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Integrability Structures in SU(2) Supersymmetric Gauge Theory with Four Flavours and AdS Black Holes

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Abstract

In this thesis we study through the Ordinary Differential Equation/Integrable Model (ODE/IM) correspondence the deformed Seiberg-Witten curve of four flavour $\mathcal{N} = 2$ SU(2) supersymmetric Yang-Mills theory. This theory, compared with the ones with fewer flavours of quarks, is very different as there is no dinamically generated scale and thus the theory is asymptotically conformal. At the ODE level this is manifested in the absence of irregular singular points in the equation and thus no Stokes phenomenon is present. While in the eyes of the AGT correspondence this is the most natural setting, for the ODE/IM correspondence this leads to the absence of the transfer matrices of integrability. However we introduce the monodromy group of the ODE into the problem and show its equivalence with the system of Q-functions. We stress how the gauge theory prepotential should be seen as the generating function of a canonical transformation on the space of monodromy data which is endowed with a symplectic structure. Using this fact we achieve a definition of the prepotential directly from the differential equation, using the Matone relation, without requiring, in principle, the usage, but only the matching with gauge theory. The same differential equation shows up as the equation describing a scalar perturbation in the five-dimensional AdS-Schwarzschild background. The applications of this is two-fold: first integrability could be used to derive Bethe ansatz equation for the quasi-normal modes of AdS black holes. Second this gravity theory is the holographic dual of four-dimensional $\mathcal{N} = 4$ SYM and the connection coefficients for the ODE can then be connected to an exact expression for the two-point function at finite temperature in this theory.

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Dear reader,

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Finally, I would like to thank you, reader, for your interest and attention. I hope you find my thesis informative, insightful and useful.

Sincerely,
Cesena, 8 March 2023
Luca Bertozzi

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Introduction

$\mathcal{N} = 2$ Supersymmetric Yang-Mills theory (SYM) can be regarded as the perfect sandbox to study non-perturbative effects in Quantum Field Theory. Unlike in $\mathcal{N} = 4$ SYM, $\mathcal{N} = 2$ SUSY is not as constraining and the theory is not necessarily conformal. However the symmetry is still constraining enough to make the exact determination of the low energy effective action possible, as first done in the celebrated paper [41] by Seiberg and Witten in 1994. This allows the study of many non-perturbative effects such as confinement and chiral symmetry breaking, potentially giving information on how this phenomena are realized in the physical world. Further, the study of Seiberg-Witten theory on the so-called Ω-background, which was first introduced as a regulator for the ADHM procedure to compute instanton contributions, opened up connections between supersymmetric gauge theories, black hole perturbation theory, integrability and conformal field theory.

In particular in the Nekrasov-Shatashvili (NS) limit of the Ω-background the theory becomes a deformation/quantization [38] of the ordinary theory and an ordinary differential equation (ODE) called the quantum Seiberg-Witten curve appears. It was recently realized that these ODE appearing in Seiberg-Witten theory are nothing else but the differential equations describing perturbations of different spin in the background of a black hole/brane or other compact objects and, depending on the matter content of the theory, it is possible to reproduce different spacetime geometries [2]. In this setting the frequency of the quasi-normal modes of the perturbation are given by quantization conditions expressible through the instanton dual period of the gauge theory.

Yet another correspondence is fundamental to our work: the ODE/IM correspondence. This first appeared in [17] as a correspondence between the spectral determinant of a certain Schroedinger equation and the eigenvalues of the Q-operators of conformal field theory. Since then it has been generalized to many other ODE and integrable model pairs. Of particular relevance are the correspondences for SU(2) NS-deformed Seiberg-Witten theory in the cases with 0, 1, 2 fundamental hypermultiplets treated in [20, 21] or the case with 3 fundamental hypermultiplets treated in [22].

Combining the ODE/IM correspondence and the connection with black holes it has been shown that the quasi-normal modes are essentially the Bethe roots of the associated integrable model [19]. This paves the way to applying integrability techniques such as the Bethe equations and the Thermodynamic Bethe Ansatz (TBA) to astrophysical problems involving also realistic black holes such as the Schwarzschild or Kerr geometries.

This thesis works focuses on the SU(2) $\mathcal{N} = 2$ SYM with four hypermultiplets in the fundamental representation of the gauge group. In this case the quantum Seiberg-Witten curve is known as Heun's equation. This theory is very different from the other theories with less hypermultiplets as there is no dinamically generated scale and the theory is conformal at high energy. This is reflected in the ODE with the absence or irregular singular points and thus no Stokes phenomenon. Usually in the ODE/IM correspondence the Stokes phenomenon is fundamental to the definition of the integrability T-functions, in fact we find no T-functions or TQ-systems. However we take into account the monodromy problem and connect it with the connection problem and the QQ-systems, deriving a more powerful and constraining functional equation for the connection coefficients. We then proceed to find the general solution to this functional equation and verify and prove that the remaining part constitutes a zero-mode. Also, we study the space of monodromy data which is endowed with a symplectic structure due to Aitiyah and Bott. We use this to define the prepotential directly from the ODE, independently of any gauge theory, as the generating function of the canonical transformation relating the so-called accessory parameter of Heun's equation and its Floquet exponents. In this picture the Matone relation of gauge theory emerges as the usual equation for the generating function and it is used to define the prepotential. We then check this definition agrees with the usual gauge theory definition up to 3 instanton order. Using this new definition we prove some exact symmetry relations of the prepotential, which are key to showing that a certain exponential of its derivatives with respect to the masses of the hypermultiplets are a zero-mode for the QQ-system.

In addition Heun's equation is also the differential equation describing a scalar perturbation in the five-dimensional AdS-Schwarzschild bacckground, holographic dual to four-dimensional $\mathcal{N} = 4$ SYM at finite temperature. We use this to show how the two-point correlator of $\mathcal{N} = 4$ SYM at finite temperature can be written in terms of the Q-functions.

The thesis is structured in the following way:

- Chapter 1 is an introduction to SU(2) Seiberg-Witten theory. First we review the pure gauge theory and find its spectrum, then we move to studying the theory with hypermultiplets and finally we turn to the Ω-background, mentioning also the AGT correspondence.

- Chapter 2 is dedicated to various aspects of the gauge/gravity dualities. First the problem of quasi-normal modes is introduced, then we turn to the AdS/CFT correspondence.

- Chapter 3 starts with a review of integrability in classical mechanics and classical field theory. Then we turn to quantum studying integrability in conformal field theory and finally introduce the ODE/IM correspondence in its first incarnation.

- Chapter 4 is devoted to the ODE/IM correspondence for Heun's equation. We derive the the fundamental functional relation satisfied by the connection coefficients: the QQ-system. Then we introduce the monodromy problem and connect it with the connection problem to find further functional equations in which also the Floquet exponents appear. We show how the holographic two-point function can be written in terms of the Q-functions and finally give the necessary identification with gauge theory, reviewing different form in which the equation sometimes appear as well as giving the dictionaries between these different forms.

- Appendix A is a review of the exact WKB method and Borel resummations.

- Appendix B is an overview of the method used to compute the prepotential from the differential equation.

- Appendix C is about the four-flavour Nekrasov-Shatashvili prepotential, its classical limit and its scaling limit to the theory with three flavours.

- Appendix D has details about the theory with three flavours and on how to derive some of its quantities and its symmetries from the four-flavour theory.

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Contents

Chapter 1

Seiberg-Witten Theory

In this chapter we review $\mathcal{N} = 2$ gauge theory with N_f fundamental hypermultiplets and the work of Seiberg and Witten [41] to find the exact form of the low energy Wilsonian effective action. First, in Section 1.1 we review the theory in the $N_f = 0$ case and in Section 1.2 we find its spectrum. Then, in Section 1.3 we extend it to the case of $N_f > 0$. Finally in Section 1.4 we introduce the Ω-background, then study the so-called Nekrasov-Shatashvili limit in which the theory becomes a deformation/quantization of ordinary Seiberg-Witten theory.

1.1 $\mathcal{N} = 2$ Pure $\mathrm{SU}(2)$ Gauge Theory

In this section we study pure, i.e. with no additional matter, SU(2) gauge theory with $\mathcal{N} = 2$ SUSY. We consider a $\mathcal{N} = 2$ vector superfield[1] $\mathcal{A}$ in the adjoint representation of SU(2). In minimal SUSY language this is composed by a chiral superfield A and a vector superfield V. We denote the field strength of the vector superfield by W^α. The action in minimal superspace can be written as

$$S = \mathrm{Im}\,\mathrm{Tr}\,\frac{\tau}{16\pi}\int \mathrm{d}^4x \left[\int \mathrm{d}^2\theta\, W^\alpha W_\alpha + \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, A^\dagger e^{-2gV} A\right], \tag{1.1}$$

having $\tau := \frac{\Theta}{2\pi} + i\frac{4\pi}{g^2}$ be the complexified coupling, which contains the instanton angle Θ and the gauge coupling g. Here the trace is taken over the gauge group Lie algebra. Notice that a non-trivial superpotential $W(A)$ is not allowed by $\mathcal{N} = 2$ SUSY. $\mathcal{N} = 2$ SUSY theories have a $\mathrm{SU}(2)_R$ R-symmetry group, whose indices of the $\frac{1}{2}$ representation we denote by $A, B, C, \ldots$. The fields of the vector multiplet are a vector boson A_μ, a

[1] this is sometimes called the $\mathcal{N} = 2$ chiral superfield in the literature, however it is not chiral since it contains Weyl spinors of opposite chirality. We will reserve the name chiral superfield for the $\mathcal{N} = 1$ case.

scalar ϕ and two Weyl spinors ψ_α^A for $A = 1, 2$. This two Weyl spinors transform in the doublet of $\mathrm{SU}(2)_R$, while the scalar and vector boson transform in the singlet.

It is useful to recast the action of (1.1) in the $\mathcal{N} = 2$ superspace language. Here we have the coordinates $\left(x^\mu, \theta_A^\alpha, \bar{\theta}_{\dot{\alpha}A}\right)$ and the $\mathcal{N} = 2$ vector superfield can be written as

$$\mathcal{A} = A(\tilde{y}, \theta_1) + \sqrt{2}\theta_2^\alpha W_\alpha(\tilde{y}, \theta_1) + \theta_2\theta_2 G(\tilde{y}, \theta_1)\,, \tag{1.2}$$

having $\tilde{y}^\mu := x^\mu + i\theta_1\sigma^\mu\bar{\theta}_1 + i\theta_2\sigma^\mu\bar{\theta}_2$ and

$$G(\tilde{y}, \theta_1) = -\frac{1}{2}\int \mathrm{d}^2\bar{\theta}_1 \left[A(\tilde{y} - i\theta_1\sigma\bar{\theta}_1, \theta_1, \bar{\theta}_1)\right]^\dagger \exp\left\{-2gV(\tilde{y} - i\theta_1\sigma\bar{\theta}_1, \theta_1, \bar{\theta}_1)\right\}. \tag{1.3}$$

In $\mathcal{N} = 2$ superspace the action can be cast in terms of a single holomorphic function of $\mathcal{A}$ called the $\mathcal{N} = 2$ *prepotential* $\mathcal{F}(\mathcal{A})$ as

$$S = \frac{1}{16\pi}\,\mathrm{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta_1\,\mathrm{d}^2\theta_2\,\mathcal{F}(\mathcal{A})\,. \tag{1.4}$$

Moreover, by expanding the prepotential around $\mathcal{A} = A$ we find the action in the minimal superspace (integrating out θ_2 and setting $\theta = \theta_1$)

$$S = \frac{1}{16\pi}\,\mathrm{Im}\int \mathrm{d}^4x\left[\int \mathrm{d}^2\theta\,\frac{\partial^2\mathcal{F}}{\partial A^a\partial A^b}W^{a\alpha}W_\alpha^b + \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\left(A^\dagger e^{-2gV}\right)^a\frac{\partial\mathcal{F}}{\partial A^a}\right]. \tag{1.5}$$

Hereon $a, b, \ldots$ will be Lie algebra indices. The choice of $\mathcal{F}(\mathcal{A}) = \frac{1}{2}\tau\,\mathrm{Tr}\{\mathcal{A}^2\}$ gives back the action (1.1).

1.1.1 The Moduli Space

In the action (1.1), after eliminating the D and F auxiliary fields we find a scalar potential

$$V(\phi) = \frac{1}{2}\,\mathrm{Tr}\left\{\left[\phi, \phi^\dagger\right]\right\}^2. \tag{1.6}$$

Clearly since $\left[\phi, \phi^\dagger\right]$ is real the potential is non-negative. We require thus $V(\phi) = 0$ to find the global minima of the potential that do not break SUSY. However this does not imply that $\phi = 0$. We call the space of gauge inequivalent vacua the *moduli space* of the theory and denote it as $\mathcal{M}$. Let ϕ_0 denote a solution to $V(\phi) = 0$, in general we have $\phi_0 = \frac{1}{2}\sum_{i=1}^3(a_i + ib_i)\sigma_i$. Here σ_i denote the Pauli matrices.

Now via a SU(2) gauge transformation we can always achieve $a_1 = a_2 = 0$[2]. We thus have to impose that $\left[\phi_0, \phi_0^\dagger\right] = 0$. This further constrains b_1 and b_2 to vanish.

[2]there is a simple intuitive way to see this. The adjoint representation of SU(2) is the vector representation of SO(3). For a vector in $\mathbb{R}^3$ we can always choose a reference frame such that the vector is aligned to the z-axis via a rotation.

Letting $a := a_3 + ib_3$, the $\mathbb{Z}_2$ Weyl subgroup of SU(2) further have us identify $a \sim -a$. A gauge invariant quantity is thus $\frac{1}{2}a^2$. Semiclassically we have that $u := \langle \mathrm{Tr}\{\phi^2\}\rangle = \frac{1}{2}a^2$, however quantum correction will change the relation between u and a. Further when $a \neq 0$ the gauge symmetry is broken down to a U(1) of the third component of $\mathfrak{su}(2)$, and the moduli space, classically, has a singularity for $u = 0$ where the full gauge symmetry is restored. Even in the quantum theory, we will find out that the moduli space is the complex u-plane, up to singularities. However in the quantum moduli space the singularity corresponding to the full gauge symmetry being restored will not be part of the connected component to $u = 0$.

1.1.2 The Wilsonian Effective Action for the Light Fields

We now want to find the low-energy Wilsonian effective action for the light (massless) fields. Since SUSY is unbroken and we have a leftover U(1) gauge symmetry after spontaneous symmetry breaking (SSB) we can write the Wilsonian effective action as the action (1.5) for a U(1) theory, i.e. with no Lie algebra indices[3] and no self-interactions:

$$S_W = \frac{1}{16\pi}\,\mathrm{Im}\int \mathrm{d}^4x \left[\int \mathrm{d}^2\theta\, \frac{\partial^2\mathcal{F}}{\partial A\partial A} W^\alpha W_\alpha + \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\, A^\dagger \frac{\partial \mathcal{F}}{\partial A}\right]. \tag{1.7}$$

Using the form of the prepotential in a U(1) gauge theory, which we found in the previous section, i.e. $\mathcal{F}(\mathcal{A}) = \frac{1}{2}\tau\,\mathrm{Tr}\{\mathcal{A}^2\}$, we interpret $\tau(a) := \mathcal{F}''(a)$ as the effective complexified coupling. Since $\mathcal{F}$ is holomorphic, the function $\tau(a)$ is harmonic. Also, expanding the action one finds that $\mathrm{Im}\,\mathcal{F}''(\phi)$ plays the role of a metric in field space in a sigma model for the scalar fields. By taking the expectation value this induces a metric in the moduli space by

$$\mathrm{d}s^2 = \mathrm{Im}\,\mathcal{F}''(a)\,\mathrm{d}a\,\mathrm{d}\bar a = \mathrm{Im}\,\tau(a)\,\mathrm{d}a\,\mathrm{d}\bar a\ . \tag{1.8}$$

Since $\tau(a)$ is harmonic it has no minimum unless it is constant, which is not. This means that the description of the metric in terms of the coordinates a and $\bar a$ on the moduli space given by (1.8) can only be local, since otherwise the metric would not be positive definite.

As computed by Seiberg in [43] the prepotential gets perturbative corrections only at one-loop because of non-renormalization theorems. The perturbative expression of the prepotential is

$$\mathcal{F}_{\mathrm{p}}(\mathcal{A}) = \frac{i}{2\pi}\mathcal{A}^2 \log\frac{\mathcal{A}^2}{\Lambda^2} \tag{1.9}$$

in terms of the dynamically generated scale Λ. Since the microscopic SU(2) theory is asymptotically free, the perturbative expression becomes a good approximation as

[3] technically there is the $\mathfrak{su}(2)$ index 3 everywhere, but it will just be suppressed

$a \to \infty$ since the theory is weakly coupled and the semiclassical relation $u \simeq \frac{1}{2}a^2$ is correct in this limit. We thus find

$$\tau(a) \simeq \frac{i}{\pi}\left(\log\frac{a^2}{\Lambda^2} + 3\right), \tag{1.10}$$

In this weak coupling regime of the theory the coordinates a and $\bar{a}$ are good coordinates.

1.1.3 Duality Transformations

We now show the theory has a strong-weak electromagnetic duality. This means that we can find a dual description where electric and magnetic degrees of freedom as well as strong and weak coupling are exchanged. We define the dual to the chiral field A via a Legendre transformation

$$A_D := \mathcal{F}'(A) \tag{1.11}$$

and a dual prepotential as

$$\mathcal{F}_D(A_D) := \min_A \{\mathcal{F}(A) - AA_D\}\,. \tag{1.12}$$

Using this definitions the second term in the action (1.5) can be rewritten as

$$S \supset \frac{1}{16\pi}\,\mathrm{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,A_D^\dagger \mathcal{F}_D'(A_D)\,. \tag{1.13}$$

Defining the dual for the $\mathcal{N}=1$ vector superfield V is a bit more involved. We notice that when performing the path integral we can either integrate over V, or integrate over W_α imposing the superspace Bianchi identity

$$\mathrm{Im}(D^\alpha W_\alpha) = 0 \tag{1.14}$$

using a Lagrange multiplier V_D, which we will show to be the dual to V. Let us call the first term in the action (1.5) S_1, this is the term we are now concerned with since we already found the dual description of the second term. In formulas what we just said means[4]

$$\int \mathcal{D}V\,\exp(iS_1) = \int \mathcal{D}W_\alpha \mathcal{D}V_D\,\exp\left(iS_1 + \frac{i}{2}\,\mathrm{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,V_D\right). \tag{1.15}$$

Notice that

$$\begin{aligned}\int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,V_D D^\alpha W_\alpha &= -\int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,D^\alpha V_D W_\alpha = \int \mathrm{d}^2\theta\,\bar{D}^2(D^\alpha V_D W_\alpha)\\ &= \int \mathrm{d}^2\theta\,(\bar{D}^2 D^\alpha V_D)W_D = -4\int \mathrm{d}^2\theta\,W_D^\alpha W_\alpha\,.\end{aligned} \tag{1.16}$$

[4]the equal signs between path integrals in this whole thesis will always be understood up to possibly infinite constants

Now we can do the path integral in W_α since it is Gaussian and find

$$\int \mathcal{D}V \exp(iS_1) = \int \mathcal{D}V_D \exp\left[\frac{i}{16\pi}\,\mathrm{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\left(-\frac{1}{\mathcal{F}''(A)}W_D^\alpha W_{D\alpha}\right)\right]. \tag{1.17}$$

Thus the dual form of the action (1.5) is

$$S_W = \frac{1}{16\pi}\,\mathrm{Im}\int \mathrm{d}^4x\left[\int \mathrm{d}^2\theta\left(-\frac{1}{\mathcal{F}''(A)}W_D^\alpha W_{D\alpha}\right) + \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\,A_D^\dagger \mathcal{F}_D'(A_D)\right] \tag{1.18}$$

$$= \frac{1}{16\pi}\,\mathrm{Im}\int \mathrm{d}^4x\left[\int \mathrm{d}^2\theta\,\mathcal{F}_D''(A) W_D^\alpha W_{D\alpha} + \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\,A_D^\dagger \mathcal{F}_D'(A_D)\right]. \tag{1.19}$$

As we claimed the action inverts the strong and weak coupling since we see that the effective coupling $\tau(a)$ is replaced by $-1/\tau(a)$ in the dual action. Furthermore we also want to show that this is indeed an EM duality. It is easy to see this when we think of what we did in ordinary spacetime. When we imposed the Bianchi identity via V_D, in spacetime the vector boson of V_D which we call A_D^μ would couple to the magnetic current $J_m^\mu = \partial_\nu(\star F)^{\nu\mu}$. This means the dual vector field V_D couples to the magnetic degrees of freedom rather than the electric.

The action can be cast as

$$S_W = \frac{1}{16\pi}\,\mathrm{Im}\left\{\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\frac{\mathrm{d}A_D}{\mathrm{d}A}W^\alpha W_\alpha + \frac{1}{32i\pi}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\left(A^\dagger A_D - A_D^\dagger A\right)\right\}. \tag{1.20}$$

In this form it contains both the fields and their duals. In this form we notice the duality transformation

$$\begin{pmatrix} A_D \\ A \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} A_D \\ A \end{pmatrix} \tag{1.21}$$

leaves the action unchanged. There is also another transformation which have no effect on the physics. It is

$$\begin{pmatrix} A_D \\ A \end{pmatrix} \mapsto \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A_D \\ A \end{pmatrix}. \tag{1.22}$$

Under this transformation the action varies as

$$\delta S = \frac{b}{16\pi}\,\mathrm{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,W^\alpha W_\alpha = -\frac{b}{16\pi}\int F\wedge F = -2\pi b k\,, \tag{1.23}$$

k being the instanton number. Since the action appears in the path integral as $\exp(iS)$, as long as $b\in\mathbb{Z}$ the transformation has no effect on the physics. This two transformations generate the group $\mathrm{SL}(2,\mathbb{Z})$ which we call the *duality group* of the theory.

1.1.4 Singularities and Monodromies of the Moduli Space

We now go on to focus on the singularities of the moduli space. In the weak coupling region, i.e. for $u \to \infty$ the perturbative expression for the prepotential (1.9) is exact. In this limit we can calculate a_D: the vacuum expectation value (VEV) of the dual field A_D

$$a_D(u) = \frac{\partial \mathcal{F}(a)}{\partial a} \simeq \frac{i}{\pi} a(u) \left(\log \frac{a^2(u)}{\Lambda^2} + 1 \right). \tag{1.24}$$

Further in this regime $u \sim \frac{1}{2} a^2$, when a goes around a clockwise path around ∞ we have that a and a_D get mapped to

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \mapsto \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix} \tag{1.25}$$

via the monodromy matrix

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}. \tag{1.26}$$

Now the question is: how many more singularities does the moduli space have? We know that ∞ is a branch point and we must have at least another branch point since the branch cut must end somewhere. Now in discussing further the number of singularities we must take into consideration a classical $\mathrm{U}(1)_R$ R-symmetry. Under this R-symmetry θ_A have R-charge -1. Clearly, from the action (1.4) we find that the prepotential must have R-charge 4 in order for this to be a symmetry. In the classical theory where we have $\mathcal{F}_{\mathrm{cl}}(\mathcal{A}) = \frac{1}{2} \tau \operatorname{Tr}\{\mathcal{A}\}^2$, thus $\mathcal{A}$ has to have R-charge 2. However the perturbative correction given by (1.9) break the $\mathrm{U}(1)_R$ symmetry to a $\mathbb{Z}_8$ symmetry. Similarly the non-perturbative correction to the prepotential are also anomalous, however they are also symmetric under the $\mathbb{Z}_8$ symmetry. The full form of the prepotential, including the non-pertubative part is given by

$$\mathcal{F}_{\mathrm{np}}(\mathcal{A}) = \mathcal{F}_{\mathrm{p}}(\mathcal{A}) + \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{\mathcal{A}} \right)^{4k} \mathcal{A}^2 . \tag{1.27}$$

Here the k-th term in the series is due to k instantons. Under this $\mathbb{Z}_8$ symmetry we have that $\phi \mapsto \exp(i\pi n/2)\phi$. This, for odd n means that $\phi^2 \mapsto -\phi^2$, which means, on the moduli space, $u \mapsto -u$. Because of the $\mathbb{Z}_8$ symmetry on the moduli space which maps $u \mapsto -u$, if u_0 is singular then so is $-u_0$. Since the only fixed points of the $\mathbb{Z}_8$ action on the moduli space are 0 and ∞ and we know ∞ is singular, if we want to have only two singularities the other one must be at 0. After all in the classical theory there was a singularity in $u = 0$, where the full gauge symmetry was restored, so one might think the same applies here. However we show that in the quantum theory the point $u = 0$ will not be singular any longer.

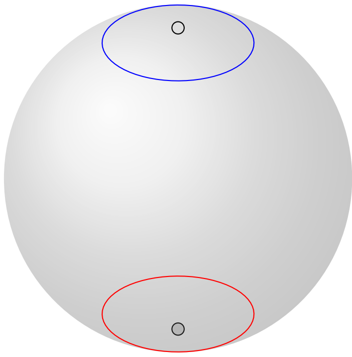

Figure 1.1: The moduli space with only two singularities. The red path can be continuously deformed, avoiding singularities, to the blue path. The orientation is however reversed.

As shown in Figure 1.1, the path enclosing ∞ clockwise can be deformed to a path which encloses the origin anticlockwise. This means that the monodromy matrix around the origin $M_0 = M_\infty$[5]. However this is not possible. In fact it would mean the full monodromy group is generated by M_∞, but a^2 is invariant under M_∞. This would imply that a^2 is a good global coordinate, but, as we saw, this is not possible if the metric is positive definite.

Since it is not possible to have 2 singularities, we should try with 3 singularities. This will mean that the singularities are at some finite $\pm u_0$ and at infinity. However another question is spontaneous now: what do this singularities correspond to? It is tempting again to assume that at least one of them could be because of the full gauge symmetry being restored. However, again this is not the case, let us show this. When the full SU(2) symmetry is restored the gauge bosons are massless. Thus, the IR theory will have SU(2) gauge symmetry and conformal invariance. Recall $u = \langle \mathrm{Tr}\, \phi^2 \rangle$, but this vanishes in a unitary CFT for all operators, except the identity. Thus $u = 0$ which implies that at finite u the singularities will not be caused by the SU(2) symmetry being restored. However the theory has solitonic excitations such as magnetic monopoles and dyons. These are BPS states and their mass is expressed in terms of the central charge of the $\mathcal{N} = 2$ SUSY algebra $Z = n_m a_D + n_e a$. Here n_m and n_e are integers expressing the magnetic and electric charge of the state respectively. We have that a BPS state satisfies

$$m^2 = 2|Z|^2 \tag{1.28}$$

We label the states by the couples (n_m, n_e). Then for example a magnetic monopole will

[5] notice that we understand the monodromy matrix at finite u anticlockwise and the monodromy matrix around infinity clockwise, this is the reason no matrix inversion appears

be denoted as $(1, 0)$. Further the mass of a state has the nice expression in vector form

$$m = \sqrt{2}\left|\begin{pmatrix} n_m & n_e \end{pmatrix}\begin{pmatrix} a_D \\ a \end{pmatrix}\right| . \tag{1.29}$$

In this notation it is clear that the monodromy matrices acting on $\begin{pmatrix} a_D & a \end{pmatrix}^T$ from the left can be considered as well as acting on $\begin{pmatrix} n_m & n_e \end{pmatrix}$ from the right. Let us consider what happens when magnetic monopoles become massless. This corresponds to $a_D = 0$. The theory describing the monopoles is just $\mathcal{N} = 2$ super QED (if you will "QMD" since it is the magnetic dual) with massless monopoles in the $\mathcal{N} = 2$ hypermultiplets. This has the beta function

$$\mu \frac{\mathrm{d}g_D}{\mathrm{d}\mu} = \frac{g_D^3}{8\pi} , \tag{1.30}$$

but the renormalization scale μ is proportional to a_D. Further considering that $\tau_D = \frac{i4\pi}{g_D^2}$ since in QED the instanton angle vanishes, we find

$$a_D \frac{\mathrm{d}}{\mathrm{d}a_D}\tau_D = -\frac{i}{\pi} . \tag{1.31}$$

After integrating this equation we obtain

$$\tau_D = -\frac{i}{\pi} \log a_D . \tag{1.32}$$

Now recall $\tau_D = \frac{\mathrm{d}(-a)}{\mathrm{d}a_D}$ thus

$$a = a_0 + \frac{i}{\pi} a_D \log a_D . \tag{1.33}$$

Since in the vicinity of u_0 we have that a_D must be a good coordinate we have that $a_D \sim c(u - u_0)$. Then using

$$a_D \sim c(u - u_0) , \qquad a \sim a_0 + \frac{i}{\pi} c(u - u_0) \log c(u - u_0) , \tag{1.34}$$

we can already find the monodromy matrix turning around the point u_0 in an anticlockwise sense

$$M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} . \tag{1.35}$$

Here the index 1 is because of the fact that we can renormalize the finite u_0 to 1. We obtained this monodromy matrix by assuming the monopoles becoming massless. We can see this also in another way. Recall expression (1.29) and the fact that the monodromy matrix acting from the left on the vector of VEV can be seen equivalently as acting from the right on the vector of magnetic and electric charges. Then going around the point

1, the BPS state that is massless must invariant under the monodromy, thus be a unit eigenvector of the matrix M_1 acting from the right, i.e.

$$\begin{pmatrix} n_m & n_e \end{pmatrix} M_1 = \begin{pmatrix} n_m & n_e \end{pmatrix} . \tag{1.36}$$

Now let us consider what happens when turning around -1. Notice that turning around infinity is the same as turning around both 1 and -1, thus

$$M_\infty = M_1 M_{-1} . \tag{1.37}$$

In this expression the ordering of the monodromy matrices seems thus far to not be dictated by anything: one could as well consider

$$M_\infty = M_{-1} M_1 . \tag{1.38}$$

Let us proceed with the ordering (1.37) and go back to the ordering issue later. From the equation we find

$$M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \tag{1.39}$$

and using the eigenvector condition again we find that the singularity is due to the $(1, -1)$ dyon becoming massless.

Let us now tackle the ordering ambiguity. Much like homotopy groups the definition of the monodromy group is dependent on the choice of a base point P in the moduli space. Suppose the choice of P gives us the ordering (1.37), then choosing $-P$ as the base point will yield ordering (1.38). The figure Figure 1.2 should help making this argument clearer.

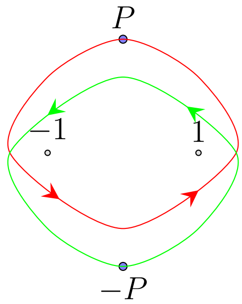

Figure 1.2: The red loop, based at P can be deformed closing around -1 first and 1 after giving ordering (1.37) for the matrices. The green loop, based at $-P$, can be deformed to give the other ordering.

Since the moduli space has the symmetry $u \mapsto -u$, under this symmetry the ordering of the monodromy matrices is exchanged. The second ordering gives

$$M_{-1} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} \tag{1.40}$$

which corresponds to the $(1,1)$ dyon becoming massless. This means the $\mathbb{Z}_2$ symmetry maps the $(1,-1)$ dyon to the $(1,1)$ dyon. At this point we introduce what we will call *democracy transformations*. So far, it looks like the magnetic monopole and the $(1,-1)$ (or $(1,1)$) dyon play a privileged role in the theory. However let us consider the monodromy matrix obtained by conjugating $M_{\pm 1}$ by M_∞^p. When conjugating M_1 we get the matrices corresponding to the $(1,-2p)$ dyons, while when conjugating M_{-1} we obtain the $(1,-2p-1)$ dyons. This means that there is a complete democracy among the dyons.

Mathematically we notice that the monodromy matrices all belong the the duality group $\mathrm{SL}(2,\mathbb{Z})$, meaning the monodromy group is represented as duality transformations. Further there is a matrix A acting by conjugation which implements the $\mathbb{Z}_2$ symmetry of the moduli space on the $\mathrm{SL}(2,\mathbb{Z})$ representation of the monodromy group. This is such that

$$M_{-1} = AM_1A^{-1} \tag{1.41}$$

and A is given by

$$A = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix} . \tag{1.42}$$

Note that $A^2 = -1$ is equivalent to the identity automorphism acting on $\mathrm{SL}(2,\mathbb{Z})$ by conjugation.

Finally for this subsection let us go back to the question of how many singularities the moduli space has. We know we have a perturbative singularity at $u = \infty$. Let us suppose we have p strong coupling singularities at $u_1, \ldots, u_p$. Then

$$M_\infty = M_{u_1} \ldots M_{u_p} . \tag{1.43}$$

In general the matrix corresponding to the (n_m, n_e) dyon becoming massless is given by

$$M(n_m, n_e) = \begin{pmatrix} 1 + 2n_m n_e & 2n_e^2 \\ -2n_m^2 & 1 - 2n_m n_e \end{pmatrix} \tag{1.44}$$

and thus each M_{u_i} is given by

$$M_{u_i} = M(n_m^{(i)}, n_e^{(i)}) \tag{1.45}$$

for some $n_m^{(i)}$ and $n_e^{(i)}$ integers for any $i = 1, \ldots, p$. Whether solutions to this problem exist for generic p is a number theory problem. It has been checked for small values of p that solutions exist only for $p = 2$ [28].

1.1.5 The Determination of the Low Energy Effective Action

Here we follow the approach of Bilal in [13] called the *differential equation approach*. We will comment later on the original approach used by Seiberg and Witten. We start by considering a general differential equation of the Schroedinger form

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}z^2}+V(z)\right]\psi(z)=0 \tag{1.46}$$

for a meromorphic $V(z)$ which has poles at $z_1,\ldots,z_p$ and possibly at infinity. The solutions lie in a two dimensional vector space spanned by a basis of two linear independent solutions ψ_1 and ψ_2. We know $V(z)$ is single-valued thus, when taking z in a closed path around a singularity z_i, the differential equation remains unchanged. However the two solutions ψ_1 and ψ_2 can still pick up a monodromy:

$$\begin{pmatrix}\psi_1(z+e^{2\pi i}(z-z_i))\\ \psi_2(z+e^{2\pi i}(z-z_i))\end{pmatrix}=M_i\begin{pmatrix}\psi_1(z)\\ \psi_2(z)\end{pmatrix}. \tag{1.47}$$

The strategy to determine the low-energy effective action exactly is as follows: first we want to find a differential equation which produces the correct monodromy matrices we found previously. Thus we think ψ_1 and ψ_2 as a_D and a respectively and z as u. Second, by solving the differential equation, we can find the solution for $a_D(u)$ and $a(u)$, which in turn can be used to recover the full non-perturbative prepotential of the low energy effective action. It is known that non-trivial constant monodromies correspond to poles that are at most second order. Thus the most general form of the potential that can produce these monodromies at $z=\pm1,\infty$ is

$$V(z)=-\frac{1}{4}\left[\frac{1-\lambda_1^2}{(z+1)^2}+\frac{1-\lambda_2^2}{(z-1)^2}-\frac{1-\lambda_1^2-\lambda_2^2-\lambda_3^2}{(z+1)(z-1)}\right]. \tag{1.48}$$

The differential equation which we obtain by the choice of the potential is well-known and can be cast in the form of a hypergeometric differential equation via the transformation

$$\psi(z)=(z+1)^{\frac{1}{2}(1-\lambda_1)}(z-1)^{\frac{1}{2}(1-\lambda_2)}f\left(\frac{z+1}{2}\right) \tag{1.49}$$

which gives us the hypergeometric equation for f (here we define $x:=\frac{z+1}{2}$)

$$x(1-x)f''(x)+[c-(a+b+1)x]f'(x)-abf(x)=0 \tag{1.50}$$

for

$$a=\frac{1}{2}(1-\lambda_1-\lambda_2+\lambda_3)\,,\qquad b=\frac{1}{2}(1-\lambda_1-\lambda_2-\lambda_3)\,,\qquad c=1-\lambda_1\,. \tag{1.51}$$

There are multiple ways in which the solutions to (1.50) can be written. A convenient choice is given in terms of the hypergeometric functions ${}_2F_1(a,b,c;x)$ by

$$\begin{aligned}f_1(x)&=(-x)^{-a}{}_2F_1\left(a,a+1-c,a+1-b;\frac{1}{x}\right),\\ f_2(x)&=(1-x)^{c-a-b}{}_2F_1(c-a,c-b,c+1-a-b;1-x)\,.\end{aligned} \tag{1.52}$$

This choice is motivated by the fact that f_1 has simple monodromy properties around $x = \infty$ ($z = \infty$) and f_2 has simple monodromy properties around $x = 1$ ($z = 1$), so they are good candidates to be identified with $a(u)$ and $a_D(u)$ respectively. Now we will use the asymptotic forms of $a_D(u)$ and $a(u)$ to further constrain the form of $V(z)$. For $z = \infty$ the potential behaves as $V(z) \sim -\frac{1}{4}\frac{1-\lambda_3^2}{z^2}$. This means the solutions behave asymptotically as either $z^{\frac{1}{2}(1\pm\lambda_3)}$ if $\lambda_3 \neq 0$ or as $\sqrt{z}$ and $\sqrt{z}\log z$ if $\lambda_3 = 0$. From the asymptotics of $a_D(u)$ and $a(u)$ we conclude thus $\lambda_3 = 0$. Similarly we consider the asymptotic form of the potential at $z = 1$ which is $V(z) \sim -\frac{1}{4}\left[\frac{1-\lambda_2^2}{(z-1)^2} - \frac{1-\lambda_1^2-\lambda_2^2}{2(z-1)}\right]$. Again using the asymptotic forms of $a_D(u)$ and $a(u)$ we find $\lambda_2 = 1$. Finally by the $\mathbb{Z}_2$ symmetry also $\lambda_1 = 1$, thus

$$V(z) = -\frac{1}{4}\frac{1}{(z+1)(z-1)}\,. \tag{1.53}$$

We can thus solve the differential equation and it turns out that the functions f_1 and f_2 do indeed satisfy the correct monodromy properties (up to some multiplicative constants, details about this can be read in [13]), thus we find

$$\begin{aligned} a_D(u) &= i\frac{u-1}{2}\,{}_2F_1\left(1/2, 1/2, 2; \frac{1-u}{2}\right), \\ a(u) &= \sqrt{2(u+1)}\,{}_2F_1\left(-1/2, 1/2, 1; \frac{2}{u+1}\right). \end{aligned} \tag{1.54}$$

Using the integral representation of the hypergeometric functions [39] we find

$$\begin{aligned} a_D(u) &= \frac{\sqrt{2}}{\pi}\int_1^u \mathrm{d}x\,\frac{\sqrt{x-u}}{\sqrt{x^2-1}}\,, \\ a(u) &= \frac{\sqrt{2}}{\pi}\int_{-1}^1 \mathrm{d}x\,\frac{\sqrt{x-u}}{\sqrt{x^2-1}}\,. \end{aligned} \tag{1.55}$$

We can thus, in principle, invert the second of these equations to find u in terms of a and then use the first to have a_D in terms of a. Then use the equation $a_D = \frac{\partial \mathcal{F}}{\partial a}$ to solve for the prepotential and thus the exact low-energy effective action. This gives only a locally valid expression of $\mathcal{F}(a)$ and different analytic continuations must be used to address different parts of the moduli space.

A Note on the Elliptic Curve Approach of Seiberg and Witten

In the original paper [41] Seiberg and Witten did not use the differential equation approach. They used a different approach based on elliptic curves which we will now justify a posteriori. Consider the integral representation given in (1.55). The integrand has square root branch cuts with branch points ± 1, u and ∞. We can take the branch cuts to run between -1 and 1 and between u and ∞. The Riemann surface of the integrand has two sheets which are connected through the branch cuts. Adding the point at infinity the Riemann surface has the topology of two 2-spheres connected by two tubes: the cuts. This is just the topology of a torus, which is the elliptic curve considered by Seiberg and Witten which we will call the *Seiberg-Witten curve*.

It is well-known that a torus has two independent cycles: one of them, call it γ_2 can be taken to run around the cut which goes from -1 to 1. The other, γ_1, goes from 1 to u on one sheet and goes back from u to 1 on the second sheet. Refer to Figure 1.3 for a visual representation.

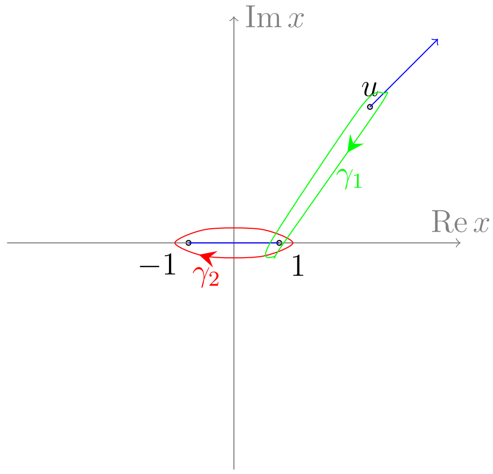

Figure 1.3: The two independent cycles on the complex x-plane.

The solutions a and a_D correspond precisely to the integration of a suitable differential λ, which we call the *Seiberg-Witten differential*, on these cycles. The differential is given by

$$\lambda = \frac{\sqrt{2}}{2\pi} \frac{\sqrt{x-u}}{\sqrt{x^2-1}} \, \mathrm{d}x \tag{1.56}$$

and the integrals, which give a_D and a, are called the *periods* of the differential:

$$\begin{aligned} a_D &= \oint_{\gamma_1} \lambda \,, \\ a &= \oint_{\gamma_2} \lambda \,. \end{aligned} \tag{1.57}$$

We will thus refer to a_D and a as the *Seiberg-Witten periods*. It is known that these periods satisfy a second-order differential equation called the Picard-Fuchs equation, which is precisely the equation we found in the differential equation approach.

1.2 The Spectrum of the Pure Gauge Theory

In this section we want to find the spectrum of $\mathcal{N} = 2$ SU(2) pure gauge theory both at strong coupling and weak coupling. The discussion will follow the paper by Ferrari and Bilal [18]. In particular we will see that there is a curve in the moduli space of particular importance called the *curve of marginal stability* (CMS) across which the spectrum of the stable particles changes and which defines in the moduli space a strong coupling regions and a weak coupling region.

We already mentioned BPS states in Section 1.1, however it is worth reviewing them again using a different notation. Let us call $\Omega := (a_D, a)$ (this should be seen as a column vector, but for typographical purpose is denoted as a row in this context). Then we notice that this is a section of bundle with fibre $\mathbb{C}^2$ associated to a $\mathrm{SL}(2,\mathbb{Z})$-bundle over the moduli space $\mathcal{M}$ via the fundamental representation of $\mathrm{SL}(2,\mathbb{Z})$. We also consider the associated bundle with fibre $\mathbb{Z}^2$ via the dual representation of $\mathrm{SL}(2,\mathbb{Z})$: the locally constant sections $p = (n_m, n_e)$ (this should be seen as a row vector since it lives in the dual representation) of this bundle correspond to BPS states and their mass is given by the $\mathrm{SL}(2,\mathbb{Z})$-invariant expression

$$m = \sqrt{2}|p\Omega| \tag{1.58}$$

in terms of the product $p\Omega := n_m a_D + n_e a$. In simpler terms under a duality transformation $M \in \mathrm{SL}(2,\mathbb{Z})$ we have

$$\begin{aligned} \Omega &\mapsto M\Omega\,, \\ p &\mapsto pM^{-1} \end{aligned} \tag{1.59}$$

and thus $p\Omega \mapsto p\Omega$. Here again we see Ω as a column vector and p as a row vector (which corresponds to saying Ω is in the fundamental representation and p in the dual). We found the exact form of Ω in the previous section. This is given by (1.54). We notice that a_D has a branch cut on the real line from $-\infty$ to -1 while a has two branch cuts on the real line from $-\infty$ to -1 and from -1 to 1.

Consider now that $\mathrm{Im}\{a_D/a\} \neq 0$, then for a given Ω the BPS states live in a two dimensional lattice in the complex Z-plane. Let us analyze a possible decay process of a BPS state $p = (n_m, n_e)$ with mass $m = \sqrt{2}|Z|$ into other states (which are not necessarily BPS) $p_i = \left(n_m^{(i)}, n_e^{(i)}\right)$ with masses $m_i \geq \sqrt{2}|Z_i|$. Since the charges are conserved we have

$$Z = \sum_i Z_i \tag{1.60}$$

and from the triangular inequality

$$|Z| \leq \sum_i |Z_i| \tag{1.61}$$

which in turn implies

$$m \leq \sum_i m_i \,. \tag{1.62}$$

Of course the decay is only possible if $m \geq \sum_i m_i$, thus, if the decay is possible we have

$$m = \sum_i m_i \,. \tag{1.63}$$

This equality can be achieved only if the states p_i are also BPS saturated (i.e. $m_i = \sqrt{2}|Z_i|$) and Z and Z_i are all aligned on the complex plane. Assuming $\text{Im}\{a_D/a\} \neq 0$ this can be achieved only when the initial vector of charges (n_m, n_e) and those of the final particles $(n_m^{(i)}, n_e^{(i)})$ are proportional, which in turn means that the initial vector of charges can be written as (qm, qn) for integers m, n and q. This state is in fact unstable under the decay to q (m, n) dyons. All the states for which n_m and n_e are relatively prime are stable for $\text{Im}\{a_D/a\} \neq 0$.

The curve $\text{Im}\{a_D/a\} = 0$ is of special importance when studying the spectrum of stable BPS states and is called the *curve of marginal stability* (CMS) and denote it as $\mathcal{C}$. When a_D/a is real the two dimensional lattice of BPS states in the complex Z-plane collapses to a one dimensional lattice. This in turn simplify greatly the conditions for the decay to be possible (in particular it is easier that the triangle inequality holds with the equals sign) and states that where before stable may become unstable. The CMS can be determined from the Picard-Fuchs equation. It is known that the ratio of the Seiberg-Witten periods $w(u) := a_D(u)/a(u)$ satisfy the equation

$$\{w, u\} = -2V(u) \,. \tag{1.64}$$

Here $\{w, u\}$ denotes the Schwartzian derivative and $V(u)$ is the potential of the Picard-Fuchs equation. Then the curve $\mathcal{C}$ is given by

$$\mathcal{C} := \{u \in \mathcal{M} |\, \text{Im}\, w(u) = 0\} \,. \tag{1.65}$$

It can be found numerically, however its precise form does not matter in what follows. What matters is that it is made of a single connected component which resembles an ellipse (however it is not an ellipse) which splits the moduli space in two connected components. The component which is connected to $u = 0$ is called the strong coupling region and denoted as $\mathcal{R}_S$, the other component, connected to ∞ is called the weak coupling region and denoted as $\mathcal{R}_W$. In Figure 1.4 a visual representation is available.

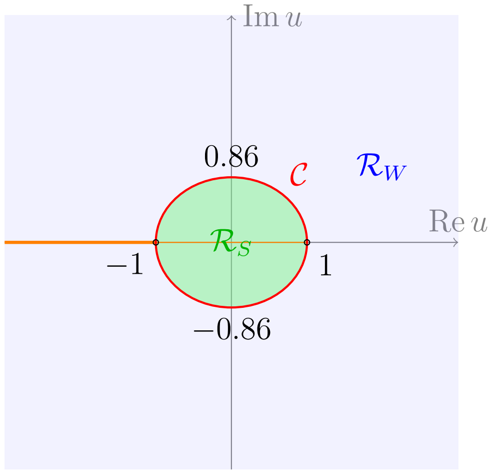

Figure 1.4: The CMS $\mathcal{C}$ (in red) divides the moduli space in a strong coupling region $\mathcal{R}_S$ (shaded green) and a weak coupling region $\mathcal{R}_W$ (shaded blue). The exact form of the CMS is not important and it has been plotted as an ellipse even though it is not. The singular points of the moduli space have been denoted by empty circles (the point at infinity however is not displayed). The two branch cuts are shown in orange with different thickness.

An interesting fact about the CMS is that massless BPS states can only exist on it. The reason is that for a massless BPS state we have $a_D/a = -n_e/n_m$ which is rational and thus real. Another important fact is that a_D/a can take any value in $[-1, 0]$ for $u \in \mathcal{C}^+$ and any value in $[0, 1]$ for $u \in \mathcal{C}^{-1}$. This can be shown from the explicit expressions of the Seiberg-Witten periods as explained in [18]. All of this means that a BPS state such that $n_e/n_m \in [-1, 1]$ will become massless somewhere on the CMS. But when a charged particle becomes massless there will be a singularity in the effective gauge coupling and thus a singularity in the Wilsonian effective action. Since there are precisely two of these singularities at $u = \pm 1$ which correspond to the magnetic monopole $(1, 0)$ and the dyon described either by $(1, -1)$ or $(1, 1)$, we know that both of these particles belong both to the strong coupling and the weak coupling spectrum since they are the only particles able to yield these singularities. Further from the absence of other singularities we conclude there are no other particles which become massless on the CMS.

We call the spectrum of stable BPS states in the strong coupling region $\mathcal{S}_S$ and the spectrum in the weak coupling region $\mathcal{S}_W$. These spectra are independent of which point of the moduli space we pick in each region. However in the strong coupling region the same BPS state might correspond in different points to two different local sections of the $\mathrm{SL}(2, \mathbb{Z})$-bundle. This is caused by the fact that the bundle is not trivial because of the singularities and branch cuts of the moduli space. To see this let us consider a local and constant section in the open set $\mathcal{R}_S^+ := \mathcal{R}_S \cap \{u \in \mathcal{M} |\, \mathrm{Im}\, u > 0\}$ which we call

$p = (n_m, n_e)$. The mass of the BPS state is given then by

$$m(u) = \sqrt{2}|p\Omega(u)|\,. \tag{1.66}$$

This changes smoothly as we move u in the strong coupling region, even when we cross the branch cut between -1 and 1. However after crossing the branch cut the mass will no longer be expressed in terms of $\Omega(u)$, but in terms of the analytic continuation $\tilde{\Omega}(u) = M_1\Omega(u)$. Then since the mass does not change the new description of the BPS state in the lower half of the strong coupling region is given by the section $\tilde{p} = pM_1^{-1}$. Thus even though there is a unique spectrum of BPS states $\mathcal{S}_S$ in the strong coupling region, we have to introduce two different set of local sections which we call $\mathcal{S}_S^+$ and $\mathcal{S}_S^-$ respectively for the regions $\mathcal{R}_S^+$ and $\mathcal{R}_S^-$.

In the weak coupling region a BPS state can be represented by a single pair of integer charges since any two points $u, u' \in \mathcal{R}_W$ can be joined by a path which does not go through any cut. We already saw that the moduli space has a $\mathbb{Z}_2$ symmetry mapping $u \mapsto u' = -u$. We want to see how this acts on the BPS states. We want the mass to be invariant under this $\mathbb{Z}_2$, thus we find the general transformation

$$\begin{aligned} \Omega(u') &= e^{i\omega} G\Omega(u)\,, \\ p' &= pG^{-1}\,, \end{aligned} \tag{1.67}$$

for some $G \in \mathrm{SL}(2,\mathbb{Z})$. The appearance of the phase $e^{i\omega}$ means that the transformation between $\Omega(u')$ and $\Omega(u)$ is not in general a duality transformation. However it is obvious that the mass of a BPS state does not change under this transformation and further it can be shown that the Lagrangian is symmetric under this U(1). In the weak coupling region as shown in [18] the transformation is given by

$$\begin{aligned} \Omega(-u) &= -i\epsilon G_{W,\epsilon}\Omega(u)\,, \\ G_{W,\epsilon} &:= \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{1.68}$$

for $\epsilon := \mathrm{sgn}(\mathrm{Im}\, u)$.

In the strong coupling region the situation is different because to go from u to $-u$ one has either pass through $\mathcal{C}$ twice or go through the branch cut. Since the spectrum changes across $\mathcal{C}$, we have to go through the cut and thus be careful which analytic continuation we choose for Ω. Again we refer to [18] for details. The result is

$$G_{S,\epsilon} := M_1^{\epsilon} G_{W,\epsilon} = \begin{pmatrix} 1 & \epsilon \\ -2\epsilon & -1 \end{pmatrix}. \tag{1.69}$$

1.2.1 Determination of the Spectrum

Now we discuss the spectrum in the weak coupling and strong coupling regions. When talking about both spectra we leave implicit the fact that both contain an everywhere massless $\mathcal{N} = 2$ abelian gauge field (photon).

The Weak Coupling Spectrum

We will show that the spectrum in the weak coupling region is composed of the massive $W^{\pm}$ bosons $(0,1)$[6] and the dyons $(n,1)$ for $n \in \mathbb{Z}$. The W bosons belong to the perturbative spectrum, thus they trivially belong to $\mathcal{S}_W$. Moreover the monopole $(1,0)$ also belongs to $\mathcal{S}_W$ since this becomes massless on the CMS. Now we apply the $\mathbb{Z}_2$ symmetry. The $\mathbb{Z}_2$ symmetry (1.68) transforms the monopole $(1,0)$ into the dyon $(1,-\epsilon)$, which thus must also belong to $\mathcal{S}_W$. Iterating this we find all the dyons $(n,1)$ for $n \in \mathbb{Z}$. The spectrum is thus invariant under the $\mathbb{Z}_2$ symmetry.

We already shown that the particles we claimed were in the spectrum are indeed in the spectrum. The only thing left to show is that they are the only particles in the spectrum. Suppose the state (n_m, n_e) is in $\mathcal{S}_W$. If $n_m = 0$ this is either the W boson or an unstable particle. So consider $n_e \neq 0$. Then again using the $\mathbb{Z}_2$ symmetry we find all the states of the form $(n_m, n_e + kn_m) =: (n_m, n_e^{(k)})$ for $k \in \mathbb{Z}$ are in the spectrum. Of course there exists a k_0 such that $n_e^{(k_0)}/n_m = n_e/n_m + k_0 \in [-1,1]$, which means that this state will become massless somewhere on the CMS. Since we know the only states that become massless on the CMS are the monopole and the $(1,-1)$ or $(1,1)$ dyon we have $n_m = 1$ and thus $n_e^{(k)} = n_e + k$, which are all the dyons we already found to be in the spectrum.

The Strong Coupling Spectrum

On the other hand the strong coupling spectrum is made only of the magnetic monopole $(1,0)$ and the dyon which is either represented as $(1,-1)$ on $\mathcal{R}_S^+$ or $(1,1)$ on $\mathcal{R}_S^-$. The proof is similar to the one for the weak coupling spectrum and can be read in detail in [18]. The only thing that makes this proof slightly more involved is the fact that the same dyons are represented by locally different sections on $\mathcal{R}_S$.

1.3 The Theory with Fundamental Matter

In this section we will see how the result of Section 1.1 is modified by adding N_f $\mathcal{N}=2$ hypermultiplets in the fundamental representation[7] of SU(2). When adding this hypermultiplets we want to preserve asymptotic freedom: the beta function is negative for $N_f \leq 3$. Further in the case $N_f = 4$ the beta function vanishes. The $\mathcal{N}=2$ hypermultiplets are composed of two $\mathcal{N}=1$ chiral multiplets, which we call Q^f and $\tilde{Q}_f$, each in the fundamental representation of SU(2) (for generic gauge groups one would be in the fundamental while the other in the antifundamental representation however for SU(2)

[6]when using this pair notation we denote collectively the particle (n_m, n_e) and its antiparticle $(-n_m, -n_e)$

[7]by fundamental matter we mean it is in the fundamental representation of SU(2)

they are isomorphic). Here the index f is the flavour index ranging from 1 to N_f, also we will call the scalars in each chiral multiplet q^f and $\tilde{q}_f$ and the Weyl spinors χ^f and $\tilde{\chi}_f$ respectively. To the action (1.1) we have to add the canonical kinetic terms and minimal gauge coupling for the hypermultiplet and the superpotential

$$W = \sqrt{2}\tilde{Q}_f A Q^f + \sum_f m_f \tilde{Q}_f Q^f \,, \tag{1.70}$$

which makes the hypermultiplets massive, each with a mass of m_f. The first term of the superpotential is required to preserve $\mathcal{N} = 2$ SUSY and it is related to the minimal gauge coupling terms for the hypermultiplets.

The classical global symmetry of the theory with massless hypermultiplets is a certain quotient of $\mathrm{O}(2N_f) \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_R$. The $\mathrm{O}(2N_f)$ is the flavour symmetry, in the case of the SU(2) gauge group it is enhanced from $\mathrm{O}(N_f)$ since the fundamental and anti-fundamental representations of SU(2) are isomorphic, thus under the flavour symmetry Q^f and $\tilde{Q}_f$ can mix. The $\mathrm{SU}(2)_R$ symmetry is the one we mentioned in Section 1.1: the scalars of each chiral multiplet Q^f and $\tilde{Q}_f$ are in the doublet of this $\mathrm{SU}(2)_R$. Finally the $\mathrm{U}(1)_R$ is also the same symmetry we mentioned in Section 1.1: the hypermultiplet scalars have R-charge 0 under this $\mathrm{U}(1)_R$. Another important fact is that the flavour symmetry group, being $\mathrm{O}(2N_f)$ and not just $\mathrm{SO}(2N_f)$, includes a "parity"[8] which we call ρ that acts on the scalars q_1 and $\tilde{q}_1$ as

$$\rho : q^1 \mapsto \tilde{q}_1 \,, \tilde{q}_1 \mapsto q^1 \tag{1.71}$$

and leaves all the other scalars invariant. For precision's sake we mention that, globally, the symmetry group is not just the product of the super-Poincaré group with $\mathrm{O}(2N_f) \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_R$. This is because, for example, the $\mathbb{Z}_2$ subgroup of $\mathrm{U}(1)_R$ is isomorphic to the subgroup of the super-Poincaré generated by $(-1)^F$. Thus the direct product would "over count" the symmetries.

To determine the classical moduli space it is sufficient, by definition, to find all the minima of the potential appearing in the action and quotient out the gauge transformations. For any N_f there is always a flat direction in the potential[9] with non-zero ϕ. Along this direction the gauge symmetry is spontaneously broken to U(1) and thus this is called the *Coulomb branch*. In the $N_f = 0$ case the moduli space consisted only of the Coulomb branch, this is similar in the $N_f = 1$ case. However for $N_f \geq 2$, when $m_f = 0$, more flat directions in the potential appear. Along these the gauge symmetry is completely broken and together they form the *Higgs branch* of the moduli space. On the Higgs branch A vanishes. The flat directions in the q^f and $\tilde{q}_f$ space are found by setting

[8] the quotes are because of the fact that this is not a parity transformation in spacetime. It is called a parity because it has determinant -1

[9] a flat direction in the potential is just a continuous line of minima, thus they correspond to the classical moduli space when gauge transformations are modded out

to zero the corresponding D-terms, imposing the superpotential is stationary and then dividing out the gauge group SU(2). As in [42] the combined operation of dividing the gauge group and setting the D-terms to vanish is equivalent to divide out by $\mathrm{SL}(2,\mathbb{C})$. It is useful to introduce $\mathrm{O}(2N_f)$ vector indices $r,s=1,\ldots,2N_f$ and denote collectively both q^f and $\tilde{q}_f$ as q^r (say the first N_f are the ones with no tilde, the remaining the ones with tilde). Then one can define the $\mathrm{SL}(2,\mathbb{C})$-invariant functions ($i$ is a fundamental representation index, recall the complexification of $\mathfrak{su}(2)$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$)

$$V^{rs}=q^r_i q^{si}\,. \tag{1.72}$$

Clearly $V^{rs}=-V^{sr}$. Moreover they also satisfy some quadratic constraint, for example in the $N_f=2$ case this is $\epsilon_{rspq}V^{rs}V^{pq}=0$. Now we still have to impose the superpotential be stationary. Since the superpotential is linear in A and $A=0$ the only non-trivial condition is $\frac{\partial W}{\partial A}=0$ which is equivalent to

$$\sum_{t=1}^{2N_f}V^{rt}V^{ts}=0\,. \tag{1.73}$$

An example of the determination of the classical moduli space for $N_f=2$ can be found in [42].

Now let us consider the quantum modification of the classical picture of the moduli space. As in the $N_f=0$ case the global $\mathrm{U}(1)_R$ is anomalous. Also, the parity transformation $\rho\in\mathrm{O}(2N_f)$ and the $\mathbb{Z}_2$ group it generates is anomalous. However a discrete $\mathbb{Z}_{4(4-N_f)}$ subgroup of this $\mathrm{U}(1)_R\times\mathbb{Z}_2$ is anomaly free for $N_f>0$[10]. This anomaly free subgroup is generated by

$$\begin{aligned}
W_\alpha &\mapsto e^{\frac{i\pi}{2(4-N_f)}}W_\alpha\Big(e^{-\frac{i\pi}{2(4-N_f)}}\theta\Big)\,,\\
A &\mapsto e^{\frac{i\pi}{4-N_f}}A\Big(e^{-\frac{i\pi}{2(4-N_f)}}\theta\Big)\,,\\
q^1 &\mapsto \tilde{q}_1\Big(e^{-\frac{i\pi}{2(4-N_f)}}\theta\Big)\,,\\
\tilde{q}_1 &\mapsto q^1\Big(e^{-\frac{i\pi}{2(4-N_f)}}\theta\Big)\,.
\end{aligned} \tag{1.74}$$

Let us only consider the Coulomb branch, the discussion regarding the Higgs branch can be found for example in [42] and will not be as important for us. Let us parametrize, as in the $N_f=0$ case, the Coulomb branch by the gauge invariant modulus $u:=\langle\operatorname{Tr}\phi^2\rangle$. As for the $N_f=0$ case the masses of the dyons and the metric are determined by Seiberg-Witten periods. The objective now is finding the exact expression of the Seiberg-Witten periods in the $N_f=1,2,3,4$ cases as we found them in the $N_f=0$ case.

[10]In the $N_f=0$ case this subgroup is $\mathbb{Z}_8$ as we saw already

1.3.1 The Seiberg-Witten Curve for any N_f

In general the Seiberg-Witten curve can be cast in as [24]

$$K(p) - \frac{\bar{\Lambda}}{2}\big(K_+(p)e^{ix} + K_-(p)e^{-ix}\big) = 0\,, \tag{1.75}$$

for

$$\begin{aligned}
\bar{\Lambda} &:= \begin{cases} \Lambda_{N_f}^{2-\frac{N_f}{2}} & \text{for } N_f = 0,1,2,3 \\ \sqrt{q} & \text{for } N_f = 4 \end{cases}\,, \\
K(p) &:= \begin{cases} p^2 - \tilde{u} & \text{for } N_f = 0,1 \\ p^2 - \tilde{u} + \frac{\Lambda_2^2}{8} & \text{for } N_f = 2 \\ p^2 - \tilde{u} + \frac{\Lambda_3}{4}\big(p + \frac{m_1+m_2+m_3}{2}\big) & \text{for } N_f = 3 \\ \big(1+\frac{q}{2}\big)p^2 - \tilde{u} + \frac{q}{4}(\sum_i m_i) + \frac{q}{8}\sum_{i<j} m_i m_j & \text{for } N_f = 4 \end{cases}\,, \\
K_+(p) &:= \begin{cases} 1 & \text{for } N_f = 0 \\ \displaystyle\prod_{i=1}^{N_+}(p+m_i) & \text{for } N_f = 1,2,3,4 \end{cases}\,, \\
K_-(p) &:= \begin{cases} 1 & \text{for } N_f = 0 \\ \displaystyle\prod_{i=N_++1}^{N_f}(p+m_i) & \text{for } N_f = 1,2,3,4 \end{cases}\,.
\end{aligned} \tag{1.76}$$

Here N_+ is any integer between 1 and N_f included, m_i are the masses of the hypermultiplets, Λ_{N_f} are the dynamically generated scales in the theory with N_f multiplets, $\tilde{u}$ is the Coulomb branch modulus u for $N_f \leq 2$ for $N_f = 3$ $\tilde{u}$ is related to the Coulomb branch modulus u in the way described in [16, 3]. For $N_f = 4$ [16], which is our main interest, we have that $q = e^{2\pi i\tau}$ for τ the coupling of the microscopic SU(2) theory. Also the relation between $\tilde{u}$ and u is, in terms of the effective U(1) coupling τ_{eff},

$$\tilde{u} = \left(\frac{\mathrm{d}\tau_{\text{eff}}}{\mathrm{d}\tau}\right)^{-1} u\,. \tag{1.77}$$

Then by integrating $p\,\mathrm{d}x$ over the two non-contractible cycles we obtain the Seiberg-Witten periods

$$\begin{aligned}
a_D &= \oint_{\gamma_1} p(x)\,\mathrm{d}x\ , \\
a &= \oint_{\gamma_2} p(x)\,\mathrm{d}x\ .
\end{aligned} \tag{1.78}$$

Further, one can introduce $y = \bar{\Lambda} K_+(p) e^{ix} - K(p)$ and obtain the Seiberg-Witten curve in the form which is found in [42]:

$$y^2 = K^2(p) - \bar{\Lambda}^2 K_+(p) K_-(p)\,. \tag{1.79}$$

1.4 The Ω-deformation and AGT Correspondence

To introduce the Nekrasov-Shatashvili deformed Seiberg-Witten theory we first need to introduce the so-called *Ω-background.* One can find a 6 dimensional $\mathcal{N} = 1$ theory whose dimensional reduction gives Seiberg-Witten in 4 dimensions[11]. Then the 6 dimensional theory is studied on a $\mathbb{R}^4$ vector bundle over the torus T^2 with a flat $\mathrm{Spin}(4) \cong \mathrm{SU}(2)_\ell \times \mathrm{SU}(2)_r$ connection, whose holonomies around the two non-contractible cycles of the torus are (r here is the radius of both cycles of the torus)

$$\left(e^{i\frac{r}{2}\,\mathrm{Re}(\epsilon_1+\epsilon_2)\sigma_3}, e^{i\frac{r}{2}\,\mathrm{Re}(\epsilon_1-\epsilon_2)\sigma_3}\right), \qquad \left(e^{i\frac{r}{2}\,\mathrm{Im}(\epsilon_1+\epsilon_2)\sigma_3}, e^{i\frac{r}{2}\,\mathrm{Im}(\epsilon_1-\epsilon_2)\sigma_3}\right). \tag{1.80}$$

Finally the right-handed part of Spin(4) is embedded in the $\mathrm{SU}(2)_R$ R-symmetry group and the limit $r \to 0$ is taken while keeping ϵ_1 and ϵ_2 finite and obtaining the Ω-background, which is used to compute the instanton contributions to the prepotential [35].

In more modern terms we can look at this through the *Alday-Gaiotto-Tachikawa (AGT) correspondence* [1]. For what concerns us the AGT correspondence relates Liouville field theory on the Riemann sphere with four punctures to the Ω-deformed SU(2) Seiberg-Witten theory in four dimensions with $N_f = 4$. The AGT dictionary states that the *Liouville coupling* b is related to the deformation parameters via

$$b = \sqrt{\frac{\epsilon_2}{\epsilon_1}}\,. \tag{1.81}$$

As usual in Liouville field theory the *background charge* is given by $Q = b + b^{-1}$ and the *central charge* is $c = 1 + 6Q^2$. In this picture the object of interest is the five point function

$$\Psi(z,t) := \langle \Phi_{2,1}(z) V_{\alpha_0}(0) V_{\alpha_t}(t) V_{\alpha_1}(1) V_{\alpha_\infty}(\infty)\rangle \tag{1.82}$$

with an insertion of the degenerate operator $\Phi_{2,1}$. The *Liouville momenta* α are related the the conformal dimension of each field via

$$\Delta = \alpha(Q - \alpha) \tag{1.83}$$

[11] a check on the number of supercharges: Seiberg-Witten in $d = 4$ dimensions with $\mathcal{N} = 2$ has a total of $\mathcal{N} 2^{[d/2]} = 8$ supercharges. A $\mathcal{N} = 1$ theory in $d = 6$ has, by the same formula, the same number of supercharges

and the conformal dimension of the degenerate field is

$$\Delta_{2,1} = -\frac{1}{2} - \frac{3}{4}\frac{\epsilon_2}{\epsilon_1}\,. \tag{1.84}$$

The field $\Phi_{2,1}$ has a null-vector at level 2 whose null vector condition becomes the BPZ equation [11, 4]

$$\left[\frac{\epsilon_1}{\epsilon_2}\frac{\partial^2}{\partial z^2} + \frac{\Delta_t}{(z-t)^2} - \frac{1-t}{z(z-t)(z-t)}t\frac{\partial}{\partial t} - \frac{2z-1}{z(z-1)}\frac{\partial}{\partial z} \right.$$
$$\left. + \frac{\Delta_0}{z^2} + \frac{\Delta_1}{(z-1)^2} - \frac{\Delta_{2,1}+\Delta_0+\Delta_t+\Delta_1-\Delta_\infty}{z(z-1)}\right]\Psi(z,t) = 0\,. \tag{1.85}$$

The conformal dimensions are related to the masses of the hypermultiplets in the gauge theory by

$$\begin{aligned} \Delta_0 &= \frac{(\epsilon_1+\epsilon_2)^2 - (m_3-m_4)^2}{4\epsilon_1\epsilon_2}\,, \quad & \Delta_t &= \frac{(\epsilon_1+\epsilon_2)^2 - (m_3+m_4)^2}{4\epsilon_1\epsilon_2}\,, \\ \Delta_1 &= \frac{(\epsilon_1+\epsilon_2)^2 - (m_1+m_2)^2}{4\epsilon_1\epsilon_2}\,, \quad & \Delta_\infty &= \frac{(\epsilon_1+\epsilon_2)^2 - (m_1-m_2)^2}{4\epsilon_1\epsilon_2}\,. \end{aligned} \tag{1.86}$$

The *Nekrasov-Shatashvili limit* (NS limit) of the Ω-background is the limit $\epsilon_2 \to 0$ while ϵ_1 remains finite. In the AGT approach, which we only carry on for the case of $N_f = 4$, in this limit, the primary insertions V_α are heavy while the insertion of the degenerate operator $\Phi_{2,1}$ is light. This suggest the semiclassical ansatz

$$\Psi(z,t) = \exp\left(-\frac{F(t,m_i)}{\epsilon_1\epsilon_2}\right)\psi(z,t)\,. \tag{1.87}$$

Then taking the limit of (1.85), keeping $\delta_I := \epsilon_2\Delta_I$ finite (for $I = 0, t, 1, \infty$), one gets

$$\left[\epsilon_1^2\frac{\partial^2}{\partial z^2} + \frac{\delta_t}{(z-t)^2} + \frac{\delta_0}{z^2} + \frac{\delta_1}{(z-1)^2} \right.$$
$$\left. - \frac{\delta_0+\delta_t+\delta_1-\delta_\infty}{z(z-1)} - \frac{(1-t)\mathcal{E}}{z(z-t)(z-t)}\right]\psi(z,t) = 0 \tag{1.88}$$

for

$$\mathcal{E} := t\frac{\partial F}{\partial t} \tag{1.89}$$

the so-called *accessory parameter* [30] of Heun's equation.

In the NS limit the theory becomes a deformation/quantization of Seiberg-Witten theory and ϵ_1 can be interpreted as $\hbar$ in the Schroedinger-like equation (1.88). This same equation can be seen as a quantization of the Seiberg-Witten curve via the correspondence principle a la Bohr: one substitutes p in the expression (1.75) with the differential operator $-i\hbar\frac{\partial}{\partial x}$ which realizes the Heisenberg algebra $[x,p]=i\hbar$ and then applies the resulting operators to a wavefunction $\Psi(x)$. This approach has been used for example in [24]. The resulting equation, for any N_f, is

$$\left[K\left(-i\hbar\frac{\partial}{\partial x}\right)-\frac{\bar{\Lambda}}{2}\left(e^{i\frac{x}{2}}K_+\left(-i\hbar\frac{\partial}{\partial x}\right)e^{i\frac{x}{2}}+e^{-i\frac{x}{2}}K_-\left(-i\hbar\frac{\partial}{\partial x}\right)e^{-i\frac{x}{2}}\right)\right]\Psi(x)=0\,. \quad (1.90)$$

Here the choice of N_+ will be made so that the equation has the form

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}+f(x)\frac{\mathrm{d}}{\mathrm{d}x}+g(x)\right]\Psi(x)=0\,. \quad (1.91)$$

Then by the definition $\Psi(x)=:\exp\left(-\frac{1}{2}\int^x f(y)\,\mathrm{d}y\right)\psi(x)$ the equation assumes the form of a Schroedinger equation

$$\left[-\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}+Q(x)\right]\psi(x)=0 \quad (1.92)$$

for a potential $Q(x)=-\frac{1}{\hbar}\left(-\frac{1}{2}\frac{\mathrm{d}f}{\mathrm{d}x}-\frac{1}{4}f^2+g\right)$.

We found that in the NS limit of the Ω-background Seiberg-Witten theory becomes deformed/quantized and a Schroedinger-like equation (1.92) arises. Techniques used to solve the Schroedinger equation become thus viable in determining the NS prepotential. Further using the ODE/IM correspondence, which we will introduce in the next chapter, it has been possible to link gauge theories and integrable models as for example in [20, 21]. Also it is known that the Schroedinger-like equations obtained from the Nekrasov-Shatashvili limit of Seiberg-Witten theory can be mapped into the equations arising in black hole perturbation theory [2]. Here different N_f correspond to different kind of black hole/brane geometries. Combining this triple correspondence between black hole perturbation theory, gauge theory and integrable models it is possible to study the three theories from a different and novel point of view [19].

Chapter 2

Gauge Theory and Gravity

This chapter will be devoted to the interplay between gauge theories and gravity. First, on the gravity side, we introduce the problem of finding *quasi-normal modes (QNM)* in various black hole geometries in Section 2.1. Then from Section 2.2 we introduce the *AdS/CFT correspondence*, which is an equivalence of two theories: one is a quantum field theory in d dimensions, the other is a theory of gravity in a $(d+1)$-dimensional space whose asymptotic boundary is d-dimensional. This correspondence is sometimes referred to as *holography* since a theory in $d+1$ dimensions is described in terms of a lower dimensional theory. Finally in Section 2.3 we give a quick review of the thermal correlators in Minkowski space then in Section 2.4 we introduce the AdS/CFT correspondence in spacetimes with Minkowskian signature. Finally, in Section 2.5 we find the holographic two-point function in the finite temperature $\mathcal{N}=4$ SYM theory in 4d dual to the AdS-Schwarzschild black hole in 5 dimensions.

2.1 Black Hole Quasi-Normal Modes

The coalescence of two astrophysical objects can be divided in three phases:

- the *inspiral* phase: when the two objects are well separated and orbit around each other,
- the *merger* phase: when the two objects merge,
- the *ringdown* phase: when the resulting remnant object relaxes emitting gravitational waves of very characteristic frequencies .

The inspiral and ringdown phases can be studied using spacetime perturbation theory, while it is generally difficult to tackle the merger phase analitically. For example the tidal response of a neutron star can be traced from the gravitational waves emitted in the late inspiral stage and expressed in terms of *Love numbers*. On the other hand *quasi-normal modes (QNM)* dominate the ringdown phase.

To introduce quasi-normal modes (see [36, 12] for more detail) let us consider an example first: an asymptotically AdS Schwarzschild black hole geometry in d dimensions:

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\,\mathrm{d}\Omega_{d-2} \tag{2.1}$$

for $\mathrm{d}\Omega_{d-2}$ the line element of the $(d-2)$-sphere and

$$f(r) := 1 + \frac{r^2}{L^2} - \left(\frac{r_0}{r}\right)^{d-3}. \tag{2.2}$$

Here L is the AdS radius and r_0 the Schwarzschild radius. Then in this spacetime we consider a conformally coupled (with coupling γ)[1] massless scalar field Φ which has action

$$S_{\text{scalar}} = \int \mathrm{d}^d x\,\sqrt{-g}\left[-g^{\mu\nu}\partial_\mu\Phi^\dagger\partial_\nu\Phi - \frac{d-2}{4(d-1)}\gamma R\Phi^\dagger\Phi\right] \tag{2.3}$$

and the equation of motion for this for the field is

$$\nabla_\mu\nabla^\mu\Phi = \frac{d-2}{4(d-1)}\gamma R\Phi\,. \tag{2.4}$$

This has to be solved coupled to Einstein equations. The black hole perturbation theory approach now consists in considering small perturbations of the metric ($h_{\mu\nu}$) and of the scalar (ϕ) around a fixed black hole background which in our example is AdS-Schwarzschild. Then the equations for the perturbations decouple and, for example, we obtain the equation for the scalar

$$\frac{1}{\sqrt{-g_{\text{BG}}}}\partial_\mu\big(\sqrt{-g_{\text{BG}}}\partial^\mu\phi\big) = \frac{\gamma(d-2)}{4(d-1)L^2}\Phi \tag{2.5}$$

for g_{BG} the determinant of the background metric. Similarly we obtain some wave equation for the metric perturbation $h_{\mu\nu}$ which will describe gravitational waves propagating in the black hole spacetime. In this particular example case we have spherical symmetry thus we use the expansion in spherical harmonics

$$\phi(x) = \sum_{l,m} e^{-i\omega t}\frac{F(r)}{r^{(d-2)/2}}Y_{lm}(\theta)\,. \tag{2.6}$$

[1]when $\gamma = 1$ the action is invariant under the Weyl transformation $g_{\mu\nu}(x) \mapsto \Omega^2(x)g_{\mu\nu}(x)$ and $\Phi(x) \mapsto \Omega^{1-d/2}(x)\Phi(x)$

Then using the tortoise coordinate r_* defined by $\mathrm{d}r_* := \mathrm{d}r\,/f(r)$ we obtain the radial equation in a Schroedinger-like form

$$\frac{\mathrm{d}^2}{\mathrm{d}r_*^2}F + \big(\omega^2 - V(r)\big)F = 0\,, \tag{2.7}$$

for

$$V(r) := f(r)\left[\frac{l(l+d-3)}{r^2} + \frac{d-2}{4}\left(\frac{(d-4)f(r)}{r^2} + \frac{2f'(r)}{r} + \frac{d\gamma}{L^2}\right)\right] \tag{2.8}$$

The catch from the previous example is that for a perturbation $\phi(t,x)$ we obtain a wave equation of the form

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + U(x)\right)\phi(t,x) = 0\,, \tag{2.9}$$

which by the Laplace transform of ϕ

$$\hat{\phi}(s,x) := \int_0^\infty \mathrm{d}t\, e^{-st}\phi(t,x) \tag{2.10}$$

becomes the non-homogeneous equation

$$\left\{-\frac{\partial^2}{\partial x^2} + s^2 + U(x)\right\}\hat{\phi}(s,x) = -\, s\phi(t,x)\Big|_{t=0} - \frac{\partial\phi(t,x)}{\partial t}\Big|_{t=0}\,. \tag{2.11}$$

The corresponding homogeneous equation has the subdominant asymptotic solutions

$$\begin{aligned}\hat{\phi}_+(s,x) &\sim e^{-sx} \text{ for } x \to \infty\,,\\ \hat{\phi}_-(s,x) &\sim e^{sx} \text{ for } x \to -\infty\,,\end{aligned} \tag{2.12}$$

for $\mathrm{Re}\, s > 0$. The Green's function for the operator in the non-homogeneous differential equation is given in terms of the Wronskian $W\left[\hat{\phi}_-, \hat{\phi}_+\right]$ of the asymptotic solutions by

$$G(s,x,x') = \frac{1}{W\left[\hat{\phi}_-, \hat{\phi}_+\right]}\hat{\phi}_-(s, \min\{x,x'\})\hat{\phi}_+(s, \max\{x,x'\})\,. \tag{2.13}$$

Let us call $\mathcal{I}(s,x) := -\, s\phi(t,x)\big|_{t=0} - \frac{\partial\phi(t,x)}{\partial t}\Big|_{t=0}$. Then the solution to the full non-homogeneous equation is

$$\hat{\phi}(s,t) = \int \mathrm{d}x'\, G(s,x,x')\mathcal{I}(s,x')\,. \tag{2.14}$$

Then we take the anti-Laplace transform

$$\begin{aligned}\phi(t,x) &= \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \mathrm{d}s\, e^{st}\hat{\phi}(s,x) \\ &= \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \mathrm{d}s\, e^{st}\int \mathrm{d}x'\, G(s,x,x')\mathcal{I}(s,x') \\ &= \sum_k e^{s_k t}\,\mathrm{Res}\left(\frac{1}{W(s)}\right)\bigg|_{s=s_k}\int \mathrm{d}x'\,\mathcal{I}(s,x')\hat{\phi}_-(s_k,\min\{x,x'\})\hat{\phi}_+(s_k,\max\{x,x'\})\,.\end{aligned} \tag{2.15}$$

Here $W(s)$ denotes the Wronskian $W\left[\hat{\phi}_-,\hat{\phi}_+\right]$ which is independent of x and the sum is over the poles of the Wronskian. Equation (2.15) shows that the perturbation is just a sum over modes with frequencies $\omega_k = -is_k$ determined by the condition

$$W(s_k) = 0\,. \tag{2.16}$$

The frequencies ω_k are not necessarily real, thus this modes are in general decaying with time. These are the *quasi-normal modes*. This approach of defining quasi-normal modes was used in [19] to show that quasi-normal modes are nothing else than the Bethe roots of an associated integrable model.

2.2 Introduction to the AdS/CFT Correspondence

In this section we will go over the basics of the AdS/CFT correspondence [32]. First we introduce the $(d+1)$-dimensional AdS_{d+1} space, which is the $(d+1)$-dimensional spacetime whose metric is

$$\mathrm{d}s^2_{\mathrm{AdS}_{d+1}} = L^2\left[-(r^2+1)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2+1} + r^2\,\mathrm{d}\Omega^2_{d-1}\right]. \tag{2.17}$$

Here L is called the *AdS radius* and $\mathrm{d}\Omega^2_{d-1}$ is the metric of the unit-radius $d-1$ dimensional sphere. The surface at $r=\infty$ is called the *border of AdS*. The full isometry group of AdS is $\mathrm{SO}(d,2)$, which is manifest from another definition of AdS: take the space $\mathbb{R}^{d,2}$ with coordinates $Y_{-1}, Y_0, \ldots, Y_d$ and metric

$$\mathrm{d}s^2_{\mathbb{R}^{d,2}} = -\,\mathrm{d}Y_{-1}{}^2 - \mathrm{d}Y_0{}^2 + \sum_{i=1}^{d}\mathrm{d}Y_i{}^2\ , \tag{2.18}$$

then AdS is the submanifold of $\mathbb{R}^{d,2}$ defined as the level set

$$\mathrm{AdS}_d := \left\{(Y_{-1},\ldots,Y_d)\in\mathbb{R}^{d,2} | -Y_{-1}^2 - Y_0^2 + \sum_{i=1}^{d} Y_d^2 = -L^2\right\}. \tag{2.19}$$

Now clearly this level set has a $\mathrm{SO}(d,2)$ symmetry, however in these coordinates the time direction is compact since it is the angle in the $Y_{-1}Y_0$-plane, thus in physical application, where we want the time direction to be non-compact we will use other parametrizations of AdS.

There is another choice of coordinates in which the metric has the form

$$\mathrm{d}s^2_{\mathrm{AdS}_{d+1}} = L^2 \frac{-\,\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2_{d-1} + \mathrm{d}z}{z^2} \tag{2.20}$$

in which the border is at $z = 0$. In these coordinate we can see explicitly the symmetry $(t, \mathbf{x}, z) \mapsto (\lambda t, \lambda \mathbf{x}, \lambda z)$. Although this coordinates cover only a portion of AdS space they are interesting because in them it appears the d-dimensional Minkowski metric.

Now the AdS/CFT conjecture postulates that all the physics in an asymptotically AdS spacetime can be described in terms of a quantum field theory that lives on the boundary. The boundary, which is given by $\mathbb{R} \times S^{d-1}$, is mapped onto itself by the $\mathrm{SO}(d,2)$ isometries of AdS. Since $\mathrm{SO}(d,2)$ is a symmetry of the theory in the bulk of AdS and it acts mapping the boundary on the boundary itself, it is also a symmetry of the boundary QFT, but $\mathrm{SO}(d,2)$ is nothing other than the conformal group in $(d-1,1)$-space, thus the QFT is in fact a conformal field theory.

How is it possible though, that a $(d+1)$-dimensional theory is equivalent to a d-dimensional theory? Let us try to count the degrees of freedom of each theory. It seems like the bulk, having an extra dimension, would have additional degrees of freedom. Consider the number of degrees of freedom at large energy in the micro-canonical ensemble: in a theory with no scale we expect that the entropy behaves as $S_{\text{boundary}} \sim V_{d-1}T^{d-1}$. Then for large temperatures compared to the radius of S^{d-1} we have $S_{\text{boundary}} \propto cT^{d-1}$, for c the central charge of the CFT. In the bulk let us only consider gravitons which are also massless particles, then the entropy of the gravitons scales as $S_{\text{gravitons}} > T^d$. This seems to be a contradiction, since clearly, for large enough T, we have $S_{\text{gravitons}} > S_{\text{boundary}}$. However it is crucial to notice that the bulk theory contains gravity and gravity gives raise to black holes. A metric for a asymptotically AdS Schwarzschild black hole is

$$\mathrm{d}s^2 = L^2 \left[-\left(1 + r^2 - \frac{2gm}{r^{d-2}}\right) \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 + r^2 - \frac{2gm}{r^{d-2}}} + r^2\, \mathrm{d}\Omega^2_{d-1} \right], \tag{2.21}$$

for m proportional the the mass of the black hole and g proportional to the Newton constant in units of the AdS radius

$$g \propto \frac{G_N^{d+1}}{L^{d-1}}\,. \tag{2.22}$$

At high energies, $T > 1/g$, the entropy we should consider is then the Bekenstein-Hawking entropy which does scale in the same way as the boundary entropy: $S_{BH} \sim$

$\frac{1}{g}T^{d-1}$. Comparing we see that the central charge c is proportional to the inverse gravitational coupling $1/g$.

This has an important consequence: a weakly coupled bulk gravitational theory corresponds to a boundary field theory with a large central charge. One important characteristic of a weakly coupled theory is that its Hilbert space has the structure of a Fock space and the energy of a multi-particle state is approximately the sum of the energies of the single particles states. The dual CFT needs to have a similar structure: this emerges naturally in large N (as in $\mathrm{U}(N)$, the gauge group) gauge theories with fields in the adjoint representation. Gauge invariant operators can be formed by taking the traces of the fundamental fields, e.g. $\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]$ or $\mathrm{Tr}[F_{\mu\nu}D_\lambda D_\sigma F^{\mu\nu}]$. Operators where a single trace appears are called *single-trace operators* while products of single trace operators are called *double-trace* (when there are two traces) or *multi-trace operators*. When we act with one of such operators on the vacuum we create a state in the field theory. In the context of AdS/CFT single-trace operators in the boundary correspond to single-particle states in the bulk and so on. An example is the stress-energy-momentum tensor $T_{\mu\nu}$ which corresponds to a graviton in the bulk.

Thus AdS/CFT connects the entropy of a black hole to the ordinary thermal entropy of field theory: this gives a statistical interpretation the the Bekenstein-Hawking entropy. In addition, since we see the black hole as an ordinary thermal state of a unitary field theory, the AdS/CFT correspondence resolves the information paradox, making the black hole time evolution unitary.

2.2.1 The Stringy Perspective on AdS/CFT

There is a general argument, based on the Feynman diagram perturbative expansion of large N gauge theories, which shows that the dimension of multi-trace operators are the sum of the dimensions of the single-trace components up to $\mathcal{O}(1/N^2)$. This same analysis also shows that the perturbative series of large N gauge theories can be arranged into the perturbative series of string theory with string coupling $g_s \sim 1/N$: that is the Feynman diagrams of large N gauge theories arrange themselves in diagrams that can be drawn on a sphere, torus and higher genus Riemann surfaces and when the genus of the surface increases we get an additional power of $1/N^2$.

From this argument we have that a large N limit is necessary to have a weakly-coupled gravity theory in the bulk. However we have another condition for the validity of the gravity approximation: in order for the graviton to be treated as a point-like object and in order for the massive string modes to be negligible we must have

$$\frac{L}{\ell_s} \gg 1 \tag{2.23}$$

for L the AdS radius and ℓ_s the string length. If this condition is not met then we should consider the full string theory in the bulk and the gravity approximation does not hold. An important feature of string theory is the existence of massive higher-spin states ($s > 2$). In a large N gauge theory we can easily write higher spin single-trace operators such as $\mathrm{Tr}\big[F_{\mu\nu} D_+^s F^{\mu\nu}\big]$ for D_+ a derivative along a null direction. These operator have relatively small scaling dimension at weak coupling, in the example $\Delta = 4 + s$. This in the bulk corresponds to particles of spin s whose mass is relatively small: these states make the gravity approximation invalid. Thus in order for the gravity approximation to hold we have to consider a strongly coupled boundary theory.

2.2.2 Scalar Field in AdS

We will now devote to an example: we consider a scalar field ϕ in Euclidean AdS with action

$$S = \int \mathrm{d}^{d+1}x \sqrt{-g}\big[(\nabla\phi)^2 + m^2\phi^2\big]\,. \tag{2.24}$$

Let us quantize this theory in the coordinates (2.17). We expect the ground state to have vanishing angular momentum, so we make the ansatz $\phi(t,r) = e^{-i\omega t}F(r)$ with the boundary condition $\lim\limits_{r\to\infty} F(r) = 0$. We obtain that

$$\phi_0(t,r) = e^{-i\Delta t}\frac{1}{(1+r^2)^{\Delta/2}} \tag{2.25}$$

for

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + (mL)^2} \tag{2.26}$$

is a solution of the equations of motion with the appropriate boundary conditions. The energy of the solution is $\omega_0 = \Delta$ and the energies of the other states differ by an integer $\omega_n = \Delta + n$ since we can get all of the other states from the action of the conformal generators and their energies are determined by the conformal algebra. The wavefunction $\phi_0(t,r)$ is localized near the center of AdS and in the limit $mL \gg 1$ it becomes sharply localized at $r = 0$. A massless particle has integer energies $\omega_n = d + n$. The case of the graviton in AdS gives a similar result since the equations are similar.

Going the the Poincaré coordinates (2.20) let us consider the problem of computing the path integral of this scalar field theory with fixed boundary conditions. Working semiclassically, if we set zero boundary conditions then the field is zero and action is zero. If we have non-zero boundary conditions the situation is more interesting. Given the translation symmetry along the t and $\mathbf{x}$ (denoting them collectively as $x = (t,\mathbf{x})$) directions we go to Fourier space and use the ansatz $\phi(x,z) = e^{ikx}f(k,z)$. The wave equation is

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + (1-d)\frac{1}{z}\frac{\mathrm{d}f}{\mathrm{d}z} - \left(k^2 + \frac{(mL)^2}{z^2}\right)f = 0\,. \tag{2.27}$$

Near the boundary, i.e. for small z, there are two independent solutions $f \sim z^{\Delta}$ and $f \sim z^{d-\Delta}$. We will put a boundary condition at $z = \epsilon$:

$$\phi(x,z)\Big|_{z=\epsilon} = \phi_0(x)\epsilon^{d-\Delta}\,, \tag{2.28}$$

then the solution that decays as $z \to \infty$ is

$$f(k,z) = e^{ikz} z^{d/2} K_\nu(kz) \tag{2.29}$$

for K a Bessel K-function and

$$\nu = \sqrt{\frac{d^2}{4} + (mL)^2}\,. \tag{2.30}$$

Then the field is given by

$$\phi(k,z) = \phi_0(k)\epsilon^{d-\Delta}\frac{f(k,z)}{f(k,\epsilon)}\,. \tag{2.31}$$

Inserting this into the action and computing we get the on-shell classical action

$$S_{\text{cl}} = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\phi_0(-k)\phi_0(k)\epsilon^{d-2\Delta}\frac{z}{f(k,\epsilon)}\frac{\mathrm{d}f(k,z)}{\mathrm{d}z}\Bigg|_{z=\epsilon}\,. \tag{2.32}$$

Transformed back into position space we get

$$S_{\text{cl}} = -\frac{2\nu\Gamma(\Delta)}{\pi^{d/2}\Gamma(\nu)}L^{d-1}\int \mathrm{d}^d x\,\mathrm{d}^d y\,\frac{\phi_0(x)\phi_0(y)}{|x-y|^{2\Delta}}\,. \tag{2.33}$$

The AdS/CFT dictionary states that this computation with fixed boundary conditions is related to the generating functional of correlation functions for the corresponding operator in the field theory. In other words for a bulk field ϕ related to the single-trace boundary operator $\mathcal{O}$ we have

$$Z_{\text{bulk}}[\phi_0(x)] = \left\langle \exp\left(\int \mathrm{d}^d x\,\phi_0(x)\mathcal{O}(x)\right)\right\rangle_{\text{boundary}}\,, \tag{2.34}$$

for

$$Z_{\text{bulk}}[\phi_0(x)] := \int_{\left\{\phi(x,\epsilon)=\phi_0(x)\epsilon^{d-\Delta}\right\}} \mathcal{D}\phi\;\exp(-S[\phi])\,. \tag{2.35}$$

The path integral taken only over the functions which satisfy the boundary condition (2.28). In the semiclassical approximation then we have $Z_{\text{bulk}}[\phi_0(x)] = \exp(-S_{\text{cl}})$ and we get the correct expression for the two-point function of dimension Δ operators in a CFT. In practice we can consider the semiclassical (saddle-point) approximation to the path integral and write (2.34) as

$$\exp(-S[\phi]) = \left\langle \exp\left(\int \mathrm{d}^d x\,\phi_0(x)\mathcal{O}(x)\right)\right\rangle\,. \tag{2.36}$$

2.3 Minkowski Space Thermal Correlators

This section will be devoted to a review of the various thermal Green functions that appear in quantum field theory in Minkowski space. Consider thus a QFT at finite temperature and let $\mathcal{O}(x)$ be a local bosonic operator in such QFT. In Minkowski space the retarded propagator is defined as

$$G_R(k) := -i \int \mathrm{d}^4x \, e^{-ik\cdot x} \theta(t) \left\langle [\mathcal{O}(x), \mathcal{O}(0)] \right\rangle \,, \tag{2.37}$$

while the advanced propagator is

$$G_A(k) := i \int \mathrm{d}^4x \, e^{-ik\cdot x} \theta(-t) \left\langle [\mathcal{O}(x), \mathcal{O}(0)] \right\rangle \,. \tag{2.38}$$

From the definitions we immediately see $G_R(k)^* = G_R(-k) = G_A(k)$. Another correlator we can consider is the Wightman function

$$G(k) := \frac{1}{2} \int \mathrm{d}^4x \, e^{-ik\cdot x} \left\langle \mathcal{O}(x)\mathcal{O}(0) + \mathcal{O}(0)\mathcal{O}(x) \right\rangle \,. \tag{2.39}$$

All other correlators can be expressed in terms of G_A, G_R and G. For example the Feynman correlator is given by

$$G_F(k) := -i \int \mathrm{d}^4x \, e^{-ik\cdot x} \left\langle T\{\mathcal{O}(x)\mathcal{O}(0)\} \right\rangle = \frac{1}{2}[G_R(k) + G_A(k)] - iG(k) \,. \tag{2.40}$$

From the spectral representation of G_R and G one can relate $G(k)$ with the imaginary part of the retarded correlator:

$$G(k) = -\coth \frac{\omega}{2T} \, \mathrm{Im} \, G_R(k) \,. \tag{2.41}$$

Therefore, if G_R is known, all of the correlators are known. For example

$$G_F(k) = \mathrm{Re} \, G_R(k) + i \coth \frac{\omega}{2T} \, \mathrm{Im} \, G_R(k) \,. \tag{2.42}$$

This multitude of Minkowski space correlators is in contrast with the situation in Euclidean QFT, where usually one only deals with the Matsubara propagator

$$G_E(k_E) := \int \mathrm{d}^4x_E \, e^{-ik_E\cdot x_E} \left\langle T_E\{\mathcal{O}(x_E)\mathcal{O}(0)\} \right\rangle \,, \tag{2.43}$$

which is only defined at discrete values of ω_E, which for bosonic operators are multiples of $2\pi T$. The Euclidean and Minkowski correlators are obviously related. The retarded correlator G_R can be analytically continued to the whole upper half ω-plane and, at complex values $2\pi i T n$ it reduces to the Euclidean propagator

$$G_R(2\pi i T n, \mathbf{k}) = -G_E(2\pi T n, \mathbf{k}) \,. \tag{2.44}$$

Similarly the advanced propagator can be continued to the lower half ω-plane and

$$G_A(-2\pi i T n, \mathbf{k}) = -G_E(-2\pi T n, \mathbf{k}) \,. \tag{2.45}$$

2.4 Minkowski AdS/CFT Prescription

In this section we review how the Euclidean AdS/CFT prescription is generalized to the Minkowski case as in [44]. In the Euclidean formulation, the classical solution is uniquely determined by the boundary value $\phi_0(x)$ and the requirement of regularity at the horizon $z = z_H$. Correspondingly the Euclidean correlator obtained using the correspondence is unique. In contrast, in Minkowski space, the requirement of regularity at the horizon is not sufficient to select a solution and a more refined boundary condition is required. This problem is thought to reflect the multitude of Minkowski space correlators and different boundary conditions at the boundary lead to different correlators.

One boundary conditions stands out physically: the *incoming wave boundary condition*. This simply states that at the horizon waves can only travel from the region outside the horizon to the region inside the horizon. We may suspect this condition corresponds to the retarded propagator, while the *outgoing wave boundary condition* corresponds to the advanced correlator.

However even after fixing the boundary conditions, it is still difficult to get the prescription (2.34) to work (after adapting it to Minkowski space, that is). Let us consider the AdS metric of the form

$$\mathrm{d}s^2 = g_{zz}\,\mathrm{d}z^2 + g_{\mu\nu}(z)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{2.46}$$

and a scalar field with action

$$S_{\text{cl}} = K\int \mathrm{d}^4x \int_{z_B}^{z_H} \mathrm{d}z\,\sqrt{-g}\big[g^{zz}(\partial_z\phi)^2 + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\big]\,, \tag{2.47}$$

for K some normalization constant, m the mass of the scalar, z_B the value of the z coordinate at the boundary of AdS and z_H at the black hole horizon. The linearized field equation for ϕ is

$$\frac{1}{\sqrt{-g}}\partial_z(\sqrt{-g}g^{zz}\partial_z\phi) + g^{\mu\nu}\partial_\mu\partial_\nu\phi - m^2\phi^2 = 0\,, \tag{2.48}$$

which has to be solved with fixed boundary condition at $z = z_B$. The solution is

$$\phi(z,x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}e^{ik\cdot x}f(k,z)\phi_0(k)\,. \tag{2.49}$$

Here $\phi_0(k)$ is determined by the boundary condition

$$\phi(z_B,x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}e^{ik\cdot x}\phi_0(k) \tag{2.50}$$

and $f(k,z)$ is the solution to the mode equation

$$\frac{1}{\sqrt{-g}}\partial_z(\sqrt{-g}g^{zz}\partial_z f(k,z)) - (g^{\mu\nu}k_\mu k_\nu + m^2)f(k,z) = 0 \tag{2.51}$$

with unit boundary value at the boundary $f(k, z_B) = 1$ and satisfying the incoming wave boundary condition at $z = z_H$. The on-shell classical action then is the boundary term

$$S_{\rm cl} = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \phi_0(-k) \mathfrak{F}(k,z) \phi_0(k) \Big|_{z_B}^{z_H} \tag{2.52}$$

for

$$\mathfrak{F}(k,z) = K\sqrt{-g} g^{zz} f(-k,z) \partial_z f(k,z) \,. \tag{2.53}$$

Then guessing the Minkowski space AdS/CFT prescription as

$$\exp(iS_{\rm cl}) = \left\langle \exp\left(i \int \mathrm{d}^4 x \, \phi_0(x) \mathcal{O}(x) \right) \right\rangle \,, \tag{2.54}$$

the Green function obtained is

$$G_R(k) = - \left. \mathfrak{F}(k,z) \right|_{z_B}^{z_H} - \left. \mathfrak{F}(-k,z) \right|_{z_B}^{z_H} \,. \tag{2.55}$$

However this quantity is real while the retarded Green's function is in general complex and the guess (2.54) turns out to be wrong. To circumvent the problem in [44] they conjectured that the correct expression for the retarded Green's function in Minkowski space is

$$G_R(k) = -2 \, \mathfrak{F}(k,z) \Big|_{z_B} \,. \tag{2.56}$$

2.5 Holographic Two-Point Function at Finite Temperature

In this section, following closely [15], we consider a holographic conformal field theory at finite temperature dual to a black hole in AdS$_5$ space. The black hole metric is

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\,\mathrm{d}\Omega_3^2 \tag{2.57}$$

for (setting the AdS radius to 1)

$$f(r) = 1 + r^2 - \frac{\mu}{r^2} \,. \tag{2.58}$$

The Schwarzschild radius is given by

$$R_+ = \sqrt{\frac{\sqrt{1+4\mu} - 1}{2}} \tag{2.59}$$

and the dimensionless parameter μ is related the the mass of the black hole M by

$$\mu := \frac{8G_N M}{3\pi}\,. \tag{2.60}$$

We consider a scalar in the bulk with mass m dual to an operator $\mathcal{O}$ on the boundary with dimension Δ. The relation between m and Δ is

$$m = \sqrt{\Delta(\Delta-4)}\,. \tag{2.61}$$

To solve the wave equation in the black hole background we expand the field as

$$\phi(t,r,\Omega) = \int \mathrm{d}\omega \sum_{\ell,\vec{m}} e^{-i\omega t} Y_{\ell,\vec{m}}(\Omega)\psi_{\omega,\ell}(r)\,, \tag{2.62}$$

here Ω denotes collectively the coordinates on S^3 and $Y_{\ell,\vec{m}}$ are the spherical harmonics on S^3. The wave equation becomes

$$\left[\frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^3 f(r)\frac{\mathrm{d}}{\mathrm{d}r}\right) + \frac{\omega^2}{f(r)} - \frac{\ell(\ell+2)}{r^2} - \Delta(\Delta-4)\right]\psi_{\omega,\ell}(r) = 0\,. \tag{2.63}$$

Since we want to compute the retarded two point function we impose the incoming wave boundary conditions at the horizon

$$\psi_{\omega,\ell}(r) \simeq (r-R_+)^{-i\frac{\omega}{2}\frac{R_+^2}{2R_+^2+1}} \quad \text{for } r \to R_+\,, \tag{2.64}$$

then on the AdS boundary we have

$$\psi_{\omega,\ell}(r) \simeq \mathcal{A}(\omega,\ell) r^{\Delta-4} + \mathcal{B}(\omega,\ell) r^{-\Delta} \quad \text{for } r \to \infty\,. \tag{2.65}$$

Then using the prescription (2.56) we get

$$G_R(\omega,\ell) = \frac{\mathcal{B}(\omega,\ell)}{\mathcal{A}(\omega,\ell)}\,. \tag{2.66}$$

This has holographic two-point function has been calculated in [15]. The differential equation (2.63) can be mapped to the Heun equation: via the map

$$\begin{aligned} z &:= \frac{r^2}{1+r^2+R_+^2}\,, \\ \psi_{\omega,\ell} &=: \left(r^3 f(r)\frac{\mathrm{d}z}{\mathrm{d}r}\right)^{-1/2} \chi_{\omega,\ell}(z)\,, \end{aligned} \tag{2.67}$$

we get

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{\frac{1}{4}-\lambda_1^2}{(z-1)^2}+\frac{\frac{1}{4}-\lambda_t^2}{(z-t)^2}+\frac{\frac{1}{4}-\lambda_0^2}{z^2}-\frac{\frac{1}{2}-\lambda_0^2-\lambda_1^2-\lambda_t^2+\lambda_\infty^2}{z(z-1)}\right.$$
$$\left.-\frac{(1-t)\mathcal{E}}{z(z-t)(z-1)}\right]\chi_{\omega,\ell}(z)=0\,. \quad (2.68)$$

This equation appear in gauge theory as the quantum Seiberg-Witten curve with $N_f=4$ (1.88): the parameters $\lambda_0,\lambda_1,\lambda_t,\lambda_\infty$ are related to the masses of the fundamental matter, $\mathcal{E}$ is the accessory parameter and t is the instanton counting parameter. The dictionary between the black hole and the gauge theory parameters is

$$\begin{aligned}
t&=\frac{R_+^2}{2R_+^2+1}\,,\\
\lambda_0&=0\,,\\
\lambda_t&=\frac{i\omega}{2}\frac{R_+}{2R_+^2+1}\,,\\
\lambda_1&=\frac{\Delta-2}{2}\,,\\
\lambda_\infty&=\frac{\omega}{2}\frac{\sqrt{R_+^2+1}}{2R_+^2+1}\,,\\
\mathcal{E}&=-\frac{\ell(\ell+2)+2(2R_+^2+1)+R_+^2\Delta(\Delta-4)}{4(R_+^2+1)}\\
&\quad+\frac{R_+^2}{1+R_+^2}\frac{\omega}{4(2R_+^2+1)}\,.
\end{aligned} \quad (2.69)$$

In [15] they use the connection coefficients computed through the conformal side of the AGT correspondence in [14] to find the holographic correlator $G_R(\omega,\ell)$ in the boundary finite temperature CFT.

Chapter 3

Integrability and the ODE/IM Correspondence

In this chapter we give first a brief review of integrability in classical mechanics in Section 3.1 and classical field theory in Section 3.2 following [45]. Then in Section 3.3 we go over the integrability structure of conformal field theory following [10, 9]. Finally, in Section 3.4, we are in the position of giving a somewhat historical introduction to the *ODE/IM correspondence* following its first incarnation [17, 8] as a correspondence between the spectral determinants of a certain Schroedinger equation and the eigenvalues of the Q-operators introduced in conformal field theory.

3.1 Lightning-fast Review of Integrability in Classical Mechanics

Let M be a $2N$-dimensional phase space with canonical coordinates (q^i, p_i) for $i = 1, \dots, N$ with Hamiltonian $H : M \longrightarrow \mathbb{R}$ and Poisson structure

$$\left\{q^i, q^j\right\} = 0\,, \quad \{p_i, p_j\} = 0\,, \quad \left\{q^i, p_j\right\} = \delta^i{}_j\,. \tag{3.1}$$

The Hamiltonian system is *Liouville integrable* if there are N independent conserved quantities $F_i : M \longrightarrow \mathbb{R}$ called the *integrals of motion* which are in involution, i.e.

$$\{F_i, F_j\} = 0\,. \tag{3.2}$$

Then consider N constants f_i and define the submanifold of the phase space M_f as the level set

$$M_f := \{x \in M \mid F_i(x) = f_i \text{ for } i = 1, \dots, N\}\,. \tag{3.3}$$

Now consider a open non-intersecting smooth path C in M_f and construct the function

$$S(q, F) := \int_C \alpha = \int_{q_0}^{q} p_i \,\mathrm{d}q^i\,, \tag{3.4}$$

here α is the canonical 1-form $\alpha := p_i \, \mathrm{d}q^i$. This is only a function of the F_i and the final point (after fixing the initial point to a convenient one) of the path C because of Stokes' theorem and the fact that on M_f the symplectic form $\omega = \mathrm{d}\alpha$ vanishes. Then we define the coordinates

$$\psi^i := \frac{\partial S}{\partial F_i} \tag{3.5}$$

so that

$$\mathrm{d}S = p_i \, \mathrm{d}q^i + \psi^i \, \mathrm{d}F_i \tag{3.6}$$

and from the fact that $\mathrm{d}^2 S = 0$ we obtain

$$\omega = \mathrm{d}F_i \wedge \mathrm{d}\psi^i \ . \tag{3.7}$$

This shows that the coordinate transformation $(q^i, p_i) \mapsto (\psi^i, F_i)$ is canonical and S is its generating function. Also all the new momenta F_i are integrals of motion and their conservation implies that $\{H, F_i\} = 0$ thus H does not depend on ψ^i. Since the equations of motion are

$$\frac{\mathrm{d}\psi^i}{\mathrm{d}t} = \frac{\partial H}{\partial F_i} \tag{3.8}$$

and $\frac{\partial H}{\partial F_i}$ is just a constant depending on the f_i, we can straightforwardly solve for ψ^i.

The manifold M_f can display non-trivial cycles, corresponding to a non-trivial topology and thus the coordinates ψ^i are in general multi-valued. Let us consider a situation in which the manifold M_f has exactly N non-trivial cycles C_i. Then we define the *action variables* as

$$I_i := \frac{1}{2\pi} \oint_{C_i} \alpha \tag{3.9}$$

which depend on the F_i. Therefore we can regard the function $S(q, F)$ as a function $S(q, I)$. Then the *angle variables* are defined as the canonical conjugate to the action

$$\theta^i = \frac{\partial S}{\partial I_i} \, . \tag{3.10}$$

Then it is a simple calculation to show

$$\oint_{C_i} \mathrm{d}\theta^j = 2\pi \delta_i{}^j \, , \tag{3.11}$$

thus the coordinates θ^i are the angle variables which parameterize the N dimensional torus T^N.

Now let L and M be matrices such that the Hamilton equations of motion can be written as

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [M, L] \, . \tag{3.12}$$

The two matrices are then said to form a *Lax pair*. Then a set of conserved quantities is given by (for natural n)

$$O_n = \mathrm{Tr}\{L^n\} = \sum_{i=1}^{N} (\lambda_i)^n \tag{3.13}$$

and in particular the eigenvalues λ_i of the matrix L are conserved: the operators $L(t)$ are *isospectral* as t varies. It is important to notice that the Lax pair is not unique, but there is a gauge freedom

$$L \mapsto gLg^{-1} \quad M \mapsto gMg^{-1} + \frac{\mathrm{d}g}{\mathrm{d}t}g^{-1} \tag{3.14}$$

for g an invertible matrix depending on the phase space variables. We will regard L and M as matrices in some Lie algebra $\mathfrak{g}$. Now if we find a Lax pair we found already N conserved quantities, however we still have to show involutivity. We introduce the notation, for any $X \in \mathfrak{g}$

$$\begin{aligned} X_1 &:= X \otimes 1 \in \mathfrak{g} \otimes \mathfrak{g}\,, \\ X_2 &:= 1 \otimes X \in \mathfrak{g} \otimes \mathfrak{g}\,, \end{aligned} \tag{3.15}$$

then it can be shown that [6] the eigenvalues of L are in involution if and only if there exists a phase space function valued in $\mathfrak{g} \otimes \mathfrak{g}$ r_{12} such that

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]\,, \tag{3.16}$$

for $r_{21} = \Pi \circ r_{12}$ and Π being the permutation operator on $\mathfrak{g} \otimes \mathfrak{g}$. Imposing the Jacobi identity on the Poisson brackets (3.16) we obtain that when r_{12} is constant the sufficient[1] condition on $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$[2]

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0\,. \tag{3.17}$$

This last equation is called the *classical Yang-Baxter equation* (CYBE) and the matrix r a *constant classical r-matrix*. An interesting situation is when we can find a family of Lax pairs parametrized by a complex number u called the *spectral parameter*.

3.2 Integrability in Classical Field Theory

To tackle integrability in classical field theory we will generalize the Lax pairs. Let us consider field theory in $(1 + 1)$-dimensional space-time and suppose we can find two

[1] a more general form of this condition has the name of *classical modified Yang-Baxter equation* and has a Casimir element in place of zero in the right-hand side

[2] the notation (3.15) extends in the one logical way, also for the r-matrix

spectral-parameter-dependent matrices L and M such that the field equations can be written as

$$\frac{\partial L}{\partial t} - \frac{\partial M}{\partial x} = [M, L]\,, \tag{3.18}$$

then we call the field theory *classically integrable.* Equation (3.18) is often called the *zero-curvature representation* of the field equations, in fact consider the $\mathfrak{g}$-valued connection 1-form called the *Lax connection*

$$\mathcal{L} := L\,\mathrm{d}x + M\,\mathrm{d}t \tag{3.19}$$

then equation (3.18) is precisely the condition for the vanishing of the curvature[3]

$$\mathrm{d}^{\mathcal{L}}\mathcal{L} = \mathrm{d}\mathcal{L} + \frac{1}{2}[\mathcal{L}, \mathcal{L}] = 0\,. \tag{3.20}$$

The condition (3.18) can also be seen as the compatibility condition of the *auxiliary problem*

$$\left(\frac{\partial}{\partial x} - L\right)\Psi = 0 \quad \left(\frac{\partial}{\partial t} - M\right)\Psi = 0\,. \tag{3.21}$$

Now we introduce the path-ordered exponential which implements the parallel transport of the Lax connection along a segment $\{t_0\} \times [s_-, s_+]$ of constant time

$$T(t, s_-, s_+, u) := \mathcal{P}\exp\left(\int_{s_-}^{s^+} L(t, x, u)\,\mathrm{d}x\right)\,. \tag{3.22}$$

Then we have

$$\frac{\partial T}{\partial t} = M(s_+)T - TM(s_-)\,. \tag{3.23}$$

We consider now the field theory on the cylinder $\mathbb{R} \times S^1$ and define the *monodromy matrix* as the holonomy of the Lax connection around the constant time cycle of the cylinder

$$T(t, u) := \mathcal{P}\exp\left(\int_0^{2\pi} L(t, x, u)\,\mathrm{d}x\right)\,, \tag{3.24}$$

then

$$\frac{\partial T}{\partial t} = [M(t, 0, u), T] \tag{3.25}$$

This means that the trace of the monodromy matrix, called the *transfer matrix*

$$t(u) := \mathrm{Tr}\{T(t, u)\} \tag{3.26}$$

[3] $\mathrm{d}^{\mathcal{L}}$ denotes the covariant exterior derivative with respect to the connection $\mathcal{L}$

is conserved for all the values of the spectral parameter u. Then by the expansion (for example if $t(u)$ is analytic near the origin)

$$t(u) = \sum_{k=0}^{\infty} t_k u^k \tag{3.27}$$

we obtain the conserved quantities t_k.

Now let us consider whether the conserved quantities are in involution. Suppose that the Lax matrix L satisfies the following relation

$$\{L_1(t,x,u), L_2(t,y,u')\} = [r_{12}(u-u'), L_1(t,x,u) + L_2(t,y,u')]\delta(x-y) \tag{3.28}$$

for some r-matrix which is only a function of the spectral parameter u and which satisfies

$$r_{12}(u-u') = -r_{21}(u'-u)\,. \tag{3.29}$$

Then it can be shown that the monodromy matrix satisfies the Poisson bracket relation

$$\{T_1(u), T_2(u')\} = [r_{12}(u-u'), T_1(u)T_2(u')] \tag{3.30}$$

from which we obtain, by taking the trace, the Poisson bracket

$$\{t(u), t(u')\} = 0\,. \tag{3.31}$$

This states that the conserved quantities t_k are all in involution. With the conditions we assumed we can again find a sufficient condition for the Jacobi identity of the Poisson brackets to be satisfies which is again called the *classical Yang-Baxter equation* (CYBE) with spectral parameter

$$\begin{aligned}[]
[r_{12}(u_1-u_2), r_{13}(u_1-u_3)] + [r_{12}(u_1-u_2), r_{23}(u_2-u_3)]& \\
+ [r_{13}(u_1-u_3), r_{23}(u_2-u_3)] = 0\,.& \qquad \text{(CYBE)}
\end{aligned}$$

The Poisson brackets (3.28) are called *ultra-local* because in them it appears the delta function, but not its derivatives. When derivatives appear (3.30) has to be modified.

3.3 Integrable Structure of Conformal Field Theory

In this section we move to quantum integrability. In particular we review the work [10, 9] and the integrability structures of 2D CFT. Recall that the conformal symmetry is generated by the energy-momentum tensor which has the mode expansion

$$T(u) = -\frac{c}{24} + \sum_{n\in\mathbb{Z}} L_{-n} e^{inu}\,, \tag{3.32}$$

expressed in terms of the operators L_n which generate the *Virasoro algebra* Vir

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}\,. \tag{3.33}$$

As usual the chiral space of states $\mathscr{H}_{\rm ch}$ is built up as the direct sum of irreducible highest weight *Verma modules* $\mathcal{V}_\Delta$

$$\mathscr{H}_{\rm ch} = \bigoplus_{\Delta} \mathcal{V}_\Delta\,, \tag{3.34}$$

whose highest weight vector $|\Delta\rangle \in \mathcal{V}_\Delta$ satisfies

$$L_n\,|\Delta\rangle = 0 \text{ for } n > 0, \quad L_0\,|\Delta\rangle = \Delta\,|\Delta\rangle\ . \tag{3.35}$$

It is known that the the universal enveloping algebra $U(\mathrm{Vir})$ contains an infinite dimensional abelian subalgebra generated by the *local integrals of motion* (LIM) I_{2k-1} that have the form

$$I_{2k-1} = \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi} T_{2k}(u)\,, \tag{3.36}$$

for T_{2k} appropriately regularized polynomials in T and its derivatives. The first few of them are [40]

$$T_2(u) = T(u)\,, \quad T_4(u) =: T^2(u):\,, \quad T_6(u) =: T^3(u) : + \frac{c+2}{12} : (T'(u))^2 :\ . \tag{3.37}$$

The T_{2k} are uniquely determined by the requirement of

$$[I_{2k-1}, I_{2k'-1}] = 0 \tag{3.38}$$

and the spin assignment[4]

$$\oint \frac{\mathrm{d}w}{2\pi i}(w-u)T\{T(w)T_{2k}(u)\} = 2kT_{2k}(u)\,. \tag{3.39}$$

The operators $I_{2k-1} : \mathcal{V}_\Delta \longrightarrow \mathcal{V}_\Delta$ can be written in terms of the L_n. For example

$$\begin{aligned} I_1 &= L_0 - \frac{c}{24}\,, \\ I_3 &= 2\sum_{n=1}^{\infty} L_{-n}L_n + L_0^2 - \frac{c+2}{12}L_0 + \frac{(5c+22)}{2880}\,. \end{aligned} \tag{3.40}$$

[4]here T is the time-ordering operator

$$T\{A(w)B(u)\} = \begin{cases} A(w)B(u) & \text{if } \mathrm{Im}\{u\} > \mathrm{Im}\{w\} \\ B(u)A(w) & \text{if } \mathrm{Im}\{u\} < \mathrm{Im}\{w\} \end{cases}$$

The objective now is simultaneously diagonalize the LIM. This problem can be though as the quantum version of the *Korteweg-de Vries (KdV) problem* as in the classical limit $c \to -\infty$ using the substitution

$$T(u) \mapsto -\frac{c}{6}U(u)\,, \quad [,] \mapsto \frac{6\pi}{ic}\{,\} \tag{3.41}$$

for periodic $U(u+2\pi) = U(u)$ we get the Poisson structure

$$\{U(u), U(v)\} = 2(U(u) + U(v))\delta'(u-v) + \delta'''(u-v)\,, \tag{3.42}$$

which is known to describe the second Hamiltonian structure of KdV, provided we take one of the infinite $I^{(\mathrm{cl})}_{2k-1}$, classical limits of the I_{2k-1}, as Hamiltonian. These are clearly the classical version of the operators (3.36).

Here we will directly go over the integrable structure of the quantum theory, however in [10] the integrability structure of KdV is studied and it is shown how the classical and quantum cases show some analogy. We start by using the *Feigin-Fuchs free-field representation* of $T(u)$

$$-\beta^2 T(u) =: \varphi'(u) : +(1-\beta^2)\varphi''(u) + \frac{\beta^2}{24}\,, \tag{3.43}$$

in terms of the free-field operator

$$\varphi(u) = iQ + iPu + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{a_{-n}}{n} e^{inu}\,, \tag{3.44}$$

and the mode operators Q, P and a_n satisfy the Heisenberg algebra

$$[Q, P] = \frac{i}{2}\beta^2\,, \quad [a_n, a_m] = \frac{n}{2}\beta^2\delta_{n+m,0}\,. \tag{3.45}$$

The parameter β is related to the central charge by

$$\beta = \sqrt{\frac{1-c}{24}} - \sqrt{\frac{25-c}{24}}\,. \tag{3.46}$$

Let $\mathscr{F}_p$ be the highest weight module of the Heisenberg algebra (3.45) whose highest weight vector $|p\rangle$ is defined by

$$P\,|p\rangle = p\,|p\rangle\ , \quad a_n\,|p\rangle = 0 \text{ for } n > 0\,. \tag{3.47}$$

and (3.43) defines the action of Vir in $\mathscr{F}_p$. In particular $\mathscr{F}_p$ is isomorphic to $\mathcal{V}_\Delta$ for

$$\Delta = \left(\frac{p}{\beta}\right)^2 + \frac{c-1}{24}\,. \tag{3.48}$$

The space $\mathscr{F}_p$ splits in the level l subspaces $\mathscr{F}_p^{(l)}$ i.e.

$$\mathscr{F}_p = \bigoplus_{l=0}^{\infty} \mathscr{F}_p^{(l)} \,, \\ L_0 \, |\psi\rangle = (\Delta + l) \, |\psi\rangle \ \text{ for any } \ |\psi\rangle \in \mathscr{F}_p^{(l)} \,. \tag{3.49}$$

The normal ordering in the Feigin-Fuchs free-field representation is the clear one of the creation and annihilation operators a_n. The operators act invariantly on the level subspace $\mathscr{F}_p^{(l)}$ thus the problem of diagonalizing them is a finite algebraic problem.

Consider now the following operator-valued matrices

$$\hat{L}_j(\lambda) := \pi_j \left[e^{i\pi PH} \mathcal{P} \exp\left(\lambda \int_0^{2\pi} \mathrm{d}u \left(V_-(u) q^{\frac{H}{2}} E + V_+(u) q^{-\frac{H}{2}} F \right) \right) \right] \tag{3.50}$$

for the vertex operators

$$V_\pm(u) :=: e^{\pm 2\varphi(u)} := \exp\left(\pm 2 \sum_{n=1}^{\infty} \frac{a_{-n}}{n} e^{inu} \right) \exp(\pm 2i(Q + Pu)) \exp\left(\mp 2 \sum_{n=1}^{\infty} \frac{a_n}{n} e^{-inu} \right) \tag{3.51}$$

and E, F and H the generators of the quantum universal enveloping algebra $U_q(\mathfrak{sl}(2,\mathbb{C}))$

$$[H, E] = 2 \,, \quad [H, F] = -2F \,, \quad [E, F] = \frac{q^H - q^{-H}}{q - q^{-1}} \,, \tag{3.52}$$

having $q = e^{i\pi\beta^2}$. Finally π_j is the spin j representation of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ and thus $\hat{L}_j(\lambda)$ is a matrix acting on the spin j representation space which we will label V_j.

The operators (3.50) are designed in such a way that they satisfy the *quantum Yang-Baxter equation* (QYBE)

$$\hat{R}_{jj'}(\lambda\mu^{-1})(\hat{L}_j(\lambda) \otimes 1)(1 \otimes \hat{L}_{j'}(\mu)) = (1 \otimes \hat{L}_{j'}(\mu))(\hat{L}_j(\lambda) \otimes 1)\hat{R}_{jj'}(\lambda\mu^{-1}) \tag{3.53}$$

for a *quantum R-matrix* which acts on the space $V_j \otimes V_{j'}$. Equation (3.53) can be checked explicitly order by order using the expansion of the path ordered exponential of (3.50).

The *quantum transfer matrices* are defined then by

$$T_j(\lambda) := \mathrm{Tr}_{V_j} \left\{ e^{i\pi PH} \hat{L}_j(\lambda) \right\} \tag{3.54}$$

and they satisfy

$$[T_j(\lambda), T_{j'}(\mu)] = 0 \tag{3.55}$$

as a consequence of (3.53). For shortness we define $T(\lambda) := \frac{1}{2}(\lambda)$. The transfer matrices satisfy the T-system

$$T_j\left(q^{\frac{1}{2}}\lambda\right) T_j\left(q^{-\frac{1}{2}}\lambda\right) = 1 + T_{j-\frac{1}{2}}(\lambda) T_{j+\frac{1}{2}}(\lambda) \tag{3.56}$$

and the fusion relations

$$T(\lambda)T_j\left(q^{j+\frac{1}{2}}\lambda\right) = T_{j-\frac{1}{2}}\left(q^{j+1}\lambda\right)T_{j+\frac{1}{2}}\left(q^{j}\lambda\right). \tag{3.57}$$

Using the fusion relations and the fact that $T_0(\lambda) = 1$ one can express recursively higher spin operators in terms of the basic one $T(\lambda)$. This has the expansion

$$T(\lambda) = 2\cos(2\pi P) + \sum_{n=1}^{\infty} \lambda^{2n} G_n \tag{3.58}$$

and the coefficients G_n define an infinite set of commuting non-local integrals of motion. It would be worth stopping for a second now to check whether the integrals we have been writing are convergent, this is done with some interesting considerations in [10, 9]. Here for what concerns us, we are interested mainly on the Q-operators and we will avoid any discussion about convergence.

The power series (3.58) defines $T(\lambda)$ as an entire function of λ^2 with an essential singularity at $\lambda^2 = \infty$. The asymptotic expansion of $T(\lambda)$ at the essential singularity determines the local IM

$$\log T(\lambda) \simeq m\lambda^{\frac{1}{1-\beta^2}} - \sum_{n=1}^{\infty} C_n I_{2n-1} \lambda^{\frac{1-2n}{1-\beta^2}}, \tag{3.59}$$

this is proved in [10] where also the constants m and C_n are defined.

We know move to our main interest: the Q-operators. Let $\mathcal{E}_\pm$ and $\mathcal{H}$ be the generators of the *q-oscillator algebra*

$$q\mathcal{E}_+\mathcal{E}_- - q^{-1}\mathcal{E}_-\mathcal{E}_+ = \frac{1}{q-q^{-1}}, \quad [\mathcal{H}, \mathcal{E}_\pm] = \pm 2\mathcal{E}_\pm. \tag{3.60}$$

Let ρ be any representation of this algebra such that the trace

$$Z(p) := \mathrm{Tr}_\rho\left\{e^{2i\pi p\mathcal{H}}\right\} \tag{3.61}$$

exists for complex p belonging to the half-plane $\mathrm{Im}\{p\} > 0$. The we can define the two operators

$$\hat{A}_\pm(\lambda) := Z^{-1}(\pm P)\,\mathrm{Tr}_\rho\left\{e^{\pm 2\pi i P\mathcal{H}}\mathcal{P}\exp\left(\lambda\int_0^{2\pi} \mathrm{d}u\, V_-(u)q^{\pm\frac{\mathcal{H}}{2}}\mathcal{E}_\pm + V_+(u)q^{\mp\frac{\mathcal{H}}{2}}\mathcal{E}_\mp\right)\right\} \tag{3.62}$$

The Q-operators are then defined

$$\hat{Q}_\pm(\lambda) := \lambda^{\pm 2P/\beta^2}\hat{A}_\pm(\lambda) \tag{3.63}$$

and they satisfy the following important relations [9]

- they commute among themselves and with the T-operators,
- they satisfy the TQ-system

$$\hat{T}(\lambda)\hat{Q}_{\pm}(\lambda) = \hat{Q}_{\pm}(q\lambda) + \hat{Q}_{\pm}(q^{-1}\lambda) , \tag{3.64}$$

- they satisfy the QQ-system

$$\hat{Q}_{+}(q^{1/2}\lambda)\hat{Q}_{-}(q^{-1/2}\lambda) - \hat{Q}_{+}(q^{-1/2}\lambda)\hat{Q}_{-}(q^{1/2}\lambda) = 2i\sin(2\pi P) , \tag{3.65}$$

- the transfer matrices can be expressed as

$$2i\sin(2\pi P)\hat{T}_j(\lambda) = \hat{Q}_{+}(q^{j+\frac{1}{2}}\lambda)\hat{Q}_{-}(q^{-j-\frac{1}{2}}\lambda) - \hat{Q}_{+}(q^{-j-\frac{1}{2}}\lambda)\hat{Q}_{-}(q^{j+\frac{1}{2}}\lambda) . \tag{3.66}$$

It is clear that the Q-operators are somewhat more fundamental than the transfer matrices, as they can be used to express the latter. Also the TQ-system can be derived from the QQ-system expressing the transfer matrices in terms of the Q-operators. The derivation of these functional equations is sketched in [9].

3.4 The ODE/IM Correspondence

The *ODE/IM correspondence* is a correspondence between certain Schroedinger-like ordinary differential equations (ODE) and some integrable models (IM). In its first incarnation [17] the correspondence related the vacuum eigenvalues of the Q-operators of conformal field theory to the spectral determinant of the Schroedinger equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + |x|^{2\alpha}\right]\psi(x) = E\psi(x) . \tag{3.67}$$

In particular, for some special values of the conformal momentum p, namely $p = \beta^2/4$, the vacuum eigenvalues of the Q-operators $\hat{Q}_+$ and $\hat{Q}_-$ essentially coincide with the spectral determinants associated with the odd and even sectors of (3.67) respectively. The parameter α of the Schroedinger equation is related to the parameter β of the CFT via

$$\alpha = \frac{1}{\beta^2} - 1 . \tag{3.68}$$

A key part of the identification is that both the spectral determinants and the Q-operators of conformal field theory satisfy the same functional relation: the QQ-system. This was shown for the spectral determinants to hold by the pioneering work [49, 48, 47].

In [8] the correspondence was generalized to hold for generic values of p. The Schroedinger equation was modified, adding an angular momentum term

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^{2\alpha} + \frac{\ell(\ell+1)}{x^2}\right]\psi(x) = E\psi(x)\,. \tag{3.69}$$

This is considered on the half-line $x > 0$. Here the relation (3.68) continues to hold and for ℓ we have the identification

$$\ell = \frac{2p}{\beta^2} - 1\,. \tag{3.70}$$

Crucially the differential equation (3.69) has a regular singular point at $x = 0$ and a irregular singularity at infinity, also it enjoys the Λ-symmetry

$$\Lambda : x \mapsto x\,, \quad E \mapsto E\,, \quad \ell \mapsto -\ell - 1 \tag{3.71}$$

and the Ω-symmetry for $q = e^{i\pi/(1+\alpha)}$

$$\Omega : x \mapsto qx\,, \quad E \mapsto q^{-2}E\,, \quad \ell \mapsto \ell\,. \tag{3.72}$$

Then we consider the unique solution $\psi_-(x)$ specified by the asymptotic condition at $x = 0$

$$\psi_-(x) \simeq \sqrt{\frac{2\pi}{1+\alpha}}(2+2\alpha)^{-\frac{2\ell+1}{2(1+\alpha)}}\frac{x^{\ell+1}}{\Gamma\left(1+\frac{2\ell+1}{2(1+\alpha)}\right)} \text{ for } x \to 0\,. \tag{3.73}$$

The solution $\psi_-(x)$ is not invariant under the Λ-symmetry. This can be though as some sort of spontaneous symmetry breaking: even though the differential equation is invariant, its solutions (the "vacuum") are in general not. Thus another solution is defined by

$$\psi_+(x) := \Lambda\psi_-(x)\,. \tag{3.74}$$

Their Wronskian

$$W[\psi_+, \psi_-] = 2i\left(q^{\ell+1/2} - q^{-\ell-1/2}\right) \tag{3.75}$$

is generically different then zero and thus these solutions are linearly independent and form a basis of the space of solutions to (3.69). Similarly for the irregular singular point $x = \infty$ we define the solution $\chi_+(x)$ by the asymptotic condition

$$\chi_+(x) \simeq x^{-\alpha/2}\exp\left(-\frac{x^{1+\alpha}}{1+\alpha}\right) \text{ for } x \to \infty \tag{3.76}$$

and again we define another solution using this time the Ω-symmetry by

$$\chi_-(x) := iq^{-1/2}\Omega\chi_+(x)\,. \tag{3.77}$$

It can be checked that $W[\chi_+, \chi_-] = 2$ and thus $\chi_-(x)$ and $\chi_+(x)$ form a basis. Using the various symmetries[5] it is possible to express the basis $\psi_\pm(x)$ in terms of the basis $\chi_\pm(x)$ using a single function $D(E, \ell)$ as

$$\psi_+(x) = -iq^{-\ell-\frac{1}{2}} D(q^{-2}E, \ell)\chi_+(x) + D(E, \ell)\chi_-(x) \,. \tag{3.78}$$

Then using the Λ-symmetry it is easy to get

$$\psi_-(x) = -iq^{-\ell-\frac{1}{2}} D(q^{-2}E, -\ell-1)\chi_+(x) + D(E, -\ell-1)\chi_-(x) \,. \tag{3.79}$$

In particular the function $D(E, \ell)$ is given by the Wronskian

$$D(E, \ell) := \frac{1}{2} W[\chi_+, \psi_+] \tag{3.80}$$

and satisfies the QQ-system

$$q^{\frac{1}{2}} D(q^2E, \ell)D(E, -\ell-1) - q^{-\ell-\frac{1}{2}} D(E, \ell)D(q^2E, -\ell-1) = q^{\ell+\frac{1}{2}} - q^{-\ell-\frac{1}{2}} \,. \tag{3.81}$$

In [8] they then use the asymptotics of the function $D(E, \ell)$ as well as its analicity properties and, crucially, the QQ-system to prove the identification with the vacuum eigenvalues of the Q-operators of conformal field theory.

Since the times of [17] the correspondence has been realized also for the excited states [7] and many more examples of correspondences between ODE and IM have been found. A relevant example to this work is the correspondences for the quantum Seiberg-Witten curves [20, 19, 22].

[5] we will do this in Chapter 4 for Heun's equation in detail, the steps are analogous

Chapter 4

Heun's Equation ODE/IM Correspondence

In this chapter we construct the ODE/IM correspondence for *Heun's equation*. We write the equation in the form (1.88) (with the identification $\delta_I := 1 - \frac{\lambda_I^2}{4}$[1])

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{\frac{1}{4}-\lambda_0^2}{z^2}+\frac{\frac{1}{4}-\lambda_t^2}{(z-t)^2}+\frac{\frac{1}{4}-\lambda_1^2}{(z-1)^2}-\frac{\frac{1}{2}+\lambda_\infty^2-\lambda_0^2-\lambda_t^2-\lambda_1^2}{z(z-1)}-\frac{(1-t)\mathcal{E}}{z(z-t)(z-1)}\right\}\chi(z)=0\,. \tag{Normal HE}$$

For convenience we define

$$V(z) := \frac{\frac{1}{4}-\lambda_0^2}{z^2}+\frac{\frac{1}{4}-\lambda_t^2}{(z-t)^2}+\frac{\frac{1}{4}-\lambda_1^2}{(z-1)^2}-\frac{\frac{1}{2}+\lambda_\infty^2-\lambda_0^2-\lambda_t^2-\lambda_1^2}{z(z-1)}-\frac{(1-t)\mathcal{E}}{z(z-t)(z-1)}\,. \tag{4.1}$$

Notice that the equation has four regular singular points at $z = 0, t, 1, \infty$. Also notice the following discrete symmetries

$$\Lambda_I : \lambda_I \mapsto -\lambda_I\,. \tag{4.2}$$

First in Section 4.1 we will derive fundamental integrability structure: the QQ-systems. In Section 4.2 we describe the monodromies of the ODE and then in the following section we show how the monodromy and connection problem are related: we give a formula for the monodromy matrices and Floquet exponents in terms of the Q-functions. Using this formula we are able to derive a novel functional equation satisfied by the Q-function (cf. (4.51)).

4.1 Connections and QQ-systems

In this section we construct specific bases of solutions to (Normal HE) which we will call the *ODE/IM bases*. Then we will find the linear relations between the ODE/IM bases in

[1] here we have introduced indices $I, J, \ldots$ to label the singular points 0, t, 1 and ∞

terms of matrices which we will call *connection matrices*. We will be able to express each of these connection matrices in terms of a single function of the moduli of the equation (Normal HE) and we will finally find this function satisfies a form of *QQ-system*.

We want to define the ODE/IM bases, as usual in the ODE/IM prescription, using some asymptotic condition at the regular singular points. It is worth spending some words on this problem: in order to determine a unique solution it is important that the asymptotic behaviours are decaying as otherwise a unique solution is not determined[2]. We therefore suppose $\lambda_I > -\frac{1}{2}$ for $I \neq \infty$ and $\lambda_\infty > \frac{1}{2}$ and define $\chi_{I,0}$ to be the unique solutions to (Normal HE) determined by the asymptotics:

$$\begin{aligned}
\chi_{0,0}(z) &:\simeq z^{\frac{1}{2}+\lambda_0} && \text{for } z \to 0\,, \\
\chi_{t,0}(z) &:\simeq (z-t)^{\frac{1}{2}+\lambda_t} && \text{for } z \to t\,, \\
\chi_{1,0}(z) &:\simeq (z-1)^{\frac{1}{2}+\lambda_1} && \text{for } z \to 1\,, \\
\chi_{\infty,0}(z) &:\simeq z^{\frac{1}{2}-\lambda_\infty} && \text{for } z \to \infty\,.
\end{aligned} \tag{4.3}$$

Then we define $\chi_{I,1}$ using the symmetries (4.2) via $\chi_{I,1}(z) := \Lambda_I \chi_{I,0}(z)$. Clearly their asymptotic behaviour can be obtained applying the Λ-symmetries to the asymptotics. Notice that if the asymptotics of (4.3) are not decaying we could always define uniquely the functions $\chi_{I,1}$ first and then apply the Λ-symmetries to define $\chi_{I,0}$.

We now can compute the Wronskians $W_I := W[\chi_{I,1}, \chi_{I,0}]$ using the asymptotics. We have $W_I = 2\lambda_I$ for $I \neq \infty$ and $W_\infty = -2\lambda_\infty$. Under the additional condition $\lambda_I \neq 0$ for $I \neq \infty$ (recall $\lambda_\infty > \frac{1}{2}$) we have that $(\chi_{I,0}, \chi_{I,1})$ form four bases of the space of solutions to (Normal HE) which we will call the *ODE/IM bases*: if we further define

$$\chi_I(z) := \begin{pmatrix} \chi_{I,0}(z) & \chi_{I,1}(z) \end{pmatrix}, \tag{4.4}$$

then any solution $\chi(z)$ of (Normal HE) can be written as

$$\chi(z) = \chi_I(z) \begin{pmatrix} C_0^{(I)} \\ C_1^{(I)} \end{pmatrix} \tag{4.5}$$

for some coordinates $\left(C_0^{(I)}, C_1^{(I)}\right) \in \mathbb{C}^2$. It is also important to note the invariance

[2]for example take the simple equation

$$\frac{\mathrm{d}^2 f(z)}{\mathrm{d}z^2} - \frac{\ell(\ell+1)}{z^2} f(z) = 0\,, \quad \ell \in \mathbb{N}\,.$$

This has solution $f(z) = C_1 z^{\ell+1} + C_2 z^{-\ell}$. Specifying the behaviour $f(z) \simeq C z^{\ell+1}$ for $z \to 0$ determines a unique solution to the equation $C_1 = C$ and $C_2 = 0$. On the other hand specifying $f(z) \simeq C z^{-\ell}$ we can only say $C_2 = C$ as $z^{\ell+1}$ is subdominant and f will satisfy the asymptotic condition for any value of C_1

properties for $I \neq J$

$$\begin{aligned}\Lambda_I \chi_{J,0} &= \chi_{J,0}\,,\\ \Lambda_I \chi_{J,1} &= \chi_{J,1}\,.\end{aligned} \tag{4.6}$$

It is now natural to wonder what are the linear relations between these four different bases. We can indeed express all these relations in terms of the functions (denoting λ_I collectively as $\vec{\lambda}$)

$$Q^{IJ}(t,\mathcal{E},\vec{\lambda}) := W[\chi_{I,0},\chi_{J,0}]\,, \quad I \neq J\,. \tag{4.7}$$

Clearly from the definition $Q^{IJ} = -Q^{JI}$, so that already we know that no more than six of these functions are independent. Also, crucially, using the invariance properties (4.6), we have

$$\begin{aligned}W[\chi_{I,1},\chi_{J,0}] &= \Lambda_I Q^{IJ}\,,\\ W[\chi_{I,0},\chi_{J,1}] &= \Lambda_J Q^{IJ}\,,\\ W[\chi_{I,1},\chi_{J,1}] &= \Lambda_I\Lambda_J Q^{IJ}\,.\end{aligned} \tag{4.8}$$

Since the expressions $\Lambda_I Q^{IJ}$ and such will happen often we introduce the notation

$$Q^{IJ}_{++} := Q^{IJ}\,,\quad Q^{IJ}_{-+} := \Lambda_I Q^{IJ}\,,\quad Q^{IJ}_{+-} := \Lambda_J Q^{IJ}\,,\quad Q^{IJ}_{--} := \Lambda_I\Lambda_J Q^{IJ}\,. \tag{4.9}$$

Then we can write, using Wronskians

$$\chi_{I,0}(z) = -\frac{Q^{IJ}_{+-}}{W_J}\chi_{J,0}(z) + \frac{Q^{IJ}_{++}}{W_J}\chi_{J,1}(z) \tag{4.10}$$

and applying the Λ_I symmetry

$$\chi_{I,1}(z) = -\frac{Q^{IJ}_{--}}{W_J}\chi_{J,0}(z) + \frac{Q^{IJ}_{-+}}{W_J}\chi_{J,1}(z)\,. \tag{4.11}$$

These relations between the bases are called *connection relations* and the functions Q^{IJ} are called the *connection coefficients* or in the integrability language *Q-functions*. Using (4.10) we immediately find

$$Q^{IJ}_{++} = W_J \lim_{z\to J}\frac{\chi_{I,0}(z)}{\chi_{J,1}(z)}\,. \tag{4.12}$$

We can write the connection relations introducing the *connections matrices*[3]

$$\mathbf{Q}^{IJ} := \frac{1}{W_J}\begin{pmatrix} -Q^{IJ}_{+-} & -Q^{IJ}_{--} \\ Q^{IJ}_{++} & Q^{IJ}_{-+}\end{pmatrix}\,, \tag{4.13}$$

then using these the connection relations become

$$\chi_I(z) = \chi_J(z)\mathbf{Q}^{IJ}\,. \tag{4.14}$$

[3]we are not explicitly writing that all of these are functions of the moduli $(t,\mathcal{E},\lambda_I)$ for convenience

To derive the QQ-system, which is the functional relation satisfied by the Q-functions, the usual approach is taking the Wronskian of (4.10) with (4.11). Since we are here using a matrix form it is convenient to introduce the *fundamental matrices of solutions* Ψ_I of (Normal HE):

$$\Psi_I(z) := \begin{pmatrix} \chi_{I,0}(z) & \chi_{I,1}(z) \\ \frac{\mathrm{d}}{\mathrm{d}z}\chi_{I,0}(z) & \frac{\mathrm{d}}{\mathrm{d}z}\chi_{I,1}(z) \end{pmatrix} \tag{4.15}$$

whose determinant is precisely the Wronskian $W[\chi_{I,0}, \chi_{I,1}] = -W_I$. We have, by linearity of the derivative

$$\Psi_I(z) = \Psi_J(z)\mathbf{Q}^{IJ} \,. \tag{4.16}$$

Then taking the determinant, we obtain the *QQ-system*

$$\det \mathbf{Q}^{IJ} = \frac{W_I}{W_J} \,, \tag{4.17}$$

which written in terms of the Q-functions reads

$$Q^{IJ}_{++}Q^{IJ}_{--} - Q^{IJ}_{-+}Q^{IJ}_{+-} = W_I W_J \,. \tag{4.18}$$

Finally notice that not all the Q-functions Q^{IJ} are independent. The following relations

$$\mathbf{Q}^{IJ} = \mathbf{Q}^{IK}\mathbf{Q}^{KJ} \tag{4.19}$$

relate them to one another and only three are independent. This is clear because we can choose one basis and express the remaining three using three Q-functions. In particular at the Q-function level (4.19) reads

$$Q^{IJ}_{++} = \frac{Q^{IK}_{-+}Q^{KJ}_{++} - Q^{IK}_{++}Q^{KJ}_{+-}}{W_K} \,. \tag{4.20}$$

4.2 Monodromies

In this section we study the monodromies of (Normal HE). We already introduced the monodromy group in Chapter 1 when speaking about Seiberg-Witten theory, but it is worth giving the definition again in this specific context. In particular we are studying (Normal HE) on the Riemann sphere with four punctures $\mathbb{CP}^1 \setminus \{0, t, 1, \infty\}$. The differential equation generates the monodromy group which is an homomorphism of the fundamental group

$$M_{z^\star} : \pi_1\big(\mathbb{CP}^1 \setminus \{0, t, 1, \infty\}\big) \longrightarrow \mathrm{SL}(2, \mathbb{C}) \tag{4.21}$$

To give a precise definition of this a base point $z^\star \in \mathbb{CP}^1 \setminus \{0, t, 1, \infty\}$ has to be chosen. Then if $(\chi_1(z), \chi_2(z))$ is a solution specified by local data at $z^\star$, the analytic continuation along a closed path depends only on the homotopy class $\gamma \in \pi_1(\mathbb{CP}^1 \setminus \{0, t, 1, \infty\})$ and it defines the monodromy matrix $(\chi_1(z), \chi_2(z))M_{z^\star}(\gamma)$. However changing the base point from $z^\star$ to z' the monodromies also change by conjugation of the holonomy $h \in \mathrm{SL}(2, \mathbb{C})$ from $z^\star$ to z'.

Now label $\gamma_I \in \pi_1(\mathbb{CP}^1 \setminus \{0, t, 1, \infty\})$ the paths which wrap around the singularity at I in anticlockwise sense and $M_I = M_{z^\star}(\gamma_I)$. We define the *monodromy data* of the differential equation as the conjugacy classes

$$\mathcal{M}_{\vec{\lambda}} := \frac{\langle M_0, M_t, M_1, M_\infty \in \mathrm{SL}(2, \mathbb{C}) \mid M_0 M_t M_1 M_\infty = I \text{ and } \operatorname{Tr} M_I = -2\cos(2\pi\lambda_I)\rangle}{(M \sim hMh^{-1} \text{ for } h \in \mathrm{SL}(2, \mathbb{C}))} . \tag{4.22}$$

Here we are working at fixed λ_I, then this space is generically 2-dimensional and by the Riemann-Hilbert correspondence $(\mathcal{E}, t)$ are a set of local coordinates on it [30]. To explicitly write the matrices M_I we need to pick a basis. The easiest job is writing M_I in the basis χ_I, we denote this by $\mathbf{M}_I^{(I)}$ and we have

$$\mathbf{M}_I^{(I)} = \begin{pmatrix} -e^{2\pi i\lambda_I} & 0 \\ 0 & -e^{-2\pi i\lambda_I} \end{pmatrix} , \tag{4.23}$$

then clearly

$$\operatorname{Tr} \mathbf{M}_I^{(I)} = \operatorname{Tr} M_I = -2\cos(2\pi\lambda_I) =: \sigma_I \tag{4.24}$$

as in the definition (4.22). Using the connection matrices we can write M_J in any basis χ_I as

$$\mathbf{M}_J^{(I)} = (\mathbf{Q}^{IJ})^{-1} \mathbf{M}_J^{(J)} \mathbf{Q}^{IJ} . \tag{4.25}$$

Any other coordinate ξ in $\mathcal{M}$ can be viewed as a function of $(\mathcal{E}, t)$. In particular inverting the relations we can see $\mathcal{E}$ as a function of (t, ξ). A standard choice for ξ is one of the three traces

$$\sigma_{0t} := \operatorname{Tr}(M_0 M_t) , \quad \sigma_{t1} := \operatorname{Tr}(M_t M_1) , \quad \sigma_{01} := \operatorname{Tr}(M_0 M_1) , \tag{4.26}$$

or one of the three *Floquet exponents* [30]

$$\sigma_{0t} =: -2\cos(2\pi\nu_{0t}) , \quad \sigma_{t1} =: -2\cos(2\pi\nu_{t1}) , \quad \sigma_{01} =: -2\cos(2\pi\nu_{01}) . \tag{4.27}$$

These variables are not of course all independent, they are related by the cubic *Fricke-Jimbo relation* [37, 34, 27, 25, 26]

$$\begin{aligned} W(\sigma_{0t}, \sigma_{t1}, \sigma_{01}) := {} & \sigma_{0t}\sigma_{t1}\sigma_{01} + \sigma_{0t}^2 + \sigma_{t1}^2 + \sigma_{01}^2 - \sigma_{0t}(\sigma_1\sigma_\infty + \sigma_0\sigma_t) - \sigma_{t1}(\sigma_0\sigma_\infty + \sigma_1\sigma_t) \\ & - \sigma_{01}(\sigma_t\sigma_\infty + \sigma_0\sigma_1) + \sigma_0^2 + \sigma_t^2 + \sigma_1^2 + \sigma_\infty^2 + \sigma_0\sigma_t\sigma_1\sigma_\infty - 4 = 0 . \end{aligned} \tag{4.28}$$

The space $\mathcal{M}_{\vec{\lambda}}$ is essentially isomorphic to the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$ connections on $\mathbb{CP}^1 \setminus \{0,t,1,\infty\}$ with fixed conjugacy classes of the monodromies around the punctures. This space admits a symplectic form due to Atiyah and Bott [5] which reads[4]

$$\Omega = \int \mathrm{Tr}(\delta A \wedge \delta A)\,. \tag{4.29}$$

In this picture the traces of monodromies correspond to the Wilson loops

$$\mathscr{W}[\gamma] := \mathrm{Tr}\left\{\mathcal{P}\exp\left(\oint_\gamma A\right)\right\}. \tag{4.30}$$

More over one has the so-called *skein-relations* [46, 37, 34] which state that for any two loops γ_i and γ_j in $\mathbb{CP}^1 \setminus \{0,t,1,\infty\}$ the Poisson bracket defined in terms of the Atiyah-Bott form satisfies

$$\{\mathscr{W}[\gamma_i], \mathscr{W}[\gamma_j]\} = \frac{1}{2}\sum_{x\in\gamma\cap\delta}\left(\mathscr{W}[\gamma^+_{i,j,x}] - \mathscr{W}[\gamma^-_{i,j,x}]\right). \tag{4.31}$$

Here $\gamma^{\pm}_{i,j,x}$ are the loops obtained by removing a small neighbourhood of the intersection point x and replacing it with two arcs making a single loop. The positive or negative sign indexes the two possible ways of doing this, a descriptive figure of the process is contained in [34, fig. 2]. Using the skein relations (see figure 4.1) it is possible to compute the Poisson bracket[5]

$$\{\sigma_{0t}, \sigma_{1t}\} = \mathrm{Tr}\big(M_t^{-1}M_0M_tM_1\big) - \sigma_{01}\,. \tag{4.32}$$

Using the relation valid for $A, B \in \mathrm{SL}(2,\mathbb{C})$

$$\mathrm{Tr}(A)\,\mathrm{Tr}(B) = \mathrm{Tr}(AB) + \mathrm{Tr}\big(AB^{-1}\big) \tag{4.33}$$

we can show that

$$\mathrm{Tr}\big(M_t^{-1}M_0M_tM_1\big) = -\sigma_{01} - \sigma_{0t}\sigma_{t1} + \sigma_t\sigma_\infty + \sigma_0\sigma_1\,, \tag{4.34}$$

thus

$$\{\sigma_{0t}, \sigma_{t1}\} = -2\sigma_{01} - \sigma_{0t}\sigma_{t1} + \sigma_t\sigma_\infty + \sigma_0\sigma_1 = -\frac{\partial W}{\partial\sigma_{01}}\,. \tag{4.35}$$

Similarly, one can compute the other Poisson brackets and show that the variables σ_{0t}, σ_{t1} and σ_{01} are not canonically conjugate.

[4]note that the tangent bundle to $\mathcal{A}_{0,4}$ is just the space of $\mathfrak{sl}(2,\mathbb{C})$-valued one forms on $\mathbb{CP}^1\setminus\{0,t,1,\infty\}$ which transform in the adjoint representation

[5]this is the Poisson bracket defined using the Atiyah-Bott form (4.29)

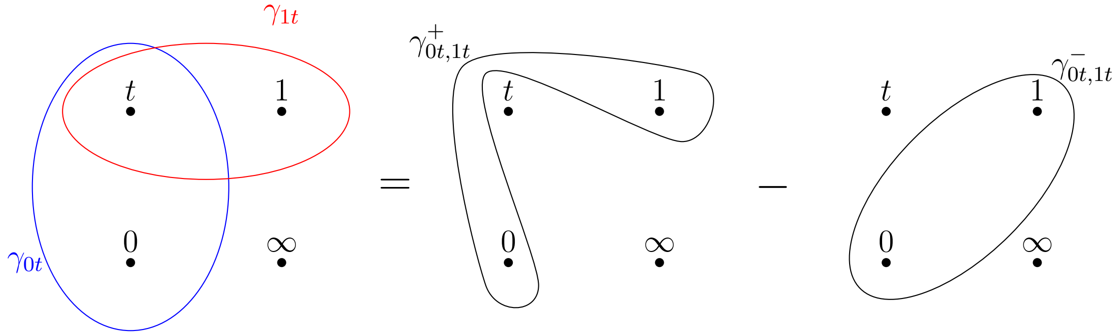

Figure 4.1: Graphical representation of the Poisson bracket $\{\sigma_{0t}, \sigma_{t1}\}$.

We will choose to use $\nu_{0t} =: \nu$ as a coordinate on $\mathcal{M}_{\vec{\lambda}}$. The direct monodromy problem for the Heun's equation is finding ν in terms of $(\mathcal{E}, t)$. It can be solved using Hill determinants. The inverse monodromy problem is finding $\mathcal{E}$ in terms of (t, ν) ($\mathcal{E}$ will also depend parametrically on the λ_I, kept fixed in $\mathcal{M}_{\vec{\lambda}}$). We can write the accessory parameter as a series in t

$$\mathcal{E}(t, \nu, \vec{\lambda}) = \sum_{n=0}^{\infty} \mathcal{E}_n(\nu, \vec{\lambda}) t^n \tag{4.36}$$

and the coefficients $\mathcal{E}_n(\nu, \vec{\lambda})$ then can be found recursively using continued fractions. The method is described in the appendix B and in [30]. One important remark is that by definition of Λ_I we have $\Lambda_I \mathcal{E} = \mathcal{E}$. This implies in turn that

$$\Lambda_I \mathcal{E}_n = \mathcal{E}_n \tag{4.37}$$

for all $n \geq 0$.

In terms of the coordinates $H := \mathcal{E}/t$ and t the Atiyah-Bott form reads [37, 31]:

$$\Omega = 2\pi i \, \mathrm{d}H \wedge \mathrm{d}t \ . \tag{4.38}$$

The canonically conjugate variable ν_D to ν with respect to the Atiyah-Bott form can be constructed in the following way. We want to first define the generating function of the canonical transformation: this will be a function $\mathcal{F}(t, \nu, \vec{\lambda})$ which we will call the *prepotential*[6]. The prepotential must satisfy the *Matone relation* [33, 23]

$$H = \frac{\partial \mathcal{F}}{\partial t} \, , \tag{4.39}$$

or in terms of $\mathcal{E}$

$$\mathcal{E} = t \frac{\partial \mathcal{F}}{\partial t} \, . \tag{4.40}$$

[6]for the moment we are defining all of these quantities just from the ODE. Their names clearly reflect the identification which we will later do with the gauge theory quantities

Now we can determine $\mathcal{E}_n(\nu, \vec{\lambda})$ in (4.36) using the recursive method of appendix B, then we integrate the Matone relation to find

$$\mathcal{F}(t, \nu, \vec{\lambda}) = \mathcal{E}_0(\nu, \vec{\lambda}) \log t + \mathcal{F}_0(\nu, \vec{\lambda}) + \sum_{n=1}^{\infty} \frac{\mathcal{E}_n(\nu, \vec{\lambda})}{n} t^n \tag{4.41}$$

in terms of some function $\mathcal{F}_0$ that does not depend on t. The conjugate variable ν_D is then given by

$$\nu_D = \frac{\partial \mathcal{F}}{\partial \nu} = \mathcal{E}_0'(\nu, \vec{\lambda}) \log t + \mathcal{F}_0'(\nu, \vec{\lambda}) + \sum_{n=1}^{\infty} \frac{\mathcal{E}_n'(\nu, \vec{\lambda})}{n} t^n \tag{4.42}$$

denoting $\mathcal{E}_n'(\nu, \vec{\lambda}) = \partial_\nu \mathcal{E}_n(\nu, \vec{\lambda})$ and $\mathcal{F}_0'(\nu, \vec{\lambda}) = \partial_\nu \mathcal{F}_0(\nu, \vec{\lambda})$. Note that we have not fully fixed ν_D yet: in fact there is a whole family of ν_D coordinates conjugate to ν, one for each choice of the function $\mathcal{F}_0(\nu, \vec{\lambda})$ which remains to be fixed to fix one specific ν_D. We will refer to

$$\mathcal{F}_{\text{inst}}(t, \nu, \vec{\lambda}) = \sum_{n=1}^{\infty} \frac{\mathcal{E}_n(\nu, \vec{\lambda})}{n} t^n \tag{4.43}$$

as the *instanton part* of the prepotential, to $\mathcal{F}_0(\nu, \vec{\lambda})$ as the *one-loop part* and to

$$\mathcal{F}_{\text{cl}}(t, \nu, \vec{\lambda}) = \mathcal{E}_0(\nu, \vec{\lambda}) \log t \tag{4.44}$$

as the *classical part*. Also since the transformation is canonical in the new Darboux coordinates (ν, ν_D) the symplectic Atiyah-Bott form reads

$$\Omega = 2\pi i \, \mathrm{d}\nu \wedge \mathrm{d}\nu_D \ . \tag{4.45}$$

Notice also how the expansion (4.36) and the method of appendix B is valid only for small t: in particular the calculation in appendix B is valid only if $|t| < 1$ and the results we found are in terms of a series in t. For large t it should be possible to use a similar approach using the different coordinate ν_{01}. Then expanding $\mathcal{E}$ as

$$\mathcal{E}(t, \nu_{01}, \vec{\lambda}) = \sum_{n=0}^{\infty} \mathcal{E}_n^{(01)}(\nu_{01}, \vec{\lambda}) t^{-n} \tag{4.46}$$

one would use the same method in the appendix B, but for $|t| > 1$[7].

[7]using a similar expansion as (B.3), but valid for in the annulus $1 < z < |t|$. This is done for example in [29]

Finally for this section we introduce a parametrization of σ_{0t}, σ_{t1} and σ_{01} as in [34][8]

$$
\begin{aligned}
\sigma_{0t} &= -2\cos(2\pi\nu)\,,\\
\sigma_{t1} &= \frac{-2\cosh(\nu_D)\sqrt{c_{0t}c_{1\infty}} + 2\cos(2\pi\nu)(\sigma_0\sigma_1+\sigma_\infty\sigma_t) + 2(\sigma_1\sigma_t+\sigma_0\sigma_\infty)}{4\sin^2(2\pi\nu)}\,,\\
\sigma_{01} &= \frac{-2\sinh(\nu_D)\sqrt{c_{0t}c_{1\infty}} + 2i\sin(2\pi\nu)(\sigma_0\sigma_1+\sigma_\infty\sigma_t-\sigma_{0t}\sigma_{t1})}{4i\sin(2\pi\nu)}\,,
\end{aligned}
\tag{4.47}
$$

for

$$
c_{IJ} := \sigma_{0t}^2 + \sigma_I^2 + \sigma_J^2 - \sigma_{0t}\sigma_I\sigma_J - 4\,. \tag{4.48}
$$

It is simple to verify that this is a parametrization of the surface $W(\sigma_{0t},\sigma_{t1},\sigma_{01}) = 0$ and that the Poisson bracket $\{\sigma_{0t},\sigma_{t1}\}$ is correctly reproduced. This parametrization in particular fixes one ν_D and thus one specific $\mathcal{F}_0(\nu,\vec{\lambda})$.

4.3 Combining Monodromies and Connections

In this section we show how the monodromy and connection problem are deeply related. In particular using (4.25) we can write the monodromy matrices in terms of the connection coefficients

$$
\mathbf{M}_J^{(I)} = \frac{1}{W_I W_J}\begin{pmatrix} e^{2\pi i\lambda_j}Q^{IJ}_{-+}Q^{IJ}_{+-} - e^{-2\pi i\lambda_J}Q^{IJ}_{++}Q^{IJ}_{--} & 2i\sin(2\pi\lambda_J)Q^{IJ}_{-+}Q^{IJ}_{--} \\ -2i\sin(2\pi\lambda_J)Q^{IJ}_{++}Q^{IJ}_{+-} & -e^{2\pi i\lambda_J}Q^{IJ}_{++}Q^{IJ}_{--} + e^{-2\pi i\lambda_J}Q^{IJ}_{-+}Q^{IJ}_{+-}\end{pmatrix}. \tag{4.49}
$$

Using this we compute the Floquet exponent taking the trace in the basis χ_I[9]

$$
\cos(2\pi\nu_{IJ}) = \frac{1}{W_I W_J}\left(\cos(2\pi(\lambda_I+\lambda_J))Q^{IJ}_{-+}Q^{IJ}_{+-} - \cos(2\pi(\lambda_I-\lambda_J))Q^{IJ}_{++}Q^{IJ}_{--}\right). \tag{4.50}
$$

This can be further simplified using the QQ-system (4.18) to write $Q^{IJ}_{-+}Q^{IJ}_{+-} = Q^{IJ}_{++}Q^{IJ}_{--} - W_I W_J$, then

$$
Q^{IJ}_{++}Q^{IJ}_{--} = -W_I W_J \frac{\cos(\pi(\lambda_I+\lambda_t+\nu_{IJ}))\cos(\pi(\lambda_I+\lambda_t-\nu_{IJ}))}{\sin(2\pi\lambda_I)\sin(2\pi\lambda_J)}\,. \tag{4.51}
$$

[8] in particular we have in terms of the coordinates (α,β) of Nekrasov, Rosly and Shatashvili: $\nu = \frac{\alpha}{2\pi i} + \frac{1}{2}$ and $\nu_D = \beta$

[9] notice that we consider (4.50) for $IJ = 0t, t1, 01$. However one could think of defining more Floquet exponents

$$
-2\cos(2\pi\nu_{IJ}) = \sigma_{IJ} = \mathrm{Tr}(M_I M_J)
$$

which would be related to the exponents we already defined in (4.27)

The functional equation (4.51) in particular is a stronger condition than the QQ-system itself, since it clearly implies the latter. In fact applying the Λ_J symmetry we obtain

$$Q^{IJ}_{-+}Q^{IJ}_{+-} = -W_I W_J \frac{\cos(\pi(\lambda_I - \lambda_J + \nu_{IJ}))\cos(\pi(\lambda_I - \lambda_J - \nu_{IJ}))}{\sin(2\pi\lambda_I)\sin(2\pi\lambda_J)}\,, \tag{4.52}$$

then using these two relations, we can check that the QQ-system is indeed satisfied. Notice that the Floquet exponents we are considering are ν_{0t}, ν_{t1} and ν_{01}, which are related by (4.51) to Q^{0t}, Q^{t1} and Q^{01} respectively. These three Q-functions are not independent: they are related by (4.20). Also recalling that $\sigma_{IJ} = -2\cos(2\pi\nu_{IJ})$ we have

$$\sigma_{IJ} = -\frac{2}{W_I W_J}\left(\cos(2\pi(\lambda_I + \lambda_J))Q^{IJ}_{-+}Q^{IJ}_{+-} - \cos(2\pi(\lambda_I - \lambda_J))Q^{IJ}_{++}Q^{IJ}_{--}\right), \tag{4.53}$$

then we use this, together with the QQ-system, to find an equivalent form of (4.51) in terms of the σ_{IJ}

$$Q^{IJ}_{++}Q^{IJ}_{--} = \frac{W_I W_J}{4}\frac{\sigma_{IJ} - 2\cos(2\pi(\lambda_I + \lambda_J))}{\sin(2\pi\lambda_I)\sin(2\pi\lambda_J)}\,. \tag{4.54}$$

4.4 Solving the Functional Equations

In this section we focus on the functional equations (4.51) and find their general solutions. We start by the general functional equation for a function Q

$$Q(\vec{x},\vec{c})Q(-\vec{x},\vec{c}) = F(\vec{x},\vec{c})\,. \tag{4.55}$$

Here $\vec{c}$ are a set of moduli and F must be an even function of $\vec{x}$ by consistency. We proceed by using the ansatz

$$Q(\vec{x},\vec{c}) = \sqrt{F(\vec{x},\vec{c})}\,\mathfrak{Q}(\vec{x},\vec{c})\,, \tag{4.56}$$

and we find that $\mathfrak{Q}(\vec{x},\vec{c})$ has to satisfy

$$\mathfrak{Q}(\vec{x},\vec{c})\mathfrak{Q}(-\vec{x},\vec{c}) = 1\,, \tag{4.57}$$

which tells us that $\mathfrak{Q}(\vec{x},\vec{c})$ is a *zero-mode* of (4.55). Then (4.56) is a general solution of (4.55) for any zero-mode $\mathfrak{Q}(\vec{x},\vec{c})$. Further, let us suppose that we can write $F(\vec{x},\vec{c})$ as

$$F(\vec{x},\vec{c}) = f(\vec{x},\vec{c})f(-\vec{x},\vec{c})\,, \tag{4.58}$$

then, a general solution to (4.55) is given by[10]

$$Q(\vec{x},\vec{c}) = f(\vec{x},\vec{c})\mathfrak{Q}(\vec{x},\vec{c})\,. \tag{4.59}$$

[10] clearly also

$$Q(\vec{x},\vec{c}) = f(-\vec{x},\vec{c})\mathfrak{Q}(\vec{x},\vec{c})$$

is a general solution. The two are related by the zero-mode $\frac{f(\vec{x},\vec{c})}{f(-\vec{x},\vec{c})}$

We focus now on Q^{0t} for which (4.51) reads explicitly

$$Q^{0t}(\vec{\lambda}_{0t},\vec{c})Q^{0t}(-\vec{\lambda}_{0t},\vec{c}) = -4\lambda_0\lambda_t\frac{\cos(\pi(\lambda_0+\lambda_t+\nu))\cos(\pi(\lambda_0+\lambda_t-\nu))}{\sin(2\pi\lambda_0)\sin(2\pi\lambda_t)}\,. \tag{4.60}$$

Here we have denoted $\vec{\lambda}_{IJ} := (\lambda_I,\lambda_J)$ and put the dependence of Q^{0t} on t, ν, λ_1 and λ_∞ into $\vec{c}$. This is the same case as the general functional equation we treated above for $\vec{x} = \vec{\lambda}_{0t}$, $\vec{c} = (t,\nu,\lambda_1,\lambda_\infty)$, $Q = Q^{0t}$ and

$$\begin{aligned} F(\vec{\lambda}_{0t},\vec{c}) &= -4\lambda_0\lambda_t\frac{\cos(\pi(\lambda_0+\lambda_t+\nu))\cos(\pi(\lambda_0+\lambda_t-\nu))}{\sin(2\pi\lambda_0)\sin(2\pi\lambda_t)} \\ &= -\frac{\Gamma(1+2\lambda_0)\Gamma(1+2\lambda_t)}{\Gamma\big(\frac{1}{2}+\lambda_0+\lambda_t+\nu\big)\Gamma\big(\frac{1}{2}+\lambda_0+\lambda_t-\nu\big)}\times \\ &\quad\times\frac{\Gamma(1-2\lambda_0)\Gamma(1-2\lambda_t)}{\Gamma\big(\frac{1}{2}-\lambda_0-\lambda_t+\nu\big)\Gamma\big(\frac{1}{2}-\lambda_0-\lambda_t-\nu\big)}\,. \end{aligned} \tag{4.61}$$

Clearly $F(\vec{\lambda}_{0t},\vec{c})$ can be written as $F(\vec{\lambda}_{0t},\vec{c}) = f(\vec{\lambda}_{0t},\vec{c})f(-\vec{\lambda}_{0t},\vec{c})$ for

$$f(\vec{\lambda}_{0t},\vec{c}) = i\frac{\Gamma(1+2\lambda_0)\Gamma(1+2\lambda_t)}{\Gamma\big(\frac{1}{2}+\lambda_0+\lambda_t+\nu\big)\Gamma\big(\frac{1}{2}+\lambda_0+\lambda_t-\nu\big)}\,. \tag{4.62}$$

We thus get for the Q-function the formula

$$Q^{0t}(t,\nu,\vec{\lambda}) = i\frac{\Gamma(1+2\lambda_0)\Gamma(1+2\lambda_t)}{\Gamma\big(\frac{1}{2}+\lambda_0+\lambda_t+\nu\big)\Gamma\big(\frac{1}{2}+\lambda_0+\lambda_t-\nu\big)}\mathfrak{Q}^{0t}(t,\nu,\vec{\lambda})\,, \tag{4.63}$$

in terms of the zero-mode $\mathfrak{Q}^{0t}$. The expression

$$\exp\left[\frac{1}{2}\left(\frac{\partial}{\partial\lambda_0}+\frac{\partial}{\partial\lambda_t}\right)(\mathcal{F}_{\rm cl}+\mathcal{F}_{\rm inst})\right] \tag{4.64}$$

constitutes a zero-mode because of the relations (4.37). It turns out that this is indeed the correct zero-mode (up to some unimportant phases possibly) and it suggests a way to fix the one-loop part of the prepotential so that

$$Q^{0t}\propto\exp\left[\frac{1}{2}\left(\frac{\partial}{\partial\lambda_0}+\frac{\partial}{\partial\lambda_t}\right)\mathcal{F}\right]. \tag{4.65}$$

The zero-mode however would need to be univoquely fixed by computing some asymptotic form of the Q-function as done in the lower N_f cases.

4.5 Holographic Two-Point Function

In this section we comment how the holographic two-point function of $\mathcal{N} = 4$ SYM at finite temperature is related to the Q-functions. In particular notice (2.66) is just the ratio of two connection coefficients. In particular making reference to [15, eq. (16), (17)] their $\chi^{\text{in}}_{\omega,\ell}$ is related to $\chi_{t,1}$ by

$$\chi^{\text{in}}_{\omega,\ell}(z) = \pm i e^{\mp i\pi\lambda_t}\chi_{t,1}(z) \tag{4.66}$$

the sign depending on the choice of a branch and we will choose the upper sign. The coefficients $\mathcal{A}$ and $\mathcal{B}$ in (17) of [15] can then be written in terms of Q^{t1} as

$$\begin{aligned}
\mathcal{A} &= e^{i\pi(\lambda_1-\lambda_t)}\frac{(1+R_+^2)^{\frac{1}{2}-\lambda_1}}{2\lambda_1}Q^{t1}_{-+}\,,\\
\mathcal{B} &= -e^{i\pi(-\lambda_1-\lambda_t)}\frac{(1+R_+^2)^{\frac{1}{2}+\lambda_1}}{2\lambda_1}Q^{t1}_{--}\,.
\end{aligned} \tag{4.67}$$

Clearly $\mathcal{B} = \Lambda_1\mathcal{A}$. Then the holographic two-point function is expressed in terms of the Q-functions by

$$G_R = -e^{-2\pi i\lambda_1}(1+R_+^2)^{2\lambda_1}\frac{Q^{t1}_{--}}{Q^{t1}_{-+}}\,. \tag{4.68}$$

The poles of the correlators are the quasi-normal modes of the black hole. Since the Q-functions are entire functions, the poles of the correlator can only be at the zeros of the Q-function Q^{t1}_{-+}, i.e. the Bethe roots of the associated integrable model. However it could also be that for some of those zeros also the numerator vanishes and the two-point function ends up being well defined.

4.6 Identification with Gauge Theory

In this section we identify the ODE quantities with their gauge theory counterparts. In particular the Floquet exponent ν should be identified with the gauge theory period a, while its dual ν_D with the instanton dual period A_D. The accessory parameter $\mathcal{E}$ should be identified with the Coulomb branch modulus u and the Matone relation emerges as the usual canonical transformation definitions, while t is identified with the instanton parameter of the gauge theory. The prepotential is identified with the gauge theory Nekrasov-Shatashvili prepotential of appendix C, in particular we checked the usual instanton expansion done with Young diagrams matches precisely the terms obtained using the method of appendix B up to 3 instanton order. In the context of the ODE we notice the appearence of multiple periods which seem to have no analogue in gauge theory, these are the remaining Floquet exponents. We think these are associated with different expansions of the prepotential at different points in the t-plane, for example see (4.46) which shows that ν_{01} is a good coordinate for $t \to \infty$. This could in principle have application for further studying the prepotential, its singularity structure and the convergence of the instantonic series.

We also write here the connection between (Normal HE) and the quantum Seiberg-Witten curve of the SU(2) theory with four flavours in its various forms which appear in the literature. Using equation (1.90) for $N_+ = 2$ and $N_f = 4$ we get

$$\left[-\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + Q(x)\right]\psi(x) = 0\,, \tag{4.69}$$

for

$$\begin{aligned} Q(x) = \frac{1}{4(-4\sqrt{q}\cos x + q + 4)^2}\Big[&-4qe^{2ix}(m_1 - m_2)^2 - 4qe^{-2ix}(m_3 - m_4)^2 \\ &+ 4\sqrt{q}e^{ix}\big(qm_1^2 + qm_2^2 - (q+8)m_1m_2 - qm_3m_4 + 8\tilde{u}\big) \\ &+ 4\sqrt{q}e^{-ix}\big(qm_3^2 + qm_4^2 - (q+8)m_3m_4 - qm_1m_2 + 8\tilde{u}\big) \\ &- \big(q^2(m_1^2 + m_2^2 + m_3^2 + m_4^2) - 24q(m_1m_2 + m_3m_4) + 16(q+4)\tilde{u}\big)\Big] \\ &+ \hbar^2 \frac{\big(\sqrt{q}(q+4)e^{ix} + \sqrt{q}(q+4)e^{-ix} - 8q\big)}{2(-4\sqrt{q}\cos x + q + 4)^2}\,. \end{aligned} \tag{4.70}$$

This is the form that appears in [24]. We can change variable to $y := ix$ so that

$$\begin{aligned} Q(y) = \frac{1}{4(-4\sqrt{q}\cosh y + q + 4)^2}\Big[&-4qe^{2y}(m_1 - m_2)^2 - 4qe^{-2y}(m_3 - m_4)^2 \\ &+ 4\sqrt{q}e^{y}\big(qm_1^2 + qm_2^2 - (q+8)m_1m_2 - qm_3m_4 + 8\tilde{u}\big) \\ &+ 4\sqrt{q}e^{-y}\big(qm_3^2 + qm_4^2 - (q+8)m_3m_4 - qm_1m_2 + 8\tilde{u}\big) \\ &- \big(q^2(m_1^2 + m_2^2 + m_3^2 + m_4^2) - 24q(m_1m_2 + m_3m_4) + 16(q+4)\tilde{u}\big)\Big] \\ &+ \hbar^2 \frac{\big(\sqrt{q}(q+4)e^{y} + \sqrt{q}(q+4)e^{-y} - 8q\big)}{2(-4\sqrt{q}\cosh y + q + 4)^2} \end{aligned} \tag{4.71}$$

and the equation becomes

$$\left[\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}y^2} + Q(y)\right]\psi(y) = 0\,. \tag{4.72}$$

We define the integrability variables:

$$\begin{aligned} q_i &:= m_i/\hbar\,, \\ p^2 &:= \tilde{u}/\hbar^2\,, \\ q &=: e^{\theta}\,, \end{aligned} \tag{4.73}$$

so that we get, dividing by $\hbar$

$$\begin{aligned}\Bigg\{\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\frac{1}{4\left(2e^y-e^{\theta/2}\right)^2\left(e^{\theta/2+y}-2\right)^2}\Big\{&-4e^\theta(q_3-q_4)^2\\
&+e^{3y}\left(8e^{\theta/2}+2e^{\frac{3}{2}\theta}+4e^{\frac{3}{2}\theta}q_1^2-32e^{\theta/2}q_1q_2-4e^{\frac{3}{2}\theta}q_1q_2+4e^{\frac{3}{2}\theta}q_2^2-4e^{\frac{3}{2}\theta}q_3q_4+32e^{\theta/2}p^2\right)\\
&+e^{y}\left(8e^{\theta/2}+2e^{\frac{3}{2}\theta}+4e^{\frac{3}{2}\theta}q_3^2-32e^{\theta/2}q_3q_4-4e^{\frac{3}{2}\theta}q_3q_4+4e^{\frac{3}{2}\theta}q_4^2-4e^{\frac{3}{2}\theta}q_1q_2+32e^{\theta/2}p^2\right)\\
&-e^{2y}\left[e^{2\theta}(q_1^2+q_2^2+q_3^2+q_4^2)-24e^\theta(q_1q_2+q_3q_4)+16(e^\theta+4)p^2+16e^\theta\right]\\
&-4e^{4y}e^\theta\left(q_1-q_2\right)^2\Big\}\Bigg\}\psi(y)=0\,. \end{aligned} \tag{4.74}$$

These variables are the $N_f=4$ analogue of the variables used in [20, 21, 22]. The equation has regular singular points at $y=\pm\infty$ and $y=\pm y_0:=\pm\big(\log 2-\frac{\theta}{2}\big)$ and the symmetries (4.2) become in these variables:

$$\begin{aligned}&\Lambda_0: q_3\mapsto q_4, && q_4\mapsto q_3\,,\\
&\Lambda_t: q_3\mapsto -q_4, && q_4\mapsto -q_3\,,\\
&\Lambda_1: q_1\mapsto -q_2, && q_2\mapsto -q_1\,,\\
&\Lambda_\infty: q_1\mapsto q_2, && q_2\mapsto q_1\,.\end{aligned} \tag{4.75}$$

Now we want to map (4.74) to the normal form of the Heun equation: this can be done via the change of variables $z:=\frac{1}{2}e^{y+\theta/2}$, $\psi(y)=:z^{-1/2}\chi(z)$. Then the equation becomes

$$\begin{aligned}\Bigg\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{1}{4z^2(z-t)^2(z-1)^2}\Big[&-q_3^2t^2-q_4^2t^2+2q_3q_4t^2+t^2\\
&+z\left(4p^2t+2q_3^2t^2+2q_4^2t^2-2q_1q_2t^2-2q_3q_4t^2-4q_3q_4t-t^2-t\right)\\
&+z^2\left(-4p^2t-4p^2-q_1^2t^2-q_2^2t^2-q_3^2t^2-q_4^2t^2+6q_1q_2t+6q_3q_4t+t^2+1\right)\\
&+z^3\left(4p^2+2q_1^2t+2q_2^2t-2q_1q_2t-2q_3q_4t-4q_1q_2-t-1\right)\\
&+z^4\left(-q_1^2+2q_2q_1-q_2^2+1\right)\Big]\Bigg\}\chi(z)=0\,. \end{aligned} \tag{4.76}$$

The singular points at $y=+\infty,-\infty$ are mapped to $z=\infty,0$ respectively, while y_0 is mapped to 1 and $-y_0$ is mapped to $t=e^\theta/4$. Introducing now the parameters

$$\begin{aligned}\lambda_0&:=\frac{q_3-q_4}{2}\,, &\lambda_t&:=\frac{q_3+q_4}{2}\,,\\
\lambda_1&:=\frac{q_1+q_2}{2}\,, &\lambda_\infty&:=\frac{q_1-q_2}{2}\end{aligned} \tag{4.77}$$

and

$$\mathcal{E} := \frac{4p^2 + 2q_3^2 t + 2q_4^2 t + 2q_1 q_2 t + 2q_3 q_4 t - 2q_3^2 - 2q_4^2 - t + 1}{4(t-1)} \tag{4.78}$$

the equation takes the form (Normal HE).

Conclusions and Outlook

In this work we presented the ODE/IM for Heun's equation and showed its connection with $\mathcal{N}=2$ supersymmetric gauge theory and AdS black holes. Crucially we found a way to define the prepotential directly from the differential equation and used this to show some important symmetry properties of its classical and instanton part (cf. (4.37)). We also showed how the monodromy problem is related to the ODE/IM constructions and used this to derive a new kind of functional equation for the Q-functions (cf. (4.51)). We also found the general solution of this functional equation and proposed a possible expression in terms of the prepotential which constitutes a possible zero-mode, fixing the exact solution of the connection problem. We also show how the holographic two-point function of finite temperature $\mathcal{N}=4$ SYM can be written in terms of the Q-functions of Heun's equation.

There are a number of reseach lines which could follow this thesis. For example in the theories with a lower number of flavours the role of the Floquet exponents is still not clear. The Floquet exponents is known to be related to the T-function in the pure gauge theory, however in the cases of 1, 2 and 3 flavours these relations still need to be investigated. In particular it is interesting to wonder if functional relations like (4.51) can exist in these theories as well and if they exists what are their implications. It should for example be possible to find such relation in the $N_f=3$ theory for the Q-function connecting the ODE/IM bases at the two regular singular points. In fact this sector of the $N_f=3$ theory is completely analogous to the theory with four flavours here studied. Moreover we would expect the Stokes phenomenon to play a prominent role.

Another interesting question is what exactly happens at the integrability structures in the decoupling limit in which we reach the $N_f=3$ theory. We already did some work in this direction which has been partly written in appenix C and D. It seems like the appearence of anti-Stokes sectors should be related to the presence of T-functions, however a definition of the T functions in the $N_f=3$ thery has still to be found, perhaps involving the Floquet exponents like in the pure gauge theory case.

It should also be possible to use the procedure here used to define the prepotential in the theories with fewer hypermultiples, obtaining analogue functional constraints on the instanton series.

Finally it is also of most interest to find the Y-system and its TBA which can be used to obtain exact numerical results. It is actually possible to define Y-functions from the Q-functions, but it is not clear how to invert the obtained Y-system into a TBA.

Appendix A

Exact WKB Method and Borel Resummations

In quantum theories one often finds a number of perturbative series which are the building blocks of many computations, but are often divergent. The series can be converted into meaningful objects by their Borel resummation in the perturbation parameter. In quantum mechanics the series of interest will be given by the exact WKB method.

Let us consider the time independent Schroedinger equation

$$-\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) + (V(x) - E)\psi(x) = 0 \tag{A.1}$$

and consider the ansatz

$$\psi(x) = \exp\left[\frac{i}{\hbar}\int^x \mathcal{P}(x')\,\mathrm{d}x'\right]. \tag{A.2}$$

Then, when this is plugged back into the Schroedinger equation we obtain the Riccati equation for the *quantum momentum* $\mathcal{P}(x)$:

$$\mathcal{P}^2(x) - i\hbar\frac{\mathrm{d}\mathcal{P}(x)}{\mathrm{d}x} = E - V(x)\,. \tag{A.3}$$

Now we expand $\mathcal{P}(x)$ in a power series in $\hbar$

$$\mathcal{P}(x) = \sum_{n=0}^{\infty} \hbar^n p_n(q) \tag{A.4}$$

and from (A.3) we find that the functions $p_n(x)$ can be computed recursively via the equation

$$p_{n+1}(x) = \frac{1}{2p_0(x)}\left(i\frac{\mathrm{d}p_n(x)}{\mathrm{d}x} - \sum_{m=1}^{n-1} p_m(x)p_{n-m}(x)\right) \tag{A.5}$$

and the initial condition

$$p_0(x) = \sqrt{E - V(x)}\,. \tag{A.6}$$

It is also convenient to split the even and odd powers in the series

$$\mathcal{P}(x) = \mathcal{P}_{\text{even}}(x) + \mathcal{P}_{\text{odd}}(x) \tag{A.7}$$

and we find that

$$\mathcal{P}_{\text{odd}}(x) = \frac{i\hbar}{2}\frac{\mathrm{d}}{\mathrm{d}x}\log \mathcal{P}_{\text{even}}(x)\,. \tag{A.8}$$

We can regard $\mathcal{P}_{\text{even}}(x)\,\mathrm{d}x$ as a meromorphic differential on the *WKB curve* Σ_{WKB} defined by

$$p^2 = E - V(x)\,. \tag{A.9}$$

Now we define the *quantum periods* as the integrals along the one-cycles $\gamma \in H_1(\Sigma_{\text{WKB}})$

$$\Pi_\gamma(\hbar) := \oint_\gamma \mathcal{P}_{\text{even}}(x)\,\mathrm{d}x = \oint_\gamma \mathcal{P}(x)\,\mathrm{d}x \ , \tag{A.10}$$

here the second equality is due to the fact that the odd powers of $\hbar$ are total derivatives and we are integrating over a cycle. We can also expand the quantum periods as

$$\begin{aligned} \Pi_\gamma(\hbar) &= \sum_{n=0}^{\infty} \hbar^{2n}\Pi_\gamma^{(n)}\,, \\ \Pi_\gamma^{(n)} &= \oint_\gamma p_n(x)\,\mathrm{d}x \ . \end{aligned} \tag{A.11}$$

We should also note that the coefficients $\Pi_\gamma^{(n)}$ depend on the moduli of the WKB curve.

Now in the exact WKB method, the quantum periods, which as far as now are a formal power series in $\hbar$, are promoted to actual functions by Borel resummation. The *Borel transform* of the quantum periods is defined as

$$\hat{\Pi}_\gamma(\xi) := \sum_{n=0}^{\infty} \frac{1}{(2n)!}\Pi_\gamma^{(n)}\xi^{2n} \tag{A.12}$$

and it is analytic in a neighbourhood of the origin of the complex ξ-plane. Moreover it can be analytically continued to a meromorphic function in the whole complex ξ-plane and the singularities are usually poles and branch cuts. The *Borel resummation* of the quantum periods is then defined as the Laplace transform

$$s(\Pi_\gamma)(\hbar) := \frac{1}{\hbar}\int_0^\infty e^{-\xi/\hbar}\hat{\Pi}_\gamma(\xi)\,\mathrm{d}\xi \qquad \hbar \in \mathbb{R}_{>0}\,. \tag{A.13}$$

If this integral is convergent for small enough $\hbar$ then the periods are said to be *Borel summable*. A more general definition, in an arbitrary direction in the complex plane is also important. We define then the *Borel resummation along a direction* φ as

$$s_\varphi(\Pi_\gamma)(\hbar) := \frac{1}{\hbar}\int_0^{e^{i\varphi}\infty} e^{-\xi/\hbar}\hat{\Pi}_\gamma(\xi)\,\mathrm{d}\xi \ . \tag{A.14}$$

Again a series is *Borel summable along the direction* φ if the integral converges for small enough $\hbar$. However note that by the substitution $\xi \mapsto e^{-i\varphi}\xi$ we obtain

$$s_\varphi(\Pi_\gamma)(\hbar) = s(\Pi_\gamma)(e^{i\varphi}\hbar) \qquad \hbar \in \mathbb{R}_{>0}\,, \tag{A.15}$$

thus $\Pi_\gamma(\hbar)$ is Borel summable along the direction φ if and only if $\Pi_\gamma(e^{i\varphi}\hbar)$ is Borel summable. Thus if we allow $\hbar$ to be complex then considering different rays in the complex $\hbar$-plane is equivalent to considering different directions in the Borel resummation.

Now recall the Borel transform might have singularities in the complex ξ-plane, this might lead to obstructions in the integrations in (A.13) and (A.14). These singularities will lead to discontinuities in the Borel resummation. To study these the *lateral Borel resummations along the direction* φ is defined

$$s_\pm(\Pi_\gamma)(e^{i\varphi}\hbar) := \lim_{\delta\to 0^+} s(\Pi_\gamma)\big(e^{i(\varphi\pm\delta)}\hbar\big) \qquad \hbar \in \mathbb{R}_{>0}\,. \tag{A.16}$$

Then the *discontinuity along the direction* φ is defined as

$$\mathrm{disc}_\varphi(\Pi_\gamma) := s_+(\Pi_\gamma)(e^{i\varphi}\hbar) - s_-(\Pi_\gamma)(e^{i\varphi}\hbar) \qquad \hbar \in \mathbb{R}_{>0}\,. \tag{A.17}$$

Appendix B

Heun Accessory Parameter using Continued Fractions

In this chapter we go over the method explained in the appendix of [30] to compute the terms of the series (4.36). It is convenient to cast (Normal HE) in the canonical form using the change of dependent variables

$$\chi(z) =: z^{\frac{1}{2}-\lambda_0}(z-t)^{\frac{1}{2}-\lambda_t}(z-1)^{\frac{1}{2}-\lambda_1}\phi(z)\,, \tag{B.1}$$

the resulting equation for ϕ is the *canonical form of Heun's equation*

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-t}\right)\frac{\mathrm{d}}{\mathrm{d}z}+\frac{\alpha\beta z-q}{z(z-1)(z-t)}\right]\phi(z)=0 \tag{HE}$$

and we have the dictionary

$$\begin{aligned}
&\alpha := 1-\lambda_0-\lambda_t-\lambda_1-\lambda_\infty\,, && \beta := 1-\lambda_0-\lambda_t-\lambda_1+\lambda_\infty\,,\\
&\gamma := 1-2\lambda_0\,, && \delta := 1-2\lambda_1\,,\\
&\epsilon := 1-2\lambda_t\,, && q := (1-t)\mathcal{E}+\frac{\gamma\epsilon+2\alpha\beta t-(\gamma+\delta)\epsilon t}{2}\,.
\end{aligned} \tag{B.2}$$

These parameters are clearly subject to the relation $\alpha+\beta+1=\gamma+\delta+\epsilon$. Then we choose a basis in the space of solutions of (HE) that diagonalizes the composite monodromy M_0M_t. Assuming $|t|<1$ the elements of this basis have the Floquet form

$$\phi_\omega(z)=\sum_{n\in\mathbb{Z}}c_n z^{n+\omega} \tag{B.3}$$

in the annulus $|t|<|z|<1$. These functions under the action of M_0M_t pick up a factor $e^{2\pi i\omega}$. Substituting $\phi_\omega(z)$ in the equation one gets the three term recursion relation

$$A_n c_{n-1}-B_n c_n+C_n t c_{n+1}=0 \tag{B.4}$$

for

$$
\begin{aligned}
A_n &:= (\omega + n - 1 + \alpha)(\omega + n - 1 + \beta)\,, \\
B_n &:= q + (\omega + n)[\epsilon + \delta t + (\omega + n - 1 + \gamma)(1 - t)]\,, \\
C_n &:= (\omega + n + 1)(\omega + n + \gamma)\,.
\end{aligned}
\tag{B.5}
$$

We make the small t series expansion of the accessory parameter q

$$
q(t) = \sum_{n=0}^{\infty} q_n t^n \tag{B.6}
$$

and the use the recursion relation to determine q_n. For $n \geq 0$ define $u_n := \frac{c_{n+1}}{c_n}$ so that (B.4) becomes

$$
u_{n-1} = \frac{A_n}{B_n - tC_n u_n}\,, \tag{B.7}
$$

then we can express u_0 as the continued fraction

$$
u_0 = \frac{A_1}{B_1 - tC_1 \frac{A_2}{B_2 - tC_2 \ldots}}\,. \tag{B.8}
$$

For $n \leq 0$ define $d_{-n} := c_n t^n$ and rewrite (B.4) as

$$
tA_{-n} d_{n+1} - B_{-n} d_n + C_{-n} d_{n-1} = 0\,, \tag{B.9}
$$

then we define $v_n := \frac{d_{n+1}}{d_n}$ and thus we obtain

$$
v_{n-1} = \frac{A_{-n}}{B_{n-} - tC_{-n} v_n}\,, \tag{B.10}
$$

so that we have the continued fraction expression for v_0

$$
v_0 = \frac{A_{-1}}{B_{-1} - tC_{-1} \frac{A_{-2}}{B_{-2} - tC_{-2} \ldots}}\,. \tag{B.11}
$$

Now we use (B.4) for $n = 0$ to obtain

$$
B_0 = t(C_0 u_0 + A_0 v_0)\,. \tag{B.12}
$$

This and the continued fraction solutions for u_0 and v_0 can be used to determine, order by order in t, q as a function of t, ω and the other parameters of the Heun's equation. For example it is very easy to compute q_0 as from (B.12) we have that B_0 has to be of order $\mathcal{O}(t)$. Thus all the $\mathcal{O}(1)$ terms must cancel each other in B_0 and

$$
q_0 = -\omega(\omega + \gamma + \epsilon - 1)\,. \tag{B.13}
$$

Similarly other terms can be computing cutting the continued fraction expressions at the right order. We have for example

$$\begin{aligned}
q_1 &= -\omega(\gamma+\delta+\omega-1) - \frac{\omega(\alpha+\omega-1)(\beta+\omega-1)(\gamma+\omega-1)}{\gamma+2\omega+\epsilon-2} \\
&\quad + \frac{(\omega+1)(\alpha+\omega)(\beta+\omega)(\gamma+\omega)}{\gamma+2\omega+\epsilon}\,, \\
q_2 &= -\frac{\omega(\alpha+\omega-1)(\beta+\omega-1)(\gamma+\omega-1)}{(\gamma+2\omega+\epsilon-2)^2}\Bigg(\frac{(\omega-1)(\alpha+\omega-2)(\beta+\omega-2)(\gamma+\omega-2)}{2(\gamma+2\omega+\epsilon-3)} \\
&\qquad - \frac{\omega(\alpha+\omega-1)(\beta+\omega-1)(\gamma+\omega-1)}{\gamma+2\omega+\epsilon-2} + \frac{(\omega+1)(\alpha+\omega)(\beta+\omega)(\gamma+\omega)}{\gamma+2\omega+\epsilon} \\
&\qquad + (\omega-1)(\gamma+\delta+\omega-2) - \omega(\gamma+\delta+\omega-1)\Bigg) \\
&\quad - \frac{(\omega+1)(\alpha+\omega)(\beta+\omega)(\gamma+\omega)}{(\gamma+2\omega+\epsilon)^2}\Bigg(-\frac{\omega(\alpha+\omega-1)(\beta+\omega-1)(\gamma+\omega-1)}{\gamma+2\omega+\epsilon-2} \\
&\qquad + \frac{(\omega+1)(\alpha+\omega)(\beta+\omega)(\gamma+\omega)}{\gamma+2\omega+\epsilon} - \frac{(\omega+2)(\alpha+\omega+1)(\beta+\omega+1)(\gamma+\omega+1)}{2(\gamma+2\omega+\epsilon+1)} \\
&\qquad - \omega(\gamma+\delta+\omega-1) + (\omega+1)(\gamma+\delta+\omega)\Bigg)\,.
\end{aligned} \tag{B.14}$$

In order to use these results for (Normal HE) it is important to relate ω and ν. First thing is that we notice the symmetry of the q_n terms

$$\omega \mapsto -\omega-\gamma-\epsilon+1\,. \tag{B.15}$$

Thus at fixed q, if ϕ_ω is a solution so is $\phi_{-\omega-\gamma-\epsilon+1}$ and this two solutions form a basis. The composite monodromy M_0M_1 in this basis is

$$\begin{pmatrix} e^{2\pi i\omega} & 0 \\ 0 & e^{-2\pi i(\gamma+\epsilon)}e^{-2\pi i\omega}\end{pmatrix} = \begin{pmatrix} e^{2\pi i\omega} & 0 \\ 0 & e^{2\pi i(2\lambda_0+2\lambda_t)}e^{-2\pi i\omega}\end{pmatrix}, \tag{B.16}$$

mapping back to χ we obtain using the monodromies of the factors in front of ϕ

$$\begin{pmatrix} e^{2\pi i(\lambda_0+\lambda_t-\omega)} & 0 \\ 0 & e^{-2\pi i(\lambda_0+\lambda_t-\omega)}\end{pmatrix} = \begin{pmatrix} -e^{2\pi i(\lambda_0+\lambda_t-\omega-1/2)} & 0 \\ 0 & -e^{-2\pi i(\lambda_0+\lambda_t-\omega-1/2)}\end{pmatrix} \tag{B.17}$$

and taking the trace the result is that

$$\nu = \omega - \lambda_0 - \lambda_t + \frac{1}{2}\,. \tag{B.18}$$

Then using the relation between q and $\mathcal{E}$ we can compute

$$\begin{aligned}\mathcal{E}_0(\nu,\lambda_I) &= -\frac{1}{4} - \nu^2 + \lambda_0^2 + \lambda_t^2\,,\\ \mathcal{E}_1(\nu,\lambda_I) &= \frac{(4\nu^2 - 4\lambda_0^2 + 4\lambda_t^2 - 1)\,(4\nu^2 + 4\lambda_1^2 - 4\lambda_\infty^2 - 1)}{8 - 32\nu^2}\,.\end{aligned} \tag{B.19}$$

While the q_0 and q_1 terms can be computed with bare hands and a bit of patience, the following Wolfram Mathematica code can be used to compute higher order terms

```
CoeffA[n_] := (\[Omega] + n - 1 + \[Alpha]) (\[Omega] + n - 1 + \[
    ↪ Beta])
CoeffB[n_] := q + (\[Omega] + n) (\[Epsilon] + \[Delta] t + (\[Omega]
    ↪  + n - 1 + \[Gamma]) (1 + t))
CoeffC[n_] := (\[Omega] + n + 1) (\[Omega] + n + \[Gamma])

MakeU0[1, next_] := CoeffA[1]/(CoeffB[1] - t CoeffC[1] next)
MakeU0[N_, next_] := MakeU0[N - 1, CoeffA[N]/(CoeffB[N] - t CoeffC[N]
    ↪  next)]
U0[N_] := MakeU0[N, 0]

MakeV0[1, next_] := CoeffC[-1]/(CoeffB[-1] - t CoeffA[-1] next)
MakeV0[N_, next_] := MakeV0[N - 1, CoeffC[-N]/(CoeffB[-N] - t CoeffA
    ↪ [-N] next)]
V0[N_] := MakeV0[N, 0]

ruleQToOrder[N_] := {q -> Sum[q[n] t^n, {n, 0, N}]}
ruleQ[-1] := {}
ruleQ[n_] := ruleQ[n] = Join[First@Solve[SeriesCoefficient[
    Series[t (CoeffC[0] U0[n + 1] + CoeffA[0] V0[n + 1]) -
      CoeffB[0] /. ruleQToOrder[n], {t, 0, n}], n] == 0, q[n]]
      /. ruleQ[n - 1] // Simplify, ruleQ[n - 1]]
```

The function `ruleQ` can be used to find the terms q_n to any finite order. This was used successfully to compute q_n up to $n = 3$. Already evaluating q_3 though takes about 18 minutes so this method does not seem viable to obtain much higher orders.

Appendix C

The Nekrasov Prepotential

In the Nekrasov-Shatashvili limit of $N_f = 4$ deformed Seiberg-Witten theory the *Nekrasov prepotential* reads[1]

$$F(t, m_1, m_2, m_3, m_4, a) = F_{\text{pert}}(t, m_1, m_2, m_3, m_4, a) + F_{\text{inst}}(t, m_1, m_2, m_3, m_4, a)\,. \quad \text{(C.1)}$$

Here

$$\begin{aligned} F_{\text{pert}}(t, m_1, m_2, m_3, m_4, a) := & -a^2 \log t + \hbar^2 \psi^{(-2)}\left(1 + \frac{2a}{\hbar}\right) + \hbar^2 \psi^{(-2)}\left(1 - \frac{2a}{\hbar}\right) \\ & - \hbar^2 \sum_{i=1}^{4} \left[\psi^{(-2)}\left(\frac{1}{2} + \frac{m_i - a}{\hbar}\right) + \psi^{(-2)}\left(\frac{1}{2} + \frac{m_i + a}{\hbar}\right)\right] \end{aligned} \quad \text{(C.2)}$$

is the perturbative part and $F_{\text{inst}}(t, m_1, m_2, m_3, m_4, a)$ is the instanton part, which is given as a series in the instanton parameter t

$$F_{\text{inst}}(t, m_1, m_2, m_3, m_4, a) = \sum_{n=1}^{\infty} c_n(m_1, m_2, m_3, m_4, a) t^n \quad \text{(C.3)}$$

and the coefficients $c_n(m_1, m_2, m_3, m_4, a)$ have a combinatorial definition in terms of Young diagrams. For example the coefficient c_1 is given by

$$c_1(m_1, m_2, m_3, m_4, a) = \frac{\hbar^2}{8(\hbar^2 - 4a^2)}\left[-\hbar^2 + 4a^2 + 4m_1 m_2\right]\left[-\hbar^2 + 4a^2 + 4m_3 m_4\right]. \quad \text{(C.4)}$$

The dual period A_D, which is the a-derivative of the Nekrasov prepotential, then reads

$$A_D(t, m_1, m_2, m_3, m_4, a) = -2a \log t + 2\hbar \log \left[\frac{\Gamma\left(1 + \frac{2a}{\hbar}\right)}{\Gamma\left(1 - \frac{2a}{\hbar}\right)}\right] + \sum_{i=1}^{4} \hbar \log \left[\frac{\Gamma\left(\frac{1}{2} + \frac{m_i - a}{\hbar}\right)}{\Gamma\left(\frac{1}{2} + \frac{m_i + a}{\hbar}\right)}\right] + \partial_a F_{\text{inst}}\,. \quad \text{(C.5)}$$

[1] the order -2 polygamma function is given by $\psi^{(-2)}(z) := \int_0^z \log \Gamma(t)\, \mathrm{d}t$

We conjecture that the instanton part of the prepotential satisfies

$$\Lambda_0 F_{\text{inst}} = F_{\text{inst}}\,, \quad \Lambda_t F_{\text{inst}} = F_{\text{inst}}\,, \quad \Lambda_1 F_{\text{inst}} = F_{\text{inst}}\,, \quad \Lambda_\infty F_{\text{inst}} = F_{\text{inst}}\,. \tag{C.6}$$

This is clearly satisfied by c_1 as given by (C.4) and has been verified up to 3 instantons. In particular this conjecture allows us to compute some functional relations satisfied by the dual period A_D, we have

$$\begin{aligned}
&A_D(t, m_2, m_1, m_3, m_4, a) = A_D(t, m_1, m_2, m_3, m_4, a)\,,\\
&A_D(t, m_1, m_2, m_4, m_3, a) = A_D(t, m_1, m_2, m_3, m_4, a)\,,\\
&A_D(t, -m_2, -m_1, m_3, m_4, a) = A_D(t, m_1, m_2, m_3, m_4, a) + h(m_1, a) + h(m_2, a)\,,\\
&A_D(t, m_1, m_2, -m_4, -m_3, a) = A_D(t, m_1, m_2, m_3, m_4, a) + h(m_3, a) + h(m_4, a)
\end{aligned} \tag{C.7}$$

for

$$h(m, a) := \hbar \log\left[\frac{\Gamma\left(\frac{1}{2} - \frac{m+a}{\hbar}\right)\Gamma\left(\frac{1}{2} + \frac{m+a}{\hbar}\right)}{\Gamma\left(\frac{1}{2} - \frac{m-a}{\hbar}\right)\Gamma\left(\frac{1}{2} + \frac{m-a}{\hbar}\right)}\right] = \hbar \log\left[\frac{\cos\left(\pi\frac{m-a}{\hbar}\right)}{\cos\left(\pi\frac{m+a}{\hbar}\right)}\right]. \tag{C.8}$$

We also write here the mass derivatives of the prepotential

$$\partial_{m_i} F = -\hbar \log\left[\Gamma\left(\frac{1}{2} + \frac{m_i - a}{\hbar}\right)\Gamma\left(\frac{1}{2} + \frac{m_i + a}{\hbar}\right)\right] + \partial_{q_i} F\,. \tag{C.9}$$

Further we can use our conjecture on the prepotential (C.6) to find some properties of the mass derivatives of the instanton part of the prepotential.

It is also useful to study the classical limit $\hbar \to 0$ of the instanton dual period A_D. This has the $\hbar$ expansion

$$A_D = 2a^{(0)}\left[-\log t + 2\pi i + \frac{1}{2}\log\left(\frac{\left(2a^{(0)}\right)^8}{\prod_{i=1}^4 (m_i^2 - (a^{(0)})^2)}\right)\right] + \sum_{i=1}^4 m_i \log\left(\frac{m_i - a^{(0)}}{m_i + a^{(0)}}\right) + \mathcal{O}(\hbar)\,. \tag{C.10}$$

We call the term of order $\hbar^0$, that is the Seiberg-Witten order, with a superscript (0) as it is usually conventional.

C.1 The $N_f = 3$ Prepotential from the Decoupling Limit

In the so-called decoupling limit in which we decouple one of the quarks by sending its mass to infinity the asymptotically conformal theory with four flavors flows into the asymptotically free theory with three flavours. Precisely this decoupling limit is the limit

$$t \to 0\,, \qquad m_4 \to \infty\,, \qquad \Lambda_3 = 4 m_4 t\,. \tag{C.11}$$

In this section we want to show how the Nekrasov prepotential for $N_f = 3$ can be obtained by decoupling q_4 from the prepotential with four flavours. We denote the full Nekrasov prepotential with three flavours as $F^{(3)}_{\text{NS}}$ and its perturbative and instanton parts as $F^{(3)}_{\text{pert}}$ and $F^{(3)}$ respectively. Note how the perturbative part of the Nekrasov prepotential of $N_f = 4$ can be split into a classical part

$$F_{\text{cl}} = -a^2 \log t \tag{C.12}$$

and a one-loop part

$$\begin{aligned} F_{\text{1-loop}} = \hbar^2 \psi^{(-2)}\left(1 + \frac{2a}{\hbar}\right) + \hbar^2 \psi^{(-2)}\left(1 - \frac{2a}{\hbar}\right) \\ - \hbar^2 \sum_{i=1}^{4} \left[\psi^{(-2)}\left(\frac{1}{2} + \frac{m_i - a}{\hbar}\right) + \psi^{(-2)}\left(\frac{1}{2} + \frac{m_i + a}{\hbar}\right)\right]. \end{aligned} \tag{C.13}$$

Then when taking the decoupling limit, intuitively, the loops of the quark that gets decoupled will enter the tree-level of $N_f = 3$. The one-loop part of the $N_f = 3$ prepotential will be

$$\begin{aligned} F^{(3)}_{\text{1-loop}} = \hbar^2 \psi^{(-2)}\left(1 + \frac{2a}{\hbar}\right) + \hbar^2 \psi^{(-2)}\left(1 - \frac{2a}{\hbar}\right) \\ - \hbar^2 \sum_{i=1}^{3} \left[\psi^{(-2)}\left(\frac{1}{2} + \frac{m_i - a}{\hbar}\right) + \psi^{(-2)}\left(\frac{1}{2} + \frac{m_i + a}{\hbar}\right)\right], \end{aligned} \tag{C.14}$$

while the tree-level will be given by the limit

$$\begin{aligned} F^{(3)}_{\text{cl}} &= -a^2 \log\left(\frac{\Lambda_3}{4\hbar}\right) \\ &+ \lim_{m_4 \to \infty} \left\{ a^2 \log \frac{m_4}{\hbar} - \hbar^2 \left[\psi^{(-2)}\left(\frac{1}{2} + \frac{m_4 - a}{\hbar}\right) + \psi^{(-2)}\left(\frac{1}{2} + \frac{m_4 + a}{\hbar}\right)\right]\right\}. \end{aligned} \tag{C.15}$$

Let us define the function

$$f(a, m_4) := a^2 \log \frac{m_4}{\hbar} - \hbar^2 \left[\psi^{(-2)}\left(\frac{1}{2} + \frac{m_4 - a}{\hbar}\right) + \psi^{(-2)}\left(\frac{1}{2} + \frac{m_4 + a}{\hbar}\right)\right] \tag{C.16}$$

which has the $m_4 \to \infty$ expansion

$$\begin{aligned} f(a, m_4) := -m_4^2\left(\log \frac{m_4}{\hbar} - \frac{3}{2}\right) - \hbar m_4 \log(2\pi) + \frac{\hbar^2}{12} \log \frac{m_4}{\hbar} \\ - 2\hbar^2 \log(A) - \frac{\hbar^2}{2} \log(2\pi) + \mathcal{O}\left(\frac{1}{m_4^2}\right) \text{ for } m_4 \to \infty \end{aligned} \tag{C.17}$$

for A the Glaisher–Kinkelin constant. Thus $f(a, m_4)$ diverges for $m_4 \to \infty$. We can however use the subtraction by differentiation scheme [50, p. 505] to regularise the limit: define

$$C(a) := \lim_{q_4 \to \infty} f(a, m_4) \tag{C.18}$$

then we take the derivative with respect to a and assume it commutes with the limit (of course it does not in reality)

$$\begin{aligned} \frac{\mathrm{d}C(a)}{\mathrm{d}a} &= \lim_{q_4 \to \infty} \partial_a f(a, m_4) \\ &= \lim_{m_4 \to \infty} \left\{ 2a \log \frac{m_4}{\hbar} + \hbar \left[\log \Gamma\left(\frac{1}{2} + \frac{m_4 - a}{\hbar} \right) - \log \Gamma\left(\frac{1}{2} + \frac{m_4 + a}{\hbar} \right) \right] \right\} = 0 \,, \end{aligned} \tag{C.19}$$

thus $C(a) =: C$ does not depend on a and it is just an infinite constant. In other words

$$F_{\mathrm{cl}}^{(3)} = -a^2 \log\left(\frac{\Lambda_3}{4\hbar} \right) + C \,. \tag{C.20}$$

Notice that this somewhat heuristic reasoning can be verified to be correct using the asymptotic expression for $f(a, m_4)$ in which the divergent and constant pieces are clearly independent of a. This procedure to take this decoupling limit is equivalent to taking the limit of the instanton dual period A_D, which is a physical quantity and has a finite limit and then integrating in a to obtain the prepotential. The integration constant is precisely this infinite constant we are getting, which has no importance for the physics and can be set to zero.

To sum up we arrived in this way to the expression for the perturbative part of the three flavor Nekrasov prepotential, which is

$$\begin{aligned} F_{\mathrm{pert}}^{(3)} = -a^2 \log\left(\frac{\Lambda_3}{4\hbar} \right) + \hbar^2 \psi^{(-2)}\left(1 + \frac{2a}{\hbar} \right) + \hbar^2 \psi^{(-2)}\left(1 - \frac{2a}{\hbar} \right) \\ - \hbar^2 \sum_{i=1}^{3} \left[\psi^{(-2)}\left(\frac{1}{2} + \frac{m_i - a}{\hbar} \right) + \psi^{(-2)}\left(\frac{1}{2} + \frac{m_i + a}{\hbar} \right) \right] \end{aligned} \tag{C.21}$$

and similarly one can take the limit of the instanton part.

Now we also want to see how the functional properties (C.7) get modified for the $N_f = 3$ instanton dual period

$$A_D^{(3)} = -2a \log\left(\frac{\Lambda_3}{4\hbar} \right) + 2\hbar \log\left[\frac{\Gamma\left(1 + \frac{2a}{\hbar}\right)}{\Gamma\left(1 - \frac{2a}{\hbar}\right)} \right] + \hbar \log\left[\prod_{i=1}^{3} \frac{\Gamma\left(\frac{1}{2} + \frac{m_i - a}{\hbar}\right)}{\Gamma\left(\frac{1}{2} + \frac{m_i + a}{\hbar}\right)} \right] + \partial_a F_{\mathrm{inst}} \,. \tag{C.22}$$

Since we are sending m_4 to infinity in the $N_f = 4$ period, we should not consider transformations which exchange m_3 and m_4 like in (C.7). However we can consider the transformation $m_3 \mapsto -m_3$ and $m_4 \mapsto -m_4$, which in the new $N_f = 3$ gauge variables reads $\Lambda_3 \mapsto e^{\pm i\pi}\Lambda_3$ and $m_3 \mapsto -m_3$. The functional relations satisfied by $A_D^{(3)}$ are then

$$\begin{aligned} A_D^{(3)}(\Lambda_3, m_2, m_1, m_3, a) &= A_D^{(3)}(\Lambda_3, m_1, m_2, m_3, a)\,, \\ A_D^{(3)}(\Lambda_3, -m_2, -m_1, m_3, a) &= A_D^{(3)}(\Lambda_3, m_1, m_2, m_3, a) + h(m_1, a) + h(m_2, a)\,, \\ A_D^{(3)}(e^{\pm i\pi}\Lambda_3, m_1, m_2, -m_3, a) &= A_D^{(3)}(\Lambda_3, m_1, m_2, m_3, a) \mp 2i\pi a + h(m_3, a)\,. \end{aligned} \tag{C.23}$$

Notice that the single $\mathbb{Z}_2$ transformation $m_3 \mapsto -m_3$, $m_4 \mapsto -m_4$ becomes, in the limit, a $\mathbb{Z}$ transformation $\Lambda_3 \mapsto e^{i\pi a}\Lambda_3$, $m_3 \mapsto -m_3$. This is because on the ODE side we have the appearance of an irregular singularity and hence anti-Stokes sectors. The term $\pm 2i\pi a$ should come from the $m_4 \to \infty$ limit of $h(m_4, a)$. This limit depends on the direction we send m_4 to infinity and it is

$$\lim_{m_4 \to e^{i\theta}\infty} h(m_4, a) = \begin{cases} 2i\pi a & \text{if } 0 < \theta < \pi \\ -2i\pi a & \text{if } \pi < \theta < 2\pi \end{cases}\,. \tag{C.24}$$

Note however that sending m_4 to infinity along the real axis (i.e. for $\theta = 0$ or $\theta = \pi$) does not give a well-defined limit.

Appendix D

The $N_f = 3$ Theory

This appendix is devoted to the different forms of the $N_f = 3$ ordinary differential equation and some details of how it is derived as the renormalization limit of the $N_f = 4$ differential equation. For a complete treatment of $N_f = 3$, including the derivation and solution of the QQ-systems see [22]. The $N_f = 3$ equation in integrability variables is

$$\frac{\mathrm{d}^2}{\mathrm{d}y_3^2}\psi(y_3) - \left[\frac{1}{4}e^{2y_3}(q_1-q_2)^2 + e^{y_3}\left(-\frac{1}{2} + 2q_1q_2 + e^{\theta_3}q_3 - 2p^2\right) + \left(e^{2\theta_3} - 6e^{\theta_3}q_3 + 4p^2\right)\right.$$
$$\left. + e^{-y_3}(8e^{\theta_3}q_3 - 4e^{2\theta_3}) + 4e^{-2y_3}e^{2\theta_3}\right]\frac{1}{(e^{y_3}-2)^2}\psi(y_3) = 0 \quad \text{(D.1)}$$

and it can be checked it enjoys the $N_f = 3$ symmetries:

$$\begin{aligned}
&\Omega_- : \theta_3 \mapsto \theta_3 + i\pi, && q_3 \mapsto -q_3\,,\\
&\Omega_+ : q_1 \mapsto -q_1, && q_2 \mapsto -q_2\,,\\
&E : q_1 \mapsto q_2, && q_2 \mapsto q_1\,.
\end{aligned} \quad \text{(D.2)}$$

First in this appendix we want to obtain (D.1) as the decoupling limit of (4.74). In integrability variables the form of the limit is

$$\theta \to -\infty\,, \qquad q_4 \to +\infty\,, \qquad e^{\theta_3} = \frac{e^{\theta}q_4}{4}\,. \quad \text{(D.3)}$$

Considering the relation

$$y_3 = y + \frac{\theta}{2}\,, \quad \text{(D.4)}$$

it can be easily checked that in the limit (D.3) we obtain (D.1) from (4.74). On the other hand (D.1) can also be mapped via the map

$$z_3 = 2e^{-y_3}\,, \qquad \psi(y_3) = z_3^{-1/2}\chi_3(z_3) \quad \text{(D.5)}$$

to the *confluent Heun equation* (CHE) in normal form

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}z_3^2}+\frac{1}{4z_3^2(z_3-1)^2}\Bigg[(q_1-q_2)^2-1+z_3\left(2e^{\theta_3}q_3+4q_1q_2-4p^2+1\right)\right.$$
$$\left.+z_3^2\left(e^{2\theta_3}-6e^{\theta_3}q_3+4p^2-1\right)+z_3^3\left(4e^{\theta_3}q_3-2e^{2\theta_3}\right)+z_3^4e^{2\theta_3}\Bigg]\right\}\chi_3(z_3)=0 \tag{Normal CHE}$$

which now has regular singular points at $z_3 = 0, 1$ and an irregular singular point at $z_3 = \infty$. We now show how the equation (4.76) flows into (D.8) in the decoupling limit. From the definition of z_3 and that of z we have that these variables are related by

$$z_3=\frac{1}{z} \tag{D.6}$$

and the functions χ and χ_3 are related by [1]

$$\chi(z)=\frac{\chi_3(z_3)}{z_3}\,. \tag{D.7}$$

Then it is easy to verify the limit of (4.76) is precisely (Normal CHE). In particular the (Normal CHE) in the variable z and the function χ reads

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{1}{4z^4(z-1)^2}\Bigg[-e^{2\theta_3}+z\left(2e^{2\theta_3}-4e^{\theta_3}q_3\right)+z^2\left(1-e^{2\theta_3}-4p^2+6e^{\theta_3}q_3\right)\right.$$
$$\left.+z^3\left(4p^2-2e^{\theta_3}q_3-4q_1q_2-1\right)+z^4\left(1-(q_1-q_2)^2\right)\Bigg]\right\}\chi(z)=0 \tag{D.8}$$

which we notice has a $1/z^4$ singularity in $z = 0$ arising from the confluence of the two $1/z^2$ singularities in $z = 0$ and $z = t$ in $N_f = 4$.

Now let us consider what happens to the solutions in the decoupling limit, in particular let us consider the asymptotic expression[2]

$$\begin{aligned}\chi_{0,0}(z) &:\simeq z^{\frac{1}{2}(1+q_4-q_3)} && \text{for } z\to 0\,,\\ \chi_{t,0}(z) &:\simeq (z-t)^{\frac{1}{2}(1+q_3+q_4)} && \text{for } z\to t\end{aligned} \tag{D.9}$$

for the subdominant solutions of (4.76). While the asymptotic solution at $z = 0$ of (D.8) is given by

$$\chi_{0,0}^{(3)}(z):\simeq z^{1+q_3}\exp\left(-\frac{e^{\theta_3}}{2z}\right) \tag{D.10}$$

[1] this can be inferred from the fact that ψ stays the same in the decoupling limit

[2] note that these have changed from (4.3) since the conditions for which (4.3) are decaying is not compatible with $q_4\to\infty$

and can be obtained as the WKB solution of the asymptotic form of the equation (D.8)

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \frac{e^{2\theta_3}}{4z^4}\left[1 + 4e^{-\theta_3}q_3 z + \mathcal{O}\big(z^2\big)\right]\right\}\chi(z) = 0\,. \tag{D.11}$$

To obtain the solution we set $f(z) := \frac{e^{2\theta_3}}{4z^4}\left[1 + 4e^{-\theta_3}q_3 z + \mathcal{O}(z^2)\right]$. Then we have

$$\sqrt{f(z)} = \frac{e^{\theta_3}}{2z^2}\left[1 + 2e^{-\theta_3}q_3 z + \mathcal{O}\big(z^2\big)\right]. \tag{D.12}$$

The WKB approximated solution is then given by

$$\chi^{(3)}_{0,0}(z) \sim \frac{1}{f^{1/4}(z)}\exp\left(-\int\sqrt{f(z)}\,\mathrm{d}z\right), \tag{D.13}$$

which gives exactly (D.10). Crucially we notice that the solution $\chi^{(3)}_{0,0}(z)$ can also be obtained as

$$\chi^{(3)}_{0,0}(z) \simeq \lim_{\substack{t\to 0\\ q_4\to\infty}} \chi_{0,1}(z)\chi_{t,0}(z) \qquad \text{for } z\to 0 \tag{D.14}$$

which gives us a recipe to compute the asymptotics of (D.8) in terms of the asymptotics of (4.76).

Let us now turn to the symmetries: one can notice how from four symmetries of $N_f = 4$ (4.75) we went to three symmetries in $N_f = 3$. The reason for this is the following: notice how the symmetries Λ_1 and Λ_∞ can be used to write the symmetries E and Ω_+ of $N_f = 3$ and thus remain unbroken. However the symmetries Λ_1 and Λ_t are broken by the decoupling limit. In fact both symmetries exchange q_3 and q_4, but the decoupling limit tells us q_4 is "special" as it enters the instanton parameter of the $N_f = 3$ theory. However the composition $\Lambda_1 \circ \Lambda_t$ remains unbroken and it is precisely the Ω_- symmetry of $N_f = 3$.

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