q^{j+\frac{1}{2}}\lambda) Q_-(q^{-j-\frac{1}{2}}\lambda) \\ & - q^{-(4j+2)p/\beta^2} Q_+(q^{-j-\frac{1}{2}}\lambda) Q_-(q^{j+\frac{1}{2}}\lambda) \end{aligned} \tag{2.14}$$
which for $j = 0$ gives the Q-functions analog of the Q-system that holds for the $\mathbf{Q}_\pm$ operators

- the Y-system for the Y-functions $Y_j(\theta) \equiv T_{j-\frac{1}{2}}(\theta) T_{j+\frac{1}{2}}(\theta)$
$$Y_j\left(\theta + i\pi\xi/2\right) Y_j\left(\theta - i\pi\xi/2\right) = \left(1 + Y_{j+\frac{1}{2}}(\theta)\right)\left(1 + Y_{j-\frac{1}{2}}(\theta)\right) \tag{2.15}$$
where we recall that $\xi = \frac{\beta^2}{1-\beta^2}$, and the θ variable is related to λ as $\lambda^{1+\xi} = e^\theta \Rightarrow \theta = \frac{1}{1-\beta^2} \ln \lambda$.

[4]It is manifest how what we have called $Q_\pm$ functions correspond to the eigenvalues of the $\mathbf{A}_\pm$ operators, rather than the $\mathbf{Q}_\pm$ ones. This is the convention used e.g. in [43], but many authors prefer to call as $Q_\pm$ functions precisely the $\mathbf{Q}_\pm$ eigenvalues. Our choice is motivated from the fact that allows for an easier identification between the formulas (2.12, 2.13, 2.14, 2.15) related to integrable models and those that will be found by analyzing the ODE.

In particular, we may already anticipate what will be identified as the ODE counterpart of these Q,T and Y functions:

- the $Q_\pm$ functions will correspond to the wronskian between the solution y_0 decaying at infinity along the positive real axis and the two solutions $\psi^\pm$ defined around $z = 0$
- the T_j functions will correspond to particular ratios of wronskians among the y_k solutions decaying in different sectors
- the Y functions will be built in the usual way as product of (the analog of) two T_j functions, and so will correspond to ratios of *four* wronskians between the y_k solutions

We now proceed to justify these identifications.

2.1.1 The TQ relation

We mainly follow [9] in our exposition, and start with the derivation of the TQ relation, which allows one to easily identify the ODE counterpart of the fundamental T function $T(\lambda) \equiv T_{\frac{1}{2}}(\lambda)$ and of the $Q_\pm$ functions, and to establish a dictionary between the quantities appearing in the differential equation and those of the integrable QFT. We begin by expanding the solution y_{-1} into the basis $\{y_0, y_1\}$

$$y_{-1}(z, E, l) = C(E, l)y_0(z, E, l) + \tilde{C}(E, l)y_1(z, E, l) \tag{2.16}$$

The coefficient of the expansion can be readily identified as ratios of wronskians among the solutions y_{-1}, y_0, y_1

$$C(E, l) = \frac{W_{-1,1}}{W_{0,1}} \qquad \tilde{C}(E, l) = -\frac{W_{-1,0}}{W_{0,1}} \tag{2.17}$$

where we used the compact notation $W_{i,j} = W[y_i, y_j] = y_i y_j' - y_j y_i'$ for the wronskians among the y_k (here $'$ stands for the z derivative). To convince oneself that the expressions (2.17) for the $C, \tilde{C}$ are correct, one can just compute e.g.

$$\begin{aligned} W_{-1,1} &= y_{-1}y_1' - y_1 y_{-1}' = \\ &= C\, y_0 y_1' + \tilde{C}\, y_1 y_1' - C\, y_1 y_0' - \tilde{C}\, y_1 y_1' = \\ &= C\, y_0 y_1' - C\, y_1 y_0' = C\, W_{0,1} \end{aligned} \tag{2.18}$$

where we used equation (2.16) to write y_{-1} in terms of the basis $\{y_0, y_1\}$ and noticed how the two terms containing the coefficient $\tilde{C}$ cancel out.
Due to the definition (2.5) of the y_k, we have that all the wronskians $W_{i,j}$ computed

between solutions that are subdominant in adjacent sectors (so with $j = i+1$) are equal among them. To find the explicit value for $W_{i,i+1}$ one can take advantage of the fact that a wronskian between two solutions does not depend on the point z where the solutions are evaluated, and use for the computation the asymptotic form (for $|z| \to \infty$) of the y_k and their derivatives. In this way we get

$$W_{i,i+1} = 2i \tag{2.19}$$

which implies $\tilde{C}(E,l) = 1$. We can thus rewrite (2.16) as

$$C(E,l)y_0(z,E,l) = y_{-1}(z,E,l) + y_1(z,E,l) \tag{2.20}$$

Taking on both sides of this expression the wronskian with the solutions $\psi_\pm$ and exploiting the relation

$$\begin{aligned} W\left[y_k, \psi^\pm\right](E,l) &= \omega^{\frac{k}{2}}\, W\left[y_0(\omega^{-k}z, \omega^{2k}E, l), \psi^\pm(z,E,l)\right] = \\ &= \omega^{\pm k(l+\frac{1}{2})}\, W\left[y_0(\omega^{-k}z, \omega^{2k}E, l), \psi^\pm(\omega^{-k}z, \omega^{2k}E, l)\right] = \\ &= \omega^{\pm k(l+\frac{1}{2})}\, W\left[y_0, \psi^\pm\right](\omega^{2k}E, l) \end{aligned} \tag{2.21}$$

where in the first equality we used the definition (2.5) of the y_k and in the second one the property (2.9) of the $\psi_\pm$, we can find

$$C(E,l)D^\mp(E,l) = \omega^{\mp(l+\frac{1}{2})}D^\mp\left(\omega^{-2}E, l\right) + \omega^{\pm(l+\frac{1}{2})}D^\mp\left(\omega^{2}E, l\right) \tag{2.22}$$

having introduced the functions

$$D^\mp(E,l) = W\left[y_0, \psi^\pm\right](E,l) \tag{2.23}$$

The similarity of this relation with the TQ relation (2.12) is manifest, with the identifications

$$C \leftrightarrow T \qquad D^\mp \leftrightarrow Q_\pm \tag{2.24}$$

providing we set

$$q = \omega \implies \beta^2 = \frac{2}{r+3}, \qquad 2p/\beta^2 = l + \frac{1}{2} \implies p = \frac{l+\frac{1}{2}}{r+3} \tag{2.25}$$

and λ is proportional to the square root of the energy $\lambda = \nu\sqrt{E}$ (we'll see in a moment how to fix the proportionality coefficient ν). This similarity between the objects appearing in the ODE and those of the integrable QFT can be transformed into an exact equality, thanks to a series of properties shared between the D^-, C and the Q^+, T functions respectively (it is sufficient to consider only D^- among the $D^\pm$, since $D^+(E,l) = D^-(E,-1-l)$). This will allow also to fix the value of ν. The properties that fully characterize D^-, C, together with the relation (2.22), are

(i) C and D^- are entire functions of E

(ii) If $l \in \mathbb{R}, l > -\frac{1}{2}$, the zeroes of D^- all lie on the positive real axis of the complex E plane

(iii) If $-\frac{r+1}{4} - 1 < l < \frac{r+1}{4}$, the zeroes of C all lie away from the positive real axis of the complex E plane

(iv) under our hypothesis of r being > 1, D^- has the large E asymptotic

$$\log D^-(E,l) \sim \frac{a_0}{2}(-E)^{\frac{r+3}{2r+2}} \quad , \quad |E| \to \infty, \quad |\arg(-E)| < \pi \tag{2.26}$$

where the coefficient a_0 is given by

$$a_0 = 2\int_0^\infty \left[\sqrt{t^{r+1}+1} - t^{\frac{r+1}{2}}\right] dt = -\frac{1}{\sqrt{\pi}}\Gamma\left(-\frac{1}{2} - \frac{1}{r+1}\right)\Gamma\left(1 + \frac{1}{r+1}\right) \tag{2.27}$$

(v) the normalisation of D^- is given by

$$D^-(E=0,l) = \frac{1}{\sqrt{\pi}}\Gamma\left(1 + \frac{2l+1}{r+3}\right)(r+3)^{\frac{2l+1}{r+3}+\frac{1}{2}} \tag{2.28}$$

As claimed in [9], these properties fix $C(E,l)$ and $D^-(E,l)$ uniquely, provided $-\frac{1}{2} < l < \frac{r+1}{4}$. The very same properties are satisfied by the functions $T(\lambda,p)$ and $Q_+(\lambda,p)$, if we replace λ^2 with νE and use the identifications (2.25) for the other parameters, proviso taking care of two minor aspects

1. the asymptotic of $Q_+(\lambda,p)$ for $|\lambda^2| \to \infty$ is

$$\log Q_+(\lambda,p) \sim \frac{r+2}{2}\,\Gamma\left(\frac{r+1}{r+3}\right)^{\frac{r+3}{r+1}} a_0\left(-\lambda^2\right)^{\frac{r+3}{2r+2}}, \quad \left|\lambda^2\right| \to \infty, \quad \left|\arg\left(-\lambda^2\right)\right| < \pi \tag{2.29}$$

which can be made to agree with the asymptotic (2.26) of D^- by choosing for the proportionality constant $\nu = \lambda^2/\sqrt{E}$ the value

$$\nu = (r+3)^{-\frac{r+2}{r+3}}\,\Gamma\left(\frac{r+2}{r+3}\right)^{-1} \tag{2.30}$$

2. the normalisation $Q_+(\lambda^2 = 0, p)$ is equal to one rather than the value (2.28) we have for D^-, so to have a perfect match between Q_+ and D^- we should multiply the latter by the inverse of (2.28), which we call α^-.

All the above considerations lead us to the exact identifications (written in terms of the parameters of the QFT):

$$Q_{\pm}(\lambda,p)|_{\beta^2} = \alpha^{\mp} D^{\mp}\left(\left(\frac{\lambda}{\nu}\right)^2, \frac{2p}{\beta^2}-\frac{1}{2}\right)\Bigg|_{r+1=2\beta^{-2}-2} \tag{2.31}$$

$$\begin{aligned} T(\lambda,p)|_{\beta^2} &= C\left(\left(\frac{\lambda}{\nu}\right)^2, \frac{2p}{\beta^2}-\frac{1}{2}\right)\Bigg|_{r+1=2\beta^{-2}-2} \\ &= \frac{W_{-1,1}}{W_{0,1}}\left(\left(\frac{\lambda}{\nu}\right)^2, \frac{2p}{\beta^2}-\frac{1}{2}\right)\Bigg|_{r+1=2\beta^{-2}-2} \end{aligned} \tag{2.32}$$

where $\alpha^{\mp} = D^{\mp}\left(0, 2p/\beta^2 - 1/2\right)^{-1}$, and we have inverted (2.25) to express E, l, r in terms of the QFT variables

$$E = \left(\frac{\lambda}{\nu}\right)^2 \qquad l = \frac{2p}{\beta^2}-\frac{1}{2} \qquad r+1 = 2\beta^2 - 2 \tag{2.33}$$

To conclude this section, we give a hint on how the properties (i)-(v) and the TQ relation may allow a complete identification of D^-. Taking account of these properties, for $r > 1$ Hadamard's factorisation theorem implies the existence of the following representation for D^-

$$D^-(E,l) = D^-(0,l)\prod_{n=0}^{\infty}\left(1-\frac{E}{E_n}\right) \tag{2.34}$$

in terms of an infinite product over its sets of zeros $\{E_n\}$. Recalling the definition (2.23) of D^- in terms of a wronskian between the solution y_0, that decays along the positive real axis, and the solution ψ^+, which goes to zero at the origin if $l > -1$, we may identify the zeros $\{E_n\}$ with the radial bound state energies of the Schrödinger equation (2.1) when the potential is confining, and interpret thus D^- as a spectral determinant of the ODE (2.1).

Apart from the constant factor $D^-(0,l)$, we can clearly see how the position of these zeros completely determine the D^- function (and the Q_+ function on the integrable side of the ODE-IM correspondence), so what is left to find in order to fix $D^-(0,l)$ are precisely the values of the $\{E_n\}$. The last relevant piece of information is that indeed properties (i)-(v) and the TQ relation allow also to locate the $\{E_n\}$, by means of a non-linear integral equation for an auxiliary function $d(E,l) = \omega^{2l+1} D^-(\omega^2 E, l)/D^-(\omega^{-2}E, l)$, usually called *Destri-de-Vega equation* or simply *NLIE* (the acronym of non-linear integral equation), of the type introduced in [45, 46]. The interested reader is referred to [9] for a detailed exposition on this aspect.

2.1.2 The full set of T functions and their fusion relations

Having seen in the previous section how the fundamental T function coincides with the ratio of two wronskians among a particular choice of the y_k solutions, one may be induced to conjecture also for the ODE analog of the other T_j functions a similar expression. This is indeed the case, as we will show in a moment.
The original route (as presented in [9]) to derive a T-system for the equation (2.1) is to consider the matrices that relate different pairs of asymptotic solutions $\{y_{k-1}, y_k\}$, taken as a basis for the differential equation. Choosing two couples $\{y_{k-1}, y_k\}$, $\{y_{k+m-1}, y_{k+m}\}$, we can express the elements of the first basis in terms of those of the second one as

$$y_{k-1}(x,E,l) = C_k^m(E,l)\, y_{k+m-1}(x,E,l) + \tilde{C}_k^m\, y_{k+m}(x,E,l) \tag{2.35}$$

$$y_k(x,E,l) = C_{k+1}^{m-1}(E,l)\, y_{k+m-1}(x,E,l) + \tilde{C}_{k+1}^{m-1}\, y_{k+m}(x,E,l) \tag{2.36}$$

or, in matrix form

$$\begin{pmatrix} y_{k-1} \\ y_k \end{pmatrix} = \mathbf{C}_k^{(m)} \begin{pmatrix} y_{k+m-1} \\ y_{k+m} \end{pmatrix}, \quad \mathbf{C}_k^{(m)} = \begin{pmatrix} C_k^m & \tilde{C}_k^m \\ C_{k+1}^{m-1} & \tilde{C}_{k+1}^{m-1} \end{pmatrix} \tag{2.37}$$

In a similar fashion of what we have shown in the previous section, we can identify the coefficients in these relations as ratios of wronskians among the solutions y_k

$$C_k^m(E,l) = \frac{W_{k-1,k+m}}{W_{k+m-1,k+m}} \qquad \tilde{C}_k^m(E,l) = -\frac{W_{k-1,k+m-1}}{W_{k+m-1,k+m}} \tag{2.38}$$

where we can notice that in all the $C_k^m, \tilde{C}_k^m$, the wronskian that sits at the denominator always involves y_k solutions that are subdominant in adjacent sectors. From the property (2.19), we then have that all these wronskians coincide, and are equal to the constant value $2i$ for any value of E, l. We may also notice that the C function that enters into the TQ relation is just C_0^1, in this new notation.
Exploiting the wronskian representation of the $C_k^m, \tilde{C}_k^m$ one can also easily derive other properties of these coefficients (in the following, where not specified, we assume the $C_k^m, \tilde{C}_k^m$ to be evaluated in (E,l)):

1. taking advantage of the equality of all wronskians appearing at the denominator of the $C_k^m, \tilde{C}_k^m$, and using the antisymmetry property $W_{i,j} = -W_{j,i}$, we can express all the $\tilde{C}_k^m$ in terms of the C_k^m

$$\tilde{C}_k^m = -C_k^{m-1} \tag{2.39}$$

   so that we can effectively consider only the C_k^m coefficients in our analysis

2. in a similar way, one can prove the equality

$$C_k^m = -C_{k+m+1}^{-m-2} \tag{2.40}$$

3. the action of the Ω symmetry on the wronskians (that can be checked by using the definition (2.5) of the y_k solutions)

$$\Omega^n \, W_{i,j}(E,l) = W_{i,j}(\omega^{2n}E,l) = W_{i+n,j+n}(E,l) \tag{2.41}$$

implies for the C_k^m coefficients

$$\Omega^n \, C_k^m(E,l) = C_k^m(\omega^{2n}E,l) = C_{k+n}^m(E,l) \tag{2.42}$$

To obtain a T-system, the authors of [9] used the following relation among the matrices $\mathbf{C}_k^{(m)}$ of change of basis

$$\mathbf{C}_k^{(m)}\mathbf{C}_{k+m}^{(n)} = \mathbf{C}_k^{(m+n)} \tag{2.43}$$

that simply express how a change of basis from $\{y_{k+m+n-1}, y_{k+m+n}\}$ to $\{y_{k+m-1}, y_{k+m}\}$, followed by another change of basis to $\{y_{k-1}, y_k\}$, is equivalent of performing the overall change in one go. Writing explicitly the components of the matrices involved, the relation reads

$$\begin{aligned}\begin{pmatrix} C_k^{m+n} & -C_k^{m+n-1} \\ C_{k+1}^{m+n-1} & -C_{k+1}^{m+n-2} \end{pmatrix} &= \begin{pmatrix} C_k^{m} & -C_k^{m-1} \\ C_{k+1}^{m-1} & -C_{k+1}^{m-2} \end{pmatrix} \begin{pmatrix} C_{k+m}^{n} & -C_{k+m}^{n-1} \\ C_{k+m+1}^{n-1} & -C_{k+m+1}^{n-2} \end{pmatrix} = \\ &= \begin{pmatrix} C_k^m C_{k+m}^n - C_k^{m-1} C_{k+m+1}^{n-1} & -C_k^m C_{k+m}^{n-1} + C_k^{m-1} C_{k+m+1}^{n-2} \\ C_{k+1}^{m-1} C_{k+m}^n - C_{k+1}^{m-2} C_{k+m+1}^{n-1} & -C_{k+1}^{m-1} C_{k+m}^{n-1} + C_{k+1}^{m-2} C_{k+m+1}^{n-2} \end{pmatrix}\end{aligned} \tag{2.44}$$

where we have used the property (2.39) to express all the coefficients $\tilde{C}_k^{(m)}$ in terms of their version without the tilde symbol on top. Specialising this relation for $n = -m$ the matrix on the left of the equality becomes the identity matrix, and choosing the equation corresponding to the lower right elements in this matrix identity[5] one gets

$$-C_{k+1}^{m-1} C_{k+m}^{-m-1} + C_{k+1}^{m-2} C_{k+m+1}^{-m-2} = 1 \tag{2.45}$$

that using the property (2.40) can be recast as

$$C_{k+1}^{m-1} C_k^{m-1} - C_{k+1}^{m-2} C_k^m = 1 \tag{2.46}$$

or also as

$$C_k^m(\omega^2 E) C_k^m(E) - C_k^{m-1}(\omega^2 E) C_k^{m+1}(E) = 1 \tag{2.47}$$

where we have used the property (2.42) in order to have all the C functions with the same lower index, and relabelled $m-1 \to m$ for later convenience. Choosing then $k = 0$,

[5] Using the upper left elements in the matrix identity at the end yields the same results. The other two off-diagonal elements give instead trivial relations, in light of properties (2.40, 2.42)

and introducing the following functions as candidate for the ODE analog of the fused T_j functions

$$C^{(n)}(E,l) \equiv C_0^{(n)}(\omega^{-n+1}E,l) = \frac{W_{-1,n}}{W_{n-1,n}}(\omega^{-n+1}E,l), \quad n=-1,0,1,2,\dots \tag{2.48}$$

we can easily rewrite (2.46) as

$$C^{(m)}(\omega E)\, C^{(m)}(\omega^{-1}E) = 1 + C^{(m-1)}(E) C^{(m+1)}(E) \tag{2.49}$$

having brought a term on the right hand side. This precisely matches the fusion relations (2.13), recalling the identification of λ with νE, and of ω with q, if we associate

$$T_{n/2}(\lambda,p) = C^{(n)}\left(\left(\frac{\lambda}{\nu}\right)^2, \frac{2p}{\beta^2}-\frac{1}{2}\right)\Bigg|_{r+1=2\beta^{-2}-2} \tag{2.50}$$

which also reproduces $T_{-1/2}=0$, $T_{1/2}=1$.
For a better intuition of the physical meaning of these objects, it is always useful to recover the representation (2.48) of the $C^{(n)}$ in terms of wronskians, which we may rewrite also as

$$C^{(n)}(E,l) = \frac{W_{-1,n}}{W_{n-1,n}}(\omega^{-n+1}E,l) = \begin{cases} \dfrac{W_{-j-1,j}}{W_{j-1,j}}(\omega E,l), & \text{if } n=2j \\ \dfrac{W_{-j-1,j+1}}{W_{j,j+1}}(E,l), & \text{if } n=2j+1 \end{cases} \tag{2.51}$$

where we have separated the even and odd values of n, in terms of a non-negative integer index $j=0,1,2,\dots$, leaving aside the "pathologic" $C^{(-1)}=0$. We notice that the different $C^{(n)}$ involve (in their numerator) the wronskian among decaying solutions in different sectors, covering all possible openings (i.e. distances, in index values) between the sectors. We see then how not only the $D^{\pm}$, but also the $C^{(n)}$, can be interpreted as spectral determinants, whose zeros in this case do not signal the presence of a radial bound state (as is for $D^{\pm}$), but of a state that decays at infinity in two particular Stokes sectors. We recall that in these formulas the wronskian in the denominator has always the same constant value of $2i$, but we have chosen to keep it in its explicit form to retain a more general form of writing, and to have always under control its physical interpretation.

In a similar fashion of what is done in [44], we could have also obtained the fusion relations (2.49) directly using only the definitions of $C^{(n)}$ in terms of wronskians, thanks to the so-called *Schouten identity*

$$W_{a,b}\, W_{c,d} + W_{a,d}\, W_{b,c} + W_{a,c}\, W_{d,b} = 0 \tag{2.52}$$

which is a direct consequence of the definition of the wronskian function. We can in fact check that any of the relations contained into the matrix relation (2.43) can be obtained from this identity by specific choices of the indices a, b, c, d, and using the basic properties of the wronskians among the y_k solutions. For example, let's say we want to reproduce the relation we used to ultimately find the fusion relation, i.e.

$$C_{k+1}^{m-1} C_{k+m}^{n-1} - C_{k+1}^{m-2} C_{k+m+1}^{n-2} = C_{k+1}^{m+n-2} \tag{2.53}$$

which in terms of wronskians reads

$$\frac{\langle k, k+m\rangle}{\langle k+m-1, k+m\rangle} \frac{\langle k+m-1, k+m+n-1\rangle}{\langle k+m+n-2, k+m+n-1\rangle} - \frac{\langle k, k+m-1\rangle}{\langle k+m, k+m-1\rangle} \frac{\langle k+m, k+m+n-1\rangle}{\langle k+m+n-2, k+m+n-1\rangle} = \\ = \frac{\langle k, k+m+n-1\rangle}{\langle k+m+n-2, k+m+n-1\rangle}$$

where we have introduced only for this section the notation $W_{i,j} = \langle i, j\rangle$ to highlight the indices related to the solutions y_k considered. In this case we can use the Schouten identity with the following choice of indices

$$a = k, \quad b = k+m, \quad c = k+m-1, \quad d = k+m+n-1$$

to get

$$\langle k, k+m\rangle\langle k+m-1, k+m+n-1\rangle + \langle k, k+m+n-1\rangle\langle k+m, k+m-1\rangle \\ + \langle k, k+m-1\rangle\langle k+m+n-1, k+m\rangle = 0$$

With some minor manipulations, this becomes

$$\langle k, k+m\rangle\langle k+m-1, k+m+n-1\rangle - \langle k, k+m-1\rangle\langle k+m, k+m+n-1\rangle = \\ = \langle k, k+m+n-1\rangle\langle k+m-1, k+m\rangle$$

where we can see that one of the wronskians on the right hand side is of the type $\langle k, k+1\rangle$, and all other wronskians match with those present in the numerators of (2.53). Thanks to property (2.19), we can then divide this expression by any two terms of the form $\langle k, k+1\rangle$ to get an equation equivalent to (2.53).

2.1.3 The Q system

Now that we have identified the ODE counterpart of the $Q_\pm$ and all the T_j functions, finding the analog of the relation (2.14) is straightforward. We begin by expanding y_0 in the basis $\psi_\pm$

$$(2l+1) y_0(z, E, l) = D^-(E, l)\psi^-(z, E, l) - D^+(E, l)\psi^+(z, E, l) \tag{2.54}$$

where we have identified $D^{\mp}(E,l)$ through their definition (2.23) and used the result $W\left[\psi^{-}(z,E,l),\psi^{+}(z,E,l)\right]=2l+1$. Applying the symmetry Ω^{k} to this equation, and recalling (2.5, 2.9), we get

$$(2l+1)y_k(z,E,l)=\omega^{-k(l+\frac{1}{2})}D^{-}(\omega^{2k}E,l)\psi^{-}(z,E,l)-\omega^{+k(l+\frac{1}{2})}D^{+}(\omega^{2k}E,l)\psi^{+}(z,E,l) \tag{2.55}$$

If we now take the wronskian between this equation evaluated for $k=-1$ and for $k=n$, shift E to $\omega^{-n+1}E$, so that on the left hand side we reproduce $C^{(n)}$, and exploit on the right hand side the bilinearity of the wronskian, we find

$$\begin{aligned}(4l+2)iC^{(n)}(E)=&\,\omega^{(n+1)(l+\frac{1}{2})}D^{-}(\omega^{n+1}E,l)D^{+}(\omega^{-n-1}E,l)\\&-\omega^{-(n+1)(l+\frac{1}{2})}D^{-}(\omega^{-n-1}E,l)D^{+}(\omega^{n+1}E,l)\end{aligned} \tag{2.56}$$

With the identifications (2.31,2.32) in mind, this strongly resembles the expression (2.14) we were seeking. To make the analogy exact one should notice how

$$D^{-}(0,l)D^{+}(0,l)=(\alpha^{-}\alpha^{+})^{-1}=(2l+1)/\sin\tfrac{2l+1}{r+3}\pi \tag{2.57}$$

where an expression for $D^{+}(0,l)$ can be found by analytically continuing from l to $-1-l$ the one (2.28) we have given for $D^{+}(0,l)$.
The above relation (2.56), specialised for $n=0$, provides also the analog of the Q-system

$$(4l+2)i=\omega^{l+\frac{1}{2}}D^{-}(\omega E,l)D^{+}(\omega^{-1}E,l)-\omega^{-l-\frac{1}{2}}D^{-}(\omega^{-1}E,l)D^{+}(\omega E,l) \tag{2.58}$$

2.1.4 Y functions and Y-system

Similarly to what we have shown in section 1.5, given the set of fused T_j functions, which in this case correspond to the $C^{(n)}$, it's immediate to build the Y ones as

$$Y_s(E,l)=C^{(s+1)}(E,l)C^{(s-1)}(E,l) \tag{2.59}$$

where $s\in\mathbb{Z}_{\geq 0}$ and $Y_0=0$ since $C^{(-1)}=0$. Thanks to the fusion relation of the $C^{(n)}$, these satisfy

$$Y_s(\omega E)Y_s(\omega^{-1}E)=(1+Y_{s+1}(E))(1+Y_{s-1}(E))\qquad s=1,2,3,\dots \tag{2.60}$$

If we want to write this set of equations in terms of the variable $\theta=\frac{1}{1-\beta^2}\ln\lambda$, we shall recall the identifications

$$\beta^2=\frac{2}{r+3}\qquad\lambda=\nu\sqrt{E} \tag{2.61}$$

so that we have, in terms of quantities related to the ODE

$$\theta=\frac{r+3}{2(r+1)}\ln E\nu^2 \tag{2.62}$$

A rescaling of the energy by a factor $\omega^k = e^{\frac{2ki\pi}{r+3}}$, will then correspond to a shift $\theta \to \theta + \frac{ik\pi}{r+1}$, and we may rewrite (2.60) as

$$Y_s\left(\theta + \frac{i\pi}{r+1}\right) Y_s\left(\theta - \frac{i\pi}{r+1}\right) = \left(1 + Y_{s+1}(\theta)\right)\left(1 + Y_{s-1}(\theta)\right) \tag{2.63}$$

Expressing the Y functions directly in terms in wronskians, we have

$$Y_s(E) = \begin{cases} \dfrac{W_{-j,j}W_{-j-1,j+1}}{W_{j-1,j}W_{j,j+1}}(E,l), & \text{if } s = 2j \\ \dfrac{W_{-j-1,j}W_{-j-2,j+1}}{W_{j-1,j}W_{j,j+1}}(\omega E,l), & \text{if } s = 2j+1 \end{cases} \tag{2.64}$$

As we have showed for the fusion relations of the T, one can derive also the Y-system directly from this wronskian representation of the Y functions, through the Schouten identity and the properties of the wronskians among the y_k solutions.
To conclude, we remark (as we have already seen when discussing the BLZ method) that, as it stands, the Y-system (2.63) is not very useful, since it involves an infinite number of Y functions, but a truncation phenomenon may occur to make the relations close among a finite number of Y. The truncation on the number of relevant Y functions will be induced in this case by a periodicity of the $D^{\pm}$ functions, coming from the monodromy of solutions around the origin; this periodicity of $D^{\pm}$, via the formula (2.14), implies an additional relation on the $C^{(n)}$ (other than the fusion relations) that allows a closure of the Y-system.
Under our assumptions of a integer value for r, despite the monodromy around the origin being non-trivial due to the angular momentum term, we have indeed a closure of the Y-system under the first $r+1$ Y functions, plus an additional $\bar{Y} \equiv C^{(r+3)}$, thanks to the relation [40]

$$C^{(r+3)} = 2\cos\left(\pi(l + \frac{1}{2})\right) + C^{(r+1)} \tag{2.65}$$

The Y-system then reads

$$\begin{aligned} Y_s\left(\theta + \frac{i\pi}{r+1}\right) Y_s\left(\theta - \frac{i\pi}{r+1}\right) &= \left(1 + Y_{s+1}(\theta)\right)\left(1 + Y_{s-1}(\theta)\right), \quad s = 1, \dots, r \\ Y_{r+1}\left(\theta + \frac{i\pi}{r+1}\right) Y_{r+1}\left(\theta - \frac{i\pi}{r+1}\right) &= \left(1 + Y_r(\theta)\right)\left(1 + e^{2\pi i l}\bar{Y}(\theta)\right)\left(1 + e^{-2\pi i l}\bar{Y}(\theta)\right) \\ \bar{Y}\left(\theta + \frac{i\pi}{r+1}\right) \bar{Y}\left(\theta - \frac{i\pi}{r+1}\right) &= \left(1 + Y_{r+1}(\theta)\right) \end{aligned} \tag{2.66}$$

This structure resembles the one of the Y-system related to an ADE theory of D_{r+3} type (recall the discussion in section 1.3), and by computing the asymptotic of the Y

functions, together with their analicity properties, can be converted into a set of TBA equations. We will not discuss this procedure here, but an explicit example of this type of analysis can be found in the following section.

2.2 An upgrade for a generic polynomial potential

In this second section we tackle a generalisation of the problem presented in the first part of this chapter. We still consider the 1D stationary Schrödinger equation as ODE, but we allow the potential to be a generic polynomial function with complex coefficients, not only the single monomial z^{r+1}. Regarding the angular momentum term, which was present in the example we already discussed, we choose to discard it to work in a simplified setting, and to align ourselves with the work [1] of K.Ito and collaborators, which has served as starting point for the main part of the original work of this thesis. Our analysis will build on the intuition developed in the previous section, in order to reproduce through standard ODE-IM techniques the Y-system presented in [1]. In the paper the authors used the results found in [44] (which is set in a different context) to directly write the the Y-functions and their functional relations, while here we will derive them through a T system.

We start off as usual by writing explicitly the ODE under consideration, namely the 1D stationary Schrödinger equation with a generic polynomial potential of degree $r+1$

$$\left(-\frac{d^2}{dz^2} + z^{r+1} + \sum_{a=1}^{r} b_a z^{r-a}\right)\psi(z, b_a) = 0 \tag{2.67}$$

where we have considered the coefficient associated to the highest power of z in the polynomial to be 1, while labelled as b_a all the other coefficients in the polynomial, b_r including the energy, and set as before $\hbar = 1$. The particular case where all the coefficients b_a are set to zero except for b_r, coincides with the situation studied in the original ODE-IM correspondence (the one presented in the previous section), when the angular term vanishes (so either $l = 0$ or $l = -1$). With respect to that case, the structure of Stokes sectors remains invariant, being these determined by the behaviour of solutions at infinity (where is the highest power in the polynomial potential that dominates), with the further simplification that, being absent the angular momentum term, now the monodromy of solutions around the origin is trivial. As done in the previous section, we first present a subdominant solution in the Stokes sector $\mathcal{S}_0$ [7]

$$y\left(z, b_a\right) \sim \frac{1}{\sqrt{2i}} z^{n_r} \exp\left(-\frac{2}{r+3} z^{\frac{r+3}{2}}\right) \tag{2.68}$$

where the power n_r is defined as

$$n_r = \begin{cases} -\frac{r+1}{4} & r+1 \text{ odd} \\ -\frac{r+1}{4} - B_{\frac{r+3}{2}} & r+1 \text{ even} \end{cases} \tag{2.69}$$

and the $B_m(b_a)$ are complex functions depending only on b_a (and not on z), determined through the series expansion

$$\left(1 - \sum_{a=1}^{r} b_a z^{-a-1}\right)^{1/2} \equiv 1 + \sum_{m=1}^{\infty} B_m z^{-m} \tag{2.70}$$

Other solutions y_k, each decaying in the related Stokes sector $\mathcal{S}_k$, are then generated by acting on y via a generalised version[6] of the symmetry Ω seen in section 2.1

$$y_k(z, b_a) \equiv \omega^{k/2}\, \Omega^k\, y(z, b_a) = \omega^{k/2}\, y(\omega^{-k} z, \omega^{-(a+1)k} b_a) \tag{2.71}$$

where

$$\Omega : (z, b_a) \to \left(\omega^{-1} z, \omega^{-a-1} b_a\right), \quad \omega = e^{\frac{2\pi i}{r+3}} \tag{2.72}$$

Since the potential is regular in $z = 0$, in this case the solutions y_k will all be regular in that point, and their monodromy will be trivial. The construction made in the previous section of the $D^{\mp}$ functions (the one corresponding to the $Q_{\pm}$ functions in the BLZ method) can't be precisely mirrored. We could still find the analog of a TQ relation by considering as Q function the y solution evaluated in the origin (along the lines of what is described in [9]), proviso we fix the value of the p parameter of the QFT to the specific value $\beta^2/4$, but we do not proceed further in this direction. Instead, we focus on building the analog of the T_j (and then the Y_j) functions, for which we can reproduce similar results of those already seen in section 2.1.
With respect to that case, what we have to be careful about is the value of the wronskian between two "adjacent" solutions y_k, y_{k+1}. Now in fact, if the degree of the polynomial potential is even, it will depend on the pair of solutions chosen

$$W_{k,k+1}(b_a) = \begin{cases} 1 & \text{if } r+1 \text{ odd} \\ \omega^{(-1)^k B_{\frac{r+3}{2}}(b_a)} & \text{if } r+1 \text{ even} \end{cases} \tag{2.73}$$

If $r+1$ is odd, the steps done in section 2.1.2 can all be reproduced without any further complication, besides taking account of the fact that now all the $C_k^m, \tilde{C}_k^m$ functions depend not only on E, but on the full set of parameters b_a, and when shifting their arguments we have to shift all parameters accordingly to the new definition of the Ω

[6] One can see that this symmetry precisely reduces to the one seen in section 2.1 if we set $b_a = 0, \forall a \neq r$, thanks to the equivalence $\omega^{-r-1} E = \omega^2 E$

symmetry. In particular, the new definition of the $C^{(n)}$ functions (the analog of the T_j) will be

$$C^{(n)}(b_a) \equiv C_0^{(n)}(\omega^{(a+1)\frac{n-1}{2}} b_a) = \frac{W_{-1,n}}{W_{n-1,n}}(\omega^{(a+1)\frac{n-1}{2}} b_a), \quad n = -1, 0, 1, 2, \dots \tag{2.74}$$

or also

$$C^{(n)}(b_a) = C_0^{(n)\,[-n+1]}(b_a) \tag{2.75}$$

having introduced the following notation for the shift of functions

$$f^{[k]}(z, b_a) = f(\omega^{-k/2} z, \omega^{-(a+1)k/2} b_a) \tag{2.76}$$

The fusion relation will read

$$C^{(n)\,[+1]}(b_a) C^{(n)\,[-1]}(b_a) = 1 + C^{(n-1)}(b_a) C^{(n+1)}(b_a) \tag{2.77}$$

As before, defining then from the $C^{(n)}$ the Y functions as

$$Y_s(b_a) = C^{(s+1)}(b_a) C^{(s-1)}(b_a), \qquad s \in \mathbb{Z}_{\geq 0} \tag{2.78}$$

one gets from (2.77) the Y-system

$$Y_s^{[+1]}(b_a) Y_s^{[-1]}(b_a) = (1 + Y_{s-1}(b_a))\,(1 + Y_{s+1}(b_a))\,, \quad s = 1, 2, 3, \dots \tag{2.79}$$

where we have $Y_0 = 0$ by definition, since $C^{(-1)} = 0$.
An important simplification with respect to to the original ODE-IM setting enters now into play, regarding the truncation of the Y-system: since in this case the monodromy around the origin is trivial, we will have $y_{k+r+3}(z, b_a) \propto y_k(z, b_a)$, and thus $C^{(r+2)} \propto C^{(-1)} = 0 \Rightarrow C^{(r+2)} = 0$, which in turn implies $Y_{r+1} = 0$. The Y-system above then truncates to the first r Y-functions, without needing any redefinition of the Y_s, leaving with an A_r type Y-system.

Moving onto the more delicate case of an even degree polynomial potential, here, due to the different value of the "nearest neighbour" wronskian $W_{k,k+1}$ depending if k is even or odd, we have to modify the relations (2.39, 2.40) that one uses to get from (2.43) to the fusion relations. The simple form of these relations relied in fact on the result that all $W_{k,k+1}$ were equal. In this case, we can update them as

$$\tilde{C}_k^m(b_a) = -\omega^{(-1)^{k+m} 2B_{\frac{r+3}{2}}(b_a)} C_k^{m-1}(b_a) \tag{2.80}$$

$$C_{k+m+1}^{-m-2}(b_a) = -\omega^{(-1)^{k-1} B_{\frac{r+3}{2}}(b_a) + (-1)^{k+m-1} B_{\frac{r+3}{2}}(b_a)} C_k^m(b_a) \tag{2.81}$$

while obviously the counterpart of the shift property 2.42 still holds, being only based on the definition of the y_k

$$\Omega^n\, W_{i,j}(b_a) = W_{i,j}(\omega^{-(a+1)n} b_a) = W_{i+n,j+n}(b_a) \tag{2.82}$$

$$\Rightarrow \Omega^n\, C^m_k(b_a) = C^m_k(\omega^{-(a+1)n} b_a) = C^m_{k+n}(b_a) \tag{2.83}$$

Another useful property is the relation

$$B(\omega^{-(a+1)k} b_a) = (-1)^k B(b_a) \tag{2.84}$$

With these properties at hand, repeating the passages that in section 2.1.2 led to the fusion relations, one can check that the equations (2.77) get modified as

$$C^{(n)\,[+1]}(b_a) C^{(n)\,[-1]}(b_a) = \omega^{B^{[+n]}_{\frac{r+3}{2}}(b_a) - B^{[-n]}_{\frac{r+3}{2}}(b_a)} + C^{(n-1)}(b_a) C^{(n+1)}(b_a) \tag{2.85}$$

where the definition of the $C^{(n)}$ is the same used for the case of $r+1$ odd. Moving onto the Y-system, this remains of the form (2.79), proviso we change the definition of the Y functions as

$$Y_s(b_a) = \omega^{-B^{[+s]}_{\frac{r+3}{2}}(b_a) + B^{[-s]}_{\frac{r+3}{2}}(b_a)} C^{(s+1)}(b_a) C^{(s-1)}(b_a), \qquad s \in \mathbb{Z}_{\geq 0} \tag{2.86}$$

All the discussion on the truncation of the Y-system done before when analyzing the case $r+1$ odd, obviously still holds.

We want now to express these Y functions in terms of wronskians, and show in particular that their form matches with that given in [1]. To this end, we first rewrite the wronskian representation (2.74) of the $C^{(n)}$ as

$$C^{(n)}(E,l) = \frac{W_{-1,n}}{W_{n-1,n}}(\omega^{(a+1)\frac{n-1}{2}} b_a) = \begin{cases} \left[\dfrac{W_{-j-1,j}(b_a)}{W_{j-1,j}(b_a)}\right]^{[+1]}, & \text{if } n = 2j \\ \dfrac{W_{-j-1,j+1}(b_a)}{W_{j,j+1}(b_a)}, & \text{if } n = 2j+1 \end{cases} \tag{2.87}$$

where we have used the shift property (2.82) of the wronskians. This is the analog of the expression (2.51) we had in section 2.1. Separating now also the Y functions in even and odd values of their index s, we see that, thanks to property (2.84):

- <u>if $s = 2j$</u>

$$\begin{aligned} -B^{[+s]}_{\frac{r+3}{2}}(b_a) + B^{[-s]}_{\frac{r+3}{2}}(b_a) &= -B_{\frac{r+3}{2}}(\omega^{-(a+1)j} b_a) + B_{\frac{r+3}{2}}(\omega^{(a+1)j} b_a) = \\ &= -(-1)^j B_{\frac{r+3}{2}}(b_a) + (-1)^{-j} B_{\frac{r+3}{2}}(b_a) = 0 \end{aligned}$$

and so we have just

$$Y_{2j}(b_a) = C^{(2j+1)}(b_a)C^{(2j-1)}(b_a) \tag{2.88}$$

that thanks to (2.87) can be rewritten as

$$Y_{2j}(b_a) = \frac{W_{-j,j}(b_a)W_{-j-1,j+1}(b_a)}{W_{j-1,j}(b_a)W_{j,j+1}(b_a)} = \frac{W_{-j,j}(b_a)W_{-j-1,j+1}(b_a)}{W_{-j-1,-j}(b_a)W_{j,j+1}(b_a)} \tag{2.89}$$

where in the second step we used the equality $W_{j-1,j}(b_a) = W_{-j-1,-j}(b_a)$ coming from (2.73) and the fact that $j-1$ and $-j-1$ share the same parity.

- if $s = 2j+1$

(i) $B^{[+s]}_{\frac{r+3}{2}}(b_a) = B_{\frac{r+3}{2}}(\omega^{-(a+1)(j+\frac{1}{2})}b_a) = (-1)^j B^{[+1]}_{\frac{r+3}{2}}(b_a) = (-1)^{-j-2}B^{[+1]}_{\frac{r+3}{2}}(b_a)$

$\Rightarrow \omega^{-B^{[+s]}_{\frac{r+3}{2}}(b_a)} = \left(W^{[+1]}_{-j-2,-j-1}(b_a)\right)^{-1}$

(ii) $B^{[-s]}_{\frac{r+3}{2}}(b_a) = B_{\frac{r+3}{2}}(\omega^{(a+1)(j+\frac{1}{2})}b_a) = (-1)^{-j-1} B^{[+1]}_{\frac{r+3}{2}}(b_a) = (-1)^{j-1}B^{[+1]}_{\frac{r+3}{2}}(b_a)$

$\Rightarrow \omega^{B^{[-s]}_{\frac{r+3}{2}}(b_a)} = W^{[+1]}_{j-1,j}(b_a)$

and thus

$$Y_{2j+1}(b_a) = \left[\frac{W_{j-1,j}(b_a)}{W_{-j-2,-j-1}(b_a)}\right]^{[+1]} C^{(2j)}(b_a)C^{(2j+2)}(b_a) \tag{2.90}$$

This form of rewriting the prefactor present in front of the $C^{(n)}$ functions is particularly useful to simplify the expression of Y_{2j+1}, once the $C^{(n)}$ have been expressed in terms of wronskians (using (2.87))

$$\begin{aligned} Y_{2j+1}(b_a) &= \left[\frac{W_{j-1,j}(b_a)}{W_{-j-2,-j-1}(b_a)}\right]^{[+1]} \left[\frac{W_{-j-1,j}(b_a)W_{-j-2,j+1}(b_a)}{W_{j-1,j}(b_a)W_{j,j+1}(b_a)}\right]^{[+1]} = \\ &= \left[\frac{W_{-j-1,j}(b_a)W_{-j-2,j+1}(b_a)}{W_{-j-2,-j-1}(b_a)W_{j,j+1}(b_a)}\right]^{[+1]} \end{aligned} \tag{2.91}$$

The expressions (2.89, 2.91) we have obtained for the even and odd-indexed Y functions in case the degree $r+1$ of the polynomial potential is even, hold also for the simpler case of $r+1$ odd, since in that situation one can trade the "nearest neighbour" wronskians that are present in the denominator of the $C^{(n)}$ (and thus also of the Y_s) functions for any other wronskian of the type $W_{k,k+1}$, being these all equal to 1. This form for the Y-functions makes manifest how each of the Y_s involves the ratio of wronskians among

only four y_k functions (i.e. are related to only four particular Stokes sectors), and can be regarded as a particular case of the cross-ratio

$$\chi_{ijkl} = \frac{W_{i,j}W_{k,l}}{W_{i,k}W_{j,l}} \tag{2.92}$$

Cross ratios of this type have been studied in [44], finding functional equations of the Y-system type, despite in a different context.
If one starts directly from the expressions (2.89, 2.91) for the Y functions, like the authors in [1] do, the existence of the Y-system can then be derived using the Schouten identity and the shift properties (2.83) of the wronskians; we have already seen in section 2.1 in a similar context how the traditional ODE-IM construction can be completely reproduced by using only the wronskians properties.

We have left for the end an important subtlety that may have been left unnoticed up to to this point due to the not very transparent notation. Looking back at the Y-system found for the present ODE model, which we rewrite here for convenience in a slightly more explicit version

$$Y_s(\omega^{-(a+1)/2}b_a)Y_s(\omega^{(a+1)/2}b_a) = (1+Y_{s-1}(b_a))\,(1+Y_{s+1}(b_a))\,, \quad s=1,2,\dots,r \tag{2.93}$$

one sees that the functions on the left hand-side are evaluated at values of the parameters b_a which are different from those at which the functions on the right hand-side are taken. Despite being the number of equations in the Y-system finite due to the truncation phenomenon, to have a closed system we should evaluate these equations at many different values of the b_a; being the number of equations equal to r, this procedure becomes more and more involved going up with the degree of the polynomial. This problem did not arise in the previous section, since then we had only the energy (other than l) as parameter.
The solution to this problem offered in [1], which is the one we will also adopt, is to abandon the idea to use directly the energy as spectral parameter, and use instead $\hbar$ for this purpose, restoring its presence into the Schrödinger equation. This will allow for a more manageable form of the functional equations, in particular of the Y-system. We will leave this task momentarily suspended, postponing it to the end of the next chapter. At that point we will be able in fact to show how the Y functions found in this "revised" ODE-IM approach that also uses $\hbar$ can be identified with objects appearing in the context of the so-called exact WKB method, a different method with respect to the ODE-IM one to obtain exact results in the analysis of the Schrödinger equation.

Chapter 3

Exact WKB method and Resurgent quantum mechanics

Having seen in the previous chapter how, through a particular type of analysis of the 1D Schrödinger equation, deeply based on broken symmetries and asymptotic properties of the solutions, mathematical objects and functional relations typical of integrable models can be reproduced in that context, we now present a different approach to tackle the study of the Schrödinger equation. This method is called *exact WKB method*, and is a revised version of the more historic WKB approximation method (developed by the scientists Wentzel, Kramers and Brillouin, from which it takes the name) aiming to obtain instead exact results. Its application can be seen as a part of a larger picture, which usually goes under the name of *resurgent* approach to quantum mechanics, a program to manage the divergent semiclassical WKB series expansion so that to still extract useful information about the Schrödinger problem, such as finding exact quantisation conditions. Despite having a different nature with respect to the ODE-IM approach, we will see that, also in this context, functional equations in the form of Y-systems arise, and the corresponding Y functions have a relevant physical meaning (we will show this explicitly for the case of a generic polynomial potential, but the procedure is far more general). This is obviously not by chance[1]; we will show at the end of the chapter how the Y-system found in section 2.2 for the polynomial potential through standard ODE-IM techniques can be brought into the one coming from WKB analysis. As anticipated at the end of the previous chapter, this will also allow to bypass the problems found in closing the ODE-IM Y-system, despite paying the price for a less direct way to extract the energy spectrum.

The exact WKB perspective is the one we have mostly followed in our developments

[1] As an historical note, the functional equations described in the articles [4–6], that inspired the first paper [8] on ODE-IM, indeed were found using WKB analysis

on the analysis of the Schrödinger equation through Y-systems and TBA equations, which are collected in sections 4 and 5, but having in mind the duality with the ODE-IM approach will allow for deeper interpretations.

3.1 The WKB ansatz

The problem we consider is the analysis of a 1-dimensional stationary Schrödinger equation on the Riemann sphere $\mathbb{P}^1$. We assume, in a local coordinate z on $\mathbb{P}^1$, the following form for the equation

$$\left(-\frac{d^2}{dz^2}+\eta^2 Q(z)\right)\psi(z,\eta)=0 \tag{3.1}$$

where $\eta=\hbar^{-1}$ and $Q(z)=V(z)-E$ is the potential function, being $E\in\mathbb{C}$ the energy and $V(z)$ the potential, which we assume to be a generic meromorphic function. With respect to the ODE-IM analyses of chapter 2, it is important to remark that here the presence of the Planck constant is retained; actually, $\hbar$ (or equivalently η) will play a central role in the WKB method, as most objects will be defined through semiclassical series expansions. We may consider from the beginning η as a complex parameter, and call $\theta=\arg(\eta)$. Under a generic coordinate transformation $z=z(\tilde{z})$, if we also redefine the wavefunction as

$$\tilde{\psi}(\tilde{z},\eta)=\psi(z(\tilde{z}),\eta)\left(\frac{dz(\tilde{z})}{d\tilde{z}}\right)^{-1/2} \tag{3.2}$$

then we can always recast the equation (3.1) in the new coordinate $\tilde{z}$ in a form like the one we started with

$$\left(\frac{d^2}{d\tilde{z}^2}-\eta^2\tilde{Q}(\tilde{z},\eta)\right)\tilde{\psi}(\tilde{z},\eta)=0 \tag{3.3}$$

but with a new expression for the potential function, that may now contain an η dependence

$$\tilde{Q}(\tilde{z},\eta)=Q(z(\tilde{z}))\left(\frac{dz(\tilde{z})}{d\tilde{z}}\right)^2-\frac{1}{2}\eta^{-2}\{z(\tilde{z});\tilde{z}\} \tag{3.4}$$

where $\{z(\tilde{z});\tilde{z}\}$ is the Schwarzian derivative

$$\{z(\tilde{z});\tilde{z}\}=\left(\frac{d^3z(\tilde{z})}{d\tilde{z}^3}/\frac{dz(\tilde{z})}{d\tilde{z}}\right)-\frac{3}{2}\left(\frac{d^2z(\tilde{z})}{d\tilde{z}^2}/\frac{dz(\tilde{z})}{d\tilde{z}}\right)^2$$

Regarding the potential function in a generic choice of local coordinate as a power series in η^{-1}, if we focus on its principal term (that in the coordinate z coincides with the potential function itself), we see that it transforms like a *meromorphic quadratic differential*. We can thus associate to the Schrödinger equation a quadratic differential, that we define in the local coordinate z as

$$\Phi_\theta=e^{2i\theta}Q(z)dz^{\otimes 2} \tag{3.5}$$

where, for a later convenience, we have chosen to include the phase coming from the factor $\eta^2 = |\eta|^2 e^{2i\theta}$ that is present in front of $Q(z)$ in (3.1). The geometry of zeros, poles and trajectories of this differential will play an important role in the following analysis. We may then from the beginning introduce the symbols P_0, P_∞ for, respectively, the set of the zeros (called *turning points*) and poles of Φ_θ, and set $P \equiv P_0 \cup P_\infty$.

In the WKB method one starts by searching for solutions of equation (3.1) in the form

$$\psi(z,\eta) = \exp\left(\int_{z_0}^{z} S(z',\eta)\, dz'\right) \tag{3.6}$$

where $z_0 \in \mathbb{P}^1$ is a chosen point. Plugging this ansatz into the Schrödinger equation, one finds an auxiliary first order differential equation for the function $S(z,\eta)$, called *Riccati equation*:

$$\frac{dS}{dz} + S^2 = \eta^2 Q(z) \tag{3.7}$$

To solve for S, one then expands it in a formal series of powers of η

$$S(z,\eta) = \sum_{n=-1}^{\infty} S_n(z)\eta^{-n} = S_{-1}(z)\eta + S_0(z) + S_1(z)\eta^{-1} + ... \tag{3.8}$$

and equating the terms with the same power of η inside (3.7) finds a set of relations that allow to compute the coefficients S_n recursively:

$$\begin{cases} S_{-1}^2 = Q(z) \\ S_n = -\frac{1}{2S_{-1}} \left(\frac{dS_{n-1}}{dz} + \sum\limits_{\substack{i+j=n-1 \\ i,j\geq 0}} S_i S_j \right), n \geq 0 \end{cases} \tag{3.9}$$

As a consequence of the first of the two equations, we have two possibilities of solution $S^{(\pm)}(z,\eta)$ for the Riccati equation, depending on the choice of the sign in front of the square root in $S_{-1} = \pm\sqrt{Q(z)}$. This choice then recursively influences also other terms in the expansion. In particular, if we separate the odd and the even powers in the expansion of $S^{(\pm)}$ as

$$S^{(\pm)}(z,\eta) = S^{(\pm)}_{odd}(z,\eta) + S^{(\pm)}_{even}(z,\eta) \tag{3.10}$$

where

$$S^{(\pm)}_{odd}(z,\eta) = \sum_{n\geq 0} S^{(\pm)}_{2n-1}\eta^{-2n+1}, \quad S^{(\pm)}_{even}(z,\eta) = \sum_{n\geq 0} S^{(\pm)}_{2n}\eta^{-2n} \tag{3.11}$$

then from (3.9) we can see that the choice of sign in S_{-1} influences only the odd terms, so that

$$S^{(-)}_{even}(z,\eta) = S^{(+)}_{even}(z,\eta), \quad S^{(-)}_{odd}(z,\eta) = -S^{(+)}_{odd}(z,\eta) \tag{3.12}$$

This separation of even and odd powers proves also to be useful to simplify the expression (3.6) of S. Focusing on the $(+)$ solution (we have just seen how the $(-)$ solution is simply related to it), and dropping the $(+)$ superscript to avoid messy notation, we can notice how the Riccati equation implies:

$$(S_{even} + S_{odd})^2 + \frac{d}{dz}(S_{even} + S_{odd}) = \eta^2 Q$$

retaining only even powers

$$2S_{even}S_{odd} + \frac{d}{dz}S_{odd} = 0$$

that is

$$S_{even} = -\frac{1}{2S_{odd}}\frac{d}{dz}S_{odd} = -\frac{1}{2}\frac{d}{dz}\log(S_{odd})$$

The relation found between S_{even} and S_{odd} implies that we can express the WKB formal ansatz (3.6) only in terms of the odd part of the expansion of S. Considering also $S^{(-)}$, by property (3.12) we can then write the following two linearly independent formal solutions of (3.1):[2]

$$\psi_\pm(z,\eta) = \frac{1}{\sqrt{S_{odd}(z,\eta)}}\exp \pm \left(\int_{z_0}^{z} S_{odd}(z',\eta)\,dz'\right) \tag{3.13}$$

Some remarks are now necessary to be pointed out:

1. Due to the definition (3.9) of the coefficients S_n, these two solutions are not meaningful at the points in which $Q(z) = 0$, that is at the turning points of the differential Φ_θ, where $S_{-1} = 0$ and all the other S_n have a singularity. What's more, since all the coefficients S_n contain a square root of $Q(z)$, the wavefunctions $\psi_\pm$ have to be considered as defined on the Riemann surface Σ of the 1-form $\sqrt{Q(z)}dz$, which is related to the Riemann sphere via a double cover $\pi : \Sigma \to \mathbb{P}^1$. The integral in (3.13) is then intended to be performed along a path in Σ.
   To visualize Σ when looking at the complex plane, we have as usual to take branch cuts connecting couples of branch points. To each point of the complex plane correspond two points on the Riemann surface, one for each of the two sheets of Σ. If we call z and z^* the points on Σ related by a change of sheet, we have the simple relation:
   $$S_{odd}(z^*,\eta) = -S_{odd}(z,\eta) \tag{3.14}$$
   This fact that the wavefunction, when projected on the complex plane, is multivalued, comes directly from the use of a semiclassical expansion. If we manage to solve the Schrödinger equation directly, without the aid of the $\hbar$ expansion, we would get a solution that may have poles, but not cuts.

[2]Notice that the solution coming from $S^{(-)}$ has been multiplied by a factor i to obtain ψ_-, just to obtain a more uniform expression for the two solutions $\psi_\pm$

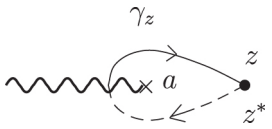


Figure 3.1: Representation of a possible choice of path γ_z on the Riemann surface Σ, used for the definition of WKB solutions normalised at a turning point. In the figure, the point a represents the turning point, with the branch cut emanating from it to the left. The two points z and z^* are related by a change of sheet, as can be inferred by the depiction of the path γ_z that join them. The picture is taken from [47].

2. Since this point, we have been quite vague about the choice of the point z_0 that we use as lower endpoint for the integration. There can be many choices for z_0, that determine the *normalization* of the WKB wavefunctions. A common choice (see for example [47]), that is the one we will also use, is the so called *"normalisation at a turning point"*. Formally, one chooses $z_0 = a$, where a is one of the turning points of Φ_θ, but what truly means (since we know $\psi_\pm$ are singular at the turning points) is:
$$\int_a^z S_{odd}(z', \eta)\, dz' = \frac{1}{2} \int_{\gamma_z} S_{odd}(z', \eta)\, dz' \tag{3.15}$$
where γ_z is a path that starts at the point z^* related to z by a change of sheet and then reaches z by rotating around the branch point a (see Figure 3.1 for a pictorial representation of γ_z).

   The above formal equivalence is guaranteed by the property (3.14) of S_{odd} to only change sign between sheets.

3. Since S_{odd} is defined via its expansion in power of η, the integration in (3.13) is defined as a term-wise integral for each coefficient in the expansion.
   Also the wavefunctions $\psi_\pm$ have to be intended as a formal series, that descends from that of S. Considering for example ψ_+ (but we have similarly for ψ_-) and restoring the original form (3.6), we can write S in its power series and then Taylor

expand to find:

$$\begin{aligned}\psi_+(z,\eta) &= \exp(s_{-1}(z)\eta)\exp(s_0(z))\exp\left(\sum_{n=1}^{\infty} s_n(z)\eta^{-n}\right) = \\ &= \exp(s_{-1}(z)\eta)\exp(s_0(z)) \\ &\times \left(1+\Big(\sum_{n=1}^{\infty} s_n(z)\eta^{-n}\Big)+\frac{1}{2!}\Big(\sum_{n=1}^{\infty} s_n(z)\eta^{-n}\Big)^2+...\right) = \\ &= \exp(s_{-1}(z)\eta)\exp(s_0(z)) \\ &\times \left(1+(s_1(z))\,\eta^{-1}+\left(s_2(z)+\frac{1}{2}s_1(z)^2\right)\eta^{-2}+...\right)\end{aligned} \tag{3.16}$$

where we have called $s_n(z) = \int_a^z S_n(z',\eta)\,dz'$, and in the last step we have grouped together the terms in the expansion with the same power of η. If the series in powers of η^{-1} present in the above expansion converged to a function near $\eta = \infty$, that can be analytically continued to the region $\{\eta \in \mathbb{R}^+, \eta \gg 1\}$, then the WKB ansatz would successfully provide a basis of solutions for the Schrödinger equation in a given region of the Riemann sphere. However, even for the simplest case of linear potential (see for example [48]), what one finds is that the coefficients $S_n(z)$, and thus also the terms in the expansion of $\psi_\pm$, are divergent.

At this point, one could think that the possibility of obtaining exact solutions of the Schrödinger equation through the WKB method is lost, and retreat to consider the expansion (3.16) just as a way to obtain semiclassical approximations[3]. This is the way the WKB method has been used when it was first introduced in 1926, and then continued to be used later on for many decades. Nevertheless, as it was started to be realised in the 1970's and 1980's, thanks to the collective work of many authors (for an historical note see the introductive section of [15]), there actually is a way to recover actual, not formal, convergent solutions starting from the WKB expression (3.16), and it is through a procedure that involves first Borel transforming $\psi_\pm$, and then Borel resumming the outcome. This is the core procedure that distinguishes the *exact* WKB method from the original one. Due to the central role played in this technique by the operations of Borel transformation and Borel resummation, we briefly review the basic notions about these concepts before applying them to the context of WKB analysis. For a more in depth analysis of the theory of Borel summability see [49].

[3]For instance, noticing that (3.9) implies $S_0 = -\frac{1}{2}\frac{d}{dz}\log(S_{-1})$, if we retain only the lowest order term in the η^{-1} expansion of (3.16) we get the usual WKB approximation

$$\psi_+(z,\eta) \simeq \frac{1}{\sqrt[4]{Q(z)}}\exp\left(\eta\int_a^z \sqrt{Q(z)}\,dz'\right)$$

3.2 Generalities on Borel transformation and Borel resummation

Given a function $F(w)$ on the complex plane, we define its *Laplace transform* $\mathcal{L}F(z)$ for $z \in \mathbb{R}^+$ as:

$$\mathcal{L}F(z) = \int_0^\infty e^{-zw} F(w) dw \tag{3.17}$$

The Laplace transform of a function is well-defined only if the integration can be performed (that is $F(w)$ is locally integrable in $[0, \infty)$) and the integral converges, so the growth of $F(w)$ along the positive real axis is less than exponential.
The Laplace transform is linear, and satisfies the property $\mathcal{L}(wF(w))(z) = -\frac{d}{dz}\mathcal{L}F(z)$, so that we can easily find its action on a generic power:

$$\mathcal{L}(w^n)(z) = (-1)^n \frac{d^n}{dz^n}\mathcal{L}(1)(z) = (-1)^n \frac{d^n}{dz^n}\left(\frac{1}{z}\right) = \frac{n!}{z^{n+1}} \tag{3.18}$$

where we have computed $\mathcal{L}(1)(z)$ directly from the definition of Laplace transformation

$$\mathcal{L}(1)(z) = \int_0^\infty e^{-zw} 1 \, dw = -\frac{e^{-zw}}{z}\Big|_{w=0}^{\infty} = \frac{1}{z}$$

Using the result (3.18) and linearity, we can then naturally define the formal Laplace transform $\tilde{\mathcal{L}}$ on power series as:

$$\tilde{\mathcal{L}}\left(\sum_{n=0}^{\infty} c_n w^n\right) = \sum_{n=0}^{\infty} c_n \tilde{\mathcal{L}}(w^n) = \sum_{n=0}^{\infty} c_n \frac{n!}{z^{n+1}} \tag{3.19}$$

Concerning power series, the *Borel transform* is introduced as the formal inverse of $\tilde{\mathcal{L}}$, that is:

$$\mathcal{B}\left(\sum_{n=1}^{\infty} c_n z^{-n}\right) = \sum_{n=1}^{\infty} c_n \mathcal{B}(z^{-n}) = \sum_{n=1}^{\infty} \frac{c_n}{(n-1)!}\sigma^{n-1} \tag{3.20}$$

where in this expression the sum doesn't start from $n = 0$ since $\tilde{\mathcal{L}} : \mathbb{C}[[w]] \longrightarrow z^{-1}\mathbb{C}[[z^{-1}]]$.
Being interested for WKB analysis in series of the form (3.16), which include the zeroth order term and possibly also have an exponential prefactor, we then extend the definition of Borel transformation to this set of series. Given a formal power series

$$f(z) = e^{sz} \sum_{n=0}^{\infty} f_n z^{-n}, \quad s \in \mathbb{C} \tag{3.21}$$

we still call, with a little abuse of notation, its Borel transform $f_B(w)$ as in (3.20), not considering the f_0 coefficient and the exponential prefactor:

$$f_B(w) \equiv \sum_{n=1}^{\infty} \frac{f_n}{(n-1)!} w^{n-1} \tag{3.22}$$

If this Borel transform converges to a function near $w = 0$, which can be analytically continued to a domain Ω containing the half line $\{w \in \mathbb{C} \,|\, \mathrm{Re}(w) \geq 0, \mathrm{Im}(w) = 0\}$ and satisfies the exponential bound

$$|f_B(w)| \leq C_1 e^{C_2|w|}, \quad w \in \Omega, \, C_{1,2} \in \mathbb{R}^+ \tag{3.23}$$

then the series f is called *Borel summable*, and we call $\mathcal{S}_B$ the space of Borel summable series of the form (3.21). The complex w plane, where the domain of $f_B(w)$ is defined, is often referred as *Borel plane*.

The notion of Borel summability is manifestly designed so that, if a series is Borel summable, then its Borel transform can be Laplace transformed back with $\mathcal{L}$ (the operator acting on functions, not $\tilde{\mathcal{L}}$ acting on series) to obtain a function in the z space where the initial formal series was defined. The whole operation, called *Borel resummation*, can be defined on the space $\mathcal{S}_B$ of Borel summable series as an operator $\mathcal{S}$, that, acting on a formal series like (3.21) gives back the so called *Borel sum* of f, a function that is analytic on $\{z \in \mathbb{R}^+, z \gg 1\}$ (notice how in the expression below the f_0 term and the exponential factor have been reinserted):

$$\mathcal{S}[f](z) = e^{sz} \left(f_0 + \int_0^\infty e^{-zw} f_B(w) dw \right) \tag{3.24}$$

As expected from a resummation operator, $\mathcal{S}$ is an operator from a space of formal series, possibly divergent, to a space of functions. Its strength however lies in a set of very interesting properties that comes with it:

1. If $f(z)$ is a convergent series and defines a function near $z = \infty$, then $f(z)$ is Borel summable and $\mathcal{S}[f](z)$ coincides with that function.

2. If $f(z)$ is instead a divergent series but is Borel summable, then the function $\mathcal{S}[f](z)$ has the same asymptotic expansion of $f(z)$ for $z \to \infty$.

3. If $f(z)$ is a formal solution of a differential equation expressed in terms of a divergent series, then $\mathcal{S}[f](z)$ is still a solution (now not divergent) of the same differential equation, and according to the previous properties it is a solution which shares the same aymptotic of $f(z)$ for $z \to \infty$.

The last property is a remarkable one, and is the primary reason for which the process of Borel resummation can be helpful when dealing with the divergent series of the WKB method.
A simplified example to showcase the effectiveness of this property (despite not still in the context of the WKB method), is the following, adapted from [15].

Example 1. Considering the ordinary differential equation

$$\left(-\frac{d}{dz}+1\right)\psi(z)=\frac{1}{z} \tag{3.25}$$

we can easily find a solution by expanding $\psi(z)$ in power series of z^{-1} to find

$$f(z)=\sum_{n=1}^{\infty}(-1)^{n-1}(n-1)!\,z^{-n} \tag{3.26}$$

This is a formal solution, that diverges at high powers of z, but if we look at its Borel transform (recall the definition in (3.20)), the divergence is cured, and the transformed series converges to a function in the w plane

$$f_B(w)=\sum_{n=1}^{\infty}(-1)^{n-1}w^{-n-1}=\frac{1}{1+w} \tag{3.27}$$

which is integrable on the positive real axis and decaying, so that $f(z)$ is Borel summable. Its Borel sum is given by

$$\int_0^{\infty}\exp\left(-zw\right)\frac{1}{1+w}\,dw \tag{3.28}$$

and if we check whether it verifies the ODE (3.25) under consideration, we see that indeed it does

$$\begin{aligned}\left(-\frac{d}{dz}+1\right)\int_0^{\infty}\exp\left(-zw\right)\frac{1}{1+w}\,dw&=\int_0^{\infty}\exp\left(-zw\right)\left(\frac{w}{1+w}+\frac{1}{1+w}\right)dw=\\&=\int_0^{\infty}\exp\left(-zw\right)dw=-\frac{e^{-zw}}{z}\Big|_{w=0}^{\infty}=\frac{1}{z}\end{aligned}$$

The process of Borel resummation has provided us an analytic solution (for $z>0$, where the Borel sum is defined) starting from a formal one. △

Up to this point we have introduced the Borel resummation procedure only for positive real values of z, but a generalisation to the full complex plane is straightforward. If $z=r\exp(i\theta)\in\mathbb{C}$, we can regard any series in z of the form (3.21) as a series in its modulus r as

$$f(z)=e^{sz}\sum_{n=0}^{\infty}f_n z^{-n}=e^{(se^{i\theta})r}\sum_{n=0}^{\infty}(f_n e^{-in\theta})r^{-n} \tag{3.29}$$

and then use the notions of Borel transform and Borel sum for series of a real variable

$$\sum_{n=1}^{\infty} \frac{f_n e^{-in\theta}}{(n-1)!} w^{n-1} = e^{-i\theta} f_B(we^{-i\theta}) \tag{3.30}$$

$$\begin{aligned} \mathcal{S}[f](z) &= e^{(se^{i\theta})r} \left(f_0 + \int_0^{\infty} e^{-rw} e^{-i\theta} f_B(we^{-i\theta}) dw \right) = \\ &= e^{sz} \left(f_0 + \int_0^{\infty e^{-i\theta}} e^{-zw'} f_B(w') dw' \right), \quad \theta = \arg(z) \end{aligned} \tag{3.31}$$

where in the last passage we have used the change of variable $w' = we^{-i\theta}$, so that in the final expression for the Borel sum, the complex variable z appears explicitly. From these formulas it appears manifest that, when z is a complex number with phase θ, requiring Borel summability of a power series in z implies that the Borel transform of the series has to converge near the origin of the complex plane, can be analytically continued along a domain Ω of the Borel plane that contains the ray $\exp(-i\theta)$, and along that ray satisfies the bound (3.23), so that the integration in (3.31) can be performed. When this criterion of summability is met for a series, we will say that the series is *Borel summable in the direction* θ.

If the Borel transform $f_B(w)$ has a singular point that lies along a certain ray $e^{-i\theta}$ of Borel plane, then the series $f(z)$ will not Borel summable along the direction θ, according to the notion of Borel summability given above, and so the Borel transform for $\arg z = \theta$ of $f(z)$ will not be defined. To overcome this barrier, what one can do is to consider, when computing the Borel sum (3.31), a slightly rotated path of integration, either clockwise or anticlockwise, introducing another type of resummation called *lateral Borel resummation*

$$\mathcal{S}_{\pm}[f](z) = e^{sz} \left(f_0 + \int_0^{\infty e^{-i\theta \pm i0^+}} e^{-zw} f_B(w) dw \right) \tag{3.32}$$

where the subscript $\pm$ specifies the two possible choices of rotation for the integration contour. The choices are not equivalent, and their difference is an instance of the so-called Stokes phenomenon. Section 3.4 is devoted to the deepening of this concept, in particular for the context of WKB analysis. Note that the choice of convention presented here for the definition of the two lateral sums $\mathcal{S}_{\pm}$ is compatible with that used in [1], but some articles may use a definition where the role of $\mathcal{S}_+$ and $\mathcal{S}_-$ are swapped.

We close this section by stating some properties of the Borel resummation operator $\mathcal{S}$ under composition with basic operations like sum and product of series (for a reference, see e.g. [50], Appendix C), that will be useful in the next part, where we will discuss the Borel summability of WKB solutions. The strategy to tackle the problem, following the line of reasoning of Iwaki and Nakanishi [47] (and before them Koike and Schäfke)

will be not to study the summability of the WKB solutions $\psi_\pm$ directly from their η expansion (3.16), but to determine whether they are summable or not through the study of the summability of the series S_{odd} built from the solutions of the Riccati equation, that we know is related to $\psi_\pm$ via (3.13). This approach will make easier the study of the summability of quantum periods, a concept closely linked to that of the WKB solutions, that will be a core part of this thesis.

– Given two series $f(z)$, $g(z)$ that are Borel summable in a certain direction θ and a constant $c \in \mathbb{C}$, then, for $\arg(z) = \theta$

$$\mathcal{S}[f + cg](z) = \mathcal{S}[f](z) + c\mathcal{S}[g](z) \quad \mathcal{S}[f \cdot g](z) = \mathcal{S}[f](z) \cdot \mathcal{S}[g](z) \tag{3.33}$$

– Borel summability is compatible with composition with convergent power series. For example, if $f(z)$ is Borel summable along θ, then, for $\arg(z) = \theta$

$$\mathcal{S}[\exp(f)](z) = \exp\left(\mathcal{S}[f](z)\right) \tag{3.34}$$

– In the WKB method, we don't have power series $f(z) = \sum_{n=0}^{\infty} f_n z^{-n}$ in the variable z with certain constant coefficients f_n, but rather we have expansions of the form $f(z, \eta) = \sum_{n=0}^{\infty} f_n(z)\eta^{-n}$, that are power series in the variable η with functions $f_n(z)$ of z as coefficients. In this case, one implements the Borel resummation operation in the η variable, and can consider the combination of this operation with other ones done instead with respect to the z coordinate. With two variables (z, η), the condition of being Borel summable in a certain direction θ (wrt η) for $f(z, \eta)$ depends on the point of the complex z plane considered, but if there is a domain U in this plane where $f(z, \eta)$ is summable $\forall z \in$ U, then, assuming $\arg(\eta) = \theta$

$$\partial_z \mathcal{S}[f](z, \eta) = \mathcal{S}[\partial_z f](z, \eta) \quad \int_{z_0}^{z} \mathcal{S}[f](z', \eta)dz' = \mathcal{S}\left[\int_{z_0}^{z} f(z', \eta)dz'\right] \tag{3.35}$$

for any $z_0 \in$ U, and the path of integration lies entirely in U (assuming obviously that the partial derivation and the integration operations can be performed on f)

3.3 Borel summability of WKB solutions

Having seen how Borel resummation can help to recover actual solutions of differential equations starting from formal ones, we would like to understand whether this resummation prescription can be used to cure the divergences of the WKB solutions (3.13). As was mentioned at the end of the previous section, instead of studying the Borel summability along a certain direction θ of the WKB solutions directly, we deduce it from that

of $S_{odd}(z,\eta)$.[4] Recalling that the series appearing in the WKB method are power series in η whose coefficients depend on the point z of the complex plane considered, what we have to check is in which points of the complex z plane the Borel resummation procedure can be carried out successfully, that is:

1. the Borel transform of S_{odd} converges to a function near the origin of the Borel plane

2. the Borel transform can be analytically continued along the ray $\exp(-i\theta)$ without encountering any singularity and retaining the controlled growth (3.23), so that it can be Laplace transformed

The first condition, sometimes called *Borel transformability* or also *pre-Borel summability*, can be readily checked to hold in any compact subset K of the complex z plane that does not include the poles of and the turning points of Φ_θ. Exploiting the recursion relation (3.9) of the terms $S_j(z)$ in the η expansion of $S(z,\eta)$, we can in fact bound their growth as (for a proof see [51], Appendix 1):

$$\sup_K |S_j(z)| < A_K C_K^j j! \quad A_K, C_K \in \mathbb{R}^+ \tag{3.36}$$

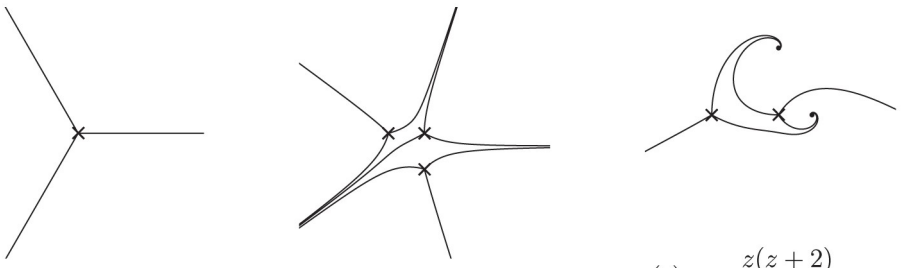
The factorial contribution, if present, gets precisely removed by the procedure of Borel transformation, making thus the Borel transform a convergent series in a neighborhood of the origin in the Borel plane.

The second condition is more subtle, and to inspect it one would like to follow the singularities of the Borel transform of S_{odd} in the Borel plane as the point in the complex z plane is varied, to see when such singularities move onto the ray $\exp(-i\theta)$.
It turns out that a crucial concept to introduce is that of *Stokes curves in the direction* θ, that are curves in the complex z plane emanating from the turning points of Φ_θ and defined by the equation:

$$\mathrm{Im}\left(e^{i\theta}\int_a^z \sqrt{Q(z')}dz'\right) = 0 \tag{3.37}$$

[4]The careful reader may have noticed that $S(z,\eta)$, and in particular $S_{odd}(z,\eta)$, are power series that include also terms with positive powers of η in their expansion, and so lie outside the class of series for which we have presented a definition of Borel resummation. However, as is the case for series that contain a constant term or an exponential prefactor, to perform Borel resummation in this situation one inspects the Borel summability condition only on the part of the series with negative powers of η, and if summability is met then reinserts the terms that have been left out when performing the Borel sum, that is

$$\mathcal{S}\left(\sum_{n=-M}^{\infty} f_n z^{-n}\right) = \sum_{n=-M}^{0} f_n z^{-n} + \mathcal{S}\left(\sum_{n=1}^{\infty} f_n z^{-n}\right), \quad M \in \mathbb{N}$$



(a) z. (b) $z(z+1)(z+i)$. (c) $\dfrac{z(z+2)}{(z-1)^2(z-2i)^2}$.

Figure 3.2: Examples of Stokes graphs in the direction $\theta = 0$ for some choices of the potential function $Q(z)$ (specified below each figure). The picture is taken from [47].

where a is the turning point considered. These curves are particular trajectories of the meromorphic quadratic differential Φ_θ that can be associated to the Schrödinger equation (3.1). Among the various types of Stokes curves, those that flow from a turning point to another are very important in the study of summability of WKB series. These are called *saddle trajectories*, and can be distinguished in *regular*, if they have different endpoints, or *closed*, if the endpoints coincide. To restrict the possibilities of Stokes trajectories related to the problem under consideration, and also simplify the analysis of Borel summability of WKB solutions, from this point on we will follow the same assumptions on Φ_θ stated in [47], namely:

- Φ_θ has at least one zero and one pole
- all the zeros of Φ_θ are simple
- the order of any pole of Φ_θ is ≥ 2

These are mild conditions, satisfied for example if $Q(z)$ is a generic polynomial of degree $m \geq 1$, provided all its turning points are distinct. If we also assume for simplicity that at most one saddle trajectory is present, then all the Stokes curves either connect two turning points (possibly also the same) or tend to approach a pole of Φ_θ. Together with the turning points and the poles of Φ_θ, Stokes curves then form a graph that is called *Stokes graph in the direction* θ and denoted as G_θ. This graph is completely determined by the choice of $Q(z)$ and of the phase $\theta = \arg(\eta)$, and the connected components in which the complex plane is divided in by its presence are called *Stokes regions*. For some examples of Stokes graphs see Figure 3.2.

Provided we keep away from Stokes lines, then we can state the following result:

Theorem 1 (Borel summability of $S_{odd}(z,\eta)$)**.** *Under the conditions assumed for Φ_θ and G_θ, for any point z in a Stokes region the formal power series $S_{odd}(z,\eta)$ is Borel summable in the direction θ.*

In [47] the theorem is only stated, and for the proof the authors refer to an article of Koike and Shäfke that has however never been published. An outline of the steps needed to perform the proof can be found in the article [16].[5] Here we explicitly write just the first few of them.
Calling $T(z,\eta) = \sum_{n=1}^{\infty} S_n(z)\,\eta^{-n}$ the part of the η expansion of S relevant for the Borel transformation procedure, one derives from the Riccati equation an equation for T, that thanks to the relations (3.9) between the coefficients $S_j(z)$ can be recast as

$$\frac{dT}{dz} + 2\eta S_{-1} T + 2S_0 T + T^2 - 2S_{-1}S_1 = 0 \tag{3.38}$$

Using the properties of Borel transformation (see for example [50], Appendix C for a brief overview of these properties)

$$(fg)_B = f_B * g_B \quad (\eta f)_B = \partial_\xi f_B \quad (\partial_z f)_B = \partial_z f_B \tag{3.39}$$

where ξ is the coordinate of the Borel plane associated to η, one gets a differential equation for the Borel transform $T_B(z,\xi)$ of T

$$\frac{\partial T_B}{\partial z} + 2\sqrt{Q(z)}\frac{\partial T_B}{\partial \xi} + 2S_0(z)T_B + T_B * T_B = 0 \tag{3.40}$$

After trading z for a more convenient coordinate, this can be converted in an integral equation for T_B. If z is chosen to be inside a Stokes region, then T_B can then be proved to be holomorphic and with the required bounded growth to be summable in a domain that in the Borel plane contains the ray $\exp(-i\theta)$.

From the above considerations it appears clear how, given a choice of $Q(z)$, to establish the Borel summability of S_{odd} and of the WKB solutions, is crucial to analyze the topology of Stokes graph associated to Φ_θ, and how this changes by varying the angle θ. To this end we collect here some of the general features of Stokes graphs, that can be found in [53, 54]. The reader is suggested to check them in the examples of Stokes graphs presented in Figure 3.2.

General features of Stokes graphs.

(i) There are exactly three Stokes curves connected to each simple turning point.[6] As θ is varied, each Stokes curve rotate around the turning point a that is its starting point.

[5] Note that the author in [16] considers for simplicity the case of a polynomial potential. There is in general in the literature a lack of papers proving the summability of S in a complete and rigorous way, and for a generic choice of $Q(z)$. Many papers, including [47], only state the results. A recent work that bridges part of this gap left open is [52].

[6] This can be easily seen from the defining expression (3.37) of a Stokes curve, by Taylor expanding

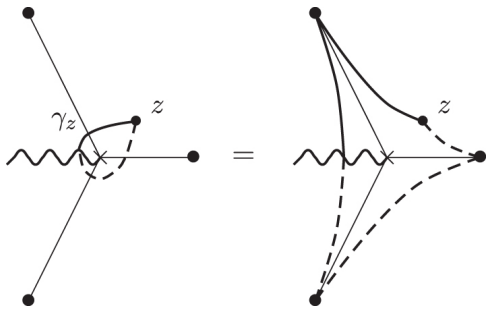


Figure 3.3: Example of modification of a path γ_z that encircles a turning point (like the one present in the WKB wavefunctions normalised at a turning point), so that it doesn't cross Stokes lines. The picture is taken from [47].

(ii) For a double pole p of Φ_θ, there are three possibilities for the topology of the Stokes curves around it, depending on the value of the residue $r_p = e^{i\theta}\text{Res}_{z=p}\sqrt{Q(z)}dz$. If r_p is purely real, the Stokes curves enter radially in p, while if it has both a real and an imaginary part that are nonzero, then they reach p following a logarithmic spiral. Finally, if r_p is purely imaginary, then there is no Stokes curve entering in p. In this last case, a closed saddle trajectory encircling the double pole is present in the graph.

(iii) For a pole of order $m \geq 3$, the Stokes trajectories reach the pole gathering along exactly $m-2$ radial directions emanating from the pole

So far we have stated the condition for the Borel summability of S_{odd}, but the WKB solutions $\psi_\pm$ imply integrating $S_{odd}(z,\eta)\,dz$ along a path on the Riemann surface Σ. If this integration path crosses a Stokes line, that is a point of not summability for S_{odd}, then summability of $\psi_\pm$ is no more guaranteed. Since we have chosen to normalise our wavefunctions around a turning point, and recalling that turning points are the sources of Stokes lines, at first glance avoiding them may seem impossible. However, there is a trick that we can still exploit.

The idea, due to Koike and Schäfke, is to take advantage of the freedom to modify the contour of integration inside each Stokes region (where S_{odd} is analytic) so that whenever we have to traverse from a Stokes region to another we don't do it trough Stokes lines, but through the poles of Φ_θ (for a graphical example see Figure 3.3). At these poles

to first order $Q(z)$ around the turning point

$$e^{i\theta}\int_a^z \sqrt{Q(z')}dz' \approx e^{i\theta}\int_a^z \sqrt{k(z'-a)}dz' = \frac{2k}{3}e^{i\theta}(z-a)^{\frac{3}{2}}, \qquad k = \frac{dQ}{dz}\Big|_{z=a}$$

and setting the imaginary part of this quantity to zero results in an equation for z with three possible solutions

the full series $S_{odd}(z,\eta)\,dz$ is divergent, but one can show (see [47], Proposition 2.8) that the divergence is only due to the first term in the η expansion, while the rest of the series, that we can call $S^{reg}_{odd}(z,\eta)\,dz \equiv \left(S_{odd}(z,\eta) - \eta\sqrt{Q(z)}\right)\,dz$ is integrable. Since the terms with positive powers of η are left unchanged in the resummation procedure, then in any situation where we have an integral of $S_{odd}(z,\eta)\,dz$, as it happens in the expression of the WKB solutions, we can separate the integration of $\eta\sqrt{Q(z)}\,dz$ from that of $S^{reg}_{odd}(z,\eta)\,dz$. If we manage to modify the contour of integration along which $S^{reg}_{odd}(z,\eta)\,dz$ is integrated so that it doesn't crosses Stokes lines, using the trick just described, then we have obtained an expression equivalent to the one we started with that we can prove to be Borel summable.
To rephrase these ideas in a more rigorous way, we first state

Theorem 2 ([47])**.** *Let p any pole of Φ_θ lying on the boundary of a Stokes region D of a given Stokes graph G_θ. Then, for any fixed $z \in D$, the formal power series defined by the integral*

$$\int_p^z S^{reg}_{odd}(z',\eta)dz' \tag{3.41}$$

is Borel summable in the direction θ as a formal power series in η^{-1} if the path of the integral (3.41) is contained in $D \cup \{p\}$.

Then we introduce the concept of *admissible paths*, in order to precisely state along which paths we can integrate $S^{reg}_{odd}(z,\eta)\,dz$ retaining Borel summability. In the following, we will call $\hat{P}_0$, $\hat{P}_\infty$ the lift of the sets P_0, P_∞ from the complex plane to the Riemann surface Σ of $\sqrt{Q(z)}\,dz$ (that we recall is the surface on which we are integrating) by using the inverse of the covering map π. In an analog way of the definition $P = P_0 \cup P_\infty$, we also set $\hat{P} \equiv \hat{P}_0 \cup \hat{P}_\infty$.

Definition 1. A path β on $\Sigma \setminus \hat{P}_0$ is said to be *admissible in the direction θ* if the projection of β to $\mathbb{P}^1$ never intersects with the Stokes graph G_θ or intersects with G_θ only at the points in P_∞.

The trick we have described of modifying a contour of integration γ so that it crosses Stokes regions only through points in P_∞, can be rephrased as the possibility of decomposing γ into a sum of admissible paths. From the general description of Stokes graphs given before however, it should appear clear that not always the trick can be performed. A relevant example is when γ crosses a saddle trajectory, since in that case the Stokes curve crossed by γ is connected at both its endpoints with points in P_0. This is why the appearance of saddle trajectories is so important when studying Borel summability of WKB series. We pack all these information in the following statement.

Lemma 1. *Let γ be a path on $\Sigma \setminus \hat{P}_0$ with end-points $\hat{z}_1, \hat{z}_2 \in \Sigma \setminus \hat{P}_0$ satisfying the following conditions:*

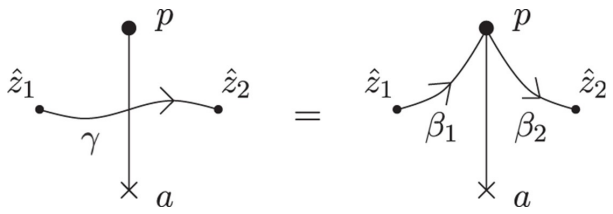


Figure 3.4: Example of decomposition of a path γ into a sum of admissible paths. In the figure, p and a stand respectively for a pole and a turning point of Φ_θ, and the line that connects them is a Stokes curve. The picture is adapted from [47].

- *both the endpoints, when projected to* $\mathbb{P}_1$*, do not lie on the Stokes graph* G_θ *or a point in* P_∞
- γ *never intersects with a saddle trajectory of* G_θ

Then, γ *has a decomposition into a finite number of paths* $\gamma = \sum_{i=1}^{N} \beta_i$ *that are admissible in the direction* θ*, and the formal series defined by the integral* $\int_\gamma S_{odd}^{reg}(z,\eta)dz$ *is Borel summable in the direction* θ*.*

A pictorial representation of the decomposition of a path into admissible ones is shown in Figure 3.4. The conditions for the Borel summability of WKB solutions $\psi_\pm$ come then as an immediate consequence of the lemma just stated.

Corollary 1 (Borel summability of WKB solutions)**.**

(a) If the Stokes graph has no saddle trajectories, then the WKB formal solutions (3.13) normalised at a turning point are Borel summable in the direction θ *at any point* z *in each Stokes region. Their Borel sums give analytic solutions of the Schrödinger equation (3.1) on each Stokes region of the* z *variable, and in the domain* $\{\eta \in \mathbb{C} \,|\, \arg(\eta) = \theta,\, |\eta| \gg 1\}$ *of the complex* η *plane. In this domain, the solutions found through Borel resummation have the same asymptotic of the WKB formal solutions for* $|\eta| \to \infty$*.*

(b) In presence of a saddle trajectory, Borel summability of the WKB solutions is not guaranteed even inside Stokes regions. To verify if Borel summability holds at a point z *inside a Stokes region, one has to check whether the path* γ_z *contained in the definition of the WKB solution considered can be decomposed into a sum of admissible paths.*

For a graphical example of how the path of integration γ_z contained into the WKB wavefunctions normalised at a turning point can be decomposed into a sum of admissible paths, in case no saddle trajectory is present, refer back to Figure 3.3.

3.4 Stokes phenomenon and connection formulas

When we perform Borel resummation to promote the WKB formal solutions of the Schrödinger equation to analytic ones, the function that we obtain remains the same if we change the value of any of the variables on which the series depends, that are z, η, as explicit variables, but also the parameters contained into the function $Q(z)$ (the parameters of the potential and the value of the energy), as implicit variables, **as long as the Borel summability condition is preserved**. If instead, changing the value of one of these variables, call it x, from a certain initial value x_{in} to another one x_{fin}, a pole of the Borel transform moves onto the ray $\exp(-i\arg(\eta))$ along which the integral of the Borel sum has to be evaluated, Borel summability is lost, and the functions obtained by doing Borel resummation at $x = x_{in}$ and $x = x_{fin}$ are in general different. This difference can be precisely captured by the computing the lateral Borel sums at the critical point where traditional Borel resummation can't be performed. The two lateral Borel sums $\mathcal{S}_\pm$ will have the same asymptotic expansion for $|\eta| \to \infty$, but will usually be different functions, since the integration paths present in their definition will not be homotopic due to the presence of the singular point between them. Their difference is defined as the *Stokes discontinuity*

$$\mathrm{disc}[f](z,\eta) = \mathcal{S}_+[f](z,\eta) - \mathcal{S}_-[f](z,\eta) \tag{3.42}$$

where $f(z,\eta)$ stands for a generic power series in η^{-1} obtained in WKB analysis. This phenomenon is called *Stokes phenomenon*, and it is at the core of the connection between exact WKB analysis and TBA equations.
An example of Stokes phenomenon is the one that affects the Borel resummed version of the WKB wavefunctions $\psi_\pm(z,\eta)$ in two adjacent Stokes regions. Fixing the value of η to have a certain phase θ, and supposing for simplicity that the Stokes graph G_θ is saddle-free (i.e. has no saddle trajectories), we know from Corollary 1 that when z lies on a Stokes curve the WKB series $\psi_\pm(z,\eta)$ are not Borel summable. From the above discussion, we then expect that in the two Stokes region that are separated by the Stokes trajectory under consideration, the resummed WKB series will be different functions, that differ by terms that vanish in the limit $|\eta| \to \infty$. This difference is encoded in the so-called *connection formulas* for the all order WKB resummed series, first found by Voros in [3], and independently by Silverstone in [17], which we state below.

Theorem 3 (Connection formulas for the resummed WKB solutions)**.** *Call C the Stokes curve, D_1 and D_2 the two Stokes regions that have C as common boundary, where D_1 (D_2) is the region that can be reached from C by rotating clockwise (counter-clockwise) around the turning point a from which the curve emanates, and choose the branch cuts on $\mathbb{P}^1$ appropriately so that C does not cross any of them (use Figure 3.5 as reference). Denote by $\Psi_\pm^{D_j}$ ($j = 1,2$) the Borel sum of the WKB series $\psi_\pm$ on the Stokes region*

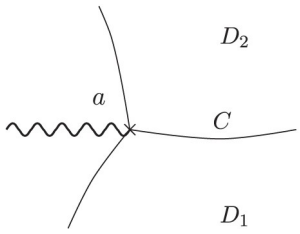


Figure 3.5: Representation of a part of Stokes graph with the notation set to match that of Theorem 3. The picture has been slightly adapted from [47].

D_j ($j = 1, 2$), normalised as in (3.15) around the turning point a. Then, the analytic continuation of $\Psi_{\pm}^{D_1}$ to D_2 across the Stokes curve C is related to $\Psi_{\pm}^{D_2}$ as follows:

$$\textit{(a)} \qquad \begin{cases} \Psi_+^{D_1} = \Psi_+^{D_2} + i\Psi_-^{D_2} \\ \Psi_-^{D_1} = \Psi_-^{D_2} \end{cases} \qquad \textit{if } \mathrm{Re}\left(e^{i\theta}\int_a^z \sqrt{Q(z')}dz'\right) \geq 0 \textit{ on } C \tag{3.43}$$

$$\textit{(b)} \qquad \begin{cases} \Psi_+^{D_1} = \Psi_+^{D_2} \\ \Psi_-^{D_1} = i\Psi_+^{D_2} + \Psi_-^{D_2} \end{cases} \qquad \textit{if } \mathrm{Re}\left(e^{i\theta}\int_a^z \sqrt{Q(z')}dz'\right) \leq 0 \textit{ on } C \tag{3.44}$$

or in a more compact form

$$\begin{pmatrix} \Psi_+^{D_1} \\ \Psi_-^{D_1} \end{pmatrix} = M \begin{pmatrix} \Psi_+^{D_2} \\ \Psi_-^{D_2} \end{pmatrix} \tag{3.45}$$

where

$$M = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \textit{ in case (a)} \quad , \quad M = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \textit{ in case (b)} \tag{3.46}$$

Voros, in his article [3], proves this theorem by establishing a link between the resummed WKB ansatz and another representation for the solutions of the Schrödinger equation, called Balian-Bloch representation [55], which is also closely related to Laplace theory:

$$\psi(z, \eta) = \eta \int_{C(z)} e^{-\eta\xi}\tilde{\psi}(z, \xi)d\xi \tag{3.47}$$

where $C(z)$ is an infinite path in the complex plane of the ξ variable.
Thanks to this connection, the Borel transform of the WKB solutions can be represented inside each Stokes region as the discontinuity of the function $\tilde{\psi}(z, \xi)$ along a certain cut in the complex ξ plane that depart from a singularities of $\tilde{\psi}(z, \xi)$, and taking the Borel sum corresponds then to computing a Laplace integral of this discontinuity along the direction of the cut. This representation allows one to perform the analytic continuation of the Borel resummed WKB solutions from a Stokes region to another in a controlled way: one

can observe how the Borel resummation procedure breaks when different cuts overlap due to the motion of the singularities from which they emanate, and through deformations of the integration contours can express the continuation of a (Borel resummed) solution through Stokes lines in terms of other (Borel resummed) WKB solutions, as stated in Theorem 3. Also Silverstone, in his historic article [17] (see also Chapter 2.4 of the book [56] for a more recent and detailed exposition), obtains the connection formulas by exploiting the use of a different representation for the wavefunctions, and then translates the results obtained in terms of the WKB solutions. In this case, the representation chosen is the so-called Langer-Cherry form of the wavefunction, that makes use of the Airy functions $\mathrm{Ai}(z)$, $\mathrm{Bi}(z)$:

$$\psi(z) = \frac{1}{\sqrt{(d\phi/dz)(z)}}\{a\,\mathrm{Ai}(\eta^{2/3}\phi(z)) + b\,\mathrm{Bi}(\eta^{2/3}\phi(z))\} \tag{3.48}$$

where a, b are constants, and the function $\phi(z)$ has to be determined. In a similar fashion of what we have done in section 3.1, one can then plug this ansatz into the Schrödinger equation to obtain an equation for the auxiliary function $\phi(z)$, and solve it by expanding $\phi(z)$ in powers of η. The main difference here with respect to the WKB ansatz, is that this form of wavefunction is not singular at the zeros of the potential function $Q(z)$, so we can obtain a (formal) expression for $\psi(z)$ that is valid in a domain containing a turning point. Once this has been done, we can then express this solution in terms of the WKB ones, on the left and on the right of the turning point (so between different Stokes regions), by matching of the asymptotic expansions. The resulting formula will tell how, chosen a form of resummed WKB solution on one side of the turning point, its analytic continuation across the turning point can be expressed in terms of the WKB basis on the other side.
Finally, a third possible way to tackle the problem is addressed in [51], where the authors use a transformation to reduce a general form of $Q(z)$ to the simple linear problem $Q(z) = z$ (that is precisely an Airy-type equation), for which the connection formulas are easily verified (thanks to the fact that Borel transformations of WKB solutions can be explicitly computed in this case). The non trivial part is then to prove, using the so-called theory of micro-local analysis, that transforming back to the initial setting the results are not altered, and thus Theorem 3 holds in a generic setting. The interested reader should refer to the articles cited for the detailed proofs.

These connection formulas are crucial in practical applications of the exact WKB method, that usually require to obtain solutions of the Schrödinger equation which are defined in more than one Stokes sector (as for example happens when searching for bound state solutions that decay at infinity on the two sides of the real z axis). What one does in these situations is finding a resummed WKB solution in one Stokes region and then analytically continuing it across neighbouring ones, using formulas (3.45, 3.46). When using the formulas repeatedly, special care has to be taken to change the normalisation

of the solutions from a turning point to another, so that the condition under which the formulas hold are always met (recall that to apply the formulas and analytically continue a solution across a Stokes curve, the solution need to be normalised around the turning point that is the base point of that specific Stokes curve). Such a change of normalisation from a turning point a_1 to another a_2, will be expressed as

$$\Psi_{\pm,a_1} = \mathcal{S}\left[\exp\left(\pm\int_{a_1}^{a_2} S_{odd}(z)dz\right)\right]\Psi_{\pm,a_2} = \mathcal{S}\left[\exp\left(\pm\frac{i\eta}{2}\Pi_{\gamma_{12}}\right)\right]\Psi_{\pm,a_2} \tag{3.49}$$

where $\Psi_{\pm,a_i}$ stands for the WKB resummed solutions normalised at the turning point a_i $(i = 1, 2)$, γ_{12} stands for a contour encircling the two turning points, and the symbol $\Pi_{\gamma_{12}}$ that we have introduced is an instance of a so called *quantum period* or *WKB period*. Formally, quantum periods are defined considering cycles γ in the first homology group $H_1(\Sigma \setminus \hat{P}) \equiv H_1(\Sigma \setminus \hat{P}; \mathbb{Z})$ as the formal series

$$\Pi_\gamma(\eta) = -\frac{i}{\eta}\oint_\gamma S_{odd}(z,\eta)\,dz \tag{3.50}$$

The exponentiated form

$$\mathcal{V}_\gamma \equiv \exp\left(i\eta\Pi_\gamma\right) = \exp\left(\oint_\gamma S_{odd}(z,\eta)\,dz\right) \tag{3.51}$$

of a quantum period, appearing in (3.49), is called *Voros symbol* or *Voros multiplier*. When using the exact WKB method to find the spectrum of a certain Schrödinger problem, due to the connection formulas (3.45, 3.46), in combination with the need of changing accordingly the normalisation of WKB solutions, the (resummed) Voros symbols enter in the derivation of exact quantisation conditions (EQC's). As anticipated before, to derive an EQC, one analytically continues a WKB resummed solution $\Psi_{D_{in}}$ that is decaying in its Stokes region D_{in} of definition, to another Stokes region D_{fin}, obtaining an expression for the analytic continuation of $\Psi_{D_{in}}$ as a linear combination of the resummed WKB solutions $\Psi_{\pm,D_{fin}}$ associated to the region D_{fin}. The bound state condition amounts then to requiring the vanishing of the coefficient in the linear combination that is multiplied to that solution, among $\Psi_{+,D_{fin}}$ and $\Psi_{-,D_{fin}}$, which is growing in D_{fin}, and typically results in a functional equation between a set (possibly also all) of the Borel resummed Voros multipliers $\mathcal{V}_{\gamma_i}$ associated to couples of turning points

$$\mathcal{Q}(\mathcal{V}_{\gamma_1}, ..., \mathcal{V}_{\gamma_n}) = 0 \tag{3.52}$$

More generally, Voros multipliers constitute part of the monodromy data associated to a given form of the Schrödinger equation. Recall that for a linear ODE in a complex domain, the monodromy group, that is the group of all 2x2 matrices describing the monodromy of its solutions along loops on the Riemann sphere $\mathbb{P}^1$, together with the

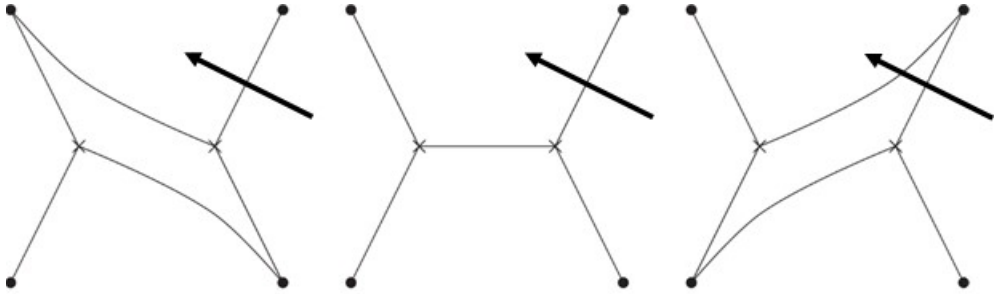


Figure 3.6: Stokes graph G_θ containing a regular saddle trajectory (in the center) and its two saddle reductions, G_θ^+ (left) and G_θ^- (right). The thick arrow marks a possible path of analytic continuation of the WKB resummed solutions between two points that in G_θ belong to adjacent Stokes regions. Note how this same path crosses a different number of Stokes regions in G_θ^+ with respect to G_θ^-, due to the topological differences between the two saddle reductions. The picture has been slightly adapted from [47].

characteristic exponents at each singular point, give a complete description of the global behaviour of the solutions of the equation.
All these reasons alone would make the quantum periods (and the related Voros multipliers) interesting objects to be analyzed through exact WKB theory, but perhaps their most important feature has been not yet mentioned: the emergence of functional equations among them. We will investigate in this direction in the next section.

3.5 The DDP formula and the emergence of functional equations

As contour integrals of the series $S_{odd}(z,\eta)\,dz$ over closed loops, the summability of quantum periods is guaranteed when their integration path does not intersect a saddle trajectory by Lemma 1 (suppose for the moment that the conditions of the Lemma hold, that is there is at most one saddle trajectory; we will generalise the results at the end). If instead the cycle associated to a given quantum period crosses a saddle trajectory, then in general the summability of the quantum period is lost, and one has to resort to lateral Borel resummations. The two possible choices $\mathcal{S}_\pm$ of lateral resummation will produce different functions due to the Stokes phenomenon (recall the discussion at the beginning of section 3.4), but the crucial observation is that their difference can be precisely represented in terms of the Voros multiplier (and thus the quantum period) associated to cycles encircling the saddle trajectory.

To understand why this should be the case, one can notice that lateral Borel resummations of a given WKB series in powers of η can be equivalently seen as usual Borel resummation of the WKB series associated to a slightly modified version of the potential

function $Q(z)$, that has in front an infinitesimal phase $\exp(\pm 2i0^+)$ (where, due to the convention we adopted for the definition of lateral resummation, the sign $-$ is related to $\mathcal{S}_+$, and the sign $+$ is related to $\mathcal{S}_-$; in [47], the opposite convention is used). This apparently innocent phase, due to the very definition of Stokes curves, implies a relevant topological modification of the Stokes graph associated to the problem, which removes the presence of the problematic saddle trajectory, "opening up" the saddle into two separate Stokes curves (use Figure 3.6 to help gaining intuition). The two Stokes graphs $G_\theta^\pm$ associated to the modified differentials $\Phi_\theta^\pm \equiv \exp(2i(\theta \pm 0^+))Q(z)dz^{\otimes 2}$ are said to be *saddle reductions* of the Stokes graph G_θ (which instead contains a saddle trajectory). The topological configuration of Stokes regions is different[7] between the two graphs $G_\theta^\pm$, and this difference influences the monodromy of WKB resummed solutions $\Psi_\pm$, when analytically continued between two points of the complex plane that in G_θ belong to adjacent Stokes regions. Here lies the origin of the difference between the two lateral resummations $\mathcal{S}_\pm$ of a quantum period that crosses a saddle trajectory of G_θ. The fact that this difference can be expressed through the resummed quantum period that encircles the saddle is then a direct consequence of the connection formulas and the relation (3.49) that expresses the change of normalisation of WKB solutions.
The above considerations can be formalised in the so called *Delabaere-Dillinger-Pham (or DDP) formula* [13], that we present below for the case where the saddle trajectory is regular. A similar one holds when the saddle trajectory is closed, but we will not need it in the forthcoming analysis, so we redirect to [47] for the description of that case. To precisely state the formula we may first introduce a bilinear form

$$(\cdot,\cdot) : H_1(\Sigma \setminus \hat{P}) \times H_1(\Sigma \setminus \hat{P}) \to \mathbb{Z} \tag{3.53}$$

that given two cycles in $H_1(\Sigma \setminus \hat{P})$ yields the number of intersections between them. The intersection number (γ_1, γ_2) between any two cycles may be defined using the right hand rule at each point of intersection between them (considering only true intersections between the cycles, that is those that occur on the same sheet of the Riemann surface Σ), where the convention is to displace the thumb along the direction of γ_1 (x direction) and the index along that of γ_2 (y direction), and assigning a $+1$ (-1) if the middle finger results pointing to the positive (negative) z direction. The intersection number is then given by the sum of these factors ± 1 evaluated at each intersection point of the two cycles. Calling θ_c the critical value of the phase of η for which the Stokes graph associated to Φ_θ has the regular saddle trajectory, we formalise the concept of "cycle encircling the saddle trajectory" as the class of cycles $\tilde{\gamma} \in H_1(\Sigma \setminus \hat{P})$, usually called *saddle class*, such that its projection on $\mathbb{P}^1$ surrounds the saddle trajectory, and we choose the orientation of those cycles such that $\mathrm{Re}\left(e^{i\theta_c} \oint_{\tilde{\gamma}} \sqrt{Q(z)}\, dz\right) < 0$. This condition uniquely fixes the value of the quantum period associated to any cycle in the saddle class.

[7] In particular, we say that G_θ^+ and G_θ^- are related by a *flip* or a *pop*, depending if the saddle trajectory in G_θ is a regular or a closed one.

We have then (for a detailed proof see [47], Appendix A):

Theorem 4 (DDP formula for a single saddle trajectory)**.** *For any $\gamma \in H_1(\Sigma \setminus \hat{P})$, the lateral Borel sums $\mathcal{S}_\pm[\mathcal{V}_\gamma]$ satisfy the following equality as analytic functions of η on a domain containing $\{\eta \in \mathbb{C} \,|\, \arg(\eta) = \theta_c,\, |\eta| \gg 1\}$:*

$$\mathcal{S}_+[\mathcal{V}_\gamma] = \mathcal{S}_-[\mathcal{V}_\gamma]\,(1 + \mathcal{S}_-[\mathcal{V}_{\tilde{\gamma}}])^{-(\tilde{\gamma},\gamma)} \tag{3.54}$$

Taking the logarithm, this equation may be rewritten in terms of the quantum periods as

$$\mathrm{disc}_{\theta_c}[\Pi_\gamma] = \mathcal{S}_+[\Pi_\gamma] - \mathcal{S}_-[\Pi_\gamma] = \frac{i}{\eta}(\tilde{\gamma},\gamma)\log\left(1 + e^{i\eta\mathcal{S}_-[\Pi_{\tilde{\gamma}}]}\right) \tag{3.55}$$

where, as standard practice, we made explicit in the notation of the Stokes discontinuity the critical angle θ_c at which is evaluated

As a typical hallmark of the Stokes phenomenon, the factor $(1 + \mathcal{S}_-[\mathcal{V}_{\tilde{\gamma}}])$ that distinguishes $\mathcal{S}_+[\mathcal{V}_\gamma]$ from $\mathcal{S}_-[\mathcal{V}_\gamma]$ is exponentially small in the limit $|\eta| \to \infty$, since the asymptotic expansion in η of $\mathcal{S}_-[\mathcal{V}_{\tilde{\gamma}}]$ contains the prefactor $\exp\left(\eta \oint_{\tilde{\gamma}} \sqrt{Q(z)}\, dz\right)$, which is decaying for large $|\eta|$ due to the assumption made when choosing the orientation of the cycles in the saddle class.
The generalisation of the DDP formula to the case when more than one saddle trajectory is present in the Stokes graph is straightforward (see for example [1]):

Theorem 5 (DDP formula for multiple saddle trajectories)**.** *Let G_{θ_c} a Stokes graph having N saddle trajectories with associated saddle classes $\tilde{\gamma}_j$, $j = 1, ..., N$. For any $\gamma \in H_1(\Sigma \setminus \hat{P})$, the lateral Borel sums $\mathcal{S}_\pm[\mathcal{V}_\gamma]$ satisfy the following equality as analytic functions of η on a domain containing $\{\eta \in \mathbb{C} \,|\, \arg(\eta) = \theta_c,\, |\eta| \gg 1\}$:*

$$\mathcal{S}_+[\mathcal{V}_\gamma] = \mathcal{S}_-[\mathcal{V}_\gamma] \prod_{j=1}^{N} \left(1 + \mathcal{S}_-[\mathcal{V}_{\tilde{\gamma}_j}]\right)^{-(\tilde{\gamma}_j,\gamma)} \tag{3.56}$$

Or equivalently, in term of quantum periods

$$\mathrm{disc}_{\theta_c}[\Pi_\gamma] = \mathcal{S}_+[\Pi_\gamma] - \mathcal{S}_-[\Pi_\gamma] = \frac{i}{\eta}\sum_{j=1}^{N}(\tilde{\gamma}_j,\gamma)\log\left(1 + e^{i\eta\mathcal{S}_-[\Pi_{\tilde{\gamma}_j}]}\right) \tag{3.57}$$

Let's stop a second to collect the thoughts. Considering for the moment fixed the choice of the potential function $Q(z)$, the resummed quantum periods depend only on η. As η, or better, its phase θ varies, saddle trajectories may appear in the Stokes graph associated to Φ_θ. A given (resummed) quantum period will be unchanged, as a function of η, until a saddle trajectory will form along its path. This means that, by

doing a scan in the values of θ, starting from an initial value and increasing θ until the topology of the Stokes graph comes back to the initial one (which will happen after a π wide interval, as can be deduced from the defining expression (3.37) of a Stokes curve), we can obtain the information of the full discontinuity structure of the quantum period considered. What's more, the DDP formula (3.57) tell us exactly how these discontinuities can be expressed in terms of the quantum periods related to the saddle classes of those saddle trajectories that appear during the scan in θ, which we may call *relevant*. Now, the important observation to make is that, if we restrict our attention to studying the discontinuity structure of the set of relevant quantum periods, then *the DDP formula becomes a closed relation*. As we will show in a moment, this, together with the knowledge of the asymptotic form of those quantum periods, is enough to obtain a set of equations, in the form of those appearing in the context of the Thermodynamic Bethe Ansatz, which can be numerically solved to obtain the quantum periods considered.

3.5.1 Encoding the discontinuity structure of quantum periods into TBA equations

> *Remark.* For the study of the TBA equations that govern the quantum periods, we will use as main reference [1]. In order to make easier the comparison between the results presented in this thesis and that of the paper, we will slightly change our notation to match that of [1]. In particular, we will call $\eta \equiv e^{\theta}$, so that what before was θ now will coincide with $\mathrm{Im}(\theta)$, and we will prefer to use $\hbar$ instead of $\eta = \hbar^{-1}$.

From now on we will focus our discussion on the choice of potential $V(z)$ that is the main subject of this thesis, namely a polynomial potential of generic degree $r+1$ ($r \in \mathbb{Z}, r > 1$) with complex coefficients. The space of parameters $\mathcal{M}$ of the potential function $Q(z) = V(z) - E$, called *moduli space*, is then formed by the coefficients of the polynomial and by the energy. The differential Φ_θ related to this choice of potential function satisfies all the hypotheses made at the beginning of section 3.3, provided we keep away from those points in $\mathcal{M}$ where two turning points coalesce. For this reason, we may from the beginning deprive $\mathcal{M}$ from such points, to obtain an auxiliary manifold that we will denote as $\mathcal{M}_0$. The Stokes graph associated to Φ_θ will then have $r+1$ distinct turning points and a single pole of order $r+5$ located at infinity[8], that is reached by Stokes curves along $r+3$ evenly spaced rays (for an example, refer back to Figure 3.2b). If for example we have that the coefficient in front of the highest power of the polynomial is 1, we can easily find the direction of these rays by approximating $Q(z) \sim z^{r+1}$ in a

[8]Note that if a potential function $Q(z)$ has a pole at infinity of order m, the associated quadratic differential Φ_θ also will have a pole at infinity, but of order $m+4$.

neighborhood of ∞ into equation (3.37), to find:

$$r_k = \left\{ z \in \mathbb{C} \,|\, \arg(z) = \frac{2\pi}{r+3}\left(k - \frac{\operatorname{Im}\theta}{\pi}\right)\right\} \tag{3.58}$$

What's more, since Φ_θ has no double poles, then only regular saddle trajectories can be present in the Stokes graph.

As already mentioned, a goal of this thesis is to obtain a set of TBA-like equations that allow us to compute the relevant quantum periods associated to a given choice of polynomial potential and energy, that is a given point in $\mathcal{M}_0$. The set of relevant quantum periods depends on the choice of point in $\mathcal{M}_0$, and, as it will be clear from the the analysis that follows, its dimension can range from a minimum of r to a maximum of $r(r+1)/2$ quantum periods, with intermediate cases. We will call *chambers* these regions of the moduli space that differ for the number and type of relevant quantum periods. Notably, the number of relevant quantum periods is always $\geq r$, that is the number of independent cycles in $H_1(\Sigma \setminus \hat{P})$. This implies that, solving the TBA equations, we will be able to compute any quantum period, and in particular those relevant for the exact quantisation conditions. The procedure of deriving the TBA equation can be done in a generic chamber, but the complexity of the task can become high as the number of relevant cycles increases, since one has to take account of the intersection number between each cycle, and most importantly collect all the discontinuities of quantum periods into a single set of closed equations. To avoid these complications, the approach we will use to tackle the problem consists in first deriving the set of TBA equations in a point of $\mathcal{M}_0$ where the task can be easily performed, and then analytically continue those result to a generic point of $\mathcal{M}_0$. This technique is the same used in [1], and before put forward in [44], despite in a different context; the main contribution of this thesis will be to develop an algorithmic procedure to efficiently perform the process of analytic continuation between chambers in $\mathcal{M}_0$, exploiting the connection between TBA equations and Y-systems. Thanks to this procedure, obtaining TBA equations related to a generic polynomial potential will be a straightforward task.

Derivation of TBA equations for the relevant quantum periods

The (non unique) point we choose in $\mathcal{M}_0$ to first derive the TBA equations is such that all the turning points z_i, $i = 1, \ldots, r+1$ are real (and distinct, since we have restricted ourselves to $\mathcal{M}_0$), and the sign of the coefficient z^{r+1} is so that the interval $[z_i, z_{i+1}]$ corresponds to a classically allowed interval[9] when i is odd, and to a classically

[9]Recall that a classically allowed (forbidden) interval is a interval of the real z axis where $Q(z) = V(z) - E$ is negative (positive).

forbidden interval when i is even, where we have chosen to label the turning points as

$$z_1 < z_2 < \ ... \ < z_{r+1}$$

The set of relevant cycles in this case can be immediately determined: we have $\sqrt{Q(z)}$ purely imaginary in the classically allowed intervals and purely real on the classically forbidden ones, so (recall formula (3.37)) we will have a saddle trajectory on the former when $\operatorname{Im}\theta = \pi/2$ (and also when $\operatorname{Im}\theta = -\pi/2$, by periodicity of Stokes curves), and on the latter when $\operatorname{Im}\theta = 0$ (or $\operatorname{Im}\theta = -\pi$). No other saddle trajectory is present for other values of $\operatorname{Im}\theta$, and so we conclude that the set of *relevant* quantum periods is formed by the periods related to cycles surrounding classically allowed and forbidden intervals. We label these periods as Π_{γ_a}, $a = 1, \dots, r$, where the cycle γ_a is taken to encircle the interval $[z_a, z_{a+1}]$.
We select then the branch cuts and the orientation of the cycles so that:

I. the zeroth order term $\Pi^{(0)}_{\gamma_a}$ of the $\hbar$ expansion of the quantum periods is real and positive (imaginary and negative) for a odd (even). In this way, if we associate to each quantum period a quantity, that we call mass for the role it will later play in the TBA equations, as

$$m_a \equiv \begin{cases} \Pi^{(0)}_{\gamma_a} = \oint_{\gamma_a} \sqrt{E - V(z)}\, dz & \text{for } a \text{ odd} \\ i\Pi^{(0)}_{\gamma_a} = i \oint_{\gamma_a} \sqrt{E - V(z)}\, dz & \text{for } a \text{ even} \end{cases} \tag{3.59}$$

then, for each period, its mass is real and positive.

II. the intersection number between two cycles γ_a, γ_b is nonzero only if $|a - b| = 1$, that is for cycles related to adjacent intervals on the real z axis, and equal to

$$(\gamma_a,\, \gamma_b) = \begin{cases} -1 & \text{for } a \text{ odd} \\ 1 & \text{for } a \text{ even} \end{cases} \tag{3.60}$$

The formation of a saddle trajectory for $\operatorname{Im}\theta = 0, -\pi$ in the *classically forbidden* intervals translates in the following discontinuity formula for the quantum periods associated to the *classically allowed* regions:

$$\frac{i}{\hbar} \operatorname{disc}_\theta[\Pi_{\gamma_a}](\hbar) = \mp \log\left(1 + e^{\mp \frac{i}{\hbar}\mathcal{S}_+[\Pi_{\gamma_{a-1}}](\hbar)}\right) \mp \log\left(1 + e^{\mp \frac{i}{\hbar}\mathcal{S}_+[\Pi_{\gamma_{a+1}}](\hbar)}\right), \tag{3.61}$$

for a odd, $\operatorname{Im}\theta = 0$ (upper sign), $-\pi$ (lower sign)

The difference in sign between the two cases above lies in the fact that the DDP formula (3.57) we presented in Theorem 5 assumes that the saddle classes $\tilde{\gamma}_j$ are chosen such that (recall the discussion before the statement of the theorem)

$$\operatorname{Re}\left(e^{i\operatorname{Im}(\theta)} \oint_{\tilde{\gamma}_j} \sqrt{Q(z)}\, dz\right) = \operatorname{Re}\left(e^{i\operatorname{Im}(\theta)}\, i \oint_{\tilde{\gamma}_j} \sqrt{E - V(z)}\, dz\right) < 0 \tag{3.62}$$

In formula (3.61), the saddle classes corresponds to the cycles $\gamma_{a\pm1}$ having even index, and due to the choice (3.59) made when defining the masses, the condition (3.62) above holds for $\gamma_{a\pm1}$ when $\mathrm{Im}\,\theta = \pi$, but is verified instead for the inverted cycles $-\gamma_{a\pm1}$ when $\mathrm{Im}\,\theta = 0$. As a cross-check of the correctness of equation (3.61), one can verify that for both values of $\mathrm{Im}\,\theta = -\arg(\hbar)$ the quantity on the right hand-side tends to zero as $|\hbar| \to 0$, as we expect from the theory of Stokes phenomena.

At $\mathrm{Im}\,\theta = \pm\pi/2$ we have a similar discontinuity formula, but now with the even and the odd cycles exchanged:

$$\frac{i}{\hbar}\,\mathrm{disc}_\theta[\Pi_{\gamma_a}](\hbar) = \mp\log\left(1+e^{\pm\frac{i}{\hbar}\mathcal{S}_+[\Pi_{\gamma_{a-1}}](\hbar)}\right) \mp \log\left(1+e^{\pm\frac{i}{\hbar}\mathcal{S}_+[\Pi_{\gamma_{a+1}}](\hbar)}\right), \qquad (3.63)$$

for a even, $\mathrm{Im}\,\theta = \pi/2$ (upper sign), $-\pi/2$ (lower sign)

Also in this case, the difference between the expression for $\mathrm{Im}\,\theta = \pi/2$ and that for $\mathrm{Im}\,\theta = -\pi/2$ lies in the convention for the saddle class. To obtain the set of TBA equations, one then introduces a set of functions $\epsilon_a(\theta)$, one for each relevant quantum period, such that

$$\begin{aligned} \epsilon_a\left(\theta + \frac{i\pi}{2} \pm i0^+\right) &= \frac{i}{\hbar}\mathcal{S}_\pm\left[\Pi_{\gamma_a}\right](\hbar), \quad \text{for } a \text{ odd} \\ \epsilon_a(\theta \pm i0^+) &= \frac{i}{\hbar}\mathcal{S}_\pm\left[\Pi_{\gamma_a}\right](\hbar), \quad \text{for } a \text{ even} \end{aligned} \qquad (3.64)$$

We will refer to the $\epsilon_a(\theta)$ as *pseudoenergies*.
The definition (3.64) allows to express the discontinuity formulas for both the even and the odd quantum periods together. All the $\epsilon_a(\theta)$ functions will in fact have a discontinuity at $\mathrm{Im}\,\theta = \pm\pi/2$, of the form[10]

$$\mathrm{disc}_\theta[\epsilon_a](\theta) = \pm L_{a-1}(\theta \mp i\pi/2) \pm L_{a+1}(\theta \mp i\pi/2), \quad a = 1, \ldots, r\ ,\ \mathrm{Im}\,\theta = \pm\pi/2 \qquad (3.65)$$

where we have introduced

$$L_a(\theta) = \log\left(1+e^{-\epsilon_a(\theta)}\right) \qquad (3.66)$$

and with the understanding that $L_0 = L_{r+1} = 0$.
From the analysis done for the quantum periods, we also know that no other discontinuities are present for the ϵ_a functions in the range $-\pi/2 < \mathrm{Im}\,\theta < \pi/2$. Then, the discontinuity structure of the ϵ_a, together with their classical limit (which can be inferred from (3.64), recalling the definition of the masses)

$$\epsilon_a(\theta) = m_a e^\theta + \mathcal{O}\left(e^{-\theta}\right), \quad \theta \to \infty \qquad (3.67)$$

[10]To derive the formulas we also used the property $\mathcal{S}\left[\Pi_{\gamma_a}\right](-\hbar) = \mathcal{S}\left[\Pi_{\gamma_a}\right](\hbar)$, which holds since Π_{γ_a} is a series in even powers of $\hbar$.

constitute a so-called Riemann-Hilbert problem, which can be encoded in the following set of TBA equations

$$\epsilon_a(\theta) = m_a e^{\theta} - \int_{\mathbb{R}} \frac{L_{a-1}(\theta')}{\cosh(\theta-\theta')}\frac{d\theta'}{2\pi} - \int_{\mathbb{R}} \frac{L_{a+1}(\theta')}{\cosh(\theta-\theta')}\frac{d\theta'}{2\pi}, \quad a = 1, \dots, r \tag{3.68}$$

or, in an equivalent form

$$\epsilon_a(\theta) = m_a e^{\theta} - K * L_{a-1}(\theta) - K * L_{a+1}(\theta). \tag{3.69}$$

where we have introduced the kernel

$$K(\theta) = \frac{1}{2\pi}\frac{1}{\cosh(\theta)} \tag{3.70}$$

These equations have been designed to precisely reproduce all the properties of the ϵ_a, namely (i) their classical limit, since as $\theta \to \infty$ the kernel in the TBA equations suppresses the contribution coming from the convolutions, and (ii) their discontinuity structure, as can be seen by analytically continuing the equations from $\mathrm{Im}\,\theta < \pi/2$ to $\mathrm{Im}\,\theta > \pi/2$, (or from $\mathrm{Im}\,\theta > -\pi/2$ to $\mathrm{Im}\,\theta < -\pi/2$), which involves picking the residues from the poles of the kernel and obtaining the $L_{a\pm1}$ terms, otherwise not present if we evaluate the TBA equations directly outside the strip $-\pi/2 \leq \mathrm{Im}\,\theta \leq \pi/2$.
Exploiting this discontinuity structure of the TBA equations, as it is common procedure, one can then also derive a Y-system. We do a smooth shift of the TBA equations in the direction $\theta \to \theta + i\pi/2$ and then add them to the equations shifted to $\theta \to \theta - i\pi/2$. Taking account of the residues from the poles of the kernel, one finds

$$\epsilon_a(\theta + i\pi/2) + \epsilon_a(\theta - i\pi/2) = -L_{a-1}(\theta) - L_{a+1}(\theta) \tag{3.71}$$

which, written in term of the Y-functions, that we introduce as

$$Y_a(\theta) = e^{-\epsilon_a(\theta)} \tag{3.72}$$

takes the form of the Y-system

$$Y_a(\theta + i\pi/2)Y_a(\theta - i\pi/2) = (1 + Y_{a-1}(\theta))(1 + Y_{a+1}(\theta)), \quad a = 1, \dots, r \tag{3.73}$$

with the understanding that $Y_0 = Y_{r+1} = 0$, as for the TBA equations. As already anticipated, the point of moduli space where we have chosen to derive the TBA equations for the quantum periods is peculiar since it involves a minimal number of relevant periods (or equivalently pseudoenergies and Y-functions). For this reason, we will refer to the chamber to which this point belongs as *minimal chamber*. This chamber, with its related TBA equations (3.68) and Y-system (3.73), will always be the starting point for the derivation of the TBA equations (or Y-system) for the quantum periods related to any polynomial potential.

A note on the use of TBA equations in combination with EQCs to compute the energy spectrum

We conclude this section with a final observation concerning the application of TBA equations to solve EQCs of the form (3.52). If we fix the parameters of the potential $V(z)$, the Voros multipliers (or the related quantum periods) entering in the EQCs depend only on $\hbar$ and the energy E. Assuming that the spectral problem yields a discrete set of energies, the EQCs then define a discrete and infinite family of curves in the $(E, \hbar)$ plane, that can be labelled via a quantum number $n = 0, 1, 2, \cdots$. Typically, these curves are interpreted as representing discrete values of the system's energy $E_n(\hbar)$ for a specific value of $\hbar$, but we can equally regard them as defining a series of discrete values $\eta_n(E)$ for the inverse Planck constant, given a choice E of energy. This second form of spectrum is often referred as *Voros spectrum*, and is related to the usual one by the consistency relation

$$E_n(\eta_n(E)) = E \tag{3.74}$$

Since the TBA equations are always referred to a given point in the moduli space, and so also a given value of the energy, the natural way to use them in combination of an EQC is for finding the values of the quantum periods at fixed energy E while varying $\theta = \log(\eta)$ until the EQC is satisfied, providing thus the Voros spectrum rather than the traditional one. The accuracy of this method can then be compared to other techniques that are commonly applied to find the spectrum of an Hamiltonian thanks to the relation (3.74); one uses the latter to compute the set of energies corresponding to a value of the inverse Planck constant $\eta_{\bar{n}}(E)$ present in the Voros spectrum previously obtained via the EQC, and verifies how close the $\bar{n}$-th energy level in such spectrum is to E. This kind of reasoning has been put into practice in [1] for several cases of cubic and quartic polynomial potentials, finding optimal agreements.

3.6 Connection with the approach of the ODE-IM correspondence

As promised at the beginning of this chapter, we now make contact between the Y-system for the quantum periods derived in section 3.5.1 and the one of section 2.2, coming instead from ODE-IM. In particular, we would like to clarify under which conditions we can push this identification also at the TBA level, that is also matching the asymptotics of the Y functions in the two cases, since this would imply a complete identification between the Y of the two approaches. Following [1], the first step to match the ODE-IM results with the one found using the exact WKB method is to restore the presence of $\hbar$ into the equation (2.67). We recall in fact that all the ODE-IM analysis has been done by setting

$\hbar = 1$. In order to do so we can apply the following coordinate transformation

$$(z, b_a) \to (\hbar^{\frac{2}{r+3}} z, \hbar^{\frac{2(a+1)}{r+3}} b_a) \tag{3.75}$$

which indeed sends the equation (2.67) into

$$\left(-\hbar^2 \frac{d^2}{dz^2} + z^{r+1} + \sum_{a=1}^{r} b_a z^{r-a} \right) \hat{\psi}(z, b_a, \hbar) = 0 \tag{3.76}$$

where we have put a hat on the solutions of the equation at a generic value of $\hbar$, in order to distinguish them from their restriction on the subspace of the parameters space where $\hbar = 1$. The two type of solutions are simply related as

$$\hat{\psi}(z, b_a, \hbar) \equiv \psi(\hbar^{-\frac{2}{r+3}} z, \hbar^{-\frac{2(a+1)}{r+3}} b_a) \tag{3.77}$$

From this expression it is manifest how the action of the Symanzik rotation Ω on the solutions $\hat{\psi}$ can be equivalently written as a rotation of the value of $\hbar$ (recall $\omega = e^{\frac{2\pi i}{r+3}}$)

$$\Omega\, \hat{\psi}(z, b_a, \hbar) = \hat{\psi}(\omega^{-1} z, \omega^{-a-1} b_a, \hbar) = \hat{\psi}(z, b_a, e^{i\pi}\hbar) \tag{3.78}$$

Starting from the set of solutions y_k used in the ODE-IM analysis of section 2.2, each of which is subdominant in a specific Stokes sector associated to the equation with $\hbar = 1$, we can find a similar set of solutions for the equation (3.76) as

$$\hat{y}_k(z, b_a, \hbar) = \omega^{k/2}\, \Omega^k\, \hat{y}(z, b_a, \hbar) = \omega^{k/2}\, \hat{y}(z, b_a, e^{i\pi k}\hbar) \tag{3.79}$$

proviso we update the notion of Stokes sector to include the $\hbar$ dependence

$$\mathcal{S}_k = \left\{ z \in \mathbb{C} \;|\quad \left| \arg(z) - \frac{2\pi}{r+3} \left(k - \frac{\operatorname{Im}\theta}{\pi} \right) \right| < \frac{\pi}{r+3} \right\}, \quad k \in \mathbb{Z} \tag{3.80}$$

where here θ has the same meaning that in the previous section, i.e. $\hbar = e^{-\theta}$. The careful reader has perhaps already noticed how at the center of each of these Stokes sectors lies one of the asymptotic directions (3.58) reached by the Stokes curves of the Stokes diagram determined by the choice of polynomial potential.
We define the wronskian $\hat{W}_{k_1,k_2}(\hbar, b_a)$ among two solutions $\hat{y}_k$ so that to match the value of the wronskian among the two solutions y_k having the same index, evaluated at the parameters choice coming from the correspondence (3.77)

$$\hat{W}_{k_1,k_2}(\hbar, b_a) = W_{k_1,k_2}(\hbar^{-\frac{2(a+1)}{r+3}} b_a) = \hbar^{\frac{2}{r+3}} (\hat{y}_{k_1} \hat{y}'_{k_2} - \hat{y}_{k_2} \hat{y}'_{k_1})(z, b_a, \hbar) \tag{3.81}$$

where as usual the $'$ stands for the z derivative. All the derivation of the (analog of) the T-system and then of the Y-system made in section 2.2 can now be repeated in the exact

same way using $\hat{y}_k$ solutions instead of the y_k ones, exploiting however the fact that for these solutions the Ω symmetry can be seen just a shift in the phase of $\hbar$, so that we can obtain functional relations among objects evaluated at different $\hbar$ values but at the same point in the moduli space. Just as an example, the shift property (2.82) for the wronskians, that then determine also the one of the C_k^m coefficients, now can be written as

$$\Omega^k \, \hat{W}_{i,j}(\hbar, b_a) = \hat{W}_{i,j}(e^{i\pi k}\hbar, b_a) = \hat{W}_{i+k,j+k}(\hbar, b_a) \tag{3.82}$$

or, equivalently, in terms of θ

$$\hat{W}_{i,j}(\theta - i\pi k, b_a) = \hat{W}_{i+k,j+k}(\theta, b_a) \tag{3.83}$$

Focusing on the Y-system, since this is the type of integrable structure we have found also through the exact WKB analysis, it will now read

$$Y_s(\theta + \frac{i\pi}{2}, b_a)Y_s(\theta - \frac{i\pi}{2}, b_a) = (1 + Y_{s+1}(\theta, b_a))(1 + Y_{s-1}(\theta, b_a)) \quad s = 1, \dots, r \tag{3.84}$$

where we have chosen to use θ instead of $\hbar$, as we will usually prefer to do, and as usual $Y_0 = Y_{r+1} = 0$. Contrarily to the Y-system (2.79) derived at $\hbar = 1$ (and also differently to an A_r Y-system of an ADE QFT), we see that here the Y-functions on the left hand-side are always shifted in their argument by the same amount, independently of the degree of the polynomial. This difference can be traced back to the use of $\hbar$ as spectral parameter instead of the energy, and to the different analytic structure of the Y-function in the two spaces. We recall in fact that in the ODE-IM correspondence, is the energy that corresponds to the spectral parameter λ used in the BLZ derivation of the Y-system for the QFT.

This Y-system matches with the one (3.73) coming from WKB analysis, but to have a complete identification between the Y_s functions appearing here and those of (3.73), we should also inspect their asymptotics for $\hbar \to 0$ (i.e. $\theta \to \infty$), so that to match them at the TBA level. The Y_s functions can be expressed in terms of cross-ratios of wronskians $\hat{W}_{k_1,k_2}$ on the same fashion of (2.89, 2.91)

$$Y_{2j}(\theta, b_a) = \frac{\hat{W}_{-j,j}\hat{W}_{-j-1,j+1}}{\hat{W}_{-j-1,-j}\hat{W}_{j,j+1}}\,(\theta, b_a) \qquad Y_{2j+1}(\theta, b_a) = \frac{\hat{W}_{-j-1,j}\hat{W}_{-j-2,j+1}}{\hat{W}_{-j-2,-j-1}\hat{W}_{j,j+1}}\,(\theta - \frac{i\pi}{2}, b_a) \tag{3.85}$$

To evaluate the asymptotic value of these objects we can use a WKB approximation for each of the y_k. In particular, we can approximate each of the wronskians $\hat{W}_{k_1,k_2}(\theta, b_a)$ as

$$\hat{W}_{k_1,k_2}(\theta, b_a) \sim \exp \pm \frac{i}{\hbar} \int_{\gamma_{k_1,k_2}} \sqrt{E - V(z)} dz \tag{3.86}$$

where γ_{k_1,k_2} is a path that reaches infinity at both its ends by collecting at one end along the asymptotic direction of the Stokes curves that lies at the center of the sector $\mathcal{S}_{k_1}$, and

at the other end along the corresponding direction in the Stokes sector $\mathcal{S}_{k_2}$, *proviso that the whole curve γ_{k_1,k_2} lies inside a Stokes region of the Stokes graph G_θ* (i.e. the topology of the Stokes graph is so that such a path exists), where G_θ is determined by the choice of the parameters b_a and by the value of θ at which the wronskian is computed [44, 57]. Actually, for each value of $\mathrm{Im}(\theta)$ where such a curve exists, the estimate continues to be good in a π wide sector in the values of $\mathrm{Im}(\theta)$, centered the value of $\mathrm{Im}(\theta)$ where the line exists.
If now, similarly to what has been done in section 3.5.1, we choose the values of the b_a parameters so that all the turning points of the potential function are distinct and lie on the real axis, one can check that indeed a desired curve γ_{k_1,k_2} exists in the interval of values $-\pi/2 < \mathrm{Im}(\theta) < \pi/2$ for $\mathrm{Im}(\theta)$, for all wronskians $\hat{W}_{k_1,k_2}(\theta, b_a)$ inside the Y_{2j}, and similarly for all wronskians $\hat{W}_{k_1,k_2}(\theta - \frac{i\pi}{2}, b_a)$ inside the Y_{2j+1}. When have then that the paths of the integrals contained in the asymptotic values of all wronskians inside any Y_s unify into a closed loop, which, thanks to contour deformation, can be identified with one of the relevant cycles γ_a encircling two turning points that are support for the loop integrals of the quantum periods entering in (3.68) [1]:

$$\begin{aligned} -\log Y_{2j}\left(\theta, b_a\right) &\sim \frac{i}{\hbar}\oint_{\gamma_{r+1-2j}} \sqrt{E - V(z)}dz \\ -\log Y_{2j+1}\left(\theta, b_a\right) &\sim \frac{1}{\hbar}\oint_{\gamma_{r-2j}} \sqrt{E - V(z)}dz \end{aligned} \tag{3.87}$$

for $|\theta| \to +\infty$, $-\pi < \theta < \pi$. Here we used the same notation that in section 3.5.1, so γ_a corresponds to the cycle encircling the interval $[z_a, z_{a+1}]$, z_a being the turning points, ordered as $z_1 < \cdots < z_{r+1}$. From these formulas, it appears clear how each of these Y_s function is associated with a corresponding cycle γ_{r+1-s}. A perfect identification of these asymptotics with those of the Y-functions coming from exact WKB analysis can be done by relabelling $Y_s \to Y_{r+1-s}$, which preserves the form of the Y-system, and recalling that in the derivation of section 3.5.1, we assumed the potential to be such that the first interval $[z_1, z_2]$ corresponded to a classically allowed interval; in the present case, that of equation (3.76), this requirement holds true if $r+1$ is even, but not for $r+1$ odd. In this second case we can imagine to invert the classically allowed and forbidden intervals by a shift $\theta \to \theta \pm i\pi/2$, which makes also the asymptotics agree with (3.64).
Now that we have found the asymptotic form of the Y functions and know their analytic structure, is a standard exercise to convert the Y-system (3.84) into a set of TBA equations (a detailed exposition on this procedure can be found in [44]). One defines the auxiliary functions (here we omit the b_a dependence)

$$l_s(\theta) \equiv \log Y_s(\theta) + m_s\, e^\theta \tag{3.88}$$

where the masses m_s are as usual just the coefficient of the lowest term in the $\hbar$ expansion of the Y_s. The l_s functions are analytic in the strip $-\pi/2 \leq \mathrm{Im}(\theta) \leq +\pi/2$, and in terms

of them the Y-system reads

$$l_s(\theta + \frac{i\pi}{2}) + l_s(\theta + \frac{i\pi}{2}) = \log\left((1 + Y_{s+1}(\theta))\,(1 + Y_{s-1}(\theta))\right) \tag{3.89}$$

By convoluting both sides with the kernel (3.70) one gets

$$\epsilon_s(\theta) = m_s e^{\theta} - K * L_{s+1}(\theta) - K * L_{s-1}(\theta) \tag{3.90}$$

where the $\epsilon_s(\theta)$ and the $L_s(\theta)$ functions are defined as in section 3.5.1, i.e.

$$\epsilon_s(\theta) = -\log Y_s(\theta) \qquad L_a(\theta) = \log\left(1 + Y_s(\theta)\right) = \log\left(1 + e^{-\epsilon_a(\theta)}\right) \tag{3.91}$$

These TBA equations perfectly match the ones (3.69) coming from exact WKB, thus allowing an identification between the Y-functions (3.85) built out of wronskians and those of section 3.5.1, which represent Borel resummed Voros symbols thanks to the definition (3.64).

As a final remark we may inspect an aspect that has not yet been discussed, namely the periodicity of the Y-functions in the minimal chamber. This can be checked using at least two different approaches:

1. Numerically, by using only the Y-system structure. One transforms the Y-system into a relation among sequences $Y_{s,n} \equiv Y_s(\theta + in\pi/2)$

$$Y_{s,n+1} = \frac{(1 + Y_{s+1,n})(1 + Y_{s-1,n})}{Y_{s,n-1}} \tag{3.92}$$

   and looks then at the first value of n to which a sequence $Y_{s,n}$ comes back to itself (given the initial conditions $Y_{s,0}, Y_{s,1}$ for all the $Y_{s,n}$).

2. Analytically, by exploiting the wronskian representation (3.85) of the Y-functions, the shift relation (3.83), and the fact that the monodromy around the origin is trivial, so that

$$\hat{W}_{i+r+3,j+r+3}(\theta) = c_i c_j \hat{W}_{i,j}(\theta) \tag{3.93}$$

   where the c_k are constants that depends only on the choice of solutions y_k (these constants are irrelevant in the Y functions since the Y involve at both the numerator and the denominator the same four solutions y_k).

In both ways, what one finds is a periodicity of $i\pi(r+3)$ in θ for all the Y-functions, which gets enhanced to $i\pi(r+3)/2$ for only the "central" Y, i.e. $Y_{\frac{r+1}{2}}$, if r is odd.

Chapter 4

Analytic continuation strikes back

In this chapter we finally introduce the main original contribution of this thesis project, namely an algorithmic procedure to find the TBA-like equations for the relevant resummed quantum periods associated to a generic polynomial potential (with distinct turning points), through algebraic manipulations of the Y-system of the minimal chamber. The justification of this algebraic procedure has been extracted from a study of the analytic continuation of the minimal chamber's TBA equations, along the lines of the work [44], applied then to the specific case of a Schrödinger equation with polynomial potential by Ito and collaborators in [1]. Despite stemming from a study of TBA equations, this procedure is completely independent of its "TBA origin", and involves from start to finish only the Y functions. In the first section of this chapter we explain the motivation of this procedure starting from the TBA equations, by considering the simplest type of potential which still allows to show all the main type of complexities that can arise in this type of calculations, namely a quartic potential. In doing so, we will skip over the details of some calculations, suggesting the interested reader to refer to the thesis work [18] of S. Franzoni, who collaborated with the author to the development of this algorithmic procedure. The style of exposition will instead focus on presenting all the essential concepts, and already push towards a Y-system perspective. We will in fact start to showcase the use of the procedure directly in the example of the quartic potential. The second part of this chapter is then devoted to the formulation of the algorithm in the general case; the reader that has become acquainted with the concepts of the first section, may use this section as a guide to practically implement the algorithm in any specific case of polynomial potential to his liking.

4.1 Reaching different chambers via analytic continuation of TBA equations

As mentioned in the introduction to this chapter, we use as exemplification model a quartic potential. This type of potential is the simplest one for which the chambers foliation of the moduli space $\mathcal{M}_0$ still contains also intermediate chambers, which one can encounter by moving in $\mathcal{M}_0$ along a path that starts from the minimal chamber, where the relevant quantum periods (and thus the ϵ_a and the Y_a functions) are 3, to the maximal one, where the periods are 6. The starting point of this type of analysis is the TBA system for the relevant quantum periods derived in section 3.5.1, specialised to the quartic case. We write it immediately followed by the corresponding Y-system:

$$\begin{aligned}
\epsilon_1^\alpha &= m_1 e^\theta - K * L_2^\alpha \\
\epsilon_2^\alpha &= m_2 e^\theta - K * L_1^\alpha - K * L_3^\alpha \\
\epsilon_3^\alpha &= m_3 e^\theta - K * L_3^\alpha
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
Y_1^{\alpha+} Y_1^{\alpha-} &= [1 + Y_2^\alpha] \\
Y_2^{\alpha+} Y_2^{\alpha-} &= [1 + Y_1^\alpha][1 + Y_3^\alpha] \\
Y_3^{\alpha+} Y_3^{\alpha-} &= [1 + Y_2^\alpha]
\end{aligned} \tag{4.2}$$

where we have left understood the θ dependence of the ϵ_a, L_a, Y_a functions and introduced the notation $\pm$ to indicate shifts of $\pm i\frac{\pi}{2}$ in θ. We also attached an index α to all the quantities containing the ϵ_a functions, as an indicator of the chamber we are considering. Moving inside the moduli space, we will then use the other Greek letters $\beta, \gamma, \delta, \ldots$ for the other chambers we will encounter along the way. Recalling the derivation made in section 3.5.1, the above form of the TBA equations hold in a particular point P of the moduli space inside the minimal chamber, i.e. describe the discontinuity relations of the relevant quantum periods associated to a particular choice of quartic potential function (for a pictorial representation of the cycles associated to the periods at P see Figure 4.1). Changing the parameters of the potential to a more general form implies describing a trajectory Γ in the moduli space. If we imagine to move along this trajectory starting from the point P, we will generally have that, as long as we remain close to P, the relevant *cycles* will be the same, but the value of the *quantum periods* built on them will start to change value. In particular their asymptotic value, i.e. their masses m_a, will start to change, departing from being real numbers and developing a complex representation, with phase ϕ_a.

Continuing then along the trajectory Γ, if we exit from the region of $\mathcal{M}_0$ corresponding to the minimal chamber, also the form of the TBA equations should be modified, to describe the fact that new relevant quantum periods enter into play (recall that the chamber foliation of $\mathcal{M}_0$ is exactly determined by the number and type of quantum

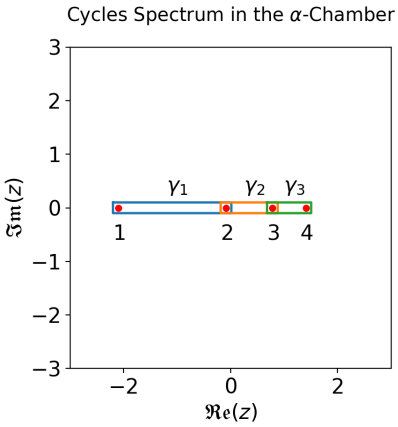


Figure 4.1: Pictorial representation of the relevant cycles for the quartic potential in the point of the α-chamber used for the derivation of the minimal chamber's TBA equations. The red dots are the turning points of the potential function.

periods that are relevant in the different points of $\mathcal{M}_0$). The need for this modification can be practically checked, point by point along Γ, by doing a π wide scan in the values of θ, and looking when two turning points which previously were not joined by a saddle trajectory for any value of θ, then start to develop one. When this phenomenon occurs, one may use a line of reasoning similar to the one used in section 3.5.1 (which is mainly based on the DDP formula) to write the new form of the TBA equations, but there is perhaps even a simpler way to do so. In fact, one can show that the TBA equations of the minimal chamber contain already in them most of the information needed to generate the "new ones". If, when still in the minimal chamber, we reabsorb the phases through a redefinition of the ϵ_a functions

$$\widetilde{\epsilon}_a(\theta) = \epsilon_a\left(\theta - i\phi_a\right) \ . \tag{4.3}$$

we may rewrite the TBA equations (4.1) in terms of modified kernels

$$K_{a,b}(\theta) \equiv \frac{1}{2\pi}\frac{1}{\cosh\left(\theta + i\Delta\phi_{a,b}\right)}, \quad \text{where} \quad \Delta\phi_{a,b} \equiv \phi_a - \phi_b \tag{4.4}$$

as (this is the expression for the quartic case, but an analog form holds for polynomial potentials with any other degree)

$$\begin{aligned}
\widetilde{\epsilon}_1^{\alpha}\left(\theta\right) &= |m_1|e^{\theta} - K_{2,1} * \widetilde{L}_2^{\alpha}\left(\theta\right) \\
\widetilde{\epsilon}_2^{\alpha}\left(\theta\right) &= |m_2|e^{\theta} - K_{1,2} * \widetilde{L}_1^{\alpha}\left(\theta\right) - K_{3,2} * \widetilde{L}_3^{\alpha}\left(\theta\right) \\
\widetilde{\epsilon}_3^{\alpha}\left(\theta\right) &= |m_3|e^{\theta} - K_{2,3} * \widetilde{L}_2^{\alpha}\left(\theta\right) \ .
\end{aligned} \tag{4.5}$$

where the L functions with a tilde on top are just $\widetilde{L}_a(\theta) \equiv L_a(\theta - i\phi_a)$, as is for the ϵ_a functions which they contain.

The convolution integrals in these equations, when written explicitly, are all of the form

$$K_{b,a} * \widetilde{L}_b(\theta) = \int_{\mathbb{R}} \frac{d\theta'}{2\pi} \frac{\widetilde{L}_b(\theta')}{\cosh(\theta - \theta' + i\Delta\phi_{b,a})} \tag{4.6}$$

One can notice that the convolution kernel $K_{b,a}$ brings together with it poles for the integrand function, located in the θ' plane (where the integral is performed) at the points θ'_p that satisfy

$$\begin{aligned} \theta - \theta'_p + i\Delta\phi_{b,a} &= i\left(\pm\frac{\pi}{2} + k\pi\right) \\ \Longrightarrow \theta'_p &= \theta + i\left[\Delta\phi_{b,a} \mp \left(\frac{\pi}{2} - k\pi\right)\right] \ , \quad k \in \mathbb{Z} \end{aligned} \tag{4.7}$$

The position of these poles depends on the phase difference $\Delta\phi_{b,a}$; in the special point P of the minimal chamber at which we originally derived the TBA equations, the masses m_a are all real and positive and thus the phase differences are all zero, so the poles θ'_p are all located away from the real axis of the θ' complex plane, where the convolution integral (4.6) is performed. However, as we move along a trajectory Γ in moduli space, we know that the phases ϕ_a, ϕ_b will vary, and if their phase difference $\Delta\phi_{b,a}$ reaches the first critical values of $+\frac{\pi}{2}$ or $-\frac{\pi}{2}$ for which the location of one of the two θ'_p poles that are closer to the real axis (those with $k = 0$ in equation (4.7)) touches the real θ' axis, then, to keep the TBA equations valid, one should modify the integration contour so that the problematic pole is avoided, i.e. pick the residue of the integrand function at that pole. This residue, that will appear on the right hand side of the TBA equation containing the convolution term (4.6) considered, is the cause of the modification of the TBA equations, and precisely signals that a new chamber has been reached. Let's see this phenomenon, which goes under the name of *wall-crossing* (since one moves across the "walls" of different chambers in $\mathcal{M}_0$), in the explicit example of the quartic potential.

Let's suppose that we choose a path Γ in the moduli space such that, keeping track of all the phases ϕ_1, ϕ_2, ϕ_3, the first wall crossing phenomenon involves

$$\Delta\phi_{2,3} < \frac{\pi}{2} \rightarrow \Delta\phi_{2,3} > \frac{\pi}{2} \tag{4.8}$$

while the other phase difference entering in the kernels of the α-chamber TBA equations (4.5), i.e. $\Delta\phi_{1,2}$, is kept $< \frac{\pi}{2}$. By scanning the Stokes graph in values of θ, one may check that after the wall-crossing, a new relevant quantum period is present in the graph, associated to the cycle (up to orientation)

$$\gamma_{23} \equiv \gamma_2 + \gamma_3 \tag{4.9}$$

Notice how the new cycle that has become relevant is built out as a combination of the

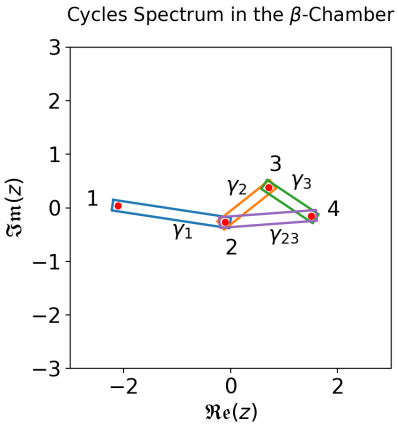


Figure 4.2: Pictorial representation of the relevant cycles for the quartic potential in a particular point of the β-chamber. The red dots are the turning points of the potential function.

two that have wall-crossed[1], and loops around the two turning points, among those encircled by γ_2, γ_3, that the two cycles do not share in common (i.e. this third cycle closes a triangle; refer to Figure 4.2 to help the intuition). We expect then to obtain four TBA equations for the four relevant resummed quantum periods $\mathcal{S}\left[\Pi_{\gamma_1}\right], \mathcal{S}\left[\Pi_{\gamma_2}\right], \mathcal{S}\left[\Pi_{\gamma_3}\right], \mathcal{S}\left[\Pi_{\gamma_{23}}\right]$, where resummation is performed in the current point of the moduli space.
Looking back at the TBA equations (4.5), since the above condition (4.8) obviously implies also that the opposite phase difference moves as $\Delta\phi_{3,2} > -\frac{\pi}{2} \to \Delta\phi_{3,2} < -\frac{\pi}{2}$, we have that two convolution terms generate a residue (in particular, the second and the third equation are the ones that need to be modified). The fact that *two* equations get modified at a single wall crossing is typical of this phenomenon (at least when only one phase difference at a time reaches a critical value), and comes just from the fact that if the TBA equation of an $\widetilde{\epsilon}_a$ function is coupled to the one of another $\widetilde{\epsilon}_b$ function through a convolution term, then the same holds true also for the TBA equation of the $\widetilde{\epsilon}_b$ function, and the two convolutions term generate a residue at the same time. After the wall crossing, i.e. in the new chamber which we may call $\beta-$chamber, the TBA equations (4.5) will read

$$\begin{aligned}
\widetilde{\epsilon}_1^{\alpha}(\theta) &= |m_1|e^{\theta} - K_{2,1} * \widetilde{L}_2^{\alpha}(\theta) \\
\widetilde{\epsilon}_2^{\alpha}(\theta) &= |m_2|e^{\theta} - K_{1,2} * \widetilde{L}_1^{\alpha}(\theta) - K_{3,2} * \widetilde{L}_3^{\alpha}(\theta) - \widetilde{L}_3^{\alpha}(\theta + i\frac{\pi}{2} + i\Delta\phi_{3,2}) \\
\widetilde{\epsilon}_3^{\alpha}(\theta) &= |m_3|e^{\theta} - K_{2,3} * \widetilde{L}_2^{\alpha}(\theta) - \widetilde{L}_2^{\alpha}(\theta - i\frac{\pi}{2} + i\Delta\phi_{2,3})
\end{aligned} \tag{4.10}$$

[1]Always happens that the new cycles appearing after wall crossing can be built starting from the pre-existing ones, since the cycles of the minimal chamber are enough to generate all the cycles in $H_1(\Sigma \setminus \hat{P})$

where we see that the evaluation of the residues coming from the kernels have generated additional terms in form of shifted $\widetilde{L}_a^\alpha$ functions. Looking at this set of equations, one may legitimately raise some observations:

1. These equations do not form a closed set, since the terms coming from the residues are evaluated at shifted values of θ. This is true, but taking account of this fact would be a straightforward task, since to close the system we may just add additional TBA equations evaluated at the appropriate shifts

2. We have performed the wall-crossing into the β-chamber, where we said a new relevant quantum period Π_{23} is present, and so we would expect a new $\widetilde{\epsilon}_a$ function, to be involved. However, we still have three of them. Where is the missing one?

3. This is perhaps less relevant, but still gives a hint that something is going on. Why have we kept the α indexes on the $\widetilde{\epsilon}_a$ functions, if we are now into the β-chamber?

The answer to the last two questions, which also explains why we did not really bother to write the other TBA equations necessary to close the system (4.10), lies in the asymptotics of the $\widetilde{\epsilon}_2^\alpha, \widetilde{\epsilon}_3^\alpha$ functions. The $\widetilde{L}_a^\alpha$ terms that have been added to their TBA equations in fact, modify their $\theta \to \infty$ (i.e. $\hbar \to 0$) asymptotic, *which will generally no more correspond to (the modulus of) a classical period*. These equations, are not the TBA equations that describe the resummed relevant quantum periods in the $\beta-$chamber, rather, as one should have expected, they just describe the analytic continuation of the $\widetilde{\epsilon}_a^\alpha$ functions, which *were* associated to the resummed quantum periods in the region of moduli space corresponding to the minimal (α) chamber (this is an instance of Stokes phenomenon on the resummed series due to changing point of resummation in the moduli space).
Despite this, we can still obtain from the set of equations (4.10) the ones associated to the $\beta-$chamber relevant quantum periods, by exploiting an easy trick. Since what we want for the $\widetilde{\epsilon}_a^\beta$ functions is to have an asymptotic form consisting in the (modulus of) a relevant classical period (as we have in the minimal chamber), i.e. $\widetilde{\epsilon}_a^\beta(\theta) \sim |m_a| e^\theta$, to get this form for the asymptote we can just bring the $\widetilde{L}_a^\alpha$ terms coming from the convolution's residue on the left hand-side of the last two equations in (4.10), and call $\widetilde{\epsilon}_2^\beta, \widetilde{\epsilon}_3^\beta$ all the terms that lie on that side of the TBA equations. This implies the definitions

$$\begin{aligned}
\epsilon_1^\beta(\theta) &\equiv \epsilon_1^\alpha(\theta) \\
\epsilon_2^\beta(\theta) &\equiv \epsilon_2^\alpha(\theta) + L_3^\alpha\left(\theta + i\frac{\pi}{2}\right) \\
\epsilon_3^\beta(\theta) &\equiv \epsilon_3^\alpha(\theta) + L_2^\alpha\left(\theta - i\frac{\pi}{2}\right).
\end{aligned} \tag{4.11}$$

which in terms of Y-functions read

$$Y_1^\beta \equiv Y_1^\alpha, \quad Y_2^\beta \equiv \frac{Y_2^\alpha}{1 + Y_3^{\alpha+}}, \quad Y_3^\beta \equiv \frac{Y_3^\alpha}{1 + Y_2^{\alpha-}} \tag{4.12}$$

Here we preferred to give the definitions in term of the "non tilded" functions to keep a more readable notation, but to obtain from these their counterpart for the $\widetilde{\epsilon}_a^\beta, \widetilde{Y}_a^\beta$ one can just shift the above ones. Notice how the ϵ_1, Y_1 functions do not get redefined, since the resummed quantum period $\mathcal{S}[\Pi_{\gamma_1}]$ is not involved in the wall crossing process. In terms of the new $\widetilde{\epsilon}_a^\beta$ functions, the TBA equations (4.10) and the corresponding Y-system (which was still (4.2), as can be checked by smooth shifting the TBA equations (4.10)[2]) can be rewritten as

$$\begin{aligned}
\widetilde{\epsilon}_1^\beta(\theta) &= |m_1|e^\theta - K_{2,1} * \widetilde{L}_2^\alpha(\theta) \\
\widetilde{\epsilon}_2^\beta(\theta) &= |m_2|e^\theta - K_{1,2} * \widetilde{L}_1^\beta(\theta) - K_{3,2} * \widetilde{L}_3^\alpha(\theta) \\
\widetilde{\epsilon}_3^\beta(\theta) &= |m_3|e^\theta - K_{2,3} * \widetilde{L}_2^\alpha(\theta)
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
Y_1^{\beta+}Y_1^{\beta-} &= [1+Y_2^\alpha] \\
Y_2^{\beta+}Y_2^{\beta-} &= [1+Y_1^\beta][1+Y_3^{\alpha++}]^{-1} \\
Y_3^{\beta+}Y_3^{\beta-} &= [1+Y_2^{\alpha--}]^{-1}
\end{aligned} \tag{4.14}$$

At this stage, these two sets of equations are still in a mixed form, namely they involve both $\alpha-$ and $\beta-$chamber's periods. To write them in terms only of $\beta-$chamber's periods we take advantage from the fact that we expect the new pseudoenergy $\widetilde{\epsilon}_{23}$ function to arise, associated to the cycle γ_{23}, that still doesn't appear in the above equations; in this way we will fix two issues at once. The most natural ansatz is to require

$$\begin{cases}
L_2^\alpha(\theta) &= L_2^\beta(\theta) + L_{23}^\beta(\theta + i\varphi_>) \\
L_3^\alpha(\theta) &= L_3^\beta(\theta) + L_{23}^\beta(\theta + i\varphi_<)
\end{cases} \tag{4.15}$$

or equivalently, in terms of Y-functions

$$\begin{cases}
1+Y_2^\alpha(\theta) &= \left[1+Y_2^\beta(\theta)\right]\left[1+Y_{23}^\beta(\theta+i\varphi_>)\right] \\
1+Y_3^\alpha(\theta) &= \left[1+Y_3^\beta(\theta)\right]\left[1+Y_{23}^\beta(\theta+i\varphi_<)\right]
\end{cases} \tag{4.16}$$

where we have inserted two phases $\varphi_>, \varphi_<$ in the values at which the new $L_{23}^\beta, Y_{23}^\beta$ are evaluated, to allow for a compatibility between the two conditions. The symbols chosen here reflect the type of wall crossing condition: we have used $\varphi_>$ ($\varphi_<$) in the equation associated to the L_a or Y_a function whose phase ϕ_a is greater (smaller), among the two that are involved in the wall-crossing. From these conditions, by using the definitions (4.11, 4.12) of the $\beta-$chamber functions in terms of the $\alpha-$chamber ones, one can extract

[2]Being the Y-system a set of algebraic relations, once it has been derived, it is valid in all the moduli space $\mathcal{M}_0$.

the definition of the new Y_{23}^{β} function

$$\begin{cases} Y_{23}^{\beta}\left(\theta+i\varphi_{>}\right) & = \dfrac{Y_{2}^{\alpha}Y_{3}^{\alpha+}}{1+Y_{2}^{\alpha}+Y_{3}^{\alpha+}} \\ Y_{23}^{\beta}\left(\theta+i\varphi_{<}\right) & = \dfrac{Y_{2}^{\alpha-}Y_{3}^{\alpha}}{1+Y_{2}^{\alpha-}+Y_{3}^{\alpha}} \end{cases} \tag{4.17}$$

where on the right hand-side we left implicit the θ dependence. The consistency of these two equations require the condition $\varphi_{>}-\varphi_{<}=\pi/2$, which makes only one of the two shifts independent. To determine it, one has to match the asymptotic form

$$\widetilde{\epsilon}_{23}^{\beta}(\theta)=-\log Y_{23}^{\beta}(\theta-i\phi_{23})\sim|m_{23}|\,e^{\theta} \tag{4.18}$$

where $m_{23}=|m_{23}|e^{i\phi_{23}}$ is the classical period built on the cycle γ_{23}

$$m_{23}\equiv\Pi_{\gamma_{23}}^{(0)}=\Pi_{\gamma_{2}}^{(0)}+\Pi_{\gamma_{3}}^{(0)}=m_{3}-im_{2} \tag{4.19}$$

In particular, we may choose

$$\varphi_{>}=\frac{\pi}{2}\;\Rightarrow\varphi_{<}=0 \tag{4.20}$$

This choice correctly reproduces the asymptote (4.18), as can be checked by evaluating $\widetilde{Y}_{23}^{\beta}(\theta)$ via (4.17) and using the α-chamber TBA equations, and implies the following definition of Y_{23}^{β}

$$Y_{23}^{\beta}\left(\theta\right)=\frac{Y_{2}^{\alpha-}Y_{3}^{\alpha}}{1+Y_{2}^{\alpha-}+Y_{3}^{\alpha}} \tag{4.21}$$

It also determines, through equations (4.15, 4.16), the form below for the TBA equations and the Y-system of the resummed quantum periods $\mathcal{S}\left[\Pi_{\gamma_1}\right],\mathcal{S}\left[\Pi_{\gamma_2}\right],\mathcal{S}\left[\Pi_{\gamma_3}\right]$ in the $\beta-$chamber

$$\begin{aligned} \widetilde{\epsilon}_{1}^{\beta} &= |m_{1}|\,e^{\theta}-K_{2,1}*\widetilde{L}_{2}^{\beta}-K_{23,1}^{+}*\widetilde{L}_{23}^{\beta} \\ \widetilde{\epsilon}_{2}^{\beta} &= |m_{2}|\,e^{\theta}-K_{1,2}*\widetilde{L}_{1}^{\beta}-K_{23,2}*\widetilde{L}_{23}^{\beta}-K_{3,2}*\widetilde{L}_{3}^{\beta} \\ \widetilde{\epsilon}_{3}^{\beta} &= |m_{3}|\,e^{\theta}-K_{2,3}*\widetilde{L}_{2}^{\beta}-K_{23,3}^{+}*\widetilde{L}_{23}^{\beta} \end{aligned} \tag{4.22}$$

$$\begin{aligned} Y_{1}^{\beta+}Y_{1}^{\beta-} &= \left[1+Y_{2}^{\beta}\right]\left[1+Y_{23}^{\beta+}\right] \\ Y_{2}^{\beta+}Y_{2}^{\beta-} &= \left[1+Y_{3}^{\beta++}\right]^{-1}\left[1+Y_{23}^{\beta++}\right]^{-1}\left[1+Y_{1}^{\beta}\right] \\ Y_{3}^{\beta+}Y_{3}^{\beta-} &= \left[1+Y_{2}^{\beta--}\right]^{-1}\left[1+Y_{23}^{\beta-}\right]^{-1} \end{aligned} \tag{4.23}$$

where we see that now only functions of the $\beta-$chamber are present, and the new cycle γ_{23} has entered into the equations (in these TBA equations, the kernels $K_{23,1}^{+},K_{23,3}^{+}$ are

just the usual kernels (4.4) with a $+\frac{\pi}{2}$ shift, that has been absorbed from the function L_{23}^{β} to which are convoluted). Despite being of the desired form, the above systems of equations are, however, not yet complete, since they are missing the equation for the $\widetilde{\epsilon}_{23}^{\beta}$ (or Y_{23}^{β}) function. The equation for the Y-function is easily derived from its definition (4.21) in terms of the Y-functions of the $\alpha-$chamber, and using the Y-system (4.2) of the latter which is known

$$\begin{aligned} Y_{23}^{\beta+}Y_{23}^{\beta-} &= \frac{Y_3^{\alpha+}Y_2^{\alpha}}{1+Y_3^{\alpha+}+Y_2^{\alpha}}\frac{Y_3^{\alpha-}Y_2^{\alpha--}}{1+Y_3^{\alpha-}+Y_2^{\alpha--}} = \\ &= \frac{\left[1+Y_2^{\alpha}\right]\left[1+Y_3^{\alpha-}\right]}{\left[1+Y_3^{\alpha+}+Y_2^{\alpha}\right]\left[1+Y_3^{\alpha-}+Y_2^{\alpha--}\right]}\left[1+Y_1^{\alpha-}\right] = \\ &= \left[1+Y_3^{\beta+}\right]^{-1}\left[1+Y_2^{\beta--}\right]^{-1}\left[1+Y_1^{\beta-}\right] \end{aligned} \tag{4.24}$$

where in the last passage we have used (4.12) to express everything in terms of Y-functions of the $\beta-$chamber. The corresponding TBA equation can be derived either by summing $\widetilde{\epsilon}_2^{\alpha}(\theta+i\Delta\phi_{2,23}-i\frac{\pi}{2})+\widetilde{\epsilon}_3^{\alpha}(\theta+i\Delta\phi_{3,23})$ so that the numerator of the tilded version of (4.21) is reproduced, and then using the TBA equations (4.22) we already have, or also directly from the above equation (4.24) for the Y_{23}^{β} function, by first rewriting it in terms of the tilded Y-functions, and then convoluting with the kernel $K=\frac{1}{2\pi}\frac{1}{\cosh(\theta)}$ in a similar fashion of what has been done in section 3.6. In both cases, the end result is

$$\widetilde{\epsilon}_{23}^{\beta} = |m_{23}|\, e^{\theta} - K_{1,23}^{-} * \widetilde{L}_1^{\beta} - K_{2,23} * \widetilde{L}_2^{\beta} - K_{3,23}^{-} * \widetilde{L}_3^{\beta} \tag{4.25}$$

We are done for the $\beta-$chamber analysis. If we want to reach the maximal chamber however, we have still some work to do.

From the calculations just performed in passing from the $\alpha-$ to the $\beta-$chamber it should appear clear how the main (and practically only) information that has to be given in input to the procedure is the evolution of the phase of the masses ϕ_a, which shall come from a numerical analysis that involves computing the classical periods at any step of movement along the curve Γ we have chosen in the moduli space. The cycles to keep track of are those which are relevant at the point of Γ which is under consideration in the current travel step. *To keep the procedure well-defined, it is important to choose a path Γ where new relevant periods appear only one at a time, i.e. we move across chambers that differ by the presence of a single new relevant quantum period.* This reflects in not having more than one couple of the phases ϕ_a present in the TBA equations crossing a critical value at each step, where for critical value we mean a value that makes kernels present in the TBA equations to produce residues. It is important to stress that this critical value is not always $\pm\frac{\pi}{2}$ as it was in the case we have already analyzed. Looking at the TBA equations (4.22) and the additional equation (4.25), we see that, due to the presence of kernels shifted of $\pm\frac{\pi}{2}$, we may have a wall-crossing when certain phase

differences overcome the value 0. In particular, the kernels to keep track of are[3]

$$K_{2,1}(\text{and the related } K_{1,2}) \rightarrow \text{its poles cross the real } \theta' \text{ axis for } \Delta\phi_{2,1} > \frac{\pi}{2} \tag{4.26}$$

$$K^{+}_{23,1}(\text{and the related } K^{-}_{1,23}) \rightarrow \text{its poles cross the real } \theta' \text{ axis for } \Delta\phi_{23,1} > 0 \tag{4.27}$$

The first of the two wall-crossing conditions was already present in the $\alpha-$chamber TBA equations (4.5), while the second is a new one. We note that, from the definition of $m_{23} = |m_{23}|e^{i\phi_{23}} = m_3 - im_2$, this second wall-crossing condition is not immediately met after just the first wall crossing (as we desire, to have the wall-crossing happen one at a time), that is when $\Delta\phi_{2,3} \gtrsim \pi/2$. Looking at these two wall crossing conditions in terms of quantum periods, we may also raise an important observation, that stands true also for the wall-crossing already done, and is indeed of general validity:

Remark. A wall crossing happens when two classical periods $\Pi^{(0)}_a, \Pi^{(0)}_b$ that are coupled though TBA (or Y-system) equations align or anti-align, that is

$$\text{Im}\left(\frac{\Pi^{(0)}_a}{\Pi^{(0)}_b}\right) = 0 \tag{4.28}$$

This condition is also stated in [1], and appears as well in the context of $\mathcal{N} = 2$ gauge theories, where it defines a locus in the moduli space that is called *curve of marginal stability*.

Let's now continue with the demonstration of a possible wall-crossing sequence to the maximal chamber for the illustrative example of the quartic potential. Having explained in detail for the first wall-crossing how the procedure of obtaining the new TBA equations and Y-system can be justified from a procedure of analytic continuation of the TBA equations, we shift at this point more towards a Y-system perspective, showcasing how one can only operate using Y-functions to derive the TBA equations in any chamber. The essentials of this type of reasoning have already appeared in the discussion done for the first wall-crossing, we just need to remove all the steps and equations that are unessential if one chooses to work mainly with Y-functions.
We separate the two remaining wall-crossing in a numbered list for sake of clarity:

1. **Second wall-crossing ($\gamma-$chamber):** $\Delta\phi_{2,1} < \frac{\pi}{2} \rightarrow \Delta\phi_{2,1} > \frac{\pi}{2}$
   After this wall-crossing, the new cycle $\gamma_{12} = \gamma_1 + \gamma_2$ becomes relevant (see Figure 4.3). We may introduce from the beginning

$$m_{12} = |m_{12}|\, e^{i\phi_{12}} \equiv \Pi^{(0)}_{\gamma_{12}} = \Pi^{(0)}_{\gamma_1} + \Pi^{(0)}_{\gamma_2} = m_1 - im_2 \tag{4.29}$$

[3]Here we did not consider kernels containing phase differences that have already wall-crossed, since picking another residue would just mean reverting a wall-crossing sequence, neither we considered kernels coupling the pseudonergy associated to the new cycle γ_{23} with those of the two cycles γ_2, γ_3 that have generated it, since a wall crossing between the related quantum periods can not happen without having first, or at the same time, another wall-crossing between the quantum periods associated to γ_2, γ_3

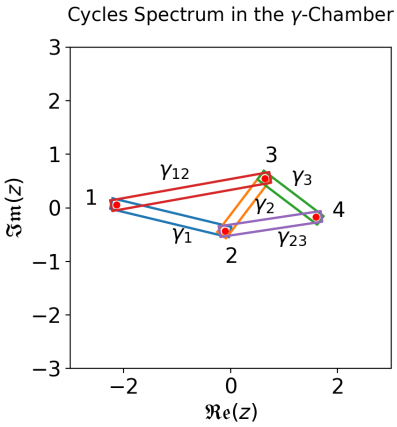


Figure 4.3: Pictorial representation of the relevant cycles for the quartic potential in a particular point of the γ-chamber. The red dots are the turning points of the potential function.

Having in mind the explanation done for the first wall-crossing, this second one implies a redefinition of the two Y-functions Y_1, Y_2 and the need to add a new Y-function Y_{12}, built from these two. The other two Y functions present in the $\beta-$chamber, namely Y_3 and Y_{23}, remain unchanged, since they are not involved in the wall-crossing. The definition of the new form for Y_1, Y_2 is the analog of (4.12), where what was Y_3 there is now Y_1, and is designed so that $\widetilde{\epsilon}_1, \widetilde{\epsilon}_2$ after the wall-crossing have still as asymptotic the modulus of the masses m_1, m_2

$$\begin{aligned} Y_1^\gamma &\equiv \frac{Y_1^\beta}{1+Y_2^{\beta-}} \qquad Y_2^\gamma \equiv \frac{Y_2^\beta}{1+Y_1^{\beta+}}, \\ Y_3^\gamma &\equiv Y_3^\beta \qquad Y_{23}^\gamma \equiv Y_{23}^\beta \end{aligned} \tag{4.30}$$

The new Y-function Y_{12}^γ is defined by requiring the conditions

$$\begin{cases} L_2^\beta(\theta) &= L_2^\gamma(\theta) + L_{12}^\gamma(\theta + i\varphi_>) \\ L_1^\beta(\theta) &= L_1^\gamma(\theta) + L_{12}^\gamma(\theta + i\varphi_<) \end{cases} \tag{4.31}$$

where compatibility among the two require $\varphi_> - \varphi_< = \frac{\pi}{2}$, and we may fix $\varphi_> = \frac{\pi}{2}$ so that we have

$$Y_{12}^\gamma \equiv \frac{Y_1^\beta Y_2^{\beta-}}{1+Y_1^\beta + Y_2^{\beta-}} \tag{4.32}$$

The choice made for $\varphi_>$ imply that the $\theta \to \infty$ asymptotic of the $\widetilde{\epsilon}_{12}^\gamma(\theta) = -\log Y_{12}^\gamma(\theta - i\phi_{12})$ coincides with $|m_{12}|\, e^\theta$.
Having at hand an expression for all the Y-functions of the γ-chamber in terms of

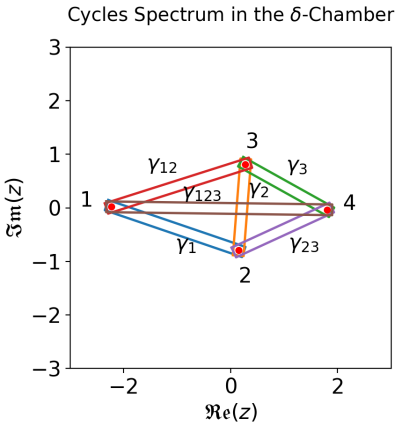


Figure 4.4: Pictorial representation of the relevant cycles for the quartic potential in a particular point of the δ-chamber. The red dots are the turning points of the potential function. Being this chamber the maximal one, one can notice how all possible simple cycles between couples of turning points appear in the spectrum of the relevant cycles.

those of the β-chamber, for which a Y-system is known, we can straightforwardly use that to derive the γ-chamber Y-system, that reads

$$\begin{aligned}
Y_1^{\gamma+}Y_1^{\gamma-} &= \left[1+Y_{23}^{\gamma+}\right]\left[1+Y_2^{\gamma--}\right]^{-1}\left[1+Y_{12}^{\gamma-}\right]^{-1}\\
Y_2^{\gamma+}Y_2^{\gamma-} &= \left[1+Y_3^{\gamma++}\right]^{-1}\left[1+Y_{23}^{\gamma++}\right]^{-1}\left[1+Y_1^{\gamma++}\right]^{-1}\left[1+Y_{12}^{\gamma++}\right]^{-1}\\
Y_3^{\gamma+}Y_3^{\gamma-} &= \left[1+Y_2^{\gamma--}\right]^{-1}\left[1+Y_{12}^{\gamma-}\right]^{-1}\left[1+Y_{23}^{\gamma-}\right]^{-1}\\
Y_{23}^{\gamma+}Y_{23}^{\gamma-} &= \left[1+Y_3^{\gamma+}\right]^{-1}\left[1+Y_2^{\gamma--}\right]^{-1}\left[1+Y_1^{\gamma-}\right]\\
Y_{12}^{\gamma+}Y_{12}^{\gamma-} &= \left[1+Y_3^{\gamma+}\right]^{-1}\left[1+Y_2^{\gamma--}\right]^{-1}\left[1+Y_1^{\gamma+}\right]^{-1}.
\end{aligned} \tag{4.33}$$

From the Y-system, since we know the asymptotics of all the $\widetilde{Y}_a^\gamma$ (they are linked to the modulus of the masses m_a), if we want we can then also obtain the TBA equations for the tilded pseudoenergies $\widetilde{\epsilon}_a^\gamma(\theta) = -\log\widetilde{Y}_a^\gamma(\theta) = -\log Y_a^\gamma(\theta - i\phi_a)$ by using the standard procedure mentioned in section 3.6

$$\begin{aligned}
\widetilde{\epsilon}_1^\gamma &= |m_1|\, e^\theta - K_{2,1} * \widetilde{L}_2^\gamma - K_{23,1}^+ * \widetilde{L}_{23}^\gamma - K_{12,1}^+ * \widetilde{L}_{12}^\gamma\\
\widetilde{\epsilon}_2^\gamma &= |m_2|\, e^\theta - K_{1,2} * \widetilde{L}_1^\gamma - K_{12,2} * \widetilde{L}_{12}^\gamma - K_{3,2} * \widetilde{L}_3^\gamma - K_{23,2} * \widetilde{L}_{23}^\gamma\\
\widetilde{\epsilon}_3^\gamma &= |m_3|\, e^\theta - K_{2,3} * \widetilde{L}_2^\gamma - K_{12,3}^+ * \widetilde{L}_{12}^\gamma - K_{23,3}^+ * \widetilde{L}_{23}^\gamma\\
\widetilde{\epsilon}_{23}^\gamma &= |m_{23}|\, e^\theta - K_{1,23}^- * \widetilde{L}_1^\gamma - K_{2,23} * \widetilde{L}_2^\gamma - K_{3,23}^- * \widetilde{L}_3^\gamma\\
\widetilde{\epsilon}_{12}^\gamma &= |m_{12}|\, e^\theta - K_{1,12}^- * \widetilde{L}_1^\gamma - K_{2,12} * \widetilde{L}_2^\gamma - K_{3,12}^- * \widetilde{L}_3^\gamma,
\end{aligned} \tag{4.34}$$

2. **Third wall-crossing ($\delta-$/maximal chamber):** $\Delta\phi_{23,1} < 0 \to \Delta\phi_{23,1} > 0$
After the wall-crossing, the new cycle $\gamma_{123} = \gamma_1 + \gamma_{23}$ becomes relevant (see Figure 4.4). We introduce as before

$$m_{123} = |m_{123}|\, e^{i\phi_{123}} \equiv \Pi^{(0)}_{\gamma_{123}} = \Pi^{(0)}_{\gamma_1} + \Pi^{(0)}_{\gamma_{23}} = m_1 + m_{23} \tag{4.35}$$

As usual, the wall-crossing imply the redefinition of only the two Y-functions involved, that in this case are Y_1 and Y_{23}, and the creation of a new Y from the two, which we label (with obvious notation) Y_{123}. The fact that in this case the wall-crossing condition is that a phase difference overcomes zero, and not $\frac{\pi}{2}$ as before, imply slightly different formulas for the Y-functions that get redefined (with respect to (4.30), here no Y-function is shifted)

$$\begin{gathered} Y_1^\delta \equiv \frac{Y_1^\gamma}{1+Y_{23}^\gamma} \qquad Y_{23}^\delta \equiv \frac{Y_{23}^\gamma}{1+Y_1^\gamma} \\ Y_2^\delta \equiv Y_2^\gamma \quad Y_3^\delta \equiv Y_3^\gamma \quad Y_{12}^\delta \equiv Y_{12}^\gamma \end{gathered} \tag{4.36}$$

These formulas can be justified as usual by analytically continuing the TBA equations of the previous chamber (the γ one), taking account of the residues that get generated and bringing them on the left hand-side of the equation in which they appear. In this case both the wall-crossing condition on the phases, that involves overcoming 0 and not $\frac{\pi}{2}$, and the formulas of $Y_1^\delta, Y_{23}^\delta$, which do not contain shifts, are ultimately due to the presence of a shift of $\pm\frac{\pi}{2}$ in the kernels $K^+_{23,1}, K^-_{1,23}$ that generate the residues. To find the expression of the new Y-function Y^δ_{123} we require the usual conditions

$$\begin{cases} L^\gamma_{23}(\theta) &= L^\delta_{23}(\theta) + L^\delta_{123}(\theta + i\varphi_>) \\ L^\gamma_1(\theta) &= L^\delta_1(\theta) + L^\delta_{123}(\theta + i\varphi_<) \end{cases} \tag{4.37}$$

Since the definitions (4.36) do not contain shifts, in this case the compatibility condition of these two equations implies $\varphi_> - \varphi_< = 0$, and we may fix $\varphi_> = 0$ so that we have

$$Y^\delta_{123} \equiv \frac{Y^\gamma_1 Y^\gamma_{23}}{1 + Y^\gamma_1 + Y^\gamma_{23}} \tag{4.38}$$

With this choice for $\varphi_>$, we have, as desired, that the $\theta \to \infty$ asymptotic of the $\widetilde{\epsilon}^\gamma_{123}(\theta) = -\log Y^\gamma_{123}(\theta - i\phi_{123})$ coincides with $|m_{123}|\, e^\theta$. As for the previous wall-crossing, the Y-system is then derived by using the expression of the Y^δ_a functions

in terms of the Y_a^γ ones, for which we have the Y-system (4.33)

$$\begin{aligned}
Y_1^{\delta+}Y_1^{\delta-} &= \left[1+Y_2^{\delta--}\right]^{-1}\left[1+Y_{23}^{\delta-}\right]^{-1}\left[1+Y_{12}^{\delta-}\right]^{-1}\left[1+Y_{123}^{\delta-}\right]^{-1}\\
Y_2^{\delta+}Y_2^{\delta-} &= \left[1+Y_1^{\delta++}\right]^{-1}\left[1+Y_{23}^{\delta++}\right]^{-1}\left[1+Y_{12}^{\delta++}\right]^{-1}\left[1+Y_3^{\delta++}\right]\left[1+Y_{123}^{\delta++}\right]^{-2}\\
Y_3^{\delta+}Y_3^{\delta-} &= \left[1+Y_2^{\delta--}\right]^{-1}\left[1+Y_{23}^{\delta-}\right]^{-1}\left[1+Y_{12}^{\delta-}\right]^{-1}\left[1+Y_{123}^{\delta-}\right]^{-1}\\
Y_{12}^{\delta+}Y_{12}^{\delta-} &= \left[1+Y_1^{\delta+}\right]^{-1}\left[1+Y_2^{\delta--}\right]^{-1}\left[1+Y_3^{\delta+}\right]^{-1}\left[1+Y_{123}^{\delta+}\right]^{-1}\\
Y_{23}^{\delta+}Y_{23}^{\delta-} &= \left[1+Y_1^{\delta+}\right]^{-1}\left[1+Y_2^{\delta--}\right]^{-1}\left[1+Y_3^{\delta+}\right]^{-1}\left[1+Y_{123}^{\delta+}\right]^{-1}\\
Y_{123}^{\delta+}Y_{123}^{\delta-} &= \left[1+Y_2^{\delta--}\right]^{-2}\left[1+Y_{23}^{\delta-}\right]^{-1}\left[1+Y_1^{\delta+}\right]^{-1}\left[1+Y_3^{\delta+}\right]^{-1}\left[1+Y_{12}^{\delta-}\right]^{-1}
\end{aligned} \tag{4.39}$$

Rewriting this Y-system for the tilded $\widetilde{Y}_a^\delta$, whose related $\widetilde{\epsilon}_a^\delta$ have as $\theta\to\infty$ asymptotic the $|m_a|$, we can derive the TBA equations

$$\begin{aligned}
\widetilde{\epsilon}_1^\delta &= |m_1|\, e^\theta - K_{2,1}*\widetilde{L}_2^\delta - K_{23,1}^+*\widetilde{L}_{23}^\delta - K_{12,1}^+*\widetilde{L}_{12}^\delta - K_{123,1}^+*\widetilde{L}_{123}^\delta\\
\widetilde{\epsilon}_2^\delta &= |m_2|\, e^\theta - K_{1,2}*\widetilde{L}_1^\delta - 2K_{123,2}*\widetilde{L}_{123}^\delta - K_{12,2}*\widetilde{L}_{12}^\delta - K_{3,2}*\widetilde{L}_3^\delta - K_{23,2}*\widetilde{L}_{23}^\delta\\
\widetilde{\epsilon}_3^\delta &= |m_3|\, e^\theta - K_{2,3}*\widetilde{L}_2^\delta - K_{12,3}^+*\widetilde{L}_{12}^\delta - K_{23,3}^+*\widetilde{L}_{23}^\delta - K_{123,3}^+*\widetilde{L}_{123}^\delta\\
\widetilde{\epsilon}_{12}^\delta &= |m_{12}|\, e^\theta - K_{1,12}^-*\widetilde{L}_1^\delta - K_{2,12}*\widetilde{L}_2^\delta - K_{3,12}^-*\widetilde{L}_3^\delta - K_{123,12}^-*\widetilde{L}_{123}^\delta\\
\widetilde{\epsilon}_{23}^\delta &= |m_{23}|\, e^\theta - K_{1,23}^-*\widetilde{L}_1^\delta - K_{2,23}*\widetilde{L}_2^\delta - K_{3,23}^-*\widetilde{L}_3^\delta - K_{123,23}^-*\widetilde{L}_{123}^\delta\\
\widetilde{\epsilon}_{123}^\delta &= |m_{123}|\, e^\theta - K_{1,123}^-*\widetilde{L}_1^\delta - 2K_{2,123}*\widetilde{L}_2^\delta - K_{23,123}^+*\widetilde{L}_{23}^\delta - K_{12,123}^+*\widetilde{L}_{12}^\delta - K_{3,123}^-*\widetilde{L}_3^\delta
\end{aligned} \tag{4.40}$$

As a check that our derivation procedure is consistent, one may compare these TBA equations with those appearing in [1], finding a perfect agreement.

An explicit example of path Γ in the moduli space that realises a wall-crossing sequence like the one described is given in [18]: we may start from the quartic potential

$$V(z) = z^4 - \frac{7}{2}z^2 + 2z + \frac{1}{5} \tag{4.41}$$

and a value for the energy of $E=0$, which satisfies the conditions of section 3.5.1, thus has a minimal chamber's form of the TBA, and then move across the chambers in $\mathcal{M}_0$ by only modifying the energy along the complex trajectory

$$\Gamma_E = \{E\in\mathbb{C}\,|\,1.5 - 1.5\,e^{i\varphi}, \varphi\in[0,\pi]\} \tag{4.42}$$

which is shown in Figure 4.5. From a physical standpoint, here the minimal and the maximal chamber correspond respectively to an energy level lying inside or above the double (asymmetric) well created by the potential, and the use of a complex path in the energy to move between these two situations is motivated by the need to avoid the coalescence of turning points that is met when the energy level is exactly at the top of

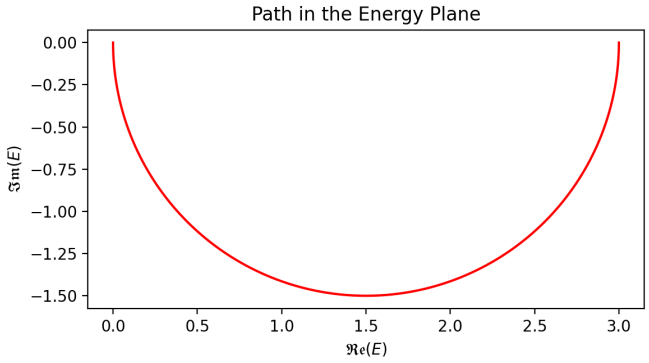


Figure 4.5: Representation in the complex energy plane of the path Γ_E used to perform the successive wall-crossings from the minimal to the maximal chamber in the explicit example of quartic potential (4.41). The picture is borrowed from [18].

the two wells.
As one may expect, all the paths connecting the initial point in the minimal chamber to a given point $\hat{P}$ in the moduli space $\mathcal{M}_0$ lead, through the procedure described in this section, to equivalent TBA equations and Y-systems, even if the paths move through different chambers to reach $\hat{P}$, and so correspond to a different wall-crossing sequence. To conclude, we now collect some relevant observations concerning the results obtained:

- Focusing on the Y-system in different chambers, one can notice how, at each chamber, some blocks of the form $[1+Y]^n$ that appear on the right hand-side of the equations get inverted. Starting from the minimal chamber, where these blocks have all $n = 1$, one arrives to the maximal one, where instead all have a negative power n (here also blocks with $n = -2$ are present, reflecting the fact that cycles may intersect more than once among them; recall that these formulas can be derived also directly via the DDP formula). What's more, in the intermediate chambers, one can notice how the blocks that still have not been inverted (i.e. have $n > 0$), hint at which are the Y-functions that have not yet wall-crossed, in the sense that if a block $[1+Y_a]^n, n > 0$ appears in the equation for a Y-function Y_b, then a wall-crossing involving the resummed quantum periods Π_a, Π_b has to be done to proceed towards the maximal chamber.

- The fact, mentioned in the previous point, that in the maximal chamber all blocks $[1+Y]^n$ appearing in the Y-system are inverted, is directly related to an important property, which holds for those Y-functions that are present from the minimal chamber (so the "fundamental" ones). We state this property, that we may call *shift-inverse rule*, in the following lemma

Lemma 2 (Shift-inverse rule)**.** *If the evolution of phase differences in a wall-crossings sequence from the minimal to the maximal chamber is such that the phases of the masses associated to the classical periods that are present since the minimal chamber respects the conditions* $\phi_i - \phi_{i\pm1} < \pi/2 \to \phi_i - \phi_{i\pm1} > \pi/2$ *(i.e. these periods either all align or anti-align, considering the whole wall-crossing procedure), then the related Y-functions respect the rule*

$$Y_a^{\text{chamber}}(\theta) = [Y_a^{\text{chamber}}(\theta \pm i\pi)]^{-1} \tag{4.43}$$

which connects the Y-functions associated to the cycles in the minimal chamber to those of the maximal chamber. Here the formula shall be used with the $+$ $(-)$ *if the phase* ϕ_a *is greater (smaller) than the neighbouring ones* $\phi_{a\pm1}$*.*

In the full-developed case of the quartic potential presented in this section (which satisfies the conditions of the Lemma), one may check that this property indeed holds, by using the expressions that relate the Y-functions chamber by chamber. For example, looking at the Y_2 function we can write

$$Y_2^{\delta} = Y_2^{\gamma} = \frac{Y_2^{\beta}}{1 + Y_1^{\beta+}} = \frac{Y_2^{\alpha}}{[1 + Y_3^{\alpha+}][1 + Y_1^{\alpha+}]} = [Y_2^{\alpha++}]^{-1} \tag{4.44}$$

where in the last step we used the Y-system of the α-chamber.
The shift-inverse property can also be justified by using the TBA equations as follows. Consider as example still the quantum cycle Π_2 and start from the TBA equations (4.5) of the minimal chamber. The one for Π_2 reads, with the kernels functions explicitly written, as

$$\widetilde{\epsilon}_2^{\alpha}(\theta) = |m_2|e^{\theta} - \int_{\mathbb{R}} \frac{d\theta'}{2\pi} \frac{\widetilde{L}_1^{\alpha}(\theta')}{\cosh(\theta - \theta' + i\Delta\phi_{1,2})} - \int_{\mathbb{R}} \frac{d\theta'}{2\pi} \frac{\widetilde{L}_3^{\alpha}(\theta')}{\cosh(\theta - \theta' + i\Delta\phi_{3,2})} \tag{4.45}$$

We perform a smooth shift $\theta \to \theta + i\pi$, modifying, in both integrals, the integration contour so that avoids being hit by the first (by distance wrt the real axis) pole of the kernel coming from below the real axis (the only that can cause problems during this procedure), whose position will become

$$\begin{aligned} \theta'_p &= \theta + i(\Delta\phi_{1,2} - \frac{\pi}{2}) + i\pi, \quad \text{for the first integral} \\ \theta'_p &= \theta + i(\Delta\phi_{3,2} - \frac{\pi}{2}) + i\pi, \quad \text{for the second integral} \end{aligned} \tag{4.46}$$

As we move then in the moduli space into the maximal chamber, the position of these poles will come back below the real axis, since the phase differences will evolve as

$$\Delta\phi_{1,2} > -\frac{\pi}{2} \to \Delta\phi_{1,2} < -\frac{\pi}{2}, \qquad \Delta\phi_{1,3} > -\frac{\pi}{2} \to \Delta\phi_{1,3} < -\frac{\pi}{2} \tag{4.47}$$

and we can restore the integration paths to run along the real axis. We see how *the initial shift in θ compensates thus for the action of the analytic continuation in the moduli space*. We may then use repeatedly the formulas (4.15, 4.31, 4.37) to write the L^α functions present on the right hand-side of the TBA equations in terms of δ-chambers ones as

$$\begin{aligned} L_3^\alpha(\theta) &= L_3^\delta(\theta) + L_{23}^\delta(\theta) + L_{123}^\delta(\theta) \\ L_1^\alpha(\theta) &= L_2^\delta(\theta) + L_1^\delta(\theta) + L_{123}^\delta(\theta) \end{aligned} \tag{4.48}$$

The TBA equation we obtain for $\widetilde{\epsilon}_2^\alpha(\theta + i\pi)$ is the same of that of $\widetilde{\epsilon}_2^\delta(\theta)$, a part for the kernel and the driving terms which are evaluated in $\theta + i\pi$ instead that in θ. This difference can be easily removed by considering the antisymmetry of these terms under a shift of $i\pi$, which produces a minus sign, that, brought on the left hand-side, allows to establish the equality

$$\widetilde{\epsilon}_2^\delta(\theta) = -\widetilde{\epsilon}_2^\alpha(\theta + i\pi) \tag{4.49}$$

which indeed translates to the inverse-shift rule for the related Y-functions.

- We may compare the idea of analytically continuing the TBA equations used in this section with the one P.Dorey and R.Tateo used in [23] to obtain excited state energies of QFTs, which we have briefly described in section 1.4. The concept is similar but with some important differences: in that case the authors performed analytic continuation along closed trajectories in the complex plane of the r parameter, and kept explicit in the TBA equations the residues coming from the convolution integrals without defining new ϵ_a functions, interpreting the different asymptotic form that these residues implied for the ϵ_a as describing excited states solutions for the pseudoenergies. In our case, the path of analytic continuation was done in the moduli space $\mathcal{M}_0$ (despite it reflected in the TBA equations through a modification of the masses, which played the role of the r parameter), along trajectories Γ that were not closed, and the residues generated in the TBA equations by this analytic continuation process were brought on the left hand-side of the equations to define new ϵ functions, since we were, at the contrary of Dorey and Tateo, interested to keep the asymptotics invariant, and always associated to classical cycles. This explains why the authors worked only with a single Y-system, while in our case we have many of them, one for each chamber.

4.2 The general algorithm using Y functions

Building on the experience shown in the previous section for the specific case of the quartic potential, we now present the general form of the algorithmic procedure to find, only through a Y-system analysis, the TBA equations for the relevant resummed quantum periods associated to any polynomial potential with distinct turning points.

Setup of the procedure

Consider a given polynomial potential $V(z)$ of degree $r+1, r>1$ and an energy level E. The parameters of the potential and the energy represent a point in the moduli space $\mathcal{M}_0$, which we may call $\hat{P}$. Build an auxiliary polynomial potential $V_0(z)$ of the same degree, and a related energy level E_0, such that the conditions considered in section 3.5.1 are satisfied, namely:

1. all the turning points are real
2. the sign of the highest order term in the polynomial is such that interval between the two leftmost turning points corresponds to a classically allowed interval

and label the turning points z_i, $i=1,\dots,r+1$ so that $z_1<z_2<\cdots<z_{r+1}$. As is for the case of the couple $(V(z),E)$, also to $(V_0(z),E_0)$ will correspond a point in the moduli space, which we call P_0. From the analysis done in section 3.5.1 we know that, among all possible cycles encircling a couple of turning points, the relevant ones in P_0 are those surrounding classically allowed and forbidden intervals, which we label as γ_a, $a=1,\dots,r$, where γ_a is taken to encircle the interval $[z_a,z_{a+1}]$ along the real axis. The orientation of these cycles, together with the choice of the branch cuts for the function $\sqrt{V(z)-E}$ defining the classical periods, are fixed by requiring that the masses associated to the γ_a (defined as in (3.59)) are real and positive, and the intersection number between two cycles γ_a,γ_b is $(\gamma_a,\gamma_b)=(-1)^a$.
Determine then a path Γ in $\mathcal{M}_0$ that connects P_0 with $\hat{P}$, chosen so that all the wall-crossing along this path happen separately, i.e. cycles that were not relevant start to become relevant one at a time. To keep track of the cycles that are relevant one may discretize the motion along Γ in small steps, and study the different types of topology (varying θ) of the Stokes graphs G_θ associated to each point of this discrete evolution in $\mathcal{M}_0$: by doing a π wide scan in θ of G_θ, the relevant cycles γ are those that during the scan happen to encircle a saddle trajectory, at a particular value of θ.
Then, the algorithm to determine the Y-system, and from that the TBA equations, in $\hat{P}$, consists first in an analysis of the evolution of the classical periods along the discretized path Γ, which is to be done via a simple numerical analysis, and next in an analytical part, that uses the information on the wall-crossing sequence coming from the numerics to effectively determine the Y-functions in $\hat{P}$. Both parts of the algorithm are to be implemented in a recursive way.

Numerical part

Starting from P_0, where the set of relevant cycles is known, and then at each step along Γ, check which are the relevant cycles and compute the classical periods on them. If there is a cycle at a certain step i which was not relevant at the step before $i-1$, this signals that a wall-crossing has happened in between the steps. As mentioned in the remark

4.1, the wall crossing is due to the alignment or anti-alignment of two of the classical periods Π_{γ_a}, Π_{γ_b} that are relevant at the step $i-1$, i.e. we might have either $\Delta\phi_{a,b}$ cross the value $\frac{\pi}{2}$ or 0, where ϕ_a is the phase of the complex mass associated to the cycle Π_{γ_a}, which is defined as in (3.59) for the cycles present from the minimal chamber, and as the classical period itself for all the new cycles that become relevant during the wall crossing procedure. If two periods align, then the new relevant classical period will coincide (up to orientation) with their sum, while if they anti-align it will be their difference. The information that has to be produced as output of the numerical analysis is the sequence of critical values $\{\Delta\phi_{a_i,b_i}\}$ that produce the different wall-crossing encountered along Γ, and, for each of these values, also the relation between the mass $m_{a_ib_i}$ of the new relevant cycle in terms of those m_{a_i}, m_{b_i} of the cycles that have generated it.

Analytical part

Similarly of what we have done for the case of the quartic potential, we may label the different chambers encountered along Γ with the symbol α_i, where i starts from 0, corresponding to the minimal chamber, and at each wall-crossing is increased by one unity, to arrive at a final value $i = n$ correponding to the chamber where $\hat{P}$ lies. After the numerical analysis has been performed, one can obtain the Y-functions and their Y-system in $\hat{P}$ by starting from the minimal chambers Y-functions $Y_a^{\alpha_0}$, whose Y-system (3.73) is known, and then applying, for each step i in the ordered wall crossing sequence $\{\Delta\phi_{a_i,b_i}\}$ (which connects the chamber α_{i-1} to the α_i one), the following procedure

1. **Find the definition of the $Y_a^{\alpha_i}$ functions**
   All the Y-functions that are not involved in the wall-crossing remain invariant, i.e. $Y_a^{\alpha_i} = Y_a^{\alpha_{i-1}}$, $a \neq a_i, b_i$. The two Y-functions Y_{a_i}, Y_{b_i} involved in the wall-crossing, and the new one $Y_{a_ib_i}$ that need to be considered (associated to the new cycle that become relevant), are instead given by

   - If $\Delta\phi_{a_i,b_i} > \frac{\pi}{2}$, then

$$Y_{b_i}^{\alpha_i} \equiv \frac{Y_{b_i}^{\alpha_{i-1}}}{1 + Y_{a_i}^{\alpha_{i-1}+}}, \qquad Y_{a_i}^{\alpha_i} \equiv \frac{Y_{a_i}^{\alpha_{i-1}}}{1 + Y_{b_i}^{\alpha_{i-1}-}} \tag{4.50}$$

$$Y_{a_ib_i}^{\alpha_i}(\theta + i\varphi) \equiv \frac{Y_{a_i}^{\alpha_{i-1}} Y_{b_i}^{\alpha_{i-1}-}}{1 + Y_{a_i}^{\alpha_{i-1}} + Y_{b_i}^{\alpha_{i-1}-}} \tag{4.51}$$

     where the phase φ in the new Y-function is determined from matching $m_{a_i} - im_{b_i}$ with $m_{a_ib_i}e^{i\varphi}$, where $m_{a_ib_i} = |m_{a_ib_i}|e^{i\phi_{a_ib_i}}$ is the mass coming from the numerical analysis, so that we have the desired $\theta \to \infty$ asymptotic

$$\widetilde{\epsilon}_{a_ib_i}^{\alpha_i}(\theta) = -\log \widetilde{Y}_{a_ib_i}^{\alpha_i}(\theta) \sim |m_{a_ib_i}|\, e^{\theta} \tag{4.52}$$

as holds similarly for all the other $\widetilde{\epsilon}_a^{\alpha_i}$ functions.
Here, the expressions for $Y_{a_i}^{\alpha_i}, Y_{b_i}^{\alpha_i}$ stem from the analytic continuation of the TBA equations of the α_{i-1} chamber, while the formula for $Y_{a_i b_i}^{\alpha_i}$ comes from requiring the ansatz

$$\begin{cases} [1+Y_{a_i}^{\alpha_{i-1}}(\theta)] &= [1+Y_{b_i}^{\alpha_i}(\theta)][1+Y_{a_i b_i}^{\alpha_i}(\theta+i\varphi)] \\ [1+Y_{b_i}^{\alpha_{i-1}}(\theta)] &= [1+Y_{a_i}^{\alpha_i}(\theta)][1+Y_{a_i b_i}^{\alpha_i}(\theta)] \end{cases} \tag{4.53}$$

- If $\Delta\phi_{a_i,b_i} > 0$, then

$$Y_{b_i}^{\alpha_i} \equiv \frac{Y_{b_i}^{\alpha_{i-1}}}{1+Y_{a_i}^{\alpha_{i-1}}}, \qquad Y_{a_i}^{\alpha_i} \equiv \frac{Y_{a_i}^{\alpha_{i-1}}}{1+Y_{b_i}^{\alpha_{i-1}}} \tag{4.54}$$

$$Y_{a_i b_i}^{\alpha_i}(\theta) \equiv \frac{Y_{a_i}^{\alpha_{i-1}} Y_{b_i}^{\alpha_{i-1}}}{1+Y_{a_i}^{\alpha_{i-1}}+Y_{b_i}^{\alpha_{i-1}}} \tag{4.55}$$

which differ from the previous case by the absence of relative shifts between Y-functions appearing in the formulas. As before, the the expressions for $Y_{a_i}^{\alpha_i}, Y_{b_i}^{\alpha_i}$ are found from the study of the analytic continuation of the TBA equations of the α_{i-1} chamber, while the formula for $Y_{a_i b_i}^{\alpha_i}$ comes from the ansatz

$$\begin{cases} [1+Y_{a_i}^{\alpha_{i-1}}(\theta)] &= [1+Y_{b_i}^{\alpha_i}(\theta)][1+Y_{a_i b_i}^{\alpha_i}(\theta)] \\ [1+Y_{b_i}^{\alpha_{i-1}}(\theta)] &= [1+Y_{a_i}^{\alpha_i}(\theta)][1+Y_{a_i b_i}^{\alpha_i}(\theta)] \end{cases} \tag{4.56}$$

2. **Derive the Y system for the $Y_a^{\alpha_i}$**
From the definitions of the $Y_a^{\alpha_i}$ in terms of the $Y_a^{\alpha_{i-1}}$, using the Y-system of the latter, and also exploiting the property (4.53) or (4.56) to simplify the calculations, the Y-system for all the $Y_a^{\alpha_i}$ functions, including the new one, may be derived.

Once the type of analysis above has been repeated for all wall-crossings conditions $\{\Delta\phi_{a_i,b_i}\}$, and thus we have obtained the Y-system in $\hat{P}$, we may derive from it the TBA equations for the shifted $\widetilde{\epsilon}_a^{\alpha_n}(\theta) = -\log Y_a^{\alpha_n}(\theta - i\phi_a)$, so that these functions, and the related all-order resummed quantum periods, can be numerically calculated. The procedure to obtain the TBA equations from the Y-system is a standard one, that involves first shifting the Y-system's equations for the $Y_a^{\alpha_n}$, and then convolute them with the kernel $K = \frac{1}{2\pi}\frac{1}{\cosh(\theta)}$, taking advantage of the knowledge of the asymptotics $\widetilde{\epsilon}_a^{\alpha_n}(\theta) \sim |m_a|\, e^{\theta}$.

Use of shift-inverse rule
In the particularly relevant case in which the point $\hat{P}$ lies in a maximal chamber, that means that the number of relevant cycles in $\hat{P}$ is equal to the maximum possible value of $\frac{r(r+1)}{2}$, and the wall crossing conditions $\{\Delta\phi_{a_i,b_i}\}$ associated to the path Γ chosen satisfy

the conditions stated in Lemma 2, then one may apply the shift-inverse rule. The rule allows to obtain directly the maximal chamber's form of the Y-functions associated to the "fundamental" cycles Π_{γ_a} that are relevant since from the minimal chamber, as (refer to the Lemma for the explanation)

$$Y_a^{\text{chamber}}(\theta) = [Y_a^{\text{chamber}}(\theta \pm i\pi)]^{-1} \tag{4.57}$$

so without needing to evolve the Y_a functions in all the intermediate steps. However, the analysis of the intermediate steps is still needed for the introduction of the new "composite" Y-functions, arising during the wall-crossing procedure.

We conclude with a final remark regarding a comparison between this version of the algorithm and the one presented in the twin work [18]. In the latter, the author used an additional construction when performing the analysis of Stokes graphs, that involves building a so-called *WKB triangulation* of the Riemann surface of the z coordinate. The motivation in [18] of this construction was twofold: on one hand, it allowed to see the wall-crossing procedure under a different perspective, that of mutations of the WKB triangulation (which can be linked to seed mutations of cluster algebras, as investigated in [47]), and on the other, the author used it to keep track of the relevant periods during the evolution in moduli space. Despite its conceptual utility, we would like to stress here that, in a minimal formulation of the algorithm, such a procedure is not necessary (and we have thus decided not to include it), since the set of relevant classical periods, including the ones that become relevant during the wall-crossing, can be determined by solely looking at the Stokes graph.

Chapter 5

Going beyond the polynomial case: the modified Mathieu equation

In an effort to extend the procedure developed in Chapter 4 to a wider class of potentials, we also investigated the so called *modified Mathieu equation*, which is a Schrödinger-type equation with periodic potential, of the form

$$\left(-\frac{d^2}{dy^2} + 2e^{2\theta}\cosh y + P^2\right)\psi(y) = 0 \tag{5.1}$$

The main interest in investigating this type of equation is a connection with a quantisation/deformation of the 4d $\mathcal{N} = 2$ supersymmetric $SU(2)$ pure gauge theory [19], whose undeformed version has been investigated by Seiberg and Witten in the seminal paper [20]. Similarly to the Seiberg-Witten theory (SW), the low energy effective version of the deformed theory can be expressed through an holomorphic prepotential $\mathcal{F}_{\mathcal{NS}}$, which in turn may be derived from a couple of cycles a, a_D that are closely related to the modified Mathieu equation (5.1) [21]. The two are in fact defined from the following differential equation, known as *Mathieu equation* (here $\Lambda \in \mathbb{R}$, while we consider $\hbar, u \in \mathbb{C}$)

$$\left(-\frac{\hbar^2}{2}\frac{d^2}{dz^2} + \Lambda^2\cos z - u\right)\psi(z) = 0 \tag{5.2}$$

as integrals of the quantum SW differential $\mathcal{P}(z) = -i\frac{d}{dz}\ln\psi(z)$

$$a(\hbar, u, \Lambda) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{P}(z;\hbar,u,\Lambda)dz, \quad a_D(\hbar, u, \Lambda) = \frac{1}{2\pi}\int_{-\arccos(u/\Lambda^2)-i0}^{\arccos\left(u/\Lambda^2\right)-i0}\mathcal{P}(z;\hbar,u,\Lambda)dz \tag{5.3}$$

and the above differential equation (5.2) easily maps into (5.1) by a change of variables $z \to -iy - \pi$ and a redefinition of the parameters as (here we use the notational convention of [21])

$$e^{\theta} = \frac{\Lambda}{\hbar} \qquad P^2 = \frac{u}{\hbar^2} \tag{5.4}$$

In a WKB fashion (recall the discussion in Chapter 3), the cycles a, a_D can be expanded in an in an asymptotic series in $\hbar$ around $\hbar = 0$ via an expansion of the SW differential contained in their definition $\mathcal{P}(z) = \sum_{n=-1}^{\infty} \hbar^n \mathcal{P}_n(z)$, implying the following modes for the cycles

$$a^{(n)}(u,\Lambda) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{P}_{2n-1}(z;u,\Lambda)dz\,, \quad a_D^{(n)}(u,\Lambda) = \frac{1}{2\pi}\int_{-\arccos(u/\Lambda^2)-i0}^{\arccos\left(u/\Lambda^2\right)-i0} \mathcal{P}_{2n-1}(z;u,\Lambda)dz \tag{5.5}$$

The lowest order term in these expansions (i.e. their classical limit) coincide with the cycles of the (undeformed) SW theory, for which we have an explicit expression in terms of the hypergeometric function ${}_2F_1$

$$\begin{aligned} a^{(0)}(u,\Lambda) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sqrt{2u - 2\Lambda^2\cos z}dz = \Lambda\sqrt{2\,(u/\Lambda^2+1)}\,{}_2F_1\left(-\frac{1}{2},\frac{1}{2},1;\frac{2}{1+u/\Lambda^2}\right) \\ a_D^{(0)}(u,\Lambda) &= \frac{1}{2\pi}\int_{-\arccos(u/\Lambda^2)-i0}^{\arccos\left(u/\Lambda^2\right)-i0} \sqrt{2u - 2\Lambda^2\cos z}dz = -i\Lambda\frac{(u/\Lambda^2-1)}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2},2;\frac{1-u/\Lambda^2}{2}\right) \end{aligned} \tag{5.6}$$

As is common when performing WKB expansions, this representations of a, a_D as asymptotic series in $\hbar$ usually diverges with a factorial growth, and one may perform Borel resummation to obtain from it convergent functions. Under the perspective of the exact WKB analysis, the asymptotic series representation of a, a_D are examples of quantum cycles on the Riemann surface Σ of the 1-form $\sqrt{2\Lambda^2\cos z - 2u}\,dz$ associated to (5.2), and one may try to use the resurgent techniques explained in Section 3 to obtain TBA equations that allow to compute the resummed version of these series.

On top of this, there is a correspondence [22] between what we have called *relevant* resummed quantum periods built on cycles of $H_1(\Sigma)$ and BPS states of SW theory: the charges Z of BPS states are in 1-1 correspondence with the values of $\arg(\hbar) = -\operatorname{Im}\theta$ where saddle trajectories appear in the Stokes graph associated to the differential equation (5.2) (generating discontinuities for the periods that cross the saddle), and we know from the theory of resurgence (recall section 3.5) that each relevant quantum period is associated to a saddle trajectory. The discontinuity structure of resummed quantum periods then tells us also the BPS spectrum of the theory.

As it is well known, the structure of the SW spectrum is strongly different depending on the point of the complex u plane considered (which plays the role of moduli space). In particular, the plane is separated into two regions by a *curve of marginal stability* [20] (a wall, in the language of Chapter 4), that intersects the real axis at the points $\pm\Lambda^2$. Expressing the central charge of the BPS states (related to their mass as $m^2 = 2|Z|^2$ [58]) via a charge's vector $\boldsymbol{\gamma}$, so that $Z_\gamma = (a^{(0)}, a_D^{(0)})\cdot\boldsymbol{\gamma}$, we can give the following description of the SW spectrum in the two chambers [58]:

- Inside the curve of marginal stability, in the so-called *strong coupling region*, the

spectrum consists of one *monopole* with charge $\boldsymbol{\gamma}_m = (0, 1)$ and one *dyon* with charge $\boldsymbol{\gamma}_m = (1, \pm 1)$ (together with their antiparticles, that carry opposite charges)

- Outside the curve of marginal stability, in the *weak coupling region*, there are instead an infinite tower of *dyons* with charges $\pm\boldsymbol{\gamma}_n$, $n \in \mathbb{Z}, n \neq 0$, where $\boldsymbol{\gamma}_n = (n, 1)$, the *monopole* $\pm\boldsymbol{\gamma}_m$, and the *W-bosons* with charge $\pm\boldsymbol{\gamma}_e$, where $\pm\gamma_e = (1, 0)$

Since the spectrum at strong coupling involves only two cycles, whereas at weak coupling an infinite number of them, we will also refer to the strong coupling region as *minimal chamber* and to the the weak coupling one as *maximal chamber*.

For the minimal chamber of the Mathieu equation (5.2), an exact WKB analysis similar to the one presented in section 3.5.1 for the case of a polynomial potential has been performed in [59], where the author extracted the TBA equations for the relevant quantum periods. However, the analytic continuation to the maximal chamber was not presented, since mentioned to be not expressible in terms of TBA equations. On the other hand, the authors of [22] had already written TBA equations in both the minimal and maximal chambers for quantities they believed associated to resummed quantum periods, supporting their conjectures with a wide use of numerical analysis. These equations were built on previous results [60] by D.Gaiotto, which in turn derived from taking a "conformal limit" of other TBA-like equations obtained in [61, 62] for a class of 4d gauge theories compactified on a circle. The kind of approach used in these papers was not that of exact WKB, involving Borel resummations of series expansions, rather was based on the construction of a set of coordinates $\mathcal{X}_\gamma$, indexed by a charge vector $\boldsymbol{\gamma}$, defined on the space of parameters of the theory. Nonetheless, these coordinates are built to be piecewise functions of $(\arg(\hbar), u)$, and jump along their discontinuities according to transformations $\mathcal{K}$ that are very similar to those we have seen in Chapter 4 for the $Y_a^{\alpha_i}$ functions built as resummed quantum periods. A "bridge" between this kind of approach and the exact WKB one in its traditional version is discussed in [63], where the authors construct, in this case for the maximal chamber, coordinates $\mathcal{X}_i$ out of wronskians of WKB solutions, and identify them with resummed cycles.
To conclude this brief (and not exhaustive) excursus on lines of research on the integrable structure of the cycle spectrum of the Mathieu equation (and its relation with SW theory), we would like to mention that the ODE-IM correspondence makes available another way to tackle this problem, without stepping into the Borel resummation procedure. In [21], the authors show how to derive from the solutions of the differential equation (5.1) a Q system, the TQ relation, and identified the quantities entering in these equations as directly related to the deformed SW cycles a, a_D. In particular, they identified the T function as directly related to the a cycle via

$$T(\theta, P^2) = T(\hbar, u, \Lambda) = 2\cosh\left(2\pi a(\hbar, u, \Lambda)\right) \tag{5.7}$$

while, in the minimal chamber, they showed how the Q-function, which in this case is the square root of Y, coincides with the exponentiated a_D cycle

$$Y(\theta, u) = Q^2(\theta, u) = e^{2\pi a_D(\hbar(\theta), u, \Lambda)} \tag{5.8}$$

obtaining TBA equations in the minimal chamber for the a_D cycle, when evaluated in u and $-u$ respectively.

Moving onto the present work, our efforts were focused on trying to apply (and perhaps adapt) to the case of the modified Mathieu equation the Y-system algorithmic procedure, founded on the analytic continuation of TBA equations, which we developed for the case of a Schrödinger equation with polynomial potential. The goal was to start with the TBA equations for the resummed quantum periods in the minimal chamber, which contains only two type of particles, thus ϵ_a functions, and then apply a procedure similar to the one shown in Chapter 4 to perform the wall-crossing to the maximal chamber; in particular we aimed to find an expression for the new ϵ_a functions that arise, associated to novel resummed quantum cycles that become relevant (interpreted as new BPS states in the theory), in terms of the ones we already had in the minimal chamber. This would mean, in a sense, "generating" the BPS spectrum at weak coupling starting from the strong coupling one, a perspective which works in the opposite direction of the physical decay of BPS states that happens when traversing the wall of marginal stability from the weak to the strong coupling region [20]. The setup of the problem in the minimal chamber is summarised in the first of the two sections that follow, where the TBA equations for the relevant quantum periods at strong coupling are stated.
The second section is instead devoted to the analysis we performed on using the the Y-system procedure in the wall-crossing. When studying the feasibility of this type of procedure, we were faced with important complications, compared to the case of the polynomial potential. These problems are mainly due to a simpler chamber structure of the moduli space of the modified Mathieu equation, which contains only one parameter u to be tuned, with respect to the more rich structure of the moduli space of a polynomial potential, that has instead a number of parameters equal to its degree, and also to the emergence of an infinite number of relevant quantum periods at weak coupling due to the entering in the picture of the infinite tower of dyons. Faced with these new challenges, we tried to engineer new solutions, and proposed two ways to tackle the problem. Some tentative work on these directions has been performed, whose outcomes we describe, but the picture is still to be clarified; the reader should regard this final line of research of the present thesis project as still open to investigations.

5.1 TBA equations in the minimal chamber

As mentioned in the introduction to this chapter, the spectrum of relevant resummed quantum cycles for the modified Mathieu equation in the minimal chamber consists in two type of cycles, which may be identified with the monopole and dyon BPS states [22, 59]. To see this through the application of the exact WKB method, as usual one shall first of all consider the arrangement of turning points of the potential function in the point of moduli space considered. Of the many related variants of the Mathieu equations, we choose here to work with the modified one (5.1), that we may rewrite using the gauge parameters $\hbar, u, \Lambda$ as (recall (5.4))

$$\left(-\frac{\hbar^2}{2}\frac{d^2}{dy^2} + \Lambda^2 \cosh y + u\right)\psi(y) = 0 \tag{5.9}$$

where $2\Lambda^2 \cosh y$ plays the role of potential and $-2u$ that of the energy. The turning points associated to this choice of potential function are determined through the condition

$$\Lambda^2 \cosh y + u = 0 \to y = \pm \operatorname{arccosh}(-u/\Lambda^2) + 2\pi i n, \quad n \in \mathbb{Z} \tag{5.10}$$

i.e. we have a couple of turning points $\pm \operatorname{arccosh}(-u/\Lambda^2)$ (one a rotation of π around the origin with respect to the other), each of them infinitely transposed along a vertical line.
If we start with a value for u that is real and $-\Lambda^2 < u < \Lambda^2$, taking e.g. u positive, then all the turning points will be aligned along the imaginary axis of the complex y plane. Since for $-1 < -u/\Lambda^2 < 1$ we have $0 < \operatorname{arccosh}(-u/\Lambda^2) < i\pi$, starting from the origin, and going up along the positive imaginary direction, the sequence of turning points will be the following

$$+\operatorname{arccosh}(-u/\Lambda^2), \quad -\operatorname{arccosh}(-u/\Lambda^2) + 2i\pi, \quad +\operatorname{arccosh}(-u/\Lambda^2) + 2i\pi, \quad \dots$$

and we have the same sequence but with opposite sign in going down the negative $\operatorname{Im} y$ direction. By doing a π wide scan in the values of $\arg(\hbar)$ (which coincides with $-\operatorname{Im}\theta$, since we consider Λ real) of the Stokes graph associated to the differential equation (5.9), one notices the appearance of a saddle trajectory in the intervals

$$\begin{gathered}[-\operatorname{arccosh}(-u/\Lambda^2) + 2\pi i n,\ +\operatorname{arccosh}(-u/\Lambda^2) + 2\pi i n], \quad \text{when } \operatorname{Im}\theta = \pm\tfrac{i\pi}{2} \\ [+\operatorname{arccosh}(-u/\Lambda^2) + 2\pi i n,\ -\operatorname{arccosh}(-u/\Lambda^2) + 2\pi i(n+1)], \quad \text{when } \operatorname{Im}\theta = 0, \pi\end{gathered} \tag{5.11}$$

The relevant quantum periods are then those whose cycle encircles these two type of intervals. Calling γ_{d_n} (γ_{m_n}) the cycle that surrounds the first (second) type of interval having the same n index value, we can write the classical limits of the related quantum

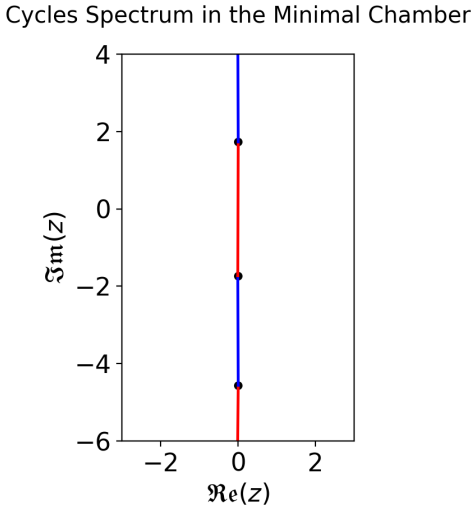


Figure 5.1: Simplified representation of the relevant cycles in the minimal chamber. Some representatives of the cycles γ_{d_n} are depicted in red, while the blue lines represent part of the γ_{m_n} cycles (the pattern then extends along the whole imaginary axis). The black dots are the turning point of the potential function.

periods as[1]

$$\begin{aligned}\Pi^{(0)}_{\gamma_{d_n}}(u,\Lambda) &\equiv -i\oint_{\gamma_{d_n}}\sqrt{2u+2\Lambda^2\cosh y}\,dy = -4\pi i a_D^{(0)}(-u,\Lambda)\\ \Pi^{(0)}_{\gamma_{m_n}}(u,\Lambda) &\equiv i\oint_{\gamma_{m_n}}\sqrt{2u+2\Lambda^2\cosh y}\,dy = -4\pi a_D^{(0)}(u,\Lambda)\end{aligned} \tag{5.12}$$

where we have made explicit also their relation with the undeformed SW cycles (5.6). For a pictorial representation of the cycles refer to Figure 5.1. These classical limits (5.12), by the correspondence between quantum periods and BPS states mentioned in the introduction, allows us to associate the dyon and monopole particles present in the strong coupling spectrum to the resummed quantum periods $\mathcal{S}[\Pi_{\gamma_d}], \mathcal{S}[\Pi_{\gamma_m}]$ respectively. If one considers the whole y plane, then there is an infinite number of periods of these two types, alternating along the imaginary axis; ultimately however, one may impose boundary conditions along the imaginary axis so that the "copies" of the periods are exactly identified, as done in [59] (or similarly in [64]). In our discussion we will usually consider the copies separately, but having in mind their identification to simplify the picture.
Similarly of what has been done in section 3.5.1, one can then introduce masses for these

[1]Recall from the definition (3.50) that the zeroth order term in the $\hbar$ expansion of a quantum period is the integral of the classical momentum $p = \sqrt{E - V(y)} = i\sqrt{2u + 2\Lambda^2\cosh y}$

periods as

$$m_{d_n} \equiv \Pi^{(0)}_{\gamma_{d_n}}(u,\Lambda) = -4\pi i a_D^{(0)}(-u,\Lambda) \qquad m_{m_n} \equiv i\Pi^{(0)}_{\gamma_{m_n}}(u,\Lambda) = -4\pi i a_D^{(0)}(u,\Lambda) \quad (5.13)$$

chosen so that they are real and positive when $u \in \mathbb{R}$, $-\Lambda^2 < u < \Lambda^2$. Thanks to the connection with SW theory, these masses are directly related to the central charges Z_γ of the associated BPS states. Exploiting the discontinuity structure of the resummed quantum periods and using the DDP formula, if one defines the ϵ functions associated to these cycles as (we will generally prefer to express the dependence of the functions in θ rather than $\hbar$, but the two are directly related via (5.4))

$$\begin{aligned} \epsilon_{d_n}\left(\theta + \frac{i\pi}{2} \pm i0^+\right) &= \frac{i}{\hbar}\mathcal{S}_\pm\left[\Pi_{\gamma_{d_n}}\right](\hbar) \\ \epsilon_{m_n}(\theta \pm i0^+) &= \frac{i}{\hbar}\mathcal{S}_\pm\left[\Pi_{\gamma_{m_n}}\right](\hbar) \end{aligned} \quad (5.14)$$

one can then derive the TBA equations in minimal chamber, which look like

$$\begin{aligned} \widetilde{\epsilon}_{d_n}(\theta) &= |m_{d_n}|\frac{e^\theta}{\Lambda} - \int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\frac{\log\left(1+e^{-\widetilde{\epsilon}_{m_n}(\theta')}\right)}{\cosh(\theta-\theta'+\Delta\phi_{m_n,d_n})} - \int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\frac{\log\left(1+e^{-\widetilde{\epsilon}_{m_{n-1}}(\theta')}\right)}{\cosh(\theta-\theta'+\Delta\phi_{m_{n-1},d_n})} \\ \widetilde{\epsilon}_{m_n}(\theta) &= |m_{m_n}|\frac{e^\theta}{\Lambda} - \int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\frac{\log\left(1+e^{-\widetilde{\epsilon}_{d_n}(\theta')}\right)}{\cosh(\theta-\theta'+\Delta\phi_{d_n,m_n})} - \int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\frac{\log\left(1+e^{-\widetilde{\epsilon}_{d_{n+1}}(\theta')}\right)}{\cosh(\theta-\theta'+\Delta\phi_{d_{n+1},m_n})} \end{aligned} \quad (5.15)$$

where we have already used the shifted $\widetilde{\epsilon}_a(\theta) \equiv \epsilon_a(\theta - i\phi_a)$ functions, so that the the above form of the TBA equations hold in all the minimal chamber, separating as usual $m_a = |m_a|e^{i\phi_a}$ and calling $\Delta\phi_{a,b} \equiv \phi_a - \phi_b$. As mentioned before, in writing these equations, we have imagined to consider all copies of the quantum periods $\mathcal{S}[\Pi_{\gamma_d}], \mathcal{S}[\Pi_{\gamma_m}]$ as separate entities; if one effectively identifies these cycles, then obtains equations completely analogous of those presented in [21, 22, 59]

$$\begin{aligned} \widetilde{\epsilon}_d(\theta) &= |m_d|\frac{e^\theta}{\Lambda} - 2\int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\frac{\widetilde{L}_m(\theta)}{\cosh(\theta-\theta'+\Delta\phi_{m,d})} \\ \widetilde{\epsilon}_m(\theta) &= |m_m|\frac{e^\theta}{\Lambda} - 2\int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\frac{\widetilde{L}_d(\theta')}{\cosh(\theta-\theta'+\Delta\phi_{d,m})} \end{aligned} \quad (5.16)$$

which contain factors of 2 in front of the integrals, reflecting the fact that, once all the copies of the cycles are identified, then the two "fundamental" cycles (which may be taken as the ones having $n = 0$ index) intersect more than once. Here, as usual, the L functions are defined as

$$\widetilde{L}_a(\theta) \equiv \log\left(1 + e^{-\widetilde{\epsilon}_a(\theta)}\right) \quad (5.17)$$

Together with these TBA equations, we can as usual write a Y-system. In particular, if we consider the different copies of the cycles as different, we have the infinite number of equations

$$\begin{aligned} Y^+_{d_n} Y^-_{d_n} &= [1+Y_{m_n}][1+Y_{m_{n-1}}] \\ Y^+_{m_n} Y^-_{m_n} &= [1+Y_{d_n}][1+Y_{m_{n+1}}] \end{aligned}, \qquad n \in \mathbb{Z} \tag{5.18}$$

that with the identification of the cycles reduces to

$$\begin{aligned} Y^+_{d} Y^-_{d} &= [1+Y_{m}]^2 \\ Y^+_{m} Y^-_{m} &= [1+Y_{d}]^2 \end{aligned} \tag{5.19}$$

As for the TBA, also this Y-system matches with the ODE-IM results obtained in [21]. In the article, the authors show how in fact how to build the following Q-system

$$1 + Q^2(\theta, u) = Q(\theta - i\pi/2, -u) Q(\theta + i\pi/2, u) \tag{5.20}$$

which, having in mind the correspondence (5.8) (proved in the same article [21]), implies the Y-system (5.19), when $Q(\theta, u)$ and $Q(\theta, -u)$ are considered associated to two different Y-functions.

5.2 Wall crossing to the maximal chamber

Let's now discuss what happens when performing a wall-crossing into the maximal chamber. As for the case of polynomial potential, looking at the TBA equations of the minimal chamber, and considering at the beginning for simplicity their reduced form, we see that a pole of the convolution's kernel hits the integration contour when either

$$\Delta\phi_{m,d} < \tfrac{\pi}{2} \to \Delta\phi_{m,d} > \tfrac{\pi}{2} \qquad \text{or} \qquad \Delta\phi_{m,d} > -\tfrac{\pi}{2} \to \Delta\phi_{m,d} < -\tfrac{\pi}{2} \tag{5.21}$$

signalling that a new chamber has been reached. Thanks to the identifications (5.12), one can notice that this wall-crossing condition is indeed the one associated to the wall between the strong and the weak coupling region of SW theory [20, 58]: the condition on the phases of the masses m_d, m_m implies as usual an alignment or anti-alignment of the related classical periods, which extends also to the SW cycles $a^{(0)}(u), a_D^{(0)}(u)$

$$0 = \mathrm{Im}\left(\frac{\Pi^{(0)}_{\gamma_d}(u)}{\Pi^{(0)}_{\gamma_m}(u)}\right) = \mathrm{Im}\left(\frac{i a_D^{(0)}(-u)}{a_D^{(0)}(u)}\right) = \mathrm{Im}\left(\frac{a^{(0)}(u)}{a_D^{(0)}(u)}\right) \tag{5.22}$$

where we have omitted the Λ dependence of the quantities, and in the last step used the relation $i a_D^{(0)}(-u) = a^{(0)}(u) - \mathrm{sgn}(\mathrm{Im}\, u) a_D^{(0)}(u)$.
If we consider in particular to follow a path on the complex u plane that starts on the region $u \in \mathbb{R}, 0 < u < \Lambda^2$ and then encircles the point Λ^2 from below (see Figure 5.2),

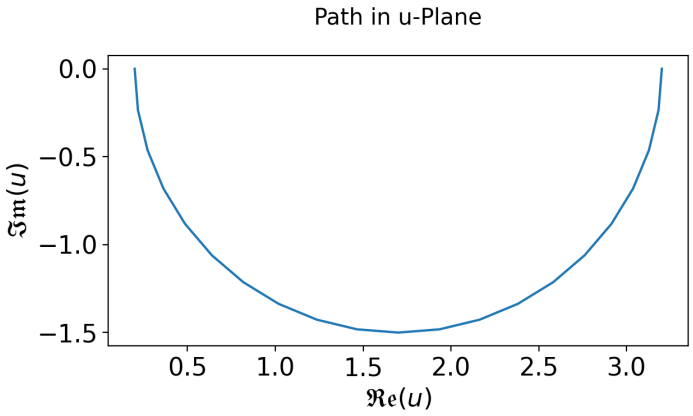


Figure 5.2: Representation in the complex u plane of the path used to perform the wall-crossing from the strong to the weak coupling region for the modified Mathieu equation (5.9)

so that coalescence of turning points is avoided, then it is the first of the two conditions (5.21) that is realised. After the wall crossing, the TBA equations get modified as

$$\begin{aligned}\widetilde{\epsilon}_d(\theta) =&|m_d|\,\frac{e^\theta}{\Lambda} - 2\int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\frac{\widetilde{L}_m(\theta')}{\cosh(\theta-\theta'+\Delta\phi_{m,d})} - 2\widetilde{L}_m\left(\theta - i\frac{\pi}{2} + \Delta\phi_{m,d}\right)\\ \widetilde{\epsilon}_m(\theta) =&|m_m|\,\frac{e^\theta}{\Lambda} - 2\int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\frac{\widetilde{L}_d(\theta')}{\cosh(\theta-\theta'+\Delta\phi_{d,m})} - 2\widetilde{L}_d\left(\theta + i\frac{\pi}{2} + \Delta\phi_{d,m}\right)\end{aligned} \tag{5.23}$$

Using the same reasoning adopted in Chapter 4, we then interpret the appearance of the residues as a Stokes phenomenon for the resummed quantum periods: if we want to have the $\widetilde{\epsilon}_a$ functions associated to a resummation procedure performed in the maximal chamber (or, equivalently, we want the asymptotic of the $\widetilde{\epsilon}_a$ to still coincide with the classical period in the whole maximal chamber), we then shall redefine the $\widetilde{\epsilon}_d(\theta), \widetilde{\epsilon}_m(\theta)$ functions so that to absorb the residues. In terms of Y-functions this implies

$$Y'_d \equiv \frac{Y_d}{[1+Y_m^-]^2}\,, \qquad Y'_m \equiv \frac{Y_m}{[1+Y_d^+]^2} \tag{5.24}$$

where we have used the symbol $'$ to distinguish the new Y functions from the old ones, and as usual the symbols $^\pm$ indicate a shift of $\pm i\frac{\pi}{2}$ in θ (where not specified the Y are evaluated in θ). We can check that the inverse-shift rule for the Y-functions (recall Lemma 2) holds for this case, since we are in the maximal chamber (we have only two chambers in the moduli space) and the conditions of the Lemma are satisfied

$$Y'_d(\theta) = [Y_d(\theta - i\pi)]^{-1} \qquad Y'_d(\theta) = [Y_d(\theta + i\pi)]^{-1} \tag{5.25}$$

With the redefinition of the Y (and thus also of the ϵ) functions, the TBA equations at weak coupling look like

$$\begin{aligned}\widetilde{\epsilon}'_d(\theta) =&|m_d|\,\frac{e^\theta}{\Lambda} - 2\int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\frac{\widetilde{L}_m(\theta')}{\cosh(\theta-\theta'+\Delta\phi_{m,d})}\\ \widetilde{\epsilon}'_m(\theta) =&|m_m|\,\frac{e^\theta}{\Lambda} - 2\int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\frac{\widetilde{L}_d(\theta')}{\cosh(\theta-\theta'+\Delta\phi_{d,m})}\end{aligned} \tag{5.26}$$

where on the left the new ϵ' functions have appeared, but on the right we still have the L functions we had before. If we want to mirror the outline of the procedure used in the case of the polynomial potential, at this point we should impose an ansatz of the form (4.15), where L_d, L_m should be expressed in terms of their new version L'_d, L'_m, plus a new L factor associated to a new cycle, built out from γ_d, γ_m, that has become relevant after the wall-crossing. However, the situation in this case is rather different.
If we look at the Stokes graph after the wall-crossing, scanning its topology in a π wide interval of values of $\mathrm{Im}\,\theta$ to see how many new cycles have become relevant, we find an infinite number of them. With respect to the polynomial case, where we could perform the wall-crossings in such a way that new relevant periods appeared once at a time, allowing us to define the Y associated to these cycles in an ordered and systematic way, here we are faced to the need of introducing an infinite number of Y-functions in one go. This could have been expected, since SW theory has a moduli space with only a single wall, separating the strong coupling region, which contains a finite number of particle types, from the weak coupling one, where due to the dyons the different species of particles are infinite. Indeed, the TBA equations at weak coupling proposed in [22], where the authors derive the equations directly in the maximal chamber by taking advantage of the results of [61, 62], do contain an infinite number of ϵ functions.

Inspecting the Stokes graph associated to the maximal chamber, we may identify from the graph to which relevant cycle corresponds each BPS state (see Figure 5.3). Keeping track of the cycles we had in the minimal chamber, associated to the monopole and dyon, as we move along the path in u space that makes us cross the curve of marginal stability, one can see that the family of cycles associated to the monopole rotates, becoming horizontal if we come back onto the real u axis, at $u > \Lambda^2$, while the ones associated to the dyons place themselves in a diagonal position (in Figure 5.3 we kept the same colors that in Figure 5.1 for these two type of cycles to better keep track of them). The family of vertical cycles (those represented in green in Figure 5.3) may be identified with the W-bosons, since they encircle the interval $[-i\pi, i\pi]$ ($+i2\pi n$ for the various copies), and recalling the expression (5.6) we have for the SW $a^{(0)}$ cycle. Finally, all the cycles that connect diagonally turning points that are further and further away may be identified with the other dyons, those with higher electric charge (here we have represented only a finite number of them, to keep the figure more readable). The fact that the picture has a $\mathbb{Z}_2$ symmetry may be addressed to the presence of both particles and antiparticles in

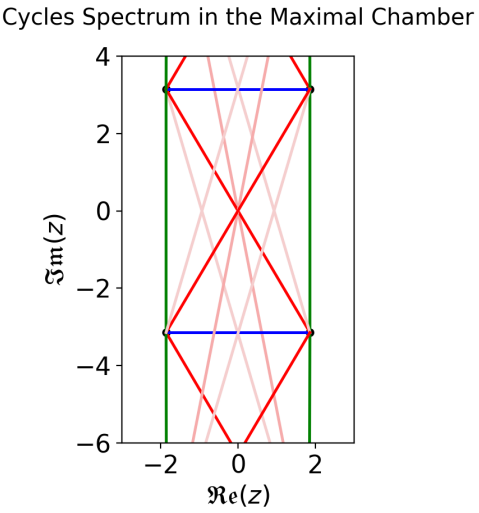


Figure 5.3: Simplified representation of the relevant cycles in the maximal chamber. For an interpretation of the cycles that are depicted in terms of BPS states, refer to the main text.

the theory.

From the point of view of the TBA equations (5.16) (or equivalently of the Y-system (5.19)), the presence of a different type of wall-crossing with respect to the polynomial case can be addressed to the presence of the factors 2 in front of the convolution integrals. To better see why it should be so, it is convenient to revert to the extended form (3.68) form of the TBA equations, where instead all the convolution integrals have no prefactor. Here we have an infinite number of ϵ functions, and the crucial fact is that they all share the same two forms (5.12) of classical limit, implying that at the curve of marginal stability they will generate a collective wall-crossing at the same time. We have then this duality in looking at the wall, either as describing a new form of single wall-crossing, or an infinite number of "usual" (of flip type, in the language of Chapter 3) wall-crossings, like the one encountered in the case of a polynomial potential, happening at the same time.
This different type of wall implies that the same procedure described in Chapter 4 cannot be applied to the case of the Mathieu equation, at least in a straightforward way. At this point one is faced with a choice: either trying to develop a new type of procedure, to deal with the novel type of wall, or searching for a way to adapt the problem so that the old algorithm can still be put into use. In our investigations we mainly followed this second route, and came forward with two types of proposals, which we list below. As anticipated in the introduction, none of the two routes has been carried out fully up to the end, due to the emergence of technical difficulties, but we believe that some of these

problematics may be clarified under a deeper investigation.

- *Approach no.1*
  Introduce a perturbation in the potential function, performing a sort of regularisation procedure of the wall-crossing phenomenon. In this case the idea is to add to the potential function of the modified Mathieu equation, which we recall is $2\Lambda^2\cosh(y) + u$, a perturbation term $V_{pert}(y, u, t_a)$, possibly depending on an additional set of parameters t_a, such that there exist a path Γ in the "extended" moduli space formed by (u, t_a) with a series of desirable properties:

  1. The path Γ need to start in a point (u^{in}, t_a^{in}) where $-\Lambda^2 < u^{in} < \Lambda^2$, $V_{pert}(u^{in}, t_a^{in}) = 0$ and end in a point (u^{fin}, t_a^{fin}) where $u^{fin} > \Lambda^2$ or $u^{fin} < -\Lambda^2$, $V_{pert}(u^{fin}, t_a^{fin}) = 0$, so that both the initial and the final point coincide with points of the "physical" moduli space, one in the minimal and one in the maximal chamber
  2. The wall-crossing phenomena encountered along Γ happen all one at a time
  3. Since we still expect the need to perform an infinite number of wall-crossing to generate an infinite number of new quantum periods, the sequence of wall-crossing along Γ shall be so that it can be predicted via an inductive reasoning, after a first set of wall-crossing is explicitly tracked along the curve

  On top of these conditions, it should be reasonable to choose a function that is not multivalued as perturbation potential, to avoid adding cuts on the y plane, and possibly also analytic. It is clear that the set of the conditions posed on $V_{pert}(y, u, t_a)$ are very stringent, and may not exist a function with all the desired properties. Of the two type of approaches we list here in fact, this one seems to be the least promising to be effectively implemented. By performing some trial tests, what appears to be the most difficult part of the procedure is not to make the wall-crossing separated, since the periodicity of the $\cosh(y)$ function can be easily broken, but to separate them in a controlled and ordered way, so that an inductive evolution of the wall-crossing procedure can be carried out successfully.

- *Approach no.2*
  In this second type of approach, we do not modify the physical formulation of the theory, but take advantage of the purely geometrical properties that characterize a sequence of single wall-crossing phenomena, when one forgets that driving the order of wall-crossing is a path in the physical moduli space. Looking back at the Y-system algorithmic procedure described in section 4.2 in fact, one can notice how the analytical part of the algorithm, which allows to find the Y functions and the Y-system in a generic chamber starting from the minimal one, solely depends on the wall-crossing sequence of the phases $\{\Delta\phi_{a_i b_i}\}$ of the masses, which has to

be determined before in a separate step of analysis and then used in input to this second part. Usually (as done e.g. for the polynomial case), one determines the sequence $\{\Delta\phi_{a_i b_i}\}$, and together with it the set of relevant classical cycles at each i-th chamber, by performing an analysis of the Stokes graph point by point along the chosen path of analytic continuation in the moduli space. In the case of the modified Mathieu equation however, where there is no path in the space u with the desired property of generating single separated wall-crossings, we could still try to make up a "fictional" wall-crossing sequence, which is in accordance with the overall physical evolution of the phases of the dyon and monopole classical periods (e.g. $\Delta\phi_{m,d} < \frac{\pi}{2} \rightarrow \Delta\phi_{m,d} > \frac{\pi}{2}$ in the case of wall crossing discussed before), but where the phases of the periods of the minimal chamber (which govern the evolution of the phases also of all other periods generated during the wall-crossings since the latter can be built from these fundamental two), are moved manually, in a self-consistent way, so that all periods we expect to be present in the maximal chamber are effectively generated during this procedure. To perform this type of analysis, obviously one chooses to work with the extended form (5.15) of the minimal chamber's TBA equations, so that the wall crossing between two cycles are of the same form studied for the polynomial case.
As for the previous approach, also in this case some trial runs of implementation of the procedure have been performed. The fictional evolution of the cycles has been driven through a numerical Python code, to produce the sequence of wall-crossings and the order at which new cycles appear along the way. Due to an easier control of the process with respect to the first type of approach, here one is able to produce indeed an evolution pattern that is self-consistent and leads to a sequence of redefinition of the Y functions (associated to the cycles) that can be written down, at least for some of the cycles (like the ones associated to the monopole and the dyon present prom the minimal chamber, or the W-boson), in an organized way. In particular, geometrical series seem to appear in these redefinition of the Y-functions, which give hope for a finite result when the number of wall-crossing is sent to infinite, so that all single wall-crossing have been done (in this case, to have manageable results, at the end one shall make the identification between copies of the same periods). A similar process of dealing with infinite flips that then leads to finite results is studied in [65], which may give some hints on the procedure.
An issue that arises in this type of fictional phase evolution, is that, being still infinite the number of new Y function to be generated, is not still clear at the end how one can manage to write the full Y-system (or TBA equations), which, from what appears by looking at the intersection between cycles in the Stokes graph of the maximal chamber (and is also supported by the TBA equations written in [22]), involves an infinite number of Y functions in any of the Y-system equations. For this reason, one would desire to find some other ways to check that, at least for the Y functions that one is able to update up to the very end of the wall-crossing

procedure, the expressions found with this fictional wall-crossing procedure is correct.
A fist consistency check is obviously that the Y functions associated to the dyon and monopole which are already present in the minimal chamber, satisfy the shift-inverse relation (5.24) at the end of the procedure. If we want to check the expression derived for other periods that appear only in the weak coupling spectrum, we should then ask if there are relations among a finite number of these resummed quantum periods which we can use as a testing ground, either involving periods resummed in the maximal or in the minimal chamber. To this aim may come into play the TQ system derived in [21]

$$T(\theta,u)=\frac{Q(\theta-i\pi/2,-u)+Q(\theta+i\pi/2,-u)}{Q(\theta,u)} \tag{5.27}$$

thanks to the identifications (5.7, 5.8), provided we relate the exact version of the SW cycles with their resummed ones (through expressions like the ones present in [22, 64]).

Conclusions and Outlook

In this thesis, we analyzed the phenomenon of wall-crossing of the TBA equations for the Borel resummed quantum periods associated to a given Schrödinger equation. In particular, we focused our attention to two selected models: first, we analyzed the case of a polynomial potential, building on the results of [1]. For this case, by looking at the evolution of the TBA equations in the moduli space from the perspective of the associated Y functions and Y-system, we were able to derive an algorithmic procedure that allows us to extract the TBA equations corresponding to a generic polynomial potential by only algebraic manipulations of the minimal chamber's Y-functions. With respect to directly following the TBA equations, along the analytic continuation process, this method has the advantage of an easy scalability to potentials with high degree, since it is much easier to put into use. As an explicit example of application of the algorithm, we have chosen to put it into use for the case of a quartic potential, but the increase in complexity in using it for a potential of degree e.g. 5 or 6 is absolutely manageable.

To further investigate on the applicability of the procedure for different types of potentials, we then moved to study the modified Mathieu equation, which represents a Schrödinger-like equation with a periodic potential. An additional reason that led us to choose this model as second candidate for the investigation is its close relation with the deformed Seiberg-Witten (SW) gauge theory, and in particular the correspondence between the relevant quantum periods of the modified Mathieu equation and the spectrum of BPS states of the SW theory. As for the polynomial case, we analyzed the spectrum of the relevant periods in the different chambers (which in this case are only of minimal and maximal type), and after having written the TBA equations for the periods in the minimal chamber, we performed the analytic continuation onto the maximal one. Due to a different type of wall separating the two chambers, or, under a different perspective, due to the impossibility of avoiding an infinite number of simple wall-crossing happening at the same time, we excluded the possibility of a straightforward application to this case of the algorithm developed in the case of a polynomial potential. We then moved onto considering possible strategies to overcome this limitation, so that the algorithm could still be applied, and proposed two different type of approaches: the first one consists in modifying the physical problem via the adding of a controlled perturbation, while the key

feature of the second approach is to imagine the wall-crossing phenomenon happening in a fictional moduli space, where the evolution of phases of the classical cycles can be manually moved in a self-consistent way, and impose that the overall evolution is consistent with that happening in the true physical moduli space. Both the two procedures are not trivial to implement, and, at the current stage of investigation, have been only partially carried out. Among the two, the second one seems perhaps to be the most promising.

Speaking about possible continuations of the analysis presented in this thesis work, the most pressing one is certainly to complete the investigation started on the modified Mathieu equation, by trying to fully perform one (or both) of the routes that we have proposed to overcome the problems faced in this first round of examination. In order to have a better understanding of the picture, a more thorough comparison with the works [61, 62] of D. Gaiotto, G.W. Moore and A. Neitzke, and with that of P. Longhi [65] is suggested. Furthermore, relations between the exact and the resummed quantum periods like the ones presented in [22, 64] may be useful to take advantage of the TQ system relation derived in [21] as a consistency check of the analytic continuation procedure.
If we leave aside the Mathieu case, there are still many models where it would be interesting to try to reproduce a similar type of analysis like the one developed in this thesis, focused on determining the Y-system in an easily tractable region of the moduli space associated to the problem, and then using it to obtain the corresponding relations in other, less trivial chambers. Perhaps the most direct generalisation would be to consider models with a polynomial potential *and* a regular singularity located at a finite point (like the one produced by the adding to the potential the angular momentum term), where a partial analysis on the analytic continuation of the TBA equations has already been presented in [42]. With the help of a Y-system driven procedure, obtaining the TBA equations in the various chambers promises to be a more easily achievable task.

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