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Minimum length metric and horizon area variation

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Abstract

Most of the efforts in unifying general relativity and quantum mechanics come out with two consequences: the presence of a minimum length scale and the non-locality of the spacetime at small scale. The qmetric, or minimum length metric, is a bitensor (it embodies non-locality) acting as a renormalized metric tensor with a minimum length built in: at large scale it approximates the classical metric tensor while the more we approach small scales the more the effects of the presence of a minimum length are relevant. After a review of the general description we construct the qmetric explicitly for Euclidean space and Minkowski spacetime, studying what happens to the area and volume elements of a geodesic congruence cross section. The relevant result is the presence of an irreducible minimum area for the cross section of a geodesic congruence emanating from a point: we can give a notion of a transverse area around any event of the spacetime upholding past results in literature. We exploit this result in the context of black hole horizon area variation, in the approximation such that the flat description can be used locally, showing that the qmetric proves that the presence of a minimum length brings with it a minimum step of area variation, i.e. a quantum of area.

Contents

Notations and conventions

During the entire thesis, unless specified, we set $c = 1$ but we keep G and $\hbar$.

We use a mostly positive metric signature $g = (-, +, +, \ldots, +)$ to describe a generic D dimensional spacetime.

We use Latin indices to express spacetime indices running from 0 to $D - 1$ while Greek indices for the purely spatial indices running from 1 to $D - 1$.

Introduction

Modern theoretical physics of fundamental interactions is built on two great pillars whose aim is to try to describe at best the four known fundamental interactions: electromagnetism, the weak and strong nuclear force and gravity. Gravity is well described by the Einstein's theory of general relativity (GR) while the others find an elegant description in the framework of the Standard Model (SM) which is a theory written in the language of quantum field theory (QFT), the most fundamental expression of quantum mechanics (QM). Despite the great success of GR and the SM in the description of the fundamental interactions of Nature they happen to not talk very well to each other. In fact GR is a classical theory: constructed in a geometrical language, this theory describes the gravitational field as the curvature of the spacetime, regarded as a Lorentzian manifold, and the role of dynamical variable of the theory is played by the spacetime metric tensor. The Einstein's field equations relate the spacetime curvature to the matter/energy content present in the spacetime:

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi G T_{ab} \tag{1}$$

making clear that "Space-time tells matter how to move; matter tells space-time how to curve"(cit. Wheeler). Problems arise when one tries to quantize GR: it turns out that one ends up with a non-renormalizable theory [1]. Several attempts have been made in order to unify GR and QM in a satisfactory way: Loop Quantum gravity, string theory, asymptotic safeness, supergravity and many others. Despite the differences among them, many of the quantum gravity formulations agree on the fact that the quantum structure of spacetime is characterized by the presence of a zero-point length [2] due to intrinsic quantum uncertainty: this minimal length, which is of the order of the Planck scale, would act as a universal regulator in quantum field theories and could avoid the formation of spacetime singularities [3]. There are also evidences on the non-local nature of quantum gravity [4]: locality seems to be inevitably unsustainable when we try to describe gravity at small scales [5]. In sight of these two hints of the quantum properties of spacetime, namely the existence of a minimum length and the unavoidable non-locality at small scales, we might try to describe the spacetime incorporating these properties in our mathematical account of it. The actual spacetime should be done this way after all, exhibiting these features in the smallest scales. In order to do so we need a non-local object which reduces to the classical local metric tensor at large scales, while the more we approach a description of the small scale structure of the spacetime, the more the presence of a minimum length becomes relevant. Such an object is called *qmetric* (*quasi metric* or *quantum metric* or *minimum-length metric*): it is a bitensor, i.e. a tensorial function of two events of spacetime embodying non-locality, defined in a way such that we can not localize with infinite accuracy an event of the spacetime as an effect of the presence of a minimum length [6].

Known results from the use of the qmetric are:

- The qmetric modified Einstein-Hilbert Lagrangian, namely the Ricci scalar computed by means of the qmetric, is actually the entropy density of the spacetime which can be seen as a relic of the quantum nature of the spacetime [7]. In particular it was shown that for a pair of null separated events the qmetric Ricci scalar becomes the flux of the heat crossing null horizons getting automatically horizon thermodynamics and opening an intriguing connection to the emergent gravity paradigm [8, 9].

- It can avoid spacetime singularities by a modification of the Raychaudhuri equation[10]
- it shows that minimum length spacetimes are effectively 2-dimensional at the Planck scale [11];
- the appearance of the notion of a minimum area around any event of the spacetime given by the coincidence limit of the cross section's area of time/space-like [12] and null [13] geodesic congruence emanating from a event.

In this work, after a general review of the qmetric description, we explicitly describe the Minkowski spacetime with the qmetric studying the volume and area elements and we try to exploit it to evaluate the variations of area of black hole horizons.

In Chapter 1 we introduce the mathematical framework we need to construct the qmetric, namely the theory of bitensors and the notion of equigeodesic surfaces.

In chapter 2 we review the construction of the qmetric in a general spacetime both for space/time-like and null-like separeted events.

In chapter 3 we explicitly construct the qmetric model for flat spaces, namely for the Euclidean space and Minkowski spacetime, studying their geometrical properties, showing the presence of the notion of a minimum transversal area.

In chapter 4 we address the problem of the variation of the area of black hole horizons applying locally the results of chapter 3 showing that the qmetric description proves that the presence of a minimum length brings with it a minimum step of area variation, i.e. a quantum of area.

Chapter 1

Bitensors

In this chapter we deal with the construction and study of particular objects called *bitensors*, which will play a central role in the construction of the qmetric. A bitensor is a tensorial function of two points of spacetime[14]: a *base point* x' and a *field point* x to which we assign respectively primed indexes $a', b', etc.$, and unprimed indexes $a, b, etc.$ We briefly review the formalism of such mathematical object and in particular we introduce two biscalars which are essential for our purposes (they are also interesting in their own): the *Synge's world function* and the *Van Vleck's determinant.*

1.1 Synge's world function and geodesic distance

Let x' a point (base point) of a D-dimensional spacetime $\mathbb{M}$ equipped with a metric tensor g_{ab}. We consider the point x (field point) in the *normal convex neighbourhood* of x' which is the set of points linked to x' by a unique geodesic. The geodesics segment β linking x and x' is described by $z^\mu(\lambda)$ where λ is an affine parameter of the geodesic that ranges from λ_0 and λ_1 such that $z(\lambda_0) = x'$ and $z(\lambda_1) = x$. Given an arbitrary point $z \in \beta$ we assign to it unprimed indexes $a, b, etc.$ We define the tangent vector to β in the point z as

$$t^a = \frac{dz^a}{d\lambda} \tag{1.1}$$

which satisfies the geodesic equation

$$t^a t^b{}_{;a} = t^a \nabla_a t^b = 0. \tag{1.2}$$

where ∇_a is the covariant derivative constructed with the spacetime metric g_{ab}. The situation is illustrated in Figure 1.1.

We define the *Synge's world function* as a scalar function both of x and x' in the following way [14, 15]

$$\Omega(x, x') = \frac{1}{2}(\lambda_1 - \lambda_0) \int_{\lambda_0}^{\lambda_1} g_{ab}(z) t^a t^b \, d\lambda \tag{1.3}$$

where the integral is performed along the geodesic segment β. By virtue of eq.(1.2) the quantity $\epsilon \equiv g_{ab}(z) t^a t^b$ is constant along β. Thus numerically $\Omega(x, x') = \epsilon(\Delta\lambda)^2/2$. We can compare the Synge's world function with the squared *geodesic distance* between x and x', namely the spacetime distance computed along the geodesic segment β given by [16]

$$\sigma^2(x, x') = \left(\int_{x'}^{x} \sqrt{g_{ab} dx^a dx^b} \right)^2 = \left(\int_{\lambda_0}^{\lambda_1} \sqrt{g_{ab} t^a t^b} d\lambda \right)^2 = \epsilon \, (\Delta\lambda)^2 \tag{1.4}$$

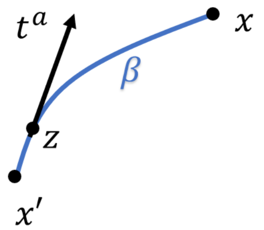


Figure 1.1: The base point x', the field point x, and the geodesic segment β that links them. The geodesic is described by parametric relations $z^a(\lambda)$ and $= t^a = dz^a/d\lambda$ is its tangent vector.

showing that $\Omega(x, x')$ is nothing but the half squared geodesic distance between the two points x, x':

$$\Omega(x, x') = \frac{1}{2}\sigma^2(x, x') \tag{1.5}$$

In particular if we are dealing with timelike geodesics we can choose the proper time τ as affine parameter and we would have $\sigma^2 = -(\Delta\tau)^2$ while choosing the proper distance l as affine parameter for spacelike geodesics we would have $\sigma^2 = (\Delta l)^2$. If the geodesic is null then we would have identically $\Omega = \sigma^2 = 0$.

1.1.1 Differentiation

Given $\Omega(x, x')$ defined as in eq.(1.3) (but the following procedure can be applied to treat generic biscalar functions) we can differentiate it with respect both to x and to x'. We indicate with primed indeces derivation w.r.t. x' and with unprimed indeces derivation w.r.t. x as well. For example we can have

$$\Omega_a = \frac{\partial\Omega}{\partial x^a} = \Omega_{,a} \tag{1.6}$$

$$\Omega_{a'} = \frac{\partial\Omega}{\partial x'^a} = \Omega_{,a'} \tag{1.7}$$

We have that Ω_a is a 1-rank tensor with respect to tensorial operations carried out in x and a scalar with respect to the ones in x'. In analogy $\Omega_{a'}$ is a 1-rank tensor with respect to tensorial operations carried out in x' and a scalar with respect to the ones in x. Iterating this procedure we can construct:

$$\Omega_{ab} = \nabla_b\Omega_a = \Omega_{a;b} \tag{1.8}$$

$$\Omega_{ab'} = \nabla_{b'}\Omega_a = \Omega_{a;b'} = \Omega_{a,b'} \tag{1.9}$$

$$\Omega_{a'b} = \nabla_b\Omega_{a'} = \Omega_{a';b} = \Omega_{a',b} \tag{1.10}$$

$$\Omega_{a'b'} = \nabla_{b'}\Omega_{a'} = \Omega_{a';b'} \tag{1.11}$$

where Ω_{ab} is a 2-rank tensor in x and a scalar in x', $\Omega_{\alpha\beta'}$ which is a 1-rank tensor both in x and in x' and so on. From the properties of partial and covariant derivatives we have the following symmetries:

$$\Omega_{ab} = \nabla_b\Omega_a = \partial_b\partial_a\Omega - \Gamma^c_{ba}\partial_c\Omega = \partial_a\partial_b\Omega - \Gamma^c_{ab}\partial_c\Omega = \nabla_a\partial_b\Omega = \Omega_{ba} \tag{1.12}$$

$$\Omega_{ab'} = \nabla_{b'}\Omega_a = \partial_{b'}\partial_a\Omega = \partial_a\partial_{b'}\Omega = \Omega_{b'a} \tag{1.13}$$

and similarly we could prove that $\Omega_{a'b'} = \Omega_{b'a'}$ and $\Omega_{a'b} = \Omega_{ba'}$.
We can now explicitly compute Ω_a. In order to do so we need to evaluate $\delta_x\Omega =$

$\Omega(x+\delta x, x') - \Omega(x, x')$ which tell us how the Synge's world function varies when the field point x moves. Under the shift δx the segment β is shifted to the unique geodesic segment $\beta + \delta\beta$ linking $x + \delta x$ to x', parameterized by $z^a(\lambda) + \delta z^a(\lambda)$ with the affine parameter λ rescaled such that it still runs from λ_0 to λ_1. Notice that $\delta z(\lambda_0) = 0$ and $\delta z(\lambda_1) = \delta x$. Expanding $\Omega(x + \delta x, x')$ at the first order in the variations we get

$$\delta_x \Omega = \Delta\lambda \int_{\lambda_0}^{\lambda_1} d\lambda \left(g_{ab} \dot{z}^a \delta \dot{z}^b + \frac{1}{2} g_{ab,c} \dot{z}^a \dot{z}^b \delta z^c \right) \tag{1.14}$$

where $\Delta\lambda = \lambda_1 - \lambda_0$, the overdot indicates differentiation w.r.t. λ and the metric and its derivatives are evaluated on β. We can integrate by parts the first term:

$$g_{ab} \dot{z} \delta \dot{z}^b = \frac{d}{d\lambda} \left(g_{ab} \dot{z}^a \delta z^b \right) - g_{ab} \ddot{z}^a \delta z^b - g_{ab,c} \dot{z}^a \dot{z}^c \delta z^b \tag{1.15}$$

and (1.14) becomes after a suitable relabelling of indeces

$$\delta_x \Omega = \Delta\lambda \left[g_{ab} \dot{z}^a \delta z^b \right]_{\lambda_0}^{\lambda_1} - \Delta\lambda \int_{\lambda_0}^{\lambda_1} d\lambda \left[g_{ab} \ddot{z}^b + \left(g_{ab,c} - \frac{1}{2} g_{cb,a} \right) \dot{z}^b \dot{z}^c \right] \delta z^a \tag{1.16}$$

With some algebraic passages it can be shown the integrand is equivalent to $g_{ab}(\dot{z}^c \dot{z}^b_{;c})$ which is vanishing because of eq.(1.2). Thus we have :

$$\delta_x \Omega = \Delta\lambda g_{ab} \dot{z}^a \delta x^b = \Delta\lambda g_{ab} t^a \delta x^b \tag{1.17}$$

hence

$$\Omega_a(x, x') = \frac{\delta_x \Omega}{\delta x^a} = \Delta\lambda g_{ab}(x) t^b(x) = t_a \Delta\lambda \tag{1.18}$$

We see that for any points $z \in \beta$ the vector $\Omega^a(z, x') = (\lambda - \lambda_0) t^a$ can be thought as a rescaled tangent vector. A similar computation leads to:

$$\Omega_{a'}(x, x') = -\Delta\lambda g_{ab}(x') t^b(x') = -t_a \Delta\lambda \tag{1.19}$$

It is interesting to compute the norm $\Omega_a \Omega^a = (\Delta\lambda)^2 t^a t_a = (\Delta\lambda)^2 \epsilon$ which we can rewrite as:

$$g^{ab}(x) \partial_a \Omega \partial_b \Omega = 2\Omega \tag{1.20}$$

and similarly

$$g^{ab}(x') \partial_{a'} \Omega \partial_{b'} \Omega = 2\Omega \tag{1.21}$$

We can recast eq.(1.20) in terms of the geodesic distance σ^2:

$$g^{ab} \partial_a \sigma^2 \partial_b \sigma^2 = 4\sigma^2. \tag{1.22}$$

A second differentiation of eq.(1.20) brings to

$$\Omega^a{}_b \Omega^b = \Omega^a{}_b \Omega^b = \Omega^a \tag{1.23}$$

which is the geodesic equation in a non-affine parametrization.

1.1.2 Geodesic structure

The study of the geodesic structure of a spacetime is particularly instructive since from the geodesic behaviour we can study the motion of freely falling probes, the causality structure of the spacetime and the possible presence of spacetime singularities [17]. The study of the geodesic structure is also relevant for the qmetric description. In particular we are interested to study the congruence of geodesics emanating from the base point x'.

Timelike/Spacelike geodesics

We choose to parametrize the timelike/spacelike geodesics with an affine parameter which corresponds to the physical geodesic distance, namely proper time or proper length, so that in this case we can rewrite the form of the unit vector tangent to a geodesic from eq.(1.18) as:

$$u^a = \frac{\partial^a \sigma^2}{2\sqrt{\epsilon\sigma^2}} \tag{1.24}$$

with $\epsilon = -1, +1$ respectively for timelike and spacelike geodesics. We introduce now the concept of *equi-geodesic surfaces*. An equi-geodesic (hyper)surface $\Sigma_{x'}(\epsilon l^2)$ is defined as the set of points at a given constant geodesic distance $\sigma^2 = \epsilon l^2$ from the base point x':

$$\Sigma_{x'}(\epsilon l^2) = \left\{ x | \epsilon\sigma^2(x, x') = \epsilon 2\Omega(x, x') = l^2, l > 0 \right\} \tag{1.25}$$

These hypersurfaces are orthogonal to the flow of geodesic emanating from x' since the normal n_a to $\Sigma_{x'}(\epsilon l^2)$ in the point x is proportional to the tangent vector u_a to the geodesic passing through x[17]:

$$n_a(x) \propto \partial_a \sigma^2(x, x') \propto u_a(x) \tag{1.26}$$

If we want the normal vector to have a unit norm then we can identify $n_a = u_a$. We can define the induced (D-1) metric $h_{\alpha\beta}$ on the equigeodesic surface restricting the coordinate displacement on Σ[18]:

$$ds^2_{|\Sigma} = g_{ab} dx^a dx^b_{|\Sigma} = g_{ab} \left(\frac{\partial x^a}{\partial y^\alpha} \right) \left(\frac{\partial x^b}{\partial y^\beta} \right) dy^\alpha dy^\beta \equiv h_{\alpha\beta} dy^\alpha dy^\beta \tag{1.27}$$

where $\{y^\alpha\}$ are (D-1) coordinates on the equigeodesic surface. Alternatively we can define the transverse metric to the tangent vector u^a in the following way:

$$h_{ab} = g_{ab} - \epsilon u_a u_b \tag{1.28}$$

which is orthogonal to u^a since:

$$u^a h_{ab} = u^a g_{ab} - \epsilon u_a u_b u^a = u_b - u_b = 0 \tag{1.29}$$

It happens that h_{ab} and $h_{\alpha\beta}$ carry the same informations. In fact since the unit normal is identified by the geodesic tangent vector we have:

$$n_a \frac{\partial x^a}{\partial y^\alpha} dy^\alpha = u_a \frac{\partial x^a}{\partial y^\alpha} dy^\alpha = 0 \tag{1.30}$$

and we can write

$$h_{\alpha\beta} = h_{ab} \frac{\partial x^a}{\partial y^\alpha} \frac{\partial x^b}{\partial y^\beta} dy^\alpha dy^\beta \tag{1.31}$$

The intrinsic and extrinsic geometry of these hypersurfaces are described respectively by the Ricci scalar R_Σ computed via the transverse metric on $\Sigma_{x'}(\epsilon l^2)$ hypersurface and by the extrinsic curvature K_{ab}. The extrinsic curvature is given by

$$K_{ab} = \nabla_a u_b = \frac{\nabla_a \nabla_b \Omega - \epsilon u_a u_b}{\sqrt{2\epsilon\Omega}} \tag{1.32}$$

We can compute the geodesics expansion in the affine parametrization as the trace of extrinsic curvature:

$$K = K^a{}_a = \frac{\nabla^a \nabla_a \Omega - 1}{\sqrt{2\epsilon\Omega}} \tag{1.33}$$

Null geodesics

The case of a null geodesic congruence is slighty different from the previous one. In this case the vector $\partial_a \sigma^2$ is a null vector and we can't use the geodesic distance σ^2 as an affine parameter since $\sigma^2 = 0$ all along a null geodesic and it fails to distinguish different points along it. We assume the null geodesics are parametrized by some affine parameter λ and the null tangent vector to the geodesics is

$$l^a = \frac{dx^a}{d\lambda} \tag{1.34}$$

If we try to determine the transverse metric as before, i.e. as $h_{ab} = g_{ab} + l_a l_b$, we will fail since in this case $l^a h_{ab} = l_b \neq 0$. In order to consistently define a transverse metric we need to define an arbitrary second null vector m^a such that $m^a l_a = -1$. In this way we define the transverse metric as:

$$h_{ab} = g_{ab} + l_a m_b + m_a l_b \tag{1.35}$$

such that $l^a h_a b = m^a h_{ab} = 0$, i.e. is orthogonal both to l^a and m^a. Taking the trace we find $h^a_a = D - 2$ and $h^a_c h^c_b = h^a_b$ showing that h_{ab} is effectively $(D-2)$-dimensional. Since the arbitrariness of m^a the transverse metric is not unique and it depends on m^a. However it turns out that quantities computed via the transverse metric are independent from the choice of the second null vector [17]. The null equigeodesic hypersurface with respect to the base point x' is defined as the set of points x satisfying $\sigma^2(x, x') = 0$, i.e. that are null separated from x'. Thus the null equigeodesic hypersurface coincides with the null cones (future and past) centered in x'. We let the null normal to the null cone to be $k_a = -\partial_a \sigma^2 = -2\partial_a \Omega$. We know that $\partial_a \Omega$ obeys a geodesic equation, thus the null normal is proportional to the null tangent vector to the geodesics: $k^a \propto l^a$. As we can arbitrarly rescale a null vector we can identify $k^a = l^a$.

Canonical observer and null affine parameters

In order to select a particular affine parameter with a physical meaning we must introduce an observer with a timelike 4-velocity V^a: any null affine parametrization λ gives us a measure of distance along the null geodesic as measured by a particular timelike observer at a certain point x of the geodesic[13, 19]. We attach a canonical observer in the point x' with a four velocity V^a such that $V^a l_a = -1$ and we let it to be parallel transported along the null geodesic. We define implicitly our affine parameter λ as [19]

$$\frac{1}{2}\partial_a \sigma^2 = \lambda l_a \tag{1.36}$$

and we can write

$$\lambda = -V^a \frac{1}{2}\partial_a \sigma^2 \tag{1.37}$$

From V^a and l^a we can construct the second null vector $m^a = V^a - (1/2)l^a$ such that:

1. $m^a V_a = -1/2$

2. $m^a l_a = -1$

Then we can define the parameter ν such that $m^a = \frac{dx^a}{d\nu}$

1.1.3 Coincidence limit

We now introduce the coincidence limit procedure which allow us to investigate the bitensors behaviour in the limit x tends to x'. We use the following notation for the coincidence limit of a generic bitensor $T_{...}(x, x')$ [14]:

$$[T_{...}(x, x')] \equiv \lim_{x \to x'} T_{...}(x, x') = T_{...}(x') \tag{1.38}$$

where $T_{...}(x')$ is now a tensor in x'. We assume that the coincidence limit is a unique tensorial function of the base point x', independent of the direction in which the limit is taken. Strictly speaking if the limit is computed by letting $\lambda_1 \to \lambda_0$ on a precise geodesic segment β the result is independent of the geodesic choice. Considering the world function, from eq.(1.3), eq.(1.18) and eq.(1.19) we have:

$$[\Omega] = [\Omega_a] = [\Omega_{a'}] = 0 \tag{1.39}$$

We can rewrite eq.(1.23) once we use eq.(1.18) as

$$(g_{ab} - \Omega_{ab})t^b = 0 \tag{1.40}$$

From the assumption that the coincidence limit must be independent from the direction in which is computed, namely t^a, we get:

$$[\Omega_{ab}] = g_{a'b'} \equiv g_{ab}(x') \tag{1.41}$$

and we also have $g_{a'b'} = [\Omega_{a'b'}] = -[\Omega_{a'b}] = -[\Omega_{ab'}]$. We can continue to differentiate the world function and compute all the coincidence limits using Synge's rule (see [14]). We have the following interesting result:

$$[\Omega_{abcd}] = -\frac{1}{3}\left(R_{a'c'b'd'} + R_{a'd'b'c'}\right) \tag{1.42}$$

where it appears the Riemann tensor. Since we can reconstruct geometrical entities via coincidence limit of (differentiation of) the Synge's world function it is natural to ask how much information about the spacetime geometry is in the world function or, alternatively, in the geodesic distance.

1.1.4 Reconstruction of the spacetime

We saw in the previous section that via coincidence limit of the world function we can determine the metric tensor and also the Riemann tensor. We can express the world function as an expansion near coincidence (see appendix B) as [16]:

$$\Omega_{a'b'} = g_{ab}(x') - \frac{1}{3}R_{acbd}(x')\Omega^{c'}\Omega^{d'} + o(R\Omega) \tag{1.43}$$

We can see that in the expansion of the world function we find both the metric tensor and the curvature tensor. Therefore the metric tensor and the world function carry the same amount of information about the spacetime geometry. Classical gravity is well described by the ten independent components of the local metric tensor but it could also be described by the biscalar Synge's world function or equivalently by the geodesic distance. However the description in terms of the metric tensor is much simpler than the other: it's not easy to identify the conditions that a biscalar must fulfill to be a geodesic distance. Moreover trying to write the Einstein-Hilbert action principle in terms of the world function would involve fourth order derivatives [16]. Thus a classical description of gravity in terms of the geodesic distance would be unnecessarily complicated. However a description based on such biscalars might be the natural way to follow in order to try to describe spacetime at very small distances, where the notion of local tensor might breaks down if quantum gravity effects are non-local. Moreover working directly with the geodesic distance it would be natural to implement a minimum length in spacetime imposing a modification of the bitensor behaviour at small distances. This way of proceeding is part of the idea that physics should deal with measurements: the notion of spacetime must arise from measurements: the outcomes of measurements can be described in terms of biscalars (geodesic distance, Green functions...) and from them we should be able to reconstruct a notion of spacetime metric [20].

1.2 Parallel propagator

We introduce now a bitensor $\Pi^a{}_{a'}$ called *parallel propagator* which allow us to parallel transport vectors from the point x to the point x' along the geodesic segment linking them [14]. Let be $A^{a'} \equiv A^a(x')$ a vector in the tangent space of the point x'. We can decompose such a vector in the tetrad frame in the point x' (see appendix A):

$$A^{a'} = A^a(x') = A^\alpha(x') e^a{}_\alpha(x') \equiv A^\alpha e^{a'}{}_\alpha \tag{1.44}$$

$$A^\alpha = A^{a'} e^\alpha{}_{a'} \tag{1.45}$$

where both Latin and Greek indexes run from 0 to $D-1$ and Latin indexes are raised and lowered with the background metric g_{ab} while Greek ones are raised and lowered with the Minkowski metric $\eta_{\alpha\beta}$. We can parallel transport this vector from x' to a point x along a geodesic segment. Parallel transport means that the tetrad frame componens A^α are kept constant. Thus we can expand the parallel transported vector in the point x as:

$$A^a = A^a(x) = A^\alpha e^a{}_\alpha(x) \equiv A^\alpha e^a{}_\alpha \tag{1.46}$$

$$A^\alpha = A^a e^\alpha{}_a \tag{1.47}$$

Since A^α must be kept constant from x' to x we can rewrite:

$$A^a = A^\alpha e^a{}_\alpha = A^{a'} e^\alpha{}_{a'} e^a{}_\alpha \tag{1.48}$$

from which we define the parallel propagator as

$$\Pi^a{}_{a'} \equiv e^a{}_\alpha e^\alpha{}_{a'} \tag{1.49}$$

and we finally can write the vector at the point x in terms of the components of the vector in the point x':

$$A^a = \Pi^a{}_{a'} A^{a'} \tag{1.50}$$

We can introduce the inverse parallel propagator $\Pi^{a'}{}_a$ since

$$\Pi^{a'}{}_a \Pi^b{}_{a'} = e^\alpha{}_a e^{a'}{}_\alpha e^b{}_\beta e^\beta{}_{a'} = \delta_a{}^{a'} \delta^b{}_{a'} = \delta_a{}^b \tag{1.51}$$

$$\Pi^{a'}{}_a \Pi^a{}_{b'} = e^\alpha{}_a e^{a'}{}_\alpha e^a{}_\beta e^\beta{}_{b'} = \delta_a{}^{a'} \delta^a{}_{b'} = \delta_{a'}{}^{b'} \tag{1.52}$$

The action of the parallel propagator can be extended on (bi)tensors of arbitrary rank: we have the occurrence of one parallel propagator for each tensorial index. We can compute the coincidence limit of the parallel propagator leading to $[\Pi^a{}_{b'}] = \delta^{a'}{}_{b'}$. We can also determin the determinant of the parallel propagator. We first notice that:

$$g^{ab}(x) = \eta^{\alpha\beta} e^a{}_\alpha(x) e^b{}_\beta(x) \to \det\{\mathbf{g}^{-1}\} = -\det\{\mathbf{e}\}^2 \tag{1.53}$$

and similarly

$$g^{ab}(x') = \eta^{\alpha\beta} e^a{}_\alpha(x') e^b{}_\beta(x') \to \det\left\{\mathbf{g'}^{-1}\right\} = -\det\{\mathbf{e'}\}^2 \tag{1.54}$$

Since we have that $\Pi^a{}_{a'} = e^a{}_\alpha e^\alpha{}_{a'}$ we can compute

$$\det\{\mathbf{\Pi}\} = \det\{\mathbf{e}\}\det\{\mathbf{e'}\}^{-1} = \frac{\sqrt{-\det \mathbf{g}^{-1}}}{\sqrt{-\det \mathbf{g'}^{-1}}} = \frac{\sqrt{-\det \mathbf{g'}}}{\sqrt{-\det \mathbf{g}}} \tag{1.55}$$

$$\det\left\{\mathbf{\Pi}^{-1}\right\} = \det\{\mathbf{e'}\}\det\{\mathbf{e}\}^{-1} = \frac{\sqrt{-\det \mathbf{g'}^{-1}}}{\sqrt{-\det \mathbf{g}^{-1}}} = \frac{\sqrt{-\det \mathbf{g}}}{\sqrt{-\det \mathbf{g'}}} \tag{1.56}$$

1.3 Van Vleck determinant

A very important biscalar function is the *Van Vleck determinant* (VVD). It appears in a large number of physics fields [19]: it plays a fundamental role in WKB approximation of quantum time evolution operator and Green functions, in the adiabatic approximation of heat kernel, in the one loop approximation of functional integrals, in the theory of caustics in geometrical optics and in focusing and de-focusing of geodesics in spacetime. For our purposes we need to restrict our attention to the role of the VVD in the description of geodesic flows. In this specific field we can give the following definition [19]: given a base point x' and a field point x in the normal convex neighbourhood of x' in a D-dimensional spacetime $\mathbb{M}$ equipped with a metric tensor g_{ab} we define the VVD $\Delta(x, x')$ as

$$\Delta(x, x') = (-1)^{D-1} \frac{\det|\Omega_{ab'}(x, x')|}{\sqrt{g(x) g(x')}} \tag{1.57}$$

where $g(x)$ and $g(x')$ are the metric determinant evaluated respectively in x and x' and $\Omega_{ab'} = \nabla_{b'}\nabla_a \Omega$ where $\Omega = \Omega(x, x')$ is the Synge's world function between x' and x. This definition turns out to be equivalent to [14]:

$$\Delta(x, x') = \det\left[\Delta^{a'}{}_{b'}\right] \tag{1.58}$$

where $\Delta^{a'}{}_{b'} = -\Pi^{a'}{}_a \Omega^a{}_{b'}$.

1.3.1 Equalities

The Van Vleck determinant satisfies a number of equations which will be useful later on. The first we mention is the following:

$$\nabla_a \left[\Delta(x, x')\Omega^a\right] = \frac{1}{2}\nabla_a \left[\Delta(x, x')\partial^a \sigma^2\right] = D\Delta(x, x') \tag{1.59}$$

where $\Omega^a = \nabla^a \Omega(x, x')$ is the derivative of the Synge's world function with respect to x.

Proof. Starting from

$$\Omega = \frac{1}{2}\Omega^c \Omega_c \tag{1.60}$$

we can differentiate twice to find

$$\Omega_a = \Omega^c \Omega_{ca} \tag{1.61}$$

$$\Omega_{ab'} = \Omega^c{}_{b'} \Omega_{ca} + \Omega^c \Omega_{cab'} \tag{1.62}$$

So we have that

$$\Delta^{a'}{}_{b'} = -\Pi^{a'}{}_a \Omega^a{}_{b'} = -\Pi^{a'}{}_a \left[\Omega^c{}_{b'} \Omega_c{}^a + \Omega^c \Omega^a{}_{b'c}\right] \tag{1.63}$$

Now we can substitute $\Omega^c{}_{b'} = -\Pi^c{}_{c'} \Delta^{c'}{}_{b'}$ and we get

$$\Delta^{a'}{}_{b'} = \Pi^{a'}{}_a \Pi^c{}_{c'} \Delta^{c'}{}_{b'} \Omega_c{}^a + \Pi^{a'}{}_a \Omega_c \nabla^c \left(\Pi^a{}_{a'} \Delta^{a'}{}_{b'}\right) \tag{1.64}$$

We introduce $\left(\Delta^{-1}\right)^{a'}{}_{b'}$ such that $\Delta^{a'}{}_{b'} \left(\Delta^{-1}\right)^{b'}{}_{c'} = \delta^{a'}{}_{c'}$ and we can write

$$\delta^{a'}{}_{b'} = \Pi^{a'}{}_a \Pi^b{}_{b'} \Omega^a{}_b + \Omega_c \left(\Delta^{-1}\right)^{c'}{}_{b'} \nabla^c \left(\Delta^{a'}{}_{c'}\right) \tag{1.65}$$

Taking the trace we find

$$D = \nabla_a \Omega^a + \Omega_c \left(\Delta^{-1}\right)^{c'}{}_{a'} \nabla^c \left(\Delta^{a'}{}_{c'}\right) \tag{1.66}$$

We see that in the second term we have $\mathrm{Tr}\left(\boldsymbol{\Delta}^{-1} \delta \boldsymbol{\Delta}\right) = \delta \det \boldsymbol{\Delta}$ and we can write:

$$D = \nabla_a \Omega^a + \Omega_a \nabla^a \ln \Delta \tag{1.67}$$

which is indeed equivalent to eq.(1.59). □

We can now provide an expression for the expansion of the VVD near coincidence. First of all by definition we have:

$$\Delta^{a'}{}_{b'} = \delta^{a'}{}_{b'} + \frac{1}{6} R^{a'}{}_{c'b'd'} \Omega^{c'} \Omega d' + o(\Omega^3) \tag{1.68}$$

from the expansion of $\Omega_{ab'}$ (see appendix B). Thus at coincidence we have $\left[\Delta^{a'}{}_{b'}\right] = \delta^{a'}{}_{b'}$. Moreover, near coincidence, we can use the approximation [14] $\det|1 + \mathbf{a}| = 1 + Tr[\mathbf{a}] + o(\mathbf{a}^2)$ to get the expansion of the Van Vleck determinant in x'

$$\Delta = 1 + \frac{1}{6} R_{a'b'} \Omega^{a'} \Omega^{b'} + o(\Omega^3) \tag{1.69}$$

with R_{ab} the Ricci tensor, showing at coincidence $[\Delta] = 1$.

1.3.2 Physical interpretation

Time/Space-like separations

For timelike/spacelike separation between x' and x we can rewrite eq.(1.59) in the following way, recalling eq.(1.24), as

$$\nabla_a \left[\Delta 2\sqrt{\epsilon\sigma^2} u^a\right] = 2D\Delta \tag{1.70}$$

and we find

$$\sqrt{\epsilon\sigma^2} u^a \nabla_a \Delta + \Delta + \Delta\sqrt{\epsilon\sigma^2} \nabla_a u^a = D\Delta \tag{1.71}$$

Choosing $s = \sqrt{\epsilon\sigma^2}$ as the affine parameter for the geodesic we find the differential equation:

$$u^a \nabla_a \Delta = \left[\frac{D-1}{s} - \nabla_a u^a \right] \Delta \tag{1.72}$$

Recalling that $U^a \nabla_a G = \frac{d}{ds} G$ for any scalar function and that $\theta \equiv \nabla_a u^a$ is the expansion of the geodesic congruence emanating from x'[17] we have:

$$\frac{1}{\Delta} \frac{d\Delta(s)}{ds} = \left[\frac{D-1}{s} - \theta \right] \tag{1.73}$$

with the initial condition $\Delta(s = 0) = 1$ (i.e. at coincidence). A direct integration gives:

$$\ln \Delta(x, x') = \int_0^{s(x)} \left[\frac{D-1}{s} - \theta \right] ds = \int_0^{s(x)} \left[\theta^{flat} - \theta \right] ds \tag{1.74}$$

where we recognize $\theta^{flat} = (D-1)/s$ as the expansion of a spacelike/timelike geodesic congruence in flat spacetime emanating from the base point x' [17]. Given in this form, the Van Vleck determinant gives us information about the ratio between the geodesic trajectories density of the geodesic congruence emanating from the point x' in the point x of a given spacetime and the density one would have in the flat spacetime [19]. In particular if $\Delta > 1$ the geodesics are expanding less rapidly than in the flat case, while if $\Delta < 1$ they are expanding more rapidly.

Null separations

We can do the same considerations also for null separation between x' and x. In this case thins are little trickier since $\partial_a \sigma^2$ is a null vector and $\sigma^2 = 0$ for any null separated points. In order to proceed with the computation we need to consider a point y near the null geodesic γ but not on it, i.e. y is time or space separated from x', and then take the limit $y \to x \in \gamma$ [5, 13]. Following the construction of section 1.1.2 we let the null geodesic be parameterized by an affine parameter λ and we let l^a be the null tangent vector to the geodesic. Picking up a canonical observer with four velocity V^a we define the second null vector $m^a = V^a - l^a/2 = dx^a/d\nu$. We select the point y as a point outside γ reachable from $x \in \gamma$ through a null geodesic segment γ' whose tangent null vector is m^a and we suitably fix $\nu(x) = 0$. We can now decompose the gradient of the squared geodesic distance as [19]:

$$\partial^a \sigma^2(x, x')_{x=y} = 2\lambda l^a(y) + 2\nu m^a(y) \tag{1.75}$$

where $l^a(y)$ and $m^a(y)$ are parallel transported from x to y along γ'. Thus we have:

$$\begin{aligned} \left(\nabla_a \partial^a \sigma^2 \right)_{x=y} &= 2 \left(\lambda \nabla_a l^a + \nu \nabla_a m^a + l^a \partial_a \lambda + m^a \partial_a \nu \right)_{x=y} = \\ &= 2 \left(\lambda \nabla_a l^a + \nu \nabla_a m^a + 2 \right)_{x=y} \end{aligned} \tag{1.76}$$

where we used the fact that $l^a \partial_a \lambda = m^a \partial_a \nu = 1$ by construction. In the limit $y \to x$ we have $\nu \to 0$ thus:

$$\nabla_a \partial^a \sigma^2(x, x') = 2 \left(\lambda \nabla_a l^a + 2 \right) \tag{1.77}$$

Therefore we can write:

$$\nabla_a \left[\Delta \partial^a \sigma^2 \right] = \partial^a \sigma^2 \nabla_a \Delta + \Delta \nabla_a \partial^a \sigma^2 = \tag{1.78}$$

$$= 2\lambda l^a \nabla_a \Delta + 2\Delta\lambda \nabla_a l^a + 4\Delta \tag{1.79}$$

Inserting this in eq.(1.59) we find:

$$l^a \nabla_a \Delta = \left[\frac{D-2}{\lambda} - \nabla_a l^a \right] \Delta \tag{1.80}$$

which is the null version of the equation (1.72). Solving the differential equation we get the solution:

$$\ln \Delta(x, x') = \int_0^{\lambda(x)} \left[\frac{D-2}{\lambda} - \theta \right] d\lambda = \int_0^{\lambda(x)} \left[\hat{\theta}^{flat} - \hat{\theta} \right] d\lambda \tag{1.81}$$

which is the null form of eq.(1.74) with $\hat{\theta} = \nabla_a l^a$ the expansion of the null geodesic congruence emanating from x' in a given spacetime and $\hat{\theta}^{flat} = (D-2)/\lambda$ the expansion we would have in the Minkowski spacetime.

Propagator in curved spacetime

The VVD is also relevant in quantum field theory in curved spacetime: in fact in any arbitrary spacetime of D dimensions the leading singular structure of the two points function associated to the d'Alambertian operator $\Box$ is given by the Hadamard form [21, 22]

$$G(x, x') = \frac{\sqrt{\Delta}}{(\epsilon\sigma^2)^{\frac{D-2}{2}}}(1 + \text{ smooth therms}) \tag{1.82}$$

thus Δ must carry information about how curvature of spacetime affects the propagation of quantum fields.

1.3.3 Maximally symmetric spaces

The VVD is known in exact form for D-dimensional maximally symmetric spacetimes. The trivial cases are the Euclidean space and the Minkowski spacetime where a direct computation from the definition gives $\Delta(x, x') = 1$ for all pairs of points x and x'. For the spherical space and the De Sitter spacetime with curvature radius a it is given by [23]:

$$\Delta(\sigma^2)^{-\frac{1}{D-1}} = \frac{\sin\left(\sqrt{\epsilon\sigma^2}/a\right)}{\sqrt{\epsilon\sigma^2}/a} \tag{1.83}$$

while for the hyperbolic space and the Anti De Sitter spacetime it is given by [24]

$$\Delta(\sigma^2)^{-\frac{1}{D-1}} = \frac{\sinh\left(\sqrt{\epsilon\sigma^2}/a\right)}{\sqrt{\epsilon\sigma^2}/a} \tag{1.84}$$

Chapter 2

The Quantum Metric

In this chapter we fully construct the qmetric for a generic spacetime. As we will see the geodesic distance/Synge's world function and the Van Vleck determinant introduced in the previous chapter are the main ingredients of this description. At the end of the chapter it is reported one of the most important results of the qmetric description which is the bridge with the so-called *emergent gravity paradigm*: the effective Ricci scalar constructed with the qmetric seems to enforce such paradigm.

2.1 Motivations for a minimum length

The present fundamental physics is dominated by the presence of three fundamental constants of nature. Since Newton proposed his theory of gravitation the Newton constant G governs the laws of gravity. In the XIX century with the formulation of Maxwell's equations the speed of light c began to have more and more importance until it was recognized as a universal constant by the theory of Special Relativity. Moreover in 1900 Planck introduced the (reduced) Planck constant $\hbar$ which rules Quantum Mechanics. Combining together this three fundamental constants we can define the so called Planck units of mass, space and time [25]:

$$M_P = \sqrt{\frac{\hbar c}{G}} \simeq 1.2 \times 10^{19} \text{ GeV} \tag{2.1}$$

$$L_P = \sqrt{\frac{\hbar G}{c^3}} \simeq 10^{-33} \text{ cm} \tag{2.2}$$

$$t_P = \sqrt{\frac{\hbar G}{c^5}} \simeq \times 10^{-43} \text{s} \tag{2.3}$$

This values identify the scale at which it is expected that the quantum nature of gravity plays a predominant role. There are several arguments which identify the length scale L_P as the minimum length scale we can actually probe : string theory (ST) and loop quantum gravity (LQG) admit the existence of a minimum length and so do other approaches to quantum gravity as asymptotically safety gravity and non-commutative geometry scenarios [25]. There is also a number of thoughts experiments which suggest the presence of a minimum measurable length such as the Heisenberg microscope [26] or measurement of black hole horizon area [27].
From now on we assume the existence of a minimum length scale L_0 which acts as a limit on the accuracy with we can actually localize events in spacetime. We assume that L_0 is of order of the Planck length L_P, namely

$$L_0 = kL_P \text{ with } k \sim O(1) \tag{2.4}$$

but it can actually be much larger. We can start to implement such a minimum length in our description of spacetime directly modifying the Lorentz invariant squared geodesic distance $\sigma^2(x,x')$ between spacetime events, which is two times the Synge's world function $\Omega(x,x')$ defined in (1.3) : in this way we do not need to rely on a specific quantum gravity theory and we can stay as generic as possible. Moreover working with Lorentz-invariant quantities implies that the notions we can derive using them can also share this invariance. A minimal length scale L_0 can be implemented for timelike/spacelike separated events with the following substitution:

$$\sigma^2(x,x') \longrightarrow S_L(\sigma^2) \text{ such that } S_L(0) = \pm L_0^2 \neq 0 \tag{2.5}$$

where $S_L(\sigma^2)$ is a generic modified geodesic distance function that remains finite in the coincidence limit $x \to x'$. This means that we can't localize spacetime events with an accuracy better than L_0: clearly we are breaking a basic axiom of any metric spaces, i.e. $\sigma^2(x,x) = 0$. Thus we expect that at energy scales at which the quantum nature of gravity/spacetime has an important role, spacetime itself is no more described by a classical metric tensor. The general prescription is that equigeodesic surfaces of a given nature (timelike, spacelike or lightlike) determined by a spacetime metric are mapped in equigeodesic surfaces of the same nature according to the qmetric. From (2.5) we clearly see that the light cone from x' is a discontinuty surface for the modified geodesic distance $S_L(\sigma^2)$ [5]: the construction of the qmetric for null separated events it is not so straightforward and deserve a separated treatment.

2.2 Construction of the qmetric: timelike/spacelike intervals

We saw in section 1.1.4 that we can reconstruct spacetime geometry by means of the coincidence limits of differentiation of the Synge's world function. With the introduction of a minimal length clearly the coincidence limit will be affected and we can use this fact to define the effective quantum metric.
We consider two events x and x' whose spacetime separation can be spacelike or timelike in a D-dimensional spacetime. This means that if u^a is the unit vector tangent to the unique geodesic segment linking the two points we have $u^a u_a = \epsilon$ where $\epsilon = -1, +1$ respectively for timelike and spacelike intervals. In order to formally define the quantum metric $q_{ab}(x,x')$ we need two mathematical inputs [28]:

1. Geodesic distances must be modified in order to stay finite in the coincidence limit. This is summarized by the replacement $\sigma^2(x,x') \to S_L(\sigma^2)$ such that $S_L(0) = \epsilon L_0^2 \neq 0$. The precise structure of $S_L(\sigma^2)$ must be determined by a complete theory of quantum gravity and so we need to be the most general as we can.

2. The modified d'Alembertian operator $\tilde{\Box}$ obtained from the qmetric must yields to a modification of two-point functions (Green functions), ruling how perturbations do propagate (then causality). In all maximally symmetric spactimes we want that the classical Green function $G(x,x') = G(\sigma^2)$ is mapped to the modified function $\tilde{G}(x,x') = \tilde{G}(\sigma^2) = G(S_L(\sigma^2))$. This would correspond to the fact that minimal length in spacetime distances would act as a universal regulator in UV divergences which affect quantum field theory. Moreover this is also a most natural prescription in order to have a metric description in the q-space which is analogous to the metric description in ordinary space, this allowing in a sense to forget that in the qmetric

space we are coming from a ordinary space. It is clear that this of point (2) can be requested only for maximally symmetric spaces. In generic spaces we can not insist on this because as long as the curvature scale becomes comparable to L_0 we can expect effects related to this dependent on the direction. As a matter of fact however, this natural request for maximally symmetric spaces is enough to fix completely the qmetric we associate to generic spacetimes. Indeed the latter simply have necessarily some further terms in the solutions to the wave equation with respect to the maximally symmetric ones, these additional terms expressing the interplay between L_0 and curvature length, differential in direction.

To implement the first point we must start from a modification of (1.22):

$$g^{ab}(x)\partial_a\sigma^2\partial_b\sigma^2 = 4\sigma^2 \longrightarrow q^{ab}\partial_a S_L \partial_b S_L = 4S_L \tag{2.6}$$

In this sense the qmetric q_{ab} is that "metric" which gives S_L as a squared geodesic distance. As we said we don't want to fix exactly the function S_L, we only require that:

(i) Minimal length condition : $S_L(0) = \epsilon L_0^2$

(ii) Identity condition : $S_{L_0=0}(\sigma^2) = \sigma^2$

We ask for a qmetric bitensor $q_{ab} = q_{ab}(x, x')$ which acts as a rank 2 tensor at field point x. The general form can be written as [3]:

$$q_{ab} = Ag_{ab} - \epsilon B u_a u_b \tag{2.7}$$

where $\epsilon = u^a u_a$, $u_a = g_{ab}u^b$ and A, B are functions of x, x' to be determined. The corresponding contra-variant form can be deduced imposing $\delta^a_b = q^{ac}q_{cb}$:

$$q^{ab} = A^{-1}g^{ab} + \epsilon Q u^a u^b = A^{-1}h^{ab} + \epsilon(Q + A^{-1})u^a u^b \tag{2.8}$$

where Q is such that $B = QA/(A^{-1}+Q)$ and $h^{ab} = g^{ab} - \epsilon u^a u^b$ is the transverse metric to u^a as defined in eq.(1.28) at the point x. Clearly, the function in parenthesis in the last term must diverge in the coincidence limit, if we want to hope that eq.(2.5) be satisfied. This ansatz is motivated by the fact that the qmetric must be a symmetric 2-rank bitensor depending on the metric tensor g_{ab} and on the vector tangent to the geodesic segment u^a. Thinking in classical terms, given two metrics related by an expression such eq.(2.7) we say the two metrics are *disformally coupled*: a detailed study of the relations between metrics disformally coupled and a prescription on how to compute geometrical entities according to a metric in terms of geometrical entities according to a disformally coupled metric can be found in [29].

To determine $\alpha \equiv A^{-1} + Q$ we substitute in (2.6) the ansatz for the qmetric (2.8) getting

$$\begin{aligned}
&\left(A^{-1}g^{ab} + \epsilon Q u^a u^b\right)\partial_a S_L \partial_b S_L = 4S_L \\
&A^{-1}g^{ab}\partial_a S_L \partial_b S_L + \epsilon Q u^a u^b \partial_a S_L \partial_b S_L = 4S_L \\
&A^{-1}g^{ab}\partial_a\sigma^2\partial_b\sigma^2\left(\frac{dS_L}{d(\sigma^2)}\right)^2 + \epsilon Q u^a u^b \partial_a\sigma^2\partial_b\sigma^2\left(\frac{dS_L}{d(\sigma^2)}\right)^2 = 4S_L \\
&4A^{-1}g^{ab}\Omega_a\Omega_b\left(\frac{dS_L}{d(\sigma^2)}\right)^2 + 4\epsilon Q\frac{\Omega^a}{\sqrt{2\epsilon\Omega}}\frac{\Omega^b}{\sqrt{2\epsilon\Omega}}\Omega_a\Omega_b\left(\frac{dS_L}{d(\sigma^2)}\right)^2 = 4S_L \\
&2A^{-1}\Omega\left(\frac{dS_L}{d(\sigma^2)}\right)^2 + 2Q\Omega\left(\frac{dS_L}{d(\sigma^2)}\right)^2 = S_L \\
&(A^{-1} + Q)\sigma^2\left(\frac{dS_L}{d(\sigma^2)}\right)^2 = S_L
\end{aligned}$$

from which we have:

$$\alpha \equiv (A^{-1} + Q) = \frac{1}{\sigma^2}\frac{S_L}{S_L'^2} \tag{2.9}$$

where a prime index means differentiation w.r.t. σ^2. We see that as long as we require that $S_L/S_L'^2$ is finite at coincidence, the function α is divergent in the coincidence limit as it should. At this point we have:

$$q^{ab} = A^{-1}h^{ab} + \epsilon\left(\frac{1}{\sigma^2}\frac{S_L}{S_L'^2}\right)u^a u^b \tag{2.10}$$

The requirement of the modification of the Green functions will fix completely the qmetric determining the value of A[28]. In order to do so we need to compute the modified covariant d'Alembertian for the qmetric. A direct application of the *matrix determinant lemma*[1] to the q metric in the form (2.7) gives:

$$\begin{aligned}\det(\mathbf{q}) &= \det(A\mathbf{g}) \times \left(1 - \epsilon B A^{-1} g^{ab} u_a u_b\right) = \\ &= A^D \det(\mathbf{g}) \times \left(1 - \frac{B}{A}\right)\end{aligned}$$

where we used the fact that $g^{ab}u_a u_b = \epsilon$ and $\epsilon^2 = 1$. Knowing that $B = \frac{QA^2}{1+QA}$, $\alpha = A^{-1} + Q$ and redefining now $q = \det(\mathbf{q})$ and $g = \det(\mathbf{g})$ we get

$$q = A^D g\left(1 - \frac{Q}{Q + A^{-1}}\right) = \frac{A^{D-1}}{\alpha} g \tag{2.11}$$

Using this one could compute $\tilde{\Box} = \frac{1}{\sqrt{-q}}\partial_a(\sqrt{-q}q^{ab}\partial_b)$, finding the result (see appendix C.1):

$$\begin{aligned}\tilde{\Box} = A^{-1}&\left[\Box_g + \frac{D-3}{2}g^{ab}\partial_a(\ln A)\partial_b + \epsilon\not{\partial}(\ln A)\not{\partial}\right] + \\ + \epsilon Q&\left[\left(\nabla_a u^a + \frac{D-1}{2}\not{\partial}\ln A\right)\not{\partial} + \not{\partial}^2\right] + \sqrt{\epsilon\sigma^2}\alpha'\not{\partial}\end{aligned} \tag{2.12}$$

where $\not{\partial} = u^a\partial_a$ and ∇_a is the covariant derivative constructed from the background metric. We consider maximally symmetric spaces in which we have a simplified form both for the ordinary d'Alembertian and the qmetric D'Alembertian(see appendix C.2):

$$\Box_g = 4\sigma^2\left\{\partial^2_{\sigma^2} + \partial_{\sigma^2}\left[\ln\left[\left(\epsilon\sigma^2\right)^{\frac{D}{2}}\Delta^{-1}\right]\right]\partial_{\sigma^2}\right\} \tag{2.13}$$

$$\tilde{\Box} = 4\alpha\sigma^2\left\{\partial^2_{\sigma^2} + \partial_{\sigma^2}\left[\ln\left[\left(\epsilon\sigma^2\right)^{\frac{D}{2}}\Delta^{-1}\sqrt{\alpha}A^{\frac{D-1}{2}}\right]\right]\partial_{\sigma^2}\right\} \tag{2.14}$$

where $\Delta = \Delta(x, x')$ is the Van Vleck determinant defined in eq.(1.57). At this point we can impose the condition on Green functions. In particular we require that the modified Green function $\tilde{G}(\sigma^2) = G(S(\sigma^2))$ in the point x is solution to $\tilde{\Box}\tilde{G} = 0$ when $\Box G = 0$, with G computed in the point $\tilde{x}$ at a geodesic distance $\sigma^2 = S$ from x' according to the metric (for simplicity we write S_L as S).
First of all we evaluate $\Box G$ at the point $\tilde{x}$ at a distance $\sigma^2 = S$ from x' and we impose $(\Box G)_{\sigma^2 = S} = 0$. Starting from eq.(2.13) we make the following substitutions:

$$\sigma^2 \to S$$
$$\Delta \to \tilde{\Delta}$$

[1] $\det(\mathbf{M} + \mathbf{u}\mathbf{v^t}) = \det(\mathbf{M}) \times (1 + \mathbf{v^t}\mathbf{M^{-1}}\mathbf{u})$ with $\mathbf{M}$ being an invertible matrix and $\mathbf{u}, \mathbf{v}$ column vectors and $\mathbf{u}\mathbf{v^t}$ is the outer product[7].

where $\tilde{\Delta} = \Delta(\tilde{x}, x')$. So we have:

$$\Box_g G(\sigma^2)_{\sigma^2=S} = 4S \left\{ \partial^2_{\sigma^2} + \partial_{\sigma^2} \left[\ln \left[(\epsilon\sigma^2)^{\frac{D}{2}} \Delta^{-1} \right] \right]_{\sigma^2=S} \partial_{\sigma^2} \right\} G(\sigma^2)_{\sigma^2=S} = \tag{2.15}$$

$$= 4S \left\{ \partial^2_S + \partial_S \left[\ln \left[(\epsilon S)^{\frac{D}{2}} \tilde{\Delta}^{-1} \right] \right] \partial_S \right\} G(S) = \Box_g G(S) \tag{2.16}$$

Using the chain rule (prime index means differentiation w.r.t. σ^2) we can write:

$$\partial_S = \frac{1}{S'} \partial_{\sigma^2} \tag{2.17}$$

$$\partial^2_S = \partial_S \left(\frac{1}{S'} \partial_{\sigma^2} \right) =$$

$$= \frac{1}{(S')^2} \left(\partial^2_{\sigma^2} - \frac{S''}{S'} \partial_{\sigma^2} \right) = \frac{1}{(S')^2} \left[\partial^2_{\sigma^2} - \partial_{\sigma^2} (\ln(S')) \partial_{\sigma^2} \right] \tag{2.18}$$

Putting all together we find

$$\Box_g G(S) = \frac{4S}{(S')^2} \left[\partial^2_{\sigma^2} + \partial_{\sigma^2} \left(\ln \frac{(\epsilon S)^{\frac{D}{2}}}{S' \tilde{\Delta}} \right) \partial_{\sigma^2} \right] G(S) \tag{2.19}$$

Having $\Box_g G(S) = 0$ is equivalent to have

$$\partial^2_{\sigma^2} G(S) = \partial_{\sigma^2} \left(\ln \frac{S' \tilde{\Delta}}{(\epsilon S)^{\frac{D}{2}}} \right) \partial_{\sigma^2} G(S) \tag{2.20}$$

Now we evaluate $\tilde{\Box} \tilde{G}(\sigma^2)$ according to the qmetric at the point x and we impose the conditon $\tilde{\Box} \tilde{G}(\sigma^2) = 0$ when $\Box G(S) = 0$. Using eq.(2.14) we have:

$$\tilde{\Box} \tilde{G}(\sigma^2) = \tilde{\Box} G(S) = 4\alpha\sigma^2 \left\{ \partial^2_{\sigma^2} + \partial_{\sigma^2} \left[\ln \left((\epsilon\sigma^2)^{\frac{D}{2}} \Delta^{-1} \sqrt{\alpha} A^{\frac{D-1}{2}} \right) \right] \partial_{\sigma^2} \right\} G(S) \tag{2.21}$$

Having $\tilde{\Box} \tilde{G}(\sigma^2) = 0$ is equivalent to have

$$\partial^2_{\sigma^2} G(S) + \partial_{\sigma^2} \left[\ln \left((\epsilon\sigma^2)^{\frac{D}{2}} \Delta^{-1} \sqrt{\alpha} A^{\frac{D-1}{2}} \right) \right] \partial_{\sigma^2} G(S) = 0 \tag{2.22}$$

Requiring this is true when $\Box G(S) = 0$ we can use eq.(2.20) and write:

$$\partial_{\sigma^2} \ln \left((\epsilon\sigma^2)^{\frac{D}{2}} \Delta^{-1} \sqrt{\alpha} A^{\frac{D-1}{2}} \frac{S' \tilde{\Delta}}{(\epsilon S)^{\frac{D}{2}}} \right) = 0 \tag{2.23}$$

which using the expression for α given by eq.(2.9) becomes

$$\partial_{\sigma^2} \ln \left(\frac{\tilde{\Delta}}{\Delta} A^{\frac{D-1}{2}} \frac{(\sigma^2)^{\frac{D-1}{2}}}{S^{\frac{D-1}{2}}} \right) = \frac{D-1}{2} \frac{d}{d\sigma^2} \ln \left(\frac{A}{S/\sigma^2} \left(\frac{\tilde{\Delta}}{\Delta} \right)^{\frac{2}{D-1}} \right) = 0 \tag{2.24}$$

whose solution is given by

$$A = \frac{S_L}{\sigma^2} \left(\frac{\Delta}{\tilde{\Delta}} \right)^{\frac{2}{D-1}} \tag{2.25}$$

where the constant of integration is fixed by the condition $A = 1$ when $S_{L_0=0}(\sigma^2) = \sigma^2$. Thus we can write the complete form of the qmetric form timelike/spacelike separated events as:

$$q_{ab} = \frac{S_L}{\sigma^2}\left(\frac{\Delta}{\tilde{\Delta}}\right)^{\frac{2}{D-1}} g_{ab} + \epsilon\left[\frac{\sigma^2 S_L'^2}{S_L} - \frac{S_L}{\sigma^2}\left(\frac{\Delta}{\tilde{\Delta}}\right)^{\frac{2}{D-1}}\right] u_a u_b \tag{2.26}$$

$$q^{ab} = \frac{\sigma^2}{S_L}\left(\frac{\tilde{\Delta}}{\Delta}\right)^{\frac{2}{D-1}} g^{ab} + \epsilon\left[\frac{S_L}{\sigma^2 S_L'^2} - \frac{\sigma^2}{S_L}\left(\frac{\tilde{\Delta}}{\Delta}\right)^{\frac{2}{D-1}}\right] u^a u^b \tag{2.27}$$

We notice that the qmetric has a singular behaviour in the limit $\sigma^2 \to 0$ while in the limit $L_0 \to 0$ (large separations) it reduces to $q_{ab}(x, x') \to g_{ab}(x)$. In contrast with ordinary classical spacetime where given a point x we can uniquely assign a value for the local metric tensor $g_{ab}(x)$ here we can't uniquely assign the value of the qmetric in the point x. In fact we need to specify also the base point x' in order to fix the value $q_{ab}(x, x')$: in the point x we have many different values for $q_{ab}(x, x')$ depending on the choice of the base point x'. The qmetric is explicitly a non-local object: indeed non-locality is required if we want to allow the existance of a minimum length scale since in a local spacetime geodesic distances are always vanishing in the coincidence limit.
We can interpret the qmetric a a renormalized metric in the sense that it is an effective metric incorporating some of non-perturbative effects of quantum gravity at Planck scales [11].

2.3 Construction of the qmetric: the null case

We saw how to construct the qmetric for points separated by timelike/spacelike geodesic intervals. The generalization to null-separated points is not straightforward: given that we want to map geodesics obtained by the metric to q-geodesics of the same nature how can we maintain the existence of a minimum length in the null case where the geodesic distance modification reads as $\sigma^2 = 0 \to S = 0$? In order to proceed we need to think about the role of affine parameters for null geodesics. While for timelike geodesics we can always choose the proper time as an affine parameter this is not true for null geodesics since we can't define a physical proper time for lightlike probes. However we can still choose the null affine parameters with a physical meaning: any null affine parametrization λ gives us a measure of distance along the geodesic measured by an observer at a certain point x of the geodesic and parallel transported along the geodesic itself [13, 19] as argued in section 1.1.2. Given a classical metric g_{ab} and two points x and x' on a null geodesic γ parametrized by the affine parameter λ we expect that the introduction of a minimum length L_0 will induce the mapping $\lambda \to \tilde{\lambda}(\lambda)$ such that $\tilde{\lambda}(x) - \tilde{\lambda}(x') = L_0$ when $\lambda(x) - \lambda(x') = 0$ i.e. when $x \to x'$. This is the null counterpart of the requirement of the modification of the geodesic distance in the timelike/spacelike case. In general we require:

1. $\tilde{\lambda}(x) = \lambda(x)$ when $L_0 = 0$
2. $\tilde{\lambda}(x) - \tilde{\lambda}(x') = L_0$ when $x \to x'$

In addition also in the null case we require the modification of the green function associated to the d'Alembertian which again reads as $\tilde{G}(\sigma^2) = G(S_L)$ in all maximally symmetric spacetimes. However this is not a trivial requirement since on the light cone the green function $G(\sigma^2)$ is singular. We will see how to deal with this issue. Now we

see how to set the framework to derive the null qmetric form.
We consider a null geodesic γ passing through x' parameterized by the affine parameter λ such that $l^a = \frac{dx^a}{d\lambda}$ is the null tangent vector to γ. We attach a canonical observer in x' with a four velocity V^a such that $V^a l_a = -1$. In this way λ is that particular affine parameter that select a particular frame in which λ itself can be considered as a distance on the null geodesic [19]. From V^a and l^a we can construct a second null vector $m^a = V^a - (1/2)l^a$ such that:

1. $m^a V_a = -1/2$
2. $m^a l_a = -1$

all along γ. The vector m^a is observer dependent since it depends on the vector V^a. The ansatz for the null qmetric reads as:

$$q_{ab} = A_\lambda g_{ab} + \left(A_\lambda - \alpha_\lambda^{-1}\right)(l_a m_b + m_a l_b) \tag{2.28}$$

$$q^{ab} = A_\lambda^{-1} g^{ab} + \left(A_\lambda^{-1} - \alpha_\lambda\right)\left(l^a m^b + m^a l^b\right) \tag{2.29}$$

where Latin indices are raised and lowered with the classical metric g_{ab}. We need to fix the two biscalars α_λ and A_λ which requires computations a little trickier than in the time/space-like case.

2.3.1 The null qmetric: fixing α_λ

In order to fix α_λ we start from defining a modified geodesic equation. Let

$$\tilde{l}^a = \frac{dx^a}{d\tilde{\lambda}} = \frac{d\lambda}{d\tilde{\lambda}} l^a \tag{2.30}$$

the q-geodesic null tangent vector. We require this modified tangent vector satisfies a geodesic equation according to the qmetric with $\tilde{\lambda}$ acting as an affine parameter, namely:

$$\tilde{l}^a \tilde{\nabla}_a \tilde{l}^b = 0 \tag{2.31}$$

where $\tilde{\nabla}_a$ is the covariant derivative constructed from the qmetric (we will call it q-covariant derivative). In particular we have:

$$\tilde{\nabla}_a \tilde{l}^b = \partial_a \tilde{l}^b + \tilde{\Gamma}^b_{ac} \tilde{l}^c \tag{2.32}$$

and

$$\tilde{\nabla}_a \tilde{l}_b = \partial_a \tilde{l}_b - \tilde{\Gamma}^c_{ab} \tilde{l}_c \tag{2.33}$$

where $\tilde{l}_c$ is given by:

$$\begin{aligned} \tilde{l}_c = q_{ac}\tilde{l}^a = \frac{d\lambda}{d\tilde{\lambda}} q_{ac} l^a &= \frac{d\lambda}{d\tilde{\lambda}} \left[A_\lambda l_c + \left(A_\lambda - \alpha_\lambda^{-1}\right) l_c m_a l^a\right] = \\ &= \frac{d\lambda}{d\tilde{\lambda}} \frac{1}{\alpha_\lambda} l_c \end{aligned} \tag{2.34}$$

We define the q-connection $\tilde{\Gamma}^b_{ac}$ from the qmetric through the same algebraic relations that link the classical connection to the classical metric:

$$\tilde{\Gamma}^b_{ac} = \frac{1}{2} q^{bl}\left(\partial_a q_{lc} + \partial_c q_{al} - \partial_l q_{ac}\right) \tag{2.35}$$

which can be rewritten as [29]:

$$\tilde{\Gamma}^b_{ac} = \Gamma^b_{ac} + \frac{1}{2} q^{bl} \left(\nabla_a q_{lc} + \nabla_c q_{al} - \nabla_l q_{ac} \right) \tag{2.36}$$

where Γ^b_{ac} and ∇_a are respectively the affine connection and the covariant derivative associated to the metric g_{ab}. The modified geodesic equation then can be written as:

$$\tilde{l}^a \tilde{\nabla}_a \tilde{l}_c = \frac{d\lambda}{d\tilde{\lambda}} l^a \tilde{\nabla}_a \left(\frac{d\lambda}{d\tilde{\lambda}} \frac{1}{\alpha_\lambda} l_c \right) = 0 \tag{2.37}$$

We need to evaluate the q-covariant derivative of the quantity inside the round brackets:

$$\begin{aligned}
\tilde{\nabla}_a \left(\frac{d\lambda}{d\tilde{\lambda}} \frac{1}{\alpha_\lambda} l_c \right) &= \nabla_a \left(\frac{d\lambda}{d\tilde{\lambda}} \frac{1}{\alpha_\lambda} l_c \right) - \frac{1}{2} q^{bl} \left(\nabla_a q_{lc} + \nabla_c q_{al} - \nabla_l q_{ac} \right) \frac{d\lambda}{d\tilde{\lambda}} \frac{1}{\alpha_\lambda} l_b = \\
&= \frac{d\lambda}{d\tilde{\lambda}} \frac{1}{\alpha_\lambda} \nabla_a l_c + l_c \partial_a \left(\frac{d\lambda}{d\tilde{\lambda}} \frac{1}{\alpha_\lambda} \right) - \frac{1}{2} q^{bl} \left(\nabla_a q_{lc} + \nabla_c q_{al} - \nabla_l q_{ac} \right) \tilde{l}_b = \\
&= \frac{d\lambda}{d\tilde{\lambda}} \frac{1}{\alpha_\lambda} \nabla_a l_c + l_c \partial_a \left(\frac{d\lambda}{d\tilde{\lambda}} \frac{1}{\alpha_\lambda} \right) - \frac{1}{2} \tilde{l}^l \left(\nabla_a q_{lc} + \nabla_c q_{al} - \nabla_l q_{ac} \right)
\end{aligned} \tag{2.38}$$

Inserting this last expression in eq.(2.37) we get:

$$\left(\frac{d\lambda}{d\tilde{\lambda}} \right)^2 \frac{1}{\alpha_\lambda} l^a \nabla_a l_c + \left(\frac{d\lambda}{d\tilde{\lambda}} \right) l^a \partial_a \left(\frac{d\lambda}{d\tilde{\lambda}} \frac{1}{\alpha_\lambda} \right) - \frac{1}{2} \frac{d\lambda}{d\tilde{\lambda}} l^a \tilde{l}^l \left(\nabla_a q_{lc} + \nabla_c q_{al} - \nabla_l q_{ac} \right) = 0 \tag{2.39}$$

We notice the first term vanishes since $l^a \nabla_a l_c = 0$ is the classical geodesic equation satisfied by construction by the tangent vector l^a with affine parameter λ. In the second term we have the directional derivative $l^a \partial_a = d/d\lambda$ so we can rewrite:

$$\left(\frac{d\lambda}{d\tilde{\lambda}} \right) \frac{d}{d\lambda} \left(\frac{d\lambda}{d\tilde{\lambda}} \frac{1}{\alpha_\lambda} \right) - \frac{1}{2} \left(\frac{d\lambda}{d\tilde{\lambda}} \right)^2 l^a l^l \left(\nabla_a q_{lc} + \nabla_c q_{al} - \nabla_l q_{ac} \right) = 0 \tag{2.40}$$

We notice that by symmetry:

$$l^a l^l \left(\nabla_a q_{lc} - \nabla_l q_{ac} \right) = 0 \tag{2.41}$$

Now we need:

$$\begin{aligned}
l^a l^l \nabla_c q_{al} &= \nabla_c \left[A g_{al} + \left(A_\lambda - \alpha_\lambda^{-1} \right) \left(l_a m_l + m_a l_l \right) \right] = \\
&= l^a l_a \nabla_c A + \nabla_c \left(A_\lambda - \alpha_\lambda^{-1} \right) \left(l^a l_a m_l l^l + l^a m_a l_l l^l \right) + \qquad (2.42) \\
&+ \left(A_\lambda - \alpha_\lambda^{-1} \right) l^a l^l \nabla_c \left(l_a m_l + m_a l_l \right) = \\
&= \left(A_\lambda - \alpha_\lambda^{-1} \right) l^a l^l \left(m_l \nabla_c l_a + l_a \nabla_c m_l + m_a \nabla_c l_l + l_l \nabla_c m_a \right) = \\
&= -2 \left(A_\lambda - \alpha_\lambda^{-1} \right) l^a \nabla_c l_a \qquad (2.43)
\end{aligned}$$

Inserting this in eq.(2.40) we find:

$$\left(\frac{d\lambda}{d\tilde{\lambda}} \right) \frac{d}{d\lambda} \left(\frac{d\lambda}{d\tilde{\lambda}} \frac{1}{\alpha_\lambda} \right) + \left(\frac{d\lambda}{d\tilde{\lambda}} \right)^2 \left(A_\lambda - \alpha_\lambda^{-1} \right) l^a \nabla_c l_a = 0 \tag{2.44}$$

Finally we notice that:

$$l^a \nabla_c l_a = \nabla_c \left(l^a l_a \right) - l^a \nabla_c l_a \tag{2.45}$$

$$l^a \nabla_c l_a = 0 \tag{2.46}$$

and we end up with the following differential equation:

$$\left(\frac{d\lambda}{d\tilde{\lambda}}\right)\frac{d}{d\lambda}\left(\frac{d\lambda}{d\tilde{\lambda}}\frac{1}{\alpha_\lambda}\right)=0 \tag{2.47}$$

wich once we require that $d\lambda/d\tilde{\lambda}\neq 0$ it is solved by:

$$\alpha_\lambda = C\left(\frac{d\tilde{\lambda}}{d\lambda}\right)^{-1} \tag{2.48}$$

The integration constant C is fixed by requiring $q_{ab}\to g_{ab}$ when $\lambda\to\infty$ i.e. in the limit $L_0\to 0$. This can happen only if $A_\lambda\to 1$ and $(A_\lambda-\alpha_\lambda^{-1})\to 0$ thus we need $\alpha_\lambda\to 1$ which is possible only if $C=1$ since in this limit $\tilde{\lambda}\to\lambda$.

2.3.2 The null qmetric: fixing A_λ

In the time/space-like case the biscalar function A in eq.(2.25) has been fixed requiring that the modified green function $\tilde{G}(\sigma^2)=G(S(\sigma^2))$ satisfies the equation $\tilde{\Box}\tilde{G}=0$ when $\Box G(S)=0$. In the null case we need to take a slightly different path since on the light cone the green function $G(\sigma^2)$ is singular and we can't perform the same straightforward computations done for the timelike/spacelike case. The key point is to evaluate $\Box G$ at a point y near the null geodesic γ but not on it, i.e. y is time or space separated from x', and then take the limit $y\to x\in\gamma$ as we have done in section 1.3.2
The first step is to determine a suitable expression for the d'Alembertian $\Box$. Considering maximally symmetric spaces in which the green function (and any biscalar) is strictly a function only of σ^2 we can write:

$$\begin{aligned}\Box G(\sigma^2) &= \nabla_a(\nabla^a G)=\nabla_a\left(\partial^a\sigma^2\frac{dG}{d\sigma^2}\right)= \\ &=\nabla_a\left(\partial^a\sigma^2\right)\frac{dG}{d\sigma^2}+\partial^a\sigma^2\partial_a\sigma^2\frac{d^2G}{d(\sigma^2)^2}\end{aligned} \tag{2.49}$$

On the null geodesic we clearly have $\partial^a\sigma^2\partial_a\sigma^2=4\sigma^2=0$ so we have:

$$\Box G(\sigma^2)=\nabla_a\left(\partial^a\sigma^2\right)\frac{dG}{d\sigma^2} \tag{2.50}$$

when $y\to x\in\gamma$. We found in eq.(1.77) that for null separated events we have:

$$\nabla_a\partial^a\sigma^2(x,x')=2\left(\lambda\nabla_a l^a+2\right) \tag{2.51}$$

therefore we can write a suitable expression for the standard d'Alembertian along null geodesic affinely parametrized by λ in maximally symmetric spaces:

$$\Box G(\sigma^2)=\left(4+2\lambda\nabla_a l^a\right)\frac{d}{d\sigma^2}G(\sigma^2) \tag{2.52}$$

Passing from classical metric g_{ab} to the qmetric q_{ab} we are mapping $\lambda\to\tilde{\lambda}(\lambda)$ and $\sigma^2\to S(\sigma^2)$ meaning that:

$$l^a\to\tilde{l}^a=\frac{d\lambda}{d\tilde{\lambda}}l^a \tag{2.53}$$

$$G(\sigma^2)\to\tilde{G}(\sigma^2)=G(S(\sigma^2)) \tag{2.54}$$

$$\Box\to\tilde{\Box}=\tilde{\nabla}_a\tilde{\nabla}^a \tag{2.55}$$

where the q-covariant derivative $\tilde{\nabla}_a$ is defined as in eq.(2.32) and in eq.(2.33). Since the null geodesic according to the metric is mapped to a null geodesic according to the qmetric we can rewrite the eq.(2.52) for the qmetric d'Alembertian as:

$$\tilde{\Box}\tilde{G}(\sigma^2) = \left(4 + 2\tilde{\lambda}\tilde{\nabla}_a \tilde{l}^a\right) \frac{d}{dS}\tilde{G}(\sigma^2) = \left(4 + 2\tilde{\lambda}\tilde{\nabla}_a \tilde{l}^a\right) \left(\frac{dG(\sigma^2)}{d\sigma^2}\right)_{\sigma^2=S} \tag{2.56}$$

From the relation in eq.(2.36) we get:

$$\begin{aligned}\tilde{\nabla}_a \tilde{l}^a &= \nabla_a \tilde{l}^a + \frac{1}{2} q^{ak}\left(\nabla_a q_{kb} + \nabla_b q_{ak} - \nabla_k q_{ab}\right) \tilde{l}^b = \\ &= \nabla_a \left(\frac{d\lambda}{d\tilde{\lambda}} l^a\right) + \frac{1}{2}\left(\frac{d\lambda}{d\tilde{\lambda}}\right) q^{ak} l^b \nabla_b q_{ak}\end{aligned} \tag{2.57}$$

Inserting in the second term the null qmetric ansatz expressed in eq.(2.28) and eq.(2.29) we find:

$$q^{ak} l^b \nabla_b q_{ak} = (D-2)\frac{d}{d\lambda}\ln A_\lambda - 2\frac{d}{d\lambda}\ln \alpha_\lambda \tag{2.58}$$

Using the form of α_λ given by eq.(2.48) we have

$$\begin{aligned}\tilde{\nabla}_a \tilde{l}^a &= \frac{d\lambda}{d\tilde{\lambda}} \nabla_a l^a + \frac{d}{d\lambda}\left(\frac{d\lambda}{d\tilde{\lambda}}\right) + \frac{1}{2}\frac{d\lambda}{d\tilde{\lambda}}\left[(D-2)\frac{d}{d\lambda}\ln A_\lambda - 2\frac{d}{d\lambda}\ln \alpha_\lambda\right] = \\ &= \frac{d\lambda}{d\tilde{\lambda}}\left[\nabla_a l^a + \frac{D-2}{2}\frac{d}{d\lambda}\ln A_\lambda\right]\end{aligned} \tag{2.59}$$

Going back to eq.(2.56) we get:

$$\tilde{\Box}\tilde{G}(\sigma^2) = \left[4 + 2\tilde{\lambda}\frac{d\lambda}{d\tilde{\lambda}}\nabla_a l^a + \tilde{\lambda}(D-2)\frac{d\lambda}{d\tilde{\lambda}}\frac{d}{d\lambda}\ln A_\lambda\right]\left(\frac{dG(\sigma^2)}{d\sigma^2}\right)_{\sigma^2=S} \tag{2.60}$$

In the classical framework we were dealing with the base point x', the field point $x \in \gamma$ at an affine distance λ from x' and an auxiliary point y at a finite squared geodesic distance $\sigma^2(y, x')$ from x'. Passing to the qmetric we can think in terms of an active interpretation of the mapping $\lambda \to \tilde{\lambda}$: the point x is mapped to the point $\tilde{x}$ at an affine distance $\tilde{\lambda}$ from x' while the auxiliary point y is mapped to the point $\tilde{y}$ at a modified geodesic distance $S(\sigma^2(y, x'))$ from the base point x'. Thus we require the following: the modified green function $\tilde{G}(\sigma^2) = G(S(\sigma^2))$ satisfies $\tilde{\Box}\tilde{G} = 0$ at the point $x \in \gamma$ when $G(\sigma^2)$ satisfies $\Box G = 0$ at the point $\tilde{x}$. In order to evaluate $\Box G$ in $\tilde{x}$ we need to start from the point $\tilde{y}$ and then take the limit $\tilde{y} \to \tilde{x}$. Following the same computations as before we find:

$$\Box G(\sigma^2)_{x=\tilde{x}} = \left[4 + 2\left(\lambda \nabla_a l^a\right)_{x=\tilde{x}}\right]\frac{d}{d\sigma^2}G(\sigma^2)_{x=\tilde{x}} = \left[4 + 2\tilde{\lambda}\left(\nabla_a l^a\right)_{x=\tilde{x}}\right]\frac{d}{d\sigma^2}G(\sigma^2)_{x=\tilde{x}} \tag{2.61}$$

If $G(\sigma^2)$ is solution of $\Box G = 0$ in $\tilde{x}$ it means that:

$$4 + 2\tilde{\lambda}\left(\nabla_a l^a\right)_{x=\tilde{x}} = 0 \tag{2.62}$$

Requiring that $\tilde{G}$ is solution of $\tilde{\Box}\tilde{G}$ in the point x it means that:

$$4 + 2\tilde{\lambda}\frac{d\lambda}{d\tilde{\lambda}}\nabla_a l^a + \tilde{\lambda}(D-2)\frac{d\lambda}{d\tilde{\lambda}}\frac{d}{d\lambda}\ln A_\lambda = 0 \tag{2.63}$$

Inserting eq.(2.62) in eq.(2.63) we get:

$$2\tilde{\lambda}\frac{d\lambda}{d\tilde{\lambda}}\nabla_a l^a - 2\tilde{\lambda}\left(\nabla_a l^a\right)_{x=\tilde{x}} + \tilde{\lambda}(D-2)\frac{d\lambda}{d\tilde{\lambda}}\frac{d}{d\lambda}\ln A_\lambda = 0 \tag{2.64}$$

Using eq.(1.80) we are able to write the geodesic expansion $\nabla_a l^a$ in terms of the Van Vleck determinant Δ:

$$\nabla_a l^a = \frac{D-2}{\lambda} + \frac{d}{d\lambda} \ln \Delta^{-1} \tag{2.65}$$

$$(\nabla_a l^a)_{x=\tilde{x}} = \frac{D-2}{\tilde{\lambda}} + \frac{d}{d\tilde{\lambda}} \ln \tilde{\Delta}^{-1} \tag{2.66}$$

where $\Delta = \Delta(x, x')$ and $\tilde{\Delta} = \Delta(\tilde{x}, x')$. Inserting back to the equation(2.64):

$$\frac{2}{\lambda} + \frac{2}{D-2} \frac{d}{d\lambda} \ln \Delta^{-1} - \frac{d\tilde{\lambda}}{d\lambda} \frac{2}{\tilde{\lambda}} + \frac{2}{D-2} \frac{d}{d\lambda} \ln \tilde{\Delta} + \frac{d}{d\lambda} \ln A_\lambda = 0$$

hence

$$\frac{d}{d\lambda} \left[\frac{\lambda^2}{\tilde{\lambda}^2} \left(\frac{\tilde{\Delta}}{\Delta} \right)^{\frac{2}{D-2}} A_\lambda \right] = 0 \tag{2.67}$$

This is solved by

$$A_\lambda = C \frac{\tilde{\lambda}^2}{\lambda^2} \left(\frac{\tilde{\Delta}}{\Delta} \right)^{-\frac{2}{D-2}} \tag{2.68}$$

where the integration constant C is fixed to be $C = 1$ requiring that $A_\lambda \to 1$ when $L_0 \to 0$.

2.3.3 The null qmetric: final form

We have fixed our null qmetric parameters to be

$$\alpha_\lambda = \frac{1}{d\tilde{\lambda}/d\lambda} \tag{2.69}$$

$$A_\lambda = \frac{\tilde{\lambda}^2}{\lambda^2} \left(\frac{\Delta}{\tilde{\Delta}} \right)^{\frac{2}{D-2}} \tag{2.70}$$

fixing the form of the qmetric for points separeted by null geodesic intervals. Thus the final form of the null qmetric reads as:

$$q_{ab} = \frac{\tilde{\lambda}^2}{\lambda^2} \left(\frac{\Delta}{\tilde{\Delta}} \right)^{\frac{2}{D-2}} g_{ab} - \left[\frac{d\tilde{\lambda}}{d\lambda} - \frac{\tilde{\lambda}^2}{\lambda^2} \left(\frac{\Delta}{\tilde{\Delta}} \right)^{\frac{2}{D-2}} \right] (l_a m_b + m_a l_b) \tag{2.71}$$

2.4 Considerations and remarks on Lorentz covariance

We fixed the form for the qmetric between two points both for timelike/spacelike and null separations. We can compare the two forms obtained. In the non-null case we started directly by a modification of the geodesic distance $\sigma^2 \to S_l(\sigma^2)$ obtaining

$$\alpha = \frac{1}{\sigma^2} \frac{S}{S'^2}. \tag{2.72}$$

We can also find the form of α by mapping the affine parameter of the classical geodesic $s = \sqrt{\epsilon\sigma^2}$ to the q-affine parameter $\tilde{s} = \sqrt{\epsilon S_l(\sigma^2}$ finding that

$$\alpha = \frac{1}{(d\tilde{s}/ds)^2} \tag{2.73}$$

which is in a form directly comparable with

$$\alpha_\lambda = \frac{1}{d\tilde{\lambda}/\lambda} \tag{2.74}$$

for null separations. We can also rewrite the bifunction A given by (2.25)

$$A = \frac{\tilde{s}^2}{s^2} \left(\frac{\Delta}{\tilde{\Delta}} \right)^{\frac{2}{d-1}} \tag{2.75}$$

and compare it to

$$A_\lambda = \frac{\tilde{\lambda}^2}{\lambda^2} \left(\frac{\Delta}{\tilde{\Delta}} \right)^{\frac{2}{d-2}} \tag{2.76}$$

for null separations. We can resume the qmetric final form:

$$q_{ab} = \frac{\tilde{s}^2}{s^2} \left(\frac{\Delta}{\tilde{\Delta}} \right)^{\frac{2}{d-1}} g_{ab} + \epsilon \left[\left(\frac{d\tilde{s}}{ds} \right)^2 - \frac{\tilde{s}^2}{s^2} \left(\frac{\Delta}{\tilde{\Delta}} \right)^{\frac{2}{d-1}} \right] u_a u_b \tag{2.77}$$

for timelike/spacelike separations while in the null case we have:

$$q_{ab} = \frac{\tilde{\lambda}^2}{\lambda^2} \left(\frac{\Delta}{\tilde{\Delta}} \right)^{\frac{2}{d-2}} g_{ab} - \left[\frac{d\tilde{\lambda}}{d\lambda} - \frac{\tilde{\lambda}^2}{\lambda^2} \left(\frac{\Delta}{\tilde{\Delta}} \right)^{\frac{2}{d-2}} \right] (l_a m_b + m_a l_b) \tag{2.78}$$

The main difference is that the null qmetric depends on the vector m^a which can be identified once we assign an observer in the point x', while in the time-/space-like case the qmetric is observer independent. This happens because given two non-null separated points their geodesic separation is uniquely determined while, for null separated events, all we have is their affine interval which is Lorentz invariant but is determined up to a constant. Thus we need to select an observer to fix this ambiguity. The fact that the null qmetric needs to specify an observer may naively make think that we are breaking Lorentz invariance. In the timelike/spacelike case the qmetric has been constructed by requiring the modification $\sigma^2 \to S_L(\sigma^2)$. Since the squared geodesic distance σ^2 is a biscalar and as such is a Lorentz invariant quantity we are introducing the minimum length L_0 in a Lorentz invariant way. The modification $\sigma^2 \to S_L$ will be the same in every frame of reference. In the null case things are little more subtle: the null qmetric is not only a non-local object but it is also sensitive to the selection of an observer, i.e. a local frame. What happens is that Lorentz invariance is preserved in the sense that whichever local observer we assign at point x' she will invariably find that same metric structure in the qmetric, in particular with L_0 being the qmetric coincidence limit value of the affine parameter taken to be time or distance according to the observer [5].

2.5 A connection to the Emergent Gravity paradigm

A key aspect that emerges when one tries to combine gravity and the quantum theory is the horizon thermality [30]. This was first noticed with the discovery of the Hawking radiation [31] and the Unruh effect [32]. The former describes the presence of a late time black body radiation in a spacetime with collapsing matter forming a black hole whose event horizon is assigned a temperature of $T_H = \hbar k/2\pi$, where k is the *surface gravity* of the black hole. The latter shows that an uniformly accelerated observer with constant magnitude of acceleration a in Minkowski spacetime describes the vacuum state of a quantum field as a thermal bath of temperature $T_U = \hbar a/2\pi$. Further developments showed that any null surfaces acting as horizons for a class of observers are endowed with termodynamical properties, namely temperature and entropy. The identification of the entropy associated to an horizon is due to Bekenstein [33] who assigned an entropy of $S_H = A/(4L_P^2)$ to an horizon of area A. The fact that a theory like General Relativity which has a purely geometrical construction gives rise to thermodynamical properties it make one doubt if there is a deeper connection between geometry and thermodynamics. Jacobson [34] showed that Einstein fields equation can be interpreted as an equation of state for the spacetime: imposing that in every local Rindler frame the laws of thermodynamic hold in presence of an horizon endowed with Unruh temperature and Bekenstein entropy one is able to derive the General Relativity fields equations. In this view it seems that General Relativity might arise as an emergent theory of an underlying more fundamental theory, just like fluidodynamics arises from the statistical mechanics of a large number of particles/molecules. We can't say nothing about these hypothetical micro degress of freedom of spacetime, only a complete theory of quantum gravity can clarify this issue. The paradigm which incorporates this view is the so called *emergent gravity paradigm*. It was shown that the Einstein fields equation can be derived through a *thermodynamic variational principle* [35]. Instead of varying the metric field in the Einstein-Hilbert action, the fields equation can be obtained from an entropy functional

$$S \propto R_{ab} n^a n^b + \text{ total divergence} \tag{2.79}$$

in which it is varied the vector field n^a, which is an arbitrary vector of constant norm (note that in this case we are dealing with pure gravity, if we want to deal also with the matter we need to add terms involving $T_{ab}n^a n^b$). We see how the qmetric justifies this entropy functional with a bottom-up approach. In fact we can ask how the qmetric would modify the Einstein-Hilbert (EH) lagrangian in a qmetric effective lagrangian. We know that the EH lagrangian is given by [17] (ignoring GHY counterterms)

$$\mathcal{L}_{EH} = R \tag{2.80}$$

where R is the Ricci scalar. Therefore we want to compute the form of the effective Ricci scalar via the qmetric.

2.5.1 The Ricci Biscalar

The first step we need to do is to compute the Ricci Biscalar $\tilde{R}(x, x')$ associated to the qmetric. Basically this object plays the same role of the Ricci scalar associated to the metric: is the simplest curvature invariant associated to any spacetime. We can compute it from the qmetric q_{ab} with the same algebraic relation that relates R to the background metric g_{ab}, but it is simpler to compute it via relations between geometrical quantities associated to disformally coupled metrics [28](see appendix D). Considering space/time-like separated events we have:

$$\tilde{R}(x, x') = A^{-1}R(x') + \epsilon(\alpha - A^{-1})\xi_d - \epsilon\alpha\xi_c \tag{2.81}$$

with

$$\xi_d = 2R_{ab}u^a u^b + K_{ab}K^{ab} - K^2 = \epsilon(R - R_\Sigma) \tag{2.82}$$

$$\begin{aligned}\xi_c = \epsilon[&2(D-1)A^{-\frac{1}{2}}\Box A^{\frac{1}{2}} + \\ &+ (D-1)(d-4)A^{-1}(\nabla\sqrt{A})^2] + (K + (D-1)u^a\nabla_a \ln\sqrt{A})u^b\nabla_b \ln \alpha A\end{aligned} \tag{2.83}$$

Using the explicit forms for the bifunction A, α after a lenghty computation we can find [28]:

$$\tilde{R}(x,x') = \underbrace{\left[\frac{\sigma^2}{S_L}\left(\frac{\Delta}{\tilde{\Delta}}\right)^{-2/(D-1)} R_\Sigma - \frac{(D-1)(D-2)}{S_L} + 4(D+1)\frac{d\ln\tilde{\Delta}}{dS_L}\right]}_{Q_0}$$
$$- \underbrace{\frac{S_L}{\epsilon\sigma^2 S_L'^2}\left[K_{ab}K^{ab} - \frac{1}{D-1}K^2\right]}_{Q_K} + \underbrace{4S_L\left[-\frac{D}{D-1}\left(\frac{d\ln\tilde{\Delta}}{dS_L}\right)^2 + 2\left(\frac{d^2\ln\tilde{\Delta}}{dS_L^2}\right)\right]}_{Q_\Delta} \tag{2.84}$$

This is an exact result, in the sense that no Taylor expansions have been used: this expression holds the key to understand non-perturbative effects of a zero point length[10]. We notice that the Ricci biscalar is completely determined by the geodesic structure of the spacetime, namely by the intrinsic curvature R_Σ, the extrinsic curvature K_{ab} and the Van Vleck determinant Δ. We also have that the Ricci biscalar reduces to the usual Ricci scalar in the $L_0 \to 0$ limit.

2.5.2 The effective Ricci scalar

Once we have the expression for the Ricci biscalar we can construct a q-Ricci scalar $\tilde{R}(x')$ in the point x' by means of the coincidence limit:

$$\tilde{R}(x') = \left[\tilde{R}(x,x')\right] = \lim_{x\to x'}\tilde{R}(x,x') = \lim_{\sigma^2\to 0}\tilde{R}(x,x') \tag{2.85}$$

In order to be able to compute this limit we must rely on Taylor expansion of several quantities appearing in the expression of $\tilde{R}(x,x')$ (see appendix E), so we need to consider a spacetime neighbourhood which is smooth enough. We notice that in (2.84) there are three distinct terms Q_0, Q_K and Q_Δ We see separately how to evaluate their coincidence limits. For the Q_0 limit using Taylor expansion We have that:

$$\left[\frac{d}{dS_L}\ln\tilde{\Delta}\right] = \left[\frac{1}{\tilde{\Delta}}\frac{d\tilde{\Delta}}{dS_L}\right] = \left[\frac{1}{\Delta(S_L)}\frac{d\Delta(S_L)}{dS_L}\right] =$$
$$= \left[\frac{1}{1+o(S_L)}\frac{d}{dS_L}\left(1+\frac{1}{6}\epsilon S_L R_{ab}u^a u^b + o(S_L^{\frac{3}{2}})\right)\right] = \left[\frac{1}{1+o(S_L)}\frac{1}{6}\epsilon R_{ab}u^a u^b + o(S_L^{\frac{1}{2}})\right] =$$
$$= \frac{1}{1+o(L_0^2)}\frac{1}{6}\epsilon R_{a'b'}u^{a'}u^{b'} + o(L_0) = \frac{1}{6}\epsilon R_{a'b'}u^{a'}u^{b'} + o(L_0) \tag{2.86}$$

where primed indeces means that the tensorial quantities are in the tangent space of the point x'. We alredy know that $[\Delta(\sigma^2)] = 1$, we can also compute:

$$\left[\tilde{\Delta}\right] = [\Delta(S_L)] = \left[1+\frac{1}{6}\epsilon S_L R_{ab}u^a u^b + o\left(S_L^{\frac{3}{2}}\right)\right] = 1+\frac{1}{6}\epsilon L_0^2 R_{a'b'}u^{a'}u^{b'} + o(L_0^3) \tag{2.87}$$

and we see that $\left[\tilde{\Delta}\right] = \Delta(\sigma^2 = L_0^2)$. Using the expansion of R_Σ we have:

$$\left[\frac{\sigma^2}{S_L}\left(\frac{\Delta}{\tilde{\Delta}}\right)^{\frac{-2}{(D-1)}} R_\Sigma -\right] =$$

$$= \left[\frac{\sigma^2}{S_L}\left(\frac{\Delta}{\tilde{\Delta}}\right)^{\frac{-2}{(D-1)}} \left(\frac{(D-1)(D-2)}{\sigma^2} + R - \frac{2\epsilon(D+1)}{3} R_{ab}u^a u^b + o(\sigma)\right)\right] =$$
$$= \left[\frac{(D-1)(D-2)}{S_L}\left(\frac{\Delta}{\tilde{\Delta}}\right)^{\frac{-2}{(D-1)}} + o(\sigma)\right] = \frac{(D-1)(D-2)}{\epsilon L_0^2}\Delta(L_0)^{\frac{2}{D-1}} \tag{2.88}$$

Hence we have:

$$[Q_0] = 4(D+1)\epsilon\frac{1}{6}R_{a'b'}u^{a'}u^{b'} + \epsilon\frac{(D-1)(D-2)}{L_0^2}\left(\Delta(L_0)^{\frac{2}{D-1}} - 1\right) + o(L_0) \tag{2.89}$$

The Q_K term provides only $o(L_0^2)$ terms (see expansion in E) as well as Q_Δ since it appears an overall factor S_L which at coincidence is ϵL_0^2. Thus we have:

$$\tilde{R}(x') = 4(D+1)\epsilon\frac{1}{6}R_{a'b'}u^{a'}u^{b'} + \epsilon\frac{(D-1)(D-2)}{L_0^2}\left(\Delta(L_0)^{\frac{2}{D-1}} - 1\right) + o(L_0) \tag{2.90}$$

The effective Ricci scalar would be given by the limit $L_0 \to 0$ of the q-Ricci scalar. The problematic part seems to be the second term of the above expression since a direct computation of the limite gives an undefined form. We can take advantage of de l'Hopital rule to get:

$$\lim_{L_0\to 0}\frac{\Delta(L_0)^{\frac{2}{D-1}} - 1}{L_0^2} = \lim_{L_0\to 0}\frac{d\Delta(L_0)^{\frac{2}{D-1}}}{dL_0^2} = \frac{1}{3(D-1)}\epsilon R_{a'b'}u^{a'}u^{b'} \tag{2.91}$$

from which we find that

$$R_{eff}(x') = \lim_{L_0\to 0}\tilde{R}(x') = \epsilon\frac{1}{6}(4(D+1) + 2(D-2))R_{a'b'}u^{a'}u^{b'} = \epsilon D R_{a'b'}u^{a'}u^{b'} \tag{2.92}$$

which fixes the form of the effective Ricci scalar. At this point if we want to define an effective Lagrangian we would get:

$$\mathcal{L}_{eff}(x) = \epsilon D R_{ab}u^a u^b \tag{2.93}$$

which is exactly, up to a constant, the entropy functional of the emergent gravity paradigm. Thus the qmetric description seems to give a natural basis for the emergent gravity paradigm. If the gravity is truly an emergent phenomenum then the quantization of the Einstein fields equation would not be the right way to construct a theory of quantum gravity.
The computation of the Ricci biscalar $\tilde{R}(x, x')$ can be generalized to null separations [9]. In this case, after computing the coincidence limit and the $L_0 \to 0$ limit, the effective Lagrangian would read as [9]:

$$\mathcal{L}_{eff}(x) = \lim_{L_0\to 0}\lim_{x\to x'}\tilde{R}(x, x') = (D-1)R_{ab}l^a l^b \tag{2.94}$$

where l^a is the null vector tangent to the null geodesic linking $x' \to x$. In this case the null effective Lagrangian is proportional to the heat flux crossing the null surface normal to l^a bringing with it the thermodynamics of horizons for free.

Chapter 3

Flat spaces

In this chapter we study the geodesic structure and we construct the qmetric for flat spaces both in the Riemannian case (Euclidean space) and in the Lorentzian case (Minkowski spacetime).

3.1 Euclidean space

We consider a D-dimensional euclidean space $\mathbb{R}^D$ endowed with a positive definite metric which in cartesian coordinates reads as

$$ds^2 = \delta_{ab} dx^a dx^b = \sum_{i=0}^{D-1} \left(dx^i\right)^2 \tag{3.1}$$

where δ_{ab} is the Kronecker delta. We select the origin as our base point $x' = 0$ and we study the structure of geodesics passing through it.

3.1.1 Geodesic structure

In the Euclidean space (where space and time coordinates are treated on an equal footing) geodesics are straight lines and there are no differences between space-/time-/null-like cases. We consider geodesic passing through the origin $x' = 0$, namely radial geodesics. Given any point x in the Euclidean space there is one and only one geodesic linking x to the origin [36]. In cartesian coordinates the geodesic distance between x and the origin is given by:

$$\sigma^2(x) \equiv \sigma^2(x, 0) = \sum_{i=0}^{D-1} \left(x^i\right)^2 \tag{3.2}$$

and we can express the unit tangent vector to the geodesic in the point x as

$$u_a(x) = \frac{\partial_a \sigma^2(x)}{2\sqrt{\sigma^2(x)}}. \tag{3.3}$$

We can now take advantage of the spherical symmetry of $\mathbb{R}^D$ to perform a coordinate transformation of the form

$$\{x^a\} \rightarrow \{\sigma, y^i\} \tag{3.4}$$

where $\{y^i\}$ are $D-1$ coordinates in the subspace orthogonal to the σ direction. We can give an explicit construction for the $D = 4$ case. In $\mathbb{R}^4$ we perform the following coordinate transformation [37]:

$$x^0 = \sigma \cos\psi \tag{3.5}$$
$$x^1 = \sigma \sin\psi \sin\theta \cos\phi \tag{3.6}$$
$$x^2 = \sigma \sin\psi \sin\theta \sin\phi \tag{3.7}$$
$$x^3 = \sigma \sin\psi \cos\theta \tag{3.8}$$

where $\psi, \theta \in [0, \pi].\phi \in [0, 2\pi]$ [37]. A direct computation verifies this change of coordinates is compatible with eq.(3.2). Computing the differentials and substituting in the line element given by eq.(3.1) we find the line element expressed in the new set of coordinates:

$$ds^2 = d\sigma^2 + \sigma^2 \left[d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] = d\sigma^2 + \sigma^2 d\Omega_3^2 \tag{3.9}$$

where $d\Omega_3^2$ is the line element on a unit 3-sphere, [37]. We can generalize to D-dimensions and we find:

$$ds^2 = d\sigma^2 + \sigma^2 d\Omega_{D-1}^2 \tag{3.10}$$

where $d\Omega_{D-1}^2$ is the line element on a unit (D-1)-sphere. In such a set of coordinates we can express the metric determinant as

$$\sqrt{\delta} = \sigma^{D-1} \times [\text{Angular terms}] \tag{3.11}$$

and in $D = 4$ we explicitly have

$$\sqrt{\delta} = \sigma^3 \sin^2\psi \sin\theta \tag{3.12}$$

We can see from the line element (3.10) that we can decompose the metric into a *longitudinal part* and a *transverse part* with respect to the direction of the σ coordinate.

3.1.2 Equigeodesic hypersurfaces

We now investigate the structure of the Equigeodesic-hypersurfaces in the space $\mathbb{R}^D$. We recall that an equigeodesic hypersurface $\Sigma_{x'}(l^2)$ is defined as the set of points which are at a given fixed squared geodesic distance l^2 from the base point x'. In our case we are considering

$$\Sigma_0(l^2) = \{x \in \mathbb{R}^D | \sigma^2(x, 0) = l^2, l > 0, l = const\} \tag{3.13}$$

hence we can write the constraint that generate these hypersurfaces:

$$f(x) \equiv \sigma^2(x) - l^2 = 0 \tag{3.14}$$

and we see that $\Sigma_0(l^2)$ is in every point orthogonal to the unit tangent vector to the geodesic passing through that point since the unit normal is given by u_a itself. We find that the induced metric on the equigeodesic hypersurface is given by

$$ds^2_{|\Sigma} = h_{\alpha\beta} dy^\alpha dy^\beta = l^2 d\Omega_{D-1}^2 \tag{3.15}$$

and its square root determinant reads as:

$$\sqrt{h} = l^{D-1} \times [\text{Angular terms}] \tag{3.16}$$

Explicitly in the case $D = 4$ we have

$$ds^2_{|\Sigma} = l^2 d\Omega_3^2 = l^2 \left[d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{3.17}$$
$$\sqrt{h} = l^3 \sin^2\psi \sin\theta \tag{3.18}$$

3.1.3 Area and Volume

We now introduce the notion of area and volume around a point. We select a point $x \in \Sigma_0(l^2)$. This point is identified by the $D-1$ coordinates y^i on the hypersurface and we can arbitrarly set $y^i(x) = 0$ fo $i = 1, \dots, D-1$. We can notice that since there is one and only one geodesic connecting every point of this surface to the origin, the coordinates y^i uniquely select equivalently a geodesic, namely the one passing through x. We can compute the differential area element in x as [17]

$$dA = \sqrt{h} d^{D-1} y = l^{D-1} d\Omega_{D-1} \tag{3.19}$$

where Ω_{D-1} is the differential of the solid angle in $D-1$ dimensions. In the explicit case of $D = 4$ we get:

$$dA(x) = l^3 \sin^2\psi \sin\theta d\psi d\theta d\phi \tag{3.20}$$

We can see in the coincidence limit $l^2 = \sigma^2 \to 0$ the area element vanishes. We can also provide a notion of volume around the point x[17]:

$$dV = \sqrt{\delta} d^D x = l^{D-1} dl d\Omega_{D-1} \tag{3.21}$$

which in the $D = 4$ case reads as

$$dV = \sigma^3 \sin^2\psi \sin\theta d\psi d\theta d\phi d\sigma \tag{3.22}$$

and we would need to integrate the variable σ between, say, $l - \epsilon \equiv l_-$ and $l + \epsilon \equiv l_+$. In the coincidence limit the volume would go to zero since every points of the geodesic collapse in the origini and so $l_- \to 0$ and $l_+ \to 0$.
Note that in the euclidean case we can also talk about area and volume around the base point surrounded by a given $\Sigma_0(l^2)$. In fact if we integrate the area element over the total solid angle we get

$$A_0(l) = l^3 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\pi} d\psi \sin^2\psi = 2\pi^2 l^3 \tag{3.23}$$

while for the volume we need also to integrate σ between 0 and l:

$$V_0(l) = \int_0^l d\sigma\sigma^3 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\pi} d\psi \sin^2\psi = \frac{\pi^2}{2} l^4 \tag{3.24}$$

and we notice in coincidence limit $A_0(0) = V_0(0) = 0$.

3.2 q-Euclidean space

We are now ready to implement a minimum length scale L_0 in the euclidean space through the modification of the squared geodesic distance:

$$\sigma^2(x) \to S_L(\sigma^2) \text{ such that } S_L(0) = L_0^2 \tag{3.25}$$

leading to the construction of a q-Euclidean space. We keep the framework of the previous section in which we set the origin to be the base point x'. We consider a point x whose classical squared geodesic distance is given by $\sigma^2(x)$ and the unit tangent vector to the geodesic passing through the x point is given by $u_a(x) = \partial_a \sigma^2/(2\sigma)$. We want to construct the qmetric in the point x with respect to the origin as base point. We recall the general form of the qmetric:

$$q_{ab} = A g_{ab} + \epsilon \left(\alpha^{-1} - A\right) u_a u_b \tag{3.26}$$

where

$$A = \frac{S_L}{\sigma^2}\left(\frac{\Delta}{\tilde{\Delta}}\right)^{\frac{2}{D-1}} \tag{3.27}$$

$$\alpha = \frac{1}{\sigma^2}\frac{S_L}{S_L'^2} \tag{3.28}$$

in which $\Delta = \Delta(x, x')$ is the Van Vleck determinant ($\tilde{\Delta} = \Delta(\tilde{x}, x')$ such that $\sigma^2(\tilde{x}, x') = S_L(\sigma^2(x, x'))$) and a prime index means differentiation with respect to σ^2. In the case of an Euclidean space we have always $\epsilon = +1$, $\Delta = \tilde{\Delta} = 1$ and $g_{ab} = \delta_{ab}$ and we can write:

$$q_{ab} = \frac{S_L}{\sigma^2}\delta_{ab} + \left(\frac{\sigma^2}{S_L}S_L'^2 - \frac{S_L}{\sigma^2}\right) u_a u_b. \tag{3.29}$$

We can directly compute the modified line element $d\tilde{s}^2$ in the following way:

$$\begin{aligned}
d\tilde{s}^2 &= q_{ab}dx^a dx^b = A\delta_{ab}dx^a dx^b + \left(\alpha^{-1} - A\right) u_a u_b dx^a dx^b = \\
&= A ds^2 + \left(\alpha^{-1} - A\right)\frac{1}{4\sigma^2}\frac{\partial\sigma^2}{\partial x^a}dx^a\frac{\partial\sigma^2}{\partial x^b}dx^b = \\
&= A\left[ds^2 - \frac{1}{2\sigma}d(\sigma^2)\frac{1}{2\sigma}d(\sigma^2)\right] + \alpha^{-1}\frac{1}{2\sigma}d(\sigma^2)\frac{1}{2\sigma}d(\sigma^2) = \\
&= A\left(ds^2 - d\sigma^2\right) + \alpha^{-1}d\sigma^2 = A\left(d\sigma^2 + \sigma^2 d\Omega^2_{D-1} - d\sigma^2\right) + \alpha^{-1}d\sigma^2 = \\
&= \alpha^{-1}d\sigma^2 + A\sigma^2 d\Omega^2_{D-1} = \frac{\sigma^2}{S_L}S_L'^2 d\sigma^2 + \frac{S_L}{\sigma^2}\sigma^2 d\Omega^2_{D-1} = \\
&= \sigma^2\left(\frac{1}{\sqrt{S_L}}\frac{dS_L}{d\sigma^2}\right)^2 d\sigma^2 + S_L d\Omega^2_{D-1} = \left(\frac{\sigma}{\sqrt{S_L}}\frac{dS_L}{d\sigma}\frac{d\sigma}{d\sigma^2}\right)^2 d\sigma^2 + S_L d\Omega^2_{D-1} = \\
&= \left(\frac{\sigma}{\sqrt{S_L}}\frac{dS_L}{d\sigma}\frac{1}{2\sigma}\right)^2 d\sigma^2 + S_L d\Omega^2_{D-1} = \left(\frac{1}{2\sqrt{S_L}}\frac{dS_L}{d\sigma}\right)^2 d\sigma^2 + S_L d\Omega^2_{D-1} = \\
&= \left(\frac{d\sqrt{S_L}}{d\sigma}\right)^2 d\sigma^2 + S_L d\Omega^2_{D-1} = \frac{d\sqrt{S_L}}{d\sigma}d\sigma\frac{d\sqrt{S_L}}{d\sigma}d\sigma + S_L d\Omega^2_{D-1}
\end{aligned}$$

and we finally get the q-line element for the q-Euclidean space:

$$d\tilde{s}^2 = d\sqrt{S_L}^2 + +S_L d\Omega^2_{D-1} \tag{3.30}$$

3.2.1 Interpretation

What does the q-line element (3.30) tell us? First of all we can notice that we can simply obtain the q-line element of the q-Euclidean space by directly substituting $\sigma \to \sqrt{S_L}$ in the classical Euclidean line element. The line element is still of the form

$$ds^2 = d\rho^2 + \rho^2\gamma_{\alpha\beta}(y)dy^\alpha dy^\beta \tag{3.31}$$

where ρ is the physical (longitudinal) distance from the base point and $h_{\alpha\beta} = \rho^2\gamma_{\alpha\beta}$ is the transverse metric telling us we are still describing a flat space. However we are describing a flat space with all the points at a classical squared geodesic distance $\sigma^2 < L_0$ from the base point removed: we actually have a "hole" of radius L_0 around the origin since we can probe "points" at least at a distance $S_L \geq L_0^2$.

3.2.2 Area and volume

We can now evaluate how the introduction of a minimum length affects the construction of areas and volumes. The relevant effects are encapsulated in the modification of the determinants square roots appearing in the formaulas for the volume and area elements. The equigeodesic surfaces are now defined as the set of points at a given fixed modified geodesic distance $S_L = S_L(\sigma^2 = l^2)$ We can define the q-area and q-volume elements as in the usual case:

$$d\tilde{\Sigma} = \sqrt{\tilde{h}} d^{D-1}y \tag{3.32}$$

$$d\tilde{V} = \sqrt{q} d^D x \tag{3.33}$$

where $\tilde{h}$ is the determinant of the modified induced metric on the equigeodesic hypersurface and q is the determinant of the qmetric. We already know that

$$\sqrt{q} = \frac{A^{\frac{D-1}{2}}}{\sqrt{\alpha}} \sqrt{g} \tag{3.34}$$

while we have that

$$\tilde{h}_{\alpha\beta} = A h_{\alpha\beta} \tag{3.35}$$

from which we find

$$\sqrt{\tilde{h}} = A^{\frac{D-1}{2}} \sqrt{h} \tag{3.36}$$

Thus the modified area and volume elements read as

$$d\tilde{\Sigma} = A^{\frac{D-1}{2}} d\Sigma = \left(\frac{\sqrt{S_L}}{l}\right)^{D-1} l^{D-1} d\Omega_{D-1} = \left(\sqrt{S_L}\right)^{D-1} d\Omega_{D-1} \tag{3.37}$$

$$d\tilde{V} = \frac{A^{\frac{D-1}{2}}}{\sqrt{\alpha}} dV = \left(\frac{\sqrt{S_L}}{\sigma}\right)^{D-1} \frac{\sigma}{\sqrt{S_L}} S_L' \sigma^{D-1} d\sigma d\Omega_{D-1} = \frac{1}{2} \sqrt{S_L}^{D-2} dS_L d\Omega_{D-1} \tag{3.38}$$

We can study what happens in the coincidence limit. With regard to the area element we have

$$\lim_{l^2 \to 0} d\tilde{\Sigma} = L_0^{D-1} d\Omega_{D-1} \tag{3.39}$$

meaning that the area element stays finite in the coincidence limit. In fact with a minimum length we have a "minimum equigeodesic hypersurface" around the base point. In [11] this procedure is used to associate an area to the base point x' (in our case the origin) as the area of this minimal equigeodesic hypersurface. If we integrate over the entire solid angle:

$$\tilde{A}_0 = L_0^{D-1} \Omega_{D-1} \tag{3.40}$$

which in the case $D = 4$ is given by

$$\tilde{A}_0 = 2\pi^2 L_0^3 \tag{3.41}$$

which is the area of the entire minimal equigeodesic surface. Anyways these area elements are orthogonal to the geodesic congruence.
Regarding the volume we can either compute the volume of the entire geodesic ball around the base point or the volume around a point on the equigeodesic surface. For the entire ball we need to integrate $S_L(\sigma^2)$ between $S_L(0)$ and $S_L(l^2)$ namely

$$\tilde{V}_0(l) = \frac{1}{2} \int_{S_L(0)}^{S_L(l^2)} dS_L \left(S_L\right)^{\frac{D-2}{2}} \int d\Omega_{D-1} = \frac{\Omega_{D-1}}{D} \left[S_L(l^2)^{D/2} - S_L(0)^{D/2}\right] \tag{3.42}$$

which using the fact that $S_L(0) = L_0^2$ we have

$$\tilde{V}_0(l) = \frac{\Omega_{D-1}}{D}\left[S_L(l^2)^{D/2} - L_0^D\right] \tag{3.43}$$

which goes to zero in the coincidence limit as can be seen in fig.3.1.
If we want to evaluate the volume around a point $x \in \Sigma_0(S_L(l^2))$ we need to integrate over an arbitrary portion of solid angle and the S_L variable must be integrated between $S_L(l_-^2)$ and $S_L(l_+^2)$ with $S_L(l_-^2) < S_L(l^2) < S_L(l_+^2)$. We would have:

$$\tilde{V}(x) = \frac{\Delta\Omega}{D}\left[S_L(l_+^2)^{D/2} - S_L(l_-^2)^{D/2}\right] \tag{3.44}$$

which also vanishes in the coincidence limit since $S_L(l_+^2) \to S_L(0) = L_0^2$ as well as $S_L(l_-^2) \to S_L(0) = L_0^2$. Despite the presence of a minimum length, the volume still vanishes in the coincidence limit highlighting the non triviality of having a finite area in the same limit.

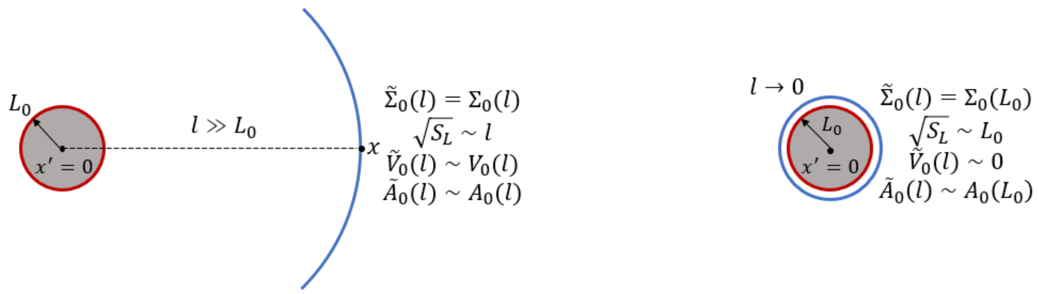


Figure 3.1: Hole in the euclidean space around the base-point x' = 0

3.2.3 Dimensional reduction

A common result of several approaches to quantum gravity is the hint that at Planck scale the space(time) becomes effectively two dimensional [38]. We expect that the qmetric description of the spacetime can reproduce such a result since the qmetric provides an effective description of the space up to the minimum length scale, incorporating quantum gravity effects. Indeed it is the case. We can define the number of effective dimensions [11, 38] as

$$D_{eff} = D + \frac{d}{d\log(l)}\left[\log\left(\frac{\tilde{V}(l)}{V(l)}\right)\right] \tag{3.45}$$

where we considering the volumes of geodesic balls in q-Euclidean and Euclidean spaces. We have that:

$$\frac{\tilde{V}(l)}{V(l)} = \frac{S_L(l^2)^{D/2} - L_0^D}{l^D} \tag{3.46}$$

of which we need to compute

$$\frac{d}{d\log(l)}\left[\log\left(\frac{\tilde{V}(l)}{V(l)}\right)\right] = l\frac{d}{dl}\left[\log\left(\frac{\tilde{V}(l)}{V(l)}\right)\right] =$$
$$= l\frac{l^D}{S_L(l^2)^{D/2} - L_0^D}\left[-\frac{D}{l^{D+1}}\left(S_L(l^2)^{D/2} - L_0^D\right) + \frac{1}{l^D}\left(\frac{D}{2}S_L^{\frac{D-2}{2}}\frac{dS_L}{dl}\right)\right] =$$
$$= -D + \frac{D}{2}l\frac{S_L^{\frac{D-2}{2}}}{S_L(l^2)^{D/2} - L_0^D}\frac{dS_L}{dl} = -D + Dl^2\frac{S_L^{\frac{D-2}{2}}}{S_L(l^2)^{D/2} - L_0^D}\frac{dS_L}{dl^2}.$$

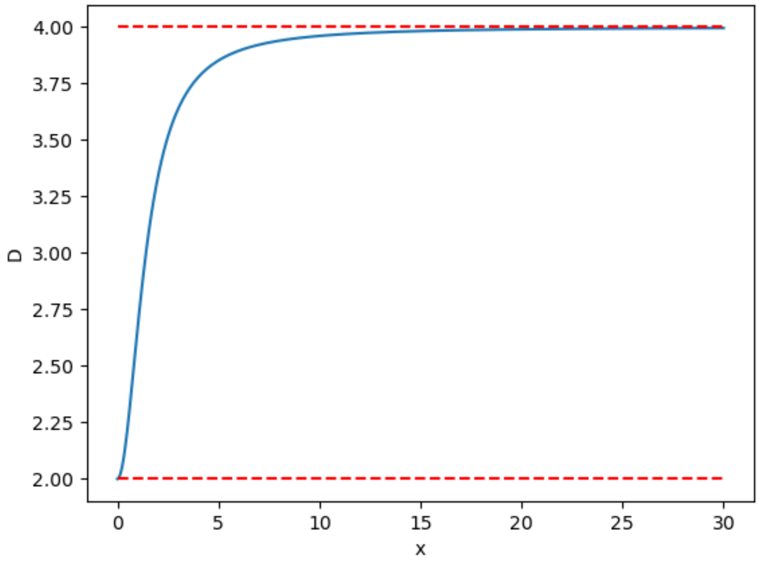


Figure 3.2: Number of effective dimensions in function of $x = \frac{l}{L_0}$ for the choice $S_L(l^2) = l^2 + L_0^2$ in a 4-dimensional Euclidean space.

Inserting this result in eq.(3.45) we find that the numbers of effective dimensions is given by

$$D_{eff} = Dl^2 \frac{S_L^{\frac{D-2}{2}}}{S_L(l^2)^{D/2} - L_0^D} \frac{dS_L}{dl^2} \tag{3.47}$$

and it is sensitive to the behaviour of the modified squared geodesic distance S_L. Using the simplest choice [11] $S_L(l^2) = l^2 + L_0^2$ in a $D = 4$ Euclidean space we find

$$D_{eff} = 4l^2 \frac{l^2 + L_0^2}{\left(l^2 + L_0^2\right)^2 - L_0^4} = 4 \frac{l^2 + L_0^2}{l^2 + 2L_0^2} = 4 \frac{1 + \left(l/L_0\right)^2}{2 + \left(l/L_0\right)^2}. \tag{3.48}$$

At large distances ($L_0 \to 0$) we have $D_{eff} = 4$ while at a very short scale ($l^2 \to 0$) we find $D_{eff} = 2$ proving that a flat Euclidean space endowed with a minimum length is effectively two-dimensional at small scales as shown in fig.(3.2).

3.3 Minkowski spacetime

We are now ready to discuss the case of a flat Lorentzian space. We consider a D-dimesional Minkowski spacetime $\mathbb{R}^{1,D-1}$ whose spacetime metric in cartesian coordinates is given by:

$$ds^2 = \eta_{ab} dx^a dx^b = -\left(dx^0\right)^2 + \sum_{i=1}^{D-1} \left(dx^i\right)^2 \tag{3.49}$$

We select the origin as our base point $x' = 0$ and we study the geodesic emanating from it. In Minkowski spacetime geodesics are simply straight lines, so we need to consider straight line passing through the origin. Considering a point x on one of such geodesics the squared geodesic distance from the origin reads as:

$$\sigma^2(x) \equiv \sigma^2(x, 0) = -\left(x^0\right)^2 + \sum_{i=1}^{D-1} \left(x^i\right)^2 \tag{3.50}$$

We can classify the geodesic according to their nature: if $\sigma^2 < 0$ we have a *timelike geodesics* (physical paths followed by free probes) while if $\sigma^2 > 0$ the geodesic is *spacelike* (unphysical paths). The case in which $\sigma^2 = 0$ refers to *null-like/light-like geodesics* (paths followed by free massless objects such free photons/light rays). In the case $\sigma^2(x) \neq 0$ we have that the unit tangent vector to the geodesic in the point x is given by

$$u_a(x) = \frac{\partial_a \sigma^2(x)}{2\sqrt{\epsilon\sigma^2(x)}} \tag{3.51}$$

where $\epsilon = -1, +1$ respectively for timelike and spacelike geodesics.

3.3.1 Time-like geodesics

Timelike geodesics lie within the past lightcone and future lightcone of the origin. we can define the proper time along a timelike geodesic as:

$$\tau^2(x) = -\sigma^2(x) \tag{3.52}$$

where $\tau(x)$ is the proper time measured by an observer moving along the given geodesic in the point x. We have $\tau > 0$ for $x^0 > 0$ and $\tau < 0$ for $x^0 < 0$. Restricting to the case of $D = 4$ we have that:

$$ds^2 = -\left(dx^0\right)^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 \tag{3.53}$$

$$\sigma^2(x) = -\left(x^0\right)^2 + \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 = -\tau^2(x) \tag{3.54}$$

We now define equigeodesic hypersurfaces $\Sigma(\tau) = \{x \in \mathbb{R}^{1,3} | \sigma^2 = -\tau_\Sigma^2 = const\}$: they are described by two-sheet hyperboloid (see fig 3.3), one in the upper half of Minkowski space and one in the lower part. Every points in such hypersurface are caracterized by the same proper time τ_Σ. We can introduce coordinates adapted to such hypersurfaces. In particular for the upper sheet of hyperboloid we can perform the following coordinate transformation:

$$x^0 = \tau \cosh(\alpha) \tag{3.55}$$

$$x^1 = \tau \sinh(\alpha) \sin(\theta) \cos(\phi) \tag{3.56}$$

$$x^2 = \tau \sinh(\alpha) \sin(\theta) \sin(\phi) \tag{3.57}$$

$$x^3 = \tau \sinh(\alpha) \cos(\theta) \tag{3.58}$$

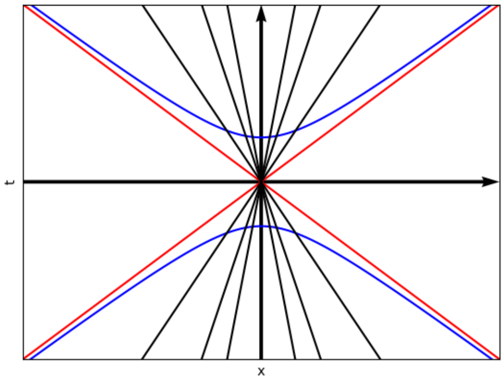
where $\theta \in [0, \pi], \phi \in [0, 2\pi]$ are the usual spherical angular coordinates. In order to understand the meaning and the regime of the α coordinate we define:

$$r^2 \equiv (x^1)^2 + (x^2)^2 + (x^3)^2 = \tau^2 \sinh^2(\alpha) \rightarrow r = \tau \sinh(\alpha) \tag{3.59}$$

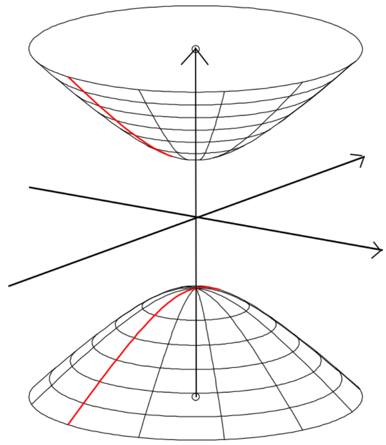
where r is the radial coordinate in the purely spatial euclidean space $\mathbb{R}^3$. Since we must have $r > 0$ we need to impose $\alpha \in [0, \infty[$ in order to have $\sinh(\alpha) \geq 0$. Moreover we also have

$$\frac{r}{x^0} = \tanh(\alpha) \rightarrow \alpha = \tanh^{-1}\left(\frac{r}{x^0}\right) \tag{3.60}$$

showing that α is equal to the rapidity $\beta \equiv tanh^{-1}(v/c)$ in units where the speed of light c is set to $c = 1$: thus α coordinate selects all the timelike geodesic passing through the origin (it does not select a unique geodesic due to the spherical symmetry of Minkowski space) along which a probe is moving at a speed $v = r/x^0 = \tanh^{-1}(\alpha)$ with respect to



(a) Examples of timelike geodesics in a D=2 Minkowski spacetime (in black). In blue it is shown an example of equigeodesic surface. The lightcone is shown in red. Note $x^0 = t$ and $x^1 = x$.



(b) Visualization of a two-sheet hyperboloid in D = 3

Figure 3.3: Timelike geodesics

a stationary frame located in the origin. In such a set of coordinates the line element reads as

$$ds^2 = -d\tau^2 + \tau^2 d\alpha^2 + \tau^2 \sinh^2(\alpha) d\Omega^2 = -d\tau^2 + \tau^2 \left(d\alpha^2 + \sinh^2(\alpha) d\Omega^2\right) \tag{3.61}$$

where $d\Omega^2 = d\theta^2 + \sin^2(\theta) d\phi^2$ is the squared line element on a unit 2-sphere. In this coordinates system the metric determinant is given by:

$$\eta \equiv \det(\eta) = -\tau_\Sigma^6 \sinh^6(\alpha) \sin^2(\theta) \tag{3.62}$$

Restricting our displacements on a equigeodesic hypersurface $\Sigma(\tau_\Sigma)$ we get the induced metric to the hypersurface:

$$ds^2_{|\Sigma} = h_{\alpha\beta} dy^\alpha dy^\beta = \tau_\Sigma^2 \left(d\alpha^2 + \sinh^2(\alpha) d\Omega^2\right) \tag{3.63}$$

which is also orthogonal in every point to the vector tangent to the geodesic passing through that point. We notice that the induced metric has explicitly a hyperbolic nature [39]. The determinant of the induced metric reads as:

$$h \equiv \det(\mathbf{h}) = \tau_\Sigma^6 \sinh^6(\alpha) \sin^2(\theta) \tag{3.64}$$

From the induced metric we can also compute the differential of the hypersurface area element on Σ around a point $x \in \Sigma$

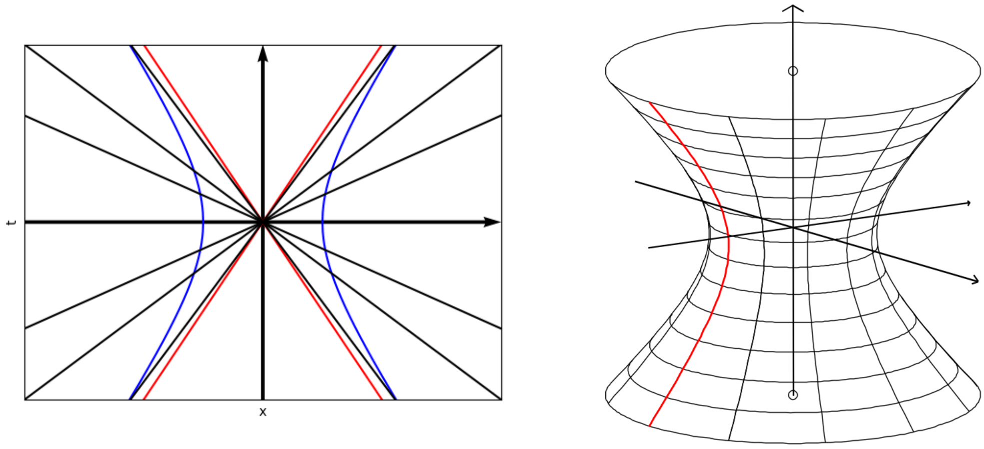
$$d\Sigma = \sqrt{h} d^3 y = \tau_\Sigma^3 \sinh^3(\alpha) d\alpha \sin(\theta) d\theta d\phi = \tau_\Sigma^3 \sinh^3(\alpha) d\alpha d\Omega \tag{3.65}$$

If we perform the coincidence limit $x \to 0$ we have that $\sigma^2 = -\tau_\Sigma^2 \to 0$ and the hypersurface area element vanishes:

$$\lim_{x \to 0} d\Sigma = \lim_{\tau_\Sigma^2 \to 0} d\Sigma = 0 \tag{3.66}$$

We can also compute the 4-volume element around a point $x \in \Sigma$ by computing:

$$dV = \sqrt{-\eta} d^4 x = \tau^3 \sinh^3(\alpha) d\tau d\alpha \sin(\theta) d\theta d\phi = \tau^3 \sinh^3(\alpha) d\tau d\alpha d\Omega \tag{3.67}$$



(a) Examples of spacelike geodesics in a D=2 Minkowski spacetime (in black). In blue it is shown an example of equigeodesic surface. The lightcone is shown in red. Note $x^0 = t$ and $x^1 = x$.

(b) Visualization of a one-sheet hyperboloid in D = 3

Figure 3.4: Spacelike geodesics

and also in this case in the coincidence limit we have a vanishing 4-volume element:

$$\lim_{x \to 0} dV = \lim_{\tau^2 \to 0} dV = 0 \tag{3.68}$$

Unlike the Euclidean case it doesn't make much sense to talk about areas and volumes around the origin since an integration over all values of α gives a divergent result both for the area and the volume.
The description in the lower half of Minkowski spacetime is the same as the one just carried on with the only difference given by the fact that the proper time τ is negative and thus $\alpha \in]-\infty, 0]$ in order to have a positive radial coordinate.

3.3.2 Space-like geodesics

Space-like geodesics lie outside the lightcones centered in the origin. In this case we can define the proper length $l(x)$ in the following way:

$$l(x) = \sqrt{\sigma^2(x)} \tag{3.69}$$

where $l(x) > 0$ is the proper distance from the origin along a spacelike geodesic in the point x. Restricting to the case $D = 4$ we have:

$$ds^2 = -\left(dx^0\right)^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx_3\right)^2 \tag{3.70}$$

$$\sigma^2(x) = -\left(x^0\right)^2 + \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 = l^2(x) \tag{3.71}$$

We see that equigeodesic hypersurfaces $\Sigma(l) = \{x \in \mathbb{R}^{1,3} | \sigma^2 = l_\Sigma^2 = const\}$ are given by a one-sheet three dimensional hyperboloid as shown in fig.(3.4)). Every point on $\Sigma(l_\Sigma)$ are at a spatial proper distance l_Σ from the origin. Following the same step previously done for the time-like case we perform the following coordinate transformations:

$$x^0 = l \sinh(\alpha) \tag{3.72}$$

$$x^1 = l \cosh(\alpha) \sin(\theta) \cos(\phi) \tag{3.73}$$

$$x^2 = l \cosh(\alpha) \sin(\theta) \sin(\phi) \tag{3.74}$$

$$x^3 = l \cosh(\alpha) \cos(\theta) \tag{3.75}$$

where $\theta \in [0, \pi], \phi \in [0, 2\pi]$ are the usual spherical angular coordinates. We can define the radial coordinate r in the 3-dimensional spatial euclidean space:

$$r^2 \equiv (x^1)^2 + (x^2)^2 + (x^3)^2 = l^2 \cosh^2(\alpha) \to r = l \cosh(\alpha) \tag{3.76}$$

and we see that r is always positive independently on the values of α which in this case is given by:

$$\alpha = tanh^{-1}\left(\frac{x^0}{r}\right) \tag{3.77}$$

and runs from $-\infty$ to $+\infty$. We can rewrite the Minkowski line element in this coordinate system and we find:

$$ds^2 = dl^2 - l^2 d\alpha^2 + l^2 \cosh^2(\alpha) d\Omega^2 \tag{3.78}$$

where $d\Omega^2 = d\theta^2 + \sin^2(\theta) d\phi^2$ is the squared line element os a unit 2-sphere. In this coordinates system the metric determinant is given by:

$$\eta \equiv \det(\eta) = -l^6 \cosh^6(\alpha) \sin^2(\theta) \tag{3.79}$$

Restricting our displacements on a equigeodesic hypersurface $\Sigma(l\Sigma)$ we get the induced metric to the hypersurface:

$$ds^2_{|\Sigma} = h_{\alpha\beta} dy^\alpha dy^\beta = l^2_\Sigma \left(-d\alpha^2 + \cosh^2(\alpha) d\Omega^2\right) \tag{3.80}$$

which is also orthogonal in every point to the vector tangent to the geodesic passing through that point. The determinant of the induced metric reads as:

$$h \equiv \det(\mathbf{h}) = l^6_\Sigma \cosh^6(\alpha) \sin^2(\theta) \tag{3.81}$$

From the induced metric we can also compute the differential of the hypersurface area element on Σ around a point $x \in \Sigma$

$$d\Sigma = \sqrt{h} d^3 y = l^3_\Sigma \cosh^3(\alpha) d\alpha \sin(\theta) d\theta d\phi = l^3_\Sigma \cosh^3(\alpha) d\alpha d\Omega \tag{3.82}$$

If we perform the coincidence limit $x \to 0$ we have that $\sigma^2 = l^2_\Sigma \to 0$ and the hypersurface area element vanishes:

$$\lim_{x\to 0} d\Sigma = \lim_{l^2_\Sigma \to 0} d\Sigma = 0 \tag{3.83}$$

We can also compute the 4-volume element around a point $x \in \Sigma$ by computing:

$$dV = \sqrt{-\eta} d^4 x = l^3 \cosh^3(\alpha) dl d\alpha \sin(\theta) d\theta d\phi = l^3 \cosh^3(\alpha) dl d\alpha d\Omega \tag{3.84}$$

and also in this case in the coincidence limit we have a vanishing 4-volume element:

$$\lim_{x\to 0} dV = \lim_{l^2 \to 0} dV = 0 \tag{3.85}$$

3.3.3 Null geodesics

The treatment of null geodesics is not so straightforward. All along the null geodesics we have:

$$\sigma^2(x) = 0 \tag{3.86}$$

which defines the lightcones (past and future) centered in the origin. Since the geodesics distance is invariantly zero all along the null paths it fails to distinguish different points and it can't be used as an affine parameter as in the previous cases. Moreover $\partial_a \sigma^2$ is a null vector and it must be proportional to the null tangent vector. In order to select a particular affine parameter with a physical meaning we must introduce an obsever with a timelike 4-velocity V^a: any null affine parametrization λ gives us a measure of distance along the null geodesic as measured by a particular timelike observer at a certain point x of the geodesic[13, 19]. We attach a canonical observer in the origin with a four velocity V^a such that $V^a l_a = -1$. Following what we have done in section 1.1.2 we have:

$$\frac{1}{2}\partial_a \sigma^2 = \lambda l_a \tag{3.87}$$

and we can write

$$\lambda = -V^a \frac{1}{2}\partial_a \sigma^2 \tag{3.88}$$

From V^a and l^a we can construct a second null vector $m^a = V^a - (1/2)l^a$ such that:

1. $m^a V_a = -1/2$
2. $m^a l_a = -1$

This second vector must be used with l^a to determine the metric transverse to the direction of the geodesic [17]:

$$h_{ab} = g_{ab} + l_a m_b + m_a l_b \tag{3.89}$$

which is effectively $D-2$ dimensional since it is subject to the orthogonality constraints $h_{ab}l^a = 0$ and $h_{ab}m^a = 0$. It may seems that the transverse metric is not unique since it depends on the choice of V^a: indeed it is true, but physical quantities computed through h_{ab} turns out to be independent from the auxiliary null vector m^a[17]. We can now introduce null coordinates u and v such that:

$$l_a = -\partial_a u \tag{3.90}$$

$$m_a = -\frac{1}{2}\partial_a v \tag{3.91}$$

Example in D = 4

Consider in four dimensions in $\{t, x, y, z\}$ coordinates a future directed null geodesic along x direction. A point p on such a geodesic is at a squared geodesic distance:

$$\sigma^2(p) = -t^2 + x^2 + y^2 + z^2 = 0 \tag{3.92}$$

from the origin. We select a stationary observer in $x = y = z = 0$ with 4-velocity $V^a = (1, 0, 0, 0)$. So we find:

$$\lambda = t \tag{3.93}$$

$$l_a = \left(-1, \frac{x}{t}, 0, 0\right) \tag{3.94}$$

hence imposing the null condition $l_a l^a = 0$ we get $x^2 = t^2$ hence $l^a = (1, \pm 1, 0, 0)$. The second null vector m^a is given by $m^a = (1/2, \mp 1/2, 0, 0)$ and the null coordinates are given by $u = t - x$ and $v = t + x$. Thus vector l^a is tangent to $u = const$ surfaces (light rays moving towards increasing values of x coordinate) while m^a is tangent to $v = const$ surfaces (light rays moving towards decreasing values of x coordinate). We can rewrite the Minkowski line element in $\{u, v, y, z\}$ coordinates and we get

$$ds^2 = -dudv + dy^2 + dz^2 \tag{3.95}$$

from which we can read the transverse metric and the metric induced on the light cone Γ

$$ds^2_{|\Gamma} = dy^2 + dz^2 \tag{3.96}$$

explicitly showing a 2-dimensional induced metric. We can again take advantage of the spherical symmetry in the purely spatial 3D Euclidean case by using $\{t, r.\theta, \phi\}$ coordinates and study radial geodesics in any direction. In this case geodesic distance is given by

$$\sigma^2 = -t^2 + r^2 = 0 \tag{3.97}$$

and the lightcones (generated by all future and past radial null geodesics) are described by $t - r = 0$ (future cone) and $t + r = 0$ (past cone). We now have that an arbitrary radial null geodesic is parametrized by :

$$l_a = \left(-1, \frac{x}{r}, , \frac{y}{r}, , \frac{z}{r}\right) = (-1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = -\partial_a(t - r) \tag{3.98}$$

and the auxiliary vector with respect to a stationary observer

$$m_a = \frac{1}{2}(-1, -\sin\theta\cos\phi, -\sin\theta\sin\phi, -\cos\theta) = -\frac{1}{2}\partial_a(t + r) \tag{3.99}$$

from which we define the null coordinates as $u = t - r$ and $v = t + r$. Using that $2\lambda = -V^a\partial_a\sigma^2$ we get $\lambda = t$. For radial null geodesic out-going from the origin $u = 0$ hence $r = t = \lambda$ while for in-going radial null geodesic $v = 0$ hence $t = \lambda = -r$. The metric in null coordinates reads as

$$ds^2 = -dudv + r^2 d\Omega^2 = -dudv + \lambda^2 d\Omega^2 \tag{3.100}$$

with $d\Omega^2$ being the squared line element on a unit 2-sphere.
We want to provide an expression for the metric in terms of the affine parameters of the geodesics tangent respectively to l^a and m^a. We know from section 1.1.2 that we have:

$$l^a = \frac{dx^a}{d\lambda} \qquad l_a = -\partial_a u \tag{3.101}$$

$$m^a = \frac{dx^a}{d\nu} \qquad m_a = -\frac{1}{2}\partial_a v \tag{3.102}$$

$$l^a m_a = -1 \tag{3.103}$$

Using eq.(3.103)we can find the following identities:

$$1 = \frac{du}{du} = \frac{dx^a}{du}\partial_a u = -\frac{dx^a}{du} l_a \to m^a = \frac{dx^a}{du} \tag{3.104}$$

$$1 = \frac{dv}{dv} = \frac{dx^a}{dv}\partial_a v = -2\frac{dx^a}{dv} m_a \to l^a = 2\frac{dx^a}{dv} \tag{3.105}$$

from which we can infer that $du = d\nu$ and $d\lambda = dv/2$. Therefore we can write the metric in the form:

$$ds^2 = -2d\lambda d\nu + \lambda^2 d\Omega^2 \tag{3.106}$$

The induced transverse metric on the lightcone is given by

$$ds^2 = \lambda^2 d\Omega^2 \tag{3.107}$$

which can be used to compute the transverse area element around a point on the light cone at an affine distance λ from the origin:

$$d\Sigma = \lambda^2 d\Omega \tag{3.108}$$

with $d\Omega = \sin\theta d\theta d\phi$.

3.4 q-Minkowski spacetime

We now study how to construct the qmetric for a $D = 4$ Minkowski spacetime. The starting point is the substitution of the squared geodesic distance by an arbitrary function:

$$\sigma^2(x) \to S_L(\sigma^2(x)) \quad \text{such that} \quad S_L(0) = \pm L_0^2 \tag{3.109}$$

preserving the nature of the spacetime interval between the point considered and the origin.

3.4.1 Time-like intervals

Consider the point x on a timelike geodesic segment which links x to the origin at a classical squared geodesic distance $\sigma^2(x)$.The timelike unit vector tangent to the geodesic is given by $u_a = \partial_a \sigma^2 / \left(2\sqrt{-\sigma^2}\right)$. Recall that for timelike separated events the qmetric reads as

$$q_{ab} = A g_{ab} - \left(\alpha^{-1} - A\right) u_a u_b \tag{3.110}$$

where A and $\alpha$are respectively given by eq.(3.27)and eq.(3.28). In the case of Minkowkski spacetime we have:

$$g_{ab} = \eta_{ab} \tag{3.111}$$

$$A = \frac{S_L}{\sigma^2} \tag{3.112}$$

$$\alpha = \frac{1}{\sigma^2} \frac{S_L}{S_L'^2} \tag{3.113}$$

where a prime means derivative with respect to σ^2. The construction of the modified line element follows the same algebraic passages done in the euclidean case, with particular attention to the presence of various minus signs appearing: keep in mind that $S_L < 0$ through the entire derivation. We also use the fact that for timelike geodesics $\tau^2 = -\sigma^2 > 0$:

$$\begin{aligned}
d\tilde{s}^2 &= q_{ab} dx^a dx^b = A \eta_{ab} dx^a dx^b - \left(\alpha^{-1} - A\right) u_a u_b dx^a dx^b = \\
&= A ds^2 - \left(\alpha^{-1} - A\right) \frac{1}{4\tau^2} \frac{\partial \tau^2}{\partial x^a} dx^a \frac{\partial \tau^2}{\partial x^b} dx^b = \\
&= A \left[ds^2 + \frac{1}{2\tau} d(\tau^2) \frac{1}{2\tau} d(\tau^2) \right] - \alpha^{-1} \frac{1}{2\tau} d(\tau^2) \frac{1}{2\tau} d(\tau^2) = \\
&= A \left(ds^2 + d\tau^2\right) - \alpha^{-1} d\tau^2 = \\
&= A \left(-d\tau^2 + \tau^2 d\alpha^2 + \tau^2 \sinh^2(\alpha) d\Omega^2 + d\tau^2\right) - \alpha^{-1} d\tau^2 = \\
&= -\alpha^{-1} d\tau^2 + A \left(\tau^2 d\alpha^2 + \tau^2 \sinh^2(\alpha) d\Omega^2\right) = \\
&= -\frac{-\tau^2}{S_L} S_L'^2 d\tau^2 + \frac{S_L}{-\tau^2} \tau^2 \left(d\alpha^2 + \sinh^2(\alpha) d\Omega^2\right) = \\
&= -\tau^2 \left(\frac{1}{\sqrt{-S_L}} \frac{dS_L}{d\sigma^2} \right)^2 d\tau^2 - S_L \left(d\alpha^2 + \sinh^2(\alpha) d\Omega^2\right) = \\
&= -\left(\frac{\tau}{\sqrt{-S_L}} \frac{dS_L}{d\tau} \frac{d\tau}{d\sigma^2} \right)^2 d\tau^2 - S_L \left(d\alpha^2 + \sinh^2(\alpha) d\Omega^2\right) = \\
&= -\left(-\frac{\tau}{\sqrt{-S_L}} \frac{dS_L}{d\tau} \frac{1}{2\tau} \right)^2 d\tau^2 - S_L \left(d\alpha^2 + \sinh^2(\alpha) d\Omega^2\right) =
\end{aligned}$$

$$
\begin{aligned}
&= -\left(-\frac{1}{2\sqrt{-S_L}}\frac{dS_L}{d\tau}\right)^2 d\tau^2 - S_L\left(d\alpha^2 + \sinh^2(\alpha)d\Omega^2\right) = \\
&= -\left(\frac{d\sqrt{-S_L}}{d\tau}\right)^2 d\tau^2 - S_L\left(d\alpha^2 + \sinh^2(\alpha)d\Omega^2\right) = \\
&= -\frac{d\sqrt{-S_L}}{d\tau}d\tau\frac{d\sqrt{-S_L}}{d\tau}d\tau - S_L\left(d\alpha^2 + \sinh^2(\alpha)d\Omega^2\right)
\end{aligned}
$$

from which we have

$$
d\tilde{s}^2 = -d\sqrt{-S_L}^2 - S_L\left(d\alpha^2 + \sinh^2(\alpha)d\Omega^2\right) \tag{3.114}
$$

which, if we introduce a modified proper time function $\tilde{\tau}^2 = -S_L$; can be rewritten as

$$
d\tilde{s}^2 = -d\tilde{\tau}^2 + \tilde{\tau}^2\left(d\alpha^2 + \sinh^2(\alpha)d\Omega^2\right). \tag{3.115}
$$

As in the euclidean case we are still describing a flat spacetime but with the points at a classical geodesic distance $\sigma^2(x) < -L_0^2$ removed. (see fig. 3.5a).

3.4.2 Space-like intervals

We can perform the same steps for a point x on a space-like geodesic. In this case we have $\sigma^2(x) = l^2(x) > 0$ and the unit spacelike vector tangent to the geodesic segment is given by $u_a = \partial_a l^2/2l$. The qmetric for spacelike separated events reads as

$$
q_{ab} = Ag_{ab} + \left(\alpha^{-1} - A\right)u_a u_b \tag{3.116}
$$

from which we can compute the modified squared line element

$$
\begin{aligned}
d\tilde{s}^2 &= q_{ab}dx^a dx^b = A\eta_{ab}dx^a dx^b + \left(\alpha^{-1} - A\right)u_a u_b dx^a dx^b = \\
&= Ads^2 + \left(\alpha^{-1} - A\right)\frac{1}{4l^2}\frac{\partial l^2}{\partial x^a}dx^a\frac{\partial l^2}{\partial x^b}dx^b = \\
&= A\left[ds^2 - \frac{1}{2l}d(l^2)\frac{1}{2l}d(l^2)\right] + \alpha^{-1}\frac{1}{2l}d(l^2)\frac{1}{2l}d(l^2) = \\
&= A\left(ds^2 - dl^2\right) + \alpha^{-1}dl^2 = \\
&= A\left(dl^2 - l^2 d\alpha^2 + l^2\cosh^2(\alpha)d\Omega^2 - dl^2\right) + \alpha^{-1}dl^2 = \\
&= \alpha^{-1}dl^2 + A\left(-l^2 d\alpha^2 + l^2\cosh^2(\alpha)d\Omega^2\right) = \\
&= \frac{l^2}{S_L}S_L'^2 dl^2 + \frac{S_L}{l^2}l^2\left(-d\alpha^2 + \cosh^2(\alpha)d\Omega^2\right) = \\
&= l^2\left(\frac{1}{\sqrt{S_L}}\frac{dS_L}{d\sigma^2}\right)^2 dl^2 + S_L\left(-d\alpha^2 + \cosh^2(\alpha)d\Omega^2\right) = \\
&= \left(\frac{l}{\sqrt{S_L}}\frac{dS_L}{dl}\frac{dl}{d\sigma^2}\right)^2 dl^2 + S_L\left(-d\alpha^2 + \cosh^2(\alpha)d\Omega^2\right) = \\
&= \left(\frac{l}{\sqrt{S_L}}\frac{dS_L}{dl}\frac{1}{2l}\right)^2 dl^2 + S_L\left(-d\alpha^2 + \cosh^2(\alpha)d\Omega^2\right) = \\
&= \left(\frac{1}{2\sqrt{S_L}}\frac{dS_L}{dl}\right)^2 dl^2 + S_L\left(-d\alpha^2 + \cosh^2(\alpha)d\Omega^2\right) = \\
&= \left(\frac{d\sqrt{S_L}}{dl}\right)^2 dl^2 + S_L\left(-d\alpha^2 + \cosh^2(\alpha)d\Omega^2\right) =
\end{aligned}
$$

$$= \frac{d\sqrt{S_L}}{dl} dl \frac{d\sqrt{S_L}}{dl} dl + S_L \left(-d\alpha^2 + \cosh^2(\alpha) d\Omega^2\right)$$

from which we have

$$d\tilde{s}^2 = d\sqrt{S_L}^2 + S_L \left(-d\alpha^2 + \cosh^2(\alpha) d\Omega^2\right) \tag{3.117}$$

which, if we introduce a modified proper length function $\tilde{l}^2 = S_L$; can be rewritten as

$$d\tilde{s}^2 = d\tilde{l}^2 + \tilde{l}^2 \left(-d\alpha^2 + \cosh^2(\alpha) d\Omega^2\right). \tag{3.118}$$

which tells us that we are still describing a flat space but with points at a geodesic distance $0 < \sigma^2 < L_0^2$ removed (see fig 3.5b).

Maximal acceleration

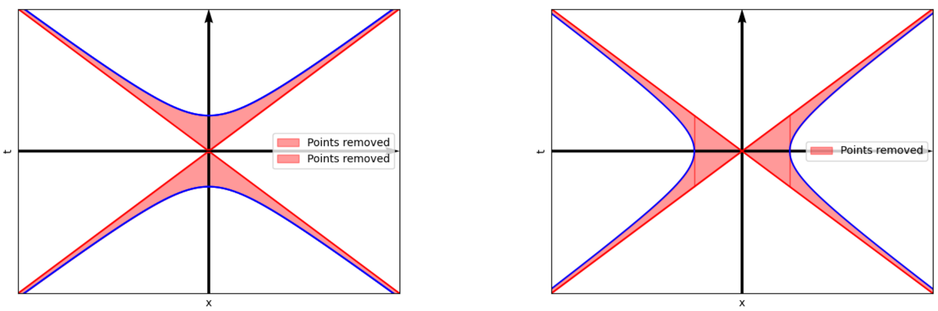
The modification of the spacelike squared geodesic distance gives us also a hint for the possible presence of a maximal value for the acceleration in the Minkowski spacetime. There are several theories which contemplate the possibility of having a finite proper maximal acceleration for an accelerating probe [40]. This fact is related to the presence of a minimal length scale[41]. The qmetric description of Minkowski spacetime allows the following arguments. The introduction of the modified squared geodesic distance S_L for spacelike separated events reveals the presence of a "minimal equigeodesic hypersurface" at a distance $S_L = +L_0^2$ from the origin. An uniformly accelerated observer along ,for example, x direction in Minkowski space follows hyperbolic paths. In particular if it is accelerating with a constant acceleration k she follows the path (in inertial coordinates $\{t, x, y, z\}$ [42]

$$x^2 - t^2 = \frac{1}{k^2} \tag{3.119}$$

describing an hyperbolic path asymptotically to the light cones $t = x$ in the future and $t = -x$ in the past. This also represents the equigeodesic surface at a distance $1/k^2$ and if this distance must be bounded from below:

$$\frac{1}{k^2} > L_0^2 \rightarrow k^2 < \frac{1}{L_0^2} \tag{3.120}$$

showing we have an upper limit for the magnitude of the proper acceleration.



(a) Points removed for timelike geodesics. (b) Points removed for spacelike geodesics.

Figure 3.5: Spacetime regions removed

3.4.3 Null-like intervals

We now consider a point p identified by the inertial coordinates (t, x, y, z) on the future light cone Γ centered in the origin where we attach a freely moving observer with 4-velocity $V^a = (1, 0, 0, 0)$ in the local frame. Using the construction of the section (3.3.3) we have :

$$l^a = \left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) \tag{3.121}$$

$$m^a = \frac{1}{2}\left(1, -\frac{x}{r}, -\frac{y}{r}, -\frac{z}{r}\right) \tag{3.122}$$

where $r^2 = x^2 + y^2 + z^2$, l^a is the null tangent vector to the null geodesic linking the origin to the point p and $m^a = V^a - l^a/2$ is the auxiliary null vector. We also have shown we can parameterize the null geodesic with the affine null parameter $\lambda = t = r$ and we can use the null coordinates $u = t - r$ and $v = t + r$ such that $l_a = -\partial_a u$ and $m_a = -\partial_a v/2$. We assume $\lambda(p) = \lambda_p > 0$ and $\lambda(0) = \lambda_0 = 0$. In order to construct the qmetric in the point p with respect to the origin we start from a modification of the null affine parameter:

$$\lambda(p) \to \tilde{\lambda}(\lambda_p) \equiv \tilde{\lambda}_p \quad \text{such that} \quad \lim_{p\to 0}\left(\tilde{\lambda}_p - \tilde{\lambda}_0\right) = L_0 \tag{3.123}$$

providing a finite affine parameter distance in the coincidence limit. We recall from eq.(2.71) the qmetric general form for null separated events:

$$q_{ab} = A_\lambda g_{ab} + \left(A_\lambda - \alpha_\lambda^{-1}\right)\left(l_a m_b + m_a l_b\right) \tag{3.124}$$

where

$$A_\lambda = \frac{\tilde{\lambda}^2}{\lambda^2}\left(\frac{\Delta}{\tilde{\Delta}}\right)^{\frac{2}{d-2}} \tag{3.125}$$

$$\alpha_\lambda = \frac{1}{d\tilde{\lambda}/d\lambda} \tag{3.126}$$

in which $\Delta(\tilde{\Delta})$ is the Van Vleck determinant evaluated in the point p ($\tilde{p}$ identified by $\lambda_{\tilde{p}} = \tilde{\lambda}_p$). In Minkowski spacetime we simply have $g_{ab} = \eta_{ab}$, $\Delta = \tilde{\Delta} = 1$. We can thus evaluate the modified squared line element for null separated events:

$$\begin{aligned}
d\tilde{s}^2 = q_{ab}dx^a dx^b &= A_\lambda \eta_{ab} dx^a dx^b + \left(A_\lambda - \alpha_\lambda^{-1}\right)\left(l_a m_b + m_a l_b\right) dx^a dx^b = \\
&= A_\lambda ds^2 + \frac{1}{2}\left(A_\lambda - \alpha_\lambda^{-1}\right)\left(\partial_a u \partial_b v + \partial_a v \partial_b u\right) dx^a dx^b = \\
&= A_\lambda\left(-du dv + \lambda^2 d\Omega^2\right) + \left(A_\lambda - \alpha_\lambda^{-1}\right) du dv = \\
&= -\alpha_\lambda^{-1} du dv + A_\lambda \lambda^2 d\Omega^2 = \\
&= -\left(\frac{d\tilde{\lambda}}{d\lambda}\right) du dv + \tilde{\lambda}^2 d\Omega^2.
\end{aligned}$$

We now search for an expression of the null qmetric involving the q-affine parameters of the q-geodesics according to the qmetric. We know that by construction a null geodesic in ordinary spacetime is still a null geodesic in the qmetric description: what change are the affine parameters. Recalling equations (2.30) and (2.34) with the expression of α_λ given by eq:(3.126) we have that the modified tangent vector $\tilde{l}^a$ is given by:

$$\tilde{l}^a = \frac{dx^a}{d\tilde{\lambda}} = \frac{d\lambda}{d\tilde{\lambda}} l^a = \alpha_\lambda l^a \tag{3.127}$$

$$\tilde{l}_a = \frac{d\lambda}{d\tilde{\lambda}}\alpha_\lambda^{-1} l_a = \frac{d\lambda}{d\tilde{\lambda}}\frac{d\tilde{\lambda}}{d\lambda} l_a = l_a \tag{3.128}$$

We now impose that the normalization given by equation (3.103) is valid also for the modified null tangent vectors with respect to the scalar product constructed from the qmetric:

$$q_{ab}\tilde{l}^a \tilde{m}^b = \tilde{l}^a \tilde{m}_a = -1 \tag{3.129}$$

where $\tilde{m}^a$ is given by:

$$\tilde{m}^a = \frac{dx^a}{d\tilde{\nu}} \tag{3.130}$$

From the normalization condition we can find that:

$$1 = \frac{d\tilde{\nu}}{d\tilde{\nu}} = \frac{dx^a}{d\tilde{\nu}}\partial_a \tilde{\nu} = \tilde{m}^a \partial_a \tilde{\nu} \rightarrow q_{ab}\tilde{l}^b = \tilde{l}_a = l_a = -\partial_a \tilde{\nu} = -\partial_a u \tag{3.131}$$

hence we have $du = d\tilde{\nu}$. From the relation of eq.(3.105) we see that:

$$dv = 2d\lambda \tag{3.132}$$

$$\frac{d\tilde{\lambda}}{d\lambda} dv = 2\frac{d\tilde{\lambda}}{d\lambda} d\lambda = 2d\tilde{\lambda} \tag{3.133}$$

from which we can rewrite the explicit form of the null qmetric for the Minkowski spacetime as

$$d\tilde{s}^2 = -2d\tilde{\lambda}d\tilde{\nu} + \tilde{\lambda}^2 d\Omega^2 \tag{3.134}$$

Since we are considering a point on a null geodesic from the origin we already know that the longitudinal part of the metric vanishes since we are describing displacements along $u = const$ or $v = const$ trajectories. What matters to us is the 2-dim. transversal part of the metric which characterize the structure of the 2-dim. space transversal to the light cone:

$$ds_\Gamma^2 = \tilde{\lambda}^2 d\Omega^2 \tag{3.135}$$

whose determinant is given by

$$\sqrt{h_\Gamma} = \tilde{\lambda}^2 \sin\theta. \tag{3.136}$$

3.4.4 Areas and volumes

We now investigate the effect of the introduction of the qmetric description in the Minkowski spacetime on the volume and area elements. We separately treat the three different spacetime separations.

Time-like separations

In the case of time-like separations the equigeodesic hypersurface is represented by a two-sheet hyperboloid of points at the same proper time distance τ_Σ from the origin. Considering the upper sheet, we select a point x on the hypersurface at a distance τ_Σ. This point is identified by the triplet $(\alpha_x, \theta_x, \phi_x)$. Using spherical symmetry we're free to rotate our frame to select $\theta_x = \phi_x = 0$.The induced metric on the hypersurface is

$$d\tilde{s}^2_{|\Sigma} = \tilde{\tau}_\Sigma^2 \left(d\alpha^2 + \sinh^2\alpha d\Omega^2\right). \tag{3.137}$$

The area element around the point x is given by:

$$d\tilde{\Sigma} = \tilde{\tau}_\Sigma^3 \sinh^2(\alpha) d\alpha d\Omega \tag{3.138}$$

which can be integrated over all the solid angle and an arbitrary finite range of α namely from α_- to α_+ such that $\alpha_- < \alpha_x < \alpha_+$ getting

$$\tilde{\Sigma}(x,\alpha_-,\alpha_+) = 4\pi\tilde{\tau}_\Sigma^3 \int_{\alpha_-}^{\alpha_+} d\alpha \sinh^2(\alpha) \tag{3.139}$$

which remains finite in the coincidence limit since $\tau_\Sigma^2 \to L_0^2$. Thus

$$\lim_{\tau_\Sigma \to 0} \tilde{\Sigma}(x,\alpha_-,\alpha_+) = 4\pi L_0^3 \int_{\alpha_-}^{\alpha_+} d\alpha \sinh^2(\alpha) \tag{3.140}$$

since in the coincidence limit the transversal coordinates are kept constant.
Regarding the volume element we have

$$d\tilde{V} = \tilde{\tau}^3 \sinh^2 \alpha d\tilde{\tau} d\alpha d\Omega \tag{3.141}$$

which additionaly must be integrated between $\tilde{\tau}_-$ and $\tilde{\tau}_+$ with $\tilde{\tau}_- < \tilde{\tau}_\Sigma < \tilde{\tau}_+$ giving

$$\tilde{V}(x,\alpha_-,\alpha_+,\tilde{\tau}_-,\tilde{\tau}_+) = \pi\left(\tilde{\tau}_+^4 - \tilde{\tau}_-^4\right) \int_{\alpha_-}^{\alpha_+} d\alpha \sinh^2 \alpha \tag{3.142}$$

and vanishes in the coincidence limit since $\tilde{\tau}_+ \to L_0$ and $\tilde{\tau}_- \to L_0$.

Space-like separations

The spacelike case is pretty similar to the timelike one. We have the induced metric

$$d\tilde{s}^2_{|\Sigma} = \tilde{l}_\Sigma^2 \left(-d\alpha^2 + \cosh^2 \alpha d\Omega^2\right) \tag{3.143}$$

from which we can compute

$$d\tilde{\Sigma} = \tilde{l}_\Sigma^3 \cosh^2 \alpha d\alpha d\Omega \tag{3.144}$$

and

$$d\tilde{V} = \tilde{l}^3 \sinh^2 \alpha d\tilde{\tau} d\alpha d\Omega \tag{3.145}$$

which can be integrated to give

$$\tilde{\Sigma}(x,\alpha_-,\alpha_+) = 4\pi\tilde{l}_\Sigma^3 \int_{\alpha_-}^{\alpha_+} d\alpha \cosh^2(\alpha) \tag{3.146}$$

$$\tilde{V}(x,\alpha_-,\alpha_+,\tilde{l}_-,\tilde{l}_+) = \pi\left(\tilde{l}_+^4 - \tilde{l}_-^4\right) \int_{\alpha_-}^{\alpha_+} d\alpha \cosh^2 \alpha \tag{3.147}$$

whose coincidence limits are given by

$$\lim_{l_\Sigma \to 0} \tilde{\Sigma}(x,\alpha_-,\alpha_+) = 4\pi L_0^3 \int_{\alpha_-}^{\alpha_+} d\alpha \cosh^2(\alpha) \tag{3.148}$$

$$\lim_{l \to 0} \tilde{V}(x,\alpha_-,\alpha_+,\tilde{l}_-,\tilde{l}_+) = 0 \tag{3.149}$$

showing again a finite transversal area and a vanishing volume in the coincidence limit.

Null-like separation

In the case of a null separations the metric induced on the lightcone Γ is

$$d\tilde{s}^2_{|\Gamma} = \tilde{\lambda}^2 d\Omega^2 \tag{3.150}$$

which can be used to compute areas elements in the point $p \in \Gamma$

$$d\tilde{\Sigma}_\Gamma(p) = \tilde{\lambda}_p^2 d\Omega \tag{3.151}$$

and after an integration over the solid angle we would have

$$\tilde{\Sigma}_\Gamma(p) = 4\pi\tilde{\lambda}_p^2. \tag{3.152}$$

In the coincidence limit we must have $\tilde{\lambda}_p - \tilde{\lambda}_0 \to L_0$ and since we set $\tilde{\lambda}_0 = 0$ we end up with $\tilde{\lambda}_p \to L_0$ and

$$\lim_{p\to 0} \tilde{\Sigma}_\Gamma(p) = 4\pi L_0^2 \tag{3.153}$$

which gives us a 2-dimensional minimum cross-sectional area around a point.

3.4.5 Dimensional reduction

Both in the timelike and spacelike case we can do the same computation of the section 3.2.3 and we find that also a Minkowski space-time endowed with a minimum length is effectively two-dimensional at very short scale length. In fact both for spacelike and timelike volume elements we have in D-dimensions

$$\frac{\tilde{V}}{V} = \frac{(\epsilon S_L)^{D/2} - L_0^D}{(\epsilon\sigma^2)^{D/2}} \tag{3.154}$$

with $\epsilon = -1, +1$ respectively in the time-like and space-like case. We can compute

$$D_{eff} = D + \frac{d}{d\log\left(\sqrt{\epsilon\sigma^2}\right)}\left[\log\left(\frac{\tilde{V}(l)}{V(l)}\right)\right] \tag{3.155}$$

getting the same result for the euclidean case

$$D_{eff} = D\,(\epsilon\sigma)^2\,\frac{(\epsilon S_L)^{\frac{D-2}{2}}}{(\epsilon S_L)^{D/2} - L_0^D}\frac{dS_L}{d\sigma^2} \tag{3.156}$$

which gives the known result in the special case $D = 4$ and $S_L = \sigma^2 + \epsilon L_0^2$:

$$D_{eff} = 4\sigma^2\frac{\epsilon\sigma^2 + L_0^2}{(\epsilon\sigma^2 + L_0^2)^2 - L_0^4} = 4\frac{\epsilon\sigma^2 + L_0^2}{\epsilon\sigma^2 + 2L_0^2} = 4\frac{1 + \left(\sqrt{\epsilon\sigma^2}/L_0\right)^2}{2 + \left(\sqrt{\epsilon\sigma^2}/L_0\right)^2}. \tag{3.157}$$

showing that for $L_0 \to 0$ we have $D_{eff} = 4$ whilefor $\sigma^2 \to 0$ we have $D_{eff} = 2$.

Chapter 4

Horizons area variation

We can apply the quantum metric construction to the following problem: given a generic null surface in the spacetime, what happen to the area of such a surface when a quantity of mass/energy crosses it? In principle we can think of this surface as a generic null surface in a generic spacetime. We know in flat spacetime a generic null surfaces can act as a horizon for a particular class of observers, *Rindler observers*, hiding regions of spacetime to them. Similar circumstances are obtained if we consider a small-enough patch of black hole horizon and an observer just hovering outside it. We can then allow matter/energy to fall in it. In particular we would like to determine if there is a minimum area variation for the black hole horizon according to the qmetric construction. This would be a signature of a quantum nature of gravity and might be probed in gravitational wave (GW) physics, since area quantization can leave imprints on GW signals from a pair of merging black holes possibly detectable in next generation (3G) of gravitational wave detectors [43].

4.1 Is Black Hole Horizon area quantized?

The first proposal of a quantization of the black hole horizon area is due to Bekenstein and Mukhanov[44, 45]. They argued that the area A of the event horizon of a black hole is quantized in units of the Planck area L_P^2:

$$A = \alpha L_P^2 N \tag{4.1}$$

where N is an integer and α is a dimensionless coefficient. Following the Sommerfeld's quantization rules of the very early quantum mechanics based on adiabatic invariants, this first suggestion was motivated by the observation that the area of black holes horizon acts as an adiabatic invariant [46]. However there is still no general agreement on the value of the proportionality constant. Very early Bekenstein suggested

$$\Delta A_{min} = 8\pi L_P^2 \tag{4.2}$$

while after, based on statistical physics consideration, Bekenstein and Mukhanov proposed

$$\Delta A_{min} = 4 \ln k L_P^2 \tag{4.3}$$

where k is an integer > 1. They argued for a value $k = 2$ while Hod [46] proposed a value of $k = 3$. Maggiore's argument based on quasinormal modes lead to the original proposal $\alpha = 8\pi$. Moreover Area Quantization is expected in Loop Quantum Gravity (LQG) [47] in which the minimum area variation is given by

$$\Delta A_{min} = 4\pi\sqrt{3}\gamma L_P^2 \tag{4.4}$$

where γ is still a free parameter, the so called Barbero-Imirzi parameter, whose value is not yet accurately fixed. We can see, although there are several arguments in favor of horizon area quantization, there is no general agreement about the value of the quantum of area.

4.2 qmetric description

In section (3.4.4) we showed that the cross-sectional (D-2)-area of a null geodesic congruence emanating from a base point x' is non vanishing in the coincidence limit:

$$A_0 = 4\pi L_0^2. \tag{4.5}$$

We now ask if the qmetric description allows us to talk about discrete variation of a null horizon area. In this we take an operational standpoint, that is we consider spacetime as a collection of coincidence events between physical particles. The variation of area we consider is thus that connected to the crossing of the horizon (with some photons going along its null generators) by a particle. We consider a lump of matter/energy falling along a geodesic towards the centre of a Schwarzschild black hole of mass M. We can express the spacetime metric in Schwarzschild coordinates $\{t, r, \theta, \phi\}$ as [17]:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{4.6}$$

where $2GM \equiv R_H$ is the radial coordinate position of the event horizon. We can consider a small enough patch of the horizon such that we can approximate it as a flat null surface and we consider some matter/energy crossing it (see fig. 4.1). We describe

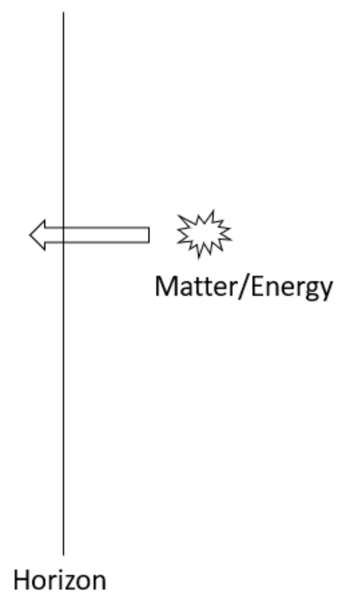
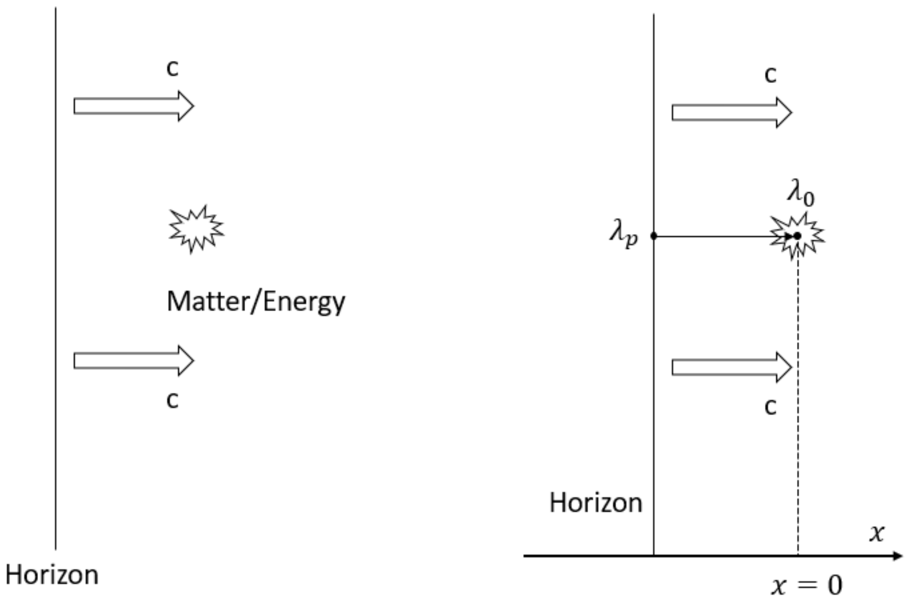


Figure 4.1: Matter/energy crossing the horizon.

this process in the matter/energy rest frame. Considering a small enough portion of spacetime, at leading order we can neglect curvature effects (which are second order in the displacements) and we are allowed to use locally the Minkowski metric. In this frame the lump of matter /energy is at rest while, considering the horizon as a null



(a) Matter rest frame

(b) Null geodesic linking the horizon to the matter/energy lump

Figure 4.2: Matter frame

surface made of radially outgoing photons, the horizon is moving towards the infalling matter at the speed of light as depicted in fig 4.2a. Each photon of the horizon is moving along outgoing null geodesic. In order to construct the qmetric description we need to fix a spacetime event which will be the base point P_0 for our description. We fix the base point to be the event *"The horizon reaches the lump of matter/energy"* and we can choose some coordinates in the matter frame such that P_0 corresponds to the origin of the frame. We also choose these coordinates such that the horizon is moving in only one direction. We denote such coordinates as $\{t, x, y, z\}$ and we let the horizon moving towards increasing values of x. The base point is thus given by:

$$P_0 = (0, 0, 0, 0) \tag{4.7}$$

We now consider the unique null geodesic segment linking the lump of matter with the horizon which is orthogonal to the horizon itself. We parameterize this null geodesic with an affine parameter λ which in the matter rest frame acquires the meaning of a spatial/time distance from the base point P_0 as shown in section 3.3.3 and fig. 4.2b. In particular in this situation the null vector tangent to the null geodesic is given by

$$l^a = \frac{dx^a}{d\lambda} = (1, 1, 0, 0) \tag{4.8}$$

and we have that the points on the null geodesic segment are parametrized by $\lambda = t = x$ for $\lambda < 0$. In such a construction the base point is individuated by $\lambda_{P_0} \equiv \lambda_0 = 0$. We identify the field point p as the event: *"The horizon is in the position $x = \lambda_p$ at the time $t = \lambda_p$ (with $\lambda_p < 0$)"* thus:

$$p = (\lambda_p, \lambda_p, 0, 0) \text{ with } \lambda_p < 0 \tag{4.9}$$

The more the horizon is approaching the matter lump, the more λ_p approaches $\lambda_0 = 0$ and we can see that the coincidence limit $p \to P_0$ is given by letting $\lambda_p \to 0$. After that

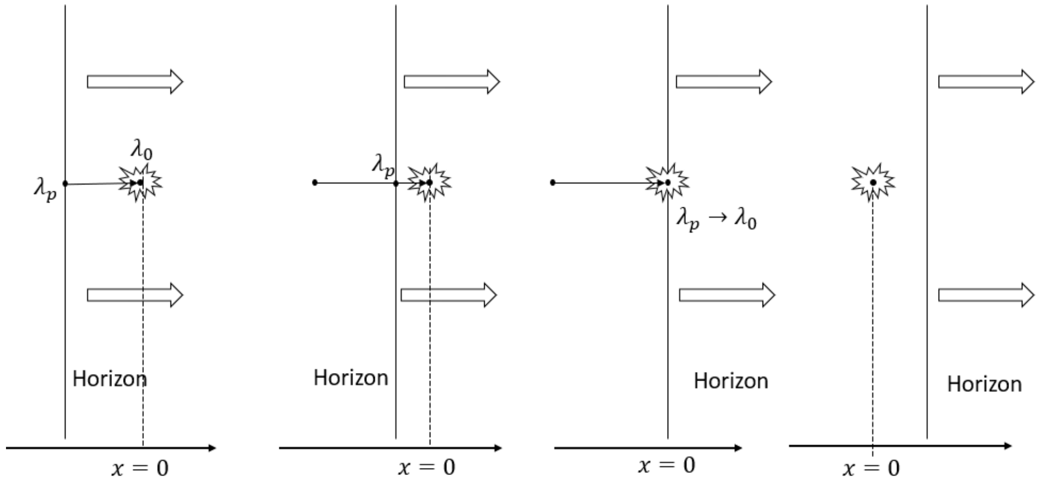


Figure 4.3: Classicl description in the matter frame.

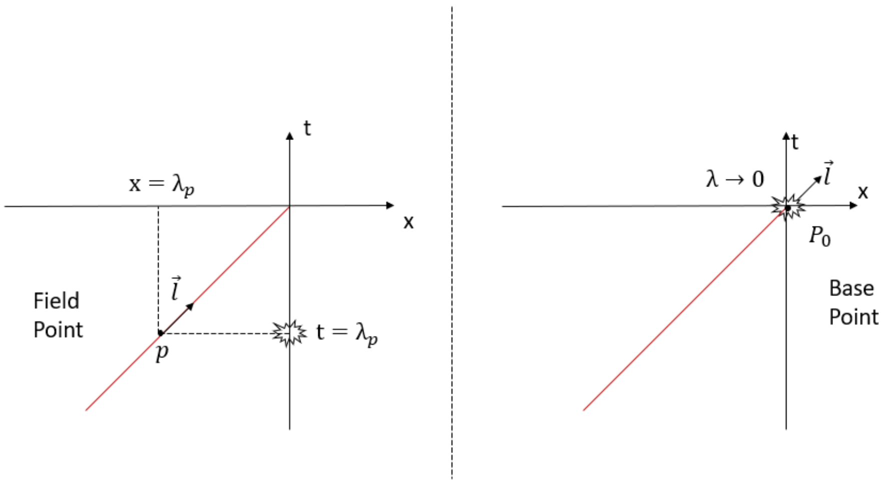


Figure 4.4: Spacetime diagram in t-x plane.

the horizon passes through the matter (the matter has just fallen inside) as shown in fig.(4.3) and fig.(4.4).

How the introduction of the qmetric changes this description? In order to understand that we need to consider the congruence of null geodesics that are pointing towards the base point P_0 and we consider the $(D-2)-$surface made by points at the same affine distance from P_0, namely $\Delta\lambda = \lambda_p - \lambda_0 = \lambda_p$ as in fig. 4.5. Basically in $D = 4$ we have a spherical surface around P_0 of radius λ_p whose area is given by

$$A(\lambda_p) = 4\pi\lambda_p^2 \tag{4.10}$$

As the horizon approaches the origin this spherical surfaces shrinks until $\lambda_p \to \lambda_0 = 0$ and we have at crossing $A_0 = 0$. Using the qmetric the description is different. With the introduction of the null qmetric we have the modification $\lambda \to \tilde{\lambda}$ such that as $\tilde{\lambda}_p \to \tilde{\lambda}_0$ we have $\left|\tilde{\lambda}_p - \tilde{\lambda}_0\right| = |\lambda_p| = L_0$ (we use the absolute value since we are considering negative values for λ_p).
As long as $\lambda_p >> L_0$ we approximately have $\lambda_p \simeq \tilde{\lambda}_p$. The more we approach the coincidence limit the more λ_p and $\tilde{\lambda}_p$ are different as discussed in section 2.3. Thus at coincidence we have $|\tilde{\lambda}_p| \to L_0$. In this way around the base point P_0 we have an

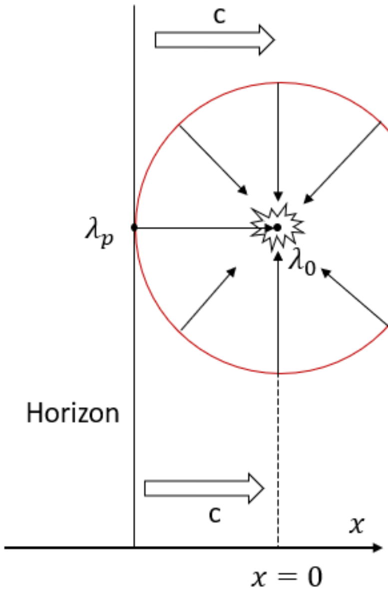


Figure 4.5: In red the 2-surface at the same affine distance from P_0.

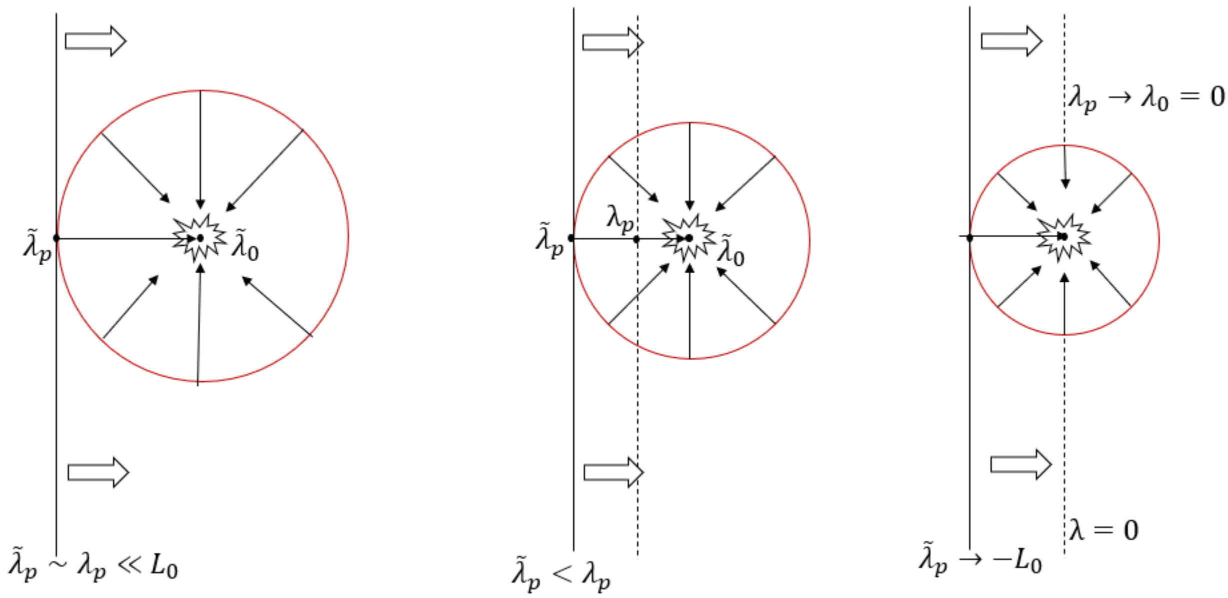


Figure 4.6: qmetric description. In dashed line the classical position of horizon, in solid line the position according to the qmetric.

irreducible transversal area coming out from the coincidence limit procedure and its value is given by $\tilde{A}_0 = 4\pi L_0^2$ as shown in section 3.4.4.

Therefore we are now able to assign two notions of area around the event P_0. In fact we have the area element given by an integration on the horizon patch that we are considering around the crossing point and the area element arising from the coincidence limit. Both the areas are irreducible: the Horizon area can only grows by means of the second laws of Black Hole (thermo-)dynamics (ignoring evaporation due to Hawking radiation) and the area A_0 on the equi-affine distant surface is irreducible according to the qmetric model. Thus this two areas must be added together leading to a minimal horizon area incrementation of:

$$\Delta A_{min} = 4\pi L_0^2 \tag{4.11}$$

showing that an effective description of a minimal length spacetime brings with it a minimum step of area variation.
At coincidence we can consider the black hole in a perturbed state: the black hole geometry is perturbed by the presence of the additional area associated to the crossing event. Therefore in order to satisfies the no-hair theorem the black hole must to relax to the unperturbed state with a spherical horizon of final area $A' \geq A + A_{min}$ if the area increment associated to the amount of the crossing matter/energy is enough to satisfies the bound. If this is not the case then the black hole must settles down to the initial state of horizon area A and the matter/energy can't be absorbed and possibly it is scattered by the horizon.

4.3 Consistency of the result

We can check the consistency of our result with a heuristic derivation coming from fundamental physical principles in addition with the requirement of a minimum length. We know General Relativity predicts the fact that a stationary observer hovering outside the horizon can't see anything passing through the horizon, due to time dilation and gravitational redshift. In the near horizon approximation this situation is equivalent to the one of an accelerating observer in Minkowski spacetime: the black hole horizon is substituted now by a Rindler Horizon. We can study what happens to a patch of this horizon when some matter/energy crosses it in the inertial frame. We can do this by means of *horizons mechanical laws* [48, 49].
Let's consider a uniformly accelerated observer with constant magnitude of acceleration k in Minkowski spacetime who is instantaneously co-moving with the inertial frame $\{t, x, y, z\}$ at the instant $(t = 0, x = x_k = 1/k, 0, 0)$. The null surface H defined by $x - t = 0$ acts as a future horizon for the accelerated observer. This surface is generated by the null vector l^a defined as:

$$l^a = \frac{dx^a}{dv} \tag{4.12}$$

where v is the null coordinate defined by $v = (x+t)/2$ which acts as an affine parameter for the vector l^a:

$$l^a \nabla_a l^b = 0. \tag{4.13}$$

The vector l^a is both tangent and normal to the horizon H. Now we consider a finite patch of the horizon Σ_0 at $v = 0$ and we parallel transport it along l^a direction to $v = \infty$. If there's no matter/energy crossing the horizon the area of this finite portion of horizon remains constant. In fact such an area can be thought of as the cross sectional area of a collimated light beam which is fired from the origin of the inertial frame along the x-direction. Since the beam will travel freely towards $v = \infty$ its expansion will always be vanishing as depicted in fig.4.7. Things are different if at some time, say at $v = v_i$, some energy/matter begins to cross the Horizon. We indicate with v_f the time when the crossing is complete. In this case the gravitational field of the energy/matter bends the light resulting in a focusing of the light rays of the light beam. In this case the cross sectional area at infinity would certainly be smaller than the initial value. Moreover in this case such light beam would not be a piece of the Rindler Horizon anymore and we can't use it to study the variation of the horizon. In order to have a constant asympotically area we need to start with a initially de-focusing light beam with a fine-tuned initially expansion such that after the matter crossing the light beam will be

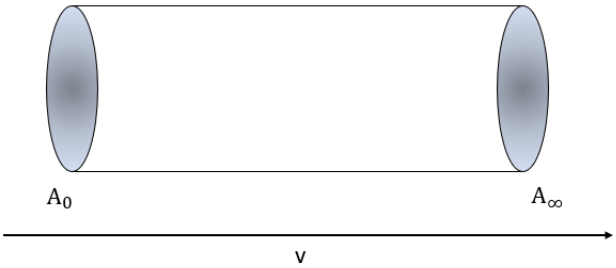


Figure 4.7: constant area.

collimated and it will be a patch of the new Rindler horizon. In other words, as shown in fig. 4.8 we need to consider a light beam that: for $0 < v < v_i$ it is expanding, for $v_i < v < v_f$ it is focused by the energy/matter crossing and for $v > v_f$ it is collimated and its area is constant.

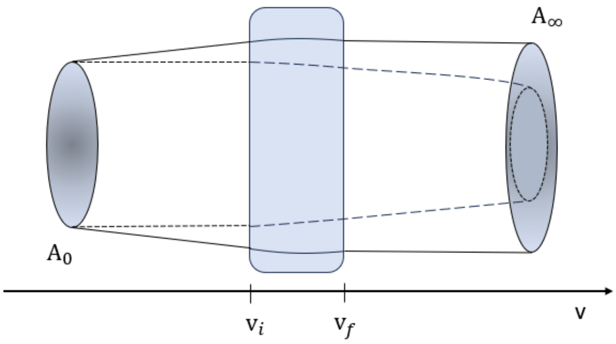


Figure 4.8: increasing area.

These are exactly the same circumstances happening for a small-enough Black Hole horizon patch: when a lump of matter/energy crosses the horizon the photons in the neighbourhood of the crossing point are focused. The new horizon patch then is obtained by light rays starting just outside the horizon with a fine tuned positive expansion such that they are collimated after the matter crossing. By equivalence principle we can study the area variation for a Rindler horizon patch in the Minkowski spacetime: this would be equivalent to the circumstances we are interested in, namely the area variation of a small enough black hole horizon patch.
We can quantitatively study the difference in the cross section area induced by the gravitational perturbation given by the crossing object. Let be T_{ab} the energy momentum tensor of the crossing energy/matter, with boundary condition $T_{ab} = 0$ as $v \to \infty$. We can relate the expansion of the light beam to the energy momentum tensor by means of the Raychaudhuri equation. In fact we have [17]:

$$\frac{d\theta}{dv} = -\frac{1}{2}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}l^a l^b \tag{4.14}$$

where θ is the beam expansion, σ_{ab} is the shear tensor, ω_{ab} is the twist and R_{ab} is the Ricci tensor. Since the beam is tangent to l^a which is surface orthogonal the twist is vanishing due to Frobenius theorem[17, 18]. We assume the contributions of θ^2 and σ_{ab}^2 are negligible. Using Einstein fields equation contracted with null vectors we have:

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G T_{ab} \to R_{ab}l^a l^b = 8\pi G T_{ab}l^a l^b \tag{4.15}$$

hence

$$\frac{d\theta}{dv} = -8\pi G T_{ab} l^a l^b \tag{4.16}$$

The expansion gives us a measure of the fractional rate of change of the cross sectional area along a null congruence [17]:

$$\theta = \frac{1}{\delta A} \frac{d\delta A}{dv} \tag{4.17}$$

Using the fact that at infinity the expansion of our light beam must be vanishing we can write

$$\theta(v) = -\int_v^\infty dv' \frac{d\theta(v')}{dv'} = 8\pi G \int_v^\infty dv' T_{ab} l^a l^b \tag{4.18}$$

At this point we can write

$$\frac{1}{\delta A} \frac{d\delta A}{dv} = 8\pi G \int_v^\infty dv' T_{ab} l^a l^b \tag{4.19}$$

and integrating with respect to dv we get

$$\delta A(v) - \delta A_0 = 8\pi G \delta A(v) \int_0^v dv'' \int_{v''}^\infty dv' T_{ab} l^a l^b \tag{4.20}$$

where A_0 is the area of the initial cross section portion Σ_0 at time $v = 0$ and $A(v)$ is the area of the evolved cross section $\Sigma(v)$ at time v. We can approximate at first order in G in the right hand side of the equation by replacing $\delta A(v)$ with $\delta A_0 = dxdy$. Integrating over the finite cross sectional portion Σ we have:

$$A(v) - A_0 = 8\pi G \int_\Sigma dydz \int_0^v dv'' \int_{v''}^\infty dv' T_{ab} l^a l^b \tag{4.21}$$

We are interested in the difference $\Delta A = A_\infty - A_0$ where $A_\infty = A(v \to \infty)$ is the area of the asympotical cross sectional finite portion Σ_∞. We can notice that in absence of a gravitational perturbation $T_{ab} = 0$ the area of Σ stays constant as expected. In the presence of a non vanishing energy momentum tensor we have

$$\Delta A = 8\pi G \int_\Sigma dydz \int_0^\infty dv'' \int_{v''}^\infty dv' T_{ab}(v') l^a l^b \tag{4.22}$$

Exchanging the integrals $\int_0^\infty dv'' \int_{v''}^\infty dv' = \int_0^\infty dv' \int_0^{v'} dv''$ we get

$$\Delta A = 8\pi G \int_\Sigma dydz \int_0^\infty dv v T_{ab}(v, y, z) l^a l^b \tag{4.23}$$

The integrals in the RHS turns out to have a key physical meaning. In fact let be ξ^a the vector tangent to the orbit of the accelerated observer. This is also the Killing vector associated to the Lorentz boosts. The combination $J_a = T_{ab}\xi^b$ is the conserved energy momentum current measured by the accelerated observer. Integrating such current over the surface Σ we get the total energy flux crossing the Rindler Horizon:

$$\Delta Q = \int_\Sigma J_a d\Sigma^a = \int_\Sigma dydz \int_0^\infty dv T_{ab} l^a \xi^b \tag{4.24}$$

Asympotically the vector ξ^b assumes the form $\xi^b = kvl^b$ where k is the observer acceleration and we can write:

$$\Delta Q = \int_\Sigma dydz \int_0^\infty dv T_{ab} l^a l^b vk \tag{4.25}$$

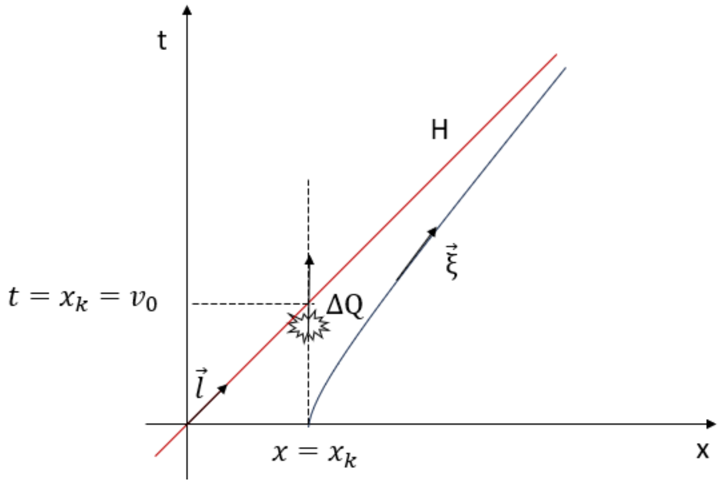


Figure 4.9: Object of energy ΔQ crossing the Rindler horizon

and we recognize its presence in eq.(4.23) and we finally have

$$\Delta A = 8\pi G \frac{\Delta Q}{k} = 8\pi G \Delta Q x_k. \tag{4.26}$$

We now consider that a physical object of energy ΔQ is dropped by the accelerated observer when she is instantaneously at rest in the inertial frame, thus at $t = 0$ and $x = x_k$ as shown in fig.4.9. This object would cross the horizon at a time $v_0 = x_k$. There is no reason why the area variation of eq.(4.26) should be bounded from below from a classical point of view. However we can ask what happens if we introduce an uncertainty Δx in the knowledge of the position of the object at time $t = 0$. We reasonably assume $\Delta x < x_k$ since we want to be sure at $t = 0$ the object is not beyond the horizon. As shown in figure 4.10 this uncertainty is reflected in an uncertainty in the crossing time Δt which can be assumed to be $\Delta t < x_k$ as well. We can now evoke Heisenberg uncertainty relation involving time and energy uncertainties:

$$\Delta E \Delta t \geq \frac{\hbar}{2} \tag{4.27}$$

hence

$$\Delta E \geq \frac{\hbar}{2\Delta t} \geq \frac{\hbar}{2x_k} \tag{4.28}$$

Inserting this relation in eq.(4.26) assuming that $\Delta Q \geq \Delta E$ we find

$$\Delta A \geq 8\pi G \frac{\hbar}{2x_k} x_k = 4\pi G\hbar = 4\pi L_P^2 \tag{4.29}$$

and identifying $L_0 = L_P$ we reproduce the result of the previous section.
The heuristic derivation of the discrete variation for the area of the horizon is independent from the mass/energy of the object crossing the horizon. However energy conservation implies that the crossing object must have an energy δm which is greater than the threshold value to which would correspond to the minimal allowed area variation. In fact let be M the mass of a black hole absorbing some energy δm. The initial Schwarzschild radius is given by $R_H = 2GM$ and the area of the event horizon is $A = 4\pi R_H^2 = 16\pi G^2 M^2$. The final area A' after the absorption must be greater than:

$$A' \geq A + \Delta A_{min} \tag{4.30}$$

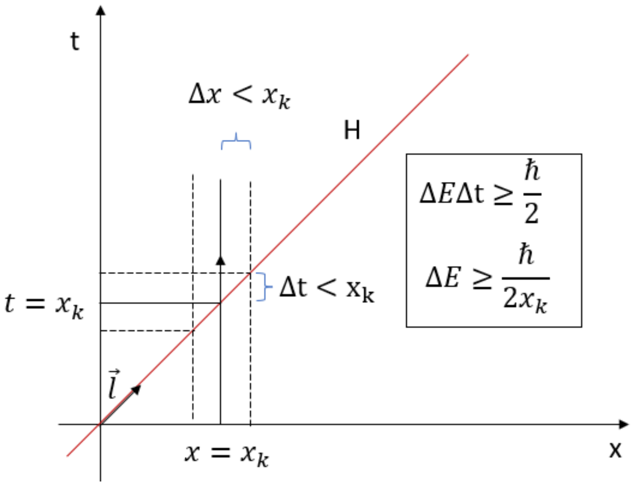


Figure 4.10: Introduction of uncertainty Δx is reflected in an uncertainty Δt

After the absorption the final mass of the black hole is $M' = M + \delta m$ to which corresponds a Schwarzschild radius of $R'_H = 2GM' = 2G(M+\delta m)$ and a horizon area of $A' = 4\pi R'^2_H = 16\pi G^2(M+\delta m)^2$. Therefore we can write:

$$16\pi G^2(M+\delta m)^2 \geq 16\pi G^2 M^2 + 4\pi L_0^2 \tag{4.31}$$

$$16\pi G^2 M^2 + 32\pi G^2 M\delta m + 16\pi G^2 \delta m^2 \geq 16\pi G^2 M^2 + 4\pi L_0^2 \tag{4.32}$$

$$\delta m^2 + 2M\delta m - \frac{L_0^2}{4G^2} \geq 0 \tag{4.33}$$

from which we have

$$\delta m \geq -M + \sqrt{M^2 + \frac{L_0^2}{4G^2}} \tag{4.34}$$

that in the approximation $M >> L_0^2/4G^2$ becomes

$$\delta m \geq \frac{L_0^2}{8G^2 M} \tag{4.35}$$

If we identify $L_0 = L_P = \sqrt{G\hbar}$ we can write:

$$\delta m \geq \frac{M_P^2}{8M} \equiv m_0 \tag{4.36}$$

where m_0 is the minimum energy that can be absorbed by a black hole of mass M. Moreover the (reduced) Compton length associated to m_0 is given by

$$\lambda_0 = \frac{\hbar}{m_0} = \frac{8M\hbar}{M_P^2} = 8MG \tag{4.37}$$

We can see the ratio between λ_0 and the Schwarzschild radius of the black hole is general and independent from the mass M:

$$\frac{\lambda_0}{R_H} = 4 \tag{4.38}$$

highlighting the fact than in the small mass regime we have the Compton length λ_c greater than the Schwarzschild radius, possibly explaining why such tiny masses can't be absorbed. Moreover this would also explain why in the heuristic derivation we don't see

the mass threshold unless invoking the conservation of energy: in the small mass regime, having $\lambda_c > R_H$ we are outside the validity of our approximation since eq.(4.28) implies $\lambda_c < x_k < R_H$ in the near horizon approximation.
It would be instructive to compute the mass threshold for a black hole of a solar mass. Being $M \sim 10^{30} kg$ and $M_P \sim 10^{-8} kg$ we find:

$$m_0 = \frac{M_P^2}{8M} \sim 10^{-46}\text{kg} \sim 10^{-11}\text{eV} \tag{4.39}$$

We can also ask what mass a black hole must have in order to admit a minimum absorbable mass of $m_0 = 1\text{eV} \sim 10^{-36}\text{kg}$:

$$M = \frac{M_P^2}{8m_0} \sim 10^{35} \times 10^{-16}\text{kg} = 10^{19}\text{kg} \tag{4.40}$$

Chapter 5

Conclusions and outlook

In this thesis we described the qmetric or minimum-length metric which is proposed as a way to implement the existence of a limiting length in the metric description of spacetime, the latter being a fundamental prediction of the union of General Relativity with the fundamental tenets of Quantum Mechanics, introducing an unavoidable non-locality in the description of the spacetime at small scales.
The qmetric is constructed in terms of tensorial quantities involving two spacetime events, namely bitensors, among which the squared geodesic distance plays a key role. Using bitensors seems to be the natural, or perhaps even obligatory, path to follow in order to take into account the non-locality of spacetime in the regime at which we can not neglect quantum gravity effects.

After an introduction to bitensors in chapter 1, we reviewed the construction of the qmetric for generic spacetimes in chapter 2. We also showed the important results regarding the effective Ricci scalar computed by means of qmetric which reproduces the entropy functional of the emergent gravity paradigm in section 2.5. In chapter 3 we explicitly constructed the qmetric for Euclidean space and for Minkowski spacetime with particular attention to the Area and Volume elements. Indeed in the qmetric description of a D-dimensional Minkowski/Euclidean space we showed that in the coincidence limit the D-dimensional volume computed on the equigeodesic surface vanishes while the $(D-1)$ area element associated to the cross section of timelike/spacelike geodesics and the $(D-2)$ area associated to the cross section of a null congruence remain finite.

In chapter 4 we used the results of chapter 3 to investigate what happens if we try to construct a qmetric description with respect to an event on a black hole horizon in the approximation of considering a small-enough patch of the horizon, allowing us to use the Minkowski metric. The qmetric predicts the existence of a discrete step in the variation of the black hole horizon area and we can actually compute it in terms of the minimum length L_0. Considering a small-enough neighbourhood of spacetime around a free falling object we showed that we can give two notions of area around a point on the event horizon. The first notion is the one associated to the point as being part of the horizon: we know that this area cannot decrease, this corresponding to the second law of black hole mechanics. The second notion arises when we perform the coincidence limit procedure using the qmetric when the given object hits the horizon: in this case there is an irreducible area appearing in the orthogonal direction due to the presence of the minimum length L_0. At the spacetime event "the free falling object crosses the horizon" it is assigned an irreducible area, the limit area coming from the qmetric description, which is added to the area of the horizon patch in which the event is located. Therefore horizon area can not grow less than the quantity:

$$\Delta A_{min} = 4\pi L_0^2 \tag{5.1}$$

This would have an intriguing consequence: only masses, or energy quanta, greater than a certain threshold (the one that would correspond to the minimum area variation) could cross the horizon and being absorbed by the black hole; the others would undergo a scattering process. This could have interesting implications in the expected signals of gravitational waves coming from merging black holes.

As a check, we also tried a back-on-the-envelope, heuristic understanding of these results in terms of first-principle physics without resorting to the full machinery of the qmetric. For a Schwarzschild black hole we considered observers hovering, static in the metric, just outside the horizon. We know we can treat them as Rindler observers in Minkowski spacetime and we studied the area variation of a patch of Rindler horizon by means of horizons mechanical laws. We found that imposing uncertainty relations we have a minimum area increment given by:

$$\Delta A_{min} = 4\pi L_P^2 \tag{5.2}$$

If we identify $L_0 = L_P$ this leads to $\Delta A_{min} = 4\pi L_0^2$ enforcing the qmetric result.

The heuristic derivation attests the qmetric prediction of a minimum area variation. It can be interesting to study the horizon mechanical law entirely within a full qmetric description starting from the modified Raychaudhury equation [10].

In this thesis our focus has been the study of the qmetric in Minkowski spacetime. The qmetric description clearly can be used in any general spacetime without any sort of symmetry. A natural continuation of the present study might be however to consider it first in the other maximally-symmetric spacetimes, i.e. general spacetimes for which the two point function depends on two events only through the quadratic interval between them, namely de Sitter and anti-de Sitter spacetimes. A further step might be the consideration of FLRW spacetimes; there are hints already of a resolution of the cosmological singularity (and of singularities in general) in the qmetric [10].

As mentioned, the existence of a minimal horizon area variation can have impact on gravitational wave signals. In particular this can be studied in the context of black hole quasi normal modes, whose signal would display some delayed echoes in case some infalling energy could be diffused by the horizon instead of being absorbed. .

Another tempting development can be the construction of a quantum field theory on a qmetric background and the study of scattering amplitudes according to the qmetric description.

Appendix A

Tetrads

We introduce an orthonormal basis, *tetrad*, of the tangent space in the point $z \in \beta$, i.e. on the geodesic segment linking x to x', $e^a{}_\alpha(z)$ with both indeces a, α running from 0 to $D-1$:

$$e^a{}_\alpha e^b{}_\beta g_{ab} = \eta_{\alpha\beta} \tag{A.1}$$

where $\eta_{ab} = diag(-1, 1, 1, \ldots, 1)$ is the Minkowski metric which it is used to raise and lower greek indeces. We also assume that the tetrad is parallely transported along the geodesic:

$$t^b \nabla_b e^a{}_\alpha = 0 \tag{A.2}$$

We have the completeness relation:

$$g^{ab} = \eta^{\alpha\beta} e^a{}_\alpha e^b{}_\beta \tag{A.3}$$

We define the dual tetrad as:

$$e^\alpha{}_a = \eta^{\alpha\beta} g_{ab} e^b{}_\beta \tag{A.4}$$

and the completenss relation takes the form:

$$g_{ab} = \eta_{\alpha\beta} e^\alpha{}_a e^\beta{}_b \tag{A.5}$$

It's easy to check that

$$e^\alpha{}_a e^a{}_\beta = \delta^\alpha{}_\beta \tag{A.6}$$

$$e^\alpha{}_a e^b{}_\alpha = \delta^a{}_b \tag{A.7}$$

Appendix B

Expansion of bitensors near coincidence

We want to express a bitensor $B_{\alpha'\beta'}(x,x')$ near coincidence as an expansion in terms of $\Omega^{\alpha'}$:

$$B_{a'b'}(x,x') = A_{a'b'} + A_{a'b'c''}\Omega^{c'} + \frac{1}{2}A_{a'b'c'd'}\Omega^{c'}\Omega^{d'} + \dots \tag{B.1}$$

where all $A_{\dots}$ are in this case tensorial functions in x'. They can be determined via coincidence limits[14]:

$$\begin{aligned} A_{a'b'} &= [B_{a'b'}] \\ A_{a'b'c'} &= [B_{a'b';c'}] - A_{a'b';c'} \\ A_{a'b'c'd'} &= [B_{a'b';c';d'}] - A_{a'b';c';d'} - A_{a'b'c';d'} - A_{a'b'cd'';c'} \end{aligned}$$

For example we have:

$$\begin{aligned} \Omega_{a'b'} &= g_{a'b'} - \frac{1}{3}R_{a'c'b'd'}\Omega^{c'}\Omega^{d'} + \dots \\ \Omega_{ab'} &= \Pi^{a'}{}_{a}\left(g_{a'b'} - \frac{1}{6}R_{a'c'b'd'}\Omega^{c'}\Omega^{d'}\right) + \dots \\ \Omega_{a'b} &= -\Pi^{b'}{}_{b}\left(g_{a'b'} + \frac{1}{6}R_{a'c'b'd'}\Omega^{c'}\Omega^{d'}\right) + \dots \\ \Omega_{ab} &= \Pi^{a'}{}_{a}\Pi^{b'}{}_{b}\left(g_{a'b'} - \frac{1}{3}R_{a'c'b'd'}\Omega^{c'}\Omega^{d'}\right) + \dots \end{aligned} \tag{B.2}$$

where $\Pi^{a'}{}_{a}$ is the parallel propagator defined in section 1.2.

Appendix C

Evaluation of qmetric d'Alembertian

C.1 General case

Let's see the computations to obtain eq.(2.12) from the definition

$$\tilde{\Box} = \frac{1}{\sqrt{-q}}\partial_a(\sqrt{-q}q^{ab}\partial_b) \tag{C.1}$$

Let be:

$$\sqrt{-q} = \frac{A^{\frac{D-1}{2}}}{\sqrt{\alpha}}\sqrt{-g} \equiv \xi\sqrt{-g} \tag{C.2}$$

Inserting the qmetric form given by eq.(2.8):

$$\tilde{\Box} = \frac{1}{\xi\sqrt{-g}}\partial_a\left[\xi\sqrt{-g}A^{-1}g^{ab}\partial_b\right] + \frac{\epsilon}{\xi\sqrt{-g}}\partial_a\left[\xi\sqrt{-g}Qu^a u^b\partial_b\right] \equiv \tilde{\Box}_1 + \tilde{\Box}_2 \tag{C.3}$$

For the first term we have:

$$\begin{aligned}\tilde{\Box}_1 &= \frac{1}{\sqrt{-g}}\partial_a\left[\sqrt{-g}A^{-1}g^{ab}\partial_b\right] + \frac{\partial_a\xi}{\xi}A^{-1}g^{ab}\partial_b = \\ &= A^{-1}\frac{1}{\sqrt{-g}}\partial_a\left[\sqrt{-g}g^{ab}\partial_b\right] + g^{ab}\partial_a A^{-1}\partial_b + \frac{\partial_a\xi}{\xi}A^{-1}g^{ab}\partial_b\end{aligned} \tag{C.4}$$

The first two terms can be rewritten as:

$$A^{-1}\frac{1}{\sqrt{-g}}\partial_a\left[\sqrt{-g}g^{ab}\partial_b\right] = A^{-1}\Box_g \tag{C.5}$$

$$g^{ab}\partial_a\left(A^{-1}\right)\partial_b = -g^{ab}A^{-2}\partial_a(A)\partial_b = -A^{-1}g^{ab}\partial_a(\ln A)\partial_b \tag{C.6}$$

For the third term we need to evaluate:

$$\begin{aligned}\frac{1}{\xi}\partial_a\xi &= \frac{D-1}{2}A^{-1}\partial_a A - \frac{1}{2\alpha}\frac{\partial\alpha}{\partial A}\partial_a A = \\ &= \frac{D-1}{2}\partial_a \ln A - \frac{1}{2}\partial_a \ln\alpha\end{aligned} \tag{C.7}$$

Putting all together we get:

$$\tilde{\Box}_1 = A^{-1}\left[\Box_g + \frac{D-3}{2}g^{ab}\partial_a(\ln A)\partial_b\right] - \frac{1}{2}A^{-1}g^{ab}\partial_a(\ln\alpha)\partial_b \tag{C.8}$$

Now we can compute $\tilde{\Box}_2$:

$$\tilde{\Box}_2 = \frac{\epsilon}{\xi\sqrt{-g}}\partial_a\left[\xi\sqrt{-g}Qu^a u^b\partial_b\right] \tag{C.9}$$

Using $u^b\partial_b \equiv \not\partial$:

$$\tilde{\Box}_2 = \frac{\epsilon}{\sqrt{-g}}\partial_a\left[Q\sqrt{-g}u^a\not\partial\right] + \epsilon\frac{\partial_a\xi}{\xi}Qu^a\not\partial \tag{C.10}$$

The first term reads as:

$$\frac{\epsilon}{\sqrt{-g}}\partial_a\left[Q\sqrt{-g}u^a\not\partial\right] = \epsilon\partial_a Qu^a\not\partial + \epsilon Q\partial_a u^a\not\partial + \epsilon Qu^a\frac{1}{\sqrt{-g}}\partial_a\sqrt{-g}\not\partial + \epsilon Q\not\partial^2 \tag{C.11}$$

Knowing that:

$$\frac{1}{\sqrt{-g}}\partial_a\sqrt{-g} = \Gamma^b_{ba} \tag{C.12}$$

we can write

$$\frac{\epsilon}{\sqrt{-g}}\partial_a\left[Q\sqrt{-g}u^a\not\partial\right] = \epsilon Q\left[\nabla_a u^a\not\partial + \not\partial^2\right] + \epsilon\not\partial Q\not\partial \tag{C.13}$$

For the second term in (C.10) we have:

$$\begin{aligned}\epsilon\frac{\partial_a\xi}{\xi}Qu^a\not\partial &= \epsilon Q\not\partial A\frac{D-1}{2}A^{-1}\not\partial - \frac{1}{2\alpha}\epsilon Q\not\partial\alpha\not\partial = \\ &= \epsilon Q\frac{D-1}{2}\not\partial\ln A\not\partial - \frac{1}{2}\epsilon Q\not\partial\ln\alpha\not\partial\end{aligned} \tag{C.14}$$

Thus:

$$\tilde{\Box}_2 = \epsilon Q\left[\left(\nabla_a u^a + \frac{d-1}{2}\not\partial\ln A\right)\not\partial + \not\partial^2\right] + \epsilon\not\partial Q\not\partial - \frac{1}{2}\epsilon Q\not\partial\ln\alpha\not\partial \tag{C.15}$$

Recalling that $\alpha = A^{-1} + Q$ for the last two terms:

$$\epsilon\not\partial Q\not\partial = \epsilon\not\partial(\alpha - A^{-1})\not\partial = \epsilon\not\partial\alpha\not\partial + \epsilon A^{-1}\not\partial\ln A\not\partial \tag{C.16}$$

$$\frac{1}{2\alpha}\epsilon Q\not\partial\alpha\not\partial = \frac{\epsilon}{2}\not\partial\alpha\not\partial - \frac{\epsilon A^{-1}}{2}\not\partial\ln\alpha\not\partial \tag{C.17}$$

So we get:

$$\tilde{\Box}_2 = \epsilon Q\left[\left(\nabla_a u^a + \frac{D-1}{2}\not\partial\ln A\right)\not\partial + \not\partial^2\right] + \frac{\epsilon}{2}\not\partial\alpha\not\partial + \frac{\epsilon A^{-1}}{2}\not\partial\ln\alpha\not\partial + \epsilon A^{-1}\not\partial\ln A\not\partial \tag{C.18}$$

Using the following identities (recalling 1.24):

$$\partial_a\alpha = \frac{\partial\alpha}{\partial\sigma^2}\frac{\partial\sigma^2}{\partial x^a} = \frac{\partial\alpha}{\partial\sigma^2}2\sqrt{\epsilon\sigma^2}u_a \tag{C.19}$$

$$\not\partial\alpha = \alpha' 2\epsilon\sqrt{\epsilon\sigma^2} \tag{C.20}$$

we clearly have

$$\frac{\epsilon}{2}\not\partial\alpha\not\partial = \alpha'\sqrt{\epsilon\sigma^2}\not\partial \tag{C.21}$$

Putting $\tilde{\Box}_1$ and $\tilde{\Box}_2$ together we get:

$$
\begin{aligned}
\tilde{\Box} = A^{-1}\left[\Box_g + \frac{D-3}{2} g^{ab}\partial_a\left(\ln A\right)\partial_b + \epsilon \not{\partial}\left(\ln A\right)\not{\partial}\right] + \\
+ \epsilon Q\left[\left(\nabla_a u^a + \frac{D-1}{2}\not{\partial}\ln A\right)\not{\partial} + \not{\partial}^2\right] + \sqrt{\epsilon\sigma^2}\alpha'\not{\partial} + \\
+ \frac{\epsilon A^{-1}}{2}\not{\partial}\left(\ln\alpha\right)\not{\partial} - \frac{A^{-1}}{2} g^{ab}\partial_a\left(\ln\alpha\right)\partial_b
\end{aligned}
\tag{C.22}
$$

We find our result since the last two terms turn out to be equal. In fact:

$$
g^{ab}\partial_a(\ln\alpha)\partial_b = g^{ab}(\ln\alpha)' 2\sqrt{\epsilon\sigma^2}u_a\partial_b = (\ln\alpha)' 2\sqrt{\epsilon\sigma^2}\not{\partial} = \epsilon\not{\partial}(\ln\alpha)\not{\partial} \tag{C.23}
$$

as shown before.

C.2 Maximally symmetric spaces

We can find a simpler form for the qmetric d'Alembertian in D-dimensional maximally symmetric spaces where all quantities are functions only of σ^2.
Let's start from the form of the standard operator $\Box_g = \nabla_a\nabla^a$. We consider a scalar test function $G = G(\sigma^2)$ and we apply the $\Box_g$ operator on it :

$$
\begin{aligned}
\Box_g G = \nabla_a\nabla^a G = \nabla_a\partial^a G = \nabla_a\left(\partial^a\sigma^2\frac{d}{d\sigma^2}G\right) = \\
= \nabla_a\left(\partial^a\sigma^2\right)\frac{d}{d\sigma^2}G + \partial^a\sigma^2\partial_a\frac{d}{d\sigma^2}G = \\
= \left(2D + \partial^a\sigma^2\partial_a\sigma^2\frac{d}{d\sigma^2}\ln\Delta^{-1}\right)\frac{d}{d\sigma^2}G + 4\sigma^2\frac{d^2}{d(\sigma^2)^2}G = \\
= 4\sigma^2\left[\left(\frac{D}{2\sigma^2} + \frac{d}{d\sigma^2}\ln\Delta^{-1}\right)\frac{d}{d\sigma^2} + \frac{d^2}{d(\sigma^2)^2}\right]G
\end{aligned}
$$

where we used the fact that

$$
\nabla_a\partial^a\sigma^2 = 2D - \frac{1}{\Delta}\partial^a\sigma^2\nabla_a\Delta = 2D + 4\sigma^2\frac{d}{d\sigma^2}\ln\Delta^{-1} \tag{C.24}
$$

coming from the following identity proved in section 1.3.1

$$
\nabla_a\left[\Delta\partial^a\sigma^2\right] = 2D\Delta \tag{C.25}
$$

We can write for timelike/spacelike separations:

$$
\frac{D}{2\sigma^2} = \frac{d}{d\sigma^2}\ln\left(\epsilon\sigma^2\right)^{\frac{D}{2}} \tag{C.26}
$$

We finally get:

$$
\Box_g = 4\sigma^2\left(\partial^2_{\sigma^2} + \partial_{\sigma^2}\ln\left[\left(\epsilon\sigma^2\right)^{\frac{D}{2}}\Delta^{-1}\right]\partial_{\sigma^2}\right) \tag{C.27}
$$

where we used a shorthand notation for the derivatives.
Now we need to work out the qmetric terms. we find that:

$$
\sqrt{\epsilon\sigma^2}\alpha'\not{\partial} = 2\alpha\sigma^2\left[\ln\alpha\right]'\frac{d}{d\sigma^2}
$$

$$
\begin{aligned}
g^{ab}\partial_a \ln A\partial_b &= 4\sigma^2 \left[\ln A\right]' \frac{d}{d\sigma^2} \\
\epsilon\not{\partial} \ln A\not{\partial} &= 4\sigma^2 \left[\ln A\right]' \frac{d}{d\sigma^2} \\
\epsilon\not{\partial}^2 = \epsilon u^a\partial_a \left(u^b\partial_b\right) &= 4\sigma^2 \left(\frac{d^2}{d(\sigma^2)^2} + \frac{1}{2\sigma^2}\frac{d}{d\sigma^2}\right) = \\
&= 4\sigma^2 \left[\frac{d^2}{d(\sigma^2)^2} + \frac{d}{d\sigma^2}\left(\ln\left(\epsilon\sigma^2\right)^{\frac{1}{2}}\right)\frac{d}{d\sigma^2}\right] \\
\epsilon\nabla_a u^a\not{\partial} &= 4\sigma^2 \left[\frac{D-1}{2\sigma^2} + \frac{d}{d\sigma^2}\left(\ln\Delta^{-1}\right)\right]\frac{d}{d\sigma^2} = \\
&= 4\sigma^2 \frac{d}{d\sigma^2}\left[\ln\left(\left(\epsilon\sigma^2\right)^{\frac{D-1}{2}}\Delta^{-1}\right)\right]\frac{d}{d\sigma^2}
\end{aligned}
$$

We see that $\epsilon\not{\partial}^2 + \epsilon\nabla_a u^a\not{\partial} = \Box_g$. Inserting all this terms in eq.(2.12) we find

$$
\tilde{\Box} = \alpha\Box_g + 2\alpha\sigma^2 \left[\ln\alpha A^{D-1}\right]' \frac{\partial}{\partial\sigma^2} \tag{C.28}
$$

which using the standard d'Alembertian form in maximally symmetric spaces found before becomes

$$
\tilde{\Box} = 4\alpha\sigma^2 \left[\partial^2_{\sigma^2} + \partial_{\sigma^2} \ln\left[\left(\epsilon\sigma^2\right)^{\frac{D}{2}}\Delta^{-1}\sqrt{\alpha}A^{\frac{D-1}{2}}\right]\partial_{\sigma^2}\right] \tag{C.29}
$$

Appendix D

Relations between geometrical quantities belonging to disformally coupled metrics

Consider two metrics $g_{ab}, \tilde{g}_{ab}$ on a given D dimensional manifold. We say that they are disformally coupled if they are related in the following way[29]:

$$\tilde{g}_{ab} = A g_{ab} - \epsilon B u_a u_b \tag{D.1}$$

where u^a is of the form eq(1.24) and $u_a = g_{ab} u^b$. In [29] are derived in details a number of relations between geometrical quantities belonging to disformally coupled metrics which are important in the derivation of quantity associated to the qmetric. In particular:

$$\tilde{h}_{ab} = A h_{ab} \text{ induced metric on orthogonal surfaces } \Sigma \text{ to } u^\alpha \tag{D.2}$$

$$\tilde{R}_\Sigma = A^{-1} R_\Sigma \text{ Induced Ricci scalar on } \Sigma \tag{D.3}$$

$$\tilde{K}_{ab} = A\sqrt{\alpha}\left[K_{ab} + (u^k \nabla_k \ln \sqrt{A}) h_{ab}\right] \tag{D.4}$$

$$\tilde{K} = \sqrt{\alpha}\left[K + (D-1) u^a \nabla_a \ln \sqrt{A}\right] \tag{D.5}$$

Using Gauss-Codazzi equations we can reconstruct $\tilde{R}$ [29]:

$$\tilde{R} = \tilde{R}_\Sigma - \epsilon\left(\tilde{K}^2 + \tilde{K}_{ab}\tilde{K}^{ab}\right) - 2\epsilon U^a \tilde{\nabla}_a \tilde{K} + 2\epsilon \tilde{\nabla}_a \tilde{a}^a \tag{D.6}$$

where $U^a = \sqrt{\alpha} u^a$ and $\tilde{a}^a = U^b \tilde{\nabla}_b U^a$. After some algebra one get the form given in eq.(2.81).

Appendix E

Useful Taylor expansions

Here are reported taylor expansions of geometrical quantity needed in the computation of the coincidence limit of the Ricci biscalar. Consider two points x, x' linked by a timelike/spacelike geodesic parametrized by $\lambda = \sqrt{\epsilon\sigma^2}$ whose tangent vector is:

$$u_\alpha = \frac{\nabla_\alpha \sigma^2}{2\sqrt{\epsilon\sigma^2}} \tag{E.1}$$

where σ^2 is the squared geodesic distance. The extrinsic curvature of the surface orthogonal to u_a is given by:

$$K_{ab} = \nabla_a u_b = \frac{\nabla_a \nabla_b(\sigma^2/2) - \epsilon u_a u_b}{\sqrt{\epsilon\sigma^2}} \tag{E.2}$$

The key expansion is the following [28]:

$$\nabla_a \nabla_b(\sigma^2/2) = g_{ab} - \frac{\lambda^2}{3}\xi_{ab} + \frac{\lambda^3}{12}u^k \nabla_k \xi_{ab} - \frac{\lambda^4}{60}\left(u^k u^l \nabla_k \nabla_l \xi_{ab} + \frac{4}{3}\xi_{ca}\xi^c{}_b\right) + o(\lambda^5) \tag{E.3}$$

where $\xi_{ab} = R_{akbl}u^k u^l$. From this expansion it follows that in D dimensions[28]:

$$\Delta = 1 + \frac{1}{6}\lambda^2 R_{ab}u^a u^b + o(\lambda^3) \tag{E.4}$$

$$K_{ab} = \frac{1}{\lambda}h_{ab} - \frac{1}{3}\lambda\xi_{ab} + \frac{1}{12}\lambda^2 u^k \nabla_k \xi_{ab} - \frac{1}{60}\lambda^3 F_{ab} + o(\lambda^4) \tag{E.5}$$

$$K = \frac{D-1}{\lambda} - \frac{1}{3}\lambda\xi + \frac{1}{12}\lambda^2 u^k \nabla_k \xi - \frac{1}{60}\lambda^3 F + o(\lambda^4) \tag{E.6}$$

$$R_\Sigma = \frac{\epsilon(D-1)(D-2)}{\lambda^2} + R - \frac{2\epsilon(D+1)}{3}\xi + o(\lambda) \tag{E.7}$$

where $F_{ab} = u^k u^l \nabla_k \nabla_l \xi_{ab} + \frac{4}{3}\xi_{ac}\xi^c{}_b$. From the expansions above we can construct the following expression:

$$\begin{aligned} K_{ab}^2 - \eta K^2 = (1-\eta(D-1))&\left[\frac{D-1}{\lambda^2} - \frac{2}{3}\xi + \frac{1}{6}\lambda u^k \nabla_k \xi - \frac{1}{30}\lambda^2\left(u^k u^l \nabla_k \nabla_l \xi - \frac{4}{3}\xi_{ab}^2\right)\right] \\ &+\frac{1}{9}\lambda^2\left(\xi_{ab}^2 - \eta\xi^2\right) + o(\lambda^3) \end{aligned}$$

We notice that in the limit $\lambda \to 0$ the above quantity is zero only if $\eta = \frac{1}{D-1}$. This is indeed the case when the coincidence limit of the Ricci biscalar is taken providing a non divergent result.

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