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CLASSIFICATION OF ISOMETRIES OF THE HILBERT METRIC

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Contents

Introduction **v**

1 Hilbert geometries **7**

1.1 Convex domains and cones . . . 7

1.1.1 Supporting hyperplanes . . . 9

1.1.2 John's Ellipsoid . . . 10

1.2 The Hilbert metric . . . 13

1.2.1 Definition and basic properties . . . 14

1.2.2 Geodesics . . . 21

1.2.3 Horospheres . . . 22

1.3 Duality . . . 30

1.4 Projective isometries . . . 38

2 Isometries of a Hilbert geometry **49**

2.1 Horofunction compactification . . . 49

2.1.1 Horofunction boundary . . . 50

2.1.2 Detour metric . . . 54

2.2 Birkhoff's version of Hilbert metric . . . 62

2.2.1 Gauge function . . . 62

2.2.2 Funk metric . . . 70

2.3 Busemann points of a Hilbert geometry . . . 79

2.4 Parts of the horofunction boundary . . . 96

2.5 Action of isometries on horofunctions . . . 106

2.5.1 Extension of isometries on the horofunction boundary . . . 107

2.5.2 Gauge-preserving and gauge-reversing maps . . . 109

2.6 The group of isometries . . . 118

Bibliography **131**

List of Figures

1.1 Configuration of ellipsoids considered in the proof of Theorem 1.1.7. . 13
1.2 Configuration of points used for the triangular inequality. 16
1.3 Configurations of lines used for the triangular inequality. 17
1.4 The quadrilateral considered to prove the convexity of metric balls. . 18
1.5 The construction of a metric ball on the left and the resulting metric ball on the right. 19
1.6 First case of the proof of Lemma 1.2.15. 24
1.7 Action of $\mathcal{I}(H, p)$ on Ω. 26
1.8 Algebraic horospheres in a particular affine chart. 29
1.9 Busemann function at a round point. 30
1.10 The pencil of hyperplanes in the hyperbolic case on the left and in the parabolic case on the right. 45
1.11 The cone on the space of directions of a C^1-point and a non-C^1-point. 47

2.1 Geometric interpretation of the gauge. 63
2.2 The gauge for non-proper cones. 64
2.3 Construction used in the proof of Proposition 2.2.8. 65
2.4 Gauge for simplicial cones. 69
2.5 Example of a Funk almost-geodesic and reverse-Funk almost-geodesic. 76
2.6 Configuration used in the proof of Theorem 2.3.1. 80
2.7 Configuration considered in the proof of Proposition 2.3.13. 86
2.8 Configuration of points considered in the proof of Theorem 2.5.11. . . 113
2.9 Configuration of simplexes considered in the proof of Theorem 2.5.11. 113

Introduction

The Hilbert metric was introduced by D. Hilbert in 1894, as a generalization of the Cayley-Klein metric on the hyperbolic space, in a letter [21] addressed to F. Klein. In this letter, Hilbert followed Klein's construction of the projective model of the hyperbolic space, and generalized the construction for generic bounded convex sets rather than ellipsoids.

A Hilbert geometry is a pair $(\Omega, \mathrm{d}_\Omega)$ given by a properly convex domain Ω, that is an open subset of the n-dimensional real projective space $\mathbb{P}^n$, which is also convex and bounded in some affine chart, and the Hilbert metric d_Ω defined on Ω.

In particular, given two distinct points x and y in Ω, there exists a unique projective line passing through them and the intersection of this line with the boundary of Ω defines exactly two points. The Hilbert distance between x and y is defined as the logarithm of the cross-ratio of x and y and the two points of the boundary of Ω obtained in this way.

Many have been interested in finding conditions for a Hilbert geometry to be isometric to the hyperbolic space of the same dimension. Among the literature, notable works come from Y. Benoist [4], J. Benzécri [5]. and H. Busemann and P. Kelly [11].

The thesis is divided in two chapters. In the first one, we present the main objects of interest of a metric space from a geometric point of view, such as geodesics and horospheres, and we begin the study of isometries, which will be ended in the second chapter. A common thread throughout the thesis is the fact that in the case in which the space underlying the Hilbert geometry is strictly convex, i.e. whose boundary contains no segment, there are many analogies with hyperbolic space. For example, it can be proved that in this case there exists a unique geodesic passing through two distinct points. In [10],[11], and [14] some first studies of these geometric objects have been carried out.

In the second part of the first chapter, we study projective isometries of a given Hilbert geometry. A projective isometry is a projective transformation that preserves the domain Ω underlying the geometry. Since every projective transformation preserves the cross-ratio of any quadruple of aligned points, it is clear that these transformations are isometries. The main tool for studying such isometries is the Jordan form of the representative with determinant ± 1 of a projective transformation. Also lots of information can be obtained from the action of these isometries on the set of horospheres. Projective isometries can be classified according to their fixed points. Again, we will remark that the classification of isometries in the case of a strictly convex domain is the same as in the hyperbolic space. In the general case, many more types of dynamics have to be considered, which play precisely on the existence of convex faces in the boundary of the domain. Among those who have dealt with the study of projective isometries we mention [3], [13], and [12].

When working with projective spaces one cannot avoid talking about duality. Even in this context, the transition to the dual space is fundamental for a complete understanding of the subject. In this context, we define the dual of a properly convex domain as the set

$$\Omega^* = \{\psi \in \mathbb{P}^* \mid \psi(x) \neq 0 \ \forall x \in \overline{\Omega}\}.$$

In the second chapter we deal with the isometries of a Hilbert geometry in all their generality, following the work of C. Walsh in [28], [24], and [30]. What stands out in particular is that either the group of the isometries coincides with the group of projective isometries, or the former is generated from the latter by means of a particular map introduced by Vinberg [27]. In the case of symmetric domains, i.e. self-dual and such that the group of projective isometries acts transitively on them, the Vinberg map induces an isometry which is an involution and which generates all the isometries together with the projective isometries. To address this study, we extend the Hilbert metric to a particular compactification of the space on which the isometries act. What is most interesting is that such compactification strongly depends on the structure of the faces of the domain and of its dual. Similarly, the action of a generic isometry is determined precisely by the action on these faces.

Chapter 1

Hilbert geometries

In this chapter, we will introduce the main characters of this thesis, the Hilbert geometries. The space we will mainly work with will be an open, bounded, convex subset of the projective space. We will equip this space with a metric structure by defining the Hilbert metric. It is important to note that we refer to Hilbert geometries because geometric properties, such as Gromov hyperbolicity, are heavily dependent on the shape of the convex set we work with, leading to distinct geometries.

1.1 Convex domains and cones

In this section, we will study convex domains in both affine and projective spaces. The contents of this section are fundamental to the thesis and include many classical notions from convex geometry and convex analysis. For a detailed exploration of the first field, we refer to [6], and for the second one we refer to [19].

In what follows, we will make use of some notions of projective geometry. For further readings, we refer to [6]. Throughout this section and the thesis, we will frequently move from projective spaces to affine spaces, and viceversa. For this reason, we will maintain two different notations when we are in the projective context and when we are in the affine context.

Throughout this thesis, we will denote by $\mathbb{P}^n$ the real n-dimensional projective space. Moreover, we will denote by $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ the natural quotient projection.

Definition 1.1.1. Let Ω be an open subset of $\mathbb{P}^n$. We say that Ω is *convex* if the intersection of Ω with any projective line is a connected set. Additionally, we say that Ω is a *properly convex domain* if it is convex and there exists an affine chart (U, ϕ_U), with $U \subseteq \mathbb{P}^n$ and $\phi_U : U \to \mathbb{R}^n$ a homeomorphism, such that $\Omega \subseteq U$ and $\phi(\Omega)$ is bounded within $\mathbb{R}^n$.

Definition 1.1.2. Let Ω be a properly convex domainand $x \in \partial\Omega$ be a point of its boundary. We say that x is a *strictly convex point* if it is not contained in the relative interior of any segment of $\overline{\Omega}$. We say that x is a C^1*-point* if x is a point of regularity C^1. Moreover, if x is a C^1-point and also strictly convex, then we say that x is a *round point*.

Definition 1.1.3. Let Ω be a properly convex domain. We say that Ω is a *strictly convex domain* when every point of the boundary $\partial\Omega$ is a strictly convex point. Similarly, we say that Ω is a *round domain* when every point of the boundary is a round point.

Let us consider the natural quotient projection $\pi : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{P}^n$. The preimage of a properly convex domain gives rise to a cone in $\mathbb{R}^{n+1}$ defined as follows.

Definition 1.1.4. Let $\mathcal{C}$ be an open, connected, and non-empty subset of $\mathbb{R}^{n+1}$. We say that $\mathcal{C}$ is a *cone* if it is invariant under the action of any positive homothety. Moreover, we say that $\mathcal{C}$ is *proper* if it contains no affine line. We say that a cone is *properly convex* if it is convex and proper.

We want to establish a connection between the projective and vectorial viewpoints. Given a properly convex domain $\Omega \subseteq \mathbb{P}^n$ we consider its preimage $\pi^{-1}(\Omega)$, which is clearly an convex open cone. Furthermore, if Ω is properly convex, then $\pi^{-1}(\Omega)$ does not contain any complete affine line. In this case, $\pi^{-1}(\Omega)$ consists of two disjoint open properly convex cones. Indeed, if $\pi^{-1}(\Omega)$ contains a line ℓ, then the intersection of any hyperplane in $\mathbb{R}^{n+1}$ with ℓ, and consequently with $\pi^{-1}(\Omega)$, is non-empty. Therefore, in any affine chart, Ω contains a point at infinity, making it unbounded.

Definition 1.1.5. Let $\Omega \subseteq \mathbb{P}^n$ be a convex domain. We define the *cone above* Ω as one of the connected components[1] of $\pi^{-1}(\Omega)$, and we denote it as $\mathcal{C}_\Omega$.

On the other hand, given an open cone $\mathcal{C} \subseteq \mathbb{R}^{n+1}$, we define the *domain under* $\mathcal{C}$ as $\Omega_{\mathcal{C}} = \pi(\mathcal{C})$. The projection π gives a correspondence between open cones in $\mathbb{R}^{n+1}$ and domains in $\mathbb{P}^n$, that maps (properly) convex open cones to (properly) convex domains.

[1]When Ω is a properly convex domain $\pi^{-1}(\Omega)$ is made of two connected components, each of which is a properly convex open cone. In what follows, nothing depends of the choice of one of the two components.

1.1.1 Supporting hyperplanes

Let $C \subsetneq A$ be a proper, non-empty, and convex open set in an affine space A. The geometric version of the Hahn-Banach Theorem [Theorem 11.4.1, [6]] asserts that for every affine subspace $L \subsetneq A$ that does not intersect C, there exists an affine hyperplane in A containing L and disjoint from C.

Given the fixed convex set C, each hyperplane H disjoint from C defines two half-spaces, one of those contains the convex and the other disjoint from it. We denote the half-space containing C as the *positive half-space*, denoted by H_+.

Applying the Hahn-Banach Theorem to all points in $A \setminus C$, we conclude that C is the intersection of open half-spaces.

Now, let us focus on a properly convex domain $\Omega \subseteq \mathbb{P}^n$. Consider a point p on the boundary $\partial\Omega$ and look at Ω in an affine chart that contains both Ω and p. Here, we can apply the Hahn-Banach Theorem to the singleton $\{p\}$ and we find that there exists an affine hyperplane in this chart containing p and disjoint from Ω. We refer to the completion in $\mathbb{P}^n$ of such a hyperplane as a *supporting hyperplane* for Ω at p. It always exists a supporting hyperplane for Ω at a point $\mathbb{P}^n$ in its boundary. We will see, in the next section, that it is not always unique.

On the other hand, we can work with the cone $\mathcal{C}_\Omega$ associated with Ω. This cone is convex, so by the Hahn-Banach Theorem, for each ray in the boundary of the cone, there is a (linear) hyperplane that contains the ray and is disjoint from $\mathcal{C}_\Omega$. Such a hyperplane is called *supporting hyperplane* for $\mathcal{C}_\Omega$. It is clear that a supporting hyperplane for Ω at a point p of its boundary is the projectivization of a supporting hyperplane for the cone $\mathcal{C}_\Omega$ at the ray in $\partial\mathcal{C}_\Omega$ corresponding to p.

To conclude this section, we notice that the closure $\overline{\Omega}$ of a properly convex domain Ω can be partitioned into *faces*. A convex subset $F \subseteq \overline{\Omega}$ is a *face* of Ω if for any two points x, y in Ω such that the open line segment from x to y intersects F in at least one point, then the closure of the segment is contained in F. The entire domain Ω and the empty-set are the *improper* faces of Ω, and any other face is a *proper* or *exposed* face of Ω and lies in the boundary.

For a point $x \in \overline{\Omega}$, the *face of* x is defined as the face F_x of maximal dimension that contains x in its relative interior. Notice that if $x \in \overline{\Omega}$ is a point of the boundary, then $F_x \subseteq \partial\Omega$ and can be determined by intersecting $\overline{\Omega}$ with all the supporting hyperplanes at x.

A point of the boundary whose face consists of a single point is called a *vertex*, and a face F contained in the boundary whose projective co-dimension is 1 is called a *facet*.

In the same way we define the *faces* of a convex cone $\mathcal{C}$. We change the notations a little bit, and call *extremal faces* the faces of $\mathcal{C}$ contained in the boundary $\partial\mathcal{C}$, and we call *extremal ray* a face consisting of a single ray. We call *extremal generator* a point contained in an extremal ray.

Notice that the faces of a (properly) convex cone are (properly) convex cones. Finally, we observe that when we consider the projection $\Omega_{\mathcal{C}}$, extremal faces correspond to exposed faces, and extreme rays correspond to vertices.

1.1.2 John's Ellipsoid

In this section, we introduce a useful construction that can be done with a bounded convex domain in $\mathbb{R}^n$. In particular, we want to define the ellipsoid of maximal volume contained in it. This notion can be extended to properly convex domains in $\mathbb{P}^n$, by considering them in an appropriate affine chart. But the definition will depend on the chosen affine chart.

An *ellipsoid* in $\mathbb{R}^n$ is the image of the unit ball B^n centered at 0 under a non-singular affine transformation $\phi : \mathbb{R}^n \to \mathbb{R}^n$. Then $\phi(x) = Ax + b$ with $\det(A) \neq 0$ and $b \in \mathbb{R}^n$. Notice that the matrix A can be chosen to be positive definite. Indeed, if we consider the polar decomposition $A = PU$, where P is positive definite and U is orthogonal, we have that $AB^n + b = PB^n + b$, since $UB^n = B^n$.

The volume of the ellipsoid $E = PB^n + b$ is

$$vol(E) = \det(P)vol(B^n).$$

Therefore, in order to study the volume of an ellipsoid, we have to investigate the behavior of the determinant of positive definite matrices. We need the following lemma to prove the main result of this section.

Lemma 1.1.6 (Minkowski's determinant inequality). *Let P and Q be two positive definite matrices of dimension n. Then*

$$\det(P+Q)^{\frac{1}{n}} \geq \det(P)^{\frac{1}{n}} + \det(Q)^{\frac{1}{n}}$$

and the equality holds if and only if there is a constant c so that $Q = cP$.

In particular if the equality holds and $\det(P) = \det(Q)$, then $P = Q$.

Proof. Since P is positive definite, it has a positive definite square root S. Then

$$\det(P+Q)^{\frac{1}{n}} = \det(S)^{\frac{2}{n}} \det(I + S^{-1}QS^{-1})$$

and

$$\det(P)^{\frac{1}{n}} + \det(Q)^{\frac{1}{n}} = \det(S)^{\frac{2}{n}}(1 + \det(I + S^{-1}QS^{-1}))$$

Let us consider the positive definite matrix $R = S^{-1}QS^{-1}$, to get the thesis it suffices to prove that $\det(I+R)^{\frac{1}{n}} \geq 1 + \det(R)^{\frac{1}{n}}$ and that the equality holds if and only if there exists $c \in \mathbb{R}$ such that $R = cI$.

Let $\lambda_1, \dots, \lambda_n$ the (real and positive) eigenvalues of R, then the two conditions above are equivalent to

$$\Big(\prod_{k=1}^{n}(1+\lambda_k)\Big)^{\frac{1}{n}} \geq 1 + (\lambda_1 \cdot \ \dots \ \cdot \lambda_n)^{\frac{1}{n}},$$

and

$$\lambda_1 = \dots = \lambda_n = c.$$

Now we compute

$$\begin{aligned}
\prod_{k=1}^{n}(1+\lambda_k) &= 1 + \sum_{k=1}^{n}\left(\sum_{1\leq i_1<\dots<i_k\leq n} \lambda_{i_1} \cdot \ \dots \ \cdot \lambda_{i_k}\right) \\
&= 1 + \sum_{k=1}^{n}\left(\binom{n}{k} \sum_{1\leq i_1<\dots<i_k\leq n} \frac{\lambda_{i_1} \cdot \ \dots \ \cdot \lambda_{i_k}}{\binom{n}{k}}\right) \\
&\geq 1 + \sum_{k=1}^{n}\left(\binom{n}{k} \prod_{1\leq i_1<\dots<i_k\leq n} (\lambda_{i_1} \cdot \ \dots \ \cdot \lambda_{i_k})^{\frac{1}{\binom{n}{k}}}\right) \\
&= 1 + \sum_{k=1}^{n}\left(\binom{n}{k} (\lambda_1 \cdot \ \dots \ \cdot \lambda_n)^{\frac{\binom{n-1}{k-1}}{\binom{n}{k}}}\right) \\
&= \sum_{k=0}^{n}\binom{n}{k}(\lambda_1 \cdot \ \dots \ \cdot \lambda_n)^{\frac{k}{n}} \\
&= (1 + (\lambda_1 \cdot \ \dots \ \cdot \lambda_n)^{\frac{1}{n}})^n,
\end{aligned}$$

where the inequality in the third line comes from the one between the arithmetic and geometric means (i.e. given $x_1, \dots, x_m \geq 0$ it holds that $\frac{x_1+\dots+x_m}{m} \geq (x_1 \dots x_m)^{\frac{1}{m}}$ with equality if and only if $x_1 = \dots = x_m$). Thus, it is an equality if and only if for all the choices of two subsets of the same size $\{i_1, \dots, i_k\}, \{j_1, \dots, j_k\} \subseteq \{1, \dots, n\}$ we have that $\lambda_{i_1} \cdot \ \dots \ \cdot \lambda_{i_k} = \lambda_{j_1} \cdot \ \dots \ \cdot \lambda_{j_k}$, this is true if and only if $\lambda_1 = \dots = \lambda_n =: c$. □

We are ready to study the volume of any ellipsoid contained in a properly convex domain and we want to show that only one of them has maximal volume.

Theorem 1.1.7. *Let C be a bounded convex domain in $\mathbb{R}^n$. Then there exists a unique ellipsoid $\mathcal{E}(C)$ of maximal volume contained in C.*

Proof. Let K be the closure of C, we prove that there exists a unique closed ellipsoid $\mathcal{E}(K)$ of maximal volume contained in K. Then, we take the interior of $\mathcal{E}(K)$ to obtain $\mathcal{E}(C)$. Let us consider the set that represents all the closed ellipsoids that are contained in K

$$\mathcal{E} = \{(A, b) \mid A \text{ positive definite matrix of size n}, \, b \in \mathbb{R}^n, \, \overline{AB^n + b} \subseteq K\}.$$

Since K has non-empty interior and it is convex, $\mathcal{E}$ is non-empty and

$$((1-t)A_1 + tA_2, (1-t)b_1 + tb_2) \in \mathcal{E} \text{ for all } (A_1, b_1), (A_2, b_2) \in \mathcal{E} \text{ and } t \in]0, 1[.$$

Moreover, $\mathcal{E}$ is compact with respect to the product topology. The function $\det : (A, b) \mapsto \det(A)$ is continuous. So, there is an element of $\mathcal{E}$ that achieve the maximum of det on $\mathcal{E}$. We want to show that such element is unique.

Let $E_1 = A_1B^n + b_1$ and $E_2 = A_2B^n + b_2$ in $\mathcal{E}$ such that $\det(A_1) = \det(\mathrm{A}_2) = \max_{(A,b)\in\mathcal{E}} \det(A)$. By the convexity of $\mathcal{E}$, the ellipsoid $E_3 = A_3B^n + b_3$, with $A_3 = \frac{1}{2}(A_1 + A_2)$ and $b_3 = \frac{1}{2}(b_1 + b_2)$, is an element of $\mathcal{E}$. We first show that $A_1 = A_2$. By Lemma 1.1.6 we have that

$$\det(A_3)^{\frac{1}{n}} \geq \frac{1}{2}(\det(A_1)^{\frac{1}{n}} + \det(A_2)^{\frac{1}{n}}) = \det(A_1)^{\frac{1}{n}}$$

so, by the maximality of $\det(A_1)$ this is an equality, hence $A_1 = A_2$.

Now, if $b_1 \neq b_2$ K contains the convex hull of $E_1 \cup E_2$. If E_1 is different from E_2, we can assume up to an affine transformation, that the two ellipsoids are $B^n - e_1$ and $B^n + e_1$, as in Figure 1.1, with e_1 the first element of the canonical basis of $\mathbb{R}^3$. Then the convex hull of $E_1 \cup E_2$ contains the ellipsoid $\mathcal{E}_3$ given by

$$\begin{pmatrix} 2 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} B^n,$$

which has volume greater than $vol(E_1)$. This contradicts the maximality of E_1. $\square$

Definition 1.1.8. Let $C \subseteq \mathbb{R}^n$ be a bounded convex domain. The *John's ellipsoid* of C is the ellipsoid $\mathcal{E}(C)$ of maximal volume contained in C.

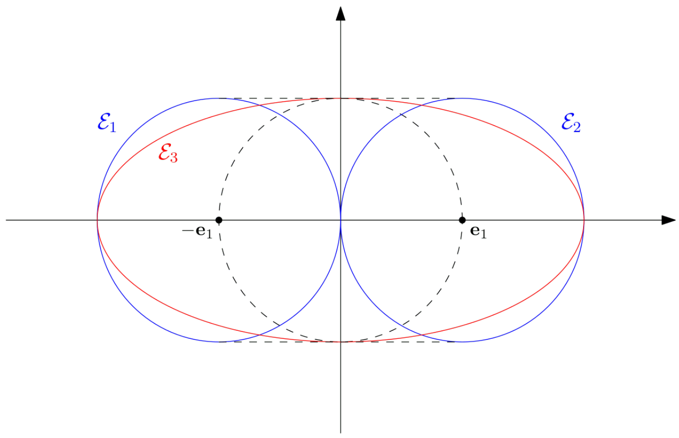


Figure 1.1: Configuration of ellipsoids considered in the proof of Theorem 1.1.7.

We want to prove that there also exists a unique ellipsoid of maximal volume centered on a fixed point, among the ellipsoids contained in a bounded convex domain. The proof of Theorem 1.1.7 can be modified to this case: we have to consider the set $\mathcal{E}_b = \{A \mid A$ positive definite matrix of size n, $\overline{AB^n + b} \subseteq K\}$ instead of $\mathcal{E}$. Then we have the following result.

Theorem 1.1.9. *Let $C \subseteq \mathbb{R}^n$ be a bounded convex domain and $b \in C$ be a point. Then there exists a unique ellipsoid $\mathcal{E}(C, b)$ centered at b, of maximal volume and contained in C.*

Definition 1.1.10. Let $C \subseteq \mathbb{R}^n$ be a bounded convex domain and $b \in C$ be a point. The *John's ellipsoid centered at b of C* is the ellipsoid $\mathcal{E}(C, b)$ of maximal volume centered at b and contained in C.

Remark 1.1.11. In the following, we will use the notion of John's ellipsoid in the context of properly convex domain in $\mathbb{P}^n$, but this notion is very extrinsic, indeed it depends on the affine chart in which we study this domain.

1.2 The Hilbert metric

In this section, we will introduce the concept of Hilbert geometry. We will define a Hilbert geometry as a properly convex domain endowed with its Hilbert metric. We will explore how Hilbert geometries serve as a generalization of hyperbolic geometry. Throughout this section, we will study classical geometric objects, such as geodesics and horospheres, highlighting both the similarities and distinctions between Hilbert and hyperbolic cases. Finally, we will analyse the group of projective isometries, revealing parallels with the group of isometries of the hyperbolic space.

1.2.1 Definition and basic properties

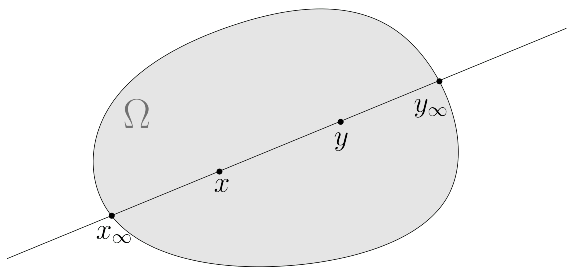
Recall that in the projective space $\mathbb{P}^n$ it is defined the *cross-ratio* of four aligned points. In this thesis, given four aligned points $x_\infty, x, y, y_\infty \in \mathbb{P}^n$, we compute their cross-ratio using the formula:

$$[x_\infty, x, y, y_\infty] = \frac{\|y - x_\infty\|}{\|x - x_\infty\|} \frac{\|x - y_\infty\|}{\|y - y_\infty\|}, \tag{1.1}$$

where the norm appearing in the right-hand side of (1.1) is the Euclidean norm within an affine chart where Ω is bounded. With abuse of notation, x_∞, x, y and y_∞ on the right-hand side denote the images of these four points in $\mathbb{R}^n$ under the homeomorphism given by the considered affine chart.

It is a well-known fact that this formula for the cross-ratio is independent of the specific affine chart used for the computation.

Let Ω be a properly convex domain. Consider two distinct points x and y within Ω. Due to proper convexity, the projective line passing through them intersects the boundary of the domain in exactly two distinct points. We call these two points x_∞ and y_∞, so that in every affine chart where Ω is bounded, x lies between x_∞ and y, and y lies between x and y_∞.



Definition 1.2.1. Let Ω be a properly convex domain. The *Hilbert distance* on Ω is defined as the function $\mathrm{d}_\Omega : \Omega \times \Omega \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$\mathrm{d}_\Omega(x, y) = \begin{cases} \log[x_\infty, x, y, y_\infty] & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

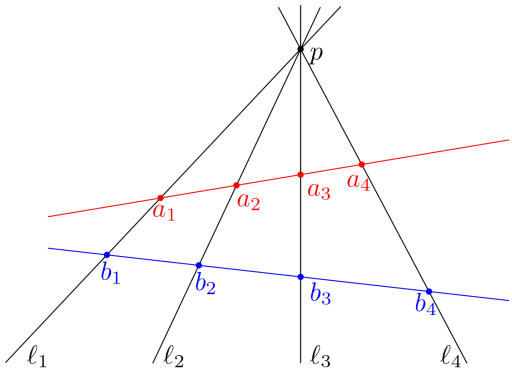
The metric space $(\Omega, \mathrm{d}_\Omega)$ is a *Hilbert geometry.*

Multiplying this expression by $\frac{k}{2}$, where k is a positive integer, one can produce a family of non-isometric distances on a fixed properly convex domain. When we consider the unit disk as domain, and multiply the Hilbert distance by $\frac{1}{2}$, we obtain the projective or Cayley-Klein model for the hyperbolic space $\mathbb{H}^n$.

The next two theorems are classical results of projective geometry, and their proofs can be found in [6].

Theorem 1.2.2 ([6, Corollary 6.1.4]). *Let ℓ_1 and ℓ_2 be two projective lines in a projective space $\mathbb{P}^n$. Given $\{a_1, a_2, a_3, a_4\}$ points in ℓ_1 and $\{b_1, b_2, b_3, b_4\}$ points in ℓ_2, there exists a projective trasformation $f \in \mathrm{PGL}(n+1, \mathbb{R})$ such that $f(a_i) = f(b_i)$ for $i = 1, 2, 3, 4$ if and only if $[a_1, a_2, a_3, a_4] = [b_1, b_2, b_3, b_4]$.*

Theorem 1.2.3 ([6, Proposition 6.5.2]). *Let $\ell_1, \ell_2, \ell_3, \ell_4 \in \mathbb{P}^2$ be four lines passing through a common point $p \in \mathbb{P}^2$. Given two sets of collinear points, $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\}$, such that $a_i, b_i \in \ell_i$, there exists a projective transformation that fixes p, and for $i = 1, 2, 3, 4$, preserves ℓ_i and maps a_i to b_i. Thus, it holds $[a_1, a_2, a_3, a_4] = [b_1, b_2, b_3, b_4]$.*



Proposition 1.2.4. *The function d_Ω is a distance.*

Proof. Since x_∞ is closer to x than to y, it follows that $\frac{\|y-x_\infty\|}{\|x-x_\infty\|} \geq 1$, and the same holds for $\frac{\|x-y_\infty\|}{\|y-y_\infty\|}$. Hence, $\mathrm{d}_\Omega(x, y) \geq 0 \ \forall \ x, y \in \Omega$. Moreover, from the definition, equality holds if and only if $x = y$.

It is clear from the definition that $\mathrm{d}_\Omega(x, y) = \mathrm{d}_\Omega(y, x)$ for all $x, y \in \Omega$.

Let $x, y, z \in \Omega$ be three distinct points. To prove the triangular inequality, we can work in the affine plane defined by these three points.

Within this plane, we consider the construction shown in Figure 1.2, with the following points:

- x_∞ and y_∞ obtained from the intersection of $\partial\Omega$ with the line through x and y, where x_∞ is the point nearest to x and y_∞ the one nearest to y;

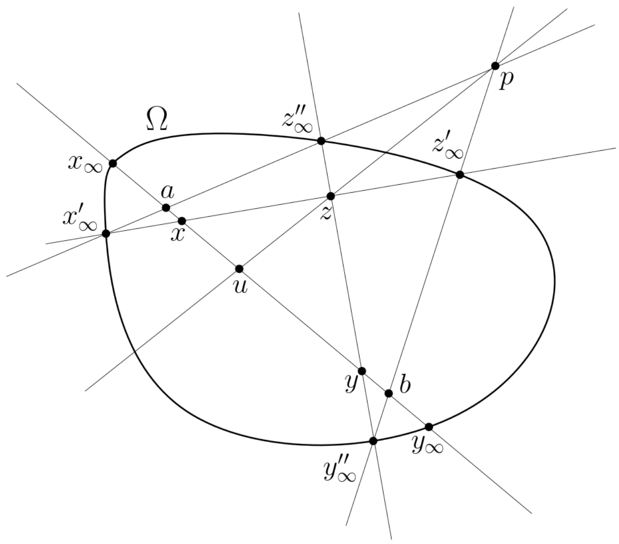


Figure 1.2: Configuration of points used for the triangular inequality.

- x'_∞ and z'_∞ obtained from the intersection of $\partial\Omega$ with the line through x and z, where x'_∞ is the one nearest to x and z'_∞ the one nearest to z;

- y''_∞ and z''_∞ obtained from the intersection of $\partial\Omega$ with the line through y and z, where y''_∞ is the one nearest to y and z''_∞ the one nearest to z;

- p obtained from the intersection (possibly at infinity) of the line through x'_∞ and z''_∞ with the line through y''_∞ and z'_∞;

- u obtained from the intersection of the line through x and y with the line through p and z;

- a obtained from the intersection of the line through x and y with the line through x'_∞ and z''_∞;

- b obtained from the intersection of the line through x and y with the line through y''_∞ and z'_∞.

Applying Theorem 1.2.3 to the two configurations of lines highlighted in Figure 1.3, we get the equations

$$[x'_\infty, x, z, z'_\infty] = [a, x, u, b] \tag{1.2}$$

$$[z''_\infty, z, y, y''_\infty] = [a, u, y, b]. \tag{1.3}$$

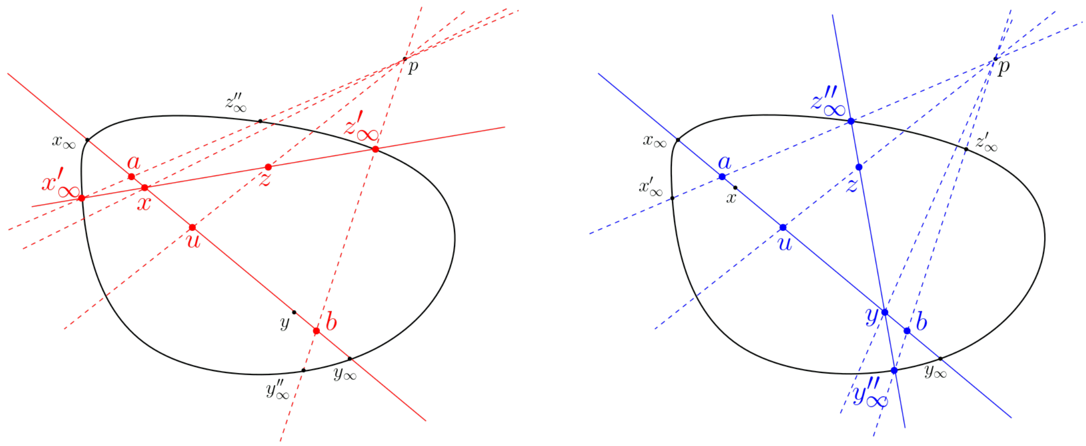


Figure 1.3: Configurations of lines used for the triangular inequality.

Furthermore, since $a,\ x,\ u,\ y$ and b are aligned in this order, we have

$$[a, x, y, b] = [a, x, u, b][a, u, y, b]. \tag{1.4}$$

Combining the equations (1.2) and (1.3) with (1.2), and applying the logarithm on both sides of each equation, we get the equation

$$\log[a, x, y, b] = \log[x'_\infty, x, z, z'_\infty] + \log[z''_\infty, z, y, y''_\infty]. \tag{1.5}$$

Since $[a, x, y, b] \geq [x_\infty, x, y, y_\infty]$, from (1.5), we conclude that the triangular inequality holds for all $x,\ y$ and z in Ω. □

Let us observe that the topology induced on Ω by the Hilbert distance d_Ω is the Euclidean topology in any affine chart. We can express the cross-ratio in 1.1 for the Hilbert distance between two distinct points $x, y \in \Omega$ as

$$[x_\infty, x, y, y_\infty] = \left(1 + \frac{\|y - x_\infty\|}{\|x - x_\infty\|}\right)\left(1 + \frac{\|y - x_\infty\|}{\|y - y_\infty\|}\right). \tag{1.6}$$

Using the notation $D = \mathrm{diam}(\Omega) = \sup\limits_{z,w \in \partial\Omega} \mathrm{d}_\Omega(z, w)$ and $d = \mathrm{d}_\Omega(x, \partial\Omega) = \inf\limits_{z \in \partial\Omega} \mathrm{d}_\Omega(x, z)$, from 1.6 we deduce that if $\|x - y\| \leq d$, then

$$0 \leq 2\log\left(1 + \frac{\|x - y\|}{D}\right) \leq \mathrm{d}_\Omega(x, y) \leq \log\left(\left(1 + \frac{\|x - y\|}{d}\right)\left(1 + \frac{\|x - y\|}{d - \|x - y\|}\right)\right).$$

Therefore, the topology induced by the Hilbert distance is equivalent to the Euclidean topology on Ω.

From this fact, it follows the next proposition.

Proposition 1.2.5. *Let Ω be a properly convex domain. The metric space $(\Omega, \mathrm{d}_\Omega)$ is proper and complete.*

To conclude this section we illustrate some useful properties of the Hilbert distance.

Proposition 1.2.6. *Let Ω and Ω' be two properly convex domains in $\mathbb{P}^n$. If $\Omega' \subseteq \Omega$, then*

$$\mathrm{d}_{\Omega'}(x, y) \geq \mathrm{d}_\Omega(x, y) \quad \forall x, y \in \Omega'.$$

Proof. This relationship is a direct consequence of the definition of the Hilbert distance. For any pair of distinct points, x and y in Ω', the projective line through them intersects $\partial\Omega'$ at two points, namely x'_∞ and y'_∞. This line intersects also $\partial\Omega$ at two points, x_∞ and y_∞, such that in any affine chart where Ω is bounded, these six points are aligned in the following order: $x_\infty, x'_\infty, x, y, y'_\infty, y_\infty$. Note that $x_\infty = x'_\infty$ or $y'_\infty = y_\infty$ might occur.

Using the formula 1.1, we get $[x'_\infty, x, y, y'_\infty] \geq [x_\infty, x, y, y_\infty]$. □

As stated in Proposition 1.2.5, the metric balls of a Hilbert geometry $(\Omega, \mathrm{d}_\Omega)$ are relatively compact. Moreover, they are also convex. Let x be a point in Ω and take $r > 0$. We denote the ball centered at x with radius r as $B_{\mathrm{d}_\Omega}(x, r)$.

Given two distinct points y and z in $B_{\mathrm{d}_\Omega}(x, r)$, let us consider a point p in the line segment joining y and z. We have to show that $\mathrm{d}_\Omega(x, p) \leq r$. As usual, we can suppose that the projective dimension of Ω is 2.

If $\mathrm{d}_\Omega(x, y) = \mathrm{d}_\Omega(x, z)$, we can consider the quadrilateral Ω' with vertices at the points where $\partial\Omega$ intersects the line through x and y and the line through x and z, as shown in Figure 1.4.

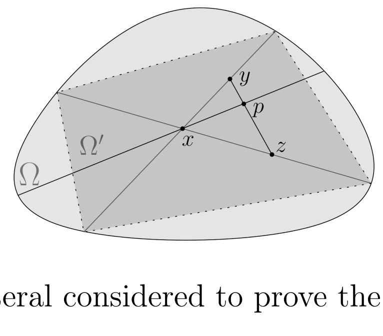


Figure 1.4: The quadrilateral considered to prove the convexity of metric balls.

We will prove in the next example that, in this case, the point p lies on the boundary of the ball $B_{\mathrm{d}_{\Omega'}}(x, \mathrm{d}_\Omega(x, y))$. By Proposition 1.2.6 $\mathrm{d}_\Omega(x, p) \leq \mathrm{d}_{\Omega'}(x, p) = \mathrm{d}_\Omega(x, y) \leq r$.

Since the distance from a distinct point increases when approaching the boundary, when $d_\Omega(x, y) > d_\Omega(x, z)$, we can find a point z' on the half-line from x to z such that $d_\Omega(x, z') = d_\Omega(x, y)$. Then the line through y and z' intersects the line through x and p in a point p' that is farther from x than p. Hence, $d_\Omega(x, p) \leq d_\Omega(x, p')$ and the results follow from the previous case applied to y, z' and p'. Similarly, when $d_\Omega(x, y) < d_\Omega(x, z)$, using the same reasoning we get the thesis.

Moreover, following what we have done to show the convexity of a metric ball, we can observe that metric balls in strictly convex domains are strictly convex.

Exemple 1.2.7. Metric balls of a polygonal domain are polygons with at least the same number of edges of the domain and at most twice this number.

Let Ω be a properly convex polygonal domain in $\mathbb{P}^2$ that is a n-gon in any affine chart where it is bounded, with $n > 2$. We work in such an affine chart. Fix a point $x \in \Omega$ and take $r > 0$.

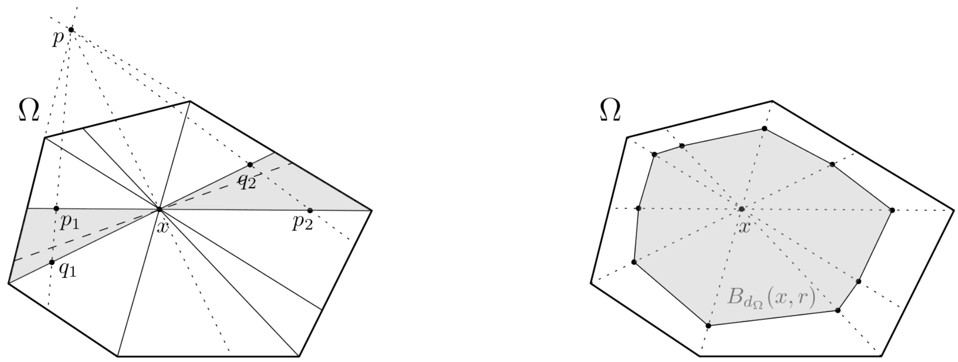


Figure 1.5: The construction of a metric ball on the left and the resulting metric ball on the right.

We can triangulate Ω by considering the triangles that arise from the lines passing through x and each vertex. Let us consider two opposite triangles. On the common line, set the point p_1 such that $d_\Omega(p_1, x) = r$ in the first triangle and the point p_2 such that $d_\Omega(p_2, x) = r$ in the second one. Define p as the intersection point of the lines containing the external edges of these two triangles.

Then, define q_1 as the intersection point of the line through p and p_1 and the internal edge of the first triangle that does not contain p_1. Similarly, define q_2 as the intersection point of the line through p and p_2 and the internal edge of the second triangle that does not contain p_2. An example of this construction is shown in Figure 1.5. This figure also shows that the number of edges of the ball depends on the position of the center of the ball.

By Theorem 1.2.3, all points lying within the segment connecting p_1 and q_1, as well as within the segment connecting p_2 and q_2, have distance r from x. In this way, on each internal edge of the triangulation, we get a point. The polygon whose boundary is the union of the closed segments joining two consecutive of these points is exactly the metric ball $B_{\mathrm{d}_\Omega}(x,r)$. Indeed, when traveling along any ray originating from x, the distance from x increases as the Euclidean distance from x increases.

This example shows that the balls of a triangle are hexagonal. This is an expression of the fact that every n-simplex with its Hilbert metric is isometric to the normed vector space $W = \mathbb{R}^{n+1} / \mathrm{Span}\,(1,\dots,1)$ endowed with the variation norm

$$\|[x]\|_{var} = \max_i x_i - \min_j x_j.$$

This property will be proved in Section 2.2.
Projective isometries

The group $\mathrm{PGL}(n+1,\mathbb{R})$ of projective transformations acts on the real projective space $\mathbb{P}^n$. Each projective transformation preserves the cross-ratio of four aligned points. Thus, any transformation that preserves a properly convex domain is an isometry for the associated Hilbert geometry.

We denote the group of isometries of the $(\Omega, \mathrm{d}_\Omega)$ (Ω, d_Ω) as $\mathrm{Isom}(\Omega, d_\Omega)$. Additionally, we define the group of *projective isometries* as

$$\mathrm{PGL}(\Omega) = \{A \in \mathrm{PGL}(n+1,\mathbb{R}) \mid A\Omega = \Omega\}.$$

This group represents the subgroup of isometries that are projective transformations.

Each linear isomorphism's projective class contains an element with determinant ± 1 and has to preserve the two connected components of $\pi^{-1}(\Omega)$. Indeed, if there exist two distinct points within the same connected component, where the first is mapped into the component to which it belongs, and the second is mapped into the other component, then the images of these two points are in an affine line that has to be contained in the image of the cone, leading to a contradiction. Hence, we have the following isomorphism

$$\mathrm{PGL}(\Omega) \cong \{A \in \mathrm{SL}^{\pm}(n+1,\mathbb{R}) \mid A\mathcal{C}_\Omega = \mathcal{C}_\Omega\} =: \mathrm{SL}^{\pm}(\Omega).$$

Where $\mathrm{SL}^{\pm}(n+1,\mathbb{R})$ is the set of non-singular matrices with determinant ± 1. In what follows we will work with $\mathrm{SL}(n+1,\mathbb{R})$, but the same arguments work also for $\mathrm{SL}^{-}(n+1,\mathbb{R})$.

Before discussing projective isometries, we introduce two important geometric objects that will be useful in the study of the group of projective isometries, geodesics and horospheres.

1.2.2 Geodesics

Given a metric space, we refer to a *geodesic* as a simple curve in which the length of the arc between any two interior points realizes their distance. Moreover, we say that a metric space is *geodesic* if for any pair of distinct point in the space there exists a geodesic joining them. A geodesic space in which there exists a unique geodesic joining any two distinct points is *uniquely geodesic.*

In this section, we study the geodesics of a Hilbert geometry.

Proposition 1.2.8. *Every Hilbert geometry* $(\Omega, \mathrm{d}_\Omega)$ *is a geodesic metric space.*

Proof. Let x and y be two distinct points in Ω. The intersection of the line through them with Ω forms a geodesic. Indeed, for any set of three ordered collinear points, the equality in the triangle inequality holds. □

From now on, we will refer to a *line* in a properly convex domain as the intersection of a projective line with the domain. Therefore, every line in a properly convex domain is a geodesic.

As a direct consequence of the proof of Proposition 1.2.4, we get the following result, which gives a characterization of geodesics in Hilbert geometries.

Proposition 1.2.9. *Let* $(\Omega, \mathrm{d}_\Omega)$ *be a Hilbert geometry and* $x,\ y \in \Omega$ *two distinct points. There exists a point* $z \in \Omega$ *that does not lie in the line through* x *and* y*, such that* $\mathrm{d}_\Omega(x,y) = \mathrm{d}_\Omega(x,z) + \mathrm{d}_\Omega(z,y)$*, if and only if* x_∞ *and* y_∞ *lie in two coplanar segments of the boundary.*

Proof. Using the notations of the proof of Proposition 1.2.4, a point $z \in \Omega$ satisfying $\mathrm{d}_\Omega(x,y) = \mathrm{d}_\Omega(x,z)+\mathrm{d}_\Omega(z,y)$ exists if and only if $a = x_\infty$ and $b = y_\infty$, so the equality in (1.5) holds. In this case, by convexity, a lies in the segment through x'_∞ and z''_∞, and b lies in the segment through y''_∞ and z'_∞. □

Remark 1.2.10. From the previous proposition, given two distinct points, $x, y \in \Omega$, there exists a unique geodesic through them, the straight line, if and only if the two points in the intersection between the line through x and y and the boundary of the domain does not lie in two coplanar segments of the boundary.

This remark yields the following corollary.

Corollary 1.2.11. *Let* $(\Omega, \mathrm{d}_\Omega)$ *be a Hilbert geometry. If* Ω *is strictly convex, then* $(\Omega, \mathrm{d}_\Omega)$ *is uniquely geodesic.*

Definition 1.2.12. Let $(\Omega, \mathrm{d}_\Omega)$ be a Hilbert geometry and $\gamma : \mathbb{R} \to \Omega$ be a geodesic. We say that γ is a *unique-geodesic* if for each pair of distinct points along its image, any other geodesic that contains them in its image is a reparametrization of γ.

1.2.3 Horospheres

In hyperbolic geometry, horospheres play a fundamental role in the study of isometries and their action on the hyperbolic space. The basic idea is of a sphere centered at a point of the boundary. This idea is formalized by considering the level sets of Busemann functions centered on geodesic rays. The boundary of the hyperbolic space is identified with the set of geodesic rays, and the level sets of the Busemann function centered at a geodesic ray depend only on the point of the boundary towards which the geodesic ray approaches.

However, in the context of Hilbert geometry, a remarkable difference arises. Here, the level sets of the Busemann function centered at a geodesic ray may depend, not only on the boundary point, but also on the geodesic ray itself.

Therefore, a new concept, known as *algebraic horospheres*, has been introduced by Cooper, Long, and Tillmann in [12]. This notion represents a significant advancement, particularly in the study of projective isometries.

Let us start with the definition of the Busemann function associated with a geodesic ray. This function has been introduced by H. Busemann in the book [10] in the generalized context of metric spaces. Here, we adjust the definition given in [10] for our aim.

Definition 1.2.13. Let $(\Omega, \mathrm{d}_\Omega)$ be a Hilbert geometry and $r : [0, +\infty[\to \Omega$ be a geodesic ray parametrized by its arc length. The *Busemann function* centered at r is the function $\beta_r : \Omega \to \mathbb{R}$ defined as

$$\beta_r(x) = \lim_{t\to\infty} \mathrm{d}_\Omega(x, r(t)) - t, \quad \forall x \in \Omega. \tag{1.7}$$

We will see below that the limit in (1.7) exists and is finite. Thus, we can define a *horosphere centered at* r as a level set of the Busemann function β_r.

By triangular inequality, the function given by $[0, +\infty[\ni t \mapsto \mathrm{d}_\Omega(x, r(t)) - t \in \mathbb{R}$ is decreasing, indeed we have that

$$\mathrm{d}_\Omega(x, r(t_2) \leq \mathrm{d}_\Omega(x, r(t_1)) + \mathrm{d}_\Omega(r(t_1), r(t_2)) = \mathrm{d}_\Omega(x, r(t_1)) + t_2 - t_1 \; \forall x \in \Omega,\; 0 \leq t_1 \leq t_2,$$

and it is bounded below by $-\mathrm{d}_\Omega(x, r(0))$. Moreover, the Busemann function is continuous since for every $x, y \in \Omega$ it holds that

$$|\beta_r(x) - \beta_r(y)| \leq \mathrm{d}_\Omega(x, y).$$

The next lemma shows that the sets of horospheres centered on distint straight geodesic rays that converges to the same C^1-point coincide. In Section 2.3, we will see that this is true for two arbitrary geodesic rays. In this case we can talk about horospheres centered at a point of the boundary.

Lemma 1.2.14. *Let $(\Omega, \mathrm{d}_\Omega)$ be a Hilbert geometry. Given a C^1-point, $p \in \partial\Omega$, and two straight geodesic rays r_1 and r_2 converging to p, the associated Busemann functions β_{r_1} and β_{r_2} differ by an additive constant.*

Proof. Let $x, y \in \Omega$ two distinct points, then

$$|\beta_{r_1}(x) - \beta_{r_2}(x) - \beta_{r_1}(y) + \beta_{r_2}(y)| \leq 2 \lim_{t\to\infty} \mathrm{d}_\Omega(r_1(p(t)), r_2(q(t))) \tag{1.8}$$

for every two reparametrizations of r_1 and r_2 given by p and q. Indeed, the followings hold

$$\beta_{r_1}(x) - \beta_{r_1}(y) = \lim_{t\to\infty} \mathrm{d}_\Omega(x, r_1(t)) - \mathrm{d}_\Omega(y, r_1(t)) = \lim_{t\to\infty} \mathrm{d}_\Omega(x, r_1(p(t))) - \mathrm{d}_\Omega(y, r_1(p(t))),$$

$$\beta_{r_2}(x) - \beta_{r_2}(y) = \lim_{t\to\infty} \mathrm{d}_\Omega(x, r_2(t)) - \mathrm{d}_\Omega(y, r_2(t)) = \lim_{t\to\infty} \mathrm{d}_\Omega(x, r_2(q(t))) - \mathrm{d}_\Omega(y, r_2(q(t))),$$

$$\mathrm{d}_\Omega(x, r_1(p(t))) - \mathrm{d}_\Omega(x, r_2(q(t))) \leq \mathrm{d}_\Omega(r_1(p(t)), r_2(q(t))),$$

$$\mathrm{d}_\Omega(y, r_2(q(t))) - \mathrm{d}_\Omega(y, r_1(p(t))) \leq \mathrm{d}_\Omega(r_1(p(t)), r_2(q(t))).$$

Combining 1.8 with the next lemma we get that

$$|\beta_{r_1}(x) - \beta_{r_2}(x) - \beta_{r_1}(y) + \beta_{r_2}(y)| = 0$$

hence, the function given by $\Omega \ni x \mapsto \beta_{r_1}(x) - \beta_{r_2}(x)$ is constant, as required. □

Lemma 1.2.15. *Let r_1, r_2 be two straight geodesic rays converging to the same point $p \in \partial\Omega$. If p is a C^1-point, then there is a reparametrization $\tilde{r}_2$ of r_2 such that*

$$\lim_{t\to\infty} \mathrm{d}_\Omega(r_1(t), \tilde{r}_2(t)) = 0.$$

Proof. Let us consider first the case when p is also a strictly convex point.

Let us extend the geodesic rays r_1 and r_2 to geodesics connecting two points of the boundary, and denote q_1 and q_2 as the endpoints respectively of r_1 and r_2 that are distinct from p.

We can work in the affine chart of $\mathbb{P}^2$ containing the two geodesics, whose intersection with Ω is bounded.

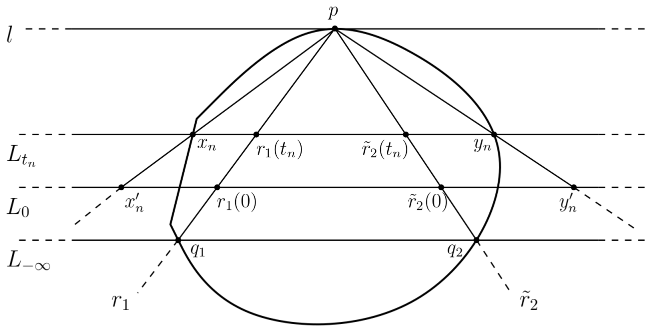


Figure 1.6: First case of the proof of Lemma 1.2.15.

Let ℓ be the supporting line at p and $L_{-\infty}$ be the line through q_1 and q_2. Since Ω is strictly convex, up to a chart's change, the intersection between ℓ and $L_{-\infty}$ is the point at infinity of the affine chart, hence we can assume that ℓ and $L_{-\infty}$ are parallel, as in Figure 1.6.

Now, we can reparametrize r_2 by setting $\tilde{r}_2(t)$ to be the intersection point between the image of r_2 and the line L_t parallel to $L_{-\infty}$, that contains the point $r_1(t)$.

Since r_1 is parametrized by arc length, also r_2 is parametrized by its arc length, by Theorem 1.2.3.

To show that $\lim_{t\to\infty} \mathrm{d}_\Omega(r_1(t), \tilde{r}_2(t)) = 0$, let us consider an arbitrary sequence $(t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ converging to $+\infty$. Let x_n and y_n the two points of $L_{t_n} \cap \partial\Omega$, x_n the one nearest to r_1 and y_n the one nearest to $\tilde{r}_2$. If we denote by x'_n the intersection point between L_0 and the line through p and x_n, and y'_n the intersection point between L_0 and the line through p and y_n, by Theorem 1.2.3 we have that

$$\mathrm{d}_\Omega(r_1(t_n), \tilde{r}_2(t_n)) = \log[x'_n, r_1(0), \tilde{r}_2(0), y'_n].$$

To conclude the proof in this first case, it suffices to observe that $\log[x'_n, r_1(0), \tilde{r}_2(0), y'_n]$ tends to 0, as $n \to \infty$. Indeed, by construction since p is a round point, $\|r_1(0) - x'_n\|$, $\|\tilde{r}_2(0) - y'_n\| \to \infty$, as $n \to \infty$.

For the second case, we suppose that p is a C^1-point and non-strictly convex. We get the reparametrization $\tilde{r}_2$ of r_2 as above. Then we can consider a properly convex domain $\Omega' \subseteq \Omega$ such that both the endpoints of the completions of r_1 and $\tilde{r}_2$

lie in $\partial\Omega'$, and p is a strictly convex and C^1-point and the supporting line at p for Ω and Ω' is the same.

By the first case, $d_{\Omega'}(r_1(t), \tilde{r}_2(t)) \to 0$ as $t \to \infty$. Since $\Omega' \subseteq \Omega$, the same is true for $d_\Omega(r_1(t), \tilde{r}_2(t))$, by Proposition 1.2.6. □

Definition 1.2.16. We define a *Busemann horospere* as a level set of a Busemann function centered at a round point.

Algebraic horospheres

Let Ω be a properly convex domain. The definition of algebraic horosphere depends on a point $p \in \partial\Omega$ and a supporting hyperplane H for Ω at p. Denote with W the linear hyperplane of $\mathbb{R}^{n+1}$ whose projection is H and fix a point $v \in \mathbb{R}^{n+1}$ such that $p = [v]$.

We denote $\mathrm{SL}(H, p)$ the subgroup of $\mathrm{SL}(n+1, \mathbb{R})$ whose matrices preserve W and have v as an eigenvector. Then, we define the subgroup $\mathcal{I}(H, p) \leq \mathrm{SL}(H, p)$ whose elements act as identity on W and preserve each plane containing v.

As usual, if we consider the projective transformation associated with an element of $\mathrm{SL}(H, p)$, we can identify the group $\mathrm{SL}(H, p)$ with its projectivized.

We want to study the group $\mathcal{I}(H, p)$ and its action on Ω. Fix a basis $(v, v_2, \dots, v_n, w)$ of $\mathbb{R}^{n+1}$ given by $v_2, \dots, v_n \in W$ and $w \in \mathbb{R}^{n+1} \setminus W$.

Let $A \in \mathcal{I}(H, p)$. Since A preserves each plane containing v, there exist $t, s \in \mathbb{R}$ such that $Aw = tv + sw$. We want to prove that $s = 1$ and therefore that A is uniquely determined by t.

For this aim, let us consider $w' \in \mathbb{R}^{n+1} \setminus (W \cup \mathrm{Span}(w))$. Since A preserves the plane spanned by v and w', there exist $t', s' \in \mathbb{R}$ such that $Aw' = t'v + s'w'$. Moreover, there exist $a, a_2, \dots, a_n, b \in \mathbb{R}$ such that $w' = av + a_2v_2 + \cdots + a_nv_n + bw$.

On one hand, we have $Aw' = av + a_2v_2 + \cdots + a_nv_n + b(tv + sw)$ and on the other we have $Aw' = (t' + s'a)v + s'(a_2v_2 + \cdots + a_nv_n + bw)$. It follows that $s = s' = 1$ and $t' = bt$. Therefore, we have

$$Aw' = (bt)v + w' \quad \text{and} \quad Aw = tv + w. \tag{1.9}$$

Once chosen $w \in \mathbb{R}^{n+1} \setminus W$, we get the group isomorphism

$$\mathbb{R} \ni t \overset{\cong}{\mapsto} A_t \in \mathcal{I}(H, p), \tag{1.10}$$

where A_t is the unique element of $\mathcal{I}(H,p)$ that satisfies $A_t w = tv + w$.

Remark 1.2.17. From (1.9) we get that if we consider $w' \in \mathbb{R}^{n+1} \setminus W$ different from w, the isomorphism $\mathbb{R} \ni s \mapsto A_s \in \mathcal{I}(H,p)$, where A_s is the unique element of $\mathcal{I}(H,p)$ that satisfies $A_s w' = sv + w'$, differ from (1.10) by a multiplicative constant.

We denote $\mathcal{S}_0 \subseteq \partial\Omega$ the subset obtained from the boundary $\partial\Omega$ by deleting p and all the line segments in $\partial\Omega$ with one endpoint at p.

Let $\mathcal{C}_\Omega$ be the cone above Ω. We want to show that if we choose $w \in \partial\mathcal{C}_\Omega \setminus W$, the isomorphism in (1.10) satisfies the following condition:

$$A_t(S_0) \setminus \{p\} \subseteq \Omega \text{ if } t > 0, \quad A_t(S_0) \setminus \{p\} \subseteq \mathbb{P}^n \setminus \overline{\Omega} \text{ if } t < 0, \quad \text{and } A_0(S_0) \subseteq \partial\Omega.$$

Let $t \in \mathbb{R}$. For each point $x \in \partial\Omega \setminus \{p\}$ such that $x \neq [w]$, we can define the intersection point q between H and the projective line through x and $[w]$.

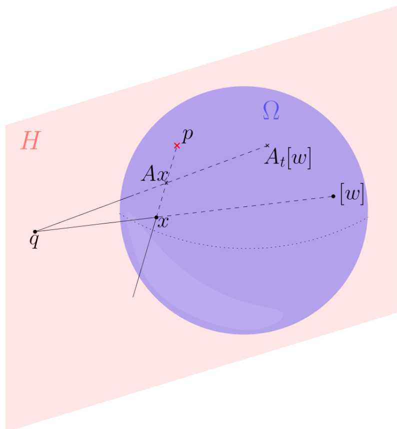


Figure 1.7: Action of $\mathcal{I}(H,p)$ on Ω.

Now, we can restrict to the 3-dimensional projective space containing $x, [w]$ and $A_t[w]$ and we can work in an affine chart where Ω is bounded. In this 3-dimensional affine chart q may be a point at infinity.

Since every element of $\mathcal{I}(H,p)$ preserves each plane passing through v, we know that the projective action of A_t preserves each projective line passing through p. Thus, $A_t(x)$ belongs to both the projective line passing through p and x and the projective line passing through q and $A_t[w]$. The convexity of Ω implies that the intersection point between these two lines is contained in Ω if $A_t[w] \in \Omega$ and it is contained in $\mathbb{P}^n \setminus \Omega$ if $A_t[w] \in \mathbb{P}^n \setminus \Omega$, see Figure 1.7. Finally, from (1.9) we know that $A_t[w] \in \Omega$ if and only if $t > 0$.

We fix such an isomorphism $\mathbb{R} \cong \mathcal{I}(H,p)$, and we denote A_t the element of $\mathcal{I}(H,p)$ corresponding to t, for all $t \in \mathbb{R}$. Moreover, denote w_Ω the point of $\partial\mathcal{C}_\Omega \setminus W$ that satisfies $A_t w_\Omega = tv + w_\Omega$ for all $t \in \mathbb{R}$.

Definition 1.2.18. Let $\Omega \subseteq \mathbb{P}^n$ be a properly convex domain. For each $t > 0$ we denote $\mathcal{S}_t$ the image of $\mathcal{S}_0$ under $A_t \in \mathcal{I}(H,p)$ and we say that $\mathcal{S}_t$ is an *algebraic horosphere centered at* (p, H).

Remark 1.2.19. Although we change $w_\Omega \in \partial\mathcal{C}_\Omega \setminus W$ in the construction of the isomorphism $\mathcal{I}(H,p) \cong \mathbb{R}$, the set $\{\mathcal{S}_t \mid t \in \mathbb{R}_{\geq 0}\}$ does not change, by Remark 1.2.17. So, the *set of algebraic horospheres* is well defined and independent of any choice.

Now, we examine the conjugacy action of $\mathrm{SL}(H,p)$ on $\mathcal{I}(H,p)$. First of all we fix a basis $(v, v_2, \dots, v_n, w)$ of $\mathbb{R}^{n+1}$, with $v_2, \dots, v_n \in W$ and $w \in \mathbb{R}^{n+1} \setminus W$. For each $B \in \mathrm{SL}(H,p)$ the image of w is $Bw = \tilde{w} + \mu w$, for some $\tilde{w} \in W$ and $\mu \neq 0$. Since B preserves W, we get that μ is an eigenvalue for B. Indeed, after a change of basis, the last row of the matrix contains 0 in every column except the last one which contains μ.

Remark 1.2.20. The eigenvalue μ does not depend on the choice of $w \in \mathbb{R}^{n+1} \setminus W$. Let us consider another $w' \in \mathbb{R}^{n+1} \setminus W$. Then, $w' = z + bw$ for some $z \in W$ and $b \in \mathbb{R} \setminus \{0\}$. If $Bw = \tilde{w} + \mu w$, for some $\tilde{w} \in W$ and $\mu \neq 0$, then $Bw' = (Bz + b\tilde{w} - \mu z) + \mu(z + bw)$. Since $Bz + b\tilde{w} - \mu z \in W$, we get that the eigenvalue μ does not depend on the choice of w.

Therefore, the eigenvalue $\mu(B)$ which satisfies that for every $w \in \mathbb{R}^{n+1} \setminus W$ there exists $\tilde{w} \in W$ such that $Bw = \tilde{w} + \mu(B)w$ is well defined. We denote $\lambda(B)$ the eigenvalue of B relative to v. Then, we define the map $\tau : \mathrm{SL}(H,p) \to \mathbb{R}$ given by

$$\tau(B) = \frac{\lambda(B)}{\mu(B)}.$$

Proposition 1.2.21. *The conjugation of* $\mathcal{I}(H,p)$ *into* $\mathrm{SL}(H,p)$ *is an automorphism of* $\mathcal{I}(H,p)$. *Moreover, we have that* $BA_tB^{-1} = A_{\tau(B)t}$ *for all* $B \in \mathrm{SL}(H,p)$ *and* $t \in \mathbb{R}$.

Proof. Fix $t \in \mathbb{R}$ and $B \in \mathrm{SL}(H,p)$. We have to show that BA_tB^{-1} acts as the identity on W and preserves each plane passing through v.

We set $w = w_\Omega$. Then, $A_t w = tv + w$ and there exists $\tilde{w} \in W$ such that $Bw = \tilde{w} + \mu(B)w$.

For each $z \in W$ it holds

$$BA_tB^{-1}(z) = B(A(B^{-1}(z))) = B(B^{-1}(z)) = z,$$

since $B^{-1}(z) \in W$. Thus, $(BA_tB^{-1})_{|W} \equiv \mathrm{Id}_W$.

Now, we have to show that for each $u \in \mathbb{R}^{n+1} \setminus W$ the point $BA_tB^{-1}(u)$ belongs to the plane spanned by u and v. Once $u \in \mathbb{R}^{n+1} \setminus W$ is fixed, there exist $z \in W$ and $b \in \mathbb{R} \setminus \{0\}$ such that $u = z + bw$. Thus, we have

$$\begin{aligned} BA_tB^{-1}(u) &= z + bBAB^{-1}(w) \\ &= z + \frac{b}{\mu(B)} BA_t(w - B^{-1}\tilde{w}) \\ &= z + \frac{b}{\mu(B)} B(tv + w - B^{-1}\tilde{w}) \\ &= z + \frac{b}{\mu(B)} (t\lambda(B)v + \tilde{w} + \mu(B)w - \tilde{w}) \\ &= \Big(\frac{\lambda(B)}{\mu(B)} bt\Big) v + u. \end{aligned} \tag{1.11}$$

Therefore, $BAB^{-1} \in \mathcal{I}(H, p)$.

From (1.11) we get also that $BAB^{-1}(w) = \dfrac{\lambda(B)}{\mu(B)} tv + w$. Hence, $BA_tB^{-1} = A_{\tau(B)t}$. □

Remark 1.2.22. From this proposition, if $B \in \mathrm{SL}(H, p) \cap \mathrm{PGL}(\Omega)$ and $A \in \mathcal{I}(H, p)$ correspond to $t \geq 0$, then we have that

$$B(\mathcal{S}_t) = BA_t(\mathcal{S}_0) = BAB^{-1}(B(\mathcal{S}_0)) = BA_tB^{-1}(\mathcal{S}_0) = \mathcal{S}_{\tau(B)t}.$$

Then any $B \in \mathrm{SL}(H, p) \cap \mathrm{PGL}(\Omega)$ such that $\tau(B) = 1$ preserves each algebraic horosphere.

Definition 1.2.23. Let $\Omega \in \mathbb{P}^n$ be a properly convex domain, $p \in \partial\Omega$ be a point of its boundary and $H \subseteq \mathbb{P}^n$ be a supporting hyperplane for Ω at p. The homomorphism $h : \mathrm{PGL}(\Omega) \cap \mathrm{SL}(H, p) \to \mathbb{R}$ given by $h(B) = \log \tau(B)$ is called *horosphere displacement function centered at* (p, H).

In the next section, we will study the group $\mathrm{PGL}(\Omega)$ and the displacement function associated with the elements which fix the center (p, H).

Proposition 1.2.24. *Let $p \in \partial\Omega$ be a round point and H be the supporting hyperplane for Ω at p. The set of algebraic horospheres centered at (p, H) coincides with* *the set of Busemann horospheres centered at p.*

Proof. Let β_γ be the Busemann function centered on a geodesic ray $\gamma : [0, +\infty[\to \Omega$ parametrized by arc length that converges to p.

To prove the assertion, we look for a particular affine chart, where the point p is a point at infinity, algebraic horospheres are given by vertical translations of the boundary and the image of γ lies on the $\mathbf{e}_n$-axis.

Let $[w] \in \partial\Omega$ the endpoint, different from p, of the geodesic extension of γ to the whole $\mathbb{R}$.

Such affine chart can be obtained taking H as the hyperplane at infinity and identifying $\mathbb{P}^n \setminus H$ with $\mathbb{R}^n$ in such a way that if $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ is the canonical basis, then p corresponds to the direction given by e_n, the image of $\gamma(0)$ is $\mathbf{e}_n$, and $[w]$ corresponds to the origin of the coordinate axes.

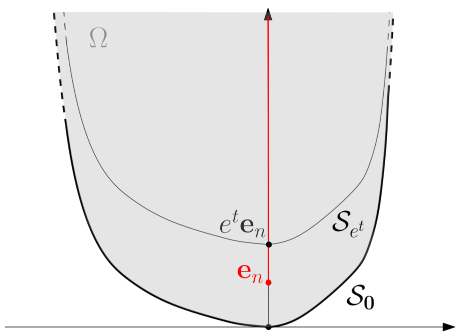


Figure 1.8: Algebraic horospheres in a particular affine chart.

We can consider the isomorphism $\mathbb{R} \ni t \mapsto A_t \in \mathcal{I}(H, p)$ given in (1.10), induced by w. Then, the action of $\mathcal{I}(H, p)$ on $\mathbb{R}^n$ is given by vertical translation $A_t(x) = x + t\mathbf{e}_n$, for every $x \in \mathbb{R}^n$.

On the other hand, the identification with $\mathbb{R}^n$ in the construction of the affine chart above, can be done so that $\gamma(t) = e^t\mathbf{e}_n$, for all $t \geq 0$.

Take $q \in \Omega$ that does not lie on the x_n-axis. Let y be the point on $\partial\Omega$ vertically below q. The straight line ℓ through $\gamma(t)$ and q has two intersections with $\partial\Omega$; denote the one on the q side by $q_\infty(t)$ and the other by $\gamma_\infty(t)$. Denote the x_n-coordinate of q by q_n, of y by y_n, and of $q_\infty(t)$ by $q_{\infty,n}(t)$. The x_n-coordinate of $\gamma_\infty(t)$ is e^{t+s}, for a well defined $s \in \mathbb{R}$. Figure 1.9 shows the considered configuration.

Theorem 1.2.3 implies that $[q_\infty(t), q, \gamma(t), \gamma_\infty(t)] = [(\underline{0}, q_{\infty,n}(t)), (\underline{0}, q_n), \gamma(t), \gamma(t+s)]$.

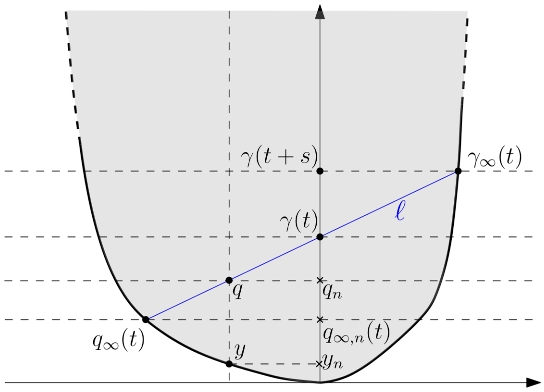


Figure 1.9: Busemann function at a round point.

Thus, we have

$$\begin{aligned}
\mathrm{d}_\Omega(\gamma(t), q) - t &= \log \Big| \frac{e^t - q_{\infty,n}(t)}{q_n - q_{\infty,n}(t)} \cdot \frac{q_n - e^{t+s}}{e^t - e^{t+s}} \cdot e^{-t} \Big| \\
&= \log \Big| \frac{1 - e^{-t} q_{\infty,n}(t)}{q_n - q_{\infty,n}(t)} \cdot \frac{e^{-(t+s)} q_n - 1}{e^{-s} - 1} \Big|.
\end{aligned}$$

Now, as $t \to \infty$, $q_\infty(t)$ tends to y, so $q_{\infty,n}(t) \to y_n$, and $s \to \infty$.

Taking the limit as $t \to \infty$ gives

$$\beta_\gamma(q) = \lim_{t\to\infty} \mathrm{d}_\Omega(\gamma(t), q) - t = -\log |q_n - y_n|.$$

It follows that the level sets of β_γ in this chart are obtained from $\partial\Omega$ by vertical translation. □

1.3 Duality

The concept of duality is essential in the study of geometric objects within projective spaces. We establish a connection between the study of objects in projective spaces and their duals. We denote $(\mathbb{P}^n)^*$ the dual projective space obtained by the projectivization of the vector space $(\mathbb{R}^{n+1})^*$ dual to $\mathbb{R}^{n+1}$. When there is no ambiguity, we will omit the dimension in the notation and use $\mathbb{P}^*$.

Definition 1.3.1. Let $\Omega \subseteq \mathbb{P}^n$ be a properly convex domain. We define the *dual* of Ω as the set

$$\Omega^* = \{\psi \in \mathbb{P}^* \mid \psi(x) \neq 0 \ \forall\, x \in \overline{\Omega}\}.$$

Under the canonical bijection between points of $\mathbb{P}^*$ and projective hyperplanes of $\mathbb{P}^n$, the projective class of a linear functional is mapped to the projectivization of its kernel. Therefore, points of Ω^* correspond to projective hyperplanes disjoint from $\overline{\Omega}$. Moreover, its boundary $\partial\Omega^*$ corresponds to the set of all supporting hyperplanes for Ω.

We will see below that the dual of a properly convex domain is a properly convex domains. Furthermore, this duality is an involution.

Remark 1.3.2. For every $p \in \partial\Omega$, the set of all the supporting hyperplanes for Ω at p forms a face of the boundary of the dual Ω^*. In fact, it corresponds to the set $\{\psi \in \Omega^* \mid \psi(p) = 0\}$, which is clearly convex.

We extend the concept of duality to convex cones in $\mathbb{R}^{n+1}$.

Definition 1.3.3. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a convex open cone. We define the *dual* of $\mathcal{C}$ as the set

$$\mathcal{C}^* = \{\varphi \in (\mathbb{R}^{n+1})^* \mid \varphi(v) > 0 \ \forall v \in \overline{\mathcal{C}} \setminus [0]_{\mathcal{C}}\},$$

where[2] $[0]_{\mathcal{C}} = \{v \in \partial\mathcal{C} \mid -v \in \partial\mathcal{C}\}$.

Given a properly convex domain $\Omega \in \mathbb{P}^n$, we can consider the cone $\mathcal{C}_\Omega$ associated with Ω. When considering the dual cone $\mathcal{C}_\Omega^*$ of $\mathcal{C}_\Omega$, it is clear that its projection $\pi(\mathcal{C}_\Omega^*)$ coincides with Ω^*, and that $\mathcal{C}_\Omega^* = \mathcal{C}_{\Omega^*}$.

On the other hand, given a non-proper convex open cone $\mathcal{C} \subseteq \mathbb{R}^{n+1}$, the dual of its projection $\Omega_{\mathcal{C}}$ is empty, even if $\mathcal{C}^*$ is non-empty.

Remark 1.3.4. Let $\mathcal{C}$ be a convex open cone. Then, its closure is

$$\overline{\mathcal{C}^*} = \bigcap_{v \in \overline{\mathcal{C}}} \{\varphi \in (\mathbb{R}^{n+1})^* \mid \varphi(v) \geq 0\}.$$

Indeed, from the continuity of each linear functional we have

$$\begin{aligned}
\overline{\mathcal{C}^*} &= \overline{\{\varphi \in (\mathbb{R}^{n+1})^* \mid \varphi(v) > 0 \ \forall v \in \overline{\mathcal{C}} \setminus [0]_{\mathcal{C}}\}} \\
&\subseteq \overline{\{\varphi \in (\mathbb{R}^{n+1})^* \mid \varphi(v) > 0 \ \forall v \in \mathcal{C}\}} \\
&\subseteq \{\varphi \in (\mathbb{R}^{n+1})^* \mid \varphi(v) \geq 0 \ \forall v \in \mathcal{C}\} \\
&= \{\varphi \in (\mathbb{R}^{n+1})^* \mid \varphi(v) \geq 0 \ \forall v \in \overline{\mathcal{C}}\}.
\end{aligned}$$

On the other hand, for any $\varphi_0 \in \{\varphi \in (\mathbb{R}^{n+1})^* \mid \varphi(v) \geq 0 \ \forall v \in \overline{\mathcal{C}}\}$ and for any $\psi \in \mathcal{C}^*$ the sequence $\left\{\left(1 - \frac{1}{n}\right)\varphi_0 + \frac{1}{n}\psi\right\}$ is contained in $\mathcal{C}^*$ and converges to φ_0.

Proposition 1.3.5. *Let $\mathcal{C}$ be a properly convex open cone. Then the dual $\mathcal{C}^*$ is also a properly convex open cone.*

[2]The notation $[0]_{\mathcal{C}}$ will be clear in the next chapter.

Proof. First, we observe that $\mathcal{C}^*$ is invariant under positive homotheties and is convex. Moreover, it is non-empty since $\mathcal{C}$ is proper. To show that $\mathcal{C}^*$ is properly convex, let us consider two linear functionals $\varphi_0, \varphi \in (\mathbb{R}^{n+1})^*$ with $\varphi \neq 0$. We have to show that the line $\mathbb{R} \ni t \mapsto \varphi_0 + t\varphi$ is not contained in $\mathcal{C}^*$. Let $v \in \mathcal{C}$ be such that $\varphi(v) \neq 0$. It is clear that we can always find a real value t such that $\varphi_0(v) + t\varphi(v) \leq 0$. So, the line $\varphi_0 + t\varphi$ $t \in \mathbb{R}$ is not contained in $\mathcal{C}^*$.

To complete the proof, we need to show that $\mathcal{C}^*$ is open. From Remark 1.3.4 we know that $\overline{\mathcal{C}^*}$ is intersection of closed half-spaces. Moreover, from Hahn-Banach theorem, we know that there is a supporting hyperplane for $\mathcal{C}^*$ passing through any point of $\partial\mathcal{C}^*$. It follows that $\varphi \in \partial\mathcal{C}^*$ if and only if there is $v \in \partial\mathcal{C} \setminus \{0\}$ such that $\varphi(v) = 0$. Since $\mathcal{C}$ is proper, we have $\mathcal{C}^* = \{\varphi \in (\mathbb{R}^{n+1})^* \mid \varphi(v) > 0 \ \forall v \in \overline{\mathcal{C}} \setminus \{0\}\}$, and hence $\mathcal{C}$ is open. □

Remark 1.3.6. If $\mathcal{C} \subsetneq \mathbb{R}^{n+1}$ is a non-proper convex cone, then, its dual has empty interior. Indeed, $\overline{\mathcal{C}^*}$ is contained in the intersection of the hyperplanes $\{\varphi \in (\mathbb{R}^{n+1})^* \mid \varphi(v) = 0\}$, for $v \in [0]_{\mathcal{C}}$.

Proposition 1.3.7. *Let $\mathcal{C}$ be a properly convex open cone. The canonical isomorphism $\mathbb{R}^{n+1} \to (\mathbb{R}^{n+1})^{**}$ maps $\mathcal{C}$ onto $\mathcal{C}^{**}$.*

Proof. The canonical isomorphism, $\iota : \mathbb{R}^{n+1} \to (\mathbb{R}^{n+1})^{**}$, maps a vector $v \in \mathbb{R}^{n+1}$ to the linear functional $\iota(v) : (\mathbb{R}^{n+1})^* \ni \varphi \mapsto \varphi(v) \in \mathbb{R}$.

If $v \in \mathcal{C}$, then for every $\varphi \in \mathcal{C}^*$, by definition, $\iota(v)(\varphi) = \varphi(v) = 0$. Hence, $\iota(\mathcal{C}) \subseteq \mathcal{C}^{**}$. Let us identify $\iota(\mathcal{C})$ with $\mathcal{C}$. From Proposition 1.3.5, both $\mathcal{C}$ and $\mathcal{C}^*$ are convex open cones. Since $\mathcal{C} \subseteq \mathcal{C}^{**}$, either $\partial\mathcal{C} = \partial\mathcal{C}^{**}$ or $\partial\mathcal{C} \cap \mathcal{C}^{**} \neq \emptyset$. In the first case, we have $\mathcal{C} = \mathcal{C}^{**}$. In the second case, we can consider $v \in \partial\mathcal{C} \cap \mathcal{C}^{**}$, and $H \subseteq \mathbb{R}^{n+1}$ be a supporting hyperplane for $\mathcal{C}$ at v. Then the functional $\varphi \in (\mathbb{R}^{n+1})^*$ defining H vanishes at v and is positive on $\mathcal{C}$. However, $v \in \mathcal{C}^{**}$ implies that $\varphi(v) > 0$, a contradiction. □

Remark 1.3.8. The open cones in $\mathbb{R}^{n+1}$ that are not properly convex are always the interior part of the product of a vector space of dimension k, for some $k \in \mathbb{N}$ and $1 \leq k \leq n$, and a properly convex cone of dimension $n + 1 - k$. In fact, if a cone $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ contains a complete affine line ℓ, then its boundary $\partial\mathcal{C}$ contains the vector line subspace given by translation of ℓ, this follows by convexity and invariance under positive homotheties. Moreover, the boundary of the cone has to contain the linear subspace spanned by all these vector lines. Let k be the dimension of the maximal linear subspace W of $\mathbb{R}^{n+1}$ contained in $\partial\mathcal{C}$. By maximality, if we cut $\mathcal{C}$ with a linear subspace of dimension $n + 1 - k$ that intersects W only at the origin, we obtain a properly convex cone $\mathcal{C}'$. Again, by convexity and homogeneity, $\mathcal{C}$ is isomorphic to the product $W \times \mathcal{C}'$.

Remark 1.3.9. The first part of the proof of Proposition 1.3.5 shows that the dual of a convex open cone is always a properly convex cone. The proper convexity of the cone with which we started gives the openness of the dual cone. If the cone is a product of a vector space of dimension k and a properly convex cone of dimension $n+1-k$, then its dual is a properly convex cone of dimension k.

Corollary 1.3.10. *Let Ω be a properly convex domain in $\mathbb{P}^n$, then the dual the Ω^* is a properly convex domain.*

We want to explore better the link between a properly convex domain and its dual. Let us start by observing that the uniqueness of a supporting hyperplane at a point of the boundary, depends on the regularity of $\partial\Omega$ at the point. In particular, it holds the following proposition.

Lemma 1.3.11 ([19, Theorem 2.7])**.** *Let $f : D \to \mathbb{R}$ be a convex function from a convex open subset D of $\mathbb{R}^n$, and $p \in int(D)$. Then the followings are equivalent:*

1. *f is differentiable at p.*
2. *$Gr(f)$ has a unique supporting hyperplane at $f(p)$, given by $\{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid y = f(p) + \nabla f(p) \cdot (x - p)\}$.*

Lemma 1.3.12 ([19, Theorem 2.8])**.** *Let $f : D \to \mathbb{R}$ be a convex and differentiable function from on an open subset D of $\mathbb{R}^n$. Then f is of class $\mathbb{C}^1$.*

Proposition 1.3.13. *Let Ω be a properly convex domain in $\mathbb{P}^n$. Then $\partial\Omega$ is of class C^1 if and only if Ω^* is strictly convex.*

Proof. Suppose that $\partial\Omega$ is of class C^1. Let $p \in \partial\Omega$ be a point of the boundary. Fix a supporting hyperplane H for Ω at p. In any affine chart that contains the point p, locally we can see $\partial\Omega$ as the graph of a convex C^1 function defined in a neighborhood of p in H. Lemma 1.3.11 implies that there exists a unique supporting hyperplane for Ω at p. Moreover, as we noticed in Remark 1.3.2, the set of all the supporting hyperplanes for Ω at p is a convex face in the boundary $\partial\Omega^*$ of the dual. From the arbitrariness of p, we conclude that Ω^* is strictly convex.

Conversely, suppose that Ω^* is strictly convex. As above, for each point $p \in \partial\Omega$ of the boundary, we can see $\partial\Omega$, in a suitable affine chart, as the graph of a convex function defined on a neighborhood of p in some supporting hyperplane. By

Lemma 1.3.11 we deduce that $\partial\Omega$ is the graph of a function that is differentiable in a neighbourhood of p. Lemma 1.3.12 implies that $\partial\Omega$ is of class C^1 at p. For the arbitrariness of p, we conclude that $\partial\Omega$ is of class C^1. □

Remark 1.3.14. In general, if a point $p \in \partial\Omega$ in the boundary of a properly convex domain is a C^1-point, by Lemma 1.3.11 we can conclude that there exists a unique supporting hyperplane fro Ω at p.

In conclusion of this section, we present the canonical isomorphism between a properly convex domain and its dual, introduced by Vinberg in [27]. The isomorphism is firstly defined between a properly convex open cone and its dual. Then, given a properly convex domain Ω, we can work with the cone $\mathcal{C}_\Omega$ associated with Ω, and project to the domains $\Omega_{\mathcal{C}} = \Omega$ and $\Omega_{\mathcal{C}^*} = \Omega^*$.

What is very useful about this isomorphism is that it is equivariant with respect to the action of the group of projective isometries.

The group of projective isometries $\mathrm{PGL}(\Omega)$ of a properly convex domain Ω acts on the left on Ω through left multiplication, and it acts on the dual Ω^* on the right through right multiplication by the transpose inverse. In particular, we have the induced actions of $\mathrm{SL}(\Omega)$ on $\mathcal{C}_\Omega$ given by

$$\begin{array}{rccll} \rho_{\mathcal{C}_\Omega} : & \mathrm{SL}(\Omega) & \longrightarrow & \mathrm{Aut}(\mathcal{C}_\Omega) & \\ & A & \mapsto & \rho(A) : & \mathcal{C}_\Omega \longrightarrow \mathcal{C}_\Omega \\ & & & & v \mapsto Av \end{array}$$

and on the dual cone $\mathcal{C}^*$ given by

$$\begin{array}{rccll} \rho_{\mathcal{C}^*_\Omega} : & \mathrm{SL}(\Omega) & \longrightarrow & \mathrm{Aut}(\mathcal{C}^*_\Omega) & \\ & A & \mapsto & \rho(A) : & \mathcal{C}^*_\Omega \longrightarrow \mathcal{C}^*_\Omega \\ & & & & \varphi \mapsto A^*\varphi \end{array}$$

Definition 1.3.15. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a properly convex open cone and $d\psi$ a parallel volume form on $(\mathbb{R}^{n+1})^*$. The *characteristic function of $\mathcal{C}$* is

$$\begin{aligned} f : \mathcal{C} &\longrightarrow \mathbb{R} \\ v &\mapsto \int_{\mathcal{C}^*} e^{-\psi(v)} d\psi \end{aligned}$$

This function is well defined up to scaling since two parallel volume forms differ by the multiplication by a constant.

Given $v \in \mathcal{C}$, we want to define a partition of $\mathcal{C}^*$ depending on v. Then, for every

$t \in \mathbb{R}$, let us consider the cross-section

$$\mathcal{C}_v^*(t) = \{\psi \in \mathcal{C}^* \mid \psi(v) = t\} = \mathcal{C}_v^* \cap V_v^*(t)$$

where $V_v^*(t) = \{\psi \in (\mathbb{R}^{n+1})^* \mid \psi(v) = t\}$. For each $v \in \mathcal{C}$ we have

$$\mathcal{C}^* = \bigsqcup_{t>0} \mathcal{C}_v^*(t)$$

and for each $s > 0$ the diffeomorphism $h_s : V_v^*(t) \ni \psi \mapsto s\psi \in V_v^*(st)$ satisfies the condition $h_s \circ h_t = h_{st}$.

Now, we can fix a volume form $d\psi_t$ on $V_v^*(t)$ for all $t \in \mathbb{R}$. Since the pull-back $h_s^* d\psi_{st}$ is a volume form on $V_v^*(t)$ and is a parallel translated of $t^{n-1} d\psi_t$, the decomposition $d\psi = d\psi_t \wedge dt$ yields:

$$\begin{aligned} f(v) &= \int_0^\infty \left(e^{-t} \int_{\mathcal{C}_v^*(t)} d\psi_t \right) dt \\ &= \int_0^\infty \left(e^{-t} t^n \int_{\mathcal{C}_v^*(1)} d\psi_1 \right) dt \\ &= (n)!\ vol(\mathcal{C}_v^*(1)) \\ &= \frac{(n+1)!}{(n+1)^{n+1}}\ vol(\mathcal{C}_v^*(n)). \end{aligned}$$

From Proposition 1.3.5, $\mathcal{C}_v^*(1)$ is bounded. Hence, this computation shows that $f(\mathcal{C}) \subseteq \mathbb{R}$.

The Vinberg's canonical bijection is defined as

$$\begin{aligned} \Phi :\ & \mathcal{C} \longrightarrow \mathcal{C}^* \\ & v \mapsto -d\ \log f(v) \end{aligned} \tag{1.12}$$

where the functional $-d\ \log f(v) : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ can be computed as

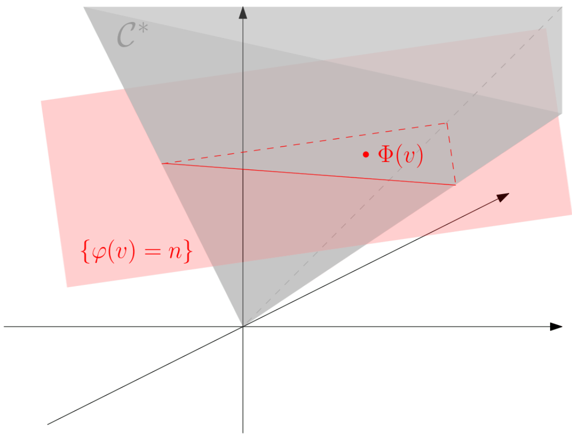
$$-d\ \log f(v)(u) = \frac{\displaystyle\int_{\mathcal{C}^*} \psi(u) e^{-\psi(v)} d\psi}{\displaystyle\int_{\mathcal{C}^*} e^{-\psi(v)} d\psi}, \quad \text{for all } u \in \mathbb{R}^{n+1}. \tag{1.13}$$

The image of Φ is in $\mathcal{C}^*$. Indeed, if $u \in \mathcal{C}$ then $\psi(u) > 0$ for every $\psi \in \mathcal{C}^*$, so $\Phi(v)(u) > 0$.

Moreover, using the same decomposition of the volume form as above, we can compute the image under Φ of a point $v \in \mathcal{C}$ as

$$\begin{aligned}\Phi(v) &= \frac{\displaystyle\int_0^\infty e^{-t}t^n\left(\int_{\mathcal{C}_v^*(1)}\psi_1 d\psi_1\right)dt}{\displaystyle\int_0^\infty e^{-t}t^n\left(\int_{\mathcal{C}_v^*(1)}\psi_1 d\psi_1\right)dt} \\ &= \frac{(n+1)!}{n!}\frac{\displaystyle\int_{\mathcal{C}_v^*(1)}\psi_1 d\psi_1}{\displaystyle\int_{\mathcal{C}_v^*(1)}d\psi_1} \\ &= n+1\ \ \mathrm{centr}(\mathcal{C}_v^*(1)) \\ &= \mathrm{centr}(\mathcal{C}_v^*(n+1)),\end{aligned} \tag{1.14}$$

where $\mathrm{centr}(\mathcal{C}_v^*(n))$ is the *centroid* (or *center of mass*) with respect to the measure defined by $d\psi_n$, of $\mathcal{C}_v^*(n)$ in the affine space defined by the cross-section $\{\psi \in \mathcal{C}^* \mid \psi(v) = n\}$. So, the computation in (1.14) gives us a geometric interpretation of the Vinberg's isomorphism.



Lemma 1.3.16. *The characteristic function f approaches $+\infty$ on $\partial\mathcal{C}$.*

Proof. Let $v_\infty \in \partial\mathcal{C}$ and $(v_n)_{n>0} \subseteq \mathcal{C}$ converging to v_∞. For every $n > 0$ the function $F_n :\ \mathcal{C}^* \longrightarrow \mathbb{R}$ given by

$$F_n(\psi) = e^{-\psi(v_n)}$$

is non-negative and the sequence $(F_n)_{n\in\mathbb{N}}$ converges uniformly on every compact set to the function $F_\infty : \psi \mapsto e^{-\psi(x_n)}$. By Beppi Levi's Theorem

$$\liminf_{n\to+\infty} f(x_n) \geq \int_{\mathcal{C}^*} F_\infty(\psi)d\psi. \tag{1.15}$$

Moreover, if the linear functional ψ_0 represents a supporting hyperplane at v_∞ and $K \in \mathcal{C}^*$ is a closed ball, then $K + \mathbb{R}_+\langle\psi_0\rangle$ is a cylinder with cross-section $K_1 = K \cap \psi_0^{-1}(c)$ for some $c \in \mathbb{R}$ and from (1.15) we have that $\lim_{n\to+\infty} f(v_n)$ exists and

$$\lim_{n\to+\infty} f(v_n) \geq \int_{K+\mathbb{R}_+\langle\psi_0\rangle} F_\infty(\psi)d\psi \geq \int_{K_1}\left(\int_0^\infty dt\right)e^{-\psi(v_\infty)}d\psi = +\infty.$$

□

Lemma 1.3.17. *Let $v \in \mathcal{C}$ be a fixed point. Denote with S_v the level set of $f : \mathcal{C} \to \mathbb{R}$ containing v. Then the tangent space at v to the level set S_v is*

$$T_vS_v = \big(\Phi(v)\big)^{-1}(n).$$

Proof. The linear functional $\Phi(v)$ is parallel to $df(v)$, then each level hyperplane of $\Phi(v)$ is parallel to the tangent space of S_v at v. Since $\Phi(v)(v) = n$, we have that $T_vS_v = \big(\Phi(v)\big)^{-1}(n)$. □

Theorem 1.3.18. *The map $\Phi : \mathcal{C} \to \mathcal{C}^*$ is a bijection.*

Proof. First, we note that, using Hölder inequality, the function $\log f$ is convex. Let $\psi_0 \in \mathcal{C}^*$, we want to show that it exists a unique $v_0 \in \mathcal{C}$ such that $\Phi(v_0) = \psi_0$. Consider the affine hyperplane

$$Q_0 = \{u \in \mathbb{R}^{n+1} \mid \psi_0(u) = n\}$$

then by Lemma 1.3.16 the function $\log f_{|Q_0}$ is convex and approaches to $+\infty$ on $\partial(Q_0 \cap \mathcal{C})$. Thus the function $f_{|Q_0\cap\mathcal{C}}$ has a unique critical point v_0.

By Lemma 1.3.17 v_0 is a critical point if and only if $T_{v_0}S_{v_0} = Q_0$. Thus $\Phi(v_0) = \psi_0$ and no other $v \in \mathcal{C}$ has the same image since there is a unique critical point. □

Let us prove that Vinberg's isomorphism is equivariant with respect to the action of $\mathrm{SL}(\Omega_\mathcal{C})$ given above. Let v be a point in $\mathcal{C}$ and $A \in \mathrm{SL}(\Omega_\mathcal{C})$. From the formula in (1.13), we have that

$$\Phi(v)(u) = \frac{\displaystyle\int_{\mathcal{C}^*} \psi(u)e^{-\psi(v)}d\psi}{\displaystyle\int_{\mathcal{C}^*} e^{-\psi(v)}d\psi}, \tag{1.16}$$

for every $u \in \mathbb{R}^{n+1}$. On the other hand, if we act on $\mathcal{C}$ with A, using the same formula, we have

$$\Phi(Av)(u) = \frac{\displaystyle\int_{\mathcal{C}^*} \psi(u)e^{-\psi(Av)}d\psi}{\displaystyle\int_{\mathcal{C}^*} e^{-\psi(Av)}d\psi}.$$

If we work with the canonical basis and its dual basis, the transposed of A given by $\psi \mapsto \big[v \mapsto \psi(Av)\big]$ is given by the multiplication by A^T. Hence, we have

$$\Phi(Av)(u) = \frac{\displaystyle\int_{\mathcal{C}^*} \psi(u)e^{-A^T\psi(v)}d\psi}{\displaystyle\int_{\mathcal{C}^*} e^{-A^T\psi(v)}d\psi}.$$

Now we can make a change of variables and, since $A^*\mathcal{C}^* = \mathcal{C}^*$, we get

$$\Phi(Av)(u) = \frac{\displaystyle\int_{\mathcal{C}^*} A^*\psi(u)e^{-\psi(v)}d\psi}{\displaystyle\int_{\mathcal{C}^*} e^{-\psi(v)}d\psi} = A^*(\Phi(v))(u),$$

since the integration is a linear operation.

Now, let us consider a properly convex domain $\Omega \subseteq \mathbb{P}^n$. The Vinberg's map $\Phi : \mathcal{C}_\Omega \to \mathcal{C}_\Omega^*$ is well defined. Moreover, from (1.16) we can see that $\Phi(\lambda v) = \frac{1}{\lambda}\Phi(v)$ holds for all $v \in \mathcal{C}_\Omega$ and $\lambda \in \mathbb{R}$. Hence, the map Φ induces a canonical isomorphism $\Psi : \Omega \to \Omega^*$ equivariant with respect to the action of the group of projective isometries $\mathrm{PGL}(\Omega)$.

Remark 1.3.19. Proposition 1.3.5 tells us that there is a canonical isomorphism ι from a properly convex open cone $\mathcal{C}$ to its bidual $\mathcal{C}^{**}$. Hence, by composing the Vinberg map $\Phi : \mathcal{C}^* \to \mathcal{C}^{**}$, with $\iota^{-1} : \mathcal{C}^{**} \to \mathcal{C}$, we obtain a canonical isomorphism $\Phi^* : \mathcal{C}^* \to \mathcal{C}$. The same construction can be done for Ω and Ω^{**}.

In chapter 2, we will see that the map Φ^* is the inverse of Φ only for a specific type of cones.

1.4 Projective isometries

The aim of this section is to classify the elements of the group of projective isometries of a properly convex domain and to study how they act on algebraic horospheres.

Recall that the group of projective isometries is

$$\mathrm{PGL}(\Omega) = \{A \in \mathrm{PGL}(n+1, \mathbb{R}) \mid A\Omega = \Omega\}$$

and it is a subgroup of the group of $\mathrm{Isom}(\Omega, \mathrm{d}_\Omega)$. In what follows we will work with $\mathrm{PGL}(\Omega)$ and with $\mathrm{SL}(\Omega)$, but the same arguments work also for $\mathrm{SL}^-(\Omega)$.

Definition 1.4.1. Let $A \in \mathrm{SL}(\Omega)$ be a projective isometry. We say that A is *elliptic* if it fixes a point in Ω. If A acts freely on Ω, we say that it is *parabolic* if every eigenvalue has modulus 1 and it is *hyperbolic* otherwise.

We can refine the definition above by introducing the *translation length* of an isometry $A \in \mathrm{PGL}(\Omega)$ as

$$t(A) = \inf_{x \in \Omega} \mathrm{d}_\Omega(x, Ax).$$

If $A \in \mathrm{PGL}(\Omega)$ is hyperbolic and the infimum in $t(A)$ is not achieved we say that A *quasi-hyperbolic.*

Lemma 1.4.2. *Let $G \leq \mathrm{PGL}(\Omega)$ be a compact subgroup of projective isometries. Then the elements of G have a common fixed point.*

Proof. Consider the set S of non-empty, compact, convex G-invariant subsets of Ω. This set is non-empty since the convex hull of the G-orbit of any point is compact. Moreover, we can consider the partial order in S given by $A < B$ if $B \supset A$. By Zorn's lemma, there is a maximal element K of S.

Suppose that the relative interior of K is non-empty. Then, we can consider the Hilbert geometry that arise from the relative interior of K. Since K is G-invariant, we can see G as a subset of $\mathrm{PGL}(K)$. Hence, the convex hull of the G-orbit of a point y in the relative interior of K is closed and contained in the relative interior of K. Thus, we found a proper G-invariant subspace of K, contradicting its maximality.

Now, suppose that K has empty relative interior. By hypothesis, K is convex, hence it consists of a single point. It follows that this point is a common fixed point for the action of every element of G. □

Proposition 1.4.3. *Let $A \in \mathrm{PGL}(\Omega)$ be a projective isometry. Then A is elliptic if and only if it is conjugate to an element of $\mathrm{O}(n+1, \mathbb{R})$.*

Proof. If A is elliptic, a point $x \in \Omega$ is fixed by A. Let us consider the image $x^* \in \Omega^*$ of x under Vinberg's bijection. Since this map is equivariant under the action of $\mathrm{PGL}(\Omega)$, the point $x^* \in \Omega^*$ is fixed by the action of A^* and A preserves the affine chart defined by x^*. Then A is an affine transformation and preserves the John's ellipsoid of Ω centered at x in the affine chart defined by x^*. Hence, A is conjugate to an element of $\mathrm{O}(n+1,\mathbb{R})$.

Conversely, if A is conjugate to an element of $\mathrm{O}(n+1,\mathbb{R})$, it generates a compact subgroup of isometries. By Lemma 1.4.2 we have that A has a fixed point in Ω. □

Let us consider now the projective isometries that are not elliptic. We denote by $r(A)$ the *spectral radius* of A and we say that the *power* of a Jordan block of A is $(|\lambda|, k)$ if k is the size of the block and λ is the eigenvalue relative to the block. We introduce a lexicographic order of the block's powers. The *power* of A is the *maximum power* $(r(A), k_{max})$, where k_{max} is the size of the most powerful blocks. Recall that the fixed points of a projective isometry A on a properly convex domain Ω correspond to eigenvectors of A contained in $\mathcal{C}_\Omega$ and relative to real positive eigenvalues. If λ is such an eigenvalue we denote by $\mathrm{Fix}(A,\lambda)$ the intersection of the projection of the relative eigenspace with Ω. The union of these sets is denoted by $\mathrm{Fix}(A)$.

The following result gives some information about the Jordan form of a non-elliptic projective isometry.

Proposition 1.4.4. *Let Ω be a properly convex domain and $A \in \mathrm{SL}(\Omega)$ non-elliptic. Then A has a most powerful Jordan block with real eigenvalue $r = r(A)$ and the set $\mathrm{Fix}(A,r)$ is non-empty and contained in the boundary $\partial\Omega$. Furthermore, if Ω is strictly convex, then there is a unique block of maximum power.*

We need some tools for the proof of Proposition 1.4.4. In particular, to study the behavior of the Jordan blocks of maximum power of $A \in \mathrm{PGL}(\Omega)$, we have to study the *$\omega$-limit set* $\omega(A,U)$ of specific subsets U of $\mathbb{P}^n$. The set $\omega(A,U)$ consists of the accumulation points of the orbits $\{A^n(x) \mid n > 0\}$ of points $x \in U$.

For this aim, let us consider the polynomial

$$h(t) = \frac{q(t)}{\prod\limits_{\lambda \in \mathcal{R}} (t-\lambda)}$$

where $q(t)$ is the minimal polynomial of A and $\mathcal{R}$ is the set of (distinct) eigenvalues relative to the Jordan blocks of maximum power. We denote $E = \mathrm{Im}\, h(A)$ and $K = \mathrm{Ker}\, h(A)$.

Exemple 1.4.5. Let us consider the projective transformation induced by the 8×8

block-matrix

$$M = \begin{pmatrix} 3 & 1 & 0 & & & & & \\ 0 & 3 & 1 & & & & & \\ 0 & 0 & 3 & & & & & \\ & & & 1 & 1 & 0 & & \\ & & & 0 & 1 & 1 & & \\ & & & 0 & 0 & 1 & & \\ & & & & & & 3 & 1 \\ & & & & & & 0 & 3 \end{pmatrix}.$$

The minimal polynomial of M is $q(t) = (3-t)^3(1-t)^3$, we have one Jordan block of maximum power and the eigenvalue relative to this block is 2. Thus, it follows that $h(t) = (3-t)^2(1-t)^3$. Moreover, we have

$$(2I-M)^2 = \begin{pmatrix} 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & & & & & \\ & & & 4 & -4 & 1 & & \\ & & & 0 & 4 & -4 & & \\ & & & 0 & 0 & 4 & & \\ & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \end{pmatrix} \text{ and } (I-M)^3 = \begin{pmatrix} -8 & -12 & -6 & & & & & \\ 0 & -8 & -12 & & & & & \\ 0 & 0 & -8 & & & & & \\ & & & 0 & 0 & 0 & & \\ & & & 0 & 0 & 0 & & \\ & & & 0 & 0 & 0 & & \\ & & & & & & -8 & -12 \\ & & & & & & 0 & -8 \end{pmatrix}.$$

If we denote by $\{v_1, \ldots, v_8\}$ the Jordan basis, we have that $E = \text{Span}\{v_1\}$ and we obtain K removing v_3 from the basis and taking the subspace generated by the remaining 7 vectors, hence $K = \text{Span}\{v_1, v_2, v_4, v_5, v_6, v_7, v_8\}$.

On the other hand, we have for every $n \in \mathbb{N}$

$$M^n = \begin{pmatrix} 3^n & n3^{n-1} & \binom{n}{2}3^{n-2} & & & & & \\ 0 & 3^n & n3^{n-1} & & & & & \\ 0 & 0 & 3^n & & & & & \\ & & & 1 & n & \binom{n}{2} & & \\ & & & 0 & 1 & n & & \\ & & & 0 & 0 & 1 & & \\ & & & & & & 3^n & n3^{n-1} \\ & & & & & & 0 & 3^n \end{pmatrix}.$$

We can divide by $\binom{n}{2}3^{n-2}$ and obtain that

$$[M]^n = \begin{bmatrix} \frac{18}{n(n-1)} & \frac{6}{n-1} & 1 & & & & & \\ 0 & \frac{18}{n(n-1)} & \frac{6}{n-1} & & & & & \\ 0 & 0 & \frac{18}{n(n-1)} & & & & & \\ & & & \frac{2}{3^{n-2}n(n-1)} & \frac{2}{3^{n-2}(n-1)} & \frac{1}{3^{n-2}} & & \\ & & & 0 & \frac{2}{3^{n-2}n(n-1)} & \frac{2}{3^{n-2}(n-1)} & & \\ & & & 0 & 0 & \frac{2}{3^{n-2}n(n-1)} & & \\ & & & & & & \frac{18}{n(n-1)} & \frac{6}{n-1} \\ & & & & & & 0 & \frac{18}{n(n-1)} \end{bmatrix}.$$

Therefore, $[M]^n(x) \to \underline{0}$ as $n \to \infty$ for all $x \in \mathbb{P}(K)$, and if $x \in \mathbb{P}^n \setminus \mathbb{P}(K)$, then $[M]^n(x) \to [v_1]$ as $n \to \infty$. It follows that $\mathbb{P}(E) = \{[v_1]\}$.

Lemma 1.4.6. *Take $A \in \mathrm{PGL}(n+1, \mathbb{R})$ and $W \subseteq \mathbb{P}^n \setminus \mathbb{P}(K)$ with non-empty interior. Then $\omega(A, W)$ is a subset of $\mathbb{P}(E)$ with non-empty interior. Moreover, the action of A on $\mathbb{P}(E)$ is conjugate into the orthogonal group.*

Proof. Let us consider the Jordan decomposition $\oplus_i V_i$ induced by the generalized eigenspaces of the complexified extension $A_{\mathbb{C}}$ of a representative $M \in \mathrm{GL}(n+1, \mathbb{R})$ of A. It is not restrictive to assume that M is a block matrix made of Jordan blocks.

Then $E_{\mathbb{C}}$ is spanned by the eigenvectors from the most powerful blocks and $K_{\mathbb{C}}$ contains the V_i relative to the blocks that do not have maximum power and it contains also the maximal proper invariant subspaces of the V_i whose blocks have maximum power.

Let k be the size of the biggest Jordan block of M relative to an eigenvalue of modulus $r = r(M)$. For every $n \in \mathbb{N}$ we can multiply M^n by $\frac{1}{\binom{n}{k-1} r^{n-k+1}}$, and obtain a matrix equivalent to M^n. So, as we saw in Example 1.4.5, after projectivizing, only the most powerful blocks contribute to the ω–limit $\omega(A, W)$.

Therefore, we consider a $k \times k$ most powerful Jordan block $\Lambda = \lambda I + N$, where I is the $k \times k$ identity matrix and N is a nilpotent $k \times k$ matrix of index k, i.e. $N^k = 0$ and $N^{k-1} \neq 0$. For any $n \geq k$

$$\Lambda^n = \lambda^n I + \binom{n}{1}\lambda^{n-1}N + \dots \binom{n}{k-1}\lambda^{n-k+1}N^{k-1}.$$

If $\lambda \in \mathbb{R}$ then for every vector v in the generalized eigenspace relative to this block, as $n \to \infty$, $\Lambda^n([v])$ converges to the projective class of the relative eigenvector, that belongs to $\mathbb{P}(E)$.

If $\lambda = r(A)e^{i\theta}$, with $\theta \in]0, 2\pi[\setminus \{\pi\}$, we consider at the same time this block and the corresponding block relative to $\overline{\lambda} = r(A)e^{-i\theta}$. Then the subspace E_λ spanned by the two respective eigenvectors is preserved by the action of A and every point of E_λ is an accumulation point of the set of orbits of points in the union of the two generalized eigenspaces relative to the complex-conjugated blocks. Applying this reasoning to every block of $A_\mathbb{C}$ we get the thesis.

Moreover, the action of A on $\mathbb{P}(E)$ is block-diagonal with eigenvalues of the same modulus $r(A)$, and each non-diagonal block represents a rotation, then the action of A on $\mathbb{P}(E)$ is conjugated into the orthogonal group. □

Proof of Proposition 1.4.4. Using the previous notations, from Lemma 1.4.6, we deduce that the set $H = \omega(A, \Omega \setminus \mathbb{P}(K))$ is a subset of $\mathbb{P}(E) \cap \overline{\Omega}$. Therefore, $\overline{\Omega} \cap \mathbb{P}(E)$ is a non-empty, compact, convex set preserved by A. According to the Brouwer Theorem [Corollary 2.15, [20]], there exists a point in $\overline{\Omega} \cap \mathbb{P}(E)$ that is fixed by A. This point corresponds to an eigenvector with a positive eigenvalue. Since the eigenvalues contained in E has modulus $r(A)$, this eigenvalue is $r(A)$. Since A is non-elliptic, we have that $\mathrm{Fix}(A, r(A)) \subseteq \partial\Omega$.

Now, suppose that Ω is strictly convex. Since A is non-elliptic, the set $\mathrm{Fix}(A, r(A))$ consists of a single point $x \in \partial\Omega$. Indeed, two points in $\mathrm{Fix}(A, r(A))$ would correspond to two eigenvectors relative to the same real eigenvalue. Then, the line segment through them would be fixed by A. But, either the segment is contained in the boundary or there is a fixed point in Ω for the action of A. In both cases we would have a contradiction.

Let d be the dimension of E. The dimension of E is the number of Jordan blocks of maximum power. So, we want to show that $d = 1$.

Suppose that $d > 1$. Then, we can consider $\Omega' = \overline{\Omega} \cap \mathbb{P}(E)$. Since $H \subseteq \overline{\Omega} \cap \mathbb{P}(E)$ has non-empty interior in $\mathbb{P}(E)$, then Ω' has dimension $d - 1$. Moreover, since A preserves both Ω and $\mathbb{P}(E)$, we have that $A_{|E}$ is a projective isometry of the Hilbert geometry on Ω'. Since, $A_{|E}$ is conjugated to a matrix in $\mathrm{O}(d)$, it represents an elliptic isometry of Ω', by Proposition 1.4.3. Then, there is a fixed point $y \in \Omega'$. It is clear that y is also a fixed point for A in $\overline{\Omega}$. Since Ω is strictly convex, the point x has to be contained in $\partial\Omega'$, where the boundary is relative to the immersion into $\mathbb{P}(E)$. Then, $x \neq y$. So, x and y are two distinct points in $\mathrm{Fix}(A, r(A))$, and we obtain a contradiction. □

Let Ω be a properly convex domain. If $A \in \mathrm{SL}(\Omega)$ is a projective isometry of parabolic or elliptic type, then the fixed points correspond to the eigenvectors whose eigenvalue is 1. Hence $\mathrm{Fix}(A) = \mathrm{Fix}(A, 1)$ and this subset is connected and convex. If A is a hyperbolic isometry, then both $\lambda_+ = r(A)$ and $\lambda_- = \frac{1}{r(A^{-1})}$ are eigenvalues relative to fixed points in the boundary. The points in $\mathrm{F}_+(A) = \mathrm{Fix}(A, \lambda_+)$ are *attracting* fixed points and points in $\mathrm{F}_-(A) = \mathrm{Fix}(A, \lambda_-)$ are *repelling* fixed points. We denote by $\mathrm{F}_0(A)$ the (possibly empty) union of the remaining fixed sets. Thus $\mathrm{Fix}(A) = \mathrm{F}_-(A) \sqcup \mathrm{F}_0(A) \sqcup \mathrm{F}_+(A)$ is a disjoint union of convex subsets of the boundary.

Remark 1.4.7. Let Ω be a properly convex domain and $A \in \mathrm{PGL}(\Omega)$. Notice that if a point $x \in \partial\Omega$ is fixed by A, then the set of supporting hyperplanes for Ω at x is preserved by A. Since this set is convex and compact, see Remark 1.3.2, from the Brouwer Theorem, we have that A presereves at least one of these supporting hyperplanes. Moreover, the supporting hyperplanes that are preserved by the action of A on Ω, correspond to the points in $\partial\Omega^*$ that are fixed by the action of A^* on Ω^*.

Remark 1.4.8. If Ω is strictly convex and $A \in \mathrm{SL}(\Omega)$ is non-elliptic, then each set of fixed points relative to an eigenvalue of A is a convex subset in the boundary, hence it has to be a single point. Therefore a parabolic isometry of a stricly convex domain has exactly one fixed point in the boundary. Moreover, a hyperbolic isometry has exactly two fixed points, an attracting one and a repelling one. Indeed, from Proposition 1.4.4 there are two such points, each with a supporting hyperplane preserved by the isometry. By strictly convexity these two hyperplanes are distinct and the intersection is a codimension-2 subspace preserved by the isometry and disjoint from $\overline{\Omega}$, hence there are no other fixed points.

Proposition 1.4.9. *Let Ω be a properly convex domain. If $A \in \mathrm{SL}(\Omega)$ is parabolic, then the size of the blocks of maximum power is odd and at least 3.*

Proof. Let us consider the Jordan block of maximum power relative to 1. Denote the size of this block by k. Then the block is $\Lambda = I + N$, where I is the $k \times k$ identity matrix and N is a nilpotent $k \times k$ matrix of index k, i.e. $N^k = 0$ and $N^{k-1} \neq 0$. For any $n \geq k$

$$\Lambda^n = I + \binom{n}{1} N + \dots \binom{n}{k-1} N^{k-1}.$$

Given $x \in \Omega \setminus \mathrm{Ker}(N^{k-1})$, let us consider $v \in \mathcal{C}_\Omega$ such that $x = [v]$, then if $k-1$ is odd

$$\lim_{n \to -\infty} \Lambda^n v = - \lim_{n \to +\infty} \Lambda^n v$$

and $\partial\mathcal{C}_\Omega$ contains two opposite points, contradicting the proper convexity. Hence k is odd. Moreover, if $k = 1$, every Jordan block of A would have size 1 and A would be in $\mathrm{O}(n+1)$, hence elliptic. □

Proposition 1.4.10. *Let* Ω *be a properly convex domain and* $A \in \mathrm{SL}(\Omega)$ *a non-elliptic isometry. Then there is a pencil of hyperplanes that is preserved by* A *whose intersection with* Ω *is a foliation and no leaf is stabilized by* A.

Proof. Thanks to the characterization of the dual of a properly convex domain, it is equivalent to find a foliation that satisfies the request of the statement and to find a projective line $\ell \subseteq \mathbb{P}^n$ in the dual projective space such that ℓ intersects $\overline{\Omega^*}$ and the action of A^* preserves $\ell \cap \overline{\Omega^*}$ and has no fixed points within $\mathbb{P}^n \setminus \Omega^*$.

The first hypothesis on ℓ implies that the center of the pencil, i.e. the intersection of all its hyperplanes, does not intersect Ω. Then, any two hyperplanes of the pencil defined by ℓ do not intersect within Ω. Therefore, the intersection of the pencil with Ω foliates Ω, as in Figure 1.10.

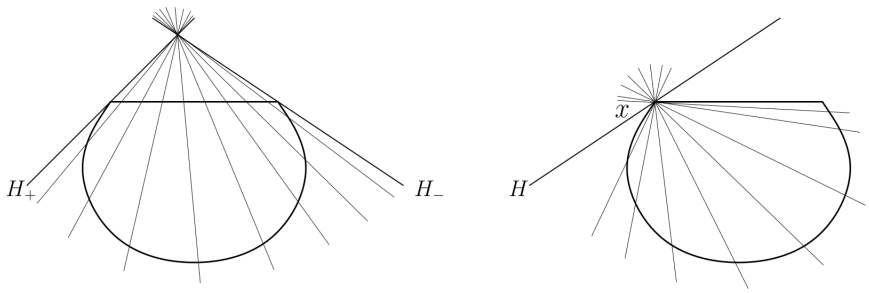


Figure 1.10: The pencil of hyperplanes in the hyperbolic case on the left and in the parabolic case on the right.

The latter hypothesis on ℓ guarantees that no leaf is stabilized by A. Indeed, each leaf of the foliation corresponds to a point of $\ell \cap \mathbb{P}^n \setminus \overline{\Omega^*}$, since this point is mapped by A^* to a different point of $\ell \cap \mathbb{P}^n \setminus \overline{\Omega^*}$ that corresponds to a different leaf of the foliation.

The equivariance of the Vinberg's map guarantees that the action of A^* on Ω^* is non-elliptic.

Therefore, if A is hyperbolic, applying the result in Proposition 1.4.4 to the action of A^* on Ω^*, we conclude that there is at least a pair made of an attracting point $H_+ \in \partial$ and a repelling point $H_- \in \partial$ for the action of A^*. Let us consider the projective line ℓ passing through H_+ and H_-. The action of A^* on $\ell \cap \mathbb{P}^n \setminus \Omega^*$ has no fixed points, since the eigenvalues relative to H_+ and H_- are distinct. Moreover, due to the convexity of Ω^*, the line ℓ intersects $\overline{\Omega^*}$.

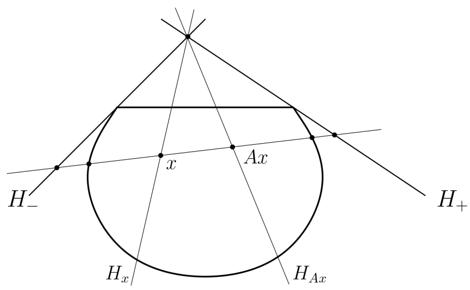
If A is parabolic, we can consider a Jordan block of A^* of maximum power relative to a fixed point $H \in \partial\Omega^*$. By Proposition 1.4.9 this block is of size at least 3, then there is a projective line in $\mathbb{P}^*$ that contains the fixed point H and is preserved by

A^*. The point H can be chosen to correspond to a supporting hyperplane at a point $x \in \partial\Omega$ fixed by A. Then the projective line that contains H is the desired one. □

Corollary 1.4.11. *The translation length of a hyperbolic isometry A acting on a properly convex domain Ω is $t(A) = \log \frac{\lambda_+}{\lambda_-}$, where λ_+ and λ_- are the maximum and minimum real eigenvalues of A.*

Proof. Let $x_+ \in \partial\Omega$ and $x_- \in \partial\Omega$ be respectively an attracting and a repelling fixed point. The action of A on the segment through x_+ and x_- has translation length $\log \frac{\lambda_+}{\lambda_-}$. Thus, $t(A) \leq \log \frac{\lambda_+}{\lambda_-}$.

To prove the opposite inequality, let us consider a foliation of Ω induced by a pencil of hyperplanes as in Proposition 1.4.10. Given a point x in Ω, denote by $H_x \in \mathbb{P}^n \setminus \overline{\Omega^*}$ be the point corresponding to the hyperplane corresponding to the leaf to which x belongs. The image Ax belongs to a hyperplane corresponding to a different leaf, denote by $H_{Ax} \in \mathbb{P}^n \setminus \overline{\Omega^*}$ the corresponding point of the dual. To compute the Hilbert distance between x and Ax, we can work in the projective plane defined by x and Ax. So, we can assume that the projective dimension of Ω is 2. Recall that the cross-ratio of four aligned points in $\mathbb{P}^2$ belonging to four lines that intersects in a common point equals the cross-ratio of the four lines, seen as points of the dual projective plane.



Thus, we have $\mathrm{d}_\Omega(x, Ax) \geq \mathrm{d}_{\Omega^*}(H_x, H_{Ax}) = \log \frac{\lambda_+}{\lambda_-}$. □

The next proposition shows that the translation length of a parabolic isometry is zero. To prove this result, we need to introduce another space associated with a properly convex domain and a point of its closure. This is the space of lines through the fixed point.

Definition 1.4.12. Let $p \in \partial\Omega$ be a point of a properly convex domain Ω. Define the line $U \subseteq \mathbb{R}^{n+1}$ to be the preimage of p under the usual quotient projection and the map $\mathcal{R}_p : \mathbb{P}^n \setminus \{p\} \to \mathbb{P}(\mathbb{R}^{n+1}/U)$ given by $\mathcal{R}_p[v] = [v + U]$. The map $\mathcal{R}_p$ is called *radial projection towards p* and the image $\mathcal{R}_p\Omega$ of Ω is called *space of directions* of Ω at p.

If $A \in \mathrm{SL}(\Omega)$ fixes the point p, then it induces $A_p \in \mathrm{SL}(\mathcal{R}_p\Omega)$ given by $A_p[v+U] = [Av+U]$. The matrix in $\mathrm{SL}(\mathcal{R}_p\Omega)$ associated with A_p is given by A by deleting the first row and column and multiplying by $\frac{1}{\sqrt[n]{\lambda}}$, where λ is the (real and positive) eigenvalue associated with the fixed point p.

The space of directions is isomorphic to the projection of the n-dimentional vector space W given by the intersection between the orthogonal $U^\perp \subseteq \mathbb{R}^{n+1}$ and the intersection of all semi-hyperspaces defined by supporting hyperplanes at p that contain $\mathcal{C}_\Omega$. Hence, its projection $\mathcal{R}_p\Omega \in \mathbb{P}^n$ is open and convex. In general, it is not properly convex, hence it is projectively equivalent to a product $\mathcal{R}_p\Omega \cong \mathbb{A}^k \times \Omega'$ that is invariant under the induced action of any projective isometry of Ω. Moreover, $\mathcal{R}_p\Omega \cong \mathbb{A}^{n-1}$ if and only p is a C^1-point.

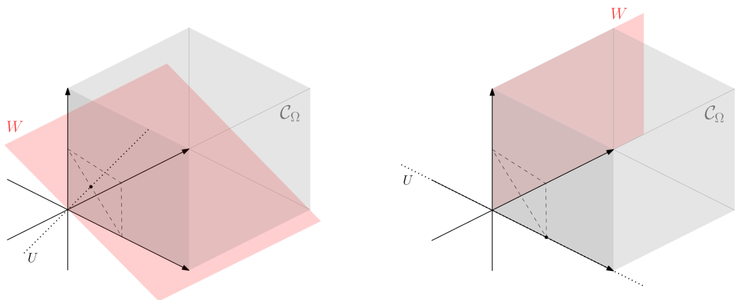


Figure 1.11: The cone on the space of directions of a C^1-point and a non-C^1-point.

Exemple 1.4.13. Let $\Omega \in \mathbb{P}^2$ be a triangle, up to a projective transformation Ω is the projectivization of the cone $\mathcal{C}_\Omega = \mathbb{R}^3_+$. Figure 1.11 shows that if p is a C^1-point, then W is an open half-plane and $\mathcal{R}_p\Omega \cong \mathbb{A}^{n-1}$, and if p is not a C^1-point, then W is an open proper cone and $\mathcal{R}_p\Omega$ is a properly convex domain of $\mathbb{P}^1$.

Proposition 1.4.14. *Let $A \in \mathrm{SL}(\Omega)$ be a parabolic isometry. Then for every $\varepsilon > 0$, there exists a point $x \in \Omega$ such that $\mathrm{d}_\Omega(x, Ax) < \varepsilon$.*

Proof. Let $p \in \partial\Omega$ be the fixed point of A. First, assume that p is a C^1-point. Choose a geodesic ray r converging to p. Then Ar is a geodesic ray. Since A preserves the foliation of Ω given by the algebraic horosperes centered at p, see Remark 1.2.22, we can parametrize r such that $r(t), Ar(t) \in \mathcal{S}_t$. Arguing as in the proof of Lemma 1.2.15, we have that $\lim_{t\to\infty} \mathrm{d}_\Omega(r(t), Ar(t)) = 0$. Hence, for any $\varepsilon > 0$ there exists a point $x \in r$ sufficiently near to p such that $\mathrm{d}_\Omega(x, Ax) < \varepsilon$.

Now, if p is not a C^1-point, then in the decomposition of space of direction $\mathcal{R}_p\Omega \cong \mathbb{A}^k \times \Omega'$ the properly convex domain Ω' is not a point. We proceed by induction on the projective dimension $n = \dim \Omega$. When $n = 1$, the result is trivially true.

For $n > 1$, the action of A_p on Ω' is non-hyperbolic and has a fixed point $w \in \Omega'$.

If $w \in \Omega'$, the preimage of w under the radial projection onto Ω' is the intersection of Ω with a projective subspace. This is a properly convex $\Omega'' \subseteq \Omega$ which is preserved by A. By induction, there is $x \in \Omega''$ with the required property.

If $w \in \partial\Omega'$, then the action on Ω' is parabolic. By induction, there is $x' \in \Omega'$ such that $\mathrm{d}_{\Omega'}(x', A_p x') < \varepsilon$. Since the projection onto Ω' given by $\mathcal{D}_p$ is projective, it preserves cross-ratios. If we choose $x \in \mathcal{D}_p^{-1}(x')$, then $\mathcal{D}_p(Ax) = A_p x'$. It follows that $\mathrm{d}_\Omega(x, Ax) = \mathrm{d}_\Omega(x', A_p x') < \varepsilon$. □

We can summarize the content of this section in the following table.

$A \in \mathrm{SL}(\Omega)$	$\mathrm{Fix}(A)$	$r(A)$	$t(A)$
elliptic	$\mathrm{Fix}(A) = \mathrm{Fix}(A, 1)$ $\mathrm{Fix}(A) \cap \Omega \neq \emptyset$	$r(A) = 1$	$t(A) = 0$ and it is achived
parabolic	$\mathrm{Fix}(A) = \mathrm{Fix}(A, 1)$ $Fix(A) \cap \Omega = \emptyset$	$r(A) = 1$	$t(A) = 0$ and it is not achived
hyperbolic	$\mathrm{Fix}(A) \supset \mathrm{Fix}(A, \lambda_+) \sqcup \mathrm{Fix}(A, \lambda_-)$ $\mathrm{Fix}(A) \cap \Omega = \emptyset$	$r(A) > 1$	$t(A) > 0$ and it is achived
quasi-hyperbolic	$\mathrm{Fix}(A) \supset \mathrm{Fix}(A, \lambda_+) \sqcup \mathrm{Fix}(A, \lambda_-)$ $\mathrm{Fix}(A) \cap \Omega = \emptyset$	$r(A) > 1$	$t(A) > 0$ and it is not achieved

Chapter 2

Isometries of a Hilbert geometry

In this chapter, we will study the group of isometries of a Hilbert geometry. The discussion is based on the work of C. Walsh and B. Lemmens contained in [30], [24], and [28] and contains some results adapted from [1] and [29]. The study of isometries of a Hilbert geometry is based on the study of their behavior on a particular compactification of the space.

Definition 2.0.1. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a properly convex open cone and $\Omega_{\mathcal{C}}$ be the domain under $\mathcal{C}$. We say that $\mathcal{C}$ is *homogeneous* if $\mathrm{PGL}(\Omega_{\mathcal{C}})$ acts transitively on $\Omega_{\mathcal{C}}$.

Definition 2.0.2. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a properly convex open cone. We say that $\mathcal{C}$ is *self-dual* if there exists an inner product of $\mathbb{R}^{n+1}$ such that under the induced identification between $\mathbb{R}^{n+1}$ and $(\mathbb{R}^{n+1})^*$ the cone $\mathcal{C}$ corresponds to its dual $\mathcal{C}^*$.

Definition 2.0.3. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a properly convex open cone. We say that $\mathcal{C}$ is *symmetric* if it is both homogeneous and self-dual.

The main theorem that will be proved in this chapter is the following.

Theorem 2.0.4. *Let* $(\Omega, \mathrm{d}_\Omega)$ *be a Hilbert geometry and* $\mathcal{C}_\Omega$ *be the cone above* Ω*. We have the following characterization of the group of isometries of* $(\Omega, \mathrm{d}_\Omega)$*.*

If $\mathcal{C}_\Omega$ *is symmetric, and non-Lorentzian, then the group of projective isometries* $\mathrm{PGL}(\Omega)$ *has index 2 into the group of isometries* $\mathrm{Isom}(\Omega, \mathrm{d}_\Omega)$*. Otherwise, we have that* $\mathrm{PGL}(\Omega) = \mathrm{Isom}(\Omega, \mathrm{d}_\Omega)$*.*

2.1 Horofunction compactification

The construction of the horofunction compactification of a semi-metric space is due to Gromov, who introduced it in [18]. In this section, we follow Gromov's construction to define the horofunction compactification of a Hilbert geometry.

2.1.1 Horofunction boundary

Let $(\Omega, \mathrm{d}_\Omega)$ be a Hilbert geometry and $\mathrm{C}(\Omega, \mathbb{R})$ be the space of continuous real-valued functions on Ω, endowed with the topology of uniform convergence on compact sets. We fix a *base point* $b \in \Omega$ and define the function $\Psi_b : \Omega \longrightarrow \mathrm{C}(\Omega, \mathbb{R})$ that maps a point $z \in \Omega$ to the continuous function $\Psi_{b,z}$ defined by

$$\begin{array}{rcll} \Psi_{b,z} & : & \Omega & \longrightarrow & \mathbb{R} \\ & & x & \mapsto & \mathrm{d}_\Omega(x, z) - \mathrm{d}_\Omega(b, z) \end{array}$$

The next two propositions show that the closure of the image of Ω under Ψ_b gives rise to a compactification of Ω.

Proposition 2.1.1. *Let $(\Omega, \mathrm{d}_\Omega)$ be a Hilbert geometry and $b \in \Omega$ be a base point. Then, $\overline{\Psi_b(\Omega)}$ is compact.*

Proof. For every $z \in \Omega$ and $x_1, x_2 \in \Omega$ we have

$$|\Psi_{b,z}(x_1) - \Psi_{b,z}(x_2)| = |\mathrm{d}_\Omega(x_1, z) - \mathrm{d}_\Omega(x_2, z)| \leq \mathrm{d}_\Omega(x_1, x_2).$$

Moreover, for every $z \in \Omega$ it holds $\Psi_{b,z}(x) \in [-\mathrm{d}_\Omega(x, b), \mathrm{d}_\Omega(x, b)]$ for all $x \in \Omega$. Hence, $\mathcal{F}_x = \{\Psi_{b,z}(x) | z \in \Omega\}$ is relatively compact for all $x \in \Omega$. By Ascoli-Arzelà Theorem [25, Theorem 47.1], the set $\{\Psi_{b,z} \mid z \in \Omega\}$ has compact closure with respect to the topology of uniform convergence on compact sets. □

Lemma 2.1.2. *Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in Ω. The sequence $(\Psi_{b,z_n})_{n\in\mathbb{N}}$ converges, up to a subsequence, to an element of $\Psi_b(\Omega)$ if and only if $(z_n)_{n\in\mathbb{N}}$ is bounded.*

Proof. Suppose that $(z_n)_{n\in\mathbb{N}}$ is bounded. Since d_Ω is proper, there is a subsequence $(z_{n_k})_{k\in\mathbb{N}}$ of $(z_n)_{n\in\mathbb{N}}$ that converges to a point $z \in \Omega$. Since for all $z, z' \in \Omega$ and for all $x \in \Omega$ it holds

$$|\Psi_{b,z} - \Psi_{b,z'}| \leq |\mathrm{d}_\Omega(x, z) - \mathrm{d}_\Omega(x, z')| + |\mathrm{d}_\Omega(b, z) - \mathrm{d}_\Omega(b, z')| \leq 2\, \mathrm{d}_\Omega(z, z'),$$

we have that Ψ_b is continuous. Then, the sequence $(\Psi_{b,z_{n_k}})_{k\in\mathbb{N}}$ converges to $\Psi_{b,z} \in \Psi_b(\Omega)$.

Conversely, suppose that $(z_n)_{n\in\mathbb{N}}$ is not bounded. Proposition 2.1.1 implies that the sequence $(\Psi_{b,z_n})_{n\in\mathbb{N}}$ converges to a point $\xi \in \overline{\Psi_b(\Omega)}$, up to a subsequence. We want to prove that $\xi \neq \Psi_{b,z}$ for all $z \in \Omega$.

We fix an arbitrary $z \in \Omega$. Since $\mathrm{d}_\Omega(z, z_n)$ converges to infinity, for every $r > 0$ there exists some $m = m(r) \in \mathbb{N}$ such that for every $n \geq m$ there is a point x_n, on the geodesic straight segment from z to z_n, such that $\mathrm{d}_\Omega(z, x_n) = r$. In this way, we obtain a sequence $(x_n)_{n\geq m}$.

The sequence $(x_n)_{n\geq m}$ is contained in the closed ball $\overline{B_{\mathrm{d}_\Omega}(z, r)}$, so it converges to a point $x \in \overline{B_{\mathrm{d}_\Omega}(z, r)}$, up to a subsequence. Moreover, for every $n \geq m$, since x_n lies on a geodesic through z and z_n, it holds the equality $\mathrm{d}_\Omega(z, z_n) = \mathrm{d}_\Omega(z, x_n) + \mathrm{d}_\Omega(x_n, z_n)$. Then $\Psi_{b,z_n}(x_n) = \Psi_{b,z_n}(z) - \mathrm{d}_\Omega(z, x_n)$. Taking the limit for $n \to \infty$ on this equality, we get $\xi(x) = \xi(z) - d(z, x)$.

Therefore, $\Psi_{b,z}(x) - \xi(x) = 2\,\mathrm{d}_\Omega(z, x) - \xi(z) - \mathrm{d}_\Omega(b, z)$. Since $x \in \overline{B_{\mathrm{d}_\Omega}(z, r)}$, if we choose $r > \frac{1}{2}(\xi(z) + \mathrm{d}_\Omega(b, z))$, we obtain that $\xi \neq \Psi_{b,z}$. □

Proposition 2.1.3. *The map Ψ_b is an embedding for every base point $b \in \Omega$.*

Proof. The map Ψ_b is injective. Indeed, for every $z, z' \in \Omega$, if $\Psi_{b,z} = \Psi_{b,z}$ then

$$\mathrm{d}_\Omega(z, z) - \mathrm{d}_\Omega(b, z) = \mathrm{d}_\Omega(z, z') - \mathrm{d}_\Omega(b, z')$$

and

$$\mathrm{d}_\Omega(z', z) - \mathrm{d}_\Omega(b, z) = \mathrm{d}_\Omega(z', z') - \mathrm{d}_\Omega(b, z').$$

So, $\mathrm{d}_\Omega(z, z') = -\,\mathrm{d}_\Omega(z', z)$ and thus $z = z'$.

To prove the continuity of Ψ_b, it suffice to observe that for every $z, z' \in \Omega$ and for every $x \in \Omega$ it holds

$$|\Psi_{b,z} - \Psi_{b,z'}| \leq |\mathrm{d}_\Omega(x, z) - \mathrm{d}_\Omega(x, z')| + |\mathrm{d}_\Omega(b, z) - \mathrm{d}_\Omega(b, z')| \leq 2\,\mathrm{d}_\Omega(z, z')$$

In order to prove that the inverse $\left(\Psi_b^{|\Psi_b(\Omega)}\right)^{-1}$ is continuous, we show that if a sequence $(\Psi_{b,z_n})_{n\in\mathbb{N}}$ in $\Psi_b(\Omega)$ converges to $\Psi_{b,z}$, where $z \in \Omega$ and $z_n \in \Omega$ for every $n \in \mathbb{N}$, then the sequence $(z_n)_{n\in\mathbb{N}}$ converges to z.

From Lemma 2.1.2, the convergence of $(\Psi_{b,z_n})_{n\in\mathbb{N}}$ to a point of $\Psi_b(\Omega)$ implies that the sequence $(z_n)_{n\in\mathbb{N}}$ is bounded. Hence, $(z_n)_{n\in\mathbb{N}}$ is contained in a compact closed ball. Recall that $(\Psi_{b,z_n})_{n\in\mathbb{N}}$ converges to $\Psi_{b,z}$ with respect to the topology of uniform convergence on compact sets. Thus, for every $\varepsilon > 0$ there exists some $\overline{n} \in \mathbb{N}$ such that $|\mathrm{d}_\Omega(z_n, z_n) - \mathrm{d}_\Omega(b, z_n) - \mathrm{d}_\Omega(z_n, z) + \mathrm{d}_\Omega(b, z)| < \varepsilon$ for every $n \geq \overline{n}$, implying that $\mathrm{d}_\Omega(z_n, z) < \frac{\varepsilon}{2}$. Hence, $(z_n)_{n\in\mathbb{N}}$ converges to z. □

Remark 2.1.4. If $b, b' \in \Omega$ are two different base points, then for every $z \in \Omega$ and for all $x \in \Omega$ it holds

$$\begin{aligned}\Psi_{b',z}(x) &= \mathrm{d}_\Omega(x,z) - \mathrm{d}_\Omega(b',z) \\ &= \mathrm{d}_\Omega(x,z) - \mathrm{d}_\Omega(b,z) - \mathrm{d}_\Omega(b',z) + \mathrm{d}_\Omega(b,z) \\ &= \Psi_{b,z}(x) - \Psi_{b,z}(b'). \end{aligned} \tag{2.1}$$

Now, we can look at the limit $\xi \in \overline{\Psi_b(\Omega)}$ of a sequence $(z_n)_{n\in\mathbb{N}}$ in Ω under the map Ψ_b. Similarly, we can look at the limit $\xi' \in \overline{\Psi_{b'}(\Omega)}$ of the same sequence under the map $\Psi_{b'}$.

What we get from (2.1) is that

$$\xi' = \xi - \xi(b') \tag{2.2}$$

where $\xi(b')$ states for the constant function $\Omega \ni x \mapsto \xi(b') \in \mathbb{R}$.

Thanks to these two propositions, we can identify Ω with its image under the map Ψ_b, where $b \in \Omega$ is a fixed base point. Moreover, the space $\overline{\Psi_b(\Omega)}$ is a compactification of the space Ω.

Henceforth, when we say that a sequence $(z_n)_{n\in\mathbb{N}}$ in Ω converges to a point $\xi \in \overline{\Psi_b(\Omega)}$ we mean that ξ is the limit of $(\Psi_{b,z_n})_{n\in\mathbb{N}}$ in the topology of uniform convergence on compact sets.

By Remark 2.1.4, different base points give rise to homeomorphic compactification. To define the horofunction compactification, we fix a base point $b \in \Omega$.

Definition 2.1.5. Let $(\Omega, \mathrm{d}_\Omega)$ be a Hilbert geometry and $b \in \Omega$ be a fixed base point. The closure of the image of Ω under the map Ψ_b is called *horofunction compactification* of Ω. We denote this space $\overline{\Omega}^\infty = \overline{\Psi_b(\Omega)}$. The set

$$\partial_\infty \Omega = \overline{\Omega}^\infty \setminus \Psi_b(\Omega)$$

is called *horofunction boundary* of Ω and its points are called *horofunctions*.

For simplicity we do not indicate the dependence on the base point in the notations, but we always assume that a base point is fixed.

Remark 2.1.6. If we considered the quotient of the set $\mathrm{C}(\Omega, \mathbb{R})$ of continuous real-valued functions, given by the identification of two functions that differs by an additive constant, we could obtain a definition independent of the base point. But it is more convenient to fix a base point and use it for all the computations.

A subset of the horofunction boundary of particular interest for our dissertation is the set of *Busemann points*. These points are those that arise from the limits of the so-called *almost-geodesics*. The next definition of almost-geodesic is the same that first appeared in the paper [26] by M. Rieffel. We call it *Rieffel almost-geodesic* to distinguish this definition from the one that we will give in Section 2.2.2.

Definition 2.1.7. Let $(\Omega, \mathrm{d}_\Omega)$ be a Hilbert geometry. A path $\gamma : \mathbb{R}_+ \to \Omega$ is a *Rieffel almost-geodesic* if for every $\varepsilon > 0$ there exists some $T = T(\varepsilon) > 0$ such that

$$|\mathrm{d}_\Omega(\gamma(0), \gamma(s)) + \mathrm{d}_\Omega(\gamma(s), \gamma(t)) - t| < \varepsilon \quad \text{for all } t \geq s \geq T.$$

Definition 2.1.8. Let $(\Omega, \mathrm{d}_\Omega)$ be a Hilbert geometry. A horofunction $\xi \in \mathcal{B}_\Omega$ is a *Busemann point* if it arises as the limit of a Rieffel almost-geodesic. In other words, ξ is a Busemann point if there exists some Rieffel almost-geodesic $\gamma : \mathbb{R}_+ \to \Omega$ such that $\xi(x) = \lim_{t \to +\infty} \Psi_{b,\gamma(t)}(x)$ for every $x \in \Omega$ and the convergence is uniform on every compact set of Ω. We denote the set of Busemann points by $\mathcal{B}_\Omega$.

Proposition 2.1.9. *Every Rieffel almost-geodesic converges to a Busemann point.*

Proof. Let $\gamma : \mathbb{R}_+ \to \Omega$ be an Rieffel almost-geodesic. It follows from the definition that for every $\varepsilon > 0$, there exists some $T > 0$ such that

$$|\mathrm{d}_\Omega(\gamma(0), \gamma(t)) - t| < \varepsilon \quad \text{for all } t \geq T. \tag{2.3}$$

Thus, $\mathrm{d}_\Omega(\gamma(0), \gamma(t)) \to +\infty$ as $t \to +\infty$. From Remark 2.1.4, it is not restrictive to assume that the base point is $b := \gamma(0)$. Lemma 2.1.2 implies that if the limit of $\Psi_{b,\gamma(t)}$ as $t \to \infty$ exists, then it is a horofunction. From the compactness of $\overline{\Psi_b(\Omega)}$, there is an increasing sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that $(\Psi_{b,\gamma(t_n)})_{n\in\mathbb{N}}$ converges to a horofunction $\xi \in \partial_\infty\Omega$. We want to prove that for every $x \in \Omega$ it holds $\lim_{t\to+\infty} \Psi_{b,\gamma(t)}(x) = \xi(x)$ and that the convergence is uniform on every compact subset of Ω.

Let $K \subseteq \Omega$ be a compact set and take $\varepsilon > 0$. For every $x \in K$, $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} \left|\Psi_{b,\gamma(t)}(x) - \xi(x)\right| &= \left|\Psi_{b,\gamma(t)}(x) - \Psi_{b,\gamma(t_n)}(x) + \Psi_{b,\gamma(t_n)}(x) - \xi(x)\right| \\ &\leq \left|\Psi_{b,\gamma(t)}(x) - \Psi_{b,\gamma(t_n)}(x)\right| + \left|\Psi_{b,\gamma(t_n)}(x) - \xi(x)\right|. \end{aligned} \tag{2.4}$$

Now, for a sufficiently large $n = n(\varepsilon) \in \mathbb{N}$ we have that

$$\left|\Psi_{b,\gamma(t_n)}(x) - \xi(x)\right| < \varepsilon, \tag{2.5}$$

independently on the choice of x in K.

Moreover, for any sufficiently large $s \in \mathbb{R}_+$ and $t \geq s$ we have for every $x \in K$

$$\begin{aligned}\left|\Psi_{b,\gamma(t)}(x) - \Psi_{b,\gamma(s)}(x)\right| &= |\mathrm{d}_\Omega(x,\gamma(t)) - \mathrm{d}_\Omega(\gamma(0),\gamma(t)) - \mathrm{d}_\Omega(x,\gamma(s)) + \mathrm{d}_\Omega(\gamma(0),\gamma(s))| \\ &\leq |\,\mathrm{d}_\Omega(x,\gamma(s)) + \mathrm{d}_\Omega(\gamma(s),\gamma(t)) - \mathrm{d}_\Omega(\gamma(0),\gamma(t)) - \mathrm{d}_\Omega(x,\gamma(s)) + \\ &\qquad + \mathrm{d}_\Omega(\gamma(0),\gamma(s))| \\ &= |\mathrm{d}_\Omega(\gamma(s),\gamma(t)) + \mathrm{d}_\Omega(\gamma(0),\gamma(s)) - \mathrm{d}_\Omega(\gamma(0),\gamma(t))| \\ &\leq |\mathrm{d}_\Omega(\gamma(s),\gamma(t)) + \mathrm{d}_\Omega(\gamma(0),\gamma(s)) - t| + |t - \mathrm{d}_\Omega(\gamma(0),\gamma(t))| \\ &< 2\varepsilon, \end{aligned} \tag{2.6}$$

where the last inequality follows from (2.3) and the definition of Rieffel almost-geodesic.

From (2.4), combined with (2.5) and (2.6), we get that for any sufficiently large $t \in \mathbb{R}_+$ it holds

$$\left|\Psi_{b,\gamma(t)}(x) - \xi(x)\right| < 3\varepsilon \quad \text{for every } x \in K.$$

Thus, $\Psi_{b,\gamma(t)}$ converges to the horofunction ξ uniformly on K. □

Remark 2.1.10. From Proposition 2.1.1, we know that every sequence of points in a properly convex domain converges to a function in the horofunction compactification, with respect to the topology of uniform convergence on compact sets. So, the pointwise limit of such a sequence is always a horofunction.

2.1.2 Detour metric

On the horofunction compactification of a Hilbert geometry $(\Omega, \mathrm{d}_\Omega)$ we define the *detour cost* as the function $H : \overline{\Omega}^\infty \times \overline{\Omega}^\infty \to \mathbb{R} \cup \{+\infty\}$ given by

$$H(\xi,\eta) = \inf_{(z_n)_{n\in\mathbb{N}} \in S} \left(\liminf_{n\to\infty} \mathrm{d}_\Omega(b, z_n) + \eta(z_n)\right) \quad \text{for all } \xi, \eta \in \overline{\Omega}^\infty$$

where S is the set of all sequences $(z_n)_{n\in\mathbb{N}} \subseteq \Omega$ that converge to ξ.

Lemma 2.1.11. *Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in Ω and $b, b' \in \Omega$ be two base points. If ξ is the limit of $(z_n)_{n\in\mathbb{N}}$ under Ψ_b and ξ' is the limit of the same sequence under $\Psi_{b'}$, then*

$$\liminf_{n\to\infty} \mathrm{d}_\Omega(b, z_n) + \xi(z_n) = \liminf_{n\to\infty} \mathrm{d}_\Omega(b', z_n) + \xi'(z_n).$$

Proof. From Remark 2.1.4, we have that $\xi' = \xi - \xi(b')$. Hence, we have

$$\begin{aligned}
\liminf_{n\to\infty} \mathrm{d}_\Omega(b', z_n) + \xi'(z_n) &= \liminf_{n\to\infty} \mathrm{d}_\Omega(b', z_n) + \xi(z_n) - \xi(b') \\
&= \liminf_{n\to\infty} \mathrm{d}_\Omega(b', z_n) - \mathrm{d}_\Omega(b, z_n) + \mathrm{d}_\Omega(b, z_n) + \xi(z_n) - \xi(b') \\
&= \liminf_{n\to\infty} \mathrm{d}_\Omega(b, z_n) + \xi(z_n) + \lim_{n\to\infty} \mathrm{d}_\Omega(b', z_n) - \mathrm{d}_\Omega(b, z_n) - \xi(b') \\
&= \liminf_{n\to\infty} (\mathrm{d}_\Omega(b, z_n) + \xi(z_n) + \xi(b') - \xi(b') \\
&= \liminf_{n\to\infty} \mathrm{d}_\Omega(b, z_n) + \xi(z_n).
\end{aligned}$$

□

Proposition 2.1.12. *Let $\xi, \eta, \nu \in \overline{\Omega}^\infty$ be three points of the horofunction compactification. The following properties hold.*

1. $H(\xi, \eta) \geq 0$,
2. $\xi(x) - \xi(y) \leq \mathrm{d}_\Omega(x, y)$ *for all* $x, y \in \Omega$,
3. $\eta(x) \leq \xi(x) + H(\xi, \eta)$ *for all* $x \in \Omega$,
4. $H(\xi, \nu) \leq H(\xi, \eta) + H(\eta, \nu)$.

Proof. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in Ω that converges to ξ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in Ω that converges to η. For every $n \in \mathbb{N}$ we have

$$\begin{aligned}
\mathrm{d}_\Omega(b, z_n) + \eta(z_n) &= \mathrm{d}_\Omega(b, z_n) + \lim_{k\to\infty} \big(\mathrm{d}_\Omega(z_n, x_k) - \mathrm{d}_\Omega(b, x_k)\big) \\
&= \lim_{k\to\infty} \big(\mathrm{d}_\Omega(b, z_n) + \mathrm{d}_\Omega(z_n, x_k) - \mathrm{d}_\Omega(b, x_k)\big) \\
&\geq \lim_{k\to\infty} \big(\mathrm{d}_\Omega(b, x_k) - \mathrm{d}_\Omega(b, x_k)\big) = 0.
\end{aligned}$$

Hence, $\liminf_{n\to\infty} \mathrm{d}_\Omega(b, z_n) + \eta(z_n) \geq 0$. Taking the infimum on all the sequences that converges to ξ, we get the firts assertion.

To prove the second one, we observe that for every x and y in Ω, we have

$$\begin{aligned}
\xi(x) &= \lim_{n\to\infty} \mathrm{d}_\Omega(x, x_n) - \mathrm{d}_\Omega(b, x_n) \\
&= \lim_{n\to\infty} \mathrm{d}_\Omega(x, x_n) - \mathrm{d}_\Omega(y, x_n) + \mathrm{d}_\Omega(y, x_n) - \mathrm{d}_\Omega(b, x_n) \\
&\leq \mathrm{d}_\Omega(x, y) + \lim_{n\to\infty} \mathrm{d}_\Omega(y, x_n) - \mathrm{d}_\Omega(b, x_n) \\
&= \mathrm{d}_\Omega(x, y) + \xi(y).
\end{aligned}$$

The third assertion follows from the second one. Indeed, for every $n \in \mathbb{N}$

$$\eta(x) \leq \Big(\mathrm{d}_\Omega(x, z_n) - \mathrm{d}_\Omega(b, z_n)\Big) + \Big(\mathrm{d}_\Omega(b, z_n) + \eta(z_n)\Big) \quad \text{for all } x \in \Omega,$$

and the conclusion follows by taking the limit infimum as $n \to \infty$ and then the infimum over all sequences that converge to ξ.

The fourth statement follows from the third one. Indeed, for all $n \in \mathbb{N}$ we have

$$\mathrm{d}_\Omega(b, z_n) + \nu(z_n) \leq \mathrm{d}_\Omega((b, z_n) + \eta(z_n) + H(\eta, \nu).$$

□

Corollary 2.1.13. *Let $\xi \in \mathcal{B}_\Omega$ be a Busemann point. Then, for every almost-geodesic $\gamma : \mathbb{R}_+ \to \Omega$ converging to ξ*

$$\lim_{t \to +\infty} \mathrm{d}_\Omega(b, \gamma(t)) + \xi(\gamma(t)) = 0.$$

Hence, we have $H(\xi, \xi) = 0$.

Proof. From Lemma 2.1.11, it is not restrictive to assume that the base point is $b := \gamma(0)$. Since γ is an almost-geodesic, for every $\varepsilon > 0$ it holds

$$\mathrm{d}_\Omega(\gamma(0), \gamma(s)) + \mathrm{d}_\Omega(\gamma(s), \gamma(t)) - t < \varepsilon, \tag{2.7}$$

for every sufficiently large s and $t \geq s$.

Moreover, from (2.3) we have

$$-\mathrm{d}_\Omega(\gamma(0), \gamma(t)) + t < \varepsilon, \tag{2.8}$$

for all sufficiently large t.

Combining (2.7) and (2.8), we get

$$\mathrm{d}_\Omega(\gamma(0), \gamma(s)) + \mathrm{d}_\Omega(\gamma(s), \gamma(t)) - \mathrm{d}_\Omega(\gamma(0), \gamma(t)) - 2\varepsilon < 0.$$

By taking the limit as $t \to \infty$, we get

$$\limsup_{s \to \infty} \mathrm{d}_\Omega(\gamma(0), \gamma(s)) + \xi(\gamma(s)) \leq 0.$$

The thesis follows from the first point of Proposition 2.1.12. □

Busemann points can be characterized using the detour cost. Indeed, the following proposition hold.

Proposition 2.1.14. *Let $\xi \in \partial_\infty \Omega$ be a horofunction. Then ξ is a Busemann point if and only if $H(\xi, \xi) = 0$.*

Proof. If ξ is a Busemann point, it follows from Corollary 2.1.13 that $H(\xi, \xi) = 0$.

Conversely, suppose that $H(\xi, \xi) = 0$. We get from Lemma 2.1.11 that for every $b' \in \Omega$

$$\inf_{(z_n)_{n\in\mathbb{N}} \in S} \left(\liminf_{n\to\infty} \mathrm{d}_\Omega(b', z_n) + \xi(z_n) - \xi(b') \right) = 0,$$

where S is the set of all sequences that converges to ξ.

Therefore, for every $\varepsilon > 0$ there exists a sequence $(z_n)_{n\in\mathbb{N}} \in S$ such that for every $m \in \mathbb{N}$ there exists some $\overline{n} \geq m$ that satisfies

$$0 \leq \mathrm{d}_\Omega(b', z_{\overline{n}}) + \xi(z_{\overline{n}}) - \xi(b') < \varepsilon.$$

Moreover, by Lemma 2.1.2, we can choose $m \in \mathbb{N}$ such that $\mathrm{d}_\Omega(b, z_{\overline{n}}) > M$, for every arbitrarily large $M \in \mathbb{R}$.

We use this property to construct an almost-geodesic that converges to ξ.

Fix $\varepsilon > 0$. Starting from the base point, we set $x_0 = b$. Then, we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ in Ω that satisfies the following condition for every $n \in \mathbb{N}$

$$|\mathrm{d}_\Omega(x_{n+1}, x_n) + \xi(x_n) - \xi(x_{n+1})| < \frac{\varepsilon}{2^{n+1}}, \tag{2.9}$$

and such that $\mathrm{d}_\Omega(b, x_n)$ monotonically diverges at $+\infty$.

The second condition guarantees that, if we consider for every $n \in \mathbb{N}$ a geodesic path from x_n and x_{n+1}, and then we concatenate these paths, the resulting path γ is defined on the whole $\mathbb{R}_+$. Moreover, if we define $t_n = \sum_{k=0}^{n-1} \mathrm{d}_\Omega(x_k, x_{k+1})$, we have that $x_n = \gamma(t_n)$. It remains to prove that γ is an almost-geodesic.

Applying the triangular inequality, since $x_0 = b$, we have

$$\mathrm{d}_\Omega(x_0, x_m) + \mathrm{d}_\Omega(x_m, x_n) - \mathrm{d}_\Omega(b, x_n) \leq 0 \quad \text{for all } m \leq n.$$

On the other hand, from Proposition 2.1.12 we have

$$\mathrm{d}_\Omega(x_0, x_m) + \mathrm{d}_\Omega(x_m, x_n) \geq \xi(x_0) - \xi(x_m) + \xi(x_m) - \xi(x_n) = -\xi(x_n). \tag{2.10}$$

Thus, combining (2.10) with (2.9) we get

$$\begin{aligned} \mathrm{d}_\Omega(x_0,x_m)+\mathrm{d}_\Omega(x_m,x_n)-t_n &= \mathrm{d}_\Omega(x_0,x_m)+\mathrm{d}_\Omega(x_m,x_n)-\sum_{k=0}^{n-1}\mathrm{d}_\Omega(x_k,x_{k+1}) \\ = \mathrm{d}_\Omega(x_0,x_m)+\mathrm{d}_\Omega(x_m,x_n) &- \Big(\sum_{k=0}^{n-1}\big(\mathrm{d}_\Omega(x_k,x_{k+1})+\xi(x_{k+1})-\xi(x_k)\big)-\xi(x_{n+1})\Big) \\ &\geq -\xi(x_{n+1})-\sum_{k=0}^{n-1}\frac{\varepsilon}{2^{k-1}}+\xi(x_{n+1}) \\ &\geq -\varepsilon. \end{aligned} \tag{2.11}$$

Therefore, we have

$$|\mathrm{d}_\Omega(x_0,x_m)+\mathrm{d}_\Omega(x_m,x_n)-t_n|<\varepsilon \quad \text{for all } m\leq n.$$

Now, we prove that for every $\delta>0$ there exists some $N_\delta\in\mathbb{N}$ such that

$$|\mathrm{d}_\Omega(x_0,x_m)+\mathrm{d}_\Omega(x_m,x_n)-t_n|<\delta \quad \text{for all } N_\delta\leq m\leq n.$$

Let us define for every $n\in\mathbb{N}$

$$a_n=\mathrm{d}_\Omega(x_0,x_n)-t_n$$

We can rewrite a_n in the following way

$$a_n=\sum_{k=0}^{n-1}\big(-\mathrm{d}_\Omega(x_0,x_k)-\mathrm{d}_\Omega(x_k,x_{k+1})+\mathrm{d}_\Omega(x_0,x_{k+1})\big),$$

and notice that $(a_n)_{n\in\mathbb{N}}$ is a decreasing sequence, since $-\mathrm{d}_\Omega(x_0,x_k)-\mathrm{d}_\Omega(x_k,x_{k+1})+\mathrm{d}_\Omega(x_0,x_{k+1})\leq 0$ for every $k\in\mathbb{N}$ and $a_{n+1}=a_n+\big(-\mathrm{d}_\Omega(x_0,x_n)-\mathrm{d}_\Omega(x_n,x_{n+1})+\mathrm{d}_\Omega(x_0,x_{n+1})\big)$. Moreover, from (2.11) we have that $(a_n)_{n\in\mathbb{N}}$ is bounded below by $-\varepsilon$. It follows that the sequence $(a_n)_{n\in\mathbb{N}}$ converges to a finite limit. So, it satisfies the Cauchy condition. Hence, for every $\delta>0$ there exists some $N_\delta\in\mathbb{N}$ such that

$$|a_m-a_n|<\delta \quad \text{for all } N_\delta\leq m\leq n.$$

Therefore, for every $N_\delta \le m \le n$ we have

$$\begin{aligned}|\mathrm{d}_\Omega(x_0,x_m)+\mathrm{d}_\Omega(x_n,x_m)-t_n| &\le |\mathrm{d}_\Omega(x_0,x_m)+\mathrm{d}_\Omega(x_n,x_m)-\mathrm{d}_\Omega(x_0,x_n)|\\ &\le \left|\mathrm{d}_\Omega(x_0,x_m)+\sum_{k=m}^{n-1}\mathrm{d}_\Omega(x_k,x_{k+1})-\mathrm{d}_\Omega(x_0,x_n)\right|\\ &= |a_m-a_n|<\delta.\end{aligned}$$

To conclude the proof, let $\delta > 0$. For every $r, s \in \mathbb{R}_+$ such that $t_{N_\delta} \le r \le s$, define $n_r \in \mathbb{N}$ such that $t_{n_r} \le r < t_{n_r+1}$ and $n_s \in \mathbb{N}$ such that $t_{n_s} \le s < t_{n_s+1}$. Then, we have

$$\begin{aligned}\mathrm{d}_\Omega(\gamma(0),\gamma(r))+\mathrm{d}_\Omega(\gamma(r),\gamma(s))-s &\ge \mathrm{d}_\Omega(\gamma(0),\gamma(s))-s\\ &\ge \mathrm{d}_\Omega(\gamma(0),)x_{n_s+1}-\mathrm{d}_\Omega(\gamma(s),x_{n_s+1})-s\\ &\ge \mathrm{d}_\Omega(\gamma(0),)x_{n_s+1}-\mathrm{d}_\Omega(\gamma(s),x_{n_s+1})+\\ &\quad -t_{n_s+1}+\mathrm{d}_\Omega(\gamma(s),x_{n_s+1})\\ &= \mathrm{d}_\Omega(\gamma(0),)x_{n_s+1}-t_{n_s+1}\\ &\ge -\delta,\end{aligned}$$

since $\gamma_{|[t_{n_s},t_{n_s+1}]}$ is a geodesic path.

On the other hand, we have

$$\begin{aligned}\mathrm{d}_\Omega(\gamma(0),\gamma(r))+\mathrm{d}_\Omega(\gamma(r),\gamma(s))-s &\le \mathrm{d}_\Omega(\gamma(0),x_{n_r})+\mathrm{d}_\Omega(x_{n_r+1},x_{n_s})+\mathrm{d}_\Omega(x_{n_s},\gamma(s))-s\\ &\le \mathrm{d}_\Omega(\gamma(0),x_{n_r})+\mathrm{d}_\Omega(x_{n_r+1},x_{n_s})+\mathrm{d}_\Omega(x_{n_s},\gamma(s))+\\ &\quad -t_{n_s}-\mathrm{d}_\Omega(x_{n_s},\gamma(s))\\ &= \mathrm{d}_\Omega(\gamma(0),x_{n_r})+\mathrm{d}_\Omega(x_{n_r+1},x_{n_s})-\sum_{k=0}^{n_s-1}\mathrm{d}_\Omega(x_{n_k},x_{n_k+1}),\end{aligned}$$

since $\gamma_{|[t_{n_s},t_{n_s+1}]}$ is a geodesic path.

It follows that for every $\delta > 0$, there exists $T_\delta > 0$ such that for every $T_\delta \le r \le s$ it holds

$$|\mathrm{d}_\Omega(\gamma(0),\gamma(r))+\mathrm{d}_\Omega(\gamma(r),\gamma(s))-s|<\delta.$$

□

The next proposition shows how to compute the detour cost from a Busemann point to a horofunction.

Proposition 2.1.15. *Let $\xi \in \mathcal{B}_\Omega$ be a Busemann point and $\eta \in \partial_\infty \Omega$ be a horofunction. Then, for every almost-geodesic $\gamma : \mathbb{R}_+ \to \Omega$ converging to ξ*

$$\lim_{t\to+\infty} \mathrm{d}_\Omega(b,\gamma(t)) + \eta(\gamma(t)) = H(\xi,\eta).$$

Proof. Let $\gamma : \mathbb{R}_+ \to \Omega$ be an almost-geodesic converging to ξ. It follows from Proposition 2.1.12 that

$$\mathrm{d}_\Omega(b,\gamma(t)) + \eta(\gamma(t)) \leq \mathrm{d}_\Omega(b,\gamma(t)) + \xi(\gamma(t)) + H(\xi,\eta) \quad \text{for all } t \in \mathbb{R}_+.$$

Taking the limit supremum as $t \to \infty$, it follows from Corollary 2.1.13 that

$$\limsup_{t\to\infty} \mathrm{d}_\Omega(b,\gamma(t)) + \eta(\gamma(t)) \leq H(\xi,\eta).$$

From the definition of detour cost we also have

$$H(\xi,\eta) \leq \limsup_{t\to\infty} \mathrm{d}_\Omega(b,\gamma(t)) + \eta(\gamma(t)).$$

□

Remark 2.1.16. Let $\xi \in \mathrm{int}(\overline{\Omega}^\infty)$ be an interior point of the horofunction compactification of Ω. Then, $\xi(\cdot) = \mathrm{d}_\Omega(\cdot,x) - \mathrm{d}_\Omega(b,x)$, for some $x \in \Omega$. We can consider the constant sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n = x$ for all $n \in \mathbb{N}$. From Proposition 2.1.12 we get that for all $\eta \in \overline{\Omega}^\infty$ it holds

$$\begin{aligned} H(\xi,\eta) &\geq \eta(x) - \xi(x) \\ &= \eta(x) - \mathrm{d}_\Omega(x,x) + \mathrm{d}_\Omega(b,x) \\ &= \mathrm{d}_\Omega(b,x) + \eta(x). \end{aligned}$$

On the other hand, by definition $\mathrm{d}_\Omega(b,x) + \eta(x) \geq H(\xi,\eta)$. Thus, we have $H(\xi,\eta) = \mathrm{d}_\Omega(b,x) + \eta(x)$.

Definition 2.1.17. Let $(\Omega, \mathrm{d}_\Omega)$ be a Hilbert geometry. The *detour metric* on the horofunction compactification of Ω is the function $\delta : \overline{\Omega}^\infty \times \overline{\Omega}^\infty \longrightarrow [0,\infty]$ defined by the symmetrization of the detour cost, so it is given by

$$\delta(\xi,\eta) = H(\xi,\eta) + H(\eta,\xi) \quad \text{for all } \xi,\eta \in \partial_\infty\Omega.$$

Proposition 2.1.18. *The detour metric restricted to the set of Busemann points $\delta_{|\mathcal{B}_\Omega\times\mathcal{B}_\Omega} : \mathcal{B}_\Omega \times \mathcal{B}_\Omega \longrightarrow [0,\infty]$ is an extended metric. Moreover, it is independnt of* *the choice of the base point.*

Proof. It is clear that δ is symmetric. From Proposition 2.1.12 we have that δ is non-negative and satisfies the triangular inequality. Corollary 2.1.13 implies that $\delta(\xi, \xi) = 0$ for every $\xi \in \mathcal{B}_\Omega$.

Now, let $b, b' \in \Omega$ be two distinct base points and $\xi, \eta \in \mathcal{B}_\Omega$ be two Busemann points with respect to the base point b. From Remark 2.1.4 we have that the corresponding Busemann points with respect to the base point b' are $\xi' = \xi - \xi(b')$ and $\eta' = \eta - \eta(b')$. Then, if $\gamma' : \mathbb{R}_+ \to \Omega$ is an almost-geodesic converging to ξ', from Proposition 2.1.15 we have

$$\begin{aligned} H(\xi', \eta') &= \liminf_{t\to\infty} \mathrm{d}_\Omega(b', \gamma'(t)) + \eta'(\gamma'(t)) \\ &= \liminf_{t\to\infty} \mathrm{d}_\Omega(b', \gamma'(t)) - \mathrm{d}_\Omega(b, \gamma'(t) + \mathrm{d}_\Omega(b, \gamma'(t) + \eta(\gamma'(t)) - \eta(b') \\ &= H(\xi, \eta) + \xi(b') - \eta(b'), \end{aligned}$$

since γ' converges under Ψ_b to ξ. With the same reasoning it can be shown that $H(\eta', \xi') = H(\xi, \eta) + \eta(b') - \xi(b')$. Hence, $H(\xi', \eta') = H(\xi, \eta)$. □

Remark 2.1.19. From Remark 2.1.16 we have that if $\xi \in \mathrm{int}(\overline{\Omega}^\infty)$ is given by $\xi(\cdot) = \mathrm{d}_\Omega(\cdot, x) - \mathrm{d}_\Omega(b, x)$, for some $x \in \Omega$, then for all $\eta \in \overline{\Omega}^\infty$ it holds $H(\xi, \eta) = \mathrm{d}_\Omega(b, x) + \eta(x)$. Therefore, if $\eta \in \mathrm{int}(\overline{\Omega}^\infty)$ is given by $\eta(\cdot) = \mathrm{d}_\Omega(\cdot, y) - \mathrm{d}_\Omega(b, y)$, for some $y \in \Omega$ then

$$\begin{aligned} \delta(\xi, \eta) &= \mathrm{d}_\Omega(b, x) + \eta(x) + \mathrm{d}_\Omega(b, y) + \xi(y) \\ &= \mathrm{d}_\Omega(b, x) + \mathrm{d}_\Omega(x, y) - \mathrm{d}_\Omega(b, y) + \mathrm{d}_\Omega(b, y) + \mathrm{d}_\Omega(y, x) - \mathrm{d}_\Omega(b, x) \\ &= 2\, \mathrm{d}_\Omega(x, y). \end{aligned} \tag{2.12}$$

On the other hand, if $\eta \in \mathcal{B}_\Omega$, then we have

$$\begin{aligned} \delta(\xi, \eta) &= \mathrm{d}_\Omega(b, x) + \eta(x) + H(\eta, \xi) \\ &= \mathrm{d}_\Omega(b, x) + \eta(x) + \inf_{(z_n)_{n\in\mathbb{N}} \in S} \big(\liminf_{n\to\infty} \mathrm{d}_\Omega(b, z_n) + \xi(z_n) \big) \\ &= \eta(x) + \inf_{(z_n)_{n\in\mathbb{N}} \in S} \big(\liminf_{n\to\infty} \mathrm{d}_\Omega(b, z_n) + \mathrm{d}_\Omega(z_n, x) \big) \\ &= +\infty, \end{aligned} \tag{2.13}$$

where S is the set of all sequences $(z_n)_{n\in\mathbb{N}} \subseteq \Omega$ that converges to η, and the last equality holds because $(z_n)_{n\in\mathbb{N}} \in S$ if and only if $\mathrm{d}_\Omega(b, z_n) \to \infty$ as $n \to \infty$, see Lemma 2.1.2.

It follows that δ defines an extended metric on the space $\mathrm{int}(\overline{\Omega}^{\infty}) \cup \mathcal{B}_{\Omega}$. From (2.12) we have that the distance between two points in $\mathrm{int}(\overline{\Omega}^{\infty})$ is finite. Moreover, the homeomorphism $\Psi_b : \Omega \to \mathrm{int}(\overline{\Omega}^{\infty})$ maps geodesics into geodesics, where b is a fixed base point. Furthermore, the pre-image of a geodesic in $\mathrm{int}(\overline{\Omega}^{\infty}) \cup \mathcal{B}_{\Omega}$ that is entirely contained in $\mathrm{int}(\overline{\Omega}^{\infty})$ is a geodesic in Ω connecting two points of the boundary. From (2.13) we have that the distance between a point in $\mathrm{int}(\overline{\Omega}^{\infty})$ and a point in $\mathcal{B}_{\Omega}$ is infinite.

Definition 2.1.20. Let $(\Omega, \mathrm{d}_{\Omega})$ be a Hilbert geometry and $\xi, \eta \in \mathcal{B}_{\Omega}$ be two Busemann points. We say that ξ and η belong to the same *part of the horofunction boundary* if $\delta(\xi, \eta) < \infty$.

In section 2.4 we will see that the partition of the set of Busemann points into parts is related to the partitions of $\partial\Omega$ and $\partial\Omega^*$ into faces.

The goals of the next sections will be to determine the set of Busemann points of a Hilbert geometry and to study the action of isometries on this set, in order to reach the proof of Theorem 2.0.4.

2.2 Birkhoff's version of Hilbert metric

The Birkhoff's version of the Hilbert metric is the extension of the Hilbert metric on a properly convex domain to the cone above the domain. Moreover, this extension can be defined also for non-proper convex cones. Birkhoff used this version of the Hilbert metric in [7] to analyse the spectral properties of linear operators. In particular, he used this metric to give a proof of Perron-Frobenius theorem.

2.2.1 Gauge function

Let $(\Omega, \mathrm{d}_{\Omega})$ be a Hilbert geometry. Using the folmula in (1.1), given two distinct points $x, y \in \Omega$ their Hilbert distance can be computed as

$$\mathrm{d}_{\Omega}(x,y) = \log \frac{\|y - x_{\infty}\|}{\|x - x_{\infty}\|} + \log \frac{\|x - y_{\infty}\|}{\|y - y_{\infty}\|} , \tag{2.14}$$

where x_{∞} and y_{∞} are the points in the intersection of line through x and y and the boundary $\partial\Omega$; we denote x_{∞} the point nearest to x and y_{∞} the point nearest to y.

The Hilbert distance d_{Ω} turns out to be the symmetrization of the semi-metric

$$\Omega \times \Omega \ni (x,y) \mapsto \log \frac{\|x - y_{\infty}\|}{\|y - y_{\infty}\|} \in \mathbb{R}_{+}, \tag{2.15}$$

Moreover, it is equivalent to do these computations in an affine chart where Ω is bounded or in a relatively compact cross-section of the cone $\mathcal{C}_\Omega$ associated with Ω.

The definition of Hilbert metric can be extended to convex cones, possibly non-proper, by considering Birkhoff's version of the Hilbert metric.

If the cone is proper, it turns out that the Birkhoff's version of the Hilbert metric restricted to a bounded cross-section of the cone coincides with the Hilbert metric on the projectivization of the cone.

The reason behind this definition is that in what follows we have to work also with non-proper cones. These cones will be associated with the faces of the dual.

Definition 2.2.1. Let $\mathcal{C}$ be a convex open cone in $\mathbb{R}^{n+1}$. We associate with $\mathcal{C}$ a pre-order $\leq_\mathcal{C}$ on $\mathbb{R}^{n+1}$ such that $u \leq_\mathcal{C} v$ if $v - u \in \overline{\mathcal{C}}$ for all $u, v \in \mathbb{R}^{n+1}$.

Remark 2.2.2. If $\mathcal{C}$ is proper $\leq_\mathcal{C}$ is a partial order. For every $v \in \mathbb{R}^{n+1}$ we have $v \leq_\mathcal{C} v$ since $v - v = 0 \in \overline{\mathcal{C}}$. The antisymmetry follows by the proper convexity. Indeed, if u and v in $\mathbb{R}^{n+1}$ are such that $u \leq_\mathcal{C} v$ and $v \leq_\mathcal{C} u$, then $u - v \in \overline{\mathcal{C}}$ and $v - u \in \overline{\mathcal{C}}$, so $u - v = v - u = 0$. Given $u, v, w \in \mathbb{R}^{n+1}$ such that $u \leq_\mathcal{C} v$ and $v \leq_\mathcal{C} w$ then $u \leq_\mathcal{C} w$, indeed $w - u = (w - v) + (v - u) \in \overline{\mathcal{C}}$, since $\mathcal{C}$ is convex and closed under the action of positive homotheties.

Definition 2.2.3. Let $\mathcal{C}$ be a convex open cone in $\mathbb{R}^{n+1}$. Given $u, v \in \mathbb{R}^{N+1}$, the *gauge* between u and v is defined as

$$M_\mathcal{C}(u/v) = \inf\{\lambda > 0 \mid u \leq_\mathcal{C} \lambda v\}.$$

Notice that if $v \in \mathcal{C}$ the value $M_\mathcal{C}(u/v)$ is finite. Otherwise, $M_\mathcal{C}(u/v) = +\infty$ when the set $\{\lambda > 0 \mid u \leq_\mathcal{C} \lambda v\}$ is empty.

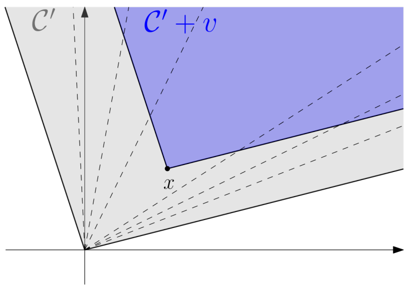


Figure 2.1: Geometric interpretation of the gauge.

This remark inspires the definition of an equivalence relation on $\overline{\mathcal{C}}$.

Definition 2.2.4. Let $u, v \in \overline{\mathcal{C}}$ be two points of the closed convex cone $\overline{\mathcal{C}}$, we say that $u \sim_\mathcal{C} v$ if and only if the two sets $\{\lambda > 0 \mid u \leq_\mathcal{C} \lambda v\}$ and $\{\mu > 0 \mid v \leq_\mathcal{C} \mu v\}$ are both non-empty. We denote the equivalence class of a point $v \in \overline{\mathcal{C}}$ as $[v]_\mathcal{C}$.

For $u, v \in \mathbb{R}^{n+1}$ and $\lambda > 0$, the condition $u \leq_{\mathcal{C}} \lambda v$ can be checked in the cone $\mathcal{C}'$ obtained from the intersection of $\mathcal{C}$ with the plane spanned by u and v. Then, as we can see in Figure 2.1, the points in $\mathcal{C}$ are all in the same equivalence class and any point in $\partial\mathcal{C}$ is equivalent to a point in $\partial\mathcal{C}$. Since each face of a cone is a closed cone, reasoning by induction, we get the following lemma.

Lemma 2.2.5. *Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a convex cone. If $x \in \overline{\mathcal{C}}$, then the the set of points in $[x]_{\mathcal{C}}$ is the relative interior of the face of x.*

Remark 2.2.6. The equivalence class of 0 is the set of points

$$[0]_{\mathcal{C}} = \{v \in \overline{\mathcal{C}} \mid -v \in \overline{\mathcal{C}}\}.$$

So, the equivalence class of the 0 in a properly convex cone $\mathcal{C}$ is $[0]_{\mathcal{C}} = \{0\}$.

On the other hand, we saw in Remark 1.3.8 that if the cone $\mathcal{C}$ is not properly convex then $[0]_{\mathcal{C}} \cong \mathbb{R}^k$ for some $1 \leq k \leq n+1$. Thus, the structure of the cone is $\mathcal{C} \cong [0]_{\mathcal{C}} \times \mathcal{C}'$, where $\mathcal{C}'$ is a properly convex cone of dimension $n + 1 - \dim[0]_{\mathcal{C}}$.

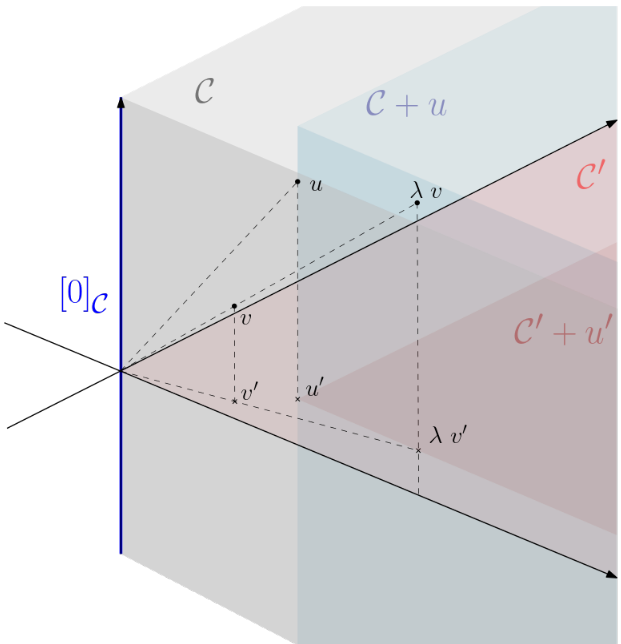


Figure 2.2: The gauge for non-proper cones.

Therefore, for every $u, v \in \mathcal{C}$, if we denote u' the projection of u onto $\mathcal{C}'$ and v' the projection of v onto $\mathcal{C}'$, we get

$$M_{\mathcal{C}}(u/v) = M_{\mathcal{C}'}(u'/v').$$

Figure 2.2 shows an example of this situation.

This remark introduces the following definition.

Definition 2.2.7. Let $\mathcal{C} \subset \mathbb{R}^{n+1}$ be an open convex cone. The *dimension of the Hilbert geometry associated with* $\mathcal{C}$ is $n - \dim[0]_{\mathcal{C}}$.

This number is indeed the projective dimension of the Hilbert geometry associated with the properly convex domain $\pi(\mathcal{C}/[0]_{\mathcal{C}})$, where $\pi : \mathbb{R}^{n+1-\dim[0]_{\mathcal{C}}} \setminus \{0\} \to \mathbb{P}n - \dim[0]_{\mathcal{C}}$ is the natural quotient map.

The next proposition shows that if u and v are contained in a bounded cross-section of the cone $\mathcal{C}$, then $M_{\mathcal{C}}(u/v)$ coincide with the argument in the logarithm of the second addend on the right side of the equation (2.14).

Proposition 2.2.8. *Let $\mathcal{C}$ be a convex open cone in $\mathbb{R}^{n+1}$ and D be a relatively compact cross-section of $\mathcal{C}$. If $u, v \in D$ are two distinct points then*

$$M_{\mathcal{C}}(u/v) = \frac{\|u - v_\infty\|}{\|v - v_\infty\|}$$

where v_∞ is the point nearest to v in the intersection between ∂D and the line through u and v.

Proof. We can work on the plane defined by u and v as in Figure 2.3. Since $v \in \mathcal{C}$, the value $\lambda = M_{\mathcal{C}}(u/v)$ is finite and $w = \lambda v - u \in \partial\mathcal{C}$. The point v_∞ is given by $v_\infty = u + \tau(v - u)$ with $\tau > 1$. Moreover, the cross-section is defined as the intersection of $\mathcal{C}$ with an affine hyperplane $\{z \in \mathbb{R} \mid \phi(z) = 1\}$, where ϕ is a linear functional. Hence, $v_\infty = \frac{1}{\phi(w)} w$ and $\lambda = \dfrac{\tau}{1-\tau} = \dfrac{\|u - v_\infty\|}{\|v - v_\infty\|}$. □

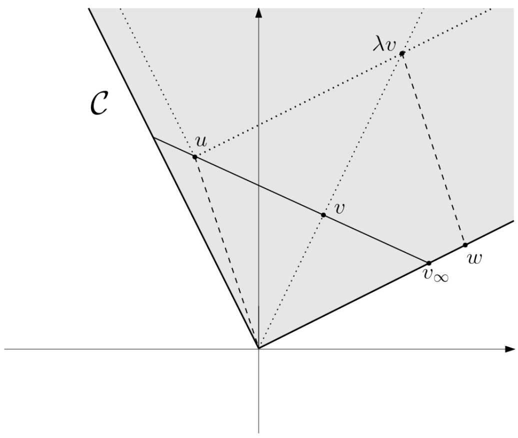


Figure 2.3: Construction used in the proof of Proposition 2.2.8.

Remark 2.2.9. In some cases, the proof of this proposition works also in the context of non-proper cones. Indeed, we only used the existence of the point $v_\infty \in \partial\mathcal{C}$. Therefore, every time this point is well-defined, the gauge can be computed in this way.

Lemma 2.2.10. *Given $v \in \partial\mathcal{C} \setminus [0]_\mathcal{C}$ and $u \in \mathcal{C}$, we parametrize the points of the open segment from v to u as $v_t = (1-t)v + tu$, for every $t \in]0,1[$. Then,*

$$\lim_{t\to 0} M_\mathcal{C}(v/v_t) = 1.$$

Proof. For every $t \in]0,1[$, we have that

$$\begin{aligned} M_\mathcal{C}(v/v_t) &= \inf\{\lambda > 0 \mid v \leq_\mathcal{C} \lambda((1-t)v + tu)\} \\ &= \inf\{\lambda > 0 \mid (\frac{1}{\lambda} - 1)v \leq_\mathcal{C} t(u-v)\} \\ &= \frac{1}{\sup\{\beta > -1 \mid \beta v \leq_\mathcal{C} t(u-v)\} + 1} \\ &= \frac{1}{t\sup\{\gamma > -\frac{1}{t} \mid \gamma v \leq_\mathcal{C} u - v\} + 1}. \end{aligned}$$

As $t \to 0$ the supremum $\sup\{\gamma > -\frac{1}{t} \mid \gamma v \leq_\mathcal{C} u - v\}$ tends to $\sup\{\delta \in \mathbb{R} \mid \delta v \leq_\mathcal{C} u - v\}$, and this supremum is finite since $v \not\leq_\mathcal{C} 0$. Then, the denominator of the last term of the equation above, tends to 1 as $t \to 0$. □

Definition 2.2.11. Let $\mathcal{C}$ be a convex open cone in $\mathbb{R}^{n+1}$. The *Birkhoff's version of the Hilbert metric* on $\mathcal{C}$ is the function $\mathcal{H}_\mathcal{C} : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ defined as

$$\mathcal{H}_\mathcal{C}(u,v) = \log M_\mathcal{C}(u/v) + \log M_\mathcal{C}(v/u) \quad \text{for all } u, v \in \mathcal{C}.$$

The original Birkhoff's version of the Hilbert distance was defined on the closure $\overline{\mathcal{C}}$ by setting $\mathcal{H}_\mathcal{C}(u,v) = \log M_\mathcal{C}(u/v) + \log M_\mathcal{C}(v/u)$, if $u \sim_\mathcal{C} v$, and $\mathcal{H}_\mathcal{C}(u,v) = +\infty$, otherwise.

Birkhoff's version of Hilbert metric on a convex cone is a metric on the set of rays of the cone. In fact, the following proposition holds.

Proposition 2.2.12. *Let $\mathcal{C}$ be a convex cone in $\mathbb{R}^{n+1}$, then for each $u, v, w \in \mathcal{C}$,*

1. *$\mathcal{H}_\mathcal{C}(u,v) \geq 0$,*
2. *$\mathcal{H}_\mathcal{C}(u,v) = \mathcal{H}_\mathcal{C}(v,u)$,*

3. $\mathcal{H}_{\mathcal{C}}(u,w) \leq \mathcal{H}_{\mathcal{C}}(u,v) + \mathcal{H}_{\mathcal{C}}(v,w)$*,*

4. $\mathcal{H}_{\mathcal{C}}(u,v) = \mathcal{H}_{\mathcal{C}}(\lambda u, \mu v)$ *for all* $\lambda, \mu > 0$*.*

Moreover, $\mathcal{H}_{\mathcal{C}}(u,v) = 0$ *if, and only if,* $u = \lambda v$ *for some* $\lambda \geq 0$*.*

Proof. To prove the first property, we notice that for each $\alpha \in \left]0, \frac{1}{M_{\mathcal{C}}(v/u)}\right[$ and $\beta \in \left]M_{\mathcal{C}}(u/v), +\infty\right[$ we have

$$\alpha v \leq_{\mathcal{C}} u \leq_{\mathcal{C}} \beta v.$$

It follows that $v \leq_{\mathcal{C}} \frac{\beta}{\alpha} v$, and hence $\frac{\beta}{\alpha} \geq 1$. Thus, $M_{\mathcal{C}}(u/v) M_{\mathcal{C}}(v/u) \geq 1$ and hence $\mathcal{H}_{\mathcal{C}}$ is non-negative. The second assertion is obvious true.

To show that $\mathcal{H}_{\mathcal{C}}$ satisfies the triangle inequality, we note that for each $\alpha \in \left]0, \frac{1}{M_{\mathcal{C}}(v/u)}\right[$ and $\gamma \in \left]0, \frac{1}{M_{\mathcal{C}}(w/v)}\right[$ we have $\alpha v \leq_{\mathcal{C}} u$ and $\gamma w \leq_{\mathcal{C}} v$, hence $\alpha\gamma w \leq_{\mathcal{C}} u$. This implies that $M_{\mathcal{C}}(w/u) \geq M_{\mathcal{C}}(v/u) M_{\mathcal{C}}(w/v)$. In the same way, it can be shown that $M_{\mathcal{C}}(u/w) \geq M_{\mathcal{C}}(u/v) M_{\mathcal{C}}(v/w)$. Thus,

$$M_{\mathcal{C}}(u/w) M_{\mathcal{C}}(w/u) \geq M_{\mathcal{C}}(u/v) M_{\mathcal{C}}(u/v) M_{\mathcal{C}}(w/v) M_{\mathcal{C}}(v/w).$$

The third assertion follows by taking the logarithm on both sides of this inequality.

To prove that the fourth property holds, take $u, v \in \mathcal{C}$ and $\lambda, \mu > 0$. It follows from the definition of the gauge that

$$M_{\mathcal{C}}(\lambda u/\mu v) = \frac{\lambda}{\mu} M_{\mathcal{C}}(u/v) \quad \text{and} \quad M_{\mathcal{C}}(\lambda u/\mu v) = \frac{\mu}{\lambda} M_{\mathcal{C}}(u/v).$$

Finally, given $u, v \in \mathcal{C}$, we have that $\frac{1}{M_{\mathcal{C}}(v/u)} v \leq_{\mathcal{C}} u \leq_{\mathcal{C}} M_{\mathcal{C}}(u/v) v$.

If $\mathcal{H}_{\mathcal{C}}(u,v) = 0$, then $M_{\mathcal{C}}(u/v) M_{\mathcal{C}}(v/u) = 1$. So, we get $v \leq_{\mathcal{C}} M_{\mathcal{C}}(v/u) u \leq_{\mathcal{C}} v$, from which we deduce that $u = \frac{1}{M_{\mathcal{C}}(v/u)} v$.

On the other hand, if $u = \lambda v$ for some $\lambda > 0$, then $\mathcal{H}_{\mathcal{C}}(u,v) = 0$ by the fourth assertion. □

Thanks to Proposition 2.2.8, we can recover the classical version of Hilbert metric on a properly convex domain Ω from Birkhoff's one by considering its restriction to a bounded cross-section of the cone $\mathcal{C}_\Omega$.

Lemma 2.2.13. *Let* $\mathcal{C}$ *be a convex open cone in* $\mathbb{R}^{n+1}$*. If* $u \in \mathbb{R}^{n+1}$ *and* $v \in \mathcal{C}$*, then*

$$M_{\mathcal{C}}(u/v) = \sup_{\varphi \in \overline{\mathcal{C}^*} \setminus \{0\}} \frac{\varphi(u)}{\varphi(v)}. \tag{2.16}$$

Proof. From the definition of the dual cone, we have that $u \leq_{\mathcal{C}} \lambda v$ if and only if $\varphi(\lambda v - u) \geq 0$ for every $\varphi \in \overline{\mathcal{C}^*}$, then it follows the equality in (2.16). □

Proposition 2.2.14. *Let $\mathcal{C}$ be a convex open cone in $\mathbb{R}^{n+1}$. The function $M_{\mathcal{C}}(\cdot/\cdot\cdot)_{|_{\mathbb{R}^{n+1}\times\mathcal{C}}}$ is continuous in both its entries.*

Proof. As we noticed in Remark 1.3.9 the dual cone $\mathcal{C}^*$ is always proper. Thus, $\mathcal{C}^*$ has a relatively compact cross-section D. From (2.16) we obtain that

$$M_{\mathcal{C}}(u/v) = \sup_{\varphi\in\overline{D}} \frac{\varphi(u)}{\varphi(v)} \quad \text{for all } u \in \mathbb{R}^{n+1},\ v \in \mathcal{C}.$$

The function $\frac{\varphi(\cdot)}{\varphi(\cdot\cdot)}$ is continuous in both its entries for every $\varphi \in D$, hence $M_{\mathcal{C}}(u/v)$ is lower-semicontinuous.

To show that $M_{\mathcal{C}}(u/v)$ is upper-semicontinuous, let us consider a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^{n+1}$ and a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$, converging respectively to $u \in \mathbb{R}^{n+1}$ and $v \in \mathcal{C}$. Since $\overline{D}$ is compact, for every $n \in \mathbb{N}$ there is a point $\varphi_n \in \overline{D}$ such that

$$M_{\mathcal{C}}(u_n/v_n) = \frac{\varphi_n(u_n)}{\varphi_n(v_n)}.$$

Up to subsequences, we can assume that the sequence $(z_n)_{n\in\mathbb{N}}$ converges to a point $\varphi \in \overline{D}$ and the sequence $\big(M_{\mathcal{C}}(u_n/v_n)\big)$ converges to its limit supremum $\frac{\varphi(u)}{\varphi(v)}$ that is less then or equal to $M_{\mathcal{C}}(u/v)$. □

To conclude this section we present a very simple proof of the fact that simplicial Hilbert geometries are isometric to normed spaces. However, the converse is also true, as shown by T. Foertsch and A. Karlsson in [15]. This paper contains also a different proof of the following result.

Definition 2.2.15. Let $(\Omega, \mathrm{d}_\Omega)$ be a Hilbert geometry. We say that it is a *polyhedral Hilbert geometry* if the domain is a polyhedra when considered in any affine chart where it is bounded. Moreover, we say that it is a *simplicial Hilbert geometry* if in any affine chart where it is bounded it is a simplex.

Let us define $V = \mathbb{R}^{n+1}/_{x\sim x+\lambda(1,\dots,1)}$, where $x \in \mathbb{R}^{n+1}$ and $\lambda \in \mathbb{R}$. We endow V with the structure of normed space defining the variation norm $\|\cdot\|_{var} : V \to \mathbb{R}$ by

$$\|[x]_\sim\|_{var} = \max_i x_i - \min_i x_i \quad \text{for all } x \in \mathbb{R}^{n+1}.$$

Proposition 2.2.16. *Let $(\Omega, \mathrm{d}_\Omega)$ be the Hilbert geometry, with Ω given by a n-simplex. Then, $(\Omega, \mathrm{d}_\Omega)$ is isometric to $(V, \|\cdot\|_{var})$.*

Proof. We can perform a change of coordinates and suppose that the cone above Ω is $\mathcal{C}_\Omega = \mathbb{R}^{n+1}_+$. Let $u, v \in \mathcal{C}_\Omega$ be two distinct points. If $\alpha = \sup\{\mu > 0 \mid \alpha v \leq_{\mathcal{C}} u\}$ and

$\beta = \inf\{\lambda > 0 \mid u \leq_{\mathcal{C}} \lambda v\}$, then $\mathcal{H}_{\mathcal{C}_\Omega}(u,v) = \log \frac{\beta}{\alpha}$ and $\alpha v \leq_{\mathcal{C}} u \leq_{\mathcal{C}} \beta v$. Therefore, for any $i = 1, \ldots, n+1$ we have $\alpha v_i \leq u_i \leq \beta v_i$. It follows that

$$\mathcal{H}_{\mathcal{C}_\Omega}(u,v) \geq \log\left(\max_i \frac{u_i}{v_i}\right) + \log\left(\min_i \frac{u_i}{v_i}\right).$$

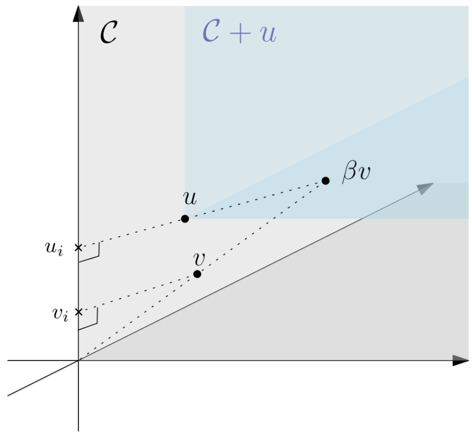


Figure 2.4: Gauge for simplicial cones.

On the other hand, as we can see in Figure 2.4 there is a closed face F of $\mathcal{C}_\Omega + u$ such that $\beta v \in F$. Since $\mathcal{C}_\Omega = \mathbb{R}_+^{n+1}$, there exists $i \in \{1, \ldots, n+1\}$ such that $F = \{z \in \mathbb{R}^{n+1} \mid z_i = u_i\}$. Hence, applying the sine rule, we get $\beta = \frac{u_i}{v_i}$. Similarly, there exists $i \in \{1, \ldots, n+1\}$ such that $\alpha = \frac{u_i}{v_i}$. Therefore, we have

$$\mathcal{H}_{\mathcal{C}_\Omega}(u,v) = \max_i(\log u_i - \log v_i) + \min_i(\log u_i - \log v_i).$$

The map $\mathbb{R}_+^{n+1} \ni u = (u_1, \ldots, u_{n+1}) \mapsto (\log u_1, \ldots, \log u_{n+1}) \in \mathbb{R}^{n+1}$ induces a bijection from the cross-section $D = \{u \in \mathbb{R}_+^{n+1} \mid \sum_i u_i = 1\}$ into V that is an isometry with respect to the Birkhoff's version of the Hilbert metric restricted to D and the variation norm on V. By Proposition 2.2.8, $(\Omega, \mathrm{d}_\Omega)$ is isometric to D with the restriction of the Birkhoff's version of the Hilbert metric, and this concludes the proof. □

For completeness, we state the following result from [15].

Proposition 2.2.17 ([15, Theorem 2]). *Let $(\Omega, \mathrm{d}_\Omega)$ be a Hilbert geometry. Then, $(\Omega, \mathrm{d}_\Omega)$ is isometric to normed space if and only if it is a simplicial Hilbert geometry.*

2.2.2 Funk metric

It is useful in what follows to introduce some definitions and notations. In particular, we will work with the *Funk (weak) metric*[1] and the *reverse-Funk (weak) metric* on a given open cone.

Definition 2.2.18. Let $\mathcal{C} \in \mathbb{R}^{n+1}$ be an open (non necessarily proper) convex cone. The Funk metric on $\mathcal{C}$ is the function $\mathcal{F}_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ given by

$$\mathcal{F}_{\mathcal{C}}(u, v) = \log M_{\mathcal{C}}(u/v) \quad \text{for all } u, v \in \mathcal{C}.$$

The reverse-Funk metric on $\mathcal{C}$ is the function $\mathcal{RF}_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ given by

$$\mathcal{RF}_{\mathcal{C}}(u, v) = \log M_{\mathcal{C}}(v/u) \quad \text{for all } u, v \in \mathcal{C}.$$

The Birkhoff's version of the Hilbert metric is the symmetrization of the Funk metric, i.e.

$$\mathcal{H}_{\mathcal{C}}(u, v) = \mathcal{F}_{\mathcal{C}}(u, v) + \mathcal{RF}_{\mathcal{C}}(u, v) \quad \text{for all } u, v \in \mathcal{C}.$$

An important fact to point out is that the Funk metric is not only non-symmetric, but can assign a negative number to a pair of points on the cone. We use the term *metric* as a convention. However, the restriction of $\mathcal{F}_{\mathcal{C}}$ to a bounded cross-section of the cone is a semi-metric, by Proposition 2.2.8.

Remark 2.2.19. As we have seen in the proof of 2.2.12 the Funk metric satisfies the following conditions

1. $\mathcal{F}_{\mathcal{C}}(\lambda u, \mu v) = \mathcal{F}_{\mathcal{C}}(u, v) + \log \lambda + \log \mu \quad$ for all $u, v \in \mathcal{C}$ and $\lambda, \mu > 0$,

2. $\mathcal{F}_{\mathcal{C}}(u, w) \leq \mathcal{F}_{\mathcal{C}}(u, v) + \mathcal{F}_{\mathcal{C}}(v, w) \quad$ for all $u, v, w \in \mathcal{C}$.

Recall that, given a Hilbert geometry $(\Omega, \mathrm{d}_\Omega)$ and a base point $b \in \Omega$, we are interested in the study of the limits in the topology of uniform convergence on compact sets of sequences of functions of the type $(\Psi_{b,z_n})_{n\in\mathbb{N}}$, where $\Psi_{b,z_n} : \Omega \to \mathbb{R}$ is given by $\Psi_{b,z_n}(x) = \mathrm{d}_\Omega(x, z_n) - \mathrm{d}_\Omega(b, z_n)$ for every $x \in \Omega$ and $(z_n)_{n\in\mathbb{N}}$ is a sequence in Ω.

The key to a complete interpretation of these limits is to work in the cone $\mathcal{C}_\Omega$ and to use the Birkhoff's version of the Hilbert metric. In fact, by Proposition 2.2.12, $\mathcal{H}_{\mathcal{C}_\Omega}$ is a distance on the set of rays of the cone $\mathcal{C}_\Omega$ above Ω, and the rays of $\mathcal{C}_\Omega$ correspond to the points of Ω. Moreover, the Hilbert distance between two points

[1] The Funk metric has been introduced by P. Funk in [16].

of Ω can be computed in any affine chart where Ω is bounded, and the images of Ω in these affine charts correspond to bounded cross-sections of the cone $\mathcal{C}_\Omega$.

Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in Ω and $b \in \Omega$ be a fixed base point. As we noticed in Remark 2.1.10, the pointwise limit of $(\Psi_{b,z_n})_{n\in\mathbb{N}}$ is a uniform limit on each compact subset of Ω, up to a subsequence. It follows that the horofunction compactification of a Hilbert geometry coincide with the set of pointwise limits of sequences in $\Psi_b(\Omega)$.

We want to show that the set of pointwise limits of sequences in $\Psi_b(\Omega)$ corresponds to the set of pointwise limits of paricular sequences of functions defined on the cone $\mathcal{C}_\Omega$.

For this aim, let us consider a sequence $(z_n)_{n\in\mathbb{N}}$ in Ω. Once fixed $x \in \Omega$ we have to study the limit of the $(\Psi_{b,z_n}(x))_{n\in\mathbb{N}}$.

If $x = b$, then $\Psi_{b,z_n}(x) = 0$. On the other hand, given $u, v \in \mathcal{C}_\Omega$ such that $[u] = x = b = [v]$, by Proposition 2.2.12 we have

$$\mathcal{H}_{\mathcal{C}_\Omega}(u, w_n) - \mathcal{H}_{\mathcal{C}_\Omega}(v, w_n) = 0 = \Psi_{b,z_n}(x).$$

If $x \neq b$, we can pick u and v in a bounded cross-section D of the properly convex cone $\mathcal{C}_\Omega$ such that $x = [u],\ b = [v]$. By Proposition 2.2.8, if for every $n \in \mathbb{N}$ we pick $w_n \in D$ such that $[w_n] = z_n$, then

$$\Psi_{b,z_n}(x) = \mathrm{d}_\Omega(x, z_n) - \mathrm{d}_\Omega(b, z_n) = \mathcal{H}_{\mathcal{C}_\Omega}(u, w_n) - \mathcal{H}_{\mathcal{C}_\Omega}(v, w_n).$$

For the arbitrariness of $x \in \Omega$, if ξ is the limit of $(\Psi_{b,z_n})_{n\in\mathbb{N}}$, then $\xi \circ \pi_{|\mathcal{C}}$ is the limit of the sequence in $\mathrm{C}(\mathcal{C}_\Omega, \mathbb{R})$ given by

$$\Big(u \mapsto \mathcal{H}_{\mathcal{C}_\Omega}(u, w_n) - \mathcal{H}_{\mathcal{C}_\Omega}(u, w_n)\Big)_{n\in\mathbb{N}}. \tag{2.17}$$

Conversely, suppose that $h \in \mathrm{C}(\mathcal{C}_\Omega, \mathbb{R})$ is the limit of a sequence

$$\Big(u \mapsto \mathcal{H}_{\mathcal{C}_\Omega}(u, w_n) - \mathcal{H}_{\mathcal{C}_\Omega}(v, w_n)\Big)_{n\in\mathbb{N}},$$

with $(w_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{C}_\Omega$. From Proposition 2.2.12, for every sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive real numbers, it holds that

$$\mathcal{H}_{\mathcal{C}_\Omega}(u, w_n) - \mathcal{H}_{\mathcal{C}_\Omega}(v, w_n) = \mathcal{H}_{\mathcal{C}_\Omega}(u, \lambda_n w_n) - \mathcal{H}_{\mathcal{C}_\Omega}(v, \lambda_n w_n).$$

Thus, we can suppose that $(w_n)_{n\in\mathbb{N}}$ lies in a bounded cross-section D of the cone. If we denote $\phi_D : \Omega \to D$ the identification between Ω and D, we get that the function

$\xi : \Omega \to \mathbb{R}$ given by

$$\xi(x) = h_{|_D} \circ \phi_D(x)$$

is the limit of $(\Psi_{b,[w_n]})_{n\in\mathbb{N}}$, with $b = [v]$.

What we obtain is that every horofunction of the Hilbert geometry $(\Omega, \mathrm{d}_\Omega)$ corresponds to the limit of a sequence in $\mathrm{C}(\mathcal{C}_\Omega, \mathbb{R})$ of the type in (2.17).

Even if the cone $\mathcal{C}_\Omega$ above a properly convex domain Ω is properly convex, as anticipated, we will need to work also with non-proper convex cones.

For this reason, we will consider for a while a pair $(\mathcal{C}, \mathcal{H}_\mathcal{C})$, where $\mathcal{C}$ is a (possibly non-proper) convex open cone in $\mathbb{R}^{n+1}$ and $\mathcal{H}_\mathcal{C}$ the Birkhoff's version of Hilbert metric on $\mathcal{C}$.

We endow the cone $\mathcal{C}$ with the Euclidean topology and the set of real valued continuous functions $\mathrm{C}(\mathcal{C}, \mathbb{R})$ with the topology of uniform convergence on compact sets.

Since $\mathcal{H}_\mathcal{C}$ is the symmetrization of $\mathcal{F}_\mathcal{C}$, given a sequence of points $(w_n)_{n\in\mathbb{N}}$ in $\mathcal{C}_\Omega$ and a base point $b \in \mathcal{C}_\Omega$, when we study the limit of the sequence of functions in (2.17), we can split the problem into two sub-problems. We can study separately the limit of

$$\Big(u \mapsto \mathcal{F}_\mathcal{C}(u, w_n) - \mathcal{F}_\mathcal{C}(b, w_n)\Big)_{n\in\mathbb{N}}. \tag{2.18}$$

and the limit of

$$\Big(u \mapsto \mathcal{R}\mathcal{F}_\mathcal{C}(u, w_n) - \mathcal{R}\mathcal{F}_\mathcal{C}(b, w_n)\Big)_{n\in\mathbb{N}}. \tag{2.19}$$

The next steps, in order to completely classify the set of Busemann points of a Hilbert geometry $(\Omega, \mathrm{d}_\Omega)$ will be to determine the set of pointwise limits of sequence in $\mathrm{C}(\mathcal{C}_\Omega, \mathbb{R})$ of the type in (2.18), to determine the set of pointwise limits of sequence in $\mathrm{C}(\mathcal{C}_\Omega, \mathbb{R})$ of the type in (2.19), and then combine these results to determine the set of pointwise limits of sequences of the type in (2.17). During the discussion, we will find some conditions in order to recognize Busemann points in $\mathcal{B}_\Omega$ among the horofunctions corresponding to the limits of sequences of the type in (2.17).

Definition 2.2.20. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be an open convex cone. Given a point $v \in \mathcal{C}$, we denote by $f_{\mathcal{C},v}$ the function from $\mathcal{C}$ to $\mathbb{R}$ defined by

$$f_{\mathcal{C},v}(u) = \mathcal{F}_\mathcal{C}(u, v) - \mathcal{F}_\mathcal{C}(b, v) \quad \text{for all } u \in \mathcal{C}.$$

We define $\mathcal{K}^\mathcal{F}_\mathcal{C} = \{f_{\mathcal{C},v} \mid v \in \mathcal{C}\}$.

Definition 2.2.21. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be an open convex cone, possibly non-proper.

Given a point $v \in \mathcal{C}$, we denote by $r_{\mathcal{C},v}$ the function from $\mathcal{C}$ to $\mathbb{R}$ defined by

$$r_{\mathcal{C},v}(u) = \mathcal{RF}_{\mathcal{C}}(u,v) - \mathcal{RF}_{\mathcal{C}}(b,v) \quad \text{for all } u \in \mathcal{C}.$$

We define $\mathcal{K}_{\mathcal{C}}^{\mathcal{RF}} = \{r_{\mathcal{C},v} \mid v \in \mathcal{C}\}$.

Proposition 2.2.22. *Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a convex open cone. Then, the closure $\overline{\mathcal{K}_{\mathcal{C}}^{\mathcal{F}}} \subseteq \mathrm{C}(\mathcal{C},\mathbb{R})$ of the set $\mathcal{K}_{\mathcal{C}}^{\mathcal{F}}$ and the closure $\overline{\mathcal{K}_{\mathcal{C}}^{\mathcal{RF}}} \subseteq \mathrm{C}(\mathcal{C}\mathbb{R})$ of the set $\mathcal{K}_{\mathcal{C}}^{\mathcal{RF}}$ are compact.*

Proof. We prove that the closure $\overline{\mathcal{K}_{\mathcal{C}}^{\mathcal{F}}}$ is compact. A similar reasoning shows that the closure $\overline{\mathcal{K}_{\mathcal{C}}^{\mathcal{RF}}}$ is compact.

It follows from the triangular inequality that $f_{\mathcal{C},v}(u) \in [-|\mathcal{F}_{\mathcal{C}}(u,b)|, |\mathcal{F}_{\mathcal{C}}(u,b)|]$ for all $u, v \in \Omega$. Hence, $(\mathcal{K}_{\mathcal{C}}^{\mathcal{F}})_u = \{f_{\mathcal{C},v}(u) \mid v \in \Omega\}$ is relatively compact for all $u \in \Omega$.

If we prove that the family of functions $\mathcal{K}_{\mathcal{C}}^{\mathcal{F}}$ is equicontinuous, then by Ascoli-Arzelà Theorem [25, Theorem 47.1], we have that $\mathcal{K}_{\mathcal{C}}^{\mathcal{F}}$ has compact closure with respect to the topology of uniform convergence on compact sets.

For every $v \in \mathcal{C}$ and $u_1, u_2 \in \mathcal{C}$ we have

$$|f_{\mathcal{C},v}(u_1) - f_{\mathcal{C},v}(u_2)| = |\mathcal{F}_{\mathcal{C}}(u_1,v) - \mathcal{F}_{\mathcal{C}}(u_2,v)| \le |\mathcal{F}_{\mathcal{C}}(u_1,u_2)|. \tag{2.20}$$

Let $u_0 \in \mathcal{C}$. We have to show that for every $\varepsilon > 0$ there exists some $\delta > 0$ such that for any $u \in B_{eucl}(u_0, \delta)$

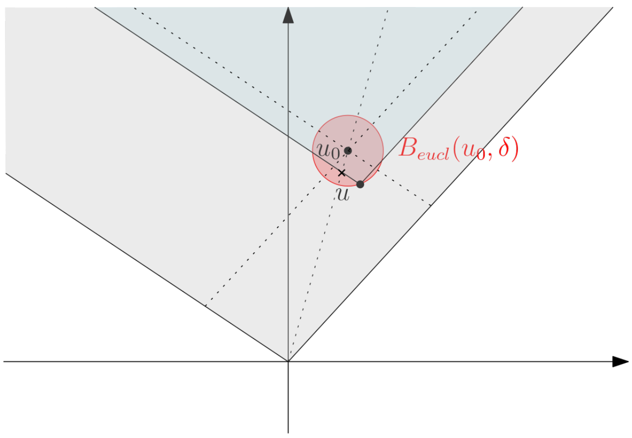
$$|f_{\mathcal{C},v}(u) - f_{\mathcal{C},v}(u_0)| < \varepsilon.$$

From (2.20) $|f_{\mathcal{C},v}(u) - f_{\mathcal{C},v}(u_0)| \le |\mathcal{F}_{\mathcal{C}}(u,u_0)|$. Then, by Remark 2.2.6, it is not restrictive to assume that $\mathcal{C}$ is proper.

We show that we can pick a sufficiently small $\delta > 0$ such that $B_{eucl}(u_0,\delta) \subseteq \mathcal{C}$ and $|\mathcal{F}_{\mathcal{C}}(u_0,u)| < \varepsilon$ for all $u \in B_{eucl}(u_0,\delta)$.

Fix an arbitrary $\delta > 0$ such that $B_{eucl}(u_0,\delta) \subseteq \mathcal{C}$. Given $u \in B_{eucl}(u_0,\delta) \subseteq \mathcal{C}$, we can work in the 2-dimensional cone obtained intersecting $\mathcal{C}$ with $\mathrm{Span}\, u, v$. Then, we have to study three different cases. The first is when $u = \lambda u_0$, with $\lambda \ge 1$, the second is when it is well defined the point u_0^∞ in the intersection of the line through u and u_0, on the side of u_0, and the third is the case when the previous conditions does not holds.

In the first case, we have $u = \lambda u_0$, with $\lambda \ge 1$. Then, we have $M_{\mathcal{C}}(u/u_0) = \lambda$. On the other hand, $\lambda \le 1+\delta$, since $u \in B_{eucl}(u_0,\delta)$. Therefore, $\mathcal{F}_{\mathcal{C}}(u,u_0) \le \log(1+\delta) < \delta$.



In the second case, from Proposition 2.2.8 we get $\mathcal{F}_{\mathcal{C}}(u,u_0) = \log\left(1+\frac{\delta}{\|u_0-u_0^\infty\|}\right) < \frac{\delta}{\|u_0-u_0^\infty\|}$.

In the third case, $u_0 \in \mathcal{C}+u$ and $u \in B_{eucl}(u_0,\delta)$. Then, $\frac{1}{1+\delta} \le M_{\mathcal{C}}(u/u_0) \le 1$.

Therefore, choosing $\delta < \min\{\varepsilon,\ \varepsilon - \mathrm{dist}(u_0,\partial\mathcal{C})\}$, it holds $|f_{\mathcal{C},v}(u) - f_{\mathcal{C},v}(u_0)| < \varepsilon$, for all $u \in B_{eucl}(u_0,\delta)$. □

We emulate the definition of horofunction given in Section 2.1. We fix a base point $b \in \mathcal{C}$. To simplify the notations, we will not specify the dependence on b in the following definitions. These notations may appear redundant, but they will be useful in the next sections.

Definition 2.2.23. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be an open convex cone, possibly non-proper. A continuous function $h \in \mathrm{C}(\mathcal{C},\mathbb{R})$ is a *Funk horofunction* if h is the pointwise limit of a sequence $(f_{\mathcal{C},v_n})_{n\in\mathbb{N}}$ with $(v_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{C}$, and $h \neq f_{\mathcal{C},v}$ for every $v \in \mathcal{C}$. We denote the set of Funk horofunctions by $\partial_\infty^{\mathcal{F}}\mathcal{C}$.

Moreover, we say that a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$ converges in Funk sense to a continuous function $h \in \mathrm{C}(\mathcal{C},\mathbb{R})$ if $(f_{\mathcal{C},v_n})_{n\in\mathbb{N}}$ converges to h.

Definition 2.2.24. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be an open convex cone, possibly non-proper. A continuous function $h \in \mathrm{C}(\mathcal{C},\mathbb{R})$ is a *reverse-Funk horofunction* if h is the pointwise limit of a sequence $(r_{\mathcal{C},v_n})_{n\in\mathbb{N}}$ with $(v_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{C}$, and $h \neq r_{\mathcal{C},v}$ for every $v \in \mathcal{C}$. We denote the set of reverse-Funk horofunctions by $\partial_\infty^{\mathcal{RF}}\mathcal{C}$.

Moreover, we say that a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$ converges in reverse-Funk sense to a continuous function $h \in \mathrm{C}(\mathcal{C},\mathbb{R})$ if $(r_{\mathcal{C},v_n})_{n\in\mathbb{N}}$ converges to h.

Definition 2.2.25. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be an open convex cone, possibly non-proper. A continuous function $h \in \mathrm{C}(\mathcal{C},\mathbb{R})$ is a *Hilbert horofunction* if h is the pointwise limit of a sequence $(r_{\mathcal{C},v_n} + f_{\mathcal{C},v_n})_{n\in\mathbb{N}}$ with $(v_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{C}$, and $h \neq r_{\mathcal{C},v} + f_{\mathcal{C},v}$ for every $v \in \mathcal{C}$. We denote the set of reverse-Funk horofunctions by $\partial_\infty^{\mathcal{H}}\mathcal{C}$.

Moreover, we say that a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$ converges in Hilbert sense to a continuous function $h \in \mathrm{C}(\mathcal{C},\mathbb{R})$ if $(r_{\mathcal{C},v_n} + f_{\mathcal{C},v_n})_{n\in\mathbb{N}}$ converges to h.

More care must be paid to the extension of the definition of almost-geodesic. In fact, in Definition 2.1.7 we require that the domain of the almost-geodesic is $\mathbb{R}_+$. But, in the case of the reverse-Funk metric, we have $\mathcal{RF}_{\mathcal{C}}(b,v) < \infty$ for every base point $b \in \mathcal{C}$ and every point $v \in \partial\mathcal{C}$ of the boundary. Since we want a straight line segment from an inner point to a point in the boundary to be an almost-geodesic, we have to change the definition. In addition, it is more convenient to give a discrete version of almost-geodesic.

Definition 2.2.26. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be an open convex cone, possibly non-proper, and $(v_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}$. We say that $(v_n)_{n\in\mathbb{N}}$ is a *Funk almost-geodesic* if there exists some $\varepsilon > 0$ such that

$$\mathcal{F}_{\mathcal{C}}(x_0,x_1) + \cdots + \mathcal{F}_{\mathcal{C}}(x_{n-1},x_n) < \mathcal{F}_{\mathcal{C}}(x_0,x_n) + \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Definition 2.2.27. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be an open convex cone, possibly non-proper, and $(v_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}$. We say that $(v_n)_{n\in\mathbb{N}}$ is a *reverse-Funk almost-geodesic* if there exists some $\varepsilon > 0$ such that

$$\mathcal{RF}_{\mathcal{C}}(x_0,x_1) + \cdots + \mathcal{RF}_{\mathcal{C}}(x_{n-1},x_n) < \mathcal{RF}_{\mathcal{C}}(x_0,x_n) + \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Definition 2.2.28. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be an open convex cone, possibly non-proper, and $(v_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}$. We say that $(v_n)_{n\in\mathbb{N}}$ is a *Hilbert almost-geodesic* if there exists some $\varepsilon > 0$ such that

$$\mathcal{H}_{\mathcal{C}}(x_0,x_1) + \cdots + \mathcal{H}_{\mathcal{C}}(x_{n-1},x_n) < \mathcal{H}_{\mathcal{C}}(x_0,x_n) + \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Remark 2.2.29. Let us consider a line segment between two points $u, v \in \mathcal{C}$. For every increasing sequence $(t_n)_{n\in\mathbb{N}} \subseteq]0,1[$ the sequence $(v_n)_{n\in\mathbb{N}}$ given by $v_n = (1-t_n)u + t_n v$, is both a Funk almost-geodesic and a reverse-Funk almost-geodesic. Indeed we prove that, given $l \leq m \leq n$, the following hold

$$\mathcal{F}_{\mathcal{C}}(v_l,v_n) = \mathcal{F}_{\mathcal{C}}(v_l,v_m) + \mathcal{F}_{\mathcal{C}}(v_m,v_n) \tag{2.21}$$

$$\mathcal{RF}_{\mathcal{C}}(v_l,v_n) = \mathcal{RF}_{\mathcal{C}}(v_l,v_m) + \mathcal{RF}_{\mathcal{C}}(v_m,v_n).$$

If the intersection between the cone and the plane spanned by u and v is a proper cone, then the segment belongs to a bounded cross-section, and from Proposition

2.2.8 we have

$$\begin{aligned}\mathcal{F}_{\mathcal{C}}(v_l, v_n) &= \log\Big(\frac{\|v_l - v_\infty\|}{\|v_n - v_\infty\|}\Big)\\ &= \log\Big(\frac{\|v_l - v_\infty\|}{\|v_m - v_\infty\|}\frac{\|v_m - v_\infty\|}{\|v_n - v_\infty\|}\Big)\\ &= \mathcal{F}_{\mathcal{C}}(v_l, v_m) + \mathcal{F}_{\mathcal{C}}(v_m, v_n),\end{aligned}$$

where v_∞ is the intersection of the line passing through u and v on the side of v, and we have

$$\begin{aligned}\mathcal{RF}_{\mathcal{C}}(v_l, v_n) &= \log\Big(\frac{\|v_l - v_\infty\|}{\|v_n - v_\infty\|}\Big)\\ &= \log\Big(\frac{\|v_l - v_\infty\|}{\|v_m - v_\infty\|}\frac{\|v_m - v_\infty\|}{\|v_n - v_\infty\|}\Big)\\ &= \mathcal{RF}_{\mathcal{C}}(v_l, v_m) + \mathcal{RF}_{\mathcal{C}}(v_m, v_n),\end{aligned}$$

where u_∞ is the intersection of the line passing through u and v on the side of u.

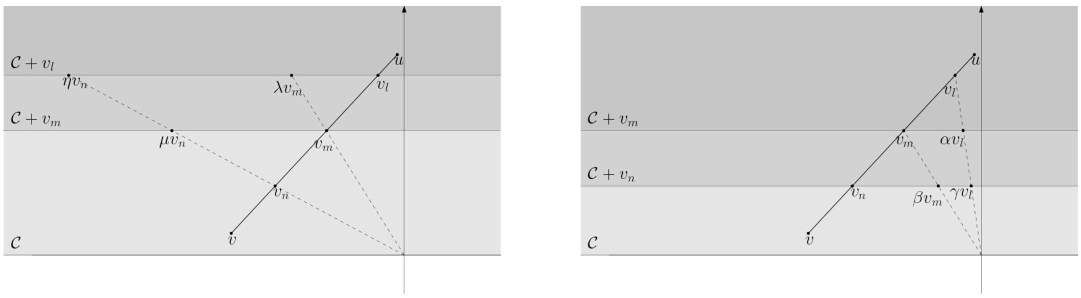


Figure 2.5: Example of a Funk almost-geodesic and reverse-Funk almost-geodesic.

If the intersection between the cone and the plane spanned by u and v is not proper, then it result to be a half-space. Define $\lambda = M_{\mathcal{C}}(v_l/v_m)$, $\mu = M_{\mathcal{C}}(v_m/v_n)$ and $\eta = M_{\mathcal{C}}(v_l/v_n)$. As we can see on the left hand side of Figure 2.5, the sine rule implies that

$$\frac{\eta}{\mu} = \frac{\|\eta v_n\|}{\|\mu v_n\|} = \frac{\|\lambda v_l\|}{\|v_l\|} = \lambda.$$

Hence, $\eta = \lambda\mu$. Similarly, as we can see on the right hand side of Figure 2.5, if we define $\alpha = M_{\mathcal{C}}(v_m/v_l)$, $\beta = M_{\mathcal{C}}(v_n/v_m)$ and $\gamma = M_{\mathcal{C}}(v_n/v_l)$, we get $\gamma = \alpha\beta$.

Finally, we can give the definition of Busemann point also in the Funk, the reverse Funk, and the Hilbert contest.

Definition 2.2.30. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be an open convex cone, possibly non-proper. A Funk horofunction $h \in \partial_\infty^{\mathcal{F}}\mathcal{C}$ is a *Funk Busemann point* if it is the limit in Funk sense of a Funk almost-geodesic. The set of Funk Busemann points is denoted by $\mathcal{B}_{\mathcal{C}}^{\mathcal{F}}$.

Definition 2.2.31. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be an open convex cone, possibly non-proper. A reverse-Funk horofunction $h \in \partial_{\infty}^{\mathcal{RF}}\mathcal{C}$ is a *reverse-Funk Busemann point* if it is the limit in reverse-Funk sense of a reverse-Funk almost-geodesic. The set of reverse-Funk Busemann points is denoted by $\mathcal{B}_{\mathcal{C}}^{\mathcal{RF}}$.

Definition 2.2.32. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be an open convex cone, possibly non-proper. A Hilbert horofunction $h \in \partial_{\infty}^{\mathcal{H}}\mathcal{C}$ is a *Hilbert Busemann point* if it is the limit in Hilbert sense of a Hilbert almost-geodesic. The set of Hilbert Busemann points is denoted by $\mathcal{B}_{\mathcal{C}}^{\mathcal{H}}$.

Lemma 2.2.33. *Let $(v_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}$ be a sequence in an open convex cone. Then $(v_n)_{n\in\mathbb{N}}$ is a Hilbert almost-geodesic if and only if it is both a Funk almost-geodesic and a reverse-Funk almost-geodesic.*

Proof. The thesis follows directly from the fact that for every $n \in \mathbb{N}$

$$\begin{aligned}\sum_{k=0}^{n-1} \mathcal{H}_{\mathcal{C}}(v_k, v_{k+1}) + \mathcal{H}_{\mathcal{C}}(v_0, v_n) = &\sum_{k=0}^{n-1} \mathcal{F}_{\mathcal{C}}(v_k, v_{k+1}) + \mathcal{F}_{\mathcal{C}}(v_0, v_n) + \\ &+ \sum_{k=0}^{n-1} \mathcal{RF}_{\mathcal{C}}(v_k, v_{k+1}) + \mathcal{RF}_{\mathcal{C}}(v_0, v_n).\end{aligned}$$

□

The following lemma shows that the definition of Busemann point given for the horofunction compactification and the one given for the Birkhoff's version are equivalent.

Lemma 2.2.34. *Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a properly convex open cone and D be a compact cross-section of $\mathcal{C}$. Denote Ω the natural projection of $\mathcal{C}$. If $\gamma : \mathbb{R}_+ \to \Omega$ is a Rieffel almost-geodesic, then there exists a sequence of points $(v_n)_{n\in\mathbb{N}} \subset D$ that is a Hilbert almost-geodesic and such that $\big([v_n]\big)_{n\in\mathbb{N}}$ belong to the image of γ. Conversely, if $(v_n)_{n\in\mathbb{N}} \in D$ is a Hilbert almost-geodesic, and $n \mapsto \mathcal{H}_{\mathcal{C}}(v_0, v_n)$ is not bounded, there exists a subsequence of $(v_n)_{n\in\mathbb{N}}$ such that $\big([v_n]\big)_{n\in\mathbb{N}}$ belongs to the image of some Rieffel almost-geodesic of Ω.*

Proof. Since D is a compact cross-section, we can work in $(\Omega, \mathrm{d}_\Omega)$, by Proposition 2.2.8.

Let $\gamma : \mathbb{R}_+ \to \Omega$ be a Rieffel almost-geodesic. For every $\varepsilon > 0$ we have, for all sufficiently large $s \leq t$

$$-\varepsilon \leq \mathrm{d}_\Omega(\gamma(0), \gamma(s)) + \mathrm{d}_\Omega(\gamma(s), \gamma(t)) - t \leq \varepsilon \quad \text{and} \quad -\varepsilon \leq \mathrm{d}_\Omega(\gamma(0), \gamma(t)) - t \leq \varepsilon.$$

Thus, starting from $t_0 = 0$, we can construct a strictly increasing sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ that satifies

$$\mathrm{d}_\Omega(\gamma(0), \gamma(t_k)) + \mathrm{d}_\Omega(\gamma(t_k), \gamma(t_{k+1})) - \mathrm{d}_\Omega(\gamma(0), \gamma(t_{k+1})) \leq \frac{1}{2^{k+1}} \quad \text{for all } k \in \mathbb{N}. \tag{2.22}$$

For $n \in \mathbb{N}$, we can take the sum for $k = 1, \ldots, n-1$ of the left hand side of (2.22). What we obtain is that

$$\mathrm{d}_\Omega(\gamma(0), \gamma(t_1)) + \cdots + \mathrm{d}_\Omega(\gamma(t_{n-1}), \gamma(t_n)) \leq \frac{1}{2} + \mathrm{d}_\Omega(\gamma(0), \gamma(t_n)) \quad \text{for all } n \in \mathbb{N}.$$

By Proposition 2.2.8, taking $v_n = D \cap \pi^{-1}(\gamma(t_n))$ for every $n \in \mathbb{N}$, we get the desired Hilbert almost-geodesic.

Let $(v_n)_{n\in\mathbb{N}}$ be a Hilbert almost-geodesic in D. We define $x_n = [v_n] \in \Omega$, for every $n \in \mathbb{N}$. Since the distance $\mathrm{d}_\Omega(x_0, x_n)$ is not bounded, up to a subsequence, we can assume that $n \mapsto \mathrm{d}_\Omega(x_0, x_n)$ is increasing. Therefore, exactly as done in the proof of Proposition 2.1.14, we can prove that the path obtained from the concatenation of the geodesic segments between the consecutive points of the sequence is a Rieffel almost-geodesic. □

In the next section we will explore the set of reverse-Funk Busemann points and the set of Funk Busemann points. The study of the first ones is very simple. Indeed, we can use the continuity of the gauge on $\mathbb{R}^{n+1} \times \mathcal{C}$, in order to study

$$\lim_{n\to\infty} \mathcal{RF}_\mathcal{C}(u, v_n) = \lim_{n\to\infty} M_\mathcal{C}(v_n/u),$$

where $u \in \mathcal{C}$ and $(v_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{C}$.

Quite the opposite, in the Funk case, we cannot use this argument to study

$$\lim_{n\to\infty} \mathcal{F}_\mathcal{C}(u, v_n) = \lim_{n\to\infty} M_\mathcal{C}(u/v_n),$$

when the sequence $(v_n)_{n\in\mathbb{N}}$ approaches the boundary.

2.3 Busemann points of a Hilbert geometry

The aim of this section is to completely characterize the set of Busemann points of a given Hilbert geometry. As we reveald, we study separately the Funk part and the reverse-Funk part of the limit in (2.17).

Theorem 2.3.1. *Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a properly convex cone. Then the following statements hold:*

1. *If a sequence $(v_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}$ converges to $v \in \overline{\mathcal{C}} \setminus \{0\}$, then it converges in reverse-Funk sense to $r_{\mathcal{C},v}$.*

2. *$\partial_\infty^{\mathcal{RF}}\mathcal{C} = \mathcal{B}_{\mathcal{C}}^{\mathcal{RF}} = \{r_{\mathcal{C},v} \mid v \in \partial\mathcal{C} \setminus \{0\}\}$.*

3. *If a sequence $(v_n)_{n\in\mathbb{N}}$ in a cross-section of $\mathcal{C}$ converges in reverse-Funk sense to $r_{\mathcal{C},v}$, then it converges to a scalar multiple of v.*

Proof. The first point follows directly from the continuity of the gauge function restricted to $\mathbb{R}^{n+1} \times \mathcal{C}$, see Lemma 2.2.14.

Let $f \in \partial_\infty^{\mathcal{RF}}\mathcal{C}$ be a horofunction. By definition, f is the reverse-Funk limit of a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$. Since $\mathcal{RF}_{\mathcal{C}}(u, \lambda v_n) - \mathcal{RF}_{\mathcal{C}}(b, \lambda v_n) = \mathcal{RF}_{\mathcal{C}}(u, v_n) - \mathcal{RF}_{\mathcal{C}}(b, v_n)$ for every $u \in \mathcal{C}$, $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$, we can assume that the sequence $(v_n)_{n\in\mathbb{N}}$ is contained in a relatively compact cross-section D of $\mathcal{C}$. Under this assumption, the sequence $(v_n)_{n\in\mathbb{N}}$ converges $\overline{v} \in \overline{D}$, up to a subsequence. The first point implies that the sequence $(v_n)_{n\in\mathbb{N}}$ converges in reverse-Funk sense to $r_{\mathcal{C},\overline{v}}$. Since $\mathcal{C}$ is endowed with the Euclidean topology and the convergence is pointwise, it follows that $f = r_{\mathcal{C},\overline{v}}$. Moreover, $\overline{v} \in \partial\mathcal{C}$ because f is a horofunction.

On the other hand, for every point $v \in \partial\mathcal{C}$ we can consider the segment that joins v to some point of the cone. By Remark 2.2.29, every ordered sequence lying on this segment and converging to v is a reverse-Funk almost-geodesic. The reverse-Funk limit of such a sequence is $r_{\mathcal{C},v}$. So, every function of the type $r_{\mathcal{C},v}$ for some $v \in \partial\mathcal{C}$ is a reverse-Funk Busemann point.

Now, let $u, v \in \mathcal{C}$ be two non-parallel points. We want to show that the functions $r_{\mathcal{C},u}$ and $r_{\mathcal{C},v}$ are distinct. As before, we can assume that u, v and b lie in the same relatively compact cross-section D of $\mathcal{C}$. We define $u_t = (1-t)u + tb$ and $v_t = (1-t)v + tb$, for every $t \in]0,1[$, as depicted in Figure . Then, we can consider the limit

$$\lim_{t\to 0}(r_{\mathcal{C},v}(u_t) - r_{\mathcal{C},u}(u_t) + r_{\mathcal{C},u}(v_t) - r_{\mathcal{C},v}(v_t)). \tag{2.23}$$

If $r_{\mathcal{C},u} = r_{\mathcal{C},v}$ the limit in (2.23) limit equals to 0. But, from Proposition 2.2.8, we have that

$$\mathcal{RF}_{\mathcal{C}}(u_t, v) = \log \frac{\|v - u_t^\infty\|}{\|u_t - u_t^\infty\|} \quad \text{and} \quad \mathcal{RF}_{\mathcal{C}}(v_t, u) = \log \frac{\|u - v_t^\infty\|}{\|v_t - v_t^\infty\|},$$

where u_t^∞ is the point of intersection between ∂D and the line through v and u_t (different from v), and v_t^∞ is the point of intersection between ∂D and the line through u and v_t (different from u). See Figure 2.6.

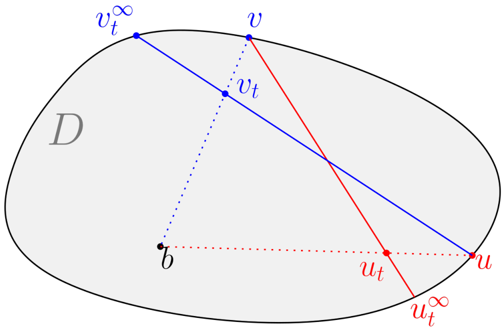


Figure 2.6: Configuration used in the proof of Theorem 2.3.1.

Hence, the limit in (2.23) equals

$$\lim_{t\to 0}\Bigg(\log \frac{\|v - u_t^\infty\|}{\|u_t - u_t^\infty\|} - \mathcal{RF}_{\mathcal{C}}(b, v) - \mathcal{RF}_{\mathcal{C}}(u_t, u) + \mathcal{RF}_{\mathcal{C}}(b, u) + \\ + \log \frac{\|u - v_t^\infty\|}{\|v_t - v_t^\infty\|} - \mathcal{RF}_{\mathcal{C}}(b, v) - \mathcal{RF}_{\mathcal{C}}(v_t, v) + \mathcal{RF}_{\mathcal{C}}(b, v)\Bigg). \tag{2.24}$$

From Lemma 2.2.10, we have

$$\lim_{t\to 0} \mathcal{RF}_{\mathcal{C}}(u_t, u) = \lim_{t\to 0} \log M_{\mathcal{C}}(u/u_t) = 0$$

and

$$\lim_{t\to 0} \mathcal{RF}_{\mathcal{C}}(v_t, v) = \lim_{t\to 0} \log M_{\mathcal{C}}(v/v_t) = 0.$$

Then the limit in (2.24) equals

$$\lim_{t\to 0}\left(\log \frac{\|v - u_t^\infty\|}{\|u_t - u_t^\infty\|} + \log \frac{\|u - v_t^\infty\|}{\|v_t - v_t^\infty\|}\right),$$

which equals

$$\lim_{t\to 0} \log \frac{\|v - u\|^2}{\|u_t - u_t^\infty\|\|v_t - v_t^\infty\|},$$

since $u_t^\infty \to u$ and $v_t^\infty \to v$ as $t \to 0$.

However, this limit equals $+\infty$ since $u \neq v$. Hence, $r_{\mathcal{C},u}$ and $r_{\mathcal{C},v}$ are distinct functions.

To prove the third assertion it suffices to observe that non-parallel limit points in $\partial\mathcal{C}$ correspond to different horofunctions; this follows from what we proved above. Therefore, a sequence in a relatively compact cross-section converges in reverse-Funk sense if and only if it converges in the usual Euclidean sense. Moreover, the usual limit of a sequence can be obtained from the reverse-Funk limit, up to the multiplication by a scalar. □

Theorem 2.3.1 implies that there is a correspondence between reverse-Funk Busemann point and the points of the boundary of the properly convex domain $\Omega_{\mathcal{C}}$ associated with $\mathcal{C}$.

The next step is to determine the set of Funk Busemann points.

As mentioned at the beginning of this chapter, in this section we will work with non-properly convex cones. The way these cones arise is the following. Given an n-dimensional properly convex domain, the set of supporting hyperplanes at a non-C^1-point corresponds to a convex subset of the boundary of the dual with projective dimension at last 1. However, the dimension of this convex is at most $n-2$. So, as we showed in Section 1.3, it can be seen as the dual of an n-dimensional convex domain which is not properly convex. Moreover, we will see that this non-properly convex domain is the projectivization of the *tangent cone* in the non-C^1-point defined below.

It is instructive, and it will turn out to be useful as well, to study the case when the cone is a half-space.

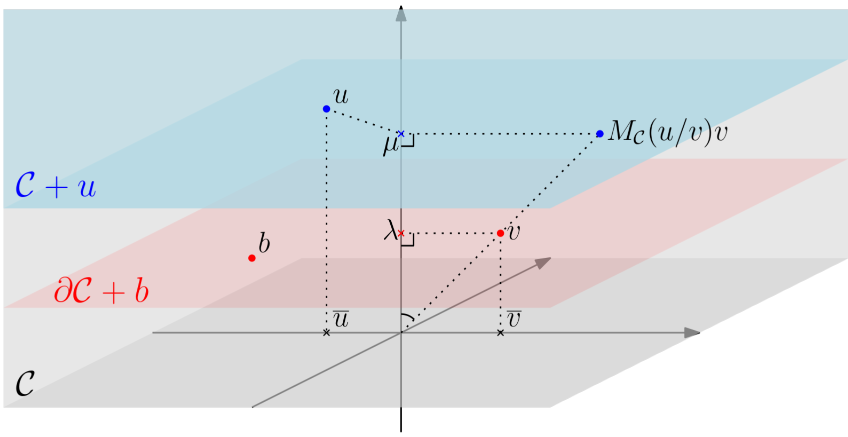
Exemple 2.3.2. In this example we study the Funk limit of sequences in a cone given by a half-space. First of all, we study the function $f_{\mathcal{C},v}$ with $v \in \mathcal{C}$.

Since $\mathcal{C} \cong \mathbb{R}^n \times \mathbb{R}_+$, all the hyperplanes that are parallel to the boundary $\partial\mathcal{C}$ are cross-sections of $\mathcal{C}$.

Since $f_{\mathcal{C},v} = f_{\mathcal{C},\lambda v}$ for all $\lambda > 0$, we can assume that v and b are contained in the same cross-section parallel to $\partial\mathcal{C}$ as in the figure below.

Thus, the point v lies in $\partial\mathcal{C} + b$ and hence $\mathcal{F}_{\mathcal{C}}(b, v) = 0$.

Let u be two points of the cone. Up to a change of coordinates, we can assume that $v = (\overline{v}, \lambda)$ with $\overline{v} \in \mathbb{R}^n$ and $\lambda > 0$, and $u = (\overline{u}, \mu)$ with $\overline{u} \in \mathbb{R}^n$ and $\mu > 0$.



The sine rule implies that $M_{\mathcal{C}}(u/v) = \frac{\mu}{\lambda}$ and hence

$$f_{\mathcal{C},v}(u) = \mathcal{F}_{\mathcal{C}}(u,v) = \log\frac{\mu}{\lambda}.$$

Therefore, for every $v, v' \in \mathcal{C}$ the functions $f_{\mathcal{C},v}$ and $f_{\mathcal{C},v'}$ coincide. It follows that $\mathcal{K}_{\mathcal{C}}^{\mathcal{F}}$ is a single point and $\partial_\infty^{\mathcal{F}}\mathcal{C} = \emptyset$.

Definition 2.3.3. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be an open convex cone. For every $v \in \partial\Omega \setminus [0]_{\mathcal{C}}$, we denote $A_{\mathcal{C}}(v)$ as the set of all horofunctions that can be obtained as the limit in Funk sense of a sequence in $\mathcal{C}$ converging to v.

Remark 2.3.4. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a non-proper cone. From Remark 2.2.6, we know that the Funk limit of a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$ equals the Funk limit of the sequence $(v_n')_{n\in\mathbb{N}}$, where v_n' is the projection of v_n on the maximal properly convex cone $\mathcal{C}'$ contained in $\mathcal{C}$, which in turn equals the Funk limit of a sequence contained in a compact cross-section D' of $\mathcal{C}'$. By compactness, this sequence converges up to a subsequence to a point in $\overline{D'}$. It follows that

$$\bigcup_{v\in\partial\mathcal{C}} A_{\mathcal{C}}(v) = \bigcup_{v\in\partial\mathcal{C}\setminus[0]_{\mathcal{C}}} A_{\mathcal{C}}(v)$$

Now, we introduce the definition of tangent cone.

Definition 2.3.5. Let $\mathcal{C}$ be an open convex cone in $\mathbb{R}^{n+1}$ and $v \in \mathcal{C}$ be a point. The *open tangent cone* at v is defined by

$$\tau(\mathcal{C}, v) = \{\lambda(u - v) \in \mathbb{R}^{n+1} \mid \lambda > 0 \text{ and } u \in \mathcal{C}\}.$$

Remark 2.3.6. Let v be a point in $\mathcal{C}$. Then, $\tau(\mathcal{C}, v)$ is an open convex cone. If $v \in \mathcal{C}$, then $\tau(\mathcal{C}, v) = \mathbb{R}^{n+1}$ and $\mathcal{C} \subseteq \tau(\mathcal{C}, v)$. Indeed, for every $z \in \mathcal{C}$, by convexity and closure under positive homotheties, we get $z+v \in \mathcal{C}$ and $z = (z+v)-v \in \tau(\mathcal{C}, v)$.

Remark 2.3.7. As we noticed in Remark 2.2.6, when we consider the equivalence relation on $\overline{\mathcal{C}}$ induced by $\leq_{\mathcal{C}}$, the equivalence class of 0 is the set

$$[0]_{\mathcal{C}} = \{v \in \overline{\mathcal{C}} \mid -v \in \overline{\mathcal{C}}\}.$$

Since $\mathcal{C}$ is convex, then $[0]_{\mathcal{C}} \subseteq \partial\mathcal{C}$. Moreover, if $v \in [0]_{\mathcal{C}}$, then $\tau(\mathcal{C}, v) = \mathcal{C}$. In fact, by convexity and closure under positive homotheties, for every $u \in \mathcal{C}$ and $\lambda > 0$ we have $\lambda(u - v) \in \mathcal{C}$, since $-v \in \overline{\mathcal{C}}$.

The tangent cone at a point of the boundary is the intersection of all the half-spaces containing the cone, defined by the supporting hyperplanes at the point. In fact, it holds the following result.

Lemma 2.3.8. *Given $v \in \partial\mathcal{C}$, we have that*

$$\tau(\mathcal{C}, v) = \{u \in \mathbb{R}^{n+1} \mid \varphi(u) > 0 \ \forall \varphi \in \mathcal{C}^* \setminus \{0\} \ s.t. \ \varphi(v) = 0\}.$$

Proof. For every $u \in \mathcal{C}$ and $\lambda > 0$, if $\varphi \in \mathcal{C}^*$ and $\varphi(v) = 0$, then $\varphi(\lambda(u - v)) = \lambda\varphi(u) > 0$. Hence, the inclusion $\subseteq$ is proved. To prove the opposite inclusion, let us consider the compact set $V = \{\varphi \in \mathcal{C}^* \setminus \{0\} \mid \varphi(v) = 0 \text{ and } \|\varphi\| = 1\}$. It suffices to prove that given $u \in \mathbb{R}^{n+1}$ such that $\varphi(u) > 0$ for all $\varphi \in V$, there exists $\mu > 0$ such that $\mu u + v \in \mathcal{C}$.

We can define $\alpha = \min_{\varphi \in V} \varphi(u)$, then $\alpha > 0$. Let us consider $\varepsilon \in \left]0, \frac{\alpha}{\|u\|}\right[$ and $W_\varepsilon = \{\psi \in \mathcal{C}^* \setminus \{0\} \mid \|\psi\| = 1 \text{ and } \exists \varphi \in V \text{ s.t. } \|\psi - \varphi\| < \varepsilon\}$. For every $\psi \in W_\varepsilon$, if $\|\psi - \varphi\| < \varepsilon$ then

$$\psi(u) = \varphi(u) + \psi(u) - \varphi(u) \geq \varphi(u) - \|\psi - \varphi\|\|u\| \geq \alpha - \varepsilon\|u\| > 0.$$

Now, let us consider $Z_\varepsilon = \{\psi \in \mathcal{C}^* \setminus \{0\} \mid \|\psi\| = 1 \text{ and } \|\psi - \varphi\| \geq \varepsilon \ \forall \varphi \in V\}$. Define $\beta = \min_{\psi \in Z_\varepsilon} \psi(u)$ and $\gamma = \min_{\psi \in Z_\varepsilon} \psi(v)$, then $\gamma > 0$.

Taking $\mu \in \left]0, \left|\frac{\gamma}{\beta}\right|\right[$ and $\psi \in Z_\varepsilon$, we have that

$$\psi(\mu u + v) = \mu\psi(u) + \psi(v) \geq \mu\beta + \gamma > 0.$$

On the other hand, for all $\psi \in W_\varepsilon$ we have that $\psi(\mu u + v) = \mu\psi(u) > 0$. Thus, $\psi(\mu u + v) > 0$ for every $\psi \in \mathcal{C}^* \setminus \{0\}$ and the proof is complete. □

Given a collection $\mathfrak{C}$ of open cones in $\mathbb{R}^{n+1}$, we denote as

$$\Gamma(\mathfrak{C}) = \{\tau(\mathcal{C}, v) \mid \mathcal{C} \in \mathfrak{C} \text{ and } v \in \partial\mathcal{C}\}$$

the set of all the cones that are tangent cones for some cone in $\mathfrak{C}$ at a boundary point.

Let $\mathcal{C}$ be an open convex cone in $\mathbb{R}^{n+1}$. Starting from the family $\{\mathcal{C}\}$ given only by $\mathcal{C}$, define $\Gamma^0(\{\mathcal{C}\})) = \Gamma(\{\mathcal{C}\})$ and for every $k \in \mathbb{N}$ define $\Gamma^{k+1}(\{C\}) = \Gamma(\Gamma^k(\{\mathcal{C}\}))$. The union of the families of cones obtained in this way is denoted as

$$T(\mathcal{C}) = \bigcup_{k=1}^{\infty} \Gamma^k(\{\mathcal{C}\}),$$

where $\Gamma_{k+1}(\{C\}) = \Gamma(\Gamma_k(\{C\}))$ for all k.

Remark 2.3.9. From Lemma 2.3.8 the tangent cone at a point of the boundary different from 0 is non-properly convex, indeed we have to intersect all the positive half-spaces defined by the supporting hyperplanes. In other words, $[0]_{\tau(\mathcal{C},v)} \supsetneq \{0\}$ for every $v \in \partial\mathcal{C} \setminus \{0\}$. Moreover, from Remark 2.3.7 we know that $\tau(\tau(\mathcal{C}, v), w) = \tau(\mathcal{C}, v)$ for every $w \in [0]_{\tau(\mathcal{C},v)}$. So, when we consider the tangent cone $\mathcal{T}'$ to a tangent cone $\mathcal{T}$ for a cone $\mathcal{C}$, the result is either the tangent cone $\mathcal{T}$ we started with or a cone such that $[0]_{\mathcal{T}'} \supsetneq [0]_{\mathcal{T}}$. Hence, the dimension of the Hilbert geometry associated to the tangent cone at a properly convex point is lower than the dimension of the Hilbert geometry of the starting cone we started from.

This remark implies that the chain of families of cones $\Gamma^0(\{\mathcal{C}\}) \subseteq \Gamma^1(\{\mathcal{C}\}) \subseteq \Gamma^2(\{\mathcal{C}\}) \subseteq \dots$ is stationary, and

$$T(\mathcal{C}) = \bigcup_{k=1}^{n} \Gamma^k(\{\mathcal{C}\}).$$

Remark 2.3.10. From Remark 2.3.6 we know that $\mathcal{C} \subseteq \tau(\mathcal{C}, v)$ for every $v \in \partial\mathcal{C}$. Therefore, $\tau(\mathcal{C}, v)^* \subseteq \mathcal{C}^*$. Moreover, $\tau(\mathcal{C}, v)^*$ is the intersection of $\mathcal{C}^*$ with the supporting hyperplane $\{\varphi \in v^* \mid \varphi(v) = 0\}$, since $\mathrm{Span}(v) \subseteq \tau(\mathcal{C}, v)$. Moreover, we know from 1.3.9, that the dimension of $\tau(\mathcal{C}, v)^*$ equals the codimension of $[0]_{\tau(\mathcal{C},v)}$. Therefore, every tangent cone for $\mathcal{C}$ correspond to a boundary face of the dual cone $\mathcal{C}^*$, and the dimension of the equivalence class of the 0 correspond to the codimension of the face.

Inductively every element of $T(\mathcal{C})$ correspond to a relatively open face of $\mathcal{C}^*$.

Now, we can come back to the study of the horofunction boundary, examining the set of Funk Buseamann points.

Lemma 2.3.11. *Let $\mathcal{C}$ be a cone in $\mathbb{R}^{n+1}$. If $w \in [0]_{\mathcal{C}}$, then for every $u, v \in \mathcal{C}$ and for every $\alpha > 0$ the followings hold*

- $\mathcal{F}_{\mathcal{C}}((1-\alpha)w + \alpha u, v) = \log \alpha + \mathcal{F}_{\mathcal{C}}(u, v)$.
- $\mathcal{F}_{\mathcal{C}}(u, (1-\alpha)w + \alpha v) = -\log \alpha + \mathcal{F}_{\mathcal{C}}(u, v)$.

Proof. Since $(1-\alpha)w \leq_{\mathcal{C}} 0$ and $0 \leq_{\mathcal{C}} (1-\alpha)w$, given $\lambda > 0$, $(1-\alpha)w + \alpha u \leq_{\mathcal{C}} \lambda v$ if and only if $u \leq_{\mathcal{C}} \frac{\lambda}{\alpha} v$. Hence, $M_{\mathcal{C}}((1-\alpha)w + \alpha u/v)v = \alpha M_{\mathcal{C}}(u/v)$. Applying the logarithm to both sides of these equations gives the first equation of the statement. The second one can be proved in the same way. □

Remark 2.3.12. Given a point $v \in \partial\mathcal{C}$, for each point $w \in \tau(\mathcal{C}, v)$, the line passing through w and v intersect the cone $\mathcal{C}$ in a segment with v as endpoint. Indeed, if this line is disjoint from $\mathcal{C}$, by the Hahn-Banach Theorem, there exists a supporting hyperplane at v containing the line. But this is a contradiction, since Lemma 2.3.8 implies that $w \notin \tau(\mathcal{C}, v)$.

Proposition 2.3.13. *Let $\mathcal{C}$ be a convex open cone in $\mathbb{R}^{n+1}$. Given $v \in \mathcal{C}$ and $w \in \partial\mathcal{C}$, denote the points on the line segment from w to v as $v_t = (1-t)w + tv$ for all $t \in]0, 1[$. Then for every $u \in \mathcal{C}$, the followins hold*

1. $\mathcal{F}_{\mathcal{C}}(u, v_t) - \mathcal{F}_{\tau(\mathcal{C},w)}(u, v_t) \longrightarrow 0$ *as* $t \to 0$,
2. $(v_t)_{t\in]0,1[}$ *converges in Funk sense to* $f_{\tau(\mathcal{C},w),v}|_{\mathcal{C}}$ *as* $t \to 0$,
3. $\overline{\{g_{|\mathcal{C}} \mid g \in \mathcal{K}^{\mathcal{F}}_{\tau(\mathcal{C},w)}\}} \subseteq \overline{\mathcal{K}^{\mathcal{F}}_{\mathcal{C}}} \cap A_{\mathcal{C}}(w)$.

Proof. To prove the first assertion, we work in the plain containing u, v and w. Since v_t approaches the boundary of $\mathcal{C}$ as $t \to 0$, for a sufficiently small t, we have from Remark 2.2.9 that

$$\mathcal{F}_{\mathcal{C}}(u, v_t) = \log \frac{\|u - a_t\|}{\|v_t - a_t\|} \quad \text{and} \quad \mathcal{F}_{\tau(\mathcal{C},w)}(u, v_t) = \log \frac{\|u - b_t\|}{\|v_t - b_t\|}$$

where a_t is the intersection point on the side of v_t of the line through u and v_t with $\partial\mathcal{C}$, and similarly, b_t is the intersection point on the side of v_t of the line through u and v_t with $\partial\tau(\mathcal{C}, w)$.

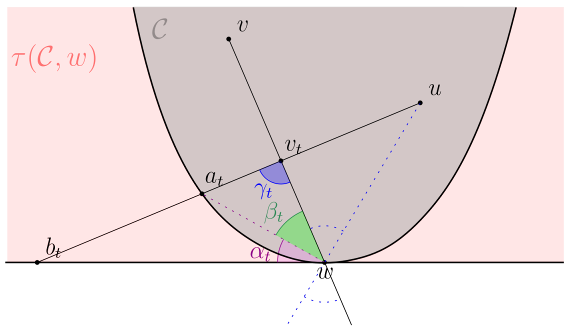


Figure 2.7: Configuration considered in the proof of Proposition 2.3.13.

Denote the angle between the segments $\overline{b_t w}$ and $\overline{a_t w}$ as α_t, the angle between the segments $\overline{a_t w}$ and $\overline{v_t w}$ as β_t, and the angle between the segments $\overline{v_t w}$ and $\overline{a_t v_t}$ as γ_t, see Figure 2.7. Using the sine rule, we get

$$\frac{\|v_t - a_t\|}{\|v_t - b_t\|} = \frac{\sin \beta_t \sin(\pi - \alpha_t - \beta_t - \gamma_t)}{\sin(\alpha_t + \beta_t) \sin(\pi - \beta_t - \gamma_t)}.$$

As $t \to 0$, $\alpha_t \to 0$, $\|u - a_t\| \to \|u - w\|$, $\|u - a_t\| \to \|u - w\|$, and γ_t tends to the angle between the segments $\overline{uw}$ and $\overline{yw}$. Then it follows the first assertion.

For every $u \in \mathcal{C}$, using the triangular inequality, we get

$$\begin{aligned}
\left| f_{\mathcal{C},v_t} - f_{\tau(\mathcal{C},w),v} \right| =& \left| \mathcal{F}_{\mathcal{C}}(u, v_t) - \mathcal{F}_{\mathcal{C}}(b, v_t) - \mathcal{F}_{\tau(\mathcal{C},w)}(u, v) + \mathcal{F}_{\tau(\mathcal{C},w)}(b, v) \right| \\
\leq& \left| \mathcal{F}_{\mathcal{C}}(u, v_t) - \mathcal{F}_{\tau(\mathcal{C},w)}(u, v_t) \right| + \left| \mathcal{F}_{\tau(\mathcal{C},w)}(u, v_t) - \mathcal{F}_{\tau(\mathcal{C},w)}(u, v) \right| + \\
&+ \left| \mathcal{F}_{\mathcal{C}}(b, v_t) - \mathcal{F}_{\tau(\mathcal{C},w)}(b, v_t) \right| + \left| \mathcal{F}_{\tau(\mathcal{C},w)}(b, v_t) - \mathcal{F}_{\tau(\mathcal{C},w)}(b, v) \right|.
\end{aligned}$$

The second point follows by applying the first one and Lemma 2.3.11.

Finally, the second assertion of this proposition implies that if $v \in \mathcal{C}$, then $f_{\tau(\mathcal{C},w),v_{|\mathcal{C}}} \in \overline{\mathcal{K}_{\mathcal{C}}^{\mathcal{F}}} \cap A_{\mathcal{C}}(w)$. Moreover, for every $z \in \tau(\mathcal{C}, w) \setminus \mathcal{C}$, from Remark 2.3.12, we deduce that there exist $v \in \mathcal{C}$ and $\lambda > 0$, such that $z = (1 - \lambda)w + \lambda v$. Lemma 2.3.11 implies that $f_{\tau(\mathcal{C},w),z_{|\mathcal{C}}} = f_{\tau(\mathcal{C},w),v_{|\mathcal{C}}}$, hence $f_{\tau(\mathcal{C},w),z_{|\mathcal{C}}} \in \overline{\mathcal{K}_{\mathcal{C}}^{\mathcal{F}}} \cap A_{\mathcal{C}}(w)$. The last statement follows from the fact that $A_{\mathcal{C}}(v)$ is closed. Indeed, if $(g_n)_{n \in \mathbb{N}} \subseteq A_{\mathcal{C}}(v)$ is a sequence converging to a Funk horofunction $g \in \partial_{\infty}^{\mathcal{F}} \mathcal{C}$. We show that $g \in A_{\mathcal{C}}(v)$.

For every $n \in \mathbb{N}$, the horofunction g_n is the limit in the Funk sense of a sequence $(v_i^n)_{i \in \mathbb{N}}$. Now, fix a decreasing basis ${U_j}_{j \in \mathbb{N}}$ of open neighbourhoods of g, and a decreasing basis ${V_j}_{j \in \mathbb{N}}$ of open neighbourhoods of v. Therefore, for every $j \in \mathbb{N}$, we can choose $n_j \in \mathbb{N}$ so that $g_{n_j} \in U_j$, and then $i_j \in \mathbb{N}$ so that $v_{i_j}^{n_j} \in V_j$ and $\mathcal{F}_{\mathcal{C}}(\cdot, v_{i_j}^{n_j}) - \mathcal{F}_{\mathcal{C}}(b, v_{i_j}^{n_j}) \in U_j$.

So, we have found that the sequence $(v_{i_j}^{n_j})_{j \in \mathbb{N}}$ converges in Funk sense to g and converges to v. □

Inspired this proposition, we want to show that

$$\mathcal{B}^{\mathcal{F}}_{\mathcal{C}} = \{f_{\mathcal{T},u|_{\mathcal{C}}} \mid \mathcal{T} \in T(\mathcal{C}) \setminus \{\mathcal{C}\}, u \in \mathcal{T}\}.$$

The next lemma is useful to prove the inclusion of $\{f_{\mathcal{T},u|_{\mathcal{C}}} \mid \mathcal{T} \in T(\mathcal{C}) \setminus \{\mathcal{C}\}, u \in \mathcal{T}\}$ into $\mathcal{B}^{\mathcal{F}}_{\mathcal{C}}$. But, it will be the key in the next section to merge the study of the reverse-Funk horofunction boundary with the study of the Funk horofunction boundary, in order to obtain the Hilbert horofunction boundary.

Lemma 2.3.14. *Let $\mathcal{C}$ be an open cone and let $v \in \partial\mathcal{C} \setminus [0]_{\mathcal{C}}$. Given $h \in \mathcal{K}^{\mathcal{F}}_{\tau(\mathcal{C},v)} \cup \mathcal{B}^{\mathcal{F}}_{\tau(\mathcal{C},v)}$, there exists a sequence in $\mathcal{C}$ that is both a Funk almost-geodesic and a reverse-Funk almost-geodesic, that converges in Funk sense to $h_{|\mathcal{C}}$ and converges in reverse-Funk sense to $r_{\mathcal{C},v}$.*

Proof. Let $(w_n)_{n\in\mathbb{N}}$ be a Funk almost-geodesic in $\tau(\mathcal{C}, v)$ converging in Funk sense to h. From this sequence we want to construct the desired one in $\mathcal{C}$ by taking $v_n = (1 - t_n)v + t_n w_n$ for an appropriate sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$, that we have to determine.

First of all, for every $n \in \mathbb{N}$, we need that $v_n \in \mathcal{C}$ and $\|v_n - v\| < \frac{1}{n}$. This is true for every sufficiently small t_n, thanks to Remark 2.3.12. Then, from Theorem 2.3.1 $(v_n)_{n\in\mathbb{N}}$ converges in reverse-Funk sense to $r_{\mathcal{C},v}$.

To prove that $(v_n)_{n\in\mathbb{N}}$ is a Funk almost-geodesic in $\mathcal{C}$ we use the fact that $(w_n)_{n\in\mathbb{N}}$ is a Funk almost-geodesic in $\tau(\mathcal{C}, v)$. So, for some $\varepsilon \geq 0$

$$\mathcal{F}_{\tau(\mathcal{C},v)}(w_0, w_1) + \cdots + \mathcal{F}_{\tau(\mathcal{C},v)}(w_{n-1}, w_n) < \mathcal{F}_{\tau(\mathcal{C},v)}(w_0, w_n) + \varepsilon \quad \text{for all } n \in \mathbb{N}. \tag{2.25}$$

Since $v \in [0]_{\tau(\mathcal{C},v)}$, from Lemma 2.3.11, we get for all $n \in \mathbb{N}$

$$\mathcal{F}_{\tau(\mathcal{C},v)}(v_n, v_{n+1}) = \log \frac{t_n}{t_{n+1}} + \mathcal{F}_{\tau(\mathcal{C},v)}(w, w_{n+1}). \tag{2.26}$$

Moreover, from Proposition 2.3.13, we can choose $(t_n)_{n\in\mathbb{N}}$ such that, once fixed t_n, t_{n+1} satisfies

$$\mathcal{F}_{\mathcal{C}}(v_n, v_{n+1}) - \mathcal{F}_{\tau(\mathcal{C},v)}(v_n, v_{n+1}) < \frac{1}{2^n}. \tag{2.27}$$

Combining (2.25), (2.26) and (2.27), we get for every $l \in \mathbb{N}$

$$\begin{aligned}\sum_{k=0}^{n} \mathcal{F}_{\mathcal{C}}(v_k, v_{k+1}) - \mathcal{F}_{\mathcal{C}}(v_0, v_{n+1}) &\leq \sum_{k=0}^{l} \mathcal{F}_{\mathcal{C}}(v_k, v_{k+1}) - \mathcal{F}_{\tau(\mathcal{C},v)}(v_0, v_{n+1}) \\ &< \sum_{k=0}^{l} \mathcal{F}_{\tau(\mathcal{C},v)}(v_k, v_{k+1}) + \sum_{k=0}^{n} \frac{1}{2^k} - \mathcal{F}_{\tau(\mathcal{C},v)}(v_0, v_{n+1}) \\ &= \sum_{k=0}^{n} \mathcal{F}_{\tau(\mathcal{C},v)}(w_k, w_{k+1}) - \mathcal{F}_{\tau(\mathcal{C},v)}(w_0, w_{n+1}) + \sum_{k=0}^{l} \frac{1}{2^k} \\ &< \varepsilon + 2.\end{aligned}$$

Hence, $(v_n)_{n\in\mathbb{N}}$ is a Funk almost-geodesic. The next step is to show that it converges in Funk sense to $h_{|_\mathcal{C}}$. It suffice to show that it is true for a dense subset S in $\mathcal{C}$. Since the topology on $\mathcal{C}$ is the Euclidean one, we can choose a countable dense set S. Therefore, from Proposition 2.3.13, we can choose $(t_n)_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$

$$\mathcal{F}_{\mathcal{C}}(s, v_n) - \mathcal{F}_{\tau(\mathcal{C},v)}(s, v_n) < \frac{1}{n}, \quad \text{for all } s \in S,$$

and

$$\mathcal{F}_{\mathcal{C}}(b, v_n) - \mathcal{F}_{\tau(\mathcal{C},v)}(b, v_n) < \frac{1}{n}.$$

Then, for every $n \in \mathbb{N}$

$$\left|\mathcal{F}_{\mathcal{C}}(s, v_n) - \mathcal{F}_{\mathcal{C}}(b, v_n) - \mathcal{F}_{\tau(\mathcal{C},v)}(s, v_n) + \mathcal{F}_{\tau(\mathcal{C},v)}(b, v_n)\right| < \frac{1}{n}.$$

From Lemma 2.3.11, we have

$$\mathcal{F}_{\tau(\mathcal{C},v)}(s, v_n) - \mathcal{F}_{\tau(\mathcal{C},v)}(b, v_n) = \mathcal{F}_{\tau(\mathcal{C},v)}(s, w) - \mathcal{F}_{\tau(\mathcal{C},v)}(b, w),$$

and from the convergence in Funk sense of $(w_n)_{n\in\mathbb{N}}$ to h, we get the convergence of $(v_n)_{n\in\mathbb{N}}$ to $h_{|_\mathcal{C}}$.

Finally, we show that $(v_n)_{n\in\mathbb{N}}$ is a reverse-Funk almost-geodesic. From Lemma 2.2.10 $\lim_{n\to\infty} r_{\mathcal{C},v}(v_n) = -\mathcal{RF}_{\mathcal{C}}(b, v)$. Since $(v_n)_{n\in\mathbb{N}}$ converges in reverse-Funk sense to $r_{\mathcal{C},v}$, for every $n \in \mathbb{N}$, once fixed t_n, for every sufficiently small t_{n+1}, we have

$$\mathcal{RF}_{\mathcal{C}}(v_n, v_{n+1}) + r_{\mathcal{C},v}(v_{n+1}) - r_{\mathcal{C},v}(v_n) < \frac{1}{2^n}.$$

Hence, for every $N \in \mathbb{N}$ we have

$$\sum_{n=0}^{N-1} \mathcal{RF}_{\mathcal{C}}(v_n, v_{n+1}) + r_{\mathcal{C},v}(v_{N+1}) - r_{\mathcal{C},v}(v_0) < \sum_{n=0}^{N-1} \frac{1}{2^k} < 2. \tag{2.28}$$

Since $r_{\mathcal{C},v}(v_{N+1}) - r_{\mathcal{C},v}(v_0) = \mathcal{RF}_{\mathcal{C}}(v_{N+1}, v) - \mathcal{RF}_{\mathcal{C}}(v_0, v) \leq \mathcal{RF}_{\mathcal{C}}(v_0, v_{N+1})$, from (2.28) we get that $(v_n)_{n\in\mathbb{N}}$ is a reverse-Funk almost-geodesic. □

Proposition 2.3.15. *Let $\mathcal{C}$ be an open cone. Given a point $v \in \partial\mathcal{C} \setminus [0]_{\mathcal{C}}$, a cone $\mathcal{T} \in T(\mathcal{C}) \setminus \{\mathcal{C}\}$ and a point $u \in \mathcal{T}$ the function $f_{\mathcal{T},u_{|\mathcal{C}}}$ belongs to the set of Funk Busemann points $\mathcal{B}^{\mathcal{F}}_{\mathcal{C}}$.*

Proof. Let $\mathcal{T} \in T(\mathcal{C})$. By definition, there exist $N \in \mathbb{N}$ and $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_N$ open cones, such that $\mathcal{T}_1 = \mathcal{C}$, $\mathcal{T}_N = \mathcal{T}$ and $\mathcal{T}_{k+1} \in \Gamma(\mathcal{T}_k)$ for $k = 1, \ldots, N-1$.

From Lemma 2.3.14, we have that $f_{\mathcal{T},u_{|\mathcal{T}_{N-1}}} \in \mathcal{K}^{\mathcal{F}}_{\mathcal{T}_{N-1}} \cup \mathcal{B}^{\mathcal{F}}_{\mathcal{T}_{N-1}}$. Repeating the argument, we get $f_{\mathcal{T},u_{|\mathcal{C}}} \in \mathcal{K}^{\mathcal{F}}_{\mathcal{C}} \cup \mathcal{B}^{\mathcal{F}}_{\mathcal{C}}$. But, since $\mathcal{T} \neq \mathcal{C}$, we have $f_{\mathcal{T},u_{|\mathcal{C}}} \in \mathcal{B}^{\mathcal{F}}_{\mathcal{C}}$. □

To prove the opposite inclusion, and conclude the classification of Funk Busemann points, we need to define the extension of a Funk horofunction to the tangent cone at a point of the boundary. This extension is crucial in order to apply an inductive argument. Indeed, Remark 2.3.9 implies that the dimension of the Hilbert geometry associated with the tangent cone at a properly convex point, is lower then the dimension of the geometry associated with the starting cone. Moreover, we noticed in Remark 2.3.4 that the set of Funk Busemann points equals the set of Busemann functions that are obtained by the Funk limit of Funk almost-geodesic that converges to points of the boundary that are not in the equivalence class of 0.

The inductive argument will stop since, as we saw in Example 2.3.2, the set of Funk Busemann points of the cone given by a half-space is empty.

Definition 2.3.16. Let $\mathcal{C}$ be an open convex cone, and $v \in \partial\mathcal{C}$ be a point in its boundary. A function $h : \mathcal{C} \to \mathbb{R}$ satisfies the *homogeneity condition* at v, if

$$h((1-\lambda)v + \lambda u) = \log \lambda + h(u) \tag{2.29}$$

for every $u \in \mathcal{C}$ and $\lambda > 0$ such that $(1-\lambda)v + \lambda u \in \mathcal{C}$.

For a function that satisfies the condition (2.29) at a point, the following extension is well-defined.

Definition 2.3.17. Let $\mathcal{C}$ be an open convex cone. Given a point $v \in \partial\mathcal{C}$ and a function $h : \mathcal{C} \to \mathbb{R}$ that satisfy the homogeneity condition at v, the *extension* of h

to $\tau(\mathcal{C}, v)$ is defined as

$$h^{|\tau(\mathcal{C},v)}(u) = -\log(\lambda) + h((1-\lambda)v + \lambda u),$$

where $\lambda > 0$ is chosen so that $(1-\lambda)v + \lambda u \in \mathcal{C}$.

Remark 2.3.18. Given a cone $\mathcal{T} \in T(\mathcal{C})$, the function $f_{\mathcal{T},u_{|\mathcal{C}}}$, with $u \in \mathcal{T}$, satisfies the homogeneity condition at every point in $[0]_{\mathcal{T}}$.

Now, we want to extend a Busemann point, obtained as the Funk limit of a Funk almost-geodesic converging to a given point of the boundary, to the tangent cone at that point. Moreover, we want to prove that this extension either is a Busemann point or is not a horofunction.

We extend the definition of detour cost given on the set of horofunctions of a Hilbert geometry, to the context of Funk horofunction. We also change the notation since we extend the definition to pairs made of a horofunction and an arbitrary function from the cone to $\mathbb{R}$.

Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a cone and $h : \mathcal{C} \to \mathbb{R}$ be a function. We define $\mu_h : \overline{\mathcal{K}_{\mathcal{C}}^{\mathcal{F}}} \to \mathbb{R} \cup \{+\infty\}$ as

$$\mu_h(\xi) = \inf_{(u_n)_{n\in\mathbb{N}} \in S} \liminf_{n\to\infty} \Big(\mathcal{F}_{\mathcal{C}}(b, u_n) + h(u_n)\Big), \quad \text{for all } \xi \in \overline{\mathcal{K}_{\mathcal{C}}^{\mathcal{F}}}, \tag{2.30}$$

where S is the set of all sequences converging in Funk sense to ξ.

Remark 2.3.19. If $h : \mathcal{C} \to \mathbb{R}$ is a Funk horofunction, we can follow the steps of Remark 2.1.16, and get that for a point $f_{\mathcal{C},v} \in \mathcal{K}_{\mathcal{C}}^{\mathcal{F}}$ with $v \in \mathcal{C}$, it holds

$$\mu_h(f_{\mathcal{C},v}) = \mathcal{F}_{\mathcal{C}}(b, v) + h(v).$$

Lemma 2.3.20. *Let $\mathcal{C}$ be a convex open cone and $v \in \partial\mathcal{C} \setminus [0]_{\mathcal{C}}$ be a point of its boundary. For every $h \in A_{\mathcal{C}}(v)$ it holds*

$$h = \inf_{\xi \in \mathcal{K}_{\tau(\mathcal{C},v)}^{\mathcal{F}}} \xi_{|\mathcal{C}} + \mu_h(\xi_{|\mathcal{C}}).$$

Moreover, given $u \in \mathcal{C}$, the infimum for $h(u)$ is achived by $f_{\tau(\mathcal{C},v),u}(u)$.

Proof. Let us fix a point $u \in \mathcal{C}$. We can consider the line segment from u to v and parametrize it with a map $\gamma : \mathbb{R}_{\geq 0} \to \mathcal{C}$ such that $\mathcal{F}_{\mathcal{C}}(u, \gamma(t)) = t$ for all $t \in \mathbb{R}_{\geq 0}$. We can do that since the Funk distance from u increase along this segment when approaching v. From Proposition 2.3.13 $\gamma(t)$ converges in Funk sense to $f_{\tau(\mathcal{C},v),u_{|\mathcal{C}}}$ as $t \to \infty$.

We claim that $h(u) - h(\gamma(t)) = t = \mathcal{F}_{\mathcal{C}}(u, \gamma(t))$. Now, if we consider the limit for $t \to \infty$, we get

$$\begin{aligned} h(u) &= \lim_{t\to\infty} \mathcal{F}_{\mathcal{C}}(u,\gamma(t)) + h(\gamma(t)) \\ &= \liminf_{t\to\infty} \mathcal{F}_{\mathcal{C}}(u,\gamma(t)) - \mathcal{F}_{\mathcal{C}}(b,\gamma(t)) + \mathcal{F}_{\mathcal{C}}(b,\gamma(t)) + h(\gamma(t)) \\ &\geq \liminf_{t\to\infty} \mathcal{F}_{\mathcal{C}}(u,\gamma(t)) - \mathcal{F}_{\mathcal{C}}(b,\gamma(t)) + \liminf_{t\to\infty} \mathcal{F}_{\mathcal{C}}(b,\gamma(t)) + h(\gamma(t)) \\ &\geq f_{\tau(\mathcal{C},v),u_{|\mathcal{C}}}(u) + \mu_h(f_{\tau(\mathcal{C},v),u_{|\mathcal{C}}}). \end{aligned}$$

To prove the claim, we consider a sequence $(v_n)_{n\in\mathbb{N}}$ converging to v, whose limit in Funk sense gives h. For any $n \in \mathbb{N}$ sufficiently large, $v_n \notin \mathcal{C} + u$, thus, the Funk distance from u of a point increase along the segment from u to v_n, when approaching v_n. As above, this segment can be parameterised by a map $\gamma_n : [0, \mathcal{F}_{\mathcal{C}}(u, v_n)] \to \mathcal{C}$ such that $\mathcal{F}_{\mathcal{C}}(u, \gamma_n(t)) = t$ for all $t \in [0, \mathcal{F}_{\mathcal{C}}(u, v_n)]$. Using the triangular inequality, we have

$$-\mathcal{F}_{\mathcal{C}}(\gamma(t),\gamma(t)) \leq \mathcal{F}_{\mathcal{C}}(\gamma(t),v_n) - \mathcal{F}_{\mathcal{C}}(\gamma(t),v_n) \leq \mathcal{F}_{\mathcal{C}}(\gamma(t),\gamma(t)). \tag{2.31}$$

For every $t \in \mathbb{R}_{\geq 0}$, the sequence $(\gamma_n(t))_{n\geq N}$ converges to $\gamma(t)$, where $N = min\{n \in \mathbb{N} \mid \mathcal{F}_{\mathcal{C}}(u, v_n) \geq t\}$. Hence, from the continuity of the gauge and the Funk convergence of $(v_n)_{n\in\mathbb{N}}$ to h, we get from (2.31) that $\mathcal{F}_{\mathcal{C}}(\gamma(t), v_n) - \mathcal{F}_{\mathcal{C}}(b, v_n)$ converges to $h(\gamma(t))$ for every $t \in \mathbb{R}_{\geq 0}$, as $n \to \infty$.

Moreover, for every $n \in \mathbb{N}$

$$\mathcal{F}_{\mathcal{C}}(u,v_n) - \mathcal{F}_{\mathcal{C}}(b,v_n) - \mathcal{F}_{\mathcal{C}}(\gamma(t),v_n) + \mathcal{F}_{\mathcal{C}}(b,v_n) = \mathcal{F}_{\mathcal{C}}(u,\gamma(t)) = t,$$

since $u, \gamma_n(t)$ and v_n are aligned in this order. Therefore, by taking $n \to \infty$, we get $h(u) - h(\gamma(t)) = t = \mathcal{F}_{\mathcal{C}}(u, \gamma(t))$.

To conclude the proof, we prove that for every $\xi \in \overline{\mathcal{K}_{\mathcal{C}}^{\mathcal{F}}}$ it holds $h \leq \xi + \mu_h(\xi)$.

Let us start by proving that for every $w \in \mathcal{C}$ we have

$$h(w) = \inf_{z\in\mathcal{C}} \Big(\mathcal{F}_{\mathcal{C}}(w,z) + \lim_{n\to\infty} \big(\mathcal{F}_{\mathcal{C}}(z,v_n) - \mathcal{F}_{\mathcal{C}}(b,v_n)\big)\Big) = \inf_{z\in\mathcal{C}} \Big(\mathcal{F}_{\mathcal{C}}(w,z) + h(z)\Big). \tag{2.32}$$

Given $w \in \mathcal{C}$, since $(v_n)_{n\in\mathbb{N}}$ converges in Funk sense to h, we have

$$\begin{aligned}\inf_{z\in\mathcal{C}}\Big(\mathcal{F}_\mathcal{C}(w,z)+h(z)\Big) &= \inf_{z\in\mathcal{C}}\Big(\mathcal{F}_\mathcal{C}(w,z)+\lim_{n\to\infty}\big(\mathcal{F}_\mathcal{C}(z,v_n)-\mathcal{F}_\mathcal{C}(b,v_n)\big)\Big)\\ &= \inf_{z\in\mathcal{C}}\Big(\lim_{n\to\infty}\big(\mathcal{F}_\mathcal{C}(w,z)+\mathcal{F}_\mathcal{C}(z,v_n)-\mathcal{F}_\mathcal{C}(b,v_n)\big)\Big)\\ &\geq \inf_{z\in\mathcal{C}}\Big(\lim_{n\to\infty}\big(\mathcal{F}_\mathcal{C}(w,v_n)-\mathcal{F}_\mathcal{C}(b,v_n)\big)\Big)=h(w)\end{aligned}$$

The inverse inequality follows by setting $z = w$.

Now, take $(u_n)_{n\in\mathbb{N}}$ in S. From (2.32), applied with $z = u_n$ for every $n \in \mathbb{N}$, as $n \to \infty$ we have

$$\begin{aligned}h(w) &\leq \liminf_{n\to\infty}\mathcal{F}_\mathcal{C}(w,u_n)-\mathcal{F}_\mathcal{C}(b,u_n)+\mathcal{F}_\mathcal{C}(b,u_n)+h(u_n)\\ &\leq \liminf_{n\to\infty}\Big(\mathcal{F}_\mathcal{C}(w,u_n)-\mathcal{F}_\mathcal{C}(b,u_n)\Big)+\liminf_{n\to\infty}\Big(\mathcal{F}_\mathcal{C}(b,u_n)+h(u_n)\Big)\\ &= \xi(w)+\liminf_{n\to\infty}\Big(\mathcal{F}_\mathcal{C}(b,u_n)+h(u_n)\Big).\end{aligned}$$

Taking the infimum over all $(u_n)_{n\in\mathbb{N}}$ in S, we get $h(w) \leq \xi(w) + \mu_h(\xi)$ for all $w \in \mathcal{C}$. □

Proposition 2.3.21. *Let $h \in \mathcal{B}^\mathcal{F}_\mathcal{C} \cap A_\mathcal{C}(v)$, with $v \in \partial\mathcal{C} \setminus [0]_\mathcal{C}$, be a Funk Busemann point. Then, h satisfies the homogeneity condition and $h^{|\tau(\mathcal{C},v)} \in \mathcal{K}^\mathcal{F}_{\tau(\mathcal{C},v)} \cap \mathcal{B}^\mathcal{F}_{\tau(\mathcal{C},v)}$.*

Proof. From Lemma 2.3.20 and Remark 2.3.18 we conclude that h satisfy the homogeneity condition at v. Since h can be written as in the statement of Lemma 2.3.20, we get that

$$h^{|\tau(\mathcal{C},v)} = \inf_{\xi\in\mathcal{K}^\mathcal{F}_{\tau(\mathcal{C},v)}} \xi + \mu_h(\xi_{|\mathcal{C}}). \tag{2.33}$$

As we saw in proof of 2.3.13, every element of $\mathcal{K}^\mathcal{F}_{\tau(\mathcal{C},v)}$ equals to $f_{\tau(\mathcal{C},v),p}$, with $p \in \mathcal{C}$. Moreover, the sequence $((1-1/n)v + 1/n\ p)_{n\in\mathbb{N}}$ converges in Funk sense to $f_{\tau(\mathcal{C},v),p}$, by Proposition 2.3.13. Proposition 2.3.13, Lemma 2.3.11, and Remark 2.3.19 imply that

$$\begin{aligned}\mu_h(f_{\tau(\mathcal{C},v),p|\mathcal{C}}) &= \lim_{n\to\infty}\mathcal{F}_\mathcal{C}(b,(1-1/n)v+1/n\ p)+h((1-1/n)v+1/n\ p)\\ &= \lim_{n\to\infty}\mathcal{F}_{\tau(\mathcal{C},v)}(b,(1-1/n)v+1/n\ p)+h((1-1/n)v+1/n\ p)\\ &= \lim_{n\to\infty}\mathcal{F}_{\tau(\mathcal{C},v)}(b,p)-\log(1/n)+h^{|\tau(\mathcal{C},v)}(p)+\log(1/n)\\ &= \mathcal{F}_{\tau(\mathcal{C},v)}(b,p)+h^{|\tau(\mathcal{C},v)}(p)\\ &= \mu_{h^{|\tau(\mathcal{C},v)}}(f_{\tau(\mathcal{C},v),p}).\end{aligned}$$

Thus, from (2.33) we have

$$h^{|\tau(\mathcal{C},v)} = \inf_{\xi\in\mathcal{K}^{\mathcal{F}}_{\tau(\mathcal{C},v)}} \xi + \mu_{h^{|\tau(\mathcal{C},v)}}(\xi) \geq \inf_{\xi\in\overline{\mathcal{K}^{\mathcal{F}}_{\tau(\mathcal{C},v)}}} \xi + \mu_{h^{|\tau(\mathcal{C},v)}}(\xi).$$

On the other hand, for any $\xi \in \mathcal{K}^{\mathcal{F}}_{\tau(\mathcal{C},v)}$ we have $h^{|\tau(\mathcal{C},v)}(u) \leq \xi(u) + \mu_{h^{|\tau(\mathcal{C},v)}}(\xi)$ for all $u \in \mathcal{C}$. Taking the limit infimum we get

$$h^{|\tau(\mathcal{C},v)} = \inf_{\xi\in\overline{\mathcal{K}^{\mathcal{F}}_{\tau(\mathcal{C},v)}}} \xi + \mu_{h^{|\tau(\mathcal{C},v)}}(\xi). \tag{2.34}$$

Now, we want to prove that there exists some $\xi \in \overline{\mathcal{K}^{\mathcal{F}}_{\tau(\mathcal{C},v)}}$ such that $h^{|\tau(\mathcal{C},v)}(u) = \xi(u) + \mu_{h^{|\tau(\mathcal{C},v)}}(\xi)$ for all $u \in \mathcal{C}$. This fact implies that $\mu_{h^{|\tau(\mathcal{C},v)}} = \mu_\xi + \mu_{h^{|\tau(\mathcal{C},v)}}$. In particular we get that $\mu_\xi(\xi) = 0$. But, since $\xi \in \overline{\mathcal{K}^{\mathcal{F}}_{\tau(\mathcal{C},v)}}$ we can follow the proof of 2.1.14 to conclude that $\xi \in \mathcal{K}^{\mathcal{F}}_{\tau(\mathcal{C},v)} \cup \mathcal{B}^{\mathcal{F}}_{\tau(\mathcal{C},v)}$. Finally, since $h^{|\tau(\mathcal{C},v)}(b) = \xi(b) = 0$, then it must be $\mu_{h^{|\tau(\mathcal{C},v)}}(\xi) = 0$. Therefore, $\xi = h^{|\tau(\mathcal{C},v)}$ and $h^{|\tau(\mathcal{C},v)} \in \mathcal{K}^{\mathcal{F}}_{\tau(\mathcal{C},v)} \cup \mathcal{B}^{\mathcal{F}}_{\tau(\mathcal{C},v)}$ as desired.

In order to complete the proof, suppose that $h^{|\tau(\mathcal{C},v)} \neq \xi + \mu_{h^{|\tau(\mathcal{C},v)}}(\xi)$ for all $\xi \in \overline{\mathcal{K}^{\mathcal{F}}_{\tau(\mathcal{C},v)}}$. Under this assumption the family $\{A_{u,\varepsilon} \mid u \in \mathcal{C}, \varepsilon > 0\}$, given by $A_{u,\varepsilon} = \{\xi \in \overline{\mathcal{K}^{\mathcal{F}}_{\tau(\mathcal{C},v)}} \mid h^{|\tau(\mathcal{C},v)}(u) + \varepsilon < \xi(u) + \mu_{h^{|\tau(\mathcal{C},v)}}(\xi)\}$, is a covering of $\overline{\mathcal{K}^{\mathcal{F}}_{\tau(\mathcal{C},v)}}$. Since we have endowed $\overline{\mathcal{K}^{\mathcal{F}}_{\tau(\mathcal{C},v)}}$ with the topology of uniform convergence on compact sets and $\mu_{h^{|\tau(\mathcal{C},v)}}$ is lower semi-continuous, then $A_{u,\varepsilon}$ is open for all $u \in \mathcal{C},\ \varepsilon > 0$. Since $\overline{\mathcal{K}^{\mathcal{F}}_{\tau(\mathcal{C},v)}}$ is compact, we get get from $\{A_{u,\varepsilon} \mid u \in \mathcal{C}, \varepsilon > 0\}$ a finite covering $\overline{\mathcal{K}^{\mathcal{F}}_{\tau(\mathcal{C},v)}} = A_{u_1,\varepsilon_1} \cup \cdots \cup A_{u_k,\varepsilon_k}$ for some $k \in \mathbb{N}$.

Therefore, if we define $\xi_i = \inf\limits_{\xi\in A_{u_i,\varepsilon_i}}$ for $i = 1, \ldots, k$, then

$$h^{|\tau(\mathcal{C},v)} = \min\{\xi_1, \ldots, \xi_k\}.$$

We claim that $h^{|\tau(\mathcal{C},v)} = \min\{\xi_1, \xi_2\}$ implies that $h^{|\tau(\mathcal{C},v)} = \xi_1$ or $h^{|\tau(\mathcal{C},v)} = \xi_2$. So, from (2.35) we get that there exists $i \in \{1, \ldots, k\}$ such that $h^{|\tau(\mathcal{C},v)} = \xi_i$. But, from the definition of A_{u_i,ε_i}, we have

$$\xi_i(u_i) \geq \varepsilon_i + h^{|\tau(\mathcal{C},v)}(u_i) > h^{|\tau(\mathcal{C},v)}(u_i),$$

leading to a contradiction.

To prove the claim suppose that $h^{|\tau(\mathcal{C},v)} = \min\{\xi_1, \xi_2\}$, with ξ_1 and ξ_2 two functions that con be written as in (2.34). This implies that $h = \min\{\xi_{1|\mathcal{C}}, \xi_{2|\mathcal{C}}\}$.

Since h is a Busemann point, we can follow the proof of 2.1.13 to get that

$\mu_h(h) = 0$. Hence, for every $m \in \mathbb{N}$, there exists a sequence $(u_n^m)_{n\in\mathbb{N}}$ in $\mathcal{C}$ that converges in Funk sense to h and such that

$$0 \leq \liminf_{n\to\infty} \mathcal{F}_\mathcal{C}(b, u_n^m) + h(u_n^m) \leq \frac{1}{m}.$$

From Lemma 2.3.20, we know that $h(u) = f_{\tau(\mathcal{C},v),u}(u) + \mu_h\big(f_{\tau(\mathcal{C},v),u|\mathcal{C}}\big)$ for every $u \in \mathcal{C}$. From Proposition 2.2.22, for every $m \in \mathbb{N}$, the sequence $(u_n^m)_{n\in\mathbb{N}}$ converges to a horofunction ξ^m, up to a subsequence. Thus for every $u \in \mathcal{C}$

$$\begin{aligned} h(u) + \frac{1}{m} &\geq \liminf_{n\to\infty} \mathcal{F}_\mathcal{C}(u, u_n^m) - \mathcal{F}_\mathcal{C}(b, u_n^m) + \mathcal{F}_\mathcal{C}(b, u_n^m) + h(u_n^m) \\ &= \liminf_{n\to\infty} \mathcal{F}_\mathcal{C}(u, u_n^m) + f_{\tau(\mathcal{C},v),u_n^m}(u_n^m) + \mu_h\big(f_{\tau(\mathcal{C},v),u_n^m|\mathcal{C}}\big) \\ &\geq \liminf_{n\to\infty} f_{\tau(\mathcal{C},v),u_n^m}(u) + \mu_h\big(f_{\tau(\mathcal{C},v),u_n^m|\mathcal{C}}\big) \\ &= \xi^m(u) + \mu_h(\xi^m). \end{aligned} \tag{2.35}$$

Applying again Proposition 2.2.22, $(\xi^m)_{m\in\mathbb{N}}$ converges to a horofunction ξ, up to a subsequence. Hence, taking the limit as $m \to \infty$, we get $h = \xi + \mu_h(\xi)$. By assumption,

$$\xi_{1|\mathcal{C}} = \inf_{\xi\in\overline{\mathcal{K}^\mathcal{F}_{\tau(\mathcal{C},v)}}} \xi_{|\mathcal{C}} + \mu_{\xi_{1|\mathcal{C}}}(\xi_{|\mathcal{C}}) \quad \text{and} \quad \xi_{2|\mathcal{C}} = \inf_{\xi\in\overline{\mathcal{K}^\mathcal{F}_{\tau(\mathcal{C},v)}}} \xi_{|\mathcal{C}} + \mu_{\xi_{2|\mathcal{C}}}(\xi_{|\mathcal{C}}).$$

Hence, we have

$$h = \inf_{\xi\in\overline{\mathcal{K}^\mathcal{F}_{\tau(\mathcal{C},v)}}} \xi_{|\mathcal{C}} + \min\{\mu_{\xi_{1|\mathcal{C}}}, \mu_{\xi_{2|\mathcal{C}}}\}(\xi_{|\mathcal{C}}).$$

As above, using the lower semi-continuous function $\min\{\mu_{\xi_1}, \mu_{\xi_2}\}$ instead of μ_h, we get that there exists $\xi \in \overline{\mathcal{K}^\mathcal{F}_{\tau(\mathcal{C},v)}}$ such that $\xi_{|\mathcal{C}} + \min\{\mu_{\xi_{1|\mathcal{C}}}, \mu_{\xi_{2|\mathcal{C}}}\}(\xi_{|\mathcal{C}})$. Suppose that $\min\{\mu_{\xi_1}, \mu_{\xi_2}\}(\xi) = \mu_{\xi_1}(\xi)$. Then, $h = \xi_{|\mathcal{C}} + \mu_{\xi_1}(\xi) \geq \xi_{1|\mathcal{C}}$ and $h \leq \xi_{1|\mathcal{C}}$, so we get $h = \xi_{1|\mathcal{C}}$. Since h satisfies the homogeneity condition (2.29) we get $h = \xi_1$. Similarly, if $\min\{\mu_{\xi_1}, \mu_{\xi_2}\}(\xi) = \mu_{\xi_2}(\xi)$ we get $h = \xi_2$. □

Corollary 2.3.22. *Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a properly convex cone. Then,*

$$\mathcal{B}^\mathcal{F}_\mathcal{C} = \{f_{\mathcal{T},w|_\mathcal{C}} \mid \mathcal{T} \in T(\mathcal{C}) \setminus \{\mathcal{C}\},\ w \in \mathcal{T}\}.$$

Proof. Let $h \in \mathcal{B}^\mathcal{F}_\mathcal{C}$ be a Funk Busemann point. By definition and Remark 2.3.4 h is the Funk limit of a Funk almost-geodesic that converges to a point $v \in \partial$. From Proposition 2.3.21 the extension of h to $\tau(\mathcal{C}, v)$ satisfies $h^{|\tau(\mathcal{C},v)} \in \mathcal{K}^\mathcal{F}_{\tau(\mathcal{C},v)} \cap \mathcal{B}^\mathcal{F}_{\tau(\mathcal{C},v)}$. If $h^{|\tau(\mathcal{C},v)} \in \mathcal{K}^\mathcal{F}_{\tau(\mathcal{C},v)}$, then we get that $h \in \{f_{\mathcal{T},w|_\mathcal{C}} \mid \mathcal{T} \in T(\mathcal{C})\setminus\{\mathcal{C}\},\ w \in \mathcal{T}\}$. Otherwise, there exists a point $v' \in\in \partial\tau(\mathcal{C}, v) \setminus [0]_{\tau(\mathcal{C},v)}$ such that $h^{|\tau(\mathcal{C},v)} \in A_{\tau(\mathcal{C},v)}(v')$. We can apply to $h^{\tau(\mathcal{C},v)}$ the same reasoning we applied to h. Thanks to Remark 2.3.9, after a finite number of steps we obtain that $h \in \{f_{\mathcal{T},w_{|\mathcal{C}}} \mid \mathcal{T} \in T(\mathcal{C}),\ w \in \mathcal{T}\}$.

Conversely, from Proposition 2.3.15 the set $\{f_{\mathcal{T},w_{|\mathcal{C}}} \mid \mathcal{T} \in T(\mathcal{C}) \setminus \{\mathcal{C}\},\ w \in \mathcal{T}\}$ is contained in $\mathcal{B}^{\mathcal{F}}_{\mathcal{C}}$. □

Finally, we merge the characterization of the set of Funk Busemann points and the set of reverse-Funk Busemann point, in order to characterize the set of Hilbert Busemann points.

Theorem 2.3.23. *Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a properly convex open cone. Then,*

$$\mathcal{B}_\Omega = \{r_{\mathcal{C},v} + f_{\mathcal{T},w_{|\mathcal{C}}} \mid v \in \partial\mathcal{C} \setminus \{0\},\ \mathcal{T} \in T(\tau(\mathcal{C}, v)) \text{ and } w \in \mathcal{T}\}.$$

Proof. Let $h \in \mathcal{B}^{\mathcal{H}}_{\mathcal{C}}$ be a Hilbert Busemann point. Then there exists a Hilbert almost-geodesic $(v_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}$ that converges in Hilbert sense to ξ. By Lemma 2.2.33 $(v_n)_{n\in\mathbb{N}}$ is both a Funk almost-geodesic and a reverse-Funk almost-geodesic. Moreover, we can assume that $(v_n)_{n\in\mathbb{N}}$ lies in a compact cross-section $D \subseteq \mathcal{C}$. Hence, the sequence $(v_n)_{n\in\mathbb{N}}$ converges, up to a subsequence to a point $v \in D$. Since ξ is a horofunction $v \in \partial D$. By Theorem 2.3.1 $(v_n)_{n\in\mathbb{N}}$ converges in reverse-Funk sense to $r_{\mathcal{C},v}$.

On the other hand, $(v_n)_{n\in\mathbb{N}}$ converges in Funk sense to $f = h - r_{\mathcal{C},v}$. By Corollary 2.3.22 $f = f_{\mathcal{T},w_{|\mathcal{C}}}$ for some cone $\mathcal{T} \in T(\tau(\mathcal{C}, v))$ and $w \in \mathcal{T}$. Therefore, $\xi = r_{\mathcal{C},v} + f_{\mathcal{T},w_{|\mathcal{C}}}$.

Conversely, let $h = r_{\mathcal{C},v} + f_{\mathcal{T},w_{|\mathcal{C}}}$ with $v \in \partial\mathcal{C} \setminus \{0\}$, $\mathcal{T} \in T(\tau(\mathcal{C}, v))$ and $w \in \mathcal{T}$. From Proposition 2.3.15$f_{\mathcal{T},w_{|\tau(\mathcal{C},v)}} \in \mathcal{K}^{\mathcal{F}}_{\tau(\mathcal{C},v)} \cup \mathcal{B}^{\mathcal{F}}_{\tau(\mathcal{C},v)}$, thus $f_{\mathcal{T},w_{|\mathcal{C}}}$ is the Funk limit of a sequence $(v_n)_{n\in\mathbb{N}}$ that is both a Funk almost-geodesic and a reverse-Funk almost-geodesic and that converges in reverse-Funk sense to $r_{\mathcal{C},v}$, and in usual sense to v, by Lemma 2.3.14. Lemma 2.2.33 implies that $(v_n)_{n\in\mathbb{N}}$ is a Hilbert almost-geodesic. Hence, $h \in \mathcal{B}^{\mathcal{H}}_{\mathcal{C}}$. □

As we saw at the beginning of this section, the Hilbert metric on a properly convex domain Ω coincide with the Birkhoff's version of the Hilbert metric on the cone $\mathcal{C}_\Omega$ above Ω, restricted to a bounded cross-section, and the set of Busemann points of a Hilbert geometry corresponds to the set of the restrictions to a bounded cross-section of Hilbert Busemann points of the cone above the domain.

Remark 2.3.24. From this theorem we get that the Busemann horospheres centered on a geodesic ray that converges to a C^1-point does not depend on the geodesic ray, but only on the point of the boundary where they converge. We recall the definition of Busemann function. Let $(\Omega, \mathrm{d}_\Omega)$ be a Hilbert geometry. Given a geodesic ray $\gamma : \mathbb{R}_+ \to \Omega$ the Busemann function centered at γ is the function $\beta_\gamma : \Omega \to \mathbb{R}$ given by

$$\beta_\gamma(x) = \lim_{t\to\infty} \mathrm{d}_\Omega(x, \gamma(t)) - t \quad \text{for all } x \in \Omega. \tag{2.36}$$

The Busemann horospheres centerd on the geodesic ray γ are the level sets of the Busemann function β_γ.

Since γ is a geodesic segment, then $t = \mathrm{d}_\Omega(\gamma(0), \gamma(t))$ for all $t \in \mathbb{R}_+$. Thus, β_γ is a Busemann point for $(\Omega, \mathrm{d}_\Omega)$ with respect to $\gamma(0)$ as base point. Hence, we can study the limit in 2.36 in a bounded cross-section D of $\mathcal{C}_\Omega$.

Suppose that γ converges to a C^1-point $z \in \partial\Omega$. Define $v \in \partial D$ such that $z = [v]$ and choose an arbitrary sequence $(v_n)_{n\in\mathbb{N}} \in D$ that converges to v and such that $[v_n]$ belongs to the image of γ for all $n \in \mathbb{N}$.

Since z is a C^1-point, then $\tau(\mathcal{C}, v) \cong \mathbb{R}^n \times \mathbb{R}_+$. Hence, $T(\tau(\mathcal{C}, v)) = \tau(\mathcal{C}, v)$. By Theorem 2.3.23 the limit of $(v_n)_{n\in\mathbb{N}}$ in Hilbert sense is $r_{\mathcal{C},v} + f_{\tau(\mathcal{C},v),w|\mathcal{C}}$ for some $w \in \tau(\mathcal{C}, v)$.

For every $x \in \Omega$, let $u \in D$ such that $[u] = x$, then

$$\beta_\gamma(x) = r_{\mathcal{C},v}(u) + f_{\tau(\mathcal{C},v),w}(u).$$

From Example 2.3.2 we know that given $w, w' \in \tau(\mathcal{C}, v)$ the difference between $f_{\tau(\mathcal{C},v),w}$ and $f_{\tau(\mathcal{C},v),w'}$ is a constant function.

Remark 2.3.25. This theorem shows also that that there exists a pair of distinct geodesic rays converging to a non-C^1-point such that the Busemann functions centered on them does not differs by an additive constant.

2.4 Parts of the horofunction boundary

In this section, we will compute the detour cost between two Busemann points of a Hilbert geometry and characterize the parts of the horofunction boundary in terms of faces of the domain and of its dual.

We extend in the obvious way the definition of detour cost to the set of Hilbert horofunctions defining

$$H(\xi, \eta) = \inf_{(v_n)_{n\in\mathbb{N}} \in S} \left(\liminf_{n\to\infty} \mathcal{H}_\mathcal{C}(b, v_n) + \eta(v_n) \right) \quad \text{for all } \xi, \eta \in \partial^\mathcal{H}_\infty \mathcal{C}$$

where S is the set of all sequences $(v_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}$ that converges in Hilbert sense to ξ.

It is clear that the detour cost between two horofunction equals the detour cost between the corresponding Hilbert horofunctions.

Let $\xi, \eta \in \mathcal{B}_\Omega$ be two Hilbert Busemann points. From Theorem 2.0.4 we know that there exist $v \in \partial\mathcal{C} \setminus \{0\}$, $\mathcal{S} \in T(\tau(\mathcal{C}, v))$ and $p \in \mathcal{S}$ such that $\xi = r_{\mathcal{C},v} + f_{\mathcal{S},p|\mathcal{C}}$. Similarly, there exists $w \in \partial\mathcal{C} \setminus \{0\}$, $\mathcal{T} \in T(\tau(\mathcal{C}, w))$ and $q \in \mathcal{T}$ such that $\eta = r_{\mathcal{C},w} + f_{\mathcal{T},q|\mathcal{C}}$.

Reasoning as in the proof of Proposition 2.1.15, we get that if $(v_n)_{n\in\mathbb{N}}$ is a Hilbert almost-geodesic converging to ξ in Hilbert sense, then

$$\begin{aligned} H(\xi,\eta) &= \lim_{n\to\infty} \mathcal{H}_\mathcal{C}(b, v_n) + \eta(v_n) \\ &= \lim_{n\to\infty} \mathcal{RF}_\mathcal{C}(b, v_n) + r_{\mathcal{C},w}(v_n) + \lim_{n\to\infty} \mathcal{F}_\mathcal{C}(b, v_n) + f_{\mathcal{T},q}(v_n). \end{aligned} \tag{2.37}$$

Moreover, from Lemma 2.2.33, we conclude that $(v_n)_{n\in\mathbb{N}}$ is both a reverse-Funk almost-geodesic and a Funk almost-geodesic. As we did in the previous section, we can study separately the Funk and the reverse-Funk part of this limit.

Recall that the face of a point $v \in \overline{\mathcal{C}}$ is the set containing those points $w \in \overline{\mathcal{C}}$ such that the line through v and w meet $\overline{\mathcal{C}}$ in a segment that contains v in its interior, and we denote it as F_v.

Proposition 2.4.1. *Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a properly convex open cone. Given a reverse-Funk almost-geodesic $(v_n)_{n\in\mathbb{N}}$ converging to a point $v \in \partial\mathcal{C} \setminus \{0\}$, and a point $w \in \partial\mathcal{C} \setminus \{0\}$, we have*

$$\lim_{n\to\infty} \mathcal{RF}_\mathcal{C}(b, v_n) + r_{\mathcal{C},w}(v_n) = \begin{cases} \mathcal{RF}_\mathcal{C}(b, v) + \mathcal{RF}_{F_v}(v, w) - \mathcal{RF}_\mathcal{C}(b, w) & \text{if } w \in F_v, \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. Theorem 2.3.1 implies that $(v_n)_{n\in\mathbb{N}}$ converges in reverse-Funk sense to $r_{\mathcal{C},v}$. In the proof of Proposition 2.1.15, we used only that the Hilbert metric satisfies the triangular inequality, thus the same argument works also to prove that

$$\lim_{n\to\infty} \mathcal{RF}_\mathcal{C}(b, v_n) + r_{\mathcal{C},w}(v_n)$$

is independent of the choice of the reverse-Funk almost-geodesic $(v_n)_{n\in\mathbb{N}}$ converging in usual sense to v and in reverse-Funk sense to $r_{\mathcal{C},v}$.

It is convenient to work with a straight line segment, that is a reverse-Funk almost-geodesic as we saw in Remark 2.2.29. To this aim, we consider the sequence $(v'_n)_{n\in\mathbb{N}}$ given by $v'_n = (1 - t_n)v + t_n b$ for all $n \in \mathbb{N}$, where $(t_n)_{n\in\mathbb{N}}$ is a decreasing sequence in $]0, 1[$ that converges to 0.

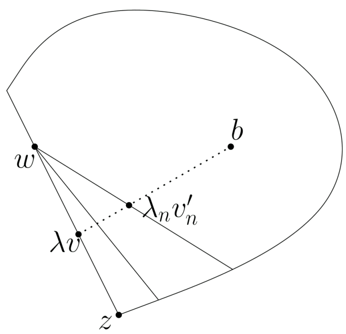
If $w = \lambda v$ for some $\lambda > 0$, then $w \in F_v$, $r_{\mathcal{C},w} = r_{\mathcal{C},v}$, and Lemma 2.2.10 thus implies that

$$\lim_{n\to\infty} \mathcal{RF}_{\mathcal{C}}(b, v_n) + r_{\mathcal{C},w}(v_n) = -\mathcal{RF}_{\mathcal{C}}(b, v) = -\mathcal{RF}_{\mathcal{C}}(b, v) - \log\lambda. \tag{2.38}$$

By the continuity of the gauge we get from (2.38) that

$$\begin{aligned}\lim_{n\to\infty} \mathcal{RF}_{\mathcal{C}}(b, v_n) + r_{\mathcal{C},w}(v_n) &= \mathcal{RF}_{\mathcal{C}}(b, v) - \mathcal{RF}_{\mathcal{C}}(b, v) - \log\lambda \\ &= \mathcal{RF}_{\mathcal{C}}(b, v) + \mathcal{RF}_{\mathcal{C}}(v, w) - \mathcal{RF}_{\mathcal{C}}(b, w).\end{aligned}$$

If $w \neq \lambda v$ for all $\lambda > 0$, and $w \in F_v$, we can find $(\lambda_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ and $\lambda \in \mathbb{R}_+$ such that λv and $(\lambda_n v_n)_{n\in\mathbb{N}}$ lie in the same relatively compact cross-section that contains w and b. Moreover, as we can see in the figure below, there exists $z \in \partial F_v$ given by the intersection of ∂F_v with the line passing through λv and w.



Thus, using the continuity of the gauge and Proposition 2.2.8, we get

$$\begin{aligned}\lim_{n\to\infty} \mathcal{RF}_{\mathcal{C}}(b, v_n) + r_{\mathcal{C},w}(v_n) &= \mathcal{RF}_{\mathcal{C}}(b, v) \lim_{n\to\infty} - \log\lambda_n + \mathcal{RF}_{\mathcal{C}}(\lambda_n v_n, w) - \mathcal{RF}_{\mathcal{C}}(b, w) \\ &= -\log\lambda + \log\frac{\|w - z\|}{\|\lambda_n v_n - z\|} - \mathcal{RF}_{\mathcal{C}}(b, w) \\ &= -\log\lambda + \mathcal{RF}_{\mathcal{C}}(\lambda v, w) - \mathcal{RF}_{\mathcal{C}}(b, w) \\ &= \mathcal{RF}_{\mathcal{C}}(\lambda v, w) - \mathcal{RF}_{\mathcal{C}}(b, w).\end{aligned}$$

If $w \neq \lambda v$ for all $\lambda > 0$, and $w \notin F_v$, then

$$\lim_{n\to\infty} \mathcal{RF}_{\mathcal{C}}(b, v_n) + r_{\mathcal{C},w}(v_n) = \mathcal{RF}_{\mathcal{C}}(b, v) + \lim_{n\to\infty} \mathcal{RF}_{\mathcal{C}}(v_n, w) - \mathcal{RF}_{\mathcal{C}}(b, v) = +\infty,$$

since $w \notin F_v$ and thus, by Lemma 2.2.5, $w \not\sim_{\mathcal{C}} v$. □

For the study of the Funk part of the limit in (2.37) we need some tools from convex analysis. In particular, we introduce the topology of *epi-convergence* on the set of proper, lowe semi-continuous, convex functions. For the details of this theory we refer to [2].

On the set of closed set of $\mathbb{R}^n \times \mathbb{R}_+$ is defined the *Painlevé-Kuratowski topology*. In that topology a sequence of closed sets $(A_n)_{n\in\mathbb{N}}$ converges to a closed set A if

$$Ls(A_n)_n = \bigcap_{n\in\mathbb{N}} \left(\overline{\cup_{k\geq n} A_k}\right) \quad \text{and} \quad Li(A_n)_n = \bigcap_{\{(n_k)_k | n_k \to \infty\})} \left(\overline{\cup_{k\geq 0} A_{n_k}}\right)$$

coincide and $A = Ls(A_n)_n = Li(A_n)_n$.

Given a function $f : \mathbb{R}^n \to \mathbb{R}$ we define its *epigraph* as the set $\text{epi}(f) = \{(x, \alpha) \mid \alpha \geq f(x)\}$. A sequence of proper, lower semi-continuous, convex functions $(f_n)_{n\in\mathbb{N}}$ is said to converge in the *epi-graph topology* to a proper, lower semi-continuous, convex function f if $(\text{epi}(f_n))_{n\in\mathbb{N}}$ converges in the Painlevé-Kuratowski topology to $\text{epi}(f)$.

Moreover, on the set of proper, lower semi-continuous, convex functions is defined an involution, called the *Fenchel-Legendre tranform* that maps a function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ to the function $f^* : (\mathbb{R}^{n+1})^* \to \mathbb{R}$ defined by

$$f^*(\varphi) = \sup_{x\in\mathbb{R}^*} \big(\varphi(x) - f(x)\big) \quad \text{for all } \varphi \in \mathbb{R}^{n+1}.$$

Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a properly convex open cone. For every $v \in \mathcal{C}$ we define the convex function $j_{\mathcal{C},v} : \mathcal{C} \to \mathbb{R}$ given by $j_{\mathcal{C},v}(u) = \exp \circ f_{\mathcal{C},v}(u)$ for all $u \in \mathcal{C}$.

Let us check that $j_{\mathcal{C},v}$ is convex. Given $u, u' \in \mathcal{C}$ and $t \in [0, 1]$, from Lemma 2.2.13, we get

$$\begin{aligned} j_{\mathcal{C},v}((1-t)u + tu') &= \frac{1}{M_{\mathcal{C}}(b/v)} \sup_{\varphi\in\mathcal{C}^*\setminus\{0\}} \frac{\varphi((1-t)u + tu')}{\varphi(v)} \\ &= \frac{1}{M_{\mathcal{C}}(b/v)} \sup_{\varphi\in\mathcal{C}^*\setminus\{0\}} \Big(\frac{(1-t)\varphi(u)}{\varphi(v)} + \frac{t\varphi(u')}{\varphi(v)}\Big) \\ &\leq \frac{1}{M_{\mathcal{C}}(b/v)} \Big((1-t) \sup_{\varphi\in\mathcal{C}^*\setminus\{0\}} \frac{\varphi(u)}{\varphi(v)} + t \sup_{\varphi\in\mathcal{C}^*\setminus\{0\}} \frac{\varphi(u')}{\varphi(v)}\Big) \\ &= (1-t)j_{\mathcal{C},v}(u) + tj_{\mathcal{C},v}(u'). \end{aligned}$$

Thus, we can compute the Fenchel-Legendre transorm of $j_{\mathcal{C},v}$. The Fenchel-Moreau Theorem [9, Theorem 4.2.1] says that on the set of convex, lower semi-continuous functions, the Fenchel-Legendre transform is an involution.

We define on a set E the indicator function of a subset $F \subseteq E$ as the function

$I_F : E \to \{0, +\infty\}$ given by

$$I_F(e) = \begin{cases} 0 & \text{if } e \in F \\ +\infty & \text{if } e \notin F \end{cases}$$

Lemma 2.4.2. *Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be an open convex cone. For all $v \in \mathcal{C}$, we have that $j^*_{\mathcal{C},v}$ is the indicator function of the set*

$$Z_{\mathcal{C},v} = \mathcal{C}^* \cap \{\varphi \in (\mathbb{R}^{n+1})^* \mid M_{\mathcal{C}}(b/v)\varphi(v) \leq 1\}.$$

Proof. We prove that $I^*_{Z_{\mathcal{C},v}} = j_{\mathcal{C},v}$. The thesis will follow from the Fenchel-Moreau Theorem.

$$\begin{aligned} I^*_{Z_{\mathcal{C},v}}(u) &= \sup_{\varphi \in (\mathbb{R}^{n+1})^*} \big(\varphi(u) - I_{Z_{\mathcal{C},v}}(\varphi)\big) \\ &= \sup_{\varphi \in Z_{\mathcal{C},v}} \varphi(u) \\ &\leq \frac{1}{M_{\mathcal{C}}(b/v)} \sup_{\varphi \in Z_{\mathcal{C},v}} \frac{\varphi(u)}{\varphi(v)} \\ &\leq \frac{1}{M_{\mathcal{C}}(b/v)} \sup_{\varphi \in \mathcal{C}^*} \frac{\varphi(u)}{\varphi(v)} \\ &= j_{\mathcal{C},v}(u), \end{aligned}$$

where the last equality follows from Lemma 2.2.13. On the other hand, from the proof of Lemma 2.2.13, we know that the infimum in (Lemma 2.16) can be computed on any corss-section of $\mathcal{C}^*$. So, we define $D_v = \{\varphi \in \mathcal{C}^* \mid \varphi(v) = \frac{1}{M_{\mathcal{C}}(b/v)}\}$. Thus, we have

$$\begin{aligned} j_{\mathcal{C},v}(u) &= \frac{1}{M_{\mathcal{C}}(b/v)} \sup_{\varphi \in D_v} \frac{\varphi(u)}{\varphi(v)} \\ &= \sup_{\varphi \in D_v} \varphi(u) \\ &\leq \sup_{\varphi \in Z_{\mathcal{C},v}} \varphi(u) \\ &= I^*_{Z_{\mathcal{C},v}}(u). \end{aligned}$$

□

Corollary 2.4.3. *Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be an open convex cone. The, the set*

$$\{j_{\mathcal{T},w} \mid \mathcal{T} \in T(\mathcal{C}), w \in \mathcal{T}\}$$

is an equi-Lipshitzian family of convex functions.

Proof. Since $\mathcal{T} \in T(\mathcal{C})$, we have from Remark 2.3.6 that $\mathcal{T}^* \subseteq \mathcal{C}^*$. Moreover, for every $\mathcal{T} \in T(\mathcal{C})$, $w \in \mathcal{T}$, and $\psi \in \mathcal{T}^*$, from Lemma 2.2.13, we have

$$\psi(w)M_{\mathcal{T}}(b/w) \geq \psi(b)$$

Therefore, $Z_{\mathcal{T},w} \subseteq \{\varphi \in \mathcal{C}^* \mid \varphi(b) \leq 1\}$ for every $\mathcal{T} \in T(\mathcal{C})$, $w \in \mathcal{T}$. Let $K = \{\varphi \in \mathcal{C}^* \mid \varphi(b) \leq 1\}$. Since $b \in \mathcal{C}$, the set K is bounded. Let $M = \sup_{\varphi \in K}\|\varphi\|$. Let us consider $u, u' \in \mathcal{T}$. Then, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |j_{\mathcal{T},w}(u) - j_{\mathcal{T},w}(u')| &\leq \left|\sup_{\varphi \in K} \varphi(u) - \sup_{\varphi \in K} \varphi(u')\right| \\ &\leq \left|\sup_{\varphi \in K} \varphi(u) - \varphi(u')\right| \\ &\leq M\|u - u'\|. \end{aligned}$$

□

Lemma 2.4.4. *Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be an open convex cone. If a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$ converges in Funk sense to $f_{\mathcal{T},w_{|\mathcal{C}}}$, for some $\mathcal{T} \in T(\mathcal{C})$ and $w \in \mathcal{T}$, then the sequence $(j^*_{\mathcal{C},v_n})_{n\in\mathbb{N}}$ converges in the epigraph topology to $j^*_{\mathcal{T},w}$.*

Proof. We define for every $n \in \mathbb{N}$ the function $g_{\mathcal{C},v_n} : \mathbb{R}^{n+1} \to \mathbb{R} \cup \{+\infty\}$ given

$$g_{\mathcal{C},v_n}(u) = \begin{cases} j_{\mathcal{C},v_n}(u) & \text{if } u \in \mathcal{C}, \\ +\infty & \text{otherwise,} \end{cases}$$

and the function $g_{\mathcal{T},w} : \mathbb{R}^{n+1} \to \mathbb{R} \cup \{+\infty\}$ given by

$$g_{\mathcal{T},w}(u) = \begin{cases} j_{\mathcal{T},w}(u) & \text{if } u \in \mathcal{C} \\ +\infty & \text{otherwise.} \end{cases}$$

Since the family $\{j_{\mathcal{T},w} \mid \mathcal{T} \in T(\mathcal{C}), w \in \mathcal{T}\}$ is equi-Lipshitzian and $(j_{\mathcal{C},v_{n|\mathcal{C}}})_{n\in\mathbb{N}}$ pointwise converges to $j_{\mathcal{T},w_{|\mathcal{C}}}$, by [2, Theorem 7.1.5] $(g_{\mathcal{C},v_n})_{n\in\mathbb{N}}$ converges to $g_{\mathcal{T},w}$ in the epigraph topology.

Now, we compute the Fenchel-Legendre transfor of $g_{\mathcal{T},w}$. For any $\varphi \in (\mathbb{R}^{n+1})^*$ we have

$$\begin{aligned} g^*_{\mathcal{T},w}(\varphi) &= \sup_{u\in\mathbb{R}^{n+1}} \Big(\varphi(u) - g_{\mathcal{T},w}(u)\Big) \\ &= \sup_{u\in\mathbb{R}^{n+1}} \Big(\varphi(u) - \max\{j_{\mathcal{T},w}, I_{\mathcal{C}}\}(u)\Big) \\ &= \max\Big\{ \sup_{u\in\mathbb{R}^{n+1},\ j_{\mathcal{T},w}(u)\le I_{\mathcal{C}}(u)} \varphi(u) - j_{\mathcal{T},w}(u)\ ,\ \sup_{u\in\mathbb{R}^{n+1},\ j_{\mathcal{T},w}(u)> I_{\mathcal{C}}(u)} \varphi(u) - I_{\mathcal{C}}(u)\Big\} \\ &= I_{Z_{\mathcal{C},w}\cup -\mathcal{C}^*}, \end{aligned}$$

since $j^*_{\mathcal{T},w} = I_{Z_{\mathcal{C},w}}$ and $I^*_{\mathcal{C}} = I_{-\mathcal{C}^*}$.

Therefore, $j^*_{\mathcal{T},w}$ and $g^*_{\mathcal{T},w}$ coincide on $\mathcal{C}^*$. Since $j^*_{\mathcal{T},w}$ equals $+\infty$ outside $\mathcal{C}^*$, we have $j^*_{\mathcal{T},w} = \max\{g^*_{\mathcal{T},w}, I_{\mathcal{C}^*}\}$. Since $(g^*_{\mathcal{C},v_n})_{n\in\mathbb{N}}$ converges in the epigraph topology, then also $(j^*_{\mathcal{C},v_n})_{n\in\mathbb{N}}$ converges in the same topology. The limit is exactly $j^*_{\mathcal{C},w}$ since the Fenchel-Legendre transform is an homeomorphism with respect to the epigraph topology [2, Proposition 7.2.11]. □

Proposition 2.4.5. *Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a properly convex open cone. Let $\mathcal{T}, \mathcal{S} \in T(\mathcal{C}) \setminus \{\mathcal{C}\}$ be two cones and let $p \in \mathcal{S}$ and $q \in \mathcal{T}$. Given a Funk almost-geodesic $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$ converging in Funk sense to $f_{\mathcal{S},p|_\mathcal{C}}$, we have*

$$\lim_{n\to\infty} \mathcal{F}_{\mathcal{C}}(b, v_n) + f_{\mathcal{S},q}(v_n) = \begin{cases} \mathcal{F}_{\mathcal{S}}(b,p) + \mathcal{F}_{\mathcal{T}}(p,q) - \mathcal{F}_{\mathcal{T}}(b,q) & \text{if } \mathcal{S} \subseteq \mathcal{T}, \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. Let A be the set of all sequences that converges in Funk sense to $f_{\mathcal{S},p|_\mathcal{C}}$. We prove that $\inf_{(v_n)_n\in A} \lim_{n\to\infty} \mathcal{F}_{\mathcal{C}}(b, v_n) + f_{\mathcal{S},q}(v_n) = \mathcal{F}_{\mathcal{S}}(b,p) + \mathcal{F}_{\mathcal{T}}(p,q) - \mathcal{F}_{\mathcal{T}}(b,q)$ when $\mathcal{S} \subseteq \mathcal{T}$, and that this infimum equals $+\infty$ otherwise.

The proof that the infimum is achived by every Funk almost-geodesic that converges to $f_{\mathcal{S},p|\mathcal{C}}$ goes exactly like the proof of Proposition 2.1.15.

So, let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}$ converging in Funk sense to $f_{\mathcal{S},p|_\mathcal{C}}$. Following the computations in the proof of Lemma 2.4.2 we get

$$\mathcal{F}_{\mathcal{C}}(b, v_n) + f_{\mathcal{T},q}(v_n) = \log \sup_{\varphi\in\mathcal{T}^*\cap U_{\mathcal{C},v_n}} \frac{1}{\varphi(q) M_{\mathcal{T}}(b/q)}. \tag{2.39}$$

From Lemma 2.4.4 the sequence $(j^*_{\mathcal{C},v_n})_{n\in\mathbb{N}}$ converges in the epigraph topology to $j^*_{\mathcal{S},p}$. So, if $\varphi \in \mathcal{C}^*$ is such that $j^*_{\mathcal{S},p}(\varphi) = +\infty$, since the convergence in the epigraph topology implies the pointwise convergence [2, Proposition 7.1.3], the sequence $(j^*_{\mathcal{C},v_n}(\varphi))_{n\in\mathbb{N}}$ converges to $+\infty$. Hence, for $n \in \mathbb{N}$ sufficiently large, we have $\varphi \in \mathcal{C}^* \setminus Z_{\mathcal{C},v_n}$. Let $U_{\mathcal{C},v_n} = \mathcal{C}^* \setminus Z_{\mathcal{C},v_n}$.

If $\mathcal{S} \nsubseteq \mathcal{T}$, we can consider $\varphi \in \mathcal{T} \setminus \int$. For this choice we have $j^*_{\mathcal{S},p}(\alpha\varphi) = +\infty$ for all $\alpha > 0$, and so $\alpha\varphi \in U_{\mathcal{C},v_n}$ for every sufficiently large $n \in \mathbb{N}$. Thus, by taking $\alpha \to 0$ we get from (2.39) that

$$\liminf_{n\to\infty} \mathcal{F}_{\mathcal{C}}(b, v_n) + f_{\mathcal{T},q}(v_n) = +\infty.$$

Now suppose that $\mathcal{S} \subseteq \mathcal{T}$. For any $\varphi \in U_{\mathcal{S},p}$ from Lemma 2.4.2$j^*_{\mathcal{S},p}(\varphi) = +\infty$. As we noticed above, for any sufficiently large $n \in \mathbb{N}$, $\varphi \in U_{\mathcal{C},v_n}$. Thus, we have $U_{\mathcal{S},p} \subseteq U_{\mathcal{C},v_n}$. From (2.39), taking the limit infimum, and following again the computation done in the proof of Lemma 2.4.2, we get

$$\begin{aligned}\liminf_{n\to\infty} \mathcal{F}_{\mathcal{C}}(b, v_n) + f_{\mathcal{T},q}(v_n) &\geq \log \sup_{\varphi\in\mathcal{T}^*\cap U_{\mathcal{S},p}} \frac{1}{\varphi(q)M_{\mathcal{T}}(b/q)} \\ &= \log\Big(M_{\mathcal{S}}(b/p) \sup_{\varphi\in\mathcal{T}^*} \frac{\varphi(p)}{\varphi(q)M_{\mathcal{T}}(b/q)}\Big) \\ &= \mathcal{F}_{\mathcal{S}}(b, p) + \mathcal{F}_{\mathcal{T}}(p, q) - \mathcal{F}_{\mathcal{T}}(b, q).\end{aligned}$$

To prove that it hols that $\inf_{(v_n)_n} \liminf_{n\to\infty} \mathcal{F}_{\mathcal{C}}(b, v_n)+f_{\mathcal{T},q}(v_n) \leq \mathcal{F}_{\mathcal{S}}(b, p)+\mathcal{F}_{\mathcal{T}}(p, q)-\mathcal{F}_{\mathcal{T}}(b, q)$, we look for a suitable sequence $(v_n)_{n\in\mathbb{N}}$ Funk converging to $f_{\mathcal{S},p|\mathcal{C}}$.

By definition of $T(\mathcal{C})$, there exists $\mathcal{S}_1, \ldots, \mathcal{S}_N$, with $N \in \mathbb{N}$, such that $\mathcal{S}_1 = \mathcal{C}$ and $\mathcal{S}_N = \mathcal{S}$. Let $v \in \partial\mathcal{S}_{N-1}$ such that $\mathcal{S}_N = \tau(\mathcal{S}_{N-1}, v)$. As in the proof of Proposition 2.3.13 there is a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive real numbers such that, if we define $v^1_n = (1 - \lambda_n)v + \lambda_n p$, we get for each $n \in \mathbb{N}$

$$\begin{gathered} v^1_n \in \mathcal{S}_{N-1}, \\ \big|\mathcal{F}_{\mathcal{S}_{N-1}}(u, v_n) - \mathcal{F}_{\mathcal{S}_N}(u, v_n)\big| < \frac{1}{n} \quad \text{for all } u \in \mathcal{S}_{N-1}, \text{ and} \end{gathered} \tag{2.40}$$

$(v^1_n)_{n\in\mathbb{N}}$ converges in Funk sense to $f_{\mathcal{S}_N,p|\mathcal{C}}$ (with respect to $\mathcal{F}_{\mathcal{S}}(\cdot,\cdot)$).

Since $v \in [0]_{\mathcal{S}}$ and hence $v \in [0]_{\mathcal{T}}$ we have from Lemma 2.3.11

$$\mathcal{F}_{\mathcal{S}_N}(u, v^1_n) = \mathcal{F}_{\mathcal{S}_N}(u, p) - \log\lambda_n \quad \text{for all } n \in \mathbb{N} \text{ and } u \in \mathcal{S}_N, \text{ and}$$

$$\mathcal{F}_{\mathcal{T}}(v^1_n, q) = \mathcal{F}_{\mathcal{T}}(p, q) + \log\lambda_n \quad \text{for all } n \in \mathbb{N}.$$

If we combine these two equalities with (2.40) we get

$$\mathcal{F}_{\mathcal{S}_{N-1}}(b, v_n^1) + \mathcal{F}_{\mathcal{T}}(v_n^1, q) < \mathcal{F}_{\mathcal{S}_N}(b, p) + \mathcal{F}_{\mathcal{T}}(p, q) + \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

Now, we can do the same construction to get a sequence $(v_n^2)_{n\in\mathbb{N}}$ in $\mathcal{S}_{N-2}$ that converges to $f_{\mathcal{S},p|\mathcal{S}_{N-1}}$ in Funk sense (with respect to $\mathcal{F}_{\mathcal{S}_{N-1}}(\cdot,\cdot)$) and such that

$$\mathcal{F}_{\mathcal{S}_{N-2}}(b, v_n^2) + \mathcal{F}_{\mathcal{T}}(v_n^2, q) < \mathcal{F}_{\mathcal{S}_{N-1}}(b, v_n^1) + \mathcal{F}_{\mathcal{T}}(v_n^1, q) + \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

We can iterate the argument again and again to get a sequence $(v_n^N)_{n\in\mathbb{N}}$ such that

$$\mathcal{F}_{\mathcal{C}}(b, v_n^N) + \mathcal{F}_{\mathcal{T}}(v_n^N, q) < \mathcal{F}_{\mathcal{S}_N}(b, p) + \mathcal{F}_{\mathcal{T}}(p, q) + \frac{N}{n} \quad \text{for all } n \in \mathbb{N}.$$

By subtracting $\mathcal{F}_{\mathcal{T}}(b, q)$ to both the sides of this inequality, and taking the limit infimum, we get

$$\liminf_{n\to\infty} \mathcal{F}_{\mathcal{C}}(b, v_n^N) + f_{\mathcal{T},q}(v_n^N) \leq \mathcal{F}_{\mathcal{S}}(b, p) + \mathcal{F}_{\mathcal{T}}(p, q) - \mathcal{F}_{\mathcal{T}}(b, q).$$

□

The previous two propositions implies the next corollary.

Corollary 2.4.6. *Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a properly convex open cone. Let $\xi = r_{\mathcal{C},v} + f_{\mathcal{S},p|\mathcal{C}}$ and $\eta = r_{\mathcal{C},w} + f_{\mathcal{T},q|\mathcal{C}}$ be two Hilbert Busemann points, with $v, w \in \mathcal{C}$, $\mathcal{S} \in T(\tau(\mathcal{C}, v))$ and $p \in \mathcal{S}$, $\mathcal{T} \in T(\tau(\mathcal{C}, w))$ and $q \in \mathcal{T}$. Then, we have*

$$\delta(\xi, \eta) = \begin{cases} \mathcal{H}_F(v, w) + \mathcal{H}_{\mathcal{S}}(p, q) & \text{if } F = F_v = F_w \text{ and } [\mathcal{S}] = [\mathcal{T}], \\ +\infty & \text{otherwise.} \end{cases}$$

From this result we have that each part of the Hilbert horofunction boundary of a cone $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ corresponds to a pair $(F, \mathcal{T})$, where F is the relative interior of a proper face of $\mathcal{C}$ and $\mathcal{T} \in T(\tau(\mathcal{C}, v))$ for some $v \in F$.

As we saw in Remark 2.3.9 each cone $\mathcal{T} \in T(\mathcal{C})$ corresponds to a face of the dual cone $\mathcal{C}^*$ and the dimension of the Hilbert geometry associated to it equals the dimension of the corresponding face of $\mathcal{C}^*$.

Therefore, each part of the Hilbert horofunction boundary of the cone corresponds to a pair (F, E^*), with F the relative interior a face of $\mathcal{C}$ and E^* a face of the dual cone F^* of F.

Let us consider a part corresponding to the pair $(F, \mathcal{T})$, defined as above, and denote $\mathcal{T}' = \mathcal{T}/[0]_{\mathcal{T}}$. Corollary 2.4.6 implies that this part endowed with its detour metric is isometric to $F \times \mathcal{T}'$ endowed with the metric $\mathcal{H}_{F\times\mathcal{T}'}$ given by

$$\mathcal{H}_{F\times\mathcal{T}'}((v,p),(w,q)) = \mathcal{H}_F(v,w) + \mathcal{H}_{\mathcal{T}'}(p,q) \quad \text{for all } (v,p),(w,q) \in F \times \mathcal{T}'.$$

Definition 2.4.7. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a properly convex open cone and let $(F, \mathcal{T})$ be a pair associated with a part of the Hilbert horofunction boundary of $\mathcal{C}$. We say that this part is a *pure Funk part* if F gives rise to a Hilbert geometry of projective dimension 0, and that it is a *pure reverse-Funk part* if $\mathcal{T}$ gives rise to a Hilbert geometry of projective dimension 0.

Corollary 2.4.6 implies that a pure part endowed with its detour metric is isometric to a Hilbert geometry.

Definition 2.4.8. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a properly convex cone. The *closure of a part* of the horofunction boundary of $\mathcal{C}$ is the intersection of the closure of the part in the set $\mathrm{C}(\mathcal{C}, \mathbb{R})$ with the set of Busemann points.

Let us consider the pure reverse-Funk part $\{r_{\mathcal{C},v} + f \mid v \in \mathrm{int}(F)\}$, with F a proper face of $\mathcal{C}$ and f the unique Funk horofunction associated with a cone in $T(F^*)$ given by an half-space. It follows from Theorem 2.3.1 applyied to the cone F that the closure of the part $\{r_{\mathcal{C},v}+f \mid v \in \mathrm{int}(F)\}$ is $\{r_{\mathcal{C},v}+f \mid v \in F\}$. Similarly, the closure of the pure Funk part $\{r_{\mathcal{C},v}+f_{\mathcal{T},p} \mid p \in \mathcal{T}\}$, with v an extremal generator and $\mathcal{T} \in T(\tau(\mathcal{C}, v))$, is the set $\{r_{\mathcal{C},v} + f_{\mathcal{S},q} \mid \mathcal{S} = \tau(\mathcal{T}, w)$ for some $w \in \partial\mathcal{T}$ and $q \in \mathcal{S}\}$.

Definition 2.4.9. A pure part is *maximal* if it is not contained in the closure of any other part.

It is clear that maximal pure reverse-Funk parts of the Hilbert horofunction boundary of $\mathcal{C}$ correspond to maximal proper faces and maximal pure Funk parts correspond to the tangents cones $\tau(\mathcal{C}, v)$ for some extremal generator $v \in \partial\mathcal{C}$.

Definition 2.4.10. The *dimension* of a pure part of the horofunction boundary is the dimension of the Hilbert geometry to which it is isomorphic. A part of dimension 0 is a *singleton* of the horofunction boundary of $\mathcal{C}$.

Remark 2.4.11. The singleton Funk parts of the horofunction boundary of a cone $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ correspond to extremal generators of $\mathcal{C}^*$. Indeed, every singleton can be written as $f_{\mathcal{T},p|\mathcal{C}}$, with $\mathcal{T} \cong \mathbb{R}^n \times \mathbb{R}_+$. As we saw in Example 2.3.2, the cross-section parallel to the boundary of $\mathcal{C}$ that contains b define an element $\varphi \in \mathcal{C}^*$ such that $\varphi(b) = 1$ and $f_{\mathcal{T},p}(u) = \varphi(u)$ for all $u \in \mathcal{C}$.

Lemma 2.4.12. *Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a properly convex cone, and let U and V be two pure parts of the Hilbert horofunction boundary of $\mathcal{C}$. Suppose that U is maximal, the intersection of the closure of U with the closure of V is non-empty, and V is not contained in the closure of U. Then, U and V are pure parts of opposite type.*

Proof. Assume that U is a pure reverse-Funk part. Then its closure is $\overline{U}^{\mathcal{B}_\Omega} = \{r_{\mathcal{C},v} + f \mid v \in F\}$, with F a maximal proper face of $\mathcal{C}$ and f the unique Funk horofunction associated with a cone in $T(F^*)$ given by an half-space. If V was a pure reverse-Funk part, since by hypothesis its closure contains a point $r_{\mathcal{C},v} + f$, with $v \in F$, then its elements would be of the form $r_{\mathcal{C},w} + f$ with w in a proper face that contains v. Since V is not contained in the closure of U, this assumption would contradict the maximality of U.

The case when U is a pure Funk part can be study in a similar way. □

Recall that every Busemann point of a Hilbert geometry corresponds to a Hilbert Busemann point of the associated cone. Thus, the partition of the set of Busemann points of a Hilbert geometry corresponds to the partition of the set of Hilbert Busemann points of the cone above the domain. Henceforth, we use the nomenclatures of this section also for the parts of the horofunction boundary of a Hilbert geometry. For example, we will use the term *Funk part* (resp. *reverse-Funk part*) for a part of the horofunction boundary of a Hilbert geometry that corresponds to a pure Funk part (resp. pure reverse-Funk part) on the cone.

In relation to what proved at the beginning of section 2.2.2, if $(\Omega, \mathrm{d}_\Omega)$ is a Hilbert geometry, a Busemann point is given by the composition of a Hilbert Busemann point of $\mathcal{C}_\Omega$ with a map ϕ_D that identifies Ω with a bounded cross-section D of the cone $\mathcal{C}_\Omega$ above Ω. We should then write an element of $\mathcal{B}_\Omega$ as $r_{\mathcal{C}_\Omega,v|_D} \circ \phi_D + f_{\mathcal{T},w|_D} \circ \phi_D$, with $v \in \partial\mathcal{C}_\Omega$, $\mathcal{T} \in T(\tau(\mathcal{C}_\Omega, v))$, and $w \in \mathcal{T}$. However, when the context is clear we will write $r_{\mathcal{C}_\Omega,v} + f_{\mathcal{T},w|\mathcal{C}_\Omega}$ instead of the correct expression above.

2.5 Action of isometries on horofunctions

In this section, we will see that the study of the horofunction boundary is fundamental for a well understanding of the isometries of a Hilbert geometry.

2.5.1 Extension of isometries on the horofunction boundary

Let us start by observing that any isometry $\Phi \in \mathrm{Isom}(\Omega, \mathrm{d}_\Omega)$ can be extended to the horofunction boundary $\partial_\infty \Omega$ by taking for $\xi \in \partial_\infty \Omega$ and $x \in \Omega$

$$\Phi(\xi)(x) = \xi(\Phi^{-1}(x)) - \xi(\Phi^{-1}(b)), \tag{2.41}$$

where $b \in \Omega$ is a fixed base point. Indeed, if $(z_n)_{n\in\mathbb{N}} \subseteq \Omega$ converges to $\xi \in \partial_\infty \Omega$, we have that

$$\begin{aligned}\lim_{n\to\infty} \Psi_{b,\Phi(z_n)}(x) &= \lim_{n\to\infty} \mathrm{d}_\Omega(x, \Phi(z_n)) - \mathrm{d}_\Omega(b, \Phi(z_n))\\ &= \lim_{n\to\infty} \mathrm{d}_\Omega(\Phi^{-1}(x), z_n) - \mathrm{d}_\Omega(\Phi^{-1}(b), z_n)\\ &= \lim_{n\to\infty} \big(\mathrm{d}_\Omega(\Phi^{-1}(x), z_n) - \mathrm{d}_\Omega(b, z_n)\big) - \big(\mathrm{d}_\Omega(\Phi^{-1}(b), z_n) - \mathrm{d}_\Omega(b, z_n)\big)\\ &= \xi(\Phi^{-1}(x)) - \xi(\Phi^{-1}(b)).\end{aligned}$$

The next lemma shows that the extension in (2.41) of an isometry in $\mathrm{Isom}(\Omega, \mathrm{d}_\Omega)$ to the horofunction boundary is an isometry if restricted to the set of Busemann points endowed with the detour metric.

Lemma 2.5.1. *Given* $\xi, \eta \in \mathcal{B}_\Omega$ *and* $\Phi \in \mathrm{Isom}(\Omega, \mathrm{d}_\Omega)$ *it holds that*

$$\delta(\Phi(\xi), \Phi(\eta)) = \delta(\xi, \eta).$$

Proof. Let $\xi, \eta \in \mathcal{B}_\Omega$ be two Busemann points. From Proposition 2.1.15 we know that the detour metric is independent of the choice of the base point. Thus, we can fix an arbitrary base point $b \in \Omega$. From the definition in (2.41) we have that a sequence $(x_n)_{n\in\mathbb{N}}$ in Ω converges to the horofunction $\Phi(\xi)$ if and only if $(\Phi^{-1}(x_n))_{n\in\mathbb{N}}$ converges to ξ. Therefore, we have

$$\begin{aligned}H(\Phi(\xi), \Phi(\eta)) &= \inf_{(x_n)_{n\in\mathbb{N}}\in S} \big(\liminf_{n\to\infty} \mathrm{d}_\Omega(b, \Phi(x_n)) + \eta(x_n) - \eta(\Phi^{-1}(b)\big)\\ &= \inf_{(x_n)_{n\in\mathbb{N}}\in S} \big(\liminf_{t\to\infty} \mathrm{d}_\Omega(\Phi^{-1}(b), x_n) - \mathrm{d}_\Omega(b, x_n) + \mathrm{d}_\Omega(b, x_n) + \eta(x_n) - \eta(\Phi^{-1}(b))\big)\\ &= H(\xi, \eta) + \xi(\Phi^{-1}(b)) - \eta(\Phi^{-1}(b)),\end{aligned}$$

where S is the set of sequences that converges to ξ. With the same reasoning, we have $H(\Phi(\eta), \Phi(\xi)) = H(\eta, \xi) + \eta(\Phi^{-1}(b)) - \xi(\Phi^{-1}(b))$. Hence, $\delta(\Phi(\xi), \Phi(\eta)) = \delta(\xi, \eta)$. □

Remark 2.5.2. Since the extension of an isometry to the horofunction compactification of Ω is an isometry on the set of Busemann points, this extension preserves the partition of $\mathcal{B}_\Omega$ into parts.

Recall that two Busemann points belong to the same part if and only if the detour distance between them is finite. Moreover, each part corresponds to a pair (F, E^*), with F the relative interior of a proper face of Ω and E^* a face of the dual F^* of F.

The next lemma shows that the extension of any isometry maps pure parts into pure parts.

Proposition 2.5.3. *Let* $(\Omega, \mathrm{d}_\Omega)$ *and* $(\Omega', \mathrm{d}_{\Omega'})$ *be two Hilbert geometries. The metric space* $(\Omega \times \Omega', \mathrm{d}_{\Omega\times\Omega'})$ *is isometric to a Hilbert geometry if and only if either the projective dimesion of* Ω *is 0 or projective dimension of* Ω' *is 0.*

Proof. Recall that $\mathrm{d}_{\Omega\times\Omega'}(x, x') = \mathrm{d}_\Omega(x) + \mathrm{d}_\Omega(x)$. If either the projective dimesion of Ω is 0 or projective dimension of Ω' is 0, it is clear that $(\Omega\times\Omega', \mathrm{d}_{\Omega\times\Omega'})$ is isometric to a Hilbert geometry. Suppose that both the projective dimesion of Ω and projective dimesion of Ω' are positive. Let ℓ_Ω be a geodesic straight line connecting an extreme point in $\partial\Omega$ to another point in $\partial\Omega$, and $\ell_{\Omega'}$ be a geodesic straight line connecting an extreme point in $\partial\Omega'$ to another point in $\partial\Omega'$. By definition every geodesic of $\Omega \times \Omega'$ is the product of a geodesic in Ω and a geodesic in Ω'. Since ℓ_Ω have an end-point in an extreme point, for every pair of point in ℓ_Ω a. The same is true for $\ell_{\Omega'}$. Therefore, $(\ell_\Omega \times \ell_{\Omega'}, \mathrm{d}_{\Omega\times\Omega'|\ell_\Omega\times\ell_{\Omega'}})$ is a uniquely geodesic metric space.

If $(\Omega \times \Omega', \mathrm{d}_{\Omega\times\Omega'})$ is isometric to a Hilbert geometry $(\Omega'', \mathrm{d}_{\Omega''})$, then the image of $\ell_\Omega \times \ell_{\Omega'}$ via that isometry is a closed and uniquely geodesic subspace of Ω''. Thus, $(\ell_\Omega \times \ell_{\Omega'}, \mathrm{d}_{\Omega\times\Omega'|\ell_\Omega\times\ell_{\Omega'}})$ is itself isometric to a Hilbert geometry. On the other hand, it is isomorphic to $\mathbb{R}^2$ with the ℓ_1-norm. By Proposition 2.2.17, $(\ell_\Omega \times \ell_{\Omega'}, \mathrm{d}_{\Omega\times\Omega'|\ell_\Omega\times\ell_{\Omega'}})$ is isometric to the 2-simplex. We saw in Example 1.2.7 that the balls of the 2-simplex are hexagonal, so we get a contradiction. □

Corollary 2.5.4. *Let* $(\Omega, \mathrm{d}_\Omega)$ *be a Hilbert geometry and let* $\Phi \in \mathrm{Isom}(\Omega, \mathrm{d}_\Omega)$*. The extension of* Φ *maps the set of pure parts of the horofuncion boundary to the set of pure parts. Moreover, it preserves the dimension and the maximality of each part.*

Proof. By Proposition 2.5.3 and Corollary 2.4.6, a pure part with its detour metric is isometric to a Hilbert geometry. Since the topology induced by the Hilbert metric is the Euclidean one, the dimensions of two isometric parts have to be the same. Moreover, the image of a maximal pure part has to be a maximal pure part. □

2.5.2 Gauge-preserving and gauge-reversing maps

It is clear that the action of the extension of an isometry on the set of parts is very important. Therefore, it will be useful in what follows, the study of the behaviour of the gauge under particular types of automorphisms of a cone.

Definition 2.5.5. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a convex cone and $\phi : \mathcal{C} \to \mathcal{C}$. We say that ϕ is *homogeneous* if for every $u \in \mathcal{C}$ and $\lambda > 0$, it holds $\phi(\lambda u) = \lambda\phi(u)$. If for every $u \in \mathcal{C}$ and $\lambda > 0$, it holds $\phi(\lambda u) = \frac{1}{\lambda}\phi(u)$, we say that ϕ is *anti-homogeneous*.

Definition 2.5.6. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a convex cone and $\phi : \mathcal{C} \to \mathcal{C}$. We say that ϕ is *isotone* if for every $u, v \in \mathcal{C}$ such that $u \leq_{\mathcal{C}} v$, it holds $\phi(u) \leq_{\mathcal{C}} \phi(v)$. If for every $u, v \in \mathcal{C}$ such that $u \leq_{\mathcal{C}} v$, it holds $\phi(v) \leq_{\mathcal{C}} \phi(u)$, we say that ϕ is *antitone*.

Definition 2.5.7. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a convex cone and $\phi : \mathcal{C} \to \mathcal{C}$. We say that ϕ is *gauge-preserving* if for every $u, v \in \mathcal{C}$ it holds that $M_{\mathcal{C}}(u/v) = M_{\mathcal{C}}(\phi(u)/\phi(v))$. If for every $u, v \in \mathcal{C}$it holds that $M_{\mathcal{C}}(u/v) = M_{\mathcal{C}}(\phi(v)/\phi(u))$, we say that ϕ is *gauge-reversing*.

Remark 2.5.8. The action of a gauge-preserving automorphism $\phi : \mathcal{C} \to \mathcal{C}$, maps pure reverse-Funk parts into pure reverse-Funkparts and pure Funk parts into pure Funk parts. Quite the opposite, a gauge-reversing automorphism $\phi : \mathcal{C} \to \mathcal{C}$, maps pure reverse-Funk parts into pure Funk parts and pure reverse-Funk parts into pure Funk parts.

We show that the image of any reverse-Funk horofunction under the action of ϕ is a reverse-Funk horofunction. The other cases can be studied in a similar way.

Assume that $\phi : \mathcal{C} \to \mathcal{C}$ is gauge-preserving. Then, for each reverse-Funk horofunction $r_{\mathcal{C},v}$, with $v \in \mathcal{C}_\Omega$, there is a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$ converging in reverse-Funk sense to $r_{\mathcal{C},v}$ and in usual sense to v. We denote Φ the projective action of ϕ and $R_{\mathcal{C},v}$ the projective action of $r_{\mathcal{C},v}$. Given $x \in \Omega$, we pick $u \in \mathcal{C}_\Omega$ such that $x = [u]$ and u belongs to the cross-section we use to identify the Hilbert horofunction boundary of $\mathcal{C}_\Omega$ and the horofunction boundary of Ω, see the construction in Section 2.2.2. Let b be the fixed base point and $\tilde{b} \in \mathcal{C}_\Omega$ such that $b = [\tilde{b}]$ and $\tilde{b}$ belongs to the cross-section specified above. Thus, we have

$$\begin{aligned}\Phi(R_{\mathcal{C},v})(x) &= R_{\mathcal{C},v}(\Phi^{-1}(x)) - R_{\mathcal{C},v}(\Phi^{-1}(b))\\ &= r_{\mathcal{C},v}(\phi^{-1}(u)) - r_{\mathcal{C},v}(\phi^{-1}(\tilde{b}))\\ &= \lim_{n\to\infty} \mathcal{RF}_{\mathcal{C}}(\phi^{-1}(u), v_n) - \mathcal{RF}_{\mathcal{C}}(b, v_n) - \mathcal{RF}_{\mathcal{C}}(\phi^{-1}(\tilde{b}), v_n) + \mathcal{RF}_{\mathcal{C}}(b, v_n)\\ &\lim_{n\to\infty} \mathcal{RF}_{\mathcal{C}}(u, \phi(v_n)) - \mathcal{RF}_{\mathcal{C}}(\tilde{b}, \phi(v_n)).\end{aligned}$$

The following result is really the key for the proof of Theorem 2.0.4. We do not include the proof of this result in the thesis since the approach used is algebraic and need a certain amount of prerequisites about Jordan algebras. However, a proof can be found in the article [22] of C. Kai.

Theorem 2.5.9 ([22, Theorem A])**.** *Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a properly convex homogeneous open cone. The composition of the Vinberg's canonical isomorphism (1.12) with the identification of $\mathcal{C}$ with its dual $\mathcal{C}^*$ is a gauge-reversing map if and only if $\mathcal{C}$ is symmetric.*

In order to prove Theorem 2.0.4, we want to show that each isometry of a Hilbert geometry either is a projective isometry or arises as the projective action of a gauge reversing automorphism. Then, we will see that symmetric cones are the only ones that admit a gauge-reversing automorphism. Theorem 2.5.9 shows that each symmetric cone admits a gauge-reversing automorphism.

We saw that the extension of a isometry maps pure parts into pure parts and maximal parts into maximal parts. Therefore, either it maps every maximal pure Funk part into a maximal pure Funk part, or there exists a (non-singleton) maximal pure Funk part that is mapped into a pure reverse-Funk part of the same dimension. We want to show that in the first case, the isometry is projective and that in the second case, the isometry arises as the projective action of a gauge-reversing automorphism.

Lemma 2.5.10. *Let $\Phi : \Omega \to \Omega$ be an isometry of the Hilbert geometry $(\Omega, \mathrm{d}_\Omega)$. Suppose that for every extreme point $z \in \partial\Omega$ there exists an extreme point $z' \in \Omega$ such that the image of any open segment $]z, x[$, with $x \in \Omega$ is an open segment $]z', y[$ for some $y \in \Omega$. Then, the isometry Φ extends continuously to the boundary $\partial\Omega$.*

Proof. Given an extreme point $z \in \partial\Omega$ we define the image of z under the extension of Φ to the boundary to be the extreme point $z' \in \partial\Omega$ in the hypothesis of the propposition.

Let us define Ω_m as the union of all the faces of Ω of projective dimension at least m, so $\Omega = \Omega_n$ and $\overline{\Omega} = \Omega_0$, where n is the projective dimension of Ω. We prove by induction that for every $m \leq n$ the followings hold

1) Φ extends continuously to Ω_m,

2) for every extreme point $z \in \partial\Omega$ and every $x \in \Omega_m$ the image of the open segment $]z, x[$ is an open segment $]z', y[$ for some extreme point $z' \in \partial\Omega$ and $y \in \Omega_m$.

If $m = n$ the two properties above hold by hypothesis.

Now, suppose that the two properties hold for every $m \leq k \leq n$. We prove that they hold also for $k = m - 1$.

Let us consider a relatively open face $F \subseteq \partial\Omega$ of dimension $m - 1$. Since F is a proper face, there exists an extreme point $z \in \partial\Omega$ such that $z \notin \overline{F}$. By inductive hypothesis, for every $x \in F$ the image of the open segment $]z, x[\subseteq \Omega_m$ is an open segment $]z', y[$ for some extreme point $z' \in \partial\Omega$ and a point $y \in \Omega_m$. We define $\Phi^{|F \cup \Omega_m}(x) = y$.

Since the relatively open faces of projective dimension $m - 1$ are disjoint, to prove that the property 1) holds, it suffices to show that $\Phi^{|F \cup \Omega_m}$ is continuous, for an arbitrary face F. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence converging to x. We claim that given $s \in\,]u', y[$ it holds that $s \in\,]u', y'[$ for every limit point y' of the sequence $(\Phi(x_n))_{n \in \mathbb{N}}$. Thus, considering a sequence $(s_n)_{n \in \mathbb{N}}$ in $]u', y'[$ convergent to y, we conclude that $y' = y$.

To prove the claim, we consider some $r \in\,]u, x[$ such that $\Phi(r) = s$ and a sequence $(r_n)_{n \in \mathbb{N}}$ such that $r_n \in\,]u, x_n[$. Since $r \in \Omega_m$, by inductive hypothesis the limit of $(\Phi(r_n))_{n \in \mathbb{N}}$ equals s and $\Phi(r_n) \in\,]u', \Phi(x_n)[$.

Now, we have to prove that the property 2) holds for Ω_{m-1}. Let $]z, x[\subseteq \Omega_{m-1}$ be an open segment with $z\partial\Omega$ an extreme point and $x \in \Omega_{m-1}$. We have to prove that there exist an extreme point $z' \in \partial\Omega$ and a point $y \in \Omega_{m-1}$ such that $\Phi(]z, x[) = \,]z', y[$. Let us consider two arbitrary points $r, s \in\,]z, x[$ such that $s \in\,]z, r[$ and two sequences $(s_n)_{n \in \mathbb{N}}$ and $(r_n)_{n \in \mathbb{N}}$ in Ω that converge respectively to s and r, and such that $s_n \in]z, r_n[$. Since these sequences belong to Ω, by hypothesis the image of $]z, r_n[$ is an open segment $]z', \Phi(r_n)[$ and $\Phi(s_n) \in\,]z', \Phi(r_n)[$ for all $n \in \mathbb{N}$. From the continuity of $\Phi^{|\Omega_{m-1}}$, we get that $\Phi(s) \in\,]z', \Phi(r)[$. By the arbitrariness of s and t, we get that $\Phi(]z, x[) \subseteq\,]z', y[$ for some $y \in \Omega$. From the continuity of Φ, the image of $]z, x[$ has to be connected, so $\Phi(]z, x[) = \,]z', y[$ for some $y \in \Omega$. ☐

Theorem 2.5.11. *Let $\Phi \in \mathrm{Isom}(\Omega, \mathrm{d}_\Omega)$ be an isometry of an Hilbert geometry $(\Omega, \mathrm{d}_\Omega)$. If Φ maps each maximal pure Funk part is mapped by the extension of Φ to a maximal pure Funk part, then Φ is a projective isometry.*

Proof. First, we show that if Φ maps each maximal pure Funk part is mapped by the extension of Φ to a maximal pure Funk part, then Φ and Φ^{-1} estends continuously to the boundary. Let $x \in \partial\Omega$ be an extreme point, and $v \in \partial\mathcal{C}_\Omega$ such that $x = [v]$. The part associated with $(F_v, \tau(\mathcal{C}_\Omega, v))$ is a maximal pure Funk part. By hypothesis, there is a maximal pure Funk part $(F_w, \tau(\mathcal{C}_\Omega, w))$, with $[w] \in \partial\Omega$ an extreme point such that Φ maps the part associated with $(F_v, \tau(\mathcal{C}_\Omega, v))$ to the part associated with

$(F_w, \tau(\mathcal{C}_\Omega, w))$.

Proposition 1.2.9 implies that the straight line through x and y is a unique-geodesic. Therefore, the image under Φ of this straight line is a unique-geodesic, and hence a strainght line, too.

Since a sequence $(x_n)_{n\in\mathbb{N}}$ in Ω converging to x converges to a horofunction in the part associated with $(F_v, \tau(\mathcal{C}_\Omega, v))$, the sequence $(\Phi(x_n))_{n\in\mathbb{N}}$ converges to a Busemann point in $(F_w, \tau(\mathcal{C}_\Omega, w))$. Theorem 2.3.23 and Theorem 2.3.1 implies that $(\Phi(x_n))_{n\in\mathbb{N}}$ converges in the Euclidean topology to $[w]$. Therefore, the isometry Φ satisfies the hypotesis of Lemma 2.5.10, and hence extends continuously to the boundary $\partial\Omega$. Using a similar argument we get that Φ^{-1} extends continuously to the boundary.

The rest of the proof goes by induction on the projective dimension n of Ω. If $n = 1$, then Ω is a segment in $\mathbb{P}^1$, denote its endpoints a and b. For each point $x \in \Omega$, there exists a unique projective trasfomation f that coincide with Φ on a, b and x. Let $y \in \Omega \setminus \{x\}$ be any other point of Ω. Since both Φ and f preserves the cross-ratio of four aligned points, the cross-ratio of $\Phi(a), \Phi(x), \Phi(b)$ and $\Phi(y)$, considered in the correct order, coincide with the cross-ratio of $f(a), f(x), f(b)$ and $f(y)$, considered in the correct order, and both coincide with the cross-ratio of a, x, b and y, considered in the correct order. From Theorem 1.2.2, there exists a unique projective transformation that maps a, x, b and y in $f(a), f(x), f(b)$ and $f(y)$, and this transformation has to coincide with Φ.

Now, suppose that the thesis holds for any Hilbert geometry of dimension $k \in \{1, \ldots, n-1\}$. Since the projective dimension of Ω is n, then there exist $n+1$ extreme points $x_0, \ldots, x_n \in \partial\Omega$ whose convex hull forms a n-simplex S in $\overline{\Omega}$. We extend the idea of the first case. Let $y \in \text{int}(S)$. Since $\{x_0, \ldots, x_n, y\}$ is a projective basis of $\mathbb{P}^n$ and Φ^{-1} extends continuously to the boundary, then also $\{\Phi(x_0), \ldots, \Phi(x_n), \Phi(y)\}$ is a projective basis of $\mathbb{P}^n$. Therefore, there exists a unique projective transformation f of $\mathbb{P}^n$ such that $f(x_i) = \Phi(x_i)$, $i = 0, \ldots, n$ and $f(y) = \Phi(y)$.

We want to prove first that $\Phi_{|S} \equiv f_{|S}$ and then that Φ and f agrees on each n-simplex given by the convex hull of $n+1$ extreme points of Ω.

For all $i \in \{0, \ldots, n\}$, let ℓ_i be the intersection of the projective line passing through x_i and y with $\overline{\Omega}$ and H_i be the intersection of the hyperplane containing $x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$. Denote y_i the intersection point of ℓ_i with H_i. Similarly, let ℓ_i' be the intersection of the projective line passing through $\Phi(x_i)$ and $\Phi(y)$ with $\overline{\Omega}$ and H_i' be the intersection with $\overline{\Omega}$ of the hyperplane containing $\Phi(x_0), \ldots, \Phi(x_{i-1}), \Phi(x_{i+1}), \ldots, \Phi(x_n)$. Denote y_i' the intersection point of ℓ_i with H_i. From Proposition 1.2.9, for each i, the line ℓ_i is the unique geodesic that has x_i as an endpoint. Since f is a projective transformation $f(\ell_i) = \ell_i'$ and $f(H_i) = H_i'$. Since $\Phi^{-1}(\ell_i')$ is a geodesic then it must be the unique-geodesic ℓ_i. Therefore, $\Phi(\ell_i) = h(\ell_i) = \ell_i'$. Moreover, $\Phi(H_i) = h(H_i) = H_i'$, since continuous extension of Φ and Φ^{-1} has to preserve the affine hull of a set of points.

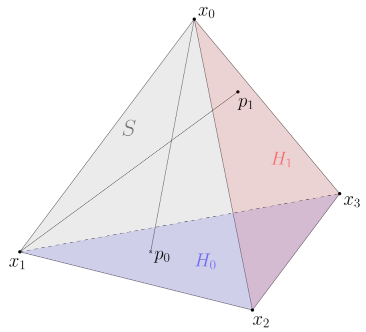


Figure 2.8: Configuration of points considered in the proof of Theorem 2.5.11.

It follows that $\Phi(y_i) = f(y_i)$. Thus, by inductive hypothesis, $\Phi_{|\mathrm{int}(H_i)}$ is a projective transformation for all i. Since f and Φ coincide on the projective bases $x_0, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n$, then $\Phi_{|\,\mathrm{int}(H_i)} = f_{|\,\mathrm{int}(H_i)}$, for all $i = 0, \dots, n$.

Now, given $p \in \mathrm{int}(S)$, we define p_0 as the intersection of the line through p and x_0 with h_0, and p_1 as the intersection of the line through p and x_1 with h_1. As above, the line through p and x_0 and the line through p and x_1 are unique-geodesic. Thus, both Φ and f map p to the intersection of the images of these two lines. It follows that $\Phi(p) = f(p)$. Hence, $\Phi_{|S} = f_{|S}$.

If $z_0, \dots, z_n$ is another set of extreme points whose convex hull is a n-simplex S' in Ω, then there exists some $i \in \{0, \dots, n\}$ such that, up to permutations of the indexes of $z_0, \dots, z_n$, the convex hull of $x_0, \dots, x_{i-1}, z_i, x_{i+1}, \dots, x_n$ is a n-simplex S'' in Ω.

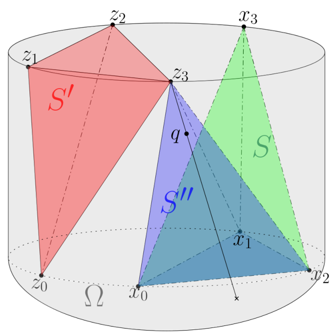


Figure 2.9: Configuration of simplexes considered in the proof of Theorem 2.5.11.

Given $q \in \mathrm{int}(S'')$, by the first case, Φ coincide with a projective transformation h on the the line through q and z_i. Since the line through q and z_i intersect $\mathrm{int}(S)$ in an open segment, h coincide with f on this line. Thus, we have $\Phi(q) = h(q) = f(q)$. From the arbitrariness of q, we conclude that $\Phi_{|S''} = f_{|S''}$. In particular

We can apply this procedure to each point in $\{z_0, \ldots, z_n\}$, to get that $\Phi_{|S'} = f_{|S'}$.

We have proved that f and Φ coincide on every n-simplex contained in Ω. The conclusion of the proof follows from Carathéodory Theorem [6, Theorem 11.1.8.6], that asserts that each point of a properly convex domain belongs to the convex hull of $n+1$ extreme points. □

Corollary 2.5.12. *Let* $(\Omega, \mathrm{d}_\Omega)$ *be a Hilbert geometry and* $\mathcal{C}_\Omega$ *be the cone above* Ω. *If* $\phi : \mathcal{C}_\Omega \to \mathcal{C}_\Omega$ *is gauge-preserving, then the projective action of* ϕ *on* Ω *is a projective isometry.*

Proof. Let $\Phi : \Omega \to \Omega$ be the map induced by ϕ. Remark 2.5.8 implies that the extension of Φ maps each reverse-Funk horofunction to a reverse-Funk horofunction and each Funk horofunction to a Funk horofunction. Therefore, from Theorem 2.5.11 we get that Φ is a projective isometry. □

Corollary 2.5.13. *Let* $L_{n+1} \subseteq \mathbb{R}^{n+1}$ *be the Lorentzian cone. Then, the group of projective isometries of the Hilbert geometry associated with* L_{n+1} *coincide with its group of isometries.*

Proof. The boundary ∂L_{n+1} of L_{n+1} has regularity C^1 and is strictly convex. It follows that every cone tangent to L_{n+1} is a half-space, and every face of L_{n+1} is a ray. Therefore, every part of the horofunction boundary is a singleton. By Theorem 2.5.11 every isometry is projective. □

Now, we study the remaining case, when there is an isometry that maps a non-singleton maximal pure Funk part to a pure reverse-Funk part.

Lemma 2.5.14. *Let* $(\Omega, \mathrm{d}_\Omega)$ *be a Hilbert geometry and* $\Phi \in \mathrm{Isom}(\Omega, \mathrm{d}_\Omega)$. *Suppose that* $x \in \partial\Omega$ *is an extreme point and that maximal pure Funk part* X *associated with* x *is not a singleton. If* $\Phi(X)$ *is a reverse-Funk part, then the line segment connecting* x *with any other extreme point* $y \in \partial\Omega$ *is contained in the boundary* $\partial\Omega$.

Proof. Let $x \in \partial\Omega$ be an extreme point satisfying the conditions of the statement. Suppose that there exists an extreme point $y \in \partial\Omega$ such that the segment $]x, y[$ is contained in Ω. We parametrize $]x, y[$ as $\gamma : \mathbb{R} \to \Omega$. Proposition 1.2.9 implies that the segment γ is a unique-geodesic.

Let X be the maximal pure Funk part associated with x. Then, each Busemann point in x is obtained as the limit of a sequence that converges in usual sense to

X. Let Y be the maximal pure Funk part associated with y. Then, each Busemann point in Y is obtained as the limit of a sequence that converges in usual sense to y.

Since both X and Y are maximal pure parts, they do not belong to the boundary of the any other part. So, the geodesic connecting a point in X to a point in Y must be contained in Ω (if it exists). It follows that the image of γ is the only unique-geodesic connecting a Busemann point in X to a Busemann point in Y. Indeed, if it was an other one, its pre-image in Ω would be a geodesic from x to y different from γ. Since the extension of Φ is an isometry, the geodesic $\Gamma = \Phi \circ \gamma$ is the only unique geodesic connecting a Busemann point in $\Phi(X)$ to a Busemann point in $\Phi(Y)$. We want to show that the fact that $\Phi(X)$ is a pure reverse-Funk part implies the existence of an other unique geodesic connecting a point in $\Phi(X)$ to a point in $\Phi(Y)$.

Let $z \in \partial\Omega$ be the endpoint of Γ in the interior of face F of Ω associated with the pure reverse-Funk part $\Phi(X)$. Denote $w \in \partial\Omega$ the other endpoint of Γ. Since $\Phi(X)$ is a non-singleton part, there is another point $z' \in \partial$ in the interior of F. Then we can consider the segment $]w, z'[$, and parametrize it as $\Gamma' : \mathbb{R} \to \Omega$. We want to prove that Γ' is a unique-geodesic. Proposition 1.2.9 tells us that this is true if there are no pair of coplanar segment in $\partial\Omega$ that contains respectively w and z'. If such segments existed then there would exist a segment parallel to the one containing z and which contains z', since z and z' belong to the interior of same face. So, we have obtained a contradiction the fact that gamma is a unique-geodesic, again by Proposition 1.2.9. So, the image of Γ' is a unique-geodesic connecting a point in $\Phi(Y)$ to a point in $\Phi(X)$. Therefore, $\Phi(X)$ can not be a pure reverse-Funk part. □

Proposition 2.5.15. *Let $(\Omega, \mathrm{d}_\Omega)$ be a Hilbert geometry and $\Phi \in \mathrm{Isom}(\Omega, \mathrm{d}_\Omega)$ be an isometry. Suppose that there is a non-singleton maximal Funk part that is mapped by Φ into a pure reverse-Funk part. Then, the isometry Φ is the projective action of a gauge-reversing automorphism of the cone $\mathcal{C}_\Omega$ associated with Ω.*

Proof. Let $b \in \Omega$ be a fixed base point. In the domain of Φ we work with the horofunction compactification of Ω associated with the base point b. In the image of Φ we work with the horofunction compactification of Ω associated with the base point $b' = \Phi(b)$. Formally, we have to compose Φ with the homeomorphism from the compactification of Ω associated with b to the compactification of Ω associated with b'. From Proposition 2.1.18 the detour metric is independent of the choice of the base point. Hence, this composition maps the non-singleton maximal Funk part in the statement into a pure reverse-Funk part.

Let $(F_v, \tau(\mathcal{C}_\Omega, v))$ be the pair associated with the non-singleton maximal pure

Funk part of the statement, where $[v] \in \partial\Omega$ is an extreme point and the dimension of the Hilbert geometry associated with $\tau(\mathcal{C}_\Omega, v)$ is positive. We denote this part V. By hypothesis, the image of this part is a pure reverse-Funk part. Let $(F, \mathcal{T})$ be the pair associated with the pure reverse-Funk part $\Phi(V)$. Thus, F is a face of $\mathcal{C}_\Omega$ and the dimension of the Hilbert geometry associated with F equals the dimension of the Hilbert geometry associated with $\tau(\mathcal{C}_\Omega, v)$, and the dimension of the Hilbert geometry associated with $\mathcal{T}$ is 0. Let $f_\mathcal{T} \in \mathcal{K}^\mathcal{F}_\mathcal{T}$ be the unique Funk horofunction associated with $(F, \mathcal{T})$.

We define $\phi : \mathcal{C}_\Omega \to \mathcal{C}_\Omega$ imposing that $[\phi(z)] = \Phi([z])$ and $f_\mathcal{T}(\phi(z)) = r_{\mathcal{C}_\Omega, v}(z)$ for each $z \in \mathcal{C}_\Omega$. The function ϕ is well defined since $f_\mathcal{T}$ is the restriction on $\mathcal{C}_\Omega$ of a linear function. By definition Φ is the projective action of ϕ. We want to show that ϕ is gauge-reversing. To this aim we start by proving that every Funk Busemann point obtained as the Funk limit of an almost geodesic converging to an extremal generator is mapped by ϕ to a reverse-Funk Busemann point.

Let $u \in \partial\mathcal{C}_\Omega$ be another extremal generator such that $\tau(\mathcal{C}, u)$ is not an half-space. We denote U the part associated with $(F_u, \tau(\mathcal{C}, u))$. Then, we show that also $\Phi(U)$ is a pure reverse-Funk part.

By Lemma 2.5.14 the segment connecting u and v belongs to $\partial\mathcal{C}_\Omega$. Thus, there is a non-empty set of hyperplanes that supports $\mathcal{C}_\Omega$ at both u and v. This set correspond to a proper face of $\mathcal{C}^*_\Omega$. We choose an extremal generator $\mathcal{S}^* \in \partial\mathcal{C}^*_\Omega$ in the boundary of this face. Let E be the face of $\mathcal{C}_\Omega$ whose point are supported by $\mathcal{S}^*$. We denote $\mathcal{S}$ the open cone (that is an half-space) whose dual cone is $\mathcal{S}^*$. Then, we consider the maximal pure reverse-Funk part associated with $(E, \mathcal{S})$. Denote W this part. Let $f_\mathcal{S}$ be the unique Funk horofunction associated with $\mathcal{S}$.

The Busemann point $r_{\mathcal{C}_\Omega, v} + f_\mathcal{S}$ belongs to both the closure of V and the closure of W, where the closure is in $\mathrm{C}(\mathcal{C}_\Omega, \mathbb{R})$. Therefore, the closure of $\Phi(V)$ and the closure of $\Phi(W)$ intersect in a point. By Lemma 2.4.12 $\Phi(W)$ is a pure Funk part. Using the same reasoning, since $r_{\mathcal{C}_\Omega, u} + f_\mathcal{S}$ belongs to both the closure of U and the closure of W, and $\Phi(W)$ is a pure reverse-Funk part, then $\Phi(U)$ has to be a pure reverse-Funk part, by Lemma 2.4.12.

Let $\mathcal{R}$ be the half-space open cone associated with $\Phi(U)$.

Let $w \in \partial\mathcal{C}_\Omega$ be the extremal generator associated with the pure Funk part$\Phi(W)$. Since $\Phi(r_{\mathcal{C}_\Omega, v} + f_\mathcal{S}) \in \overline{\Phi(W)}^{\mathcal{B}_\Omega} \cap \overline{\Phi(V)}^{\mathcal{B}_\Omega}$ then we have

$$\Phi(r_{\mathcal{C}_\Omega, v} + f_\mathcal{S}) = r_{\mathcal{C}_\Omega, w} + f_\mathcal{T}. \tag{2.42}$$

Similarly, since $\Phi(r_{\mathcal{C}_\Omega,u} + f_{\mathcal{S}}) \in \overline{\Phi(W)}^{\mathcal{B}_\Omega} \cap \overline{\Phi(U)}^{\mathcal{B}_\Omega}$ then we have

$$\Phi(r_{\mathcal{C}_\Omega,u} + f_{\mathcal{S}}) = r_{\mathcal{C}_\Omega,w} + f_{\mathcal{R}}. \tag{2.43}$$

Therefore, we get from (2.42) and the definition of ϕ that

$$\phi(f_{\mathcal{S}}) = \phi(r_{\mathcal{C}_\Omega,v} + f_{\mathcal{S}}) - \phi(r_{\mathcal{C}_\Omega,v}) = r_{\mathcal{C}_\Omega,w} + f_{\mathcal{T}} - f_{\mathcal{T}}. \tag{2.44}$$

Now, from (2.43) and (2.44) we get

$$\phi(r_{\mathcal{C}_\Omega,u}) = f_{\mathcal{R}}. \tag{2.45}$$

Let f be a Funk Busemann point obtained as the Funk limit of an almost geodesic converging to u. Then, the horofunction $r_{\mathcal{C}_\Omega,u} + f$ is contained in the closure of U. As we proved above, $\Phi(U)$ is a pure reverse-Funk part, then $\phi(r_{\mathcal{C}_\Omega,u} + f) = r_{\mathcal{C}_\Omega,p} + f_{\mathcal{R}}$ for some $p \in \partial\mathcal{C}_\Omega$. Therefore, we get from (2.45) that

$$\phi(f) = \phi(r_{\mathcal{C}_\Omega,u} + f) - \phi(r_{\mathcal{C}_\Omega,u}) = r_{\mathcal{C}_\Omega,p} + f_{\mathcal{R}} - f_{\mathcal{R}} = r_{\mathcal{C}_\Omega,p}. \tag{2.46}$$

Hence, we have proved that ϕ maps each Funk Busemann point, obtained as the Funk limit of a Funk almost-geodesic converging to an extremal generator, to a reverse-Funk Busemann point.

Now, let $v_1, v_2 \in \mathcal{C}_\Omega$ be two non-parallel points of the cone above Ω. Then, there exists a bounded cross-section D of $\mathcal{C}_\Omega$ that contains both v_1 and v_2. From Proposition 1.2.9, there exists a geodesic ray $\gamma : \mathbb{R}_{\geq 0} \to D$ from v_1 that passes through v_2 and converges to an extreme point of the boundary of the cross-section. Then, following the argument in Remark 2.2.29 we have

$$\mathcal{F}_{\mathcal{C}}(v_1, v_2) = \mathcal{F}_{\mathcal{C}}(v_1, \gamma(t)) - \mathcal{F}_{\mathcal{C}}(b, \gamma(t)) - \mathcal{F}_{\mathcal{C}}(v_2, \gamma(t)) + \mathcal{F}_{\mathcal{C}}(b, \gamma(t)) \quad \text{for all } t \in \mathbb{R}_{\geq 0}. \tag{2.47}$$

Hence, if f is the Funk limit of $\gamma(t)$, we get from (2.47) that $\mathcal{F}_{\mathcal{C}}(v_1, v_2) = f(v_1) - f(v_2)$. Let $r = \phi(f)$, we know from the previous part of the proof that r is a reverse-Funk Busemann point. Moreover, from the definition of the push-forward of an isometry, and for the first assumption on the base points of the horofunction boundary, we have $r = f \circ \phi^{-1}$. Then, we have

$$\mathcal{F}_{\mathcal{C}}(v_1, v_2) = f(v_1) - f(v_2) = r(\phi(v_1)) - r(\phi(v_2)) \leq \mathcal{RF}_{\mathcal{C}}(\phi(v_1), \phi(v_2)).$$

On the other hand, since ϕ preserves the Hilbert distance, applying this last inequality, we get

$$\mathcal{F}_{\mathcal{C}}(v_1, v_2) = \mathcal{H}_{\mathcal{C}}(v_1, v_2) - \mathcal{F}_{\mathcal{C}}(v_2, v_1) \geq \mathcal{H}_{\mathcal{C}}(v_1, v_2) - \mathcal{RF}_{\mathcal{C}}(v_2, v_1) = \mathcal{RF}_{\mathcal{C}}(v_1, v_2).$$

Applying the exponential we get that if $v_1, v_2 \in \mathcal{C}_\Omega$ are not parallel, then $M_{\mathcal{C}}(v_1/v_2) = M_{\mathcal{C}}(\phi(v_2)/\phi v_1)$.

If $v_1 = \lambda v_2$ for some $\lambda > 0$, we get form the linearity of $f_{\mathcal{T}}$ and the definition of ϕ, that $M_{\mathcal{C}}(v_1/v_2) = M_{\mathcal{C}}(\phi(v_2)/\phi v_1)$. Hence, the map ϕ is gauge-reversing. □

Theorem 2.5.16. *Let* $(\Omega, \mathrm{d}_\Omega)$ *be a Hilbert geometry. Every element of* $\mathrm{Isom}(\Omega, \mathrm{d}_\Omega)$ *is either a projective isometry or arises from the projective action of a gauge-reversing automorphism of the cone* $\mathcal{C}_\Omega$ *above* Ω. *Moreover, the subgroup of projective isometries* $\mathrm{PGL}(\Omega)$ *is a normal subgroup of* $\mathrm{Isom}(\Omega, \mathrm{d}_\Omega)$ *and has index at most 2.*

Proof. Let us consider an isometry $\Phi : \Omega \to \Omega$. Then, either every maximal pure Funk part is mapped to a maximal pure Funk part, or there is a non-singleton maximal pure Funk part that is mapped to a pure reverse-Funk part of the same dimension. In the former case, we get from Theorem 2.5.11 that Φ is a projective isometry. In the latter case we get from Proposition 2.5.15 that Φ is the projective action of a gauge-reversing automorphism of $\mathcal{C}_\Omega$. Since every linear automorphism is gauge-preserving and the composition of a gauge-reversing map with a gauge-preserving map is a gauge-reversing map, and vice versa, the subgroup is normal. Moreover, the composition of two gauge-reversing maps is a gauge-preserving map, and hence its projective action is a projective isometry, by Corollary 2.5.12. Therefore, the index of $\mathrm{PGL}(\Omega)$ into $\mathrm{Isom}(\Omega, \mathrm{d}_\Omega)$ is at most 2. □

2.6 The group of isometries

In order to prove Theorem 2.0.4, we have to show that if the cone above the domain of a Hilbert geometry admits a gauge-reversing map, then it is homogeneous and self- dual. We start by proving the first condition.

The function that maps a pair of points of a properly convex cone $\mathcal{C} \subseteq \mathbb{R}^{n+1}$, $(u, v) \in \mathcal{C} \times \mathcal{C}$ to the real number $\mathrm{d}^T_{\mathcal{C}}(u, v) := \log \max\{M_{\mathcal{C}}(u/v), M_{\mathcal{C}}(v/u)\}$ is a metric[2]. Since u and v belong to $\mathcal{C}$, both $M_{\mathcal{C}}(u/v)$ and $M_{\mathcal{C}}(v/u)$ are finite numbers. Then, given $u, v \in \mathcal{C}$, if $M_{\mathcal{C}}(u/v) \geq 1$ we have $\mathrm{d}^T_{\mathcal{C}}(u, v) \geq 0$, if instead $M_{\mathcal{C}}(u/v) < 1$, then $v \in \mathcal{C} + u$ and hence $M_{\mathcal{C}}(v/u) \geq 1$. In both cases, we have $\mathrm{d}^T_{\mathcal{C}}(u, v) \geq 0$. In

[2]This metric is known as Thompson metric. We refer to [28] for further information.

the same way, we get that $\mathrm{d}_{\mathcal{C}}^{T}(u,v)=0$ implies $u=v$. The symmetry of $\mathrm{d}_{\mathcal{C}}^{T}$ is clear. Finally, given $u,\ v$ and w in $\mathcal{C}$, we have

$$M_{\mathcal{C}}(u/v)\leq M_{\mathcal{C}}(u/z)M_{\mathcal{C}}(z/v)\quad\text{and}\quad M_{\mathcal{C}}(v/u)\leq M_{\mathcal{C}}(v/z)M_{\mathcal{C}}(z/u).$$

It follows that $M_{\mathcal{C}}(u/v)\leq\max\{M_{\mathcal{C}}(u/z),M_{\mathcal{C}}(z/u)\}\max\{M_{\mathcal{C}}(z/v),M_{\mathcal{C}}(v/z)\}$ and $M_{\mathcal{C}}(v/u)\leq\max\{M_{\mathcal{C}}(u/z),M_{\mathcal{C}}(z/u)\}\max\{M_{\mathcal{C}}(z/v),M_{\mathcal{C}}(v/z)\}$. Therefore, $\mathrm{d}_{\mathcal{C}}^{T}$ satisfies the triangular inequality.

Moreover, from Proposition 2.2.8 we get

$$\mathrm{d}_{\mathcal{C}}^{T}(u,v)=\begin{cases}\log\max\left\{1+\dfrac{\|u-v\|}{\|v-v_{\infty}\|},1+\dfrac{\|u-v\|}{\|u-u_{\infty}\|}\right\} & \text{if } u\neq\lambda v \text{ for all } \lambda>0,\\ \log\|u-v\| & \text{if } \exists\lambda>0 \text{ such that } u=\lambda v,\end{cases}$$

where u_{∞} and v_{∞} the intersection points of the line through u and v with $\partial\mathcal{C}$, defined such that u belong to the segment between u_{∞} and v and v belong to the segment between u and v_{∞}. Therefore, the metric $\mathrm{d}_{\mathcal{C}}^{T}$ is Lipschitz equivalent to the Euclidean metric on any ball of finite radius contained in $\mathcal{C}$ and the topology induced by $\mathrm{d}_{\mathcal{C}}^{T}$ is the Euclidean one. These facts give rise to the following proposition.

Proposition 2.6.1. *Let $\phi:\mathcal{C}\to\mathcal{C}$ be a gauge-reversing map. Then, ϕ is a bijection. Moreover, ϕ is differentiable almost-everywhere on $\mathcal{C}$, with respect to the Lebesgue measure.*

Proof. By definition, ϕ is an isometry of $(\mathcal{C},\mathrm{d}_{\mathcal{C}}^{T})$. Therefore, it is injective and continuous. From the invariance of domain theorem, we get that $\phi(\mathcal{C})$ is an open. Since $\mathcal{C}$ is connected, it suffice to show that $\phi(\mathcal{C})$ is closed.

Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $\phi(\mathcal{C})$ converging to $v\in\mathcal{C}$. Then the sequence $(\phi^{-1}(v_n))_{n\in\mathbb{N}}$ in $\mathcal{C}$ satisfies the Cauchy criterion. Since $(\mathcal{C},\mathrm{d}_{\mathcal{C}}^{T})$ is complete, $(\phi^{-1}(v_n))_{n\in\mathbb{N}}$ converges to some point $u\in\mathcal{C}$. From continuity, we get that $\phi(u)=v$.

Since ϕ is 1-Lipschitz with respect to $\mathrm{d}_{\mathcal{C}}^{T}$, and $\mathrm{d}_{\mathcal{C}}^{T}$ is Lipschitz equivalent to the Euclidean metric on any ball of finite radius, from Rademacher's Theorem [17, Theorem 3.2], we deduce that ϕ is differentiable almost everywhere within every ball of finite radius. Since $\mathcal{C}$ endowed with the Euclidean topology is second countable, ϕ is differentiable in the whole $\mathcal{C}$. □

Corollary 2.6.2. *Assume there exists a gauge-reversing map $\phi:\mathcal{C}\to\mathcal{C}$. Then, for almost all v in $\mathcal{C}$, there exists a gauge-reversing map $\phi_v:\mathcal{C}\to\mathcal{C}$ that fixes v, has derivative $D_v\phi_v=-\mathrm{Id}$ at v, and is an involution.*

Proof. By Proposition 2.6.1, the map ϕ is bijective and ϕ is differentiable almost everywhere in $\mathcal{C}$. So, it is well-defined ϕ^{-1}, which is also gauge-reversing.

Let v be a point of $\mathcal{C}$ where ϕ is differentiable. The map $(-D_v\phi)^{-1}$ is linear, and hence gauge-preserving. Indeed, for any linear map f and any $u, u' \in \mathcal{C}$, we have that $u \leq_{\mathcal{C}} u'$ implies $u' - u \in \overline{\mathcal{C}}$ that implies $f(u' - u) \in \overline{\mathcal{C}}$ that implies, by linearity that $f(u) \leq_{\mathcal{C}} f(u')$.

So the map $\phi_v : \mathcal{C} \to \mathcal{C}$ defined by $\phi_v := (-D_v\phi)^{-1} \circ \phi$ is gauge-reversing. By the chain rule $D_v\phi_v = -\operatorname{Id}$.

On the other hand, since ϕ_v is gauge-reversing, ϕ_v is anti-homogeneous. Indeed, for each $u \in \mathcal{C}$ and $\lambda > 0$, we have

$$\begin{aligned} M_{\mathcal{C}}(\frac{1}{\lambda}\phi_v(u)/\phi_v(\lambda u)) &= \frac{1}{\lambda}M_{\mathcal{C}}(\phi_v(u)/\phi_v(\lambda u)) \\ &= \frac{1}{\lambda}M_{\mathcal{C}}(\lambda u/u) \\ &= M_{\mathcal{C}}(u/u) = 1. \end{aligned}$$

Then, $\frac{1}{\lambda}\phi_v(u) \leq_{\mathcal{C}} \phi_v(\lambda u)$. In a similar way one can prove that $M_{\mathcal{C}}(\phi_v(\lambda u)/\frac{1}{\lambda}\phi_v(u)) = 1$ and hence that $\frac{1}{\lambda}\phi_v(u) = \phi_v(\lambda u)$.

Therefore, we have $\phi_v(v + \lambda v) = \frac{\phi_v(v)}{1+\lambda}$ for all $\lambda > 0$. This implies that $D_v\phi_v(v) = -\phi_v(v)$. Therefore, $\phi_v(v) = v$.

The map $\phi_v \circ \phi_v$ is gauge-preserving, and hence linear. Moreover, its derivative at v is Id. It follows that $\phi_v \circ \phi_v = \operatorname{Id}$. □

Lemma 2.6.3. *Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a properly convex open cone and Ω be the projection of $\mathcal{C}$. Assume that there exists a gauge-reversing map $\phi : \mathcal{C} \to \mathcal{C}$. Then,* $\operatorname{PGL}(\Omega)$ *acts transitively on pairs of points that are collinear with an extreme point of $\partial\Omega$.*

Proof. Let $x, y \in \Omega$ be two points collinear with an extreme point $a \in \partial\Omega$. Let z be the mid-point, in the Hilbert metric on Ω, between x and y on the straight line joining them. By Corollary 2.6.2, we can find a sequence $(z_n)_{n\in\mathbb{N}}$ in Ω converging to z and such that, for all $n \in \mathbb{N}$, there exists a gauge-reversing map $\phi_{z_n} : \mathcal{C} \to \mathcal{C}$ that fixes some representative $\widetilde{z}_n \in C$ of z_n, and has derivative $-\operatorname{Id}$ at $\widetilde{z}_n$.

Assume that x lies between a and y. For each $n \in \mathbb{N}$, define L_n the line in Ω passing through a and z_n, and x_n and y_n within L_n be such that z_n is the mid-point, in the Hilbert metric on Ω, between x_n and y_n on the straight line joining them. By definition, $(x_n)_{n\in\mathbb{N}}$ converges to x and $(y_n)_{n\in\mathbb{N}}$ converges to y.

Since a is an extreme point, by Proposition 1.2.9, for all $n \in \mathbb{N}$, L_n is the unique geodesic passing through x_n and y_n. Therefore, $\phi_{z_n}(L_n)$ is the unique geodesic passing through $\phi(x_n)$ and $\phi(y_n)$. Hence, $\phi_{z_n}(L_n)$ is a straight line. Since $\phi_{z_n}(z_n) = z_n$ and $D_{z_n}\phi = -\operatorname{Id}$, we have that $\phi_{z_n}(x_n) = y_n$ for all $n \in \mathbb{N}$.

From Corollary 2.6.2, there exists a sequence w_n in Ω converging to y such that, for all $n \in \mathbb{N}$, there is a gauge-reversing map $\phi_{w_n} : C \to C$ that fixes a representative $\widetilde{w}_n \in C$ of w_n. For each $n \in \mathbb{N}$, the map $f_n : C \to C$ defined by

$$f_n := \phi_{w_n} \circ \phi_{z_n}$$

is gauge-preserving. By Corollary 2.5.12, the action of f_n on Ω is a projective transformation, for all $n \in \mathbb{N}$.

Observe that the sequences $(y_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ have the same limit, and that $\phi_{w_n}(w_n) = w_n$ converges to y. Since $\{\phi_{w_n}\}$ is 1-Lipschitz, for all $n \in \mathbb{N}$, we get that $\phi_{w_n}(y_n)$ converges to y. But $\phi_{w_n}(y_n) = f_n(x_n)$ for all $n \in \mathbb{N}$, and $(x_n)_{n\in\mathbb{N}}$ converges to x. We conclude that $(f_n(x))_{n\in\mathbb{N}}$ converges to y. This implies that the sequence $(f_n)_{n\in\mathbb{N}}$ lie in some bounded subset of $\mathrm{PGL}(\Omega)$. Therefore, up to a subsequence, there is a projective isometry $f \in \mathrm{PGL}(\Omega)$ that is the limit of $(f_n)_{n\in\mathbb{N}}$. Since $(f_n(x))_{n\in\mathbb{N}}$ converges to y, we have $f(x) = y$.

If instead y lies between x and a, we can aplly a similar reasoning to show that there is a projective isometry that maps x to y. □

Proposition 2.6.4. *Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a properly convex open cone. Suppose that there exists a gauge-reversing map $\phi : \mathcal{C} \to \mathcal{C}$. Then, $\mathcal{C}$ is homogeneous.*

Proof. Let x and y be points in the domain Ω under $\mathcal{C}$. Suppose that there is an extremal point $z \in \partial\mathcal{C}$ such that x and y are aligned in Ω with the extreme point z. It follows from Lemma 2.6.3 that there is a projective isometry that maps x to y.

Given two generic points x and y in Ω, we notice that since Ω is the convex hull of its extremal generator, by Carathéodory Theorem [6, Theorem 11.1.8.6], we can pass from x to y with a finite number of projective isometries. □

Now, we want to prove that if a cone admits a gauge-reversing map onto itself, then the cone has to be self-dual. The idea is to find a symmetric, positive definite bilinear form on $\mathbb{R}^{n+1}$ for which the cone is a domain of positivity. We refer to [23] for further readings.

Definition 2.6.5. Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be an open cone and $\beta : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ be a symmetric, non-degenerate bilinear form. Then, $\mathcal{C}$ is a *domain of positivity for β* if $\beta(u, v) > 0$ for all $u, v \in \mathcal{C}$ and if every time that $u \in \mathbb{R}^{n+1}$ satisfies $\beta(u, v) > 0$ for all $v \in \overline{\mathcal{C}} \setminus \{0\}$, we have that $u \in \mathcal{C}$.

A domain of positivity of a positive definite symmetric bilinear form is self-dual with respect to the induced inner product. Indeed, the first condition tells us that $\mathcal{C}$ is contained in the image of $\mathcal{C}^*$ under the identification of $(\mathbb{R}^{n+1})^*$ with $\mathbb{R}^{n+1}$ induced by β. The second condition gives the opposite inclusion.

Now, given $\mathcal{C} \subseteq \mathbb{R}^{n+1}$, assume that there exists a gauge-reversing map $\phi : \mathcal{C} \to \mathcal{C}$. Let Ω be the projection of $\mathcal{C}$. As we mentioned at the beginning of this chapter, the behaviour of an isometry of Ω is strictly linked to its action on the parts of the horofunction boundary of Ω.

Therefore, we fix a base point $b \in \Omega$. From Corollary 2.6.2 and Proposition 2.1.18, we can choose $\tilde{b} \in [b]$ such that there exists a gauge-reversing map ϕ_b on $\mathcal{C}$ so that $\phi_b(\tilde{b}) = \tilde{b}$, $D_{\tilde{b}}\phi = -\operatorname{Id}$, and $\phi_b^2 = \operatorname{Id}$.

Henceforth, we assume that the gauge-reversing function $\phi : \mathcal{C} \to \mathbb{R}$ is an involution that fixes $\tilde{b}$ and that the differential of ϕ at $\tilde{b}$ is $-\operatorname{Id}$.

We are looking for a symmetric, positive definite bilinear form $\beta : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ such that $\mathcal{C}$ is a domain of positivity for β.

We saw in Remark 2.4.11, that the singleton Funk parts of the horofunction boundary are the restriction to the cone of linear functionals. Moreover, from 2.5.4 we know that the extension of an isometry to the horofunction boundary maps pure parts to pure parts, and preserves the dimension of each part. In Remark 2.5.8 we saw that if ϕ is a gauge-reversing map, then its extension maps each Funk horofuntion to a reverse-Funk horofunction, and each $reverse - Funk$ horofunction to a Funk horofunction.

Recall that $r_{\mathcal{C},u}(\cdot) = \mathcal{RF}_{\mathcal{C}}(\cdot, u) - \mathcal{RF}_{\mathcal{C}}(\tilde{b}, u)$ is defined on $\mathcal{C}$ and is a reverse-Funk Busemann point. It follows that given an extremal generator $u \in \mathcal{C}$, the extension of ϕ maps $r_{\mathcal{C},u}$ to a Funk horofunction, and each pure Funk part associated with $r_{\mathcal{C},u}$ to a pure part of the same dimension. Since each Funk horofunction is mapped by ϕ to a reverse-Funk horofunction, each pure Funk part associated with $r_{\mathcal{C},u}$ is mapped by ϕ to a pure reverse-Funk part of the same dimension. It follows that the image of $r_{\mathcal{C},u}$ under ϕ is a Funk singleton. Therefore, by Remark 2.4.11 we have that $\exp(r_{\mathcal{C},u} \circ \phi)$ is the restriction to $\mathcal{C}$ of a linear function.

Hence, for each extremal generator $u \in \partial\mathcal{C}$ it is well-defined the linear function $h_u : \mathbb{R}^{n+1} \to \mathbb{R}$ given by the extension of $\exp(r_{\mathcal{C},u} \circ \phi)$ to the whole $\mathbb{R}^{n+1}$.

The candidate to become the desired bilinear form is the function β defined on the set of pairs made of an extremal generators of $\mathcal{C}$ and a point of $\mathbb{R}^{n+1}$ and given by

$$\beta(u, v) = h_u(v) M_{\mathcal{C}}(u/\tilde{b}),$$

for $v \in \mathbb{R}^{n+1}$ and $u \in \partial\mathcal{C}$ an extremal generators.

The next steps are to extend β to the whole $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, to prove that β is a positive definite symmetric bilinear form, and that $\mathcal{C}$ is a domain of positivity for β.

Since $\mathcal{C}$ is open, there is a basis $(u_0, \dots, u_n)$ of $\mathbb{R}^{n+1}$ made of extremal generator of $\mathcal{C}$. Thus, we can define β to the whole $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ by taking $\beta(w, v) = \sum_{i=0}^{n} w_i \beta(u_i, v)$ for all $w = \sum_{i=0}^{n} w_i u_i$ in $\mathbb{R}^{n+1}$, with $w_i \in \mathbb{R}$ for $i = 0, \dots, n$.

Lemma 2.6.6. *This definition of β to the whole $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ does not depend of the chosen basis of extremal generator. Moreover, β is a symmetric bilinear form.*

Proof. Let $(u_0, \dots, u_n)$ and $(u'_0, \dots, u'_n)$ be two bases of extremal generators. Given $v, w \in \mathbb{R}^{n+1}$ we can write $v = \sum_i v_i u_i$ and $w = \sum_{i=0}^{n} w_i u_i = \sum_{i=0}^{n} w'_i u'_i$, with $v_i, w_i, w'_i \in \mathbb{R}$ for $i = 0, \dots, n$. Then, we have

$$\sum_i w_i \beta(v, u_i) = \sum_{i,j} w_i v_j \beta(u_j, u_i), \tag{2.48}$$

$$\sum_i w'_i \beta(v, u'_i) = \sum_{i,j} w_i v_j \beta(u_j, u'_i). \tag{2.49}$$

Now, we notice that $\beta(u_j, u_i) = \beta(u_i, u_j)$ and $\beta(u_j, u'_i) = \beta(u'_i, u_j)$. In particular, it holds that given two extremal generators $u, u' \in \partial\mathcal{C}$, then $\beta(u, u') = \beta(u', u)$. Indeed, if we pick a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$ that converges to u, for each $n \in \mathbb{N}$ we get $h_{u_n} = j_{\mathcal{C},\phi(u_n)}$, since ϕ is a gauge-reversing involution that fixes b. Recall that $j_{\mathcal{C},\phi(u_n)}(\cdot) = \dfrac{M_{\mathcal{C}}(\cdot/\phi(u_n))}{M_{\mathcal{C}}(\tilde{b}/\phi(u_n))}$. Thus, the sequence $(j_{\mathcal{C},\phi(u_n)})_{n\in\mathbb{N}}$ pointwise converges to h_u. Moreover, since $\{j_{\mathcal{C},z} \mid z \in \mathcal{C}\}$ is an equi-Lipschitzian family, see Lemma 2.4.3, we have that $(j_{\mathcal{C},\phi(u_n)}(u'_n))_{n\in\mathbb{N}}$ converges to $h_u(u')$ for any sequence $(u'_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$ converging to u. From the continuity of the gauge, we get

$$\begin{aligned}
\beta(u', u) &= h_u(u') M_{\mathcal{C}}(u/\tilde{b}) \\
&= \lim_{n\to\infty} j_{\mathcal{C},\phi(u_n)}(u'_n) M_{\mathcal{C}}(u_n/\tilde{b}) \\
&= \lim_{n\to\infty} j_{\mathcal{C},\phi(u_n)}(u'_n) M_{\mathcal{C}}(\tilde{b}/\phi(u_n)) \\
&= \lim_{n\to\infty} M_{\mathcal{C}}(u'_n/\phi(u_n)).
\end{aligned} \tag{2.50}$$

Since ϕ is gauge-reversing, then $\lim_{n\to\infty} M_{\mathcal{C}}(u'_n/\phi(u_n)) = \lim_{n\to\infty} M_{\mathcal{C}}(u_n/\phi(u'_n))$ and a similar reasoning gives $\lim_{n\to\infty} M_{\mathcal{C}}(u_n/\phi(u'_n)) = \beta(u, u')$.

Therefore, from (2.48) and (2.49), and the linearity of β with respect to its first entry, we get

$$\sum_i w_i \beta(v, u_i) = \sum_{i,j} w_i v_j \beta(u_i, u_j) = \sum_j v_j \beta(w, u_j), \tag{2.51}$$

$$\sum_i w'_i \beta(v, u'_i) = \sum_{i,j} w_i v_j \beta(u'_i, u_j) = \sum_j v_j \beta(w, u_j). \tag{2.52}$$

Hence, β is well defined and it is clearly bilinear. The reasoning above that shows that $\beta(u, u') = \beta(u', u)$ for each pair of extremal generator $u, u' \in \partial\mathcal{C}$, gives the symmetry of β.

Now, suppose that β is degenerate. Then there exists a point $v \in \mathbb{R}^{n+1}$ such that $\beta(u, v) = 0$ for every extremal generator $u \in \partial\mathcal{C}$. Since $M_{\mathcal{C}}(u/\tilde{b}) > 0$ for every $u \in \partial\mathcal{C}$, it follows that $h_u(v) = 0$. Since $\mathcal{C}$ is proper, also $\mathcal{C}^*$ is a proper cone, then there is a basis $(h_0, \dots, h_n)$ of $(\mathbb{R}^{n+1})^*$ made of extremal generator of $\mathcal{C}^*$. Remark 2.4.11 implies that each extremal generator of $\mathcal{C}^*$ gives a Funk horofunction. The reasoning above applied to ϕ^{-1}, gives that for each $i = 0, \dots, n$ there exists an extremal generator u_i of $\mathcal{C}$ such that $h_{u_i} = h_i$, because ϕ is gauge-reversing. It follows that $v = 0$. □

Lemma 2.6.7. *$\mathcal{C}$ is a domain of positivity for the non-degenerate symmetric bilinear form β.*

Proof. We first shows that $\mathcal{C}$ is a domain of positivity for β. Let $v \in \mathcal{C}$ and $u \in \partial\mathcal{C}$ be an extremal generator of $\mathcal{C}$. Then, we have

$$\beta(u, v) = h_u(v) M_{\mathcal{C}}(u/\tilde{b}) = \frac{M_{\mathcal{C}}(u/\phi(v))}{M_{\mathcal{C}}(u/\tilde{b})} M_{\mathcal{C}}(u/\tilde{b}) = M_{\mathcal{C}}(u/\phi(v)) > 0. \tag{2.53}$$

Moreover, the coordinates of every point of $\mathcal{C}$ with respect to a basis of $\mathbb{R}^{n+1}$ made of extremal generators are positive, since $\mathcal{C}$ is proper. Thus, $\beta(u, v) > 0$ for all $u, v \in \mathcal{C}$.

Now, if $v \in \mathbb{R}^{n+1}$ is such that $\beta(u, v) > 0$ for all $u \in \overline{\mathcal{C}} \setminus \{0\}$, then for every extremal generator $u \in \partial\mathcal{C}$ we have $h_u(v) > 0$. As we saw in the proof of Lemma 2.6.6 there exists a basis of $(\mathbb{R}^{n+1})^*$ made of linear functionals of the type h_u, with $u \in \partial\mathcal{C}$ an extremal generator. Then, $\varphi(v) > 0$ for all $\varphi \in \mathcal{C}^*$ and hence $v \in \mathcal{C}$. □

It remains to prove that β is positive definite. Since $\mathcal{C}$ is a non-empty domain of positivity for β, then β is positive definite or indefinite. The next lemma will be used to show that β cannot be indefinite.

Lemma 2.6.8. *Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a domain of positivity for a symmetric indefinite non-degenerate bilinear form. Then, there exists a point $v \in \partial\mathcal{C} \setminus \{0\}$ such that $\beta(v, v) = 0$.*

Proof. Since β is a simmetric non-degenerate bilinear form, by the Spectral Theorem, there exists a basis $(u_i)_{i=0,\dots,n}$ of $\mathbb{R}^{n+1}$ made of eigenvectors that diagonalize the matrix associated with β and orthogonal with respect to β. Then, if we denote $\sigma : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ the function that maps for each $i \in \{0, \dots, n\}$ the element u_i to $-u_i$ if $\beta(u_i, u_i) = -1$ and maps u_i to u_i otherwise. We denote $\alpha : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ the bilinear form given by $\alpha = \beta \circ (\sigma \times \mathrm{Id})$. Thus, the basis $(u_i)_{i=0,\dots,n}$ is orthonormal with respect to α and $\alpha(v, v) > 0$ for each $v \in \mathbb{R}^{n+1} \setminus \{0\}$.

Now, define the function $f : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ given by

$$f(v) = \frac{\beta(v, v)}{\alpha(v, v)} \quad \text{for all } v \in \mathbb{R}^{n+1} \setminus \{0\}.$$

Since f is homogeneous, we have that

$$\inf_{v \in \overline{\mathcal{C}} \setminus \{0\}} f(v) = \inf_{v \in \overline{D}} f(v),$$

where D is a bounded cross-section of the properly convex cone $\mathcal{C}$. Then, there exists a point $z \in \overline{\mathcal{C}} \setminus \{0\}$ that minimizes f on $\overline{\mathcal{C}} \setminus \{0\}$. Since $\mathcal{C}$ is a domain of positivity of β we have that $\beta(z, z) \geq 0$. We wish to prove that $\beta(z, z) = 0$.

Each $v \in \mathbb{R}^{n+1} \setminus \{0\}$ can be written as $\sum_i y_i v_i$, and

$$\frac{\partial}{\partial u_i}\beta(v, v) = \frac{\partial}{\partial u_i} \sum_i {v_i}^2 \beta(u_i, u_i) = 2v_i \ \operatorname{sgn}(\beta(u_i, u_i)).$$

Therefore, $\nabla\beta(v, v) = 2\sigma(v)$ and similarly $\nabla\alpha(v, v) = 2v$, with respect to the inner product given by α. It follows that for each $v \in \mathbb{R}^{n+1} \setminus \{0\}$ it holds

$$\begin{aligned} \nabla f(v) &= \frac{\nabla\beta(v, v)}{\alpha(v, v)} + \frac{\beta(v, v)\nabla\alpha(v, v)}{{\alpha(v, v)}^2} \\ &= \frac{2\alpha(v, v)\sigma(v) - 2\beta(v, v)v}{\alpha(v, v)^2}. \end{aligned} \tag{2.54}$$

Moreover, $D_z f(v) = \alpha(\nabla f(z), v)$ and $D_z f(v) \geq 0$ for all $v \in \mathcal{C}$ since z is a minumum over the convex $\mathcal{C}$. By definition $\beta(\sigma(\nabla f(z)), v) = \alpha(\nabla f(z), v)$, then $\beta(\sigma(\nabla f(z)), v) \geq 0$ for all $v \in \mathcal{C}$. Since $\mathcal{C}$ is a domain of positivity, we have that $\sigma(\nabla f(z)) \in \overline{\mathcal{C}}$, by [23, Theorem 1]. Therefore, we have $\beta(\nabla f(z), \nabla f(z)) =$

$\beta(\sigma(\nabla f(z)), \sigma(\nabla f(z))) \geq 0$, because $\mathcal{C}$ is a domain of positivity for β.

Since we have assumed β indefinite, there exists a point v in the open cone $\mathcal{C}$ such that $\alpha(v,v) > \beta(v,v)$. Then, $f(v) < 1$, and for the minimality of $f(z)$ we have that $\alpha(z,z) > \beta(z,z)$.

If we denote with z_+ and z_- the components of z in Span $\{u_i \mid \beta(u_i,u_i) = 1\}$ and Span $\{u_i \mid \beta(u_i,u_i) = -1\}$ respectively, we get that

$$\begin{aligned}\beta(z,z) &= \alpha(z_+,z_+) - \alpha(z_-,z_-), \qquad \text{and}\\ \alpha(z,z) &= \alpha(z_+,z_+) + \alpha(z_-,z_-).\end{aligned}$$

Since $\alpha(z,z) > \beta(z,z)$, we get $\alpha(z_-,z_-) > 0$, and since $\beta(z,z) \geq 0$, we get $\alpha(z_+,z_+) \geq \alpha(z_-,z_-)$.

Now, from (2.54), since $\sigma(z) = z_+ - z_-$ and $z = z_+ + z_-$, we get

$$\begin{aligned}\beta(\nabla f(z), \nabla f(z)) &= \frac{4}{\alpha(v,v)^4}\beta\Big(\alpha(z,z)\sigma(z) - \beta(z,z)z, \alpha(z,z)\sigma(z) - \beta(z,z)z\Big)\\ &= \frac{16}{\alpha(v,v)^4}\beta\Big(-\alpha(z_+,z_+)z_- + \alpha(z_-,z_-)z_+, -\alpha(z_+,z_+)z_- + \alpha(z_-,z_-)z_+\Big)\\ &= \frac{16}{\alpha(v,v)^4}\Big(\alpha(z_+,z_+)^2\beta(z_-,z_-) + \alpha(z_-,z_-)^2\beta(z_+,z_+)\Big)\\ &= \frac{16}{\alpha(v,v)^4}\alpha(z_+,z_+)\alpha(z_-,z_-)\Big(\alpha(z_-,z_-) - \alpha(z_+,z_+)\Big).\end{aligned}$$

Since we have proved that $\beta(\nabla f(z), \nabla f(z)) \geq 0$ and $alpha(z_-,z_-) - \alpha(z_+,z_+) \leq 0$, we get $-\beta(z,z) = alpha(z_-,z_-) - \alpha(z_+,z_+) = 0$. Hence, we have $\beta(z,z) = 0$ as desired. ☐

In order to prove that β is positive definite, we need also the following lemma.

Lemma 2.6.9. *The projective action of ϕ has b as unique fixed point.*

Proof. Since the derivative of ϕ at $\tilde{b}$ is $-\mathrm{Id}$, then $\tilde{b}$ is an isolated fixed point. We show that also b is an isolated fixed point for the projective action of ϕ. If there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in Ω coverging to b of points fixed by the projective action of ϕ, then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$ of points fixed by ϕ such that $[u_n] = x_n$ for all $n \in \mathbb{N}$. Indeed, for $n \in \mathbb{N}$, for all $u \in \pi^{-1}(x_n)$, there is $\lambda_u > 0$ such that $\phi(u) = \lambda_u u$. The map ϕ is gauge-reversing, then by Proposition **??** it is anti-homogeneous, then $\phi(\sqrt{|\lambda_u|}u) = \frac{1}{\sqrt{|\lambda_u|}}\lambda_u u = \sqrt{|\lambda_u|}u$. Since ϕ is gauge-reversing if $u_n \leq_{\mathcal{C}} \tilde{b}$ then $\tilde{b} \leq_{\mathcal{C}} u_n$, and if $\tilde{b} \leq_{\mathcal{C}} u_n$ then $u_n \leq_{\mathcal{C}} \tilde{b}$. It follows that, for all $n \in \mathbb{N}$ either $u_n = \tilde{b}$ or $u_n \not\leq_{\mathcal{C}} \tilde{b}$ and $u_n \not\geq_{\mathcal{C}} \tilde{b}$. If the latter condition holds for some subsequence $(u_{k_n})_{n\in\mathbb{N}}$, then for each $n \in \mathbb{N}$ there exists a bounded cross-section of $\mathcal{C}$ that contains both the u_{k_n} and $\tilde{b}$. So, $(u_{k_n})_{n\in\mathbb{N}}$ converges to $\tilde{b}$, since $(x_n)_{n\in\mathbb{N}}$ converges to b. But, this contradicts the fact that $\tilde{b}$ is an isolated fixed point. Therefore, b is an isolated fixed point for the projective action of ϕ. We have to show that b is the unique fixed point.

Assume that there is another fixed point $b' \in \Omega$. For $\alpha \in]0,1[$ consider the set

$$Z_\alpha = \{x \in \Omega \mid \mathrm{d}_\Omega(b,x) = \alpha\, \mathrm{d}_\Omega(b,b') \text{ and } \mathrm{d}_\Omega(b',x) = (1-\alpha)\, \mathrm{d}_\Omega(b,b')\},$$

which is invariant under the projective action of ϕ, compact, convex and non-empty, since $\alpha b' + (1-\alpha)b \in Z_\alpha$. By Brouwer theorem, for each $\alpha \in]0,1[$ there is a point in Z_α that is fixed by the projective action of ϕ. Since α can be chosen as small as we want, we can find a sequence in Ω, made of points fixed by the projective action of ϕ that converges to b. But this contradicts the fact that b is an isolated fixed point. □

Proposition 2.6.10. *The symmetric non-degenerate bilinear form β is positive definite.*

Proof. We want to show that $\beta(v,) > 0$ for all $v \in \overline{\mathcal{C}} \setminus \{0\}$. Then, the conclusion will follows from Lemma 2.6.8, since β cannot be negative definite, because $\mathcal{C} \neq \emptyset$. So, we write $v = \sum_i v_i u_i$, where $(u_i)_{i=0,\dots,n}$ is a basis made of extremal generators of $\mathcal{C}$ and $v_i > 0$. Since $\mathcal{C}$ is a domain of positivity for β we have $\beta(u_i,u_j) \geq 0$ for all $i,j = 0,\dots,n$. We show that $\beta(u_i,u_i) > 0$ for all $i = 0,\dots,n$. Fix for all $i = 0,\dots,n$. From (2.50) we get that $\beta(u_i,u_i) = \lim_{n\to\infty} M_{\mathcal{C}}(u_n/\phi(u_n))$ for any $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$ converging to u.

From the triangular inequality and the assumptions that $\phi(b) = b$ and $\phi^2 = \mathrm{Id}$, we get

$$\begin{aligned}\beta(u_i,u_i) &= \lim_{n\to\infty} M_{\mathcal{C}}(u_n/\phi(u_n)) \\ &\leq \lim_{n\to\infty} M_{\mathcal{C}}(u_n/b) M_{\mathcal{C}}(b/\phi(u_n)) \\ &= \lim_{n\to\infty} M_{\mathcal{C}}(u_n/b)^2 \\ &= M_{\mathcal{C}}(u/b)^2. \end{aligned} \tag{2.55}$$

We wish to get the inverse inequality in (2.55) to obtain $\beta(u_i,u_i) = M_{\mathcal{C}}(u/b)^2$.

Therefore, we prove that $M_{\mathcal{C}}(u_n/\phi(u_n)) \geq M_{\mathcal{C}}(u_n/b) M_{\mathcal{C}}(b/\phi(u_n))$.

Notice that, once fixed $n \in \mathbb{N}$, the set

$$S = \{v \in \mathcal{C} \mid M_{\mathcal{C}}(u_n/\phi(u_n)) \geq M_{\mathcal{C}}(u_n/b)M_{\mathcal{C}}(b/\phi(u_n))\}$$

is non-empty, since it contains u, closed, since the gauge is a continuous function, invariant under ϕ and closed by the action of positive homotheties. Thus, the projective action of ϕ on S has a fixed point in $\pi(S)$ by Brouwer Theorem. Lemma 2.6.9 implies that the fixed point must be b. □

Proposition 2.6.11. *Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be a properly convex open cone. If there exists a gauge-reversing map $\phi : \mathcal{C} \to \mathcal{C}$, then $\mathcal{C}$ is self-dual*

Proof. Proposition 2.6.10 implies that β is a positive definite symmetric non-degenerate bilinear form, and Lemma 2.6.7 implies that $\mathcal{C}$ is a domain of positivity of β. As we noticed, the definition of domain of positivity implies that $\mathcal{C}$ is self-dual with respect to β. □

Now, we are ready to prove Theorem 2.0.4. Recall that the statement says that if $(\Omega, \mathrm{d}_\Omega)$ is a Hilbert geometry, then the group of projective isometries $\mathrm{PGL}(\Omega)$ coincides with the group of isometries $\mathrm{Isom}(\Omega, \mathrm{d}_\Omega)$ if the cone $\mathcal{C}_\Omega$ above Ω is non-symmetric or Lorentzian, and has index 2 in $\mathrm{Isom}(\Omega, \mathrm{d}_\Omega)$ otherwise.

Theorem 2.5.9 implies that if $\mathcal{C}_\Omega$ is symmetric, there exists a gauge-reversing automorphism $\phi : \mathcal{C}_\Omega \to \mathcal{C}_\Omega$. From Remark 2.5.8, we know that if there is a non-singleton maximal pure Funk part in the Hilbert horofunction boundary of the cone $\mathcal{C}_\Omega$, then ϕ maps this part to a pure reverse-Funk part. Therefore, its projective action is not a projective transformation, by Remark 2.5.8. We want to show that there always exists a non-singleton maximal pure Funk part in the Hilbert horofunction boundary of a non-Lotentzian symmetric cone. This is equivalent to showing that the domain under a non-Lotentzian symmetric cone is never both strictly convex and with a boundary of regularity C^1.

First of all we state some important results about symmetric cones. In particular, symmetric convex cones where classified, up to change of coordinates, by M. Koecher in [23, Chapter V] using the classification of Jordan algebras. From this classification arises that, in any dimension, the only symmetric cone whose projection is strictly convex is the Lorentzian cone.

On the other hand, from the results of A. Borel in [8], we get that if the cone $\mathcal{C}_\Omega$ above a properly convex domain Ω is symmetric, then the domain Ω is *divisible*, i.e. there exists a discrete subgroup of $\mathrm{PGL}(\Omega)$ that acts cocompactly on Ω.

Finally, a theorem of Y. Benoist [4, Theorem 1.1], says that if a properly convex domain is divisible and its boundary has regularity C^1, then the domain is strictly convex. Benoist actually proved that also the converse is true and furthermore, every divisible strictly convex domain is Gromov-hyperbolic.

From all this results we get the following fact.

Fact 1. In any dimension, the only symmetric properly convex cone whose projection is strictly convex and has boundary of regularity C^1 is the Lorentzian cone.

Proof of Theorem 2.0.4. Theorem 2.5.16 implies that every element of $\mathrm{Isom}(\Omega, \mathrm{d}_\Omega)$ is either a projective isometry or arises from the projective action of a gauge-reversing automorphism of the cone $\mathcal{C}_\Omega$ above Ω. Moreover, the subgroup of projective isometries $\mathrm{PGL}(\Omega)$ has index at most 2 into the group of isometries $\mathrm{Isom}(\Omega, \mathrm{d}_\Omega)$.

Proposition 2.6.4 and Proposition 2.6.11 implies that if a cone admits a gauge-reversing automorphism, then it is symmetric. It follows that if $\mathcal{C}_\Omega$ is not symmetric, then $\mathrm{PGL}(\Omega) = \mathrm{Isom}(\Omega, \mathrm{d}_\Omega)$.

If the cone $\mathcal{C}_\Omega$ is Lorentzian, from Corollary 2.5.13 we get that every isometry has to be projective. Thus, also in this case the group of projective isometries coincide with the group of isometries.

Now, suppose that $\mathcal{C}_\Omega$ is symmetric and non-Lorentzian. Then, the Vinberg's isomorphism between $\mathcal{C}_\Omega$ and $\mathcal{C}_\Omega^*$ induces a gauge-reversing automorphism $\phi : \mathcal{C}_\Omega \to \mathcal{C}_\Omega$, by Theorem 2.5.9. From Fact 1, we know that in the Hilbert horofunction boundary of $\mathcal{C}_\Omega$ there is a non-singleton pure Funk part. The gauge-reversing map ϕ maps this part to a pure reverse-Funk part, by Remark 2.5.8. Therefore, the projective action of ϕ is not a projective isometry. It follows from Theorem 2.5.16 that the group of projective isometries has index 2 in the group of isometries. $\square$

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