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Limits of Classical Electrodynamics: Retarded Potentials, Accelerated Charges, Fields and Radiation Reaction

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A mamma e papà, senza i quali non sarei mai arrivato fin qui, e a tutti coloro che sono rimasti al mio fianco durante questo lungo cammino.

«*From a long view of the history of mankind — seen from, say, ten thousand years from now — there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics.*»

— Richard P. Feynman,
The Feynman Lectures on Physics

Sommario

Lo scopo di questa tesi è quello di analizzare e calcolare il campo elettromagnetico prodotto da una particella carica accelerata, studiandone le caratteristiche, sia nel limite relativistico che in quello a basse velocità. Successivamente viene trattata la potenza irradiata da tale particella, ponendo particolare attenzione all'effetto di frenamento che la radiazione stessa imprime sulla carica. Infine, si pone come obbiettivo quello di mostrare come tale analisi conduca a conclusioni paradossali del tutto inconsistenti, indicando la necessità di una nuova teoria quantistica dell'atomo.

Abstract

The aim of this thesis is to analyze and to compute the electromagnetic field arising from an accelerating charged particle, both at relativistic and at low speed limit. In particular it will focus on the radiated power and, consequently, on the radiation reaction experienced by the charge itself. Finally, it shows how this fact could imply paradoxical non-physical consequences, indicating the necessity of a fundamentally new quantum atomic theory.

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Introduction

Electrodynamics was effectively formulated in its modern form in 1865, when J.C. Maxwell synthesized the entire theory of electromagnetism in a set of four equations describing the relation between charge distributions and the fields they produce. His description established such a powerful interconnection between electric and magnetic fields that A. Einstein in 1905, with his Special Relativity, proved they are actually two different manifestations of the same and unique physical reality.

To complete this picture, the Lorentz force finally explained how the fields influence the charge distribution dynamics. Classical electrodynamics usually allows computing fields arising from a given charge distribution or, on the contrary, permits to find the distribution dynamics in presence of a known electromagnetic field.

Despite this fact, charges and fields can interact with each other in a self-consistent way that could be relevant in some physical cases, like, for instance, the classical atomic model based on the interaction between electrons and a charged nucleus.

It is a well-known fact that an accelerated point-like charge emits radiation, but this radiation carries off energy and momentum from the moving charge itself producing a sort of self-damping effect. Classical electrodynamics allows to study this process; however, it leads to pathological results that exhibit causality violation and divergent runaway solutions, proving the inconsistency of the theory itself. Only modern quantum theory can provide an effective solution for this problem.

Chapter 1

Electric and Magnetic Fields

1.1 Point-like Sources Approximation

Point-like charges cannot physically exist, however it is possible to find a limit distribution that satisfies our desired features.
We expect these expressions to be the most suitable to resolve our request

$$\rho(\boldsymbol{x},t) = \frac{q}{4\pi r_0^3} f\left(\frac{|\boldsymbol{x}-\boldsymbol{r}(t)|}{r_0}\right), \tag{1.1.1a}$$

$$\boldsymbol{j}(\boldsymbol{x},t) = \frac{q}{4\pi r_0^3} f\left(\frac{|\boldsymbol{x}-\boldsymbol{r}(t)|}{r_0}\right)\dot{\boldsymbol{r}}(t), \tag{1.1.1b}$$

for some constant r_0, where $\rho(\boldsymbol{x},t)$ is the charge density distribution, $\boldsymbol{j}(\boldsymbol{x},t)$ is the related current density distribution, while $\boldsymbol{r}(t)$ is the charge's position and $\boldsymbol{x}$ the observation point. The real function f has to satisfy some other requirements

$$\int_0^\infty ds\, s^2 f(s) = 1, \tag{1.1.2a}$$

$$f(s) \to 0, \quad s \gg 1. \tag{1.1.2b}$$

Condition (1.1.2a) ensures that the integral throughout space of $\rho(\boldsymbol{x},t)$ is limited and equal to charge q, while (1.1.2b) limits the distribution around an arbitrarily small ball of radius r_0 around $\boldsymbol{r}(t)$, describing a rigid sphere moving with a known trajectory.
Lastly, the distributions must solve the continuity equation

$$\frac{\partial\rho(\boldsymbol{x},t)}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j}(\boldsymbol{x},t) = 0, \tag{1.1.3}$$

that is due to the charge conservation principle.

Proof. We are going to verify that our assumptions on (1.1.1a) and (1.1.1b) are well founded by checking they are compatible with the continuity equation (1.1.3).
Calculating each contribute of (1.1.3) with these distributions, we obtain

$$\frac{\partial\rho(\boldsymbol{x},t)}{\partial t} = \frac{q}{4\pi r_0^3}\frac{df}{dt}\left(\frac{|\boldsymbol{x}-\boldsymbol{r}(t)|}{r_0}\right) \tag{1.1.4}$$

$$= -\frac{q}{4\pi r_0^4} f'\left(\frac{|\boldsymbol{x}-\boldsymbol{r}(t)|}{r_0}\right)\frac{\boldsymbol{x}-\boldsymbol{r}(t)}{|\boldsymbol{x}-\boldsymbol{r}(t)|}\cdot\dot{\boldsymbol{r}}(t),$$

where f' stands for the derivative of the function f with respect to its argument, and

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{j}(\boldsymbol{x},t) &= \frac{q}{4\pi r_0^3}\boldsymbol{\nabla}\cdot\left[f\left(\frac{|\boldsymbol{x}-\boldsymbol{r}(t)|}{r_0}\right)\dot{\boldsymbol{r}}(t)\right] \\ &= \frac{q}{4\pi r_0^3} f'\left(\frac{|\boldsymbol{x}-\boldsymbol{r}(t)|}{r_0}\right)\dot{\boldsymbol{r}}(t)\cdot\boldsymbol{\nabla}\frac{|\boldsymbol{x}-\boldsymbol{r}(t)|}{r_0} \\ &= \frac{q}{4\pi r_0^4} f'\left(\frac{|\boldsymbol{x}-\boldsymbol{r}(t)|}{r_0}\right)\frac{\boldsymbol{x}-\boldsymbol{r}(t)}{|\boldsymbol{x}-\boldsymbol{r}(t)|}\cdot\dot{\boldsymbol{r}}(t),\end{aligned} \tag{1.1.5}$$

confirming the compatibility with (1.1.3).

The value of r_0 can be chosen arbitrarily small to take care of the sizing that we are searching for. We are interested to treat the source as a point-like charge, ensuring that all charge is spatially located at $\boldsymbol{r}(t)$ at all times t. This approximation is stated through the following condition

$$|\boldsymbol{x}-\boldsymbol{r}(t)| \gg r_0, \tag{1.1.6}$$

which formally corresponds to the limit $r_0 \to 0$.

Applying this limit to the distributions (1.1.1), it can be shown that

$$\lim_{r_0\to 0}\frac{q}{4\pi r_0^3} f\left(\frac{|\boldsymbol{x}-\boldsymbol{r}(t)|}{r_0}\right) = q\delta(\boldsymbol{x}-\boldsymbol{r}(t)), \tag{1.1.7}$$

where $\delta(\boldsymbol{s})$ denotes the three dimensional Dirac delta function.

This fact can intuitively be proven by noting that the delta results from the limit of a Gaussian-shaped distribution centered at $\boldsymbol{r}(t)$ as $r_0 \to 0$; the condition (1.1.6) makes the function more and more peaked around $\boldsymbol{r}(t)$ until it becomes a delta-shaped distribution.

The two distributions can now be easily recast as the following expressions

$$\rho(\boldsymbol{x},t) = q\delta(\boldsymbol{x}-\boldsymbol{r}(t)), \tag{1.1.8a}$$

$$\boldsymbol{j}(\boldsymbol{x},t) = q\delta(\boldsymbol{x}-\boldsymbol{r}(t))\dot{\boldsymbol{r}}(t), \tag{1.1.8b}$$

that evidently respect the continuity equation (1.1.3).

Proof. Proving that these relations are compatible with the continuity equation (1.1.3) is quite simple. Taking the partial derivative with respect to time of (1.1.8a), one gets

$$\begin{aligned}\frac{\partial\rho(\boldsymbol{x},t)}{\partial t} &= q\frac{\partial\delta(\boldsymbol{x}-\boldsymbol{r}(t))}{\partial t} \\ &= -q\dot{\boldsymbol{r}}(t)\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{r}(t)).\end{aligned} \tag{1.1.9}$$

Since $\boldsymbol{\nabla}\cdot\dot{\boldsymbol{r}}(t)=0$,

$$\frac{\partial\rho(\boldsymbol{x},t)}{\partial t} = -\boldsymbol{\nabla}\cdot q\dot{\boldsymbol{r}}(t)\delta(\boldsymbol{x}-\boldsymbol{r}(t)), \tag{1.1.10}$$

consistently with the continuity equation (1.1.3), only if

$$\boldsymbol{j}(\boldsymbol{x},t) = q\delta(\boldsymbol{x}-\boldsymbol{r}(t))\dot{\boldsymbol{r}}(t). \tag{1.1.11}$$

1.2 Liénard-Wiechert Potentials

Finding an exact solution to the Maxwell's equations (1.2.1) for a point-like source is surely a non-trivial problem

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}(\boldsymbol{x}, t) = 4\pi\rho(\boldsymbol{x}, t), \tag{1.2.1a}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B}(\boldsymbol{x}, t) = 0, \tag{1.2.1b}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{x}, t) = -\frac{1}{c}\frac{\partial \boldsymbol{B}(\boldsymbol{x}, t)}{\partial t}, \tag{1.2.1c}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B}(\boldsymbol{x}, t) = \frac{4\pi}{c}\boldsymbol{j}(\boldsymbol{x}, t) + \frac{1}{c}\frac{\partial \boldsymbol{E}(\boldsymbol{x}, t)}{\partial t}. \tag{1.2.1d}$$

The electric and magnetic fields, respectively $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{B}(\boldsymbol{x}, t)$, can be easily derived from the electromagnetic potentials $\phi(\boldsymbol{x}, t)$ and $\boldsymbol{A}(\boldsymbol{x}, t)$ through

$$\boldsymbol{E}(\boldsymbol{x}, t) = -\boldsymbol{\nabla}\phi(\boldsymbol{x}, t) - \frac{1}{c}\frac{\partial \boldsymbol{A}(\boldsymbol{x}, t)}{\partial t}, \tag{1.2.2a}$$

$$\boldsymbol{B}(\boldsymbol{x}, t) = \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{x}, t). \tag{1.2.2b}$$

Proof. Since (1.2.1b) holds, $\boldsymbol{B}(\boldsymbol{x}, t)$ can be defined as the curl of a vector function $\boldsymbol{A}(\boldsymbol{x}, t)$ as we can observe in (1.2.2b), in such a way that

$$\boldsymbol{B}(\boldsymbol{x}, t) = \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{x}, t). \tag{1.2.3}$$

Therefore, (1.2.1c) can be cast as the following relation

$$\boldsymbol{\nabla} \times \left(\boldsymbol{E}(\boldsymbol{x}, t) + \frac{1}{c}\frac{\partial \boldsymbol{A}(\boldsymbol{x}, t)}{\partial t}\right) = 0. \tag{1.2.4}$$

The quantity within the brackets has a vanishing curl, then it can be considered as the gradient of a scalar function called electric potential $\phi(\boldsymbol{x}, t)$, proving in this way the relation (1.2.2a).

The problem now lies in which equation should the potentials resolve in order to respect the Maxwell's equations (1.2.1).

To answer this question it is necessary to observe that electric and vector potential cannot be chosen in a unique manner. Indeed, if we introduce a basically new pair of potentials

$\phi'(\boldsymbol{x},t)$ and $\boldsymbol{A}'(\boldsymbol{x},t)$ such that

$$\phi(\boldsymbol{x},t) \to \phi'(\boldsymbol{x},t) = \phi(\boldsymbol{x},t) - \frac{1}{c}\frac{\partial\Lambda(\boldsymbol{x},t)}{\partial t}, \tag{1.2.5a}$$

$$\boldsymbol{A}(\boldsymbol{x},t) \to \boldsymbol{A}'(\boldsymbol{x},t) = \boldsymbol{A}(\boldsymbol{x},t) + \boldsymbol{\nabla}\Lambda(\boldsymbol{x},t), \tag{1.2.5b}$$

where $\Lambda(\boldsymbol{x},t)$ is a generic real function named gauge function, we observe that the fields derived from $\phi'(\boldsymbol{x},t)$ and $\boldsymbol{A}'(\boldsymbol{x},t)$ are the same derived from $\phi(\boldsymbol{x},t)$ and $\boldsymbol{A}(\boldsymbol{x},t)$, according to the relations (1.2.2).

The transformations (1.2.5) are the so-called gauge transformations and they confer a sort of controlled ambiguity to the definition of the electromagnetic potentials. This freedom allows us to choose a new set of potentials $\phi(\boldsymbol{x},t)$ and $\boldsymbol{A}(\boldsymbol{x},t)$ in such a way that the condition below is satisfied

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}(\boldsymbol{x},t) + \frac{1}{c}\frac{\partial\phi(\boldsymbol{x},t)}{\partial t} = 0, \tag{1.2.6}$$

that corresponds to the well-known Lorentz gauge choice.

Under this assumption, the gauge function $\Lambda(\boldsymbol{x},t)$ must respect the D'Alembert homogeneous wave equation (1.2.7), thus being harmonic

$$\nabla^2\Lambda(\boldsymbol{x},t) - \frac{1}{c^2}\frac{\partial^2\Lambda(\boldsymbol{x},t)}{\partial t^2} = 0. \tag{1.2.7}$$

This choice permits to modify the Maxwell's equations (1.2.1) that, therefore, can be cast in a simpler and most symmetrical way

$$\nabla^2\phi(\boldsymbol{x},t) - \frac{1}{c^2}\frac{\partial^2\phi(\boldsymbol{x},t)}{\partial t^2} = -4\pi\rho(\boldsymbol{x},t), \tag{1.2.8a}$$

$$\boldsymbol{\nabla}^2\boldsymbol{A}(\boldsymbol{x},t) - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}(\boldsymbol{x},t)}{\partial t^2} = -\frac{4\pi}{c}\boldsymbol{j}(\boldsymbol{x},t). \tag{1.2.8b}$$

Proof. Combining the relations (1.2.2a) and (1.2.2b) with the Maxwell's equations (1.2.1), one obtains the following statements

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A}(\boldsymbol{x},t)\right) = -4\pi\rho(\boldsymbol{x},t), \tag{1.2.9a}$$

$$\boldsymbol{\nabla}^2\boldsymbol{A}(\boldsymbol{x},t) - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}(\boldsymbol{x},t)}{\partial t^2} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A}(\boldsymbol{x},t) + \frac{1}{c}\frac{\partial\phi(\boldsymbol{x},t)}{\partial t}\right) = -\frac{4\pi}{c}\boldsymbol{j}(\boldsymbol{x},t). \tag{1.2.9b}$$

The four initial Maxwell's equations (1.2.1) are now reduced to only two relations where the potentials represent the problem's unknowns.
Considering the Lorentz gauge choice (1.2.6), the equations (1.2.8) follow immediately.

Equations (1.2.8) can be resolved by a pair of electromagnetic potentials of this form

$$\phi(\boldsymbol{x}, t) = \int d^3x' \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} \rho\left(\boldsymbol{x}', t - \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right), \tag{1.2.10a}$$

$$\boldsymbol{A}(\boldsymbol{x}, t) = \int d^3x' \frac{1}{c|\boldsymbol{x}-\boldsymbol{x}'|} \boldsymbol{j}\left(\boldsymbol{x}', t - \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right). \tag{1.2.10b}$$

These are called retarded potentials and they represent a possible solution to the Maxwell's equations in presence of charges and currents; they can be obtained through the introduction of Green functions in order to solve the inhomogeneous wave equations in (1.2.8), but that will not be proven here. They tell us a crucial aspect: all the electromagnetic quantities which one would measure in a position $\boldsymbol{x}$ at time t are actually generated in the past, as far back as it takes for a light signal from the source element location $\boldsymbol{x}'$ to reach the observer in $\boldsymbol{x}$ at time t. This means that any electromagnetic signal travels through the space at the speed of light, neither faster nor slower.
At this point, it is natural to wonder what form should the retarded potentials assume to describe the electromagnetic field arising from a point-like source. To give an answer and for the sake of compactness it is useful to introduce two pieces of notation namely

$$\boldsymbol{n}(\boldsymbol{x}, t) = \frac{\boldsymbol{x}-\boldsymbol{r}(t)}{|\boldsymbol{x}-\boldsymbol{r}(t)|}, \tag{1.2.11a}$$

$$\boldsymbol{\beta}(t) = \frac{\dot{\boldsymbol{r}}(t)}{c}, \tag{1.2.11b}$$

where $\boldsymbol{n}(\boldsymbol{x}, t)$ represents the unit vector pointing towards the straight line that connects the charge's position $\boldsymbol{r}(t)$ to the observation point $\boldsymbol{x}$, while $\boldsymbol{\beta}(t)$ stands for the velocity of the charge in unit of c. The figure (1.2.1) on the next page provides a clear pictorial representation of the examined physical situation.
Inserting the expressions of charge and current densities (1.1.8) in (1.2.10), yields

$$\phi(\boldsymbol{x}, t) = \frac{q}{1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x}, t^*)} \frac{1}{|\boldsymbol{x}-\boldsymbol{r}(t^*)|}, \tag{1.2.12a}$$

$$\boldsymbol{A}(\boldsymbol{x}, t) = \frac{q}{1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x}, t^*)} \frac{\boldsymbol{\beta}(t^*)}{|\boldsymbol{x}-\boldsymbol{r}(t^*)|}, \tag{1.2.12b}$$

where $t^* = t^*(\boldsymbol{x}, t)$ is a scalar field, well defined as the unique solution of

$$t^* = t - \frac{|\boldsymbol{x} - \boldsymbol{r}(t^*)|}{c}. \tag{1.2.13}$$

For the sake of clarity, from now on we will denote $t^*(\boldsymbol{x}, t)$ as t^* only. Relations (1.2.12) are better known as the Liénard-Wiechert potentials and they represent the retarded potentials for a point-like charge. Furthermore, we notice how they perfectly reflect the retardation effect proving that the electromagnetic field calculated in $\boldsymbol{x}$ at time t has been actually produced in the past by the charge located in $\boldsymbol{r}(t^*)$ at time t^*.

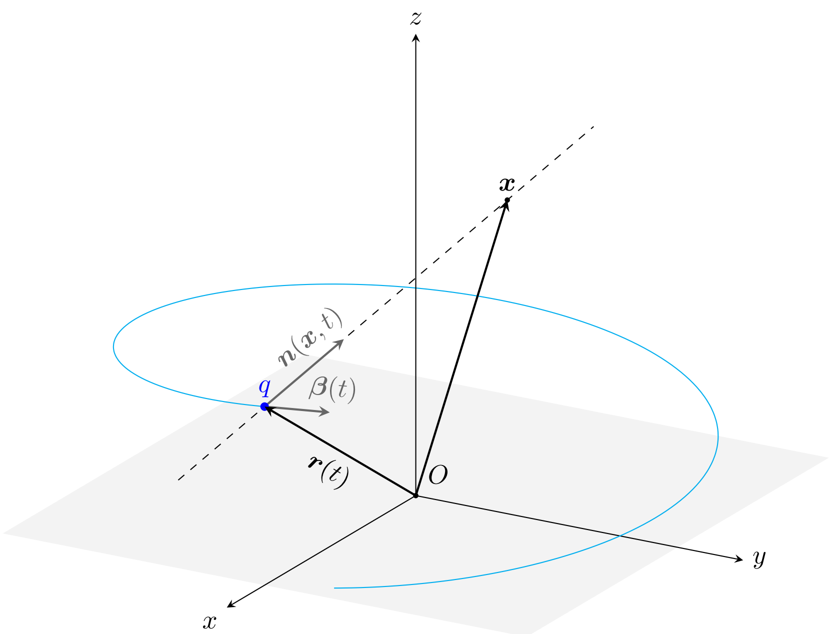

Figure 1.2.1: A moving charge q, its velocity $\boldsymbol{\beta}(t)$ and the unit vector $\boldsymbol{n}(\boldsymbol{x}, t)$.

Proof. Proof is quite long, hence it will be divided into two different steps.

Step 1. We start to compute $\phi(\boldsymbol{x}, t)$, inserting (1.1.8a) into (1.2.10a)

$$\begin{aligned}
\phi(\boldsymbol{x}, t) &= \int d^3x' \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} \rho\left(\boldsymbol{x}', t - \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}\right) \\
&= \int d^3x' \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} \int dt' \rho(\boldsymbol{x}', t') \delta\left(t' - t + \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}\right) \\
&= q \int d^3x' \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} \int dt' \delta(\boldsymbol{x} - \boldsymbol{r}(t'))\delta\left(t' - t + \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}\right) \\
&= q \int dt' \int d^3x' \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} \delta(\boldsymbol{x} - \boldsymbol{r}(t'))\delta\left(t' - t + \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}\right)
\end{aligned} \tag{1.2.14}$$

$$= q \int dt' \frac{1}{|\boldsymbol{x} - \boldsymbol{r}(t')|} \delta \left(t' - t + \frac{|\boldsymbol{x} - \boldsymbol{r}(t')|}{c} \right),$$

where we have passed the inverse modulus of the distance through the time integral, changing the integration order. Same calculations can be made analogously to the potential vector, inserting (1.1.8b) into (1.2.10b)

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{x}, t) &= \int d^3x' \frac{1}{c|\boldsymbol{x} - \boldsymbol{x}'|} \boldsymbol{j} \left(\boldsymbol{x}', t - \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c} \right) \\ &= \int d^3x' \frac{1}{c|\boldsymbol{x} - \boldsymbol{x}'|} \int dt' \boldsymbol{j}(\boldsymbol{x}', t') \delta \left(t' - t + \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c} \right) \\ &= q \int d^3x' \frac{1}{c|\boldsymbol{x} - \boldsymbol{x}'|} \int dt' \dot{\boldsymbol{r}}(t') \delta(\boldsymbol{x} - \boldsymbol{r}(t')) \delta \left(t' - t + \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c} \right) \\ &= q \int dt' \int d^3x' \frac{1}{c|\boldsymbol{x} - \boldsymbol{x}'|} \dot{\boldsymbol{r}}(t') \delta(\boldsymbol{x} - \boldsymbol{r}(t')) \delta \left(t' - t + \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c} \right) \\ &= q \int dt' \frac{1}{c|\boldsymbol{x} - \boldsymbol{r}(t')|} \dot{\boldsymbol{r}}(t') \delta \left(t' - t + \frac{|\boldsymbol{x} - \boldsymbol{r}(t')|}{c} \right). \end{aligned} \tag{1.2.15}$$

Step 2. Now we have two expressions depending each one on a similar integral over time. It cannot be resolved easily and it is necessary to find the value of t' for which the delta function's argument vanishes. This corresponds to resolve the equation

$$t^* = t - \frac{|\boldsymbol{x} - \boldsymbol{r}(t^*)|}{c}. \tag{1.2.16}$$

It is clear that a unique solution $t^*(\boldsymbol{x}, t)$ for this equation exists, remembering that $|\dot{\boldsymbol{r}}(t^*)| < c$ and $t^* < t$ are always verified. Thinking backwards, a spherical front coming from the spatial infinity at time $-\infty$ and passing through $\boldsymbol{x}$ at time t could intercept the point-like charge's trajectory only once, precisely at time t^*.

Upon establishing the uniqueness of t^*, we can find a solution for the integrals in (1.2.14) and (1.2.15). One has to apply the following composition rule in order to solve them

$$\delta(f(\xi)) = \sum_j \frac{\delta(\xi - \xi_j)}{|f'(\xi_j)|}, \tag{1.2.17}$$

where $f(\xi)$ is a generic function of ξ while ξ_j are the f zeros. As proven before, there exists a unique zero for our delta function argument, hence

$$\delta \left(t' - t + \frac{|\boldsymbol{x} - \boldsymbol{r}(t')|}{c} \right) = \left| \frac{d}{dt'} \left(t' - t + \frac{|\boldsymbol{x} - \boldsymbol{r}(t')|}{c} \right) \right|^{-1} \delta(t' - t^*(\boldsymbol{x}, t)) \tag{1.2.18}$$

$$= \left|1 - \boldsymbol{\beta}(t') \cdot \frac{\boldsymbol{x} - \boldsymbol{r}(t')}{|\boldsymbol{x} - \boldsymbol{r}(t')|}\right|^{-1} \delta(t' - t^*(\boldsymbol{x}, t))$$
$$= \frac{\delta(t' - t^*(\boldsymbol{x}, t))}{|1 - \boldsymbol{\beta}(t') \cdot \boldsymbol{n}(\boldsymbol{x}, t')|}.$$

Inserting this expression into the integrals, (1.2.12a) and (1.2.12b) follow immediately.

1.3 Point-like Charge Fields

Upon obtaining the expressions of the electromagnetic potentials, we can finally derive the electric and the magnetic field produced by a point-like charge through the relations between fields and potentials shown in (1.2.2a) and (1.1.2b)

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{x},t) = &\frac{q}{(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))^3}\frac{1}{|\boldsymbol{x}-\boldsymbol{r}(t^*)|^2}\\ &\Bigg\{(1-|\boldsymbol{\beta}(t^*)|^2)(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*))\\ &+\frac{|\boldsymbol{x}-\boldsymbol{r}(t^*)|}{c}\boldsymbol{n}(\boldsymbol{x},t^*)\times\Big[(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*))\times\dot{\boldsymbol{\beta}}(t^*)\Big]\Bigg\},\end{aligned} \tag{1.3.1a}$$

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{x},t) = &\frac{q}{(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))^3}\frac{1}{|\boldsymbol{x}-\boldsymbol{r}(t^*)|^2}\\ &\boldsymbol{n}(\boldsymbol{x},t^*)\times\Bigg\{(1-|\boldsymbol{\beta}(t^*)|^2)(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*))\\ &+\frac{|\boldsymbol{x}-\boldsymbol{r}(t^*)|}{c}\boldsymbol{n}(\boldsymbol{x},t^*)\times\Big[(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*))\times\dot{\boldsymbol{\beta}}(t^*)\Big]\Bigg\}.\end{aligned} \tag{1.3.1b}$$

Although these expressions are rather lengthy and complicated, it is interesting to discuss some relevant physical implications. We can observe that

$$\boldsymbol{B}(\boldsymbol{x},t) = \boldsymbol{n}(\boldsymbol{x},t^*)\times\boldsymbol{E}(\boldsymbol{x},t), \tag{1.3.2}$$

hence the electric and magnetic fields are always orthogonal to each other and the related wave propagates towards $\boldsymbol{n}(\boldsymbol{x},t^*)$ evaluated at the retarded time t^*.
Both the fields finally depend on the particle's position at this particular instant, reflecting again the fact that electromagnetic signals travel through space with a limited speed equal to the light's one.

Proof. For the sake of clarity, proof will be shown in several steps.
Step 1. We will prove (1.3.1) using variational calculus. First of all, we will apply the variational operator δ (beware not to confuse it with the delta function) to the electric potential (1.2.12a), varying it under an infinitesimal variation of $\boldsymbol{x}$ and t

$$\begin{aligned}\delta\phi(\boldsymbol{x},t) &= \delta\left[\frac{q}{1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)}\frac{1}{|\boldsymbol{x}-\boldsymbol{r}(t^*)|}\right]\\ &= -\frac{q}{1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)}\frac{1}{|\boldsymbol{x}-\boldsymbol{r}(t^*)|^2}\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\delta(\boldsymbol{x}-\boldsymbol{r}(t^*))\end{aligned} \tag{1.3.3}$$

$$+\frac{q}{(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))^2}\delta(\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))\frac{1}{|\boldsymbol{x}-\boldsymbol{r}(t^*)|}.$$

Remaining variations in (1.3.3) will be calculated singularly. The first variation term is

$$\delta(\boldsymbol{x}-\boldsymbol{r}(t^*)) = \delta\boldsymbol{x} - c\boldsymbol{\beta}(t^*)\delta t^*, \tag{1.3.4}$$

where δt^* needs to be expressed in terms of $\boldsymbol{x}$ and t as it will become clearer later. The second term is quite tricky; exploiting Leibniz product rule, we see that

$$\begin{aligned}\delta(\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)) &= \boldsymbol{\beta}(t^*)\cdot\delta\boldsymbol{n}(\boldsymbol{x},t^*) + \boldsymbol{n}(\boldsymbol{x},t^*)\cdot\delta\boldsymbol{\beta}(t^*)\\ &= \boldsymbol{\beta}(t^*)\cdot\delta\boldsymbol{n}(\boldsymbol{x},t^*) + \boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\delta t^*\\ &= \boldsymbol{\beta}(t^*)\cdot\frac{\mathbb{1}-\boldsymbol{n}(\boldsymbol{x},t^*)\otimes\boldsymbol{n}(\boldsymbol{x},t^*)}{|\boldsymbol{x}-\boldsymbol{r}(t^*)|}\cdot(\delta\boldsymbol{x}-c\boldsymbol{\beta}(t^*)\delta t^*)\\ &+ \boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\delta t^*.\end{aligned} \tag{1.3.5}$$

In the last statement the unit vector variation rule occurs; indeed, given a unit vector $\boldsymbol{n}=\boldsymbol{r}/r$, its variation corresponds to

$$\begin{aligned}\delta\boldsymbol{n} = \delta\left(\frac{\boldsymbol{r}}{r}\right) &= \frac{\delta\boldsymbol{r}}{r} - \frac{\boldsymbol{r}}{r^2}\delta|\boldsymbol{r}|\\ &= \mathbb{1}\frac{\delta\boldsymbol{r}}{r} - \frac{\boldsymbol{r}}{r^2}\otimes\left(\frac{\boldsymbol{r}}{r}\cdot\delta\boldsymbol{r}\right)\\ &= (\mathbb{1}-\boldsymbol{n}\otimes\boldsymbol{n})\cdot\frac{\delta\boldsymbol{r}}{r},\end{aligned} \tag{1.3.6}$$

where $\otimes$ denotes the dyadic tensor product, whose result can be interpreted as a 3×3 matrix. In this case we used (1.2.11a) to identify $\boldsymbol{r}$ and, consequently, $\delta\boldsymbol{r}$ is given by (1.3.4). Proceeding with the proof, one inserts (1.3.5) and (1.3.4) into (1.3.3), getting

$$\begin{aligned}\delta\phi(\boldsymbol{x},t) = &-\frac{q}{1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)}\frac{1}{|\boldsymbol{x}-\boldsymbol{r}(t^*)|^2}\boldsymbol{n}(\boldsymbol{x},t^*)\cdot(\delta\boldsymbol{x}-c\boldsymbol{\beta}(t^*)\delta t^*)\\ &+\frac{q}{(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))^2}\frac{1}{|\boldsymbol{x}-\boldsymbol{r}(t^*)|}\Bigg[\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\delta t^*\\ &+\boldsymbol{\beta}(t^*)\cdot\frac{\mathbb{1}-\boldsymbol{n}(\boldsymbol{x},t^*)\otimes\boldsymbol{n}(\boldsymbol{x},t^*)}{|\boldsymbol{x}-\boldsymbol{r}(t^*)|}\cdot(\delta\boldsymbol{x}-c\boldsymbol{\beta}(t^*)\delta t^*)\Bigg].\end{aligned} \tag{1.3.7}$$

In order to make reading easier we shall introduce

$$\alpha = 1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*), \tag{1.3.8a}$$

$$r = |\boldsymbol{x} - \boldsymbol{r}(t^*)|. \tag{1.3.8b}$$

Using these pieces of notations, the electric potential (1.2.12a) and its variation (1.3.7) become respectively

$$\phi(\boldsymbol{x}, t) = \frac{q}{\alpha r}, \tag{1.3.9a}$$

$$\begin{aligned} \delta\phi(\boldsymbol{x}, t) = \frac{q}{\alpha r}\bigg[&- \frac{\boldsymbol{n}(\boldsymbol{x}, t^*)}{r} \cdot (\delta\boldsymbol{x} - c\boldsymbol{\beta}(t^*)\delta t^*) \\ &+ \boldsymbol{\beta}(t^*) \cdot \frac{\mathbb{1} - \boldsymbol{n}(\boldsymbol{x}, t^*) \otimes \boldsymbol{n}(\boldsymbol{x}, t^*)}{\alpha r} \cdot (\delta\boldsymbol{x} - c\boldsymbol{\beta}(t^*)\delta t^*) \\ &+ \frac{\boldsymbol{n}(\boldsymbol{x}, t^*)}{\alpha} \cdot \dot{\boldsymbol{\beta}}(t^*)\delta t^* \bigg]. \end{aligned} \tag{1.3.9b}$$

Step 2. Variation (1.3.7) may be written in terms of δt and $\delta\boldsymbol{x}$ to find an explicit expression for the gradient $\boldsymbol{\nabla}\phi(\boldsymbol{x}, t)$, indeed it can be derived from $\delta\phi(\boldsymbol{x}, t)$ imposing $\delta t = 0$ and formally "dividing" it by $\delta\boldsymbol{x}$.
However, it is firstly necessary to extract a relation between those differentials. Reminding the retarded time definition (1.2.13) and varying it, yields

$$\begin{aligned} 0 &= \delta\left(t^* - t + \frac{|\boldsymbol{x} - \boldsymbol{r}(t^*)|}{c}\right) \\ &= \delta t^* - \delta t + \frac{\boldsymbol{x} - \boldsymbol{r}(t^*)}{c|\boldsymbol{x} - \boldsymbol{r}(t^*)|} \cdot (\delta\boldsymbol{x} - c\boldsymbol{\beta}(t^*)\delta t^*) \\ &= \delta t^* \left(1 - c\boldsymbol{\beta}(t^*) \cdot \frac{\boldsymbol{x} - \boldsymbol{r}(t^*)}{c|\boldsymbol{x} - \boldsymbol{r}(t^*)|}\right) - \delta t + \frac{\boldsymbol{x} - \boldsymbol{r}(t^*)}{c|\boldsymbol{x} - \boldsymbol{r}(t^*)|} \cdot \delta\boldsymbol{x} \\ &= \alpha\delta t^* - \delta t + \frac{1}{c}\boldsymbol{n}(\boldsymbol{x}, t^*) \cdot \delta\boldsymbol{x}. \end{aligned} \tag{1.3.10}$$

Therefore, despite δt could vanish, δt^* does not, indeed

$$\delta t^*|_t = -\frac{1}{\alpha c}\boldsymbol{n}(\boldsymbol{x}, t^*) \cdot \delta\boldsymbol{x}. \tag{1.3.11}$$

Finally, it is possible to compute $\boldsymbol{\nabla}\phi(\boldsymbol{x}, t)$ and the equation (1.3.9b), at $\delta t = 0$, becomes

$$\begin{aligned} \delta\phi(\boldsymbol{x}, t)|_t = \frac{q}{\alpha r}\bigg[&- \frac{\boldsymbol{n}(\boldsymbol{x}, t^*)}{r} \cdot \left(\delta\boldsymbol{x} + c\boldsymbol{\beta}(t^*)\frac{1}{\alpha c}\boldsymbol{n}(\boldsymbol{x}, t^*) \cdot \delta\boldsymbol{x}\right) \\ &+ \boldsymbol{\beta}(t^*) \cdot \frac{\mathbb{1} - \boldsymbol{n}(\boldsymbol{x}, t^*) \otimes \boldsymbol{n}(\boldsymbol{x}, t^*)}{\alpha r} \cdot \left(\delta\boldsymbol{x} + c\boldsymbol{\beta}(t^*)\frac{1}{\alpha c}\boldsymbol{n}(\boldsymbol{x}, t^*) \cdot \delta\boldsymbol{x}\right) \\ &- \frac{\boldsymbol{n}(\boldsymbol{x}, t^*)}{\alpha} \cdot \dot{\boldsymbol{\beta}}(t^*)\frac{1}{\alpha c}\boldsymbol{n}(\boldsymbol{x}, t^*) \cdot \delta\boldsymbol{x} \bigg] \end{aligned} \tag{1.3.12}$$

$$
\begin{aligned}
&= \frac{q}{\alpha r}\bigg[-\frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{r} - \frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha r} \cdot \boldsymbol{\beta}(t^*)\boldsymbol{n}(\boldsymbol{x},t^*) \\
&+ \boldsymbol{\beta}(t^*) \cdot \frac{\mathbb{1} - \boldsymbol{n}(\boldsymbol{x},t^*) \otimes \boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha r} \\
&+ \boldsymbol{\beta}(t^*) \cdot \frac{\mathbb{1} - \boldsymbol{n}(\boldsymbol{x},t^*) \otimes \boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha^2 r} \cdot \boldsymbol{\beta}(t^*)\boldsymbol{n}(\boldsymbol{x},t^*) \\
&- \frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha^2 c} \cdot \dot{\boldsymbol{\beta}}(t^*)\boldsymbol{n}(\boldsymbol{x},t^*)\bigg] \cdot \delta\boldsymbol{x},
\end{aligned}
$$

where we have factorized the common term $\delta\boldsymbol{x}$ in such a way as to simplify the following gradient derivation.

As we have seen earlier, $\delta\phi(\boldsymbol{x},t)|_t = \boldsymbol{\nabla}\phi(\boldsymbol{x},t) \cdot \delta\boldsymbol{x}$, hence we have

$$
\begin{aligned}
\boldsymbol{\nabla}\phi(\boldsymbol{x},t) = \frac{q}{\alpha r}\bigg[&-\frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{r} - \frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha r} \cdot \boldsymbol{\beta}(t^*)\boldsymbol{n}(\boldsymbol{x},t^*) \\
&+ \boldsymbol{\beta}(t^*) \cdot \frac{\mathbb{1} - \boldsymbol{n}(\boldsymbol{x},t^*) \otimes \boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha r} \\
&+ \boldsymbol{\beta}(t^*) \cdot \frac{\mathbb{1} - \boldsymbol{n}(\boldsymbol{x},t^*) \otimes \boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha^2 r} \cdot \boldsymbol{\beta}(t^*)\boldsymbol{n}(\boldsymbol{x},t^*) \\
&- \frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha^2 c} \cdot \dot{\boldsymbol{\beta}}(t^*)\boldsymbol{n}(\boldsymbol{x},t^*)\bigg].
\end{aligned}
\tag{1.3.13}
$$

Once computed the electric potential gradient, we are going to try writing it in a more pleasant way. We will deal with the dyadic products in (1.3.13)

$$
\begin{aligned}
\boldsymbol{\beta}(t^*) \cdot (\mathbb{1} - \boldsymbol{n}(\boldsymbol{x},t^*) \otimes \boldsymbol{n}(\boldsymbol{x},t^*)) &= \boldsymbol{\beta}(t^*) \cdot \mathbb{1} - \boldsymbol{\beta}(t^*) \cdot \boldsymbol{n}(\boldsymbol{x},t^*) \otimes \boldsymbol{n}(\boldsymbol{x},t^*) \qquad (1.3.14\text{a}) \\
&= \boldsymbol{\beta}(t^*) - (\boldsymbol{\beta}(t^*) \cdot \boldsymbol{n}(\boldsymbol{x},t^*))\boldsymbol{n}(\boldsymbol{x},t^*) \\
&= \boldsymbol{\beta}(t^*) - (1-\alpha)\boldsymbol{n}(\boldsymbol{x},t^*), \\
(\boldsymbol{\beta}(t^*) - (1-\alpha)\boldsymbol{n}(\boldsymbol{x},t^*)) \cdot \boldsymbol{\beta}(t^*) &= |\boldsymbol{\beta}(t^*)|^2 - (1-\alpha)\boldsymbol{n}(\boldsymbol{x},t^*) \cdot \boldsymbol{\beta}(t^*) \qquad (1.3.14\text{b}) \\
&= |\boldsymbol{\beta}(t^*)|^2 - (1-\alpha)^2.
\end{aligned}
$$

Inserting these results into (1.3.13) yields

$$
\begin{aligned}
\boldsymbol{\nabla}\phi(\boldsymbol{x},t) = \frac{q}{\alpha r}\bigg[&-\frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{r} - \frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha r} \cdot \boldsymbol{\beta}(t^*)\boldsymbol{n}(\boldsymbol{x},t^*) \\
&+ \frac{\boldsymbol{\beta}(t^*) - (1-\alpha)\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha r} \\
&+ \frac{|\boldsymbol{\beta}(t^*)|^2 - (1-\alpha)^2}{\alpha^2 r}\boldsymbol{n}(\boldsymbol{x},t^*) \\
&- \frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha^2 c} \cdot \dot{\boldsymbol{\beta}}(t^*)\boldsymbol{n}(\boldsymbol{x},t^*)\bigg],
\end{aligned}
\tag{1.3.15}
$$

where we have to rearrange some terms in order to keep the expression more compact. Finally, performing some other calculations, we deduce the most suitable formulation for the gradient that corresponds to be

$$\begin{aligned}\boldsymbol{\nabla}\phi(\boldsymbol{x},t) = &-\frac{q}{\alpha^3 r^2}\Big[\alpha^2\boldsymbol{n}(\boldsymbol{x},t^*) - \alpha\boldsymbol{\beta}(t^*) + 2\alpha(1-\alpha)\boldsymbol{n}(\boldsymbol{x},t^*) \\ &- |\boldsymbol{\beta}(t^*)|^2\boldsymbol{n}(\boldsymbol{x},t^*) + (1-\alpha)^2\boldsymbol{n}(\boldsymbol{x},t^*) \\ &+ \frac{r}{c}\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\boldsymbol{n}(\boldsymbol{x},t^*)\Big] \\ = &-\frac{q}{\alpha^3 r^2}\Big[-\alpha\boldsymbol{\beta}(t^*) - |\boldsymbol{\beta}(t^*)|^2\boldsymbol{n}(\boldsymbol{x},t^*) + \boldsymbol{n}(\boldsymbol{x},t^*) \\ &+ \frac{r}{c}\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\boldsymbol{n}(\boldsymbol{x},t^*)\Big].\end{aligned} \tag{1.3.16}$$

Step 3. In order to ultimate the electric field derivation, we need to find the partial derivative of the potential vector $\boldsymbol{A}(\boldsymbol{x},t)$ with respect to time.

Let's apply the variational operator δ to the potential vector this time, as done before with the scalar electric potential.

As we can notice in (1.2.12b), the potential vector can be written in terms of $\phi(\boldsymbol{x},t)$ in such a way as to have

$$\boldsymbol{A}(\boldsymbol{x},t) = \phi(\boldsymbol{x},t)\boldsymbol{\beta}(t^*), \tag{1.3.17}$$

hence, computing its variation, we shall obtain

$$\begin{aligned}\delta\boldsymbol{A}(\boldsymbol{x},t) &= \delta(\phi(\boldsymbol{x},t)\boldsymbol{\beta}(t^*)) \\ &= \delta\phi(\boldsymbol{x},t)\boldsymbol{\beta}(t^*) + \phi(\boldsymbol{x},t)\delta\boldsymbol{\beta}(t^*) \\ &= \delta\phi(\boldsymbol{x},t)\boldsymbol{\beta}(t^*) + \phi(\boldsymbol{x},t)\dot{\boldsymbol{\beta}}(t^*)\delta t^*.\end{aligned} \tag{1.3.18}$$

We have already deduced $\delta\phi(\boldsymbol{x},t)$ in (1.3.9b), then it remains to extract its variation at constant $\boldsymbol{x}$, taking (1.3.10) and setting $\delta\boldsymbol{x} = 0$.

Summarizing our results, the retarded time variations and the related derivatives with respect to t and $\boldsymbol{x}$ are thus

$$\delta t^*|_{\boldsymbol{x}} = \frac{\delta t}{\alpha} \quad \rightarrow \quad \frac{\partial t^*}{\partial t} = \frac{1}{\alpha}, \tag{1.3.19a}$$

$$\delta t^*|_t = -\frac{1}{\alpha c}\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\delta\boldsymbol{x} \quad \rightarrow \quad \boldsymbol{\nabla}t^* = -\frac{1}{\alpha c}\boldsymbol{n}(\boldsymbol{x},t^*). \tag{1.3.19b}$$

Combining (1.3.19a) and (1.3.9b) into (1.3.18) and imposing $\delta \boldsymbol{x} = 0$, one gets

$$\begin{aligned}\delta \boldsymbol{A}(\boldsymbol{x},t)|_{\boldsymbol{x}} &= \delta\phi(\boldsymbol{x},t)|_{\boldsymbol{x}}\, \boldsymbol{\beta}(t^*) + \phi(\boldsymbol{x},t)\dot{\boldsymbol{\beta}}(t^*)\frac{\delta t}{\alpha} \\ &= \phi(\boldsymbol{x},t)\dot{\boldsymbol{\beta}}(t^*)\frac{1}{\alpha}\delta t + \boldsymbol{\beta}(t^*)\frac{q}{\alpha r}\Bigg[-\frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{r}\cdot(-c\boldsymbol{\beta}(t^*)\frac{\delta t}{\alpha}) \\ &+ \boldsymbol{\beta}(t^*)\cdot\frac{\mathbb{1}-\boldsymbol{n}(\boldsymbol{x},t^*)\otimes\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha r}\cdot(-c\boldsymbol{\beta}(t^*)\frac{\delta t}{\alpha}) \\ &+ \frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha}\cdot\dot{\boldsymbol{\beta}}(t^*)\frac{\delta t}{\alpha}\Bigg] \\ &= \phi(\boldsymbol{x},t)\dot{\boldsymbol{\beta}}(t^*)\frac{1}{\alpha}\delta t + \boldsymbol{\beta}(t^*)\frac{q}{\alpha r}\Bigg[\frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{r}\cdot c\boldsymbol{\beta}(t^*)\frac{\delta t}{\alpha} \\ &- \boldsymbol{\beta}(t^*)\cdot\frac{\mathbb{1}-\boldsymbol{n}(\boldsymbol{x},t^*)\otimes\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha r}\cdot c\boldsymbol{\beta}(t^*)\frac{\delta t}{\alpha} + \frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha}\cdot\dot{\boldsymbol{\beta}}(t^*)\frac{\delta t}{\alpha}\Bigg].\end{aligned} \tag{1.3.20}$$

We shall "divide" by δt to derive $\partial_t \boldsymbol{A}(\boldsymbol{x},t)$ and simplify these intricate terms, putting inside the expression for $\phi(\boldsymbol{x},t)$ in (1.3.9a)

$$\begin{aligned}\frac{\partial \boldsymbol{A}(\boldsymbol{x},t)}{\partial t} &= \frac{q}{\alpha^2 r}\dot{\boldsymbol{\beta}}(t^*) + \boldsymbol{\beta}(t^*)\frac{q}{\alpha r}\Bigg[\frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha r}\cdot c\boldsymbol{\beta}(t^*) \\ &- \boldsymbol{\beta}(t^*)\cdot\frac{\mathbb{1}-\boldsymbol{n}(\boldsymbol{x},t^*)\otimes\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha^2 r}\cdot c\boldsymbol{\beta}(t^*) + \frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha^2}\cdot\dot{\boldsymbol{\beta}}(t^*)\Bigg] \\ &= \frac{q}{\alpha^2 r}\dot{\boldsymbol{\beta}}(t^*) + \boldsymbol{\beta}(t^*)\frac{q}{\alpha r}\Bigg[\frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha r}\cdot c\boldsymbol{\beta}(t^*) \\ &- \frac{\boldsymbol{\beta}(t^*)-(1-\alpha)\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha^2 r}\cdot c\boldsymbol{\beta}(t^*) + \frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha^2}\cdot\dot{\boldsymbol{\beta}}(t^*)\Bigg].\end{aligned} \tag{1.3.21}$$

Upon removing the dyadic product, we have to multiply and divide everything by a factor $-c/(\alpha r)$; the reason will be much clearer later. Therefore, we will obtain

$$\begin{aligned}\frac{\partial \boldsymbol{A}(\boldsymbol{x},t)}{\partial t} &= \frac{-qc}{\alpha^3 r^2}\Bigg[-\frac{r\alpha}{c}\dot{\boldsymbol{\beta}}(t^*) - \boldsymbol{\beta}(t^*)\frac{r\alpha}{c}\Bigg(\frac{c(1-\alpha)}{r} + \frac{\boldsymbol{n}(\boldsymbol{x},t^*)}{\alpha}\cdot\dot{\boldsymbol{\beta}}(t^*) \\ &- c\frac{|\boldsymbol{\beta}(t^*)|^2-(1-\alpha)^2}{\alpha r}\Bigg)\Bigg] \\ &= \frac{-qc}{\alpha^3 r^2}\Bigg[-\frac{r\alpha}{c}\dot{\boldsymbol{\beta}}(t^*) - \boldsymbol{\beta}(t^*)\alpha(1-\alpha) + \boldsymbol{\beta}(t^*)|\boldsymbol{\beta}(t^*)|^2 \\ &- \boldsymbol{\beta}(t^*)(1-\alpha)^2 - \frac{r}{c}\boldsymbol{\beta}(t^*)\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\Bigg] \\ &= \frac{-qc}{\alpha^3 r^2}\Bigg[-\frac{r\alpha}{c}\dot{\boldsymbol{\beta}}(t^*) - \boldsymbol{\beta}(t^*)(1-\alpha) + \boldsymbol{\beta}(t^*)|\boldsymbol{\beta}(t^*)|^2 \\ &- \frac{r}{c}\boldsymbol{\beta}(t^*)\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)\Bigg].\end{aligned} \tag{1.3.22}$$

This statement concludes the derivation of the potential vector partial time derivative.
Step 4. Lastly, we can compute the electric field using $\nabla\phi(\boldsymbol{x},t)$ and $\partial_t \boldsymbol{A}(\boldsymbol{x},t)$ through the relation (1.2.2a) shown previously. We have to sum in this way the relations (1.3.16) and (1.3.22), substituting the effective value of α within the brackets

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{x},t) = \frac{q}{\alpha^3 r^2}\bigg[&-\boldsymbol{\beta}(t^*) + \boldsymbol{\beta}(t^*)(\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)) - |\boldsymbol{\beta}(t^*)|^2\boldsymbol{n}(\boldsymbol{x},t^*) + \boldsymbol{n}(\boldsymbol{x},t^*) \\ &+ \frac{r}{c}(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*))\boldsymbol{n}(\boldsymbol{x},t^*) - \frac{r}{c}\dot{\boldsymbol{\beta}}(t^*) - \frac{r}{c}\boldsymbol{\beta}(t^*)(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)) \\ &+ \frac{r}{c}\dot{\boldsymbol{\beta}}(t^*)(\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)) - \boldsymbol{\beta}(t^*)(\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)) + \boldsymbol{\beta}(t^*)|\boldsymbol{\beta}(t^*)|^2\bigg] \\ = \frac{q}{\alpha^3 r^2}\bigg\{&-\boldsymbol{\beta}(t^*) - |\boldsymbol{\beta}(t^*)|^2\boldsymbol{n}(\boldsymbol{x},t^*) + \boldsymbol{n}(\boldsymbol{x},t^*) + \boldsymbol{\beta}(t^*)|\boldsymbol{\beta}(t^*)|^2 \\ &+ \frac{r}{c}\bigg[(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*))\boldsymbol{n}(\boldsymbol{x},t^*) - \dot{\boldsymbol{\beta}}(t^*) + \dot{\boldsymbol{\beta}}(t^*)(\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)) \\ &- \boldsymbol{\beta}(t^*)(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*))\bigg]\bigg\},\end{aligned} \tag{1.3.23}$$

in which we have made explicit calculations for the sake of completeness.
Evidently, the last expression is quite chaotic, then we rewrite it more symmetrically

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{x},t) = \frac{q}{\alpha^3 r^2}\bigg\{&(\boldsymbol{\beta}(t^*) - \boldsymbol{n}(\boldsymbol{x},t^*))(|\boldsymbol{\beta}(t^*)|^2 - 1) \\ &+ \frac{r}{c}\bigg[(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*))\boldsymbol{n}(\boldsymbol{x},t^*) - \dot{\boldsymbol{\beta}}(t^*) + \dot{\boldsymbol{\beta}}(t^*)(\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*)) \\ &- \boldsymbol{\beta}(t^*)(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*))\bigg]\bigg\}.\end{aligned} \tag{1.3.24}$$

It is now clear why we have multiplied and divided all by that factor $-c/(\alpha r)$.
This formulation is still rather complicated, hence we will rearrange it through the following triple cross product property

$$\begin{aligned}\boldsymbol{a}\times[(\boldsymbol{a}-\boldsymbol{b})\times\boldsymbol{c}] &= (\boldsymbol{a}-\boldsymbol{b})(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot(\boldsymbol{a}-\boldsymbol{b})) \\ &= \boldsymbol{a}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}|\boldsymbol{a}|^2 + \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b}).\end{aligned} \tag{1.3.25}$$

Setting $\boldsymbol{a} = \boldsymbol{n}(\boldsymbol{x},t^*)$, $\boldsymbol{b} = \boldsymbol{\beta}(\boldsymbol{x},t^*)$ and $\boldsymbol{c} = \boldsymbol{\beta}(\dot{\boldsymbol{x}},t^*)$ yields

$$\begin{aligned}\boldsymbol{n}(\boldsymbol{x},t^*)\times[(\boldsymbol{n}(\boldsymbol{x},t^*) - \boldsymbol{\beta}(t^*))\times\dot{\boldsymbol{\beta}}(t^*)] &= \boldsymbol{n}(\boldsymbol{x},t^*)(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*)) \\ &\quad - \boldsymbol{\beta}(t^*)(\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\dot{\boldsymbol{\beta}}(t^*))\end{aligned} \tag{1.3.26}$$

$$- \dot{\boldsymbol{\beta}}(t^*) + \dot{\boldsymbol{\beta}}(t^*)(\boldsymbol{n}(\boldsymbol{x}, t^*) \cdot \boldsymbol{\beta}(t^*)).$$

We insert now this triple cross product into the electric field in (1.3.24), getting

$$\boldsymbol{E}(\boldsymbol{x}, t) = \frac{q}{\alpha^3 r^2}\Bigg\{(\boldsymbol{\beta}(t^*) - \boldsymbol{n}(\boldsymbol{x}, t^*))(|\boldsymbol{\beta}(t^*)|^2 - 1) + \frac{r}{c}\Big[\boldsymbol{n}(\boldsymbol{x}, t^*) \times (\boldsymbol{n}(\boldsymbol{x}, t^*) - \boldsymbol{\beta}(t^*)) \times \dot{\boldsymbol{\beta}}(t^*)\Big]\Bigg\}, \tag{1.3.27}$$

where we shall substitute the (1.3.8) to the terms depending on r and α to obtain the conclusive expression for the electric field, which is the same presented in (1.3.1a).

Step 5. In this last step, we aim to find the magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$. From the definition of $\boldsymbol{A}(\boldsymbol{x}, t)$ in (1.2.2b) we can develop these calculations

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{x}, t) &= \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{x}, t) \\ &= \boldsymbol{\nabla} \times (\phi(\boldsymbol{x}, t)\boldsymbol{\beta}(t^*)) \\ &= \phi(\boldsymbol{x}, t)\boldsymbol{\nabla} \times \boldsymbol{\beta}(t^*) - \boldsymbol{\beta}(t^*) \times \boldsymbol{\nabla}\phi(\boldsymbol{x}, t).\end{aligned} \tag{1.3.28}$$

Since we have already carried out the gradient $\boldsymbol{\nabla}\phi(\boldsymbol{x}, t)$, we only need to get the curl of $\boldsymbol{\beta}(t^*)$, that, in vector notation, turns out to be

$$\boldsymbol{\nabla} \times \boldsymbol{\beta}(t^*) = \boldsymbol{\nabla} t^* \times \dot{\boldsymbol{\beta}}(t^*) = \frac{1}{\alpha c}\dot{\boldsymbol{\beta}}(t^*) \times \boldsymbol{n}(\boldsymbol{x}, t^*),$$

in which $\boldsymbol{\nabla} t^*$ has already been found in (1.3.19b).

Upon calculating this curl, we have all the elements needed to derive the magnetic field arising from a point-like moving charge. Substituting the above relation and the gradient (1.3.16) into (1.3.28), one gets

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{x}, t) &= \frac{q}{\alpha^3 r^2}\Bigg\{\alpha\frac{r}{c}\dot{\boldsymbol{\beta}}(t^*) \times \boldsymbol{n}(\boldsymbol{x}, t^*) + \boldsymbol{\beta}(t^*) \times \Bigg[- \alpha\boldsymbol{\beta}(t^*) - |\boldsymbol{\beta}(t^*)|^2\boldsymbol{n}(\boldsymbol{x}, t^*) \\ &\quad + \boldsymbol{n}(\boldsymbol{x}, t^*) + \frac{r}{c}\boldsymbol{n}(\boldsymbol{x}, t^*) \cdot \dot{\boldsymbol{\beta}}(t^*)\boldsymbol{n}(\boldsymbol{x}, t^*)\Bigg]\Bigg\} \\ &= \frac{q}{\alpha^3 r^2}\Bigg\{\frac{r}{c}\dot{\boldsymbol{\beta}}(t^*) \times \boldsymbol{n}(\boldsymbol{x}, t^*) - (\boldsymbol{\beta}(t^*) \cdot \boldsymbol{n}(\boldsymbol{x}, t^*))\frac{r}{c}\dot{\boldsymbol{\beta}}(t^*) \times \boldsymbol{n}(\boldsymbol{x}, t^*) \\ &\quad + \boldsymbol{\beta}(t^*) \times \Bigg[- |\boldsymbol{\beta}(t^*)|^2\boldsymbol{n}(\boldsymbol{x}, t^*) + \boldsymbol{n}(\boldsymbol{x}, t^*) \\ &\quad + \frac{r}{c}\boldsymbol{n}(\boldsymbol{x}, t^*) \cdot \dot{\boldsymbol{\beta}}(t^*)\boldsymbol{n}(\boldsymbol{x}, t^*)\Bigg]\Bigg\}\end{aligned} \tag{1.3.29}$$

$$
\begin{aligned}
&= \frac{q}{\alpha^3 r^2}\boldsymbol{n}(\boldsymbol{x},t^*) \times \Bigg[(|\boldsymbol{\beta}(t^*)|^2 - 1)\boldsymbol{\beta}(t^*) \\
&+ \frac{r}{c}\left(-\dot{\boldsymbol{\beta}}(t^*) + (\boldsymbol{\beta}(t^*) \cdot \boldsymbol{n}(\boldsymbol{x},t^*))\dot{\boldsymbol{\beta}}(t^*) - (\boldsymbol{n}(\boldsymbol{x},t^*) \cdot \dot{\boldsymbol{\beta}}(t^*))\boldsymbol{\beta}(t^*)\right)\Bigg].
\end{aligned}
$$

This expression is quite similar to the electric field's one, hence we insert some terms proportional to $\boldsymbol{n}(\boldsymbol{x},t^*)$ inside the square brackets in order to extract the connection between them, since all the products $\boldsymbol{n}(\boldsymbol{x},t^*) \times \boldsymbol{n}(\boldsymbol{x},t^*)$ vanish anyway

$$
\begin{aligned}
\boldsymbol{B}(\boldsymbol{x},t) = \frac{q}{\alpha^3 r^2}\boldsymbol{n}(\boldsymbol{x},t^*) \times \Bigg[&(|\boldsymbol{\beta}(t^*)|^2 - 1)(\boldsymbol{\beta}(t^*) - \boldsymbol{n}(\boldsymbol{x},t^*)) \\
&+ \frac{r}{c}\Bigg(\boldsymbol{n}(\boldsymbol{x},t^*)(\boldsymbol{n}(\boldsymbol{x},t^*) \cdot \dot{\boldsymbol{\beta}}(t^*)) - \dot{\boldsymbol{\beta}}(t^*) + (\boldsymbol{\beta}(t^*) \cdot \boldsymbol{n}(\boldsymbol{x},t^*))\dot{\boldsymbol{\beta}}(t^*) \\
&- (\boldsymbol{n}(\boldsymbol{x},t^*) \cdot \dot{\boldsymbol{\beta}}(t^*))\boldsymbol{\beta}(t^*)\Bigg)\Bigg].
\end{aligned}
\tag{1.3.30}
$$

Reminding (1.3.26) and verifying (1.3.2), we conclude the proof.

1.4 Regimes and Field's Approximations

The electric and magnetic field expressions (1.3.1) are rather intricate and applying them to calculations is not so simple, hence we need to approximate the fields, possibly without loosing any relevant contribute contextually to the physical regime where such calculations are done. A logical relevant assumption on those fields could be done distinguishing different speed's limits.

When the particle's speed is much slower than the light's one, the non-relativistic limit occurs. This limit is stated through the condition

$$|\boldsymbol{\beta}(t^*)| \ll 1. \tag{1.4.1}$$

Consequently, under this assumption, $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{B}(\boldsymbol{x}, t)$ will take the following form

$$\boldsymbol{E}(\boldsymbol{x}, t) = \frac{q}{|\boldsymbol{x} - \boldsymbol{r}(t^*)|^2}\left\{\boldsymbol{n}(\boldsymbol{x}, t^*) + \frac{|\boldsymbol{x} - \boldsymbol{r}(t^*)|}{c}\boldsymbol{n}(\boldsymbol{x}, t^*) \times \left[\boldsymbol{n}(\boldsymbol{x}, t^*) \times \dot{\boldsymbol{\beta}}(t^*)\right]\right\}, \tag{1.4.2a}$$

$$\boldsymbol{B}(\boldsymbol{x}, t) = \frac{q}{|\boldsymbol{x} - \boldsymbol{r}(t^*)|^2}\left\{\boldsymbol{\beta}(t^*) + \frac{|\boldsymbol{x} - \boldsymbol{r}(t^*)|}{c}\dot{\boldsymbol{\beta}}(t^*)\right\} \times \boldsymbol{n}(\boldsymbol{x}, t^*). \tag{1.4.2b}$$

Proof. We will take the limit of the field relations in (1.3.1) as $|\boldsymbol{\beta}| \to 0$, precising that the limit's computing is independent of the choice of evaluating $\boldsymbol{\beta}$ both in retarded or present time. Firstly, we have to calculate the value of this limit for some useful terms that occur in our fields expressions (1.3.1), then

$$1 - \boldsymbol{\beta} \cdot \boldsymbol{n} \to 1, \quad |\boldsymbol{\beta}| \to 0, \tag{1.4.3a}$$

$$(1 - |\boldsymbol{\beta}|^2)(\boldsymbol{n} - \boldsymbol{\beta}) \to \boldsymbol{n}, \quad |\boldsymbol{\beta}| \to 0. \tag{1.4.3b}$$

We just require these two relations to prove the electric field (1.4.2a), simply by substituting them into (1.3.1a). Dealing with the second term within the brackets of (1.3.1b) and manipulating the related triple cross product, we have

$$\begin{aligned}
\boldsymbol{n} \times [\boldsymbol{n} \times ((\boldsymbol{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}})] &= [((\boldsymbol{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}) \times \boldsymbol{n}] \times \boldsymbol{n} \\
&= [\boldsymbol{n} \times (\dot{\boldsymbol{\beta}} \times \boldsymbol{n}) - \boldsymbol{\beta} \times (\dot{\boldsymbol{\beta}} \times \boldsymbol{n})] \times \boldsymbol{n} \\
&= [\dot{\boldsymbol{\beta}} - \boldsymbol{n}(\boldsymbol{n} \cdot \dot{\boldsymbol{\beta}}) - \dot{\boldsymbol{\beta}}(\boldsymbol{\beta} \cdot \boldsymbol{n}) + \boldsymbol{n}(\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}})] \times \boldsymbol{n} \\
&= \dot{\boldsymbol{\beta}}(1 - \boldsymbol{\beta} \cdot \boldsymbol{n}) \times \boldsymbol{n} \to \dot{\boldsymbol{\beta}} \times \boldsymbol{n}, \quad |\boldsymbol{\beta}| \to 0.
\end{aligned} \tag{1.4.4}$$

while the first term of (1.3.1b) becomes

$$\boldsymbol{n} \times \left[(1 - |\boldsymbol{\beta}|^2)(\boldsymbol{n} - \boldsymbol{\beta})\right] \to \boldsymbol{\beta} \times \boldsymbol{n}, \quad |\boldsymbol{\beta}| \to 0, \tag{1.4.5}$$

proving the expression (1.4.2b) for the magnetic field.

The quasi-static regime corresponds to a non-relativistic limit in which the time it takes for electromagnetic signals to reach the observer is smaller compared to the time it takes for the charge to change its position by the same spatial amount.
In other words, while signals have reached the observer walking a distance $|\boldsymbol{x} - \boldsymbol{r}(t)|$, the charge has covered a negligible one compared to that of the signals. Mathematically, this condition can be stated through the inequalities

$$|\boldsymbol{x} - \boldsymbol{r}(t)| \gg |\dot{\boldsymbol{r}}(t)| \frac{|\boldsymbol{x} - \boldsymbol{r}(t)|}{c} \gg |\ddot{\boldsymbol{r}}(t)| \left[\frac{|\boldsymbol{x} - \boldsymbol{r}(t)|}{c}\right]^2 \gg \dots \tag{1.4.6}$$

According to this assumption, retardation is also small in such a way that $t^* \to t$, indeed if (1.4.6) holds, then

$$\boldsymbol{r}\left(t + \frac{|\boldsymbol{x} - \boldsymbol{r}(t)|}{c}\right) \approx \boldsymbol{r}(t) + \frac{|\boldsymbol{x} - \boldsymbol{r}(t)|}{c} \dot{\boldsymbol{r}}(t), \tag{1.4.7}$$

by truncating the Taylor expansion at the first order.
We therefore conclude that $\boldsymbol{r}(t)$ varies little in the time lapse between t and t^*; the same conclusions can be drawn by analyzing the higher order derivatives of $\boldsymbol{r}(t)$, indeed

$$\begin{aligned} \dot{\boldsymbol{r}}\left(t + \frac{|\boldsymbol{x} - \boldsymbol{r}(t)|}{c}\right) &\approx \dot{\boldsymbol{r}}(t) + \frac{|\boldsymbol{x} - \boldsymbol{r}(t)|}{c} \ddot{\boldsymbol{r}}(t), \\ \ddot{\boldsymbol{r}}\left(t + \frac{|\boldsymbol{x} - \boldsymbol{r}(t)|}{c}\right) &\approx \ddot{\boldsymbol{r}}(t) + \frac{|\boldsymbol{x} - \boldsymbol{r}(t)|}{c} \dddot{\boldsymbol{r}}(t), \\ \dddot{\boldsymbol{r}}\left(t + \frac{|\boldsymbol{x} - \boldsymbol{r}(t)|}{c}\right) &\approx \dots \end{aligned} \tag{1.4.8}$$

Imposing this condition in (1.4.2), we get respectively the Coulomb and the Biot-Savart time evaluated expressions for the electric and the magnetic field

$$\boldsymbol{E}(\boldsymbol{x}, t) = \frac{q}{|\boldsymbol{x} - \boldsymbol{r}(t)|^2} \boldsymbol{n}(\boldsymbol{x}, t), \tag{1.4.9a}$$

$$\boldsymbol{B}(\boldsymbol{x}, t) = \frac{q}{|\boldsymbol{x} - \boldsymbol{r}(t)|^2} \boldsymbol{\beta}(t) \times \boldsymbol{n}(\boldsymbol{x}, t), \tag{1.4.9b}$$

of course, the retarded time t^* has been replaced with the present time t, making field effects instantaneous.

Another useful approximation is the radiation regime defined by

$$|\ddot{\boldsymbol{r}}(t)|\frac{|\boldsymbol{x}-\boldsymbol{r}(t)|}{c} \gg |\dot{\boldsymbol{r}}(t)|. \tag{1.4.10}$$

This condition is satisfied either when the particle strongly accelerates or when the distance between the particle and the observer is sufficiently great. In this case, each first term in (1.3.1a) and (1.3.1b), dependent on the velocity, is neglected compared to the second ones depending on the acceleration. In addition, the first terms, proportional to the inverse-square of the distance from the charge, decay more rapidly than the second ones, depending only on the inverse of the distance. Thus $\boldsymbol{E}(\boldsymbol{x},t)$ and $\boldsymbol{B}(\boldsymbol{x},t)$ become

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{x},t) = & \frac{q}{(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))^3}\frac{1}{c|\boldsymbol{x}-\boldsymbol{r}(t^*)|} \\ & \left\{\boldsymbol{n}(\boldsymbol{x},t^*)\times\left[(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*))\times\dot{\boldsymbol{\beta}}(t^*)\right]\right\},\end{aligned} \tag{1.4.11a}$$

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{x},t) = & \frac{q}{(1-\boldsymbol{\beta}(t^*)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))^3}\frac{1}{c|\boldsymbol{x}-\boldsymbol{r}(t^*)|} \\ & \boldsymbol{n}(\boldsymbol{x},t^*)\times\left\{\boldsymbol{n}(\boldsymbol{x},t^*)\times\left[(\boldsymbol{n}(\boldsymbol{x},t^*)-\boldsymbol{\beta}(t^*))\times\dot{\boldsymbol{\beta}}(t^*)\right]\right\}.\end{aligned} \tag{1.4.11b}$$

Finally, we underline how, in this regime, both the fields are perpendicular to each other and to the unit vector $\boldsymbol{n}(\boldsymbol{x},t^*)$

$$\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\boldsymbol{E}(\boldsymbol{x},t) = 0, \tag{1.4.12a}$$

$$\boldsymbol{n}(\boldsymbol{x},t^*)\cdot\boldsymbol{B}(\boldsymbol{x},t) = 0. \tag{1.4.12b}$$

These relations prove that electromagnetic waves produced by an accelerated charge propagate towards $\boldsymbol{n}(\boldsymbol{x},t^*)$ evaluated at the retarded time t^*.

Chapter 2

Poynting Vector and Power Emission

2.1 Poynting Vector

The Poynting vector describes the directional flow of electromagnetic energy and its magnitude corresponds to the amount of energy flowing through a unit area transverse to the flow per unit time. Its definition is intimately linked to Poynting's theorem which states the principle of conservation of energy for an electromagnetic system.
The Poynting vector for a point-like accelerated charge is given by the formula

$$\boldsymbol{S}(\boldsymbol{x},t) = \frac{c}{4\pi}|\boldsymbol{E}(\boldsymbol{x},t)|^2\boldsymbol{n}(\boldsymbol{x},t^*), \tag{2.1.1}$$

where we have considered the radiation regime fields (1.4.11).

Proof. The Poynting vector of a general electromagnetic field is defined as

$$\boldsymbol{S}(\boldsymbol{x},t) = \frac{c}{4\pi}\boldsymbol{E}(\boldsymbol{x},t) \times \boldsymbol{B}(\boldsymbol{x},t), \tag{2.1.2}$$

Now we have to recall the relation (1.3.2) and to take into account the radiation regime, so that the previous expression becomes

$$\begin{aligned}\boldsymbol{S}(\boldsymbol{x},t) &= \frac{c}{4\pi}\boldsymbol{E}(\boldsymbol{x},t) \times (\boldsymbol{n}(\boldsymbol{x},t^*) \times \boldsymbol{E}(\boldsymbol{x},t)) \\ &= \frac{c}{4\pi}\left[\boldsymbol{n}(\boldsymbol{x},t^*)|\boldsymbol{E}(\boldsymbol{x},t)|^2 - \boldsymbol{E}(\boldsymbol{x},t)(\boldsymbol{E}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x},t^*))\right] \\ &= \frac{c}{4\pi}|\boldsymbol{E}(\boldsymbol{x},t)|^2\boldsymbol{n}(\boldsymbol{x},t^*),\end{aligned} \tag{2.1.3}$$

proving the relation (2.1.1).

The Poynting vector and, consequently, the energy flow is directed towards $\boldsymbol{n}(\boldsymbol{x},t^*)$ evaluated at the retarded time t^*, which means that also the energy flow exhibits retardation; the energy flow measured at time t seems to be produced by a fictitious particle located at $\boldsymbol{r}(t^*)$ instead by the physical particle which stays at $\boldsymbol{r}(t)$, therefore any present interaction between radiation and matter is actually conveyed by a past configuration of the universe itself. Keeping in mind all these facts, it seems logical to express the Poynting vector in terms of the retarded time as shown below

$$\boldsymbol{S^*}(\boldsymbol{x}, t^*) = \boldsymbol{S}(\boldsymbol{x}, t(\boldsymbol{x}, t^*))\frac{\partial t}{\partial t^*}(\boldsymbol{x}, t^*) = \boldsymbol{S}(\boldsymbol{x}, t(\boldsymbol{x}, t^*))\alpha(\boldsymbol{x}, t^*), \tag{2.1.4}$$

such that the energy radiated by the charge at the point $\boldsymbol{x}$ through a solid angle δo along the direction $\boldsymbol{n}(\boldsymbol{x}, t^*)$ in a time δt^* is

$$\delta W = \boldsymbol{S^*}(\boldsymbol{x}, t^*) \cdot \boldsymbol{n}(\boldsymbol{x}, t^*)|\boldsymbol{x} - \boldsymbol{r}(t^*)|^2 \delta o \delta t^*, \tag{2.1.5}$$

which has to be equal to the detected energy expressed by

$$\delta W = \boldsymbol{S}(\boldsymbol{x}, t) \cdot \boldsymbol{n}(\boldsymbol{x}, t^*(\boldsymbol{x}, t))|\boldsymbol{x} - \boldsymbol{r}(t^*(\boldsymbol{x}, t))|^2 \delta o \delta t, \tag{2.1.6}$$

where we precise that these relations simply follow from the definition of the Poynting vector itself.

2.2 Differential and Total Radiated Power

In the previous section we stated that the energy emitted by the particle is given by the (2.1.5). That emission corresponds to an equal loss of kinetic energy, which is inevitably dissipated throughout space by the electromagnetic field. Considering the emitted energy (2.1.5), the differential radiated power corresponds to

$$\frac{dP^*}{do}(\boldsymbol{x}, t^*) = \boldsymbol{S^*}(\boldsymbol{x}, t^*) \cdot \boldsymbol{n}(\boldsymbol{x}, t^*)|\boldsymbol{x} - \boldsymbol{r}(t^*)|^2, \tag{2.2.1}$$

and, for an accelerated charge, it becomes

$$\begin{aligned}\frac{dP^*}{do}(\boldsymbol{x}, t^*) = \frac{q^2}{4\pi c}\frac{1}{(1-\boldsymbol{\beta}(t^*)\cdot \boldsymbol{n}(\boldsymbol{x}, t^*))^5}& \\ \left[\left|(\boldsymbol{n}(\boldsymbol{x}, t^*) - \boldsymbol{\beta}(t^*)) \times \dot{\boldsymbol{\beta}}(t^*)\right|^2 - \left|\boldsymbol{n}(\boldsymbol{x}, t^*) \cdot (\boldsymbol{\beta}(t^*) \times \dot{\boldsymbol{\beta}}(t^*))\right|^2\right]&.\end{aligned} \tag{2.2.2}$$

It is fundamental to keep in mind that this formula expresses the emission rate in terms of the retarded time t^* and, quantitatively, it differs from the detection rate by a factor $\alpha(\boldsymbol{x}, t^*)$, comparing (2.1.5) to the (2.1.6).

Proof. We have to combine the (2.2.1) with the (2.1.4), obtaining

$$\begin{aligned}\frac{dP^*}{do}(\boldsymbol{x}, t^*) &= \boldsymbol{S^*}(\boldsymbol{x}, t^*) \cdot \boldsymbol{n}(\boldsymbol{x}, t^*)|\boldsymbol{x} - \boldsymbol{r}(t^*)|^2 \\ &= \boldsymbol{S}(\boldsymbol{x}, t(\boldsymbol{x}, t^*))\frac{\partial t}{\partial t^*}(\boldsymbol{x}, t^*) \cdot \boldsymbol{n}(\boldsymbol{x}, t^*)|\boldsymbol{x} - \boldsymbol{r}(t^*)|^2 \\ &= \frac{c}{4\pi}|\boldsymbol{E}(\boldsymbol{x}, t(\boldsymbol{x}, t^*))|^2 \frac{\partial t}{\partial t^*}(\boldsymbol{x}, t^*)|\boldsymbol{x} - \boldsymbol{r}(t^*)|^2 \\ &= \frac{c}{4\pi}|\boldsymbol{E}(\boldsymbol{x}, t^*)|^2 \alpha(\boldsymbol{x}, t^*)|\boldsymbol{x} - \boldsymbol{r}(t^*)|^2,\end{aligned} \tag{2.2.3}$$

in which we have used the definition of $\alpha(\boldsymbol{x}, t^*)$ in (1.3.8a) as the inverse of (1.3.19a). Putting inside the electric field (1.4.11a), we can verify (2.2.2).

Expression (2.2.2) is certainly not trivial, however it can be simplified in some particular physical cases; for instance, assuming the non-relativistic limit in which $|\boldsymbol{\beta}(t^*)| \ll 1$, the emission rate (2.2.2) can be written as

$$\frac{dP^*}{do}(\boldsymbol{x}, t^*) = \frac{q^2}{4\pi c}\left|\boldsymbol{n}(\boldsymbol{x}, t^*) \times \dot{\boldsymbol{\beta}}(t^*)\right|^2. \tag{2.2.4}$$

The angular distribution of (2.2.4) with respect to the polar angle between $\boldsymbol{n}(\boldsymbol{x}, t^*)$ and $\dot{\boldsymbol{\beta}}(t^*)$ has been plotted in the figure (2.2.1) below

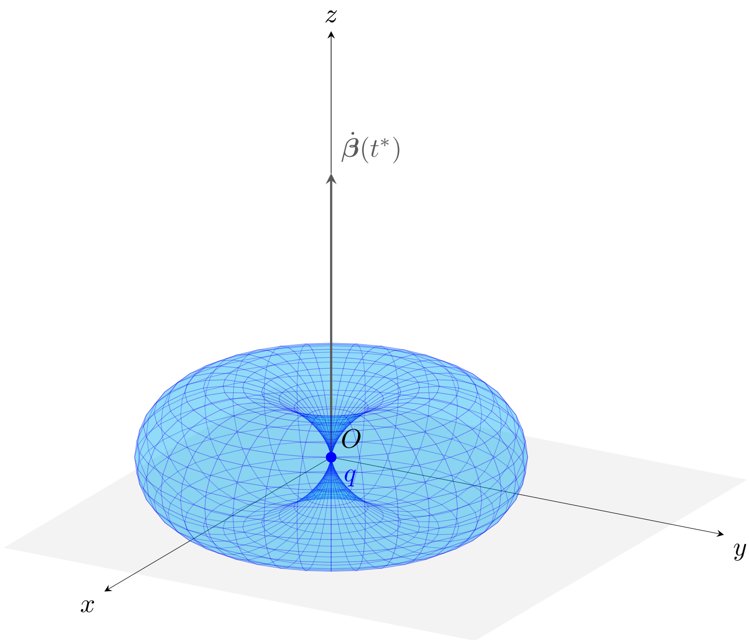

Figure 2.2.1: Emission rate of a slow moving accelerated charge q located at the origin.

Figure (2.2.1) shows that the radiation pattern resembles a doughnut, without any contribution parallel or antiparallel to $\dot{\boldsymbol{\beta}}(t^*)$ and a maximum transversal to the acceleration. Furthermore, both an accelerated or a decelerated motion have exactly the same pattern. Once the general differential radiated power (2.2.2) has been calculated, it is natural to wonder what expression the related total radiated power should assume through a spherical shell of radius $|\boldsymbol{x} - \boldsymbol{r}(t^*)|$ around the particle position $\boldsymbol{r}(t^*)$.
Integrating (2.2.2) throughout the shell, we find the following relation

$$P^*(t^*) = \frac{2q^2}{3c} \frac{1}{(1 - |\boldsymbol{\beta}(t^*)|^2)^3} \left[|\dot{\boldsymbol{\beta}}(t^*)|^2 - |\boldsymbol{\beta}(t^*) \times \dot{\boldsymbol{\beta}}(t^*)|^2 \right]. \tag{2.2.5}$$

This expression is the so-called Liénard generalization of the Larmor formula, indeed, if we assume the non-relativistic regime, equation (2.2.5) reduces to

$$P^*(t^*) = \frac{2q^2}{3c} |\dot{\boldsymbol{\beta}}(t^*)|^2, \tag{2.2.6}$$

which effectively corresponds to the celebrated Larmor formula that describes the power emitted by a slowly moving point-like charge; of course, it could be derived alternatively by integrating (2.2.4). Lastly, in case of a slightly variable source, retardation becomes almost negligible in such a way that $t^* \to t$, as $c \to \infty$.

Proof. We will now compute the total radiated power through the shell at a fixed t^*. Firstly, we notice that the differential emission rate depends on $\boldsymbol{x}$ only through $\boldsymbol{n}(\boldsymbol{x}, t^*)$

$$\frac{dP^*}{do}(\boldsymbol{x}, t^*) = \frac{dP^*}{do}(\boldsymbol{n}(\boldsymbol{x}, t^*), t^*), \tag{2.2.7}$$

hence the total radiated power over the spherical shell can be computed straightforwardly by integrating throughout the $\boldsymbol{n}$'s orientations

$$P^*(t^*) = \oint d^2\boldsymbol{n} \frac{dP^*}{do}(\boldsymbol{n}, t^*). \tag{2.2.8}$$

Substituting the differential emission rate (2.2.2), yields

$$P^*(t^*) = \frac{q^2}{4\pi c} \oint d^2\boldsymbol{n} \frac{1}{(1 - \boldsymbol{\beta} \cdot \boldsymbol{n})^5} \left\{ \left[(\boldsymbol{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]^2 - \left[\boldsymbol{n} \cdot (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})\right]^2 \right\}, \tag{2.2.9}$$

where the time dependence has been omitted for the sake of clarity. Now we are going to rearrange the summed terms within the integral so that

$$\begin{aligned} \left[(\boldsymbol{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]^2 - \left[\boldsymbol{n} \cdot (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})\right]^2 &= \left[\boldsymbol{n} \times \dot{\boldsymbol{\beta}} - \boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}\right]^2 - \left[\boldsymbol{n} \cdot (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})\right]^2 \\ &= (\boldsymbol{n} \times \dot{\boldsymbol{\beta}})^2 + (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})^2 - 2(\boldsymbol{n} \times \dot{\boldsymbol{\beta}}) \cdot (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}) \\ &- (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}) \cdot \boldsymbol{n} \otimes \boldsymbol{n} \cdot (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}) \\ &= (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}) \cdot (\mathbb{1} - \boldsymbol{n} \otimes \boldsymbol{n}) \cdot (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}) \\ &+ \dot{\boldsymbol{\beta}} \cdot (\mathbb{1} - \boldsymbol{n} \otimes \boldsymbol{n}) \cdot \dot{\boldsymbol{\beta}} - 2\boldsymbol{n} \cdot (\dot{\boldsymbol{\beta}} \times (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})). \end{aligned} \tag{2.2.10}$$

The integral in (2.2.9), which we will call $I(\boldsymbol{\beta}, \dot{\boldsymbol{\beta}})$, can be cast as it follows

$$\begin{aligned} I(\boldsymbol{\beta}, \dot{\boldsymbol{\beta}}) &= \oint d^2\boldsymbol{n} \frac{1}{(1 - \boldsymbol{\beta} \cdot \boldsymbol{n})^5} \Big[(\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}) \cdot (\mathbb{1} - \boldsymbol{n} \otimes \boldsymbol{n}) \cdot (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}) \\ &+ \dot{\boldsymbol{\beta}} \cdot (\mathbb{1} - \boldsymbol{n} \otimes \boldsymbol{n}) \cdot \dot{\boldsymbol{\beta}} - 2\boldsymbol{n} \cdot (\dot{\boldsymbol{\beta}} \times (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}))\Big] \\ &= (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}) \cdot \oint d^2\boldsymbol{n} \frac{\mathbb{1} - \boldsymbol{n} \otimes \boldsymbol{n}}{(1 - \boldsymbol{\beta} \cdot \boldsymbol{n})^5} \cdot (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}) \\ &+ \dot{\boldsymbol{\beta}} \cdot \oint d^2\boldsymbol{n} \frac{\mathbb{1} - \boldsymbol{n} \otimes \boldsymbol{n}}{(1 - \boldsymbol{\beta} \cdot \boldsymbol{n})^5} \cdot \dot{\boldsymbol{\beta}} - \oint d^2\boldsymbol{n} \frac{2\boldsymbol{n}}{(1 - \boldsymbol{\beta} \cdot \boldsymbol{n})^5} \cdot (\dot{\boldsymbol{\beta}} \times (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})) \\ &= (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}) \cdot \mathbb{I}_2(\boldsymbol{\beta}, \dot{\boldsymbol{\beta}}) \cdot (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}) \end{aligned} \tag{2.2.11}$$

$$+ \dot{\boldsymbol{\beta}} \cdot \mathbb{I}_2(\boldsymbol{\beta}, \dot{\boldsymbol{\beta}}) \cdot \dot{\boldsymbol{\beta}} - 2\boldsymbol{I}_1(\boldsymbol{\beta}, \dot{\boldsymbol{\beta}}) \cdot (\dot{\boldsymbol{\beta}} \times (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})).$$

At this point, we aim to find the value of $\boldsymbol{I}_1(\boldsymbol{\beta}, \dot{\boldsymbol{\beta}})$ and $\mathbb{I}_2(\boldsymbol{\beta}, \dot{\boldsymbol{\beta}})$ below

$$\boldsymbol{I}_1(\boldsymbol{\beta}, \dot{\boldsymbol{\beta}}) = \oint d^2\boldsymbol{n} \frac{\boldsymbol{n}}{(1 - \boldsymbol{\beta} \cdot \boldsymbol{n})^5}, \tag{2.2.12a}$$

$$\mathbb{I}_2(\boldsymbol{\beta}, \dot{\boldsymbol{\beta}}) = \oint d^2\boldsymbol{n} \frac{\mathbb{1} - \boldsymbol{n} \otimes \boldsymbol{n}}{(1 - \boldsymbol{\beta} \cdot \boldsymbol{n})^5}. \tag{2.2.12b}$$

Once computed these integrals, it is not difficult to insert them into (2.2.11), extracting the total radiated power which we purposed to determine.
Firstly, we notice that $\boldsymbol{I}_1(\boldsymbol{\beta}, \dot{\boldsymbol{\beta}})$ and $\mathbb{I}_2(\boldsymbol{\beta}, \dot{\boldsymbol{\beta}})$ can be written in the following manner

$$\boldsymbol{I}_1(\boldsymbol{\beta}, \dot{\boldsymbol{\beta}}) = \frac{1}{4} \boldsymbol{\nabla}_{\boldsymbol{\beta}} \oint d^2\boldsymbol{n} \frac{1}{(1 - \boldsymbol{\beta} \cdot \boldsymbol{n})^4}, \tag{2.2.13a}$$

$$\mathbb{I}_2(\boldsymbol{\beta}, \dot{\boldsymbol{\beta}}) = \oint d^2\boldsymbol{n} \frac{1}{(1 - \boldsymbol{\beta} \cdot \boldsymbol{n})^5} \mathbb{1} - \frac{1}{12} \boldsymbol{\nabla}_{\boldsymbol{\beta}} \otimes \boldsymbol{\nabla}_{\boldsymbol{\beta}} \oint d^2\boldsymbol{n} \frac{1}{(1 - \boldsymbol{\beta} \cdot \boldsymbol{n})^3}, \tag{2.2.13b}$$

where the first integral is identified by a row vector, while the second by a 3×3 matrix.
As we can observe, it is sufficient to compute the same unique integral

$$J_p(\boldsymbol{\beta}) = \oint d^2\boldsymbol{n} \frac{1}{(1 - \boldsymbol{\beta} \cdot \boldsymbol{n})^p}, \tag{2.2.14}$$

that can be solved in polar spherical coordinates by employing a reference frame in which $\boldsymbol{\beta}$ stays along the polar axis, as shown immediately

$$\begin{aligned} J_p(\boldsymbol{\beta}) = \oint d^2\boldsymbol{n} \frac{1}{(1 - \boldsymbol{\beta} \cdot \boldsymbol{n})^p} &= \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \frac{1}{(1 - |\boldsymbol{\beta}| \cos\theta)^p} \\ &= 2\pi \int_{-1}^{1} du \frac{1}{(1 - |\boldsymbol{\beta}| u)^p} \\ &= \frac{2\pi}{p - 1} \frac{1}{|\boldsymbol{\beta}|} \left(\frac{1}{(1 - |\boldsymbol{\beta}|)^{p-1}} - \frac{1}{(1 + |\boldsymbol{\beta}|)^{p-1}} \right), \end{aligned} \tag{2.2.15}$$

where the variable change $u = \cos\theta$ has been performed.
This formula finally allows to evaluate the above integrals and a straight computation provides the following expressions

$$\boldsymbol{I}_1(\boldsymbol{\beta}, \dot{\boldsymbol{\beta}}) = \frac{4\pi}{3} \frac{5 + |\boldsymbol{\beta}|^2}{(1 - |\boldsymbol{\beta}|^2)^4} \boldsymbol{\beta}, \tag{2.2.16a}$$

$$\mathbb{I}_2(\boldsymbol{\beta}, \dot{\boldsymbol{\beta}}) = \frac{8\pi}{3} \frac{(1 + 2|\boldsymbol{\beta}|^2)\mathbb{1} - 3\boldsymbol{\beta} \otimes \boldsymbol{\beta}}{(1 - |\boldsymbol{\beta}|^2)^4}, \tag{2.2.16b}$$

in which we have directly calculated the relations (2.2.13), substituting the related integrals given by (2.2.14) at different p's.
Substituting these expressions into (2.2.11), we shall obtain

$$
\begin{aligned}
I(\boldsymbol{\beta}, \dot{\boldsymbol{\beta}}) &= (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}) \cdot \frac{8\pi}{3} \frac{(1+2|\boldsymbol{\beta}|^2)\mathbb{1} - 3\boldsymbol{\beta} \otimes \boldsymbol{\beta}}{(1-|\boldsymbol{\beta}|^2)^4} \cdot (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}) \\
&\quad + \dot{\boldsymbol{\beta}} \cdot \frac{8\pi}{3} \frac{(1+2|\boldsymbol{\beta}|^2)\mathbb{1} - 3\boldsymbol{\beta} \otimes \boldsymbol{\beta}}{(1-|\boldsymbol{\beta}|^2)^4} \cdot \dot{\boldsymbol{\beta}} - \frac{8\pi}{3} \frac{5+|\boldsymbol{\beta}|^2}{(1-|\boldsymbol{\beta}|^2)^4} \boldsymbol{\beta} \cdot (\dot{\boldsymbol{\beta}} \times (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})) \\
&= \frac{8\pi}{3} \frac{|\dot{\boldsymbol{\beta}}|^2 - |\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}|^2}{(1-|\boldsymbol{\beta}|^2)^3},
\end{aligned}
\tag{2.2.17}
$$

consequently, inserting this result into (2.2.9), we conclude the proof.

Chapter 3

Radiation Reaction and Abraham-Lorentz Force

3.1 Abraham-Lorentz Force

In this chapter we aim to analyze the self-force experienced by a particle, following to its radiative emission. It is evident that handling problems in electrodynamics neglecting the emission of radiation can be of only approximate validity, indeed the emitted radiation carries off energy and momentum from the particle influencing its motion. This reaction is negligible compared to the external fields effect, so much that it is usually ignored. Furthermore, a complete satisfactory treatment of the radiation reaction does not exist in classical electrodynamics.

Despite all these facts, it is possible to provide partial solutions workable within limited areas and that is what we are going to deal with. A first approach could be to exploit the Lorentz force evaluating it at the particle instantaneous position, but it would not make any sense, because the fields (1.3.1) are not defined at that location.

For the sake of simplicity, we set the non-relativistic limit where the particle's speed is much slower than the light's one. Neglecting the emission of radiation, Newton's second law of dynamics states that

$$\boldsymbol{F}_e = m\dot{\boldsymbol{v}}, \tag{3.1.1}$$

where $\boldsymbol{F}_e$ is the external force acting on the particle, while $\dot{\boldsymbol{v}}$ is its acceleration.

It is now possible to include the radiation reaction in a Newtonian picture by introducing a frictional force $\boldsymbol{F}_r$, in response to the power emission given by the Larmor result (2.2.6). Newton's law should be then recast as shown below

$$\boldsymbol{F}_e + \boldsymbol{F}_r = m\dot{\boldsymbol{v}}, \tag{3.1.2}$$

where, selecting the most suitable expression for $\boldsymbol{F}_r$, we shall obtain the Abraham-Lorentz equation of motion

$$\boldsymbol{F}_e = m(\dot{\boldsymbol{v}} - \tau\ddot{\boldsymbol{v}}), \qquad \tau = \frac{2q^2}{3mc^3}. \tag{3.1.3}$$

This equation includes in some effective approximate way the radiative contribution to the particle's motion. Since it depends on the third derivative of the position with respect to time, it runs counter to the normal requirements of a dynamical equation of motion and this fact leads to many critical issues, making this equation unreliable, except in some physical cases.

Proof. We will determine the form of the frictional force $\boldsymbol{F}_r$ by considering the derivative of the particle's kinetic energy with respect to time

$$\dot{K} = \boldsymbol{F}_e \cdot \boldsymbol{v} - P, \tag{3.1.4}$$

where $-P$ represents the negative Larmor power contribute (2.2.6) due to radiative effects. Upon establishing the power balance, we aim to impute all these effects to a unique frictional force $\boldsymbol{F}_r$ in such a way that

$$\dot{K} = \boldsymbol{F}_e \cdot \boldsymbol{v} + \boldsymbol{F}_r \cdot \boldsymbol{v}. \tag{3.1.5}$$

In order to find the most suitable expression for this sort of radiation reaction force, we may write $-P$ as the dot product of the latter $\boldsymbol{F}_r$ with the velocity $\boldsymbol{v}$. In this way, the power loss turns out to be

$$\begin{aligned} -P = -\frac{2q^2}{3c^3}\dot{\boldsymbol{v}} &= -\frac{2q^2}{3c^3}\left[\frac{1}{2}(\ddot{\boldsymbol{v}^2}) - \ddot{\boldsymbol{v}} \cdot \boldsymbol{v}\right] \\ &= m\tau\ddot{\boldsymbol{v}} \cdot \boldsymbol{v} - \tau\ddot{K}, \end{aligned} \tag{3.1.6}$$

in which we have introduced the constant $\tau = 2q^2/3mc^3$, proportional to the time taken by a light signal to cross the classical particle radius q^2/mc^2. Finally, the relation (3.1.5) can be written as

$$\dot{K} = \boldsymbol{F}_e \cdot \boldsymbol{v} + m\tau\ddot{\boldsymbol{v}} \cdot \boldsymbol{v} - \tau\ddot{K}. \tag{3.1.7}$$

The last term is not yet expressed as we were wondering, hence we aim to check under what conditions it can be neglected, taking into account the other summed terms.
At the beginning, we have to verify in which cases the condition below is satisfied

$$\tau|\ddot{K}| \ll |\dot{K}|, \tag{3.1.8}$$

hence, applying the chain rule, yields

$$\frac{\tau|\ddot{K}|}{|\dot{K}|} = \frac{\tau|\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{v}} + \boldsymbol{v} \cdot \ddot{\boldsymbol{v}}|}{|\boldsymbol{v} \cdot \dot{\boldsymbol{v}}|} \ll 1. \tag{3.1.9}$$

At this point, we should compare each contribute singularly to finally get the following application requirements

$$\tau|\dot{\boldsymbol{v}}| \ll |\boldsymbol{v}|, \tag{3.1.10a}$$

$$\tau|\ddot{\boldsymbol{v}}| \ll |\dot{\boldsymbol{v}}|. \tag{3.1.10b}$$

Considering the non-relativistic limit, one may obtain that

$$\tau^2|\ddot{\boldsymbol{v}}| \ll \tau|\dot{\boldsymbol{v}}| \ll |\boldsymbol{v}| \ll c. \tag{3.1.11}$$

Now we have to verify the following inequality to complete our request

$$\ddot{K} \ll m|\boldsymbol{v} \cdot \ddot{\boldsymbol{v}}|, \tag{3.1.12}$$

then, if this condition holds, one has that

$$m\boldsymbol{v} \cdot \ddot{\boldsymbol{v}} - \ddot{K} = -m|\dot{\boldsymbol{v}}|^2 \approx m\boldsymbol{v} \cdot \ddot{\boldsymbol{v}} \tag{3.1.13}$$
$$\rightarrow m\boldsymbol{v} \cdot \ddot{\boldsymbol{v}} < 0.$$

Lastly, neglecting the $\tau\ddot{K}$ term in (3.1.7), we can identify the force $\boldsymbol{F}_r$ with

$$\boldsymbol{F}_r = \frac{2q^2}{3c^3}\ddot{\boldsymbol{v}}, \tag{3.1.14}$$

where we precise that $\boldsymbol{F}_r \cdot \boldsymbol{v} < 0$, definitely proving its frictional behaviour.

At this point it is natural to discuss the application limits of the radiation reaction comparing it to the other external forces, that we estimate to be extremely more intense than the previous one, as suggested by (3.1.10b). The Abraham-Lorentz equation (3.1.3) therefore reduces to the so-called Landau-Lifschitz form of the radiation reaction

$$\boldsymbol{F}_e + \tau\dot{\boldsymbol{F}}_e \approx m\dot{\boldsymbol{v}}. \tag{3.1.15}$$

The external forces, which we suppose to be electromagnetic, are given by the Lorentz force

$$\boldsymbol{F}_e = q\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}\right), \tag{3.1.16}$$

and, since in this context $|\boldsymbol{F}_r| \ll |\boldsymbol{F_e}|$, one would get that the radiation reaction force is appreciable only when the following condition is verified

$$\frac{\tau q}{mc}|\boldsymbol{B}| \ll 1. \tag{3.1.17}$$

Proof. First of all, we are allowed to neglect the self-force inside the Abraham-Lorentz equation of motion (3.1.3), and, performing a derivative with respect to time, we find

$$\dot{\boldsymbol{F}}_e \approx m\ddot{\boldsymbol{v}} \approx \frac{1}{\tau}\boldsymbol{F}_r, \tag{3.1.18}$$

that, together with the (3.1.3), proves (3.1.15). Combining it with (3.1.16), yields

$$\boldsymbol{F}_r \approx \tau q \left(\dot{\boldsymbol{E}} + \frac{1}{c}\dot{\boldsymbol{v}} \times \boldsymbol{B} + \frac{1}{c}\boldsymbol{v} \times \dot{\boldsymbol{B}}\right), \tag{3.1.19}$$

moreover, since $\boldsymbol{F}_r$ is pretty small, we can assume anyway that

$$\dot{\boldsymbol{v}} \approx \frac{q}{m}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}\right), \tag{3.1.20}$$

therefore, equation (3.1.19) turns out to be

$$\boldsymbol{F}_r \approx \tau q \left[\dot{\boldsymbol{E}} + \frac{q}{mc}\boldsymbol{E} \times \boldsymbol{B} + \frac{q}{mc^2}(\boldsymbol{v} \times \boldsymbol{B}) \times \boldsymbol{B} + \frac{1}{c}\boldsymbol{v} \times \dot{\boldsymbol{B}}\right]. \tag{3.1.21}$$

When $|\boldsymbol{\beta}| \ll 1$, the previous relation becomes

$$\boldsymbol{F}_r \approx \tau q \left[\dot{\boldsymbol{E}} + \frac{q}{mc}\boldsymbol{E} \times \boldsymbol{B}\right], \tag{3.1.22}$$

consequently the condition $|\boldsymbol{F}_r| \ll q|\boldsymbol{E}|$ guarantees the (3.1.17) by comparing, singularly, each contribute of $\boldsymbol{F}_r$.

3.2 Runaway Solutions and Causality Violation

In this section, we will consider the previously anticipated difficulties arising from the Abraham-Lorentz equation of motion (3.1.3) with the aim of exploring possible pathological solutions and to discuss its behaviour in some relevant cases.
At the beginning, we set the external force $\boldsymbol{F}_e = 0$, hence the equation of motion gives

$$\tau \ddot{\boldsymbol{v}} = \dot{\boldsymbol{v}}. \tag{3.2.1}$$

This equation leads to runaway solutions which do not have any physical meaning, indeed one finds that

$$\dot{\boldsymbol{v}}(t) = \boldsymbol{a}_0 e^{t/\tau}. \tag{3.2.2}$$

Except when $\boldsymbol{a}_0 = 0$, this solution is clearly not acceptable since it would violate the energy conservation principle. Even a small initial acceleration $\boldsymbol{a}_0$ could give rise to an infinite accelerated motion and the term $\tau|\dot{\boldsymbol{v}}|$ would turn to be much greater than the speed $|\boldsymbol{v}|$, losing any validity on the assumptions made to build our self-force (3.1.10a). Moreover, when the acceleration is constant, it does not seem to be any radiation reaction, even if the particle continues to emit radiation, showing a contradictory behaviour.
It is clear that the Abraham-Lorentz force could be useful only in the domain where the radiative term represents a small perturbation in the state of motion of the particle, as suggested by the Landau-Lifshitz equation of motion (3.1.15).
The situation becomes even worse when a time-dependent force $\boldsymbol{F}_e \neq 0$ is applied. In this condition, the laws of motion lead to violations of the principle of causality, indeed

$$\boldsymbol{F}_e(t) = m\dot{\boldsymbol{v}} - m\tau\ddot{\boldsymbol{v}}, \tag{3.2.3}$$

that, in this case, can be solved by the following integral expression for the acceleration

$$\dot{\boldsymbol{v}}(t) = \frac{1}{m\tau}\int_t^{\infty} dt' e^{-(t'-t)/\tau}\boldsymbol{F}_e(t'). \tag{3.2.4}$$

It is evident that the acceleration at time t depends on an exponentially damped $\boldsymbol{F}_e$, evaluated at all future times $t' > t$; the particle basically pre-accelerates, violating causality.

Proof. The Abraham-Lorentz equation of motion (3.2.3), in presence of an external force, can be solved by the ansatz

$$\dot{\boldsymbol{v}}(t) = e^{t/\tau}\boldsymbol{u}(t), \tag{3.2.5}$$

which derivative with respect to time yields the relation

$$\begin{aligned} \ddot{\boldsymbol{v}} &= \frac{1}{\tau}e^{t/\tau}\boldsymbol{u} + e^{t/\tau}\dot{\boldsymbol{u}} \\ \dot{\boldsymbol{v}} - \tau\ddot{\boldsymbol{v}} &= -\tau e^{t/\tau}\dot{\boldsymbol{u}}. \end{aligned} \tag{3.2.6}$$

Comparing it to the (3.2.3), we observe that

$$\dot{\boldsymbol{u}}(t) = -\frac{e^{-t/\tau}}{m\tau}\boldsymbol{F}_e(t). \tag{3.2.7}$$

A straightforward integration gives the result

$$\dot{\boldsymbol{u}}(t) = \dot{\boldsymbol{u}}(t_0) - \frac{1}{m\tau}\int_{t_0}^{t} dt' e^{-t'/\tau}\boldsymbol{F}_e(t'), \tag{3.2.8}$$

hence, the acceleration $\dot{\boldsymbol{v}}(t)$ can be written as

$$\dot{\boldsymbol{v}}(t) = e^{t/\tau}\dot{\boldsymbol{u}}(t_0) - \frac{1}{m\tau}\int_{t_0}^{t} dt' e^{-(t'-t)/\tau}\boldsymbol{F}_e(t'). \tag{3.2.9}$$

Imposing $\dot{\boldsymbol{u}}(t_0) = 0$ and $t_0 = +\infty$, we prove the (3.2.4).

Understanding the pre-acceleration phenomenon would be easier through a practical example. We take, for instance, an external force built as it follows

$$\boldsymbol{F}_e(t) = \theta(t-T)\boldsymbol{F}_0, \tag{3.2.10}$$

where $\theta(s)$ denotes the Heaviside step function and $\boldsymbol{F}_0$ is a constant vector. If we compute $\dot{\boldsymbol{v}}(t)$ for $t < T$ using the formula (3.2.4), we shall obtain

$$\dot{\boldsymbol{v}}(t) = \frac{1}{m\tau}\int_{T}^{\infty} dt' e^{-(t'-t)/\tau}\boldsymbol{F}_0 = \frac{1}{m}e^{-(T-t)/\tau}\boldsymbol{F}_0, \tag{3.2.11}$$

clearly, there is a non-vanishing acceleration even if the force has not been applied yet, violating causality. Definitely, Abraham-Lorentz force is pathological.
Although the proposed derivation of the Abraham-Lorentz force, while plausible, is nei-

ther rigorous nor fundamental, providing a satisfactory description of the radiation reaction would certainly involve the most intimate aspects of the particles related to their internal structure, that a classical theory cannot examine without falling in ambiguities. In the next section we will see the main example which represents the definite decline of the classical electrodynamics and the consequent birth of quantum physics.

3.3 The Classical Atom

In this conclusive section we will deal with a well-known practical example of the radiation reaction which makes manifest the failure of classical electrodynamics to describe atomic physics. We think about the simplest atom as an electrical system constituted by a fixed charged nucleus and an electron orbiting around it. The electron accelerates and, as it radiates, it spirals into the nucleus. We aim to find at least an approximate solution for the electron's equation of motion, taking into account the radiation reaction, shown in the previous section, as a small perturbation to the unperturbed circular trajectory due to the scalar Coulomb's potential arising from the nucleus.
Given a central force $\boldsymbol{F}_e$, it can be written as the gradient of a scalar potential energy $U(|\boldsymbol{x}|)$ in such a way that

$$\boldsymbol{F}_e = -\boldsymbol{\nabla} U(|\boldsymbol{x}|) = -U'(|\boldsymbol{x}|)\frac{\boldsymbol{x}}{|\boldsymbol{x}|}, \tag{3.3.1}$$

where $U'(s)$ stands for the derivative of $U(s)$ with respect to its argument. In the context of the classical Bohr hydrogen atom, this potential energy is Coulombic and equal to

$$U(\boldsymbol{x}) = -\frac{e^2}{|\boldsymbol{x}|}, \tag{3.3.2}$$

where e denotes the elementary charge. The electron's instantaneous position in plane polar coordinates is expressed by the relations below

$$\boldsymbol{r}(t) = a_0\left(1 - \frac{6\tau e^2}{m a_0^3}t\right)^{\frac{1}{3}}\boldsymbol{e}_\rho, \tag{3.3.3a}$$

$$\theta(t) = \sqrt{\frac{m a_0^3}{9\tau^2 e^2}}\left(1 - \sqrt{1 - \frac{6\tau e^2}{m a_0^3}t}\right), \tag{3.3.3b}$$

where m is the reduced mass of the proton-electron system and a_0 is the Bohr radius.
It is clear how these relations describe a spiral orbit centered on the nucleus simply by observing the radial position's decline, directly proportional to time t. The plot of the electron's resized motion described by the laws (3.3.3) is shown in figure (3.3.1).

Proof. Proof will be divided into two steps.
Step 1. First of all, we calculate the total time derivative of the force (3.3.1) evaluated

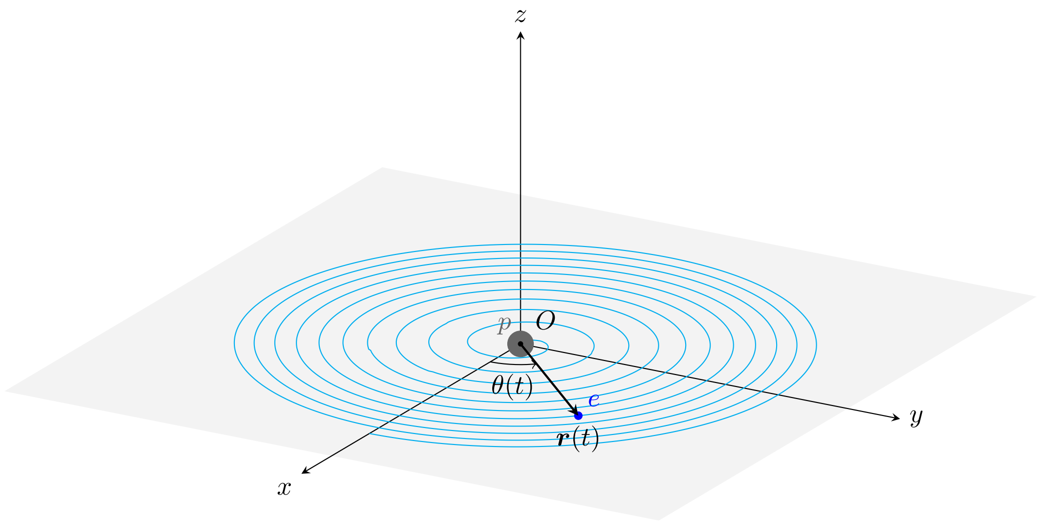

Figure 3.3.1: The electron e, located at $\boldsymbol{r}(t)$, spirals towards the proton p in the origin.

at the particle position $\boldsymbol{r}(t)$; the reason will be soon much clearer

$$\dot{\boldsymbol{F}}_e = -U'(|\boldsymbol{r}|)\frac{\dot{\boldsymbol{r}}}{|\boldsymbol{r}|} + \left[\frac{U'(|\boldsymbol{r}|)}{|\boldsymbol{r}|} - U''(|\boldsymbol{r}|)\right]\frac{\boldsymbol{r}\cdot\dot{\boldsymbol{r}}}{|\boldsymbol{r}|^2}\boldsymbol{r}. \tag{3.3.4}$$

Similarly, by (3.1.2), the derivative of the kinetic energy K with respect to time is

$$\dot{K} = \boldsymbol{F}_e\cdot\dot{\boldsymbol{r}} + \boldsymbol{F}_r\cdot\dot{\boldsymbol{r}} \approx \boldsymbol{F}_e\cdot\dot{\boldsymbol{r}} + \tau\dot{\boldsymbol{F}}_e\cdot\dot{\boldsymbol{r}}, \tag{3.3.5}$$

where we have considered the Landau-Lifschitz form of the radiation reaction (3.1.15). Substituting (3.3.1) and (3.3.4) into the above relation, yields

$$\dot{K} = -U'(|\boldsymbol{r}|)\frac{\boldsymbol{r}\cdot\dot{\boldsymbol{r}}}{|\boldsymbol{r}|} - \tau U'(|\boldsymbol{r}|)\frac{|\dot{\boldsymbol{r}}|^2}{|\boldsymbol{r}|} + \tau\left[\frac{U'(|\boldsymbol{r}|)}{|\boldsymbol{r}|} - U''(|\boldsymbol{r}|)\right]\left(\frac{\boldsymbol{r}\cdot\dot{\boldsymbol{r}}}{|\boldsymbol{r}|}\right)^2. \tag{3.3.6}$$

Normally, the central field (3.3.2) would guarantee circular orbits, but the radiation reaction modifies them to become a spiral centered on the nucleus.

The last term in (3.3.6) is proportional to $\boldsymbol{r}\cdot\dot{\boldsymbol{r}}$ which is actually small if we introduce an adiabatic approximation that allows to treat the motion as a succession of nearly circular orbits at each time t, accordingly to our previous assumptions on the radiation reaction force applicability.

Quantitatively, this approximation is stated through the condition

$$\tau\frac{\boldsymbol{r}\cdot\dot{\boldsymbol{r}}}{|\boldsymbol{r}|^2} \ll 1. \tag{3.3.7}$$

In light of this assumption, we neglect the last term in (3.3.6), obtaining

$$\begin{aligned}
\dot{K} &= -U'(|\boldsymbol{r}|)\frac{\boldsymbol{r}\cdot\dot{\boldsymbol{r}}}{|\boldsymbol{r}|} - \tau U'(|\boldsymbol{r}|)\frac{|\dot{\boldsymbol{r}}|^2}{|\boldsymbol{r}|} \\
\dot{K} &= -\dot{U} - \tau U'(|\boldsymbol{r}|)\frac{|\dot{\boldsymbol{r}}|^2}{|\boldsymbol{r}|} \\
\dot{E} &= -\frac{\tau U'(|\boldsymbol{r}|)}{|\boldsymbol{r}|}\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}},
\end{aligned} \tag{3.3.8}$$

where, in the last statement, we used the definition of the mechanical energy $E = K + U$. We will exploit this definition of the classical mechanical energy to write it extensively

$$E = K + U = \frac{1}{2}m|\dot{\boldsymbol{r}}|^2 - \frac{e^2}{|\boldsymbol{r}|}, \tag{3.3.9}$$

hence, computing $\dot{E}$, we have

$$\dot{E} = m\ddot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} + \frac{e^2}{|\boldsymbol{r}|^3}\boldsymbol{r}\cdot\dot{\boldsymbol{r}}. \tag{3.3.10}$$

Combining the last relation with (3.3.8), we get the following law of motion

$$m\ddot{\boldsymbol{r}} = -\frac{e^2}{|\boldsymbol{r}|^3}\boldsymbol{r} - \tau\frac{U'(|\boldsymbol{r}|)}{|\boldsymbol{r}|}\dot{\boldsymbol{r}}, \tag{3.3.11}$$

where we observe, on the left, the classical Coulomb force joined, on the right, by a frictional term proportional to the particle's velocity.

Step 2. Let's perform a basis change, from the canonical to the plane polar one. The new basis is related to the previous one through the transformations

$$\boldsymbol{e}_\rho = \cos\theta\boldsymbol{e}_1 + \sin\theta\boldsymbol{e}_2 \tag{3.3.12a}$$

$$\boldsymbol{e}_\theta = -\sin\theta\boldsymbol{e}_1 + \cos\theta\boldsymbol{e}_2. \tag{3.3.12b}$$

The position is now expressed by $\boldsymbol{r} = \rho\boldsymbol{e}_\rho$, while $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ become respectively

$$\dot{\boldsymbol{r}} = \dot{\rho}\boldsymbol{e}_\rho + \rho\dot{\theta}\boldsymbol{e}_\theta, \tag{3.3.13a}$$

$$\ddot{\boldsymbol{r}} = (\ddot{\rho} - \rho\dot{\theta}^2)\boldsymbol{e}_\rho + (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta})\boldsymbol{e}_\theta. \tag{3.3.13b}$$

which represent the velocity and the acceleration in polar plane coordinates.

Taking into account these expressions and projecting the laws of motion (3.3.11) on the polar basis, we shall obtain

$$m(\ddot{\rho} - \rho\dot{\theta}^2) = -\frac{e^2}{\rho^2} - \tau\dot{\rho}\frac{U'(\rho)}{\rho}, \tag{3.3.14a}$$

$$m(\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta}) = -\tau U'(\rho)\dot{\theta}, \tag{3.3.14b}$$

where we have to compute the derivative of the potential (3.3.2) with respect to its argument as shown below

$$m(\ddot{\rho} - \rho\dot{\theta}^2) = -\frac{e^2}{\rho^2} - \tau\dot{\rho}\frac{e^2}{\rho^3}, \tag{3.3.15a}$$

$$m(\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta}) = -\tau\frac{e^2}{\rho^2}\dot{\theta}. \tag{3.3.15b}$$

Equations (3.3.15) are still rather complicated, therefore we need to remind the adiabatic approximation, in which we suppose the orbit remains nearly circular at all times; consequently we have

$$\dot{\theta}^2 \approx \frac{e^2}{m\rho^3}. \tag{3.3.16}$$

Considering the $\dot{\theta}$ formula and performing some calculations, one can rearrange the relation (3.3.15b), getting

$$\dot{\rho} = -\frac{2\tau e^2}{m\rho^2}, \tag{3.3.17}$$

that can be integrated in order to extract the radial coordinate

$$\rho(t) = \rho_0\left(1 - \frac{6\tau e^2}{m\rho_0^3}t\right)^{\frac{1}{3}}. \tag{3.3.18}$$

Finally, the initial radius ρ_0 naturally identifies with the Bohr radius a_0, allowing us to prove the expression (3.3.3a).

At this point, it remains to check the angle time dependence (3.3.3b) that simply follows by integrating the (3.3.16), in which the radius is given by the formula (3.3.18). Furthermore, we arbitrarily set the initial angle to be $\theta_0 = 0$, thus

$$\theta(t) = \theta_0 + \sqrt{\frac{e^2}{m\rho_0^3}}\int_0^t dt'\left(1 - \frac{6\tau e^2}{m\rho_0^3}t'\right)^{-\frac{1}{2}} \tag{3.3.19}$$

$$= \sqrt{\frac{m\rho_0^3}{9\tau^2 e^2}} \left(1 - \sqrt{1 - \frac{6\tau e^2}{m\rho_0^3} t}\right),$$

then, identifying the Bohr radius as done before, we prove the (3.3.3b).

One may ask what would happen to the electron's angular momentum, which can be derived from the results obtained before

$$\boldsymbol{l}(t) = \boldsymbol{l}_0 \left(1 - \frac{6\tau e^2}{m a_0^3} t\right)^{\frac{1}{6}}; \tag{3.3.20}$$

evidently, this quantity is no longer conserved, contrary to predictions in the absence of radiative effects.

Proof We aim to derive the angular momentum (3.3.20) of the spiraling electron. Without any frictional term, the angular momentum would be an integral of motion, but this is not the case.

The angular momentum is defined as $\boldsymbol{l} = \boldsymbol{r} \times m\dot{\boldsymbol{r}}$, then, keeping in mind the Landau-Lifschitz equation of motion (3.1.15) and combining them together, we compute its derivative with respect to time that corresponds to be

$$\begin{aligned} \dot{\boldsymbol{l}} &= \boldsymbol{r} \times \boldsymbol{F}_e + \boldsymbol{r} \times \boldsymbol{F}_r \\ &\approx \boldsymbol{r} \times \boldsymbol{F}_e + \tau \boldsymbol{r} \times \dot{\boldsymbol{F}}_e \\ &\approx -\tau U'(|\boldsymbol{r}|) \frac{\boldsymbol{r}}{|\boldsymbol{r}|} \times \dot{\boldsymbol{r}}. \end{aligned} \tag{3.3.21}$$

We have to write it in polar coordinates, analogously to what we did in the last proof

$$\begin{aligned} \dot{\boldsymbol{l}} &= -\tau U'(|\boldsymbol{r}|) \frac{\boldsymbol{r}}{m|\boldsymbol{r}|} \times m\dot{\boldsymbol{r}} \\ &= -\tau U'(\rho) \frac{\boldsymbol{r}}{m\rho} \times m\dot{\boldsymbol{r}} \\ &= -\tau \frac{e^2}{m\rho^3} \boldsymbol{l}. \end{aligned} \tag{3.3.22}$$

In order to solve this equation, it is necessary to insert the $\rho(t)$ formula (3.3.18), hence

$$\dot{\boldsymbol{l}} = -\frac{\tau e^2}{m\rho_0^3} \left(1 - \frac{6\tau e^2}{m\rho_0^3} t\right)^{-1} \boldsymbol{l}, \tag{3.3.23}$$

which leads to

$$\boldsymbol{l}(t) = \boldsymbol{l}_0 \exp\left(-\int_0^t dt' \frac{\tau e^2}{m\rho_0^3}\left(1 - \frac{6\tau e^2}{m\rho_0^3}t'\right)^{-1}\right). \tag{3.3.24}$$

Extracting the expression for $\boldsymbol{l}(t)$, a straightforward computation gives

$$\begin{aligned} \boldsymbol{l}(t) &= \boldsymbol{l}_0 \exp\left(-\int_0^t dt' \frac{\tau e^2}{m\rho_0^3}\left(1 - \frac{6\tau e^2}{m\rho_0^3}t'\right)^{-1}\right) \\ &= \boldsymbol{l}_0 \exp\left[\frac{1}{6}\ln\left(1 - \frac{6\tau e^2}{m\rho_0^3}t\right)\right] \\ &= \boldsymbol{l}_0 \left(1 - \frac{6\tau e^2}{m\rho_0^3}t\right)^{\frac{1}{6}}. \end{aligned} \tag{3.3.25}$$

Finally, identifying ρ_0 with the Bohr radius a_0, we find (3.3.20).

Results shown in (3.3.3a) and (3.3.3b) are effective only under well defined conditions, when the radiation's effects constitute a small correction to the Coulomb's circular motion. As we can notice from the results (3.3.8) and (3.3.22), the characteristic time over which the energy E and the angular momentum $\boldsymbol{l}$ change is of the order of $1/\dot{\theta}(0)^2\tau$. This time has to be much longer than the orbital period $2\pi/\dot{\theta}(0)$, hence one has to verify that

$$\dot{\theta}(0)\tau \ll 1, \tag{3.3.26}$$

which, numerically, corresponds to the following expression

$$\left(\frac{e^2}{ma_0c^2}\right)^{\frac{3}{2}} \approx 10^{-7}, \tag{3.3.27}$$

confirming the applicability of this model to the Bohr hydrogen atom, where the values for τ and a_0 are respectively 6.3×10^{-24} s and 5.3×10^{-9} cm.

At this point, we are able to compute the time T necessary to the atom to break down

$$T = \frac{a_0^3 m^2 c^3}{4e^4} \approx 1.6 \times 10^{-11}\,\mathrm{s}, \tag{3.3.28}$$

which is incredibly long compared to the atomic scales: the electron spins around the proton about 2×10^5 times before colliding with it.

It is worth observing that, although the angular momentum (3.3.20) decreases during the electron's motion, the kinetic energy grows freely, indeed

$$K = \frac{1}{2}m|\dot{\boldsymbol{r}}|^2 \approx \frac{1}{2}m\rho^2\dot{\theta}^2, \tag{3.3.29}$$

then, considering (3.3.16) and (3.3.18), yields

$$K \approx \frac{e^2}{2a_0}\left(1 - \frac{6\tau e^2}{ma_0^3}t\right)^{-\frac{1}{3}}, \tag{3.3.30}$$

that becomes infinite when t approaches to T, highlighting the runaway behaviour of this solution. Clearly, the radiated energy does not arise from the electron's kinetic energy, which increases over time, but rather from its potential one that decreases, being dependent on the inverse of position. Moreover, the emitted radiation progressively intensifies its frequency as the electron's orbit decays, leading to continuous emission spectra only whereas the experimental evidence shows instead discrete spectral lines.

Definitely, as far as we know, if electrodynamics could always be freely applied, matter in our universe would not exist. The intrinsic atoms' instability, together with the runaway solutions and the causality violation represent the most critical issues of the classical electrodynamics, defining its ultimate limits. Early quantum atomic theory will overcome the spiraling electron's continuous radiation emission, through the introduction of fixed, quantized orbits, to which a discrete set of possible energy values corresponds. However, for a definitive solution to the question of the self-force, we will have to wait for the development of quantum electrodynamics, which would have been able to deal with fundamental particles by describing their electromagnetic interactions in terms of quantum fields. In conclusion, self-force has always been a controversial topic and, today, it is still being object of research, even in the context of the general relativity, where the same identical question emerges from gravity.

Bibliography

[1] Maxwell, James Clerk (1873) *A Treatise on Electricity and Magnetism* Dover.

[2] Lifshitz, Evgeny; Landau, Lev D. (1980) *The Classical Theory of Fields (4th ed.)*

[3] Griffiths, David J. (1998) *Introduction to Electrodynamics (3rd ed.)* Prentice Hall.

[4] Jackson, J. D. (1999) *Classical Electrodynamics (3rd ed.)* Wiley.

[5] Feynman, Richard P. (2005) *The Feynman Lectures on Physics (2nd ed.)* Addison-Wesley.

[6] Panofsky, Wolfgang K. H.; Phillips, Melba (2005). *Classical Electricity and Magnetism (2nd ed.)* Dover.

[7] Olsen, James D.; McDonald, Kirk T. (2005) *Classical Lifetime of a Bohr Atom* Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544.

[8] Zangwill, A. (2013). *Modern Electrodynamics (1st ed.)* Cambridge. Butterworth-Heinemann.

[9] Franklin, Joel (2017) *Classical Field Theory* Cambridge University Press.

[10] Grabovsky, David (2021) *Self-force and Radiation Reaction* SPS Spring 2021.