Alma Mater Studiorum · University of Bologna

School of Science
Department of Physics and Astronomy
Master Degree in Physics

Quintessence, fifth forces and string theory

Supervisor:
Prof. Francisco Manuel Soares Verissimo Gil Pedro

Submitted by:
Elisa Medda

Academic Year 2022/2023

For the love of physics

Quintessence, 5th forces and String Theory

Elisa Medda

Abstract

This thesis focuses on the study of the dynamical systems of late time cosmology including scalar field driven dark energy models. Light gravitationally coupled scalars are known to mediate fifth forces that lead to deviations of Newton's gravitational inverse square law. These bounds severely constrain dynamical dark energy models. In this thesis we explore a recently proposed mechanism, easily embedded in the moduli sector of string compactifications, that exploits field space curvature to evade fifth force bounds. This mechanism can be realised with an axion and its dilaton in the presence of a small axion-matter coupling. We perform a systematic study of the dynamics of systems and of their ultraviolet embedding into string theory.

Contents

Introduction

One of the main challenging problems in today physics is the cosmological constant problem.

The Standard Model (SM) is nowadays the most powerful theory that explains the behaviour of particles and is in good agreement with most of the experimental results. Nevertheless, it lacks in many aspects, requiring new physics to be developed. Indeed, the SM is a theory which does not incorporate all the four fundamental forces, this is due to the fact that gravity works at very different energies in respect to weak, strong and electromagnetic forces, making the theory harder to incorporate and leading us to the necessity of developing a new theory. It was the 1915-1916 when Einstein completed his theory of General Relativity (GR), assuming that our universe was static. The belief of a static universe was rooted on the observation that the velocities of the stars were much smaller than the velocity of the light. However, he thought that his original theory didn't present a static solution, which brought Einstein to add a new parameter Λ, the cosmological constant, in the equations, but it was later discovered by Friedman that the Einstein field equations indeed provided the possibility of a static universe as solution, leading to the discard of this additional term. Nowadays, there are several experimental measurements that confirm that we live in an accelerated universe, e.g. the first evidence for a cosmological constant from type Ia supernova observations [1] or the more recent Planck survey [2] which proved again that ordinary matter is only around 5% of the total content of the universe, another 25% is given by Dark Matter and an astounding 70% is instead **Dark Energy**.

However a profound discrepancy arise when looking at what the theory *naturally* predicts and what observations instead show us: this is the aforementioned **cosmological constant problem**. We know that the cosmological constant Λ is defined by the energy density of the vacuum which can be estimated to be $\Lambda_P \sim (10^{18}\ \text{GeV})^4$, while from cosmological observation we have that $\Lambda_{exp} \leq (10^{-12}\ \text{GeV})^4$. Thus, $\Lambda_P/\Lambda_{exp} \sim 10^{120}$, which implies an huge gap which can be explained only if one allows for a fine tuned value of the bare constant in the QFT.

In the past decades several attempts have been developed in order to solve the cosmological constant problem. For example, by the introduction of Supersymmetry (SUSY), where at each SM particle is associated a super-partner of opposite spin statistics, we have that the extra contributions help mitigate the discrepancy bringing $\Lambda_P/\Lambda_{exp} \sim 10^{60}$ which still requires a fine tuning, where Λ_P is the SUSY breaking scale which we are assuming at the TeV scale. In this thesis we will focus on a different approach to solve the cosmological constant problem, which is called Quintessence. This model is based on the idea that Λ is considered as a dynamical field varying in time.

Hence, in this work we start, in chapter 1, with a review of the Standard Model, paying attention on its problems. In detail, we will argue about the absence of gravity, one of the four fundamental forces, and the consequences of that. In the end of the chapter we will recall one of the most fascinating mysteries of physics, which is the cosmological constant problem, since in this thesis we want to talk about one of the possible explanations, using the quintessence model. Then, in chapter 2 we introduce String Theory, describing the main concept and the bosonic string. Thus, we add the fermions and we discuss about Superstring Theory, where we implement the theory of supersymmetry with the one of the strings. In detail, we are going to focus on Type IIB String Theory. Therefore, we will discuss about compatification in extra dimensions, and using the Kaluza-Klein reduction we will be able to discuss about a particular form of a potential that will be useful for the development of the next chapters. In chapter 3 we are going to argue the main body of what then later will be analysed in more details. In particular, we are going to describe the evolution of the universe as a first order dynamical system, starting from a theory with only one scalar field to a more complete theory with two scalar fields. Finally, in chapter 4 we are going to face the model that was used for the main results of this thesis. The model is the Relaxed Model, from a recent work in [3], where we have a scalar field, a supergravity sector coupled with the SM and a scale invariance, from which we construct our EFT. In the final part of this chapter we will explain the scalar tensor theory and the Brans-Dicke model.

Lastly, in chapter 5, we explore the Brans-Dicke model but with an extra scalar field and an axion source. Here we repeat the same kind of study of chapter 3, with particular attention to the study of the critical points and their stability. In detail, we are going to analyze two different type of potentials, one which is only an exponential and than second one, known as the relaxed model described in chapter 4, which is given by an exponential times a polynomial term, which allows for a local minimum granting new possible cosmological evolutions.

Some Conventions

Special Relativity

- metric signature: $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$.
- Minkowski spacetime $\mathbb{R}^{1,3}$.
- 4-vectors $x^\mu = (t, \vec{x})$, where $x^0 = t$.

Constant of Nature

- $c = \hbar = e = 1$, i.e. all the units are expressed as kg $\sim$ m^{-1}
- Reduced Planck mass, $M_P^2 = 1/(8\pi G) = (2.4 \times 10^{18}\ \text{GeV})^2$

Chapter 1

Standard Model and Cosmology

The Standard Model (SM) is a theory which explains the interaction between three fundamental forces (weak, strong and electromagnetic) and classifies all the elementary particles that we know today, showing a very high precision and agreement with experimental data. With respect to the other three forces, gravity is working at very different scales of energy, which make it hard to find an unified quantum theory for all the four forces. Moreover, another lack of the Standard Model is a natural explanations of Dark Energy.

In this chapter we will briefly discuss about the SM and some of its limit, following [4–8], and for the rest of this chapter we will focus on the Cosmological Constant problem.

1.1 The Standard Model and its problems

The Standard Model is a 4D spacetime QFT which describes the most general renormalisable field theory constituted by:

- *Matter particles*, where we have quarks

$$\begin{pmatrix} u^i \\ d^i \end{pmatrix}_L , \quad u^i_R , \quad d^i_R , \qquad i = 1, 2, 3 ,$$

  with $u^i = (u, c, t)$ and $d^i = (d, s, b)$, and leptons

$$\begin{pmatrix} \nu^i \\ e^i \end{pmatrix}_L , \quad \nu^i_R , \quad e^i_R ,$$

  with $\nu^i = (\nu_e, \nu_\mu, \nu_\tau)$ and $e^i = (e^-, \mu^-, \tau^-)$. They all have spin $\hbar/2$, where the subscript L/R denotes the left/right handed components, and it is important to note that quarks feel the strong interaction and carry colours while leptons don't.

- *Interaction particles*

$$\underbrace{SU(3)_C}_{\text{Strong}} \times \underbrace{SU(2)_L \times U(1)_Y}_{\text{Electro-Weak}}$$

  where

  (a) $SU(3)_C$ is the **strong interaction**, with 8 gluons that are the mediators in the QCD theory. Here the subscript C means colour.

(b) $SU(2)_L \times U(1)_Y$ is the **electro-weak interaction** . The subscript L is due to the fact that the SM is chiral, i.e. doesn't preserve parity and differentiates between *left*- and *right*-handed particles, while Y means hypercharge.

- *The Higgs particle*, which is a scalar field with spin $s = 0$. Its potential has a Mexican hat shape, see Fig.1.1, which via the Higgs mechanism gives mass to all particles.

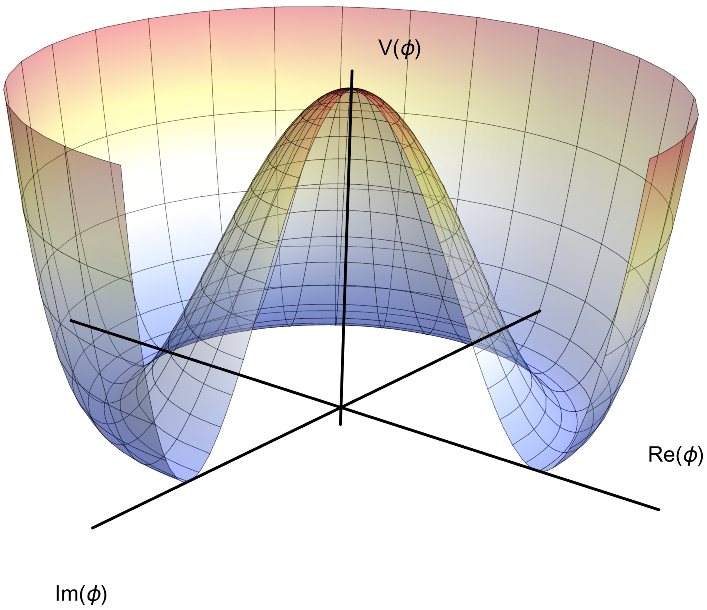


Figure 1.1: Mexican-hat potential in function of the field ϕ.

Moreover, the Higgs particle is responsible for the breaking of the SM

$$SU(2)_L \times U(1)_Y \to U(1)_{EM} \,,$$

since it gives masses to three of the four gauge bosons of $SU(2)_L \times U(1)_Y$ which combine into the $W^{\pm}$ and Z^0 bosons, while leaving one massless boson, which represents the photon of the $U(1)_{EM}$ group.

We present the full spectrum with corresponding representation in Tab.1.1.
The structure of the SM Lagrangian is

$$\mathcal{L}_{SM} = \mathcal{L}_{gauge} + \mathcal{L}_{matter} + \mathcal{L}_{Higgs} + \mathcal{L}_{Yukawa}, \tag{1.1}$$

where $\mathcal{L}_{gauge}$ has the standard form of

$$\mathcal{L}_{gauge} = -\sum_i \frac{1}{4g_i^2} F^{(i)\mu\nu} F^{(i)}_{\mu\nu} \qquad i \text{ runs over } U(1),\ SU(2) \text{ and } SU(3)\,, \tag{1.2}$$

with

$$F_{\mu\nu} = \mathrm{i}\left[D_\mu, D_\nu\right], \qquad \text{where} \quad D_\mu = \partial_\mu - \mathrm{i}A_\mu\,. \tag{1.3}$$

particle	**field**	**spin($\hbar$)**	$(SU(3)_C, SU(2)_L)_{U(1)_Y}$
quarks	$\begin{pmatrix} u^i \\ d^i \end{pmatrix}_L$	$\frac{1}{2}$	$(\mathbf{3},\mathbf{2})_{\frac{1}{6}}$
	u^i_R	$\frac{1}{2}$	$(\bar{\mathbf{3}},\mathbf{1})_{-\frac{2}{3}}$
	d^i_R	$\frac{1}{2}$	$(\bar{\mathbf{3}},\mathbf{1})_{\frac{1}{3}}$
leptons	$\begin{pmatrix} \nu^i \\ e^i \end{pmatrix}_L$	$\frac{1}{2}$	$(\bar{\mathbf{1}},\mathbf{2})_{-\frac{1}{2}}$
	ν^i_R	$\frac{1}{2}$	$(\bar{\mathbf{1}},\mathbf{1})_1$
	e^i_R	$\frac{1}{2}$	$(\bar{\mathbf{0}},\mathbf{1})_1$
Higgs	$\begin{pmatrix} \phi^- \\ \phi^0 \end{pmatrix}$	0	$(\mathbf{1},\mathbf{2})_{-\frac{1}{2}}$
gluons	G^a	1	$(\mathbf{8},\mathbf{1})_0$
W bosons	W^b	1	$(\mathbf{1},\mathbf{3})_0$
B boson	B	1	$(\mathbf{1},\mathbf{1})_0$

Table 1.1: Particles of the Standard Model.

The matter Lagrangian is instead given by the Dirac Lagrangian for fermions as

$$\mathcal{L}_{matter} = \mathrm{i}\sum_j \bar{\Psi}_j \not{D}_j\, \Psi_j\,, \qquad \text{with} \quad \Big(D_j\Big)_\mu = \partial_\mu - \mathrm{i}R_j(A_\mu), \tag{1.4}$$

where $R_j(A_\mu)$ is the representation of A_μ. For convention, instead of the Dirac spinors we are using 4-spinors that are left-handed

$$\Psi = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}, \qquad \alpha = 1, 2\,.$$

Moreover, it is important to define the Higgs Lagrangian term, which has an important role in the theory of SM

$$\mathcal{L}_{Higgs} = -\Big(D_\mu\phi\Big)^\dagger D^\mu\phi + m_\phi^2|\phi|^2 - \lambda_\phi|\phi|^4, \tag{1.5}$$

where the Mexican hat potential is

$$V(\phi) = -\frac{1}{2}m_H^2\phi^\dagger\phi + \frac{1}{4}\lambda_\phi(\phi^\dagger\phi)^2, \qquad \lambda_\phi > 0\,, \tag{1.6}$$

where λ_ϕ is a self interaction term. The potential has a minimum at $|\phi| = v \simeq 174$ GeV, that leads to a spontaneous gauge symmetry breaking (SSB).

Lastly, we can define the Yukawa Lagrangian

$$\mathcal{L}_{Yukawa} = -\sum_{j,k}\lambda_{jk}\bar{\Psi}_j\Psi^c_k\,\phi + \text{h.c.}\,, \tag{1.7}$$

where Ψ_k^c is the charge conjugated spinor and the Yukawa couplings λ_{ij} are responsible for the masses of the spinors.

1.1.1 Quantum Gravity

One of the main shortcomings of the SM is the absence of gravity, which is treated only from a classical point of view. In detail, SM describes three of the four fundamental forces at the quantum level, which means that gravity can only be treated as an effective field theory, only valid at scales smaller than the Planck scale (i.e. $M_P \sim 10^{19}$ GeV).

Now, let us take the SM Lagrangian, where $\mathcal{L}_{SM}$ is the Lagrangian for the matter fields Ψ, and adding in a minimal way gravity we have

$$\int d^4x\, \mathcal{L}_{SM}(\eta_{\mu\nu}, \Psi) \to \int d^4x \sqrt{-g}\left(\frac{M_P^2}{2}\mathcal{R}(g_{\mu\nu}) + \mathcal{L}_{SM}(g_{\mu\nu}, \Psi) - \Lambda\right), \tag{1.8}$$

where Λ is the cosmological constant, $\mathcal{R}$ is the Ricci scalar, M_P is the Planck constant and $g = \det g_{\mu\nu}$ is the determinant of the metric.

We can now develop this theory thinking at the SM at which we add another gauge theory.

Einstein's gravity is a theory of a dynamical metric, $g_{\mu\nu}(x)$, that in particular situation can be seen as a background metric $\langle g_{\mu\nu}\rangle$ plus a small perturbation $\delta g_{\mu\nu}$. Thus, expanding the metric $g_{\mu\nu}$ around flat space we obtain [9]

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}\,, \tag{1.9}$$

where $\delta g_{\mu\nu}$ is very small and it can be written in terms of a linear perturbation of the graviton $h_{\mu\nu}$, which is a massless spin-2 field

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_P}\,, \tag{1.10}$$

where $h_{\mu\nu}$ is a gauge potential analogous to A_μ.

Hence, we can rescale the covariant derivative of a vector field as

$$D_\mu v_\nu = \partial_\mu v_\nu - \Gamma^\rho_{\mu\nu} v_\rho \qquad \text{with} \qquad \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})\,, \tag{1.11}$$

and we can write the curvature tensor $\mathcal{R}$ symbolically as

$$\mathcal{R} \sim [\partial - \Gamma, \partial - \Gamma]\,. \tag{1.12}$$

Thus, rescaling $h \to h/M_P$ the Lagrangian takes the symbolic form

$$h\,\partial^2 h + \frac{h}{M_P}(\partial h)^2 + \left(\frac{h}{M_P}\right)^2 (\partial h)^2 + \ldots \quad . \tag{1.13}$$

Differently from the gauge theory structure in this case we have that the coupling has mass dimension -1, and without going into details, we have that the Fadeev-Popov procedure and the introduction of ghosts works as in gauge theory. Hence, substituting the expansion of the metric in equation (1.8), we obtain

$$\mathcal{L}_{kin} \subset \frac{1}{M_P} h_{\mu\nu} T^{\mu\nu}\,. \tag{1.14}$$

1.1.2 Hierarchy Problem

The Hierarchy Problem is at the basis of the research of new physics at LHC distances.

The central idea of this problem is to explain the totally different energy scales associated with the SM and with gravity

$$M_{ew} \approx 10^2 \text{ GeV}\,, \qquad M_P = \sqrt{\frac{Gh}{c^3}} \approx 10^{19} \text{ GeV}\,, \tag{1.15}$$

which means $M_{ew}/M_P \sim 10^{-15}$, i.e. the weakness of gravity when compared with the other interactions. Indeed, there are two sides to analyze of the hierarchy problem.

1. Why is $M_{ew} \ll M_P$ at tree level?

This first question bring us to the so called *hierarchy problem*, which can be solved with an extensions of the SM. Once solved this problem another natural question is:

2. Why is the hierarchy stable under quantum corrections?

Which is more complicated to solve and it is called *technical hierarchy problem.* In fact, one loop contributions to the Higgs mass μ are quadratically divergent [10]

$$\mu^2_{phys} = \mu^2_{tree} + (\delta\mu)^2\,, \tag{1.16}$$

where

$$(\delta\mu)^2 = \left(\frac{\Lambda}{16\pi}\right)^2 \Big[-6y_t^2 + \frac{1}{4}(9g^2 + 3g'^2) + 6\lambda_\phi \Big]\,, \tag{1.17}$$

where Λ is the cutoff of the theory, y_t is the top Yukawa coupling, g, g' are the $SU(2)$ and $U(1)$ gauge couplings and λ_ϕ is the Higgs self-coupling.

1.1.3 Cosmological Constant Problem

The cosmological constant problem is one of the main problems of the theory of the Standard Model [11–14]. The problem is based on the huge discrepancy between the measured value of the cosmological constant and the expected one.

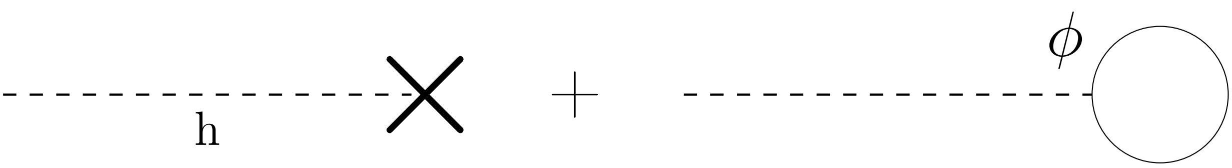
From QFT, due to locality and unitarity, we know that the vacuum has an energy which can be calculated by computing the vacuum loop diagrams for each particle species. Taking back the term (1.14), what is crucial is that the cosmological constant gives rise to a term of the energy momentum tensor as

$$T^{\mu\nu} = -\eta^{\mu\nu}\Lambda\,, \tag{1.18}$$

then, if Λ is different from zero we can have a tadpole term

$$\mathcal{L} \subset -\frac{1}{M_P} h_{\mu\nu}\eta^{\mu\nu}\Lambda\,, \tag{1.19}$$

which will give us a loop diagram, assuming gravity is coupled with a scalar field ϕ



that has the meaning of a correction to Λ as

$$\Lambda = \Lambda_0 + \frac{c_\lambda}{16\pi^2}\Lambda = \Lambda_0 + \delta\Lambda \tag{1.20}$$

with $\delta\Lambda \simeq 10^{120}(\mathrm{meV})^4$ which is in total contrast with the experimental value $\Lambda_{\mathrm{exp}} \simeq \mathrm{meV}^4$ and thus requires a fine tuning of the bare constant Λ_0.

1.2 Cosmology

At large scales our universe looks **isotropic**, which means that it is the same in all the direction, and **homogeneous**, i.e. the same everywhere [15–20], this bring us to the so called *Cosmological Principle*, which can be proven.

Suppose to take two intersected circles as in Fig.1.2, each of them, thanks to isotropy, has a density which is constant, i.e. $\rho_A = const.$, $\rho_B = const.$, thus also the intersected region is constant and we have $\rho_A = \rho_B$, which implies that we have homogeneity of the space.

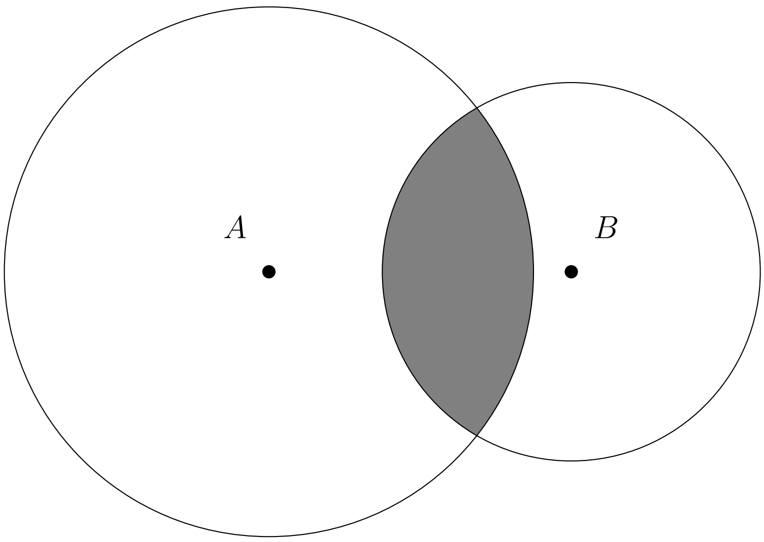


Figure 1.2: Representation of two intersected circles with an isotropic density to prove the Cosmological Principle.

Homogeneity is what allows us to have a cosmological time which is taken globally and not only locally. Thus, assuming that our universe is homogeneous and isotropic, the dynamics of our universe is described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\times \mathrm{d}l^2 = -\,\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2/R_0^2} + r^2\,\mathrm{d}\Omega^2\right], \tag{1.21}$$

where $a(t)$ is the scale factor, which describes the expansion of the universe, and dl^2 is the symmetric 3−space. In detail, $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$ and the coordinates r, θ and ϕ are called the *comoving* coordinates [21]. Moreover, for the value of k we can have

$$k = \begin{cases} 0 & \mathbb{E}^3 \\ +1 & \mathbb{S}^3 \\ -1 & \mathbb{H}^3 \end{cases},$$

respectively for flat, spherical and hyperbolic spatial slices with curvature radius R_0.

The differential equations for the scale factor and matter density comes from the Einstein's equation

$$G_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \,, \tag{1.22}$$

where G is the Newton constant, $G_{\mu\nu}$ is the Einstein tensor, $\mathcal{R}_{\mu\nu}$ is the Riemann curvature tensor, Λ is the cosmological constant, which can be interpreted as a vacuum energy and thus be part of ρ, the energy density, and lastly, $T_{\mu\nu}$ is the energy momentum tensor. Precisely, if we consider a perfect fluid we have that

$$T^{\mu}_{\nu} = \mathrm{diag}(-\rho, p, p, p), \tag{1.23}$$

where p is the pressure density and ρ is the total energy density of our universe, given respectively by radiation, neutrinos, baryons and cosmological constant as

$$\rho \equiv \rho_\gamma + \rho_\nu + \rho_b + \rho_\Lambda. \tag{1.24}$$

It is important to define also the Hubble parameter

$$H = \frac{\dot{a}}{a} \,, \tag{1.25}$$

where the dot means a derivative in respect to time, $\dot{a} = \frac{\mathrm{d}a}{\mathrm{d}t}$. Thus, from the Einstein's equation we can write the two independent equations as

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2 R_0^2} + \frac{\Lambda}{3} \,, \tag{1.26}$$

$$\dot{H} = -4\pi G(p+\rho) + \frac{k}{a^2 R_0^2} - \frac{\Lambda}{3} \,. \tag{1.27}$$

An important feature to note is that the space-time remains invariant if we rescale simultaneously a, r, R_0 in the following way

$$a \to \lambda\, a, \qquad r \to \lambda^{-1}\, r, \qquad R_0 \to \lambda^{-1}\, R_0 \,, \tag{1.28}$$

which allow us to set $a(t_0) \equiv 1$, where $t = t_0$ is today time, without loss of generality. Moreover, due to the Bianchi identities $\nabla_\mu T^{\mu\nu} = 0$, we obtain the continuity equation

$$\dot{\rho}_\gamma = -3H(\rho_\gamma + P_\gamma), \tag{1.29}$$

where γ is a constant in the range $0 \leq \gamma < 2$; thus if $\gamma = 4/3$ we have radiation and with $\gamma = 1$ we have dust. Furthermore, since we know that our universe is in a accelerated expansion, which means $\ddot{a} > 0$, we need to satisfy

$$\rho_\gamma + 3P_\gamma < 0. \tag{1.30}$$

Thus, we can classify sources by their equation of state

$$\omega \equiv \frac{P}{\rho} = \begin{cases} 0, & \text{matter} \\ \frac{1}{3}, & \text{radiation} \\ -1, & \text{dark energy} \end{cases} \,,$$

and we can write the densities scale as

$$\rho \propto a^{-3(1+\omega)}. \tag{1.31}$$

Now, considering a flat universe, i.e. with $k = 0$, we can define the critical density ρ_c

$$\rho_c = \frac{3H_0^2}{8\pi G} = 1.9 \times h^2 \times 10^{-29}\mathrm{g\,cm}^{-3}\,. \tag{1.32}$$

Thus, it is more convenient to use the following dimensionless density parameters

$$\Omega_{i,0} \equiv \frac{\rho_{i,0}}{\rho_{c,0}} \qquad \text{with } i = r, m, \Lambda... \tag{1.33}$$

where for simplicity we will just drop out the index 0 in Ω.

Thus, the Friedmann equation takes the form

$$\frac{H^2}{H_0^2} = \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda, \tag{1.34}$$

where $\Omega_k \equiv -kc^2/(R_0 H_0)^2$ is the curvature density parameter, with $\Omega_k < 0$ when $k > 0$.

Now, considering eq.(1.34) at present time, $a(t_0) \equiv 1$, we obtain

$$1 = \Omega_r + \Omega_m + \Omega_\Lambda + \Omega_k. \tag{1.35}$$

In order to determine the composition of the universe we need to measure the parameters occurring in the Friedmann equation.

1.3 ΛCDM model

We are now going to describe a model of the compositions of our universe, called the ΛCDM model [22–24]. Today we know that our universe is constituted by matter, radiation and dark energy, which in this model is assumed to be given by a cosmological constant. The two dominant components are given by

$$\Omega_\Lambda = 0.69 \qquad \text{and} \qquad \Omega_m = 0.31, \tag{1.36}$$

where the dark energy comprises almost the 70% of the total amount of energy of our universe. The evidence of the existence of Λ comes from information of the distribution of galaxies and from direct measurement of Type Ia supernovae [1]. In detail, from the CMB [2], we infer that our universe is spatially flat, which means that the total energy density must be critical, and so it must contain more than just regular and dark matter. Secondly, we deduced that our universe is constantly accelerating whose evidence come from calculating the distance-redshift of supernovae, since the expansion rate of the universe has changed over time also the distance-redshift relation has changed [1]. Moreover, a smaller contribution to the total amount of energy in the universe comes from photons and neutrinos

$$\Omega_\gamma \sim 5 \times 10^{-5} \qquad \Omega_\nu \sim 3.4 \times 10^{-5}\,, \tag{1.37}$$

and, since there are no evidence of the curvature of our universe, we have

$$|\Omega_k| < 0.01\,. \tag{1.38}$$

The collection of Ω_Λ, Ω_m, Ω_γ and Ω_ν is called the *Standard Cosmological Model* or ΛCDM model, where Λ stands for the cosmological constant and CDM denotes cold dark matter which we will not analyze here.

Despite the $\Lambda-$CDM is a very good model, according with the cosmology community, there are different difficulties that arise, see [25].

One of the main problem is connected with the fact that most of the vacuum associated with the quantum field theories predict an huge cosmological constant Λ which is of 100 orders larger than the expected one, which arise in a fine-tuning problem.

Another unsolved problem is related to the question of why the densities associated with dark energy and dark matter are approximately equal today.

1.4 Quintessence

As stated in the previous section, from the Cosmic Microwave Background (CMB) and other experimental evidence, we know that 70% of our universe is constituted by Dark Energy, which is characterized by the equation of state parameter

$$\omega = \frac{P}{\rho} < -\frac{1}{3}, \tag{1.39}$$

which means that it has negative pressure. One of the candidates for Dark Energy is the cosmological constant Λ with a value of ω equal to -1. However, despite the years today we are not yet able to find a way to explain why the vacuum energy associated with the particles of the Standard Model (SM) is far away from what we found for the vacuum related with Dark Energy [11, 12]. A possible explanation would be to consider that Λ is *not a constant* but a time varying value. A natural way to introduce a time dependency is given by the Quintessence model [26–29], i.e. one considers a scalar field ϕ with a potential $V(\phi)$, such that its dynamics drives the expansion of our universe today.

A simple example of $V(\phi)$ is given by

$$V(\phi) = M^{4+\alpha}\phi^{-\alpha} \qquad \text{with } \alpha > 0, \tag{1.40}$$

where M is a constant with mass unit equals to one. This type of potential takes the name of *runaway*, discussed in more detail in the context of String Theory.

In detail, the acceleration of the universe can be explained by this slowly varying scalar field along a potential, which is similar to the inflaton in inflationary cosmology [30, 31].

Thus, suppose to take a perfect barotropic fluid (i.e. the density is only a function of the pressure) we can write the action of the quintessence field ϕ in the presence of non-relativistic matter, as

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\left(\frac{1}{2}M_P^2\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) + S_m, \tag{1.41}$$

where S_m is the matter action, g is the determinant of the metric $g = \det g_{\mu\nu}$, M_P is the reduced mass of Planck and $\mathcal{R}$ is the Ricci scalar. Hence, we can write the pressure and the energy density of ϕ respectively as

$$P_\phi = \frac{\dot{\phi}^2}{2} - V(\phi)\,, \qquad \rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi)\,, \tag{1.42}$$

and consequently, the Dark Energy equation of state will be given by

$$\omega = \frac{P_\phi}{\rho_\phi} = \frac{\frac{\dot{\phi}^2}{2} - V(\phi)}{\frac{\dot{\phi}^2}{2} + V(\phi)}, \tag{1.43}$$

and ϕ satisfy the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0, \tag{1.44}$$

where $V_\phi \equiv \frac{\mathrm{d}V}{\mathrm{d}\phi}$. One of the main advantages of the quintessence model is that it provides several possible scenarios for the evolution of the universe, adding the possibility that we are now in a transient phase, that it is slowly evolving over enormous time scales. However, there are also several problems, such as **fine-tuning**, related with the fact that we need that the field has to be in the point where $V(\phi_0) \simeq \rho_0 \simeq (0.003\ \text{eV})^4$ today. Another problem is to match the **phenomenology constraints** given by the fact that quintessence field must be weakly coupled to ordinary matter, otherwise we should have some long-range forces, which has not been seen experimentally. [26]. Lastly, let us address the so called *"why now?"* problem. In fact, in order for the dark energy to dominated the energy density content of the universe one should have a very peculiar epoch in which radiation and matter energy densities are comparable with dark energy. There is an anthropic answer to this questions, which says that our universe happens to be in this optimal value because otherwise we would not be here to witness it. However, for many physicist this is not a satisfying explanation and thus there is a search for a natural trigger of this phenomenon, i.e. one could study the dynamical system of the cosmological evolution equation and explain our phase through the reaching of a fixed point, as we will analyse in Chap.5.

Chapter 2

Introduction to String Theory

String Theory is today one of the most promising theory that can explain the unification of all the forces in nature, including gravity, in an unique theory.

In detail, String Theory can be seen as an extension of the Standard Model, since it naturally include also non abelian gauge interactions, fundamental scalars and gravity. Moreover, String Theory allow us to have also others important results, such as a realization of holography in terms of AdS/CFT correspondence [32].

Unfortunately, as today, String Theory is a theory where experimental confirmations are hard to gather, since it probes really high energies and thus requires us to look at scenarios such as the early universe or limits in which gravity becomes important, e.g. in black holes. Furthermore, we are considering particles as excitations of 1D strings, which in turn requires extra dimension, that are hard to experimentally prove, due to the usual really small scales at which they become evident. Lastly, String Theory is still not well fully understood, lacking a complete formulation and a lot of work and effort is currently being put into developing the theory.

In the following discussion we will mainly follow the work done in [33–42].

2.1 What is a String

The basic concept of String Theory is to consider particle not as the fundamental building blocks but as excitation of a one dimensional extended object which is called *string*, as it is shown in Fig.2.1

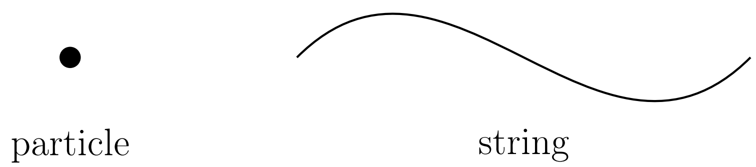


Figure 2.1: Schematic representation of a point-like particle and an open string.

A string can either be *open* or *closed* and different theories can be developed with this content.

When a string, which is a one dimensional object, evolves in time it generates a two dimensional surface Σ in spacetime, which is called *worldsheet*, analogously as the worldline for point particles. Open strings correspond to worldsheet with boundaries while closed strings without boundaries.

To describe a string we need two variables, one for the direction along the string σ and one to parameterize a time-like direction t. Thus, if we take into consideration a D-dimensional Minkowski space $\mathcal{M}^D$, we can write D functions $X^\mu(t,\sigma)$ as

$$(t,\sigma) \to x^\mu = X^\mu(t,\sigma)\,, \qquad \mu = 0, ..., D-1 \tag{2.1}$$

that can be represented through Fig.2.2.

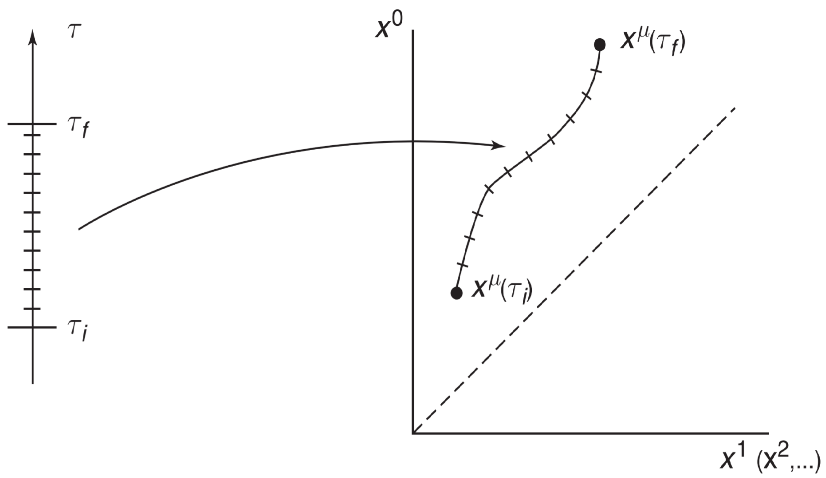


Figure 2.2: Systematic representation of the function $X^\mu(t,\sigma)$ that provide an embedding of a surface Σ into D-dimensional spacetime called *target space*.

Moreover, it is also important to introduce the generalization of worldsheet, the worldvolume, as the volume of spacetime occupied by an extended p-dimensional object, which are typical building blocks in String Theory constructions such as p-branes, see Fig.2.3.

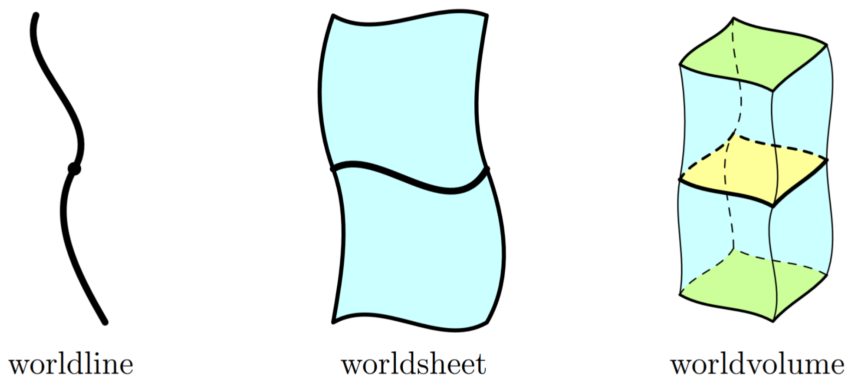


Figure 2.3: Representation of a worldline, a worldsheet and a worldvolume in comparison. Source [42].

2.1.1 Bosonic Strings

From now on we will focus on bosonic string theory, and later generalize to include supersymmetry on the worldsheet and thus super string theory.

We already know that the action of a relativistic point particle is given by

$$S = -m \int_{\alpha} \mathrm{d}s \,, \tag{2.2}$$

where α is the worldline and $\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}X^\mu\,\mathrm{d}X^\nu$ is the interval. By analogy, the dynamics of a string of length ℓ is defined by the action, which is a reparametrization-invariant expression for a surface embedded in $\mathcal{M}^D$, called the Nambu-Goto action, whose expression is

$$\begin{aligned} S_{NG}[X^\mu] &= -\frac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}^2\xi\,\sqrt{-h} \\ &= -\frac{1}{2\pi\alpha'}\int \mathrm{d}t \int \mathrm{d}\sigma \left[\left(\frac{\partial X^\mu}{\partial t}\frac{\partial X_\mu}{\partial \sigma}\right)^2 - \left(\frac{\partial X^\mu}{\partial t}\frac{\partial X_\mu}{\partial t}\right)\left(\frac{\partial X^\mu}{\partial \sigma}\frac{\partial X_\mu}{\partial \sigma}\right)\right]^{1/2}, \end{aligned} \tag{2.3}$$

where $\xi^\nu = (t, \sigma)$ are the compact coordinates, and h is the determinant of the metric on the worldsheet $h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu$, with α, β equals to t or σ, while α' is called the Regge Slope, which gives the string tension $T = \frac{1}{2\pi\alpha'}$, with dimension of Planck's constant divided by length squared. In addition, it is important to define the mass and the length of strings, given by

$$M_s = \frac{1}{\sqrt{\alpha'}}, \qquad l_s = 2\pi\sqrt{\alpha'}. \tag{2.4}$$

In order to easily quantise the action it is usually employed the Polyakov action

$$S_P = -\frac{T}{2}\int_\Sigma \mathrm{d}^2\xi\,\sqrt{-h}\,h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\,, \tag{2.5}$$

which is equivalent to the Nambu-Goto action.

A Quantisation of this system gives us an infinite set of decoupled harmonic oscillators which correspond to the oscillation modes of the string, which we will analyse in section 2.2.

2.1.2 Equations of motion and Symmetries

Now, it is more convenient to look at the worldsheet theory as a 2D QFT with metric given by $h_{\alpha\beta}$ and D free scalars X^μ. Thus, we can now write the action in the following way

$$S_P = -\frac{T}{2}\int \mathrm{d}^2\xi\,\sqrt{-h}\,(\partial X)^2\,, \tag{2.6}$$

where $(\partial X)^2 = h^{\alpha\beta}\,\partial_\alpha X^\mu \partial_\beta X^\nu\,\eta_{\mu\nu}$. Now, we are able to see that with the action in this form we have three main symmetries

1. **Diffeomorphism**: it transforms the 2d coordinates while leaving X^μ invariant

$$\xi^\alpha \to \xi'^\alpha(\xi) \tag{2.7}$$

$$X^\mu(\xi) \to X'^\mu(\xi') = X^\mu(\xi)\,, \tag{2.8}$$

$$g_{\alpha\beta}(\xi) \to g'_{\alpha\beta}(\xi') = \frac{\partial\xi^\gamma}{\partial\xi'^\alpha}\frac{\partial\xi^\delta}{\partial\xi'^\beta}\,g_{\gamma\delta}(\xi)\,. \tag{2.9}$$

2. ***D*-dimensional Poincaré invariance**: it transforms the X^μ field leaving the coordinates ξ^α invariant

$$X^\mu(\xi) \to X'^\mu = \Lambda^\mu{}_\nu X^\nu(\xi) + V^\nu \tag{2.10}$$
$$g_{\alpha\beta}(\xi) \to g'_{\alpha\beta}(\xi) = g_{\alpha\beta}(\xi)\,. \tag{2.11}$$

where $\Lambda \in SO(1, D-1)$.

3. **Weyl invariance under local rescaling**: only the 2d metric transforms as

$$g_{\alpha\beta}(\xi) \to g'_{\alpha\beta}(\xi) = e^{2\omega(\xi)}\, g_{\alpha\beta}(\xi)\,, \tag{2.12}$$

where ω is a function $\in \mathbb{R}$.

Moreover, we need to gauge-fix the local invariance to be able to quantize the theory, thus, we can use the reparametrization of two coordinates of ξ^α and Weyl invariance in order to remove the degrees of freedom of $g_{\alpha\beta}$ by imposing

$$g_{\alpha\beta} = \eta_{\alpha\beta}\,, \tag{2.13}$$

and the Polyakov action with this gauge is given by

$$S_P = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi\ \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\nu \eta_{\mu\nu}. \tag{2.14}$$

Furthermore, for simplicity it is more helpful to use the energy-momentum tensor

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\partial S}{\delta g_{\mu\nu}}\,, \tag{2.15}$$

and in detail, for an isotropic fluid we will have $T_{\mu\nu} = \mathrm{diag}(\rho, p, ..., p)$. Furthermore, it is convenient to use the following normalization when we are talking about the string on the worldsheet

$$T^{\alpha\beta} = \frac{-4\pi}{\sqrt{-h}}\frac{\delta S_P}{\delta h_{\alpha\beta}} = -\frac{1}{\alpha'}\left(G^{\alpha\beta} - \frac{1}{2}h^{\alpha\beta}h^{\gamma\delta}G_{\gamma\delta}\right)\,. \tag{2.16}$$

Thus, it easy to see that the equation of motion for $h_{\alpha\beta}$ is given by

$$T^{\alpha\beta} = 0\,. \tag{2.17}$$

The wave equation can be easily solved using the light cone coordinates $\xi^{L/R}$

$$\xi^{L,R} = t \pm \sigma, \qquad \partial_{L/R} = \frac{1}{2}(\partial_t \pm \partial_\sigma), \tag{2.18}$$

where $\xi^{L,R}$ is defined as the left/right-moving worldsheet coordinates. In this way we obtain the following worldsheet metric

$$\mathrm{d}s^{\,2} = -\,\mathrm{d}t^{\,2} + \mathrm{d}\sigma^{\,2} = -\,\mathrm{d}\xi^L\,\mathrm{d}\xi^R\ . \tag{2.19}$$

Hence, the equations of motion are given by the Virasoro constraints

$$\partial_L\partial_R X^\mu = 0, \qquad (\partial_{L/R}X)^2 = 0\,, \tag{2.20}$$

where the solutions of the first equation has the form

$$X^\mu(\xi^L, \xi^R) = X^\mu_L(\xi^L) + X^\mu_R(\xi^R)\,, \tag{2.21}$$

solved by a separation of variables, thus, from the second equation $(\partial X_{R,L})^2 = 0$ we remove one left and one right mover.

It is possible to see that we still have two reparametrisations:

1. **Conformal Transformations**

$$\xi^R \to \xi'^R(\xi^R) \qquad \xi^L \to \xi'^L(\xi^L). \tag{2.22}$$

Only on two dimensions we have conformal symmetries, since the reparametrisation (2.22) preserves the conformal structure of the two dimensional metric, i.e. the angles.

2. **Constant Shifts**, between X^μ_L and X^μ_R.

Hence, from this freedom we can fix two of the D total fields, first introducing the light-cone field coordinates

$$X^\pm = \frac{1}{\sqrt{2}}\left(X^0 \pm X^1\right) , \tag{2.23}$$

from which we gauge fix as

$$X^+ \equiv t\,, \tag{2.24}$$

while X^- is almost entirely defined by the remaining X^i fields, with $i = 2, ..., D_1$, from the Virasoro constraint (2.20) as

$$\partial_{L/R} X^-_{L/R} = \frac{1}{2}\left(\partial_{L/R} X^i_{L/R}\right)^2 , \tag{2.25}$$

leaving as the only degree of freedom the center of mass

$$x^-(t) = \frac{1}{\ell}\int_0^\ell \mathrm{d}\sigma\, X^-(\xi)\,, \tag{2.26}$$

and since it evolves linearly with constant momentum we obtain

$$p_- = -p^+ = \frac{\partial \mathcal{L}}{\partial\left(\partial_t\, x^-\right)} = -\frac{\ell}{2\pi\alpha'}\,, \tag{2.27}$$

where the minus sign arise from the spacetime metric in light-cone coordinates

$$\eta_{++} = \eta_{--} = 0 \quad \eta_{+-} = \eta_{-+} = -1\,, \quad \eta_{ij} = \delta_{ij}\,. \tag{2.28}$$

Thus, we find that the string length can be rewritten in terms of the momentum and of α' as

$$\ell = 2\pi\alpha' p^+\,. \tag{2.29}$$

2.1.3 Oscillator expansions

It is convenient to expand the functions $X^i_{L,R}$ in oscillator modes, imposing the following boundary conditions

$$X^\mu(t,\sigma) = X^\mu(t,\sigma+\ell), \tag{2.30}$$

which follow from periodicity of σ of a string of length ℓ. Thus, the corresponding expansion is

$$X^i_L(\xi^R) = \frac{x^i}{2} + \frac{p^i}{2p^+}\xi^R + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}-\{0\}}\frac{\alpha^i_n}{n}e^{-2\pi\mathrm{i}n\xi^R/\ell}\,, \tag{2.31}$$

$$X^i_R(\xi^L) = \frac{x^i}{2} + \frac{p^i}{2p^+}\xi^L + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}-\{0\}}\frac{\tilde{\alpha}^i_n}{n}e^{-2\pi\mathrm{i}n\xi^L/\ell}\,, \tag{2.32}$$

which implies

$$X^i(t,\sigma) = x^i + \frac{p^i}{p^+}t + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\left(\frac{\alpha_n^i}{n}e^{-2\pi\mathrm{i}n\xi^R/\ell} + \frac{\tilde{\alpha}_n^i}{n}e^{-2\pi\mathrm{i}n\xi^L/\ell}\right), \tag{2.33}$$

where x^i and p^i describe the coordinates and the momentum of the motion of the center of mass, instead $\alpha_n = (\alpha_n)^*$ and $\tilde{\alpha}_n = (\tilde{\alpha}_n)^*$ describe the amplitudes of the oscillations on the string.

2.2 Quantum level

Now we can promote the worldsheet degrees of freedom in operators

$$[x^-, p^+] = -\mathrm{i} \quad [x^i, p_j] = \mathrm{i}\delta^i_j, \qquad [\alpha^i_m, \alpha^j_n] = [\tilde{\alpha}^i_m, \tilde{\alpha}^j_n] = m\,\delta^{ij}\delta_{m,-n}, \quad [\alpha^i_m, \tilde{\alpha}^j_n] = 0\,. \tag{2.34}$$

The Hamiltonian in terms of the oscillators can be written as

$$H = \sum_{i=2}^{D-1}\frac{p_i^2}{2p^+} + \frac{1}{\alpha' p^+}\left[\sum_i\sum_{n>0}(\alpha^i_{-n}\alpha^i_n + \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n) + E_0 + \tilde{E}_0\right], \tag{2.35}$$

where E_0 and $\tilde{E}_0$ are the zero point energies.

Thus, now we can introduce the *Number operator*

$$N \equiv \sum_i\sum_{n>0}\alpha^i_{-n}\alpha^i_n\,, \qquad \tilde{N} \equiv \sum_i\sum_{n>0}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n\,, \tag{2.36}$$

from which we can deduct, looking at string states at each level starting from the **vacuum state**

$$\alpha^i_n\,|0;q\rangle = \tilde{\alpha}^i_n\,|0;q\rangle = 0\,, \quad \forall n>0, \implies N = \tilde{N} = 0\,, \tag{2.37}$$

where we have spin zero. It is important to note that our parametrization is invariant under σ translations, which are generated by the σ-momentum operator

$$P_\sigma = \int_0^\ell \mathrm{d}\sigma\,\Pi_i\partial_\sigma X^i = \frac{2\pi}{l}(N - \tilde{N}), \tag{2.38}$$

with Π_μ the momenta conjugate to X^μ

$$\Pi^i(t,\sigma) = \frac{\partial\mathcal{L}}{\partial\left(\partial_t X_i\right)} = \frac{1}{2\pi\alpha'}\partial_t X^i(t,\sigma). \tag{2.39}$$

From eq.(2.38) we can see that physical states must satisfy

$$N = \tilde{N}\,, \tag{2.40}$$

which is called the *level matching constraint*. Equation (2.40) is also the only relationship between left and right movers, since the quantum evolution is defined by independent Hamiltonians. Hence, from the mass shell condition we can define the mass of the particles as

$$M^2 = -p^2 = 2p^+p^- - p^i p_i\,, \tag{2.41}$$

since $p^- = \mathrm{i}\partial_{x^+} = \mathrm{i}\partial_t = H$ we can also write the mass condition as

$$M^2 = 2p^+ H - p_i p_i \,. \tag{2.42}$$

Moreover, knowing the form of the Hamiltonian we can find that the masses increase with the number of oscillators in the corresponding string state

$$\frac{\alpha' M^2}{2} = N + \tilde{N} + E_0 + \tilde{E}_0 \,, \tag{2.43}$$

from here it is easy to see that

$$E_0 = \tilde{E}_0 = \sum_{i=2}^{D-1} \frac{1}{2} \sum_{n=1}^{\infty} n. \tag{2.44}$$

2.2.1 Zeta Function Regularization

The Zeta function is defined, in the limit $\epsilon \to 0$ a

$$\begin{aligned} Z(\epsilon) &= \frac{1}{2} \sum_{n=0}^{\infty} n \, e^{-n\epsilon} = -\frac{1}{2} \frac{d}{d\epsilon} \sum_{n=0}^{\infty} e^{-n\epsilon} \\ &= -\frac{1}{2} \frac{d}{d\epsilon} \Big(\frac{1}{1 - e^{-\epsilon}} \Big) = -\frac{1}{2} \Big(-\frac{1}{\epsilon^2} + \frac{1}{12} + \ldots \Big) \,, \end{aligned} \tag{2.45}$$

from which one regularise taking the non singular part, hence, we are able to see that the zero point energies give us the following prescription

$$E_0 = \tilde{E}_0 = \sum_{i=2}^{D} \lim_{\epsilon \to 0} Z(\epsilon)_{\text{reg}} = -\frac{D-2}{24} \,, \tag{2.46}$$

where the D that preserves the full Lorentz invariance and takes the name of critical dimension, and will be now defined via the analysis of the spectrum. Looking at the light particles spectrum, they correspond to the smallest number of oscillators, i.e.

$$N = \tilde{N} = 0 \,, \qquad |k\rangle \,, \qquad \frac{\alpha' M^2}{2} = -\frac{D-2}{12} \,, \tag{2.47}$$

$$N = \tilde{N} = 1 \,, \qquad \alpha^i_{-1} \alpha^j_{-1} \, |k\rangle \,, \qquad \frac{\alpha' M^2}{2} = 2 - \frac{D-2}{12} \,. \tag{2.48}$$

In the light-cone gauge it is Lorentz invariant only under SO$(D-2)$. In fact, for massless particles we can write the D-momentum as $(E, E, 0, 0, \ldots)$ and in this case the little group is SO$(D-2)$. while massive particles, where the D-momentum can be written as $P = (M, 0, 0, \ldots)$, are invariant under the little group SO$(D-1)$. Thus, the critical dimension for closed strings turns out to be, taking (2.48) and requiring it to be a massless state

$$\frac{\alpha' M^2}{2} = 2 - \frac{D-2}{12} = 0 \implies D = 26 \,. \tag{2.49}$$

Hence we have that in general the mass of any state can be written as

$$D = 26 \implies \frac{\alpha' M^2}{2} = N + \tilde{N} - 2 \,, \tag{2.50}$$

which means that we can write the light spectrum as in Tab.2.1, which contains a tachyon T, the traceless metric tensor $G_{\mu\nu}$, the Kalb-Ramond antisymmetric tensor $B_{\mu\nu}$ and the dilaton ϕ.

$\boldsymbol{N}$, $\boldsymbol{\tilde{N}}$	state	field	$\boldsymbol{\alpha' M^2}$	representation
$N = \tilde{N} = 0$	$\vert k\rangle$	T	-4	$\mathbf{1}$
$N = \tilde{N} = 1$	$\alpha^i_{-1}\alpha^j_{-1}\vert k\rangle$	$G_{\mu\nu}$, $B_{\mu\nu}$, ϕ	0	$\mathbf{299}_S$, $\mathbf{274}_A$, $\mathbf{1}$

Table 2.1: The vacuum and the first states in the closed string spectrum

As it is immediate to see, the theory presents a *tachyonic* state and one should in principle be worried about it, but what it actually means is that the theory is placed on a local maximum. One way to fix this will be to introduce supersymmetry on the worldsheet, which will also allow for fermions in the spectrum, even if this last passage is not automatic but requires some care.

Let us now stress that the dilaton state ϕ vacuum expectation value is responsible for the string coupling g_s as

$$g_s = \langle e^{-\phi} \rangle \,, \tag{2.51}$$

which is an example of how all physical quantities in String Theory are not free parameters but are dynamically generated.

2.2.2 Open Strings

Differently from the closed strings, the open strings correspond to the case of worldsheet with boundaries, and they are strings with endpoints. It is important to recall the theories with open strings include closed strings, since two open strings can always merge into one closed one but it is not true the viceversa, thus they share the same local worldsheet structure, see Fig.2.4.

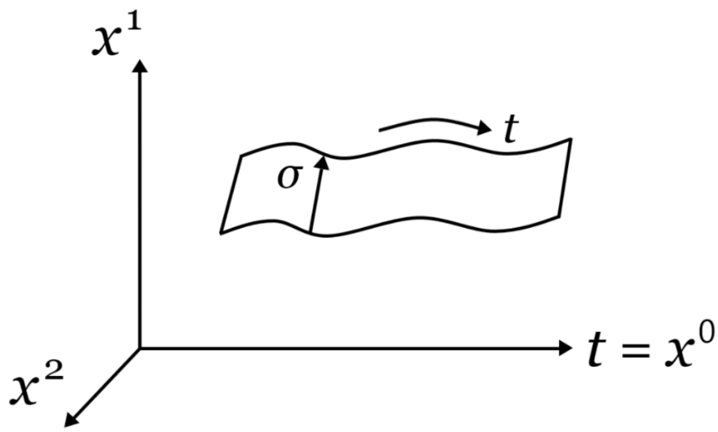


Figure 2.4: Representation of an open string. Source [4].

The procedure to quantise them is similar to the one of the closed string with the difference that now we have $\sigma \in (0, \ell)$, which implies that there is no symmetry under σ-translations. Moreover, we know that the local dynamics on the open string must be identical to the closed one, which implies that the only physical excitation are the fields $X^i(t,\sigma)$, with Hamiltonian

$$H = \frac{1}{2}\int_0^\ell \mathrm{d}\sigma \left(2\pi\alpha'\Pi^i\Pi_i + \frac{1}{2\pi\alpha'}\partial_\sigma X^i\,\partial_\sigma X_i\right) . \tag{2.52}$$

The general solution is given by

$$X_L^i(\xi^R) = \frac{x^i}{2} + \frac{p_i}{2p^+}\xi^R + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_\nu \frac{\alpha_\nu}{\nu}e^{-\mathrm{i}\pi\nu\xi^R/\ell} , \tag{2.53}$$

$$X_R^i(\xi^L) = \frac{x^i}{2} + \frac{p_i}{2p^+}\xi^L + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_\nu \frac{\tilde{\alpha}_\nu}{\nu}e^{-\mathrm{i}\nu\xi^L/\ell} . \tag{2.54}$$

The index ν needs to be fixed by the boundary conditions. It is important to note that differently from the closed string we don't have a two in the exponent of the expansion, since we have the interval of σ as a $\mathbf{Z}_2$ quotient of a circle with length 2ℓ. Thus, from the variational principle we can write

$$\delta S_P = -\frac{1}{2\pi\alpha'}\left(\int_{-\infty}^{\infty}\mathrm{d}t\,(\delta X^\mu\partial_\sigma X_\mu)\Big|_{\sigma=0}^{\sigma=l} - \int_\Sigma \mathrm{d}^2\xi\,\delta X^\mu\partial_a\partial^a X_\mu\right). \tag{2.55}$$

Hence, the first term is what will give us the boundary conditions that fix the index ν, while the second term gives the equations of motion identical to the closed strings. There are two types of boundary conditions

- **Neumann BCs**:
$$\partial_\sigma X^i = 0 \qquad \text{at } \sigma = 0,\, \ell \ , \tag{2.56}$$
and in this case the string end moves freely.

- **Dirichlet BCs**:
$$\delta X_i = 0 \tag{2.57}$$
and with this choice we have that the open string is confined to lie in a fixed hyperplane.

Let us focus now on the so called Neumann-Neumann (NN) boundary conditions, since for a generic Poincaré invariant theory δX^i has no constraints, and one needs to introduce branes, and so the open string sector of closed strings, in order to talk about other possible choices. The boundary condition at $\sigma = 0$ implies

$$\alpha_\nu^i = \tilde{\alpha}_\nu^i , \tag{2.58}$$

that means that left-moving waves are reflected at the end into right-moving one and same for the opposite. Moreover, the condition on $\sigma = \ell$ instead implies

$$\alpha_\nu^i \sin(\pi\nu) = 0 \implies \nu \in \mathbb{Z} . \tag{2.59}$$

Thus, following the same procedure of the closed string we find that the mass formula is

$$\alpha' M^2 = N - 1 \qquad \text{where } N = \sum_i \sum_{n>0} \alpha_{-n}^i \alpha_n^i \tag{2.60}$$

Hence, we can write again the lightest mode as shown in Tab.2.2, containing a tachyon T and a $U(1)$ gauge field A_μ. Again the tachyonic field signals a problem related to being in a local maximum and thus instability of the theory.

N, $\tilde{N}$	state	field	$\alpha' M^2$	representation
$N = 0$	$\|0\rangle$	T	-1	**1**
$N = 1$	$\alpha^i_{-1} \|0\rangle$	A_μ	0	**24**

Table 2.2: The vacuum and the first state in the open string spectrum for a bosonic string theory.

2.3 Brief introduction to Supersymmetry

Let us now briefly introduce the main concepts of supersymmetry, which will be the main ingredient in order to develop the so called Super String Theories, taking inspiration from [43].

2.3.1 Supersymmetry algebra and representations

First we make a small recall about the Poincaré group, that acts on spacetime as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \, x^\nu + a^\mu \,, \tag{2.61}$$

where $\Lambda^T \eta \Lambda = \eta$ is the Lorentz transformation. And the generators for the Poincaré group are the $M^{\mu\nu}$ and P^σ with algebra

$$\Big[P^\mu, P^\nu\Big] = 0 \,, \tag{2.62}$$

$$\Big[M^{\mu\nu}, P^\sigma\Big] = \mathrm{i}\Big(P^\mu \eta^{\nu\sigma} - P^\nu \eta^{\mu\sigma}\Big) \,, \tag{2.63}$$

$$\Big[M^{\mu\nu}, M^{\rho\sigma}\Big] = \mathrm{i}\Big(M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho}\Big) \,, \tag{2.64}$$

where

$$(M^{\rho\sigma})^\mu{}_\nu = \mathrm{i}(\eta^{\mu\nu}\delta^\rho_\nu - \eta^{\rho\mu}\delta^\sigma_\nu) \,. \tag{2.65}$$

Thus, now we would like to write a supersymmetric extension of the Poincaré algebra, in order to do so we need to introduce the **graded algebras**,

$$O_a O_b - (-1)^{\eta_a \eta_b} O_b O_a = \mathrm{i}\, C^e_{ab} O_e \,, \tag{2.66}$$

where O_a are operators of a Lie algebra, and η_a is equal to zero for a bosonic generator or one for a fermionic generator.

For supersymmetry the generators are P^μ, $M^{\mu\nu}$ and the spinor generators Q^A_α, $\bar{Q}^A_{\dot{\alpha}}$, with $A = 1, ..., \mathcal{N}$. In the case of $\mathcal{N} = 1$ we can speak about *simple* SUSY, otherwise of *extended* SUSY. For simplicity we will focus only on the simple SUSY case.

Hence, the SUSY algebra is given by

$$\Big[Q_\alpha, M^{\mu\nu}\Big] = (\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta \,, \tag{2.67}$$

$$\Big[Q_\alpha, P^\mu\Big] = \Big[\bar{Q}^{\dot\alpha}, P^\mu\Big] = 0 \,, \tag{2.68}$$

$$\Big\{Q_\alpha, Q_\beta\Big\} = 0 \,, \tag{2.69}$$

$$\Big\{Q_\alpha, \bar{Q}_{\dot\beta}\Big\} = 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu \,. \tag{2.70}$$

or in the extended version with $\mathcal{N} > 1$, we add the label $A, B = 1, 2, ..., \mathcal{N}$, and the algebra is the same of the simple SUSY except for

$$\Big\{Q^A_\alpha, \bar{Q}_{\dot\beta B}\Big\} = 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu \delta^A_B \,, \tag{2.71}$$

$$\Big\{Q^A_\alpha, Q^B_\beta\Big\} = \epsilon_{\alpha\beta} Z^{AB} \,, \tag{2.72}$$

where $Z^{AB} = -Z^{BA}$ is the central charge, that commute with all the generators,

$$\Big[Z^{AB}, P^\mu\Big] = \Big[Z^{AB}, M^{\mu\nu}\Big] = \Big[Z^{AB}, Q^A_\alpha\Big] = \Big[Z^{AB}, Z^{CD}\Big] = \Big[Z^{AB}, T_a\Big] = 0 \,. \tag{2.73}$$

2.3.2 Superfields, Chiral and Vector Superfield

We know that every continuous group G defines a manifold $\mathcal{M}$ as

$$\Lambda : G \longrightarrow \mathcal{M}, \qquad \{g = e^{i\alpha_a T^a}\} \longrightarrow \{\alpha_a\} \tag{2.74}$$

where $dimG = dim\mathcal{M}$. Thus, taking two groups G and H we can define a **coset** as G/H, where $g \in G$ is identified by $g \cdot h$, with $h \in H$.

For example suppose to take the case of Minkowski=Poincaré/Lorentz= $\{\omega^{\mu\nu}, a^\mu\}/\{\omega^{\mu\nu}\}$.

Thus, in the case of $\mathcal{N} = 1$, we can define the superspace to be the coset

$$\text{Super Poincaré/Lorentz} = \{\omega^{\mu\nu}, a^\mu, \theta^\alpha, \bar\theta_{\dot\alpha}\} \tag{2.75}$$

For a general scalar superfield $S(x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha})$ we can make an expansion in θ_α and $\bar\theta_{\dot\alpha}$ and we obtain

$$\begin{aligned} S(x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha}) =& \varphi(x) + \theta\psi(x) + \bar\theta\bar\chi(x) + \theta\theta M(x) + \bar\theta\bar\theta N(x) + (\theta\sigma^\mu\bar\theta)V_\mu(x) \\ &+ (\theta\theta)\bar\theta\bar\lambda(x) + (\bar\theta\bar\theta)\theta\rho(x) + (\theta\theta)(\bar\theta\bar\theta)D(x). \end{aligned} \tag{2.76}$$

Under a Poincaré transformation we have that $S(x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha})$ transforms as

$$S(x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha}) \to e^{-\mathrm{i}\left(\epsilon Q + \bar\epsilon\bar Q\right)}\, S\, e^{\mathrm{i}\left(\epsilon Q + \bar\epsilon\bar Q\right)}, \tag{2.77}$$

and as an Hilbert vector it transforms in the following way

$$\begin{aligned} S(x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha}) \to& e^{\mathrm{i}\left(\epsilon \mathcal{Q} + \bar\epsilon\bar{\mathcal{Q}}\right)}\, S(x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha}) \\ &= S\big(x^\mu - \mathrm{i}c(\epsilon\sigma^\mu\bar\theta) + \mathrm{i}c^*(\theta\sigma^\mu\bar\epsilon), \theta + \epsilon, \bar\theta + \bar\epsilon\big), \end{aligned} \tag{2.78}$$

where ϵ is a parameter, $\mathcal{Q}$ is a representation of the spinorial generators Q_α that acts on θ and $\bar{\theta}$, while c is a constant.

Thus, we have the following translations in x^μ, θ_α and $\bar{\theta}_{\dot{\alpha}}$

$$\mathcal{Q}_\alpha = -\mathrm{i}\frac{\partial}{\partial\theta^\alpha} - c\,(\sigma^\mu)_{\alpha\dot{\beta}}\,\bar{\theta}^{\dot{\beta}}\frac{\partial}{\partial x^\mu}, \tag{2.79}$$

$$\bar{\mathcal{Q}}_{\dot{\alpha}} = \mathrm{i}\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + c^*\theta^\beta\,(\sigma^\mu)_{\beta\dot{\alpha}}\,\frac{\partial}{\partial x^\mu}, \tag{2.80}$$

$$P^\mu = -\mathrm{i}\partial_\mu. \tag{2.81}$$

The constant c is determined by

$$\left\{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot{\alpha}}\right\} = 2(\sigma^\mu)_{\alpha\dot{\alpha}}\,P_\mu \implies \mathrm{Re}\{c\} = 1. \tag{2.82}$$

Moreover, it is useful to define a covariant derivative

$$\mathcal{D}_\alpha \equiv \partial_\alpha + \mathrm{i}(\sigma^\mu)_{\alpha\dot{\beta}}\;\bar{\theta}^{\dot{\beta}}\partial_\mu, \qquad \bar{\mathcal{D}}_{\dot{\alpha}} \equiv -\bar{\partial}_{\dot{\alpha}} - \mathrm{i}\theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\;\partial_\mu, \tag{2.83}$$

which satisfy

$$\{\mathcal{D}_\alpha, \mathcal{Q}_\beta\} = \{\mathcal{D}_\alpha, \bar{\mathcal{Q}}_{\dot{\beta}}\} = \{\bar{\mathcal{D}}_{\dot{\alpha}}, \mathcal{Q}_\beta\} = \{\bar{\mathcal{D}}_{\dot{\alpha}}, \bar{\mathcal{Q}}_{\dot{\beta}}\} = 0. \tag{2.84}$$

Hence, we can define a **chiral superfield** such that it satisfies $\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0$. Thus, let us define

$$y^\mu \equiv x^\mu + \mathrm{i}\theta\sigma^\mu\bar{\theta}, \tag{2.85}$$

if $\Phi = \Phi(y, \theta, \bar{\theta})$, we can write

$$\begin{aligned}
\bar{D}_{\dot{\alpha}}\Phi &= -\bar{\partial}_{\dot{\alpha}}\Phi - \frac{\partial\Phi}{\partial y^\mu}\frac{\partial y^\mu}{\partial\bar{\theta}_{\dot{\alpha}}} - \mathrm{i}\theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\partial_\mu\Phi \\
&= -\bar{\partial}_{\dot{\alpha}}\Phi - \partial_\mu\Phi(-\mathrm{i}\theta\sigma^\mu)_{\dot{\alpha}} - \mathrm{i}\theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\partial_\mu\Phi \\
&= -\bar{\partial}_{\dot{\alpha}}\Phi = 0,
\end{aligned} \tag{2.86}$$

which means that Φ depends only on y and θ, i.e. in components

$$\Phi(y^\mu, \theta^\alpha) = \varphi(y^\mu) + \sqrt{2}\theta\psi(y^\mu) + \theta\theta F(y^\mu)\,, \tag{2.87}$$

or in terms of x^μ

$$\begin{aligned}
\Phi(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) = &\,\varphi(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) + \mathrm{i}\theta\sigma^\mu\bar{\theta}\partial_\mu\varphi(x) \\
&- \frac{\mathrm{i}}{\sqrt{2}}(\theta\theta)\partial_\mu\psi(x)\sigma^\mu\bar{\theta} - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial_\mu\partial^\mu\varphi(x)\,.
\end{aligned} \tag{2.88}$$

A **vector** superfield is defined s.t. it satisfy the following condition

$$V(x, \theta, \bar{\theta}) = V^\dagger(x, \theta, \bar{\theta})\,, \tag{2.89}$$

where $V(x, \theta, \bar{\theta})$ is defined as

$$\begin{aligned}
V(x, \theta, \bar{\theta}) = &\,C(x) + \mathrm{i}\theta\chi(x) + \frac{\mathrm{i}}{2}\theta\theta(M(x) + \mathrm{i}N(x)) - \frac{\mathrm{i}}{2}\bar{\theta}\bar{\theta}(M(x) - \mathrm{i}N(x)) \\
&+ \theta\sigma^\mu\bar{\theta}V_\mu(x) + \mathrm{i}\theta\theta\bar{\theta}\big(-\mathrm{i}\bar{\lambda}(x) + \frac{\mathrm{i}}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\big) \\
&- i\bar{\theta}\bar{\theta}\theta\big(\mathrm{i}\lambda(x) - \frac{\mathrm{i}}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\big) + \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})\big(D - \frac{1}{2}\partial_\mu\partial^\mu C\big).
\end{aligned} \tag{2.90}$$

Now, noting that if Λ is a chiral superfield , then $\mathrm{i}(\Lambda - \Lambda^\dagger)$ is a vector superfield. Hence, we can choose properly the component of Λ through the so called Wess-Zumino gauge, under which the vector superfield become

$$V_{WZ}(x,\theta,\bar\theta) = (\theta\sigma^\mu\bar\theta)V_\mu(x) + (\theta\theta)(\bar\theta\bar\lambda(x)) + (\bar\theta\bar\theta)(\theta\lambda(x)) + \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)D(x). \tag{2.91}$$

2.3.3 4D SUSY Lagrangians

The most general Lagrangian for a chiral superfield can be written as

$$\mathcal{L} = K(\Phi,\Phi^\dagger)\Big|_D + \left(W(\Phi)\Big|_F + \text{ h.c.}\right), \tag{2.92}$$

where K is the **Kähler potential** and W is the **Super-potential**, and it is an holomorphic function of Φ. In order to obtain a renormalizable theory we must ask $[\mathcal{L}] = 4$, and we know that

$$[\Phi] = [\varphi] = 1, \quad [\psi] = \frac{3}{2} \implies [\theta] = -\frac{1}{2}, \quad [F] = 2\,, \tag{2.93}$$

and in order to have $[\mathcal{L}] = 4$ we need to ask

$$[K] \leq 2, \quad [W] \leq 3\,. \tag{2.94}$$

Thus, the most simple case is the one where

$$K = \Phi^\dagger\Phi, \qquad W = \alpha + \lambda\Phi + \frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3, \tag{2.95}$$

whose Lagrangian is called Wess-Zumino model

$$\begin{aligned}\mathcal{L} &= \Phi^\dagger\Phi\Big|_D + \left[\left(\alpha + \lambda\Phi + \frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3\right)\Big|_F + \text{ h.c.}\right] \\ &= \partial^\mu\varphi^*\partial_\mu\varphi - \mathrm{i}\bar\psi\bar\sigma^\mu\partial_\mu\psi + FF^* + \left(\frac{\partial W}{\partial\varphi}F + \text{ h.c.}\right) - \frac{1}{2}\left(\frac{\partial^2 W}{\partial\varphi^2}\psi\psi + \text{ h.c.}\right).\end{aligned} \tag{2.96}$$

2.4 Superstrings theory

There are several problems related with the bosonic string theory:

- We have a vacuum which is unstable, since it correspond to a space time tachyon.
- It doesn't contains fermions, which are necessary to describe the real world

Superstring theory arise by considering a string action with local supersymmetry in $2D$. Hence, we consider fermionic fields $\Psi^\mu(\xi)$ that are the superpartner of the fields $X^\mu(\xi)$, and an additional worldsheet gravitino $\Psi_a(\xi)$, partner of the metric g_{ab}. From now on we will focus only in the closed strings case, namely Type II theories.

Thus, in $4D$ we need to add fermionic worldsheet coordinates θ_α and the Lorentz transformations

$$\theta_\alpha \to S_\alpha{}^\beta\theta_\beta \qquad \text{with } S = e^{-\epsilon^{ab}\frac{[\gamma_a,\gamma_b]}{4}}\,, \tag{2.97}$$

with

$$\gamma^0 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \{\gamma^a, \gamma^b\} = -2\eta^{ab}\,. \tag{2.98}$$

The local dynamics of fermionic fields correspond to free oscillations, and since we know that physical observables are either periodic or anti- periodic, due to the fact that fermions enter into observables always in a quadratic way, we can define two types of boundary conditions in superstrings

Neveu-Schwarz	NS	$\Psi^i_L(t+\sigma+\ell) = -\Psi^i_L(t+\sigma)$
Ramond	R	$\Psi^i_L(t+\sigma+l) = \Psi^i_L(t+\sigma)$

where we focus only on the left-movers since right-movers are analogous and decoupled except for the level matching which we will impose at the end. Hence, we have an arbitrary choice for either right and left sectors which bring us to a total four types of closed strings sectors: **NS-NS**, **NS-R**, **R-R** and **R-NS**. It is possible to verify that the modular invariance forces us to have these as different sectors on the same theory.

2.4.1 Neveu-Schwarz (NS)

The expansion for the NS sector is given by

$$\Psi^i_L(\xi^L) = \mathrm{i}\sqrt{\frac{\alpha'}{2}} \sum_{r\in\mathbb{Z}} \Psi^i_{r+1/2}\, e^{-2\pi \mathrm{i}\xi^L\left(r+\frac{1}{2}\right)/\ell}, \tag{2.99}$$

with anticommutator

$$\{\Psi^i_{m+\frac{1}{2}}, \Psi^j_{n+\frac{1}{2}}\} = \delta^{ij}\delta_{m+\frac{1}{2},-\left(n+\frac{1}{2}\right)}\,. \tag{2.100}$$

From now we are going to proceed in the same way of the bosonic string. Thus, first we define the Hamiltonian for the fermion in the NS sector as

$$H^{F_{NS}}_L = \frac{1}{\alpha' p^+}\left[\sum_{r=0}^{\infty}\left(r+\frac{1}{2}\right)\Psi^i_{-r-1/2}\Psi^i_{r+1/2} + E^{F_{NS}}_0\right], \tag{2.101}$$

with the value of E_0 given by

$$E^{F_{NS}}_0 = -\frac{1}{2}\sum_{i=2}^{D}\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right), \tag{2.102}$$

following from the zeta function $Z(\epsilon)$ regularization (reg), written as

$$Z_\alpha = \frac{1}{2}\sum_{n=0}^{\infty}(n+\alpha) \overset{\text{reg}}{=} -\frac{1}{24} + \frac{1}{4}\alpha(1-\alpha) \qquad \text{for } 0 \le \alpha \le 1, \tag{2.103}$$

we can obtain the value of E_0

$$E^{F_{NS}}_0 = \sum_{i=2}^{D} \overset{reg}{\lim_{\epsilon\to 0}} Z_{1/2}(\epsilon) \overset{\text{reg}}{=} -\frac{1}{48}(D-2). \tag{2.104}$$

Thus, we can define the number operator, fermionic and bosonic, as

$$N_{F_{NS}} = \sum_i \sum_{r=0}^{\infty} \left(r + \frac{1}{2}\right) \Psi^i_{-r-1/2} \Psi^i_{r+1/2}\,, \qquad N_B = \sum_i \sum_{n>0} \alpha^i_{-n} \alpha^i_n\,. \tag{2.105}$$

Hence, summing also the bosonic contribute, we can write the total Hamiltonian as

$$H_L^{NS} = \frac{1}{4p^+} \sum_i p_i^2 + \frac{1}{\alpha' p^+} \left(N_{F_{NS}} + N_B - \frac{D-2}{16}\right)\,. \tag{2.106}$$

Where we have that the mass term of the left moving sector is

$$\left(M_L^{NS}\right)^2 = 2p^+ H_L - \frac{1}{2} \sum_i p_i^2 \implies \frac{\alpha' \left(M_L^{NS}\right)^2}{2} = N_{F_{NS}} + N_B - \frac{D-2}{16}\,. \tag{2.107}$$

Looking at the light spectrum we obtain

$$N_{NS} + N_B = 0, \qquad |0, p\rangle_{NS}\,, \qquad \frac{\alpha' \left(M_L^{NS}\right)^2}{2} = -\frac{D-2}{16}\,, \tag{2.108}$$

$$N_{NS} + N_B = \frac{1}{2}\,, \qquad \Psi^i_{-1/2} |0, p\rangle_{NS}\,, \qquad \frac{\alpha' \left(M_L^{NS}\right)^2}{2} = \frac{1}{2} - \frac{D-2}{16}\,. \tag{2.109}$$

In the end the following states need to combine with the physical states of the right sector, which will give us a total mass of

$$M^2 = M_L^2 + M_R^2 = 2M_L^2\,, \tag{2.110}$$

due to the fact that we must satisfy the level matching $M_L^2 = M_R^2$. Again, requiring that the states must be accommodated into the little groups we can say that the critical dimension is $D = 10$, which from now on we will take as fixed. We can finally present the light spectrum in Tab.2.3, consisting in one tachyon and a vector of SO(8) in the $\mathbf{8}_V$ representation.

$N^{NS} + N^B$	**state**	$\frac{\alpha'(M_L^{NS})^2}{2}$	**SO(8)**
0	$\|0\rangle$	$-1/2$	$\mathbf{1}$
1	$\Psi^i_{-1/2}\|0\rangle$	0	$\mathbf{8}_V$

Table 2.3: Light states spectrum for the NS sector of a closed string.

2.4.2 Ramond (R)

From here we will proceed analogously as for the explanation of the Neveu-Schwarz sector. Thus, the expansion for the R sector is given by

$$\Psi_L^i\left(\xi^L\right) = \mathrm{i}\sqrt{\frac{\alpha'}{2}} \sum_{r\in\mathbb{Z}} \Psi^i_r e^{-2\pi \mathrm{i} r \xi^L/\ell}\,, \tag{2.111}$$

with anticommutation relations

$$\left\{\Psi^i_m, \Psi^j_n\right\} = \delta^{ij}\delta_{m,-n}\,. \tag{2.112}$$

While the Hamiltonian has the form

$$H_L^{F_R} = \frac{1}{2\alpha' p^+}\left(\sum_{r=1}^{\infty} r\ \Psi^i_{-r}\Psi^i_r + E_0^{F_R}\right), \tag{2.113}$$

with

$$E_0^{F_R} = -\frac{1}{2}(D-2)\sum_{r=1}^{\infty} r \overset{\text{reg}}{=} \frac{1}{24}(D-2) \to E_0^{F_R} \overset{\text{reg}}{=} \frac{1}{3}\,. \tag{2.114}$$

Hence, the total Hamiltonian and the $\alpha'\left(M_L^R\right)^2$ term are given by

$$H_L^R = \frac{1}{4p^+}\sum_i p_i^2 + \frac{1}{\alpha' p^+}\Big(N_{F_R} + N_B\Big), \tag{2.115}$$

$$\frac{\alpha'(M_L^R)^2}{2} = N_{F_R} + N_B\,. \tag{2.116}$$

For completeness we introduce the number operator

$$N_{F_R} = \sum_{r=1}^{\infty} r\,\Psi^i_{-r}\Psi^i_r\,, \qquad N_B = \sum_{n>0}\alpha^i_{-n}\alpha^i_n\,. \tag{2.117}$$

Differently from the NS sector we can have Ψ_0^i that does not contribute to the string energy

$$\Psi^i_r\,|0\rangle = 0 \qquad \forall i,\, \forall r > 0\,, \tag{2.118}$$

which means that we can have a degeneracy in the groundstate, i.e. fermion zero mode operators relating different states.

Recalling the anticommutation relation, for the ground state we obtain

$$\{\Psi^i_0, \Psi^j_0\} = \delta^{ij}, \tag{2.119}$$

which means that it must be a representation of the Clifford Algebra, and Ψ_0^i behave as a Dirac matrices of the SO(8) symmetry group, since $D = 10$. Finally, we can write the lightest mode as in Tab.2.4, consisting in two massless opposite chirality spinors in the $\mathbf{8}_C$ and $\mathbf{8}_S$ spinor representations of SO(8).

state	$\alpha'(M_{R,L})^2/2$	SO(8)
$\lvert\mathbf{8}_C\rangle$	0	$\mathbf{8}_C$
$\lvert\mathbf{8}_S\rangle$	0	$\mathbf{8}_S$

Table 2.4: Light states spectrum for the R sector of a closed string.

2.4.3 Modular Invariance, Type IIA/IIB

To obtain the physical spectrum we need to match left and right states, each in the NS and R spectrum, thus we need to ask

$$M_L^2 = M_R^2\,, \tag{2.120}$$

and impose modular invariance. In fact, when one considers amplitudes, one receives contributions by all the possible inequivalent worldsheet geometries. An example is given by the simple torus with circle dimensions R_1 and R_2, which can be viewed as a closed string of length R_1 propagating for a distance R_2 or viceversa, mapped by the exchange $t \leftrightarrow \sigma$, see Fig.2.5.

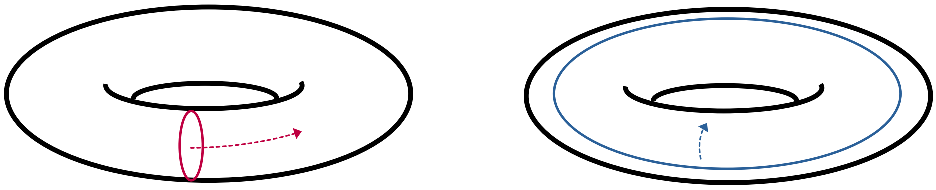


Figure 2.5: A torus geometry of the worldsheet can be viewed as a closed string of radius R_1 propagating for a length R_2 or viceversa.

Thus, we can see a 2-torus as 2d real plane with coordinate $z = \sigma + \mathrm{i}t$, with identifications $z \sim z + \ell$ and $z \sim z + \mathrm{i}r\ell$ with $r = r_1 + \mathrm{i}r_2$ the complex structure and ℓ the string length, meaning that all possible inequivalent geometries are labelled by r. Now, we can factorize the partition functions, of a closed string, in right and left sector

$$Z(r) = (4\pi^2\alpha' r_2)^{-4}\,\mathrm{Tr}_{\mathcal{H}_L}\, q^{N+E_0}\,\mathrm{Tr}_{\mathcal{H}_R}\, q^{-\tilde{N}+E_0}\,, \tag{2.121}$$

with $q = e^{2\pi \mathrm{i} r}$, and one can thus easily show that the modular group of the 2-torus is given by $\mathrm{SL}(2,\mathbb{Z})$, with general transformation given by

$$r \to \frac{ar+b}{cr+d}\,, \qquad ad - bc = 1\,. \tag{2.122}$$

Moreover, we can further split the partition function for left movers in NS and R sector, and after few calculation (see [41]) we have the bosonic contribution

$$\mathrm{Tr}_{\ \mathrm{bos}}\, q^{N_B+E_0^B} = \eta(r)^{-8}\,, \tag{2.123}$$

and fermionic contribution

$$\mathrm{Tr}_{NS}\, q^{N_{F_{NS}}+E_0^{F_{NS}}} = \frac{\theta\begin{bmatrix}0\\0\end{bmatrix}^4}{\eta^4}\,, \qquad \mathrm{Tr}_{R}\, q^{N_{F_R}+E_0^{F_R}} = \frac{\theta\begin{bmatrix}1/2\\0\end{bmatrix}^4}{\eta^4}\,, \tag{2.124}$$

where we have the Dedekind η function defined as

$$\eta(r) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,, \tag{2.125}$$

and the θ function given by

$$\theta\begin{bmatrix}\phi\\\psi\end{bmatrix} = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\phi)^2} e^{2\pi\mathrm{i}(n+\phi)\psi}\,. \tag{2.126}$$

Moreover, we find that the partition functions mixes different fermionic coordinates since that are related by

$$-\frac{\theta\begin{bmatrix}0\\0\end{bmatrix}^4}{\eta^4} \xrightarrow{r\to r+1} \frac{\theta\begin{bmatrix}0\\1/2\end{bmatrix}^4}{\eta^4} \xrightarrow{r\to -1/r} \frac{\theta\begin{bmatrix}1/2\\0\end{bmatrix}^4}{\eta^4}\,. \tag{2.127}$$

The following result give us the important information that *modular invariance* forces the theory to contain string states of different sectors simultaneously. From here we can build two partition functions

$$Z_\pm \equiv \frac{1}{2\eta^4}\left(\theta\begin{bmatrix}0\\0\end{bmatrix}^4 - \theta\begin{bmatrix}0\\1/2\end{bmatrix}^4 - \theta\begin{bmatrix}1/2\\0\end{bmatrix}^4 \pm \theta\begin{bmatrix}1/2\\1/2\end{bmatrix}^4\right), \tag{2.128}$$

which combined for left and right sectors define two modular invariant theories

$$\textbf{Type IIB}:\ Z_+\bar{Z}_+, \qquad \textbf{Type IIA}:\ Z_+\bar{Z}_-\,. \tag{2.129}$$

Lastly, one must employ the so called *GSO-projection*, which removes tachyonic states. This is done via the definition of the fermionic sign operator $(-1)^F$ which commutes with all fermion oscillators

$$(-1)^F\psi^i_\nu = \psi^i_\nu(-1)^F\,, \tag{2.130}$$

allowing us to rewrite the NS and R contributions as

$$\frac{1}{2\eta^4}\left(\theta\begin{bmatrix}0\\0\end{bmatrix}^4 - \theta\begin{bmatrix}0\\1/2\end{bmatrix}^4\right) = \mathrm{Tr}_{H_{NS}}\left(q^{N_{F_{NS}}+E_0^{F_{NS}}}\mathcal{P}^F_-\right), \tag{2.131}$$

$$\frac{1}{2\eta^4}\left(\theta\begin{bmatrix}1/2\\0\end{bmatrix}^4 \pm \theta\begin{bmatrix}1/2\\1/2\end{bmatrix}^4\right) = \mathrm{Tr}_{H_R}\left(q^{N_{F_R}+E_0^{F_R}}\mathcal{P}^F_\pm\right), \tag{2.132}$$

with fermion even/odd projector defined as

$$\mathcal{P}^F_\pm = \frac{1\pm(-1)^F}{2}\,, \tag{2.133}$$

which finally gives the partition function with the glueing prescription of left and right movers as schematically given by

$$Z_+\bar{Z}_\pm = \mathrm{Tr}_{NS}\,\mathrm{Tr}^*_{NS} - \mathrm{Tr}_{NS}\,\mathrm{Tr}^*_{R\pm} - \mathrm{Tr}_{R-}\,\mathrm{Tr}^*_{NS} + \mathrm{Tr}_{R-}\,\mathrm{Tr}^*_{R\pm}\,. \tag{2.134}$$

Let us address that the minus sign incidentally denotes the spacetime fermions, showing as we anticipated that their presence is not automatically given by supersymmetry on the worldsheet but rather is a consequence of the GSO projection.

From now on we will focus only on Type IIB String Theory. Hence, the massless spectrum is given by

Sector	$\vert\rangle_L\otimes\vert\rangle_R$	SO(8)	$10D$ field
NS-NS	$\mathbf{8}_V\otimes\mathbf{8}_V$	$\mathbf{1}+\mathbf{28}_V+\mathbf{35}_V$	ϕ, B_{MN}, G_{MN}
NS-R	$\mathbf{8}_V\otimes\mathbf{8}_C$	$\mathbf{8}_S+\mathbf{56}_S$	$\lambda^1_\alpha, \Psi^1_{M\alpha}$
R-NS	$\mathbf{8}_C\otimes\mathbf{8}_V$	$\mathbf{8}_S+\mathbf{56}_S$	$\lambda^2_\alpha, \Psi^2_{M\alpha}$
R-R	$\mathbf{8}_C\otimes\mathbf{8}_C$	$\mathbf{1}+\mathbf{28}_C+\mathbf{35}_C$	a, C_{MN}, C_{MNPQ}

Table 2.5: Massless spectrum for the Type IIB string.

Type IIB action

Now we can write the action for Type IIB strings in $10D$ in the Einstein frame

$$S_{IIB}^{10D} = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\,\sqrt{-G}\,\Big[e^{-2\phi}\Big(R + 4\,\partial_M\phi\partial^M\phi - \frac{1}{2}|H_3|^2\Big) - \frac{1}{2}|F_1|^2 - \frac{1}{2}|\tilde{F}_3|^2 - \frac{1}{2}|\tilde{F}_5|^2\Big] + \int C_4\wedge H_3\wedge F_3\,, \tag{2.135}$$

where $2\kappa_{10}^2 = (2\pi)^7(\alpha')^4$, R is the Ricci scalar, and the last integral takes the name of Chern-Simons action, and we have defined

$$F_p = \mathrm{d}C_p\,,\quad H_3 = \mathrm{d}B_2\,,\quad \tilde{F}_3 = F_3 - C_0H_3\,,\quad \tilde{F}_5 = F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3\,. \tag{2.136}$$

It is crucial to note that the action (2.135) is incomplete and one needs to include the following condition of self duality

$$\tilde{F}_5 = F_5\,. \tag{2.137}$$

It is important to pay our attention also to the symmetries of Type IIB string theory, since they play a very crucial role [44].

- $\mathbf{SL(2,\mathbb{R})}$: defining $\varphi = C_0 + \mathrm{i}e^{-\phi}$ and $G_3 = F_3 - \varphi H_3$ we have

$$\varphi \to \frac{a\varphi + b}{c\varphi + d},\qquad G_3 \to \frac{G_3}{c\varphi + d}\qquad \text{with } ad - bc = 1. \tag{2.138}$$

  and this symmetry is broken by α' and g_s corrections which in turn is recovered as $\mathrm{SL}(2,\mathbb{Z})$ at the full perturbative level.

- **Scale invariance**:

  Given some weights ν and ω we have

$$G_{MN} \to \lambda^\nu G_{MN},\quad \varphi \to \lambda^{2(\omega-\nu)}\varphi,\quad B_2 \to \lambda^{2\nu-\omega}B_2,\quad C_2 \to \lambda^\omega C_2,\quad C_4 \to \lambda^{2\nu}C_4\,. \tag{2.139}$$

  Thus, the bulk action transforms as

$$S_{bulk}^0 \to \lambda^{4\nu}S_{bulk}^0\,. \tag{2.140}$$

2.4.4 String Compatifications

The concept of *compatification* is a very old idea in physics. The first physicist that came out with this notion was Kaluza (years before also Nordstrom tried) in 1921, which tried to unify gravity with electromagnetism by assuming the existence of an extra dimension. Later in 1926, Klein noted that this extra dimension needs to be finite in order to to not be observed. Even if the original idea of Kaluza-Klein did not work out in the end the concept is now used in String Theory.

As we previously saw, for consistency we need a $10D$ spacetime, and thus, using the concept of compatification, we can divide these manifold in $\mathcal{M}_{10-d}$, which is the non-compact spacetime, and $\mathcal{M}_d$ which is instead the internal compact one as a product

$$\mathcal{M}_{10} = \mathcal{M}_{10-d}\times\mathcal{M}_d\,. \tag{2.141}$$

The most interesting case is the one where $d = 6$ since we live in a $4D$ spacetime, and thus one needs to find a suitable compactified manifold. Important to note is that the compatification scale is $M_c = \frac{1}{R} \ll M_s$, where R is the internal length associated with the internal space.

A way to connect string theory with particle physics is to consider a string vacuum with $N = 1$ supersymmetric version of the SM in a $4D$ spacetime. Thus, we will take geometries of the form

$$\mathcal{M}_{10} = \mathcal{M}_4 \times Y_6, \tag{2.142}$$

where $\mathcal{M}_4$ is the $4D$ Minkowski space and Y_6 is a compact Calabi-Yau manifold, which determines the amount of SUSY left intact.

The $4D$ action S_{eff} can be derived by computation of the string scattering amplitudes, which is complicate since it means that we must know the CFT correlation functions. Another way to construct S_{eff} is to use the **Kaluza-Klein reduction**, valid only in the large volume limit, i.e. when $Y_6 \gg l_s$, where we combine the $10D$ actions which are constitute by Type II bulk SUGRA action, the Dirac-Born-Infeld and the Chern-Simons action which are governing the dynamics of the D-branes.

2.4.5 Calabi-Yau threefolds

Let us now follow the discussion and notations done in [45]. We have that the $10D$ Lorentz group, when we are considering strings propagation in the spacetime, decomposes as

$$Spin(1,9) \to Spin(1,3) \times Spin(6), \tag{2.143}$$

or in terms of representation

$$\mathbf{16} \in Spin(1,9) \to (\mathbf{2},\mathbf{4}) \otimes (\bar{\mathbf{2}},\bar{\mathbf{4}}). \tag{2.144}$$

Since we want the minimal supersymmetry in 4D, Y_6 must have a reduced structure group SU(3) $\in Spin(6)$. This means that we can reduce more the representation $\mathbf{4}$ as $\mathbf{4} \to \mathbf{3} \otimes \mathbf{1}$. This procedure is very important, since from here we obtain an invariant spinor η (the singlet $\mathbf{1}$), which is also globally defined. Since, Type II string theories have two supersymmetries in $10D$, Calabi-Yau compatifications lead to two supersymmetries in $4D$.

Furthermore, if we take into consideration the Laplacian of a scalar field ϕ, this one will be split as

$$\Delta_{10}\phi = (\Delta_4 + \Delta_6)\phi = (\Delta_4 + m^2)\phi = 0, \tag{2.145}$$

where $\Delta_6\phi = m^2\phi$, it means that ϕ is an eigenfunction of Δ_6. In particular, it is important to note that from (2.145) we have that the massless modes of Δ_4 correspond to the zero modes of Δ_6. These zero modes are in a one-to-one correspondence with the harmonic forms on Y_6, which are in turn in a one-to-one correspondence with elements of the cohomology groups $H^{p,q}(Y_6)$ of dimensions $\dim H^{p,q}(Y_6) = h^{p,q}(Y_6)$, called the Hodge numbers, where (p,q) denotes the number of holomorphic and anti-holomorphic differentials of the harmonic forms.

Hence, deforming the Calabi-Yau metric $g_{i\bar{\jmath}}$, with $i, \bar{\jmath} = 1, .., 3$, we obtain a deformation on the complex structure δg_{ij} and deformations of the Kähler form $\delta g_{i\bar{\jmath}}$

$$\delta g_{i\bar{\jmath}} = \mathrm{i} v^a (\omega_a)_{i\bar{\jmath}} \qquad \text{with}\, a = 1, ..., h^{1,1}\,, \quad \omega_a \in H^{1,1}\,, \tag{2.146}$$

where v^a denote $h^{1,1}$ moduli, which in the effective action appear as a scalar field. Similarly, the deformation of the complex structure

$$\delta g_{ij} = \frac{i}{\|\Omega_3\|^2} \bar{z}^k (\bar{\chi}_k)_{i\bar{\imath}\bar{\jmath}} \ (\Omega_3)^{\bar{\imath}\bar{\jmath}}{}_j \qquad \text{with } \ k = 1, ..., h^{1,2} , \quad \bar{\chi}_k \in H^{1,2} , \tag{2.147}$$

where Ω_3 is the holomorphic $(3,0)$-form with normalisation $\|\Omega\|^2 \equiv \frac{1}{3!}\Omega_{ijk}\bar{\Omega}^{ijk}$. In detail, v^a and $\bar{z}^k$ are the coordinates of a moduli space $\mathcal{M}_k^{h^{1,1}} \times \mathcal{M}_{cs}^{h^{1,2}}$. The metric on $\mathcal{M}_{cs}^{h^{1,2}}$ is a special Kähler metric with a Kähler potential given by

$$K_{cs} = -\ln\Big[-\mathrm{i}\int_{Y_6} \Omega_3 \wedge \overline{\Omega}_3\Big] = -\ln\left[\mathrm{i}\left(\bar{Z}^K \mathcal{F}_K - Z^K \bar{\mathcal{F}}_K\right)\right] , \tag{2.148}$$

having expanded in the sympletic basis $(\alpha_K, \beta^L) \in H^3(Y)$, with $K,\, L = 1, ..., h^3$, as

$$\Omega_3(z) = Z^K(z)\,\alpha_K - \mathcal{F}_L(z)\,\beta^L \qquad \text{with} \quad \mathcal{F}_L(z) = \partial_{Z^L}\mathcal{F}(Z(z)) , \tag{2.149}$$

where Z, $\mathcal{F}$ are holomorphic functions of the complex structure moduli z, and in particular $\mathcal{F}$ is a prepotential, which means that $M_{cs}^{h^{1,2}}$ is a so called *special* Kähler manifold.

Furthermore, the Kähler -form J can be complexified as

$$J_c = J + \mathrm{i}B_2 = t^a \omega_a , \qquad \omega_a \in H^{h^{1,1}} , \tag{2.150}$$

where B_2 is the Kalb-Ramond NS two-form of type II string theories and the t^a are the complex Kähler moduli introduced by this procedure. Moreover, the associated manifold $\mathcal{M}_k^{h^{1,1}}$ is as well a special Kähler manifold, with a Kähler potential and a prepotential $F(t)$ given by

$$K_k = -\ln \mathcal{K} , \qquad \mathcal{K} = \mathcal{K}_{abc} v^a v^b v^c , \qquad F(t) = \mathcal{K}_{abc} t^a t^b t^c , \tag{2.151}$$

where we have the triple intersection number $\mathcal{K}_{abc} = \int \omega_a \wedge \omega_b \wedge \omega_b$.

2.4.6 Kaluza-Klein reduction of Type IIB on Y

Let us take the massless bosonic spectrum of type IIB in $D = 10$ defined in (2.135), thus the low energy effective action can be rewritten using differential forms, taking $\kappa_{10} = 1$, as

$$\begin{aligned} S_{IIB}^{10} = &-\int \Big(\frac{1}{2}\hat{R} \star \mathbf{1} + \frac{1}{4}\,\mathrm{d}\hat{\phi} \wedge \star\, \mathrm{d}\hat{\phi} + \frac{1}{4} e^{-\hat{\phi}} \hat{H}_3 \wedge \star \hat{H}_3\Big) \\ &- \frac{1}{4}\int \Big(e^{2\phi} \hat{F}_1 \wedge \star \hat{F}_1 + e^{\hat{\phi}} \hat{\tilde{F}}_3 \wedge \star \hat{\tilde{F}}_3 + \frac{1}{2} \hat{\tilde{F}}_5 \wedge \star \hat{\tilde{F}}_5\Big) - \frac{1}{4}\int \hat{C}_4 \wedge \hat{H}_3 \wedge \hat{F}_3 , \end{aligned} \tag{2.152}$$

where $\star$ means the Hodge operator and we are now assuming the notation of an hat to distinguish 10D fields from the 4D ones. The fields in the action (2.152) are defined as

$$\hat{H}_3 = \mathrm{d}\hat{B}_2 , \qquad \hat{F}_p = \mathrm{d}\hat{C}_{p-1} , \qquad \hat{\tilde{F}}_3 = \hat{F}_3 - \hat{C}_0 \hat{H}_3 , \tag{2.153}$$

$$\hat{\tilde{F}}_5 = \hat{F}_5 - \frac{1}{2}\hat{H}_3 \wedge \hat{C}_2 + \frac{1}{2}\hat{B}_2 \wedge \hat{F}_3 . \tag{2.154}$$

Hence, it is important to say that we have the self- duality condition

$$\hat{\tilde{F}}_5 = \star \hat{\tilde{F}}_5 , \tag{2.155}$$

which is imposed at the level of the equations of motion.

Hence, the metric is in block-diagonal form

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + g_{i\bar{\jmath}}\mathrm{d}y^i\,\mathrm{d}\bar{y}^{\bar{\jmath}} \qquad \mu,\nu = 0,...,3\,, \quad i,\bar{\jmath} = 1,2,3\,, \tag{2.156}$$

with $g_{\mu\nu}$ the metric of the 4D space and $g_{i\bar{\jmath}}$ the metric of the Calabi-Yau manifold. Now, we need to expand in terms of the harmonic forms of Y_6, stopping at the zero mode since we are interested in the massless modes of $D = 4$, obtaining

$$\hat{B}_2 = B_2(x) + b^a(x)\omega_a, \qquad \hat{C}_2 = C_2(x) + c^a(x)\omega_a, \tag{2.157}$$

$$\hat{C}_4 = D_2^a(x)\wedge\omega_a + V^K(x)\wedge\alpha_K - U_K(x)\wedge\beta^K + \rho_a(x)\tilde{\omega}^a, \qquad K = 0,...,h^{1,2}\,, \tag{2.158}$$

where ω^a are harmonic of type $(1,1)$ and $\tilde{\omega}^a$ are harmonic of type $(2,2)$, that constitute a basis for $H^{2,2}(Y_6)$, while $b^a(x), c^a(x)$ and $\rho_a(x)$ are $4D$ scalar fields. Hence, we can write the following $\mathcal{N} = 2$ multiplets for Type IIB SUGRA compactified on a Calabi-Yau manifold as shown in Tab.2.6, where the self duality of $\hat{\tilde{F}}_5$ (2.155) allow us to eliminate half of the degrees of freedom in $\hat{C}_4$, and conventionally one chooses to eliminate D_2^a and U_K in favor of ρ_a and V^K.

Multiplet	Number	Components
gravity	1	$(g_{\mu\nu}, V^0)$
vector	$h^{1,2}$	(V^K, z^K)
hyper	$h^{1,1}$	(v^a, b^a, c^a, ρ_a)
double-tensor	1	(B_2, C_2, ϕ, C_0)

Table 2.6: $\mathcal{N} = 2$ multiplets for Type IIB SUGRA on a Calabi-Yau.

Thus, using (2.153), (2.154), (2.157) and (2.158) we obtain the following *new* effective action in 4D with $\mathcal{N} = 2$ SUSY

$$S^4_{IIB} = \int -\frac{1}{2}R\star 1 + \frac{1}{4}\,\mathrm{Re}\,(\mathcal{M}_{KL})\;F^K\wedge F^L + \frac{1}{4}\,\mathrm{Im}\,(\mathcal{M}_{KL})\;F^K\wedge\star F^L$$
$$- g_{k\bar{l}}\;\mathrm{d}z^k\wedge\star\,\mathrm{d}\bar{z}^{\bar{l}} - h_{AB}\;\mathrm{d}q^A\wedge\star\,\mathrm{d}q^B\;, \tag{2.159}$$

where $F^K = \mathrm{d}V^K$, $M(z) = M(\mathcal{F}(z))$ the gauge kinetic matrix giving the couplings which can be expressed in terms of the prepotential $\mathcal{F}(z)$, $g_{k\bar{l}}$ is the special Kähler metric for the complex structure moduli while h_{AB} is a quaternionic metric for the q^A fields which incorporate all the hypermultiplet fields. Thus, the moduli space is

$$\mathcal{M}_{\mathcal{N}=2} = \mathcal{M}_Q^{4(h^{1,1}+1)}\times\mathcal{M}_{SK}^{2h^{1,2}}\,. \tag{2.160}$$

2.4.7 Calabi-Yau orientifolds in IIB

In order to ensure the consistency of compactifications it is important to define the Calabi-Yau orientifold compactifications.

Starting from Type IIB String Theory in $D = 10$ with $\mathcal{N} = 2$, i.e. 32 supercharges, one can obtains an $\mathcal{N} = 2$ in $D = 4$ EFT by compactifying on a Calabi-Yau threefold

Y_6. However, we know that our world is described by a chiral theory, which $\mathcal{N}=2$ is not, and in order to re-obtain the SM limit we need to further break SUSY to $\mathcal{N}=1$ using BPS D-branes, which preserves only half of the SUSY, and/or orientifold planes.

Thus, we introduce modding of the Calabi-Yau manifold via a composition of the worldsheet parity operator Ω_p with an internal isometry σ which acts only on Y_6. Moreover, since σ must be a holomorphic isometry in order to preserve $\mathcal{N}=1$ SUSY, it leaves J invariant, while possibly acting non trivially on the 3-form Ω_3, defining two different classes of orientifold

$$\textbf{O3/O7-planes}: \quad \mathcal{O}_- = (-1)^{F_L}\Omega_p\,\sigma^* \qquad \text{with} \qquad \sigma^*\Omega_3 = -\Omega_3\,, \tag{2.161}$$
$$\textbf{O5/O9-planes}: \quad \mathcal{O}_+ = \Omega_p\,\sigma^* \qquad \text{with} \qquad \sigma^*\Omega_3 = \Omega_3\,, \tag{2.162}$$

where F_L is the left fermionic number. Let us now focus in O3/O7-planes, which act on the 10D fields as

$$\sigma^* \,:\, \left(\hat{G}\,,\hat{B}_2\,,\hat{\phi}\,,\hat{C}_0\,,\hat{C}_2\,,\hat{C}_4\right) \to \left(\hat{G}\,,-\hat{B}_2\,,\hat{\phi}\,,\hat{C}_0\,,-\hat{C}_2\,,\hat{C}_4\right)\,, \tag{2.163}$$

which then will split the cohomology groups into even/odd parity $H^{p,q} = H^{p,q}_+ \oplus H^{p,q}_-$, leading to the new deformations as

$$J = v^{a+}(x)\omega_{a+}, \quad \delta g_{ij} = \frac{i}{\|\Omega_3\|^2}\bar{z}^{k-}\left(\bar{\chi}_{k-}\right)_{i\bar{i}\bar{j}}\left(\Omega_3\right)^{\bar{i}\bar{j}}_{j}\,, \quad \hat{B}_2 = b^{a-}\omega_{a-}\,, \tag{2.164}$$
$$\hat{C}_4 = D_2^{a_+}\wedge\omega_{a_+} + V^{k_+}\wedge\alpha_{k_+} - U_{k_+}\wedge\beta^{k_+} + \rho_{a_+}\wedge\tilde{\omega}^{a_+}, \tag{2.165}$$

where $\omega_{a\pm}\in H^{1,2}_{\pm}$, $\bar{\chi}_{k_-}\in H^{1,2}_{-}$, $a_\pm = 1,...,h^{1,1}_{\pm}$ and $k_\pm = 1,...,h^{1,2}_{\pm}$, which can be arranged in supermultiples of $\mathcal{N}=1$ supergravity in $D=4$, shown in Tab.2.7.

Multiplet	Number	Components
gravity	1	$g_{\mu\nu}$
vector	$h^{2,1}_+$	V^{K_+}
chiral	$h^{2,1}_-$	z^{k_+}
	1	(ϕ, C_0)
	$h^{1,1}_-$	(b^{a_-}, c^{a_-})
chiral/linear	$h^{1,1}_+$	(v^{a_+}, ρ_{a_+})

Table 2.7: $\mathcal{N}=1$ 4D spectrum of Type IIB with $O3/O7$

Hence, we can write the action for $\mathcal{N}=1$, in terms of the Kähler potential K, holomorphic superpotential W and the holomorphic gauge-kinetic coupling functions f as

$$S^{(4)} = -\int \frac{1}{2}R\star\mathbf{1} + K_{A\bar{B}}\mathcal{D}M^A\wedge\star\mathcal{D}\overline{M}^{\bar{B}} + \frac{1}{2}\,\mathrm{Re}\{f_{KL}\}F^K\wedge\star F^L + \frac{1}{2}\,\mathrm{Im}\{f_{KL}\}F^K\wedge F^L + V\,, \tag{2.166}$$

where $F^K = \mathrm{d}V^K$, $\mathcal{D}$ is the gauge covariant derivative, and the scalar potential is written as

$$V = e^K(K^{I\bar{J}}D_IW\overline{D_jW} - 3|W|^2) + \frac{1}{2}\left[(\mathrm{Re}\{f\})^-1\right]^{KL}\mathfrak{D}_K\mathfrak{D}_L, \tag{2.167}$$

where we have the covariant derivative $D_AW = \partial_AW + (\partial_AK)W$ and $\mathfrak{D}$ are the D-terms. Again, we need to find the correct chiral coordinates on the space of scalar fields, finding that that the complex structure moduli are already correct for this purpose, while the others read

$$\varphi = C_0 + \mathrm{i}e^{-\phi}\,, \qquad G^a = c^{a_-} - \varphi b^{a_-}\,, \tag{2.168}$$

$$T_{a_+} = \frac{3\mathrm{i}}{2}\rho_{a_+} + \frac{3}{4}\mathcal{K}_{a_+b_+c_+}v^{b_+}v^{c_+} - \frac{3\mathrm{i}}{4(\varphi-\bar{\varphi})}K_{a_+b_-c_-}G^b - (G-\bar{G})^{c_-}\,, \tag{2.169}$$

with the triple intersection numbers

$$K_{a_+b_+c_+} = \int \omega_{a_+}\wedge\omega_{b_+}\wedge\omega_{c_+}\,, \qquad K_{a_+b_-c_-} = \int \omega_{a_+}\wedge\omega_{b_-}\wedge\omega_{c_-}\,. \tag{2.170}$$

Hence, the Kähler potential reads

$$K = K_{cs}(z,\bar{z}) + K_k(\varphi,T,G)\,, \tag{2.171}$$

$$K_{cs} = -\ln\left[-\mathrm{i}\int\Omega(z)\wedge\bar{\Omega}(\bar{z})\right], \tag{2.172}$$

$$\begin{aligned} K_k &= -\ln\left[-\mathrm{i}(\varphi-\bar{\varphi})\right] - 2\ln\left[\frac{1}{6}\mathcal{K}(\varphi,T,G)\right] \\ &= -\ln\left[-\mathrm{i}(\varphi-\bar{\varphi})\right] - 2\ln\mathcal{V}\,, \end{aligned} \tag{2.173}$$

with

$$\mathcal{V} \equiv \mathrm{Vol}(Y_6) = \frac{1}{6}\mathcal{K}\,. \tag{2.174}$$

It is important to notice that the Kähler potential for the Kähler moduli enjoys a no-scale structure of the type

$$\partial_AK_k\,\partial_{\bar{B}}K_k\,(K_k)^{A\bar{B}} = 4\,, \tag{2.175}$$

with $(K_k)^{A\bar{B}}$ the inverse of $(K_k)_{A\bar{B}} = \partial_A\partial_{\bar{B}}K_k$, where the derivative is $\partial_A = \partial_{M^A}$.

We are now ready to discuss the potential. The theory as it is at this stage as an identically null superpotential, thus $V = 0$, which means that all the moduli are massless. This is an huge problem, since if that would be the case they should mediate a new long range 5th force which is ruled out by data. In order to generate such potential one can turn on background fluxes for the form G_3 which induces some F-terms, with the Gukov-Vafa-Witten superpotential [46] given by

$$W_{\mathrm{GVW}} = \int_{Y_6}\Omega_3(z)\wedge G_3\,. \tag{2.176}$$

As it is immediate to see, since $\Omega_3(z)$ is a function of the complex structure only, while G_3 introduces a dependence on the axio-dilaton φ, only these fields get a mass, leaving all the Kähler moduli massless, requiring some extra step in order to fully stabilise the model. One possibility is to use non-perturbative effects such as instanton corrections to the superpotential. In fact, one first assumes that all the complex structure moduli z and the axio-dilaton φ are already stabilised at some higher scale, giving the vev of the GVW superpotential $W_0 = \langle W_{\rm GVW}\rangle$. Then, one considers some instanton effect to correct the superpotential, whose origin can be multiple as for example by the introduction of Dp-branes in the Calabi-Yau manifold. We will now present a simplified model called **KKLT** [47] where our Y_6 has an unique Kähler modulus, i.e. $h^{1,1} = 1$, with the chiral coordinate given by $T = \tau + {\rm i}a$. In this simple case the volume can be written as

$$\mathcal{V} \sim \tau^{3/2} \sim \left(T + \bar{T}\right)^{3/2} \implies K = -3\ln\tau\,, \tag{2.177}$$

and schematically the superpotential is given by

$$W = W_0 + A\, e^{-\mathfrak{a}T}\,, \tag{2.178}$$

where $A = A(z)$ is to be thought as a function of the complex structure moduli, but since they are already stabilised is effectively a constant, and $\mathfrak{a}$ is a parameter which is related to the different origin of this instanton correction. Notice that the exponential nature of this corrections justifies the procedure of decoupling between the complex structure (and axio-dilaton) sector and the Kähler moduli sector. With these it is now possible to construct the scalar potential using (2.167), which gives

$$V \sim \frac{e^{-\mathfrak{a}\tau}}{\tau}\,, \tag{2.179}$$

which one finds to be consistent only if $|W_0| \ll 1$. Let us add that this model in the end will give an AdS minimum and an uplift mechanism is required in order to reach dS, one example is given by the addition of anti D3-branes in the form of a nilpotent goldstino [48–51]. Lastly, let us address that that there is another viable construction for Kähler moduli known as **Large Volume Scenario** (LVS) [52, 53] which uses at least two moduli and employs α' corrections to generate the potential, along the instanton correction of the type (2.178). This model takes the name from its consistency condition, i.e. $\mathcal{V} \gg 1$, since it strongly uses expansions in $1/\mathcal{V}$. One advantage of this construction is that an exponentially small W_0 is no longer required, leaving some freedom on its choice, but it still suffer from an AdS minimum, needing again some uplift mechanism.

Chapter 3

Dark Energy and Dynamical Systems

In this chapter we will first describe how to express the evolution of the universe as a first order dynamical system, following [54–57]. Secondly, we will introduce a so called relaxed model which suppresses heavy particles' contributions, based on the work of [3, 58], which will be the base for our work and analysis in Chap.5.

3.1 Dynamical system for the cosmological evolution

Let us now describe how it's possible to explain the law of evolution obtained from the action as a first order dynamical system. We start by defining a theory with only one scalar field and then we will explore the case of two fields to end up with the case of two scalar fields with one of them coupling with matter, which is the one that we are going to analyze in detail in Chap.5.

3.1.1 Derivation with one Scalar Field

We would like to study the dynamics of quintessence on the flat Friedmann-Lemaitre-Robertson-Walker (FLRW) universe, with line element

$$\mathrm{d}s^{\,2} = -\,\mathrm{d}t^{\,2} + a^2(t)\,\mathrm{d}\vec{x}^{\,2}\ , \tag{3.1}$$

where $a(t)$ is the scale factor (see Chap.1). We are assuming an universe that contains a barotropic equation of state

$$p_\gamma = (\gamma - 1)\rho_\gamma \qquad \text{where} \qquad 0 \leq \gamma \leq 2. \tag{3.2}$$

Thus the action is given by

$$S[g_{\mu\nu}, \phi] = \int \mathrm{d}^4x\,\sqrt{-g}\,\left(\frac{1}{2}\mathcal{R} - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right)\,, \tag{3.3}$$

where ϕ is a scalar field $\phi = \phi(t)$, whose dependence is solely on the time since we assume isotropy of the space. For simplicity, let us focus on an exponential potential energy density of the form

$$V(\phi) = V_0\,e^{-\lambda\kappa\phi}\,, \tag{3.4}$$

where V_0 and λ are constants and $\kappa^2 = 8\pi G$ has mass dimension -1. This type of potential comes out naturally in string or Kaluza-Klein type models. The evolution equations in this scenario are given by

$$\dot{H} = -\frac{1}{2}\Big(\rho_\gamma + p_\gamma + \dot{\phi}^2\Big) = -\frac{1}{2}\Big(\gamma\rho_\gamma + \dot{\phi}^2\Big)\,, \tag{3.5}$$

$$\dot{\rho}_\gamma = -3H(\rho_\gamma + p_\gamma) = -3H\gamma\rho_\gamma\,, \tag{3.6}$$

$$\ddot{\phi} = -3H\dot{\phi} - \frac{\mathrm{d}V}{\mathrm{d}\phi}\,, \tag{3.7}$$

where in the first and second equation we have used the equation of state eq.(3.2), and we have that H is the Hubble parameter, with the constraint

$$H^2 = \frac{1}{3}\Big(\rho_\gamma + \frac{1}{2}\dot{\phi}^2 + V(\phi)\Big)\,. \tag{3.8}$$

We now define the following dimensionless variables

$$x \equiv \frac{\dot{\phi}}{\sqrt{6}H}\,, \qquad y \equiv \frac{\sqrt{V}}{\sqrt{3}H}\,, \tag{3.9}$$

and moreover, we recall that

$$\mathrm{d}N = \frac{\mathrm{d}a}{a} \qquad \text{and} \qquad \frac{\mathrm{d}N}{\mathrm{d}t} = H\,, \tag{3.10}$$

where N is the number e-foldings defined as $N \equiv \log a$. Thus, we can write the evolution equations, using the dimensionless variables, as a plane autonomous system

$$\begin{aligned} x' \equiv \frac{\mathrm{d}x}{\mathrm{d}N} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}N} = \frac{\dot{x}}{H} = \frac{1}{\sqrt{6}H}\left(\frac{\ddot{\phi}}{H} - \frac{\dot{\phi}}{H^2}\dot{H}\right) \\ = -3x + \lambda\sqrt{\frac{3}{2}}y^2 + \frac{3}{2}x\Big[2x^2 + \gamma(1 - x^2 - y^2)\Big], \end{aligned} \tag{3.11}$$

$$\begin{aligned} y' \equiv \frac{\mathrm{d}y}{\mathrm{d}N} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}N} = \frac{\dot{y}}{H} = \frac{1}{\sqrt{3H}}\left(\frac{V_\phi}{2\sqrt{V}H} - \frac{\sqrt{V}}{H^2}\dot{H}\right) \\ = -\lambda\sqrt{\frac{3}{2}}xy + \frac{3}{2}y[2x^2 + \gamma(1 - x^2 - y^2)]\,, \end{aligned} \tag{3.12}$$

with $V_\phi = \frac{\mathrm{d}V}{\mathrm{d}\phi}$. For the late universe physics it is important to define the equation of state for the scalar sector. Hence, using the definition in (1.43), we have

$$\omega_\phi = \frac{p_\phi}{\rho_\phi} = \frac{x^2 - y^2}{x^2 + y^2}. \tag{3.13}$$

Moreover, using the energy density in the scalar tensor

$$\Omega_\phi \equiv \frac{\rho_\phi}{3H^2} = x^2 + y^2, \tag{3.14}$$

we can bound the variables x^2 and y^2 as $0 \le x^2 + y^2 \le 1$, as a result of which we have trajectories inside of an unit disc. Due to the symmetry under reflection we will consider $y \ge 0$.

Description	$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{\Omega_\phi}$	$\boldsymbol{\omega_\phi}$	**Existence Conditions**
$\mathcal{K}_\pm$ Kinetic Dom.	± 1	0	1	1	for all k and λ
$\mathcal{F}$ Fluid Dom.	0	0	0	undefined	for all k and λ
$\mathcal{SD}$	$\frac{k}{\sqrt{6}}$	$\sqrt{\frac{1-k^2}{6}}$	1	$\frac{k^2}{3}-1$	$k^2 < 6$
$\mathcal{S}$ Scaling Sol.	$\sqrt{\frac{3}{2}\frac{\gamma}{\lambda}}$	$\sqrt{\frac{3(2-\gamma)\gamma}{2\lambda^2}}$	$\frac{3\gamma}{\lambda^2}$	$\gamma - 1$	$\lambda^2 > 3\gamma$

Table 3.1: Fixed points of the system with one scalar field, assuming $k_2 > 0$.

From these equations, imposing $x' = 0 = y'$, we can find five fixed points, called *critical points* [55] which are given in Tab.3.1

From Tab.3.1, we can see that we have different solutions at which correspond to different domination eras. We have the case where $\omega_\phi = \Omega_\phi = 1$, which is the *Kinetic Domination*, $\mathcal{K}_\pm$, since only x is different from zero, and these solutions are expected to be relevant at early times. We also have a solution where $\Omega_\phi = 0$ and ω_ϕ is undefined, which is the *Fluid Domination*, $\mathcal{F}$, where there is no contributions from x and y. We have other two solutions where x, y are both different from zero, which are the $\mathcal{SD}$, *Scalar field dominated solution*, with $\Omega_\phi = 1$, and $\mathcal{S}$ and the *Scaling solution*, where $\Omega_\phi = 3\gamma/\lambda^2$. Differently from the $\mathcal{SD}$ solution the $\mathcal{S}$ solution depends on both parameters λ and γ, and this could be a possible global attractor solution for $\lambda^2 > 3\gamma$.

3.1.2 Stability of the critical points with one Scalar Field

With the purpose of studying the stability of the critical points $\hat{x}$, $\hat{y}$, we add a small linear perturbation around the solutions as

$$x = \hat{x} + u\,, \qquad y = \hat{y} + v\,, \tag{3.15}$$

and, once we substitute these equations in (3.11) and (3.12) we obtain at first order in u and v

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \mathcal{M} \begin{pmatrix} u \\ v \end{pmatrix} \tag{3.16}$$

with general solution given by

$$u = u_+ e^{m_+ N} + u_- e^{m_- N} \tag{3.17}$$

$$v = v_+ e^{m_+ N} + v_- e^{m_- N} \tag{3.18}$$

where we have that $m_\pm$ are the eigenvalues of $\mathcal{M}$ and $u_\pm$ and $v_\pm$ the coefficients of the series expansion. Since our general solution has the form of an exponential, in order to achieve a stability we must require that the real part of both eigenvalues is negative.

After few calculations it is possible to obtain all the different solutions for the different domination era, again here we are studying general solutions, without fixing the value of λ and γ, see Tab.3.2.

	μ_+	μ_-	Conditions
$\mathcal{K}_\pm$	$\sqrt{\frac{3}{2}}(\sqrt{6}\pm\lambda)$	$3(2-\gamma)$	saddle for $\lambda > \sqrt{6}$ and $\lambda < -\sqrt{6}$
$\mathcal{F}$	$-\frac{3(2-\gamma)}{2}$	$\frac{3\gamma}{2}$	saddle for $0<\gamma<2$
$\mathcal{SD}$	$\frac{\lambda^2-6}{2}$	$\lambda^2-3\gamma$	stable for $\lambda^2<3\gamma$
$\mathcal{S}$	$-\frac{3(2-\gamma)}{4}\left[1+\sqrt{1-\frac{8\gamma(\lambda^2-3\gamma)}{\lambda^2(2-\gamma)}}\right]$	$-\frac{3(2-\gamma)}{4}\left[1-\sqrt{1-\frac{8\gamma(\lambda^2-3\gamma)}{\lambda^2(2-\gamma)}}\right]$	stable for $3\gamma<\lambda^2<\frac{24\gamma^2}{9\gamma-2}$

Table 3.2: Study of the stability of the critical points of the system with one scalar field.

Thus, from Tab.3.2, the only possible late-time attractor solution are the one given by $\mathcal{SD}$, i.e. Scalar field dominated solution, and by $\mathcal{S}$, the Scaling solution. In the first case the condition to have stability is that $\lambda^2 < 3\gamma$, which means that the stability exists for sufficiently flat potential. While, for the Scaling solution, this is a global attractor solution, where the energy density is proportional to the one of the barotropic fluid.

In detail, the critical points that we find are still λ and γ dependent, since we did not impose any constraint on them, and in the end we will have different qualitative evolution. In the following discussion we are going to show just a particular case where $\gamma = 1$, i.e. dust domination era, and $\lambda = 1$. In this case we obtain the evolution as in Fig.3.1.

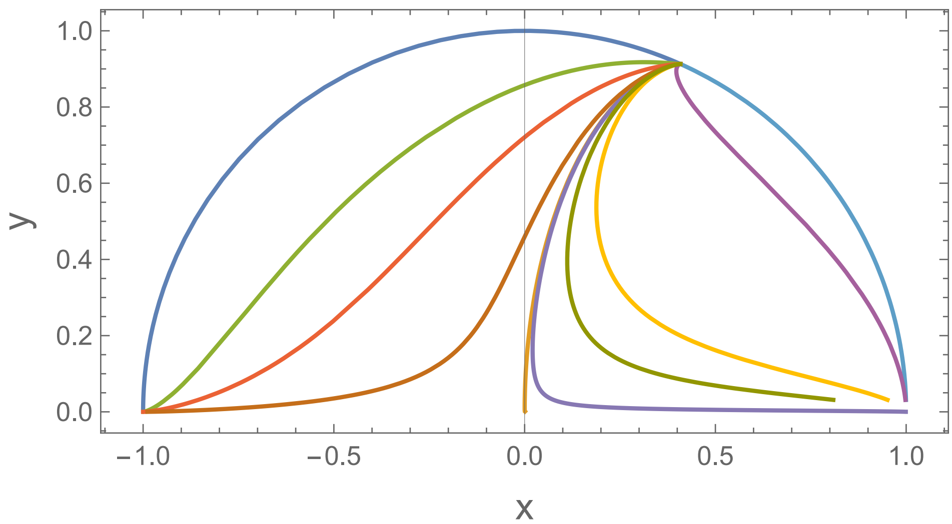


Figure 3.1: Phase plane with $\lambda = 1$ and $\gamma = 1$. The late time attractor is given by $x = \sqrt{1/6}$ and $y = \sqrt{5/6}$, which correspond to the kinetic domination solution.

3.1.3 Derivation with two Scalar Fields

In this section we will analyze an analogous case of the previous one but adding a second real scalar field, ϕ_2 and asking that the potential still depends only on ϕ_1, which means that we have a shift symmetry in the ϕ_2 direction, i.e. an axionic field, see [54, 57]. Moreover, we assume the presence of only gravitational interaction, thus, the action can now be written as

$$S[g_{\mu\nu},\phi] = \int \mathrm{d}^4x\,\sqrt{-g}\,\left(\frac{1}{2}\mathcal{R} - \frac{1}{2}(\partial\phi_1)^2 - -\frac{1}{2}f(\phi_1)^2(\partial\phi_2)^2 - V(\phi_1)\right)\,, \tag{3.19}$$

where $f(\phi_1) = f_1 = e^{-k_1\phi_1}$ is the kinetic coupling of ϕ_2, that as the potential $V = V_0 e^{-k_2\phi_1}$, depends only the field ϕ_1. In this case we have a modification and an extra equation of motion, leading to

$$\dot{H} = -\frac{1}{2}\Big(\gamma\rho_\gamma + \dot{\phi_1}^2 + f^2\dot{\phi_2}^2\Big), \tag{3.20}$$

$$\dot{\rho}_\gamma = -3\gamma H\rho_\gamma, \tag{3.21}$$

$$\ddot{\phi}_1 + 3H\dot{\phi}_1 - ff_1\dot{\phi}_2^2 + V_1 = 0, \tag{3.22}$$

$$\ddot{\phi}_2 + 3H\dot{\phi}_2 + 2\frac{f_1}{f}\dot{\phi}_2\dot{\phi}_1 = 0, \tag{3.23}$$

where $f_1(\phi_1)$ is the derivative of f in respect to ϕ_1. Thus, as in the previous case we have the constraint given by the Friedman equation

$$H^2 = \frac{1}{3}\Big(\rho_\gamma + \frac{1}{2}\dot{\phi}_1^2 + \frac{1}{2}f^2(\phi_1)\dot{\phi}_1^2 + V(\phi_1)\Big) \tag{3.24}$$

From here, we can define new dimensionless variables, which we will use to study our dynamical system

$$x_1 \equiv \frac{\dot{\phi}_1}{\sqrt{6}H}, \qquad x_2 \equiv \frac{f\dot{\phi}_2}{\sqrt{6}H}, \qquad y \equiv \frac{\sqrt{V}}{\sqrt{3}H}, \tag{3.25}$$

from which we can take the derivative in respect to N, obtaining

$$x_1' = 3x_1(x_1^2 + x_2^2 - 1) + \sqrt{\frac{3}{2}}(-2k_1x_2^2 + k_2y^2) - \frac{3}{2}x_1\gamma(x_1^2 + x_2^2 + y^2 - 1), \tag{3.26}$$

$$x_2' = 3x_2(x_1^2 + x_2^2 - 1) + \sqrt{6}k_1x_1x_2 - \frac{3}{2}\gamma x_2(x_1^2 + x_2^2 + y^2 - 1), \tag{3.27}$$

$$y' = -\sqrt{\frac{3}{2}}k_2x_1y - \frac{3}{2}\gamma y(x_1^2 + x_2^2 + y^2 - 1) + 3y(x_1^2 + x_2^2), \tag{3.28}$$

with $k_1 \equiv -\frac{f_1}{f}$ and $k_2 = -\frac{V_1}{V}$.

For the study of the evolution of the universe we define the equation of state

$$\omega_\phi = \frac{x_1^2 + x_2^2 - y^2}{x_1^2 + x_2^2 + y^2}, \tag{3.29}$$

with energy density

$$\Omega_\phi = x_1^2 + x_2^2 + y^2. \tag{3.30}$$

From which we have the constraints

$$-1 \leq x_1 \leq 1\,, \qquad -1 \leq x_2 \leq 1\,, \qquad 0 \leq x_3 \leq 1\,, \tag{3.31}$$

thus the parameter space is half of a 3−disk, with unit radius centred in the origin. From here, imposing equals to zero the equations x_1', x_2' and y', we obtain the critical points in Table 3.3.

Description	x_1	x_2	x_3	Ω_ϕ	ω_ϕ	Existence Conditions
$\mathcal{K}_\pm$ Kinetic Dom.	± 1	0	0	1	1	$k_2 > k_1 > 0$
$\mathcal{F}$ Fluid Dom.	0	0	0	0	undefined	$k_2 > k_1 > 0$
$\mathcal{G}$ Geodesic	$\frac{k_2}{\sqrt{6}}$	0	$\sqrt{\frac{6-k_2^2}{6}}$	1	$\frac{k_2^2-3}{3}$	$\sqrt{6} \geq k_2 > k_1 > 0$
$\mathcal{S}$ Scaling Sol.	$\frac{\sqrt{3/2}}{k_2}$	0	$\frac{\sqrt{3/2}}{k_2}$	$\frac{3}{k_2^2}$	-1	$k_2 > \sqrt{6} > k_1 > 0$ or $\sqrt{6} \geq k_2 \geq \sqrt{\frac{3}{2}} > k_1 > 0$
$\mathcal{NG}$ Non-Geodesic	$\frac{\sqrt{6}}{2k_1+k_2}$	$\mp\frac{\sqrt{-6+2k_1k_2+k_2^2}}{2k_1+k_2}$	$\sqrt{\frac{2k_1}{2k_1+k_2}}$	1	$1-\frac{4k_1}{2k_1+k_2}$	$\sqrt{6} \geq k_2 \geq \sqrt{2} \geq \frac{6-k_2^2}{2k_2} > k_1 > 0$ or $k_2 \geq \sqrt{6},\quad k_2 > k_1 > 0$

Table 3.3: Fixed points of the system with two scalars fields, assuming $k_2 > 0$.

In this Table, as in the case of only one scalar field, we have different possible eras. The Kinetic domination $\mathcal{K}_\pm$, as in the previous case, correspond to the case where only x_1 is different from zero and is relevant at the early stage of the universe. We also have the Fluid Domination, $\mathcal{F}$, which is in agreement with the previous study case of one field, i.e. there are no contributions from none of the fields. Moreover, we have a Geodesic solution, $\mathcal{G}$, called in this way since the system evolves along a geodesic trajectory. Furthermore, the more interesting solution that can be relevant for the study of the evolution of the universe is the Scaling solution, $\mathcal{S}$, where we have the same contribution for the kinetic and potential energy of the field ϕ_1. In this era, we have a value of $\omega_\phi = -1$ which is the one that we expected from the theory, and a value of Ω_ϕ that depends on the parameter k_2. Lastly, we have a so called Non-Geodesic solution, $\mathcal{NG}$, where we have a contribution from all the variables, hence, the evolution does not follow a geodesic in the field space. Again, this solution is in agreement with the one in literature.

3.1.4 Stability of the critical points with two Scalar Fields

We can follow the same procedure done for one field, add a small linear perturbation around the solutions $(\hat{x}_i, \hat{y})$ obtaining

$$x_i = \hat{x}_i + u_i\,, \qquad y = \hat{y} + u_3\,, \tag{3.32}$$

and using the equations (3.26), (3.27) and (3.28) we obtain at linear order

$$u' = \mathcal{M}u\,. \tag{3.33}$$

After few calculations, we can find that the eigenvalues of M_{ij}, denoting as μ, which are given in Table.3.4.

	μ			Conditions
$\mathcal{K}_{\pm}$	$\pm\sqrt{6}k_1$	$\frac{1}{2}(6 \mp \sqrt{6}k_2)$	$3(2-\gamma)$	saddle
$\mathcal{F}$	$-3+\frac{3}{2}\gamma$	$-3+\frac{3}{2}\gamma$	$\frac{3}{2}\gamma$	saddle
$\mathcal{S}$	$-\frac{3}{2}+\frac{3k_1}{k_2}$	$-\frac{3}{4}\left(1+\frac{\sqrt{24-7k_2^2}}{k_2}\right)$	$-\frac{3}{4}\left(1-\frac{\sqrt{24-7k_2^2}}{k_2}\right)$	$\gamma=1,\ k_2>2k_1,\ k_2>\sqrt{3}$
$\mathcal{G}$	$\frac{1}{2}(k_2^2-6)$	$k_1k_2+\frac{k_2^2}{2}-3$	$k_2^2-3\gamma$	$\gamma=1\,, k_1>0\,, 0<k_2<\sqrt{3}$ or $\gamma=1\,, k_1>0\,.0<k_2<\sqrt{k_1^2+6}-k_1$
$\mathcal{NG}$	$\simeq -3$	$\simeq \frac{1}{2}\left(-3+\sqrt{3}\sqrt{27-8k_1k_2-4k_2^2}\right)$	$\simeq \frac{1}{2}\left(-3-\sqrt{3}\sqrt{27-8k_1k_2-4k_2^2}\right)$	$\sqrt{6+k_1^2}-k_1 \leq k_2 \leq 2k_1$

Table 3.4: Study of the stability of the critical points of the system with two scalar fields.

Today, we know that the expected value for the energy density and the equation of state are $\omega_\phi \simeq -1$ and $\Omega_\phi \simeq 0.7$, and thus, we can see that none of the solutions in Tab.3.3 is a valid one, with the only exception given by the Scaling solution $\mathcal{S}$, but only for $\Omega_\phi < 1$. Moreover, for the condition of stability we must ask that all the eigenvalues of the matrix $\mathcal{M}$ are negative. But, as it is possible to see from Tab.3.4, the Kinetic and the Fluid solutions are both a saddle point. While, the other solutions can satisfy the stability conditions under some range for the values of the parameters.

In detail, the critical points are parameters dependent, and we can show in Fig.3.2 a particular case where $k_1 = 300$ and $k_2 = 1$, i.e. matter domination, and the system evolves towards the $\mathcal{NG}$ solution.

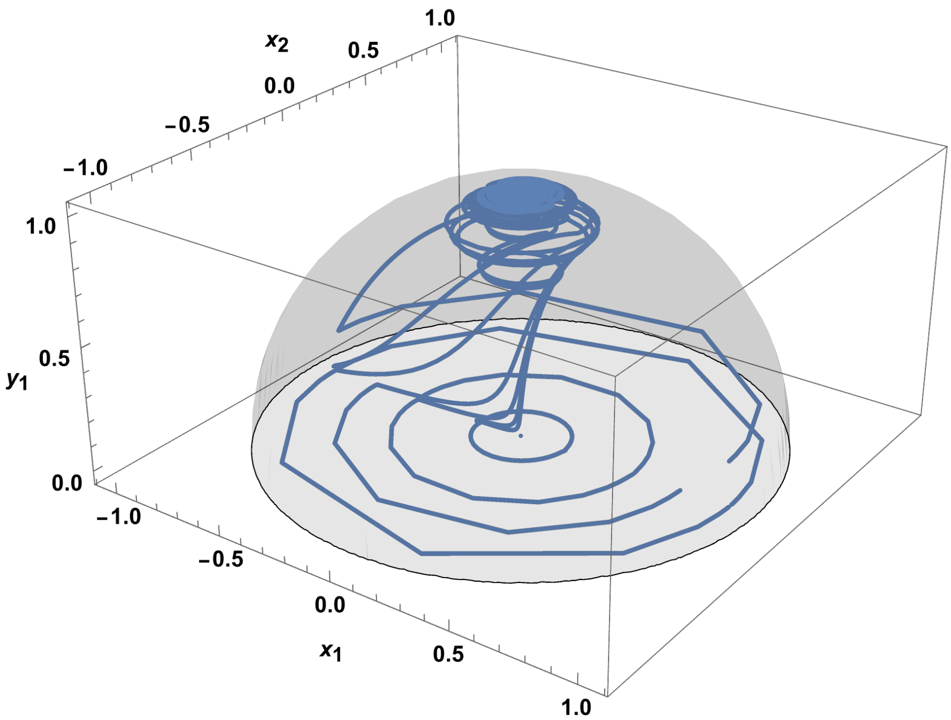


Figure 3.2: Phase plane with $k_1 = 300$ and $k_2 = 1$. The late time attractor correspond to the Non-Geodesic solution.

Chapter 4

Relaxed Dark Energy Model

One of the most challenging research topic in today physics is the cosmological constant problem. Despite the years, today we are not yet able to find a way to explain why the vacuum energy associated with the particles of the Standard Model (SM) is far away from what we found for the vacuum related with Dark Energy.

In order to face this problem let us consider a 4D model that it is naturally relaxed, for which the gravitational response of heavy particles' vacuum energies is strongly suppressed [3]. In chapter 5 we will use this model in order to modify the dynamical system and study the new evolution of the universe.

4.1 Relaxed model

The following models contain three main ingredients, a so called *relaxation* mechanism conducted by a scalar field ϕ, a supergravity sector coupled to the Standard Model and lastly an accidental approximate scale invariance that is displayed through the presence of a low energy dilaton supermultiplet. Thus, we construct a Lagrangian where at low energy we have a non-supersymmetric sector coupled to supergravity, with the inclusion of the relaxation mechanism that, when integrating out the usual SM fields their vacuum energy contributions are removed dynamically.

Hence, we can construct the EFT, using specific types of constrained superfields for each particle that doesn't have an explicit superpartner.

- The nilpotent chiral multiplet X, i.e. satisfying $X^2 = 0$, containing the spin 1/2 goldstino G and the UV SUSY breaking parameter F^X. Namely, given $X = (\chi, \psi, F^X)$ one finds that $\chi = \frac{\psi\psi}{2F^X}$ [48–51].
- The left invariant superfield Φ, satisfying the condition that $\bar{D}(X\bar{\Phi}) = 0$ represents the scalar fields as the relaxon ϕ.

Assuming that the theory has an accidental scale invariance one can then expand in powers of $\frac{1}{\tau}$ with $T = \frac{1}{2}(\tau + \mathrm{i}a)$, while the shift symmetry of the axion a forbids any dependence on T of the superpotential W. Moreover, we need to have an homogeneous function of the chiral coordinate T indicating that one could choose it accordingly such that the leading expression of the expansion is of homogeneous degree one, allowing us to present all these corrections as τ^{-k} with k non-negative integer.

The Kähler potential thus becomes

$$K \simeq -3M_p^2 \ln(P)\,, \qquad \text{with} \quad P(\tau, X, \overline{X}, \Phi, \bar{\Phi}) = \tau - k + \frac{h}{\tau} + \ldots\,, \tag{4.1}$$

and

$$W \simeq \omega_0(\Phi) + X\omega_X(\Phi,\bar{\Phi}), \tag{4.2}$$

k, h and W are chosen such that they are consistent with the constraints $X^2 = X(\bar{\Phi} - \Phi) = 0$ giving

$$k = \frac{1}{M_p^2}\Big[\mathfrak{K}(\Phi,\bar{\Phi},\ln\tau) + [X\mathfrak{K}_X(\Phi,\bar{\Phi},\ln\tau) + h.c.] + \overline{X}X\mathfrak{K}_{X\overline{X}}(\Phi,\bar{\Phi},\ln\tau)\Big], \tag{4.3}$$

from which one can easily see that

$$[X] = 1 \implies [\mathfrak{K}] = 2,\ [\mathfrak{K}_X] = 1,\ [\mathfrak{K}_{X\overline{X}}] = 0\,. \tag{4.4}$$

The scalar potential V_F is

$$V_F = e^{K/M_p^2}\left[K^{\bar{A}B}\overline{D_AW}D_BW - \frac{3|W|^2}{M_p^2}\right]. \tag{4.5}$$

Working to leading nontrivial order in $1/\tau$ we drop h/τ and higher orders

$$P \simeq \tau - k = \tau - \frac{\mathfrak{K}}{M_p^2}, \tag{4.6}$$

and the Kähler metric

$$K_{A\bar{B}} \simeq \frac{3M_p^2}{P^2}\begin{pmatrix} 1 & -k_{\overline{X}} \\ -k_X & Pk_{X\overline{X}} + k_Xk_{\overline{X}}\end{pmatrix}, \quad K^{\bar{B}A} \simeq \frac{P}{3M_p^2}\begin{pmatrix} P + k^{\overline{X}X}k_{\overline{X}}k_X & -k^X \\ -k^{\overline{X}} & k^{\overline{X}X}\end{pmatrix}, \tag{4.7}$$

where $z^A := \{T, X\}$ and subscripts on k indicate differentiation.

The kinetic terms for the physical scalars $z^I := \{T,\phi\}$ are given by the second derivatives of K

$$-\frac{L_{\text{kin scal}}}{\sqrt{-g}} = K_{T\overline{T}}\partial^\mu\overline{T}\partial_\mu T + K_{\phi\bar{\phi}}\partial^\mu\bar{\phi}\partial_\mu\phi + (K_{\phi\overline{T}}\partial^\mu\overline{T}\partial_\mu\phi + h.c.), \tag{4.8}$$

and the scalar potential is given by

$$V_F \simeq \frac{1}{P^2}\left[\frac{1}{3}\mathfrak{K}^{\overline{X}X}|\omega_X|^2 + \left(\frac{\mathfrak{K}^{\overline{X}X}\mathfrak{K}_{X\overline{T}}}{M_P^2}\omega_0\overline{\omega_X} + \text{ h.c. }\right) - \frac{3\mathfrak{K}_{T\overline{T}}}{1 + 2\mathfrak{K}^{X\overline{X}}\mathfrak{K}_X\mathfrak{K}_{\overline{X}}/M_p^2}\frac{|\omega_0|^2}{M_p^4}\right]. \tag{4.9}$$

4.1.1 Scalar potential

Let us analyze the scalar potential in detail

Case: $\mathfrak{K}_{X\overline{T}} = 0$

When we have that all ω_X-ω_0 cross terms vanish, the potential V_F (4.9) simplifies to

$$V_F \simeq \frac{1}{P^2}\left(\frac{1}{3}\mathfrak{K}^{\overline{X}X}|\omega_X|^2 - \frac{3\mathfrak{K}_{T\overline{T}}}{1 + 2\mathfrak{K}^{X\overline{X}}\mathfrak{K}_X\mathfrak{K}_{\overline{X}}/M_p^2}\frac{|\omega_0|^2}{M_p^4}\right), \tag{4.10}$$

and we know that the large contribution we want to avoid comes from the ω_X term. Let us suppose that it assumes the following form

$$\omega_X = g(\Phi\overline{\Phi} - v^2) = g(\Phi^2 - v^2), \tag{4.11}$$

in the proximity of $\Phi = v$, thus favouring the relaxon ϕ to minimise this term to zero, thus rendering ω_0 the dominant contribution. Moreover, assuming $\mathfrak{K}_{\Phi\bar{\Phi}}$ to be order unity, we have that the kinetic term of ϕ will be of the form $Z(\partial\phi)^2$, with $Z \sim \tau^{-1}$, giving a mass of order

$$m_\phi^2 \sim \frac{V_{\Phi\bar{\Phi}}}{Z_{\Phi\bar{\Phi}}} \sim \frac{(gv)^2}{\tau}\,, \tag{4.12}$$

Moreover, given with M and $\mu_0 = |\omega_0|^{1/3}$ two UV scales, the τ-dependence of ordinary particles in this scenario is given by

$$m_{\text{TeV}} \sim \frac{M}{\sqrt{\tau}}\,, \tag{4.13}$$

while, the value of the potential at his minimum (with $\omega_X = 0$ and $\mathfrak{K}_{X\overline{T}} = 0$) is

$$V_{\min} \sim \frac{\mathfrak{K}_{T\overline{T}}|\omega_0|^2}{\tau^2 M_p^4} \sim \frac{\epsilon^5 M^2 \mu_0^3}{\tau^4 M_P^2} \sim \left(\frac{m_{\text{TeV}}^2}{M_P}\right)^4\,, \tag{4.14}$$

where the factor ϵ^5 comes from the stabilisation mechanism of τ and will be analyzed in detail later.

In conclusion, $\omega_X = 0$ turns out to be a consequence of the non essential simplifying assumption $\mathfrak{K}_{X\overline{T}} = 0$.

Case: $\mathfrak{K}_{X\overline{T}} \neq 0$

Let us now study the low energy potential for τ after ϕ is already stabilised at his minimum.

Supposing that the only dependence on ϕ is in $|\omega_X|$, from V_F we can immediately write the minimum condition

$$\omega_X = -\frac{3\mathfrak{K}_{X\overline{T}}\,\omega_0}{M_p^2} \sim \frac{M\mu_0^3}{\tau M_P^2}\,, \tag{4.15}$$

thus ω_X is non-zero in the general case but is further suppressed by τ and by the Planck mass. From this information we can write the potential written as

$$V_F \simeq -\frac{3|\omega_0|^2}{\tau^2 M_P^4}\left[\mathfrak{K}^{\overline{X}X}\mathfrak{K}_{X\overline{T}}\mathfrak{K}_{\overline{X}T} - \frac{3\mathfrak{K}_{T\overline{T}}}{1 + 2\mathfrak{K}^{X\overline{X}}\mathfrak{K}_X\mathfrak{K}_{\overline{X}}/M_p^2}\frac{|\omega_0|^2}{M_p^4}\right] \equiv \frac{U}{\tau^4}\,, \tag{4.16}$$

where $U = U(\ln\tau)$ depends on the underlying UV details. Moreover, this shows that it is not mandatory to ask independence on ϕ, since in the scalar potential the leading term will always be of order τ^{-4}.

4.1.2 Auxiliary fields

We are now concerned on the consistency of the nilpotent non-linear realization, since we need that its F-term must be large. Using (4.15) we find the auxiliary field as

$$F^{\overline{X}} = e^{\frac{K}{2M_P^2}} K^{\overline{X}X} W_X \sim \frac{\omega_X}{\tau_{3/2}} \sim \frac{M\mu_0^3}{\tau^{3/2}M_p^2}\,, \tag{4.17}$$

which allow us to introduce the mass scale μ_X for scalar superpartner as $\mu_X^2 = F^X$, which must satisfy the bound given by phenomenology

$$\mu_X^2 \sim \frac{M\mu_0^3}{\tau^{3/2}M_p^2} > (10^4 \text{ GeV})^2\,, \tag{4.18}$$

along the SM scale which is given by

$$m_{\text{TeV}} \sim \frac{M}{\tau^{1/2}} \sim 10^3 \text{ GeV}\,, \tag{4.19}$$

and the value of the potential in the minimum given by

$$V_{\text{min}} \sim \frac{\epsilon^5 M^2 \mu_W^6}{\tau^4 M_p^4} \sim \left(10^{-11} \text{ GeV}\right)^2\,, \tag{4.20}$$

which can be combined as the bounds

$$\frac{\mu_X^2}{m_{\text{TeV}}} \sim \frac{\mu_0^3}{MM_P^2\sqrt{\tau}} \gtrsim 100\,, \qquad \frac{V_{\text{min}}}{\mu_X^4} \sim \frac{\epsilon^5}{\tau} \lesssim 10^{-60}\,. \tag{4.21}$$

This expressions of $m_{\text{TeV}} \sim M/\tau^{1/2}$ and $V_{\text{min}} = \epsilon^5 m_{\text{vac}}^4$ will determine the three input parameters μ_0, M, τ, thus, if we assume $M \sim M_P \sim 10^{18}$ GeV we find from (4.19)

$$m_{\text{TeV}} \sim \frac{10^{18} \text{ GeV}}{\sqrt{\tau}} \sim 10^3 \text{ GeV} \implies \tau \sim 10^{30}\,, \tag{4.22}$$

which in turn, applied to (4.21) gives

$$\frac{\mu_X^2}{m_{\text{TeV}}} \sim 10^{-15}\left(\frac{\mu_0}{M_P}\right)^3 \gtrsim 100 \implies \mu_0 \gtrsim 10^5 M_P\,, \tag{4.23}$$

$$\frac{V_{\text{min}}}{\mu_X^4} \sim 10^{30}\epsilon^5 \lesssim 10^{-60} \implies \epsilon \lesssim 10^{-5}\,. \tag{4.24}$$

Moreover, SM superpartners acquire masses through their coupling to F^X, while the gravitino responds to the total invariant order parameter F as

$$F^2 = K_{A\overline{B}} F^A \overline{F}^{\overline{B}} = e^{K/M_P^2} K^{A\overline{B}} D_A W \overline{D_B W} \sim e^{K/M_P}\frac{|W|^2}{M_P^2} \sim \left(\frac{\mu_0^3}{\tau^{3/2}M_P}\right)^2\,, \tag{4.25}$$

giving

$$m_{3/2} \sim \frac{F}{M_p} \sim \frac{\mu_0^3}{M_p^2\tau^{3/2}}\,, \tag{4.26}$$

which using our estimates gives

$$m_{3/2} \gtrsim 10^{-14} \text{ GeV}\,. \tag{4.27}$$

Lastly, let us concern with the relaxon mass, considering that in order for the relaxation to take place in the low energy EFT implies that m_ϕ needs to be small accordingly, i.e. smaller than the lightest particle whose vacuum energy contribution is dangerous, namely the electron, leading to

$$m_\phi \sim \frac{gv}{\sqrt{\tau}} \lesssim m_e \sim y_e m_{\text{TeV}} \sim \frac{y_e M}{\sqrt{\tau}} \simeq 0.5 \text{ MeV} \implies gv \lesssim m_e\sqrt{\tau} \sim y_e M\,, \tag{4.28}$$

where y_e is the Yukawa coupling for the electron, which using our estimates leads to

$$gv \lesssim 10^{13} \text{ GeV} \ll M \sim M_P\,. \tag{4.29}$$

Summing up, we identify three different EFTs:

1. **SM EFT**: valid up to the TeV scales, the SM degrees of freedom are here realized non-linearly since their super partner lie at above scales and are integrated out.

2. **ϕ EFT**: valid below the last massive dangerous SM particle, i.e. the electron, where the SM degrees of freedom are integrated out but the relaxon is still dynamical.

3. **Gravity SUSY sector**: valid up to the mass of the relaxon, here the gravitational sector must be a 4D EFT and is relevant for astrophysical and cosmological test.

We provide a picture to understand the different scales in Fig.4.1, based on an analogous figure in [3].

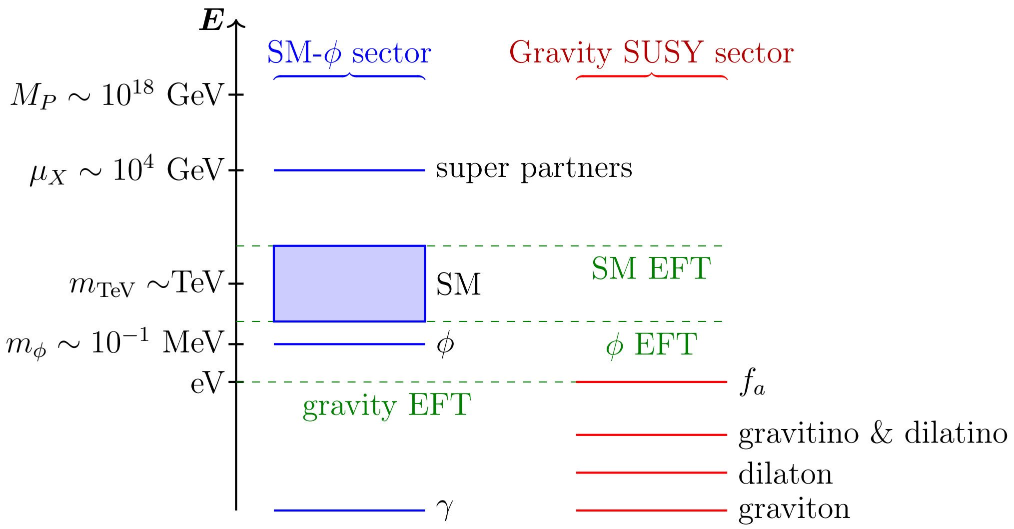


Figure 4.1: Scales involved in the relaxation models, with the two different sectors, the SM-relaxon and the gravity SUSY one. We have highlighted three important thresholds (dashed green) which denotes three different EFT regimes: SM, relaxon and gravity. Figure inspired from [3]

4.1.3 Axio-Dilaton

We now must examine the axio-dilaton multiplet and how they modify the physical observable. The kinetic term for the chiral field T is given by

$$-\frac{L_{\text{kin}}}{\sqrt{-g}} = \frac{3M_p^2}{4}\frac{(\partial\tau)^2 + (\partial a)^2}{\tau}\,, \tag{4.30}$$

where, canonically normalizing the field τ we get

$$-L_{\rm kin} = \frac{1}{2}(\partial\chi)^2 + \frac{3M_p^2}{4} f^2 (\partial a)^2 \,, \tag{4.31}$$

where $f = e^{-k_1\chi/M_p}$, leading to the canonical normalization

$$\tau = \tau_0 \, e^{k_1\chi/M_p} \,, \qquad k_1 = \sqrt{\frac{2}{3}} \,, \tag{4.32}$$

leading to, using our estimates done in section 4.1.2

$$\tau \sim 10^{30} \implies \chi \sim 70 \,. \tag{4.33}$$

Moreover, we know from (4.16) that the potential for the dilaton is given by

$$V_F \simeq \frac{U(\ln\tau)}{\tau^4} = U(\chi)\, e^{-k_2\chi/M_p} \qquad \text{with} \qquad k_2 \equiv 4\sqrt{\frac{2}{3}} = 4k_1 \,. \tag{4.34}$$

Since the potential is an exponential, its derivatives near the minimum are of the same order of the potential itself

$$V_0' \equiv V'(\chi_0) \sim -\frac{k_2 V_0}{M_p} \qquad V_0'' \equiv V''(\chi_0) \sim \frac{k_2^2 V_0}{M_p^2}. \tag{4.35}$$

Furthermore, it is important to note that in our model building we are to set V_0 to the present scale of Dark Energy density implying that the Hubble scale H_0 should be mainly driven by the potential V, in fact $H_0^2 \simeq V_0/(3M_p^2)$ which means that $V_0 \simeq (10^{-2}\ \text{eV})^4$ which implies that the dilaton mass is of order $m_\chi^2 \sim V_0'' \sim H_0^2$ where $H_0 \sim 10^{-32}$ eV.

Now, focusing on the axion a one can see that the axion kinetic term in (4.31) can be viewed as a χ dependent axion decay constant f_a, that evaluated with our estimates leads to

$$f_a \sim \frac{M_p}{\tau} \sim 10^3\ \text{eV}. \tag{4.36}$$

In String Theory f_a is similarly suppressed with $f_a/M_p \propto 1/\tau \propto g_s$, where g_s is the string coupling, however, it is relevant to note that this "new physics" doesn't play any role when it is applied to energies much smaller than f_a.

4.2 String inspired τ and stabilisation

Type IIB flux compactifications on Calabi-Yau orientifolds allow us in a explicit way to connect the extra dimensional models to the string vacua [47, 52, 53, 59]. They enjoy the accidental scaling invariance of supergravity which follows from the $SL(2,\mathbb{Z})$ symmetry of the axio-dilaton in the 10D EFT of type IIB. Thus, an important role is played by the moduli, since they are naturally light. In fact, several moduli can arise, and the exact number depends on the underlying property of the compactified 6D manifold, but there appear always a modulus associated to the overall volume of the extra dimensions Ω_6

$$\mathcal{V} = M_s^6\, \Omega_6 \,, \tag{4.37}$$

where M_s is the string mass scale. One can interpret this modulus $\mathcal{V}$ as related to the field τ, since they both are required to be large, in fact $\mathcal{V} \gg 1$ is a consistency requirement for the supergravity EFT to be under control and one expands in powers of $\mathcal{V}^{-1}$. It is relevant to see that the volume modulus is the one that allow us to define the size of the string and $4D$ Planck scales as

$$M_s \sim \frac{M_p}{\mathcal{V}^{1/2}}, \tag{4.38}$$

and, assuming the 6D compactified manifold has an homogeneous size L, we can define the Kaluza-Klein scale as

$$M_{KK} = \frac{1}{L} \sim \frac{M_s}{\mathcal{V}^{1/6}} \sim \frac{M_P}{\mathcal{V}^{2/3}}\,. \tag{4.39}$$

Typical Kähler potential in Type IIB is given by

$$K = -2\ln\mathcal{V}\,, \tag{4.40}$$

while the one we used in our model is given by

$$K = -3\ln\tau\,, \tag{4.41}$$

thus, if we identify

$$\mathcal{V} \sim \tau^{3/2}, \tag{4.42}$$

we recover the kinetic terms described in (4.30), and the mass scales now read

$$M_S = \frac{M_P}{\tau^{3/4}}\,, \qquad M_{KK} = \frac{M_P}{\tau}\,. \tag{4.43}$$

Using our estimated value of $\tau \sim 10^{30}$ we will obtain

$$\mathcal{V} \sim 10^{45}\,, \quad M_S \sim 10^{-5}\ \text{GeV}\,, \quad M_{KK} \sim 10^{-12}\ \text{GeV}\,, \tag{4.44}$$

however, one should impose that M_S and M_{KK} must be at UV scales, i.e.

$$M_S \gtrsim 10^4\ \text{GeV} \implies \tau \lesssim 10^{18}\,, \tag{4.45}$$

$$M_{KK} \gtrsim 10^4\ \text{GeV} \implies \tau \lesssim 10^{14}\,, \tag{4.46}$$

even if the constraint from M_{KK} can be relaxed if one allows for asymmetric compactifications, this bounds seems to rule out any possible identification of the type (4.42). One possible way out is to employ warping [59–61], which happens when the 4D metric has an overall scale factor which depends on the internal manifold coordinates y as

$$g_{\mu\nu}(x,y) = e^{2A(y)} g^{(0)}_{\mu\nu}(x)\,, \tag{4.47}$$

which would introduce extra suppression factors on the mass scales, justifying the stricter bounds on $\mathcal{V}$ which translates to bounds on τ. Let us remark that the nilpotent superfield provide a description of anti D3-branes near the tip of the throat generated by warping, giving a mechanism for SUSY breaking which we employed in this relaxed model [62–65].

4.3 Scalar Tensor Theory

Let us now describe the scalar-tensor theory of gravity and the definition of the parameterized post Newtonian parameters which will give us the test for gravity of our model, based on solar system data [66]. We will follow the derivation done in [58].

4.3.1 Brans-Dicke model

The Brans-Dicke theory is the simplest theory that satisfy the equivalence principle. The main concept of this theory is that the Brans-Dicke scalar couples to matter through the Jordan frame metric

$$\tilde{g}_{\mu\nu} = A^2(\phi) g_{\mu\nu} \,, \tag{4.48}$$

that it is related to the Einstein frame metric by

$$A(\phi) = e^{\mathfrak{g}\,\phi/M_p} \,, \tag{4.49}$$

where $A(\phi)$ is the Weyl rescaling factor and $\mathfrak{g}$ is associated to the Brans-Dicke coupling ω by $2\mathfrak{g}^2 = 1/(3+2\omega)$, that comes from [67].

The Brans-Dicke model is a specific physical model in which a light scalar ϕ couples to matter and gravity via a Lagrangian density of the form

$$\mathcal{L} = -\sqrt{-g}\left(\frac{M_P^2}{2}\mathcal{R} + \frac{1}{2}(\partial\phi)^2 + V(\phi)\right) + \mathcal{L}_m(\tilde{g}_{\mu\nu}, \tilde{\psi}) \,, \tag{4.50}$$

where $\mathcal{R}$ is the Ricci scalar, $M_P^{-2} = 8\pi G$, $\tilde{\psi}$ is a representative matter field and ϕ is the Einstein frame scalar field. It is important to note that $\mathcal{L}_m$, which is the matter Lagrangian, does not contain any ϕ dependence except that through the conformal factor $A(\phi)$, and we are further assuming that there is no mass term for ϕ. Hence, one can build the matter stress-energy tensor in the Jordan and Einstein frame as

$$\tilde{T}^{\mu\nu} := \frac{2}{\sqrt{-\tilde{g}}}\frac{\delta S_m}{\delta \tilde{g}_{\mu\nu}} \,, \qquad T^{\mu\nu} := \frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g_{\mu\nu}} \,, \tag{4.51}$$

and using (4.48) we can relate $T^{\mu\nu}$ with $\tilde{T}^{\mu\nu}$ by

$$\tilde{g}_{\mu\nu} = A^2\, g_{\mu\nu} \implies g_{\mu\nu} = A^{-2}\, \tilde{g}_{\mu\nu} \,, \tag{4.52}$$

$$T^{\mu\nu} = \frac{2A^4}{\sqrt{-\tilde{g}}}\frac{\delta S_m}{\delta \tilde{g}_{\mu\nu}} A^2 = A^6\, \tilde{T}^{\mu\nu} \,. \tag{4.53}$$

Let us note that in the case we are considering a perfect fluid, thus we have

$$T^{\mu\nu} = (\rho + p) U^\mu U^\nu + p g^{\mu\nu} \,, \tag{4.54}$$

and, an analogous formula is valid for $\tilde{T}^{\mu\nu}$, where ρ is the density, p is the pressure and U^μ is the 4-velocity of the observer co-moving with the fluid, with normalization given by $g_{\mu\nu}U^\mu U^\nu = -1$. Hence, from equation (4.53) we can write

$$\rho = g_{\mu\nu}T^{\mu\nu} = g_{\mu\nu}A^6\tilde{T}^{\mu\nu} = A^{-2}\tilde{g}_{\mu\nu}A^6\tilde{T}^{\mu\nu} = A^4\tilde{\rho} \,, \tag{4.55}$$

and the same discussion can be done for the pressure. Thus, we can write the dilaton equation of motion as

$$\Box\phi = -\frac{\mathfrak{g}}{M_p} g_{\mu\nu}T^{\mu\nu} \,, \tag{4.56}$$

while the Einstein's equation takes the form

$$\mathcal{R}_{\mu\nu} + \frac{1}{M_p^2}\partial_\mu\phi\,\partial_\nu\phi + \frac{1}{M_p^2}\Big(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\Big) = 0 \,. \tag{4.57}$$

In the non-relativistic limit, for which $\rho \gg p$ and $\mathcal{R}_{tt} \simeq \nabla^2\Phi$ where $\Phi \sim g_{tt}$ is the Newtonian potential, (4.56) and (4.57) becomes respectively

$$\nabla^2\phi - \frac{\mathfrak{g}\,\rho}{M_p} \simeq 0\,, \tag{4.58}$$

$$\nabla^2\Phi - \frac{\rho}{2M_p^2} = \nabla^2\Phi - 4\pi G\rho \simeq 0\,, \tag{4.59}$$

which immediately tells us that ϕ and Φ satisfy the same equation, leading to the general solution

$$\varphi = \varphi_\infty + 2\mathfrak{g}\,\rho\,\Phi\,, \tag{4.60}$$

with $\varphi \equiv \phi/M_p$ is the dimensionless field. Hence, if we assume that we are in the presence of a spherical symmetric star of constant density ρ, in the exterior the solution would be given by

$$\varphi = \varphi_\infty - \frac{2\mathfrak{g}GM}{r} \equiv \varphi_\infty + \frac{\tilde{\varphi}}{r}\,, \tag{4.61}$$

while an analogous description for the interior of the star is difficult to find due to the dependence of $\rho \propto \phi$ through the scale factor $A(\phi)$ as one can see from (4.55).

It is important to remark that in this theory, the metric on the kinetic term of the test particles, is the Jordan frame metric $\tilde{g}_{\mu\nu}$, which means that particles move along its geodesic and not the one derived from $g_{\mu\nu}$. Thus, the parameterized post Newtonian (PPN) framework shows the deviation from the standard behaviour obtained from General Relativity and the results predicted using $\tilde{g}_{\mu\nu}$ are given by

$$\tilde{g}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -\left[1 - \frac{2GM}{r} + 2\,(\beta_{PPN} - \gamma_{PPN})\left(\frac{GM}{r}\right)^2 + \ldots\right]\mathrm{d}t^2 + $$
$$+\left[1 + 2\gamma_{PPN}\left(\frac{GM}{r}\right) + \ldots\right]\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,, \tag{4.62}$$

where the Schwarzschild solution is recovered by the choice $\gamma_{PPN} = \beta_{PPN} = 1$, where the Cassini probe data [68] gives the bound

$$|\gamma_{PPN} - 1| < 2.3 \times 10^{-5}\,, \tag{4.63}$$

where $\gamma_{PPN} = 1$ is the predicted of GR. Now, we want to show that the Weyl scaling factor $A(\phi)$ is the one that dominates at leading order in GM/r. First, let us take (4.57) without any external source

$$\mathcal{R}_{\mu\nu} + \partial_\mu\varphi\,\partial_\nu\varphi = 0\,, \tag{4.64}$$

and using the solution in (4.61) and metric parameterized as

$$\mathrm{d}s^2 = -e^{2u(r)}\,\mathrm{d}t^2 + e^{2v(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,, \tag{4.65}$$

we obtain

$$R_{rr} = u'' + (u')^2 - u'v' - \frac{2v'}{r} = -(\varphi')^2 = -\frac{\tilde{\varphi}^2}{r^4}\,, \tag{4.66}$$

$$R_{tt} = e^{2(u-v)}\left[-u'' - (u')^2 + u'v' - \frac{2u'}{r}\right] = 0\,, \tag{4.67}$$

$$R_{\theta\theta} = -1 + e^{-2v}\left[1 + r(u' - v')\right] = 0\,, \tag{4.68}$$

thus, from (4.66) and (4.67) we find

$$u + v = -\frac{\tilde{\varphi}^2}{4r^2}, \tag{4.69}$$

with the conditions that when $r \longrightarrow \infty$ we have $u + v \longrightarrow 0$. Hence, from (4.68) we can write a solution for $u(r)$ in powers of r^{-1} as

$$u(r) = -\frac{l}{r} - \frac{l^2}{r^2} + \frac{3l\tilde{\varphi}^2 - 16l^3}{12r^3} + \mathcal{O}\left(r^{-4}\right), \tag{4.70}$$

where l is an integration constant. Now, expanding in powers of r^{-1} the metric components we get

$$e^{2u} \simeq 1 - \frac{2l}{r} + \frac{l\tilde{\varphi}^2}{2r^3} + \mathcal{O}\left(r^{-4}\right), \tag{4.71}$$

$$e^{2v} \simeq 1 + \frac{2l}{r} + \frac{8l^2 - \tilde{\varphi}^2}{r^2} + \frac{16l^3 - 3l\tilde{\varphi}^2}{2r^3} + \mathcal{O}\left(r^{-4}\right), \tag{4.72}$$

while the Weyl factor turns out to be

$$A(r) = A_\infty \left[1 + \frac{a_1}{r} + \frac{a_2}{r^2} + \mathcal{O}\left(r^{-3}\right)\right]. \tag{4.73}$$

Thus, the metric will be

$$\tilde{g}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = A_\infty^2 \left[1 + \frac{2a_1}{r} + \frac{a_1^2 + 2a_2}{r^2} + \mathcal{O}\left(r^{-3}\right)\right]\left(e^{2u}\,\mathrm{d}t^2 + e^{2v}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\right), \tag{4.74}$$

from which one can rescale the time coordinate and define the new coordinate $\tilde{r}$, as

$$t \to \tilde{t} \equiv A_\infty t\,, \qquad r \to \tilde{r} \equiv A(r)\,r = A_\infty\left[r + a_1 + \frac{a_2}{r} + \mathcal{O}\left(r^{-2}\right)\right], \tag{4.75}$$

which inverted is

$$t = \frac{\tilde{t}}{A_\infty}\,, \qquad r = \frac{\tilde{r}}{A_\infty} - a_1 - \frac{a_2 A_\infty}{\tilde{r}}\,, \tag{4.76}$$

from which the metric will be

$$\begin{aligned}\tilde{g}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = &-\left[1 - \frac{2(l-a_1)}{\tilde{r}}A_\infty + \mathcal{O}\left(\tilde{r}^{-2}\right)\right]\mathrm{d}\tilde{t}^{\,2} + \\ &+\left[1 + \frac{2(l+a_1)}{\tilde{r}}A_\infty + \mathcal{O}\left(\tilde{r}^{-2}\right)\right]\mathrm{d}\tilde{r}^{\,2} + \tilde{r}^2\,\mathrm{d}\Omega^2\,. \end{aligned}\tag{4.77}$$

If we now define $GM = (l - a_1)A_\infty$, $\tilde{a}_1 \equiv a_1 A_\infty$ and $\tilde{a}_2 \equiv a_2 A_\infty^2$ we have

$$g_{\tilde{t}\tilde{t}} \simeq -\left(1 - \frac{2GM}{\tilde{r}}\right), \qquad g_{\tilde{r}\tilde{r}} \simeq 1 + \frac{2\left(GM + 2\tilde{a}_1\right)}{\tilde{r}}, \tag{4.78}$$

and comparing it with (4.62) we can find the form of γ_{PPN}

$$\gamma_{PPN} = 1 + \frac{2\tilde{a}_1}{GM} = \frac{l + a_1}{l - a_1}\,. \tag{4.79}$$

In our case, $\tilde{\varphi} = -2\mathfrak{g}GM$ which means that $\varphi = \varphi_\infty - \frac{2\mathfrak{g}l}{r}$ and since $A = e^{\mathfrak{g}\varphi}$ and $A_\infty = e^{\mathfrak{g}\varphi_\infty}$, we have

$$A = e^{\mathfrak{g}\varphi} = e^{\mathfrak{g}\varphi_\infty} e^{-2\mathfrak{g}l/r} = A_\infty e^{-2\mathfrak{g}l/r}\,, \tag{4.80}$$

and $a_1 = -2\mathfrak{g}^2 l$. Substituting in γ_{PPN} we have

$$\gamma_{PPN} = \frac{1-2\mathfrak{g}^2}{1+2\mathfrak{g}^2} = \frac{\omega+1}{\omega+2}\,, \tag{4.81}$$

with $\frac{1}{2\mathfrak{g}^2} = 2\omega + 3$, from [67], whose bounds from (4.63) are

$$|\mathfrak{g}| < 5.8 \times 10^{-6} \quad \text{or} \quad |\omega + 2| < 43478.3\,. \tag{4.82}$$

Let us remark that in order to compute the β_{PPN} parameter one should keep track more carefully of the r^{-2} terms in the expansions which will eventually lead to $\alpha_2 = 1/2\alpha_1^2$ and $\alpha_1 = -4\mathfrak{g}^2 l$. Hence, when the Einstein-frame metric is Schwarzschild we have

$$\beta_{PPN} = \frac{l^2 + l\alpha_1 + 1/2\alpha_2}{(l + 1/2\alpha_1^2)} = 1, \tag{4.83}$$

where in the last equation we have used $\alpha_2 = \frac{1}{2}\alpha_1^2$.

4.3.2 Axio-Dilaton cosmological model

Now, let us analyze in detail a simple example in which on top of the usual Brans-Dicke scalar we add an axion to the field content. We will take the chiral super field $T = \frac{1}{2}(\tau + \mathrm{i}a)$, for which the Kähler potential is

$$K = -3\ln\left(T + \overline{T}\right) = -3\ln\tau\,, \tag{4.84}$$

and it is to see that Kähler metric is given by

$$K_{T\overline{T}} = \partial_T \partial_{\overline{T}} K = -3\,\partial_T \Big(T + \overline{T}\Big)^{-1} = \frac{3}{\tau^2}\,, \tag{4.85}$$

and

$$\Gamma^T_{TT} = K^{\overline{T}T} K_{TT\overline{T}} = \frac{\tau^2}{3}\partial_T \frac{3}{(T+\overline{T})^3} = -\frac{2}{\tau}\,. \tag{4.86}$$

Now, we choose as particular scale factor

$$A = e^{K/6} \implies A = \frac{1}{\sqrt{\tau}}\,. \tag{4.87}$$

In detail, we have that

$$T = \frac{1}{2}(\tau + \mathrm{i}a)\,, \tag{4.88}$$

for which

$$K = 3 - \ln\big(T + \bar{T}\big) = -3\ln(\tau) \implies A = \frac{1}{\sqrt{\tau}}\,, \tag{4.89}$$

and, asking that $g_{\mu\nu}T^{\mu\nu} = -\rho$, in absence of a scalar potential for τ and a, we obtain

$$\Box T - \frac{2}{\tau}\partial_\mu T \partial^\mu T + \frac{\tau}{6M_p^2}\rho = 0\,, \tag{4.90}$$

with real and imaginary parts given by

$$\Box a - \frac{2}{\tau}\partial_\mu \tau \partial^\mu a = 0\,, \tag{4.91}$$

$$\Box \tau - \frac{1}{\tau}\Big(\partial_\mu \tau \partial_\mu \tau - \partial_\mu a \partial^\mu a\Big) + \frac{\tau\rho}{3M_p^2} = 0\,. \tag{4.92}$$

Hence, canonically normalizing as $\tau = e^{\zeta\phi}$, where we have that $\zeta = \sqrt{2/3}$, (4.92) will give us

$$\Box \phi = -\frac{\rho}{3\zeta M_P^2}\,, \tag{4.93}$$

which compared with (4.58) gives us the value of

$$\mathfrak{g} = -\frac{1}{\sqrt{6}} = -0.408\,, \tag{4.94}$$

which will be our benchmark value for the numerical analysis in the next chapter. Moreover, there we will analyze in detail a similar model in which we will add the findings of section 4.1 in order to study its cosmological evolution as a dynamical system.

Chapter 5

Solutions with Axio-Dilaton and Matter Coupling

In this chapter we will focus on the case where we consider two scalar fields, that are the dilaton and the axion, with an axion source, following the work [3]. In detail we are going to study the evolution of the universe as a first order dynamical system, i.e. we are going to generalize the procedure as in Chap.3.

We will take into consideration the relaxed model but in the gravity sector, i.e. only with the degrees of freedom of the axio-dilaton multiplet, which has a non-minimal kinetic term, the coupling between the axion and matter and of course gravity.

5.1 Axion Dilaton Cosmology

Now we are going to study the dilaton evolution, in the presence of a specific potential $V(\chi)$, which provides a particular type of quintessence model.

The class of models which are relevant to our analysis contains a coupling between the Einstein gravity and two scalar fields, χ and a, which are called the dilaton and the axion respectively, from which we can build the following action

$$S[g_{\mu\nu}, \chi] = \int \mathrm{d}^4x \sqrt{-g} \left(\frac{M_p^2}{2}\mathcal{R} - \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}f(\chi)^2(\partial a)^2 - V(\chi) \right) + S_m(\tilde{g}_{\mu\nu}, \Psi) \,, \quad (5.1)$$

$\mathcal{R}$ is the Ricci scalar, $f(\chi)$ is the kinetic coupling, defined as $f = e^{-k_1\chi/M_P}$.

Hence, assuming a spatially flat FLRW spacetime, the field equations describing cosmology obtained by varying the action, setting $\hat{\chi} = \chi/M_P$, are

$$\mathcal{R}_{\mu\nu} + e^{-2k_1\hat{\chi}}\partial_\mu a\, \partial_\nu a + \partial_\mu\hat{\chi}\, \partial_\nu\hat{\chi} + \frac{1}{M_p^2}\left[V(\hat{\chi})g_{\mu\nu} + T_{\mu\nu} - \frac{1}{2}g^{\lambda\rho}T_{\lambda\rho}g_{\mu\nu}\right] = 0 \,, \quad (5.2)$$

$$\Box\hat{\chi} + k_1\, e^{-2k_1\hat{\chi}}\, \partial_\mu a\, \partial^\mu a + \frac{1}{M_p^2}\left(-\frac{\partial V}{\partial\hat{\chi}} + \mathfrak{g}\, T\right) = 0 \,, \quad (5.3)$$

$$\left(\ddot{a} + 3H\dot{a} - \frac{2k_1}{M_p^2}\dot{\hat{\chi}}\dot{a}\right) e^{-2k_1\hat{\chi}} - \frac{J}{3M_p^2} = 0 \,, \quad (5.4)$$

where $\frac{\delta S_m}{\delta a} = J$, is the axion source, defined proportional to the number density of the baryonic matter n_b and the axion-matter coupling g_a, $J = g_a n_b$. In addition, it is also important to define the Friedmann equation

$$H^2 = \frac{\rho}{3M_p^2} = \frac{1}{M_p^2}\Big[\rho_f + \rho_a + \frac{\dot{\hat{\chi}}^2}{2} + V\Big]\,, \tag{5.5}$$

which is a constraint and where ρ_f is the density of the cosmological fluid, given by

$$\rho_f = \rho_m + \rho_{rad} = \rho_{DM} + \rho_b + \rho_{rad} \sim \rho_{DM} + \rho_b\,, \tag{5.6}$$

that satisfies

$$\rho_f = \rho_m(\chi) + \rho_{\rm rad}\,, \qquad P_f = P_{\rm rad} = \frac{1}{3}\rho_{\rm rad}\,, \tag{5.7}$$

where we are stressing the dependence of ρ_m on the dilaton field which is obtained considering that in the Einstein frame the number density is given by $\sqrt{-g}\,n$ giving $n \sim \exp(-3N)$ and that the mass is

$$m(\hat{\chi}) = \tilde{m}\,A(\hat{\chi}) \sim e^{-\frac{1}{2}k_1\hat{\chi}}\,, \implies \rho_m(\hat{\chi}) = n\,m(\hat{\chi}) = \rho_{m0}e^{-3N-\frac{1}{2}k_1\hat{\chi}}\,. \tag{5.8}$$

For completeness, one can then derive the variation of the Hubble constant as

$$\dot{H} = -\frac{1}{2M_P^2}\left(\rho_m + \frac{4}{3}\rho_{\rm rad} + \rho_a + \frac{\dot{\hat{\chi}}^2}{2}\right)\,. \tag{5.9}$$

Thus, the dilaton equation can be written as

$$\ddot{\hat{\chi}} + 3H\dot{\hat{\chi}} + k_1 e^{-2k_1\hat{\chi}}\dot{a}^2 + \frac{1}{M_p^2}\Big[V'(\hat{\chi}) + \mathfrak{g}\rho_m(\hat{\chi})\Big] = 0\,. \tag{5.10}$$

While, the scalar potential is assumed to be

$$V = V_0\, e^{-k_2\hat{\chi}}\,, \tag{5.11}$$

where V_0 is a constant, which can be seen as a particular case of (4.16), with $U = V_0$.

Now, if we choose this specific potential and the initial conditions, we can study the evolution of the energies of radiation, matter and the scalar potential. Suppose we take the case of a small and negative $\mathfrak{g}$, what we obtain in the case of $\mathfrak{g} = -10^{-5}$, $k_1 = \sqrt{\frac{2}{3}}$ and $k_2 = 4k_1$, is the evolution in Fig.5.1.

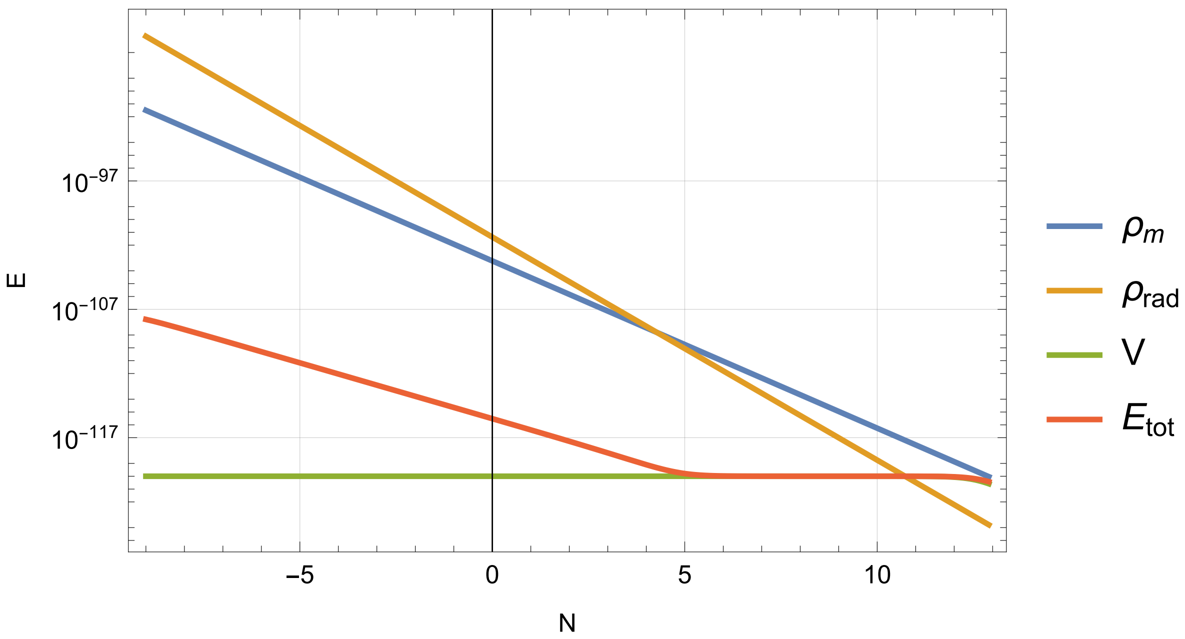


Figure 5.1: Log-Log plot of the energy densities (E) of matter (blue), radiation (orange), dilaton potential (green) and total (red) energy in function of the number of e-folds (N), in the case with no axion evolution, and with $\mathfrak{g} = -10^{-5}$, where we are taking an exponential potential.

Where we have been taking the following initial conditions

$$\rho_{m0} = 10^{-90}, \quad \rho_{\rm rad0} = 10^{-84}, \quad V_0 = 10^{-120}\, e^{-k_2 \hat{\chi}_0}, \quad \hat{\chi}_0 = 74, \quad \rho_{\chi 0} = 10^{-108.7}. \quad (5.12)$$

Thus, from this evolution we can clearly see that at a certain point all the dilaton kinetic energy becomes small and just the potential energy dominates, remaining a constant. This point correspond approximately to the change from radiation to matter domination. However, if one allows the system to evolve for longer we will reach the point where the matter energy density becomes comparable with the potential and the dilaton's total energy which start to decay at the same rate of ρ_m, as one can see from Fig.5.2. This is to be interpreted as the damping effect coming from the term $\mathfrak{g}\rho_m$ from (5.10) which no longer compensates the other terms leading to the runway of the field $\hat{\chi}$.

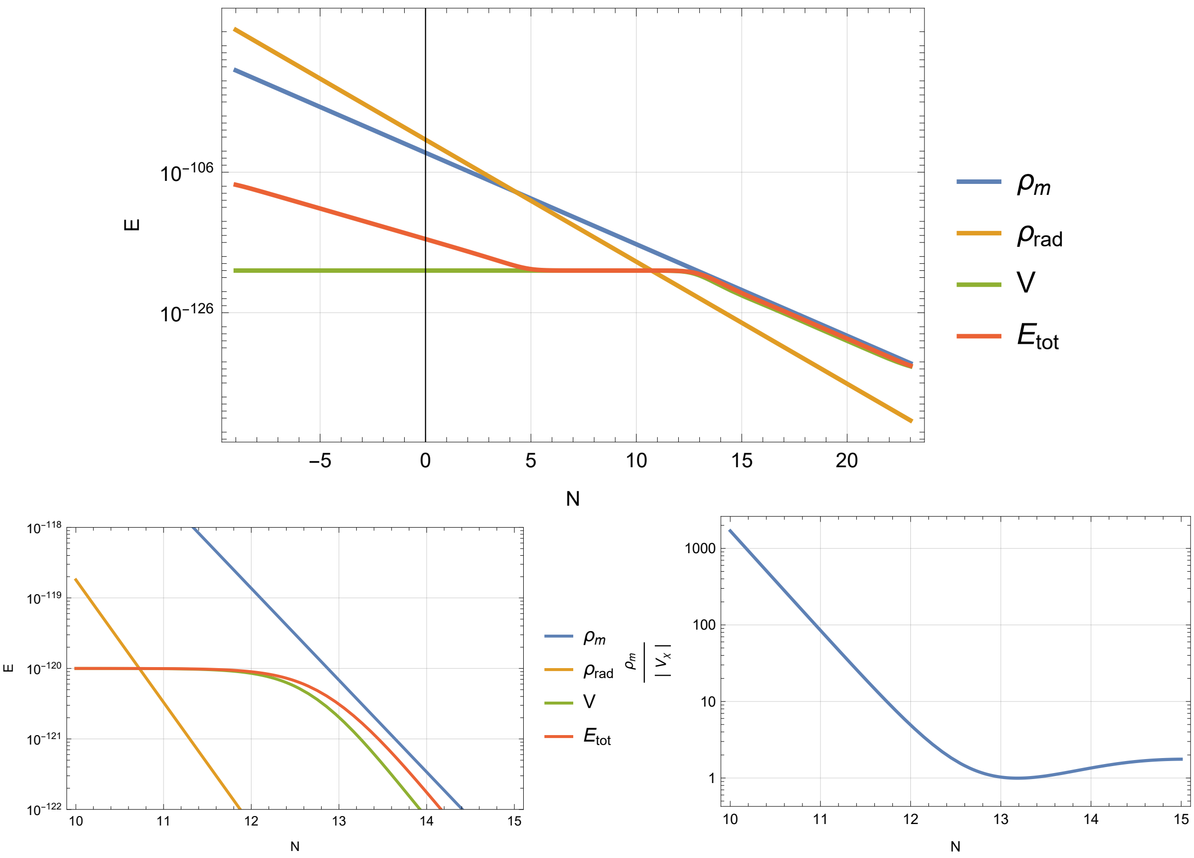


Figure 5.2: Log-Log plot of the energy densities (E) of matter(blue), radiation (orange), dilaton potential (green) and total energy (red) in function of the number of e-folds (N), in the case with no axion evolution, and with $\mathfrak{g} = -10^{-5}$, taking the case of an exponential potential. With a particular zoom on the final part of the plot and a zoom in the ratio between the matter radiation and potential in function of the number of e-folds .

Now, instead, suppose we take the case of the axion evolution equal to zero again but with a value of $\mathfrak{g} = -\frac{1}{2}k_1 \sim -0.408$, where again we are considering $k_1 = \sqrt{\frac{2}{3}}$ and $k_2 = 4k_1$, which corresponds to the case of an *axio-dilaton model*, as phenomenologically argued in section 4.1. In this particular scenario, we can make the same type of study that we have done before. Thus, we obtain the evolution in Fig.5.3, where for the evolution of the densities we have the initial conditions

$$\rho_{m0} = 10^{-96.5}, \quad \rho_{\text{rad}0} = 10^{-92}, \quad V_0 = 10^{-120}\, e^{-k_2\hat{\chi}_0}, \quad \hat{\chi}_0 = 74, \quad \rho_{\chi 0} = 10^{-102}, \quad (5.13)$$

and now the total energy of the dilaton (orange line) does not become only potential energy at a certain point and the potential (green line) itself is not a constant, which means that we have an attractor scaling evolution, see Chap.3. In detail, from Fig.5.4, we can see that if we let evolve the system, the matter energy density becomes comparable with the total energy of the dilaton, which decay at the same rate of ρ_m. Hence, again we have a damping effect due to the $g\rho_m$ term.

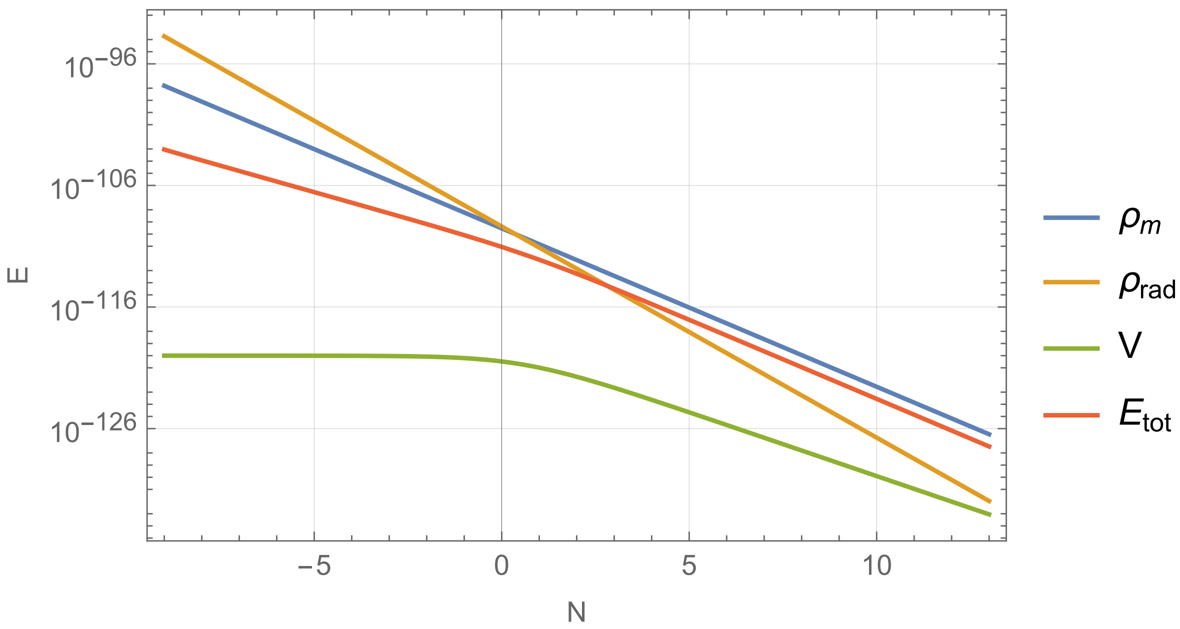


Figure 5.3: Log-Log plot of the energy densities (E) of matter (blue), radiation (orange), dilaton potential (green) and total (red) energy in function of the number of e-folds (N), in the case with no axion evolution, and with $\mathfrak{g} = -0.408$, where we are taking an exponential potential.

In the end we can also study the case where we take an axion evolution which is now different from zero. We keep the value of $\mathfrak{g} \simeq -0.408$, since is the most relevant one from the phenomenological point of view. Again, we considered the value of $k_1 = \sqrt{\frac{2}{3}}$, $k_2 = 4k_1$, and we take the following initial conditions

$$\rho_{m0} = 10^{-90}, \quad \rho_{\text{rad}0} = 10^{-84}, \quad V_0 = 10^{-120}\, e^{-k_2 \hat{\chi}_0}, \quad \hat{\chi}_0 = 74, \quad \rho_{a0} = \rho_{\chi 0} = 10^{-98}\,, \tag{5.14}$$

What we obtain is the evolution in Fig.5.5. From this plot we can see that the evolutions for the energies density of matter and of radiation are the same, since the equations are not changed. The biggest differences are that now we have a representation of the evolution of the total axion energy. Which it is smaller in respect to the total energy of the dilaton in the radiation era, while is bigger in the matter era. Noting that the axion energy executes a crossover

$$\rho_a \propto a^{-2} \qquad \text{to} \qquad \rho_a \propto a^{-3}\,, \tag{5.15}$$

when we pass from the radiation to the matter era. In detail, we can note that the energy density of the dilaton potential remains almost constant until when the energy density of the axion becomes bigger than the one of the dilaton.

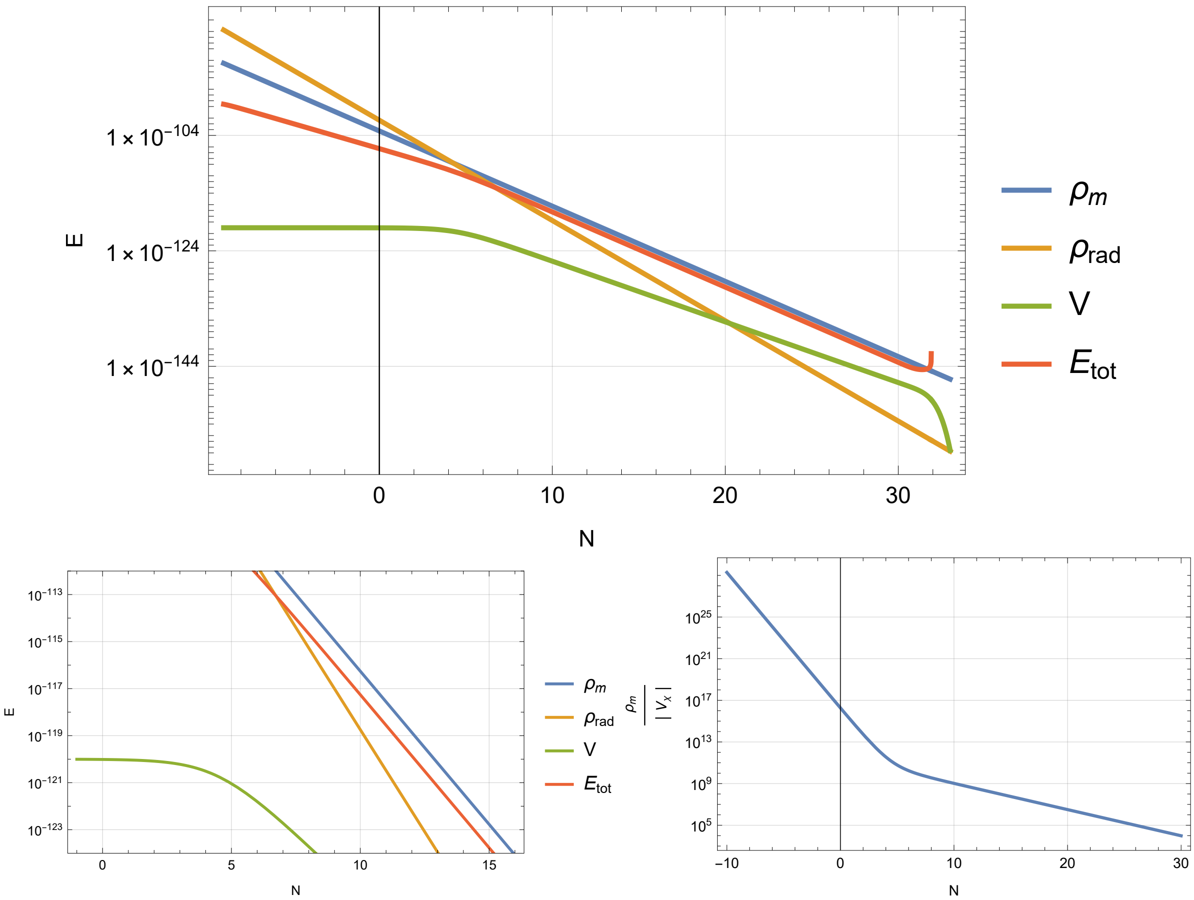


Figure 5.4: Log-Log plot of the energy densities (E) of matter(blue), radiation (orange), dilaton potential (green) and total energy (red) in function of the number of e-folds (N), in the case with no axion evolution, and with $\mathfrak{g} = -0.408$, taking the case of an exponential potential. With a particular zoom on the final part of the plot.

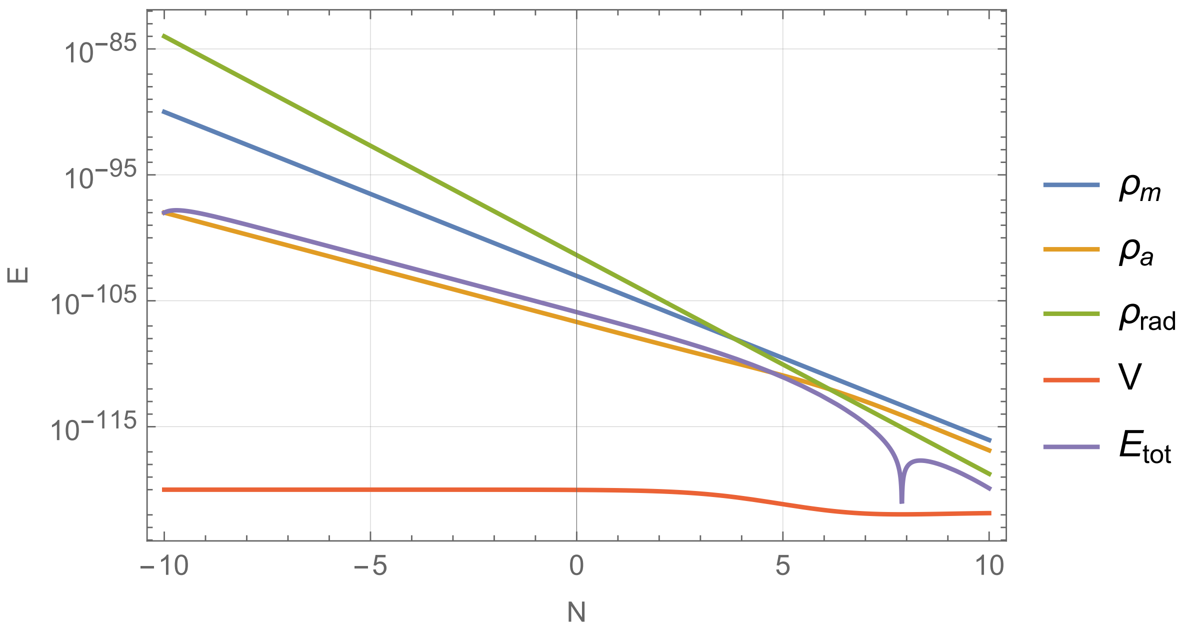


Figure 5.5: Log-Log plot of the energy densities (E) of matter (blue), radiation (green), dilaton potential (dark orange), total axion energy density (light orange) and total (purple) dilaton energy density in function of the number of e-folds (N), in the case with axion evolution equal to one, and with $\mathfrak{g} = -0.408$, where we are taking an exponential potential.

5.2 Dynamical system of the axio-dilaton model

Let us now employ the techniques of section 3.1 in order to develop a generalisation to the axio-dilaton model and derive its dynamical system.

5.2.1 Analytical derivation

An important fact to note from the action (5.1) is that the field a is a flat direction of V which only couples kinetically to χ. The equations of motion, taking for simplicity a single fluid, and only the case of ρ_m, are given by

$$\begin{cases} \ddot{\chi} = -3H\dot{\chi} + f_\chi f \dot{a}^2 - V_\chi - \frac{\mathfrak{g}\rho_m}{M_p}, \\ \ddot{a} = -3H\dot{a} - 2\frac{f_\chi}{f}\dot{\chi}\dot{a} + \frac{J}{3M_p f^2}, \end{cases} \tag{5.16}$$

Recalling that

$$J = g_a n_b, \tag{5.17}$$

and where V_χ define the derivative of the potential in respect to χ.

Thus, in order to study the dynamical evolution of the universe, we can define the following dimensionless variables

$$x_1 \equiv \frac{\dot{\chi}}{\sqrt{6}M_p H}, \quad x_2 \equiv \frac{f\dot{a}}{\sqrt{6}M_p H}, \quad x_3 \equiv \frac{\sqrt{V}}{\sqrt{3}M_p H}, \quad z \equiv \sqrt{\frac{J}{3\sqrt{6}f}\frac{1}{HM_p}}\,. \tag{5.18}$$

Thus, recalling that

$$x' \equiv \frac{dx}{dt}\frac{dt}{dN} = \frac{\dot{x}}{H}, \tag{5.19}$$

noting that "dot" means derivation in respect to time, while "prime" means derivation in respect to N. Hence, we can see that the dimensionless variables, where we consider only ρ_m for the sake of pursuing an analytic study, obey the following dynamical equations

$$\begin{aligned}
x_1' &= \sqrt{6}\left(-k_1 x_2^2 + \frac{1}{2}k_2 x_3^2\right) - \sqrt{\frac{3}{2}}\,\mathfrak{g}\left(1 - x_1^2 - x_2^2 - x_3^2\right) + \frac{3}{2}x_1\left(-1 + x_1^2 + x_2^2 - x_3^2\right)\,, \\
x_2' &= \sqrt{6}\,k_1 x_1 x_2 + z^2 + \frac{3}{2}x_2\left(-1 + x_1^2 + x_2^2 - x_3^2\right)\,, \\
x_3' &= -\sqrt{\frac{3}{2}}\,k_2 x_3 x_1 + \frac{3}{2}x_3\left(1 + x_1^2 + x_2^2 - x_3^2\right)\,, \\
z' &= \frac{1}{H M_p\sqrt{3\sqrt{6}}}\left(\frac{\dot{J}}{2\sqrt{J}H\sqrt{f}} + \dot{\chi}\frac{k_1}{2M_p}\frac{\sqrt{J}}{H\sqrt{f}} - \frac{\sqrt{J}}{\sqrt{f}H^2}\dot{H}\right) \\
&= \sqrt{\frac{3}{2}}\,k_1 z x_1 + \frac{3}{2}z\left(x_1^2 + x_2^2 - x_3^2\right)\,,
\end{aligned} \tag{5.20}$$

where

$$J = n_B\, g_a \implies \dot{J} = g_a\,\dot{n}_B = -3 g_a n_{B,0}\frac{\dot{a}}{a^4} = -3H g_a n_B = -3HJ\,, \tag{5.21}$$

taking $n_B = n_{B0}\,a^{-3}$.

Using the definitions in (5.20) we can re-write the Friedmann equation as

$$H^2 = \frac{1}{3}\left(\rho_m + \frac{1}{2}\dot{a}^2 f^2 + \frac{1}{2}\dot{\chi}^2 + V\right) \implies \frac{\rho_m}{3H^2} = 1 - \sum_{i=1}^{3} x_i^2\,. \tag{5.22}$$

It is also important to define the equation of state for the scalar sector, which is given by

$$\omega_{\chi+a} = \frac{p_\chi + p_a}{\rho_\chi + \rho_a} = \frac{p_{\chi+a}}{\rho_{\chi+a}} = \frac{x_1^2 + x_2^2 - x_3^2}{x_1^2 + x_2^2 + x_3^2}\,. \tag{5.23}$$

Now, since the total energy density of χ is $\rho_\chi = \frac{\dot{\chi}^2}{2} + V \geq 0$ and of a is $\rho_a = \frac{\dot{a}^2}{2}f^2 \geq 0$, we can define the energy density for the dilaton and the axion as

$$\Omega_\chi \equiv \frac{\rho_\chi}{3H^2} = x_1^2 + x_3^2\,, \tag{5.24}$$

$$\Omega_a \equiv \frac{\rho_a}{3H^2} = x_2^2\,, \tag{5.25}$$

and for simplicity we will call then sum $\Omega_{\chi+a} = \Omega_\chi + \Omega_a$. From the definition of the energy density we find the following constraints

$$0 \leq x_1^2 + x_3^2 \leq 1\,, \qquad 0 \leq x_2^2 \leq 1\,, \tag{5.26}$$

and, in addition, our system is symmetric under reflection, which allows us to write the total bounds as

$$-1 \leq x_1 \leq 1\,, \quad -1 \leq x_2 \leq 1\,, \quad 0 \leq x_3 \leq 1\,, \quad z \geq 0\,, \quad k_2 > k_1 > 0 > \mathfrak{g}\,, \tag{5.27}$$

getting the fixed points as in Table 5.1, where we defined the constants

$$A = 9 + 4k_1 \left[9\mathfrak{g} + \left(-6 + \mathfrak{g}^2\right) k_1 - 4\mathfrak{g}k_1^2 + 4k_1^3\right] , \tag{5.28}$$

$$B_\pm = \frac{1}{6}\left(-3 - 2\mathfrak{g}\, k_1 + 4k_1^2 \mp \sqrt{A + 12k_1^4}\right) , \tag{5.29}$$

$$C_\pm = \frac{1}{k_1^2}\left(-3 + 2k_1^2\right)\left(3 - 4k_1^2 \pm \sqrt{A}\right) , \tag{5.30}$$

$$D_\pm = \frac{\sqrt{-3 + 2k_1\left(-\mathfrak{g} + k_1\right)\left(3 - 2\mathfrak{g}k_1 + 4k_1^2 \mp \sqrt{A}\right)\sqrt{-4\mathfrak{g}^2 - \frac{2\mathfrak{g}\left(12 - 6k_1^2 \mp \sqrt{A}\right)}{k_1} + C_\pm}}}{2 \times 3^{1/4}\sqrt{-6 + 4k_1\left(-\mathfrak{g} + k_1\right)}} . \tag{5.31}$$

These fixed points can be compared to those of Table 3.3.

	x_1	x_2	x_3	z	$\Omega_{\chi+a}$	$\omega_{\chi+a}$	Existence Conditions
$\mathcal{K}_-$	-1	0	0	0	1	1	$k_2 > k_1 > 0 > \mathfrak{g}$
$\mathcal{K}_+$	1	0	0	0	1	1	$k_2 > k_1 > 0 > \mathfrak{g}$
$\mathcal{F}_1$	$-\sqrt{\frac{2}{3}}\,\mathfrak{g}$	0	0	0	$\frac{2}{3}\mathfrak{g}^2$	1	$k_2 > k_1 > 0 > \mathfrak{g} \geq -\sqrt{\frac{3}{2}}$
$\mathcal{G}$	$\frac{k_2}{\sqrt{6}}$	0	$\sqrt{\frac{6-k_2^2}{6}}$	0	1	$\frac{-3+k_2^2}{3}$	$\sqrt{6} \geq k_2 > k_1 > 0 > \mathfrak{g} > -k_2$
$\mathcal{S}$	$\frac{\sqrt{\frac{3}{2}}}{\mathfrak{g}+k_2}$	0	$\frac{\sqrt{\frac{3}{2}+\mathfrak{g}(\mathfrak{g}+k_2)}}{(\mathfrak{g}+k_2)}$	0	$\frac{3+\mathfrak{g}(\mathfrak{g}+k_2)}{(\mathfrak{g}+k_2)^2}$	$-\frac{\mathfrak{g}(\mathfrak{g}+k_2)}{3+\mathfrak{g}(\mathfrak{g}+k_2)}$	$k_2 > \sqrt{6} > k_1 > 0 > \mathfrak{g} \geq -\frac{k_2}{2} + \frac{1}{2}\sqrt{-6+k_2^2}$ or $\sqrt{6} \geq k_2 \geq \sqrt{\frac{3}{2}} > k_1 > 0 > \mathfrak{g} \geq \frac{3-\sqrt{6}k_2}{\sqrt{6}}$
$\mathcal{F}_2$	$-\frac{\sqrt{3/2}}{\mathfrak{g}-2k_1}$	$\pm\frac{\sqrt{-\frac{3}{2}+\mathfrak{g}(\mathfrak{g}-2k_1)}}{\lvert\mathfrak{g}-2k_1\rvert}$	0	0	$\frac{\mathfrak{g}}{\mathfrak{g}-2k_1}$	1	$k_2 > k_1 > 0 > k_1 - \sqrt{\frac{3}{2}+k_1^2} \geq \mathfrak{g}$
$\mathcal{F}_3$	$\frac{3+2(\mathfrak{g}-2k_1)k_1 \pm \sqrt{A}}{2\sqrt{6}k_1}$	$\frac{\sqrt{-4\mathfrak{g}^2 - \frac{2\mathfrak{g}(12-6k_1^2 \pm \sqrt{A})}{k1} \pm C_\pm}}{2\sqrt{3}}$	0	$D_\pm$	$B_\pm$	1	$\mathfrak{g} < 0 < \sqrt{3(\sqrt{2}-1)} < k_1 < \sqrt{\frac{3}{2}}\,,\ \frac{\sqrt{6(3-2k_1^2)}}{k_1} + \frac{-9+4k_1^2}{2k_1} \leq \mathfrak{g} < \frac{-3+2k_1^2}{2k_1}$ or $\left(-\sqrt{\frac{3}{2}} < \mathfrak{g}\,,\ k_1 = \sqrt{\frac{3}{2}} \text{ or } k_1 > \sqrt{\frac{3}{2}}\right)\,,\ \ k_2 > k_1 > 0 > \mathfrak{g}$
$\mathcal{NG}$	$\frac{\sqrt{6}}{2k_1+k_2}$	$\mp\frac{\sqrt{-6+2k_1k_2+k_2^2}}{2k_1+k_2}$	$\sqrt{\frac{2k_1}{2k_1+k_2}}$	0	1	$1 - \frac{4k_1}{2k_1+k_2}$	$\sqrt{6} \geq k_2 \geq \sqrt{2} \geq \frac{6-k_2^2}{2k_2} > k_1 > 0 > \mathfrak{g}$ or $k_2 \geq \sqrt{6},\ \ k_2 > k_1 > 0 > \mathfrak{g}$

Table 5.1: Fixed points of the system with two scalars fields in the presence of a axion-matter coupling.

In detail we have different domination eras corresponding to different critical points from Tab.5.1.

- Kinetic energy domination, $\mathcal{K}_\pm$:

  The first two points in the table correspond to the case with an energy density given by $\Omega_{\chi+a} = 1$ and $\omega_{\chi+a} = 1$, which is a *Kinetic energy domination*, $\mathcal{K}_\pm$, since the only contribution comes from x_1, that as we know from the definition is proportional to the kinetic term of χ.

- Fluid domination, $\mathcal{F}_1$:

  While, from the table it is also possible to see a solution that it is a *Fluid domination* $\mathcal{F}_1$, where $\Omega_{\chi+a}$, for small value of $\mathfrak{g}$, is approximately zero, i.e. the scalar sector does not contribute in a relevant form. Moreover, differently from the previous one found, we have a non zero contribution from $\mathfrak{g}$, which give us a non zero value for the kinetic term, which in turn allow for a well defined value of $\omega_{\chi+a}$, cf. Table 3.3.

- Fluid domination, $\mathcal{F}_i$, with $i = 2, 3$:

  This is a new solution in respect to the one already present in literature. In fact, differently from $\mathcal{F}_1$, we have a contribution also from the kinetic term of the field a. In detail, the solution $\mathcal{F}_3$ has a value of x_2 and z different from zero, which means that in addition to the solution $\mathcal{F}_2$ we have a contribution also from the source J. Moreover, the value of $\omega_{\chi+a}$ remains equal to 1 in both cases.

- Geodesic, $\mathcal{G}$:

  For the critical point $\mathcal{G}$, the system evolves along a geodesic trajectory. In detail we have that χ slow rolls down the slope of its potential, and a remain the same, $x_2 = 0$. This solution is in agreement with the one already present in literature and we don't have any contribution from the presence of the axion-matter coupling.

- Scaling solution, $\mathcal{S}$:

  For this solution, as we expected from the case that we have already analyzed in previous sections, we have a contribution equal to zero from the field a, and only x_1 and x_3 are different from zero, which means that the axion field doesn't add anything to the system and remains static. Moreover, we have some modification of the solutions, due to the presence of $\mathfrak{g}$, which give us new type of possible values. In detail, we can see that we can have an interesting value of $\omega_{\chi+a}$, since there is a range of $\mathfrak{g}$ for while $\omega < \frac{1}{3}$. Suggesting the possibility that $\mathcal{S}$ can support accelerated expansion.

- Non-Geodesic solution, $\mathcal{NG}$:

  In this solution we have different contributions from all the variables, and the evolution is along a non-geodesic trajectory in field space. Moreover, this solution is in agreement from the one found in literature.

5.2.2 Stability of the critical points

Another important aspect is to determine the stability of the critical points. In order to check this we shall study the evolution of a small perturbation around the solutions $(\hat{x}_i, \hat{z})$, with $i = 1, 2, 3$, found as

$$x_i = \hat{x}_i + u_i \,, \qquad z = \hat{z} + u_4 \,, \tag{5.32}$$

thus, using (5.32) in (5.20) we obtain

$$u' = \mathcal{M}\, u \,, \tag{5.33}$$

where $u = (u_1,\, u_2,\, u_3,\, u_4)$, with general solution given by

$$u_i = \sum_{j=1}^{4} u_{ij}\, e^{m_{ij} N} \,, \tag{5.34}$$

	m_{i1}	m_{i2}	m_{i3}	m_{i4}	Stability Conditions
$\mathcal{K}_-$	-2	$3-\sqrt{6}\mathfrak{g}$	7	$\frac{1}{2}$	unstable
$\mathcal{K}_+$	-1	2	$\frac{5}{2}$	$3+\sqrt{6}\mathfrak{g}$	unstable
$\mathcal{F}_1$	$\frac{\mathfrak{g}}{3}\left(3\mathfrak{g}-\sqrt{6}\right)$	$\frac{2\mathfrak{g}^2-3}{2}$	$\frac{6\mathfrak{g}^2-4\sqrt{6}\,\mathfrak{g}-9}{6}$	$\frac{6\mathfrak{g}^2-4\sqrt{6}\,\mathfrak{g}-9}{6}$	unstable
$\mathcal{G}$	$\frac{31}{6}$	5	$\frac{7}{3}$	$\frac{4\sqrt{6}\,\mathfrak{g}+23}{3}$	unstable
$\mathcal{S}$	$\frac{3}{2}\frac{\sqrt{6}-3\mathfrak{g}}{4\sqrt{6}+3\mathfrak{g}}$	$-3\frac{\sqrt{6}+3\mathfrak{g}}{4\sqrt{6}+3\mathfrak{g}}$	$-\frac{3}{2}\frac{48+18\sqrt{6}\,\mathfrak{g}+9\mathfrak{g}^2+\mathfrak{r}}{\left(4\sqrt{6}+3\mathfrak{g}\right)^2}$	$-\frac{3}{2}\frac{48+18\sqrt{6}\,\mathfrak{g}+9\mathfrak{g}^2-\mathfrak{r}}{\left(4\sqrt{6}+3\mathfrak{g}\right)^2}$	unstable
$\mathcal{F}_2$	$\frac{3}{2}\frac{\sqrt{6}-3\mathfrak{g}}{2\sqrt{6}-3\mathfrak{g}}$	$-3\frac{\sqrt{6}+3\mathfrak{g}}{2\sqrt{6}-3\mathfrak{g}}$	$\frac{3}{2}\frac{-12+3\sqrt{6}\,\mathfrak{g}+\mathfrak{r}}{\left(2\sqrt{6}-3\mathfrak{g}\right)^2}$	$\frac{3}{2}\frac{-12+3\sqrt{6}\,\mathfrak{g}-\mathfrak{r}}{\left(2\sqrt{6}-3\mathfrak{g}\right)^2}$	unstable
$\mathcal{F}_3$	$\frac{\mathfrak{g}}{3}\left(3\mathfrak{g}-\sqrt{6}\right)$	λ_1	λ_2	λ_3	unstable
$\mathcal{NG}$	$\frac{1}{2}\left(-1+\mathrm{i}\sqrt{39}\right)$	$\frac{1}{2}\left(-1-\mathrm{i}\sqrt{39}\right)$	1	$\frac{1}{2}\left(2+\sqrt{6}\,\mathfrak{g}\right)$	unstable

Table 5.2: Study of the stability of the critical points of the system with two scalar fields in the presence of an axion-matter coupling.

where u_{ij} are coefficients and m_{ij} are the eigenvalues of the matrix $\mathcal{M}$. For the case with $k_1=\sqrt{2/3}$, $k_2=4k_1$ we find the eigenvalues in Tab.5.2 with the definitions

$$-\mathfrak{r}^2 = 144\sqrt{6}g^5+4023g^4+7812\sqrt{6}g^3+40674g^2+15664\sqrt{6}g+10944\,, \tag{5.35}$$

$$\mathfrak{l}^2 = 1296-424\sqrt{6}g-1722g^2+1044\sqrt{6}g^3-1152g^4+72\sqrt{6}g^5\,, \tag{5.36}$$

and λ_i with $i=1,2,3$ solutions of the third order equation

$$\begin{aligned}
&23219011584\,\mathfrak{g}^4+2592\,\mathfrak{g}^2\Bigg(432\left(\sqrt{24\,\mathfrak{g}^2+76\sqrt{6}\,\mathfrak{g}+1}-13\right)\lambda_i \\
&+373248\left(25\sqrt{24\,\mathfrak{g}^2+76\sqrt{6}\,\mathfrak{g}+1}+189\right)-\lambda_i^2\Bigg) \\
&-216\sqrt{6}\,\mathfrak{g}\Bigg(\left(\sqrt{24\,\mathfrak{g}^2+76\sqrt{6}\,\mathfrak{g}+1}-3\right)\lambda_i^2+864\left(16\sqrt{24\,\mathfrak{g}^2+76\sqrt{6}\,\mathfrak{g}+1}+127\right)\lambda_i \\
&+746496\left(53\sqrt{24\,\mathfrak{g}^2+76\sqrt{6}\,\mathfrak{g}+1}+757\right)\Bigg)+432\left(\sqrt{24\,\mathfrak{g}^2+76\sqrt{6}\,\mathfrak{g}+1}+19\right)\lambda_i^2 \\
&+279936\left(5\sqrt{24\,\mathfrak{g}^2+76\sqrt{6}\,\mathfrak{g}+1}+59\right)\lambda_i-1612431360\left(\sqrt{24\,\mathfrak{g}^2+76\sqrt{6}\,\mathfrak{g}+1}+1\right) \\
&+2239488\sqrt{6}\,\mathfrak{g}^3\left(864\left(\sqrt{24\,\mathfrak{g}^2+76\sqrt{6}\,\mathfrak{g}+1}+44\right)+\lambda_i\right)+\lambda_i^3=0\,,
\end{aligned} \tag{5.37}$$

which we omit to not clutter the table.

Thus, for the stability we need that the real part of all the eigenvalues, of the matrix $\mathcal{M}$, to be negative, however as can be seen from Tab.5.2 none of these solutions is stable. Moreover, one can perform a generic study leaving k_1, k_2 and $\mathfrak{g}$ as free parameters in order to find for which values the stability conditions are satisfied, from which we find that the only stable point is given by the geodesic $\mathcal{G}$ with conditions

$$0<k_2\leq\sqrt{\frac{3}{2}}\,,\ -k_2<\mathfrak{g}<0<k_1<k_2 \quad \text{or} \quad \sqrt{\frac{3}{2}}<k_2<\sqrt{3}\,,\ -k_2<\mathfrak{g}<0<k_1<\frac{3-k_2^2}{k_2}\,. \tag{5.38}$$

These new instabilities are to be understood as a consequence of the new variable z, since if we make the limit $\mathfrak{g} \to 0$ we recover the systems described in [54].

From now one we are going to set $M_P = 1$ to simplify the notation.

5.2.3 Case $x_2 \equiv 0$, small $\mathfrak{g}$

Let us now focus on a simplified case where we have no axion evolution and we set $\dot{a} = 0$. Which means that the equations of motion in eq.(5.16), are now given by

$$\begin{cases} \ddot{\chi} = -3H\dot{\chi} - V_\chi - \mathfrak{g}\rho_m, \\ \dot{a} \equiv 0 \implies J = 0 \end{cases} \tag{5.39}$$

thus, from here we have only two variables different from zero, due to the fact that $x_2 \propto \dot{a}$ and $z \propto \sqrt{J}$, so $x_2 \equiv 0$ and $z \equiv 0$. Thus, in this particular case, with $k_1 = \sqrt{\frac{2}{3}}$, $k_2 = 4k_1$ and $\mathfrak{g} = -10^{-5}$, the dynamical system coordinates in (5.20) evolves to the fixed point

$$\begin{aligned} x_1 &= \frac{\chi'}{\sqrt{6}} = 0.375\,, \\ x_2 &= 0\,, \\ x_3 &= \frac{\sqrt{V}}{\sqrt{3}H} = 0.375\,, \\ z &= 0\,, \end{aligned} \tag{5.40}$$

which corresponds to the solution $\mathcal{S}$, shown in Figure 5.6.

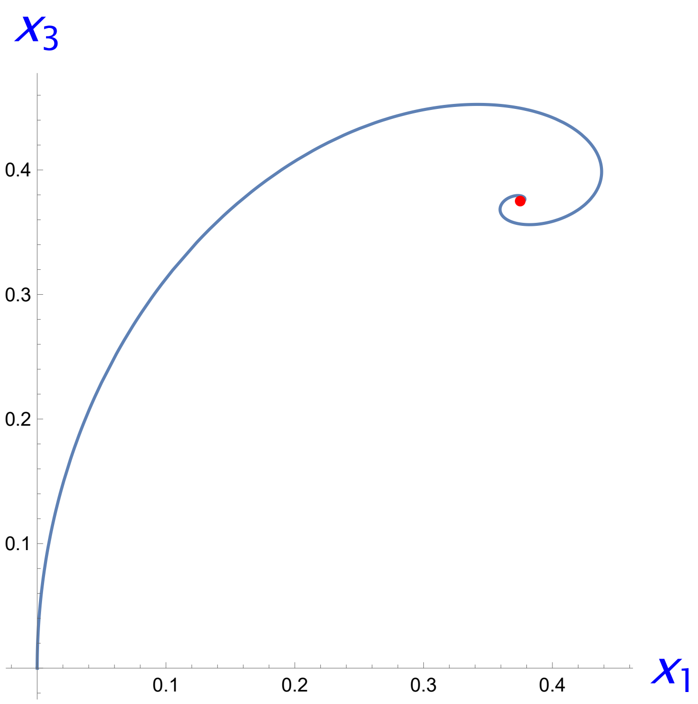


Figure 5.6: Fixed points x_1 and x_3, in the case of $\mathfrak{g} = -10^{-5}$, corresponding to the solution $\mathcal{S}$.

Now, it is important to note that even if in Tab.5.2 we have that none of the critical points is in agreement with the stability conditions, once we impose $x_2 \equiv 0$ and $z \equiv 0$, we are studying a two dimensional case and we have stability, as it is shown in Fig.5.6.

5.2.4 Case $x_2 \equiv 0$, "phenomenological" $\mathfrak{g}$

Now, we want to repeat the same study of 5.2.3, where we have $\dot{a} = 0$, and taking the case of $\mathfrak{g} \simeq -0.408$. Which means that we have again the requirement of a negative value for $\mathfrak{g}$, and this type of model is the most phenomenology successful one, as we explained in section 4.3.2. In detail we can have a lower bound on the axion-matter coupling, which turns out to be very small, and these kind of coupling can easily evade solar-system bounds, since the axion itself is not detectable, or through the "homeopathy" mechanism [58]. Hence, the equations of motion transform as in (5.39) and again the value of x_2 and z are zero.

Hence, we now consider the particular case where the parameters are given by $k_1 = \sqrt{\frac{2}{3}}$ and $k_2 = 4k_1$; thus, the dynamical system coordinates evolve to the values

$$\begin{aligned} x_1 &= \frac{\dot{\chi}}{\sqrt{6}H} = 0.428, \\ x_2 &= 0, \\ x_3 &= \frac{\sqrt{V}}{\sqrt{3}H} = 0.202, \\ z &= 0, \end{aligned} \tag{5.41}$$

which again correspond to the solution $\mathcal{S}$, as it is shown in Fig.5.7.

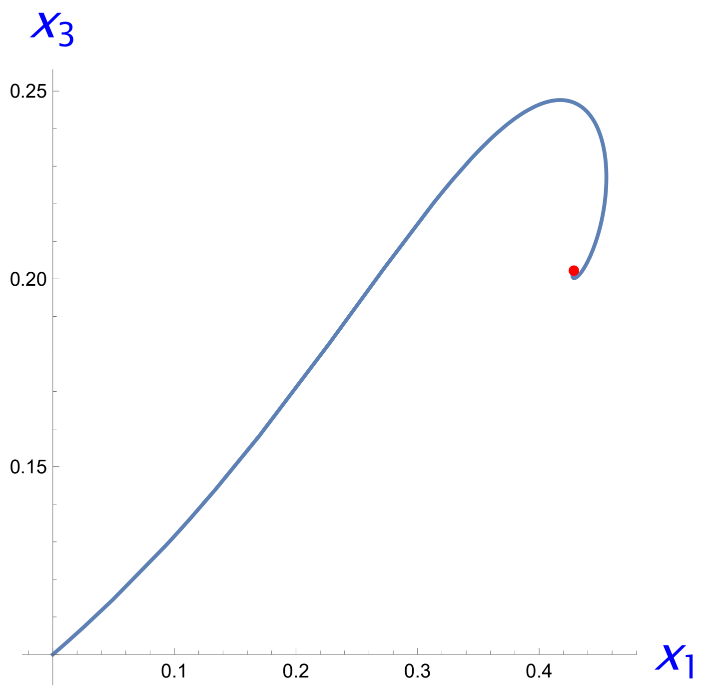


Figure 5.7: Fixed points x_1 and x_3, in the case of $\mathfrak{g} = -0.408$, corresponding to the solution $\mathcal{S}$.

The critical points change due to the different value of $\mathfrak{g}$ but they reach again the scaling solution, through a different trajectory. Again from Table 5.2 we don't expect it to have any stable solution at all, but by imposing $\dot{a} \equiv 0$, which in turn implies $J = 0$, allow us to reach this stable point, bypassing the instability generated by the dimensionless variable z.

5.2.5 Case $x_2 \neq 0$

In this particular case we would like to study the scenario where we now allow the axion to evolve, again considering the value of $\mathfrak{g} \simeq -0.408$. Thus, we can study two different cases, one where the axion source J is set to vanish and another one where we consider it non-zero.

- $J = 0$:

  With this choice, the equations of motion now read

$$\begin{cases} \ddot{\chi} = -3H\dot{\chi} + f_\chi f \dot{a}^2 - V_\chi - \mathfrak{g}\rho_m, \\ \ddot{a} = -3H\dot{a} - 2\frac{f_\chi}{f}\dot{\chi}\dot{a}, \end{cases} \tag{5.42}$$

  hence, choosing $k_1 = \sqrt{\frac{2}{3}}$ and $k_2 = 4k_1$, we find that the dynamical system coordinates evolve to the values

$$\begin{aligned} x_1 &= \frac{\dot{\chi}}{\sqrt{6}H} = 0.428, \\ x_2 &= 0, \\ x_3 &= \frac{\sqrt{V}}{\sqrt{3}H} = 0.202, \\ z &\equiv 0, \end{aligned} \tag{5.43}$$

  which is the Scaling solution, as we expected from Tab.5.2, since $\mathcal{S}$ is the only one with all negative real parts m_i in the case of $z = 0$, i.e. neglecting the only eigenvalue with a positive real part. Hence, we have the result that we expected, since it is in agreement with the one in Chapter 3.1.4

- $J \neq 0$:

  The equations of motion are again the full system given in (5.16). In this case, there is no stability, since in this case we have that $J \neq 0$, hence, $z \neq 0$, and so all the eigenvalues of the Tab.5.2 are to be taken in consideration and the Scaling solution now develop an eigenvalue with positive real part. More generally, if one considers solutions with $z = 0$, i.e. all except the $\mathcal{F}_3$ solution, the linear equation for the z perturbation u_4 becomes decoupled from the other u_i variables with $i = 1, 2, 3$ as

$$u_4' = \left[\sqrt{\frac{3}{2}} k_1 \hat{x}_1 + \frac{3}{2} \left(\hat{x}_1^2 + \hat{x}_2^2 - \hat{x}_3^2 \right) \right] u_4 \,, \tag{5.44}$$

which directly gives the eigenvalue

$$m_4 = \sqrt{\frac{3}{2}} k_1 \hat{x}_1 + \frac{3}{2}\left(\hat{x}_1^2 + \hat{x}_2^2 - \hat{x}_3^2\right) = \begin{cases} \frac{1}{2} & \text{for} \quad \mathcal{K}_+ \,, \mathcal{F}_1 \,, \\ \frac{5}{2} & \text{for} \quad \mathcal{K}_- \,, \\ \frac{31}{6} & \text{for} \quad \mathcal{G} \,, \\ \frac{9}{14} & \text{for} \quad \mathcal{S} \,, \\ \frac{9}{10} & \text{for} \quad \mathcal{F}_2 \,, \\ 1 & \text{for} \quad \mathcal{NG} \,, \end{cases} \tag{5.45}$$

where the numerical value is calculated with the parameters $k_1 = \sqrt{\frac{2}{3}}$, $k_2 = 4k_1$ and $\mathfrak{g} = -\frac{1}{2}k_1$, giving us immediately positive real numbers which implies an instability of the fixed point. Moreover, let us note that the cases where $x_3 = 0$, i.e. $\mathcal{K}_\pm$, $\mathcal{F}_i$ with $i = 1, 2, 3$, will give analogously a decoupled equation for the u_3 perturbation as

$$u_3' = \left[-\sqrt{\frac{3}{2}} k_2 \hat{x}_1 + \frac{3}{2}\left(1 + \hat{x}_1^2 + \hat{x}_2^2\right)\right] u_3 \,, \tag{5.46}$$

which gives the eigenvalue

$$m_3 = -\sqrt{\frac{3}{2}} k_2 \hat{x}_1 + \frac{3}{2}\left(1 + \hat{x}_1^2 + \hat{x}_2^2\right) = \begin{cases} 7 & \text{for} \quad \mathcal{K}_- \,, \\ -1 & \text{for} \quad \mathcal{K}_+ \,, \\ \frac{1}{3} & \text{for} \quad \mathcal{F}_1 \,, \\ -\frac{3}{5} & \text{for} \quad \mathcal{F}_2 \,, \\ \frac{23+5\mathrm{i}\sqrt{71}}{12} & \text{for} \quad \mathcal{F}_3 \,, \end{cases} \tag{5.47}$$

which shows that $\mathcal{F}_3$ that was the only solution still available from the previous argument is now ruled out, thus showing that none of the fixed points is stable with this choice of parameters.

5.3 Yoga Potential

We now explore a specific type of potentials that deviates from the simple exponential, which will allow for different cosmological evolution, following [3]. Now, suppose to take a scalar potential of the form (4.16) that is given by

$$V(\hat{\chi}) = U(\hat{\chi}) e^{-k_2 \hat{\chi}} \qquad \text{with} \qquad U(\hat{\chi}) \simeq U_0 \left[1 - u_1 \hat{\chi} + \frac{u_2}{2} \hat{\chi}^2\right] , \tag{5.48}$$

where U is a polynomial or rational function of $\hat{\chi}$ used to stabilize τ, whose particular form is such that we find the minimum

$$\hat{\chi}_{\text{min}} = \frac{k_2 u_1 + u_2 - \sqrt{k_2^2\left(u_1^2 - 2u_2\right) + u_2^2}}{k_2 u_2} \,, \tag{5.49}$$

with the requirement $u_1^2 \leq 2u_2$ which ensures the presence of the minimum, $V'(\hat{\chi}_{\text{min}}) = 0$ and $V''(\hat{\chi}_{\text{min}}) > 0$, and positivity of V for all values of $\hat{\chi}$. Let us also note that the potential presents a maximum at

$$\hat{\chi}_{\text{max}} = \frac{k_2 u_1 + u_2 + \sqrt{k_2^2\left(u_1^2 - 2u_2\right) + u_2^2}}{k_2 u_2} \,, \tag{5.50}$$

after which the field $\hat{\chi}$ incurs in a runaway. At $\hat{\chi} = \hat{\chi}_{\min}$ we set a minimum for U, since any change in χ results in a variation of the particles masses through $A(\chi)$, because they are set by phenomenological data at present day, recombination and nucleosynthesis, hence, we choose $U_0 \sim \epsilon^5 M_p^4$ for $\epsilon \simeq 1/30$, and in order to obtain present time cosmology, again setting $M_P = 1$, we have

$$V \sim \epsilon^5 e^{k_2 \hat{\chi}} \implies \hat{\chi} \sim 75 \implies \tau \sim e^{k_1 \hat{\chi}} \sim 10^{26} \, . \tag{5.51}$$

Thus, a possible choice of parameters is given by

$$k_1 = \sqrt{\frac{2}{3}} \, , \quad k_2 = 4k_1 \, , \quad u_1 = 0.027027 \quad u_2 = 0.00036523 \, , \quad \epsilon = \frac{1}{30} \, , \tag{5.52}$$

which reproduces the following values

$$\hat{\chi}_{\min} = 74.0159 \, , \quad \hat{\chi}_{\max} = 74.5963 \, , \quad V_{\min} = 7.62522 \times 10^{-118} \, , \quad \tau_{\min} = 1.76215 \times 10^{26} \, . \tag{5.53}$$

Now, one could try again to recast the equation of motion into a linear dynamical system, however the task is difficult due to the presence of this new potential. In fact, it could be interpreted as a field dependent exponent [28] as

$$V(\hat{\chi}) = U_0 \, e^{-\lambda(\hat{\chi}) \, \hat{\chi}} \, , \qquad \text{with} \quad \lambda = k_2 - \frac{1}{\hat{\chi}} \ln \left[1 - u_1 \hat{\chi} + \frac{u_2}{2} \hat{\chi}^2 \right] \, , \tag{5.54}$$

which means that when we take the derivative in respect to N we obtain

$$\frac{V'}{V} = \lambda \hat{\chi}' + \lambda' \hat{\chi} = \hat{\chi}' \left(\lambda + \lambda_{\hat{\chi}} \hat{\chi} \right) = \hat{\chi}' \left(k_2 + \frac{u_1 - u_2 \hat{\chi}}{1 - u_1 \hat{\chi} + \frac{u_2}{2} \hat{\chi}^2} \right) \tag{5.55}$$

$$\sim x_1 \left(k_2 + \frac{u_1 - u_2 \hat{\chi}}{1 - u_1 \hat{\chi} + \frac{u_2}{2} \hat{\chi}^2} \right) \, , \tag{5.56}$$

which thus leads to a modification of the equation of motion of x_3', in addition to the obvious correction to x_1' due to the derivative of V. This introduces a direct dependence on $\hat{\chi}$ which is not part of the dynamical variables we defined, and thus forbid us an easy solution. One possibility would be to study asymptotic regions in which this exponent λ becomes constant, and can be treated simply. In the next section we will numerically analyse the model in order to see if fixed points exist and if they allow for a correct phenomenology. In detail, we can see the main differences between the exponential and the Yoga potentials from Fig.5.8, where it is visible the local minimum of the Yoga potential.

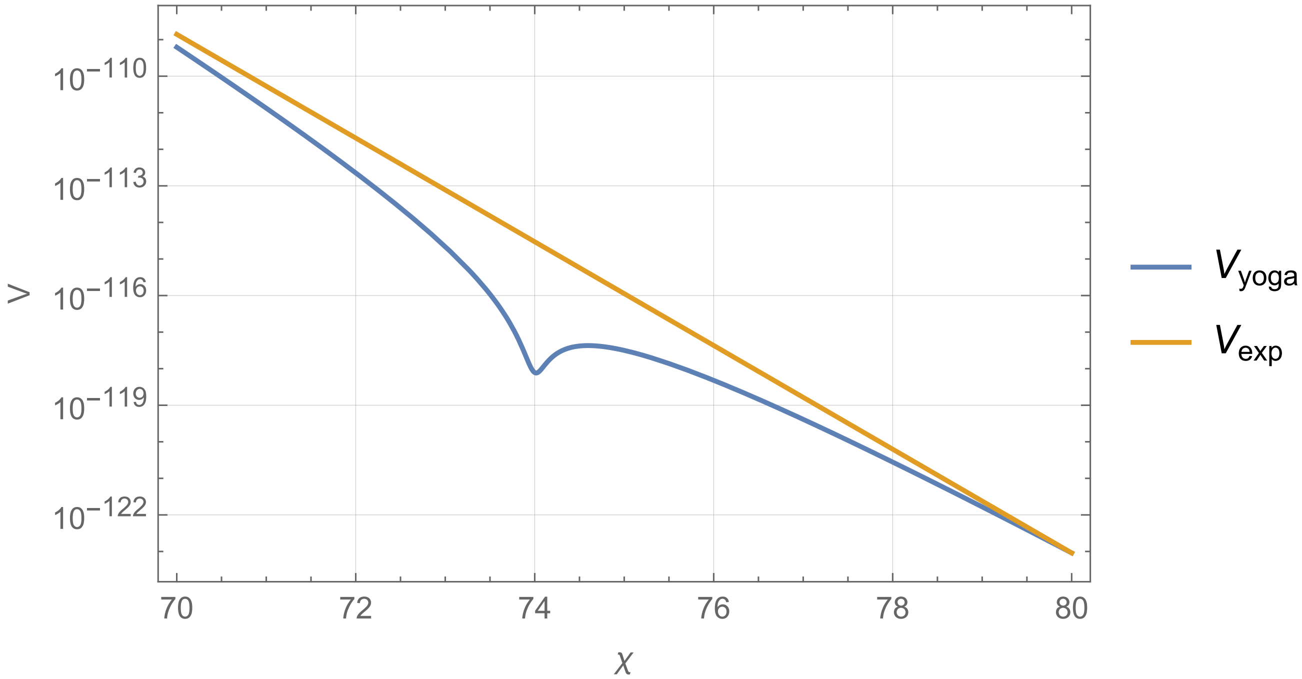


Figure 5.8: Potential in function of χ. In detail, the Yoga potential (blue) in comparison with the exponential potential (orange).

5.3.1 No axion evolution, $\mathfrak{g}$ small

We would like to study now the simplified case where no axion evolution is allowed, as we previously did with the exponential potential. However, differently from before, we are not able to extract analytical information since now is not possible to write a linear dynamical system, but we will proceed by setting our initial conditions and letting the system evolve, ultimately we will analyze the possible critical points found. Hence, we take the value of $\mathfrak{g}$ equals to -10^{-5}, since we want that the dilaton $\hat{\chi}$ stays approximately in the local minimum of the potential in order not to go into a runaway, as it is possible to see in Fig.5.8.

Studying the evolution of the energies of radiation, matter and of the scalar potential, as it is shown in Fig.5.9, we can see that the dilaton is trapped at late time. Thus, taking $k_1 = \sqrt{\frac{2}{3}}$ and $k_2 = 4k_1$, and the initial conditions

$$\rho_{m0} = 10^{-90}, \quad \rho_{\text{rad}0} = 10^{-84}, \quad \hat{\chi}_0 = 74, \quad \rho_{\chi 0} = 10^{-108.7}, \tag{5.57}$$

we are able to find the following critical points

$$\begin{aligned} x_1 &= 0, \\ x_2 &= 0, \\ x_3 &= 1, \\ z &= 0. \end{aligned} \tag{5.58}$$

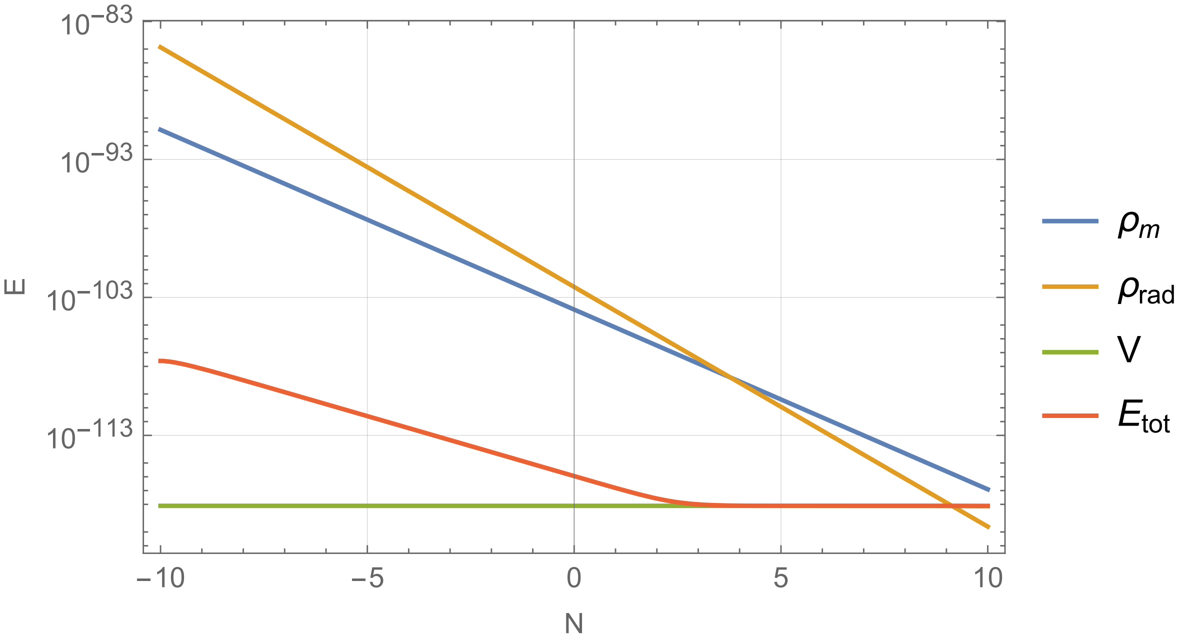


Figure 5.9: Log-Log plot of the energies densities (E) of matter (blue), radiation (orange), dilaton potential (green) and total dilaton energy density (red) in function of e-folds (N), in the case with no axion evolution, with $\mathfrak{g} = -10^{-5}$ and where we are taking the Yoga potential.

Where $x_2 = 0$ and $z = 0$, since we are now studying the case without axion evolution, thus, as in the case of the exponential potential we have the equations of motions as in eq.(5.39). But now, differently from the previous case, we have that the kinetic term of the dilaton is zero and we only have the potential term. In detail, from the Fig.5.10, it is possible to see that the evolution of the system reach a stable critical point, which is given by $x_3 = 1$ and $x_1 = 0$.

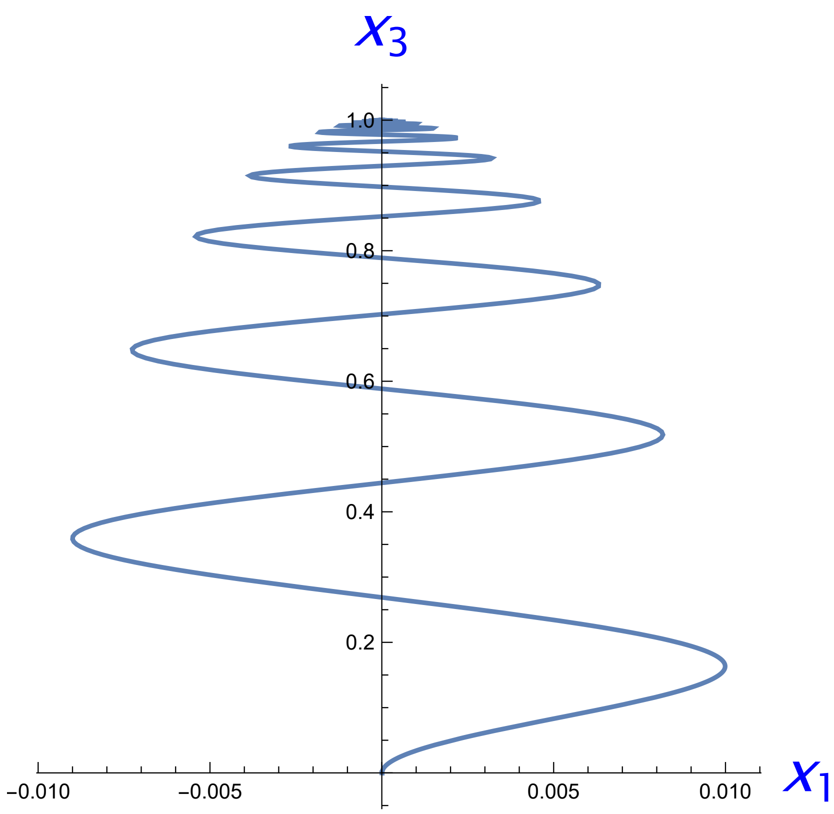


Figure 5.10: Fixed points x_1 and x_3 with the value of $\mathfrak{g}$ equals to -10^{-5}, corresponding to the solution where in the end all the energy is given only by the dilaton potential.

Thus, since we know that $x_1 \propto \dot{\chi}$ and $x_3 \propto \sqrt{V}$ we only have potential energy in the end. Hence, from eq.(5.22) we can see that the energy densities in this case are zero and so the kinetic energies of the axion a and the dilaton χ, thus, we have that $H^2 \propto V$.

5.3.2 No axion evolution, case $\mathfrak{g} = -0.408$

Now we would like to study the case again of no axion evolution, but with a value of $\mathfrak{g}$ which is given by -0.408, that, as we already explained in the previous section, is the most phenomenologically successful one.

Now, studying the evolution of the energie densities we obtain Fig.5.7, where it is possible to see that now, differently from Fig.5.9, in the end we don't have that the total energy becomes only potential energy, instead, we have that, in the moment in which we transition from radiation to matter domination era, the potential is not constant but increases to then reduce itself. Where it is important to note that we used the following initial conditions

$$\rho_{m0} = 10^{-96.5}, \qquad \rho_{\text{rad}0} = 10^{-92}, \qquad \hat{\chi}_0 = 74, \qquad \rho_{\chi 0} = 10^{-105}. \tag{5.59}$$

Thus, now we can take the case of $k_1 = \sqrt{\frac{2}{3}}$ and $k_2 = 4k_1$, in this way we find that the evolution ends in the point

$$\begin{aligned} x_1 &= 0.377, \\ x_2 &= 0, \\ x_3 &= 0.198, \\ z &= 0. \end{aligned} \tag{5.60}$$

As in the case of the exponential potential, we only have the values of x_1 and x_3 different from zero. In detail, we have that the point that it is reached by the values of x_1 and x_3 it is very close to the one of the exponential potential case, see eq.(5.41).

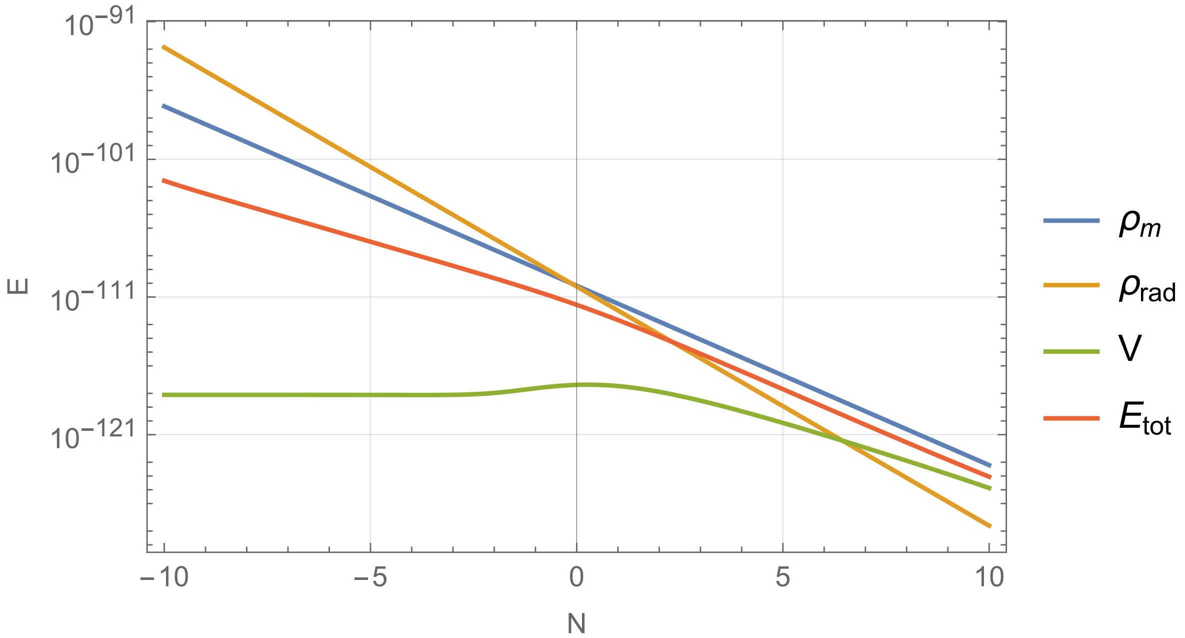


Figure 5.11: Log-Log plot of the energies densities (E) of matter (blue), radiation (orange), dilaton potential (green) and dilaton total energy (red) in function of e-folds (N), in the particular case of no axion evolution, with $\mathfrak{g} = -0.408$ and the Yoga potential.

In detail, we have again $x_2 = 0$ due to the fact that we are taking the case without axion evolution, which implies in the equations of motion the absence of J, and thus $z = 0$.

5.3.3 Axion evolution $\neq 0$, case $\mathfrak{g} = -0.408$

Now we will focus on the particular scenario with an axion evolution different from zero and a value of $\mathfrak{g} = -0.408$. Thus, the equations of motion are the one of eq.(5.16), with the Yoga potential instead of the exponential one. Now, we can study the evolution of the energies densities of radiation, matter, the kinetic and the potential term for the dilaton and the kinetic term of the axion, finding Fig.5.12.

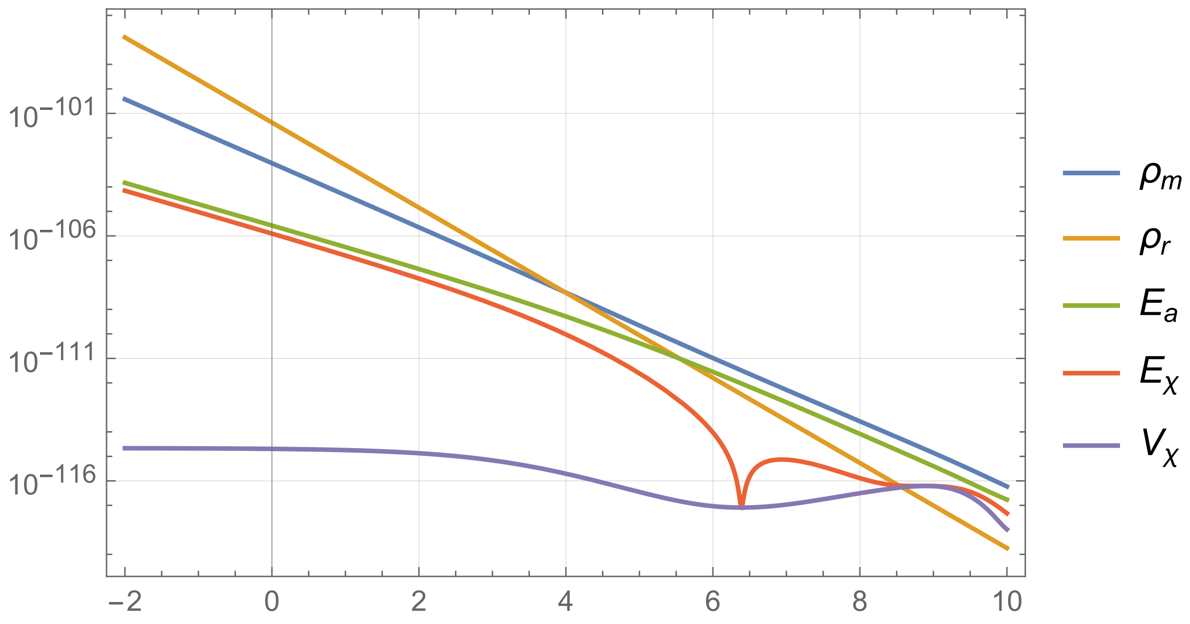


Figure 5.12: Log-Log plot of the energies densities (E) of matter (blue), radiation (orange), dilaton potential (purple), dilaton total energy (red) and kinetic term of the axion (green) in function of e-folds (N), in the particular case of no axion evolution, with $\mathfrak{g} = -0.408$ and the Yoga potential.

Choosing as initial conditions the following

$$\rho_{m0} = 10^{-90}, \qquad \rho_{\text{rad}0} = 10^{-84}, \qquad \hat{\chi}_0 = 74,$$
$$\rho_{\chi 0} = \rho_{a0} = 10^{-98}, \qquad J_0 = 2.3\rho_{m0} \times 10^{-26}\,. \tag{5.61}$$

From this plot we can see that the evolution of the axion's energy density has the same behaviour of the dilaton total energy density as long as we have radiation domination. After the starting of the matter domination we can see that the kinetic energy density goes to zero and become only potential, this is due to the fact that we are oscillating in the minimum.

Thus, taking the case of $k_1 = \sqrt{\frac{2}{3}}$, $k_2 = 4k_1$, the dynamical system coordinates evolve in

$$\begin{aligned} x_1 &\sim 0, \\ x_2 &\sim 0, \\ x_3 &= 1, \\ z &\sim 0, \end{aligned} \tag{5.62}$$

which is the case of no kinetic energies for the axion and the dilaton, all the kinetic energy of the dilaton goes to the potential term. We already saw this type of scenario in the case of no axion evolution, with a $\mathfrak{g} = -10^{-5}$ and with the Yoga potential.

Discussion and conclusions

In this thesis we studied the dynamics of the late universe, where the observational evidence for accelerated expansion indicates the need for an unknown component called dark energy. According to the most recent data, about 70% of the universe's energy density is in the form of dark energy. The simplest explanation for dark energy would be a cosmological constant, though this proposal presents several shortcomings. The main subject of this work is a class of dynamical DE models usually called quintessence. In these models a scalar field rolling down a (typically) flat potential provides an adequate description of the late universe. Such models while relatively easy to build from a field theory point of view are difficult to embed into string theory and are often in tension with solar system tests of gravity.

The idea of this work is to consider a coupled dilaton-axion sector that can be found in string compactifications to give rise to dark energy. In particular we analyse the model recently proposed in [3], and dubbed Yoga dark energy, where the scalar potential driving the universe's late time expansion features a local minimum allowing for new end states of the universe where the model transitions from a quintessence to a cosmological constant, finding that with the right choice of initial data the system evolves to an apparently stable solution.

In order to better understand the dynamics of the two field system we performed an analytic study when the potential is a pure exponential, finding modifications of the known solutions [57]. These modifications are due to the presence of both a dilaton matter coupling (always present due to the Weyl rescaling from the Jordan to the Einstein frame) and of an axion matter coupling (an additional ingredient added for compatibility with fifth force constraints). The results are presented in Tab.3.3, most notably the presence of two new fluid dominated fixed points $\mathcal{F}_2$ and $\mathcal{F}_3$, and a shift in the location of the known ones by factors of the axion-matter coupling $\mathfrak{g}$.

Moreover, phenomenologically today we expect a value of the energy density $\omega \simeq -1$ and for the equation of state $\Omega \simeq 0.7$, see [69]. Thus, we can see that for certain value of $\mathfrak{g}$ this could be possible for the modified Scaling solution in Tab.5.1.

However, we can also note that differently from the previous study in [57], the stability conditions derived from the eigenvalues of the linear order perturbation matrix are greatly modified. In particular, looking at Tab.5.2, where we study the particular value of $k_1 = \sqrt{\frac{2}{3}}$ and $k_2 = 4k_1$, we can see that we no solution is stable, since the extra variable z induces a fundamental instability, which is manifested by the presence of an eigenvalue with positive real part. Moreover, this particular study can be generalized leaving the parameters k_1, k_2 and $\mathfrak{g}$ as free, which were instead previously inferred from phenomenological and string inspired considerations. With this new freedom what we found is that only the geodesic solution is stable under a subset of the parameter space, which we recall is unmodified by the addition of the axion-matter coupling. These considerations suggest that if the potential is to be considered as a simple exponential our universe could be currently only in a transient phase which in turn generates a plethora of questions such as why the universe is in this exact state at this stage of its evolution which remains unanswered.

In addition, in the particular case of the Yoga potential we were not able to perform a systematic search and a further development left for the future would be the search for an analytic study of the solutions, in order to perform a similar analysis that we already done in the exponential potential, and thus, to find the expected critical points. The richer structure of the Yoga potential implies that an analytic study of the dynamical system is more involved since it is impossible eliminate a direct dependence on the dilaton field from the equations of motion, rendering the system non-linear. Hence, one should search for new/different dimensionless variables that would be able to simplify the system allowing for a systematic search of critical point and thus a classification. In fact, in our case we were only able to perform a numerical analysis based on empirical data, trying to analyse if a possible choice of starting state would eventually lead to a stable solution at late times, finding that indeed this is possible if one has that the axion source contribution interplay with the matter energy density stops the rolling of the dilaton enough such that it gets trapped inside of the local minimum, otherwise one would incur in a runway and in the limit of large χ the field dependent exponent will then tend to become a constant and thus recovering the exponential case analysed with its associated critical points that we argued to be of little phenomenological relevance.

Lastly, we note that in recent years several doubts arose about the existence of de Sitter vacua in String Theory [70, 71], that were formulated as the so called no dS conjecture, which states that

$$M_p \frac{|V'|}{V} \gtrsim c \qquad \text{or} \qquad M_P^2 \frac{V''}{V} \lesssim -\tilde{c}\,,$$

with $\mathcal{O}(1)$ coefficients c and $\tilde{c}$. In particular this bound forbids dS vacua.

In particular this bound forbids both exact dS vacua, i.e. a true cosmological constant, and flat potentials that support geodesic evolution. This prompts the question: if this conjecture is true, what is the dynamics behind the observed accelerated expansion of the universe? What one can find from this work is that the critical points already found in previous works in literature [54–57] receive corrections due to the presence of the axion coupling, and on top of these known points new solutions emerge which enriches the landscape of possibilities. In fact, previous studies found that the current state of the universe could not be a critical point and thus it required us to be in a transient phase, while with the new freedom given by the parameters of the axio-dilaton system together with matter coupling new possibilities arise, and at the same time satisfying the bounds on PPN parameters in accordance with data.

In conclusion, our study poses a new intriguing development which aims to reconcile Quintessence models derived from an underline String Theory UV completion to experimental data from solar system test, avoiding heavy modification of the PPN parameters, alongside constraints from several present and upcoming detections.

Acknowledgements

I would like to thank my supervisor Professor Francisco Gil Pedro for his availability and for everything that he teach me during the last months. I would also like to thank my family and all my friends that support me in these years.

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