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Effects of Loop Corrections on String Inflation

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Abstract

In this Master Thesis we study the effects of loop corrections to the inflationary potential of the *Kähler moduli inflation* model.
After reviewing basic concepts of String Theory and String Compactifications, focusing in particular on the Large Volume Scenario (LVS) in Type IIB String Theory, we introduce the model of Kähler moduli inflation, and the cosmological predictions thereof. We then consider two forms of open-string loop effects: Kaluza-Klein and Winding corrections. After comparing their relative magnitude, we study the inflationary potential arising upon inclusion of these corrections. We constrain the values of the parameters for which it can still support slow-roll inflation, and find the preferred range of the compactification volume. Afterwards, we analyze the post-inflationary dynamics of the model in two possible scenarios, and find a consistent prediction for the number of e-foldings of inflation in both cases. This yields a unique prediction for the scalar spectral index, which has to be compared with the experimental value obtained considering the amount of Dark Radiation predicted by our model.

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Contents

1 Introduction

One of the hardest and most compelling theoretical challenges of the last 50 years has been the formulation of a theory of Quantum Gravity. The aim is a theory that embeds all the gravitational phenomena in a coherent quantum framework which is free of non-renormalizable Ultra-Violet (UV) divergences. These naturally show up in a Quantum Field Theory including gravity, since the gravitational coupling has the dimension of an inverse square mass [1]. Despite many efforts, no renormalizable theory of quantum gravity has been found within the framework of QFT. This led to the idea that QFT may not be the ultimate UV-complete framework, but it might have to be modified at scales approaching the gravitational one.

One of the most promising frameworks for a complete theory of quantum gravity is *String Theory*. In fact, it is a *finite* theory, completely free of UV divergences, and naturally includes the graviton in its spectrum. Throughout the years, different versions of String Theory have been studied, starting from the theory of the *Bosonic String*, which, as the name suggests, only includes bosonic degrees of freedom, up to various types *Superstring* theories, which incorporate fermions through the inclusion of *Supersymmetry* (SUSY).

A common feature of all the versions of string theory is the requirement of a number of spacetime dimensions which is higher than what we currently observe experimentally. In particular, we will focus on Type-IIB superstring theory, which needs 10 spacetime dimensions for consistency. Although this looks like a problem, it may actually be a point of strength of the theory. In fact, string theory solves it by supposing that the 6 extra spatial dimensions are not extended, but rather compactified in a specific kind of manifold, called a *Calabi-Yau threefold*. The 10d theory is brought down to a 4d Effective Field Theory (EFT) through a procedure called Kaluza-Klein reduction. During this *compactification* procedure, many scalar fields are produced, usually called *moduli* Moduli fields have a deep geometrical meaning, since their Vacuum Expectation Values (VEVs) control the shape and size of the compactification space. However, from a 4d EFT point of view these fields are *flat directions* of the scalar potential, hence their VEVs will be undefined. This is a problem, since the number of such moduli could be very high, up to hundreds or thousands, and the unfixed VEVs will act as a free parameters in the EFT. This can be solved by considering additional UV effects which produce *stabilizing* terms in the 4d $\mathcal{N} = 1$ SUSY EFT. One kind of moduli, called *complex structure moduli*, is usually stabilized by switching on 3-form fluxes on the Calabi-Yau, while the other kind, called Kähler moduli, needs to be stabilized through quantum corrections.

There are various paradigms to stabilize Kähler moduli, all differing in the kind of quantum effects considered. We will mainly focus on the Large Volume Scenario (LVS) of Type-IIB string theory, in which the stabilization is achieved balancing perturbative corrections to the Kähler potential and non-perturbative effects of the superpotential. The resulting scalar potential has a non-supersymmetric Anti-de Sitter minimum, which we can *lift* up to Minkowski or de Sitter with appropriate uplifting terms. The stabilized

potential still exhibits *exponentially flat* directions. Therefore, it seems perfect to support *slow-roll inflation*. The inflaton field is represented by (the canonically normalized version of) one of the so-called 'small' or 'blow-up' Kähler moduli. Geometrically, these represent the volumes of co-dimension-2 surfaces, called 4-cycles, that resolve (blow-up) point-like singularities of the Calabi-Yau. Since these 4-cycles locally resolve singular points, their volume is *small* compared to the overall volume of the Calabi-Yau, which is taken to be exponentially large in the LVS setting (hence the name).
The resulting model of inflation is called *Kähler moduli inflation* [2], and it has been widely studied in the last decades. Sensible predictions for the cosmological parameters have been formulated studying the post-inflationary evolution of the system [3].
The aim of this work is to study the effect of *open string loop* corrections to the Kähler potential of the model of Kähler moduli inflation. We will analyze the conditions under which the corrected potential can still support slow-roll inflation. After that, we will study the post-inflationary dynamics of the system, and try to extract a prediction for the number of e-foldings N_e and the spectral index n_s, under a general choice of the parameters.
The thesis is structured as follows. In Section 2 we review the basic ideas behind String Theory and Superstring Theory, with particular interest in the spectrum of the theories. In Section 3, we will study the procedure of Kaluza-Klein reduction and introduce the concept of Calabi-Yau spaces, focusing mainly on their moduli space. In Section 4 we review the most important quantum corrections to the 4d EFT and introduce the LVS setting for moduli stabilization. In Section 5 we first review the original model of Kähler moduli inflation and then add loop corrections to it and study their effects. We will conclude in Section 6.

2 String Theory and Superstring Theory

String theory is a powerful theory which naturally incorporate General Relativity in a quantum setting. In this section we will review the bosonic string theory and its quantization, and then move to Superstring theory. We mainly follow the books [4, 5, 6].

2.1 Review of the Classical Bosonic String

The foundational idea behind String Theory is that the fundamental physical entity is no longer the point particle, but rather the *string*: an extended 1-dimensional object. The reason behind this choice lies in the elimination of the UV divergences in loops, specifically in quantum gravity. In fact, while in point particle scatterings there exists a precise *interaction point*, in a string scattering, the interaction gets *smeared out*, hence no interaction point is present and strings appear to *merge* smoothly. This solves the divergence problem for the following reason. In the case of a point-particle, we can make a loop to be infinitely small, i.e. take the interaction points to be infinitely close. This translates in energies tending to infinity, which can give rise to non-renormalizable divergences. In string theories, instead, string loops can never go to zero size, thus completely eliminating UV divergences, see Fig.2.1.

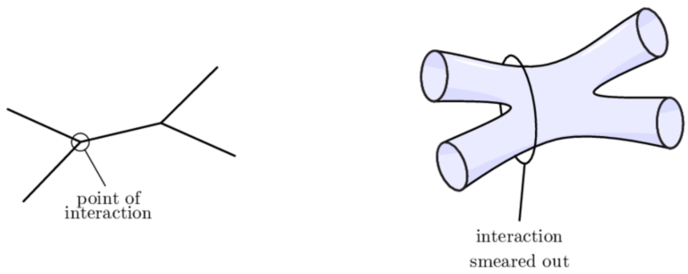

Figure 2.1: Point-Particle Scattering (left) vs (closed) string scattering (right). Credits to [7] for the image.

To understand why string theory naturally includes gravity, we need to show how to describe the propagation and evolution of a string through space-time. While the space-time trajectory of a zero-dimensional point particle was described by a one-dimensional *world-line*, the space-time trajectory of a one-dimensional string is described by a two-dimensional surface called *world-sheet*. To properly study it, though, we have to embed the world-sheet in a *target space*, whose coordinates are usually indicated by X^μ. This embedding is specified by the set of functions

$$X^\mu = X^\mu(\sigma) \tag{2.1}$$

with $\mu = 0, ..., D-1$ where D is the dimension of the target space $\mathcal{M}$. The world-sheet Σ is parameterized by coordinates which are usually indicated as $\sigma = (\tau, \sigma)$ and indexed as σ^a with $a = 1, 2$. The most commonly used world-sheet action to describe the propagation of a free string in a Minkowski target space is the so-called *Polyakov action*:

$$S_P = -\frac{T_s}{2} \int_\Sigma \sqrt{-h}\, h^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}\, \mathrm{d}\sigma \mathrm{d}\tau \tag{2.2}$$

where h_{ab} is the *world-sheet metric* and $h = \det(h_{ab})$. T_s is the string tension, which is given by

$$T_s = \frac{1}{2\pi\alpha'} \tag{2.3}$$

where α' is an important parameter called *Regge slope* and it can be interpreted in terms of the *string length* l_s [1]:

$$l_s = \sqrt{\alpha'} \tag{2.4}$$

One can see the Polyakov action (2.2) as the action of a 2d QFT featuring D free scalar fields X^μ. This field theory has the following symmetries:

1) *Poincaré symmetry*: in this case it can be viewed as a symmetry of the internal space of the fields X^μ:

$$X^\mu \mapsto X'^\mu = \Lambda^\mu_\nu X^\nu + A^\mu \tag{2.5}$$

with $\Lambda \in SO(1, D-1)$ and A^μ a constant D-vector.

2) *General world-sheet diffeomorphisms*: under a general differentiable transformation of the world-sheet coordinates $\sigma^a \mapsto \sigma'^a = \sigma'^a(\tau, \sigma)$:

$$X^\mu(\tau,\sigma) \mapsto X'^\mu(\tau',\sigma') \tag{2.6}$$

$$h_{ab}(\tau,\sigma) \mapsto h'_{ab}(\tau',\sigma') = \frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}h_{cd}(\tau,\sigma) \tag{2.7}$$

3) *2-dimensional Weyl invariance*: the theory is invariant under local rescaling of the world-sheet metric:

$$h_{ab}(\tau,\sigma) \mapsto h'_{ab}(\tau,\sigma) = e^{2\omega}h_{ab}(\tau,\sigma) \tag{2.8}$$

with $\omega = \omega(\tau,\sigma)$ arbitrary function of the world-sheet coordinates.

Among these symmetries, the most characteristic and constraining one is the Weyl invariance. It can be shown that given a similar theory featuring p-dimensional dynamical objects described by world-hypersurface coordinates $\{\sigma^1, ..., \sigma^{p+1}\}$, the only case in which such a theory has a Weyl invariance is $p = 1$.
While Poincaré invariance is a consequence of the choice of a flat target space, both diffeomorphism and Weyl invariance are intrinsic properties of the world-sheet action. Combining the two, it is always possible to make a gauge choice for the world-sheet metric such that the world-sheet is flat. In fact, one can see that under a Weyl transformation $h_{ab} \mapsto h'_{ab} = e^{2\omega}h_{ab}$, the curvature scalar of the world-sheet $\mathcal{R}$ transforms as:

$$\sqrt{-h'}\mathcal{R}[h'] = \sqrt{-h}(\mathcal{R}[h] - 2\Box\omega) \tag{2.9}$$

where $\Box = \nabla^a\partial_a$, being ∇^a the covariant derivative on the world-sheet. Therefore, to make this vanish, one can choose a function $\omega(\tau,\sigma)$ such that:

$$\Box\omega = \frac{1}{2}\mathcal{R}[h] \tag{2.10}$$

[1]The definition of l_s in terms of α' vary in literature of some factors of $\sqrt{2}$ or 2π. Here we stick to the convention of [4].

This is called *flat gauge*, in fact in two dimensions, the Riemann tensor is:

$$\mathcal{R}_{abcd} = \frac{1}{2}(h_{ac}h_{bd} - h_{ad}h_{bc})\mathcal{R} \tag{2.11}$$

then choosing ω as in (2.10) makes the Riemann tensor vanish and hence, the world-sheet flat. Once in flat gauge, an appropriate choice of coordinates can make the world-sheet metric be a Minkowski metric:

$$h_{ab} = \mathrm{diag}(-1, 1) \tag{2.12}$$

Equations of motion

Let us now consider the equations of motion deriving from the Polyakov action. First, we can see the variation of the action with respect to the world-sheet metric:

$$\delta_h S_P = -\frac{1}{4\pi\alpha'}\int_\Sigma \sqrt{-h}\,\delta h^{ab}\left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2}h_{ab}\partial^c X^\mu \partial_c X_\mu\right)\,\mathrm{d}\sigma\mathrm{d}\tau \tag{2.13}$$

We can then define the energy-momentum tensor on the world-sheet as:

$$T^{ab} := -\frac{4\pi}{\sqrt{-h}}\frac{\delta S_P}{\delta h_{ab}} = -\frac{1}{\alpha'}\left(\partial^a X^\mu \partial^b X_\mu - \frac{1}{2}h^{ab}\partial^c X^\mu \partial_c X_\mu\right) \tag{2.14}$$

making the equation of motion deriving from (2.13) simply:

$$T_{ab} = 0 \tag{2.15}$$

On the other hand, varying the action with respect to the fields X^μ gives:

$$\delta_X S_P = \frac{1}{2\pi\alpha'}\int_\Sigma \sqrt{-h}\,\delta X^\mu \Box X_\mu\,\mathrm{d}\sigma\mathrm{d}\tau - \frac{1}{2\pi\alpha'}\int_{-\infty}^{+\infty}\sqrt{-h}\,\left[\delta X^\mu \partial^\sigma X_\mu\right]_{\sigma=0}^{\sigma=l_s}\,\mathrm{d}\tau \tag{2.16}$$

where we supposed that the range of the world-sheet variables is:

$$-\infty < \tau < \infty \quad \text{and} \quad 0 \le \sigma \le l_s \tag{2.17}$$

To evaluate the boundary term, we have to impose boundary conditions on the fields $X^\mu(\sigma)$. Different choices of these boundary conditions can be made. then one can choose the so-called *Neumann boundary conditions*:

$$\partial_\sigma X^\mu(\tau, 0) = \partial_\sigma X^\mu(\tau, l_s) = 0 \tag{2.18}$$

these describe a freely propagating *open string* whose endpoints move ate the speed of light. On the other hand, one can also impose *periodic boundary conditions*:

$$X^\mu(\tau, 0) = X^\mu(\tau, l_s), \quad \partial_\sigma X^\mu(\tau, 0) = \partial_\sigma X^\mu(\tau, l_s), \quad h^{ab}(\tau, 0) = h^{ab}(\tau, l_s) \tag{2.19}$$

describing *closed strings*, and the absence of boundary of the world-sheet. It is immediate to see that with both choices of the boundary conditions (2.18) and (2.19) the boundary term of the variation (2.16) vanishes. Therefore, the equations of motion from (2.16) simply become

$$\Box X^\mu(\tau, \sigma) = 0 \tag{2.20}$$

We can solve these equations in flat gauge (2.12), where they simplify to:

$$(\partial_\sigma^2 - \partial_\tau^2) X^\mu = 0 \tag{2.21}$$

It is more convenient to use *light-cone coordinates* :

$$\sigma^\pm = \tau \pm \sigma \tag{2.22}$$

so that

$$\Box = -4\partial_+\partial_- \quad \text{with} \quad \partial_\pm = \frac{\partial}{\partial \sigma^\pm} \tag{2.23}$$

In light-cone coordinates the equations of motion become:

$$\partial_+\partial_- X^\mu = 0 \tag{2.24}$$

We can decouple the equations by choosing the ansatz:

$$X^\mu(\sigma^+, \sigma^-) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-) \tag{2.25}$$

we will refer to X_L^μ as the *left-moving* part of X^μ and to X_R^μ as the *right-moving* one. Let us first solve these in the case of closed strings. The boundary conditions (2.19) impose that both X^μ and $\partial_\sigma X^\mu$ are periodic, hence also $\partial_+ X_L^\mu$ and $\partial_- X_R^\mu$ must be periodic. Therefore they can be expanded in a Fourier series. Upon integration, this gives the mode decompositions:

$$X_L^\mu(\sigma^+) = \frac{\bar{x}^\mu}{2} + l_s^2 p^\mu \sigma^+ + i\frac{l_s}{\sqrt{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2in\sigma^+} \tag{2.26}$$

$$X_R^\mu(\sigma^-) = \frac{\bar{x}^\mu}{2} + l_s^2 p^\mu \sigma^- + i\frac{l_s}{\sqrt{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in\sigma^-} \tag{2.27}$$

By convention the constant terms are chosen to be equal and such that their sum is equal to $\bar{x}^\mu$, where:

$$\bar{x}^\mu = \frac{1}{l_s} \int_0^{l_s} X^\mu(\sigma, \tau = 0)\, \mathrm{d}\sigma \tag{2.28}$$

is the target-space position of the center of mass of the string at $\tau = 0$. Here p^μ appears as an integration constant, but it can be easily interpreted as the target-space momentum of the center of mass of the string at $\tau = 0$:

$$p^\mu = \frac{1}{l_s} \int_0^{l_s} \partial_\tau X^\mu(\sigma, \tau = 0)\, \mathrm{d}\tau \tag{2.29}$$

The oscillator modes α_n^μ and $\tilde{\alpha}_n^\mu$ are independent, but the reality of X^μ imposes that:

$$(\alpha_n^\mu)^* = \alpha_{-n}^\mu \quad \text{and} \quad (\tilde{\alpha}_n^\mu)^* = \tilde{\alpha}_{-n}^\mu \tag{2.30}$$

The situation with open strings is similar, but not the same. Enforcing Neumann boundary conditions (2.18) leads to non-independent oscillator modes in the mode expansion $\tilde{\alpha}_n^\mu = \alpha_n^\mu$. Thus, one can see that the final result for the overall X^μ is given by:

$$X^\mu = \bar{x}^\mu + 2l_s^2\, p^\mu \tau + \sqrt{2} i l_s \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma) \tag{2.31}$$

Before moving on, it can be said that, analyzing the variation (2.16), there is yet another boundary condition for which the boundary term vanishes, that is

$$\delta X^\mu(\tau, 0) = \delta X^\mu(\tau, l_s) = 0 \tag{2.32}$$

This is called *Dirichlet* boundary condition, and differently from the Neumann and periodic boundary conditions, these describe a string whose endpoints are neither free or coinciding, but rather confined to a fixed hyperplane. These 'membranes' where open string endpoints lie are called *D-branes* (where the D stands for Dirichlet). These branes are space-filling, meaning that they are extended in various spatial dimensions. We call *Dp-brane* a D-brane filling p spatial dimensions. Since the target space must be stationary, D-branes should always fill the time dimension, so from a space-time perspective Dp-branes are $(p+1)$-dimensional objects. Notice that it is not mandatory that we use Dirichlet boundary conditions for all the dimensions of the target space. In general, for open strings, the boundary conditions will be a combination of Neumann and Dirichlet, e.g. Dirichlet boundary condition on X^i and Neumann on X^μ for $\mu \neq i$.

2.2 Quantization of the Bosonic String

We want to see how to quantize the Polyakov action. We will only show the so-called 'old covariant' approach, which has been improved on by more recent approaches such as BRST quantization. For a detailed description of these alternative approaches see [4]. We start from the flat-gauge Polyakov action for the 2d QFT:

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma (\partial_\tau X^\mu \partial_\tau X_\mu - \partial_\sigma X^\mu \partial_\sigma X_\mu)\, \mathrm{d}\tau \mathrm{d}\sigma \tag{2.33}$$

Notice first of all that, since we fixed the flat gauge, Weyl invariance is lost. Moreover, even if this is a theory of D free scalars in 2d, one of them, X^0, has the wrong sign in the kinetic term. We define the conjugate momenta Π_μ to the canonical variables X^μ as:

$$\Pi_\mu = \frac{\delta S_P}{\delta \dot{X}^\mu} = \frac{1}{2\pi\alpha'} \partial_\tau X_\mu \tag{2.34}$$

Our aim is to perform canonical quantization, so we impose canonical equal-time commutation relations between the fields and the momenta:

$$[\Pi_\mu(\tau,\sigma), X^\nu(\tau,\sigma')] = -i\,\delta(\sigma-\sigma')\,\delta^\nu_\mu \tag{2.35}$$

$$[\Pi_\mu, \Pi_\nu] = [X^\mu, X^\nu] = 0 \tag{2.36}$$

Taking into consideration the expansion of X^μ in terms of oscillator modes (the sum of (2.26) and (2.27) for the closed string and (2.31) for the open string), and considering the analogue expansion for Π_μ, which can be directly obtained from that of X_μ by means of (2.34), we can write down the canonical commutators of the oscillator modes:

$$[p^\mu, \bar{x}^\nu] = -i\eta^{\mu\nu}, \quad [\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m,-n}\,\eta^{\mu\nu}, \quad [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\delta_{m,-n}\,\eta^{\mu\nu} \tag{2.37}$$

Fixing the momentum p, we can focus only on the algebra of the oscillator modes. Because of the relations (2.30) we can rewrite it as

$$\left[\alpha^\mu_m, (\alpha^\nu_n)^\dagger\right] = m\delta_{m,n}\,\eta^{\mu\nu} \tag{2.38}$$

and similarly for the $\tilde{\alpha}^\mu_n$. at this point we are able to perform the construction of a Fock space using the oscillateor modes as creation and annihilation operators. We define the vacuum state as $|0,0;p\rangle$ such that

$$\alpha^\mu_n\,|0,0;p\rangle = \tilde{\alpha}^\mu_n\,|0,0;p\rangle = 0 \quad \forall\,\mu = 0,...,D-1,\; n>0 \tag{2.39}$$

and the whole Hilbert space is built acting on $|0,0;p\rangle$ with creation operators α^μ_{-n} and $\tilde{\alpha}^\mu_{-n}$ for $n>0$. Looking at the commutation relations (2.38) it is obvious that the modes α^0_n and $\tilde{\alpha}^0_n$ have the wrong sign in the commutation relations. This will lead to the presence in the spectrum of non-physical states which must be eliminated. This can be done imposing a condition defining a physical state based on gauge invariance, in a very similar way to the Gupta-Bleuler condition in QED.

By fixing the gauge and eliminating the degrees of freedom of h_{ab}, we have also eliminated one of the equations of motion, namely (2.15). This must be imposed as a constraint of the theory. Using light-cone coordinates (2.22), we notice that:

$$T_{+-} = T_{-+} = 0 \tag{2.40}$$

identically, so we just have to impose that

$$T_{++} = T_{--} = 0 \tag{2.41}$$

One can check from the definition (2.14) that

$$T_{++} = \partial_+ X^\mu_L \partial_+ X_{L\mu} \quad \text{and} \quad T_{--} = \partial_- X^\mu_R \partial_- X_{R\mu} \tag{2.42}$$

This leads to the definition of the so called *Virasoro operators*, which are the coefficients of the Laurent series expansion of T_{ab}:

$$L_m = \frac{1}{4\pi\alpha'} \int_0^{l_s} T_{--} e^{-2im\sigma}\, \mathrm{d}\sigma = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \alpha_{m-n} \cdot \alpha_n \tag{2.43}$$

$$\tilde{L}_m = \frac{1}{4\pi\alpha'} \int_0^{l_s} T_{++} e^{-2im\sigma}\, \mathrm{d}\sigma = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n \tag{2.44}$$

where, for simplicity, we set

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{l_s}{\sqrt{2}} p^\mu \tag{2.45}$$

For the open string, one can use the same definition, recalling the identification of the oscillator modes. These operators satisfy the so-called *Virasoro Algebra*:

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m,-n} \quad \text{with} \quad A(m) = \frac{m^3 - m}{12} D \tag{2.46}$$

and similarly for $\tilde{L}_m$. We may now define the physical state condition naively as follows. A state $|\chi\rangle$ is physical if

$$L_m |\chi\rangle = 0 \quad \forall m \geq 0 \tag{2.47}$$

While for $m > 0$ this condition works fine, for $m = 0$ we encounter some divergence problems. The reason of these problems is an ambiguity in the ordering of the operators in the definition of L_0. In fact, explicitly we have:

$$L_0 = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \alpha_{-n} \cdot \alpha_n = \frac{1}{2}\alpha_0^2 + \sum_{n\neq 0} \alpha_{-n} \cdot \alpha_n \tag{2.48}$$

however, in the final sum we have a clear ordering ambiguity since n can assume both positive and negative values. This is the core reason of the divergence, which can be resolved simply by normal ordering. In fact, we can write:

$$L_0 =: L_0 : + a \quad \text{where} \quad a = -\frac{1}{2}(D-2) \sum_{n=1}^{\infty} n \tag{2.49}$$

To eliminate the divergence, now explicitly present in the factor a, one regularizes the sum using Riemann ζ-function. This gives:

$$L_0 =: L_0 : + a \quad \text{with} \quad a = \frac{D-2}{24} \tag{2.50}$$

This suggests that the correct way to impose the condition on the physical states is to say that $|\chi\rangle$ is physical if:

$$(L_m - a\delta_{m,0}) |\chi\rangle = 0 \quad \forall m \geq 0 \tag{2.51}$$

A similar condition holds for $\tilde{L}_m$ as well.

2.3 Bosonic String Spectrum

We are interested in the spectrum of the bosonic string arising from the quantization sketched in the previous section.

Open String Spectrum

Let us start off describing the spectrum of *open strings*. This is the simplest case, since we only have *one* set of creation/annihilation operators α_n^μ. The open string vacuum state will be indicated as $|0;p\rangle$. This is such that:

$$\alpha_n^\mu |0;p\rangle = 0 \; \forall n > 0 \quad \text{and} \quad \hat{p}^\mu |0;p\rangle = p^\mu |0;p\rangle \tag{2.52}$$

where by $\hat{p}^\mu$ we here mean the target-space momentum operator. On the string vacuum, the physical condition (2.51) is automatically satisfied for all $m > 0$, because of (2.52) and (2.43). The condition for $m = 0$, on the other hand, can be cast as:

$$\left(\alpha' p^2 + \sum_{n>0} [\alpha_{-n} \cdot \alpha_n] - a\right) |0;p\rangle = 0 \tag{2.53}$$

where we used $\alpha_0^\mu = \frac{l_s}{\sqrt{2}} p^\mu = \frac{\sqrt{\alpha'}}{\sqrt{2}} p^\mu$. For the mass-shell condition we write $M^2 = -p^2$ and thus we get the mass of the open string vacuum state:

$$M^2 = -\frac{a}{\alpha'} < 0 \tag{2.54}$$

This means that the ground state of the open string in bosonic string theory is a *tachyon*, indicating that this theory is not completely stable. Moving to the first excited state, we have to consider all the states of the form:

$$\epsilon_\mu \alpha_{-1}^\mu |0;p\rangle \tag{2.55}$$

where ϵ_μ is a generic polarization vector. Considering the physical state condition for L_0 we get that:

$$(L_0 - a)\epsilon_\mu \alpha_{-1}^\mu |0;p\rangle = (\alpha' p^2 + 1 - a)\epsilon_\mu \alpha_{-1}^\mu |0;p\rangle = 0 \tag{2.56}$$

which implies that

$$M^2 = -p^2 = \frac{1-a}{\alpha'} \tag{2.57}$$

Moreover, the condition for $m = 1$ is also non-trivial. It reduces to:

$$L_1 \epsilon_\mu \alpha_{-1}^\mu |0;p\rangle = \epsilon_\mu p^\mu |0;p\rangle = 0 \tag{2.58}$$

which means that the polarization of the state must be transversal to the momentum. Finally, we get that the norm of the state is

$$\langle 0;p| \alpha_1^\nu \epsilon_\nu \epsilon_\mu \alpha_{-1}^\mu |0;p\rangle = \langle 0;p|0;p\rangle \, \epsilon_\mu \epsilon^\mu = \epsilon_\mu \epsilon^\mu \tag{2.59}$$

where we used the commutation relations (2.37). Therefore we choose ϵ^μ to be a real vector [2].

There are three possible cases to consider. If $a > 1$ we say the string theory is *supercritical*: the mass of the first excited state is $M^2 < 0$, so it is a second tachyon, and p^μ is space-like; this implies, by (2.58), that there exist physical states with ϵ^μ time-like and hence with negative norm, which is pathological. If instead $a < 1$ we have $M^2 > 0$ for the first excited state and all the following ones. This case is not very interesting phenomenologically speaking, since we would end up in the final spectrum with a tachyon and a massive vector: this case is called *sub-critical*. The most interesting case, which we will stick to, happens when $a = 1$ and it is called *critical string theory*. In this case we have $M^2 = 0$ and also $p^2 = 0$, which means that the polarizations allowed are one longitudinal, $\epsilon^\mu \parallel p^\mu$, and $(D-2)$ transversal ones. The longitudinal polarization brings about zero-norm states, consistently with what happens in QED upon quantization of a massless vector field. The choice of $a = 1$, also fixes the number of dimensions in which a sensible bosonic string theory can take place. In fact from (2.51) we see that for a to be 1, we must set $D = 26$. As a side note, the condition $a = 1$ is also necessary for Weyl anomaly cancellation upon BRST quantization. We will not treat this argument and refer the interested reader to [4] for a thorough derivation.

Sticking to the critical case, we have ruled out negative-norm states by imposing the physical condition, but still have null states. These can be eliminated if we take the quotient Hilbert space:

$$\mathcal{H} = \mathcal{H}_{phys}/\mathcal{H}_{null} \tag{2.60}$$

Notice finally that we can rewrite the mass-shell condition $(L_0 - 1)\,|phys\rangle = 0$ introducing the *level* operator:

$$N := \sum_{n>0} \alpha_{-n} \cdot \alpha_n \tag{2.61}$$

so that the mass-shell relation can be written in an operatorial way as

$$M^2 = \frac{N-1}{\alpha'} \tag{2.62}$$

which makes it explicit that in the open string spectrum we have a tachyon for $N = 0$, a massless vector for $N = 1$ and massive tensor fields for $N > 1$.

Closed String Spectrum

The case of the *closed string* is very similar. The same conditions on a can be found analyzing the spectrum. Focusing only on the critical case $a = 1$, $D = 26$ we can write down the physical state conditions for $m = 0$ as:

$$(L_0 - 1)\,|phys\rangle = (\tilde{L}_0 - 1)\,|phys\rangle = 0 \tag{2.63}$$

[2] We can always do that, since we can re-define α_{-1}^μ

We can combine them to obtain:

$$(L_0 - \tilde{L}_0)\,|phys\rangle = (L_0 + \tilde{L}_0 - 2)\,|phys\rangle = 0 \tag{2.64}$$

Recalling the expression for L_0 and $\tilde{L}_0$ and the definition of N (2.61) (an analogous definition for $\tilde{N}$ can be given), we get two conditions called respectively *level matching* and *mass-shell* condition:

$$(N - \tilde{N})\,|phys\rangle = 0 \quad \text{and} \quad \left(\frac{\alpha'}{2}p^2 + N + \tilde{N} - 2\right)|phys\rangle = 0 \tag{2.65}$$

The level matching condition tells us that in the closed string spectrum, there appear only states in which the number of right-moving and left-moving excitations match. On the other hand, the mass-shell condition gives us an expression for the mass of each state:

$$M^2 = \frac{2(N + \tilde{N} - 2)}{\alpha'} \tag{2.66}$$

Hence, also for the closed string the vacuum state $|0,0;p\rangle$, for which $N = \tilde{N} = 0$ is a *tachyon*. On the other hand, the first physical excited state has $N = \tilde{N} = 1$ due to the level-matching condition, and it is a massless spin-2 state. Using group-theoretical arguments, one finds that this spin-2 state is a *reducible representation* of $SO(D-2)$ and can be split into three irreducible representations. The first is a symmetric, traceless tensor $G_{\mu\nu}$ generically called the *graviton*, then we have an anti-symmetric tensor $B_{\mu\nu}$ called *Kalb-Ramond* field and finally we have a scalar field, the *dilaton* φ. Hence, the spectrum of the closed string includes a tachyon at $N = \tilde{N} = 0$, the graviton $G_{\mu\nu}$, the Kalb-Ramond field $B_{\mu\nu}$ and the dilaton φ, which are all massless, at $N = \tilde{N} = 1$, and massive excited states for $N = \tilde{N} > 1$.

2.4 Target-Space Action

If we return to a target-space perspective, it is possible to write down an action based on the spectrum we found. We will focus on the closed-string spectrum to illustrate how this process works. In $D = 26$, excluding the tachyon degrees of freedom, we can write the simplest action for the massless spectrum of the closed string as:

$$S_{26} = \frac{1}{\kappa^2}\int_{\mathcal{M}} \sqrt{-G}e^{-2\varphi}\left[\mathcal{R}[G] - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\varphi\partial^\mu\varphi\right]\mathrm{d}^{26}x \tag{2.67}$$

where $H_{\mu\nu\lambda}$ can be thought of as the field-strength tensor of $B_{\mu\nu}$, which in differential-form language is expressed as:

$$H = dB \tag{2.68}$$

with d the exterior derivative operator.
Let us comment on this action. First of all, the value of the constant κ^2 can be changed

with a proper shift of the dilaton. In general we define it as $\kappa^2 = c\alpha'^{12}$, but the constant c can be reabsorbed into the dilaton. Notice then that this action is not canonically-normalized: the Einstein-Hilbert term has a factor of $e^{-2\varphi}$ and the dilaton kinetic term has the wrong sign. To fix this, we just Weyl-transform the graviton as:

$$G_{\mu\nu} \mapsto \tilde{G}_{\mu\nu} = e^{\frac{\varphi}{6}} G_{\mu\nu} \tag{2.69}$$

which modifies (2.67) as:

$$S_{26} = \frac{1}{\kappa^2} \int_{\mathcal{M}} \sqrt{-\tilde{G}} \left[\mathcal{R}[\tilde{G}] - \frac{1}{12} e^{-\frac{\varphi}{3}} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{6} \partial_\mu \varphi \partial^\mu \varphi \right] \mathrm{d}^{26}x \tag{2.70}$$

We call this *Einstein frame*, where the Planck scale is manifestly fixed and the Einstein-Hilbert term has a canonical form, whereas the form (2.67) is called *string frame* action. Notice that taking (2.67) in the case $G_{\mu\nu} = \eta_{\mu\nu}$ and $B_{\mu\nu} = 0$ as well as $\varphi = 0$ gives us a solution describing the background in which our original 2d QFT (2.2) lives. This suggests that a straightforward generalization of the world-sheet Polyakov action would arise replacing $\eta_{\mu\nu}$ with a general target-space metric $G_{\mu\nu}(X)$:

$$S = -\frac{1}{4\pi\alpha'} \int_\Sigma \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) \, \mathrm{d}^2\sigma \tag{2.71}$$

Similar modifications appear when we turn on the fields B and φ. The former introduces new interaction terms in the 2d QFT of the X^μ, while the role of the latter is subtler. It appears as the coefficient of the world-sheet Einstein-Hilbert term:

$$S_P \supset \frac{1}{4\pi} \int_\Sigma \varphi \sqrt{-h} \, \mathcal{R}[h] \, \mathrm{d}^2\sigma \tag{2.72}$$

This is crucial for what follows, since it will prove the fact that φ governs the string coupling.
To see that, let us consider the the *Polyakov path integral*, which is the path integral built from the Polyakov action. To make it simple, we will suppose the target-space metric is Minkowski and we only take a non-zero dilaton as a modification to the action (2.2). In Euclidean world-sheet time, we can write the path integral as:

$$\mathcal{Z} = \int DX \, Dg \, e^{-S} \tag{2.73}$$

The action featured in this definition is the sum of two pieces:

$$S = S_X + \lambda\chi \tag{2.74}$$

where:

$$S_X = \frac{1}{4\pi\alpha'}\int_\Sigma \sqrt{g}g^{ab}\partial_a X^\mu \partial_b X_\mu \, \mathrm{d}^2\sigma \tag{2.75}$$

$$\chi(\Sigma) = \frac{1}{4\pi}\int_\Sigma \sqrt{g}\,\mathcal{R}[g]\,\mathrm{d}^2\sigma \tag{2.76}$$

where we used an Euclidean world-sheet metric g_{ab}. Notice that the second piece of (2.74), considering the expression for the Euler number (2.76), has exactly the same form as in (2.72). This suggests that we identify the apparently arbitrary coefficient λ exactly with the dilaton φ.

The most interesting aspect of this is considering what happens to the action (2.74) when we add to the world-sheet a closed string loop. From a purely topological perspective, adding a closed string loop is equivalent to adding a *handle* to the world-sheet, raising by 1 the *genus* of the world-sheet topology. For a closed, oriented surface the Euler number can be written as:

$$\chi = 2 - 2g \tag{2.77}$$

where g is the genus of the surface. Therefore, adding a handle modifies the Euler number as $\chi \mapsto \chi - 2$. From the perspective of the path integral, this means multiplying by an overall factor of $e^{2\lambda} = e^{2\varphi}$. Going back to the physical perspective of a closed-string loop we then expect a *string coupling* factor, g_s, each time a closed string is emitted and reabsorbed, so a closed string loop should come with an overall factor of g_s^2. But then it is immediate to identify the string coupling with the exponential of the dilaton:

$$g_s = e^\varphi \tag{2.78}$$

2.5 Superstring Theory

Up to now, we found string theory is promising. First of all, it naturally includes the graviton upon quantization, making it a good candidate for a quantum gravity theory. Second, it appears to be a finite theory, with no UV divergences or need of renormalization. However, as it stands, bosonic string theory has some deep problems:

i) *It does not include fermions.* In fact all the open- or closed-string states we discussed are bosons, since the creation operators are only bosonic. Of course, we know there are fermions in nature, so we need a way to include them.

ii) *The ground state is a tachyon.* This indicates the overall theory is unstable, as the ground state could decay through tachyon condensation [8]. However, in the context of bosonic string theory there is no way (that we know of) to get rid of tachyonic states.

iii) $D = 26$. The number of spacetime dimensions of critical string theory is way bigger than the one we observe.

As it is, the theory seems phenomenologically unviable. However, introducing *Supersymmetry* in the world-sheet theory will solve problems (i) and (ii) and improve on problem (iii). Let us sketch how.

World-sheet Supersymmetry

First of all, we have to include some fermionic world-sheet coordinates that we will call θ_α. It is easy to verify that with a flat world-sheet metric, the theory is Lorentz-symmetric if we transform the world-sheet variables as:

$$\sigma^a \mapsto \sigma'^a = \Lambda^a_b \sigma^b \quad \text{and} \quad \theta_\alpha \mapsto \theta'_\alpha = S^\beta_\alpha \theta_\beta \tag{2.79}$$

where:

$$\Lambda = e^{i\omega^{ab} J_{ab}} \quad \text{and} \quad S = e^{i\omega^{ab}\Sigma_{ab}} \tag{2.80}$$

with J_{ab} being the 2d Lorentz generator in vector representation, while Σ_{ab} the 2d Lorentz generator in spinorial representation, and ω^{ab} the parameter tensor. As in 4d, we can write Σ_{ab} in terms of 2d γ-matrices:

$$\Sigma^{ab} = \frac{i}{4}\left[\gamma^a, \gamma^b\right] \tag{2.81}$$

where the γ-matrices satisfy the usual Clifford algebra [3]:

$$\{\gamma^a, \gamma^b\} = -2\eta^{ab} \tag{2.82}$$

As it appears manifestly from (2.81), Σ^{ab} is anti-symmetric, which constrains the parameter space to only one dimension, so that $\omega_{ab} \propto \epsilon_{ab}$, the 2d Levi-Civita symbol. The matrix S can be proven to be *real*, which implies that θ_α is real too. The condition $\theta^* = \theta$ is equivalent, in 2d, to the Majorana condition, so that θ_α is a Majorana fermion, see appendix B of [5] for details.
To introduce SUSY in the world-sheet action, we promote all the scalar fields in the theory to *superfields* $X^\mu(\sigma) \mapsto Y^\mu(\sigma, \theta)$ such that:

$$Y^\mu(\sigma, \theta) = X^\mu(\sigma) + \bar{\theta}\psi^\mu(\sigma) + \frac{1}{2}\bar{\theta}\theta B^\mu(\sigma) \tag{2.83}$$

with $\bar{\theta} = \theta^\dagger \gamma^0$. As it is clear from (2.83), ψ^μ is a fermion and B^μ an auxiliary field. The SUSY generators on the world-sheet can be expressed in a very similar way to 4d SUSY:

$$Q_\alpha = \frac{\partial}{\partial \bar{\theta}^\alpha} + i(\gamma^a\theta)_\alpha \partial_a \tag{2.84}$$

[3]Notice that the signature of our world-sheet metric is $(-,+)$, and since we want to preserve the nice property $(\gamma^0)^2 = \mathbb{1}$, we take the minus sign outside the commutator.

It is an easy exercise to prove they satisfy the SUSY algebra:

$$\{Q_\alpha, \bar{Q}^\beta\} = -2i(\gamma^a)_\alpha^\beta \partial_a \tag{2.85}$$

hence, the SUSY transformation of the superfield Y^μ is akin to the usual four-dimensional one:

$$\delta_\xi Y^\mu = (\bar{\xi} Q) Y^\mu \tag{2.86}$$

Considering relation (2.86) component-wise, we get the SUSY transformations of the individual component fields [4]:

$$\delta_\xi X^\mu = \bar{\xi} \psi^\mu \tag{2.87}$$
$$\delta_\xi \psi^\mu = -i\gamma^a \xi \partial_a X^\mu + \xi B^\mu \tag{2.88}$$
$$\delta_\xi B^\mu = -i\bar{\xi}\gamma^a \partial_a \psi^\mu \tag{2.89}$$

Our purpose now is writing down a generalization of the Polyakov action in a SUSY-invariant way. To do that, it is useful to introduce the so-called *supercovariant derivative* on the world-sheet:

$$D_\alpha = \frac{\partial}{\partial \bar{\theta}^\alpha} - i(\gamma^a \theta)_\alpha \partial_a \tag{2.90}$$

which is such that:

$$\{D_\alpha, Q_\beta\} = 0 \tag{2.91}$$

This is useful because we can now easily make our Polyakov action SUSY-invariant just by substituting $X^\mu \mapsto Y^\mu$, $\partial_a \mapsto D_\alpha$ and $\mathrm{d}^2\sigma \mapsto \mathrm{d}^2\sigma \mathrm{d}^2\theta$:

$$S = \frac{i}{4\pi\alpha'} \int \bar{D}^\alpha Y^\mu \, D_\alpha Y_\mu \, \mathrm{d}^2\sigma \mathrm{d}^2\theta = -\frac{1}{2\pi\alpha'} \int \left[\partial_a X^\mu \partial^a X_\mu - i\bar{\psi}^\mu \not{\partial} \psi_\mu - B^\mu B_\mu\right] \mathrm{d}^2\sigma \tag{2.92}$$

At the end of the day, since the auxiliary field B_μ can be eliminated through its equation of motion, the final result of the supersymmetrization the world-sheet is just the addition to the theory of a fermion for each scalar.

Quantization of the Superstring

Let us now quantize the new SUSY-invariant Polyakov action in a similar way as we did for the bosonic string. We start from the action (2.92) from which we eliminate B^μ through its equations of motion [5]. First of all, we notice that this action is already in *flat gauge*, being $h_{ab} = \mathrm{diag}(-1, 1)$. Thus, we have to impose by hand the equation of motion

[4]Notice how the transformation of the field B^μ under SUSY is a total world-sheet derivative. This is the common behaviour of the auxiliary fields

[5]The Euler-Lagrange equations for this field simply read $B^\mu = 0$

for h_{ab}, which is the same as (2.15), as a constraint. In this case, the energy-momentum tensor of the supersymmetric world-sheet theory is:

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu + \frac{i}{2}\bar{\psi}^\mu \gamma_{\{a}\partial_{b\}}\psi_\mu - \frac{1}{2}h_{ab}\left(\partial_c X^\mu \partial^c X_\mu + \frac{i}{2}\bar{\psi}^\mu \not{\partial}\psi_\mu\right) \tag{2.93}$$

where the curly brackets indicate symmetrization of the indices. We also reckon that, because of SUSY, the world-sheet metric h_{ab} must have a fermionic partner, usually called world-sheet *gravitino* χ_a, which in expression (2.92) is set to zero because of the flat gauge. From a more general analysis, one can derive its equations of motion, which are given by the vanishing of the current:

$$J^a = \frac{1}{2}\gamma^b\gamma^a\psi^\mu\partial_b X_\mu \tag{2.94}$$

Therefore $J_a = 0$ is the second constraint we must implement.
The decomposition in oscillator modes of the bosonic part of the action turns out to be the same as before, so we will focus on the fermionic part. We can rewrite the part of the action involving ψ^μ as:

$$S_F = \frac{i}{\pi\alpha'}\int (\psi_- \cdot \partial_+\psi_- + \psi_+ \cdot \partial_-\psi_+)\, \mathrm{d}^2\sigma \tag{2.95}$$

where we set $\gamma^\pm = \gamma^0 \pm \gamma^1$. Before solving the equations of motion which stem from this action, we have to address the question of the choice of boundary conditions for the fermions. Since all the fermionic observables are always built from bilinears, a sign is not detectable. This means that we may choose the sign of the fermion to stay the same at each revolution around the string, in which case we talk about *Ramond* (R) boundary condition, or we may set it to change at each revolution, in which case we have *Neveu-Schwarz* (NS) boundary conditions. Since we have two fermions and two boundary conditions we have a total of 4 sectors, identified by the choice of the boundary conditions on each of the fermions:

$$\psi_+(\sigma + l_s) = \psi_+(\sigma), \qquad \psi_-(\sigma + l_s) = \psi_-(\sigma) \qquad \text{(RR)} \tag{2.96}$$
$$\psi_+(\sigma + l_s) = \psi_+(\sigma), \qquad \psi_-(\sigma + l_s) = -\psi_-(\sigma) \qquad \text{(RNS)} \tag{2.97}$$
$$\psi_+(\sigma + l_s) = -\psi_+(\sigma), \qquad \psi_-(\sigma + l_s) = \psi_-(\sigma) \qquad \text{(NSR)} \tag{2.98}$$
$$\psi_+(\sigma + l_s) = -\psi_+(\sigma), \qquad \psi_-(\sigma + l_s) = -\psi_-(\sigma) \qquad \text{(NSNS)} \tag{2.99}$$

Now we can expand the fermionic fields in oscillator modes as for their bosonic counterparts, but we have to take care for the different sectors. We will have:

$$\psi_+^\mu = \sum_{r\in\mathbb{Z}+\nu}\tilde{\psi}_r^\mu e^{-2ir(\tau+\sigma)} \quad \text{and} \quad \psi_-^\mu = \sum_{r\in\mathbb{Z}+\nu}\psi_r^\mu e^{-2ir(\tau-\sigma)} \tag{2.100}$$

where

$$\nu = \begin{cases} 0 \text{ for R} \\ \frac{1}{2} \text{ for NS} \end{cases} \tag{2.101}$$

plus the usual reality condition:

$$(\psi_r^\mu)^* = \psi_{-r}^\mu \quad \text{and} \quad (\tilde{\psi}_r^\mu)^* = \tilde{\psi}_{-r}^\mu \tag{2.102}$$

Notice that for the open string one only has the RR and NSNS sectors, since the other two would be inconsistent with the structure of the open string, for details see for example [9]. We can immediately write down the algebra of the oscillator modes which are derived from equal-time commutators of the fields:

$$[\alpha_m^\mu, \alpha_n^\nu] = m\, \delta_{m,-n}\eta^{\mu\nu}, \quad \{\psi_r^\mu, \psi_s^\nu\} = \delta_{r,-s}\eta^{\mu\nu} \tag{2.103}$$

and similarly for their right-moving counterparts. Once again, we define the Virasoro operators both for the bosons and for the fermions in a similar way:

$$L_m = \frac{1}{2\pi\alpha'}\int_0^{l_s} e^{im\sigma}T_{++}\mathrm{d}\sigma, \quad G_r = \frac{\sqrt{2}}{2\pi\alpha'}\int_0^{l_s} e^{ir\sigma}J_+\mathrm{d}\sigma \tag{2.104}$$

These can be written in terms of the oscillator modes and satisfy the so-called *super-Virasoro* algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m,-n} \tag{2.105}$$
$$\{G_r, G_s\} = 2L_{r+s} + B(r)\delta_{r,-s} \tag{2.106}$$
$$[L_m, G_r] = \left(\frac{m}{2} - r\right) G_{m+r} \tag{2.107}$$

where

$$A(m) = (m^3 - m)\frac{D}{8} \quad \text{and} \quad B(r) = (4r^2 - 1)\frac{D}{8} \tag{2.108}$$

The physical state conditions, now are written as:

$$(L_m - a\delta_{m,0})\,|phys\rangle = 0, \quad G_r\,|phys\rangle = 0 \tag{2.109}$$

for all $m, r \geq 0$. Notice that there is no ordering ambiguity in G_0 because of its fermionic nature. Therefore, We have that:

$$a = \begin{cases} 0 \text{ for the R sector} \\ \frac{D-2}{16} \text{ for the NS sector} \end{cases} \tag{2.110}$$

In the R sector case, the contribution of the fermions precisely cancels the one from the bosons, while in the NS case, the non-trivial boundary condition spoils the cancellation.

Let us focus on the open superstring spectrum, and begin analyzing the NSNS sector. The vacuum state is $|0;p\rangle$, and the creation operators are α^{μ}_{-n} and ψ^{μ}_{-r} for $n, r > 0$. Now, let us impose the physical state condition for L_0 on the vacuum:

$$(L_0 - a)\,|0;p\rangle = (\alpha' p^2 + N^{\alpha} + N^{\psi} - a)\,|0;p\rangle \tag{2.111}$$

where we defined the level operators as:

$$N^{\alpha} = \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m \quad N^{\psi} = \sum_{r \in \mathbb{N}+\frac{1}{2}} r\psi_{-r} \cdot \psi_r \tag{2.112}$$

Then, at the level zero, we have once again a scalar with a mass:

$$M^2 = -\frac{a}{\alpha'} \tag{2.113}$$

Then we have a target-space spinor at level 1/2 given by $\epsilon_{\mu}\psi^{\mu}_{-1/2}\,|0,p\rangle$, with mass

$$M^2 = \frac{1}{\alpha'}\left(\frac{1}{2} - a\right) \tag{2.114}$$

Now, we want this to be massless, in order to be in a critical string theory, so we set $a = \frac{1}{2}$ which means that the critical number of dimensions for superstring theory is $D = 10$.
For the RR sector of the open string, the only difference from a superficial point of view is that the sum of N^{ψ} in (2.112) runs over the integers. This means that L_0 is independent of ψ^{μ}_0, which does not affect the mass. This means that ψ^{μ}_0 is a generator of the Little group. Therefore, each mass eigenstate must carry a representation of the Clifford algebra (2.103) that ψ^{μ}_0 satisfies. Therefore, every state in the RR sector of the theory must be a target-space spinor. In particular, we must have a vacuum $|\alpha;p\rangle$ which is a massless fermion. Since $a = 0$ here, we cannot infer from this argument the number of critical dimensions in this sector. It is possible to prove, through central-charge vanishing arguments [5], that $D = 10$ also for this sector.
The quantization and spectrum for the closed superstring is very similar and we will not cover it here. We just remark that exactly as in the bosonic string case, for critical dimensions $D = 10$ it contains the graviton, together with its superpartner the *gravitino*.

2.6 GSO Projection and Type II String Theories

The introduction of world-sheet supersymmetry solved the problem of the absence of fermionic degrees of freedom in the spectrum. However, in the NSNS sector of the open string we still have a tachyon. The way we can get rid of it is through an appropriate projection, called Gliozzi-Scherk-Olive (GSO) projection. The form of this projector is:

$$P_{GSO} = \frac{1}{2}(1 + (-1)^F) \tag{2.115}$$

where F is the fermion number. Concretely, one defines the fermion number of the tachyon state to be odd, in order to eliminate it:

$$(-1)^F |0;p\rangle = -|0;p\rangle \tag{2.116}$$

This implies that:

$$P_{GSO} |0;p\rangle = 0 \tag{2.117}$$

Therefore, restricting the Hilbert space to the image of the GSO projector, we get a theory which has no tachyons.
This projection also has the property of restricting the possible number of consistent superstring theories. In particular, it restricts the number of Type-II string theories, i.e. string theories with $\mathcal{N} = 2$ SUSY. In fact, acting with the GSO projector on the closed superstring spectrum, and imposing the absence of the tachyon, we only get two inequivalent and consistent Type-II string theories.
In Type-IIA string theory, we only keep states such that $(-1)^F = 1$ on the left-moving modes, $(-1)^F = 1$ on right-moving NS states and $(-1)^F = -1$ on right-moving R states. With these constraints on the spectrum, we get the following field content of Type-IIA string theory:

1. One scalar field (dilaton) φ, one symmetric traceless tensor (graviton) $g_{\mu\nu}$ and an anti-symmetric tensor (Kalb-Ramond tensor) $B_{\mu\nu}$

2. Two spinors with opposite chirality and two vector spinors (gravitinos) with opposite chirality

3. A 1-form field C_1 and a 3-form field C_3, corresponding to D0- and D2-brane states

Type-IIB string theory has a simpler constraint on states, requiring all the states to satisfy $(-1)^F = 1$. The field content of Type-IIB string theory is:

1. One scalar field (dilaton) φ, one symmetric traceless tensor (graviton) $g_{\mu\nu}$ and an anti-symmetric tensor (Kalb-Ramond tensor) $B_{\mu\nu}$

2. Two spinors of the same chirality and two vector-spinors of the opposite chirality with respect to the spinors (gravitinos)

3. One 0-form field C_0, a 2-form field C_2 and a 4-form field C_4 with self-duality condition: $F_5 = *F_5$ where $F_5 = dC_4$

The clearest difference between the two theories is that Type-IIB string theory is chiral: one of the two chiralities is preferred over the other.

3 String Compactification

Including supersymmetry and implementing the right projections solved the problems of the absence of fermions and of tachyons in the spectrum. One 'problem' is still standing: the large number of spacetime dimensions. As we will see in this section, in string theory, this is not regarded as a problem, but rather as a feature and source of predictions. In fact, the key insight is that one can *compactify* the extra dimensions, supposing they form a specific kind of 6-dimensional compact manifold with a series of topological and geometrical properties which reflect in phenomenological models.
In this section, we start by showing the 10d action for Type-IIB string theory, then we move to the main aspects of Kaluza-Klein compactifications. We analyze the structure of Calabi-Yau manifolds and their mathematical properties, and eventually study the 4d supergravity EFT resulting from the compactification of Type-IIB string theory on a Calabi-Yau. In this section we follow mainly the references [10, 11, 6].

3.1 10-dimensional Action for Type-IIB String Theory

Among all the possible different superstring theories and generalizations thereof (e.g. M-theory), the one that today has produced the most promising phenomenological results is Type-IIB string theory, whose field content we have exposed at the end of the previous section. The 10d action of the bosonic part of Type-IIB string theory has the following form:

$$S_{10} = S_K + S_{CS} + S_{LOC} \tag{3.1}$$

Let us analyze each of the components separately[6]. First of all, we have S_K, which contains all the kinetic terms of the fields:

$$S_K = \frac{1}{2\kappa_{10}^2}\int_{\mathcal{M}} \sqrt{-g}\left[e^{-2\varphi}\left(\mathcal{R} + 4(\partial\varphi)^2 - \frac{1}{2\cdot 3!}H_3^2\right) - \frac{1}{2}F_1^2 - \frac{1}{2\cdot 3!}\tilde{F}_3^2 - \frac{1}{4\cdot 5!}\tilde{F}_5^2\right] \mathrm{d}^{10}x \tag{3.2}$$

where $2\kappa_{10}^2 = (2\pi)^7\alpha'^4$. H_3 is the 3-form field strength potential of the Kalb-Ramond 2-form field B_2 as expressed in (2.68), while in general:

$$F_i = dC_i \tag{3.3}$$

are the field strengths of the form-fields C_i. The square of a differential form in (3.1) has the following meaning:

$$F_i^2 = F_i \wedge *F_i \tag{3.4}$$

[6]For simplicity, from now on we will use differential-form notation, see appendix B.4 of [5] for a complete review.

where $*$ is the Hodge-star operator. In (3.2) we have a generalization of these form-field strengths, defined as:

$$\tilde{F}_3 = F_3 - C_0 \wedge H_3 \quad \text{and} \quad \tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3 \tag{3.5}$$

which are invariant under the gauge transformations:

$$\begin{cases} C_2 \mapsto C_2' = C_2 + d\lambda_1 \\ C_4 \mapsto C_4' = C_4 + \frac{1}{2}\lambda_1 \wedge H_3 \end{cases} \tag{3.6}$$

where λ_1 is a generic 1-form. The second piece of (3.1) is called *Chern-Simons* action, and it collects terms that do not involve the metric. In this case we have:

$$S_{CS} = \frac{1}{4\kappa_{10}^2} \int_{\mathcal{M}} e^{\varphi} C_4 \wedge H_3 \wedge F_3 \tag{3.7}$$

The final part of the Type-IIB action collects the contributions of various localized objects, like the various D-branes that may be present in the model. For example, if we have a D3-brane, we can express part of the localized action as:

$$S_{LOC} \supset S_{D3} = \frac{1}{2\pi^3 \alpha'^2} \int_{D3} C_4 - T_3 \int_{D3} \sqrt{-g}\, \mathrm{d}^4\zeta \tag{3.8}$$

where ζ are coordinates parametrizing the world-volume of the D3-brane and T_3 is the tension of the D-brane. In general a Dp-brane has tension:

$$T_p = \frac{e^{(p-3)\frac{\varphi}{4}}}{(2\pi)^p \alpha'^{\frac{(p-1)}{2}}} \tag{3.9}$$

Notice that the local part of the action is not complete as is. It must be extended including the gauge fields living on the D-branes as open string states and their fermionic superpartners (the so-called gauginos). Including these and the pullback of B_2 on the brane, one gets the *Dirac-Born-Infeld* (DBI) action, completely characterizing the dynamics of Dp-branes:

$$S_{DBI} = -T_p \int_{Dp} e^{-\varphi} \sqrt{-\det(g_{IJ} + \mathcal{F}_{IJ})}\, \mathrm{d}^{(p+1)}\zeta \tag{3.10}$$

where ζ_I are coordinates on the world-volume of the Dp-brane, g_{IJ} is the pullback of the 10d metric to the $(p+1)$-dimensional brane and:

$$\mathcal{F}_{IJ} = B_{IJ} + 2\pi\alpha' F_{IJ} \tag{3.11}$$

where B_{IJ} and F_{IJ} are the pullback on the brane respectively of the Kalb-Ramond tensor, and of the field-strength tensor of the gauge fields living on the D-brane.

3.2 Kaluza-Klein Compactification

Now that we have our 10d action, we need a way to convert this into a 4d EFT by *compactifying* six of the dimensions and leaving the other four extended. The way to do this is to perform a so-called *Kaluza-Klein dimensional reduction*.
We can give a simple example of such a process considering the so called Kaluza-Klein theory. This is a straightforward generalization of the Einstein-Hilbert action to 5 space-time dimensions, of which 4 are extended and one is taken to be isomorphic to a S^1 geometry of radius r.

$$S_{KK} = \frac{M_5^3}{2} \int_{\mathcal{M}} \sqrt{-g^{(5)}} \mathcal{R}_5 \, \mathrm{d}y \, \mathrm{d}^4 x \tag{3.12}$$

where $\mathcal{M} = \mathbb{R}^4 \times S^1$, $y \in [0, 2\pi r)$ is the coordinate parameterizing the S^1 geometry, and the prefactor M_5 is the 5d reduced Planck mass. The first thing we do is writing the 5d metric tensor as:

$$g_{MN}^{(5)} = \begin{pmatrix} g_{\mu\nu} + \frac{2}{M_P^2}\phi^2 A_\mu A_\nu & \frac{\sqrt{2}}{M_P}\phi^2 A_\mu \\ \frac{\sqrt{2}}{M_P}\phi^2 A_\nu & \phi^2 \end{pmatrix} \tag{3.13}$$

with $M, N = 0, ..., 4$; $\mu, \nu = 0, .., 3$, and the parameter M_P is for the moment a parameter of the dimension of a mass, which we will later identify as the 4d reduced Planck mass. Next, since the overall spacetime is a product of a compact dimension times a non-compact part, we expect that our fields $g_{\mu\nu}, A_\mu$ and ϕ admit a decomposition in Fourier modes of the kind:

$$\phi(x, y) = \sum_{n=0}^{\infty} \phi_n(x) \cos\left(\frac{ny}{r}\right) + \sum_{n=1}^{\infty} \tilde{\phi}_n(x) \sin\left(\frac{ny}{r}\right) \tag{3.14}$$

What is found is that only one of these modes is massless, namely $\phi_0(x)$, while the others are an infinite tower of massive modes, called *Kaluza-Klein* (KK) modes, that in this case have a mass of $m_n = \frac{n}{r}$. The mode ϕ_0 has no mass, and no potential term in the dimensionally-reduced EFT: it represents a flat direction of the scalar potential, namely a *modulus*. Since it has no potential, its vacuum expectation value (VEV) is not fixed. This is a very general feature of these zero-modes emerging from Kaluza-Klein compactifications. Performing the same expansion for all the fields in the decomposition (3.13), integrating over S^1 in the action and keeping only the massless modes in our EFT (the cut-off of the EFT is, in fact, the mass of the first massive KK state), it is easy to convince oneself that the result is [7]:

$$S_{EFT} = \int \sqrt{-g} \, \phi \left[\frac{M_P^2}{2} R - \frac{1}{4}\phi^2 F_{\mu\nu}F^{\mu\nu} + \frac{M_P^2}{3} \frac{(\partial\phi)^2}{\phi^2} \right] \mathrm{d}^4 x \tag{3.15}$$

where R is the 4-dimensional Ricci scalar and $F_{\mu\nu}$ is the field strength of the vector field A_μ. We have effectively reduced the 5d Einstein-Hilbert action to a 4d EFT. This

[7] We drop the label '0' for the zero-states, as they are the only ones we are interested in.

theory not only features gravity in 4d, but also a scalar field ϕ and a U(1) gauge field A_μ. Notice that the action (3.15) can be brought to Einstein frame just by rescaling the metric $g_{\mu\nu} \mapsto \frac{g_{\mu\nu}}{\phi}$. We have found an EFT for each fixed value of r, the radius of the S^1. Therefore, we expect a flat direction in the potential corresponding to the freedom in the choice of the radius. We recognize that the only field in the final EFT that can play this role is ϕ. Therefore we regard ϕ as a scalar field *governing the size* of the extra dimension: this is a general property of the moduli, which characterize the geometrical properties of the extra dimensions. The result in (3.15) has made the assumption that:

$$M_P^2 = 2\pi r M_5^3 \tag{3.16}$$

So, interpreting M_P as the 4d Planck mass to make sense of the final EFT, this means that the 5d version of the Planck mass scales as:

$$M_5 = \frac{M_P^{2/3}}{\text{Vol}^{1/3}} \tag{3.17}$$

where Vol is the volume of the extra dimensions. This implies that if we consider a *large compactification volume*, the effects of gravity would be much stronger as the new gravitational energy scale would be M_5 and no longer M_P.

Now we have to apply the concept of a Kaluza-Klein reduction to our 10d superstring theory. The essential step to take is the choice of the compact geometry. First of all, we know we want to go from 10d down to 4d, which means we need to use a 6d manifold for the compact dimensions. These are real dimensions, but as we will see in a moment, it may be more useful to consider a *complex manifold*, with 3 complex dimensions. The main difference is that a complex manifold is furnished of a *complex structure* J, which may be interpreted as a map

$$J : T_P^* \to T_P^* \tag{3.18}$$

corresponding, roughly speaking, to a 'multiplication times i' in the cotangent space T_P^* of every point P of the manifold, with the crucial property that [8]:

$$J^2 = -\mathbb{1} \tag{3.19}$$

Moreover, our manifold should have a metric, and this metric has to be compatible with J. This automatically classifies our complex manifold as a *Kähler manifold*, which also implies the existence of a real function K of the local variables $(z^i, \bar{z}_{\bar{\jmath}})$ such that:

$$g_{i\bar{\jmath}} = \frac{\partial^2 K}{\partial z^i \partial z^{\bar{\jmath}}} \tag{3.20}$$

[8]Actually, this is not enough for the manifold to be a complex manifold. A manifold with a complex map (3.18) with property (3.19) is called a *almost complex manifold*. To be a full-fledged complex structure, J must have vanishing Nijenhuis tensor [12].

The compatibility condition of the Kähler metric and the complex structure lets us lower one index of the complex structure form J, so we can write it as a 2-form:

$$J = ig_{i\bar{\jmath}} dz^i \wedge d\bar{z}^{\bar{\jmath}} \tag{3.21}$$

called the *Kähler form*. It is clear that, given the Kähler form, the metric is immediately determined and vice versa. There are other two properties we want our Kähler manifold to have. If we set all the fields but the 10d metric tensor g_{MN} to zero, we want this complex manifold to be a solution of the vacuum Einstein equation:

$$\mathcal{R}_{MN} = 0 \tag{3.22}$$

This condition is called *Ricci flatness* and it is common to many simple geometries like the flat tori T^n.

The final request for our manifold is that our 4d theory, upon compactification, preserves at least the minimal supersymmetry from the 10d theory. This requirement is equivalent to asking for the existence in the 4d EFT of at least one massless spinor. From our description of the Kaluza-Klein reduction process, this boils down to asking for the spinor to have a non-vanishing zero-mode in the expansion. It can be shown that this condition is immediately satisfied if we ask for the existence of a *covariantly constant* spinor in the compactification space.

We should express these requirements in a more geometrical way, in order to have simpler constraints on the manifold. It can be proved that having a covariantly constant spinor and Ricci flatness corresponds to having a $SU(3)$ *holonomy group*. In general, a complex manifold of complex dimension n will have a holonomy group which is a subset of $SO(2n)$. A Kähler manifold always has holonomy group $U(n) \subset SO(2n)$. Imposing that this Kähler manifold is Ricci-flat is equivalent to imposing that the actual holonomy group of the Kähler manifold is $SU(n)$, and this immediately implies the existence of a covariantly constant spinor.

To recap, we want the extra dimensions to form a compact, Kähler manifold with $SU(3)$ holonomy group. Such a class of manifolds exists: they are called *Calabi-Yau* manifolds.

3.3 Calabi-Yau Manifolds

We want to give a formal definition of a Calabi-Yau manifold and analyze its structure. We will be brief in this description and won't dive too much in the formalities of Calabi-Yau geometries and constructions. A more in-depth analysis can be found in [13, 14]. Formally, a Calabi-Yau n-fold is a compact, connected Kähler manifold whose *first Chern class* vanishes. Let us understand what this means. The tangent bundle of a Kähler manifold can be viewed, due to the $U(n)$ holonomy, as a complex vector bundle whose curvature is specified by the Riemann tensor $R_{i\bar{\jmath}}{}^k{}_l$ of the Kähler manifold itself.

Therefore, the curvature 2-form on the tangent bundle of the manifold X can be written as:

$$R(T_X) = dz^i \wedge d\bar{z}^{\bar{j}} R_{i\bar{j}}{}^k{}_l \tag{3.23}$$

This allows us to define the *Chern multi-form*:

$$c(X) = \det(\mathbb{1} + R(T_X)) \tag{3.24}$$

where the determinant refers to the matrix indices and the product of forms is, of course, the wedge product. This can be expanded into a series of $(2k)$-forms $c_k(X)$ called kth *Chern class*:

$$c(X) = 1+c_1(X)+c_2(X)+... = 1+\text{tr}R(T_X)+\text{tr}(R(T_X)\wedge R(T_X)-2\text{tr}R(T_X)^2)+... \tag{3.25}$$

Going back to the definition of the Calabi-Yau, then, we can say that a compact Kähler manifold Y is a Calabi-Yau n-fold if $c_1(Y)$ is *exact*, i.e. there exists a 1-form ω such that $c_1(Y) = d\omega$. This can be expressed, in a more mathematical way, that $c_1(Y)$ is zero *in cohomology*, i.e. it sits in the same cohomology class as the null 2-form. The main reason behind this formal definition of a Calabi-Yau is that Chern classes are *topological invariants*, they are independent of smooth variations of the metric, and this will be crucial for us later on. In fact, this means that deforming in a smooth way the metric of a Calabi-Yau does not destroy the Calabi-Yau structure itself.
How does this formal definition relate to the definition based on the $SU(n)$ homology given in the previous section? We know that a Kähler manifold X for sure has a $U(n) = U(1) \times SU(n)$ holonomy. It is possible to prove that the field-strength related to the $U(1)$ bundle can be expressed as:

$$F_{i\bar{j}} = -2iR_{i\bar{j}} = -2i\text{tr}R(T_X) = -2ic_1(X) \tag{3.26}$$

Therefore, if c_1 vanishes (in cohomology) then also $F_{i\bar{j}}$ vanishes. Thus if $c_1 = 0$ there is no $U(1)$ bundle, and the holonomy group is simply $SU(n)$. This result is formalized and expanded upon by the *Yau* theorem:
Let Y be a Kähler manifold. If $c_1(Y)$ vanishes in cohomology, then there exists a Ricci-flat metric in the same cohomology class. This metric is unique and it is called Calabi-Yau metric.

Hodge diamond of a Calabi-Yau

Let us analyze some consequences of these definitions. First of all, recall that on a complex manifold, the de Rham cohomology group H^k, i.e. the set of cohomology classes of differential k-forms, admits a so-called *Hodge decomposition*:

$$H^k = \bigoplus_{p+q=k} H^{p,q} \tag{3.27}$$

where $H^{p,q}$ is called *Dolbeault cohomology group*, and it contains the cohomology classes of (p,q)-forms on the complex manifold. A (p,q)-form is a differential form having p holomorphic components and q anti-holomorphic components. One of the most characterizing features of a complex manifold are the so-called *Hodge numbers*:

$$h^{p,q} := \dim\, H^{p,q} \tag{3.28}$$

These indicate the number of independent (p,q)-forms defined on the manifold. Clearly, $p+q \leq 2n$ where n is the complex dimension of the manifold. The Hodge numbers are not all independent. In fact, they obey the *Hodge duality*:

$$h^{p,q} = h^{q,p} \tag{3.29}$$

and also the *Serre symmetry*:

$$h^{p,q} = h^{(n-p),(n-q)} \tag{3.30}$$

The Hodge numbers of a complex manifold are usually displayed in what is called a *Hodge diamond*, where the two conditions (3.29) and (3.30) together give rise to horizontal and vertical mirror symmetries. We display the general Hodge diamond for a complex 3-fold in Fig.3.1.

$$\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & & & h^{3,3} & & &
\end{array}$$

Figure 3.1: Hodge diamond for a general complex 3-fold.

Because of the symmetries, only 6 out of the 16 Hodge numbers are independent. Turning to Calabi-Yau manifolds, we see that the vanishing of the first Chern class or, equivalently, the $SU(3)$ holonomy, further constrains the number of independent Hodge numbers. In fact, the Hodge diamond of a Calabi-Yau has a very simple structure, as shown in Fig. 3.2.

As we can see, the only independent Hodge numbers of a Calabi-Yau 3-fold are $h^{1,1}$ and $h^{2,1}$. Two crucial properties, imposed by the defining holonomy condition of

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Figure 3.2: Hodge diamond for a Calabi-Yau 3-fold.

the Calabi-Yau are $h^{1,0} = h^{2,0} = 0$, which means that there are no 1-forms and no holomorphic or anti-holomorphic 2-forms, and $h^{3,0} = h^{0,3} = 1$, which means there exists precisely one holomorphic $(3,0)$-form, which we indicate as Ω [9].

3.4 Moduli Spaces of Calabi-Yau Three-folds

Yau's theorem states that the Calabi-Yau metric, once we fix the complex structure and cohomology class, is unique. However, what happens if we *deform* the metric $g_{i\bar{j}}$? Do such deformations exist that maintain the Ricci-flatness of the resulting Calabi-Yau? By the topological invariance of the first Chern class, we know the answer is yes. However we now want to classify them and see what these deformations mean from a more physical point of view.
There are two kind of variations we can perform on the metric of a Calabi-Yau Y:

$$g_{i\bar{j}} dz^i d\bar{z}^{\bar{j}} \mapsto g_{i\bar{j}} dz^i d\bar{z}^{\bar{j}} + \delta g_{i\bar{j}} dz^i d\bar{z}^{\bar{j}} + \delta g_{ij} dz^i dz^j + \text{h.c.} \tag{3.31}$$

Because of Yau's theorem, these variations of the metric preserving Ricci-flatness must be accompanied by either a change in the Kähler cohomology class or a change in the complex structure. We can interpret a change in the metric of the kind $\delta g_{i\bar{j}}$ directly as a change of the Kähler form J because of (3.21). These variations can be shown to be in one-to-one correspondence with the space of harmonic $(1,1)$-forms, and can be expanded as:

$$\delta g_{i\bar{j}} = i v^k (\omega_k)_{i\bar{j}} \quad \text{with } k = 1, ..., h^{1,1} \tag{3.32}$$

where ω_k is a basis of harmonic $(1,1)$-forms on the Calabi-Yau, and v^k are the so-called *Kähler moduli*, as they appear from deformations of the Kähler structure. On the other

[9]The existence and uniqueness of the holomorphic 3-form Ω are a consequence of what is sometimes called the Bogomolov theorem, and can be seen as alternative defining conditions of a Calabi-Yau 3-fold.

hand, variations of the metric of the form δg_{ij} violate the hermiticity; therefore they must be coupled with a change in the complex structure. We can establish a one-to-one correspondence between the space of these so-called *complex structure deformations* and the space of $(1,2)$-forms as:

$$\delta g_{ij} = \frac{i}{||\Omega||^2}\bar{U}^a(\bar{\chi}_a)_{i\bar{k}\bar{l}}\Omega^{\bar{k}\bar{l}}{}_j \quad \text{with } a = 1,..,h^{1,2} \tag{3.33}$$

here we used the holomorphic $(3,0)$-form Ω whose indices have been raised through the metric, its 'norm' given by $||\Omega||^2 = \frac{1}{6}\Omega_{ijk}\bar{\Omega}^{ijk}$ and we introduced a basis $(\bar{\chi}_a)$ of $H^{1,2}$.Always recall that $h^{2,1} = h^{1,2}$ so this argument can be easily converted to $H^{2,1}$ by simple conjugation. The (complex) coefficients of this expansion, U^a, are called *complex structure moduli*.

The Kähler moduli v^k and the complex structure moduli U^a span the *moduli space* of the Calabi-Yau, which can be factorized in a Kähler moduli space and a complex structure moduli space, at least in a simple compactification model:

$$\mathcal{M}_{moduli} = \mathcal{M}_K \times \mathcal{M}_{CS} \tag{3.34}$$

Both these moduli spaces are Kähler manifolds, and their metric can be expressed through an appropriate Kähler potential. Starting from the complex structure case, we have that:

$$K_{CS} = -\ln\left[-i\int_Y \Omega\wedge\bar{\Omega}\right] \tag{3.35}$$

where we see $\Omega = \Omega(U)$ as a function of the complex structure moduli.

On the other hand, we have the Kähler moduli space $\mathcal{M}_K$, which can be spanned by the Kähler moduli v^k. Nevertheless, it is more useful to introduce a different set of Kähler variables with an explicit geometrical meaning. First of all, we expand the Kähler form J in a harmonic basis ω_i of $(1,1)$-forms:

$$J = t^i\omega_i \tag{3.36}$$

with $i = 1,...,h^{1,1}$. It can be seen that the overall volume of the Calabi-Yau can be expressed as:

$$\mathcal{V} = \frac{1}{6}\int_Y J\wedge J\wedge J = \frac{1}{6}k_{ijk}t^it^jt^k \tag{3.37}$$

where k_{ijk} are *intersection numbers* on the Calabi-Yau given by:

$$k_{ijk} = \int_Y \omega_i\wedge\omega_j\wedge\omega_k \tag{3.38}$$

This way, it is easy to interpret the Kähler variables t^i as volumes of the 2-cycles D_i, duals of the 2-forms ω_i, which form a basis of divisors for the Calabi-Yau. The correct variables to use from an EFT perspective, however, are are the volumes of the associated 4-cycles. We indicate them as τ_i and define them as:

$$\tau_i = \frac{\partial \mathcal{V}}{\partial t^i} = \frac{1}{2} k_{ijk} t^j t^k \tag{3.39}$$

These variables represent the volume of 4-cycles Σ_i which are Poincaré-dual to the 2-cycles D_i. As such, they are real variables, as opposed to the complex structure moduli which are complex. Thus, we can complexify the τ_i adding, as imaginary part, the projection of the RR 4-form C_4 on the 4-cycle Σ_i:

$$b_i = \int_{\Sigma_i} C_4 \tag{3.40}$$

The fields b_i are *axions*, as they enjoy a shift symmetry around the respective 4-cycle. The complex version of the Kähler moduli are:

$$T_i = \tau_i + i b_i \tag{3.41}$$

One can see the volume of the Calabi-Yau as a function of the T_i variables. Then, the Kähler potential for these moduli is given by:

$$K_K = -2 \ln \mathcal{V} \quad \text{with } \mathcal{V} = \mathcal{V}(T_i) \tag{3.42}$$

In all the compactification models, there is always a non-geometrical modulus, the *dilaton* φ. It is a real scalar field that can be complexified adding the RR 0-form C_0 as its imaginary part. This new field is usually referred to as the *axio-dilaton*:

$$S = e^{-\varphi} + i C_0 \tag{3.43}$$

Notice that the dilaton appears in (3.43) as $e^{-\varphi}$, which, as argued in (2.78), corresponds to $\frac{1}{g_s}$. In fact, one often finds the string coupling written as:

$$g_s = \frac{1}{\text{Re}(S)} \tag{3.44}$$

The overall Kähler potential for the moduli space of the Calabi-Yau, then, reads:

$$K = -2 \ln \mathcal{V} - \ln\big(S + \bar{S}\big) - \ln \left[-i \int_Y \Omega \wedge \bar{\Omega} \right] \tag{3.45}$$

This will be the tree-level Kähler potential of the low-energy EFT, which is going to be a 4d supergravity theory. Notice that, since we started from a Type-II string theory and SUSY is unbroken in the compactification process, for now this will be a $\mathcal{N} = 2$ SUGRA EFT. In fact, one can prove that the Kähler potential (3.45) can be written in terms of a holomorphic *prepotential*.

3.5 Orientifold Projection

To break this $\mathcal{N}=2$ EFT down to $\mathcal{N}=1$ one can do two things. Either we introduce localized sources in the form of D-branes, or we perform an *orientifold projection*. Introducing D-branes in a consistent way is not immediate. In fact, they carry a particular charge in the form of their tension (3.9) we need to neutralize. Therefore, we need to introduce together with them some objects carrying a *negative* tension which neutralizes the charge tadpole. An obvious candidate would be anti D-branes ($\bar{D}$-branes), which are the charge-conjugated states of D-branes. However, suppose we have introduced a D3-brane somewhere in our Calabi-Yau to brake SUSY, then we introduce a $\overline{D3}$-brane to compensate the charge. Since these branes are not fixed in the extra dimensions, but can freely move around the surface of the Calabi-Yau, they would be statically attracted towards each other, and when they meet they would annihilate. Therefore, we need something which carries the opposite charge of D-branes, but cannot annihilate them upon contact. The most interesting class of objects of this kind are *O-planes*, which arise naturally upon orientifold projections.
Let us first of all define an *orientifold symmetry*. It is a composition of two symmetries:

1. *The world-sheet orientation reversal* P_Σ which exchanges right- and left-moving modes;

2. *An internal symmetry* σ of the Calabi-Yau leaving the 4d Minkowski spacetime untouched.

Since we want our final theory to be a $\mathcal{N}=1$ SUSY theory, the transformation σ must be a *holomorphic involution* of the Calabi-Yau Y. This means that σ must leave the Kähler form J untouched. Nevertheless, it can act non-trivially on the holomorphic 3-form Ω. In fact, we can have only two possible overall orientifold actions depending on how σ acts on Ω [15]:

1. $\tilde{\sigma}\Omega = -\Omega$ which gives rise to the overall orientifold symmetry $\mathcal{O} = (-1)^{F_L} P_\Sigma \sigma$. Implementing this kind of orientifold projection produces *O3/O7-planes*

2. $\tilde{\sigma}\Omega = \Omega$ which gives rise to the overall orientifold symmetry $\mathcal{O} = P_\Sigma \sigma$. Implementing this kind of orientifold projection produces *O5/O9-planes*

where we indicated with $\tilde{\sigma}$ the pullback of the action of σ to the holomorphic 3-form Ω and F_L is the number of target-space fermions in the left-moving sector. Once we established the orientifold symmetry $\mathcal{O}$, the *orientifold projection* $\pi_{\mathcal{O}}$ reduces naturally the overall Calabi-Yau Y to its quotient with respect to $\mathcal{O}$:

$$\pi_{\mathcal{O}} : Y \to Y/\mathcal{O} \tag{3.46}$$

we refer to the quotient space as a *Calabi-Yau orientifold*.
Op-planes arise in the Calabi-Yau orientifold at points in which the orientifold projection

is singular. It can be shown that they carry a *negative tension* which is a rational multiple of the tension of the corresponding Dp-brane. We will not dive in too much detail, however the presence of the localized O-planes breaks some of the total supersymmetry in the 4d EFT, which becomes a $\mathcal{N} = 1$ SUSY theory.

4 Moduli Stabilization

We found that upon compactification a certain number of moduli fields arise. Their VEVs control the shape and complex structure of the Calabi-Yau, and even the coupling of the theory. As an important remark, the number of moduli of a Calabi-Yau can be very high, up to $\sim \mathcal{O}(1000)$, as can be seen in certain examples from the database [16].Until their VEVs are unfixed, these act as free parameters and the theory has no use. This brings about the problem of *moduli stabilization*. In fact, for now the effective theory admits no superpotential, and therefore the scalar potential for all the moduli is identically zero. Our goal is to produce such a superpotential considering some extra effects on the Calabi-Yau orientifold that will eventually fix the VEV of these fields. First we analyze how to stabilize the complex structure moduli and the axio-dilaton by turning on fluxes on the Calabi-Yau. Then we move to Kähler moduli stabilization, in particular focusing on the Large Volume Scenarios (LVS). To do that, we first have to make an aside on quantum corrections to the Kähler potential, where we will also discuss the effects of string loops.

4.1 Type IIB Moduli Stabilization via Fluxes

A simple (yet not completely free [10]) way to generate a superpotential for the complex structure moduli in Type-IIB is turning on *3-form fluxes*.
To understand what this means let us first give a brief introduction to p-form fluxes in general. If we have a differential p-form F_p and a localized source, one can see that the integral of F_p on a p-cycle Σ^p encircling the source is:

$$\int_{\Sigma^p} F_p = e \tag{4.1}$$

This is called a background *p-form flux*. The idea is very similar to Gauss's Law in Electromagnetism. In fact, if we have a localized charge Q, the flux of the electromagnetic field through a surface surrounding the source is precisely given by its charge:

$$\int_{\Sigma^2} F_2 = Q \tag{4.2}$$

[10] See for example [17] for an introduction to what is called the *tadpole problem* of complex structure moduli stabilization.

Now, because of Bianchi identities, since for us $F_p = dC_{p-1}$, we have that $dF = 0$. This implies that also the fluxes e must be *constant*. In particular, we will impose *Dirac quantization* condition on the fluxes:

$$\frac{1}{2\pi\alpha'}\int_{\Sigma^p} F_p = 2\pi n \quad \text{with } n \in \mathbb{Z} \tag{4.3}$$

which give a different vacuum for our theory for each choice of n.
In the case at hand, we will turn on 3-form fluxes. In particular, we will have $F_3 = dC_2$ where C_2 is the RR 2-form of Type-IIB, and $H_3 = dB_2$. They can be combined to form the 3-form field:

$$G_3 = F_3 + iSH_3 \tag{4.4}$$

where S is the axio-dilaton defined in (3.43). Turning on the fluxes for G_3 on the Calabi-Yau orientifold Y induces a superpotential called *Gukov-Vafa-Witten superpotential*:

$$W_{GVW} = \int_Y G_3 \wedge \Omega \tag{4.5}$$

where Ω is the holomorphic 3-form of the Calabi-Yau. Notice that $W_{GVW} = W_{GVW}(S, U)$ since G_3 and Ω do not depend on the Kähler moduli.
Now that we have a non-zero superpotential, we also have a non-zero scalar potential. For a $\mathcal{N} = 1$ supergravity EFT we can write the F-term scalar potential as:

$$V = e^K \left[K^{S\bar{S}} D_S W D_{\bar{S}} \bar{W} + K^{a\bar{b}} D_a W D_{\bar{b}} \bar{W} + K^{ij} D_i W D_j \bar{W} - 3|W|^2\right] \tag{4.6}$$

where for us K will be the Kähler potential defined as in (3.45), and $W = W_{GVW}$. In (4.6) we used the following notations:

$$K^{\alpha\beta} = (K^{-1})^{\alpha\beta} \quad \text{where} \quad K_\alpha = \partial_\alpha K \tag{4.7}$$

with α, β being generic indices spanning over all the moduli. The Kähler derivatives in (4.6) are given by:

$$D_S := \frac{\partial}{\partial S} + K_S \tag{4.8}$$

$$D_a := \frac{\partial}{\partial U^a} + K_a \tag{4.9}$$

$$D_i := \frac{\partial}{\partial \tau_i} + \frac{1}{2}\frac{\partial K}{\partial \tau_i} \tag{4.10}$$

Since W, defined as in (4.5) is independent of the Kähler moduli, we can rewrite the general potential (4.6) as:

$$V = e^K \left[K^{SS} D_S W D_S \bar{W} + K^{a\bar{b}} D_a W D_{\bar{b}} \bar{W} + (K^{ij} K_i K_j - 3)|W|^2\right] \tag{4.11}$$

Looking at the structure of the Kähler potential for the Kähler moduli (3.42) and converting the definition of the volume (3.37) in a function of the τ_i, it is immediate to find out that

$$K^{ij}K_iK_j = 3 \tag{4.12}$$

This is called *no-scale structure* of the scalar potential, because the contribution to the scalar potential by the Kähler moduli identically vanishes. Now the scalar potential takes the form:

$$V = e^K \left[K^{SS}D_SWD_S\bar{W} + K^{a\bar{b}}D_aWD_{\bar{b}}\bar{W}\right] \tag{4.13}$$

The Kähler moduli enter this potential only through the exponential e^K, which does not modify the position of the minima. Hence, one can find the vacuum expectation value for the complex structure moduli U^a and the axio-dilaton S simply studying the equation

$$D_aW = D_SW = 0 \tag{4.14}$$

Notice that this minimum is supersymmetric. We have stabilized the complex structure moduli and the axio-dilaton, while the Kähler moduli are still unfixed. From now on, we will consider the U^a and S as stabilized and sitting at their minimum. With this, the Gukov-Vafa-Witten potential becomes just a constant that we indicate as:

$$W_0 := \left\langle \int_Y G_3 \wedge \Omega \right\rangle \tag{4.15}$$

4.2 Quantum Corrections to the Low-Energy EFT

Since we are dealing with a $\mathcal{N} = 1$ SUSY theory, the *non-renormalization theorems* hold. They state that while the Kähler potential gets corrections at any perturbative level (K_P), as well as at non-perturbative level (K_{NP}), the superpotential only gets non-perturbative corrections (W_{NP}). Therefore, including the effects of quantum corrections in our EFT we'll have:

$$K = K_{Tree} + K_P + K_{NP} \tag{4.16}$$

$$W = W_{Tree} + W_{NP} \tag{4.17}$$

Let us first of all treat the non-perturbative corrections. In general, we expect these to be very small compared to perturbative corrections, since they are *exponentially suppressed*. However, since the superpotential does not receive perturbative corrections at all, these are the dominant corrections to W. They arise as a series of negative exponentials of the Kähler moduli:

$$W_{NP} = \sum_{i=1}^{h^{1,1}} \sum_n A_i^{(n)} e^{-na_iT_i} \tag{4.18}$$

where, in general, $A_i^{(n)} = A_i^{(n)}(S, U^a)$ and some of these coefficients can be vanishing. Nonetheless, one often disregards this dependence while stabilizing Kähler moduli, as the complex structure moduli and the axio-dilaton have been fixed already by stronger tree-level effects. In our study we will always work in a regime in which $a_i\tau_i \gg 1$, so that we can simply keep the dominant term for each modulus in (4.18):

$$W_{NP} \simeq \sum_{i=1}^{h^{1,1}} A_i e^{-a_i T_i} \tag{4.19}$$

Where do these non-perturbative corrections come from? There are mainly two sources of non-perturbative corrections of the superpotential: *Euclidean D3-brane instantons* which yield $a_i = 2\pi$ and *gaugino condensation* on wrapped stacks of N D7-branes yielding $a_i = 2\pi/N$. We will not consider non-perturbative corrections to the Kähler potential, since these are generally subleading with respect to the perturbative ones.

Let us instead move to the *perturbative corrections* of the Kähler potential. These come from two different contributions: the expansion in powers of α', usually called α'-corrections, and the expansion in powers of g_s, usually called string-loop corrections:

$$K_P = \delta_{(\alpha')}K + \delta_{(g_s)}K \tag{4.20}$$

Let us show the dominant α' correction here, and reserve the next section to g_s loop corrections.

The leading α'-correction to K is of order $\mathcal{O}(\alpha'^3)$ and comes from a correction proportional to $\mathcal{R}^4$ of the Einstein-Hilbert term of the action (3.1).[11] Upon compactification, this correction reflects on the Kähler potential as follows:

$$K_{Tree} + \delta_{(\alpha')}K = -2\ln\left[\mathcal{V} + \frac{\xi}{2g_s^{3/2}}\right] = -2\ln\mathcal{V} - \frac{\xi}{g_s^{3/2}\mathcal{V}} + \mathcal{O}\left(\frac{1}{\mathcal{V}^2}\right) \tag{4.21}$$

where

$$\xi = -\frac{\chi(Y)\zeta(3)}{2(2\pi)^3} \tag{4.22}$$

with $\chi(Y) = 2(h^{1,1} - h^{1,2})$ the Euler characteristic of the Calabi-Yau, and $\zeta(3) \simeq 1.2$ Apéry's constant. It is important to stress that the α' expansion is in the *inverse volume*, and thus can be controlled only in the case in which $\mathcal{V} \gg 1$ as it is visible from (4.21).

4.3 Loop Corrections to the Kähler potential

As mentioned above, K receives perturbative corrections from string loops, and these can be modelled as a power expansion in the string coupling g_s. As of today, we know of

[11] We will not display the full form of the 10d correction here, see [18] for the complete derivation.

two kinds of open string 1-loop corrections which are produced by different UV effects. The first kind is called *Kaluza-Klein* (KK) corrections, and originate as the tree-level exchange, in the closed string channel, of Kaluza-Klein modes between O3/D3- and O7/D7-branes or parallel D7-branes. We then have *winding* (W) corrections, originating form the tree-level exchange of closed strings wound around non-contractible 1-cycles located at the intersection locus between different stacks of D7 branes. Therefore, we can write the overall loop correction to the Kähler potential as:

$$\delta_{(g_s)}K = \delta_{(g_s)}K^{KK} + \delta_{(g_s)}K^{W} \tag{4.23}$$

No explicit calculation of these corrections has been performed on a Calabi-Yau, but we do have some results for fluxless toroidal orientifolds [19]. Based on those calculations and on dimensional arguments Berg, Haack and Pajer (BHP) made an educated guess about the form of these loop corrections on Calabi-Yau's as well [20]. What they conjectured is that the contributions to the Kähler potential of string loops have the following functional form at 1-loop level in Einstein frame:

$$\delta_{(g_s)}K^{KK}_{1-loop} = \sum_{i=1}^{h^{1,1}} \frac{g_s \mathcal{C}_i^{KK} f(t_j)}{\mathcal{V}} \tag{4.24}$$

$$\delta_{(g_s)}K^{W}_{1-loop} = \sum_{i=1}^{h^{1,1}} \frac{\mathcal{C}_i^{W}}{\mathcal{V} g(t_j)} \tag{4.25}$$

where f and g are homogeneous functions of degree 1 of the 2-cycles volumes t_j. The loop constants $\mathcal{C}_i^{KK}$ and $\mathcal{C}_i^{W}$ are in general possibly complicated functions of the complex structure moduli, which are unknown to us. However, we consider them simply constants since complex structure moduli have been fixed by tree-level effects. It was estimated based on QFT arguments [21], that the KK correction to the Kähler potential should go as (setting for simplicity $h^{1,1} = 1$):

$$\delta_{(g_s)}K^{KK}_{1-loop} \sim \frac{\mathcal{C}^{KK} g_s}{\tau_E} = \frac{\mathcal{C}^{KK} g_s^2}{\tau_{st}} \tag{4.26}$$

while the winding correction goes as:

$$\delta_{(g_s)}K^{W}_{1-loop} \sim \frac{\mathcal{C}^{W}}{\tau_E^2} = \frac{\mathcal{C}^{W} g_s^2}{\tau_{st}^2} \tag{4.27}$$

where we used the fact that $\tau_E = \tau_{st}/g_s$ where τ_E is in Einstein frame and τ_{st} in the string frame. Now, we do know that τ_{st} is the volume of a 4-cycle Σ_4 in string units. This means that:

$$\text{Vol}(\Sigma) \sim \tau_{st} l_s^4 \sim \tau_{st} \alpha'^2 \tag{4.28}$$

then:

$$\frac{1}{\tau_{st}} \sim \mathcal{O}(\alpha'^2) \tag{4.29}$$

Therefore, based on (4.26) and (4.27) we have:

$$\delta_{(g_s)} K^{KK}_{1-loop} \sim \mathcal{O}(g_s^2 \alpha'^2) \tag{4.30}$$

$$\delta_{(g_s)} K^{W}_{1-loop} \sim \mathcal{O}(g_s^2 \alpha'^4) \tag{4.31}$$

An interesting thing to point out is that, in order to have control over the EFT, we must require the volumes of the 4-cycles to be *larger than* 1 in string units. This is the condition for our moduli fields to be *inside the Kähler cone*, where we know we can trust the EFT. Therefore, if we require that $\tau_{st} \gg 1$ what we get is that:

$$\frac{\delta_{(g_s)} K^{KK}_{1-loop}}{\delta_{(g_s)} K^{W}_{1-loop}} \sim \tau_{st} \gg 1 \tag{4.32}$$

This means, as argued in [22], that in a regime where we can trust our EFT, the KK corrections should be *dominant* with respect to the W corrections, at least at the level of the Kähler potential. Moreover, one should in general expect KK corrections to arise independently from the precise topology of the CY. In fact KK modes are an intrinsic feature of *string compactifications*, whereas winding corrections, from a purely microscopic point of view, only arise when the topology of the CY allows two stacks of intersecting D7-branes and, even more specifically, a non-contractible 1-form which wraps the intersection site.

4.4 Loop Corrections to the Scalar Potential

Looking at the expressions (4.24) and (4.25) where the volume is expressed in terms of the 2-cycles t_i as in (3.37) we can say that $\delta_{(g_s)} K^{KK}_{1-loop}$ is a homogeneous function of t_i of degree $n = -2$ while $\delta_{(g_s)} K^{W}_{1-loop}$ is a homogeneous function of t_i of degree $n = -4$. This peculiar form of the 1-loop corrections is such that the scalar potential exhibits an *extended no-scale structure*, i.e. the first KK correction to the scalar potential vanishes. Let us analyze the open string 1-loop corrections to the scalar potential and settle which are the dominant ones. For the time being, since we are interested only in the perturbative corrections to the scalar potential, let us set:

$$K = K_{Tree} + \delta_{(g_s)} K \quad \text{and} \quad W = W_0 \tag{4.33}$$

The scalar potential, with this choice of K and W is simply given by:

$$V = V_{Tree} + \delta_{(g_s)} V \tag{4.34}$$

with

$$\delta_{(g_s)}V = (K^{ij}K_iK_j - 3)\frac{|W_0|^2}{\mathcal{V}^2} \tag{4.35}$$

We can write the correction $\delta_{(g_s)}V$ as:

$$\delta_{(g_s)}V = \delta V^{KK} + \delta V^{W} \tag{4.36}$$

The extended no-scale structure theorem proved by Cicoli et al. in [21] tells us that the first contribution from δV^{KK} is zero. As a matter of fact, we can write:

$$\delta V^{KK}_{1-loop} = \delta V^{KK}_{\mathcal{O}(g_s^2\alpha'^2)} + \delta V^{KK}_{\mathcal{O}(g_s^4\alpha'^4)} + \mathcal{O}(g_s^6\alpha'^6) \tag{4.37}$$

and one finds that:

$$\delta V^{KK}_{\mathcal{O}(g_s^2\alpha'^2)} = -\frac{|W_0|^2}{\mathcal{V}^2}\frac{n}{4}(n+2)\delta_{(g_s)}K^{KK}_{1-loop} \tag{4.38}$$

where n is the degree of the homogeneous function $\delta_{(g_s)}K^{KK}_{1-loop}$. However, as we said, $n = -2$, implying:

$$\delta V^{KK}_{\mathcal{O}(g_s^2\alpha'^2)} = 0 \tag{4.39}$$

The expression for the winding correction to the scalar potential is the same, but this time the degree of the homogeneous function is $n = -4$. This does not lead to a cancellation, so the first perturbative correction we get to V is a winding correction at 1-loop level:

$$\delta V^{W}_{1-loop} = -2\frac{|W_0|^2}{\mathcal{V}^2}\delta_{(g_s)}K^{W}_{1-loop} \tag{4.40}$$

As we argued in (4.31) we know $\delta_{(g_s)}K^{KK}_{1-loop} \sim \mathcal{O}(g_s^2\alpha'^4)$ so that this is true for the corresponding correction to V as well:

$$\delta V^{W}_{1-loop} = \delta V^{W}_{\mathcal{O}(g_s^2\alpha'^4)} + \mathcal{O}(g_s^4\alpha'^8) \tag{4.41}$$

Therefore, overall, the scalar potential has the following form with leading-order winding corrections:

$$V = V_{Tree} + \delta V^{W}_{\mathcal{O}(g_s^2\alpha'^4)} \tag{4.42}$$

However, as we argued in the previous section, winding corrections may be less common than KK corrections, and may be negligible by construction. In [23] it was argued that corrections with the same functional scaling as winding corrections are present in general scenarios based on QFT arguments. It must be said that the corrections found therein are not of the same kind as those considered by the BHP conjecture. In fact, there are moduli and closed string fields running in loops, suggesting that, from a string theory perspective, such corrections arise from *closed string loops*, rather than open string loops in the closed-string channel as considered in [20]. It must also be said that, from a purely EFT perspective, one cannot see the difference between open- and closed-string

modes, as they are all treated equally as fields. However, as we argued in Sec 2.4 around equation (2.77), adding a closed string loop should carry a factor of g_s^2. Therefore, at the moment it is unclear whether the effects described in [23] are dominant over second-order open-string KK corrections. To be more precise and assess all the string coupling constants, invisible from an EFT perspective, one should perform a world-sheet calculation of closed-string loops on some simple toroidal manifolds, as [19] did for open-string loops and try to generalize it to Calabi-Yau's as well.
In the case we happened not to have winding corrections by construction, the most relevant correction to the scalar potential would be the next term in the expansion of the open-string 1-loop correction δV_{1-loop}^{KK}:

$$V = V_{Tree} + \delta V_{\mathcal{O}(g_s^4\alpha'^4)}^{KK} \tag{4.43}$$

We can see that:

$$\delta V_{\mathcal{O}(g_s^4\alpha'^4)}^{KK} \sim \frac{|W_0|^2}{\mathcal{V}^2}\frac{(C^{KK})^2 g_s^4}{\tau_{st}^2} \tag{4.44}$$

Looking at the general form (4.24) of the open-string 1-loop KK corrections to the Kähler potential, we expect that at open-string 2-loop level, the correction should go as [22] (as well as at closed-string 1-loop level as estimated in [23]):

$$\delta_{(g_s)} K_{2-loop}^{KK} \sim \frac{\mathcal{D}^{KK} g_s^2}{\tau_E^2} = \frac{\mathcal{D}^{KK} g_s^4}{\tau_{st}^2} \sim \mathcal{O}(g_s^4\alpha'^4) \tag{4.45}$$

Therefore, in the potential we expect the effects (4.43) and (4.45) to be competing. So, one could say that:

$$\delta V_{\mathcal{O}(g_s^4\alpha'^4)}^{KK} \sim \left[(\mathcal{C}^{KK})^2 + \mathcal{D}^{KK}\right] \frac{|W_0|^2}{\mathcal{V}^2}\frac{g_s^4}{\tau_{st}^2} \tag{4.46}$$

which in Einstein frame becomes:

$$\delta V_{\mathcal{O}(g_s^4\alpha'^4)}^{KK} \sim [(\mathcal{C}^{KK})^2 + \mathcal{D}^{KK}]\frac{|W_0|^2}{\mathcal{V}^2}\frac{g_s^2}{\tau_E^2} \tag{4.47}$$

4.5 KKLT Moduli Stabilization

Now that we have talked about the effects that can be used to stabilize the Kähler moduli, let us give an example of such a moduli stabilization models. One of the most studied techniques for moduli stabilization is the Kachru-Kallosh-Linde-Trivedi (KKLT) model, first introduced in [24]. The KKLT moduli stabilization scheme uses a tree-level Kähler potential, and switches on non-perturbative effects in the superpotential as in (4.19) for all the Kähler moduli. For simplicity, let us treat the case in which $h^{1,1} = 1$, so that we only have one modulus, $T = \tau + ib$, to stabilize. In this case, the Kähler potential and superpotential will be:

$$K = -3\ln\left(T + \bar{T}\right) \tag{4.48}$$

$$W = W_0 + A\, e^{-aT} \tag{4.49}$$

Inserting these in the general formula for the scalar potential (4.6), having previously fixed the complex structure moduli and axio-dilaton, yields to [10]:

$$V = \frac{|aA|^2}{6\tau} e^{-2a\tau} + \frac{a|A|^2}{2\tau^2} e^{-2a\tau} + \frac{a\mathrm{Re}\left(AW_0^* e^{-ib}\right)}{2\tau^2} e^{-a\tau} \tag{4.50}$$

Upon fixing the axionic part b to its minimum, we can minimize the potential. One can prove that the minimum is supersymmetric, so it can be computed simply imposing the F-flatness condition $D_T W = 0$. This yields the result:

$$W_0 = -Ae^{-a\tau}\left(1 + \frac{2}{3}a\tau\right) \tag{4.51}$$

Since $a\tau \gg 1$ to justify the use of a tree-level Kähler potential and only the first non-perturbative correction, we must have that $|W_0|$ is *exponentially* small. There have been explicit constructions of models in which $|W_0|$ is as low as 10^{-120} [25]. General considerations of *flux vacuum statistics* based on the Bousso-Polchinski model [26], seem to suggest that it is possible to tune the flux quanta so that $|W_0|$ is exponentially small. With this requirement satisfied, we see that τ is stabilized by the implicit solution of equation (4.51) and its qualitative behaviour is:

$$\langle\tau\rangle \sim -\frac{1}{a}\ln|W_0| \tag{4.52}$$

Once we have a VEV for τ we can also look for the mass spectrum of the theory. We find that (reinstating an appropriate power of M_P):

$$m_T \sim m_{3/2}\ln\left(\frac{M_P}{m_{3/2}}\right) \tag{4.53}$$

where

$$m_{3/2} = e^{\frac{K}{2}}|W_0|\, M_P \tag{4.54}$$

is the *gravitino mass*.

4.6 LVS Moduli Stabilization

An alternative method to stabilize Kähler moduli is the so-called Large Volume Scenario (LVS). As the name says, this stabilization method has the peculiarity that the *volume* of the Calabi-Yau gets stabilized at an exponentially large value. This introduces strong hierarchies between scales that could be of interest for various reasons, like low-energy SUSY breaking, and a better control over the EFT. Let us first of all show how an LVS moduli stabilization works with a simple 2-moduli field example. Then we are going to expand on that, computing the mass spectrum of the theory and finally generalizing the model to a n-moduli case.

Introducing LVS: a Simple Example

Let us follow, as we study our first LVS model, the track of [27], one of the first papers to develop the EFT for LVS compactifications. In LVS, one considers specific Calabi-Yau's that have two kinds of Kähler moduli: the so-called *big* moduli, $T_B = \tau_B + ib_B$, which regulate the (complexified) volumes of 'big' 4-cycles determining the global shape and size of the Calabi-Yau, and the so-called *small* moduli, $T_s = \tau_s + ib_s$ which are the (complexified) volumes of 'small' or local 4-cycles. In particular, we will consider, as small cycles, the so-called *blow-up* cycles, particular 4-cycles that resolve (blow up) geometrical point-like singularities of the Calabi-Yau. The simplest possible model of a Calabi-Yau suited for this kind of compactification is the complex manifold $\mathbb{CP}^4_{[1,1,1,6,9]}[18]$, also called a *Swiss Cheese* Calabi-Yau because of its structure containing a big and a small cycle, the latter resembling a hole in a Swiss cheese. In the specific case of $\mathbb{CP}^4_{[1,1,1,6,9]}[18]$, the volume can be written as:

$$\mathcal{V} = \frac{1}{9\sqrt{2}} \left(\tau_B^{\frac{3}{2}} - \tau_s^{\frac{3}{2}} \right) \tag{4.55}$$

Now, we consider the Kähler potential corrected with the leading-order α' correction, as we did in (4.21):

$$K = -2 \ln \left[\frac{1}{9\sqrt{2}} \left(\tau_B^{\frac{3}{2}} - \tau_s^{\frac{3}{2}} \right) + \frac{\xi}{2 g_s^{\frac{3}{2}}} \right] \tag{4.56}$$

where ξ is the same as (4.22). Differently from the KKLT case, we consider non-perturbative corrections to the superpotential for the small modulus only:

$$W = W_0 + A_s \, e^{-a_s T_s} \tag{4.57}$$

As we discussed in general, the non-perturbative corrections can arise as a result of either instantons on Euclidean D3-branes, in which case $a_s = 2\pi$ or from gaugino condensation on N-stacks of D7-branes, then $a_s = 2\pi/N$. Notice that we can reabsorb the overall factor $\frac{1}{9\sqrt{2}}$ by a simple rescaling of the fields, accompanied by a proper redefinition of the constants A_s and W_0, so that we can simply write:

$$K = -2 \ln \left[\left(\tau_B^{\frac{3}{2}} - \tau_s^{\frac{3}{2}} \right) + \frac{\xi}{2 g_s^{\frac{3}{2}}} \right] \tag{4.58}$$

$$W = W_0 + A_s \, e^{-a_s T_s} \tag{4.59}$$

We postpone to Appendix A the precise calculations for the model, and we limit ourselves to presenting the main results. Plugging the Kähler potential (4.58) and the superpotential (4.59) in the general expression of the F-term scalar potential (4.6), and fixing the axions at their minima, we get the following form of the scalar potential:

$$V = \frac{8(a_s A_s)^2 \sqrt{\tau_s}\, e^{-2a_s\tau_s}}{3\mathcal{V}} - \frac{4a_s A_s W_0 \tau_s e^{-a_s\tau_s}}{\mathcal{V}^2} + \frac{3\hat{\xi}|W_0|^2}{4\mathcal{V}^3} \tag{4.60}$$

where for convenience we set:

$$\hat{\xi} \equiv \frac{\xi}{g_s^{3/2}} \tag{4.61}$$

The α' correction breaks SUSY at the minimum, so we expect a non-supersymmetric vacuum. Therefore, to find it we have to minimize the potential (4.60) with respect to τ_s and $\mathcal{V}$, setting

$$\frac{\partial V}{\partial \tau_s} = 0 \tag{4.62}$$

$$\frac{\partial V}{\partial \mathcal{V}} = 0 \tag{4.63}$$

We can solve (4.62) for $\mathcal{V}$, obtaining:

$$\frac{1}{\mathcal{V}} = \frac{1}{3}\frac{a_s A_s}{W_0\sqrt{\tau_s}}\left(\frac{1-4a_s\tau_s}{1-a_s\tau_s}\right)e^{-a_s\tau_s} \tag{4.64}$$

From (4.64) we see that:

$$\mathcal{V} \sim e^{a_s\tau_s} \tag{4.65}$$

Therefore, the volume can be *exponentially large* depending on the value of $a_s\tau_s$. On the other hand, inserting (4.64) in (4.63), we get an equation for the VEV of τ_s:

$$\tau_s^{\frac{3}{2}} = \frac{3\hat{\xi}}{32}\left[\frac{\left(\frac{1-4a_s\tau_s}{1-a_s\tau_s}\right)^2}{\frac{1-4a_s\tau_s}{1-a_s\tau_s}-1}\right] \tag{4.66}$$

Therefore, using the definition (4.61) we see that τ_s gets stabilized around:

$$\tau_s \sim \frac{\xi^{\frac{2}{3}}}{g_s} \tag{4.67}$$

so that, at the minimum of the potential we have:

$$\mathcal{V} \sim e^{\frac{1}{g_s}} \tag{4.68}$$

Thus, in a perturbative regime of string theory, when $g_s \ll 1$, we get that $\mathcal{V} \gg 1$. Plugging the results we just got into the scalar potential, we find that its minimum is negative:

$$\langle V \rangle < 0 \tag{4.69}$$

this means that this theory, as it is, predicts an Anti-de Sitter (AdS) vacuum. Since we know that we live in a (slightly) de Sitter (dS) spacetime, with a positive (but very small) cosmological constant, this is not phenomenologically viable. However, there exist various ways to *lift* the vacuum from a non-SUSY AdS to a non-SUSY dS vacuum. To do so, one has to add an *uplifting term* to the scalar potential, δV_{up}. This term must be sufficiently small not to spoil the moduli stabilization, but sufficiently large to effectively make the minimum of the potential positive. Usually, one adds a term of the form:

$$\delta V_{up} = \frac{D}{\mathcal{V}^\gamma} \quad \text{with } D > 0 \tag{4.70}$$

where γ is an exponent that can vary in a range $\gamma \in [1, 3]$, depending on the specific UV effects considered to perform the uplift. In fact, there are multiple microscopic effects that can produce such a term, like $\overline{D}3$-brane uplifting, in which the uplifting term in the 4d effective action is given by an anti-D3-brane positioned at the end of a heavily warped Klebanov-Strassler throat of the Calabi-Yau, as explained in [24], or T-brane uplifting where the positive contribution to the vacuum energy comes from non-zero F-terms of hidden sector matter fields [28]. We will not enter in too much detail here about the uplifting mechanism of the potential, as it does not modify significantly the model and we will neglect the uplifting term altogether for now.
Let us instead go back to the model. We see that, setting $M_P = 1$, the gravitino mass is given by:

$$m_{3/2} = e^{\frac{K}{2}} W = \frac{1}{\mathcal{V}} e^{-\frac{\hat{\xi}}{\mathcal{V}}} [W_0 + A_s e^{-a_s T_s}] \simeq \frac{W_0}{\mathcal{V}} \tag{4.71}$$

as $e^{-\frac{\hat{\xi}}{\mathcal{V}}} \simeq 1$ and $\frac{A_s e^{-a_s T_s}}{\mathcal{V}} \ll 1$. It is important to stress that in LVS models $W_0 \sim \mathcal{O}(1)$, or in any case it does not need to bee exponentially small as opposed to KKLT models. Notice also that, to have $m_{3/2} \sim 1\text{TeV}$ we have to stabilize the volume at around $\mathcal{V} \sim \mathcal{O}(10^{15})$ which is not unreasonably large given (4.68).

Mass Spectrum for the Swiss Cheese Model

From the expression of the Kähler potential of the Swiss Cheese model (4.58), we see that there is a mixing between τ_s and τ_B in the volume. To find the physical mass eigenstates, we need to diagonalize the mass matrix once we have canonically normalized the fields. To do that, first of all we compute the mass matrix from the potential:

$$M_{ij} = \frac{1}{2} \frac{\partial^2 V}{\partial \tau^i \partial \tau^j} \tag{4.72}$$

In our case, upon expanding the vacuum expressions (4.64) and (4.67) in powers of $\varepsilon = \frac{1}{4a_s\tau_s}$ (see Appendix A), we find, to second order in ε and at leading order in $\mathcal{V}$:

$$M_{ij} = \begin{pmatrix} \frac{27|W_0|^2\hat{\xi}}{16\mathcal{V}^{13/3}}\left[1+2\varepsilon\right] & -\frac{9a_s|W_0|^2\hat{\xi}}{8\mathcal{V}^{11/3}}\left[1-5\varepsilon+4\varepsilon^2\right] \\ -\frac{9a_s|W_0|^2\hat{\xi}}{8\mathcal{V}^{11/3}}\left[1-5\varepsilon+4\varepsilon^2\right] & \frac{3a_s^2|W_0|^2\hat{\xi}}{4\mathcal{V}^3}\left[1-3\varepsilon+6\varepsilon^2\right] \end{pmatrix} \tag{4.73}$$

Before going on diagonalizing it, though, we need to canonically normalize the fields. To do that one has to multiply M_{ij} by the inverse Kähler metric (A.7):

$$\tilde{M}_{ij} = (K^{-1}M)_{ij} = \frac{a_s\langle\tau_s\rangle|W_0|^2\hat{\xi}}{2\mathcal{V}^3}\begin{pmatrix} -9\left[1-7\varepsilon\right] & -6a_s\mathcal{V}^{2/3}\left[1-5\varepsilon+16\varepsilon^2\right] \\ -\frac{6\mathcal{V}^{1/3}}{\langle\tau_s\rangle^{1/2}}\left[1-5\varepsilon+4\varepsilon^2\right] & \frac{4a_s\mathcal{V}}{\langle\tau_s\rangle^{1/2}}\left[1-3\varepsilon+6\varepsilon^2\right] \end{pmatrix} \tag{4.74}$$

Notice first of all that different powers of the volume introduce in the mass matrix (4.74) important hierarchies between terms. This tells us that in this simple model, we will have a great mass hierarchy between the two mass eigenstates. If we expect such a hierarchy, we can easily find the eigenvalues as:

$$m_\Phi^2 \simeq \mathrm{tr}[\tilde{M}] \simeq \frac{2|W_0|^2a_s^2\hat{\xi}\sqrt{\langle\tau_s\rangle}}{3\mathcal{V}^2} \sim \left(\frac{W_0\ln\mathcal{V}}{\mathcal{V}}\right)^2 \tag{4.75}$$

$$m_\chi^2 \simeq \frac{\det\left[\tilde{M}\right]}{\mathrm{tr}[\tilde{M}]} = \frac{81|W_0|^2\hat{\xi}}{16a_s\langle\tau_s\rangle\mathcal{V}^3} \sim \frac{W_0^2}{\mathcal{V}^3\ln\mathcal{V}} \tag{4.76}$$

Here we called Φ the heavy degree of freedom, and χ the light one. Notice that:

$$\frac{m_\Phi}{m_\chi} \sim \sqrt{\mathcal{V}\ln\mathcal{V}} \sim \mathcal{O}(10^8) \tag{4.77}$$

for $\mathcal{V} \sim 10^{14}$. Φ is much heavier than the gravitino, while χ is much lighter, as we can see from (4.71). If we go on and compute the mixing of τ_B and τ_s with Φ and χ we get that, expanding $\tau_i = \langle\tau_i\rangle + \delta\tau_i$, with $i = B, s$, and finding the eigenvectors of $\tilde{M}$:

$$\delta\tau_B = \left[\sqrt{6}\,\langle\tau_B\rangle^{1/4}\langle\tau_s\rangle^{3/4}(1-2\varepsilon)\right]\frac{\Phi}{\sqrt{2}} + \left[\sqrt{\frac{4}{3}}\,\langle\tau_B\rangle\right]\frac{\chi}{\sqrt{2}} \sim \mathcal{O}(\mathcal{V}^{1/6})\Phi + \mathcal{O}(\mathcal{V}^{2/3})\chi \tag{4.78}$$

$$\delta\tau_s = \left[\frac{2\sqrt{6}}{3}\langle\tau_B\rangle^{3/4}\langle\tau_s\rangle^{1/4}\right]\frac{\Phi}{\sqrt{2}} + \left[\frac{\sqrt{3}}{a_s}(1-2\varepsilon)\right]\frac{\chi}{\sqrt{2}} \sim \mathcal{O}(\mathcal{V}^{1/2})\Phi + \mathcal{O}(1)\chi \tag{4.79}$$

We can see from (4.78) and (4.79) that the dominant component of τ_B is χ, while the dominant component of τ_s is Φ. Therefore, we can say that the *volume modulus*, which is mostly given by $\tau_B^{3/2}$, is very light, while the *small modulus* τ_s is quite heavy. The large volume gives rise to important hierarchies in the mass spectrum. In fact one can see that:

$$m_{\mathcal{V}} \sim m_\chi \ll m_{3/2} \ll m_\Phi \sim m_{\tau_s} \tag{4.80}$$

This is an appealing feature for *modular cosmology*. In fact, τ_s is so heavy that it can evade the *cosmological moduli problem*.
This problem is related to the reheating temperature when we have moduli-dominated epoch following inflation. Since moduli emerge from the compactification of the 10d graviton down to 4d, their interactions tend to be Planck-suppressed. This implies that they are usually long-lived. Moreover, the reheating temperature depends on the mass m_τ of the modulus which last dominates the energy density of the early Universe [10]:

$$T_{RH} \sim \sqrt{\frac{m_\tau}{M_P}} m_\tau \sim 1\text{GeV} \left(\frac{m_\tau}{10^6\text{GeV}}\right)^{3/2} \tag{4.81}$$

Therefore, if the mass of the modulus is too small, the reheating temperature drops very low, and clashes with observations related to BBN [29, 30]. Then since $\Phi \sim \tau_s$ is way heavier than $m_{3/2}$ it is immune to the cosmological moduli problem. The problem, however, persists with $\chi \sim \tau_B$ which results way lighter than the gravitino mass.

General LVS Setting

We want to enlarge our vision of LVS models, and consequently we can generalize what we did in the two previous sections to ta more general case. Following the notation of [3] we consider the case in which we have one big cycle τ_1 and $(n-1)$ small cycles $\tau_2, ..., \tau_n$ with the hierarchy:

$$\tau_1 \gg \tau_i \quad \forall i = 2, ..., n \tag{4.82}$$

We consider the case in which we can write the volume of the Calabi-Yau in a 'Swiss-Cheese fashion':

$$\mathcal{V} = \alpha \left(\tau_1^{3/2} - \sum_{i=2}^{n} \lambda_i \tau_i^{3/2}\right) \tag{4.83}$$

where α is an overall constant and the λ_i are intersection numbers that can be thought as $\mathcal{O}(1)$ constants, see [31] for an explicit construction.
Since we are interested in LVS moduli stabilization, we consider the α'-corrected Kähler potential:

$$K = -2 \ln \left[\mathcal{V} + \frac{\hat{\xi}}{2}\right] \tag{4.84}$$

where $\hat{\xi}$ is defined as in (4.61). We switch on the non-perturbative corrections to the superpotential for all the small moduli:

$$W = W_0 + \sum_{i=2}^{n} A_i \, e^{-a_i T_i} \tag{4.85}$$

Once we stabilized the axions to their minima, we retrieve a scalar potential which is very close to (4.60), as all the mixed terms are suppressed by stronger exponential factors:

$$V = \sum_{i=2}^{n} \left[\frac{8(a_i A_i)^2 \sqrt{\tau_i}}{3\alpha\lambda_i \mathcal{V}} e^{-2a_i\tau_i} - \frac{4a_i A_i W_0 \tau_i}{\mathcal{V}^2} e^{-a_i\tau_i} \right] + \frac{3\hat{\xi} W_0^2}{4\mathcal{V}^3} \tag{4.86}$$

Notice that the negative sign between the first two terms of the potential arises because of the minimization of the axionic fields b_i. As we did for the 2-moduli case, we now turn to minimize this potential with respect to τ_i and $\mathcal{V}$. First, the minimization with respect to τ_i yields (see Appendix B for a three-moduli example):

$$a_i A_i e^{-a_i\tau_i} = 3\alpha\lambda_i W_0 \frac{\sqrt{\tau_i}}{\mathcal{V}} \left(\frac{1 - a_i\tau_i}{1 - 4a_i\tau_i} \right) \tag{4.87}$$

Once again we find that:

$$\mathcal{V} \sim e^{a_i\tau_i} \quad \text{or} \quad a_i\tau_i \sim \ln \mathcal{V} \tag{4.88}$$

which justifies the assumption that $\mathcal{V} \gg 1$. Fixing all the small moduli to their minima, we can write down a scalar potential for the volume only:

$$V(\mathcal{V}) = -\frac{3W_0^2}{2\mathcal{V}^3} \left[\alpha \sum_{i=2}^{n} \left(\frac{\lambda_i}{a_i^{3/2}} \right) (\ln \mathcal{V})^{3/2} - \frac{\hat{\xi}}{2} \right] \tag{4.89}$$

As we discussed in the previous section, we shall add to this potential an uplifting term as in (4.70), so that it has a Minkowski minimum [12]. The final scalar potential for LVS, therefore, will be:

$$V_{LVS} = V + \delta V_{up} = \sum_{i=2}^{n} \left[\frac{8(a_i A_i)^2 \sqrt{\tau_i}}{3\alpha\lambda_i \mathcal{V}} e^{-2a_i\tau_i} - \frac{4a_i A_i W_0 \tau_i}{\mathcal{V}^2} e^{-a_i\tau_i} \right] + \frac{3\hat{\xi} W_0^2}{4\mathcal{V}^3} + \frac{D}{\mathcal{V}^\gamma} \tag{4.90}$$

with $1 \leq \gamma \leq 3$. The minimization of the potential (4.90) once again furnishes mass scales of the model. We will have that:

$$m_{\tau_i} \simeq \frac{W_0 \ln \mathcal{V} M_P}{\mathcal{V}} \tag{4.91}$$

$$m_{\mathcal{V}} \simeq \frac{W_0 M_P}{\mathcal{V}^{3/2}\sqrt{\ln \mathcal{V}}} \tag{4.92}$$

and once again we have that $m_{\mathcal{V}} \ll m_{\tau_i}$, so that $\mathcal{V}$ is the lightest of all the geometrical moduli.

[12] The constant D can be also tuned so that the minimum is a de Sitter minimum with a very small cosmological constant, as we observe today.

5 Kähler Moduli Inflation with Loop Corrections

5.1 Original model

Looking at the scalar potential (4.90) one can easily see that the directions corresponding to the small cycle moduli, albeit not flat, are *almost flat*. What we mean by that is that the direction of τ_i, which was flat at tree level, has been lifted by *exponentially suppressed* terms. The resulting potential for one of the small moduli, upon fixing all the others to their minima, can be seen in Fig.5.1. This inspired [2] to propose a model

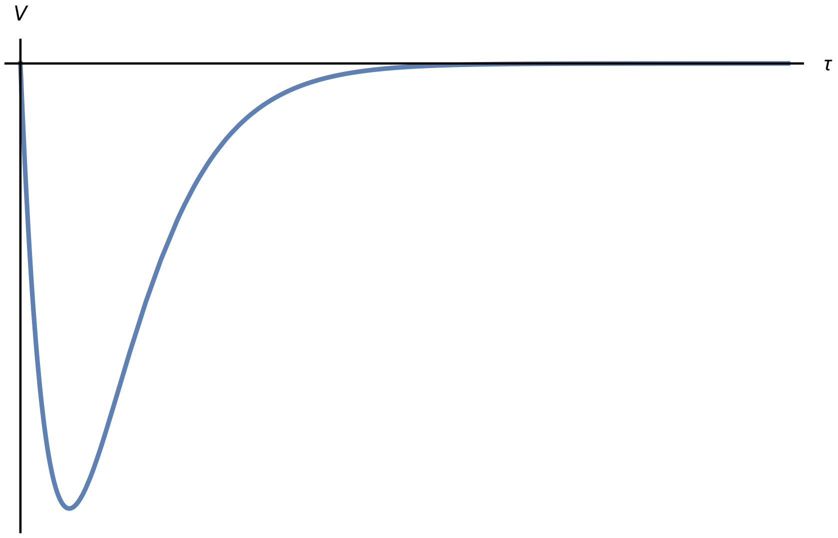

Figure 5.1: Plot of the (non-uplifted) scalar potential for a generic small modulus τ, upon fixing all the others to their minima. For display purposes we set $\mathcal{V} \sim 10^5$, $W_0 \sim 1$ and $a \sim 2\pi$. It can be clearly seen that the exhibits a large plateau upon displacing τ from its minimum.

for slow-roll inflation based on this potential, which is called *Kähler moduli inflation*. In this model the inflaton field is given by a canonically normalized blow-up modulus, which, incidentally, inspires the alternative name of this model: *Blow-Up inflation*.
In this section we will study the original model for blow-up inflation, and subsequently analyze, following [3], the predictions this model produces by studying its post-inflationary dynamics. After that, we will add *open string loop corrections* to the potential and see how the predictions of the model get modified.

Inflationary Potential and Parameters

As we said, the role of the inflaton (the scalar field driving inflation) is played by a (canonically normalized) small 4-cycle, say τ_n. First of all, we want to write the potential for this field as it gets displaced from its minimum. Starting from the potential (4.86) for all the moduli, we fix $\tau_2, ..., \tau_{n-1}$ to their minima and simply consider the contribution of τ_n, writing it as:

$$V(\tau_n) = V_0 - \frac{4a_n A_n W_0 \tau_n}{\mathcal{V}^2} e^{-a_n \tau_n} \tag{5.1}$$

where:

$$V_0 = \frac{\beta W_0^2}{\mathcal{V}^3} \tag{5.2}$$

sets the scale for inflation. The β factor appearing in (5.2) is a constant depending on the geometry of the Calabi-Yau and the uplifting mechanism that we will derive precisely later on.

Notice that, upon writing the expression (5.1) of the potential, we dropped the first term which appears in (4.86), since it is doubly exponentially suppressed with respect to the second one. The resulting potential has exactly the shape displayed in Fig.5.1, featuring an extended plateau as $\tau_n \gg 1$. Looking at the Kähler potential (4.84) with the volume defined as in (4.83), we see that τ_n is not canonically normalized. In fact, one computes that:

$$K_{n\bar{n}} = \frac{3\lambda_n}{8\mathcal{V}\sqrt{\tau_n}} \tag{5.3}$$

Therefore, to properly compute the inflationary parameters we have to canonically normalize it. We define:

$$\phi = \sqrt{\frac{4\lambda_n}{3\mathcal{V}}}\, \tau_n^{3/4} \tag{5.4}$$

the canonically normalized *inflaton* field. We can rewrite (5.1) in therms of ϕ as:

$$V(\phi) = V_0 - \frac{4a_n A_n W_0}{\mathcal{V}^2} \left(\frac{3\mathcal{V}}{4\lambda_n}\right)^{2/3} \phi^{4/3}\, e^{-\left[a_n \left(\frac{3\mathcal{V}}{4\lambda_n}\right)^{2/3} \phi^{4/3}\right]} \tag{5.5}$$

and we see the potential still has its characteristic exponential suppression. Now, recall the definition of the slow-roll parameters ϵ and η in terms of the potential[13]:

$$\epsilon = \frac{1}{2}\left(\frac{V'}{V}\right)^2 \tag{5.6}$$

$$\eta = \frac{V''}{V} \tag{5.7}$$

[13]Here we set $M_P = 1$, we will often do this implicitly when computing inflationary parameters.

where the prime indicates a derivative with respect to ϕ. After computing them, we re-express them in terms of the geometrically meaningful field τ_n:

$$\epsilon = \frac{32(a_n A_n)^2 \mathcal{V}^3}{3\beta^2 W_0^2 \lambda_n} \sqrt{\tau_n}(1 - a_n\tau_n)^2\, e^{-2a_n\tau_n} \tag{5.8}$$

$$\eta = -\frac{4a_n A_n \mathcal{V}^2}{3\lambda_n \beta W_0 \sqrt{\tau_n}}[1 - 9a_n\tau_n + 4(a_n\tau_n)^2]\, e^{-a_n\tau_n} \tag{5.9}$$

Notice that both ϵ and η are exponentially suppressed, moreover, for a sufficiently large value of τ_n:

$$\epsilon \ll \eta \ll 1 \tag{5.10}$$

We will later compute precisely what are the predictions of this model, but for now the slow-roll condition seems to work. In addition to this, we can also look at what are the expressions for the *scalar spectral index* n_s and the number of e-foldings N_e in terms of τ_n. First of all we have that:

$$n_s - 1 = 2\eta - 6\epsilon \tag{5.11}$$

so, plugging in the expressions (5.8) and (5.9) we get:

$$n_s - 1 = -\tfrac{8a_n A_n \mathcal{V}^2}{3\lambda_n \beta W_0 \sqrt{\tau_n}} e^{-a_n\tau_n} \left[1 - 9a_n\tau_n + 4(a_n\tau_n)^2 - \tfrac{24 a_n A_n \mathcal{V}}{\beta W_0}\tau_n(1 - a_n\tau_n)^2 e^{-a_n\tau_n}\right] \tag{5.12}$$

On the other hand, the number of e-foldings can be computed as:

$$N_e = \int_{\phi_E}^{\phi_*} \frac{V}{V'}\, \mathrm{d}\phi \tag{5.13}$$

where ϕ_E is the inflaton field evaluated ate the *end* of inflation, while ϕ_* is the inflaton field computed at the *pivot scale*, i.e., at the time when its wavelength exits the comoving horizon. Using (5.4) we can express this as an integral over τ_n:

$$N_e = \frac{3\beta W_0 \lambda_n}{16\mathcal{V}^2 a_n A_n} \int_{\tau_n^E}^{\tau_n^*} \frac{e^{a_n\tau_n}}{\sqrt{\tau_n}(a_n\tau_n - 1)}\, \mathrm{d}\tau_n \tag{5.14}$$

where τ_n^E corresponds to ϕ_E and τ_n^* to ϕ_*. In the large volume limit, $a_n\tau_n^* > a_n\tau_n^E \sim \ln\mathcal{V} \gg 1$, so that this integral can be expressed in terms of the error function $\mathrm{erf}(x)$:

$$N_e = \frac{3\beta W_0 \lambda_n}{8\mathcal{V}^2 a_n^{3/2} A_n} \left[\frac{e^{a_n\tau_n}}{\sqrt{a_n\tau_n}} + i\sqrt{\pi}\,\mathrm{erf}(i\sqrt{a_n\tau_n})\right]_{\tau_n^E}^{\tau_n^*} \tag{5.15}$$

We can now expand the error function asymptotically for $a_n\tau_n \gg 1$

$$i\sqrt{\pi}\,\mathrm{erf}(\sqrt{a_n\tau_n}) \simeq -\frac{e^{a_n\tau_n}}{\sqrt{a_n\tau_n}}\left(1 + \frac{1}{2a_n\tau_n}\right) \tag{5.16}$$

so that the expression (5.15) reduces to:

$$N_e = \frac{3\beta W_0 \lambda_n}{16\mathcal{V}^2 a_n A_n} \left[\frac{e^{a_n \tau_n}}{(a_n \tau_n)^{3/2}}\right]_{\tau_n^E}^{\tau_n^*} \simeq \frac{3\beta W_0 \lambda_n}{16\mathcal{V}^2 a_n A_n} \frac{e^{a_n \tau_n^*}}{(a_n \tau_n^*)^{3/2}} \tag{5.17}$$

where we neglected the term in τ_n^E due to the presence of the exponential. This expression allows us to write ϵ and η in terms of N_e, simply substituting the exponential:

$$\epsilon \simeq \left(\frac{3\lambda_n}{8a_n^{3/2}}\mathcal{V}\right) \frac{1}{N_e^2 \sqrt{a_n \tau_n}} \tag{5.18}$$

$$\eta = -\frac{1}{N_e}\left[1 + \mathcal{O}\left(\frac{1}{a_n \tau_n}\right)\right] \tag{5.19}$$

To see how well this model performs in terms of concrete predictions, we'll have to study the post-inflationary dynamics of the system.

Volume modulus displacement during inflation

Before discussing the evolution of the system when inflation ends, let us show that the volume VEV gets shifted during inflation and compute this shift.
As shown in (4.80), the volume modulus is the lightest of all the Kähler moduli. Therefore, during inflation, it experiences a *vacuum misalignment*, due to the fact that the potential of the volume modulus depends on the inflaton. We are interested in the determination of this displacement, which will be of crucial importance for the calculation of the number of e-foldings in the context of the post-inflationary evolution of the system.
First of all, we need to make sure the volume direction does not experience a runaway due to inflation. To do that, we need to impose that:

$$R \equiv \frac{\lambda_n a_n^{-3/2}}{\sum_{i=2}^n \lambda_i a_i^{-3/2}} \ll 1 \tag{5.20}$$

Values of $R \sim 0.1 - 0.01$ are easily achievable by an appropriate choice of the non-perturbative effects used to stabilize the potential.
Now we turn to the potential in terms of the volume. We shall start by finding the value of D, such that that the post-inflationary vacuum of the potential is Minkowski. To do that, consider the volume potential (4.89) to which we add the uplifting term (4.70):

$$V = -\frac{3W_0^2}{2\mathcal{V}^3}\left[\alpha \sum_{i=2}^n \left(\frac{\lambda_i}{a_i^{3/2}}\right)(\ln \mathcal{V})^{3/2} - \frac{\hat{\xi}}{2}\right] + \frac{D}{\mathcal{V}^\gamma} \tag{5.21}$$

To simplify things, let us rewrite this in terms of:

$$\psi = \ln \mathcal{V} \tag{5.22}$$

$$P = \alpha \sum_{i=2}^n \lambda_i a_i^{-3/2} = \frac{\alpha}{R}\lambda_n a_n^{3/2} \tag{5.23}$$

so that:

$$V = -\frac{3}{2}W_0^2 e^{-3\psi}\left(P\psi^{3/2} - \frac{\hat{\xi}}{2}\right) + De^{-\gamma\psi} \tag{5.24}$$

We will now set $\gamma = 2$. This choice is made for simplicity sake, and it can be seen that the particular choice of γ does not influence significantly the final result. Let ψ_m be the value of ψ at its post-inflationary minimum, then we impose the two conditions:

$$V(\psi_m) = 0 \tag{5.25}$$

$$\frac{\partial V}{\partial \psi}(\psi_m) = 0 \tag{5.26}$$

where (5.26) is just the condition for ψ_m to be the value at which the minimum is realized, while (5.25) imposes that such minimum is Minkowski. We can expand them as:

$$-\frac{3}{2}W_0^2\left(P\psi_m^{3/2} - \frac{\hat{\xi}}{2}\right)e^{-\psi_m} + D = 0 \tag{5.27}$$

$$\frac{3}{2}W_0^2\left(3P\psi_m - \frac{3}{2}P\psi_m^{1/2} - \frac{3}{2}\hat{\xi}\right)e^{-\psi_m} - 2D = 0 \tag{5.28}$$

Combining these two equations we get:

$$\psi_m^{3/2} - \frac{3}{2}\psi_m^{1/2} - \frac{\hat{\xi}}{2P} \tag{5.29}$$

which implicitly fixes the value of ψ_m. We can invert this in terms of $\hat{\xi}$:

$$\frac{\hat{\xi}}{2} = P\left(\psi_m^{3/2} - \frac{3}{2}\psi_m^{1/2}\right) \tag{5.30}$$

and substitute it in (5.27) to get the tuned value of D:

$$D = \frac{9}{4}W_0^2 P e^{-\psi_m}\psi_m^{1/2} \tag{5.31}$$

Therefore, we can substitute (5.31) in (5.24) to get:

$$V(\psi) = -\frac{3}{4}W_0^2 e^{3\psi}\left[2P\psi^{3/2} - \hat{\xi} - 3P\psi_m^{1/2}e^{(\psi-\psi_m)}\right] \tag{5.32}$$

Let us now turn to determine the shift in the volume field during inflation. In order to do that, we consider the inflationary potential, where we fix all the field except $\mathcal{V}$ and τ_n fixed at their minima. We can therefore write this as:

$$V_{inf} = -\frac{3W_0^2}{2\mathcal{V}^3}\left[\alpha\sum_{i=2}^{n-1}\left(\frac{\lambda_i}{a_i^{3/2}}\right)(\ln\mathcal{V})^{3/2} - \frac{\hat{\xi}}{2}\right] + \frac{D}{\mathcal{V}^2} - \frac{4a_n A_n W_0 \tau_n}{\mathcal{V}^2}\,e^{-a_n\tau_n} \tag{5.33}$$

We can regard (5.33) as a potential for $\mathcal{V}$, as τ_n slow-rolls. During most of inflation, we'll have that $e^{a_n\tau_n} \gg \mathcal{V}$, so that we neglect the last term. We now rewrite V_{inf} in terms of ψ:

$$V_{inf}(\psi) \simeq -\frac{3}{4}W_0^2 e^{-3\psi}\left[2P(1-R)\psi^{3/2} - \hat{\xi} - 3P\psi_m^{1/2}e^{(\psi-\psi_m)}\right] \tag{5.34}$$

and let ψ_{inf} be the minimum of ψ during inflation, i.e. the value of ψ that minimizes (5.34). We can determine its value imposing the minimization condition:

$$\frac{\partial V_{inf}}{\partial \psi}(\psi_{inf}) = 0 \tag{5.35}$$

This produces the following equation:

$$(1-R)\psi_{inf}^{3/2} - \frac{1}{2}(1-R)\psi_{inf}^{1/2} - \psi_m^{1/2}e^{(\psi_{inf}-\psi_m)} - \frac{\hat{\xi}}{2P} = 0 \tag{5.36}$$

which determines ψ_{inf} implicitly. What we are really interested in, though, is the *shift* of the volume minimum during inflation:

$$\Delta\psi = \psi_{inf} - \psi_m \tag{5.37}$$

To compute this it is useful to write the inflationary potential as:

$$V_{inf}(\psi) = V(\psi) + \Delta V(\psi) \tag{5.38}$$

where:

$$\Delta V(\psi) = \frac{3}{2}W_0^2 e^{-3\psi} PR\psi^{3/2} \tag{5.39}$$

and $V(\psi)$ is as in (5.32). Then, simply from a Taylor expansion, we can get the value for $\Delta\psi$ as:

$$\Delta\psi = -\frac{\Delta V'(\psi_m)}{V''(\psi_m)} = 4R\frac{\psi_m + \frac{\hat{\xi}}{2P}\psi_m^{1/2}}{2\psi_m - 1} \simeq 2R\psi_m \tag{5.40}$$

where in the last equality we used (5.30) and the large volume approximation. Now, the interesting range of volumes for this inflationary model is $\mathcal{V} \sim 10^5 - 10^6$, and we previously imposed that $R \sim 0.1 - 0.01$. This means that for us $\Delta\psi \sim 0.1 M_P$. Now, we are interested in the displacement of the canonically normalized volume field, which can be seen to be:

$$\frac{\chi}{M_P} = \sqrt{\frac{2}{3}}\ln\mathcal{V} = \sqrt{\frac{2}{3}}\psi \tag{5.41}$$

so that we can express the volume displacement during inflation in Planck units as:

$$Y \equiv \frac{\Delta\chi}{M_P} = \sqrt{\frac{2}{3}}\Delta\psi = 2\sqrt{\frac{2}{3}}R\psi_m \sim 0.1 \tag{5.42}$$

We can compute the shift in the minimum of the other small Kähler moduli in a similar way. Since we know $a_i\tau_i \sim \ln\mathcal{V}$, we expect:

$$a_i\Delta\tau_i \sim \Delta\psi = 2R\psi_m \quad \text{for } i = 2, ..., n-1 \tag{5.43}$$

However, their canonically normalized fields are:

$$\frac{\sigma_i}{M_P} = \sqrt{\frac{4\lambda_i}{3\mathcal{V}}}\tau_i^{3/4} \tag{5.44}$$

which means that we expect a displacement of the order:

$$\Delta\sigma_i \sim \frac{M_P}{\sqrt{\mathcal{V}}} \sim M_s \ll \Delta\chi \tag{5.45}$$

thus, their displacement is negligible. As a final remark, we can compute the value of β in the definition (5.2). To do that, we notice that:

$$V_0 = V_{inf}(\psi_m) \simeq \frac{1}{2}V''(\psi_m)\Delta\psi^2 + \Delta V(\psi_m) \simeq \frac{3}{2}W_0^2 PRe^{-3\psi_m}\psi_m^{3/2} \tag{5.46}$$

so that:

$$\beta = \frac{3}{2}PR\psi_m^{3/2} = \frac{3}{2}PR(\ln\mathcal{V})^{3/2} \tag{5.47}$$

For our range of volumes, and typical choices of the constants $\beta \sim \mathcal{O}(1)$.

Estimate of the number of e-foldings

Let us illustrate how we can estimate the number of e-foldings N_e of inflation within this model. To do that, we have to make some assumptions about the structure of our Calabi-Yau. We will follow [3], in which it was assumed that the 4-cycle whose volume is regulated by τ_n is wrapped by a stack of D7 branes, on which the Standard Model gauge fields live. If this is the case, the inflaton τ_n has a preferred decay channel to these fields. It can be shown, see [32], that the decay rate of the inflaton is about:

$$\Gamma_{\tau_n} \sim 0.1\frac{m_{\tau_n}^3}{M_s^2} \sim \frac{W_0^3}{10}\frac{(\ln\mathcal{V})^3}{\mathcal{V}^2}M_P \tag{5.48}$$

where we have re-inserted the Planck mass, used (4.91) and the fact that $M_s \sim M_P/\sqrt{\mathcal{V}}$. Let us analyze what happens at the end of inflation. As inflation ends, the inflaton field and the lightest of the geometric moduli, i.e. the volume modulus, find themselves displaced from their minimum. Therefore, they start oscillating coherently around their post-inflationary VEV. The energy density associated with these oscillations redshifts as matter, which means that there will be a first period of matter domination immediately

following the end of inflation, in which the energy density of the Universe is dominated by the oscillations of the inflaton. When the inflaton decays, it transfers its energy to gauge fields, which redshift as radiation. Therefore, a period of radiation domination starts, caused by the decay products of the inflaton field. If at this point the volume has not decayed yet (in our present case we will assume a Planck-suppressed coupling of the volume), there will be a time of matter-radiation equality, as the energy density of the volume oscillations becomes dominant over that of radiation. Then, a second epoch of matter domination starts, this time given by the coherent oscillations of the volume. This lasts until the volume decays and we have a new era of radiation domination.
To find the overall number of e-foldings of inflation, we will use the following formula, first derived in [33]:

$$N_e + \frac{1}{4}(1-3\omega_{RH})N_{RH} + \frac{1}{4}N_{\mathrm{mod}} \simeq 57 + \frac{1}{4}\ln r + \frac{1}{4}\ln\left(\frac{\rho_*}{\rho_E}\right) \tag{5.49}$$

where N_{RH} is the number of e-foldings between the end of inflation and the decay of the inflaton, ω_{RH} is the effective equation of state parameter during the reheating period, N_{mod} is the number of e-foldings of domination of a modulus displaced from its minimum, $r = 16\epsilon_*$ is the tensor-to-scalar ratio, ρ_* is the energy density at the scale of horizon exit, and ρ_E is the energy density at the end of inflation. We will make some assumptions to simplify the formula.
First of all, our model is a slow-roll model for inflation, therefore $\rho_* \simeq \rho_E$ due to the plateau; therefore we can neglect the last term of (5.49). We also work in the hypothesis of sudden thermalization of the decay products. Moreover, we will have *two* epochs of moduli domination, respectively the inflaton ϕ and the volume $\mathcal{V}$. The final formula we will use to determine the number of e-foldings between horizon exit and the end of inflation, N_e, is:

$$N_e \simeq 57 + \frac{1}{4}\ln r - \frac{1}{4}N_\phi - \frac{1}{4}N_{\mathcal{V}} \tag{5.50}$$

where N_ϕ is the number of e-foldings of inflaton domination at the end of inflation and $N_{\mathcal{V}}$ is the number of e-foldings of volume domination. All we have to do, then, is find a way to express N_ϕ and $N_{\mathcal{V}}$ in terms of the parameters of our model.
First of all, we know that the energy density stored in the inflaton field at the time of the end of inflation, t_E, is approximately equal to the inflation scale, so:

$$\rho_\phi(t_E) \simeq V_0 = \frac{M_P^4 W_0^2 \beta}{\mathcal{V}^3} \tag{5.51}$$

Also, we can estimate the energy density of the volume field as:

$$\rho_{\mathcal{V}}(t_E) \simeq \frac{1}{2}m_{\mathcal{V}}^2 \psi_{inf}^2 \simeq \frac{M_P^4 W_0^2 Y^2}{\mathcal{V}^3 \ln \mathcal{V}} \tag{5.52}$$

We can easily see that the energy density of the inflaton dominates, in fact:

$$\frac{\rho_{\mathcal{V}}(t_E)}{\rho_\phi(t_E)} \simeq \frac{Y^2}{\beta \ln \mathcal{V}} \equiv \theta^2 \ll 1 \tag{5.53}$$

where we used (5.42). Thus, immediately after the end of inflation, a period of matter domination starts in which the energy density is dominated by the oscillations of the inflaton field. We want to find out how long it lasts in terms of e-foldings. To do that, we first of all find the value of the Hubble ratio at the end of inflation using (5.51):

$$H(t_E) \simeq \frac{M_P W_0 \sqrt{\beta}}{\mathcal{V}^{3/2}} \tag{5.54}$$

Notice that $H(t_E) \sim m_{\mathcal{V}}$ (4.92), which means that the volume modulus starts oscillating immediately after the end of inflation. the energy density associated with these oscillations also redshifts as matter, which implies that the ratio (5.53) is *constant* until the inflaton decays.
The decay happens at a time t_ϕ such that:

$$H(t_\phi) \sim \Gamma_{\tau_n} \tag{5.55}$$

Therefore, we can compute the number of e-foldings of inflaton domination as:

$$N_\phi = \ln\left(\frac{a(t_\phi)}{a(t_E)}\right) \tag{5.56}$$

Since during matter domination:

$$\rho(t) \sim a^{-3}(t) \tag{5.57}$$

then we can rewrite N_ϕ as:

$$N_\phi = \frac{1}{3}\ln\left(\frac{\rho_\phi(t_E)}{\rho_\phi(t_\phi)}\right) \tag{5.58}$$

We can now express $\rho_\phi(t_E)$ in terms of $H(t_E)$ and $\rho_\phi(t_\phi)$ in terms of $H(t_\phi)$ and, using (5.55) we get:

$$N_\phi \simeq \frac{2}{3}\ln\left(\frac{H(t_E)}{\Gamma_{\tau_n}}\right) \simeq \frac{2}{3}\ln\left(\frac{10\sqrt{\beta\mathcal{V}}}{W_0^2(\ln \mathcal{V})^3}\right) \tag{5.59}$$

At this point, the inflaton has decayed, an era of radiation domination starts. Since the volume oscillations continue to redshift as matter, there will be a time t_{eq} of matter-radiation equality. After that, the volume modulus oscillations will come to dominate the energy density of the Universe until it eventually decays. Therefore we can express the duration of the period of *volume domination* as:

$$N_{\mathcal{V}} \simeq \frac{2}{3}\ln\left(\frac{H(t_{eq})}{\Gamma_{\mathcal{V}}}\right) \tag{5.60}$$

To estimate this, we need the decay rate of the volume modulus. In this particular case we'll use:

$$\Gamma_{\mathcal{V}} \simeq \frac{m_{\mathcal{V}}^3}{16\pi M_P^2} \simeq \frac{W_0^3}{16\pi \mathcal{V}^{9/2}(\ln \mathcal{V})^{3/2}} M_P \tag{5.61}$$

as originally found in [34, 35]. On the other hand, we need to find the the Hubble ratio at t_{eq}. To do that, notice first of all that:

$$H(t_\phi) \simeq H(t_E) e^{-\frac{3}{2}N_\phi} \simeq H(t_E) \frac{W_0^2 (\ln \mathcal{V})^3}{10\sqrt{\beta \mathcal{V}}} \tag{5.62}$$

Then we can determine the time of matter-radiation equality imposing:

$$\rho_{\mathcal{V}}(t_{eq}) = \rho_{rad}(t_{eq}) \tag{5.63}$$

We can rewrite this as:

$$\rho_{\mathcal{V}}(t_\phi) \left(\frac{a(t_\phi)}{a(t_{eq})} \right)^3 = \rho_{rad}(t_\phi) \left(\frac{a(t_\phi)}{a(t_{eq})} \right)^4 \tag{5.64}$$

Since, as we said, the ratio between $\rho_{\mathcal{V}}$ and ρ_ϕ is constant until ϕ decays, $\rho_{\mathcal{V}}(t_\phi)/\rho_{rad}(t_\phi) = \theta^2$. Thus, we get that $a(t_\phi)/a(t_{eq}) = \theta^2$, yielding $\rho(t_{eq}) \simeq \rho_{rad}(t_\phi)\theta^8$. This implies that:

$$H(t_{eq}) \simeq H(t_\phi)\theta^4 = H(t_E) \frac{W_0^2 (\ln \mathcal{V})^3 \theta^4}{10\sqrt{\beta \mathcal{V}}} \tag{5.65}$$

Therefore inserting (5.65) and (5.61) in (5.60) we can determine:

$$N_{\mathcal{V}} \simeq \frac{2}{3} \ln \left(\frac{16\pi \mathcal{V}^{5/2} (\ln \mathcal{V})^{5/2} Y^4}{10\beta^2} \right) \tag{5.66}$$

where we used (5.53) to express θ in terms of Y.
As a final remark, we can compute the final reheating temperature T_{RH}. To do that, we want to compute $H(t_{\mathcal{V}})$, i.e. the Hubble ratio at the time of the decay of the volume modulus. Clearly:

$$H(t_{\mathcal{V}}) \simeq \Gamma_{\mathcal{V}} \simeq \frac{M_P W_0^3}{16\pi \mathcal{V}^{9/2} (\ln \mathcal{V})^3/2} \tag{5.67}$$

This can be used to derive the reheating temperature using the relation:

$$3M_P^2 H^2(t_{\mathcal{V}}) \simeq \frac{\pi^2}{30} g_* T_{RH}^4 \tag{5.68}$$

where g_* is the effective number of degrees of freedom which thermalize. For a volume in the range of $\mathcal{V} \sim 10^5 - 10^6$ and $g_* \sim 100$ this gives $T_{RH} \gtrsim 10^3$ GeV, which shows there is no tension with the success of Big Bang Nucleosynthesis, and the cosmological moduli problem is evaded.

Determination of the inflationary parameters

Now that we have our expressions for $N_{\mathcal{V}}$ and N_ϕ we can use some general values of the parameters to retrieve a prediction for the spectral index n_s. First of all, we have to impose that the scalar perturbations at the scale of horizon exit match the observed amplitude of the primordial density fluctuations. Using the notation of [36], the power spectrum of scalar perturbations can be written as:

$$\Delta_s^2(k) = A_s \left(\frac{k}{k_*}\right)^{n_s-1} \tag{5.69}$$

where A_s is the amplitude of scalar perturbations, and k_* is the mode at horizon exit. The value of A_s has been determined by [37] to be:

$$A_s = (2.105 \pm 0.030) \times 10^{-9} \tag{5.70}$$

It is possible to see that the power spectrum can also be expressed in terms of the potential V as:

$$\Delta_s^2(k) = \frac{1}{24\pi^2}\frac{V}{\epsilon}\bigg|_{\phi=\phi(k)} \tag{5.71}$$

Evaluating this expression at the scale of horizon exit $\phi_* = \phi(k_*)$, substituting ϵ with ist definition (5.6) and comparing it to (5.69), one retrieves the relation:

$$\frac{V^3}{V'^2}\bigg|_{\phi=\phi_*} = 12\pi^2 A_s \equiv \hat{A}_s \simeq 2.5 \times 10^{-7} \tag{5.72}$$

which is the COBE normalization condition our model has to match. In our case, we can write this expressing it in terms of τ_n:

$$\left(\frac{g_s e^{K_{CS}}}{8\pi}\right)\frac{3\lambda_n \beta^3 W_0^2}{64\sqrt{\tau_n}(1-a_n\tau_n)^2}\left(\frac{W_0}{a_n A_n}\right)^2 \frac{e^{2a_n\tau_n}}{\mathcal{V}^6} = 2.5 \times 10^{-7} \tag{5.73}$$

Notice that we included an extra factor in front of the potential containing g_s and $e^{K_{CS}}$, see Appendix A of [38] for a precise derivation. The idea is that it encapsulates the contribution of the stabilized complex structure moduli and dilaton to the scalar potential in Einstein frame.
Now we can substitute (5.17) in (5.73), and working in the limit $a_n\tau_n \gg 1$ we obtain:

$$\tau_n \simeq 7.3 \times 10^{-14}\left(\frac{6\pi\lambda_n}{g_s\beta e^{K_{CS}}}\right)^2 \frac{\mathcal{V}^4}{W_0^4 a_n^4}\frac{1}{N_e^4} \tag{5.74}$$

We can then substitute in (5.74) in the expression (5.18) and get the nice result that ϵ does not depend on N_e:

$$\epsilon \simeq 3.7 \times 10^6 \left(\frac{g_s\beta e^{K_{CS}}}{16\pi}\right)\frac{W_0^2}{\mathcal{V}^3} \tag{5.75}$$

This implies that the tensor-to-scalar ratio $r = 16\epsilon$ in this model is independent of the post-inflationary history of the Universe at leading order. However, we have to be careful about the choice of the parameter since these must be such that the approximation $a_n\tau_n \gg 1$ holds. Therefore, combining this and (5.74) we get an upper bound for r depending on N_e

$$r \ll 3.1 \times 10^{-3}\sqrt{\frac{6\pi}{g_s\beta e^{K_{CS}}}}\frac{\lambda_n^{3/2}}{W_0 a_n^3 N_e^3} \tag{5.76}$$

so taking $a_n = 2\pi$, $g_s \sim 0.1$ and $e^{K_{CS}} = 1$,$W_0, \beta, \lambda_n \sim \mathcal{O}(1)$, we get:

$$r \ll 10^{-4}N_e^{-3} \tag{5.77}$$

which yields $r \ll 5 \times 10^{-10}$ for $N_e = 60$ and $r \ll 10^{-9}$ for $N_e = 40$. Therefore it is safe to assume a value of $r \sim 10^{-10}$, which makes it is undetectable for this model. Now, if we turn to the spectral index n_s, we can write it in terms of N_e as:

$$n_s = 1 + 2\eta - 6\epsilon \simeq 1 - \frac{2}{N_e} \tag{5.78}$$

since, as we have previously seen $\eta \gg \epsilon$. Then, to get a nice prediction for the spectral index, we just need an estimate of the number of e-foldings of the model. To have that, we'll have to perform a choice for the various parameters. The preferred range for the volume in Kähler inflation turns out to be around $\mathcal{V} \sim 10^5 - 10^6$, moreover generic choices of the parameters in (5.59) ($W_0, \beta \sim \mathcal{O}(1)$) yield $N_\phi \simeq 1$. The same choice of parameters, with also $Y \sim 0.1$, plugged in (5.66) returns $N_\mathcal{V} \simeq 25$. Thus:

$$N_e \simeq 57 + \frac{1}{4}\ln r - \frac{1}{4}N_\phi - \frac{1}{4}N_\mathcal{V} \simeq 57 - 2.5\ln(10) - 0.25 - 6 \simeq 45 \tag{5.79}$$

Plugging this result in (5.78) yields:

$$n_s \simeq 0.955 \tag{5.80}$$

This value for n_s is about $2\sigma - 3\sigma$ lower than the experimental value by Planck 2018 [37]

$$n_s = 0.9665 \pm 0.0038 \qquad (68\%\ \text{CL}) \tag{5.81}$$

In the next section, we are going to include in this model *loop corrections* of the kind we introduced in Sec.4.3-4.4 in the Kähler potential, and study how these corrections modify these predictions.

5.2 Inclusion of Loop Corrections

We hereby consider a simpler Swiss-cheese structure for our model, which includes only two small cycles which we call τ_s and τ_ϕ respectively, so that the volume of the Calabi-Yau can be written as:

$$\mathcal{V} = \alpha\left(\tau_B^{3/2} - \lambda_s\tau_s^{3/2} - \lambda_\phi\tau_\phi^{3/2}\right) \tag{5.82}$$

where we simply set $n = 3$ in (4.83). As usual in the LVS setting, for stabilization purposes, we include the first α'-correction to the Kähler potential, as in (4.84) and turn on non-perturbative effects for the small-cycle moduli in the superpotential:

$$W = W_0 + A_s\, e^{-a_s T_s} + A_\phi\, e^{-a_\phi T_\phi} \tag{5.83}$$

At this stage, we have a scalar potential which is similar to the one in (4.86) but with only two small moduli. We can write the inflationary potential using τ_ϕ as the inflaton, and it turns out to be:

$$V(\tau_\phi) = V_0 - \frac{4a_\phi A_\phi W_0 \tau_\phi}{\mathcal{V}^2}\, e^{-a_\phi \tau_\phi} \tag{5.84}$$

exactly as we had above, where V_0 is still given by (5.2).
We now aim to include loop corrections to the scalar potential for our specific model. As argued in Sec 4.4, loop corrections to the scalar potential for a single modulus take the form (4.40) if we consider winding corrections, or (4.47) if we consider Kaluza-Klein corrections. In [21], a general formula for leading-order KK corrections was derived in terms of the derivatives of the tree-level Kähler potential K_{Tree}:

$$\delta_{(g_s)} V^{KK} \simeq \frac{W_0^2}{\mathcal{V}^2} \sum_{i=1}^{h^{1,1}} (g_s C_i^{KK})^2 \frac{\partial^2 K_{Tree}}{\partial T_i^2} \tag{5.85}$$

Then, in our specific case, considering the volume expressed as in (5.82), we can get the KK correction for one specific small modulus τ_i ($i = s$ or ϕ) as:

$$\delta_{(g_s)} V_{\tau_i}^{KK} \simeq \frac{W_0^2}{\mathcal{V}^2} (g_s C_i^{KK})^2 \left(\frac{3\alpha\lambda_i}{2\mathcal{V}\sqrt{\tau_i}} + \frac{9\alpha^2\lambda_i^2\tau_i}{2\mathcal{V}^2} \right) \tag{5.86}$$

Since we expect a volume $\mathcal{V} \gg \tau_i^{3/2}$, the relevant 1-loop KK correction will be given by:

$$\delta_{g_s} V_{\tau_i}^{KK} \simeq \frac{W_0^2}{\mathcal{V}^2} (g_s C_i^{KK})^2 \frac{3\alpha\lambda_i}{2\mathcal{V}\sqrt{\tau_i}} \tag{5.87}$$

As argued in Sec.4.4, this correction is expected to be at the same order as a hypothetical *2-loop KK correction*. Therefore, the full KK correction to the potential may be written as:

$$\delta_{(g_s)} V_{\tau_i}^{KK} \sim \frac{W_0^2}{\mathcal{V}^3} g_s^2 \left[(C_i^{KK})^2 + D_i^{KK} \right] \frac{1}{\sqrt{\tau_i}} \tag{5.88}$$

On the other hand, if we consider winding corrections, we have the formula (4.40) which relates the correction of the scalar potential to the winding correction of the Kähler potential which is conjectured by BHP to be of the form (4.25). Therefore, we can write that, for a small-cycle τ_i in our model, this correction amounts to:

$$\delta_{(g_s)} V_{\tau_i}^{W} \sim -2 \frac{W_0^2}{\mathcal{V}^3} \frac{C_i^W}{\sqrt{\tau_i}} \tag{5.89}$$

Comparing(5.88) and (5.89) we see that they are very similar to each other. We can therefore write a unique form of the open-string loop corrections we expect as:

$$\delta_{(g_s)}V \simeq -\frac{W_0^2}{\mathcal{V}^3}\frac{c_{\text{loop}}}{\sqrt{\tau_i}} \tag{5.90}$$

where we set:

$$c_{\text{loop}} \sim \begin{cases} C_i^W & \text{for W corrections} \\ -g_s^2\left[(C_i^{KK})^2 + D_i^{KK}\right] & \text{for KK corrections} \end{cases} \tag{5.91}$$

Notice that we factored an overall minus sign in the correction, which gets compensated in the KK case by an additional minus sign in the definition of c_{loop}, we will come back to the importance of this sign later. For now, it is worth noting that while we have two different effects, we only included *one* correction in the final expression of the potential. The reason is the following: if winding corrections are present, these are *dominant* over the Kaluza-Klein corrections, since the latter are g_s^2-suppressed. On the other hand, winding corrections may be eliminated by construction, so if we do not allow for them to be present, the only relevant ones are KK corrections.
Adding these corrections computed for τ_ϕ to the inflationary potential yields:

$$V \simeq V_0 - \frac{4a_\phi A_\phi W_0 \tau_\phi}{\mathcal{V}^2}\, e^{-a_\phi\tau_\phi} - \frac{W_0^2 c_{\text{loop}}}{\mathcal{V}^3\sqrt{\tau_\phi}} \tag{5.92}$$

Notice that the first term is *exponentially suppressed*, while the last term only has a suppression of $\tau_\phi^{-1/2}$. Moreover, during inflation, we displace the field τ_ϕ from its minimum, located at $a_\phi\tau_\phi \sim \ln\mathcal{V}$, which means that during inflation $e^{-a_\phi\tau_\phi} \ll \frac{1}{\mathcal{V}}$, accounting for the extra $1/\mathcal{V}$ factor in the second term. Therefore, the first term does not contribute significantly to the inflationary potential, and we can write directly that:

$$V \simeq V_0 - \frac{W_0^2 c_{\text{loop}}}{\mathcal{V}^3\sqrt{\tau_\phi}} \tag{5.93}$$

and recalling (5.2), we can express this as:

$$V \simeq V_0\left(1 - \frac{c_{\text{loop}}}{\beta}\frac{1}{\sqrt{\tau_\phi}}\right) \tag{5.94}$$

As we can see from Fig 5.2, the potential still exhibits a plateau, but it is *less flat* than its uncorrected counterpart in Kähler moduli inflation.

Now we shall rewrite this in terms of the canonically normalized inflaton field, ϕ. Exactly like in the uncorrected case, we have:

$$\phi = \sqrt{\frac{4\lambda_\phi}{3\mathcal{V}}}\,\tau_\phi^{3/4} \tag{5.95}$$

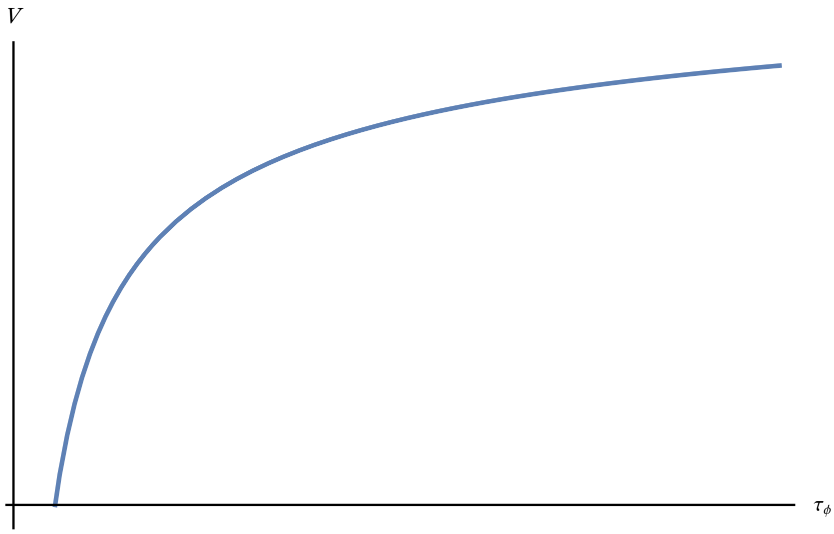

Figure 5.2: Plot of the corrected potential (5.94). For display purposes, the following values for the parameters were used: $\mathcal{V} = 10^4$, $c_{\text{loop}} = 0.01$, $W_0 = 10$, $\beta = 1$.

So that we can rewrite (5.94) as:

$$V = V_0 \left(1 - c_{\text{loop}} \frac{b}{\phi^{2/3}}\right) \tag{5.96}$$

where we collected all the constants in:

$$b = \frac{1}{\beta} \left(\frac{4\lambda_\phi}{3\mathcal{V}}\right)^{1/3} \tag{5.97}$$

Inflationary parameters and constraints

The potential (5.96) is different from the uncorrected potential (5.5) in that it does not have an exponentially flat direction, but rather an inverse-power suppression. The question, then, is whether this may still be a good potential to support slow-roll inflation. To understand this, we have to compute the slow-roll parameters and see under which conditions on c_{loop} and b they satisfy the inflationary requirements.

Before doing that, let us impose some bounds on the range of values τ_ϕ can take not to spoil the consistency of the theory. First of all, τ_ϕ must be a *small modulus*, meaning that it should not exceed the value of τ_B, otherwise the theory completely loses its geometrical sense. This means that:

$$\tau_\phi \ll \tau_B \sim \mathcal{V}^{2/3} \tag{5.98}$$

Inverting the relation (5.95) for τ_ϕ and plugging the result in (5.98), we get a condition on ϕ:

$$\phi \ll \sqrt{\frac{4\lambda_\phi}{3}} \tag{5.99}$$

For generic choices of $\lambda_\phi \sim \mathcal{O}(1)$ we get:

$$\phi \ll 1 \tag{5.100}$$

In particular, this also means that the inflaton at the scale of horizon exit should never exceed 1 in Planck units:

$$\phi_* \ll 1 \tag{5.101}$$

Therefore, we should rely on choices of the parameters b and c_{loop} such that these conditions are satisfied. Now, if we compute the inflationary parameters in terms of ϕ we get:

$$\epsilon = \frac{1}{2}\left(\frac{V'}{V}\right)^2 \simeq \frac{2}{9} b^2 c_{\text{loop}}^2 \phi^{-10/3} \tag{5.102}$$

$$\eta = \frac{V''}{V} \simeq -\frac{10}{9} b c_{\text{loop}} \phi^{-8/3} \tag{5.103}$$

$$n_s - 1 = 2\eta - 6\epsilon \simeq -\frac{20}{9} b c_{\text{loop}} \phi^{-8/3} - \frac{4}{3} b^2 c_{\text{loop}}^2 \phi^{-10/3} \tag{5.104}$$

Moreover, we can also write down the number of e-foldings as a function of ϕ_* as:

$$N_e(\phi_*) = \int_{\phi_E}^{\phi_*} \frac{V}{V'}\, \mathrm{d}\phi \simeq \frac{9}{16} \frac{\phi_*^{8/3}}{b c_{\text{loop}}} \tag{5.105}$$

where we only kept the upper limit of the integral since we suppose $\phi_* \gg \phi_E$.
From these definitions, we immediately see one first requirement we must impose on c_{loop}. We know η must be negative, otherwise we would get a spectral index *larger than one*, we have to impose that $b c_{\text{loop}} > 0$, but since b is defined as (5.97), and λ_ϕ is an intersection number which is positive, then $b > 0$ and therefore also c_{loop} must be positive. Now, looking at (5.91) we notice that, in the case of winding corrections, this simply amounts to requiring that C_ϕ^W is positive. On the other hand, if we consider Kaluza-Klein corrections, then this imposes non-trivial conditions on the 2-loop constant:

$$-\left[\left(C_\phi^{KK}\right)^2 + D_\phi^{KK}\right] > 0 \implies D_\phi^{KK} < 0 \quad \text{and } |D_\phi^{KK}| > (C_\phi^{KK})^2 \tag{5.106}$$

Notice that in the parameter ϵ we have a factor of $\phi^{-10/3}$. Because of the bound (5.100), we see that this quantity may be quite large. Therefore, since during inflation we have that $\epsilon \ll 1$, it must be the factor $(c_{\text{loop}} b)^2$ that make the slow-roll parameter small enough for inflation. Let us then consider the values of c_{loop} and $\mathcal{V}$ for which inflation is feasible. First of all, consider relation (5.105). This gives us a relation between $\phi_*, \mathcal{V}$ and c_{loop}. Considering for simplicity a number of e-foldings around 50 (which is midway between the Kähler moduli inflation prediction and the standard cosmological range centered in 55), we get:

$$\frac{(\mathcal{V}\phi_*^8)^{1/3}}{c_{\text{loop}}} \sim 100 \tag{5.107}$$

Now let us consider again the COBE normalization condition (5.72), but this time for our corrected potential (5.96):

$$\frac{9V_0}{4c_{\text{loop}}^2}\frac{\phi_*^{10/3}}{b^2} = 2.5 \times 10^{-7} \tag{5.108}$$

Including in V_0 the prefactor containing the contribution from the complex structure moduli as we did in (5.73), we get, as an order of magnitude (neglecting $\mathcal{O}(1)$ constants):

$$\left(\frac{g_s e^{K_{CS}}}{8\pi}\right)\frac{W_0^2}{\mathcal{V}^{7/3}}\frac{\phi_*^{10/3}}{c_{\text{loop}}^2} \sim 10^{-7} \tag{5.109}$$

The prefactor in parenthesis in (5.109) is of order 10^{-2} if we take $g_s \sim 0.1$. Multiplying and dividing by $\mathcal{V}^{1/3}$ the left-hand side, we get:

$$\frac{W_0^2(\mathcal{V}\phi_*^8)^{1/3}\phi_*^{2/3}}{\mathcal{V}^{8/3}c_{\text{loop}}^2} \sim 10^{-5} \tag{5.110}$$

substituting (5.107) in (5.110) we get:

$$\frac{W_0^2\phi_*^{2/3}}{\mathcal{V}^{8/3}c_{\text{loop}}} \sim 10^{-7} \tag{5.111}$$

Now, we already know that $\phi_* \ll 1$ for consistency, therefore, to get the maximal value for $\mathcal{V}$ we can see what happens if we set $\phi_* \sim \mathcal{O}(0.1)$. Then we would have:

$$\frac{W_0^2}{\mathcal{V}^{8/3}c_{\text{loop}}} \sim 10^{-6} \tag{5.112}$$

Then the allowed range for the volume depends heavily on c_{loop} and W_0. We will talk about constraints on W_0 in a moment, but for now let us assume that W_0 can get as high as $\mathcal{O}(10)$. In this case, we have that:

$$\frac{1}{\mathcal{V}^{8/3}c_{\text{loop}}} \sim 10^{-8} \tag{5.113}$$

What are natural choices for the order of magnitude of the parameter c_{loop}? If we considered winding corrections, the only way to make c_{loop} small is to suppose some hidden suppressing factor inside C_ϕ^W, like some inverse power of 2π. Instead, considering Kaluza-Klein corrections, the constant c_{loop} contains an explicit factor of g_s^2 as it is visible from (5.91). In that case, for $\mathcal{C}_\phi^{KK} \sim D_\phi^{KK} \sim \mathcal{O}(1)$ we would have a naturally small loop constant of order $c_{\text{loop}} \sim \mathcal{O}(10^{-2})$. if we plug this value of c_{loop} in (5.113) we get a maximal value for the volume:

$$\mathcal{V} \sim 10^4 \tag{5.114}$$

Notice how this is pretty small with respect to the range of volumes we expected in the standard version of Kähler moduli inflation. Relaxing some of the previous assumptions ($W_0 \sim \mathcal{O}(1-10), c_{\text{loop}} \sim \mathcal{O}(0.1-0.01)$) we get a suitable range for the inflationary volume:

$$\mathcal{V} \sim 10^3 - 10^4 \tag{5.115}$$

Notice that in this calculation we have assumed $\phi_* \sim \mathcal{O}(0.1)$. As we will see, this is precisely the order of magnitude of the value for ϕ_* we will find taking into account the number of e-foldings N_e from the post-inflationary dynamics. For the moment, we can see that plugging in (5.105) $c_{\text{loop}} \sim 0.01$, $\mathcal{V} \sim 10^4$, $N_e \simeq 50$ and solving for ϕ_* we get $\phi_* \simeq 0.3$. This allows us to have an estimate for the tensor-to-scalar ratio r as well. For a range of e-foldings between $N_e \sim 40-60$ we always get $\phi_* \sim 0.29 - 0.34$, which ultimately gives:

$$r = 16\epsilon_* \sim (3-5) \times 10^{-5} \tag{5.116}$$

notice that this is much higher than the tensor-to-scalar ratio in the uncorrected Kähler moduli inflation, which was around $r \sim 10^{-10}$.

Tadpole constraint on W_0

Looking at the relation (5.112) we immediately see that, the higher the value for W_0 is, the higher the volume $\mathcal{V}$. This may suggest that we could retrieve the same range of values for $\mathcal{V}$ as in the uncorrected case if we consider a model with a large W_0. However, this is not the case. In fact we will now prove a constraint on W_0 emerging from D3-brane charge tadpole cancellation, which was introduced in [39]. This constraint is given as a lower bound on the D3-charge Q_3 from O3/O7-planes and D7-branes in the model in order to cancel the contribution of D3-branes (for example used for uplifting [40]). In our normalization of the potential we can write this bound as:

$$-Q_3 \geq 4\pi \frac{g_s W_0^2}{2} \tag{5.117}$$

We can read this as a bound on the maximal value of W_0^2 which appears in (5.112):

$$W_0^2 \leq -\frac{Q_3}{2\pi g_s} \tag{5.118}$$

As a side note, an anti-D3 brane used for uplifting has a *negative* D3-charge, which means that $Q_3 < 0$, therefore (5.118) is only concerned with the absolute value of the D3 charge coming from non-D3 sources. The maximal value for $|Q_3|$ reported in [39] is around $-Q_3 \sim 250$. For a more generic case, we can simply set $-Q_3 \sim 100$. This translates into a bound on W_0:

$$W_0^2 \lesssim \mathcal{O}(100) \tag{5.119}$$

thus motivating that (5.114) is really the maximal volume we can achieve in this model. As a side note, in [40] the tadpole constraint is used to get a consistency condition on the warping factors in the Klebanov-Strassler throat (without considering backreactions) for anti-D3 brane uplifting. This may be used to constrain the volume of our model from below and used as a consistency check. If the value of the volume we get happened to be too low, this would just be an indication that the uplifting could not be achieved by anti-D3 brane in warped throats, but with other mechanisms, like D-term effects [41] or dilaton-dependent non-perturbative effects [42, 43].

5.3 Post-Inflationary Dynamics in Loop-Corrected Inflation

We now turn to get precise predictions from our loop-corrected model for Kähler moduli inflation. We will analyze two *scenarios*. *Scenario I* is quite similar to what we previously displayed in the uncorrected case of Kähler moduli inflation. The inflaton 4-cycle whose volume is controlled by τ_ϕ is wrapped by a stack of hidden sector D7-branes (so that T_ϕ-dependent non-perturbative effects can be generated), while the Standard Model (SM) is realized on another stack of D7-branes wrapped around an additional blow-up mode, which we shall call τ_{SM}, that is expected to be stabilised by loop corrections. *Scenario II*, instead, represents the case in which the inflaton 4-cycle is not wrapped by any D7-brane, and again τ_{SM} is an additional blow-up modulus which is stabilised by loops and supports the SM D7-stack. In this case, the preferred decay channel of the inflaton field is much more suppressed with respect to Scenario I, as we will see shortly.

Let us explain more in detail why the 4-cycle carrying the SM cannot be stabilized via non-perturbative effects. Hereby we give a brief idea of why, and refer the interested reader to [44]. Suppose the SM lives on a stack of D7-branes wrapping the 4-cycle τ_{SM}, which we complexify as $T_{SM} = \tau_{SM} + ib_{SM}$. Now, if the effects stabilizing the Kähler modulus T_{SM} were non-perturbative, we would have a term in the superpotential going as:

$$W_{SM} \sim e^{-a_{SM}T_{SM}} \tag{5.120}$$

However, T_{SM} becomes *charged* under the $U(1)_Y$ of the SM, which implies that the expression (5.120) is not gauge-invariant. The way to make it gauge invariant is to write it as:

$$W_{SM} \sim \mathcal{O}_Y e^{-a_{SM}T_{SM}} \tag{5.121}$$

where $\mathcal{O}_Y$ is an operator which is charged under $\mathrm{U}(1)_Y$ in such a way that the product with $e^{-a_{SM}T_{SM}}$ is overall gauge invariant. This means that $\mathcal{O}_Y$ must be a product (or a combination of products) of the SM fields which are charged under $\mathrm{U}(1)_Y$. However, since the $\mathrm{U}(1)_Y$ symmetry is not broken at the string scale, we must have that the VEV of the SM fields must be zero. Thus, $\langle \mathcal{O}_Y \rangle = 0$ and therefore $W_{SM} = 0$. Therefore, there must be some perturbative effect stabilizing the SM-carrier 4-cycle. We will follow [45] and suppose the SM cycle (which we simply call τ_s here) is stabilized through *loop effects*, so that:

$$V_{\text{loop}}(\tau_s) = \frac{W_0^2}{\mathcal{V}^3}\left(\frac{\mu_1}{\sqrt{\tau_s}} - \frac{\mu_2}{\sqrt{\tau_s} - \mu_3}\right) \tag{5.122}$$

where μ_1 and μ_2 only depend on the complex structure moduli and can be thought as constants in this case, while μ_3 may also depend on some other small cycle, but we suppose it fixed at the scale where these corrections are relevant. We delay the details of the calculations of the stabilization and mixing of the fields to Appendix C.

Loop-enhanced Higgs coupling

Before considering both these scenarios, we will retrieve a loop-modified coupling of the volume modulus to the Higgs scalar which will vastly enhance its decay rate. To do that, we will follow the track of [45], where this coupling was originally discovered.
Consider the Higgs mass matrix at the Kaluza-Klein scale M_{KK} on the D-brane hosting the SM. The KK scale is such that for energies below M_{KK} we have a 4d SUSY EFT which can be used to run the elements of the Higgs mass matrix down to the SUSY-breaking scale $m_{3/2}$. We know that the scale at which SUSY is broken is set by the F-terms of the Kähler moduli T_i, in particular one has that:

$$\frac{m_{3/2}}{M_P} \sim \frac{F_T}{T} \tag{5.123}$$

Moreover, when the MSSM is realized with D7-branes, the gaugino mass $m_{1/2}$ is proportional to the gravitino mass:

$$m_{1/2} \sim m_{3/2} \tag{5.124}$$

These considerations suggest that at least *some* of the entries of the Higgs mass matrix must be of order $m_{3/2}^2$. At loop-level, the scale of the Higgs mass matrix can be suppressed by a loop factor $\tilde{c}$, but at the same time it gets enhanced by the logarithm resulting from the running of the coupling. For example, we may have the gaugino mass contribution to the Higgs matrix as:

$$m_g^2 \sim \tilde{c}\, m_{1/2}^2 \ln\left(\frac{M_{KK}}{m_{3/2}}\right) \tag{5.125}$$

Below $m_{3/2}$, SUSY is broken, and the Higgs mass matrix loses one linear combination of the two Higgs doublets of the MSSM. The remaining combination has its mass fixed by the determinant of the Higgs mass matrix which is then fine-tuned to a small value. We can express the main contributions to the Higgs mass as:

$$m_H^2 \sim m_{3/2}^2 \left[c_0 + \tilde{c} \ln \left(\frac{M_{KK}}{m_{3/2}} \right) \right] \tag{5.126}$$

We know that in this model the SM wraps a small cycle, whose volume is about $\mathcal{O}(1-10)$ in string units. Therefore, the KK scale is close to the string scale:

$$M_{KK} \sim M_s \sim \frac{M_P}{\sqrt{\mathcal{V}}} \tag{5.127}$$

On the other hand, the SUSY-breaking scale is lower :

$$\frac{m_{3/2}}{M_P} \sim \frac{W_0}{\mathcal{V}} \tag{5.128}$$

Substituting these two relations in (5.126) we get:

$$m_H^2 \sim \left(\frac{W_0}{\mathcal{V}} \right)^2 \left[c_0 + \tilde{c} \ln \left(\frac{\sqrt{\mathcal{V}}}{W_0} \right) \right] M_P^2 \tag{5.129}$$

We now canonically normalize the volume modulus as in (5.41) and expand χ around its VEV:

$$\chi = \langle \chi \rangle + \hat{\chi} \tag{5.130}$$

Since the gravitino mass, at least in the model at hand, is way higher than the observed value of the Higgs mass, we know there must be a severe fine-tuning, which translates into the constant c_0 being very small. This is not a bad thing in the present case. In fact, the fine tuning is such that the *logarithmic term dominates*. Therefore, in the Lagrangian we will have a term which goes as:

$$\mathcal{L} \supset \left(\frac{m_{3/2}^2 \tilde{c}}{2} \sqrt{\frac{2}{3}} \right) \frac{\hat{\chi}}{M_P} h^2 \sim m_{3/2}^2 \tilde{c} \frac{\hat{\chi}}{M_P} h^2 \tag{5.131}$$

where h is the Higgs scalar field. This is a trilinear coupling, which gives rise to a decay channel of $\mathcal{V}$ into a pair of h. Its decay rate parametrically scales as:

$$\Gamma(\chi \to hh) \sim \frac{\tilde{c}^2 \, m_{3/2}^4}{m_{\mathcal{V}} M_P^2} \sim (\tilde{c}\mathcal{V})^2 \frac{m_{\mathcal{V}}^3}{M_P^2} \tag{5.132}$$

There is a small catch, though. The volume, during inflation, due to its mixing with the inflaton, is not constant. The volume modulus gets displaced from its minimum $\langle \chi \rangle$ to its inflationary minimum. This displacement is usually indicated as:

$$Y = \frac{\Delta \chi}{M_P} \tag{5.133}$$

It's been calculated that $Y \sim \mathcal{O}(0.1)$ for un-corrected Kähler moduli inflation [3], however, the scalar potential for the volume does not get any important loop correction, which means this should also be valid for the loop-corrected case. Therefore, we may say that during inflation there is no fine tuning of m_H, which instead scales as:

$$m_H^2 \sim \tilde{c}\, m_{3/2}^2 \frac{\Delta\chi}{M_P} \tag{5.134}$$

This means that during inflation and immediately after its end:

$$m_H^2 > m_{\mathcal{V}}^2 \qquad \text{as long as} \qquad \mathcal{V} > \frac{M_P}{\tilde{c}\Delta\chi} \sim \mathcal{O}(10^3) \tag{5.135}$$

where we estimated $\tilde{c} \simeq \frac{1}{16\pi^2} \sim \mathcal{O}(10^{-2})$. So the decay of the volume modulus into the Higgs seems to be kinematically forbidden. However, we do know that the volume modulus, at the end of inflation, starts oscillating as soon as :

$$H_{OSC} \sim m_{\mathcal{V}} \tag{5.136}$$

and decays only when:

$$H_{DEC} \sim \Gamma_{\chi \to hh} \tag{5.137}$$

Since we are in matter domination and the amplitude of the oscillations redshifts exactly like H, we can estimate the Hubble ratio H_{EQ} at the time when $m_H \simeq m_{\mathcal{V}}$. This is as simple as computing:

$$H_{EQ} = \frac{\Delta\chi^{(EQ)}}{\Delta\chi^{(OSC)}} H_{OSC} \simeq \frac{M_P^2}{\tilde{c}\, \mathcal{V}^{5/2} \Delta\chi^{(OSC)}} \tag{5.138}$$

And we can compare this to the decay-time H:

$$\frac{H_{EQ}}{H_{DEC}} \sim \frac{M_P}{\tilde{c}^3 \Delta\chi^{(OSC)}} \sim 10^7 \tag{5.139}$$

assuming $\Delta\chi^{(OSC)} = Y M_P \sim 0.1 M_P$. This tells us that the two masses become comparable way before $\mathcal{V}$ decays, and when this happens $m_H < m_{\mathcal{V}}$ due to the fine tuning, making the decay kinematically allowed.

Scenario I: Reheating from volume mode decay

In this subsection we simply follow what we already did in the study of the post-inflationary dynamics for the uncorrected Kähler moduli inflation model, but with the loop-enhanced Higgs coupling for the volume. This means that we are going to use a decay rate for the volume modulus which is (5.132), which can be expressed explicitly in terms of the volume as:

$$\Gamma_{\mathcal{V}} \simeq \frac{\tilde{c}^2 \mathcal{V}^2 m_{\mathcal{V}}^3}{M_P^2} \simeq \frac{\tilde{c}^2 W_0^3}{(\ln \mathcal{V})^{3/2} \mathcal{V}^{5/2}} M_P \tag{5.140}$$

while we will keep the inflaton decay rate Γ_{τ_ϕ} the same as (5.48). We can easily see that the inflaton decays before the volume, since:

$$\frac{\Gamma_{\tau_\phi}}{\Gamma_{\mathcal{V}}} \sim \frac{(\ln \mathcal{V})^{9/2}}{\tilde{c}^2} \sqrt{\mathcal{V}} \gg 1 \tag{5.141}$$

Therefore, the history of the Universe in this scenario will be the same as the one in the uncorrected case: after inflation we will have a (brief) period in which the energy density is dominated by the coherent oscillations of the inflaton; after that the inflaton decays and a radiation-dominated era begins while the volume modulus is still oscillating; at some point we will have matter-radiation equality and from that moment onward the oscillations of the volume will dominate, until it ultimately decays and a second era of radiation domination starts. The expression for the number of e-foldings of inflaton-domination will be exactly the same as in the uncorrected case, (5.59). The same is true for the expression of $H(t_{eq})$ which ultimately depends only on the ratio of the energy densities, θ (5.53), which we assume is unaffected by the loop corrections. Therefore, the only expression that differs from the previously studied case is the final expression of $N_{\mathcal{V}}$. We will have:

$$N_{\mathcal{V}} \simeq \frac{2}{3} \ln \left(\frac{H(t_{eq})}{\Gamma_{\mathcal{V}}} \right) \simeq \frac{2}{3} \ln \left(\frac{\theta^4 (\ln \mathcal{V})^{9/2} \sqrt{\mathcal{V}}}{10 \tilde{c}^2} \right) \simeq \frac{2}{3} \ln \left(\frac{(\ln \mathcal{V})^{5/2} \sqrt{\mathcal{V}} Y^4}{10 \beta^2 \tilde{c}^2} \right) \tag{5.142}$$

where in the first equality we used the value of $H(t_{eq})$ we computed in (5.65) and the expression (5.140) for $\Gamma_{\mathcal{V}}$, while in the second one the definition of θ in terms of Y (5.53). We can evaluate N_ϕ and $N_{\mathcal{V}}$ in this scenario for a generic choice of the parameters. One thing we have to be careful about, however, is the choice of W_0. In fact, if we consider (5.59), plugging in $W_0^2 \sim 100$ and $\mathcal{V} \sim 10^4$ gives a *negative result*. This would mean that the inflaton field decays *before* the end of inflation. This is a pathological situation and we shall avoid this region. This is solved if we take smaller values of $W_0 \sim \mathcal{O}(1)$, which inevitably also lowers the volume. The result is that in the best-case scenario, in which $W_0 \sim 1$ and $\mathcal{V} \sim 10^4$ still (which may be achieved lowering the loop constant to $c_{\text{loop}} \sim 10^{-3}$), $N_\phi \simeq 0.25$. For what concerns the volume, we take $\beta \sim 1$, $\tilde{c} \sim 10^{-2}$ and $Y \sim 0.1$, and get $N_{\mathcal{V}} \simeq 4$. Therefore, we can get the total number of e-foldings predicted by this scenario using the formula (5.50):

$$N_e \simeq 57 + \frac{1}{4} \ln r - \frac{1}{4} N_\phi - \frac{1}{4} N_{\mathcal{V}} \simeq 57 - 2.5 - 0.1 - 1 \simeq 53 \tag{5.143}$$

where we used the value of $r \simeq 4 \times 10^{-5}$ which is in the range (5.116).

Scenario II: Inflaton decay rate

The scenario in which the inflaton cycle is not wrapped by any D7-brane was studied in detail in [45]. Since the inflaton field does not have a direct decay channel to the gauge fields living upon the D7-branes, the decay rate, as we will see, is drastically smaller.

Let us first of all retrieve the decay rate of τ_ϕ in the dominant decay channel. In this scenario, the SM has to be attached to D7-branes wrapping some other blow-up cycle, τ_{SM}. As argued in [45], the dominant decay channel of τ_ϕ is the kinetic-coupling induced decay to SM gauge fields. This happens through the mixing of τ_ϕ and τ_{SM} in the Kähler potential. Given that τ_{SM} is a local blow-up mode as τ_s, its mixing with τ_ϕ has the same volume scaling as the mixing of τ_s with τ_ϕ. We shall therefore focus just on the volume form (5.82) which shows that the canonically normalized counterparts of τ_ϕ and τ_s, which are ϕ and σ respectively, will have a small mixing suppressed by powers of the volume. Moreover, since the SM lives of D7-branes wrapped around τ_{SM}, the latter will be the (real part of the) gauge kinetic function of the SM gauge fields:

$$\mathcal{L} \supset \tau_{SM}\, \mathrm{tr}\left[F_{\mu\nu}F^{\mu\nu}\right] \tag{5.144}$$

which shows the coupling of τ_{SM} to the gauge fields A_μ. Expanding τ_{SM} around its VEV as $\tau_{SM} = \langle\tau_{SM}\rangle + \delta\tau_{SM}$, and canonically normalizing the gauge fields, (5.144) becomes:

$$\mathcal{L} \supset \frac{1}{2}\,\mathrm{tr}\left[F^c_{\mu\nu}F^{\mu\nu}_c\right] + \frac{1}{2}\frac{\delta\tau_{SM}}{\langle\tau_{SM}\rangle}\,\mathrm{tr}\left[F^c_{\mu\nu}F^{\mu\nu}_c\right] \tag{5.145}$$

where the label c indicates canonical normalization. We will understand this label in what follows not to make the notation too heavy. The mixing of the various 4-cycles in this case is derived precisely in Appendix C [14] Therefore, we have:

$$\mathcal{L} \supset \frac{1}{2}\frac{\delta\phi}{\sqrt{2}\,\langle\tau_{SM}\rangle}(\vec{v}_\phi)_s\,\mathrm{tr}\left[F_{\mu\nu}F^{\mu\nu}\right] \tag{5.146}$$

where $(\vec{v}_\phi)_s$ controls the mixing between τ_ϕ and τ_{SM}, and is derived in (C.24). Therefore, we can plug in the approximate result and get:

$$\mathcal{L}_{\phi-A} \sim \frac{1}{2\sqrt{2}\,\langle\tau_{SM}\rangle}\left(\frac{4(m_{12}m_{31}+m_{22}m_{33})\,\langle\tau_\phi\rangle^{1/4}}{\sqrt{6\lambda_\phi}m_{22}(m_{22}-m_{33})\,\langle\tau_B\rangle^{3/4}}\right)\delta\phi\,\mathrm{tr}\left[F_{\mu\nu}F^{\mu\nu}\right] \tag{5.147}$$

Now, looking at the expression of the elements of the mass matrix in (C.10)-(C.18) we see that $m_{22}m_{33} \gg m_{12}m_{31}$ and also $m_{22} \gg m_{33}$ (see (C.26)). Then we can rewrite this as:

$$\mathcal{L}_{\phi-A} \simeq \frac{m_{32}\,\langle\tau_\phi\rangle^{1/4}}{\sqrt{3\lambda_\phi}m_{22}\,\langle\tau_{SM}\rangle\,\langle\tau_B\rangle^{3/4}}\delta\phi\,\mathrm{tr}\left[F_{\mu\nu}F^{\mu\nu}\right] \simeq \frac{\sqrt{3\lambda_\phi}\,\langle\tau_\phi\rangle^{3/4}}{2\,\langle\tau_B\rangle^{3/4}}\delta\phi\,\mathrm{tr}\left[F_{\mu\nu}F^{\mu\nu}\right] \tag{5.148}$$

[14] In Appendix C, the role of τ_{SM} is played by a generic τ_s.

We can therefore write down the decay rate of the inflaton field to SM gauge fields following [45] as:

$$\Gamma(\phi \to AA) \simeq \frac{3\lambda_\phi N_g}{64\pi}\left(\frac{\langle\tau_\phi\rangle}{\langle\tau_B\rangle}\right)^{3/2}\frac{m_{\tau_\phi}^3}{M_P^2} \simeq \frac{3\lambda_\phi N_g W_0^3 (\ln\mathcal{V})^{9/2}}{8\pi a_\phi^{3/2}\mathcal{V}^4} M_P \tag{5.149}$$

where N_g is the number of available decay channels, i.e. the number of gauge bosons the inflaton can decay to. This is the dominant decay channel of the inflaton field in Scenario II, and in what follows we will refer to (5.149) simply as Γ_ϕ.

Scenario II: Reheating from inflaton decay

Let us now turn to derive the post-inflationary evolution of Scenario II. We will follow exactly the same steps we followed in Scenario I and in the uncorrected case before that, but using the decay rate (5.149) for the inflaton. Notice that this decay rate is much more suppressed with respect to the one we used in Scenario I. In fact, it was observed in [45] that it may be so suppressed that the volume modulus actually decays *before* the inflaton does in this scenario. While for the range of volumes considered in the aforementioned article that was most certainly true, it is not in our present case. In fact, consider the ration between the Γ_ϕ (5.149) and $\Gamma_\mathcal{V}$ (5.140):

$$\frac{\Gamma_{\tau_\phi}}{\Gamma_\mathcal{V}} \simeq \frac{3\lambda_\phi N_g (\ln\mathcal{V})^6}{8\pi\tilde{c}^2 a_\phi^{3/2}\mathcal{V}^{3/2}} \tag{5.150}$$

Now, for a generic choice of the parameters ($N_g \sim 10, \tilde{c} \sim 0.01$, $a_\phi \sim 2\pi$, $\lambda_\phi \sim 1$) and taking our maximal volume $\mathcal{V} \sim 10^4$, we get that:

$$\frac{\Gamma_{\tau_\phi}}{\Gamma_\mathcal{V}} \sim 10^2 \tag{5.151}$$

It must be noted that if we had, like in the uncorrected Kähler moduli inflation case, a range of volumes of $\mathcal{V} \sim 10^5 - 10^6$ then the ratio (5.150) would be close to 1, and depending on the specific choices of the constants could also be smaller. However, in our case the history of the Universe will follow a similar path to that we already encountered, however we will see that the oscillations of the inflaton will last way longer than before. In fact, we can plug our new decay rate in the usual formula (5.59):

$$N_\phi \simeq \frac{2}{3}\ln\left(\frac{H(t_E)}{\Gamma_\phi}\right) \simeq \frac{2}{3}\ln\left(\frac{8\pi a_\phi^{3/2}\sqrt{\beta}\mathcal{V}^{5/2}}{3\lambda_\phi N_g W_0^2(\ln\mathcal{V})^{9/2}}\right) \tag{5.152}$$

Now we need the expression of $H(t_{eq})$ which can always be obtained from (5.65):

$$H(t_{eq}) \simeq \theta^4 H(t_\phi) \simeq H(t_E)\frac{3\lambda_\phi N_g W_0^2(\ln\mathcal{V})^{9/2}\theta^4}{8\pi a_\phi^{3/2}\sqrt{\beta}\mathcal{V}^{5/2}} \tag{5.153}$$

This is where the phenomenology of this scenario differs from the ones we have seen before. In fact, consider the ratio:

$$\frac{H(t_{eq})}{\Gamma_{\mathcal{V}}} \simeq \frac{3\lambda_\phi N_g (\ln \mathcal{V})^4 Y^4}{8\pi\beta^2 a_\phi^{3/2} \tilde{c}^2 \mathcal{V}^{3/2}} \tag{5.154}$$

where we have used relation (5.53) to rewrite θ in terms of the displacement Y. Plugging in (5.154) reasonable values of the parameters ($N_g \sim 10$, $a_\phi \sim 2\pi$, $\beta \sim 1$, $Y \sim 0.1$, $\lambda_\phi \sim 1$ and $\tilde{c} \sim 0.01$), for our range of volumes $\mathcal{V} \sim 10^3 - 10^4$ we get:

$$\frac{H(t_{eq})}{\Gamma_{\mathcal{V}}} \sim 10^{-2} - 10^{-3} \tag{5.155}$$

This means that the volume modulus decays *before* the time of matter-radiation equality, while the Universe is still dominated by the radiation produced by the inflaton decay. Thus, in this scenario the coherent oscillations of the volume modulus never come to dominate the energy density of the Universe, so that we will have to set:

$$N_{\mathcal{V}} = 0 \tag{5.156}$$

Therefore, we can now use the formula (5.50) to get an estimate of the number of e-foldings in this scenario. Plugging in the generic values $N_g \sim 10, a_\phi \sim 2\pi, \lambda_\phi \sim 1, \beta \sim 1, W_0 \sim 1 - 10$ in (5.152) we get $N_\phi \simeq 10.6 - 8$, which we can roughly round to $N_\phi \simeq 10$. Therefore, we will get:

$$N_e \simeq 57 + \frac{1}{4}\ln r - \frac{1}{4}N_\phi \simeq 57 - 2.5 - 2.5 \simeq 52 \tag{5.157}$$

where we set $N_{\mathcal{V}} = 0$ identically as discussed above and $r \simeq 4 \times 10^{-5}$ which is the central value of (5.116).

5.4 Inflationary Parameters in the Loop-Corrected Case

Now that we have a solid prediction for the number of e-foldings in both scenarios, we can finally look at the predicted values for the inflationary parameters. First of all we can determine the value of the field ϕ at the end of inflation, ϕ_E imposing that:

$$\epsilon(\phi_E) \simeq \frac{2}{9} b^2 c_{\text{loop}}^2 \phi_E^{-10/3} \simeq 1 \tag{5.158}$$

Using (5.97) for $\mathcal{V} \sim 10^4$ and $\beta \sim \lambda_\phi \sim 1$, and setting $c_{\text{loop}} \sim 10^{-2}$ and we get:

$$\phi_E \simeq 0.007 \tag{5.159}$$

which is much lower than the value we expect for ϕ_*. This justifies keeping only the dominant upper bound in the expression of (5.105). Now we can simply use the relation (5.105) and invert it to obtain:

$$\phi_* \simeq \left(\frac{16}{9} b c_{\text{loop}} N_e\right)^{3/8} \tag{5.160}$$

We now simply have to insert in this formula the values for N_e we computed from the post-inflationary evolution in the two possible scenarios (5.143) and (5.157). Setting $N_e \simeq 53$ as prescribed by Scenario I, (5.160) gives [15]

$$\phi_* \simeq 0.32 \tag{5.161}$$

Inserting this in (5.104) will give us the prediction for the spectral index in Scenario I:

$$n_s \simeq 0.9764 \tag{5.162}$$

Similarly, plugging in (5.160) the value $N_e \simeq 52$ from Scenario II, the result is very close:

$$\phi_* \simeq 0.33 \tag{5.163}$$

which yields a spectral index of:

$$n_s \simeq 0.9759 \tag{5.164}$$

Therefore, accounting for the variability of some parameters, we can roughly estimate an overall predicted value of the spectral index of:

$$n_s \simeq 0.976 \tag{5.165}$$

Notice that, while the predicted value in the original Kähler moduli inflation model without loop corrections is about 2σ *below* the value measured by the Planck collaboration in 2018 [37](5.81), the value we found including the loop corrections is *above* (5.81) by about 2σ. However, this does not mean that this model is completely wrong. In fact, it must be said that the Planck value from n_s arises from a *fit* of the data, which takes as input a cosmological model, which is the ΛCDM standard model of cosmology. In particular, one of the assumptions underlying the value of n_s of (5.81) is that there is *no dark radiation* in the history of the Universe. As we will see below, this is indeed the case in Scenario I, while Scenario II predicts a small, but non negligible amount of Dark Radiation, which yields to a change in the effective number of neutrino species ΔN_{eff} which may actually *close the gap* between our predicted value of n_s and the one inferred from CMB data in the presence of extra dark radiation. In fact, the Planck collaboration in 2015 performed an analysis of the variation of n_s with respect to the amount of Dark Radiation [46]. It was found that the effect of the inclusion of $\Delta N_{eff} > 0$ was to *increase* the spectral index. They found:

$$n_s = 0.983 \pm 0.006 \quad (68\%\,\text{CL}) \quad \text{for} \quad \Delta N_{eff} = 0.39 \tag{5.166}$$

[15]Throughout these calculations we assume $\mathcal{V} \sim 10^4, \beta \sim \lambda_\phi \sim 1$, and $c_{\text{loop}} \sim 10^{-2}$.

Kähler cone constraints

Let us now remark some technical aspects regarding the *Kähler cone* of this model. Since we did not specify explicitly the intersection numbers of the Calabi-Yau we are using, we shall treat what follows simply as an indicative discussion about how close to the boundary of the Kähler cone we are getting. First of all, we shall convert back ϕ_*, (which we indicatively take as 0.3) to τ_ϕ^* inverting relation (5.95):

$$\tau_\phi^* \simeq \left(\frac{3\mathcal{V}}{4}\right)^{2/3} \phi_*^{4/3} \simeq \left(\frac{3}{4}\right)^{2/3} \tau_B \phi_*^{4/3} \tag{5.167}$$

where in the second equality we used the fact that $\mathcal{V}^{2/3} \sim \tau_B$. Now we convert back τ_ϕ and τ_B to the dual 2-cycles we call t_ϕ and t_B as:

$$\tau_\phi \sim t_\phi^2 \tag{5.168}$$

$$\tau_B \sim t_B^2 \tag{5.169}$$

where the precise relation depends on the intersection numbers of the divisors basis. Then we can rewrite (5.167) in terms of the 2-cycles:

$$(t_\phi^*)^2 \sim t_B^2 \phi_*^{4/3} \tag{5.170}$$

which translates into:

$$\frac{t_\phi^*}{t_B} \sim \phi_*^{2/3} \sim 0.4 \tag{5.171}$$

Now, considering that usually τ_B has a $\mathcal{O}(10)$ coefficient multiplying t_B^2, see for example the explicit construction of [31], we can safely state that $t_\phi \ll t_B$ at all times during inflation, hence we are not getting close to the boundary of the Kähler cone.

5.5 Dark Radiation from moduli decays

The effective number of neutrino species N_{eff} is a standard indicator of the fraction of the total energy density that lies in the photon plasma at the time it is measured. It constrains the fraction of energy density which, after inflation, is transferred to hidden sector fields redshifting as radiation, i.e. Dark Radiation. At the CMB temperature, we can determine N_{eff} as [34]:

$$\rho_{TOT} = \rho_\gamma \left(1 + \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} N_{eff}\right) \tag{5.172}$$

If there was dark radiation either at CMB or BBN times, this would lead to a non-trivial fraction of the total energy density of the Universe being stored in a plasma which is *decoupled* from the photon bath. Then, from (5.172) we would be able to see a non-zero shift of N_{eff}:

$$\Delta N_{eff} = N_{eff} - N^0_{eff} \tag{5.173}$$

where, with N^0_{eff} we indicate the number of Standard Model effective neutrino species during the evolution of the Universe (with *zero* Dark Radiation). The most relevant component of the Dark Radiation which is predicted by string-inflation models is made of *non-thermally produced axions*. These are usually the axionic counterparts of the Kähler moduli which take part in inflation (e.g. the inflaton and the volume modulus for Kähler moduli inflation). In [34, 35, 47, 48] a formula was derived which relates ΔN_{eff} with the *branching ratio* of the decay of a modulus φ to its axions a:

$$\Delta N_{eff} \simeq 6.1 \left(\frac{11}{g_*^4 g_{*,S}^{-3}}\right)^{1/3} B(\varphi \to aa) \simeq 6.1 \left(\frac{11}{g_*^4 g_{*,S}^{-3}}\right)^{1/3} \frac{\rho_{DR}}{\rho_{SM} + \rho_{DR}}\bigg|_{T=T_{RH}} \tag{5.174}$$

where g_* is the effective number of degrees of freedom of the energy density and $g_{*,S}$ the effective number of degrees of freedom of the entropy density. The branching ratio is defined as:

$$B(\varphi \to aa) = \frac{\Gamma(\varphi \to aa)}{\Gamma_\varphi} \tag{5.175}$$

where Γ_φ is the total decay rate of φ. The Planck collaboration observations of the CMB and large scale structure [37], in 2018 constrained ΔN_{eff} to:

$$\Delta N_{eff} \lesssim 0.2 - 0.4 \tag{5.176}$$

depending on the specific data-set used. Following [45], we are going to evaluate ΔN_{eff} in both of our post-inflationary scenarios. In the present model we have two moduli fields that actively play a role during inflation: the volume $\mathcal{V}$ and the inflaton ϕ. The volume modulus drives reheating in Scenario I, while the inflaton does so in Scenario II. Let us analyze them separately.

Scenario I: Dark radiation from volume decay

If we do not immediately stabilize the axions b_i to their minima, they are regarded as dynamical fields and appear in the kinetic term of the Lagrangian with the same coefficients as their corresponding 4-cycle fields τ_i. In the specific case of the volume mode τ_B we have:

$$\mathcal{L} \supset K_{BB}(\partial_\mu \tau_B \partial^\mu \tau_B + \partial_\mu b_B \partial^\mu b_B) = \frac{3}{4\tau_B^2}\partial_\mu \tau_B \partial^\mu \tau_B + \frac{3}{4\tau_B^2}\partial_\mu b_B \partial^\mu b_B \tag{5.177}$$

where we used K_{BB} which we retrieve in Appendix B and Appendix C. Now we can canonically normalize the fields and expand them around their VEVs to get the trilinear coupling between τ_B and b_B, see for example [34]. The resulting decay rate is then given by:

$$\Gamma(\mathcal{V} \to b_B b_B) = \frac{1}{48\pi} \frac{m_{\mathcal{V}}^3}{M_P^2} \tag{5.178}$$

This has to be compared with the loop-enhanced decay rate of the volume modulus to Higgs fields retrieved in (5.132). We get that:

$$\frac{\Gamma(\mathcal{V} \to hh)}{\Gamma(\mathcal{V} \to b_B b_B)} \simeq 48\pi(\tilde{c}\mathcal{V})^2 \gg 1 \tag{5.179}$$

since $10^4 \sim \mathcal{V} \gg 1/\tilde{c} \sim 10^2$. This means that the decay channel of the volume to its axions is heavily suppressed, and therefore we get a negligible branching ratio. Therefore, (almost) no dark radiation is produced by the decay of the volume modulus. So for Scenario I we have $\Delta N_{eff} \simeq 0$.

Scenario II: Dark radiation from inflaton decay

We will now perform a similar analysis for the inflaton field in Scenario II. The first thing we notice is that the axion of the inflaton field has *the same mass* of the inflaton itself. The reason for that is that we are using a supersymmetric non-perturbative stabilization for this particular field: the α'-correction which stabilizes the volume does not depend on T_ϕ and neither do the loop corrections stabilizing τ_{SM}. Therefore, the decay of ϕ into a pair of b_ϕ is kinematically forbidden. Another important thing to notice is that, as shown in [45], the decay rate to the axions of the other moduli is *the same* ass the decay rate to the respective 4-cycle field, and the dominant decay amplitude is given by the *kinetic terms*. Therefore, we shall work out only the decay rates to the 4-cycles through the kinetic terms. To do that, we have to expand the kinetic term of the Lagrangian up to third order around the VEV of the 4-cycles τ_i, and only keep the terms on which $\delta\tau_\phi$ appear:

$$\mathcal{L} \supset (\partial_{\tau_\phi} K_{ij})\delta\tau_\phi \partial_\mu \delta\tau^i \partial^\mu \delta\tau^j \tag{5.180}$$

What we can do now is eliminate the derivatives. Since we are looking for a tree-level decay, the products of the decay will be *on shell*. Therefore, we can integrate by parts (5.180) and use Klein-Gordon equation to substitute the derivatives with the respective *masses*:

$$\mathcal{L} \supset \frac{1}{2} K_{\phi ij}(m_{\tau_\phi}^2 - m_i^2 - m_j^2)\delta\tau_\phi \delta\tau^i \delta\tau^j \tag{5.181}$$

Since we are considering the specific case $i = j \neq \phi$, and the inflaton field is the *heaviest*, we can simply replace the difference in (5.181) all simply with $m_{\tau_\phi}^2$:

$$\mathcal{L}_{\phi ij} \sim m_{\tau_\phi}^2 K_{\phi ij} \delta\tau_\phi \delta\tau^i \delta\tau^j \tag{5.182}$$

Now we have to work out the decay rate to the volume modulus. To do that, we simply set $i = j = B$ in (5.182). first of all, we keep the whole formula of the volume and compute:

$$\partial_{\tau_\phi} K_{BB} \sim \frac{\sqrt{\tau_\phi}}{\tau_B^{7/2}} \tag{5.183}$$

so that the term in the Lagrangian will be:

$$\mathcal{L}^K_{\phi BB} \sim m^2_{\tau_\phi} \frac{\tau_B \sqrt{\tau_\phi}}{\mathcal{V}^3} \delta\tau_\phi \delta\tau_B^2 \tag{5.184}$$

Upon canonical normalization of the fields, we get a decay amplitude of:

$$\Gamma(\phi \to \mathcal{V}\mathcal{V}) \simeq \Gamma(\phi \to b_B b_B) \simeq \frac{\tau_\phi^{9/2}}{\mathcal{V}^4} M_P \tag{5.185}$$

The exact formula can be retrieved taking care of all the $\mathcal{O}(1)$ factors we neglected here, see appendix D of [45], and we get:

$$\Gamma(\phi \to b_B b_B) \simeq \frac{3\lambda_\phi W_0^3 a_\phi^3 \tau_\phi^{9/2}}{64\pi \mathcal{V}^4} M_P \tag{5.186}$$

To retrieve the decay rate to the small modulus τ_{SM} we follow the same procedure. The result we get is very similar, but enhanced by a factor of 4:

$$\Gamma(\phi \to b_{SM} b_{SM}) \simeq \frac{3\lambda_\phi W_0^3 a_\phi^3 \tau_\phi^{9/2}}{16\pi \mathcal{V}^4} M_P \tag{5.187}$$

We can notice that the decay rate of the inflaton to the SM gauge fields (5.149) is $\Gamma(\phi \to b_B b_B)$ but enhanced by a factor of $8N_g$ where N_g is the number of SM gauge bosons. Therefore, by setting $\Gamma \equiv \Gamma(\phi \to b_B b_B)$ we can finally compute the branching ratio of the inflaton to dark radiation as:

$$B(\phi \to DR) \simeq \frac{(1+4)\Gamma}{(8N_g + 5)\Gamma} \simeq \frac{5}{8N_g} \simeq 0.05 \tag{5.188}$$

where we have used $N_g = 12$. All the other decay rates are subleading and can be ignored. Therefore, we are finally able to get a precise prediction for the difference in the effective number of neutrino species:

$$\Delta N_{eff} \simeq 0.05 \times 6.1 \left(\frac{11}{g_*^4 g_{*,S}^{-3}} \right)^{1/3} \simeq 0.14 \tag{5.189}$$

where we used $g_* = g_{*,S} = 106.75$, since the reheating temperature of our model is roughly 10^3 GeV [49].

6 Conclusions

In this thesis we studied the effects of string-loop corrections to the famous Kähler moduli inflation model in the LVS setting of Type IIB string compactification.
We began with a fast and general overview of string theory, starting from the dynamics and spectrum of bosonic strings up to the inclusion of supersymmetry and the formulation of superstring theory. We studied how the latter solves the problem of the absence of fermions in the spectrum and how it eliminates the tachyonic ground states of bosonic strings, but still predicts the presence of 10 spacetime dimensions. This is dealt with through compactification. We presented the Kaluza-Klein dimensional reduction mechanism and introduced the geometric spaces on which such compactifications take place in the context of String Theory: Calabi-Yau threefolds. After studying the topological and geometrical properties of Calabi-Yau's, we noticed how, upon compactification, many (possibly thousands) moduli fields are produced. At tree level, they represent flat directions of the scalar potential and therefore their VEV is a free parameter. The procedure to fix these VEVs is called moduli stabilization. We have seen how the inclusion of quantized 3-form fluxes on the Calabi-Yau managed to stabilize the complex structure moduli and the dilaton field, but left the Kähler moduli unfixed. To stabilize the latter, we had to include quantum corrections to the Kähler potential and superpotential of the 4d EFT, stemmed from different UV effects. Because of the non-renormalization theorems of supersymmetric theories, we could only introduce non-perturbative corrections to the superpotential, while the Kähler potential receives perturbative corrections. These are of two kinds, the α'-corrections descend from higher-derivative terms of the 10d theory, while g_s-corrections originate from string loops. We showed the form of these loop corrections to the Kähler potential as postulated by the BHP conjecture [20], and studied how these propagated to the scalar potential. We then turned to applying what we found and displayed the two most-used moduli stabilization paradigms: KKLT and LVS.
The main focus was on the latter, since the scalar potential of LVS theories exhibits almost flat directions, which can support a model for slow-roll inflation called Kähler moduli inflation. First we showed the original model and the predictions deriving from the study of its post-inflationary dynamics following [3]. We found the value for the spectral index $n_s = 0.955$ the model predicts, and showed it is below the experimental value [37] by about 2σ.
At this point we introduced loop corrections to the scalar potential for the inflaton field, and argued that they 'spoil' the almost flat direction. However, if the parameter c_{loop} is small enough, inflation is still feasible. We worked out the inflationary parameters and found first of all that the expected range of volumes for this corrected model is much lower than the preferred range of the standard version of Kähler moduli inflation: $\mathcal{V}_{\text{cor}} \sim 10^3 - 10^4 \ll \mathcal{V}_{\text{std}} \sim 10^5 - 10^6$. Then, we turned to analyzing the post-inflationary evolution of this model in two different scenarios. We called Scenario I the case in

which which the inflaton 4-cycle is wrapped by D7-branes supporting a hidden sector, and Scenario II the case in which it is not wrapped by D7-branes at all. We used the recently-found loop-enhanced coupling of the volume modulus to Higgs scalars [45] in order to compute the corresponding decay rate.
In Scenario I the situation was very similar to what happened in the original model, with the exception of a faster decay of the volume, leading to a higher number of e-foldings $N_e^{(SI)} \simeq 53$. On the other hand, in Scenario II we worked out the decay rate of the inflaton field to visible sector gauge fields through the mixing with the loop-stabilized modulus. Eventually, we computed the post-inflationary dynamics of this scenario. The oscillations of the volume modulus never comes to dominate the energy density of the Universe, since it decays in the era still dominated by the radiation produced by the decay of the inflaton. The number of e-foldings we found was surprisingly close to that of Scenario I: $N_e^{(SII)} \simeq 52$.
With these results, we managed to find a solid prediction for the spectral index $n_s \simeq 0.976$ and for the tensor-to-scalar ratio $r \simeq 4 \times 10^{-5}$.
This time, the value of the spectral index is higher with respect to the experimental value by Planck 2018 [37] of about 2σ. However, the Planck analysis does not include Dark Radiation in the cosmological model it uses to fit the data. We showed that our model predicts a non-zero difference in the number of effective neutrino species: $\Delta N_{eff} \simeq 0.14$. This is within bounds, but high enough to make a difference in the expansion of the Universe which is within reach of upcoming CMB experiments [50]. The Planck collaboration performed a similar analysis for a varying value of ΔN_{eff} in 2015 [46], and found that higher values of ΔN_{eff} tend to *increase* the central value for n_s. Therefore, to see whether the model produces an acceptable prediction for the spectral index, one should redo the fit of Planck data inserting by hand our value of ΔN_{eff} (5.189), similarly to what has been done in [51] for Fiber Inflation.
Another interesting development would be to explicitly build a model of a Calabi-Yau supporting loop-corrected blow-up inflation. This would mean to explicitly specify the intersection numbers λ_i, choose a flux vacuum configuration by fixing the 3-form flux quantization yielding a value for W_0, and explicitly characterize the quantum effects stabilizing the Kähler moduli. Such a construction has already been carried out, for example in [31], and one would just have to adapt that model to support a loop-stabilized 4-cycle.
Another interesting and wider direction which might be explored is that of *closed string loop corrections* to the Kähler potential. We mentioned briefly in section 4.3 that from an EFT point of view, some of the effects involving, for example, the graviton running in the loop have already been studied. A more detailed analysis, based on world-sheet calculations of bulk loop corrections should be considered. A possible way to follow may be an approach similar to [19], in which toroidal orientifolds were used as compactification spaces, and the effects found therein then generalized to Calabi-Yau's. These effects, if present, may produce corrections to the Kähler potential comparable in magnitude

to second-order Kaluza-Klein open-string corrections, which, in the absence of winding corrections, are the dominant loop effects considered here.

A Calculations for the LVS Swiss Cheese

Given the Kähler potential and the superpotential

$$K = -2\ln\left[\left(\tau_B^{\frac{3}{2}} - \tau_s^{\frac{3}{2}}\right) + \frac{\xi}{2g_s^{\frac{3}{2}}}\right] \tag{A.1}$$

$$W = W_0 + A_s\, e^{-a_s T_s} \tag{A.2}$$

First we expand the Kähler potential in the volume for $\mathcal{V} \gg 1$ as suggested by (4.21):

$$K \simeq -2\ln\mathcal{V} - \frac{\hat{\xi}}{\mathcal{V}} \tag{A.3}$$

So that we can compute its derivatives:

$$K_B = \frac{1}{2}\frac{\partial K}{\partial \tau_B} = -\frac{3}{2}\sqrt{\tau_B}\left[\frac{1}{\mathcal{V}} - \frac{1}{2}\frac{\hat{\xi}}{\mathcal{V}^2}\right] \tag{A.4}$$

$$K_s = \frac{1}{2}\frac{\partial K}{\partial \tau_s} = \frac{3}{2}\sqrt{\tau_s}\left[\frac{1}{\mathcal{V}} - \frac{1}{2}\frac{\hat{\xi}}{\mathcal{V}^2}\right] \tag{A.5}$$

Supposing that $\tau_B \gg \tau_s$ so that $\mathcal{V} \simeq \tau_B^{\frac{3}{2}}$ the leading order Kähler metric can be written as:

$$K_{ij} = \begin{pmatrix} \frac{3}{4\tau_B^2} & -\frac{9\sqrt{\tau_s}}{8\tau_B^{5/2}} \\ -\frac{9\sqrt{\tau_s}}{8\tau_B^{5/2}} & \frac{3}{8\sqrt{\tau_s}\tau_B^{3/2}} \end{pmatrix} \tag{A.6}$$

and, inverting this, we get:

$$K^{ij} \equiv (K^{-1})^{ij} = \begin{pmatrix} \frac{4\tau_B^2}{3} & 4\tau_B\tau_s \\ 4\tau_B\tau_s & \frac{8\sqrt{\tau_s}\tau_B^{3/2}}{3} \end{pmatrix} \tag{A.7}$$

Now we can compute the scalar potential using the inverse Kähler metric (A.7) and the Kähler derivatives computed from the superpotential (A.2) and the derivatives of the Kähler potential (A.4) and (A.5). Inserting them in the general formula (4.6). After stabilizing the axionic parts of the moduli, the final expression is:

$$V = \frac{8(a_sA_s)^2\sqrt{\tau_s}\, e^{-2a_s\tau_s}}{3\mathcal{V}} - \frac{4a_sA_sW_0\tau_s e^{-a_s\tau_s}}{\mathcal{V}^2} + \frac{3\hat{\xi}|W_0|^2}{4\mathcal{V}^3} \tag{A.8}$$

To minimize this, we impose:

$$\begin{cases} \frac{\partial V}{\partial \tau_s} = 0 \\ \frac{\partial V}{\partial \mathcal{V}} = 0 \end{cases} \tag{A.9}$$

The first equation of (A.9) yields:

$$\frac{W_0}{\mathcal{V}}(1-a_s\tau_s) = \frac{a_s A_s}{3\sqrt{\tau_s}}(1-4a_s\tau_s)e^{-a_s\tau_s} \tag{A.10}$$

We can invert this to get an expression for $e^{-a_s\tau_s}$ at the minimum :

$$e^{-a_s\tau_s} = \frac{3|W_0|}{a_s A_s}\frac{\sqrt{\tau_s}}{\mathcal{V}}\left(\frac{1-a_s\tau_s}{1-4a_s\tau_s}\right) \tag{A.11}$$

Similarly, from the second equation of (A.9), one can get:

$$-4(a_s A_s)^2\sqrt{\tau_s}e^{-2a_s\tau_s} + \frac{12a_s A_s|W_0|\tau_s e^{-a_s\tau_s}}{\mathcal{V}} - \frac{27|W_0|^2\hat{\xi}}{8\mathcal{V}^2} = 0 \tag{A.12}$$

Substituting in (A.12) the volume one can find from (A.10), we get an equation for τ_s, which can be cast as:

$$\tau_s^{\frac{3}{2}} = \frac{3\hat{\xi}}{32}\left[\frac{\left(\frac{1-4a_s\tau_s}{1-a_s\tau_s}\right)^2}{\frac{1-4a_s\tau_s}{1-a_s\tau_s}-1}\right] \tag{A.13}$$

At this point, one can expand the expressions (A.11) and (A.13) in powers of $\varepsilon = \frac{1}{4a_s\tau_s}$. Keeping up to order ε^2, we get:

$$e^{-a_s\tau_s} \simeq \frac{3}{4}\frac{|W_0|}{a_s A_s}\frac{\sqrt{\tau_s}}{\mathcal{V}}(1-3\varepsilon-3\varepsilon^2+\mathcal{O}(\varepsilon^3)) \tag{A.14}$$

$$\tau_s^{\frac{3}{2}} \simeq \frac{\hat{\xi}}{2}(1+2\varepsilon+9\varepsilon^2+\mathcal{O}(\varepsilon^3)) \tag{A.15}$$

Now we shall compute the mass matrix by evaluating the Hessian of the potential (A.8) at the minimum. Let us then compute the second derivative of the potential:

$$\frac{\partial^2 V}{\partial\tau_B^2} = \frac{27|W_0|^2\hat{\xi}}{8\tau_B^{13/2}}(1+2\varepsilon) \tag{A.16}$$

$$\frac{\partial^2 V}{\partial\tau_B\partial\tau_s} = -\frac{9a_s|W_0|^2\hat{\xi}}{4\tau_B^{11/2}}(1-5\varepsilon+4\varepsilon^2) \tag{A.17}$$

$$\frac{\partial^2 V}{\partial\tau_s^2} = \frac{3a_s^2|W_0|^2\hat{\xi}}{2\tau_B^{9/2}}(1-3\varepsilon+6\varepsilon^2) \tag{A.18}$$

We can get the mass matrix setting $\mathcal{V} \simeq \tau_B^{3/2}$:

$$M_{ij} = \frac{1}{2}\frac{\partial^2 V}{\partial\tau^i\partial\tau^j} = \begin{pmatrix} \frac{27|W_0|^2\hat{\xi}}{16\mathcal{V}^{13/3}}[1+2\varepsilon] & -\frac{9a_s|W_0|^2\hat{\xi}}{8\mathcal{V}^{11/3}}[1-5\varepsilon+4\varepsilon^2] \\ -\frac{9a_s|W_0|^2\hat{\xi}}{8\mathcal{V}^{11/3}}[1-5\varepsilon+4\varepsilon^2] & \frac{3a_s^2|W_0|^2\hat{\xi}}{4\mathcal{V}^3}[1-3\varepsilon+6\varepsilon^2] \end{pmatrix} \tag{A.19}$$

where we neglected additional terms which are suppressed by higher powers of $\mathcal{V}$. Notice that the fields τ_B and τ_s are not canonically normalized. As a matter of fact, their kinetic term is given by:

$$\mathcal{L}_K(\tau) \sim K_{ij}\partial\tau^i\partial\tau^j \tag{A.20}$$

Therefore, to get the physical degrees of freedom, we have to canonically normalize them. As a result of the canonical normalization of the fields, the physical state mass matrix becomes:

$$\tilde{M}_{ij} = (K^{-1}M)_{ij} = \frac{a_s\langle\tau_s\rangle|W_0|^2\hat{\xi}}{2\mathcal{V}^3}\begin{pmatrix} -9\left[1-7\varepsilon\right] & -6a_s\mathcal{V}^{2/3}\left[1-5\varepsilon+16\varepsilon^2\right] \\ -\frac{6\mathcal{V}^{1/3}}{\langle\tau_s\rangle^{1/2}}\left[1-5\varepsilon+4\varepsilon^2\right] & \frac{4a_s\mathcal{V}}{\langle\tau_s\rangle^{1/2}}\left[1-3\varepsilon+6\varepsilon^2\right]\end{pmatrix} \tag{A.21}$$

B Mixing in the case of two Small Moduli

In this appendix we work out the mixing of the canonically normalized 4-cycles in the case of 3-moduli Kähler inflation, following [32] and neglecting the loop corrections, since they do not add a significant contribution in our specific case of interest. The volume of the Calabi-Yau has the expression:

$$\mathcal{V} = \alpha\left(\tau_B^{3/2} - \lambda_s\tau_s^{3/2} - \lambda_\phi\tau_\phi^{3/2}\right) \tag{B.1}$$

the Kähler potential has the usual LVS form (4.84) and the superpotential is:

$$W = W_0 + A_s e^{-a_s T_s} + A_\phi e^{-a_\phi T_\phi} \tag{B.2}$$

We can write down the Kähler metric at leading order as:

$$K_{ij} = \frac{9}{8\tau_B^2}\begin{pmatrix} \frac{2}{3} & -\lambda_s\varepsilon_s & -\lambda_\phi\varepsilon_\phi \\ -\lambda_s\varepsilon_s & \frac{\lambda_s}{3\varepsilon_s} & \lambda_s\lambda_\phi\varepsilon_s\varepsilon_\phi \\ -\lambda_\phi\varepsilon_\phi & \lambda_s\lambda_\phi\varepsilon_s\varepsilon_\phi & \frac{\lambda_\phi}{3\varepsilon_\phi}\end{pmatrix} \tag{B.3}$$

where

$$\varepsilon_i = \sqrt{\frac{\tau_i}{\tau_B}} \quad \text{for } i = s,\phi \tag{B.4}$$

which means that $\varepsilon_i \ll 1$. The inverse Kähler metric can therefore easily be computed as:

$$(K^{-1})^{ij} = 4\tau_B^2\begin{pmatrix} \frac{1}{3} & \varepsilon_s^2 & \varepsilon_\phi^2 \\ \varepsilon_s^2 & \frac{2}{3\lambda_s}\varepsilon_s & \varepsilon_s^2\varepsilon_\phi^2 \\ \varepsilon_\phi^2 & \varepsilon_s^2\varepsilon_\phi^2 & \frac{2}{3\lambda_\phi}\varepsilon_\phi\end{pmatrix} \tag{B.5}$$

Now we shall work out the scalar potential in order to get the mass matrix. After axion minimization, it takes the following form:

$$V = \frac{\gamma_s \sqrt{\tau_s} e^{-2a_s\tau_s}}{\mathcal{V}} + \frac{\gamma_\phi \sqrt{\tau_\phi} e^{-2a_\phi\tau_\phi}}{\mathcal{V}} - \frac{\mu_s \tau_s e^{-a_s\tau_s}}{\mathcal{V}^2} - \frac{\mu_\phi \tau_\phi e^{-a_\phi\tau_\phi}}{\mathcal{V}^2} + \frac{\nu}{\mathcal{V}^3} \tag{B.6}$$

where we used:

$$\gamma_i = \frac{8(a_i A_i)^2}{3\alpha\lambda_i} \tag{B.7}$$

$$\mu_i = 4W_0 A_i a_i \tag{B.8}$$

$$\nu = \frac{3\hat{\xi} W_0^2}{4} \tag{B.9}$$

where $\hat{\xi}$ is defined as in (4.61). With this potential, we can work out the VEV of the moduli exactly as we did for the 2-moduli case in Appendix A. The results are:

$$a_i \langle\tau_i\rangle \simeq \left(\frac{4\nu}{J}\right)^{2/3} \tag{B.10}$$

$$\langle\mathcal{V}\rangle = \left(\frac{\gamma_i}{2\mu_i}\right) \sqrt{\langle\tau_i\rangle}\, e^{a_i\langle\tau_i\rangle} \tag{B.11}$$

for $i = s, \phi$ with:

$$J = \sum_{i=s,\phi} \left(\frac{\mu_i^2}{\gamma_i a_i^{3/2}}\right) \tag{B.12}$$

At this stage, we can expand all the moduli around their VEV:

$$\tau_i = \langle\tau_i\rangle + \delta\tau_i \tag{B.13}$$

for $i = B, s, \phi$. Then, the lagrangian at leading order in $\delta\tau_i$ will be:

$$\mathcal{L} = K_{ij}\partial_\mu\delta\tau^i\partial^\mu\delta\tau^j - \langle V\rangle - \frac{1}{2}V_{ij}\delta\tau^i\delta\tau^j + \mathcal{O}(\delta\tau^3) \tag{B.14}$$

Now we shall properly canonically normalize the perturbations around the VEV. We call in general σ_i the canonically normalized fields τ_i [16]. We will have to perform a change of basis, writing:

$$\begin{pmatrix}\delta\tau_B\\ \delta\tau_s\\ \delta\tau_\phi\end{pmatrix} = \vec{v}_B\frac{\delta\sigma_B}{\sqrt{2}} + \vec{v}_s\frac{\delta\sigma_s}{\sqrt{2}} + \vec{v}_\phi\frac{\delta\sigma_\phi}{\sqrt{2}} \tag{B.15}$$

[16] In the main text we will use χ as the canonically normalized volume $\mathcal{V}$, ϕ as the canonically normalized inflaton τ_ϕ, and σ for the canonically normalized small cycle τ_s

such that the Lagrangian (B.14) takes the form:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \delta\sigma_i \partial^\mu \delta\sigma^i - \langle V \rangle - \sum_i \frac{m_i^2}{2}\delta\sigma_i^2 \tag{B.16}$$

This is possible if the vectors $\vec{v}_i$ and the square masses m_i^2 are chosen to be the eigenvectors and eigenvalues of the canonically-normalized mass matrix:

$$M_{ij}^2 = \frac{1}{2}(K^{-1})_i^k V_{kj} \tag{B.17}$$

satisfying an orthonormality condition:

$$K_{kl}(\vec{v}_i)^k(\vec{v}_j)^l = \delta_{ij} \tag{B.18}$$

We can compute the Hessian of the potential evaluated at the minima at leading order in $1/\tau_B$:

$$V_{ij} = \frac{\alpha^{-3}}{\langle\tau_B\rangle^{13/2}} \begin{pmatrix} c - c_s \langle\tau_s\rangle^{3/2} - c_\phi \langle\tau_\phi\rangle^{3/2} & -\frac{4}{27}a_s c_s \langle\tau_B\rangle \langle\tau_s\rangle^{3/2} & -\frac{4}{27}a_\phi c_\phi \langle\tau_B\rangle \langle\tau_\phi\rangle^{3/2} \\ -\frac{4}{27}a_s c_s \langle\tau_B\rangle \langle\tau_s\rangle^{3/2} & \frac{8}{81}a_s^2 c_s \langle\tau_B\rangle^2 \langle\tau_s\rangle^{3/2} & 0 \\ -\frac{4}{27}a_\phi c_\phi \langle\tau_B\rangle \langle\tau_\phi\rangle^{3/2} & 0 & \frac{8}{81}a_\phi^2 c_\phi \langle\tau_B\rangle^2 \langle\tau_\phi\rangle^{3/2} \end{pmatrix} \tag{B.19}$$

where we defined:

$$c = \frac{99\nu}{4} \tag{B.20}$$

$$c_i = \frac{81\mu_i^2}{16\gamma_i} \tag{B.21}$$

for $i = s, \phi$. Then we can get the canonically normalized mass matrix applying (B.17). To be more concise, we are going to set $A_s = A_\phi = \lambda_s = \lambda_\phi = W_0 = 1$, and reinsert them later on.

$$M_{ij}^2 = \frac{\alpha^{-3}}{\langle\tau_B\rangle^{9/2}} \begin{pmatrix} -9(a_s\langle\tau_s\rangle^{5/2} + a_\phi\langle\tau_\phi\rangle^{5/2})(1-7\delta) & 6a_s^2\langle\tau_B\rangle\langle\tau_s\rangle^{5/2}(1-5\delta) & 6a_\phi^2\langle\tau_B\rangle\langle\tau_\phi\rangle^{5/2}(1-5\delta) \\ -6a_s\sqrt{\langle\tau_B\rangle}\langle\tau_s\rangle^2(1-5\delta) & 4a_s^2\langle\tau_B\rangle^{3/2}\langle\tau_s\rangle^2(1-3\delta) & 6a_\phi^2\langle\tau_s\rangle\langle\tau_\phi\rangle^{5/2} \\ -6a_\phi\sqrt{\langle\tau_B\rangle}\langle\tau_\phi\rangle^2(1-5\delta) & 6a_s^2\langle\tau_\phi\rangle\langle\tau_s\rangle^{5/2} & 4a_\phi^2\langle\tau_B\rangle^{3/2}\langle\tau_\phi\rangle^2(1-3\delta) \end{pmatrix} \tag{B.22}$$

We want to find the eigenvalues and eigenvectors of this matrix. To do that, we impose the eigenvalue relation :

$$M^2\vec{v}_i = m_i^2\vec{v}_i \tag{B.23}$$

and the normalization condition (B.18). This yields the eigenvectros:

$$\vec{v}_B = \begin{pmatrix} \langle\tau_B\rangle \\ a_s^{-1} \\ a_\phi^{-1} \end{pmatrix} \quad \vec{v}_s = \begin{pmatrix} \langle\tau_B\rangle^{1/4}\langle\tau_s\rangle^{3/4} \\ \langle\tau_B\rangle^{3/4}\langle\tau_s\rangle^{1/4} \\ \langle\tau_B\rangle^{-3/4}\langle\tau_s\rangle^{7/4} \end{pmatrix} \quad \vec{v}_\phi = \begin{pmatrix} \langle\tau_B\rangle^{1/4}\langle\tau_\phi\rangle^{3/4} \\ \langle\tau_B\rangle^{-3/4}\langle\tau_\phi\rangle^{7/4} \\ \langle\tau_B\rangle^{3/4}\langle\tau_\phi\rangle^{1/4} \end{pmatrix} \tag{B.24}$$

Therefore, from (B.15) we can retrieve the combinations that give rise to the canonically normalized fields:

$$\frac{\delta\tau_B}{\langle\tau_B\rangle} \simeq \mathcal{O}(1)\delta\sigma_B + \mathcal{O}(\epsilon)\delta\sigma_s + \mathcal{O}(\epsilon)\delta\sigma_\phi \tag{B.25}$$

$$\frac{\delta\tau_s}{\langle\tau_s\rangle} \simeq \mathcal{O}\left(\frac{1}{\ln\mathcal{V}}\right)\delta\sigma_B + \mathcal{O}\left(\frac{1}{\epsilon}\right)\delta\sigma_s + \mathcal{O}(\epsilon)\delta\sigma_\phi \tag{B.26}$$

$$\frac{\delta\tau_\phi}{\langle\tau_\phi\rangle} \simeq \mathcal{O}\left(\frac{1}{\ln\mathcal{V}}\right)\delta\sigma_B + \mathcal{O}(\epsilon)\delta\sigma_s + \mathcal{O}\left(\frac{1}{\epsilon}\right)\delta\sigma_\phi \tag{B.27}$$

where:

$$\epsilon = \left(\frac{\langle\tau_s\rangle}{\langle\tau_B\rangle}\right)^{3/4} = \left(\frac{\langle\tau_\phi\rangle}{\langle\tau_B\rangle}\right)^{3/4} \ll 1 \tag{B.28}$$

This means that, as we discovered in the two-mosuli case, each $\delta\tau_i$ is mostly $\delta\sigma_i$, with small mixings among the fields determined by the corresponding components of the respective eigenverctors.

C Loop-Stabilization of a Small Modulus

We want to study the stabilization and mixing of a three-moduli system τ_B, τ_ϕ and τ_s with the usual Swiss-cheese structure of the volume:

$$\mathcal{V} = \tau_B^{3/2} - \lambda_\phi \tau_\phi^{3/2} - \lambda_s \tau_s^{3/2} \tag{C.1}$$

and the usual LVS Kähler potential (4.84). This time, however, we only turn on the non-perturbative effects for τ_ϕ, so that:

$$W = W_0 + A_\phi\, e^{-a_\phi T_\phi} \tag{C.2}$$

Being the Kähler potential formally identical to the one in Appendix B we conclude that also the Kähler metric at leading order will have the same form:

$$K_{ij} = \frac{3}{4\tau_B^2}\begin{pmatrix} 1 & -\frac{3\lambda_\phi\sqrt{\tau_\phi}}{2\sqrt{\tau_B}} & -\frac{3\lambda_s\sqrt{\tau_s}}{2\sqrt{\tau_B}} \\ -\frac{3\lambda_\phi\sqrt{\tau_\phi}}{2\sqrt{\tau_B}} & \frac{\lambda_\phi\sqrt{\tau_B}}{2\sqrt{\tau_\phi}} & \frac{3\lambda_\phi\lambda_s\sqrt{\tau_\phi\tau_s}}{2\tau_B} \\ -\frac{3\lambda_s\sqrt{\tau_s}}{2\sqrt{\tau_B}} & \frac{3\lambda_\phi\lambda_s\sqrt{\tau_\phi\tau_s}}{2\tau_B} & \frac{\lambda_s\sqrt{\tau_B}}{2\sqrt{\tau_s}} \end{pmatrix} \tag{C.3}$$

as well as its inverse:

$$K^{ij} \equiv (K^{-1})^{ij} = \begin{pmatrix} \frac{4\tau_B^2}{3} & 4\tau_B\tau_\phi & 4\tau_B\tau_s \\ 4\tau_B\tau_\phi & \frac{8\tau_B^{3/2}\sqrt{\tau_\phi}}{3\lambda_\phi} & 4\tau_\phi\tau_s \\ 4\tau_B\tau_s & 4\tau_\phi\tau_s & \frac{8\tau_B^{3/2}\sqrt{\tau_s}}{3\lambda_s} \end{pmatrix} \tag{C.4}$$

The form of the potential, this time will be different from the usual LVS one. For the 4-cycles τ_B and τ_ϕ we will have the usual form (upon axion minimization):

$$V_{LVS}(\mathcal{V},\tau_\phi) = \frac{8(A_\phi a_\phi)^2\sqrt{\tau_\phi}\,e^{-2a_\phi\tau_\phi}}{3\lambda_\phi\mathcal{V}} - \frac{4A_\phi a_\phi \tau_\phi e^{-a_\phi\tau_\phi}}{\mathcal{V}^2} + \frac{3\hat{\xi}W_0^2}{4\mathcal{V}^3} \tag{C.5}$$

on the other hand, the other small modulus τ_s will be stabilized by a perturbative potential:

$$V_{\text{loop}}(\mathcal{V},\tau_s) = \frac{W_0^2}{\mathcal{V}^3}\left(\frac{\mu_1}{\sqrt{\tau_s}} - \frac{\mu_2}{\sqrt{\tau_s}-\mu_3}\right) \tag{C.6}$$

where we regard μ_1, μ_2 and μ_3 as constants. Now we stabilize the moduli and expand them around their VEV, as we did in (B.13). The Lagrangian will have the same form as (B.14):

$$\mathcal{L} = K_{ij}\partial_\mu\delta\tau^i\partial^\mu\delta\tau^j - \langle V\rangle - \frac{1}{2}V_{ij}\delta\tau^i\delta\tau^j + \mathcal{O}(\delta\tau^3) \tag{C.7}$$

We now have to diagonalize this by canonically normalizing the fields. First of all, let us compute the Hessian matrix of the potential, V_{ij}:

$$V_{ij} = \begin{pmatrix} \frac{9(11W_0^2(\mu_1\tilde{\mu}+\mu_2\sqrt{\tau_s})+3W_0^2\tilde{\mu}\lambda_\phi\tau_\phi^{3/2}\sqrt{\tau_s})}{4\tilde{\mu}\tau_B^{13/2}\sqrt{\tau_s}} & -\frac{9W_0^2\lambda_\phi a_\phi\tau_\phi^{3/2}}{2\tau_B^{11/2}} & \frac{9W_0^2(\mu_1\tilde{\mu}^2-\mu_2\tau_s)}{4\tilde{\mu}^2\tau_s^{3/2}\tau_B^{11/2}} \\ -\frac{9W_0^2\lambda_\phi a_\phi\tau_\phi^{3/2}}{2\tau_B^{11/2}} & \frac{3W_0^2\lambda_\phi a_\phi\tau_\phi^{3/2}}{\tau_B^{9/2}} & \frac{9\lambda_\phi\sqrt{\tau_\phi}(W_0^2(\mu_2\tau_s-\mu_1\tilde{\mu}^2)-3W_0^2\lambda_s\tau_s^2\tilde{\mu}^2)}{4\tilde{\mu}^2\tau_s^{3/2}\tau_B^6} \\ \frac{9W_0^2(\mu_1\tilde{\mu}^2-\mu_2\tau_s)}{4\tilde{\mu}^2\tau_s^{3/2}\tau_B^{11/2}} & \frac{9\lambda_\phi\sqrt{\tau_\phi}(W_0^2(\mu_2\tau_s-\mu_1\tilde{\mu}^2)-3W_0^2\lambda_s\tau_s^2\tilde{\mu}^2)}{4\tilde{\mu}^2\tau_s^{3/2}\tau_B^6} & \frac{W_0^2(3\mu_1\tilde{\mu}^3-\mu_2(\mu_3-3\sqrt{\tau_s})\tau_s)}{4\tilde{\mu}^3\tau_s^{5/2}\tau_B^{9/2}} \end{pmatrix} \tag{C.8}$$

where $\tilde{\mu} = \mu_3 - \sqrt{\tau_s}$, where all the fields are computed at their minima (not shown here for brevity). Now we want to construct the canonically normalized mass matrix again using (B.17). We write the result as:

$$M_{ij}^2 = \begin{pmatrix} \frac{m_{11}}{\mathcal{V}^3} & \frac{m_{12}}{\mathcal{V}^{7/3}} & \frac{m_{13}}{\mathcal{V}^{7/3}} \\ \frac{m_{21}}{\mathcal{V}^{8/3}} & \frac{m_{22}}{\mathcal{V}^2} & \frac{m_{23}}{\mathcal{V}^3} \\ \frac{m_{31}}{\mathcal{V}^{8/3}} & \frac{m_{32}}{\mathcal{V}^3} & \frac{m_{33}}{\mathcal{V}^2} \end{pmatrix} \tag{C.9}$$

Where:

$$m_{11} = \frac{3W_0^2\left[-6\tilde{\mu}^2\sqrt{\tau_s}\lambda_\phi a_\phi\tau_\phi^{5/2} + 14\tilde{\mu}^2\mu_1 + 11\tilde{\mu}\mu_2\sqrt{\tau_s} - 3\mu_2\tau_s\right]}{2\tilde{\mu}^2\sqrt{\tau_s}} \tag{C.10}$$

$$m_{12} = 6W_0^2\lambda_\phi a_\phi^2\tau_\phi^{5/2} \tag{C.11}$$

$$m_{13} = \frac{W_0^2[6\tilde{\mu}^3\mu_1 - 3\tilde{\mu}\mu_2\tau_s + \mu_2\tau_s(3\sqrt{\tau_s}-\mu_3)]}{2\tilde{\mu}^3\tau_s^{3/2}} \tag{C.12}$$

$$m_{21} = -6W_0^2 a_\phi\tau_\phi^2 \tag{C.13}$$

$$m_{22} = 4W_0^2 a_\phi^2\tau_\phi^2 \tag{C.14}$$

$$m_{23} = -\frac{\tau_\phi[18W_0^2\lambda_s\tilde{\mu}^3\tau_s^2 + W_0^2(\mu_2\tau_s(\mu_3 - 3\sqrt{\tau_s}) - 12\tilde{\mu}^3\mu_1 + 3\tilde{\mu}(2\tilde{\mu}^2\mu_1 + \mu_2\tau_s))]}{2\tilde{\mu}^3\tau_s^{3/2}} \quad (C.15)$$

$$m_{31} = \frac{3W_0^2(\tilde{\mu}^2\mu_1 - \mu_2\tau_s)}{\lambda_s\tilde{\mu}^2\tau_s^2} \quad (C.16)$$

$$m_{32} = \frac{3\lambda_\phi W_0^2\left[2\lambda_s\tau_s^2\tilde{\mu}^2 a_\phi^2\tau_\phi^{5/2} + \sqrt{\tau_\phi}(\mu_2\tau_s - \mu_1\tilde{\mu}^2)\right]}{\lambda_s\tilde{\mu}^2\tau_s} \quad (C.17)$$

$$m_{33} = \frac{W_0^2(3\tilde{\mu}^3\mu_1 + \mu_2\tau_s(3\sqrt{\tau_s} - \mu_3))}{3\lambda_\phi\tilde{\mu}^3\tau_s^2} \quad (C.18)$$

To canonically normalize the fields and diagonalize the mass matrix, we have to find the eigenvalues m_i^2 and the normalized eigenvectors $\vec{v}_i$ of M^2 satisfying the relation (B.18), where $i = B, \phi, s$. Only then we can write:

$$\begin{pmatrix}\delta\tau_B\\ \delta\tau_\phi\\ \delta\tau_s\end{pmatrix} = (\vec{v}_B)\frac{\delta\sigma_B}{\sqrt{2}} + (\vec{v}_\phi)\frac{\delta\sigma_\phi}{\sqrt{2}} + (\vec{v}_s)\frac{\delta\sigma_s}{\sqrt{2}} \quad (C.19)$$

Imposing the conditions we stated above, we find the eigenvalues:

$$m_B^2 = \frac{m_{11}m_{22}m_{33} - m_{13}m_{22}m_{31} - m_{12}m_{21}m_{33}}{\mathcal{V}^3 m_{22}m_{33}} \quad (C.20)$$

$$m_\phi^2 = \frac{m_{22}}{\mathcal{V}^2} \quad (C.21)$$

$$m_s^2 = \frac{m_{33}}{\mathcal{V}^2} \quad (C.22)$$

and the corresponding eigenvectors:

$$\vec{v}_B \simeq \begin{pmatrix}\frac{2\tau_B}{\sqrt{3}}\\ -\frac{2m_{21}}{\sqrt{3}m_{22}}\\ \frac{-2m_{31}}{\sqrt{3}m_{33}}\end{pmatrix} \quad (C.23)$$

$$\vec{v}_\phi \simeq \begin{pmatrix}\frac{4m_{12}\tau_B^{1/4}\tau_\phi^{1/4}}{\sqrt{6\lambda_\phi}m_{22}}\\ -\frac{4\tau_B^{3/4}\tau_\phi^{1/4}}{\sqrt{6\lambda_\phi}}\\ \frac{4(m_{12}m_{31}+m_{22}m_{32})\tau_\phi^{1/4}}{\sqrt{6\lambda_\phi}m_{22}(m_{22}-m_{33})\tau_B^{3/4}}\end{pmatrix} \quad (C.24)$$

$$\vec{v}_s \simeq \begin{pmatrix}\frac{4m_{13}\tau_B^{1/4}\tau_\phi^{1/4}}{\sqrt{6\lambda_s}m_{22}}\\ \frac{4(m_{13}m_{21}+m_{22}m_{33})\tau_s^{1/4}}{\sqrt{6\lambda_s}m_{33}(m_{33}-m_{22})\tau_B^{3/4}}\\ \frac{4\tau_B^{3/4}\tau_s^{1/4}}{\sqrt{6\lambda_s}}\end{pmatrix} \quad (C.25)$$

Notice that, with this stabilization we get that:

$$\frac{m_\phi^2}{m_s^2} = \frac{m_{22}}{m_{33}} \sim a_\phi^2 \tau_\phi^2 \tau_s^2 \gg 1 \tag{C.26}$$

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