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SYMMETRIES OF q-CHARACTERS OF REPRESENTATIONS OF QUANTUM AFFINE ALGEBRAS

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My work has always tried to unite
the true with the beautiful
and when I had to choose one or the other,
I usually chose the beautiful.

H. Weyl

Introduction

In this thesis we address the q-characters of representations of quantum affine algebras, especially their symmetries under the Weyl group action defined recently by Frenkel and Hernandez in [10]. This subject gives us the opportunity to study the theory of characters in the *classical* setting of semisimple Lie algebras and the representation theory of quantum affine algebras.

Given a finite-dimensional representation V of a simple Lie algebra $\mathfrak{g}$, the character $\chi(V)$ associated to V is a Laurent polynomial in the ring

$$\mathscr{Y} := \mathbb{Z}[y_i^{\pm 1}]_{i \in I},$$

where each indeterminate y_i corresponds to a fundamental weight for $\mathfrak{g}$. $\chi(V)$ encodes the weights of the representation and their multiplicities; moreover the resulting map $\chi : Rep(\mathfrak{g}) \to \mathscr{Y}$ is injective, providing a description of the Grothendieck ring $Rep(\mathfrak{g})$ of the category of finite-dimensional $\mathfrak{g}$-modules. An action of the Weyl group $\mathscr{W}$ of $\mathfrak{g}$ on the target ring $\mathscr{Y}$ is naturally defined and it is a classical result that the image of χ, isomorphic to $Rep(\mathfrak{g})$, is equal to the subring of $\mathscr{W}$-invariants of $\mathscr{Y}$, i.e.

$$\chi : Rep(\mathfrak{g}) \simeq \mathscr{Y}^{\mathscr{W}}.$$

The situation is more complicated when we pass to quantum groups, that are Hopf algebras obtained as deformations of the universal enveloping algebra of $\mathfrak{g}$. We deal with the quantum affine algebra $\mathscr{U}_q(\hat{\mathfrak{g}})$ corresponding to a simple Lie algebra $\mathfrak{g}$. $\hat{\mathfrak{g}}$ denotes the affinization of $\mathfrak{g}$, an infinite-dimensional Lie algebra built from $\mathfrak{g}$, which is in particular a Kac-Moody algebra. There are various motivations to study the very rich representation theory of quantum affine algebras. Even in this setting there is a notion of character, that is the q-character defined by Frenkel and Reshetikhin in [11]. The principle for the definition of these objects is the same as in the classical setting. However, because of the higher complexity of representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$, this time the q-character $\chi_q(V)$ for a $\mathcal{U}_q(\hat{\mathfrak{g}})$-module V is a Laurent polynomial in the ring $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*}$. Though more complicated, q-characters share some good properties of their classical counterpart, for example

$$\chi_q : \mathcal{K}_0(\mathcal{C}) \to \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*}$$

is injective, where $\mathcal{K}_0(\mathcal{C})$ denotes the "reduced" Grothendieck ring for $\mathcal{U}_q(\hat{\mathfrak{g}})$ (we have to impose a relation on short exact sequences of modules). Nevertheless, for long time it was believed that a quantum analogue of the action of the Weyl group of $\mathfrak{g}$ on the image of q-characters did not exist. For example Chari in [4] found some remarkable automorphisms of $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*}$ related to simple reflections of $\mathcal{W}$, but those automorphisms generate only a braid group action and in addition the subring of invariants does not correspond to the image of q-characters. Screening operators, which are usually used to describe the image of the q-character morphism, do not define a group action. Hence, the article by Frenkel and Hernandez is surprising from this point of view. Indeed they find automorphisms Θ_i, $i \in I$, defined on a direct sum Π of completions of $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*}$ that verifies the relations of the Coxeter presentation of the Weyl group (i.e. involution and braid relations). Furthermore, we can characterize $\mathcal{K}_0(\mathcal{C})$ as being isomorphic to the subring of $\mathcal{W}$-invariants in $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*} \subset \Pi$. This seems to be a crucial step in the theory of q-characters and may lead to interesting discoveries in topics related to quantum affine algebras, such as cluster algebras and quiver varieties.

The dissertation is organized in the following way. In Chapter 1 we collect the required notations to deal with simple Lie algebras and we prove the main results concerning the ordinary characters. In Chapter 2 we present the representation theory of quantized universal enveloping algebras and quantum affine algebras, emphasizing the differences at level of category of finite-dimensional modules between the classical setting and the quantum setting. We give the definition of the q-character morphism and we prove many of its properties. In Chapter 3 we follow the construction by Frenkel and Hernandez of the action of the Weyl group on the target ring of q-characters, trying to make it clearer for a reader at a first approach to this theory. We deal especially with the proofs for Lie type $A_1 \times A_1$ and A_2. In Chapter 4 we present the proof of the result about the $\mathscr{W}$-invariance of the image of q-characters. To do so, we introduce the screening operators. We also provide two appendices to complete the overview on characters. In Appendix A we address the formulas by Freudenthal, Konstant and Weyl to compute characters and multiplicities of $\mathfrak{g}$-modules. In Appendix B we construct the q-character using the notion of R-matrix. This is actually the way Frenkel and Reshetikhin first used to define χ_q.

Throughout the thesis we provide examples and step-by-step computations to clarify the content. This work has the intent to be a self-contained guide to the studying of q-characters and to the reading of the article by Frenkel and Hernandez. We underline some open questions in the concluding part of the memoir.

Introduzione

In questa tesi ci occupiamo dei q-caratteri delle rappresentazioni delle algebre quantiche affini, in particolare delle loro simmetrie sotto l'azione del gruppo di Weyl definita recentemente da Frenkel e Hernandez in [10].

Data una reppresentazione finito-dimensionale V di un'algebra di Lie semplice $\mathfrak{g}$, il carattere $\chi(V)$ associato a V è un polinomio di Laurent nell'anello

$$\mathscr{Y} := \mathbb{Z}[y_i^{\pm 1}]_{i \in I},$$

dove ogni variabile y_i corrisponde a un peso fondamentale per $\mathfrak{g}$. $\chi(V)$ codifica i pesi della rappresentazione e le loro molteplicità; inoltre la mappa indotta $\chi \ : Rep(\mathfrak{g}) \to \mathscr{Y}$ è iniettiva, fornendo quindi una descrizione dell'anello di Grothendieck $Rep(\mathfrak{g})$ della categoria dei $\mathfrak{g}$-moduli di dimensione finita.
Un'azione del gruppo di Weyl $\mathscr{W}$ di $\mathfrak{g}$ sull'anello $\mathscr{Y}$ è definita in modo naturale ed è un risultato classico il fatto che l'immagine di χ, isomorfa a $Rep(\mathfrak{g})$, è uguale al sotto-anello dei $\mathscr{W}$-invarianti di $\mathscr{Y}$, i.e.

$$\chi \ : Rep(\mathfrak{g}) \simeq \mathscr{Y}^{\mathscr{W}}.$$

La situazione è più complicata quando si passa ai gruppi quantici che sono algebre di Hopf ottenute come deformazioni dell'algebra universale inviluppante di $\mathfrak{g}$. Trattiamo l'algebra quantica affine $\mathscr{U}_q(\hat{\mathfrak{g}})$ relativa a un'algebra di Lie semplice $\mathfrak{g}$. $\hat{\mathfrak{g}}$ indica la versione affine di $\mathfrak{g}$, cioè un'algebra di Lie di dimensione infinita costruita a partire da $\mathfrak{g}$ e che è in particolare un'algebra di Kac-Moody. Ci sono varie ragioni per studiare la ricca teoria delle rappresentazioni delle algebre quantiche affini. Anche in questo contesto c'è una nozione di carattere, cioè il q-carattere definito da Frenkel e Reshetikhin in [11]. Il principio per la definizione di questi oggetti è lo stesso che nel caso classico. Tuttavia, a causa della maggiore complessità delle rappresentazioni di $\mathscr{U}_q(\hat{\mathfrak{g}})$, questa volta il q-carattere $\chi_q(V)$ per un $\mathscr{U}_q(\hat{\mathfrak{g}})$-modulo V è un polinomio di Laurent nell'anello $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I,a\in\mathbb{C}^*}$. Benché più complicati, i q-caratteri condividono alcune buone proprietà delle loro controparti classiche, ad esempio

$$\chi_q \ : \mathscr{K}_0(\mathscr{C}) \to \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I,a\in\mathbb{C}^*}$$

è iniettivo, dove $\mathscr{K}_0(\mathscr{C})$ denota l'anello di Grothendieck "ridotto" di $\mathscr{U}_q(\hat{\mathfrak{g}})$ (dobbiamo imporre delle relazioni sulle successioni esatte corte).

Ciononostante, si è creduto per molto tempo che non esistesse un analogo quantico dell'azione del gruppo di Weyl di $\mathfrak{g}$ sull'immagine dei q-caratteri. Per esempio, Chari in [4] aveva trovato degli automorfismi interessanti di $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I,a\in\mathbb{C}^*}$ relativi alle riflessioni semplici di $\mathscr{W}$, ma tali automorfismi generavano solo un'azione del gruppo di trecce e inoltre il sotto-anello degli invarianti non corrispondeva all'immagine dei q-caratteri. Gli operatori di screening, usati di solito per descrivere l'immagine dei q-caratteri, non definiscono un'azione del gruppo. Quindi l'articolo di Frenkel e Hernandez è sorprendente da questo punto di vista. Infatti loro hanno trovato degli automorfismi Θ_i. $i \in I$, definiti su una somma diretta Π di completamenti di $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I,a\in\mathbb{C}^*}$ che verificano le relazioni della presentazione di Coxeter del gruppo di Weyl (cioè involuzione e relazioni di trecce). Inoltre, si può caratterizzare $\mathscr{K}_0(\mathscr{C})$ come sotto-anello di $\mathscr{W}$-invarianti in $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I,a\in\mathbb{C}^*} \subset \Pi$. Questo sembra essere un passo importante nella teoria dei q-caratteri e potrebbe portare a scoperte interessanti in ambiti connessi alle algebre quantiche affini, quali le algebre cluster e le varietà quiver. Il lavoro è organizzato nel modo seguente. Nel Capitolo 1 diamo le notazioni necessarie per trattare le algebre di Lie semplici e dimostriamo i risultati principali sul carattere classico. Nel Capitolo 2 presentiamo la teoria delle algebre inviluppanti quantiche e delle algebre quantiche affini, evidenziando le differenze a livello di categoria di moduli di dimensione finita tra il contesto classico e quello quantico. Diamo la definizione di q-carattere e proviamo diverse delle sue proprietà. Nel Capitolo 3 seguiamo la costruzione di Frenkel e Hernandez dell'azione del gruppo di Weyl sul target dei q-caratteri, cercando di renderla più chiara per un lettore alle prime armi con questa teoria. Trattiamo in particolare le dimostrazioni per i tipi di Lie $A_1 \times A_1$ e A_2. Nel Capitolo 4 presentiamo la dimostrazione del risultato sulla $\mathscr{W}$-invarianza dell'immagine dei q-caratteri. Per fare questo introduciamo gli operatori di screening. Forniamo anche due appendici per completare il panorama sui caratteri. Nell'Appendice A ci occupiamo delle formule di Freudenthal, Konstant e Weyl per calcolare i caratteri e le molteplicità dei $\mathfrak{g}$-moduli. Nell'Appendice B costruiamo i q-caratteri usando la nozione di R-matrice. In effetti questa e la maniera usata inizialmente da Frenkel e Reshetikhin per definire χ_q. In tutta la tesi procuriamo esempi e calcoli step-by-step per chiarire il contenuto. Questo lavoro ha l'intento di essere una guida autonoma allo studio del q-carattere e alla lettura dell'articolo di Frenkel e Hernandez. Nelle conclusioni evidenziamo alcune domande aperte.

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Chapter 1

Classical version

In this chapter we recall some basic facts about the characters of representations of classical Lie algebras. For reminders on classical Lie theory one can see [14]. We will work always on the field of complex numbers $\mathbb{C}$. Here some useful notation:

- $\mathfrak{g}$ is a finite-dimensional simple Lie algebra, let ℓ be its rank, $I = \{1, \dots, \ell\}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We denote by $[\cdot, \cdot]$ the bracket on $\mathfrak{g}$;

- $(\cdot, \cdot)$ the normalized invariant bilinear form on $\mathfrak{g}$ that induces also an inner product on the dual of $\mathfrak{h}$ (see [17] for the construction of this bilinear form). For our purpose, since we will deal mostly with $\mathfrak{sl}_n$, the bilinear form $(\cdot, \cdot)$ is just the Killing form normalized so that the square length of the maximal root is 2, e.g. for $\mathfrak{sl}_3$ the constant of normalization is 6;

- $\mathfrak{sl}_2 = \langle e, h, f \rangle$, where

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; h := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1.1}$$

- Δ the set of roots, Δ^+ and Δ^- respectively the sets of positive and negative roots, $\mathscr{S} = \{\alpha_i\}_{i \in I}$ the set of simple roots, that is a base for Δ;

- $\{\omega_i\}_{i \in I}$ the set of fundamental weights;

- $\mathscr{P}$ and $\mathscr{P}^+$ respectively the lattice of integral weights and dominant integral weights. For every $\lambda \in \mathscr{P}^+$ we denote $V(\lambda)$ the finite-dimensional simple representation of highest weight λ and we recall that these are (up to isomorphism) all the finite-dimensional simple modules of $\mathfrak{g}$;

- $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$ is the Cartan decomposition of $\mathfrak{g}$, i.e. $\mathfrak{g}_\alpha$ is the root space relative to α. Fixed a basis $\mathscr{S}$ of Δ, for a choice of $e_i \in \mathfrak{g}_\alpha$, we can find an element $f_i \in \mathfrak{g}_{-\alpha}$ such that

$$\mathfrak{sl}_2(\alpha_i) := \langle e_i, h_i, f_i\rangle,\ h_i = [e_i, f_i], \tag{1.2}$$

  is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$. It is called the $\mathfrak{sl}_2$-triple associated with α_i. Moreover, we denote by $\preceq$ the Chevalley partial order on $\mathscr{P}$, i.e. $\lambda \preceq \mu$ if and only if $(\mu-\lambda) \in \mathbb{Z}_{\geq 0}\mathscr{S}$;

- $C = (C_{ij})_{1\leq i,j\leq \ell}$ the *Cartan matrix* of $\mathfrak{g}$,

$$C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = \alpha_j(h_i)$$

- s_i is the simple reflection associated with the simple root α_i. s_i is the reflection about the hyperplane perpendicular to α_i and it acts on $\xi \in \mathfrak{h}^*$ as $s_i(\xi) = \xi - \xi(h_i)\alpha_i$;

- $\mathscr{W}$ is the Weyl group of $\mathfrak{g}$. It is the subgroup of $O(\mathfrak{h}^*)$ generated by the simple reflections s_i for all $i \in I$. In particular, it is a Coxeter group with presentation

$$\langle s_1, \dots, s_l | s_i^2 = 1,\ (s_i s_j)^{m_{ij}} = 1,\ i \neq j\rangle,$$

  where the $(m_{ij})_{i,j\in I}$ depend on the Cartan matrix of $\mathfrak{g}$. We get also the following braid relations R_{ij}:

$C_{ij}C_{ji}$	m_{ij}	R_{ij}
0	2	$s_i s_j = s_j s_i$
1	3	$s_i s_j s_i = s_j s_i s_j$
2	4	$s_i s_j s_i s_j = s_j s_i s_j s_i$
2	6	$s_i s_j s_i s_j s_i s_j = s_j s_i s_j s_i s_j s_i$

Table 1.1: The braid relations

- *Rep*$(\mathfrak{g})$ is the Grothendieck ring of the category of finite-dimensional representations of $\mathfrak{g}$, that is the ring whose elements are isomorphism classes of finite-dimensional $\mathfrak{g}$-modules with sum given by direct sum and product given by tensor product.

Given a finite-dimensional representation V of $\mathfrak{g}$, it can be decomposed as $V = \bigoplus_{\lambda \in \mathscr{P}(V)} V_\lambda$, where for any $\lambda \in \mathfrak{h}^*$ the weight space is $V_\lambda = \{v \in V | h \cdot v = \lambda(h)v\}$ and the weights of the representation are $\mathscr{P}(V) = \{\lambda \in \mathfrak{h}^* | V_\lambda \neq 0\}$. We consider $\mathbb{Z}[\mathscr{P}]$, the group ring of weights with integer coefficients and $\{y_i\}_{i \in I}$, a family of algebraically independent elements that generates $\mathbb{Z}[\mathscr{P}]$ (y_i corresponds to ω_i). Thus, if a weight is of the form $\lambda = \sum_{i \in I} m_i \omega_i$, the corresponding element in $\mathbb{Z}\left[y_i^{\pm 1}\right]_{i \in I}$ is $y_\lambda = \prod_{i \in I} y_i^{m_i}$. Recall that $m_i = \lambda(h_i)$. We consider the Grothendieck ring Rep$(\mathfrak{g})$ of the category of finite-dimensional representations of $\mathfrak{g}$ and we define the *character* χ : Rep$(\mathfrak{g}) \to \mathbb{Z}[\mathscr{P}] = \mathbb{Z}\left[y_i^{\pm 1}\right]_{i \in I}$ as the map that attaches to each isomorphism class of representation $[V]$ the polynomial

$$\chi([V]) = \sum_{\lambda \in \mathscr{P}(V)} dim(V_\lambda) y_\lambda.$$

χ is a ring homomorphism, as two isomorphic representations have the same weights with same multiplicities, hence the characters are equal. Notice that it is a common abuse of notation to write $\chi(V)$ instead of $\chi([V])$ and we will follow this convention of representation theorists.

Example 1.0.1

Consider $\mathfrak{g} = \mathfrak{sl}_2$ *and the fundamental representation* $V_1 = \mathbb{C}^2$*. Here* $\mathfrak{h}$ *is one* *dimensional and the weights of the representation are* ± 1*, so the character is*

$$\chi(V_1) = y + y^{-1}.$$

Example 1.0.2
We can generalize the previous example to the case $\mathfrak{g} = \mathfrak{sl}_{n+1}$ *and the natural representation on* $V = \mathbb{C}^{n+1} = V(\omega_1)$*. We fix a basis* $\{v_i\}_{i=1,\dots,n+1}$ *of* $\mathbb{C}^{n+1}$*. We choose a Cartan subalgebra* $\mathfrak{h} = \langle \mathfrak{h}_i \rangle_{i=1}^{n}$*, where* $\mathfrak{h}_i = e_{ii} - e_{i+1,i+1}$ *and* e_{ij} *is the elementary matrix with* 1 *in position* (i,j) *and all other entries being* 0*. The weights of this representation are* $\{\omega_1, \omega_2 - \omega_1, \dots, -\omega_n\}$*. Then* $V_{\omega_1} = \langle v_1 \rangle$*,* $V_{\omega_2-\omega_1} = \langle v_2 \rangle, \dots, V_{-\omega_n} = \langle v_{n+1} \rangle$*, so the weights spaces are all one-dimensional. We obtain*

$$\chi(V) = y_1 + y_2 y_1^{-1} + y_3 y_2^{-1} + \cdots + y_n^{-1}.$$

Example 1.0.3
We consider again $\mathfrak{g} = \mathfrak{sl}_2$ *and the adjoint representation* $V_2 = \mathfrak{sl}_2$*. In this case the set of weights is* $\Delta \cup \{0\}$*,* $\Delta = \{\pm\alpha\}$*. The weight spaces are the root spaces* $\mathfrak{g}_\alpha = \langle e \rangle$*,* $\mathfrak{g}_{-\alpha} = \langle f \rangle$ *and* $\mathfrak{h} = \langle h \rangle$*. Since* $\alpha = 2\omega$*, we have,*

$$\begin{aligned}\chi(V_2) &= dim(\mathfrak{h}) y_0 + dim(\mathfrak{g}_\alpha) y_\alpha + dim(\mathfrak{g}_{-\alpha}) y_{-\alpha} \\ &= 1 + y_{2\omega} + y_{-2\omega} \\ &= 1 + y^2 + y^{-2}.\end{aligned}$$

Let us explore some fundamental properties of the character. Our reference here are [2] and [3]. Since the Weyl group acts on $\mathfrak{h}^*$, there is an induced action on the ring containing the image of the character. Recalling that we can express the simple roots in terms of fundamental weights by the Cartan matrix, we can write the action of the simple reflections on the fundamental weights as $s_i \cdot \omega_j = \omega_j - \delta_{ij} \sum_{k \in I} C_{ki} \omega_k$. Thus, the action on the monomials in $\mathbb{Z}\left[y_i^{\pm 1}\right]_{i \in I}$ is

$$s_i \cdot y_j = y_j a_i^{-\delta_{ij}}, \; a_i = \prod_{k \in I} y_k^{C_{ki}}. \tag{1.3}$$

Proposition 1.0.1
Let V *and* V' *be two finite-dimensional representation of* $\mathfrak{g}$*. Then,*

1. *$\chi(V)$ is invariant under the action of the Weyl group $\mathscr{W}$;*

2. *$V \simeq V'$ if and only if $\chi(V) = \chi(V')$.*

Proof. 1. We need to show that the Weyl group permutes the weights of V and preserves the multiplicities of the weights. Actually, for the first fact we can restrict to the case of a simple representation $V = V(\lambda)$. Let us call $\mathscr{P}(\lambda)$ the set of weights of such a representation and pick μ in $\mathscr{P}(\lambda)$, a corresponding vector $u \in V_\mu$ and define $m := \mu(h_i) \in \mathbb{Z}$. We only need to show that $\mathscr{P}(\lambda)$ is stable under the action of the simple reflections as they generate the group. $V(\lambda)$ is sum of its finite-dimensional $\mathfrak{sl}_2(\alpha_i)$-submodules (as defined in (1.2)) that we may call F_i, so u lies in a F_i. We set,

$$x = f_i^m \cdot u, \qquad m \geq 0$$
$$x = e_i^{-m} \cdot u, \qquad m < 0.$$

We notice that $x \in V_{\mu - m\alpha_i} = V_{s_i(\mu)}$, so it rests to show that x is non-zero. This is immediate since u is a non-zero vector in the weight space $F_i(m)$ and $e_i^{-m} : F_i(-m) \to F_i(m)$ and $f_i^m : F_i(m) \to F_i(-m)$ are isomorphisms. To conclude, it suffices to check that $dim(V_\mu) = dim(V_{s_i(\mu)})$. It is easy to see that $\forall i \in I$, f_i and e_i are locally nilpotent as endomorphisms of V, thus we can define $\sigma_i := exp(e_i)exp(-f_i)exp(e_i)$, which is in fact an automorphism of V. As in the previous discussion, V_μ lies in some F_i and σ_i acts on this submodule sending each weight in its opposite. On F_i, V_μ has weight m and $V_{s_i(\mu)}$ has weight $-m$, so we have $\sigma_i(V_\mu) \subset V_{s_i(\mu)}$ and $\sigma_i(V_{s_i(\mu)}) \subset V_\mu$. Thus, since σ_i is an automorphism, $dim(V_\mu) \leq dim(V_{s_i(\mu)}) \leq dim(V_\mu)$. Since $\mathscr{W}$ is generated by the simple reflections, for each $w \in \mathscr{W}$ we have

$$dim(V_\mu) = dim(V_{w \cdot \mu}). \tag{1.4}$$

2. We have already observed that two isomorphic representations have the same character. Conversely, we prove by induction on $dim(V)$ that two

representations with the same character are isomorphic. If $dim(V)=0$, then $\chi(V)=0=\chi(V')$, so $V'=\{0\}$ too. Note that since the characters of V and V' are the same then $\mathscr{P}(V)=\mathscr{P}(V')$. If $dim(V)>0$ then $\mathscr{P}(V)\neq\varnothing$ and since the set of weights is finite, we can take $\mu\in\mathscr{P}(V)$ such that for all $i\in I$ the element $\mu+\alpha_i$ is not a weight. We take $v\in V_\mu$ non-zero and from our choice of μ we have in particular that v is a highest weight vector. By Weyl's Theorem on complete reducibility, every finite-dimensional $\mathfrak{g}$-module is semisimple, so we can consider the decomposition $V=V_1\oplus V_2$, where V_1 is the highest weight simple module generated by v and V_2 is a direct summand module. We also have $V'=V_1'\oplus V_2'$, where V_1' is simple with highest weight μ. Thus, being V_1 and V_1' isomorphic, $\chi(V_1)=\chi(V_1')$ and by hypothesis and point (1) we have $\chi(V_2)=\chi(V_2')$. So, using the inductive hypothesis, we obtain $V_2\simeq V_2'$ and we can conclude that V is isomorphic to V'.

□

Remark 1.0.1
As a consequence of Proposition 1.0.1, the character map is ***injective****.*

Further properties of character and important formulas to compute it and multiplicities are provided in Appendix A. We are now interested in understanding the image of the character χ and it will turn out that it is the subring $\mathbb{Z}[\mathscr{P}]^{\mathscr{W}}$ of invariant elements under the Weyl group action. By (3) of Proposition 1.0.1 we already have the inclusion

$$\chi(Rep(\mathfrak{g}))\subset\mathbb{Z}[y_i^{\pm1}]_{i\in I}^{\mathscr{W}}. \tag{1.5}$$

Thus, we need to prove the other inclusion. We refer to [2] and [14]. Given an element $f=\sum a_\lambda y_\lambda$ in $\mathbb{Z}[\mathscr{P}]$, we say that a monomial $a_\lambda y_\lambda$ is maximal if the coefficient a_λ is non-zero and if λ is maximal in $\mathscr{P}$ with respect to the Chevalley partial order. We recall that if $\lambda\in\mathscr{P}^+$, then for every $w\in\mathscr{W}$ we have $w(\lambda)\preceq\lambda$. Then, it is clear that in $S(y_\lambda):=\sum_{w\in\mathscr{W}}y_{w\cdot\lambda}$ the only maximal element is y_λ and that $S(y_\lambda)$ belongs to $\mathbb{Z}[\mathscr{P}]^{\mathscr{W}}$. For f to be in $\mathbb{Z}[\mathscr{P}]^{\mathscr{W}}$ it is necessary that for all $w\in\mathscr{W}$, the coefficients a_λ and $a_{w(\lambda)}$ are equal. Also, for every $\lambda \in \mathscr{P}$, the orbit $\mathscr{W}\lambda$ intersects $\mathscr{P}^+$ only in one point, which means that for an element $\mu \in \mathscr{P}$ there exists only one $w \in \mathscr{W}$ such that $w \cdot \mu \in \mathscr{P}^+$ (we say that $\mathscr{P}^+$ is a fundamental domain). So, we can rewrite

$$f = \sum_{\lambda \in \mathscr{P}^+} a_\lambda S(y_\lambda). \tag{1.6}$$

Theorem 1.0.1
The character homomorphism of $Rep(\mathfrak{g})$ in $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$ induces an isomorphism of rings:

$$Rep(\mathfrak{g}) \simeq \mathbb{Z}[y_i^{\pm 1}]_{i \in I}^{\mathscr{W}}.$$

Proof. As pointed out above, we are left to show that the subring of $\mathscr{W}$-invariants in $\mathbb{Z}[\mathscr{P}]$ is the image of $Rep(\mathfrak{g})$ through the character map. Thus, let us consider $f = \sum_{\lambda \in \mathscr{P}^+} a_\lambda (\sum_{w \in \mathscr{W}} y_{w \cdot \lambda}) \in \mathbb{Z}[\mathscr{P}^{\mathscr{W}}]$. We fix a $\lambda \in \mathscr{P}^+$ maximal among the dominant weights that appear in f such that $a_\lambda \neq 0$. For such λ we call M_f the set of dominant weights μ such that $\mu \preceq \lambda$. Notice that M_f is finite! (as we have recalled above). We set $f' = f - a_\lambda \chi(V(\lambda))$. f' is still $\mathscr{W}$-invariant. The dominant μ in $\chi(V(\lambda))$ verify $\mu \prec \lambda$, so $M_{f'} \subseteq M_f$, but in f' the monomial of weight λ does not appear, so the inclusion is proper, i.e. $M_{f'} \subset M_f$. Hence, thanks to finiteness of M_f we can argue by induction and conclude that f' can be written as a $\mathbb{Z}$-linear combination of characters, so the same goes for f. We have to show that the base step of the induction holds. To do so, suppose that the cardinality of M_f is one. This means that $f = \sum_{w \in \mathscr{W}} y_{w \cdot \lambda}$, with λ a minimal dominant weight. From the structure of simple finite-dimensional modules of $\mathfrak{g}$, we know that the weights appearing in $V(\lambda)$ are only those the form $w \cdot \lambda$, by the minimality of λ. The multiplicity of each $w \cdot \lambda$ is $dim(V(\lambda)_{w \cdot \lambda}) = dim(V(\lambda)_\lambda) = 1$, by (1.4). Therefore, $\chi(V(\lambda)) = \sum_{w \in \mathscr{W}} y_{w \cdot \lambda} = f$, so the base step of induction is proved and the proof is complete. $\square$

Chapter 2

Quantum affine algebras and their q-characters

In this chapter we collect some fundamental results about the representation theory of the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ and we present the *q-characters* of Frenkel and Reshetikhin defined in [11]. We will provide examples for $\mathcal{U}_q(\widehat{\mathfrak{sl}_2})$ and $\mathcal{U}_q(\widehat{\mathfrak{sl}_3})$. In what follows, unless otherwise specified, the parameter q is a non-zero complex number, not root of unity.

2.1 Quantum universal enveloping algebras

2.1.1 Definition of $\mathcal{U}_q(\mathfrak{g})$

We first start with a finite-dimensional simple Lie algebra $\mathfrak{g}$. The quantum universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ is a q-deformation of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. It can be presented by generators x_i^+, x_i^-, $k_i^{\pm 1}$, $1 \le i \le l$ (where l is the rank of classical Lie algebra $\mathfrak{g}$) and by relations which are quantum analogous of the Serre presentation of the Lie algebra $\mathfrak{g}$ (they will be given below in the case of quantum affine algebras, so we do not repeat it here).

A crucial property of $\mathcal{U}_q(\mathfrak{g})$ is that it has a structure of a Hopf algebra, which allows to define tensor products and duals of representations.

Example 2.1.1

The associative algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ *is defined over* $\mathbb{C}$ *by generators*

$$x^+,\ x^-,\ k,\ k^{-1}$$

subject to the following relations:

$$kx^+ = q^2x^+k \tag{2.1}$$

$$kx^- = q^{-2}x^-k \tag{2.2}$$

$$[x^+, x^-] = \frac{k-k^{-1}}{q-q^{-1}} \tag{2.3}$$

$$kk^{-1} = 1 = k^{-1}k. \tag{2.4}$$

It has the structure of Hopf algebra with the comultiplication given by

$$\Delta:\ \mathcal{U}_q(\mathfrak{sl}_2) \to \mathcal{U}_q(\mathfrak{sl}_2) \otimes \mathcal{U}_q(\mathfrak{sl}_2)$$

defined on the generators as

$$\begin{aligned}
\Delta(x^+) &= x^+ \otimes 1 + k \otimes x^+ \\
\Delta(x^-) &= x^- \otimes k^{-1} + 1 \otimes x^- \\
\Delta(k) &= k \otimes k.
\end{aligned}$$

The counit

$$\epsilon:\ \mathcal{U}_q(\mathfrak{sl}_2) \to \mathbb{C}$$

is defined as

$$\begin{aligned}
\epsilon(x^+) &= 0 \\
\epsilon(x^-) &= 0 \\
\epsilon(k) &= 1.
\end{aligned}$$

Finally the antipodal map is

$$S:\ \mathcal{U}_q(\mathfrak{sl}_2) \to \mathcal{U}_q(\mathfrak{sl}_2)$$

which assigns

$$
\begin{aligned}
S(x^+) &= -x^+k^{-1} \\
S(x^-) &= -kx^- \\
S(k) &= k^{-1}.
\end{aligned}
$$

In particular, S is an antimorphism of algebras, i.e. $S(ab) = S(b)S(a)$ for all $a, b \in \mathscr{U}_q(\mathfrak{sl}_2)$.

We will use the following standard notation:

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}, \quad n \in \mathbb{Z}.$$

2.1.2 Finite dimensional representations of $\mathscr{U}_q(\mathfrak{g})$

In this section we illustrate the structure of finite-dimensional simple modules of $\mathscr{U}_q(\mathfrak{g})$. Our reference for this section is Chapter 5 of [15]. Let V be a finite-dimensional $\mathscr{U}_q(\mathfrak{g})$-module. Given a weight $\mu \in \mathscr{P}$ we define the corresponding weight space as follows:

$$V_\mu := \{v \in V | k_i \cdot v = q^{\mu(h_i)}v, \forall i \in I\}.$$

We say that V is of type **1** if $V = \bigoplus_{\mu \in \mathscr{P}} V_\mu$. A non-zero vector in the weight space V_λ is called a highest weight vector if $x_i^+ \cdot v = 0$ for all $i \in I$. If, in addition, $V = \mathscr{U}_q(\mathfrak{g}) \cdot v$ for a certain highest weight vector v, we say that V is a highest weight module with highest weight λ. All highest weight module are of type **1**. As in the classical theory, for any $\lambda \in \mathscr{P}$ we define the Verma module $M(\lambda)$ to be the quotient of $\mathscr{U}_q(\mathfrak{g})$ by the left ideal generated by $\{x_i^+, k_i - q^{\lambda(h_i)}1\}_{i \in I}$. Then, $M(\lambda)$ is a highest weight representation with highest weight λ and highest weight vector the class of 1. The weight space corresponding to λ is one-dimensional and $M(\lambda)$ has a unique irreducible quotient $V(\lambda)$. As we expect, $V(\lambda)$ is finite dimensional if and only if λ is dominant. Then the following holds:

Theorem 2.1.1
For each dominant weight $\lambda \in \mathscr{P}^+$ the simple $\mathscr{U}_q(\mathfrak{g})$-module $V(\lambda)$ has finite dimension. Each finite-dimensional simple $\mathscr{U}_q(\mathfrak{g})$-module is isomorphic to exactly one $V(\lambda)$ with λ dominant.

Proof. See Theorem 5.10 in [15]. □

In this setting we have the analogous of Weyl's Theorem on complete reducibility:

Theorem 2.1.2
Suppose q is transcendental over $\mathbb{Q}$, then every finite-dimensional $\mathscr{U}_q(\mathfrak{g})$-module is semisimple.

Proof. See Theorem 5.17 in [15]. □

Example 2.1.2
The representation theory of $\mathscr{U}_q(\mathfrak{sl}_2)$ for q not root of unity is analogous to the one of $\mathfrak{sl}_2$, namely the finite-dimensional simple modules of type **1** *are parameterized by $\mathbb{N}$. In particular, for each integer $n \geq 0$ there is a simple $\mathscr{U}_q(\mathfrak{sl}_2)$-module V_n with basis $\{v_0, \dots, v_n\}$ such that for all $0 \leq i \leq n$:*

$$\begin{aligned} k \cdot v_i &= q^{n-2i} v_i \\ x^+ \cdot v_i &= [n-i+1]_q v_{i-1} \\ x^- \cdot v_i &= [i+1]_q v_{i+1}, \end{aligned}$$

where $v_{-1} = v_{n+1} = 0$. For example, on the two-dimensional $\mathscr{U}_q(\mathfrak{sl}_2)$-module V_1 generated by v_0, v_1 the action is given by

	x^+	x^-	k
v_0	0	v_1	qv_0
v_1	v_0	0	$q^{-1}v_1$

.

Example 2.1.3

In this example we show what "goes wrong" if we admit q to be a root of unity. We consider the algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ and a finite-dimensional $\mathcal{U}_q(\mathfrak{sl}_2)$-module V. Let μ be a weight in $\mathcal{P}$ and let V_μ be the corresponding weight space. Then, we have that $x^+ \cdot V_\mu \subset V_{\mu+2}$. More generally, taking $0 \neq v \in V_\mu$, for every $n \in \mathbb{N}$ we find $(x^+)^n \cdot v \in V_{\mu+2n}$. Since V is finite-dimensional, there exists a positive integer m such that $(x^+)^m \cdot v \neq 0$, but $(x^+)^{m+1} \cdot v = 0$. In particular, if we assume V to be simple, the observation above is the key point to show the existence of a highest weight vector. Now, assume q to be a root of unity of order p, i.e. $q^p = 1$, $p \in \mathbb{N}$. We have two different situations depending on the dimension of V. If $dim(V) \geq p$, and $V_\mu \neq \{0\}$, then we get:

$$\begin{tikzcd} V_\mu = V_{\mu+2p} \arrow[r, "x^+"] & V_{\mu+2} \arrow[r, "x^+"] & \dots \arrow[r, "x^+"] & V_{\mu+2(p-1)} \arrow[lll, bend left, "x^+"] \end{tikzcd} ,$$

so in this case we do not find a vector in V_μ such that after iterated actions of x^+ it goes to 0. While, if $dim(V) < p$, then we can argue as in the non root of unity case.

2.1.3 Characters for $\mathcal{U}_q(\mathfrak{g})$

We define the character $\chi(V)$ of a finite-dimensional representation V of $\mathcal{U}_q(\mathfrak{g})$ as for representations of $\mathfrak{g}$, by encoding the dimension of the weight spaces of V:

$$\chi(V) = \sum_{\mu \in \mathcal{P}} \dim(V_\mu) y_\mu \in \mathbb{Z}[y_i^{\pm 1}]_{i \in I}.$$

It is known that the main results for the character theory of $\mathfrak{g}$ hold for $\mathcal{U}_q(\mathfrak{g})$: the characters define an injective ring morphism on the Grothendieck ring $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$ of finite-dimensional representations of $\mathcal{U}_q(\mathfrak{g})$, and the image of the character morphism in $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$ is exactly the subring of invariants for the Weyl group action.

The situation changes completly when we work with quantum affine algebras. This what we explain below.

2.2 Quantum affine algebras and their representations

2.2.1 Definition of $\mathscr{U}_q(\hat{\mathfrak{g}})$

Let $\mathfrak{g}$ be a simple Lie algebra (with notations as in the previous chapter). Given C, the Cartan matrix of $\mathfrak{g}$, we call B the symmetric matrix $B = DC$, where $D = \mathrm{diag}(r_1, \dots, r_l)$ and the r_i's are relatively prime integers. For $i \neq j$, if $C_{ij} = C_{ji} = -1$, then $r_i = r_j = 1$, if $C_{ij} < -1$, then $r_i = 1$ and $r_j = -C_{ij}$.

Notice that in our examples $\mathfrak{g} = \mathfrak{sl}_n$, so C is symmetric and $B = C$. We set $q_i := q^{r_i}$. In the following definition we will denote C also the Cartan matrix of the affine algebra $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}$. This means that $C \in \mathrm{Mat}_{(l+1),(l+1)}(\mathbb{C})$. The affine algebra $\hat{\mathfrak{g}}$ is defined as

$$\hat{\mathfrak{g}} := (\mathbb{C}[t^{\pm 1}] \otimes \mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where c is a central element and d is a derivation element. Then the bracket works as follows:

$$[(t^k \otimes x) + \lambda c + \mu d, (t^h \otimes y) + \beta c + \nu d] = (t^{k+h} \otimes [x, y] + \mu k t^k \otimes y - \nu h t^h \otimes x) + \\ + k\delta_{k,-h} \frac{K(x, y)}{h^\vee},$$

where $K(\cdot, \cdot)$ denotes the Killing form on $\mathfrak{g}$ and $h^\vee$ is the dual Coxeter number associated to the Weyl group of $\mathfrak{g}$. The component

$$\mathfrak{L}\mathfrak{g} := \mathbb{C}[t^{\pm 1}] \otimes \mathfrak{g}$$

is called loop algebra.

In the following, we will work with the Lie subalgebra without derivation element, that is $\mathfrak{L}\mathfrak{g} \oplus \mathbb{C}c$ as it will be sufficient for our purposes. By a common abuse of terminology, it will be also called the affine Lie algebra and denoted also by $\hat{\mathfrak{g}}$.

For example, for $\mathfrak{sl}_2$ we have $\widehat{\mathfrak{sl}_2} = \left(\mathbb{C}[t^{\pm 1}] \otimes \mathfrak{sl}_2\right) \oplus \mathbb{C}c$ and $\mathfrak{L}\mathfrak{sl}_2$ is generated by:

$$e_0 := t \otimes f,\ f_0 := t^{-1} \otimes e$$
$$e_1 := 1 \otimes e,\ f_1 := 1 \otimes f.$$

The Cartan matrix of $\widehat{\mathfrak{sl}_2}$ is

$$C = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

Now we turn to the quantum analogue of the affine Lie algebra.

Definition 2.2.1
The quantum affine algebra $\mathscr{U}_q(\hat{\mathfrak{g}})$ in the Drinfeld-Jimbo realization is an associative algebra over $\mathbb{C}$ with generators $x_i^{\pm}$, $k_i^{\pm}$ ($i = 0, \ldots, l$) and relations:

$$k_i k_i^{-1} = k_i^{-1} k_i = 1,$$
$$k_i k_j = k_j k_i,$$
$$k_i x_j^{\pm} = q^{\pm B_{ij}} x_j^{\pm} k_i,$$
$$[x_i^+, x_j^-] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{r=0}^{1-C_{ij}} (-1)^r \begin{bmatrix} 1 - C_{ij} \\ r \end{bmatrix}_{q_i} (x_i^{\pm})^r x_j^{\pm} (x_i^{\pm})^{1-C_{ij}-r} = 0, \text{ for } i \neq j.$$

The algebra $\mathscr{U}_q(\hat{\mathfrak{g}})$ has a structure of a Hopf algebra with the comultiplication Δ, antipode S and counit ϵ defined on the generators as follows:

$$\Delta(k_i) = k_i \otimes k_i,$$
$$\Delta(x_i^+) = x_i^+ \otimes 1 + k_i \otimes x_i^+,$$
$$\Delta(x_i^-) = x_i^- \otimes k_i^{-1} + 1 \otimes x_i^-,$$

$$S(x_i^+) = -x_i^+ k_i, \quad S(x_i^-) = -k_i^{-1} x_i^-, \quad S(k_i^{\pm 1}) = k_i^{\mp 1},$$
$$\epsilon(x_i^+) = \epsilon(x_i^-) = 0, \ \epsilon(k_i) = 1.$$

Remark 2.2.1
The algebra $\mathcal{U}_q(\mathfrak{g})$ is a subalgebra of $\mathcal{U}_q(\hat{\mathfrak{g}})$ obtained taking the generators $x_i^\pm$, $k_i^{\pm 1}$, $i \in I = \{1, \ldots, l\}$.

There is also another presentation of $\mathcal{U}_q(\hat{\mathfrak{g}})$, which is in fact a theorem by Drinfeld, see [8] for the statement and [7],[1], [6] for the proof.

Theorem 2.2.1
The algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ has a realization as the algebra with generators $x_{i,n}^\pm$ ($i \in I$, $n \in \mathbb{Z}$), $k_i^{\pm 1}$ ($i \in I$), $h_{i,n}$ ($i \in I$, $n \in \mathbb{Z} \setminus \{0\}$) and central elements $c^{\pm \frac{1}{2}}$, with the following relations:

$$k_i k_j = k_j k_i, \ k_i h_{n,j} = h_{n,j} k_i,$$
$$k_i x_{j,n}^\pm = q^{\pm B_{ij}} x_{j,n}^\pm k_i, \tag{2.5}$$
$$[h_{i,n}, x_{j,m}^\pm] = \pm \frac{1}{n} [n B_{ij}]_q c^{\mp |n|/2} x_{j,n+m}^\pm,$$
$$x_{i,n+1}^\pm x_{j,m}^\pm - q^{\pm B_{ij}} x_{j,m}^\pm x_{i,n+1}^\pm = q^{\pm B_{ij}} x_{i,n}^\pm x_{j,m+1}^\pm - x_{j,m+1}^\pm x_{i,n}^\pm, \tag{2.6}$$
$$[h_{i,n}, h_{j,m}] = \delta_{n,-m} \frac{1}{n} [n B_{ij}]_q \frac{c^n - c^{-n}}{q - q^{-1}}, \tag{2.7}$$
$$[x_{i,n}^+, x_{j,m}^-] = \delta_{ij} \frac{c^{(n-m)/2} \phi_{i,n+m}^+ - c^{-(n-m)/2} \phi_{i,n+m}^-}{q_i - q_i^{-1}}, \tag{2.8}$$
$$\sum_{\pi \in \Sigma_s} \sum_{k=0}^{s} (-1)^k \begin{bmatrix} s \\ k \end{bmatrix}_{q_i} x_{i,n_{\pi(1)}}^\pm \ldots x_{i,n_{\pi(k)}}^\pm x_{j,m}^\pm x_{i,n_{\pi(k+1)}}^\pm \ldots x_{i,n_{\pi(s)}}^\pm = 0, \ s = 1 - C_{ij},$$

for all sequences of integers $n_1, \ldots, n_s$, and $i \neq j$, where Σ_s is the symmetric group on s letters, C is the Cartan matrix of $\mathfrak{g}$ and $\phi_{i,n}^\pm$ are determined by the formula

$$\sum_{n=0}^{\infty} \phi_{i,\pm n}^\pm u^{\pm n} = k_i^{\pm 1} exp\left(\pm (q - q^{-1}) \sum_{m=1}^{\infty} h_{i,\pm m} u^{\pm m} \right). \tag{2.9}$$

Example 2.2.1
We can write down the relations for the quantum affine algebra $\mathcal{U}_q(\widehat{\mathfrak{sl}_2})$. It is the

associative algebra over $\mathbb{C}$ with generators $\{x_i^+, x_i^-, k_i^{\pm 1}\}_{i=0,1}$ and relations:

$$k_i k_i^{-1} = k_i^{-1} k_i = 1$$
$$k_i k_j = k_j k_i$$
$$k_i x_i^{\pm} = q^{\pm 2} x_i^{\pm} k_i$$
$$k_i x_j^{\pm} = q^{\mp 2} x_j^{\pm} k_i,\ for\ i \neq j$$
$$[x_i^+, x_i^-] = \frac{k_i - k_i^{-1}}{q - q^{-1}}$$
$$[x_0^+, x_1^-] = [x_1^+, x_0^-] = 0$$
$$x_i^{\pm} x_j^{\pm 2} - (q^2 + q^{-2}) x_j^{\pm} x_i^{\pm} x_j^{\pm} + x_j^{\pm 2} x_i = 0,\quad for\ i \neq j.$$

As we have seen in the general case, there is also another presentation of $\mathscr{U}_q(\widehat{\mathfrak{sl}_2})$ with generators $x_m^{\pm}$ ($m \in \mathbb{Z}$), $k^{\pm 1}$, h_m ($m \in \mathbb{Z} \setminus \{0\}$), central elements $c^{\pm \frac{1}{2}}$ and relations:

$$kk^{-1} = k^{-1}k = 1 \tag{2.10}$$
$$[k, h_m] = 0 \tag{2.11}$$
$$kx_m^{\pm} = q^2 x_m^{\pm} k \tag{2.12}$$
$$[h_m, x_{m'}^{\pm}] = \pm \frac{1}{m} [2m]_q c^{\mp \frac{|m|}{2}} x_{m+m'}^{\pm} \tag{2.13}$$
$$x_{n+1}^{\pm} x_m^{\pm} - q^{\pm 2} x_m^{\pm} x_{i,n+1}^{\pm} = q^{\pm 2} x_n^{\pm} x_{m+1}^{\pm} - x_{m+1}^{\pm} x_n^{\pm}, \tag{2.14}$$
$$[x_n^+, x_m^-] = \frac{c^{(n-m)/2} \phi_{n+m}^+ - c^{-(n-m)/2} \phi_{n+m}^-}{q - q^{-1}}. \tag{2.15}$$

2.2.2 Finite-dimensional representations of $\mathscr{U}_q(\hat{\mathfrak{g}})$

It is known by Chari and Pressley that $c^{\frac{1}{2}}$ acts as the identity on any finite-dimensional representation of $\mathscr{U}_q(\hat{\mathfrak{g}})$.

Definition 2.2.2
A representation of $\mathscr{U}_q(\hat{\mathfrak{g}})$ is called of type 1 if $c^{\frac{1}{2}}$ acts as the identity on V and if V is a type 1 module as $\mathscr{U}_q(\mathfrak{g})$-module.

A vector $v \in V$ is called a highest weight vector if

$$x_{i,m}^{+} \cdot v = 0, \quad \phi_{i,m}^{\pm} \cdot v = \psi_{i,m}^{\pm} v, \quad c^{\frac{1}{2}} v = v,$$

for some complex numbers $\psi_{i,m}^{\pm}$. A type 1 representation V is a highest weight representation if $V = \mathcal{U}_q(\hat{\mathfrak{g}}) \cdot v$, for some highest weight vector v. In that case the set $(\psi_{i,m}^{\pm})_{i \in I, m \in \mathbb{Z}}$ is called the highest weight.

For every $i \in I$ we define the formal series $\Phi_i^{+}(u) \in \mathcal{U}_q(\hat{\mathfrak{g}})[[u]]$ and $\Phi_i^{-}(u) \in \mathcal{U}_q(\hat{\mathfrak{g}})[[u^{-1}]]$ as

$$\Phi_i^{\pm}(u) := \sum_{m \geq 0} \phi_{i,m}^{\pm} u^{\pm m}.$$

If V is a simple module of type 1 with highest weight $(\psi_{i,m}^{\pm})_{i \in I, m \in \mathbb{Z}}$, we can define for all $i \in I$ the formal series $\Psi_i^{+} \in \mathbb{C}[[u]]$ and $\Psi_i^{-} \in \mathbb{C}[[u^{-1}]]$ as

$$\Psi_i^{\pm}(u) := \sum_{m \geq 0} \psi_{i,m}^{\pm} u^{\pm m}.$$

Then we get naturally that

$$\Phi_i^{+}(u) \cdot v = \Psi_i^{+}(u) v$$

for some highest weight vector v. We will refer to $\Psi_i^{\pm}$ as an *eigen-value*. We can state the following important result due to Chari and Pressley (see [5])

Theorem 2.2.2 (Classification of simple $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules)

1. *Every finite-dimensional irreducible representation of type 1 of $\mathcal{U}_q(\hat{\mathfrak{g}})$ is a highest weight representation.*

2. *Let V be a finite-dimensional irreducible representation of $\mathcal{U}_q(\hat{\mathfrak{g}})$ of type* **1** *and highest weight $(\psi_{i,m}^{\pm})_{i \in I, m \in \mathbb{Z}}$. Then, there exists a unique I-tuple $\mathbf{P} = (P_i)_{\in I}$, $P_i \in \mathbb{C}[u]$ of polynomials of constant term 1 such that*

$$\Psi_i^{\pm}(u) = q_i^{deg(P_i)} \frac{P_i(uq_i^{-1})}{P_i(uq_i)} \in \mathbb{C}[u^{\pm 1}]. \tag{2.16}$$

Assigning to V the set $\mathbf{P}$ defines a bijection between the set of I-tuples of polynomial is $\mathbb{C}[u]$ with constant term 1 and the set of isomorphism classes of finite-dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ of type $\mathbf{1}$. We will denote the simple module associated to $\mathbf{P}$ by $V(\mathbf{P})$.

3. *Let $\mathbf{P}$ and $\mathbf{Q}$ be two I-tuples of polynomials as above and let $v_{\mathbf{P}}$ and $v_{\mathbf{Q}}$ be highest weight vectors of $V(\mathbf{P})$ and $V(\mathbf{Q})$ respectively. Then, we may denote by $\mathbf{P} \otimes \mathbf{Q}$ the I-tuple $(P_i Q_i)_{i \in I}$. Then $V(\mathbf{P} \otimes \mathbf{Q})$ is isomorphic to a quotient of the submodule of $V(\mathbf{P}) \otimes V(\mathbf{Q})$ generated by $v_{\mathbf{P}} \otimes v_{\mathbf{Q}}$.*

Definition 2.2.3
If $\mathbf{P}$ is such that $P_j(u) = 1$ if $i \neq j$ and $P_i(u) = (1 - ua)$, then we denote $V(\mathbf{P})$ by $V_i(a)$ and we call it a fundamental module. Another useful notation is the following. Let us consider the ring $\mathcal{Y} := \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^*}$. If $\mathbf{P} = (P_i)_{i \in I}$, $P_i(u) = \prod_{a \in \mathbb{C}^*}(1 - ua)^{u_{i,a}(m)}$, then we denote $L(m) = V(\mathbf{P})$, where $m = \prod_{i,a} Y_{i,a}^{u_{i,a}(m)}$. Notice that $\sum_{a \in \mathbb{C}^*} u_{i,a}(m) = deg(P_i)$. The fundamental module $V_i(a)$ is also denoted by $L(Y_{i,a})$.

The proof of the following Corollary can be found in [5] and is based on the fact that every polynomial in $\mathbb{C}[u]$ can be factorized in a product of linear polynomials.

Corollary 2.2.2.1
Any irreducible finite-dimensional $\mathcal{U}_q(\hat{\mathfrak{g}})$-module occurs as a quotient of the submodule of the tensor product $V_{i_1}(a_1) \otimes \cdots \otimes V_{i_n}(a_n)$, generated by the tensor product of the highest weight vectors. The parameters $(i_1, a_1), \ldots, (i_n, a_n)$ are uniquely determined by the module up to permutation.

Example 2.2.2
We will turn the two dimensional simple module described in 2.1.2 into a $\mathcal{U}_q(\widehat{\mathfrak{sl}_2})$-module by the evaluation morphism. The idea is that we "forget" the loop component of $\widehat{\mathfrak{sl}_2}$ by evaluating the Laurent polynomials on a certain scalar $a \in \mathbb{C}^*$.

More precisely, for any $a \in \mathbb{C}^*$, we define

$$
\begin{aligned}
ev_a :\ \mathcal{U}_q(\widehat{\mathfrak{sl}_2}) &\to \mathcal{U}_q(\mathfrak{sl}_2)\\
x_m^+ &\mapsto q^{-m}a^{-m}k^m x^+\\
x_m^- &\mapsto q^{-m}a^{-m}x^- k^m\\
c^{\pm\frac{1}{2}} &\mapsto 1.
\end{aligned}
$$

If we call φ the representation of $\mathcal{U}_q(\mathfrak{sl}_2)$ on V_1, then $\varphi \circ ev_a$ is a representation of $\mathcal{U}_q(\widehat{\mathfrak{sl}_2})$ on V_1 and we may call this module $V_1(a^{-1}q^{-1})$ (the reason will be clear later). In order to determine the weights of $V_1(a^{-1}q^{-1})$, we should understand the evaluation of the $\phi_m^\pm$, $(n \in \mathbb{Z})$. Since $\phi_0^\pm = k^{\pm 1}$, we know the action on $V_1(a^{-1}q^{-1})$ for $m = 0$. We observe that

$$ev_a([x_r^+, x_s^-]) = q^{-(r+s)}a^{-(r+s)}k^r x^+ x^- k^s - q^{-(r+s)}a^{-(r+s)}x^- k^s k^r x_1^+.$$

On the other hand,

$$[x_r^+, x_s^-] = \frac{c^{\frac{r-s}{2}}\phi_{r+s}^+ - c^{\frac{-(r-s)}{2}}\phi_{r+s}^-}{q-q^{-1}},$$

so if we suppose $m = r+s > 0$, we have that $\phi_m^- = 0$. Thus,

$$ev_a(\phi_m^+) = (q-q^{-1})(q^{-m}a^{-m}k^r x^+ x^- k^s - q^{-m}a^{-m}x^- k^s k^r x_1^+), \quad (m > 0).$$

Now we can calculate

$$\phi_m^+ \cdot v_0 = (q-q^{-1})a^{-m}v_0, \quad (m > 0),$$

thanks to the fact that the action of $\mathcal{U}_q(\mathfrak{sl}_2)$ on v_0 is known. Similarly we get

$$\phi_m^+ \cdot v_1 = -(q-q^{-1})a^{-m}v_1, \quad (m > 0).$$

Moreover we are able to calculate the series

$$\begin{aligned}
\Phi^+(z)\cdot v_0 &= \sum_{m\geq 0}\phi^+_m z^m\cdot v_0\\
&= q\ v_0+\sum_{m>0}(q-q^{-1})a^{-m}z^m v_0\\
&= \left(q+\frac{z(q-q^{-1})a^{-1}}{1-za^{-1}}\right)v_0\\
&= \left(q\ \frac{1-zq^{-2}a^{-1}}{1-za^{-1}}\right)v_0\\
&= q\ \frac{P_0(zq^{-1})}{P_0(zq)}v_0,\quad P_0(z):=1-q^{-1}a^{-1}z
\end{aligned}$$

for v_0; while for v_1

$$\begin{aligned}
\Phi^+(z)\cdot v_1 &= \sum_{m\geq 0}\phi^+_m z^m\cdot v_1\\
&= q^{-1}-\sum_{m>0}(q-q^{-1})a^{-m}z^m v_1\\
&= q^{-1}\ \frac{P_1(zq)}{P_1(zq^{-1})}v_1\quad P_1(z):=1-qa^{-1}z.
\end{aligned}$$

The same calculation, made for the variable z^{-1}, shows that

$$\begin{aligned}
\Phi^-(z)\cdot v_0 &= q\ \frac{P_0(zq^{-1})}{P_0(zq)}v_0,\\
\Phi^-(z)\cdot v_1 &= q^{-1}\ \frac{P_1(zq)}{P_1(zq^{-1})}v_1.
\end{aligned}$$

Thus, calling $\gamma_1^+(z):=q\ \frac{P_0(zq^{-1})}{P_0(zq)}$ and $\gamma_2^+(z):=q^{-1}\ \frac{P_1(zq)}{P_1(zq^{-1})}$, we have the eigenspaces

$$V(\gamma_1^+)=\mathbb{C}v_0\quad V(\gamma_2^+)=\mathbb{C}v_1.$$

Proposition 2.2.1

Let $V=V(\mathbf{P})$ a simple finite-dimensional $\mathcal{U}_q(\hat{\mathfrak{g}})$-module, $\mathbf{P}=(P_i)_{i\in I}$. Then V is a finite-dimensional $\mathcal{U}_q(\mathfrak{g})$-module of highest weight $\sum_i deg(P_i)\omega_i$.

Proof. The action of the $(k_i^{\pm 1})_{i\in I}\in\mathcal{U}_q(\mathfrak{g})$ on the highest weight vector $v\in V$ can be deduced from the action of the $\phi^{\pm 1}_{i,m}$ by evaluating the expansion

$$k_i\cdot v+\sum_{n>0}\phi^+_{i,n}z^n\cdot v=q_i^{deg(P_i)}\frac{P_i(zq_i^{-1})}{P_i(zq_i)}$$

in $z = 0$. Since each P_i has constant term 1 , we find $k_i \cdot v = q_i^{deg(P_i)}$, which means that v has weight $\sum_i deg(P_i)\omega_i$ (indeed, recall that for a $\mathcal{U}_q(\mathfrak{g})$-module, saying that v has weight ω means that $k_i \cdot v = q_i^{\omega(h_i)}$). This is enough to conclude. $\square$

Definition 2.2.4

Since $\mathcal{U}_q(\mathfrak{g}) \subset \mathcal{U}_q(\hat{\mathfrak{g}})$ is a subalgebra, every $\mathcal{U}_q(\hat{\mathfrak{g}})$-module can be restricted to a $\mathcal{U}_q(\mathfrak{g})$-module. We call the restriction map

$$res : Rep(\mathcal{U}_q(\hat{\mathfrak{g}})) \to Rep(\mathcal{U}_q(\mathfrak{g})).$$

2.3 Definition and properties of χ_q

We aim at presenting the character theory of finite-dimensional representations of quantum affine algebras. The usual definition is too naive for the following reason: the evaluation representations discussed in Example 2.2.2 are not isomorphic for different evaluation parameters a. However, they have the same character $y + y^{-1}$ if it is defined by taking into account the eigenvalues of the $k_i^{\pm 1}$ only. Indeed, in the $\mathcal{U}_q(\widehat{\mathfrak{sl}_2})$ case, explicit computations show that the trace of the operator $\Phi^+(u)$ is independent of the indeterminate u, e.g. $tr_{V_1(a)}(\Phi^+(u)) = tr_{V_1(a)}(k^+)$. Hence the ordinary character morphism does not distinguish the isomorphism classes of simple modules. That is why another definition should be used.

In this section we present the definition of q-characters, which was given by Edward Frenkel and Nikolai Reshetikhin in [11]. Actually the first definition given by Frenkel and Reshetikhin uses the R-matrix, but we prefer to present here the q-characters from a pure representation-theoretic point of view. We will discuss the other definition in Appendix B. From the defining relations of $\mathcal{U}_q(\hat{\mathfrak{g}})$, we know that the $\phi_{i,m}^{\pm}$ commute with each other on finite-dimensional representations (as $c^{\frac{1}{2}}$ acts by 1 on finite-dimensional representations). Then it is possible to decompose any $\mathcal{U}_q(\hat{\mathfrak{g}})$-module into a direct sum of common generalized eigenspaces

$$V = \oplus V_{(\gamma_{i,m}^{\pm})}$$

where

$$V_{(\gamma^{\pm}_{i,m})} := \{v \in V | (\phi^{\pm}_{i,m} - \gamma^{\pm}_{i,m})^p \cdot v = 0,\ \exists p \in \mathbb{N}, \forall i, m\}.$$

For a collection of eigenvalues $(\gamma^{\pm}_{i,m})$, we associate for all i the generating function

$$\gamma^{\pm}_i(u) = \sum_{m>0} \gamma^{\pm}_{i,\pm m} u^{\pm m}.$$

We consider a series $\gamma^{\pm}_i(u)$ as an eigenvalue of $\Phi^{\pm}_i(u)$.

Proposition 2.3.1 (General form of the eigenvalues)
The eigenvalues $\gamma^{\pm}_i(u)$ of $\Phi^{\pm}_i(u)$ on any finite-dimensional representation of $\mathcal{U}_q(\hat{\mathfrak{g}})$ have the form

$$\gamma^{\pm}_i(u) = q_i^{deg(Q_i)-deg(R_i)} \frac{Q_i(uq_i^{-1})R_i(uq_i)}{Q_i(uq_i)R_i(uq_i^{-1})}, \tag{2.17}$$

where

$$Q_i(u) = \prod_{a\in\mathbb{C}^*} (1-ua)^{q_{i,a}} \text{ and } R_i(u) = \prod_{a\in\mathbb{C}^*} (1-ua)^{r_{i,a}}$$

are polynomials.

Proof. This is first proved by direct inspection in the sl_2-case as it is known that for this special type, any simple finite-dimensional module is isomorphic to a tensor product of evaluation representations. Then the general case is obtained by a reduction to the sl_2-case, by considering subalgebras that are isomorphic to the quantum affine algebra $\mathcal{U}_q(\widehat{\mathfrak{sl}_2})$ □

Definition 2.3.1
The q-character is a the map $\chi_q : Rep(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathbb{Z}[Y^{\pm 1}_{i,a}]_{i\in I, a\in C^*} =: \mathcal{Y}$ defined by

$$\chi_q(V) = \sum_{\gamma^{\pm}_i} dim(V_{\gamma^{\pm}_i}) \prod_{a\in\mathbb{C}^*} Y_{i,a}^{q_{i,a}-r_{i,a}}. \tag{2.18}$$

Proposition 2.3.2
χ_q : $Rep(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathbb{Z}[Y^{\pm 1}_{i,a}]_{i\in I, a\in C^*}$ is a ring morphism. In particular, for any V, W finite-dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ we have:

- $\chi_q(V \oplus W) = \chi_q(V) + \chi_q(W)$

- $\chi_q(V \otimes W) = \chi_q(v)\chi_q(W)$.

Theorem 2.3.1

Let us call $\beta : \mathbb{Z}[Y_{i,a}^{\pm}]_{i\in I, a\in\mathbb{C}^*} \to \mathbb{Z}[y_i^{\pm}]_{i\in I}$ the ring homomorphism which "forgets" the spectral parameters $a \in \mathbb{C}^*$, that is $\beta(Y_{i,a}^{\pm}) = y_i^{\pm}$. Recall the restriction map of Definition 2.2.4. Then the following diagram is commutative:

$$\begin{array}{ccc} Rep(\mathscr{U}_q(\hat{\mathfrak{g}})) & \xrightarrow{\chi_q} & \mathbb{Z}[Y_{i,a}^{\pm}]_{i\in I, a\in\mathbb{C}^*} \\ \downarrow{\scriptstyle res} & & \downarrow{\scriptstyle \beta} \\ Rep(\mathscr{U}_q(\mathfrak{g})) & \xrightarrow{\chi} & \mathbb{Z}[y_i^{\pm}]_{i\in I} \end{array}$$

Proof. Since all the maps involved are ring homomorphisms, we only need to prove that $\beta(\chi_q(V)) = \chi(res(V))$ for any finite-dimensional $\mathscr{U}_q(\hat{\mathfrak{g}})$-module V. By a similar argument as the one used in Proposition 2.2.1, one can prove that each weight space in the decomposition of V as $\mathscr{U}_q(\hat{\mathfrak{g}})$-module is contained in a weight space for the decomposition as $\mathscr{U}_q(\mathfrak{g})$-module. For example, if the weight $\gamma_i^{\pm}(u)$ for the action of $\Phi_i^{\pm}(u)$ has corresponding polynomials $Q_i(u) = \prod_{a\in\mathbb{C}^*}(1-ua)^{q_{i,a}}$ and $R_i(u) = \prod_{a\in\mathbb{C}^*}(1-ua)^{r_{i,a}}$, then the weight for the restricted action is $\sum_i(deg(Q_i) - deg(R_i))\omega_i$. Hence for any $\lambda \in \mathscr{P}$, the $\mathscr{U}_q(\mathfrak{g})$-weight space V_λ contains all the $V_{(\gamma)}$ such that $\omega((\gamma)) = \sum_i(deg(Q_i) - deg(R_i))\omega_i = \lambda$. In particular, in the character of V the monomial $\prod_i y_i^{deg(Q_i)-deg(R_i)} = y_\lambda$ will appear.

So, suppose we have $\chi_q(V) = \sum_{(\gamma)} dim(V_{(\gamma)}) \prod_{a\in\mathbb{C}^*} Y_{i,a}^{q_{i,a}-r_{i,a}}$, then

$$\begin{aligned} \beta(\chi_q(V)) &= \sum_{(\gamma)} dim(V_{(\gamma)}) \prod_i y_i^{deg(Q_{\gamma,i})-deg(R_{\gamma,i})} \\ &= \sum_{\lambda\in\mathscr{P}} \sum_{(\gamma),\omega(\gamma)=\lambda} dim(V_{(\gamma)}) \prod_i y_i^{deg(Q_{\gamma,i})-deg(R_{\gamma,i})} \\ &= \sum_\lambda dim(V_\lambda) \prod_i y_i^{\lambda(h_i)} \\ &= \sum_\lambda dim(V_\lambda) y_\lambda \\ &= \chi(res(V)). \end{aligned}$$

$\square$

Definition 2.3.2

For any monomial $m = \prod_{i\in I, a\in\mathbb{C}^*} Y_{i,a}^{u_{i,a}(m)} \in \mathscr{Y}$, we can assign a weight $\omega(m)$ in $\mathscr{P}^+$ by

$$\omega(m) := \sum_{i\in I, a\in\mathbb{C}^*} u_{i,a}(m)\omega_i.$$

Recalling Definition (2.2.3), we notice that the weight of the monomial m corresponding to the weight $\mathbf{P}$ (that is $V(\mathbf{P}) = L(m)$), is $\omega(m) = \sum_{i\in I} deg(P_i)\omega_i$.

Example 2.3.1

We write the character of the two-dimensional representation obtained by evaluation in Example 2.2.2. In order to compute the character, observe that the roles of the polynomials Q and R of Definition (2.3.1) for γ_1^+ and γ_2^+ are played respectively by P_0, 1 and 1, P_1. So, for γ_1^+ we have the degrees $q_{q^{-1}a^{-1}} = 1$, $r_{q^{-1}a^{-1}} = 0$, $q_{qa^{-1}} = 0$, $r_{qa^{-1}} = 1$. We can finally compute the q-character:

$$\chi_q(V_1(a^{-1}q^{-1})) = Y_{a^{-1}q^{-1}} + Y_{a^{-1}q}^{-1}. \tag{2.19}$$

Example 2.3.2

In this example we compute the q-character of the evaluation representation of dimension three for $\mathscr{U}_q(\widehat{\mathfrak{sl}_3})$. First, we consider the Drinfeld presentation for $\mathscr{U}_q(\widehat{\mathfrak{sl}_3})$, i.e. it is the algebra generated by $x_{i,r}^{\pm}$ $(i \in I,\ r \in \mathbb{Z})$, $k_i^{\pm 1}$, $(i \in I)$, $h_{i,r}$ $(i \in I,\ r \in \mathbb{Z}\setminus\{0\})$, $I = \{1,2\}$ and central element $c^{\pm\frac{1}{2}}$, subject to the relations as in Theorem 2.2.1. Recall from Example (1.0.2) that the classical character for the simple 3-dimensional representation of $\mathfrak{sl}_3$ is

$$\chi = y_1 + y_2 y_1^{-1} + y_2^{-1}.$$

In particular, the weights are ω_1, $\omega_2 - \omega_1$ and $-\omega_2$, all with multiplicity one. We associate to them the generators of the corresponding weight spaces, v_0, v_1, v_2; v_0 is the highest weight vector. To construct the representation for $\mathscr{U}_q(\widehat{\mathfrak{sl}_3})$ we will mimic the action on $\mathbb{C}^3 = \langle v_0, v_1, v_2\rangle$ of $\mathfrak{sl}_3$ and $\mathscr{U}_q(\mathfrak{sl}_3)$. We choose $\mathfrak{h} = \mathbb{C}h_1 \oplus \mathbb{C}h_2$ as a Cartan subalgbra of $\mathfrak{sl}_3$, where $h_1 = e_{11} - e_{22}$, $h_2 = e_{22} - e_{33}$. That is,

$\mathfrak{sl}_3 = \mathfrak{h} \oplus \langle e_1, e_2, f_1, f_2 \rangle$. *Then, h_1 and h_2 act on $\mathbb{C}^3$ as follows:*

$$\begin{aligned}
h_1 \cdot v_0 &= \omega_1(h_1)v_0 = v_0 \\
h_1 \cdot v_1 &= (\omega_2 - \omega_1)(h_1)v_1 = -v_1 \\
h_1 \cdot v_2 &= -\omega_2(h_1)v_2 = 0 \\
h_2 \cdot v_0 &= \omega_1(h_2)v_0 = 0 \\
h_2 \cdot v_1 &= (\omega_2 - \omega_1)(h_2)v_1 = v_1 \\
h_2 \cdot v_2 &= -\omega_2(h_2)v_2 = -v_2.
\end{aligned}$$

Thus, as k_1 and k_2 are the q-analogues of h_1 and h_2 respectively, in our representation of $\mathscr{U}_q(\widehat{\mathfrak{sl}_3})$ we require:

	k_1	k_2
v_0	qv_0	v_0
v_1	$q^{-1}v_1$	qv_1
v_2	v_2	$q^{-1}v_2$

.

Since we are constructing an evaluation representation, it seems reasonable to make the $x_i^{\pm 1}$'s act as follows:

$$x^+_{1,r} = a^r U, \quad x^-_{1,r} = b^r V \tag{2.20}$$

$$x^+_{2,r} = c^r W, \quad x^-_{2,r} = d^r Y, \tag{2.21}$$

where U, V, W, Y are endomorphisms of $\mathbb{C}^3$ and a, b, c, d are scalars. In particular, for $r = 0$, we have that U, V, W, Y represent the action of e_1, f_1, e_2, f_2. If we fix the basis v_0, v_1, v_2, then we can define

$$U := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$W := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Now we want to find the relations between a, b, c and d. In our case, the relation (2.6) gives us

$$x^+_{1,r+1}x^+_{2,m} - q^{-1}x^+_{2,m}x^+_{1,r+1} = q^{-1}x^+_{1,r} + x^+_{2,m+1} - x^+_{2,m+1}x^+_{1,r}$$

which with our assumptions is equivalent to

$$a^{r+1}Uc^mW - q^{-1}c^mWa^{r+1}U = q^{-1}a^rUc^{m+1}W - c^{m+1}Wa^rU.$$

So, we obtain

$$a = q^{-1}c.$$

Using the same relation with the $x^-_{i,r}$'s we find

$$b = q^{-1}d.$$

Now we use relation (2.8) to get

$$[x^+_{1,r}, x^-_{1,r'}] = \frac{\phi^+_{1,r+r'} - \phi^-_{1,r+r'}}{q - q^{-1}}.$$

If $r + r' > 0$, then

$$\phi^+_{1,r+r'} = (q - q^{-1})(a^r b^{r'} UV - b^{r'} a^r VU). \tag{2.22}$$

In particular, acting with $\phi^+_{i,m}$ both on v_0 and v_1 we can deduce that $a = b$. Using the other relations found above, we can say also that $c = d$. To sum up, for a choice of $a \in \mathbb{C}^*$ we have

$$b = a,\ c = qa,\ d = qa.$$

Equation 2.22 allows us to write the action of $\phi^+_{1,m}$ for $m > 0$ on the basis vector, namely

$$\begin{aligned}
\phi^+_{1,m} \cdot v_0 &= (q - q^{-1})a^m v_0 \\
\phi^+_{1,m} \cdot v_1 &= -(q - q^{-1})a^m v_1 \\
\phi^+_{1,m} \cdot v_2 &= 0
\end{aligned}$$

With the same steps we find

$$\begin{aligned}
\phi_{2,m}^{+} \cdot v_0 &= 0 \\
\phi_{2,m}^{+} \cdot v_1 &= (q-q^{-1})(qa)^m v_1 \\
\phi_{2,m}^{+} \cdot v_2 &= -(q-q^{-1})(qa)^m v_2.
\end{aligned}$$

We can now use the development (2.9) of the $\phi_{1,m}^{+}$'s to describe the weights of the representation. We have

$$\begin{aligned}
\Phi_1^{+}(z) \cdot v_0 &= \sum_{m\geq 0} \phi_{1,m}^{+} z^m \cdot v_0 \\
&= k_1 \cdot v_0 + (q-q^{-1}) \sum_{m>0} a^m z^m v_0 \\
&= \left(q + (q-q^{-1}) \frac{az}{1-az}\right) v_0 \\
&= q \frac{Q_0(zq^{-1})}{Q_0(zq)} v_0,
\end{aligned}$$

where $Q_0(z) := 1 - aq^{-1}z$. *We also have* $\Phi_2^{+} \cdot v_0 = v_0$, *so we say that* $v_0 \in V_{\gamma^+}$, $\gamma^+ = (\gamma_1^+, \gamma_2^+)$, $\gamma_1^+(z) = q\frac{Q_0(zq^{-1})}{Q_0(zq)}$ *and* $\gamma_2^+(z) = 1$. *So, the monomial corresponding to* γ^+ *is just* $Y_{1,aq^{-1}}$. *Repeating the same steps we find:*

$$\begin{aligned}
\Phi_1^{+} \cdot v_1 &= q^{-1} \frac{1-q^2az}{1-az} v_1 \\
\Phi_2^{+} \cdot v_1 &= q \frac{1-q^{-1}az}{1-qaz} v_1,
\end{aligned}$$

hence the first polynomial is $1-aqz$ *and the second one is* $1-az$. *Considering also the degrees of q, we have that the monomial corresponding to* $V_\mu = \langle v_1 \rangle$, $\mu = (q^{-1}\frac{1-q^2az}{1-az}, q\frac{1-q^{-1}az}{1-qaz})$ *is* $Y_{1,aq}^{-1} Y_{2,a}$. *Finally,*

$$\begin{aligned}
\Phi_1^{+} \cdot v_2 &= v_2 \\
\Phi_2^{+} \cdot v_2 &= q \frac{1-q^{-1}az}{1-qaz} v_2,
\end{aligned}$$

hence the first polynomial is 1 *and the second one is* $1-aq^2z$. *So the monomial for* $V_\epsilon = \langle v_2 \rangle$, $\epsilon = (1, q\frac{1-q^{-1}az}{1-qaz})$ *is* Y_{2,aq^2}^{-1}. *These calculations let us compute the q-character:*

$$\chi_q(V) = Y_{1,aq^{-1}} + Y_{1,aq}^{-1} Y_{2,a} + Y_{2,aq^2}^{-1}.$$

We can notice that the q-character is really the q-analogue of the classical character. Moreover, the weights associated to each monomial in the expression are exactly ω_1, $\omega_2 - \omega_1$ and $-\omega_2$, i.e. the same weights as in the classical version.

We explore now some important properties of the q-character.

Proposition 2.3.3
Let $U, V, W \in Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$ be some representations such that there is a short exact sequence of $\mathcal{U}_q(\hat{\mathfrak{g}})$-module

$$0 \to V \xrightarrow{i} W \xrightarrow{\pi} U \to 0.$$

Then $\chi_q(W) = \chi_q(V) + \chi_q(U)$.

Proof. Recall that $\chi_q(W) = \sum_\gamma dim(W_\gamma)m_\gamma$, where $\gamma = (\gamma_{i,m})_{i\in I, m\in\mathbb{Z}}$ and W_γ is the generalized eigenspace relative to γ for the action of the ϕ_i's. We can consider a basis B_V for V and complete it with a complement B'_W to a basis of the entire W, obtaining $B_W = B_V \sqcup B'_W$. In particular, as the projection "kills" V, we have $\pi(B_W) = \pi(B'_W)$ and, as π is surjective, $\pi(B'_W)$ is a basis for U. Then we may call $B_U = \pi(B'_W)$. Now, on the basis $B_W = B_V \sqcup B'_W$, we represent ϕ_i with the matrix

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where A is the matrix of the restriction of ϕ_i to the sub-module V on the basis B_V. We show that C represents the action of ϕ_i on U. Indeed, if we consider $u \in B_U$, then there exists $\tilde{u} \in B'_W$ such that $\pi(\tilde{u}) = u$. Now

$$\phi_i(u) = \phi_i(\pi(\tilde{u})) = \pi(\phi_i(\tilde{u})) = \pi\begin{pmatrix} x \\ y \end{pmatrix} = \pi(y),$$

where we have used that π is a morphism of module (so it commutes with ϕ_i) and x, y are the components of $\phi_i(\tilde{u})$ respectively in B_V and B'_W. So $\phi_i(u) = \pi(y)$ is just the vector made of the components of $\phi_i(\tilde{u})$ in C. So we can represent ϕ_i as

$$\begin{pmatrix} \phi_i|_V & B \\ 0 & \phi_i|_U \end{pmatrix}. \tag{2.23}$$

Now we prove that $dim(W_\gamma) = dim(V_\gamma) + dim(U_\gamma)$. To do this, we obtain from (2.23) that the characteristic polynomial of ϕ_i on W is equal to the product of the characteristic polynomial of ϕ_i on V by the characteristic polynomial on U.

So, in particular, the algebraic multiplicity of γ in W is the sum of the multiplicities in V and U, i.e. $dim(W_\gamma) = dim(V_\gamma) + dim(U_\gamma)$. □

Proposition 2.3.4
For every finite-dimensional representation V of $\mathcal{U}_q(\hat{\mathfrak{g}})$ there exists a finite, strictly increasing sequence of sub-representations

$$\{0\} = V_s \subset V_{s-1} \subset \cdots \subset V_0 = V$$

such that the quotients $G_i = V_i/V_{i+1}$ are irreducibles $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules, for $0 \leq i \leq s-1$. Such a sequence is called Jordan-Hölder sequence. The length of the sequence and the quotients submodules depend only on the representation V up to permutation. Moreover

$$\chi_q(V) = \sum_{i=0}^{s-1} \chi_q(G_i).$$

This means that it is possible to write in a unique way the q-character of every $\mathcal{U}_q(\hat{\mathfrak{g}})$-module as a linear combination with integer coefficients of q-characters of simple $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules.

Proof. If V is irreducible, then $\{0\} \subset V$ is the Jordan-Hölder sequence. For V reducible, we proceed by induction on the dimension of V. If $dim(V) = 0$, there is nothing to prove. Otherwise, let V' be a maximal proper submodule of V, so that V/V' is irreducible. By inductive hypothesis, V' admits a Jordan-Hölder sequence and we can complete it on the right with V in order to get the result (as V/V' is irreducible). Now, for each $i = 0, \ldots, s-1$, we have the short exact sequence $0 \to V_{i+1} \to V_i \to G_i \to 0$, so by Proposition 2.3.3 $\chi_q(V_i) = \chi_q(V_{i+1}) + \chi_q(G_i)$. Thus, $\chi_q(V) = \chi_q(V_0) = \chi_q(V_1) + \chi_q(G_0) = \chi_q(V_2) + \chi_q(G_1) + \chi_q(G_0) = \cdots = \sum_{i=0}^{s-1} \chi_q(G_i)$. We are left to prove the uniqueness. Suppose we have another Jordan-Hölder sequence, so that $\chi_q(V) = \sum_{i=0}^{r-1} m_i \chi_q(W_i)$, the W_i

being irreducible, $m_i \in \mathbb{Z}$. Since the q-character is uniquely defined, it must be $\sum_i m_i = s$ and the set of factors G_i has to be equal to the set of W_i, counted with multiplicities. Thus, up to permutation, the length of the sequence and the quotient submodules are unique. $\square$

Lemma 2.3.1

Let $V = V(\mathbf{P})$ be a simple finite-dimensional $\mathcal{U}_q(\hat{\mathfrak{g}})$- module, $\mathbf{P} = (P_i)_{i\in I}$. Let $\chi_q(V) = \sum_\gamma dim(V_\gamma)m_\gamma$ be its character and let $m = \sum_i deg(P_i)\omega_i$ be the highest weight of V as a $\mathcal{U}_q(\mathfrak{g})$-module. Then, the q-character has the expression

$$\chi_q(V) = m + \textit{lower weight monomials},$$

where the order on the weights is the Chevalley order defined in Chapter 1.

Proof. In the expression of the q-character we have a sum on the weights γ of the representation and each monomial m_γ has a weight $\omega(m_\gamma)$ as in Definition (2.3.2). In particular, $\omega(m_\gamma)$ is a weight for V regarded as a $\mathcal{U}_q(\mathfrak{g})$-module. Then, since m corresponds to the highest weight, it follows that the other monomials have a lower weight. Moreover, the multiplicity of the highest weight is one, thus the coefficient of m in the expression of q-character is 1. $\square$

The category of finite-dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ is not semisimple, namely it is not true that every short exact sequence of $\mathcal{U}_q(\hat{\mathfrak{g}})$-module splits. For example, we can consider $\mathcal{U}_q(\widehat{\mathfrak{sl}_2})$ and its module

$$W := V_1(1) \otimes V_1(q^2).$$

We can prove that W contains a unique proper submodule W' of dimension three isomorphic to $V_2(q)$ and it is not semisimple. However, there is a short exact sequence

$$0 \to W' \to W \to \mathbb{1} \to 0.$$

Thus, by Proposition 2.3.3, we have $\chi_q(W) = \chi_q(\mathbb{1}) + \chi_q(W')$, but W is not isomorphic to $\mathbb{1} \oplus W'$. In other words, the category of finite-dimensional $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules is not semisimple. In particular, χ_q is not injective on the ring $Rep(\mathcal{U}_q(\hat{\mathfrak{g}})) =$

$\bigoplus \mathbb{Z}[V]$, where the sum is over all the finite-dimensional $\mathcal{U}_q(\hat{\mathfrak{g}})$-module.
We would like to have an injective character, so we define the *reduced* Grothendieck ring by quotienting $Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$ by the ideal J generated by the relations ($[W] = [V]+[U]$) where $W, V, U \in Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$ such that there is an exact sequence of $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules $0 \to V \to W \to U \to 0$. We get

$$\begin{aligned}\mathcal{K}_0(\mathcal{C}) &:= \Big(\bigoplus_{V module} \mathbb{Z}[V]\Big)\Big/J \\ &= \bigoplus_{V simple} \mathbb{Z}[V].\end{aligned}$$

Now, thanks to Proposition 2.3.3 we obtain that the q-character map passes to quotient, i.e.

$$\chi_q : \mathcal{K}_0(\mathcal{C}) \to \mathbb{Z}\left[Y_{i,a}^{\pm 1}\right]_{\substack{i\in I\\ a\in\mathbb{C}^*}}$$

is well defined.

Theorem 2.3.2
The q-character $\chi_q : \mathcal{K}_o(\mathcal{C}) \to \mathbb{Z}\left[Y_{i,a}^{\pm 1}\right]_{\substack{i\in I\\ a\in\mathbb{C}^*}}$ is an injective ring morphism.

Proof. Recalling Definition (2.2.3), each simple $\mathcal{U}_q(\hat{\mathfrak{g}})$-module is a $L(m)$ for a monomial $m \in \mathbb{Z}\left[Y_{i,a}^{\pm 1}\right]_{i\in I, a\in\mathbb{C}^*}$ of the form $m = \prod_{i,a} Y_{i,a}^{u_{i,a}(m)}$.
We notice that proving the injectivity of χ_q is equivalent to prove that the elements $(\chi_q(L(m)))_m$ are linearly independent in $\mathbb{Z}\left[Y_{i,a}^{\pm 1}\right]_{i\in I, a\in\mathbb{C}^*}$. By definitions (2.3.2) and (2.2.3), the monomial m has weight $\lambda := \omega(m) = \sum_i deg(P_i)\omega_i$. In particular, by Proposition 2.2.1, λ is the highest weight vector for $L(m)$ regarded as a $\mathcal{U}_q(\mathfrak{g})$ and its multiplicity (i.e. the dimension of the corresponding eigenspace) is 1. To show the linear independence, let us consider the equation

$$A = \sum_m a_m \chi_q(L(m)) = 0,$$

where the a_m are almost all zero. The q-character of each $L(m)$ is of the form

$$\chi_q(L(m)) = m + \text{ lower weight monomials.} \tag{2.24}$$

So, if we consider a monomial $\tilde{m}$ such that $a_{\tilde{m}} \neq 0$ and $\tilde{m}$ is maximal in A, then it must appear in A only one time; moreover it identifies uniquely the corresponding simple module. Thus, since $A=0$, we have $a_{\tilde{m}}=0$ and this is a contradiction. $\square$

Remark 2.3.1

Theorem 2.3.2 implies that the reduced Grothendieck ring $\mathcal{K}_0(\mathcal{C})$ is commutative, i.e. $[V]\cdot[W]=[W]\cdot[V]$, even if in $Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$ in general $V\otimes W$ and $W\otimes V$ are not isomorphic (in technical language, the category $Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is not braided).

From the representation theory of $\mathcal{U}_q(\hat{\mathfrak{g}})$ we know that given a monomial $m=\prod_{i,a} Y_{i,a}^{u_{i,a}}$, $u_{i,a}\geq 0$, it is possible to associate to it the so-called *standard* module

$$M(m):=\bigotimes (V_i(a))^{u_{i,a}}, \tag{2.25}$$

where $V_i(a):=L(Y_{i,a})$ are fundamental modules. At the level of the category, different orders of the tensor product will give non isomorphic representations in general. But we will only use the class of $M(m)$ in the Grothendieck ring, which is commutative, so the order is not important in the following.

Proposition 2.3.5

The q-characters of the fundamental modules are algebraically independent.

Proof. We prove equivalently that the q-characters of standard modules are linearly independent in $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{\substack{i\in I\\ a\in\mathbb{C}^*}}$. Indeed, assume we have the polynomial equation $A=Q(\chi_q(V_{i_1}(a_{i_1})),\dots,\chi_q(V_{i_N}(a_{i_N}))=0$, then each monomial which appears in A is of the form

$$\begin{aligned} Q_j &= \chi_q(V_{i_1}(a_{i_1}))^{n_1}\cdots\chi_q(V_{i_N}(a_{i_N}))^{n_N} \\ &= \chi_q(V_{i_1}(a_{i_1})^{\otimes n_1}\otimes\cdots,\otimes V_{i_N}(a_{i_N})^{\otimes n_N}) \\ &= \chi_q(M(m)), \quad m=\prod_{k=1}^{N} Y_{i_k,a_{i_k}}^{n_k}. \end{aligned}$$

Thus, the algebraic combination $A=0$ can be rewritten as a linear combination $A'=\sum_m \alpha_m\chi_q(M(m))=0$. For a standard module as (2.25), the q-character is $\chi_q(M(m))=\prod_{i,a}\prod_{j=1}^{u_{i,a}}\chi_q(V_j(a))$ and by (2.24) we deduce that the maximal weight term in $\chi_q(M(m))$ is $\prod_{i,a} Y_{i,a}^{u_{i,a}}$ with multiplicity 1, that is exactly m. But m identifies uniquely the standard module $M(m)$. So if $\chi_q(M(m))$ appears in the linear combination with m of maximal weight, only the term $\alpha_m \chi_q(M(m))$ can contribute to the multiplicity of m in the sum. Hence α_m has to be zero in order to have $A' = 0$, and we obtain a contradiction if the linear combination is non-trivial. $\square$

Let us call, for convenience, $X_{i,a}$ the q-character of the fundamental module $L(Y_{i,a})$. By the previous Proposition, we have that the ring generated by the characters of fundamental modules is exactly $\mathbb{Z}[X_{i,a}]_{\substack{i\in I\\ a\in\mathbb{C}^*}}$, as we have shown the algebraic independence of the $(X_{i,a})_{i,a}$. Then we have the following

Theorem 2.3.3
$\chi_q(\mathscr{K}_o(\mathscr{C})) = \mathbb{Z}[X_{i,a}]_{\substack{i\in I\\ a\in\mathbb{C}^*}}$.

Proof. Let us call $A = \mathbb{Z}[X_{i,a}]_{i\in I, a\in\mathbb{C}^*}$ the subring of $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*}$ generated by the q-characters of fundamental modules. We need to show that $\chi_q(\mathscr{K}_o(\mathscr{C})) = A$. By definition, we have the inclusion $A \subset \chi_q(\mathscr{K}_0(\mathscr{C}))$. For the opposite inclusion it suffices to prove that the q-character of a simple module can be written as a linear combination (with integer coefficients) of the $(X_{i,a})_{i\in I, a\in\mathbb{C}^*}$, as $\mathscr{K}_o(\mathscr{C})$ is generated as a ring by the simple modules. To do so, we recall Corollary (2.2.2.1), i.e. every simple module $V(\mathbf{P})$ of $\mathscr{U}_q(\hat{\mathfrak{g}})$ is isomorphic to a subquotient of a tensor product of fundamental representations. Let us call W_0 such a tensor product of fundamental representations, $W_0 = V_{i_1}(a_{i_1}) \otimes \cdots \otimes V_{i_k}(a_{i_k})$ and $v_{i_1}, \ldots, v_{i_k}$ the corresponding highest weight vectors. So, we can assume that $V(\mathbf{P}) \simeq G/H$, where G is the submodule of W_0 generated by the tensor product of highest weight vectors, i.e. $G = \mathscr{U}_q(\hat{\mathfrak{g}}) \cdot (v_{i_1} \otimes \cdots \otimes v_{i_k})$, while H is a submodule of G. We obtain two short exact sequences:

$$0 \to H \to G \to V(\mathbf{P}) \to 0 \tag{2.26}$$

and

$$0 \to G \to W_0 \to W_0/G \to 0. \tag{2.27}$$

By Proposition 2.3.3, (2.26) implies $\chi_q(G) = \chi_q(V(\mathbf{P})) + \chi_q(H)$, while (2.27) implies $\chi_q(W_0) = \chi_q(G) + \chi_q(W_0/G)$. As a result, we get

$$\chi_q(W_0) = \chi_q(V(\mathbf{P})) + \chi_q(W_0/G) + \chi_q(H). \tag{2.28}$$

Let m be the highest weight monomial in $\chi_q(V(\mathbf{P}))$. Then $\omega(m) = \sum_i deg(P_i)\omega_i$, for $\mathbf{P} = (P_i)_{i\in I}$. We finish our proof by induction on the weight of such monomials by the Chevalley order. Firstly, observe that the induction makes sense. Indeed, the base step holds, since the fundamental weights are minimal in $(\mathscr{P}^+, \preceq)$ and if $\omega(m) = \omega_i$, then $V(\mathbf{P}) = L(m) = V_i(a)$ for some $a \in \mathbb{C}^*$, so by definition $\chi_q(V_i(a)) = X_{i,a}$. Now, the monomial corresponding to a simple module has weight in $\mathscr{P}^+$, so there exists an index i such that $\omega(m) \succ \omega_i$ and moreover for every $\lambda \in \mathscr{P}^+$ there are only finitely many μ such that $\lambda \succ \mu \succ \omega_i$. By definition of W_0 and by the structure of q-characters of fundamental modules we have that

$$\chi_q(W_0) = \prod_{j=1}^{k} Y_{i_j,a_{i_j}} + \text{lower weight monomials}.$$

Let us call $m = \prod_{j=1}^{k} Y_{i_j,a_{i_j}}$ the leading term. So we assume as inductive hypothesis that for every simple module $L(m')$ with $\omega(m') \prec \omega(m)$, $\chi_q(L(m')) \in A$. In (2.28), the maximal monomial in the LHS is m and it appears also in $\chi_q(V(\mathbf{P}))$ in the RHS. So, writing $\chi_q(W_0/G)+\chi_q(H)$ as a linear combination with integer coefficients of q-characters of simple modules (by Proposition 2.3.4) we obtain that these simple modules must have leading monomial of weight strictly lower than $\omega(m)$. Thus, by inductive hypothesis $\chi_q(W_0/G) + \chi_q(H)$ is an element in A. Also $\chi_q(W_0)$ is in A since it is a product of $X_{i,a}$'s. Hence we conclude that $\chi_q(V(\mathbf{P}))$ is in A. $\square$

Chapter 3

The Weyl group action

This Chapter and the next one are the core of the thesis as we present the new results by Frenkel and Hernandez which appear in [10]. We follow the construction of the Weyl group action on the image of q-characters as the authors do in the article and we try to explicit and explain some points in a more didactic way and with additional technical details. The general ideal is the following. We consider the q-analogues of the monomials a_i, $i \in I$, of Equation 1.3, i.e. the elements of $\mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*}$ defined by

(3.1)

$$A_{i,a} := Y_{i,aq_i^{-1}} Y_{i,aq_i} \left(\prod_{\{j\in I | C_{ji}=-1\}} Y_{j,a} \prod_{\{j\in I | C_{ji}=-2\}} Y_{j,aq^{-1}} Y_{j,aq} \prod_{\{j\in I | C_{ji}=-3\}} Y_{j,aq^{-2}} Y_{j,a} Y_{j,aq^2} \right)^{-1}.$$

Indeed, by Definition 2.3.2, one can see that the weight of $A_{i,a}$ is α_i for every $a \in \mathbb{C}$. The action of each simple refection $s_i \in \mathcal{W}$, $i \in I$, is denoted by Θ_i. A delicate aspect of the definition of Θ_i is that it involves infinite series. Therefore we need to extend the ring $\mathcal{Y}$ with some completions $\tilde{\mathcal{Y}}^w$ indexed by the elements w of $\mathcal{W}$ and define Θ_i on the direct sum Π of these completions. This allow us to deal with the q-difference equation

$$\Sigma_{i,a} = 1 + A_{i,a}^{-1} \Sigma_{i,aq_i^{-2}} \tag{3.2}$$

which has unique solution $\Sigma_{i,a}$ in these completions. Finally, the automorphisms Θ_i's are defined by $\Theta_i(Y_{j,a}^{\pm 1}) := Y_{j,a}^{\pm 1}$, if $i \neq j$ and

$$\Theta_i(Y_{i,a}) := Y_{i,a} A_{i,aq_i^{-1}}^{-1} \frac{\Sigma_{i,aq_i^{-3}}}{\Sigma_{i,aq_i^{-1}}}. \tag{3.3}$$

Observe that in the limit $q \to 1$ the fraction cancels out and we are left with the classical action (1.3).

3.1 Some technical considerations

In this section we present some completions of $\mathscr{Y}$ and the q-difference equation defined above.

3.1.1 Completions

We denote by $\mathscr{Y}$ the ring of Laurent polynomials $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^*}$ and by $\mathscr{M}$ the multiplicative group of monomials. Then, there is a group homomorphism $\omega : \mathscr{M} \to \mathscr{P}$ which assigns to each monomial $m = \prod_{i,a} Y_{i,a}^{u_{i,a}(m)}$ the weight

$$\omega(m) := \sum_{i \in I, a \in \mathbb{C}^*} u_{i,a}(m) \omega_i. \tag{3.4}$$

Notice that $\omega(Y_{i,a}) = \omega_i$ and $\omega(A_{i,a}) = \alpha_i$.

In the new action of the Weyl group on the image of q-characters defined by Frenkel and Hernandez, the elements $(\Sigma_{i,a})_{i \in I, a \in \mathbb{C}^*}$ appear and these are defined as collections of formal series in the $A_{i,a}^{\pm 1}$. Thus, in order to give sense to this action, we need to extend the ring $\mathscr{Y}$. Since we are talking about completions, we should clarify what type of convergence we are considering.

Given a weight $\lambda \in \mathfrak{h}^*$, we call the *cone under* λ the set

$$D_\lambda := \{\lambda - \sum_{i \in I} n_i \alpha_i | n_i \geq 0\}. \tag{3.5}$$

Definition 3.1.1

A sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathscr{Y}$ converges (formally) to $u \in \mathscr{Y}$ if

(1) There exists a finite union of cones D_λ so that for any n, the weights of the monomials in u_n belong to this finite union.

(2) $\forall \omega \in \mathscr{P}$ there exists a positive integer N_ω such that $\forall n \geq N_\omega$ we have

$$|u_n - u|_\omega = 0,$$

where $|\cdot|_\omega$ denotes the sum of all the monomials of weight ω (as defined in 3.4).

As for a given ω and a cone D_λ, there is a finite number of weight $\gamma \succeq \omega$ which are in D_λ, the definition implies the following: $\forall \omega \in \mathscr{P}$ there exists a positive integer N_ω such that $\forall n \geq N_\omega$ and $\forall \gamma \succeq \omega$ we have

$$|u_n - u|_\gamma = 0.$$

Definition 3.1.2

A Cauchy sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathscr{Y}$ is a sequence such that

(1) There exists a finite union of cones D_λ so that for any n, the weights of the monomial in u_n belong to this finite union.

(2) $\forall \omega \in \mathscr{P}$ there exists a positive integer N_ω such that $\forall n,\ m \geq N_\omega$ we have

$$|u_n - u_m|_\omega = 0.$$

As above, the definition implies that $\forall \omega \in \mathscr{P}$ there exists a positive integer N_ω such that $\forall n,\ m \geq N_\omega$ and $\forall \gamma \succeq \omega$ we have

$$|u_n - u_m|_\gamma = 0.$$

Remark 3.1.1

Thanks to Definition 3.1.1 of convergence on $\mathscr{Y}$, we endow $\mathscr{Y}$ with the structure of topological ring. Indeed, having defined the notion of limit, we have in particular defined what a closed subset is.

We recall the definition of category $\mathscr{O}$ for a semisimple Lie algebra $\mathfrak{g}$. We say that a $\mathfrak{g}$-module V belongs to the category $\mathscr{O}$ if it can be decomposed as a direct sum of finite-dimensional weight spaces $V = \oplus_{\mu\in\mathfrak{h}^*} V_\mu$ and if there exist a finite number of cones $D_{\lambda_1},\ldots,D_{\lambda_s}$ such that if $V_\mu \neq 0$, then $\mu \in D_{\lambda_1}\cup\cdots\cup D_{\lambda_s}$. For example, Verma modules and finite-dimensional $\mathfrak{g}$-modules are in the category $\mathcal{O}$.

Example 3.1.1

In the $\mathbb{Z}$-graded case ($\mathfrak{g} = \mathfrak{sl}_2$), we have $\mathcal{Y} = \mathbb{Z}[y^{\pm 1}]$. If we consider the characters for finite-dimensional representations of $\mathfrak{sl}_2$ as seen in Chapter 1, we do not need any formal series. However, we may want to adjoin to $\mathcal{Y}$ the characters of infinite-dimensional representations of $\mathfrak{sl}_2$ from the category $\mathcal{O}$, for example Verma modules. In this case we have to admit some series. Let us consider the sequence $(u_n)_{n\in\mathbb{N}}$, $u_n = 1 + y^{-2} + \cdots + y^{-2n}$. Then, $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, as we can choose $N_\omega = 0$, if $\omega \in \mathbb{Z}_{\geq 0}$ and $N_\omega = -\omega$, if $\omega \in \mathbb{Z}_{<0}$. Then, $\lim_{n\to\infty} u_n = \sum_{n\geq 0} y^{-2n} \in \mathbb{Z}[[y^{-1}]]$ and it is the character of the Verma module of highest weight 0. Hence, characters of representations from the category $\mathcal{O}$ belong to a completion of $\mathbb{Z}[y^{\pm 1}]$. It is well known that series might admit various developments. For example the limit above can be expressed also as

$$\sum_{n\geq 0} y^{-2n} = \frac{1}{1-y^{-2}} = -y^2 \sum_{n\geq 0} y^{2n} \in \mathbb{Z}[[y]].$$

The idea of having more than one expression for the same series is fundamental in what follows. We can also observe that if we want an action of the Weyl group on $\mathcal{Y}$, we have to extend our ring in more than one "direction". Indeed, if we take the simple reflection $s \in \mathcal{W} = \{e,s\}$ it acts on $\sum_{n\geq 0} y^{-2n}$ sending it in $\sum_{n\geq 0} y^{2n} \in \mathbb{Z}[[y]]$. It is not possible to take $\mathbb{Z}[[y^{\pm 1}]]$ as a completion of the ring because here the product of two series is not defined. A way to solve this problem is to define a variation of the category $\mathcal{O}$, say $\mathcal{O}_s$. Its objects are the modules whose weight spaces are non-zero only for those weights whose image under s is in a union of cones D_{λ_j} (as in the definition of category $\mathcal{O}$). Thus we should consider the character of modules in $\mathcal{O}_s$ which belongs to another completion of $\mathcal{Y}$. Then the action of the Weyl group is well defined on the direct sum of all these completions.

In particular, the link with category $\mathcal{O}$ justifies the following

Definition 3.1.3
Given an element $w \in \mathcal{W}$, we define $\tilde{\mathcal{Y}}^w$ as the free $\mathbb{Z}$ module consisting of formal series

$$\sum_{m \in S} a_m\, m, \quad a_m \in \mathbb{Z}, \tag{3.6}$$

where S is any subset of the group $\mathcal{M}$ with the property that for any series as (3.6) there exist some integral weights $\lambda_1, \ldots, \lambda_N$ such that for any m appearing in the series we have

$$w \cdot \omega(m) \in \bigcup_{j=1,\ldots,N} D_{\lambda_j}. \tag{3.7}$$

We require also that for any $\omega \in \mathcal{P}$,

$$|\{m \in S | \omega(m) = \omega \text{ and } a_m \neq 0\}| < \infty. \tag{3.8}$$

We give $\tilde{\mathcal{Y}}^w$ the structure of topological ring as we did with $\mathcal{Y}$, keeping in mind the condition on the support of every sequence in $\tilde{\mathcal{Y}}^w$. Moreover, the topology on $\mathcal{Y}$ is the subspace topology induced by $\tilde{\mathcal{Y}}^w$.

Example 3.1.2
We consider the case $\mathcal{U}_q(\widehat{\mathfrak{sl}_2})$. Let us call $u_n = 1 + A_a^{-1} + \cdots + A_a^{-1} \cdot \cdots \cdot A^{-1}_{aq^{-2(n-1)}}$. Then the sequence is a Cauchy one (using exactly the same argument as in the previous example). The limit is:

$$\Sigma_a = \lim_{n \to \infty} u_n = \sum_{k \geq 0} \prod_{0 < j \leq k} A^{-1}_{aq^{-2j+2}}. \tag{3.9}$$

This limits belongs to $\tilde{\mathcal{Y}}^e$. However, Σ_a can also be "expanded" in $\tilde{\mathcal{Y}}^s$ as

$$\Sigma_a = -\sum_{k > 0} \prod_{0 < j \leq k} A_{aq^{2j}}. \tag{3.10}$$

as it satisfies the same q-difference equation

$$\Sigma_a = 1 + A_a^{-1} \Sigma_{aq^{-2}}.$$

Remark 3.1.2

In Example 3.1.2, the expansion (3.9) of Σ_a belongs to $\tilde{\mathcal{Y}}^e$ since each monomial has weight $-k\alpha$, $k \geq 0$, so $e \cdot (-k\alpha) = -k\alpha \preceq \alpha$; the expansion (3.10) of Σ_a belongs to $\tilde{\mathcal{Y}}^s$ as each monomial has weight $k\alpha$ for $k \geq 0$ and we have $s \cdot (k\alpha) = -k\alpha \preceq \alpha$.

For a general $\mathcal{U}_q(\hat{\mathfrak{g}})$, the ring $\tilde{\mathcal{Y}}^e$ contains

$$\mathcal{Y} \underset{\mathbb{Z}[A_{i,a}^{-1}]_{i\in I, a\in\mathbb{C}^*}}{\otimes} \mathbb{Z}[[A_{i,a}^{-1}]]_{i\in I, a\in\mathbb{C}^*}.$$

Now we prove Lemma 2.2 from [10].

Lemma 3.1.1

$\tilde{\mathcal{Y}}^w$ is a complete topological ring with respect to the topology defined above and it is a completion of $\mathcal{Y}$ with respect to this topology.

Proof. We want to show that, for a fixed $w \in \mathcal{W}$, $\tilde{\mathcal{Y}}^w$ is a complete topological ring. We explain the proof for $w = e$ the neutral element (this is analogue for the other cases). Thus, we consider $(u_n)_{n\in\mathbb{N}}$ a Cauchy sequence in $\tilde{\mathcal{Y}}^e$ and we prove that it admits a limit. By the definition of Cauchy sequences in this space, for any weight ω, the sequence $|u_n|_\omega$ is stationary. Let us denote by $|u|_\omega$ the corresponding limit. Then, $u := \sum_{\omega\in\mathcal{P}} |u|_\omega$ is in $\tilde{\mathcal{Y}}^e$ from the point (1) in the definition of Cauchy sequences (the weights of the monomial in u belong to a fixed finite union of cones). Now, by the construction of u, the sequence u_n converges to u for the given topology. □

Dealing with a completion of $\mathcal{Y}$, we want to be able to write the weights of formal series. To do so, we construct also completions of the ring $\mathbb{Z}[y_i^{\pm 1}]_{i\in I}$. For every $\alpha = \sum_{i\in I} c_i\alpha_i \in \Delta$ we write its correspondent monomial as $a_\alpha := \prod_{i\in I} a_i^{c_i}$. Then we define

$$(\mathbb{Z}[y_i^{\pm 1}]_{i\in I})^w := \mathbb{Z}[y_i^{\pm 1}]_{i\in I} \underset{\mathbb{Z}[a^{\pm 1}_{w^{-1}(\alpha_i)}]_{i\in I}}{\otimes} \mathbb{Z}((a^{-1}_{w^{-1}(\alpha_i)}))_{i\in I}.$$

There is a ring morphism

$$\tilde{\omega}_w : \tilde{\mathcal{Y}}^w \to (\mathbb{Z}[y_i^{\pm 1}]_{i\in I})^w \tag{3.11}$$

given by the extension of the composition of the assignment $\tilde{\omega}_w(Y_{i,a}) = y_i$ and the weight morphism.

3.1.2 q-difference equation

Now we report Section 2.4 from [10], trying to give additional details.
The ring $\tilde{\mathscr{Y}}^w$ admits automorphisms τ_a for $a \in \mathbb{C}^*$, defined by $\tau_a(Y_{i,b}) = Y_{i,ab}$.

Definition 3.1.4
$\tilde{G}^w$ is the subgroup of the group of invertible elements of $\tilde{\mathscr{Y}}^w$ consisting of elements of the form $A \cdot S$ for $A \in \mathscr{M}$ and S of the form

$$S = \pm 1 + \sum_{m \in \mathscr{M}, w \cdot \omega(m) < 0} a_m m,$$

$a_m \in \mathbb{Z}$. In particular, there is a group homomorphism $\tilde{G}^w \to \mathscr{P}$ that sends $A \cdot S$ to the weight $\omega(A)$ of A.

Lemma 3.1.2
Let $\chi \in \tilde{\mathscr{Y}}^w$, $\psi \in \tilde{G}^w$ with weight in Δ and let r be an integer different from 0. Then the unique solution of

$$\chi = \psi \tau_{q^{-r}}(\chi)$$

is $\chi = 0$.

Proof. Let ν be the maximal weight in χ. We have two cases. If $\omega(\psi) \in \Delta^-$, we may call this weight $-\alpha$, where α is a positive root. Then in the equation we have that the maximal weight in the LHS is ν, while in the RHS is $\nu - \alpha$ and $\nu \succ \nu - \alpha$, so this is a contradiction. In the second case, the weight of ψ is a positive root. Since ψ is invertible, we can equivalently consider the equation $\psi^{-1}\chi = \tau_{q^{-r}}(\chi)$, where now the the weight of ψ^{-1} is a negative root. So we can apply the previous argument and conclude. $\square$

3.1.3 The ring Π

Here we follow almost verbatim the subsection 2.5 of [10]. The ring π is defined as

$$\pi := \bigoplus_{w\in\mathscr{W}} (\mathbb{Z}[y_i^{\pm 1}]_{i\in I})^w \tag{3.12}$$

and in particular $\mathbb{Z}[y_i^{\pm 1}]_{i\in I}$ embeds diagonally in π. The action of simple reflections s_i, $i \in I$, can be extended naturally to $s_i^w : (\mathbb{Z}[y_j^{\pm 1}]_{j\in I})^w \to (\mathbb{Z}[y_j^{\pm 1}]_{j\in I})^{ws_i}$. Thus, we obtain:

Lemma 3.1.3
The action of the Weyl group $\mathscr{W}$ on $\mathbb{Z}[y_i^{\pm 1}]_{i\in I}$ generated by the simple reflections $(s_i)_{i\in I}$ extends naturally to π, with s_i acting on each direct summand $(\mathbb{Z}[y_j^{\pm 1}]_{j\in I})^w$ as s_i^w.

We define

$$\Pi := \bigoplus_{w\in\mathscr{W}} \tilde{\mathscr{Y}}^w,$$

the q-analogue of π. There is a diagonal embedding $\mathscr{Y} \hookrightarrow \Pi$ and there exist automorphisms τ_a, $a \in \mathbb{C}^*$. The projection $\Pi \to \tilde{\mathscr{Y}}^w$ for each $w \in \mathscr{W}$ is denoted E_w and the weight map $\Pi \to \pi$ is denoted $\tilde{\omega}$. It restricts to $\tilde{\omega}_w$ defined in (3.11) on each summand $\tilde{\mathscr{Y}}^w$. Notice that the completion has the topology of direct sum.

Definition 3.1.5
For $i \in I$ and $a \in \mathbb{C}^*$, let

(3.13)
$$\Sigma^+_{i,a} := \sum_{k\geq 0}\prod_{0<j\leq k} A^{-1}_{i,aq_i^{-2j+2}} = 1 + A^{-1}_{i,a} + A^{-1}_{i,a}A^{-1}_{i,aq_i^{-2}} + \ldots = 1 + A^{-1}_{i,a}(1 + A^{-1}_{i,aq_i^{-2}}(1 + \ldots)),$$

where the $k = 0$ term in the series is defined to be 1;

(3.14)
$$\Sigma^-_{i,a} := -\sum_{k>0}\prod_{0<j\leq k} A_{i,aq_i^{2j}} = -A_{i,aq_i^2} - A_{i,q_i^2}A_{i,q_i^4} - \ldots = -A_{i,aq_i^2}(1 + A_{i,aq_i^4}(1 + \ldots)).$$

Given $w \in \mathscr{W}$, we define an element $\Sigma^w_{i,a} \in \tilde{\mathscr{Y}}^w$ as follows:

$$\Sigma^w_{i,a} = \Sigma^+_{i,a}, \qquad \textit{if } w(\alpha_i) \in \Delta^+;$$
$$\Sigma^w_{i,a} = \Sigma^-_{i,a}, \qquad \textit{if } w(\alpha_i) \in \Delta^-.$$

We set $\Sigma_{i,a} := (\Sigma^w_{i,a})_{w \in \mathscr{W}} \in \Pi$.

Recalling Example 3.1.2, we can see how the definition 3.1.5 makes sense. Indeed, in this case we have $\Delta^+ = \{\alpha_1\}$, $\mathscr{W} = \{e, s_1\}$ and $e(\alpha_1) = \alpha_1$, so $\Sigma^e_{1,a} = \Sigma^+_{1,a}$, while $s_1(\alpha_1) = -\alpha_1 \in \Delta^-$, so $\Sigma^{s_1}_{i,a} = \Sigma^-_{i,a}$. This is coherent on what we observed in Remark 3.1.2. The following result about elements $\Sigma^w_{i,a}$, $w \in \mathscr{W}$ is consequence of Lemma 3.1.2.

Lemma 3.1.4
The q-differences equation $\Sigma^w_{i,a} = 1 + A^{-1}_{i,a}\Sigma^w_{i,aq^{-2}}$ has unique solution $\Sigma^w_{i,a}$ in $\tilde{\mathscr{Y}}^w$. Moreover $\Sigma^w_{i,a}$ is invertible in $\tilde{\mathscr{Y}}^w$ and $\Sigma_{i,a}$ is invertible in Π.

Proof. The property of being invertible descends by definition, as $\Sigma^w_{i,a}$ is a formal series with constant term 1. Now, suppose by contradiction that there exists also another solution for the q-differences equation, say $\Omega^w_{i,a} = 1 + A^{-1}_{i,a}\Omega^w_{i,aq^{-2}}$. Then, substracting on both sides of the equation, we find

$$(\Sigma^w_{i,a} - \Omega^w_{i,a}) = A^{-1}_{i,a}(\Sigma^w_{i,aq^{-2}} - \Omega^w_{i,aq^{-2}}). \tag{3.15}$$

$A^{-1}_{i,a}$ has weight $-\alpha_i \in \Delta$, $\chi := \Sigma^w_{i,a} - \Omega^w_{i,a} \in \tilde{\mathscr{Y}}^w$ and $(\Sigma^w_{i,aq^{-2}} - \Omega^w_{i,aq^{-2}}) = \tau_{q^{-2}}(\chi)$. Thus, Equation 3.15 verifies the hypothesis of Lemma 3.1.2, so $\Sigma^w_{i,a} - \Omega^w_{i,a} = 0$, i.e. there is only one possible solution. $\square$

3.2 Definition and involution property

Recall that for each $w \in \mathscr{W}$, $\tilde{\mathscr{Y}}^w$ is a complete topological ring which is a completion of $\mathscr{Y}$. We denote by $\mathscr{Y}^w$ the image of $\mathscr{Y}$ in $\tilde{\mathscr{Y}}^w$ and of the corresponding image in Π. Here we give the precise definition of the action of the Weyl group.

Definition 3.2.1

For every $i \in I$, for every $w \in \mathscr{W}$ we define the ring homomorphism

$$\Theta_i^w : \tilde{\mathscr{Y}}^w \longrightarrow \tilde{\mathscr{Y}}^{ws_i}$$

by $\Theta_i^w(Y_{j,a}^{\pm 1}) := Y_{j,a}^{\pm 1}$ if $i \neq j$ and

$$\Theta_i^w(Y_{i,a}) := Y_{i,a} A^{-1}_{i,aq_i^{-1}} \frac{\Sigma^{ws_i}_{i,aq_i^{-3}}}{\Sigma^{ws_i}_{i,aq_i^{-1}}}. \tag{3.16}$$

Lemma 3.2.1

The homomorphisms $\Theta_i^w : \tilde{\mathscr{Y}}^w \to \tilde{\mathscr{Y}}^{ws_i}$, $i \in I$ and $w \in \mathscr{W}$, extend uniquely to a continuous homomorphism

$$\Theta_i : \Pi \to \Pi$$

by taking the direct sum over all $w \in \mathscr{W}$.

See Lemma 3.2 in [10] for the proof.

In order to prove that the action just defined is a genuine Weyl group action, we need to check that the Θ_i's are involutions and that they satisfies the braid group relations. In this section we present the proof of the first property.

Proposition 3.2.1

For all $i \in I$ the endomorphism Θ_i verifies

$$\Theta_i^2 = Id.$$

Proof. It suffices to show that $\Theta_i^2 = Id$ on $\mathscr{Y}^w$ for all $w \in \mathscr{W}$ (that is on the image of $\mathscr{Y}$ in each $\tilde{\mathscr{Y}}^w$) as Θ_i is continuous, so the property extend on the entire Π. Moreover, since Θ_i is a homomorphism, we only need to show that the property holds on $Y_{i,a}$ seen as an element in $\mathscr{Y}^w$. So, we want to prove that

$$\Theta_i^{ws_i} \circ \Theta_i^w(Y_{i,a}) = Y_{i,a}, \tag{3.17}$$

which by Definition 3.16 is equivalent to prove that

$$\Theta_i^{ws_i}\left(Y_{i,a} A^{-1}_{i,aq_i^{-1}} \frac{\Sigma^{ws_i}_{i,aq_i^{-3}}}{\Sigma^{ws_i}_{i,aq_i^{-1}}} \right) = Y_{i,a}. \tag{3.18}$$

Let us analyse the action of $\Theta_i^{ws_i}$ on each factor.

The action on $Y_{i,a}$ is given by definition. By convenience, let us call

$$X_j = \left(\prod_{\{j\in I | C_{ji}=-1\}} Y_{j,a} \prod_{\{j\in I | C_{ji}=-2\}} Y_{j,aq^{-1}} Y_{j,aq} \prod_{\{j\in I | C_{ji}=-3\}} Y_{j,aq^{-2}} Y_{j,a} Y_{j,aq^2} \right),$$

the factor with parameter j which appear in the definition of $A_{i,a}$. Since $\Theta_i^{ws_i}$ is the identity on $Y_{j,b}$ for all $b \in \mathbb{C}^*$:

$$\begin{aligned}
\Theta_i^{ws_i}(A_{i,a}^{-1}) &= \Theta_i^{ws_i}(Y_{i,aq_i}^{-1} Y_{i,aq_i^{-1}}^{-1} X_j) \\
&= Y_{i,aq_i}^{-1} A_{i,a} \frac{\Sigma_{i,a}^w}{\Sigma_{i,aq_i^{-2}}^w} Y_{i,aq_i^{-1}}^{-1} A_{i,aq_i^{-2}} \frac{\Sigma_{aq_i^{-2}}^w}{\Sigma_{aq_i^{-4}}^w} X_j \\
&= A_{i,aq_i^{-2}} A_{i,a} Y_{i,aq_i}^{-1} Y_{i,aq_i^{-1}}^{-1} X_j \frac{\Sigma_{i,a}^w}{\Sigma_{i,aq_i^{-4}}^w} \\
&= A_{i,aq_i^{-2}} A_{i,a} A_{i,a}^{-1} \frac{\Sigma_{i,a}^w}{\Sigma_{i,aq_i^{-4}}^w} \\
&= A_{i,aq_i^{-2}} \frac{\Sigma_{i,a}^w}{\Sigma_{i,aq_i^{-4}}^w}.
\end{aligned}$$

One can observe that both $1-\Sigma_{i,a}^w$ and $\Theta_i^{ws_i}(\Sigma_{i,a}^{ws_i})$ are solutions of the q-difference equation

$$\Theta_i^{ws_i}(\Sigma_{i,a}^{ws_i}) = 1 + A_{i,aq_i^{-2}} \frac{\Sigma_{i,a}^w}{\Sigma_{i,aq_i^{-4}}} \Theta_i^{ws_i}(\Sigma_{i,aq_i^{-2}}^{ws_i})$$

in $\tilde{\mathscr{Y}}^w$. As the element $A_{i,aq_i^{-2}} \frac{\Sigma_{i,a}^w}{\Sigma_{i,aq_i^{-4}}}$ belongs to $\tilde{G}^w$, using Lemma 3.1.2 we can conclude that

$$\Theta_i^{ws_i}(\Sigma_{i,a}^{ws_i}) = 1 - \Sigma_{i,a}^w = -A_{i,a}^{-1} \Sigma_{i,aq_i^{-2}}^w. \tag{3.19}$$

Thus, combining all the results above, we get

$$\Theta_i^{ws_i} \circ \Theta_i^w(Y_{i,a}) = Y_{i,a} A_{i,aq_i^{-1}}^{-1} \frac{\Sigma_{i,aq_i^{-3}}^{ws_i}}{\Sigma_{i,aq_i^{-1}}^{ws_i}} \cdot A_{i,aq_i^{-2}} \frac{\Sigma_{i,a}^w}{\Sigma_{i,aq_i^{-4}}^w} \cdot \frac{-A_{i,aq_i^{-3}}^{-1} \Sigma_{i,aq_i^{-5}}^w}{-A_{i,aq_i^{-1}}^{-1} \Sigma_{i,aq_i^{-3}}^w} = Y_{i,a}.$$

□

In the next section we prove the braid group relations between the Θ_i's.

Lemma 3.6 from [10] shows that

$$\tilde{\omega}_{ws_i} \circ \Theta_i^w = s_i \circ \tilde{\omega}_w. \tag{3.20}$$

3.3 Braid relations

In order to prove that the automorphisms defined in (3.16) generate in fact an action of the Weyl group on the ring Π, we are left to prove that they satisfy the braid group relations (R_{ij}) for $i \neq j$ as in Table 1.1. Since the Θ_i's are continuous by Lemma 3.2.1, it suffices to show the braids relations on the elements of $\mathscr{Y}^w \subset \Pi$ for all $w \in W$. Moreover, as Θ_i's are ring morphisms, we only need to prove that the braids relations are satisfied on the generators of $\mathscr{Y}^w$, that is the $Y_{k,a}$. In this section, instead of writing equations for each w-component in Π, we combine all the components in a single equation in Π. However, it is possible to switch from a presentation to the other as each braid relation is of the form $\Theta_i\Theta_j\dots\Theta_i = \Theta_j\Theta_i\dots\Theta_j$ or $\Theta_i\Theta_j\dots\Theta_i\Theta_j = \Theta_j\Theta_i\dots\Theta_j\Theta_i$. These automorphisms map $\tilde{\mathscr{Y}}^w$ to $\tilde{\mathscr{Y}}^{w'}$ where $w' = ws_is_j\dots s_i = ws_js_i\dots s_j$ or $w' = s_js_i\dots s_js_i = ws_is_j\dots s_is_j$. Since the map $\mathscr{W} \to \mathscr{W}$ given by $w \mapsto w'$ is a bijection (left multiplication), we can always recover the component equation for each $w \in \mathscr{W}$. Thus, every element will be considered as a sum of its components in $\tilde{\mathscr{Y}}^w$ for all $w \in W$. For example, by $Y_{i,a}$ we mean the element of Π given by the sum of each copy of $Y_{i,a} \in \mathscr{Y}^w$ for all $w \in \mathscr{W}$. Similarly, each equation should be seen as an equation in Π, that is a collection of equations in $\tilde{\mathscr{Y}}^w$ for all $w \in \mathscr{W}$. So, to sum up the results above and fix the notation, we have:

$$\Theta_i(Y_{i,a}) = Y_{i,a}A^{-1}_{i,aq_i^{-1}}\frac{\Sigma_{i,aq_i^{-3}}}{\Sigma_{i,aq_i^{-1}}}, \quad \Theta_i(A^{-1}_{i,a}) = A_{i,aq_i^{-2}}\frac{\Sigma_{i,a}}{\Sigma_{i,aq_i^{-4}}}, \tag{3.21}$$

$$\Theta_i(\Sigma_{i,a}) = 1 - \Sigma_{i,a} = -A^{-1}_{i,a}\Sigma_{i,aq_i^{-2}}. \tag{3.22}$$

3.3.1 Some invariant elements

Definition 3.3.1

For $i \in I$, $a \in \mathbb{C}^*$ and $k \geq 0$, define the element of $\mathscr{Y}$

$$\begin{aligned} T_{i,a}^{(k)} &:= Y_{i,a} Y_{i,aq_i^{-2}} \cdots Y_{i,aq_i^{2(1-k)}} (1 + A_{i,aq_i}^{-1}(1 + A_{i,aq_i^{-1}}^{-1}(1 + \ldots (1 + A_{i,aq_i^{3-2k}})))) \\ &= Y_{i,a} \cdots Y_{i,aq_i^{2(1-k)}} \sum_{0 \leq \alpha \leq k} V_{i,aq_i}^{(\alpha)}, \end{aligned}$$

where

$$V_{i,a}^{(\alpha)} := (A_{i,a} A_{i,aq_i^{-2}} \cdots A_{i,aq_i^{2-2\alpha}})^{-1}, \ \alpha > 0, \quad V_{i,a}^{(0)} := 1.$$

Moreover,

$$V_{i,a}^{(\alpha)} := \left(V_{i,aq_i^{-2\alpha}}^{(-\alpha)} \right)^{-1}, \ \alpha < 0.$$

For example, $T_{i,a}^{(1)} = Y_{i,a}(1 + A_{i,aq_i}^{-1})$. We have that

$$\begin{aligned} V_{i,a}^{(\alpha+1)} &= (A_{i,a} A_{i,aq_i^{-2}} \cdots A_{i,aq_i^{-2\alpha}})^{-1} \\ &= (A_{i,a} A_{i,aq_i^{-2}} \cdots A_{i,aq_i^{2-2\alpha}})^{-1} A_{i,aq_i^{-2\alpha}}^{-1} = V_{i,a}^{(\alpha)} A_{i,aq_i^{-2\alpha}}^{-1} && (3.23) \\ &= A_{i,a}^{-1} (A_{i,aq_i^{-2}} \cdots A_{i,aq_i^{-2\alpha}})^{-1} = A_{i,a}^{-1} V_{i,aq_i^{-2}}^{(\alpha)}. && (3.24) \end{aligned}$$

By Definition 3.1.5, it follows that

$$\Sigma_{i,a}^{w} = \begin{cases} \sum_{\alpha \geq 0} V_{i,a}^{(\alpha)}, \text{ if } w(\alpha_i) \in \Delta^+, \\ -\sum_{\alpha < 0} V_{i,a}^{(\alpha)}, \text{ if } w(\alpha_i) \in \Delta^-. \end{cases} \tag{3.25}$$

Proposition 3.3.1

The elements $T_{i,a}^{(k)}$ are fixed by Θ_i.

3.3.2 Reduction to rank 2 Lie algebras

We have to prove the braid relation (R_{ij}) for $i \neq j$ on each $Y_{k,a}$ for $k \in I$ and $a \in \mathbb{C}^*$. We can denote this $(R_{ij}(Y_{k,a}))$. By definition, if $k \notin \{i, j\}$, then $\Theta_i(Y_{k,a}) = \Theta_j(Y_{k,a}) = Y_{k,a}$, so $(R_{ij}(Y_{k,a}))$ holds. So, it suffices to show that $(R_{ij}(Y_{i,a}))$ and $(R_{ij}(Y_{j,a}))$ are satisfied. Again by definition, applying Θ_i and Θ_j on $Y_{i,a}$ or on $Y_{j,a}$ we obtain an expression in the $Y_{i,a}, Y_{j,a}, A_{i,b}, A_{j,b}$. Thus, we can consider the rank 2 Lie algebra $\mathfrak{g}_{ij}$ associated to the roots α_i and α_j (by Serre's Theorem), replace our variables $Y_{i,a}$, $Y_{j,a}$, $A_{i,b}$, $A_{j,b}$ with the corresponding of type $\mathfrak{g}_{ij}$ and consider the action of Θ_i and Θ_j corresponding to $\mathfrak{g}_{ij}$. Hence, if (R_{ij}) holds for $\mathfrak{g}_{ij}$, then it holds also for $\mathfrak{g}$. Then, it suffices to prove the braids relations for simple Lie algebras of type $A_1 \times A_1$, A_2, B_2, G_2. However, since the proofs for cases B_2 and G_2 are very technical and use the same ideas as in case A_2 (second proof), we do not report them here. They appear in sections 4.7 and 4.8 of [10].

3.3.3 Type $A_1 \times A_1$

In this case we have $C_{ij} = C_{ji} = 0$. We only need to prove the relation $(R_{ij}(Y_{i,a}))$, since $(R_{ij}(Y_{j,a}))$ can be obtained applying the automorphism that exchanges i and j.

Proposition 3.3.2

$$\Theta_j\Theta_i(Y_{i,a}) = \Theta_i\Theta_j(Y_{i,a}).$$

Proof. In this case, for all $a \in \mathbb{C}^*$, $A_{i,a} = Y_{i,aq^{-1}}Y_{a,q} \in \mathbb{Z}[Y_{i,a}^{\pm1}]_{i\in I, a\in\mathbb{C}^*}$, so $\Theta_j(A_{i,a}) = A_{i,a}$. As $\Sigma_{i,a}$ is an expression in the $A_{i,a}$, we have that Θ_j fixes also $\Sigma_{i,a}$. So, we get:

$$\Theta_j\Theta_i(Y_{i,a}) = \Theta_i(Y_{i,a}) = \Theta_i\Theta_j(Y_{i,a}),$$

that is, $(R_{ij}(Y_{i,a}))$ is satisfied. The proof of $(R_{ij}(Y_{j,a}))$ is completely symmetric. So for this type we are done. □

3.3.4 Image of the $\Sigma_{j,a}$

For the other Lie types we have $C_{ij} < 0$. To deal with these cases we need some results. Let us define for all $a \in \mathbb{C}^*$,

$$A_{ij,a} := \begin{cases} A_{i,aq_i^{-1}} & \text{if } C_{ij} = -1, \\ A_{i,aq^{-2}}A_{i,a} & \text{if } C_{ij} = -2, \\ A_{i,aq}A_{i,aq^{-1}}A_{i,aq^{-3}} & \text{if } C_{ij} = -3. \end{cases}$$

Then, we have that

$$\Theta_i(A_{j,a}) = A_{j,a}A_{ij,a}\frac{\Sigma_{i,aq_i^{-2-C_{ij}}}}{\Sigma_{i,aq_i^{-2+C_{ij}}}}. \tag{3.26}$$

This can be proved case by case, for example if $C_{ij} = -1$ we have:

$$\begin{aligned}\Theta_i(A_{j,a}) &= \Theta_i(Y_{j,aq_i}Y_{j,aq_i^{-1}}Y_{i,a}^{-1})\\ &= Y_{j,aq_i}Y_{j,aq_i^{-1}}\Theta_i(Y_{i,a}^{-1})\\ &= Y_{j,aq_i}Y_{j,aq_i^{-1}}Y_{i,a}^{-1}A_{i,aq_i^{-1}}\frac{\Sigma_{i,aq_i^{-1}}}{\Sigma_{i,aq_i^{-3}}}\\ &= A_{j,a}A_{ij,a}\frac{\Sigma_{i,aq_i^{-2-C_{ij}}}}{\Sigma_{i,aq_i^{-2+C_{ij}}}}.\end{aligned}$$

In particular,

$$\Theta_i(\Sigma_{j,a}) = \Theta_i(1 + A_{j,a}^{-1}\Sigma_{j,aq_i^{-2}}) = 1 + A_{j,a}^{-1}A_{ij,a}^{-1}\frac{\Sigma_{i,aq_i^{-2+C_{ij}}}}{\Sigma_{i,aq_i^{-2-C_{ij}}}}\Theta_i(\Sigma_{j,aq_j^{-2}}).$$

We also have

$$\Theta_i(\Sigma_{j,a}) = \frac{\Sigma_{ij,a}}{\Sigma_{i,a}^{(j)}}, \text{ where } \Sigma_{i,a}^{(j)} := \begin{cases}\Sigma_{i,aq_i^{-2-C_{ij}}} & \text{if } C_{ji} = -1,\\ \Sigma_{i,aq^{-2}}\Sigma_{i,aq_i^{-4}} & \text{if } C_{ji} = -2,\\ \Sigma_{i,aq_i^{-3}}\Sigma_{i,aq_i^{-5}}\Sigma_{i,aq_i^{-7}} & \text{if } C_{ji} = -3,\end{cases} \tag{3.27}$$

and $\Sigma_{ij,a}$ is the unique solution of the q-difference equation

$$\Sigma_{ij,a} = \Sigma_{i,a}^{(j)} + A_{j,a}^{-1}A_{ij,a}^{-1}\Sigma_{ij,aq_j^{-2}}. \tag{3.28}$$

3.3.5 Type A_2

In this case we have $C_{ij} = C_{ji} = -1$ for $i \neq j$, thus the braid relation (R_{ij}) is of the form

$$\Theta_i\Theta_j\Theta_i = \Theta_j\Theta_i\Theta_j. \tag{3.29}$$

It is sufficient to prove $(R_{ij}(Y_{i,a}))$ as $(R_{ij}(Y_{j,a}))$ is obtained by just switching i and j and this is an automorphism of A_2.

Proposition 3.3.3

$$\Theta_i\Theta_j\Theta_i(Y_{i,a}) = \Theta_j\Theta_i\Theta_j(Y_{i,a}).$$

First proof. As $\Theta_j(Y_{i,a}) = Y_{i,a}$, we have to show that

$$\Theta_i\Theta_j\Theta_i(Y_{i,a}) = \Theta_j\Theta_i(Y_{i,a}).$$

On the RHS, using formulas (3.26) and (3.27) we find

$$\begin{aligned}\Theta_j\Big(Y_{i,a}A^{-1}_{i,aq^{-1}}\frac{\Sigma_{i,aq^{-3}}}{\Sigma_{i,aq^{-1}}}\Big) &= Y_{i,a}A^{-1}_{i,aq^{-1}}A^{-1}_{j,aq^{-2}}\frac{\Sigma_{i,aq^{-4}}}{\Sigma_{i,aq^{-2}}}\frac{\Sigma_{ji,aq^{-3}}}{\Sigma_{j,aq^{-4}}}\frac{\Sigma_{j,aq^{-2}}}{\Sigma_{ji,aq^{-1}}}\\ &= Y_{i,a}Y^{-1}_{i,a}Y^{-1}_{i,aq^{-2}}Y_{j,aq^{-1}}Y^{-1}_{j,aq^{-3}}Y^{-1}_{j,aq^{-1}}Y_{i,aq^{-2}}\frac{\Sigma_{ji,aq^{-3}}}{\Sigma_{ji,aq^{-1}}}\\ &= Y^{-1}_{j,aq^{-3}}\frac{\Sigma_{ji,aq^{-3}}}{\Sigma_{ji,aq^{-1}}}.\end{aligned}$$

We have to prove that

$$\Theta_i(\Sigma_{ji,a}) = \Sigma_{ji,a}, \tag{3.30}$$

so that the following diagram commutes:

$$\begin{tikzcd} Y_{i,a} \arrow[r, mapsto, "\Theta_i"] \arrow[d, mapsto, "\Theta_j"'] & Y_{i,a}A^{-1}_{i,aq^{-1}}\frac{\Sigma_{i,aq^{-3}}}{\Sigma_{i,aq^{-1}}} \arrow[r, mapsto, "\Theta_j"] & Y^{-1}_{j,aq^{-3}}\frac{\Sigma_{ji,aq^{-3}}}{\Sigma_{ji,aq^{-1}}} \arrow[d, mapsto, "\Theta_i"] \\ Y_{i,a} \arrow[r, mapsto, "\Theta_i"'] & Y_{i,a}A^{-1}_{i,aq^{-1}}\frac{\Sigma_{i,aq^{-3}}}{\Sigma_{i,aq^{-1}}} \arrow[r, mapsto, "\Theta_j"'] & Y^{-1}_{j,aq^{-3}}\frac{\Sigma_{ji,aq^{-3}}}{\Sigma_{ji,aq^{-1}}} \end{tikzcd}$$

Recall that by definition $\Sigma_{ji,a}$ is the solution of

$$\Sigma_{ji,a} = \Sigma_{j,aq^{-1}} + A^{-1}_{i,a}A^{-1}_{j,aq^{-1}}\Sigma_{ji,aq^{-2}}. \tag{3.31}$$

By Lemma 3.1.2,

$$\Sigma_{i,a}\Sigma_{j,aq} = \Sigma_{ij,aq} + A^{-1}_{j,aq}\Sigma_{ji,a} \tag{3.32}$$

since both sides of (3.32) are solutions of the q-difference equation:

$$U(a) = A^{-1}_{j,aq}A^{-1}_{i,a}U(aq^{-2}) + (\Sigma_{i,a} + \Sigma_{j,aq} - 1).$$

So, in particular we have

$$\Sigma_{ji,a} = A_{j,aq}(-\Sigma_{ij,aq} + \Sigma_{j,aq}\Sigma_{i,a}).$$

By (3.27) we have $\Theta_i(\Sigma_{j,a}) = \Sigma_{ij,a}\Sigma^{-1}_{i,aq^{-1}}$, so applying Θ_i on both sides of (3.32), we find:

$$-A^{-1}_{i,a}\Sigma_{i,aq^{-2}}\Sigma_{ij,aq}\Sigma^{-1}_{i,a} = -\Sigma_{j,aq}A^{-1}_{i,a}\Sigma_{i,aq^{-2}} + A^{-1}_{j,aq}A^{-1}_{i,a}\Theta_i(\Sigma_{ji,a})\Sigma_{i,aq^{-2}}\Sigma^{-1}_{i,a}.$$

Thus, we have

$$\begin{aligned}\Theta_i(\Sigma_{ji,a}) &= (-A^{-1}_{i,a}\Sigma_{i,aq^{-2}}\Sigma_{ij,aq}\Sigma^{-1}_{i,a} + \Sigma_{j,aq}A^{-1}_{i,a}\Sigma_{i,aq^{-2}})A_{j,aq}A_{i,a}\Sigma^{-1}_{i,aq^{-2}}\Sigma_{i,a} \\ &= A_{j,aq}(-\Sigma_{ij,aq} + \Sigma_{j,aq}\Sigma_{i,a}) \\ &= \Sigma_{ji,a}.\end{aligned}$$

□

Alternative proof. This proof is significant, though less natural, as it can be generalized to the cases B_2 and G_2. Firstly, we compute the w-component $E_w(\Sigma_{ji,a})$ of the solution of (3.31) using the elements $V^{(\alpha)}_{k,a}$ defined in Definition 3.3.1. We find:

(3.33)

$$E_w(\Sigma_{ji,a}) = \begin{cases} \sum_{0\le\beta\le\alpha} V^{(\alpha)}_{j,aq^{-1}}V^{(\beta)}_{i,a} & \text{if } w(\alpha_j)\in\Delta^+ \text{ and } w(\alpha_i+\alpha_j)\in\Delta^+, \\ -\sum_{0\le\beta,\alpha<\beta} V^{(\alpha)}_{j,aq^{-1}}V^{(\beta)}_{i,a} & \text{if } w(\alpha_j)\in\Delta^- \text{ and } w(\alpha_i+\alpha_j)\in\Delta^+, \\ -\sum_{\beta<0,\beta\le\alpha} V^{(\alpha)}_{j,aq^{-1}}V^{(\beta)}_{i,a} & \text{if } w(\alpha_j)\in\Delta^+ \text{ and } w(\alpha_i+\alpha_j)\in\Delta^-, \\ \sum_{\alpha<\beta<0} V^{(\alpha)}_{j,aq^{-1}}V^{(\beta)}_{i,a} & \text{if } w(\alpha_j)\in\Delta^- \text{ and } w(\alpha_i+\alpha_j)\in\Delta^-. \end{cases}$$

The root $\alpha_i+\alpha_j$ appears because it is the weight of $A_{i,a}A_{j,aq^{-1}}$ in equation (3.31), while the root α_j appears as it is the weight of $A_{j,aq^{-1}}$ in the q-differences equation (3.2) satisfied by $\Sigma_{j,aq^{-1}}$ which appears in (3.31). Let us prove the first case in (3.33), so we assume $w(\alpha_j)\in\Delta^+$ and $w(\alpha_i+\alpha_j)\in\Delta^+$. Using (3.25), we have:

$$\begin{aligned}
E_w(\Sigma_{ji,a}) &= E_w(\Sigma_{j,aq^{-1}} + \Sigma_{j,aq^{-3}}A^{-1}_{i,a}A^{-1}_{j,aq^{-1}} + \Sigma_{j,aq^{-5}}A^{-1}_{i,a}A^{-1}_{i,aq^{-2}}A^{-1}_{j,aq^{-1}}A^{-1}_{j,aq^{-3}} + \dots)\\
&= \sum_{\alpha\geq 0} V^{(\alpha)}_{j,aq^{-1}} + \left(\sum_{\alpha\geq 0} V^{(\alpha)}_{j,aq^{-3}}\right) A^{-1}_{j,aq^{-1}}A^{-1}_{i,a} + \dots\\
&= \sum_{\alpha\geq 0} V^{(\alpha)}_{j,aq^{-1}} \cdot 1 + \left(\sum_{\alpha\geq 0} V^{(\alpha+1)}_{j,aq^{-1}}\right) A^{-1}_{i,a} + \dots\\
&= \sum_{\alpha\geq 0} V^{(\alpha)}_{j,aq^{-1}} V^{(0)}_{i,a} + \sum_{\alpha\geq 1} V^{(\alpha)}_{j,aq^{-1}} V^{(0)}_{i,a}V^{(1)}_{i,a} + \dots\\
&= \sum_{0\leq\beta\leq\alpha} V^{(\alpha)}_{j,aq^{-1}} V^{(\beta)}_{i,a}.
\end{aligned}$$

The third and fourth case are similar. To prove the other equalities we proceed in the same fashion, using the proper development of (3.25). So, for the second case we get:

$$\begin{aligned}
E_w(\Sigma_{ji,a}) &= -\sum_{\alpha<0} V^{(\alpha)}_{j,aq^{-1}} - \left(\sum_{\alpha<0} V^{(\alpha)}_{j,aq^{-1}}\right) A^{-1}_{j,aq^{-1}}A^{-1}_{i,a} - \dots\\
&= \left(-\sum_{\alpha<0} V^{(\alpha)}_{j,aq^{-1}}\right) V^{(0)}_{i,a} + \left(-\sum_{\alpha<1} V^{(\alpha)}_{j,aq^{-1}}\right) V^{(0)}_{i,a}V^{(1)}_{i,a} + \dots\\
&= -\sum_{\beta\geq 0,\alpha<\beta} V^{(\alpha)}_{j,aq^{-1}} V^{(\beta)}_{i,a}.
\end{aligned}$$

As in the previous proof, we only need to show that equation (3.30) holds, i.e.

$$\Theta_i(E_w(\Sigma_{ji,a})) = E_{ws_i}(\Sigma_{ji,a}). \tag{3.34}$$

We notice that since $s_i(\alpha_j) = \alpha_i + \alpha_j$, the element $w \in \mathscr{W}$ satisfies the first condition in (3.33) if and only if ws_i does the same. So, we rewrite the first equality in (3.33) as

$$E_w(\Sigma_{ji,a}) = \sum_{\alpha\geq 0} \left(V^{(\alpha)}_{j,aq^{-1}} Y^{-1}_{i,aq^{-1}} \cdots Y^{-1}_{i,aq^{2(1-\alpha)}}\right) T^{\alpha}_{i,aq^{-1}}. \tag{3.35}$$

In each term of (3.35) the second factor is invariant under Θ_i^w by Proposition 3.3.1, while the first factor is fixed by Θ_i^w as it is a monomial in $Y^{\pm 1}_{j,b}$ (indeed in $V^{(\alpha)}_{j,aq^{-1}}$ the factors $Y_{i,aq^{-1}} \cdots Y_{i,aq^{2(1-\alpha)}}$ appear). Thus, we have proved (3.34) for

$w \in \mathscr{W}$ satisfying first condition of (3.33). The case w satisfying the last condition of (3.33) is analogous. Observe that w satisfies the third condition if and only if ws_i satisfies the third and vice versa. If we consider $\sum_{0\leq\beta,\alpha<\beta} V^{(\alpha)}_{j,aq^{-1}} V^{(\beta)}_{i,a}$, we can rewrite it as $\sum_{\alpha<0}\left(V^{(\alpha)}_{j,aq^{-1}}\sum_{\beta\geq 0}V^{(\beta)}_{i,a}\right)+\sum_{\alpha\geq 0}\left(V_{j,aq^{-1}}\sum_{\beta>\alpha}V^{\beta}_{i,a}\right)$ and consider each term. If $\alpha<0$, each term in the summation can be written as

$$V^{(\alpha)}_{j,aq^{-1}}\sum_{\beta\geq 0}V^{(\beta)}_{i,a} = V^{(\alpha)}_{j,aq^{-1}}\Sigma^{w}_{i,a}.$$

Recall that for $\alpha<0$ $V^{(\alpha)}_{j,aq^{-1}} = (V^{(\alpha)}_{j,aq^{-2\alpha-1}})^{-1}$, thus it has a factor $Y^{-1}_{i,aq^{-1}}\cdots Y^{-1}_{i,aq^{-2\alpha-1}}$. Hence, applying Θ^{w}_{i} on this factor, we find a product of the image of $Y_{i,*}$, i.e there are $Y^{-1}_{i,*}$, $A_{i,*}$ and fractions of $\Sigma_{i,*}$ (we indicate with $*$ a general spectral parameter) . If we consider the image of the entire $V^{(\alpha)}_{j,aq^{-1}}$ we obtain that the $Y_{j,*}$'s (fixed by the action) together with the $Y^{-1}_{i,*}$'s compose $V^{(\alpha)}_{j,aq^{-1}}$, the $A_{i,*}$'s form $V^{(\alpha)}_{i,aq^{-2}}$, the fractions of $\Sigma_{i,*}$ cancel out leaving just $\frac{\Sigma^{ws_i}_{i,aq^{-2\alpha-2}}}{\Sigma^{ws_i}_{i,aq^{-2}}}$. So, recalling (3.21):

$$\begin{aligned}\Theta^{w}_{i}(V^{(\alpha)}_{j,aq^{-1}}\Sigma^{w}_{i,a}) &= V^{(\alpha)}_{j,aq^{-1}}V^{(\alpha)}_{i,aq^{-2}}\frac{\Sigma^{ws_i}_{i,aq^{-2\alpha-2}}}{\Sigma^{ws_i}_{i,aq^{-2}}}(-A^{-1}_{i,a}\Sigma^{ws_i}_{i,aq^{-2}})\\ &= V^{(\alpha)}_{j,aq^{-1}}(-V^{(\alpha+1)}_{i,a})\Sigma^{ws_i}_{i,aq^{-2\alpha-2}}\\ &= V^{(\alpha)}_{j,aq^{-1}}\sum_{\beta\leq\alpha}V^{(\beta)}_{i,a}.\end{aligned}$$

If $\alpha\geq 0$, we have

$$V^{(\alpha)}_{j,aq^{-1}}\sum_{\alpha<\beta}V^{(\beta)}_{i,a} = V_{j,aq^{-1}}V^{(\alpha+1)}_{i,a}\Sigma^{w}_{i,aq^{-2(\alpha+1)}},$$

where $V^{(\alpha)}_{j,aq^{-1}}V^{(\alpha+1)}_{i,a}$ has a factor $Y^{-1}_{i,aq}Y_{i,aq^{-1}}\cdots Y^{-1}_{i,aq^{-2\alpha-1}}$. Thus,

$$\begin{aligned}\Theta^{w}_{i}(V_{j,aq^{-1}}V^{(\alpha+1)}_{i,a}\Sigma^{w}_{i,aq^{-2(\alpha+1)}}) &= V_{j,aq^{-1}}V^{(\alpha+1)}_{i,a}(V^{(\alpha+2)}_{i,a})^{-1}\frac{\Sigma^{ws_i}_{i,a}}{\Sigma^{ws_i}_{i,aq^{-2\alpha-4}}}(-A^{-1}_{i,aq^{-2(\alpha+1)}}\Sigma^{ws_i}_{i,aq^{-2(\alpha+2)}})\\ &= -V^{(\alpha)}_{j,aq^{-1}}\Sigma^{ws_i}_{i,a}\\ &= V^{(\alpha)}_{j,aq^{-1}}\sum_{\beta<0}V^{\beta}_{i,a}.\end{aligned}$$

Hence, we have

$$\begin{aligned}\Theta_i^w\left(-\sum_{0\le\beta,\alpha<\beta}V_{j,aq^{-1}}^{(\alpha)}V_{i,a}^{(\beta)}\right)&=-\Theta_i^w\left(\sum_{\alpha<0}\left(V_{j,aq^{-1}}^{(\alpha)}\sum_{\beta\ge0}V_{i,a}^{(\beta)}\right)+\sum_{\alpha\ge0}\left(V_{j,aq^{-1}}\sum_{\beta>\alpha}V_{i,a}^{\beta}\right)\right)\\&=-\sum_{\alpha<0}\left(V_{j,aq^{-1}}^{(\alpha)}\sum_{\beta\le\alpha}V_{i,a}^{(\beta)}\right)-\sum_{\alpha\ge0}\left(V_{j,aq^{-1}}^{(\alpha)}\sum_{\beta<0}V_{i,a}^{(\beta)}\right)\\&=-\sum_{\beta<0,\beta\le\alpha}V_{j,aq^{-1}}^{(\alpha)}V_{i,a}^{(\beta)}\\&=E_{ws_i}(\Sigma_{ji,a}).\end{aligned}$$

Hence, we have that (3.34) holds for w satisfying the second condition. Since Θ_i is an involution, (3.34) holds also for w satisfying the third condition of (3.33). So we are done with type A_2. $\square$

Chapter 4

Subring of invariants under Weyl group action

In this chapter we report the important result from Section 5.2 of [10], namely that the image of the q-character homomorphism in $\mathscr{Y}$ coincides with the subring of invariants of $\mathscr{Y} \subset \Pi$ under the action of the Weyl group defined in Chapter 3. However, since the proof of this fact relies on some algebraic objects called *screening operators*, in the following section we recall their definition.

4.1 Screening operators

For this section we have referred to [11]. We start by defining the free $\mathscr{Y}$-module

$$\tilde{\mathscr{Y}}_i := \bigoplus_{a \in \mathbb{C}^*} \mathscr{Y} \otimes S_{i,a},$$

where the $S_{i,a}$'s are just formal variables. Then $\mathscr{Y}_i$ is defined to be the quotient of $\tilde{\mathscr{Y}}_i$ by the submodule generated by the relation

$$S_{i,aq_i^2} = A_{i,aq_i} S_{i,a}, \tag{4.1}$$

so $\mathscr{Y}_i$ is a free $\mathscr{Y}$-module too. We call $p : \tilde{\mathscr{Y}}_i \to \mathscr{Y}_i$ the canonical projection. Thanks to the relation above, we have

$$\mathscr{Y}_i \simeq \bigoplus_{a \in (\mathbb{C}^*/q_i^{2\mathbb{Z}})} \mathscr{Y} \otimes S_{i,a}.$$

The linear operator $\tilde{S}_i : \mathscr{Y} \to \tilde{\mathscr{Y}}_i$ is defined as

$$\tilde{S}_i \cdot Y_{j,a} = \delta_{ij} Y_{i,a} S_{i,a}$$

and we require it to be a derivation, i.e. $\tilde{S}_i(ab) = (\tilde{S}_i \cdot a)b + a(\tilde{S}_i \cdot b)$. We notice that:

$$\tilde{S}_i(1 \cdot 1) = (\tilde{S}_i \cdot 1)1 + 1(\tilde{S}_i \cdot 1) = 2(\tilde{S}_i \cdot 1),$$

so

$$\tilde{S}_i \cdot 1 = 0. \tag{4.2}$$

Moreover, using the Leibniz rule and (4.2), we have

$$\tilde{S}_i \cdot Y_{j,a}^{-1} = -\delta_{ij} Y_{i,a}^{-2} \tilde{S}_i \cdot Y_{i,a} = -\delta_{ij} Y_{i,a}^{-1} S_{i,a}.$$

Finally we define $S_i : \mathscr{Y} \to \mathscr{Y}_i$, the i-th *screening operator* as

$$\begin{tikzcd} \mathscr{Y} \arrow[r, "\tilde{S}_i"] \arrow[rr, bend right, "S_i"'] & \tilde{\mathscr{Y}}_i \arrow[r, "p"] & \mathscr{Y}_i \end{tikzcd} .$$

It was conjectured in [11] and proved in [9] that

Theorem 4.1.1
The image of the homomorphism χ_q equals the intersection of the kernels of the operators S_i, $i \in I$.

4.2 The subring of $\mathscr{W}$-invariants

In [10] the authors give an interpretation of the screening operators in terms of the action of the Weyl group defined in Chapter 3. Recall that we denote by $\mathscr{Y}^w$ the image of $\mathscr{Y}$ in $\tilde{\mathscr{Y}}^w$ and in Π. We define $\boldsymbol{\Theta}_i$ as the composition of the isomorphism $\mathscr{Y} \simeq \mathscr{Y}^{s_i}$ and the morphism $\Theta_i^{s_i} : \mathscr{Y}^{s_i} \to \tilde{\mathscr{Y}}^e$, so

$$\boldsymbol{\Theta}_i(Y_{j,a}^{\pm 1}) = Y_{j,a}^{\pm 1} \text{ if } i \neq j \quad \text{and } \boldsymbol{\Theta}_i(Y_{i,a}) = Y_{i,a} A_{i,aq_i^{-1}}^{-1} \frac{\Sigma^e_{i,aq_i^{-3}}}{\Sigma^e_{i,aq_i^{-1}}}. \tag{4.3}$$

We consider a formal variable h and we set

$$\tilde{\Sigma}_{i,a} := h\Sigma^e_{i,a} \text{ for } i \in I, \ a \in \mathbb{C}^*$$

Then, we can rewrite the definition of $\boldsymbol{\Theta}_i$ as

$$\begin{aligned} \boldsymbol{\Theta}_i(Y_{i,a}) &= Y_{i,a} - h\frac{Y_{i,a}}{\tilde{\Sigma}_{i,aq_i^{-1}}}, \\ \boldsymbol{\Theta}_i(Y_{i,a}^{-1}) &= Y_{i,a}^{-1} + h\frac{Y_{i,a}^{-1} A_{i,aq_i^{-1}}}{\tilde{\Sigma}_{i,aq_i^{-3}}}. \end{aligned}$$

We set $\mathscr{Y}_{i,h} := \mathscr{Y}\left[\tilde{\Sigma}_{i,a}^{-1}, h\right]_{a\in\mathbb{C}^*}$ and we can use the formulas above to define some ring homomorphisms

$$\boldsymbol{\Theta}_{i,h} : \mathscr{Y} \to \mathscr{Y}_{i,h}.$$

$\tilde{\Sigma}_{i,a}$ satisfies the q-difference equation $\tilde{\Sigma}_{i,a} = h + A_{i,a}^{-1}\tilde{\Sigma}_{i,aq_i^{-2}}$ that, if $h \to 0$, becomes $\tilde{\Sigma}_{i,a} = A_{i,a}^{-1}\tilde{\Sigma}_{i,aq_i^{-2}}$. Let us set

$$S_{i,a} = -\tilde{\Sigma}^{-1}_{i,aq_i^{-1}}.$$

Then, for $h \to 0$ we have

$$S_{i,aq_i^2} = A_{i,aq_i} S_{i,a},$$

which is exactly the same relation in (4.1). Moreover we call $\mathscr{Y}_i = \mathscr{Y}\left[\tilde{\Sigma}_{i,a}^{-1}\right]_{i\in I, a\in\mathbb{C}^*}$, and we set the derivation $S_i : \mathscr{Y} \to \mathscr{Y}_i$ as

$$S_i(Y_{j,a}) = -\delta_{ij} Y_{i,a} \tilde{\Sigma}^{-1}_{i,aq_i^{-1}}, \ S_i(Y_{j,a}^{-1}) = \delta_{ij} Y_{i,a} \tilde{\Sigma}^{-1}_{i,aq_i^{-1}}.$$

With the identification $S_{i,a} = -\tilde{\Sigma}^{-1}_{i,aq_i^{-1}}$ we obtain that the definition of $\mathscr{Y}_i$ given in this section is the same as in the previous one and the same goes for the operators S_i. By the definition of S_i, we find $\mathbf{\Theta}_{i,h}(Y_{i,a}) = Y_{i,a} + hS_i(Y_{i,a})$. More in general, thanks to the Leibniz rule, we have that

$$\mathbf{\Theta}_{i,h} = Id + hS_i + h^2(...) + h^3(...) + ... \tag{4.4}$$

For example

$$\mathbf{\Theta}_{i,h}(Y_{i,a}Y_{i,b}) = (Y_{i,a} + hS_i(Y_{i,a}))(Y_{i,b} + hS_i(Y_{i,b}))$$

$$= Y_{i,a}Y_{i,a} + hS_i(Y_{i,a}Y_{i,b}) + h^2 S_i(Y_{i,a})S_i(Y_{i,b}).$$

Hence, we find the screening operator S_i in the limit of a one-parameter deformation of the automorphism $\mathbf{\Theta}_i$. This observation is crucial to prove the following

Theorem 4.2.1
The image of the q-character homomorphism χ_q in $\mathscr{Y} \hookrightarrow \Pi$ is equal to the subring of invariants of the diagonal subspace $\mathscr{Y} \hookrightarrow \Pi$ under the action of Θ_i, $i \in I$, i.e.

$$Im(\chi_q) = \bigcap_{i \in I} \mathscr{Y}^{\Theta_i}.$$

Equivalently, recalling that χ_q is injective, the Grothendieck ring of the category of finite-dimensional representations of $\mathscr{U}_q(\hat{\mathfrak{g}})$ is isomorphic to the subring of invariants in $\mathscr{Y}$ of the action of the Weyl group on Π.

Proof. We start by showing that the elements in $Im(\chi_q)$ are invariant under the Weyl group action. We fix $i \in I$ and we consider $P \in Im(\chi_q)$, then by Corollary 5.7 in [9] P is a polynomial in $Y_{i,a}(1 + A^{-1}_{i,aq_i})$ and in $Y^{\pm 1}_{j,a}$ for $i \neq j$ and some $a \in \mathbb{C}^*$. Now, by definition, $\Theta_i(Y^{\pm 1}_{j,a}) = Y^{\pm 1}_{j,a}$ and, since $Y_{i,a}(1 + A^{-1}_{i,aq_i}) = T^{(1)}_{i,a}$ and by Proposition 3.3.1 $T^{(1)}_{i,a}$ is Θ_i-invariant, we have that P is Θ_i-invariant. We can repeat the same argument for every $i \in I$, so $Im(\chi_q) \subset \bigcap_{i \in I} \mathscr{Y}^{\Theta_i}$. For the opposite inclusion, let P be a polynomial in $\bigcap_{i \in I} \mathscr{Y}^{\Theta_i}$. Then, $\mathbf{\Theta}_i(P) = P$ and so $\mathbf{\Theta}_{i,h}(P) = P$ too. By the development in (4.4), this means that $S_i(P) = 0$ for all $i \in I$, that is $P \in \bigcap_{i \in I} Ker_{\mathscr{Y}} S_i$. Thanks to Theorem 4.1.1 we conclude that $P \in Im(\chi_q)$, so $\bigcap_{i \in I} \mathscr{Y}^{\Theta_i} \subset Im(\chi_q)$. □

Chapter 5

Final remarks

In conclusion, the theory of q-characters is much more complex than the classical analogue and still many investigations have to be done. For example, at this point we do not have a quantum version of the characters and multiplicities formulas presented in Appendix A.

Concerning the work of Frenkel and Hernandez, some questions arise. Is it possible to find automorphisms Θ_i corresponding to the elements of a general Coxeter group? Another possible direction concerns the so-called *quantum* Grothendieck ring $\mathscr{K}_t(\mathscr{C})$ of the category of finite-dimensional $\mathscr{U}_q(\hat{\mathfrak{g}})$-modules. We give here a short description of this ring and we refer to [13]. In Chapter 2 we showed that the Grothendieck ring $\mathscr{K}_0(\mathscr{C})$ is commutative (see Remark 2.3.1) and this lies on the fact that the product in $\mathscr{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^*}$ is commutative. Now, if we impose some relations on the product of the variables, we loose the commutativity of $\mathscr{K}_0(\mathscr{C})$. To make an example, we can consider $\mathscr{U}_q(\widehat{\mathfrak{sl}_2})$. Here, we have that $\chi_q([L(Y_{1,q^p})]) = Y_{1,q^p} + Y_{1,q^{p+2}}^{-1}$ and $\chi_q([L(Y_{1,q^p}Y_{1,q^{p+2}})]) = Y_{1,q^p}Y_{1,q^{p+2}} + Y_{1,q^p}Y_{1,q^{p+4}}^{-1} + Y_{1,q^{p+2}}^{-1}Y_{1,q^{p+4}}^{-1}$. Then

$$[L(Y_{1,q^p})] \cdot [L(Y_{1,q^{p+2}})] = [L(Y_{1,q^{p+2}})] \cdot [L(Y_{1,q^p})] = [L(Y_{1,q^p}Y_{1,q^{p+2}})] + [\mathbb{1}]. \tag{5.1}$$

We introduce the quantum torus $(\mathscr{Y}_t, *)$ that is the ring of polynomials in the variables $Y_{1,a}^{\pm 1}$, $a \in \mathbb{C}^*$, with relations

$$Y_{1,q^p} * Y_{1,q^s} = t^{\mathscr{N}(p,s)} Y_{1,q^s} * Y_{1,q^p}, \ p, s \in \mathbb{Z}$$

where for example $\mathcal{N}(p, p+2) = -2$. So, we have $Y_{1,q^p} * Y_{1,q^{p+2}} = t^{-2} Y_{1,q^{p+2}} * Y_{1,q^p}$. Hence, in the quantum Grothendieck ring, equation 5.1 becomes

$$[L(Y_{1,q^p})] * [L(Y_{1,q^{p+2}})] = t^{-2}[L(Y_{1,q^{p+2}})] * [L(Y_{1,q^p})] + (1 - t^{-2})\mathbb{1}, \tag{5.2}$$

while

$$[L(Y_{1,q^{p+2}})] * [L(Y_{1,q^p})] = t^{2}[L(Y_{1,q^{p+2}})] * [L(Y_{1,q^p})] + (1 - t^{2})\mathbb{1}. \tag{5.3}$$

The t-deformation of the Grothendieck ring carries a deformation of the q-character, that is the q,t-character. So, one may ask if it is possible to generalize the Weyl group action defined in [10] to describe the symmetries of q,t-characters.

These are just a few remarks that show how q-characters and their symmetries are related to other topics like combinatorics of Coxeter groups, cluster algebras and quiver varieties.

Appendix A

Some characters formulas

In this appendix we would like to present some results about the classical theory of characters, namely the Freudenthal's, Konstant's and Weyl's characters formulas. Showing the proofs of these results is an opportunity to study some important objects in the theory of semisimple Lie algebras, such as the universal Casimir element, and to manipulate representations. We point out that there are not q-analogous of these formulas, hence computing q-characters and dimension of representations of quantum affine algebras is more difficult than in the classical setting. However, in the view of the importance of the Weyl group action for character formulas in the classical setting, one can expect that the deformed Weyl group action presented in this thesis may play such an important role in the future.

Our main reference for this part are [14] and [12].

A.1 Freudenthal's formula

We recall the notations of Chapter 1, so let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra of rank l, let $\mathfrak{h}$ be a fixed Cartan subalgebra, $\Delta^{\pm}$ the sets of positive/negative roots, $\Delta = \Delta^+ \sqcup \Delta^-$ the set of roots. We denote by $\mathscr{P}$ the group of weights and for every λ dominant integral weight we denote by $V(\lambda)$ the unique (up to isomorphism) finite-dimensional simple $\mathfrak{g}$-module of highest weight λ. Let V_μ be the weight space corresponding to μ for every $\mu \in \mathfrak{h}^*$, then we denote by $m_\mu := dim(V_\mu)$ the multiplicity of μ in V. Recall that the adjoint representation for $\mathfrak{g}$ is $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, defined by $ad_x(y) = [x, y]$, for every $x, y \in \mathfrak{g}$. Then, the Killing form on $\mathfrak{g}$ is a symmetric bilinear form defined by $K(x, y) := tr(ad_x \cdot ad_y)$. Since $\mathfrak{g}$ is semisimple, K is non degenerate and the same goes for its restriction to $\mathfrak{h}$. Thus, K and $K|_\mathfrak{h}$ induce respectively isomorphisms $\mathfrak{g} \simeq \mathfrak{g}^*$ and $\mathfrak{h} \simeq \mathfrak{h}^*$. Moreover, the isomorphism ν induced between $\mathfrak{h}$ and its dual let us define a symmetric bilinear non degenerate form on $\mathfrak{h}^*$ by $(\lambda, \mu) := K(t_\lambda, t_\mu)$, $t_\lambda = \nu^{-1}(\lambda)$. We want to prove the following

Theorem A.1.1 (Freudenthal's formula)
Let $V = V(\lambda)$ be a simple $\mathfrak{g}$-module of highest weight $\lambda \in \mathscr{P}^+$. If $\mu \in \mathscr{P}$, then the multiplicity of μ in V is given recursively by:

$$((\lambda+\rho, \lambda+\rho) - (\mu+\rho, \mu+\rho))\, m_\mu = 2 \sum_{\alpha \in \Delta^+} \sum_{i=1}^{\infty} m_{\mu+i\alpha}(\mu + i\alpha, \alpha), \tag{A.1}$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$.

A.1.1 A universal Casimir element

We build a basis for $\mathfrak{g}$ and a dual basis with respect to the Killing form. First, we choose a basis for $\mathfrak{h}$, say $\{h_1, .., h_l\}$ (associated to a base $\mathscr{S}$ of the root system), and we take the dual basis $\{k_1, ..., k_l\}$ with respect to the Killing form. Now, for every root $\alpha \in \Delta$ we pick an element x_α in the root space $\mathfrak{g}_\alpha$ and we consider z_α the unique element in $\mathfrak{g}_{-\alpha}$ such that $K(x_\alpha, z_\alpha) = 1$. Note that z_α is different from the traditional y_α which realizes the $\mathfrak{sl}_2$-triple associated to α. However, there is a relation between z_α and y_α, namely $z_\alpha = (\alpha, \alpha) y_\alpha / 2$. We call S_α the Lie algebra generated by x_α, z_α, t_α, isomorphic to $\mathfrak{sl}_2(\alpha)$.
$\{h_i, i \in I, x_\alpha, \alpha \in \Delta\}$ and $\{k_i, i \in I, z_\alpha, \alpha \in \Delta\}$ are two basis for $\mathfrak{g}$, dual with respect to K. We can define a universal Casimir element as

$$C_\mathfrak{g} := \sum_{i \in I} h_i k_i + \sum_{\alpha \in \Delta} x_\alpha z_\alpha \ \in \mathscr{U}(\mathfrak{g}).$$

One can prove that $C_{\mathfrak{g}}$ does not depend on the choice of the basis of $\mathfrak{g}$. By the universal property of $\mathscr{U}(\mathfrak{g})$, every representation ψ of $\mathfrak{g}$ on a vector space V extends uniquely to a representation of $\mathscr{U}(\mathfrak{g})$ and we have that $\psi(C_{\mathfrak{g}})$ and $\psi(x)$ commute as elements in $\mathfrak{gl}(V)$, for all $x \in \mathfrak{g} \hookrightarrow \mathscr{U}(\mathfrak{g})$. If V is irreducible, Schur Lemma and the remark above imply that $\psi(C_{\mathfrak{g}})$ acts on every V as the multiplication by a scalar. From now on, we consider V as in the hypothesis of Theorem A.1.1, with the representation given by $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$. Then, $\phi(C_{\mathfrak{g}}) = \sum_{i\in I} \phi(h_i)\phi(k_i) + \sum_{\alpha\in\Delta} \phi(x_\alpha)\phi(z_\alpha)$.

A.1.2 The trace of $\phi(x_\alpha)\phi(z_\alpha)$ on V_μ

Given a root α and a weight $\mu \in \mathscr{P}$, it is well known that $\phi(z_\alpha)|_{V_\mu}$ sends V_μ in $V_{\mu-\alpha}$, while $\phi(x_\alpha)|_{V_{\mu-\alpha}}$ sends $V_{\mu-\alpha}$ in V_μ. Thus, the composition $\phi(x_\alpha)\phi(z_\alpha)$ defines an element in $\mathfrak{gl}(V_\mu)$. The *canonical* $\mathfrak{sl}_2(\alpha)$-triple generated by x_α,y_α,$h_\alpha = [x_\alpha, y_\alpha]$ has a module of highest weight n for every $n \in \mathbb{N} \cup \{0\}$. A canonical basis for this module is built as follows. Let v_0 be a highest weight vector, then for every $i = 1, ..., n$, $v_i := 1/i!y_\alpha^i \cdot v_0$. Then $\{v_0, ..., v_n\}$ is a basis for the module. Now, we build a similar basis for the simple S_α-module of highest weight n. We just replace each v_i by $w_i := i!((\alpha,\alpha)/2)^i v_i$. The action is given by:

$$\begin{aligned} t_\alpha \cdot w_i &= (n-2i)\frac{(\alpha,\alpha)}{2} w_i; \\ z_\alpha \cdot w_i &= w_{i+1} \\ x_\alpha \cdot w_i &= i(n-i+1)\frac{(\alpha,\alpha)}{2} w_{i-1} \quad w_{-1} = 0. \end{aligned}$$

As a result,

$$x_\alpha z_\alpha \cdot w_i = (i+1)(n-i)\frac{(\alpha,\alpha)}{2} w_i. \tag{A.2}$$

Now we fix a root α, a weight $\mu \in \mathscr{P}(V)$ such that $\mu+\alpha \notin \mathscr{P}(V)$ and we denote by m the integer $\mu(h_\alpha) = 2(\mu,\alpha)/(\alpha,\alpha)$. We have that the α-string of weights through μ is $\mu,\ \mu-\alpha, ..., \mu-m\alpha$. By Weyl's Theorem on complete reducibility, the S_α-module $W = V_\mu \oplus ... \oplus V_{\mu-m\alpha}$ can be decomposed as a direct sum of simple modules, each of these having highest weight $(\mu-i\alpha)(h_\alpha) = m-2i$. For

$0 \leq i \leq [m/2]$, let n_i be the number of occurrences of $V(m-2i)$ as component of W. A moment of thought about structure of representations of $\mathfrak{sl}_2$ makes us realize that

$$\text{(A.3)} \qquad m_{\mu-i\alpha} = n_0 + \ldots + n_i, \quad \text{so} \quad n_i = m_{\mu-i\alpha} - m_{\mu-(i-1)\alpha}.$$

It is easy to understand $n_0 = m_\mu$, since $V(m) = V(\mu(h_\alpha))$ is the only module with a weight μ (and it is the highest). To compute n_1, we have to count how many simple modules with highest weight $m-2$ appear in the decomposition of W. $V(m-2)$ has weights $m-2$, $m-4\ldots$,$-m+2$ and each weight space is one-dimensional. So we remove one dimension to each $V_{\mu-i\alpha}$ for $i = 1,..,m-1$. The only other module that has weight $m-2$ is $V(m)$, so $dim(V_{\mu-\alpha}) = m_{\mu-\alpha}$ is exhausted by the occurrences of $V(m)$ and $V(m-2)$. With this idea we obtain (A.3). For each $0 \leq k \leq [m/2]$ we want to compute $tr(\phi(x_\alpha)\phi(z_\alpha)|_{V_{\mu-k\alpha}})$. The only summands in the decomposition of W intersecting $V_{\mu-k\alpha}$ are the $V(m-2i)$ for $0 \leq i \leq k$. Since the weight spaces of simple $\mathfrak{sl}_2$-modules are one-dimensional, each $V(m-2i)$ has the weight space corresponding to $\mu - k\alpha$ spanned by the vector w_{k-i} (in notation above). Then, by formula (A.2), we find

$$\text{(A.4)} \qquad \phi(x_\alpha)\phi(z_\alpha)w_{k-i} = (m-i-k)(k-i+1)\frac{(\alpha,\alpha)}{2}w_{k-i}.$$

For each $0 \leq i \leq k$ the weight $m-2k$ appears as a weight for $V(m-2i)$ with multiplicity 1, hence there are n_i linearly independent vectors in $V(m-2k)$ on which $\phi(x_\alpha)\phi(z_\alpha)$ acts as in (A.4). For i ranging in $0,..,k$ we find that the matrix of $\phi(x_\alpha)\phi(z_\mathfrak{g})$ on a proper basis has $n_0 + \ldots + n_k = m_{\mu-k\alpha}$ diagonal entries $(m-i-k)(k-i+1)\frac{(\alpha,\alpha)}{2}$. Therefore, recalling that $m/2=(\mu,\alpha)/(\alpha,\alpha)$,

(A.5)

$$\begin{aligned}
tr(\phi(x_\alpha)\phi(z_\alpha)|_{V_{\mu-k\alpha}}) &= \sum_{i=0}^{k} n_i(m-i-k)(k-i+1)\frac{(\alpha,\alpha)}{2}\\
&= \sum_{i=0}^{k}(m_{\mu-i\alpha}-m_{\mu-(i-1)\alpha})(m-i-k)(k-i+1)\frac{(\alpha,\alpha)}{2}\\
&= \sum_{i=0}^{k} m_{\mu-i\alpha}(m-2i)\frac{(\alpha,\alpha)}{2}\\
&= \sum_{i=0}^{k} m_{\mu-i\alpha}(\mu-i\alpha,\alpha).
\end{aligned}$$

Now, if $m/2<k\leq m$, by some technical observations the argument above can be replicated, with the result that (A.5) holds also in this case. Until now we have argued for μ such that $\mu+\alpha$ is not a weight. However if this is not the case, we can just take the final term $\mu+r\alpha$ play the role of μ and repeat the same reasoning. Finally, using the convention $m_\mu=0$ if $V_\mu=\{0\}$ and some more manipulations, we obtain

$$tr_{V_\mu}(\phi(x_\alpha)\phi(z_\alpha))=\sum_{i=0}^{\infty} m_{\mu+i\alpha}(\mu+i\alpha,\alpha). \tag{A.6}$$

A.1.3 Freudenthal's formula

In the previous subsection we found the way a "half" of the Casimir element acts on a weights space V_μ. Now we focus on what is left. As said above, since ϕ is irreducible, $\phi(C_{\mathfrak{g}})|_{V_\mu}=cId_{V_\mu}$ for a certain scalar c. We also know that h_i acts as $\phi(h_i)|_{V_\mu}=\mu(h_i)Id_{V_\mu}=K(t_\mu,h_i)Id_{V_\mu}$ for every $i=1,...,l$. Since the h_i's are a basis for $\mathfrak{h}$, we can write $t_\mu=\sum_i a_ih_i$, for some $a_i\in\mathbb{C}$. Then,

$$\begin{aligned}
\mu(h_i) &= \sum_j a_jK(h_j,h_i);\\
\mu(k_i) &= K(t_\mu,k_i)=\sum_j a_jK(h_j,k_i)=a_i,
\end{aligned}$$

where the last equality is due to duality. Hence, $(\mu,\mu) = K(t_\mu, t_\mu) = \sum_{ij} a_i a_j K(h_i, h_j) = \sum_i \mu(h_i)\mu(k_i)$. Thus,

$$\sum_i tr_{V_\mu}(\phi(h_i)\phi(k_i)) = m_\mu(\mu,\mu).$$

As a result:

$$\begin{aligned} cm_\mu &= tr_{V_\mu}(\phi(C_{\mathfrak{g}})) \\ &= m_\mu(\mu,\mu) + \sum_{\alpha\in\Delta}\sum_{i=1}^{\infty} m_{\mu+i\alpha}(\mu+i\alpha,\alpha), \end{aligned}$$

where we start the sum with $i = 1$ as $m_\mu(\mu,\alpha)$ and $m_\mu(\mu,-\alpha)$ appear and cancel out. The formula above is still true when μ is a general weight, not necessarily in $\mathscr{P}(V)$. In that case we have $\sum_{\alpha\in\Delta}\sum_{i=1}^{\infty} m_{\mu+i\alpha}(\mu+i\alpha,\alpha) = 0$. Indeed, as μ is not a weight for the representation, the α-string through μ of elements $\mu+i\alpha$ must have all positive i or all negative i. In the latter case the summand for α is 0, in the former one should proceed with a bit more technical argument. Denoting s_α the simple reflection associated with α, by Proposition 1.0.1, we have $m_{\mu+i\alpha} = m_{s_\alpha\cdot(\mu+i\alpha)} = m_{\mu-(m+i)\alpha}$. Moreover

$$(\mu+i\alpha,\alpha) + (\mu-(m+i)\alpha,\alpha) = (2\mu - m\alpha,\alpha) = 0,$$

so

$$(\mu+i\alpha,\alpha) = -(\mu-(m+i)\alpha,\alpha).$$

This means that, for general $\alpha\in\Delta$ and $\mu\in\mathscr{P}$ we have

$$\sum_{i=-\infty}^{\infty}(\mu+i\alpha,\alpha)m_{\mu+i\alpha} = 0.$$

This implies that

$$(\mu,\alpha)m_\mu + \sum_{i=1}^{\infty} m_{\mu+i\alpha}(\mu+i\alpha,\alpha) = \sum_{i=1}^{\infty}(\mu-i\alpha,-\alpha)m_{\mu-i\alpha}.$$

We use this result to find:

$$\begin{aligned}
cm_\mu &= m_\mu(\mu,\mu) + \sum_{\alpha\in\Delta^+}\sum_{i=1}^{\infty} m_{\mu+i\alpha}(\mu+i\alpha,\alpha) + \sum_{\alpha\in\Delta^+}\sum_{i=1}^{\infty} m_{\mu-i\alpha}(\mu-i\alpha,-\alpha)\\
&= m_\mu(\mu,\mu) + \sum_{\alpha\in\Delta^+}\sum_{i=1}^{\infty} m_{\mu+i\alpha} + \sum_{\alpha\in\Delta^+} m_\mu(\mu,\alpha) + \sum_{\alpha\in\Delta^+}\sum_{i=1}^{\infty} m_{\mu+i\alpha}(\mu+i\alpha,\alpha)\\
&= m_\mu(\mu,\mu) + m_\mu 2(\mu,\rho) + 2\sum_{\alpha\in\Delta^+}\sum_{i=1}^{\infty} m_{\mu+i\alpha}(\mu+i\alpha,\alpha)
\end{aligned}$$

(A.7)

$$= m_\mu(\mu,\mu+2\rho) + 2\sum_{\alpha\in\Delta^+}\sum_{i=1}^{\infty} m_{\mu+i\alpha}(\mu+i\alpha,\alpha).$$

In order to find Freudenthal's formula, we are left to compute the constant c. This can be done by recalling that we know the multiplicity of the highest weight, i.e. $m_\lambda = 1$. λ being the highest weight means also that $m_{\lambda+i\alpha} = 0$ for all $i \geq 1$. Hence, by (A.7), we find

$$c = (\lambda,\lambda+2\rho) = (\lambda+\rho,\lambda+\rho) - (\rho,\rho).$$

Plugging the value of c in (A.7) and writing $(\mu,\mu+2\rho) = (\mu+\rho,\mu+\rho) - (\rho,\rho)$ we finally obtain

$$((\lambda+\rho,\lambda+\rho) - (\mu+\rho,\mu+\rho))\, m_\mu = 2\sum_{\alpha\in\Delta^+}\sum_{i=1}^{\infty} m_{\mu+i\alpha}(\mu+i\alpha,\alpha).$$

A.2 Harish-Chandra Isomorphism

Following the approach of [14] for the proof of Weyl's characters formula, we have the opportunity to present some important results about invariant polynomial functions on a Lie algebra $\mathfrak{g}$ and to state the famous Theorem of isomorphism of Harish-Chandra.

A.2.1 Invariant polynomial functions

Given a finite-dimensional vector space V, we denote by $\mathbb{C}[V]$ the algebra of polynomial functions on V which is, formally, the symmetric algebra on the dual vector space, i.e. $Sym(V^*)$. If we fix a basis $\{f_1,...,f_n\}$ of V^*, then $\mathbb{C}[V]$ is identified with the algebra of polynomial in the n variables $f_1...,f_n$. We continue to use the same notations as in the previous section. So we will consider $\mathbb{C}[\mathfrak{g}]$ and $\mathbb{C}[\mathfrak{h}]$. As the weight lattice $\mathscr{P}$ spans $\mathfrak{h}^*$, then the polynomials in the weights span $\mathbb{C}[\mathfrak{h}]$. Then, the Weyl group acts on the polynomial functions and we denote by $\mathbb{C}[\mathfrak{h}]^{\mathscr{W}}$ the subalgebra of $\mathscr{W}$-invariant polynomial functions on $\mathfrak{h}$. We consider now the entire Lie algebra $\mathfrak{g}$ and the group of inner automorphisms of $\mathfrak{g}$ that acts on $\mathbb{C}[\mathfrak{g}]$. Namely, $G := Inn(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is the group generated by elements $exp(ad_x)$ for all $x \in \mathfrak{g}$ such that ad_x is nilpotent (we say that x is ad-nilpotent). G acts on $\mathbb{C}[\mathfrak{g}]$ as:

$$(\sigma \cdot f)(x) = f(\sigma^{-1} \cdot x)$$

for every $\sigma \in G$ and every $x \in \mathfrak{g}$. We construct a special class of invariant polynomial functions which are very used in invariant theory. Let us consider the irreducible $\mathfrak{g}$-module $V = V(\lambda)$, $\lambda \in \mathscr{P}^+$ and the corresponding representation $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$. For an element $z \in \mathfrak{n}_+ = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$ we consider $\sigma = exp(ad_z)$ which is well defined as $\mathfrak{n}_+$ is a nilpotent subalgebra of $\mathfrak{g}$. Then we define a new representation of $\mathfrak{g}$ on the same vector space V by $\phi^\sigma(x)(v) := \phi(\sigma \cdot x)(v)$, for every $x \in \mathfrak{g}$. Let v_o be the highest weight vector for V with the representation given by ϕ, then one can check that v_0 is a maximal vector of weight λ also for ϕ^σ. This implies that there exists an isomorphism of $\mathfrak{g}$-modules $\psi_\sigma : (V,\phi) \to (V,\phi^\sigma)$. In particular, $\psi_\sigma(\phi(x)(v)) = \phi^\sigma(x)(\psi_\sigma(v))$, for $v \in V$. So, for all $x \in \mathfrak{g}$ we have $\phi(x) = \psi_\sigma^{-1}\phi^\sigma(x)\psi_\sigma$, hence $\phi(x)$ and $\phi^\sigma(x)$ are similar and this implies they have the same trace. Even more is true, namely for any k positive integer, $tr(\phi(x)^k) = tr(\phi^\sigma(x)^k)$. Now, we can consider $\phi(x)$ as a linear polynomial in the coordinate functions, then $\phi(x)^k$ is a polynomial too and the same goes for the trace of these endomorphisms. Hence we have that $x \mapsto tr(\phi(x)^k)$ is a polynomial function invariant under the G action.

Let f be a function in $\mathbb{C}[\mathfrak{g}]$. Then, if we consider $f|_\mathfrak{h}$ we find an element in $\mathbb{C}[\mathfrak{h}]$. Suppose $f \in \mathbb{C}[\mathfrak{g}]^G$, then in particular f is invariant under the action of $\tau_\alpha := exp(ad_{x_\alpha})exp(ad_{-y_\alpha})exp(ad_{x_\alpha}) \in G$ for $\alpha \in \Delta$. Note that, since $\mathfrak{h} \simeq \mathfrak{h}^*$ via Killing form for example, we can consider the elements of $\mathscr{W}$ as automorphisms of $\mathfrak{h}$. Now, one can prove that $\tau_\alpha|_{\mathfrak{h}} = s_\alpha$ (by regarding $\mathfrak{h} = Ker(\alpha) \oplus \mathbb{C}\alpha$). Thus, $f \in \mathbb{C}[\mathfrak{g}]^G$ implies $f|_{\mathfrak{h}}$ is s_α-invariant, so $f \in \mathbb{C}[\mathfrak{h}]^{\mathscr{W}}$. This defines a homomorphism of algebras:

$$\Theta : \mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]^{\mathscr{W}}.$$

Theorem A.2.1 (Chevalley's restriction Theorem)
Θ *is an isomorphism.*

Proof. See Theorem 23.1 and the Appendix of Section 23.3 in [14]. □

A.2.2 Harish-Chandra's Isomorphism

In this section we deal with the action of the centre $\mathscr{Z}$ of the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ on Verma modules for a fixed semisimple Lie algebra $\mathfrak{g}$. Observe that an automorphism of $\mathfrak{g}$ extends uniquely to an automorphism of $\mathscr{U}(\mathfrak{g})$ so $G = Inn(\mathfrak{g})$ acts on $\mathscr{U}(\mathfrak{g})$.

Lemma A.2.1

$$\mathscr{Z} = \mathscr{U}(\mathfrak{g})^G$$

Proof. See Lemma 23.2 in [14]. □

In particular, the universal Casimir element defined in Section A.1 is central and hence G-invariant. Let $\mathfrak{h}$ be a fixed Cartan subalgebra of $\mathfrak{g}$, let λ be a functional in $\mathfrak{h}^*$ and $M(\lambda)$ the associated Verma module. Then $M(\lambda) = \mathscr{U}(\mathfrak{g})\cdot v_0$ for v_0 maximal vector. Now, consider $z \in \mathscr{Z}$ and let us check the action of $\mathscr{U}(\mathfrak{g})$ on $z\cdot v_0$. We have $x_\alpha \cdot z \cdot v_0 = z \cdot x_\alpha \cdot v_0 = 0$ and $h\cdot z\cdot v_0 = z\cdot h\cdot v_0 = \lambda(h) z\cdot v_0$, since z is central and v_0 is a highest weight vector. Thus, $z\cdot v_0$ is another highest weight vector and by the fact that $M(\lambda)_\lambda$ is one-dimensional, there exists a complex number $\chi_\lambda(z)$ such that

$$z \cdot v_0 = \chi_\lambda(z) v_0.$$

The notation reminds the one for the characters of representations used in all this work, actually the algebra homomorphism

$$\chi_\lambda : \mathscr{Z} \to \mathbb{C}$$

defined as above is called the *character* determined by λ.

If we call $V^{\lambda,z} = \{v \in M(\lambda) | z \cdot v = \chi_\lambda(z)v\}$, then by the property of z to be central we have that $V^{\lambda,z}$ is a submodule and it contains v_0, hence $V^{\lambda,z} = M(\lambda)$. This means that z acts on the entire $M(\lambda)$ as the multiplication by $\chi(\lambda)(z)$.

Definition A.2.1
Let $\lambda, \mu \in \mathfrak{h}^*$. We write $\lambda \sim \mu$ and we say that λ and μ are linked if $\lambda + \rho$ and $\mu+\rho$ are $\mathscr{W}$-conjugated, where we recall that $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. Note that be linked is an equivalence relation.

Theorem A.2.2 (Harish-Chandra)
Let $\lambda, \mu \in \mathfrak{h}^*$. $\chi_\lambda = \chi_\mu$ if and only if $\lambda \sim \mu$.

We do not present the proof of this result because it is very long and complex for an appendix; however, we would like to point out that in the proof the so-called Harish-Chandra's Isomorphism appear and before closing the section we explain briefly how it works.

Theorem A.2.3 (Harish-Chandra's Isomorphism)
There exists an isomorphism of algebras

$$\Psi : \mathscr{Z} \to \mathbb{C}[\mathfrak{h}]^{\mathscr{W}}.$$

Remark A.2.1
Thanks to PBW Theorem and the fact that $\mathfrak{h}$ is abelian, we can identify $\mathbb{C}[\mathfrak{h}]$ with $\mathscr{U}(\mathfrak{h})$.

In order to build the isomorphism, we first fix a basis of $\mathfrak{g}$ made of h_i, $i = 1,..,l$, x_α, y_α for $\alpha \in \Delta$, $h_i = h_{\alpha_i}$ for a base of the root system $\mathscr{S} = \{\alpha_1, ..., \alpha_l\}$. We order the basis elements taking the y_α's first, then h_i's and finally x_α's ($\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$). This allow us to have a PBW basis for $\mathscr{U}(\mathfrak{g})$. We define a map that take monomials in the h_i in themselves and everything else to 0. This extends uniquely to a map

$$\xi : \mathscr{U}(\mathfrak{g}) \to \mathscr{U}(\mathfrak{h}), \tag{A.8}$$

which is basically the projection on $\mathscr{U}(\mathfrak{h})$. One can prove that $\chi_\lambda(z) = \lambda(\xi(z))$ for every $z \in \mathscr{Z}$. Next, we define a map $\mathfrak{h} \to \mathscr{U}(\mathfrak{h})$ by sending h_i to $h_i - 1$. This extends uniquely to

$$\eta : \mathscr{U}(\mathfrak{h}) \to \mathscr{U}(\mathfrak{h})$$

and it is clearly an automorphism as $\eta^{-1} : h_i \mapsto h_i + 1$. Then, we define

$$\Psi = \eta \circ \xi|_{\mathscr{Z}} : \mathscr{Z} \to \mathscr{U}(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}]$$

and with some efforts one can prove that $Im(\Psi)$ is $\mathscr{W}$-invariant. It is easy to see that Ψ is injective, while the surjectivity is not obvious. As a result we have that $\Psi : \mathscr{Z} \to \mathbb{C}[\mathfrak{h}]^{\mathscr{W}}$ is an isomorphism. Moreover,

$$\chi_\lambda(z) = (\lambda + \rho)(\Psi(z)).$$

A.3 Weyl's formula

We consider a finite-dimensional Lie algebra $\mathfrak{g}$ and a fixed Cartan subalgebra $\mathfrak{h}$. In this section we want to manipulate formal characters, as we have done during all this work. Since we want to deal with characters of infinite-dimensional modules (this is a problem we have already seen) it is convenient to interpret the group ring $\mathbb{Z}[\mathscr{P}]$ as the set of $\mathbb{Z}$-valued functions defined on $\mathscr{P}$ and with finite support. From this point of view, the single weight $\lambda \in \mathscr{P}$ corresponds to the function ϵ_λ, defined as $\epsilon_\lambda(\mu) = \delta_{\lambda\mu}$. Moreover, the product of two functions is given by the convolution, i.e. $(f * g)(\mu) = \sum_{\lambda+\nu=\mu} f(\lambda)g(\nu)$. We consider now a bigger set of functions, namely the set $\mathfrak{X}$ of $\mathbb{Z}$-valued functions defined on $\mathfrak{h}^*$ whose support is contained in a finite union of cones under some $\lambda \in \mathfrak{h}^*$ (see Definition 3.5). We can easily see that $\mathfrak{X}$ with the operations of sum and convolution has the structure of commutative, associative

$\mathbb{C}$-algebra. In particular, it contains characters of Verma modules. The elementary functions ϵ_λ, $\lambda \in \mathfrak{h}^+$ are a basis for the algebra $\mathfrak{X}$ and the neutral element is ϵ_0. Notice that $\epsilon_\lambda * \epsilon_\mu = \epsilon_{\lambda+\mu}$. The Weyl group of $\mathfrak{g}$ acts on $\mathfrak{h}^*$, hence it acts on $\mathfrak{X}$ by $w \cdot \epsilon_\lambda = \epsilon_{w \cdot \lambda}$. An interesting element of $\mathfrak{X}$ is the so-called Konstant function p. For any $\lambda \in \mathfrak{h}^*$, $p(\lambda)$ counts the number of sets of non-negative integers $\{k_\alpha, \alpha \in \Delta^+\}$ such that $-\lambda = \sum_{\alpha\in\Delta^+} k_\alpha \alpha$. For example, recalling that the weights of the Verma module of highest weight μ are all the weights of the form $\mu - \sum_{\alpha\in\Delta^+} k_\alpha \alpha$, then $p(\lambda) = \chi(M(0))(\lambda)$ (regarding elements of $\mathbb{Z}[\mathscr{P}]$ as functions). In particular, $p \in \mathfrak{X}$. We will also use the Weyl function $q := \prod_{\alpha\in\Delta^+}(\epsilon_{\alpha/2} - \epsilon_{-\alpha/2})$, where by $\prod$ we denote successive convolutions. Now we present without the proofs some simple, yet useful, results about the functions that we have just defined. For a reference, Section 24.1 in [14].

Lemma A.3.1

1. *$\sigma \cdot q = (-1)^{det(\sigma)} q$, for every $\sigma \in \mathscr{W}$.*

2. *$q * p * \epsilon_{-\rho} = \epsilon_0$.*

3. *$\chi(M(\lambda))(\mu) = p(\mu - \lambda) = (p * \epsilon_\lambda)(\mu)$.*

4. *$q * \chi(M(\lambda)) = \epsilon_{\lambda+\rho}$.*

We define now a family of $\mathfrak{g}$-modules. We say that a $\mathfrak{g}$-module V belongs to $\mathfrak{M}_\lambda$ if V is a direct sum of weight spaces, the action of $z \in \mathscr{Z}$ on V is given by scalars $\chi_\lambda(z)$ for the fixed $\lambda \in \mathfrak{h}^*$, the character of V belongs to $\mathfrak{X}$. Verma module of highest weight λ and their submodules, quotient and isomorphic images belongs to $\mathfrak{M}_\lambda$.

Lemma A.3.2
If $V \in \mathfrak{M}_\lambda$ then V posses at least one maximal vector.

Proof. The third condition in the definition of $\mathfrak{M}_\lambda$ implies that there exists a weight for V, say μ, such that for all $\alpha \in \Delta^+$, $\mu + \alpha$ is not a weight. Then, a nonzero vector in V_μ is maximal (as x_α will "kill" it). □

Proposition A.3.1
Let $\lambda \in \mathfrak{h}^*$. Then:

1. *$M(\lambda)$ has a Jordan-Hölder sequence $\{0\} = H_0 \subset H_1 \subset \ldots \subset H_m = M(\lambda)$.*

2. *Each $G_i = H_i/H_{i-1}$ (called composition factor) is of the form $V(\mu)$ for $\mu \in \Theta(\lambda)$, where $\Theta(\lambda) = \{\mu \in \mathfrak{h}^* | \mu \prec \lambda,\ \mu \sim \lambda\}$.*

3. *$V(\lambda)$ occurs only once as a composition factor of $M(\lambda)$.*

Proof.
1. If $M(\lambda)$ is irreducible, then it is $V(\lambda)$, so we are done. Otherwise, let V be a proper submodule of $M(\lambda)$. In particular by the remark above, $V \in \mathfrak{M}_\lambda$ and by the Lemma above it has a maximal vector. The weight corresponding to the maximal vector cannot be λ (as V is a proper submodule), so it is a weight $\mu \prec \lambda$. Hence, V contains a homomorphic image W of $M(\mu)$. Moreover, by the discussion about the central characters, we know that every $z \in \mathscr{Z}$ acts on the entire $M(\lambda)$ by $\chi_\lambda(z)$, but V contains W, thus $\chi_\lambda = \chi_\mu$. So, by Harish-Chandra Theorem A.2.2, $\lambda \sim \mu$, that is $\mu \in \Theta(\lambda)$. Now we continue by arguing on $Z(\lambda)/W$ and W and the process will end by weights and multiplicities reasons.

2. By (1), the composition factors are irreducible highest weight modules, so each of them has a maximal vector.

3. $M(\lambda)$ is one-dimensional and this concludes.

$\square$

We would like to use Lemma A.3.1 on characters of $\mathfrak{g}$-modules. To do so, we want to write any character as a $\mathbb{Z}$-linear combination of characters of Verma modules. Here is why it is possible. By Proposition A.3 (and by the same argument as in Proposition 2.3.4) we know that for any $\lambda \in \mathfrak{h}^*$ we have $\chi(M(\lambda)) = \chi(V(\lambda)) + \sum_{\mu \in \Theta(\lambda)} d_\mu \chi(V(\mu))$, $d_\mu \in \mathbb{Z}_{\geq 0}$. We can order the elements in $\Theta(\lambda)$ as $\{\mu_1, ..., \mu_t\}$ by the condition $\mu_i \prec \mu_j$ implies $i \leq j$ and setting $\lambda = \mu_t$. Notice that $\mu_i \in \Theta(\lambda)$ implies $\Theta(\mu_i) \subset \Theta(\lambda)$ as the "linked" relation and the order relation have the transitive property. Hence, thanks to Proposition A.3 each $\chi(M(\mu_i))$ can be written as a $\mathbb{Z}$-linear combination of the $\chi(V(\mu_j))$ for $j \leq i$ (with $\chi(V(\mu_i))$ appearing with coefficient one). Thus, the matrix expressing the characters of $M(\mu_i)$, $i = 1, ..., t$, in terms of the $V(\mu_i)$'a is a lower triangular one with integer coefficients and ones on the diagonal, so it can be inverted. Therefore we can write the characters of each $V(\mu_i)$ as a $\mathbb{Z}$-linear combination of $\chi(M(\mu_i))$. We have proved the following

Corollary A.3.0.1
Let $\lambda \in \mathfrak{h}^*$. Then $\chi(V(\lambda))$ is a $\mathbb{Z}$-linear combination $\sum_{\mu \in \Theta(\lambda)} c_\mu \chi(M(\mu))$.

Now we consider $\lambda \in \mathscr{P}^+$, hence $dim(V(\lambda))$ is finite and by Theorem 1.0.1 we have $w \cdot \chi(V(\lambda)) = \chi(V(\lambda))$ for all $w \in \mathscr{W}$.

Consider the expression of $\chi(V(\lambda))$ as in Corollary A.3.0.1. We can apply the convolution with the Weyl function q and (4) from Lemma A.3.1 to get

$$q * \chi(V(\lambda)) = \sum_{\mu \in \Theta(\lambda)} c_\mu \epsilon_{\mu+\rho}. \tag{A.9}$$

Now, we act on the expression above with a general element w in the Weyl group. By the invariance of characters and by (1) of Lemma A.3.1 we have

$$(-1)^{det(w)} q * \chi(V(\lambda)) = \sum_{\mu \in \Theta(\lambda)} c_\mu \epsilon_{w \cdot (\mu+\rho)}.$$

Denote $sgn(w) = (-1)^{det(w)}$. We can rewrite the previous equation as

$$sgn(w)\left(\epsilon_{\lambda+\rho} + \sum c_\mu \epsilon_{\mu+\rho}\right) = \sum c_\mu \epsilon_{w \cdot (\mu+\rho)}.$$

This implies that for μ such that $w^{-1} \cdot \mu = \lambda + \rho$ the coefficient satisfies $c_\mu = sgn(w)$. Since $\mathscr{W}$ permutes the $\mu + \rho$ transitively ($\lambda \sim \mu$), we have:

$$q * \chi(V(\lambda)) = \sum_{w \in \mathscr{W}} sgn(w) \epsilon_{w \cdot (\lambda+\rho)}. \tag{A.10}$$

Recalling that we are regarding $\chi(V(\lambda))$ as function on $\mathscr{P}$ and the notation m_μ for the multiplicity of the weight μ in $V(\lambda)$, we have:

$$\begin{aligned} m_\mu = \chi(\lambda)(\mu) &= q * p * \epsilon_{-\rho} * \chi(V(\lambda))(\mu) \\ &= p * \epsilon_{-\rho} \sum_{w \in \mathscr{W}} sgn(w) \epsilon_{w \cdot (\mu+\rho)} \\ &= p * \left(\sum_{w \in \mathscr{W}} sgn(w) \epsilon_{w \cdot (\mu+\rho) - \rho} \right)(\mu) \\ &= \sum_{w \in \mathscr{W}} sgn(w) p * \epsilon_{w \cdot (\mu+\rho)}(\mu) \\ &= \sum_{w \in \mathscr{W}} p\left(\mu - w \cdot (\lambda+\rho) + \rho\right). \end{aligned}$$

We have just proved the following

Theorem A.3.1 (Konstant's formula)
Let $\lambda \in \mathscr{P}^+$. Then the multiplicities of $V(\lambda)$ are given by the formula

$$m_\mu = \sum_{w \in \mathscr{W}} p\left(\mu - w \cdot (\lambda+\rho) + \rho\right).$$

Lemma A.3.3

$$q = \sum_{w \in \mathscr{W}} sgn(w) \epsilon_{w \cdot \rho}.$$

Proof. We apply Equation A.10 in the case $\lambda = 0$, so $\chi(V(\lambda)) = \epsilon_0$ and we conclude. □

The elements ϵ_μ correspond to the variables y_μ in our notation of previous chapters about characters. Recalling this and the new expression for q, we obtain the following

Theorem A.3.2 (Weyl's formula)
Let $\lambda \in \mathscr{P}^+$. Then

$$\chi(V(\lambda)) = \frac{\sum_{w\in\mathscr{W}} sgn(w) y_{w\cdot(\lambda+\rho)}}{\sum_{w\in\mathscr{W}} sgn(w) y_{w\cdot\rho}}.$$

We can deduce from this an interesting formula for computing the dimensions of finite-dimensional simple $\mathfrak{g}$-modules. We follow the exposition in [16].

Theorem A.3.3 (Weyl dimension formula)
Let $\lambda \in \mathscr{P}^+$. Then the dimension of the simple module of highest weight λ can be computed as

$$dim(V(\lambda)) = \prod_{\alpha\in\Delta^+} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}.$$

Proof. Here we consider $\mathbb{Z}[\mathscr{P}]$ as in the previous chapters of our work, i.e. as the polynomials with integer coefficients in the variables y_λ, $\lambda \in \mathscr{P}$. We start by a consideration about the Weyl function $q = \prod_{\alpha\in\Delta^+}(y_{\alpha/2} - y_{-\alpha/2})$. We observe that $y_{-\rho}\cdot q = \prod_{\alpha\in\Delta^+}(1-y_{-\alpha})$. Hence we have

$$\sum_{w\in\mathscr{W}} sgn(w) y_{w\cdot\rho} = q = y_\rho \prod_{\alpha\in\Delta^+}(1-y_{-\alpha}).$$

For each $\mu \in \mathfrak{h}^*$ we define a function $F_\mu : \mathbb{Z}[\mathscr{P}] \to \{f : \mathbb{C} \to \mathbb{C}\}$ by the assignment $y_\mu \mapsto e^{t(\lambda,\mu)}$. By the previous remark on the Weyl function we rewrite the Weyl character formula as

$$y_\rho \prod_{\alpha\in\Delta^+}(1-y_{-\alpha})\chi(V(\lambda)) = \sum_{w\in\mathscr{W}} sgn(w) y_{w\cdot(\lambda+\rho)} \tag{A.11}$$

We apply the map F_ρ on both sides of this equation, obtaining

$$\begin{aligned}
e^{t(\rho,\rho)} \prod_{\alpha\in\Delta^+}\left(1-e^{-t(\rho,\alpha)}\right) F_\rho(\chi(V(\lambda))) &= \sum_{w\in\mathscr{W}} sgn(w) e^{t(w\cdot(\lambda+\rho)),\rho} \\
&= \sum_{w\in\mathscr{W}} sgn(w) e^{t(\lambda+\rho,w^{-1}(\rho))} \\
&= F_{\lambda+\rho}(\sum_{w\in\mathscr{W}} sgn(w) y_{w^{-1}\cdot\rho}) \\
&= F_{\lambda+\rho}(\sum_{w\in\mathscr{W}} sgn(w) y_{w\cdot\rho}).
\end{aligned}$$

Now, consider equation (A.11) in the case $\lambda = 0$, i.e. $\chi(V(0)) = 1$. This tells us that $y_\rho \prod_{\alpha\in\Delta^+}(1-y_{-\alpha}) = \sum_{w\in\mathscr{W}} sgn(w) y_{w\cdot\rho}$. Hence we get

$$e^{t(\rho,\rho)}F_\rho(\chi(V(\lambda))) = \frac{F_{\lambda+\rho}(y_\rho \prod_{\alpha\in\Delta^+}(1-y_{-\alpha}))}{\prod_{\alpha\in\Delta^+}(1-e^{-t(\rho,\alpha)})}$$
$$= e^{t(\lambda+\rho,\rho)} \prod_{\alpha\in\Delta^+} \frac{\left(1-e^{-t(\lambda+\rho,\alpha)}\right)}{(1-e^{-t(\rho,\alpha)})}.$$

We consider the limit $t \to 0$ and we apply De l'Hopital Theorem to get

$$\sum_{\mu\in\mathscr{P}(V)} dim(V(\lambda)_\mu) = \prod_{\alpha\in\Delta^+} \frac{(\lambda+\rho,\alpha)e^{-t(\lambda+\rho,\alpha)}}{(\rho,\alpha)e^{-t(\rho,\alpha)}}.$$

We evaluate the equation above in $t = 0$ and we find

$$dim(V(\lambda)) = \sum_{\mu\in\mathscr{P}(V)} dim(V(\lambda)_\mu) = \prod_{\alpha\in\Delta^+} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}.$$

□

Example A.3.1

We use the Weyl dimension formula to compute the dimension of a general finite-dimensional simple module of $\mathfrak{sl}_3$*. We use the same notation as above in the Appendix.* $I = \{1,2\}$*,* α_1, α_2 *are the simple roots,* ω_1, ω_2 *are the fundamental weights. Using the Cartan matrix of type* A_2 *we have that the positive roots are* $\alpha_1 = 2\omega_1 - \omega_2$*,* $\alpha_2 = -\omega_1 + 2\omega_2$*,* $\alpha_1 + \alpha_2 = \omega_1 + \omega_2$*. Moreover,* $\rho = \omega_1 + \omega_2$ *and (this is true in general)* $\rho(h_1) = \rho(h_2) = 1$*. Let* $\lambda = m_1\omega_1 + m_2\omega_2 \in \mathscr{P}^+$*,* $m_1, m_2 \in \mathbb{Z}_{\geq 0}$*. Having this, we can compute easily the terms that appear in the Weyl dimension formula and obtain*

$$dim(V(\lambda)) = \frac{(m_1+1)(m_2+1)(m_1+m_2+2)}{2}.$$

Example A.3.2

Let us compute the character of the simple $\mathfrak{sl}_3$*-module* $V(\omega_1)$ *(that is the natural representation on* $\mathbb{C}^3$*) using the Weyl character formula (WCF). Here the data we need (have a look to the previous example).*

$\rho = \omega_1 + \omega_2$*;* $\mathscr{W} = \{id, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$ *(thanks to the braid relations listed* *in Table 1.1); the signs of the reflections, i.e. the signs of their determinants, can be deduced by the fact that $\mathscr{W}$ is isomorphic to the symmetric group on three elements S_3. Therefore, $\{id, s_1s_2, s_2s_1\}$ have sign $+1$, while $\{s_1, s_2, s_1s_2s_1\}$ have sign -1. The action of $\mathscr{W}$ on ρ and $\omega_1+\rho$ is shown in the table below:*

	id	s_1	s_2	s_1s_2	s_2s_1	$s_1s_2s_1$
ρ	$\omega_1+\omega_2$	$-\omega_1+2\omega_2$	$2\omega_1-\omega_2$	$-2\omega_1+\omega_2$	$\omega_1-2\omega_2$	$-\omega_1-\omega_2$
$\omega_1+\rho$	$2\omega_1+\omega_2$	$-2\omega_1+3\omega_2$	$3\omega_1-\omega_2$	$-3\omega_1+2\omega_2$	$\omega_1-3\omega_2$	$-\omega_1-2\omega_2$

Thus, we are able to write down the character following WCF:

$$\chi(V(\omega_1)) = \frac{y_1^2y_2 - y1^{-2}y_2^3 - y_1^3y_2^{-1} + y_1^{-3}y_2^2 + y_1y_2^{-3} - y_1^{-1}y_2^{-2}}{y_1y_2 - y_1^{-1}y_2^2 - y_1^2y_2^{-1} + y_1^{-2}y_2 + y_1y_2^{-2} - y_1^{-1}y_2^{-1}}. \tag{A.12}$$

However, by Example 1.0.2 we also know that

$$\chi(V(\omega_1)) = y_1 + y_2y_1^{-1} + y_2^{-1} \tag{A.13}$$

and multiplying the denominator in (A.12) by expression (A.13), one finds the numerator in (A.12), so the two expressions of the character are in fact the same.

Appendix B

q-characters through R-matrix

In this Appendix we would like to present a different construction of the q-character which relies on the so-called universal R-matrix of $\mathcal{U}_q(\hat{\mathfrak{g}})$. Actually this is the original way that Frenkel and Reshetikhin used in [11] to explain the q-character. Indeed, this point of view, apparently less natural than the one presented in Chapter 2, is very meaningful and related to other topics such as quantum integrable models and braided categories. However, to give a complete and satisfying exposition would require much more than an appendix; therefore we choose to give some key ideas, but to skip many details, proofs and calculations. The references are [18] for the general part on universal R-matrix, [11] for the transfer matrix and the construction of q-character, [7] for an explicit construction of the universal R-matrix for $\mathcal{U}_q(\hat{\mathfrak{g}})$.

B.1 The universal R-matrix

We start by the definition of universal R-matrix. Let $\mathscr{A}$ be a Hopf algebra with comultiplication Δ. We denote by τ the *flip*, i.e. the automorphism of $\mathscr{A}^{\otimes 2}$ that switches the two copies of $\mathscr{A}$. For the comultiplication we will use the Sweedler notations, that is $\Delta(x) = \sum_i x_{(1)} \otimes x_{(2)}$, for any $x \in \mathscr{A}$. We need also the following convention: if $R = \sum_i s_i \otimes t_i \in \mathscr{A}^{\otimes 2}$, then in $\mathscr{A}^{\otimes 3}$ we set

$$R_{12} = \sum_i s_i \otimes t_i \otimes 1;$$
$$R_{13} = \sum_i s_i \otimes 1 \otimes t_i;$$
$$R_{23} = \sum_i 1 \otimes s_i \otimes t_i.$$

Definition B.1.1
With the above notation, we say that $\mathscr{A}$ is a braided (or quasi-triangular) Hopf algebra if there exists an invertible element $R \in \mathscr{A} \otimes \mathscr{A}$ that verifies:

$$\Delta^{op}(x) := \tau \circ \Delta(x) = R\Delta(x)R^{-1}, \ \forall x \in \mathscr{A}; \tag{B.1}$$
$$(\Delta \otimes id)(R) = R_{13}R_{23}; \tag{B.2}$$
$$(id \otimes \Delta)(R) = R_{13}R_{12}. \tag{B.3}$$

Such an element is called a universal R-matrix.

A trivial example is a co-commutative Hopf algebra, that is a Hopf algebra such that $\tau \circ \Delta = \Delta$. Then $R = 1 \otimes 1$ is a universal R-matrix.

Remark B.1.1
The universal R-matrix satisfies the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \ \in \mathscr{A}^{\otimes 3}. \tag{B.4}$$

This can be proved using the defining relations of R, that is

$$\begin{aligned} R_{12}R_{13}R_{23} &= R_{12}(\Delta \otimes id)(R) \\ &= ((\tau \circ \Delta) \otimes id)(R)R_{12} \\ &= (\tau \otimes id) \circ (\Delta \otimes id)(R)R_{12} \\ &= (\tau \otimes id)(R_{13}R_{23})R_{12} \\ &= R_{23}R_{13}R_{12}. \end{aligned}$$

The existence of a universal R-matrix is related to the structure of the category of $\mathscr{A}$-modules. For any V, W $\mathscr{A}$-modules, the comultiplication let us endow $V \otimes W$ with the structure of $\mathscr{A}$-module. We are interested in understanding when the *flip* map $\tau_{V,W} : V \otimes W \to W \otimes V$ that switches the components is a morphism of modules and when it is an isomorphism. For example, for the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$ ($\Delta(x) = x \otimes 1 + 1 \otimes x$, $x \in \mathfrak{g}$), the flip of two $\mathscr{U}(\mathfrak{g})$-modules is an isomorphism. The situation is different for the quantized universal enveloping algebra, for example $\mathscr{U}_q(\mathfrak{sl}_2)$. Here we can consider the representation V_1 of Example 2.1.2 and study the tensor product $V_1 \otimes V_1$. We check if τ_{V_1,V_1} is a morphism of modules, i.e. if it commutes with action of $\mathscr{U}_q(\mathfrak{sl}_2)$. Using the same notation as in Example 2.1.2, we compute

$$\begin{aligned}
\tau_{V_1,V_1} \cdot x^- \cdot (v_0 \otimes v_1) &= \tau_{V_1,V_1} \cdot \Delta(x^-) \cdot (v_0 \otimes v_1) \\
&= \tau_{V_1,V_1} \cdot (x^- \otimes k^{-1} + 1 \otimes x^-)(v_0 \otimes v_1) \\
&= \tau_{V_1,V_1} \cdot (v_1 \otimes q^{-1} v_0 + v_0 \otimes 0) \\
&= q^{-1} v_0 \otimes v_1.
\end{aligned}$$

While

$$\begin{aligned}
x^- \cdot \tau_{V_1,V_1} \cdot (v_0 \otimes v_1) &= \Delta(x^-) \cdot (v_1 \otimes v_0) \\
&= v_1 \otimes v_1.
\end{aligned}$$

Hence, in this case τ_{V_1,V_1} is not even a morphism of modules! Here the R-matrix comes to help. Indeed, if the Hopf algebra $\mathscr{A}$ admits a universal R-matrix R, for every V, W $\mathscr{A}$-modules we can define

$$c^R_{V,W}(v \otimes w) := \tau_{V,W}(R(v \otimes w)).$$

More precisely, if $R = \sum_i s_i \otimes t_i$, then $c^R_{V,W}(v \otimes w) = \sum_i t_i w \otimes s_i v$. Then we have the following

Proposition B.1.1

Let $\mathscr{A}$ be a braided Hopf algebra with R-matrix R and V, W, U three $\mathscr{A}$-modules. Then

- $c^R_{V,W}$ *is an isomorphism of $\mathscr{A}$-modules;*

- *the hexagon equality holds, i.e. the following diagram is commutative*

$$\begin{tikzcd}
 & U\otimes W\otimes V \arrow[r, "c^R_{U,W}\otimes id"] & W\otimes U\otimes V \arrow[dr, "id\otimes c^R_{U,V}"] & \\
U\otimes V\otimes W \arrow[ur, "id\otimes c^R_{V,W}"] \arrow[dr, "c^R_{U,V}\otimes id"'] & & & W\otimes V\otimes U \\
 & V\otimes U\otimes W \arrow[r, "id\otimes c^R_{U,W}"'] & V\otimes W\otimes U \arrow[ur, "c^R_{V,W}\otimes id"'] &
\end{tikzcd}$$

Actually computing the R-matrix is non trivial at all. Consider again the case of $\mathcal{U}_q(\mathfrak{sl}_2)$, then an explicit formula is given by

$$R=\left(\sum_{r\geq 0}\frac{((q^{-1}-q)x^+\otimes x^-)^r}{q^{\frac{r(r-1)}{2}}[r]_q!}\right)e^{-ht_\infty},$$

where formally we pose $q=e^h$ and t_∞ is an element in $\mathfrak{h}\otimes\mathfrak{h}$ with certain properties (we do not present more detail in this context). Observe that a priori the element e^{-ht_∞} belongs to a completion of $\mathcal{U}_q(\mathfrak{sl}_2)^{\otimes 2}$ as the exponential has a development as a series. However, when we consider $c^R_{V,W}$ for finite-dimensional modules V, W, since $x^\pm$ are nilpotent, we obtain a well defined morphism of modules, so in this case the completion does not really pose any problem.

Then, for the $\mathcal{U}_q(\mathfrak{sl}_2)$-module V_1 considered above, the isomorphism $c^R_{V_1,V_1}$ is expressed on the basis $\{v_0\otimes v_0, v_0\otimes v_1, v_1\otimes v_0, v_1\otimes v_1\}$ by the matrix

$$C=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & q^{-1} & 0\\ 0 & q^{-1} & 1-q^{-2} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

In the case of quantum affine algebras $\mathcal{U}_q(\hat{\mathfrak{g}})$, we have seen that in general there is no isomorphism between $V\otimes W$ and $W\otimes V$, for example the case $V_1(1)\otimes V_1(q^2)$ which is not isomorphic to $V_1(q^2)\otimes V_1(1)$. Therefore the category of $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules is not braided and there is no universal R-matrix in the ordinary sense.

B.2 The affine case

We define some subalgebras of $\mathcal{U}_q(\hat{\mathfrak{g}})$.

- $\mathcal{U}_q(\mathfrak{b}_+)$ (respectively $\mathcal{U}_q(\mathfrak{b}_-)$) is generated in the Drinfeld-Jimbo realization by elements x_i^+, $k_i^{\pm 1}$ (respectively x_i^-, $k_i^{\pm 1}$) for $i = 0, ..., l$, with the usual relations;

- $\mathcal{U}_q(\hat{\mathfrak{n}}_+)$ (respectively $\mathcal{U}_q(\hat{\mathfrak{n}}_-)$) is generated in the Drinfeld realization by $x_{i,n}^+$ (respectively $x_{i,n}^-$) for $i = 1, ..., l$ with usual relations;

- let us extend $\mathcal{U}_q(\hat{\mathfrak{g}})$ with elements $\tilde{k}_i$, $i = 1, .., l$, subject to relations $\tilde{k}_i \tilde{k}_j = \tilde{k}_j \tilde{k}_i$, $\tilde{k}_i h_{j,n} = h_{j,n} \tilde{k}_i$, $\tilde{k}_i x_{j,m}^{\pm} = q^{\pm(\omega_i, \alpha_j)} x_{j,m}^{\pm} \tilde{k}_i$, $k_j = \prod_{i \in I} \tilde{k}_i^{C_{ij}}$. Observe that the $\tilde{k}_i$'s are the fundamental weights-analogue of the k_i's. This larger algebra is denoted by $\mathcal{U}_q(\hat{\mathfrak{g}})'$. In particular, inside $\mathcal{U}_q(\hat{\mathfrak{g}})'$ we consider the subalgebra generated by $x_{i,m}^{\pm}$, $\tilde{k}_i^{\pm}$, $h_{i,n}$ for $i \in I$, $m \in \mathbb{Z}_{\leq 0}$, $n \in \mathbb{Z}_{<0}$. We denote this subalgebra by $\mathcal{U}_q(\tilde{\mathfrak{g}})$. Notice that $\mathcal{U}_q(\mathfrak{b}_-) \subset \mathcal{U}_q(\tilde{\mathfrak{g}})$;

- $\mathcal{U}_q(\tilde{\mathfrak{h}}) \subset \mathcal{U}_q(\tilde{\mathfrak{g}})$ is the subalgebra generated by $\tilde{k}_i^{\pm}$, $h_{i,m}$ for $i \in I$ and $m < 0$;

- $\mathcal{U}_q(\tilde{\mathfrak{n}}_+)$ (respectively $\mathcal{U}_q(\tilde{\mathfrak{n}}_-)$) is the subalgebra of $\mathcal{U}_q(\tilde{\mathfrak{g}})$ generated by the $x_{i,n}^-$ (respectively $x_{i,n}^+$) for $i \in I$ and $n \leq 0$.

For $\mathcal{U}_q(\hat{\mathfrak{g}})$ there is an element called universal R-matrix which belongs to a larger algebra than the tensor square of $\mathcal{U}_q(\hat{\mathfrak{g}})$ and in fact it does not provide the braided property on $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules. However it is of great interest. Let us denote R the R-matrix in the complete topological Hopf algebra $(\mathcal{U}_q(\hat{\mathfrak{g}}) \otimes \mathcal{U}_q(\tilde{\mathfrak{g}}))[[h]]$ that verifies the conditions in the definition of universal R-matrix. Moreover, we have a factorization $R = R_{re}^+ R_{im} R_{re}^- K$. We want to define a ring homomorphism $\chi_q : Rep(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathcal{U}_q(\tilde{\mathfrak{h}})[[z]]$ and we obtain this by the composition of two morphisms. We recall that for this part we follow [11]. So, we consider a finite dimensional representation $\pi_V : \mathcal{U}_q(\hat{\mathfrak{g}}) \to End(V)$. We define

$$\hat{f}_V = \pi_V((z)) \otimes id : \big(\mathcal{U}_q(\hat{\mathfrak{g}})((z)) \otimes \mathcal{U}_q(\tilde{\mathfrak{g}})\big)[[h]] \to \big(End(V)((z)) \otimes \mathcal{U}_q(\tilde{\mathfrak{g}})\big)[[h]],$$

which on Drinfeld generators gives

$$\hat{f}_V(x^{\pm}_{i,r} \otimes x) = z^r \pi_V(x^{\pm}_{i,r}) \otimes x,\ \hat{f}_V(h_{i,m} \otimes x) = z^m \pi_v(h_{i,m}) \otimes x,$$
$$\hat{f}_V(c^{\frac{1}{2}} \otimes x) = id_V \otimes x,\ \hat{f}_V(k_i \otimes x) = \pi_V(k_i) \otimes x,\ \hat{f}_V(\phi^{\pm}_{i,r} \otimes x) = z^r \pi_V(\phi^{\pm}_{i,r}) \otimes x.$$

This shows that we can restrict $\hat{f}_V$ to get

$$f_{V,m} : \mathcal{U}_q(\mathfrak{b}_+)_m \otimes \mathcal{U}_q(\mathfrak{b}_-) \to z^m End(V) \otimes \mathcal{U}_q(\mathfrak{b}_-), \quad \text{for } m \geq 0.$$

If we call $\mathfrak{U} = \prod_{m\geq 0} \mathcal{U}_q(\mathfrak{b}_+)_m \otimes \mathcal{U}_q(\mathfrak{b}_-)$ and $\mathfrak{B} = \prod_{m\geq 0} z^m End(V) \otimes \mathcal{U}_q(\mathfrak{b}_-)$, then "glueing" together the maps $f_{V,m}$ we obtain

$$f_V : \mathfrak{U} \to \big(End(V) \otimes \mathcal{U}_q(\mathfrak{b}_-)\big)[[z]] = \mathfrak{B},$$

that is the restriction of $\hat{f}_V$ to $\mathfrak{U}$.

As mentioned above, $R = R^+_{re} R_{im} R^-_{re} K \in \big(\mathcal{U}_q(\hat{\mathfrak{g}})((z)) \otimes \mathcal{U}_q(\tilde{\mathfrak{g}})\big)[[h]]$. Now, one can prove that the R_{im} component of R belongs to $\mathfrak{U}$, while $a := (\pi_V \otimes id)[[h]](K) \in End(V) \otimes \mathcal{U}_q(\mathfrak{b}_-)$. Hence, we can set

$$\hat{L}_V := \hat{f}_V(R) \in \big(End(V)((z)) \otimes \mathcal{U}_q(\tilde{\mathfrak{g}})\big)[[h]], \tag{B.5}$$

$$L_V := f_V(R_{im})a \in \big(End(V) \otimes \mathcal{U}_q(\mathfrak{b}_-)\big)[[z]]. \tag{B.6}$$

We deduce from the classical trace $tr_V : End(V) \to \mathbb{C}$ a linear application $tr'_V \big(End(V) \otimes \mathcal{U}_q(\tilde{\mathfrak{g}})\big)[[h]] \to \mathcal{U}_q(\tilde{\mathfrak{g}})[[h]]$. Then, applying tr'_V on each term of the formal series in z, we have

$$\hat{Tr}_V : \big(End(V)((z)) \otimes \mathcal{U}_q(\tilde{\mathfrak{g}})\big)[[h]] \to \mathcal{U}_q(\tilde{\mathfrak{g}})((z))[[h]], \tag{B.7}$$

$$Tr_v : \big(End(V) \otimes \mathcal{U}_q(\mathfrak{b}_-)\big)[[z]] \to \mathcal{U}_q(\mathfrak{b}_-)[[z]]. \tag{B.8}$$

Hence, it makes sense to define the *transfer matrices*

(B.9) $$\hat{t}_V := \hat{Tr}_V(\hat{L}_V) \in \mathcal{U}_q(\tilde{\mathfrak{g}})((z))[[h]];$$

(B.10) $$t_V := Tr(L_V) \in \mathcal{U}_q(\mathfrak{b}_-)[[z]],$$

which allow us to define the following group morphisms:

(B.11) $$\hat{\nu}_q : Rep(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathcal{U}_q(\tilde{\mathfrak{g}})((z))[[h]],$$

(B.12) $$\nu_q : Rep(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathcal{U}_q(\mathfrak{b}_-)[[z]].$$

Remark B.2.1
Thanks to the Yang-Baxter equation satisfied by R, one can prove that transfer matrices commute with each other. Moreover, the subalgebra of $\mathcal{U}_q(\hat{\mathfrak{g}})$ generated by all the coefficients that appear in all the transfer matrices is commutative and is called the Baxter subalgebra.

As $\mathbb{C}$-vector space, $\mathcal{U}_q(\tilde{\mathfrak{n}})$ decomposes as

$$\mathcal{U}_q(\tilde{\mathfrak{g}}) = \mathcal{U}_q(\tilde{\mathfrak{n}}_+) \otimes \mathcal{U}_q(\tilde{\mathfrak{h}}) \otimes \mathcal{U}_q(\tilde{\mathfrak{n}}_-).$$

Recalling the notation $\epsilon : \mathcal{U}_q(\hat{\mathfrak{g}}) \to \mathbb{C}$ for the counit in the structure of Hopf algebra of $\mathcal{U}_q(\hat{\mathfrak{g}})$, we define the augmentation ideal of $\mathcal{U}_q(\tilde{\mathfrak{n}}_-)$ as

$$\left(\mathcal{U}_q(\tilde{\mathfrak{n}}_-)\right)_0 = \mathcal{U}_q(\tilde{\mathfrak{n}}_-) \cap Ker(\epsilon|_{\mathcal{U}_q(\tilde{\mathfrak{g}})}),$$

and similarly the augmentation ideal $\left(\mathcal{U}_q(\tilde{\mathfrak{n}}_+)\right)_0$ of $\mathcal{U}_q(\tilde{\mathfrak{n}}_+)$. So, by the definition of ϵ, we have $\mathcal{U}_q(\tilde{n}) = \left(\mathcal{U}_q(\tilde{\mathfrak{n}}_-)\right)_0 \oplus \mathbb{C}1$. Therefore, we obtain

$$\mathcal{U}_q(\tilde{\mathfrak{g}}) = \mathcal{U}_q(\tilde{\mathfrak{h}}) \oplus \left(\mathcal{U}_q(\tilde{\mathfrak{g}}) \cdot \mathcal{U}_q(\tilde{\mathfrak{n}}_+)_0\right) \oplus \left(\mathcal{U}_q(\tilde{\mathfrak{n}}_-)_0 \cdot \mathcal{U}_q(\tilde{\mathfrak{g}})\right).$$

Hence, the following projection maps are well defined:

(B.13) $$\hat{h}_q : \mathcal{U}_q(\tilde{\mathfrak{g}})((z))[[h]] \to \mathcal{U}_q(\tilde{\mathfrak{h}})((z))[[h]]$$

(B.14) $$h_q : \mathcal{U}_q(\tilde{\mathfrak{g}})[[z]] \to \mathcal{U}_q(\tilde{\mathfrak{h}})[[z]].$$

This leads us to the important

Definition B.2.1
We call q-character morphism the composition

$$\chi_q = h_q \circ \nu_q : Rep(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathcal{U}_q(\tilde{\mathfrak{h}})[[z]].$$

We denote $\hat{\chi}_q$ the application

$$\hat{\chi}_q = \hat{h}_q \circ \hat{\nu}_q : Rep(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathcal{U}_q(\tilde{\mathfrak{h}})((z))[[h]].$$

Actually, for every $V \in Rep(\mathcal{U}_q(\tilde{\mathfrak{g}}))$ the two maps $\hat{\chi}_q$ and χ_q coincide, thanks to the projection $\hat{h}_q$. Recall that at the beginning of Section 2.2 we called B the symmetrization of the Cartan matrix C associated to $\mathfrak{g}$. We call $B(q)$ the matrix with entries $B(q)_{ij} = [B_{ij}]_q$ and $\tilde{B}(q)$ the inverse of $B(q)$. Then, an explicit formula for the q-character of V is

$$\chi_q(V) = Tr_V\left(e^{-(q-q^{-1})\sum_{m>0,i,j\in I}\frac{m}{[m]_q}\tilde{B}_{ij}(q^m)z^m\pi_V(h_{i,m})\otimes h_{j,-m}}a\right), \tag{B.15}$$

where a has been defined above. The commutativity of the Baxter subalgebra implies the commutativity of the Grothendieck ring.
Now, where is the connection between the q-character just defined (which is in $\mathcal{U}_q(\tilde{\mathfrak{h}})[[z]]$) and the one seen in the previous chapters (which is in $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*}$)? We set

$$Y_{i,a} := q^{2(\rho,\omega_i)}\tilde{k}_i^{-1}e^{-(q-q^{-1})\sum_{m>0}\tilde{h}_{i,-m}z^m a^m}$$

for $a \in \mathbb{C}^*$. Then, by some long calculations, proving the algebraic independence of the $Y_{i,a}^{\pm 1}$, one can find that $\chi_q(V)$ defined by transfer matrices is a polynomial in the $Y_{i,a}^{\pm 1}$, $i \in I$, $a \in \mathbb{C}^*$.

We want to point out a link with Appendix A. Reading Section 3.2 from [11], we see that the projection map $h_q : \mathcal{U}_q(\tilde{\mathfrak{g}})[[z]] \to \mathcal{U}_q(\tilde{\mathfrak{h}})[[z]]$ is called a *quantum analogue* of Harish-Chandra morphism (see Theorem A.2.2). The deep motivation to call h_q like this lies on the theory of W-algebras, but a precise presentation of these objects would require the space and time for a new and different master thesis. However, the idea is that there exists an injective morphism, let us call it φ, from $Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$ to the center of $\mathcal{U}_q(\hat{\mathfrak{g}})$ at the critical level, i.e. for $c = q^{-h^\vee}$, $h^\vee$ being the dual Coxeter number for $\mathfrak{g}$. We denote $\mathscr{Z}(\mathscr{U}_q(\hat{\mathfrak{g}}))_c$ the center of the algebra at the critical level. Then, for every $V \in Rep(\mathscr{U}_q(\hat{\mathfrak{g}}))$ there is a corresponding transfer matrix $t_V \in \mathscr{U}_q(\mathfrak{b}_-)[[z]]$ and by the injectivity of φ, t_V corresponds bijectively $\varphi(V)$. Now, seeing t_V as an element in $\mathscr{Z}(\mathscr{U}_q(\hat{\mathfrak{g}}))_c$, we can indeed interpret the projection h_q on the Cartan factor as the quantum analogue of the map ξ in (A.8).

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