Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE
Corso di Laurea in Matematica

AN INTRODUCTION TO THE KODAIRA DIMENSION OF ALGEBRAIC VARIETIES

Tesi di Laurea in Geometria

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Anno Accademico 2022-2023

Introduction

Named after Kunihiko Kodaira, founder of the Japanese school of algebraic geometry and the first Japanese mathematician to ever win the Fields Medal in 1954, the Kodaira dimension is the most basic birational invariant in algebraic geometry. The main goal of this thesis will be to state the definition of the Kodaira dimension of a variety and compute it for irreducible smooth projective hypersurfaces and curves.

We begin the exposition with a rapid review of the construction of the category of algebraic varieties defined over an algebraically closed field. We introduce the concept of structure sheaf of a variety and remind that the categorical product of varieties differs from the topological product of their underlying topological spaces.

In Chapter 1 we define the construction of an algebraic vector bundle. We introduce the first immediate consequences, transition functions and sections, which we will show contain in fact equivalent information as the vector bundle itself. We also show that a lot of endofunctors of the category of vector spaces can also be applied to vector bundles: for example, to define the dual of a vector bundle, or the tensor product of two vector bundles.

Next, in Chapter 2 we study algebraic vector bundles of rank 1, called line bundles. Line bundles are very important in algebraic geometry since they govern all rational maps to projective space (in particular, all possible embeddings in projective space). Line bundles over a fixed variety X also have the natural structure of a group, called the Picard Group of X. Furthermore, we define the canonical line bundle of a variety. The canonical bundle is invariant under isomorphism of varieties; it is important because it and its powers are the only line bundles on an algebraic variety which are intrinsically defined.

Finally, in Chapter 3 we have the necessary tools to define the Kodaira dimension of an irreducible projective variety. We also prove the birational invariance of the Kodaira dimension for irreducible smooth projective varieties. Afterwards we explicitly compute the canonical bundles of affine and projective smooth irreducible hypersurfaces, as well as for the blowup of a point in $\mathbb{A}^2$; we get a formula for the Kodaira dimension of an irreducible smooth projective hypersurface, based on its degree. We finish the dissertation with a complete Kodaira dimension classification of irreducible projective smooth curves, based on their (geometrical) genus. To do this we develop the terminology of divisors in order to define the degree of a line bundle. We can then employ the powerful Riemann-Roch theorem (for curves) for the classification. We conclude with an example of how to construct a variety of a fixed (algebraic) dimension of all possible Kodaira dimensions, by considering products of curves, thanks to the additivity of the Kodaira dimension.

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Chapter 0

Algebraic Varieties

In this section we introduce the category of quasi-projective varieties over a fixed algebraically closed field k. This will by no means be a thorough introduction to the subject; the reader is expected to have some familiarity with algebraic geometry. A good introduction on which this work is partly based is [6]. Some basic familiarity with the theory of sheaves will be given for granted; we use the terminology found in [7, Section 7]. The goal of this chapter is to introduce clear definitions and notation, as well as some facts that will be used later.

While algebraic varieties can be defined over any field (with important applications!) within the context of this thesis the symbol k will always denote an algebraically closed field. This gives us access to *Hilbert's Nullstellensatz* [6, 2.3]. $\mathbb{A}^n$ will denote the affine n-space, k^n with the Zariski topology, and similarly the projective n-space $\mathbb{P}^n := \mathbb{P}^n(k)$ will also be equipped with the Zariski topology.

Definition 0.0.1. A *quasi-projective variety* (or simply, a *variety*) is a locally closed subset of $\mathbb{P}^n$. That is, it is the intersection of an open subset and a closed subset of $\mathbb{P}^n$.

Unless stated otherwise, we do not require varieties to be (topologically) irreducible. We give this temporary definition of a *morphism* of varieties: given a variety $V \subseteq \mathbb{P}^n$, a map $f : V \to \mathbb{P}^m$ is a morphism of varieties if for all $p \in V$ there exist homogeneous polynomials of same degree $F_0, \ldots, F_m \in k[x_0, \ldots, x_n]$ such that the map

$$\begin{aligned} V &\longrightarrow \mathbb{P}^m \\ q &\longmapsto [F_0(q) : \ldots : F_m(q)] \end{aligned}$$

is well defined at p (not all polynomials vanish) and agrees with f in some neighbourhood of p. An *isomorphism* of varieties is a bijective morphism whose inverse is also a morphism.

Zariski-closed subsets of $\mathbb{A}^n$ are quasi-projective varieties: we may imagine X as the intersection of the zero locus in $\mathbb{P}^n$ of the homogenisation of its defining ideal $\mathbb{I}(X)$ with the affine chart U_0.

Definition 0.0.2. A variety is said to be *affine* if it is isomorphic as a quasi-projective variety to some Zariski-closed subset X of $\mathbb{A}^n$ (for any n). The *coordinate ring* $k[V]$ of an affine variety V is defined to be the coordinate ring $k[X]$ of any Zariski-closed subset X of affine space isomorphic to V. More precisely, it consists of all functions $f : V \to k$ which are pullbacks of functions in $k[X]$ under the isomorphism. This does not depend on the choice of isomorphism or Zariski closed subset X.

Definition 0.0.3. A variety is said to be *projective* if it is isomorphic as a quasi-projective variety to some Zariski-closed subset Y of $\mathbb{P}^n$.

To define better define morphisms between varieties, we first introduce *regular functions*:

Definition 0.0.4. Let $U \subseteq V$ be an open subset of an affine variety. A function $f : U \to k$ is *regular at a point* $p \in U$ if there exist polynomials $g, h \in k[V]$ such that $h(p) \neq 0$ and f agrees with $\frac{g}{h}$ in a neighbourhood of p. f is *regular* on U if it is regular at every point of U.

We introduce the first important fact: affine open subsets of any quasi-projective variety form a basis for the Zariski topology. We can introduce the following more general definition:

Definition 0.0.5. Let $U \subseteq V$ be an open subset of a variety. A function $f : U \to k$ is *regular at a point* $p \in U$ if there exists an affine open set containing p on which f is regular at p. f is *regular* on U if it is regular at every point of U. We denote the set of all regular functions on U by $\mathcal{O}_V(U)$.

If V is affine, this definition agrees with the previous one. The next important fact is that, for affine varieties, $\mathcal{O}_V(V) = k[V]$ (see [6, 4.3]). We can finally introduce the definition we will use for *morphisms of varieties*:

Definition 0.0.6. A map $\phi : V \to W$ between quasi-projective varieties is a *morphism of varieties* if for all $p \in V$ there exist affine open neighbourhoods $U \subseteq V$ of p, $U' \subseteq W$ of $\phi(p)$ such that $\phi(U) \subseteq U'$ and $\phi|_U$ is given by a set of regular functions in the coordinates of U. Informally, a morphism of varieties is a map locally given by polynomials.

Remark 0.0.7. Morphisms of varieties are continuous in the Zariski topology. We have thus defined the category of algebraic varieties, whose objects are quasi-projective varieties and whose arrows are morphisms of varieties.

Remark 0.0.8. Unlike the Euclidean topology, the product Zariski topology on $\mathbb{A}^1 \times \mathbb{A}^1$ is strictly weaker than the Zariski topology on $\mathbb{A}^2$. The categorical product of varieties is not the topological product of varieties; it is instead given by the Segre map [6, 5.3]. When we write the product $V \times W$ of varieties, we will always mean the categorical product of varieties, which comes with the categorical projections π_1, π_2.

We introduce some more facts:

- The intersection of affine varieties is affine.
- The product of affine varieties is affine; the product of projective varieties is projective.
- There are no non-constant regular functions on projective varieties.
- A smooth variety is irreducible if and only if it is connected.

Let V be a variety, consider the set $\mathcal{O}_V(U)$ of regular functions on U for each open subset U of V. We summarize the local nature of regular functions with the following properties:

1. Every $\mathcal{O}_V(U)$ is a ring (in fact, a k-algebra) with respect to pointwise addition and multiplication.

2. If a function is regular on U, it is regular on all its open subsets; if $U' \subseteq U$ is open, then the restriction to U' defines a ring homomorphism $\mathcal{O}_V(U) \to \mathcal{O}_V(U')$.

3. Let $\{U_i\}_{i \in I}$ be an open cover of U, $\{f_i\}_{i \in I}$ a family of functions such that f_i is defined and regular on U_i and f_i, f_j agree on the intersection $U_i \cap U_j$ for all $i, j \in I$. Then they define a unique function f regular on U, and the f_i are the restrictions of this function to the sets U_i.

4. If $F : V \to W$ is a morphism of varieties, $U \subseteq W$ is an open set, then for any $f \in \mathcal{O}_W(U)$, the *pullback* is a regular function

$$f \circ F \in \mathcal{O}_V(F^{-1}(U)).$$

The first three properties mean that regular functions on a variety V satisfy the definition of a *sheaf of k-algebras*. We denote this sheaf by $\mathcal{O}_V$ and call it the *structure sheaf* of V. The pair $(V, \mathcal{O}_V)$ is a *ringed space*. Property 4 states that every morphism of algebraic varieties induces a morphism of sheaves of k-algebras from $\mathcal{O}_W$ to $f^*\mathcal{O}_V$.

Chapter 1

Vector Bundles

Vector bundles are a very natural construction for manifolds and varieties. In fact vector bundles can be defined on arbitrary topological spaces; we adapt this concept to algebraic varieties. Topological, differentiable and complex manifolds are very similar to varieties in this regard, in the sense that the entire contents of this chapter can be translated with minimal effort to the language of manifolds (or even to the simple topological space case).

1.1 Vector Bundles

Let B be a fixed variety, which we will call the *base space*.

Definition 1.1.1. [5, § 2] A *vector bundle* ξ over B consists of the following:

1. A variety $E = E(\xi)$ called the *total space*,

2. A surjective morphism $\pi : E \to B$ called the *projection map*,

3. For each $b \in B$ we call the preimage $\pi^{-1}(b) := E_b$ the *fibre* over b and require it to have the structure of a vector space over k.

The structure satisfies the following property: there exists an open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of B such that there exist isomorphisms φ_λ which make the following diagram commute:

$$\begin{tikzcd} \pi^{-1}(U_\lambda) \arrow[rr, "\varphi_\lambda"] \arrow[dr, "\pi"'] & & U_\lambda \times \mathbb{A}^n \arrow[dl, "\pi_1"] \\ & U_\lambda & \end{tikzcd}$$

where π_1 is the categorical projection to the first factor. We require that the restriction to each fibre $\varphi_\lambda|_{E_b} : E_b = \pi^{-1}(b) \to \{b\} \times \mathbb{A}^n$ is a linear isomorphism (of vector spaces) for all b in U_λ. We call the pair $(U_\lambda, \varphi_\lambda)$ a *local trivialisation*; we will refer to U_λ as a *trivialising open*.

If it is possible to choose U to be equal to the entire base space, we will call ξ a *trivial vector bundle*.

Again, we stress that $U_\lambda \times \mathbb{A}^n$ is the categorical product of varieties as in Remark 0.0.8. Note that the structure of a vector space over k is compatible with that of a variety, since:

1. They can be thought of as k^n and thus equipped with the Zariski topology;

2. Vector space operations and linear functions are given by linear polynomials in their coordinates, so they define morphisms of varieties.

Thus it makes sense to consider morphisms of varieties which restrict to linear functions between vector spaces.

Remark 1.1.2. In general the dimension n of each fibre is allowed to be a (locally constant) function of b; in most cases of interest, and in the context of this thesis, n will be constant and we will call it the *rank* of the vector bundle.

Remark 1.1.3. We may assume the trivialising opens to be affine. Indeed, If U is a trivialising open, then all of its open subsets will also be trivialising. Since affine open sets form a base for the Zariski topology, affine opens cover U, thus immediately B has a cover of open affine trivialising sets.

Even though bundles can be defined with fibres not necessarily isomorphic to vector spaces, from now when we mention bundles in this thesis we will always refer to algebraic vector bundles.

Definition 1.1.4. [5, §3] A *morphism of vector bundles* ξ, η over the same variety B is a morphism ϕ between the total spaces $\phi : E(\xi) \to E(\eta)$ which maps every fibre $E_b(\xi)$ linearly into the corresponding vector space $E_b(\eta)$. If ϕ is additionally an isomorphism of varieties which defines a linear isomorphism when restricted to each fibre, we say that ξ is *isomorphic* to η.

Example 1.1.5. A trivial bundle defined earlier can thus be interpreted as a bundle isomorphic to $B \times \mathbb{A}^n$, with projection map $\pi(b, v) = b$, and vector structure on the fibres defined by

$$t_1(b, v) + t_2(b, w) = (b, t_1 v + t_2 w) \qquad \forall\, t_1, t_2 \in k.$$

As in most branches of mathematics, we are not interested in the differences between isomorphic vector bundles, and will often identify them.

1.2 Transition Functions

Let ξ be an algebraic vector bundle of rank n over the variety B, $\{U_\lambda\}_{\lambda\in\Lambda}$ a covering of affine trivialising opens. For each $\alpha, \beta \in \Lambda$ the composition

$$\varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{A}^n \to (U_\alpha \cap U_\beta) \times \mathbb{A}^n$$

is an isomorphism of varieties which restricts to a linear isomorphism when evaluated at each point $b \in U_\alpha \cap U_\beta$. Thus we have morphisms of varieties $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(k)$ such that

$$\varphi_\alpha \circ \varphi_\beta^{-1}(b, v) = (b, g_{\alpha\beta}(b)v) \qquad \forall\, b \in U_\alpha \cap U_\beta,\ v \in \mathbb{A}^n = k^n$$

Definition 1.2.1. [1, 3.1.6] The maps $g_{\alpha\beta}$ are called *transition maps* for the bundle ξ with respect to the open cover $\{U_\lambda\}_{\lambda\in\Lambda}$.

We introduce the following notation for trivialising open sets:

$$U_{\lambda\mu} := U_\lambda \cap U_\mu$$

Remark 1.2.2. $GL_n(k)$ has a natural structure of affine variety: since the determinant is a polynomial, $GL_n(k)$ is the complement of the hypersurface $\mathbb{V}(\det) \subseteq \mathbb{A}^{n^2}$ and is isomorphic to $\mathbb{V}((x_{n^2+1} \cdot \det) - 1) \subseteq \mathbb{A}^{n^2+1}$. Since the intersection of affine varieties is still affine, each transition function $g_{\alpha\beta} : U_{\alpha\beta} \to GL_n(k)$ defines a morphism of affine varieties.

Remark 1.2.3. Given a local trivialisation (U, φ) of a vector bundle ξ of rank n, what are the other possible trivialisations $\varphi' : \pi^{-1}(U) \to U \times \mathbb{A}^n$ defined on $\pi^{-1}(U)$? The composition $\varphi' \circ \varphi^{-1} : U \times \mathbb{A}^n \to U \times \mathbb{A}^n$ must preserve each vector space $\{p\} \times \mathbb{A}^n$ for all $p \in U$ and be a linear isomorphism when restricted to these spaces. Thus all possible trivialisations (U, φ') are of the form

$$\varphi'(p) = (\mathrm{id}, \chi) \circ \varphi : \pi^{-1}(U) \to U \times \mathbb{A}^n$$

where $\chi : U \to GL_n(k)$ is a morphism of varieties (which is a linear isomorphism when evaluated at each $p \in U$).

Remark 1.2.4. Transition functions are not unique! By the previous remark, if $g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$ is a transition function for ξ with respect to the open cover $\{U_\lambda\}_{\lambda \in \Lambda}$, then

$$\chi_\alpha\big|_{U_{\alpha\beta}} \circ g_{\alpha\beta} \circ \chi_\beta^{-1}\big|_{U_{\alpha\beta}}$$

are still transition functions for ξ for any fixed morphisms $\chi_\alpha : U_\alpha \to GL_n(k)$, $\chi_\beta : U_\beta \to GL_n(k)$ (of which we considered the pointwise inverse) for all $\alpha, \beta \in \Lambda$.

Proposition 1.2.5. *[1, 3.1.8]*

(i) Let $\{g_{\alpha\beta}\}_{\alpha,\beta \in \Lambda}$ be transition functions of an algebraic vector bundle ξ of rank n with respect to the trivialising open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of B. We

have:

$$g_{\alpha\alpha}(b) = I_n \qquad \forall\, b \in U_\alpha \tag{1.1}$$
$$g_{\beta\alpha}(b) = g_{\alpha\beta}(b)^{-1} \qquad \forall b \in U_{\alpha\beta} \tag{1.2}$$
$$g_{\alpha\beta}(b)g_{\beta\gamma}(b) = g_{\alpha\gamma}(b) \qquad \forall b \in U_{\alpha\beta\gamma} \tag{1.3}$$

(ii) Viceversa, suppose we have an open affine cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of B, morphisms $g_{\alpha\beta} : U_{\alpha\beta} \to GL_n(k)$ satisfying the previous properties. Then there exists a unique vector bundle (up to isomorphism) ξ over B which has $\{g_{\alpha\beta}\}_{\alpha,\beta\in\Lambda}$ as transition functions with respect to the open cover $\{U_\lambda\}_{\lambda\in\Lambda}$.

Proof. (i) This follows immediately from the properties of the trivialisations $\{(U_\lambda, \varphi_\lambda)\}_{\lambda\in\Lambda}$:

$$\varphi_\alpha \circ \varphi_\alpha^{-1} = \mathrm{id} \qquad \text{in } U_\alpha \times \mathbb{A}^n$$
$$\varphi_\beta \circ \varphi_\alpha^{-1} = (\varphi_\alpha \circ \varphi_\beta^{-1})^{-1} \qquad \text{in } U_{\alpha\beta} \times \mathbb{A}^n$$
$$(\varphi_\alpha \circ \varphi_\beta^{-1}) \circ (\varphi_\beta \circ \varphi_\gamma^{-1}) = \varphi_\alpha \circ \varphi_\gamma^{-1} \qquad \text{in } U_{\alpha\beta\gamma} \times \mathbb{A}^n.$$

(ii) We define the total space E by

$$E = \left(\coprod_{\lambda\in\Lambda} U_\lambda \times \mathbb{A}^n\right)_{/\sim}$$

where $\sim$ is the equivalence relation identifying $(p, v) \in U_\alpha \times \mathbb{A}^n$ with $(q, w) \in U_\beta \times \mathbb{A}^n$ if and only if $p = q \in U_{\alpha\beta}$ and $v = g_{\alpha\beta}(p)w$. Properties (1.1)-(1.3) guarantee that this is indeed an equivalence relation. No two distinct elements of $U_\alpha \times \mathbb{A}^n$ are $\sim$-related: we have a surjective map $\pi : E \to B$ such that $\pi^{-1}(U_\alpha) = (U_\alpha \times \mathbb{A}^n)_{/\sim}$ and a natural bijection $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{A}^n$. The latter restricts to a bijection $\pi^{-1}(p) \to \{p\} \times \mathbb{A}^n$ for all $p \in U_\alpha$, so we can define a vector space structure on $\pi^{-1}(p)$ induced by the bijection. We still need to show that this structure does not depend on the trivialisation we used. On the intersections $U_{\alpha\beta} \times \mathbb{A}^n$ we get $\varphi_\alpha \circ \varphi_\beta^{-1}(p, v) = (p, g_{\alpha\beta}(p)v)$ by construction. Then for any $w_1, w_2 \in \mathbb{A}^n$ let $v_i = g_{\alpha\beta}(p)w_i,\ i = 1, 2$, we have

$$\begin{aligned}\varphi_\alpha^{-1}(p, v_1 + v_2) &= \varphi_\alpha^{-1}(p, g_{\alpha\beta}(p)w_1 + g_{\alpha\beta}(p)w_2)\\ &= \varphi_\alpha^{-1}(\varphi_\alpha \circ \varphi_\beta^{-1}(p, w_1 + w_2))\\ &= \varphi_\beta^{-1}(p, w_1 + w_2)\end{aligned}$$

so our structure is well defined. The last thing is to show that E is a variety. This is not easy, but we can sketch how the local structure of a ringed space on E is defined. We assign to each $\pi^{-1}(U_\alpha)$ the structure of an affine variety given by its bijection φ_α with $U_\alpha \times \mathbb{A}^n$, and declare this to be an open cover of E. The transition maps recover how the pieces of the cover are glued together; in particular they define the restrictions $\mathcal{O}_E(\pi^{-1}(U_\alpha)) \to \mathcal{O}_E(\pi^{-1}(U_{\alpha\beta}))$, $\mathcal{O}_E(\pi^{-1}(U_\beta)) \to \mathcal{O}_E(\pi^{-1}(U_{\alpha\beta}))$.

Suppose we have another vector bundle $\tilde{\pi} : \tilde{E} \to B$ with the same transition functions with respect to trivialisations $\tilde{\varphi}_\alpha : \tilde{\pi}^{-1}(U_\alpha) \to U_\alpha \times \mathbb{A}^n$. For each $\alpha \in \Lambda$ we define $L_\alpha : \pi^{-1}(U_\alpha) \to \tilde{\pi}^{-1}(U_\alpha)$ by $L_\alpha = \tilde{\varphi}_\alpha^{-1} \circ \varphi_\alpha$. Clearly L_α is an isomorphism linear in each fibre, and $\tilde{\pi} \circ L_\alpha = \pi$. Finally $L_\alpha \equiv L_\beta$ on $\pi^{-1}(U_{\alpha\beta})$, since $\tilde{\varphi}_\alpha^{-1} \circ \varphi_\alpha \equiv \tilde{\varphi}_\beta^{-1} \circ \varphi_\beta$ if and only if $\varphi_\alpha \circ \varphi_\beta^{-1} = \tilde{\varphi}_\alpha \circ \tilde{\varphi}_\beta^{-1}$ and this is true since E and $\tilde{E}$ have the same transition maps. We get a well defined isomorphism of bundles $L : E \to \tilde{E}$ by gluing together the L_α.

□

Remark 1.2.6. If the transition functions for two vector bundles ξ_1, ξ_2 over the same variety are defined for two different open covers $\{U_\lambda\}_{\lambda\in\Lambda}, \{U_\delta\}_{\delta\in\Delta}$, we can still determine if they are isomorphic. We need to consider the open cover consisting of pairwise intersections, $\{U_\lambda \cap U_\delta\}_{\lambda\in\Lambda,\delta\in\Delta}$ on which we have the restrictions of the transition functions for the two bundles. If the conditions of Proposition 1.2.5 still hold for these restrictions, then the two bundles are isomorphic.

1.3 Constructions on Vector Bundles

Definition 1.3.1. [5, §3] Let ξ be a bundle over the variety B of rank n, B' be any variety. Any morphism $f : B' \to B$ induces the *pullback bundle* $f^*\xi$ over B' defined as follows:

1. The total space E' is the fibred product $B' \times_B E$, that is, the closed subset of the variety $B' \times E$ defined by

$$B' \times_B E = \{(x, e) \in B' \times E \;\mid\; f(x) = \pi(e)\};$$

,

2. The projection map $\pi' : E' \to B'$ is defined by $\pi'(x, e) = x$ (the restriction of the canonical projection);

3. We have the following commutative diagram:

$$\begin{tikzcd} E' \arrow[r, "\hat{f}"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

where $\hat{f}(x, e) = e$. The vector space structure on each fibre $E'_x = \pi'^{-1}(x)$ is defined by

$$t_1(x, e_1) + t_2(x, e_2) = (x, t_1e_1 + t_2e_2) \qquad \forall\, t_1, t_2 \in k,\; e_1, e_2 \in \pi^{-1}(f(x))$$

so $\hat{f}$ carries each fibre E'_x isomorphically onto $E_{f(x)}$;

4. If (U, φ) is a local trivialisation for ξ, then by setting $U' = f^{-1}(U)$, defining

$$\varphi' : \pi'^{-1}(U') \to U' \times k^n \qquad \varphi'(x, e) = (x, \pi_2 \circ \varphi\,(e)) = (x, v)$$

where $\varphi(e) = (f(x), v) \in U \times \mathbb{A}^n$, $\pi_2 : U \times \mathbb{A}^n \to \mathbb{A}^n$ is the projection to the second factor. Then (U', φ') is a local trivialisation for $f^*\xi$.

It follows immediately that if ξ is trivial, then $f^*\xi$ is trivial.

Example 1.3.2. If ξ is a vector bundle over B, $\overline{B} \subseteq B$ a subvariety, $\iota : \overline{B} \hookrightarrow B$ the inclusion map, then the *restriction* of ξ to $\overline{B}$ is the vector bundle $\iota^*\xi$.

Remark 1.3.3. It is clear by how the trivialisations of the pullback bundle are defined that if $g_{\lambda\mu} : U_{\lambda\mu} \to GL_n(k)$ is a transition function for ξ, then

$$f^* g_{\lambda\mu} := (g_{\lambda\mu} \circ f) : f^{-1}(U_{\lambda\mu}) \longrightarrow GL_n(k)$$

is a transition function for $f^*\xi$.

There exist many important constructions on vector spaces which produce new vector spaces. For example, if V, W are vector spaces over k:

1. The direct sum $V \oplus W$;
2. The vector space $\hom_k(V, W)$ of linear functions $f : V \to W$;
3. The dual vector space $V^\vee = \hom_k(V, k)$;
4. The tensor product $V \otimes W$;
5. The n-th exterior product $\Lambda^n V$;

and many others. We will now show that all of these constructions can also be applied to vector bundles.

Definition 1.3.4. Let $\mathcal{V}$ be the category whose objects are all finite dimensional vector spaces over the field k, and whose arrows are all isomorphisms between such spaces. A (covariant) *functor* $T : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is an operation which assigns:

1. to each pair $V, W \in \mathcal{V}$ a vector space $T(V, W) \in \mathcal{V}$;
2. to each pair $f : V \to V'$, $g : W \to W'$ of isomorphisms, a new isomorphism $T(f, g) : T(V, W) \to T(V', W')$ such that:
   (a) $T(\mathrm{id}_V, \mathrm{id}_W) = \mathrm{id}_{T(V,W)}$;

(b) $T(f_1 \circ f_2, g_1 \circ g_2) = T(f_1, g_1) \circ T(f_2, g_2)$.

Remark 1.3.5. The above conditions imply $T(f^{-1}, g^{-1}) = T(f, g)^{-1}$.

The concept of a functor $T : \mathcal{V} \times \ldots \times \mathcal{V} \to \mathcal{V}$ in n variables is defined similarly. Note that all of the previous examples are functors in one or two variables.

Fix a functor T (in two variables), given two vector bundles ξ, ξ' over a variety B of rank r_1, r_2 which have E_p, E'_p as fibres, we want to define a new vector bundle with $T(E_p, E'_p)$ as fibre over p for all $p \in B$. To do this we focus on the assignment $T(f, g)$ for any two isomorphisms of vector spaces f, g from E_p, respectively E'_p to any vector space. Let $(U_\lambda, \varphi_\lambda)$, $(U_\lambda, \varphi'_\lambda)$ be local trivialisations around p for each bundle on the same neighbourhood. When restricted to each fibre, they define linear isomorphisms from E_p, E'_p to $\{p\} \times \mathbb{A}^{r_1} = \{p\} \times k^{r_1}$, respectively $\{p\} \times k^{r_2}$. With an abuse of notation, we denote by $T(\varphi_\lambda, \varphi'_\lambda)$ the function

$$T(\varphi_\lambda, \varphi'_\lambda) : \coprod_{p \in U_\lambda} \{p\} \times T(E_p, E'_p) \longrightarrow \coprod_{p \in U_\lambda} \{p\} \times T(k^{r_1}, k^{r_2})$$

which when restricted to each fibre $T(E_p, E'_p)$ is the map $T(\varphi_\lambda|_{E_p}, \varphi'_\lambda|_{E'_p})$. Given other trivialisations around p (U_μ, φ_μ), (U_μ, φ'_μ), consider the map on $U_{\lambda\mu} \times T(k^{r_1}, k^{r_2})$ (for now, this is a set product and not a product of varieties!) to itself

$$\begin{aligned}
\big(T(\varphi_\lambda, \varphi'_\lambda) \circ T(\varphi_\mu, \varphi'_\mu)^{-1}\big)(p, T(v_1, v_2)) = \\
&= \big(T(\varphi_\lambda, \varphi'_\lambda) \circ T(\varphi_\mu^{-1}, \varphi_\mu'^{-1})\big)\,(p, T(v_1, v_2)) \\
&= \big(T(\varphi_\lambda \circ \varphi_\mu^{-1}, \varphi'_\lambda \circ \varphi_\mu'^{-1})\big)\,(p, T(v_1, v_2)) \\
&= (p, T(g_{\lambda\mu}, g'_{\lambda\mu}) T(v_1, v_2))
\end{aligned}$$

where $g_{\lambda\mu}, g'_{\lambda\mu}$ are the transition maps for ξ, respectively ξ'. The equations make sense on the restrictions to each fibre: on those restrictions we used the functoriality of T. Now if we show that

$$T(g_{\lambda\mu}, g'_{\lambda\mu}) : U_{\lambda\mu} \longrightarrow GL(T(k^{r_1}, k^{r_2}))$$

is indeed a morphism of varieties for all $\lambda, \mu \in \Lambda$, we are done thanks to Proposition 1.2.5. We will see that, for all our relevant examples, the coefficients of $T(g_{\lambda\mu}, g'_{\lambda\mu})$ are given by products, inverses, sums or compositions of the coefficients of $g_{\lambda\mu}, g'_{\lambda\mu}$. Since regular functions are closed with respect to these operations, the coefficients of $T(g_{\lambda\mu}, g'_{\lambda\mu})$ are also (locally) regular functions; in other words, it is a morphism of varieties.

Definition 1.3.6. [5, §3.6] Let $T : \mathcal{V}\times\ldots\times\mathcal{V} \to \mathcal{V}$ be a functor in n variables whose assignment $T(f_1, \ldots, f_n)$ of isomorphsisms depends rationally (in the sense $\frac{p}{q}$, where p, q are polynomials in multiple variables) on the coefficients of $f_1, \ldots, f_n$. Let $\xi_1, \ldots, \xi_n$ be vector bundles on the same base variety B with transition maps $g^{(1)}_{\lambda\mu}, \ldots, g^{(n)}_{\lambda\mu}$ with respect to a trivialising open cover $\{U_\alpha\}_{\alpha\in\Lambda}$. Then we define the vector bundle $T(\xi_1, \ldots, \xi_n)$ to be the bundle with transition maps $T(g^{(1)}_{\lambda\mu}, \ldots, g^{(n)}_{\lambda\mu})$ with respect the open cover $\{U_\alpha\}_{\alpha\in\Lambda}$. Its fibres are $E_b = T(E_b(\xi_1), \ldots, E_b(\xi_n))$ for each $b \in B$.

Example 1.3.7. Let ξ, ξ' be two vector bundles over the variety B of rank r_1, respectively r_2. The *tensor product bundle* $\xi\otimes\xi'$ is the vector bundle over B given by the tensor product functor. The fibres over each point $p \in B$ is $E_p \otimes E'_p$. The way the functor acts on linear isomorphisms is given by the *Kronecker product* [1, 1.E.6]. If $A \in M_{r_1}(k)$, $B \in M_{r_2}(k)$ are two matrices, the Kronecker product $A \otimes B$ is the (block) matrix

$$A \otimes B := \begin{pmatrix} a_{11}B & \ldots & a_{1r_1}B \\ \vdots & \ddots & \vdots \\ a_{r_1 1}B & \ldots & a_{r_1 r_1}B \end{pmatrix} \in M_{r_1 r_2}(k)$$

(the Kronecker product is defined on non-invertible matrices, or even non-square matrices).

Example 1.3.8. *The dual bundle* $\xi^\vee$ of a bundle ξ over B of rank r is given by the dual functor; its fibres are $E_p^\vee$. The dual functor is actually contravariant: it assigns every linear function $f : V \to W$ to its transpose $f^\vee : W^\vee \to V^\vee$. This is one of the reasons why we considered the category of finite vector spaces with as arrows only linear isomorphisms: it does not really matter if the functor is covariant or contravariant (the other reason being that transition functions are given by invertible matrices on the fibres, so it is not restrictive to consider only isomorphisms). Since we want a function $T(f) : V^\vee \to W^\vee$, our assignment will be $T(f) = (f^\vee)^{-1}$.

Example 1.3.9. The *i-th exterior product bundle* $\bigwedge^i \xi$ of a bundle ξ over B of rank r is given by the i-th exterior product functor; its fibres are $\bigwedge^i E_p$. The functor assigns to each $f \in \hom_k(V, W)$ the function $\bigwedge^i f \in \hom_k(\bigwedge^i V, \bigwedge^i W)$ defined by

$$\bigwedge^i f(v_1 \wedge \ldots \wedge v_i) = f(v_1) \wedge \ldots \wedge f(v_i).$$

It is easy, yet tedious, to show that the coefficients of $\bigwedge^i f$ are just differences of products of the coefficients of f. We are mainly interested in the highest exterior product of ξ, $\bigwedge^r \xi$. This yields a vector bundle of rank one, and the functor assigns to each isomorphism f its determinant $\det f$.

1.4 Sections

Definition 1.4.1. [6, 8.3] Let ξ be a vector bundle with base space B, $U \subseteq B$ an open subset. A *section* of ξ over U is a morphism

$$s : U \to E(\xi)$$

such that

$$\pi \circ s = \iota_{U \hookrightarrow B}$$

that is, which takes each $b \in U$ into the corresponding fibre E_b. We denote the set of all sections over U by $\mathcal{E}(U)$. We call the elements of $\mathcal{E}(B)$ the *global sections* of ξ.

Remark 1.4.2. Every vector bundle admits a unique global *zero section*. Indeed, the map $B \xrightarrow{s} E$ defined by

$$b \mapsto 0_b$$

where 0_b is the zero vector in E_b, is clearly a morphism. Since $\pi \circ s$ is the identity on B, we may interpret this map as an embedding of B into E.

Every vector bundle thus admits at least one section over each open set, namely the restriction of its zero section to each open. If $s_1, s_2 \in \mathcal{E}(U)$ are two sections over U of a vector bundle ξ on B, $\lambda \in k$, then s_1+s_2, $\lambda\cdot s_1$ are also sections over U: sections are morphism of varieties, that is, locally defined by a set of regular functions in the coordinates of U, which are closed under addition and scalar multiplication. Furthermore, for any regular function $f \in \mathcal{O}_B(U)$ the product fs is still a section over U. Indeed, this map is defined by $b \mapsto f(b) \cdot s(b)$, where $f(b) \in k$ simply acts by multiplication on the vector $s(b) \in E_b(\xi)$.

The sections of an algebraic vector bundle are a sheaf; more specifically, they satisfy the following definition:

Definition 1.4.3. Let $(X, \mathcal{R})$ be a ringed space. A *sheaf of* $\mathcal{R}$*-modules* is the datum of a sheaf $\mathcal{E}$ over X with the structure of an $\mathcal{R}(U)$-module on $\mathcal{E}(U)$ for all $U \subseteq X$ open subsets such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{R}(U) \times \mathcal{E}(U) \arrow[r, "\bullet"] \arrow[d, "\rho_{UV} \times \sigma_{UV}"'] & \mathcal{E}(U) \arrow[d, "\sigma_{UV}"] \\
\mathcal{R}(V) \times \mathcal{E}(V) \arrow[r, "\bullet"] & \mathcal{E}(V)
\end{tikzcd}$$

where V is any open subset of U, ρ_{UV}, σ_{UV} are the sheaf restriction maps of $\mathcal{R}$, respectively $\mathcal{E}$, and $\bullet$ is the module multiplication. We call $\mathcal{E}$ an $\mathcal{R}$-*module.*

The sheaf of sections $\mathcal{E}$ of an algebraic vector bundle over a variety B is an $\mathcal{O}_B$-module: for each open set $U \subseteq B$, $\mathcal{E}(U)$ is a module over the ring $\mathcal{O}_B(U)$ of regular functions on U. If U is a trivialising open, then a section on U is a morphism

$$\begin{aligned} U &\longrightarrow \pi^{-1}(U) \cong U \times \mathbb{A}^n \\ b &\longmapsto (b, f_1(b), \ldots, f_n(b)) \end{aligned}$$

where each f_i is a regular function from U to k. We have an isomorphism of $\mathcal{O}_B(U)$-modules

$$\mathcal{E}(U) \cong \mathcal{O}_B(U)^{\oplus n}$$

These isomorphisms commute with sheaf restriction maps: any open subset V of a trivialising open U is still trivialising, so we still have the isomorphism of $\mathcal{O}_B(V)$-modules $\mathcal{E}(V) \cong \mathcal{O}_B(V)^{\oplus n}$ commuting with the respective sheaf restrictions. The sheaf of sections of an algebraic vector bundle thus also satisfies the following important definition:

Definition 1.4.4. Let $(X, \mathcal{R})$ be a ringed space, $\mathcal{E}$ an $\mathcal{R}$-module. $\mathcal{E}$ is *locally free of rank* r if there exists an open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ such that for all $\lambda \in \Lambda$ we have an isomorphism of sheaves

$$\mathcal{E}|_{U_\lambda} \cong \left(\mathcal{R}|_{U_\lambda}\right)^{\oplus r}.$$

If $\mathcal{E}$ is locally free of rank 1, we call it an *invertible sheaf.*

Notice the difference in notation between an isomorphism of sheaves (more precisely, an isomorphism of $\mathcal{R}|_U$-modules) $\phi : \mathcal{E}|_U \leftrightarrow (\mathcal{R}|_U)^{\oplus r}$, and the corresponding isomorphism of $\mathcal{R}(U)$-modules $\phi_U : \mathcal{E}(U) \leftrightarrow \mathcal{R}(U)^{\oplus r}$ commuting with the sheaf restriction maps.

Lemma 1.4.5. *Let ξ be a vector bundle of rank r over a variety B, $\mathcal{E}$ its sheaf of sections, $(U_\lambda, \varphi_\lambda)$ (U_μ, φ_μ) local trivialisations, $g_{\lambda\mu}$ the corresponding transition function, $s \in \mathcal{E}(U)$ where $U := U_\lambda \cup U_\mu$. Then*

$$(\pi_2 \circ \varphi_\lambda \circ s|_{U_{\lambda\mu}})(b) = g_{\lambda\mu}(b)\left((\pi_2 \circ \varphi_\mu \circ s|_{U_{\lambda\mu}})(b)\right) \qquad \forall\, b \in U_{\lambda\mu}$$

where $\pi_2 : U_{\lambda\mu} \times \mathbb{A}^r \to \mathbb{A}^r$ is the projection to the second factor, so we are comparing vector-valued functions $U_{\lambda\mu} \to \mathbb{A}^r$. In sheaf terminology, if ϕ^λ, ϕ^μ are the sheaf isomorphisms $\phi^\lambda : \mathcal{E}|_{U_\lambda} \leftrightarrow (\mathcal{O}_B|_{U_\lambda})^{\oplus r}$, respectively $\phi^\mu : \mathcal{E}|_{U_\mu} \leftrightarrow (\mathcal{O}_B|_{U_\mu})^{\oplus r}$, then

$$(\phi^\lambda_{U_{\lambda\mu}} \circ \sigma_{U,U_{\lambda\mu}})(s) = g_{\lambda\mu}(\phi^\mu_{U_{\lambda\mu}} \circ \sigma_{U,U_{\lambda\mu}})(s)$$

where $\sigma_{U,U_{\lambda\mu}}$ is the sheaf restriction $U \to U_{\lambda\mu}$ in $\mathcal{E}$.

Proof. The lemma follows from the commutativity of the following diagram:

$$\begin{tikzcd}
 & & U_{\lambda\mu} \times \mathbb{A}^r \arrow[r, "\pi_2"] \arrow[dd, "(b,v)\mapsto(b,g_{\lambda\mu}(b)v)"] & \mathbb{A}^r \\
U_{\lambda\mu} \arrow[r, "s|_{U_{\lambda\mu}}"] & \pi^{-1}(U_{\lambda\mu}) \arrow[ur, "\varphi_\lambda|_{U_{\lambda\mu}}"] \arrow[dr, "\varphi_\mu|_{U_{\lambda\mu}}"'] & & \\
 & & U_{\lambda\mu} \times \mathbb{A}^r \arrow[r, "\pi_2"] & \mathbb{A}^r
\end{tikzcd}$$

□

This lemma states that we can define *transition maps* for the sheaf of sections of a vector bundle, and that these are precisely the same transition maps we previously defined for vector bundles.

Theorem 1.4.6. *Let X be a variety, $r \in \mathbb{N}$. We have a bijection (up to bundle and sheaf isomorphisms):*

$$\left\{ \begin{array}{c} \textit{Algebraic vector bundles} \\ \textit{over } X \textit{ of rank } r \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Locally free sheaves of} \\ \mathcal{O}_X\textit{-modules of rank } r \end{array} \right\}$$
$$E(\xi) \xrightarrow{\pi} X \qquad \longmapsto \qquad \mathcal{E} \textit{ sheaf of sections of } \xi$$

Proof. The assignment from each vector bundle to its sheaf of sections, which we have just shown to be a locally free $\mathcal{O}_X$-module, is clear. We now show the reverse assignment.

Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r; we have an open cover of X $\{U_\lambda\}_{\lambda\in\Lambda}$ such that

$$\mathcal{E}|_{U_\lambda} \cong (\mathcal{O}_X|_{U_\lambda})^{\oplus r} \qquad \forall\, \lambda \in \Lambda.$$

Let us denote these isomorphisms of sheaves by ϕ^λ for all λ in Λ. For each non-disjoint pair U_λ, U_μ we have two isomorphisms of $\mathcal{O}_X(U_{\lambda\mu})$-modules

$$\begin{aligned} \mathcal{E}(U_{\lambda\mu}) &\cong (\mathcal{O}_X(U_{\lambda\mu}))^{\oplus r} \qquad \text{via } \phi^\lambda_{U_{\lambda\mu}}, \\ \mathcal{E}(U_{\lambda\mu}) &\cong (\mathcal{O}_X(U_{\lambda\mu}))^{\oplus r} \qquad \text{via } \phi^\mu_{U_{\lambda\mu}}. \end{aligned}$$

By composing the first isomorphism with the inverse of the second, we get an isomorphism of $(\mathcal{O}_X(U_{\lambda\mu}))^{\oplus r}$ with itself. When evaluated at each point $x \in U_{\lambda\mu}$, this gives a morphism

$$g_{\lambda\mu} : U_{\lambda\mu} \longrightarrow GL_r(k).$$

We argue that these are the transition maps of a unique vector bundle (up to isomorphism) by Proposition 1.2.5. The conditions of the proposition follow immediately by the properties of sheaf restriction maps and how they commute with sheaf morphisms. By Lemma 1.4.5 which we just showed, $\mathcal{E}$ is indeed the sheaf of sections of this vector bundle. □

In conclusion, when thinking about vector bundles we can either define the total space and its trivialisations, the sheaf of sections or the transition functions: they all contain equivalent information, and we can use whichever is most helpful case by case. Because of this, a frequent abuse of notation in Algebraic Geometry is to use the same symbol to denote both a vector bundle and its sheaf of sections. We will also adopt this convention.

Chapter 2

Line Bundles

Definition 2.0.1. A *line bundle* is an algebraic vector bundle of rank 1.

We know from our previous discussion that the sheaf of sections of a line bundle is an invertible sheaf.

2.1 Examples of Line Bundles

Let X be a variety.

The Trivial Bundle

The *trivial line bundle* over X is

$$\begin{aligned} X \times \mathbb{A}^1 &\xrightarrow{\pi} X \\ (p, \lambda) &\longmapsto p. \end{aligned}$$

Its sections are the morphisms $p \mapsto (p, f(p))$, in the sense that giving a section of the trivial line bundle over an open set U is the same as giving a regular function $f : U \to \mathbb{A}^1$. Thus we can identify the sheaf of sections with the structure sheaf $\mathcal{O}_X$ of the variety. If X is projective, then $\mathcal{O}_X(X)$ consists precisely of locally constant functions $X \to k$. In particular, if X is projective and connected, then $\mathcal{O}_X(X) = k$.

The transition functions of the trivial line bundle are as follows: since $\{X\}$ is already a trivialising open cover, on any pair of open subsets they are just the identity on all points. We denote the trivial line bundle over X (and its sheaf of sections!) by $\mathcal{O}_X$.

We have already seen the trivial vector bundle $X \times \mathbb{A}^n$: its sheaf of sections is isomorphic to $\mathcal{O}_X^{\oplus n}$ and its transition maps are also just the identity on all points for all pairs of opens.

For the next two examples we fix an embedding $X \hookrightarrow \mathbb{P}^n$.

The Tautological Line Bundle

All varieties embedded in $\mathbb{P}^n$ have a natural line bundle inherited from the embedding, called the *tautological bundle*. Indeed, because the points in $\mathbb{P}^n$ are precisely the lines through the origin in $\mathbb{A}^{n+1}$, we associate to each point $p = [x_0 : \ldots : x_n] \in \mathbb{P}^n$ its corresponding line $\ell_p := \{(tx_0, \ldots, tx_n) \mid t \in k\}$ in $\mathbb{A}^{n+1}$. More precisely, we construct the bundle over X as follows: consider the set

$$E = \{(p, x) \mid p \in X,\ x \in \ell_p\} \subseteq X \times \mathbb{A}^{n+1} \subseteq \mathbb{P}^n \times \mathbb{A}^{n+1}$$

together with the natural projection

$$\pi := \pi_1 : E \to X.$$

E is a variety, since its defining property can be locally written as polynomial equations in the coordinates of X and $\mathbb{A}^{n+1}$. The affine charts

$$\begin{aligned} U_i = \{[p_0 : \ldots : p_n] \in X \mid p_i \neq 0\} &= \left\{ \left[\frac{p_0}{p_i} : \ldots : \underbrace{1}_{i} : \ldots : \frac{p_n}{p_i} \right] \in X \right\} \\ &= \{[q_0 : \ldots : q_n] \in X \mid q_i = 1\} \end{aligned}$$

trivialise the bundle. Again, we stress that the tautological bundle depends on the embedding of X in a particular projective space; in other words, the pullback of the tautological bundle under an isomorphism may fail to be tautological.

Tautological line bundles over a connected projective variety X have no nonzero global sections at all (unless X is just a single point in $\mathbb{P}^n$). Indeed a global section on X defines a morphism

$$\begin{aligned} X &\longrightarrow \mathbb{A}^{n+1} \\ p &\longmapsto (f_0(p), \ldots, f_n(p)) \end{aligned}$$

where each $(f_0(p), \ldots, f_n(p))$ lies on the line through the origin corresponding to p and $f_0, \ldots, f_n$ are regular functions on X. But since all regular functions on a connected projective variety are constant, f_i must be zero functions: the only point lying on all lines through the origin is the origin 0. Thus X admits no nonzero global sections.

We study the transition functions of the tautological bundle: on the affine charts, the trivialising maps are:

$$\begin{aligned} \varphi_i : \pi^{-1}(U_i) &\longrightarrow U_i \times \mathbb{A} \\ (p, (x_0, \ldots, x_n)) &\longmapsto (p, x_i) \end{aligned}$$

since $p = [p_0/p_i : \ldots : \underbrace{1}_{i} : \ldots : p_n/p_i]$, and either $x_i = 0$ or

$$\ell_p \ni x = (x_0, \ldots, x_n) = x_i \left(\frac{x_0}{x_i}, \ldots, \underbrace{1}_{i}, \ldots, \frac{x_n}{x_i} \right) = x_i \left(\frac{p_0}{p_i}, \ldots, \underbrace{1}_{i}, \ldots, \frac{p_n}{p_i} \right)$$

so x is the point on the line ℓ_p identified by x_i. Thus, since $x_j/x_i = p_j/p_i$ (and p_j/p_i does not depend on the choice of representatives for p),

$$\begin{aligned} \varphi_i \circ \varphi_j^{-1} : U_{ij} \times k &\longrightarrow U_{ij} \times k \\ (p, \lambda) &\longmapsto \left(p, \frac{p_j}{p_i} \lambda \right) \end{aligned}$$

and the transition functions are

$$\begin{aligned} g_{ij} : U_{ij} &\longrightarrow GL_1(k) \\ p &\longmapsto \frac{p_j}{p_i}. \end{aligned}$$

We denote the tautological line bundle over X by $\mathcal{O}_X(-1)$.

The Hyperplane Bundle

The *hyperplane bundle* H on a variety X is defined to be the dual of the tautological bundle (as in Example 1.3.8): its fiber H_p over a point $p \in X$ is the space of one-dimensional linear functionals on the line $\ell_p \subseteq k^{n+1}$ corresponding to p. Again it is important to realise that, by definition, the hyperplane bundle also depends on the embedding of X in a specific projective space.

The hyperplane bundle has many global sections: consider any linear homogenous polynomial $\sum_{i=0}^n a_i x_i$. It defines a linear functional on k^{n+1} which, when restricted to a line ℓ through the origin, defines a linear functional on the line. We have a well defined global section:

$$\begin{aligned} X &\longrightarrow H \\ p &\longmapsto \left(p, \left.\sum_{i=0}^n a_i x_i\right|_{\ell_p}\right). \end{aligned}$$

If X is projective (and again, not finite), the converse is also true: given any global section, the coefficients a_i of the linear functionals when restricted to the lines must vary continuously with the points of X; in other words, we have morphisms (regular functions) $a_i : X \to k$, so the a_i are constant and we have a homogenous polynomial of degree one.

Finally, we study the transition functions of the hyperplane bundle. The trivialising maps are:

$$\begin{aligned} \varphi_i : \pi^{-1}(U_i) &\longrightarrow U_i \times k \\ \left(p, \left.\sum_{j=0}^n a_j x_j\right|_{\ell_p}\right) &\longmapsto \left(p, \frac{a_0 p_0}{p_i} + \ldots + a_i \cdot 1 + \ldots + \frac{a_n p_n}{p_i}\right) \end{aligned}$$

where $p = [p_0 : \ldots : p_i : \ldots : p_n] = [p_0/p_i : \ldots : \underbrace{1}_{i} : \ldots : p_n/p_i]$. Indeed since we are dealing with linear functionals over a line, we can identify them by their value on a chosen (nonzero) point. Since we do not want this evaluation to depend on choices of representatives for p, evaluating in p_k/p_i is good choice. Now, since in U_{ij}

$$\sum_{k=0}^{n} \frac{a_k p_k}{p_j} = \frac{p_i}{p_j} \sum_{k=0}^{n} \frac{a_k p_k}{p_i}$$

the transition maps are

$$\begin{aligned} g_{ij} : U_{ij} &\longrightarrow GL_1(k) \\ p &\longmapsto \frac{p_i}{p_j} = \left(\frac{p_j}{p_i}\right)^{-1}, \end{aligned}$$

in accordance with how we defined transitions functions for the dual of a line bundle. From now on we denote the hyperplane bundle over X by $\mathcal{O}_X(1)$.

2.2 The Picard Group

We recall the tensor product of bundles we described in Example 1.3.7. Since the tensor product of one-dimensional vector spaces is again one-dimensional, the tensor product of line bundles is also a line bundle. Its transition maps are given by the Kronecker product of the transition maps of the bundles, which for the one-dimensional case is just

$$f \otimes g = fg.$$

Analogously, the dual of a line bundle (Example 1.3.8) is also a line bundle. Its transition maps are given by $(f^\vee)^{-1} = f^{-1}$, the inverse of the transition maps of the original line bundle.

The immediate consequence is that isomorphism classes of line bundles over a fixed variety together with the tensor product of bundles form a group. The identity element is the isomorphism class of trivial line bundle. The group is abelian by the commutativity of the tensor product (up to isomorphism).

Definition 2.2.1. [7, 7.9.1] Given a ringed space $(X, \mathcal{R})$, the abelian group of isomorphism classes of invertible sheaves of $\mathcal{R}$-modules on X is called the *Picard group* of X and is denoted $\mathrm{Pic}(X)$.

Recall the correspondence between line bundles and invertible sheaves we showed in Theorem 1.4.6.

Example 2.2.2. Let $X = \mathbb{P}^n$, in the previous section we defined the line bundles $\mathcal{O}_{\mathbb{P}^n}(0) := \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n}(1)$. We can define $\mathcal{O}_{\mathbb{P}^n}(d)$ for any $d \in \mathbb{Z}$, by considering the group generated by these line bundles (in fact this is the entire Picard group of $\mathbb{P}^n$, but showing this is non-trivial). $\mathcal{O}_{\mathbb{P}^n}(d)$ has global sections for all $d \geq 0$: we can interpret the fibres as all homogenous polynomial functionals of degree d over the lines in $\mathbb{A}^{n+1}$ (the d-tensor product of the linear functionals with themselves). Thus the global sections of $\mathcal{O}_{\mathbb{P}^n}(d)$ are precisely the homogenous polynomials of degree d in $k[x_0, \ldots, x_n]$, following an analogous argument as in the hyperplane case). The fibres of $\mathcal{O}_{\mathbb{P}^n}(d)$ for negative d do not have an easy interpretation, and the bundles have no nonzero global sections.

The transition maps for $\mathcal{O}_{\mathbb{P}^n}(d)$, $d \in \mathbb{Z}$ with respect to the standard affine charts are:

$$\begin{aligned} g_{ij} : U_{ij} &\longrightarrow GL_1(k) \\ p &\longmapsto \left(\frac{p_i}{p_j}\right)^d \end{aligned}$$

2.3 The Canonical Line Bundle

As in differential and complex geometry, the concept of a *tangent space at a point* is a very important construction in algebraic geometry. A formal definition of tangent space of a variety X not depending on a particular embedding $X \hookrightarrow \mathbb{P}^n$ requires abstract algebraic tools outside the scope of this thesis; the reason why this however is crucial is that the tangent space is a vector bundle TX over X, and we want it to be invariant under the pullback of an isomorphism (that is, if $f : Y \to X$ is an isomorphism, then TY should be isomorphic to $f^*(TX)$). Similarly to the theory of manifolds, the dual *cotangent bundle* $\Omega_X := T^*X$ is of much more interest than the tangent bundle itself; we will limit ourselves to just stating a proposition which guarantees the existence and uniqueness of Ω_X.

Proposition 2.3.1. *Let X be a variety. There exist a unique sheaf Ω_X of $\mathcal{O}_X$-modules (up to isomorphism) such that:*

1. *For any affine open subset $U \subseteq X$, if*

$$\mathcal{O}_X(U) = k[U] = \frac{k[x_1, \ldots, x_m]}{(f_1, \ldots, f_r)}$$

   then

$$\mathcal{O}_U^{\oplus r} \xrightarrow{\substack{\text{Jacobian matrix} \\ \text{of the } f_i}} \mathcal{O}_U^{\oplus m} \longrightarrow \Omega_X|_U \longrightarrow 0$$

   is an exact sequence of sheaves of $\mathcal{O}_U$-modules;

2. *For any $U \subseteq X$ affine open subset as above, for any non-invertible $g \in k[U]$, denote $U_g := \{x \in U \mid g(x) \neq 0\}$, then U_g is affine and*

$$\mathcal{O}_X(U_g) = k[U_g] = \frac{k[x_1, \ldots, x_m, y]}{(f_1, \ldots, f_r, yg - 1)}.$$

   The restriction morphism $\sigma : \Omega_X(U) \to \Omega_X(U_g)$ fits into the following commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_X(U)^{\oplus r} \arrow[r, "Jac(f_1,\ldots,f_r)"] \arrow[d, "\binom{I_r}{0}"] & \mathcal{O}_X(U)^{\oplus m} \arrow[r] \arrow[d, "\binom{I_m}{0}"] & \Omega_X(U) \arrow[r] \arrow[d, "\sigma"] & 0 \\
\mathcal{O}_X(U_g)^{\oplus (r+1)} \arrow[r, "Jac(f_1,\ldots,f_r,yg-1)"'] & \mathcal{O}_X(U_g)^{\oplus (m+1)} \arrow[r] & \Omega_X(U_g) \arrow[r] & 0
\end{tikzcd}$$

Moreover:

- *If X is smooth of dimension n, then Ω_X is locally free of rank n;*
- *For any affine open subset $U \subseteq X$ $\Omega_X|_U \cong \Omega_U$.*

Proof. See [2, II.8]. □

Remark 2.3.2. Complements of hypersurfaces in an affine space U are a basis for the Zariski topology of U. To see this, first we consider U as a closed subset of affine m-space $\mathbb{V}(f_1, \ldots, f_r)$: then any open in U is of the form $\mathbb{V}(f_1, \ldots, f_r) \setminus \mathbb{V}(g_1, \ldots, g_l)$ where the $g_i \in k[U]$ are non-invertible. Then clearly it is covered by $\bigcup (\mathbb{V}(f_1, \ldots, f_r) \setminus \mathbb{V}(g_i)) = \bigcup U_{g_i}$. Thus we have correctly defined sheaf restriction maps $\sigma : \Omega_X(U) \to \Omega_X(U')$ for all open subsets $U' \subseteq U$.

We denote the exterior product of Ω_X by $\Omega_X^k := \bigwedge^k \Omega_X$, so $\Omega_X = \Omega_X^1$.

Definition 2.3.3. Let X be a smooth variety of dimension n. The *canonical line bundle* ω_X is the highest exterior power of the cotangent vector bundle,

$$\omega_X := \Omega_X^n.$$

The elements of the fibres of ω_X over each point $p \in X$ are the algebraic n-forms $\alpha \cdot (dx_1 \wedge \ldots \wedge dx_n)\big|_p$ for $\alpha \in k$. The canonical line bundle on a variety X is the most important line bundle because it (and its powers, that is, the subgroup of $\operatorname{Pic}(X)$ generated by ω_X) are the only line bundles that are intrinsically defined on X. In other words, if $f : Y \to X$ is an isomorphism of varieties, then $f^*\omega_X \cong \omega_Y$.

Example 2.3.4. The cotangent bundle on affine space $\mathbb{A}^n$ is trivial. Since $\mathbb{A}^n$ is already affine and its coordinate ring is just $k[x_1, \ldots, x_n]$, by Proposition 2.3.1 we have the isomorphism of sheaves $\mathcal{O}_{\mathbb{A}^n}^{\oplus n} \cong \Omega_{\mathbb{A}^n}^1$. This implies that its exterior products are also trivial, so in particular $\omega_{\mathbb{A}^n} \cong \mathcal{O}_{\mathbb{A}^n}$.

Example 2.3.5. Let us compute the transition maps of $\omega_{\mathbb{P}^1}$. On the affine charts we have isomorphisms

$$\begin{aligned} \tilde{\varphi}_0 : U_0 \to \mathbb{A}^1 \qquad & [x_0 : x_1] \mapsto \frac{x_1}{x_0} =: s \\ \tilde{\varphi}_1 : U_1 \to \mathbb{A}^1 \qquad & [x_0 : x_1] \mapsto \frac{x_0}{x_1} =: t. \end{aligned}$$

Since the canonical bundle is intrinsic, these isomorphisms induce an isomorphism between the trivial $\omega_{\mathbb{A}^1} \cong \mathcal{O}_{\mathbb{A}^1}$ and the restriction of $\omega_{\mathbb{P}^1}$ to the affine charts. Let us denote

$$\begin{aligned} \mathcal{O}_{U_0} &\cong \omega_{\mathbb{P}^1}|_{U_0} & \qquad \mathcal{O}_{U_1} &\cong \omega_{\mathbb{P}^1}|_{U_1} \\ 1 &\mapsto ds & 1 &\mapsto dt \end{aligned}$$

the isomorphisms induced on the sheaves of sections. On the intersection $U_0 \cap U_1$ we have

$$s = \frac{x_1}{x_0} = -\left(\frac{x_0}{x_1}\right)^{-1} = t^{-1}$$

which implies

$$ds = -\frac{1}{t^2}\,dt.$$

Thus the corresponding transition map g_{01} is

$$g_{01} = \frac{1}{t^2} = \left(\frac{x_1}{x_0}\right)^2.$$

so $\omega_{\mathbb{P}^1}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-2)$ (we can ignore the minus sign by Remark 1.2.4).

Example 2.3.6. Analogous computations show that the canonical bundle of projective n-space is

$$\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-(n+1)).$$

Remark 2.3.7. Let X, Y be varieties, $X \times Y$ their product, p_1, p_2 the categorical projections. Then

$$\Omega^1_{X\times Y} \cong p_1^*\Omega_X \oplus p_2^*\Omega_Y$$

by [2, Exercise II.8.3]. Since the determinant of a diagonal block matrix is just the product of the determinants of the blocks, we get

$$\omega_{X\times Y} \cong p_1^*\omega_X \otimes p_2^*\omega_Y.$$

2.4 Rational Maps to Projective Space

Definition 2.4.1. [6, 7.2] Let X, Y be algebraic varieties, U, U' be dense open subsets of X. Two morphisms of varieties $U \xrightarrow{\varphi} Y$, $U' \xrightarrow{\varphi'} Y$ are *equivalent* in they agree on the intersection $U \cap U'$. It is easy to check that this is an equivalence relation.

Remark 2.4.2. If X is irreducible, then any non-empty open subset of X is dense.

Definition 2.4.3. A rational map $X \dashrightarrow Y$ is an equivalence class of morphisms defined on dense open subsets of X as above.

We think of a rational map as a morphism defined only on a dense open set, and we do not concern ourselves with the particular open set on which it is defined. A rational map should be interpreted as a morphism "defined almost everywhere". Despite its name, a rational map is not an actual mapping, which is the reason why we use a broken arrow to denote it. One must take care when composing rational maps that the image of (a representative of) φ_1 is dense in Y in order to define a composition

$$X \overset{\varphi_1}{\dashrightarrow} Y \overset{\varphi_2}{\dashrightarrow} Z.$$

Rational maps are extremely important in Algebraic Geometry; varieties are often studied up to *birational equivalence* instead of up to isomorphism.

Definition 2.4.4. Let X, Y be algebraic varieties. We call them *birationally equivalent* if there exist mutually inverse rational maps

$$X \overset{F}{\dashrightarrow} Y \qquad Y \overset{G}{\dashrightarrow} X.$$

In other words, there exist dense open subsets $U \subseteq X$ and $V \subseteq Y$ which are isomorphic. We denote birational varieties by $X \sim_{\text{bir}} Y$.

Line bundles and their global sections govern all rational maps to $\mathbb{P}^n$. An understanding of all the possible ways in which a variety may be mapped to projective space is tantamount to a complete understanding of all line bundles on the variety.

Definition 2.4.5. Let L be a line bundle over X. Its global sections have the structure of a vector space, called the *complete linear system of* L $H^0(X,L)$. A *linear system relative to* L is a linear subspace of $H^0(X,L)$.

Theorem 2.4.6. *[6, 8.5] Let X be an irreducible variety.*

Let L be a line bundle over X, $\{s_0,\dots,s_n\}$ a basis for a linear system relative to L. Then

$$\begin{aligned} X &\dashrightarrow \mathbb{P}^n \\ x &\longmapsto [s_0(x):\ldots:s_n(x)] \end{aligned}$$

is a rational map. If U is a dense open subset of X on which F makes sense as a morphism of varieties $f: U \to \mathbb{P}^n$, then

$$f^*\mathcal{O}_{\mathbb{P}^n}(1) \cong \iota^* L$$

where $\iota: U \hookrightarrow X$ is the inclusion map.

Conversely, if X is also smooth, under the same notation as before, let $X \overset{F}{\dashrightarrow} \mathbb{P}^n$ be a rational map. Then there exists a line bundle L over X such that the rational map is given by a set of global sections of L, and $f^*\mathcal{O}_{\mathbb{P}^n}(1) = \iota^* L$.

Proof. First we need to make sense of the expression $[s_0(x):\ldots:s_n(x)]$. Consider a trivialisation (V,φ) of L around x; we identify the sections s_i

$$\begin{array}{ccccc} V & \xrightarrow{s_i} & \pi^{-1}(V) & \xrightarrow{\varphi} & V\times\mathbb{A}^1 \\ x & \mapsto & s_i(x) & \mapsto & (x,\tilde{s}_i(x)) \end{array}$$

with the regular functions $\tilde{s}_i : V \to \mathbb{A}^1$. These functions depend on the choice of local trivialisation, and we know that this dependence is given by the transition functions of L (Lemma 1.4.5). Thus the $\tilde{s}_i(x)$ are determined pointwise up to multiplication by the same nonzero scalar: the notation

$$[s_0(x):\ldots:s_n(x)] := [\tilde{s}_0(x):\ldots:\tilde{s}_n(x)]$$

is well defined unless $\tilde{s}_i(x) = 0\ \forall\, i = 0,\dots,n$. Unfortunately there is nothing to prevent the sections from simultaneously vanishing, so the map

$$\begin{aligned} X &\overset{F}{\dashrightarrow} \mathbb{P}^n \\ x &\longmapsto [s_0(x):\ldots:s_n(x)] \end{aligned}$$

is only rational and not an everywhere-defined morphism of varieties. Indeed, the *base locus* consisting of the points on which all the sections vanish is the intersection of closed sets

$$\bigcap_{i=0}^{n} s_i^{-1}(0).$$

where 0 denotes the closed subvariety of $E(L)$ consisting of the zero vector of each fibre. Since X is irreducible, the complement of the base locus U is open and dense, and F is a well defined morphism of varieties on U. Now, a couple of remarks. We assumed linear independence of the $n+1$ sections to study maps to $\mathbb{P}^n$; linearly dependent sections will define a map whose image can be embedded in a lower dimensional projective space. What happens if we choose a different basis of sections $\{s'_1, \ldots, s'_n\}$ for the same linear system? It is easy to check that the two maps can be transformed into each other by an *automorphism of* $\mathbb{P}^n$; that is, a linear change in coordinates. The base locus on which the new rational map is not defined will still be the same.

The last thing to show is $f^*\mathcal{O}_{\mathbb{P}^n}(1) \cong \iota^* L$. Consider the bundle $f^*\mathcal{O}_{\mathbb{P}^n}(1)$: we define the *pullback of a (global) section* $s \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ to be the map

$$\begin{aligned} f^*s : U &\longrightarrow U \times_{\mathbb{P}^n} \mathcal{O}_{\mathbb{P}^n}(1) = f^*\mathcal{O}_{\mathbb{P}^n}(1) \\ p &\longrightarrow (p, s(f(p))). \end{aligned}$$

We study the pullbacks of the global sections $x_k,\ k = 0, \ldots, n$, which form a basis for the global sections of $\mathcal{O}_{\mathbb{P}^n}(1)$ (the linear polynomials). We get, on the trivialisations $(U'_i = f^{-1}(U_i), \varphi'_i)$ corresponding to the trivialisations (U_i, φ_i) of $\mathcal{O}_{\mathbb{P}^n}(1)$:

$$\varphi'_i(f^*x_k(p)) = \varphi'_i(p, x_k([f_0(p) : \ldots : f_n(p)])) = \left(p, \frac{f_k(p)}{f_i(p)}\right)$$

where $f_l,\ l = 0, \ldots, n$ denote the components of f (they are regular functions determined up to a nonzero scalar and, under the same notation as before, $f_l(p) = \tilde{s}_l(p)$). This yields transition maps for the sections

$$\begin{aligned} f^*g_{ij} : f^{-1}(U_{ij}) &\longrightarrow GL_1(k) \\ p &\longmapsto \frac{f_i(p)}{f_j(p)} = \frac{\tilde{s}_i(p)}{\tilde{s}_j(p)} \end{aligned}$$

in accordance with Remark 1.3.3. By definition of f, we can actually interpret $f^{-1}(U_i) = f^{-1}(U_i \cap f(U)) = X \setminus s_i^{-1}(0)$. These opens cover $U = \bigcap_{i=0}^{n} s_i^{-1}(0)$ and trivialise $\iota^* L$: more precisely, if (V, φ) is a trivialisation for $\iota^* L$ inducing the correspondence $s_l \mapsto \tilde{s}_l$, then the composition

$$\begin{array}{ccccc} \pi^{-1}(V \cap f^{-1}(U_i)) & \overset{\varphi}{\to} & (V \cap f^{-1}(U_i)) \times A^1 & \to & (V \cap f^{-1}(U_i)) \times A^1 \\ E_p \ni e & \mapsto & (p, \lambda) & \mapsto & \left(p, \frac{\lambda}{\tilde{s}_i(p)}\right) \end{array}$$

is still a trivialisation inducing the correspondence $s_l \mapsto \frac{\tilde{s}_l}{\tilde{s}_i}$. Thus we get transition functions for the sheaf of sections of $\iota^* L$ on opens $(V_1 \cap f^{-1}(U_i)) \cap (V_2 \cap f^{-1}(U_j))$

$$\frac{\tilde{s}_l}{\tilde{s}_i} \cdot \frac{\tilde{s}_i}{\tilde{s}_j} = \frac{\tilde{s}_l}{\tilde{s}_j} \implies g_{ij}(p) = \frac{\tilde{s}_i(p)}{\tilde{s}_j(p)}$$

(by remebering that the ratio $\frac{\tilde{s}_l}{\tilde{s}_k}$ does not depend on the choice of trivialisation), proving $\iota^* L \cong f^* \mathcal{O}_{\mathbb{P}^n}(1)$.

Now let F be a rational map $X \overset{F}{\dashrightarrow} \mathbb{P}^n$, let U be a dense open subset of X on which F makes sense as a morphism of varieties $f : U \to \mathbb{P}^n$. We study the line bundle on U defined by the pullback $f^* \mathcal{O}_{\mathbb{P}^n}(1)$. Consider the pullbacks of $x_k,\ k = 0, \ldots, n$. For all $p \in U$ we have (separately on each open of the cover $\{f^{-1}(U_i)\},\ i = 0, \ldots, n$)

$$[f^* x_0(p) : \ldots : f^* x_n(p)] = \left[\frac{f_0(p)}{f_i(p)} : \ldots : \frac{f_n(p)}{f_i(p)}\right] = [f_0(p) : \ldots : f_n(p)] = f(p)$$

so f agrees on U with the map we previously defined for the set of global sections $\{f^* x_0, \ldots, f^* x_n\}$ of $f^* \mathcal{O}_{\mathbb{P}^n}(1)$. $f^* \mathcal{O}_{\mathbb{P}^n}(1)$ is a bundle over only the dense open set U; however the bundle can be extended uniquely to the whole of X. This is because a rational map from a smooth variety to a projective one always admits a representative U of codim 2 [4, 4.1.16], and there is a bijection between the Picard groups of X and U for such an open [2, II.6.5.b]. We denote this extension by $F^* \mathcal{O}_{\mathbb{P}^n}(1)$; then clearly $f^* \mathcal{O}_{\mathbb{P}^n}(1)$ is the restriction $\iota^*(F^* \mathcal{O}_{\mathbb{P}^n}(1))$ where $\iota : U \hookrightarrow X$ is the inclusion map. $\square$

Remark 2.4.7. In the proof we said that, for a line bundle L on X, if $s \in H^0(X, L)$, the open $X \setminus s^{-1}(0)$ trivialises the bundle. More generally, any line bundle with a nowhere vanishing global section (such as the restriction of L to $X \setminus s^{-1}(0)$) is trivial. We can define the map

$$\begin{aligned} X \times \mathbb{A}^1 &\longrightarrow E(L) \\ (p, \lambda) &\longmapsto \lambda s(p) \end{aligned}$$

which is an isomorphism of bundles. The same argument holds for any bundle of rank r, provided we have r pointwise linearly independent global sections.

Definition 2.4.8. A line bundle L is *very ample* if the rational map determined by its complete linear system is an everywhere defined morphism that defines an isomorphism of varieties with its image; that is, a (closed) embedding into projective space. A line bundle M is *ample* if there exists an $n > 0$ such that $M^{\otimes n}$ is very ample.

As a corollary to Theorem 2.4.6, we may interpret a very ample line bundle over X as the hyperplane bundle for a particular embedding of X into projective space.

Lemma 2.4.9. *Let L be a line bundle on an irreducible projective variety X which admits a non-zero global section. Then $L^\vee$ admits a non-zero global section if and only if L is trivial.*

Proof. Suppose s_1, s_2 are non-zero global sections of L, respectively $L^\vee$. Then the product $s_1 s_2$ defines a non-zero global section of $\mathcal{O}_X$ because the union of the two zero locuses cannot be the whole of X. But a non-zero global section of $\mathcal{O}_X$ is just a non-zero constant, since X is projective. Thus s_1 is actually nowhere vanishing: L has a nowhere vanishing global section. $\square$

Chapter 3

The Kodaira Dimension

3.1 Definition

Let X be a variety, $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules. We denote by $H^0(X, \mathcal{E}) := \mathcal{E}(X)$ the vector space of its global sections.

Remark 3.1.1. If X is projective then $H^0(X, \mathcal{E})$ is finite-dimensional [2, III.5.2], and we denote its dimension by

$$h^0(X, \mathcal{E}) := \dim H^0(X, \mathcal{E}).$$

Definition 3.1.2. Let X be a smooth, projective, irreducible variety of dimension 1. The *geometric genus* of X is

$$g(X) := h^0(X, \omega_X).$$

Remark 3.1.3. A smooth, projective, irreducible variety X of dimension 1 over $k = \mathbb{C}$ has the natural structure of a *Riemann Surface*. In this case the geometric genus of X agrees with the topological definition of genus by Hodge theory [8].

Definition 3.1.4. [3, 2.1.1] Let L be a line bundle over an irreducible variety X. We define the *semi-group of L*

$$\mathbb{N}(L) := \{m \in \mathbb{N} \mid H^0(X, L^{\otimes m}) \neq \{0\}\}$$

where $L^{\otimes 0} = \mathcal{O}_X$.

Remark 3.1.5. If X is irreducible, then $\mathbb{N}(L)$ is indeed a semigroup: the trivial line bundle has constant global sections; if $s_1 \in H^0(X, L^{\otimes m_1})$, $s_2 \in H^0(X, L^{\otimes m_2})$ are nonzero sections, then $s_1 \otimes s_2 = s_1 \cdot s_2$ is a nonzero global section in $H^0\left(X, L^{\otimes(m_1+m_2)}\right)$. Since X is irreducible, so the union of the zero locuses of s_1, s_2 cannot be the whole of X.

Definition 3.1.6. The *exponent* $e(L)$ of a line bundle L over an irreducible projective variety is the greatest common divisor of the elements of $\mathbb{N}(L)$. By the previous remark, all sufficiently big multiples of $e(L)$ are in $\mathbb{N}(L)$.

For all $m \in \mathbb{N}(L)$ we have a rational map $X \overset{F_m}{\dashrightarrow} \mathbb{P}^{d_m}$, where $d_m = h^0(X, L^{\otimes m}) - 1$, defined by the complete linear system of L. We denote

$$Y_m := \overline{F_m(X)} \subseteq \mathbb{P}^{d_m}$$

where $F_m(X)$ is the image of the rational map F_m on any dense open subset on which it is defined as a morphism. Since the image of a morphism is not necessarily a variety, we consider its Zariski closure in $\mathbb{P}^{d_m}$.

Remark 3.1.7. If L is ample, then $Y_m \cong X$ for a sufficiently large m.

Definition 3.1.8. [3, 2.1.3] Let X be an normal variety[1], L a line bundle over X. If $\mathbb{N}(L) \neq \{0\}$, the *Iitaka dimension* of L is

$$\kappa(L) = \max_{m \in \mathbb{N}(L)} (\dim Y_m).$$

where Y_m is the closure of the image of X under any rational map defined by the complete linear system of $L^{\otimes m}$, as defined above.

If instead $H^0(X, L^{\otimes m}) = \{0\}$ for all $m > 0$, then we set

$$\kappa(L) = -\infty.$$

Remark 3.1.9. The dimension of the (closure of the) image of a variety X under any morphism is at most $\dim X$. Thus for any line bundle L on X

$$\kappa(L) = -\infty \qquad \text{or} \qquad 0 \leq \kappa(L) \leq \dim X.$$

[1] All irreducible smooth varieties are normal.

Definition 3.1.10. [3, 2.1.5] Let X be a smooth, projective variety. The *Kodaira dimension* of X is $\kappa(\omega_X)$, which we will also denote by $\kappa(X)$.

Example 3.1.11. The canonical bundle of $\mathbb{P}^n$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-(n+1))$. We know that this bundle does not admit any nonzero global sections, and neither do its powers $\mathcal{O}_{\mathbb{P}^n}(-m(n+1))$ for $m > 0$. Thus the Kodaira dimension of the projective n-space is

$$\kappa(\mathbb{P}^n) = -\infty \qquad \forall\, n \in \mathbb{N} \setminus \{0\}.$$

Proposition 3.1.12. *Let X be an irreducible projective variety, L be a line bundle on X with $\kappa := \kappa(L) \in \mathbb{N}$, Y_m be the closure of the image of X under a rational map defined by any complete linear system of $L^{\otimes m}$. Then there exists an $M > 0$ such that*

$$\dim Y_m = \kappa \qquad \forall\, m > M,\ m \in \mathbb{N}(L).$$

Proof. Let e be the exponent of L, replace L with $L^{\otimes e}$. Let $J \in \mathbb{N}$ be such that $j \in \mathbb{N}(L)$ for all $j \geq J$. Let $k \in \mathbb{N}(L)$ be such that $\dim Y_k = \kappa$, set $M = k + J$. For $j \geq J$, let $0 \neq s \in H^0(X, L^{\otimes j})$, consider the map

$$H^0(L^{\otimes k}) \xrightarrow{\otimes s} H^0(L^{\otimes k+j}).$$

This is an injection (as per Remark 3.1.5, or more generally by how the sections of a tensor product of bundles are defined: the tensor product of sections). We thus get the corresponding projection

$$\begin{tikzcd}
 & \mathbb{P}^{d_{k+j}} \arrow[d, dashed] \\
X \arrow[ur, dashed] \arrow[r, dashed] & \mathbb{P}^{d_k}
\end{tikzcd}$$

factorizing any rational map defined by linear system of $L^{\otimes k}$. So

$$\kappa \geq \dim Y_{k+j} \geq \dim Y_k = \kappa.$$

$\square$

Definition 3.1.13. Let X be an irreducible smooth projective variety. The *m-th plurigenus P_m* of X is

$$P_m := h^0(X, \omega_X^{\otimes m}).$$

Proposition 3.1.14. *[3, 2.1.37] The Kodaira dimension of an irreducible smooth projective variety X is the rate of growth of its plurigenera; that is,*

$$\kappa(X) = \min \left\{ k \in \mathbb{N} \mid \left(\frac{P_m}{m^k} \right)_{m \in \mathbb{N}^+} \text{ is bounded} \right\}.$$

Remark 3.1.15. If $P_m = 0$ for all $m > 0$, then $\kappa(X) = -\infty$. This is in a way still consistent with the formula in Proposition 3.1.14, since

$$\lim_{k \to -\infty} \sup_{m>0} \frac{P_m}{m^k} = \lim_{k \to -\infty} \sup_{m>0} \frac{0}{m^k} = 0.$$

We will now prove the birational invariance of the Kodaira dimension for smooth varieties.

Proposition 3.1.16. *Let X be an irreducible variety, $V \subseteq X$ a non-empty open set, $L \in \operatorname{Pic}(X)$, $\mathcal{L}$ its sheaf of sections. Then the restriction map*

$$\mathcal{L}(X) \xrightarrow{\sigma} \mathcal{L}(V)$$

is injective.

Proof. Let $s \in \mathcal{L}(X)$ be a global section whose image under σ is the zero section in $\mathcal{L}(V)$: s vanishes on V. Since the zero locus of s is a closed subset of X containing V, it is actually the whole of X (V is dense since X is irreducible). So s is the zero section in $\mathcal{L}(X)$. $\square$

Proposition 3.1.17. *Let X be a smooth irreducible variety, $V \subseteq X$ an open subset such that $\operatorname{codim}(X \setminus V, X) \geq 2$, $L \in \operatorname{Pic}(X)$, $\mathcal{L}$ its sheaf of sections. Then the restriction map*

$$\mathcal{L}(X) \xrightarrow{\sigma} \mathcal{L}(V)$$

is bijective.

Proof. The idea of the proof is that since $\operatorname{codim}(X \setminus V, X) \geq 2$, a section on V can be uniquely extended to a section on X, proving surjectivity. For a proof of this fact in the case when X is affine and $L = \mathcal{O}_X$ see [2, II 6.3.A]. The general case follows by taking an affine open cover of X which trivialises L. □

Theorem 3.1.18. *[2, II 8.19] Let $X \sim_{\mathrm{bir}} Y$ be birational irreducible smooth projective varieties. Then for every integer $m \geq 0$ one has*

$$P_m(X) = P_m(Y)$$

and consequently

$$\kappa(X) = \kappa(Y).$$

Proof. Given the birational map $X \dashrightarrow Y$, let V be the largest open subset of X for which there is a morphism $f : V \to Y$ representing the rational map. Note that f is not necessarily an isomorphism; in any case there is a dense open subset U of V on which f restricts to an isomorphism with its image $f(U)$, which is open in Y. The morphism f induces a morphism of sheaves $f^*\Omega^1_Y \to \Omega_V$; since these are locally free sheaves of the same rank, we get an induced map on the exterior powers $f^*\omega_Y \to \omega_V$, and consequently morphisms $f^*\omega_Y^{\otimes m} \to \omega_V^{\otimes m}$ for all $m \in \mathbb{N} \setminus \{0\}$. We have a morphism on the spaces of global sections $f^* : \omega_Y^{\otimes m}(Y) \to \omega_V^{\otimes m}(V)$ commuting with the following diagram:

$$\begin{tikzcd}
\omega_Y^{\otimes m}(Y) \arrow[r, "f^*"] \arrow[d, hook] & \omega_V^{\otimes m}(V) \arrow[d, hook] \\
\omega_Y^{\otimes m}(f(U)) \arrow[r, "\sim"] & \omega_V^{\otimes m}(U)
\end{tikzcd}$$

where the vertical arrows are the sheaf restrictions, which are injections by Proposition 3.1.16. Thus necessarily $f^* : \omega_Y^{\otimes m}(Y) \to \omega_V^{\otimes m}(V)$ is injective. Now we show that the restriction $\omega_X^{\otimes m}(X) \to \omega_X^{\otimes m}(V) \cong \omega_V^{\otimes m}(V)$ is a bijection. By [2, II 4.7] $\operatorname{codim}(X \setminus V, X) = 2$, so we can apply Proposition

3.1.17. Combining our results, we get an injective map $\omega_Y^{\otimes m}(Y) \to \omega_X^{\otimes m}(X)$, hence $P_m(Y) \leq P_m(X)$. We obtain the reverse inequality by symmetry, so $P_m(X) = P_m(Y)$.

The equality $\kappa(X) = \kappa(Y)$ follows from by Proposition 3.1.14. $\square$

Theorem 3.1.19 (Hironaka's desingularization theorem)**.** *[6, 7.1] Assume* $\operatorname{char} k = 0$*. Let X be a variety. There exists a smooth variety $\tilde{X}$ and a projective birational morphism $\tilde{X} \xrightarrow{\pi} X$. Furthermore, π may be assumed to be an isomorphism on the smooth locus of X, and if X is a projective variety, then so is $\tilde{X}$. $\tilde{X}$ is called a* smooth model *for X.*

Definition 3.1.20. Assume $\operatorname{char} k = 0$. Let X be an irreducible projective variety, $\tilde{X}$ a smooth model for X. The Kodaira dimension of X is

$$\kappa(X) := \kappa(\tilde{X}).$$

Remark 3.1.21. Let X be an irreducible projective variety. A direct consequence of Hironaka's desingularization theorem is that if X_1, X_2 are two smooth models for X, then they are birationally invariant. Indeed, the birational maps are isomorphisms on the smooth locus of X, so we can compose them. This, together with Theorem 3.1.18 ensures that the Definition 3.1.20 is well defined.

Theorem 3.1.22. *Assume* $\operatorname{char} k = 0$*. The Kodaira dimension is a birational invariant.*

Proof. All open subesets of irreducible varieties are dense, thus we may always compose birational equalities between two varieties and their smooth models and conclude by Theorem 3.1.18. $\square$

3.2 Basic Examples

The last two sections are the applied part of this thesis: after finally stating the definition of the Kodaira dimension, we compute it for some basic cases. In this section we explicitly calculate the canonical bundle for hypersurfaces and the blowup of a point in $\mathbb{A}^2$.

Hypersurfaces

Proposition 3.2.1. *Let $X \subseteq \mathbb{A}^n$ be a smooth irreducible hypersurface defined by a polynomial f. Then*

$$\omega_X \cong \mathcal{O}_X.$$

Proof. On X

$$0 = df = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} dx_i,$$

let us denote $X_i = X \cap \{\frac{\partial f}{\partial x_i} \neq 0\}$. We argue that the canonical bundle ω_X is trivial. To do this, we construct isomorphisms $\mathcal{O}_{X_i} \cong \omega_{X_i}$ through the Poincaré residues of f:

$$\begin{aligned} \mathcal{O}_{X_i} &\longrightarrow \omega_{X_i} = \omega_X|_{X_i} \\ 1 &\longmapsto \frac{(-1)^{i-1}}{\partial f/\partial x_i} dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n. \end{aligned}$$

These are indeed isomorphisms with the trivial bundle since they describe a nonzero section for ω_{X_i}, $i = 1, \ldots, n$. The X_i are an affine cover (since they are complements of $\mathbb{V}(\frac{\partial f}{\partial x_i})$ in X) and cover X (otherwise if $\frac{\partial f}{\partial x_i}(p) = 0$ for all i, then $\sum \frac{\partial f}{\partial x_i}(p)$=0, but X is smooth). Suppose $i < j$, on the intersections V_{ij} we get

$$\begin{aligned} 0 = df &= df \wedge dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n \\ &= \left(\frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial x_j} dx_j \right) \wedge dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n \\ &= (-1)^{i-1} \frac{\partial f}{\partial x_i} dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n \\ &\quad + (-1)^{j-2} \frac{\partial f}{\partial x_j} dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \end{aligned}$$

so

$$\frac{(-1)^{j-1}}{\partial f/\partial x_j} dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n = \frac{(-1)^{i-1}}{\partial f/\partial x_i} dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$$

which means that the transition maps are the identity. $\square$

Proposition 3.2.2. *Let $X \subseteq \mathbb{P}^n$ be a projective smooth irreducible hypersurface defined by the homogenous polynomial F of degree d. Then*

$$\omega_X \cong \mathcal{O}_X(-n-1+d).$$

Proof. We know by the previous affine case that $\omega_X|_{U_j}$ is trivial on the affine charts U_i. In particular we defined a nowhere vanishing global section $s_j \in H^0(U_j \cap X, \omega_X|_{U_j})$ by gluing together the Poincaré residues of $F|_{U_j}$. We will compute the transition maps relative to $U_j \cap U_k$ for these sections. First we introduce some notation:

$$f_j(y_0, \dots, \widehat{y_j}, \dots, y_n) = F\left(\frac{x_0}{x_j}, \dots, \underbrace{1}_{j}, \dots, \frac{x_n}{x_j}\right) = F|_{U_j}$$

where $y_l = \frac{x_l}{x_j}$. We get, on the intersections U_{jk} $(i \neq j)$, for $f_j(y_0, \dots, \widehat{y_j}, \dots, y_n)$, $f_k(z_0, \dots, \widehat{z_k}, \dots, z_n)$:

$$y_l = \frac{x_l}{x_j},\ z_l = \frac{x_l}{x_k} \implies y_l = \frac{z_l}{z_j},\ z_l = \frac{y_l}{y_k} \qquad l \neq j, k$$
$$y_k = \frac{x_k}{x_j} = z_j^{-1}$$
$$F\left(\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j}\right) = \left(\frac{x_k}{x_j}\right)^d F\left(\frac{x_0}{x_k}, \dots, \frac{x_n}{x_k}\right)$$

so

$$dy_l = \frac{1}{z_j} dz_l - \frac{z_l}{z_j^2} dz_j = \frac{z_j dz_l - z_l dz_j}{z_j^2} \qquad l \neq j, k$$
$$dy_k = -\frac{1}{z_j^2} dz_j$$
$$f_j(y_0, \dots, \widehat{y_j}, \dots, y_n) = z_j^{-d} f_k(z_0, \dots, \widehat{z_k}, \dots, z_n).$$

We will the interpret the transition maps for ω_X in $(U_j \cap X_i) \cap (U_k \cap X_i)$, since there we have an explicit expression for the sections s_j, s_k. In order for our notation to make sense, we need to suppose $i \neq j, k$: indeed F has $n+1$ partial derivatives, but the restrictions f_j only have n. Under such conditions, we have

$$\left.\frac{\partial F}{\partial x_i}\right|_{U_j} = \frac{\partial\, F|_{U_j}}{\partial y_i}.$$

We now employ the chain rule ($i \neq k$):

$$\frac{\partial f_j}{\partial y_i} = \sum_{\substack{m=1 \\ m\neq k}}^{n} \frac{\partial f_j}{\partial z_m}\frac{\partial z_m}{\partial y_i} = \frac{\partial f_j}{\partial z_i}\frac{\partial z_i}{\partial y_i} = z_j^{-d}\frac{\partial f_k}{\partial z_i}\frac{1}{y_k} = z_j^{(1-d)}\frac{\partial f_k}{\partial z_i}$$

We finally compute the transition map relative to $(U_j \cap X_i) \cap (U_k \cap X_i)$, $i < j < k$ (in the other cases they will just differ by a sign, which we can ignore):

$$\begin{aligned} s_j|_{X_i} &= \frac{(-1)^{i-1}}{\partial f_j/\partial y_i} dy_0 \wedge \ldots \wedge \widehat{dy_i} \wedge \ldots \wedge \widehat{dy_j} \wedge \ldots \wedge dy_k \wedge \ldots \wedge dy_n \\ &= \frac{(-1)^{i-1}}{z_j^{(1-d)}\frac{\partial f_k}{\partial z_i}} \frac{z_j dz_0 - z_0 dz_j}{z_j^2} \wedge \ldots \widehat{dz_i} \ldots \widehat{dz_j} \ldots \frac{-dz_j}{z_j^2} \ldots \wedge \frac{z_j dz_n - z_n z_j}{z_j^2} \\ &= \frac{(-1)^{i-1}}{z_j^{(1-d)}\frac{\partial f_k}{\partial z_i}} \frac{(-1)^{k-j}}{z_j^n} dz_0 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_j \wedge \ldots \wedge \widehat{dz_k} \wedge \ldots \wedge dz_n \\ &= (-1)^{k-j} z_j^{(-n-1+d)}\ s_k|_{X_i}. \end{aligned}$$

Thus the transition maps are

$$g_{jk} = z_j^{-n-1+d} = \left(\frac{x_j}{x_k}\right)^{-n-1+d}$$

that is, $\omega_X \cong \mathcal{O}_X(-n-1+d)$. □

If $-n-1+d > 0$, then ω_X is very ample: the (rational) maps defined by the positive powers of the hyperplane bundle $\mathcal{O}_X(m)$, $m > 0$ are the Veronese maps ν_m [6, 5.1], which are embeddings $\mathbb{P}^n \xhookrightarrow{\nu_m} \mathbb{P}^k$ (where $k = \binom{n+m}{n} - 1$).

Corollary 3.2.3. *Let $X \subseteq \mathbb{P}^n$ be a smooth irreducible projective hypersurface defined by a homogenous polynomial of degree d. Then*

$$\kappa(X) = \begin{cases} -\infty & d < n+1 \\ 0 & d = n+1 \\ \dim(X) = n-1 & d > n+1 \end{cases}$$

Remark 3.2.4. Every irreducible variety is birational to a hypersurface in $\mathbb{P}^n$ for some $n \in \mathbb{N}$ [2, I.4.9]. However the hypersurface might not be smooth.

Blowup of a point

The *blowup at the origin of* $\mathbb{A}^n$ is the set

$$B = B_p(\mathbb{A}^n) := \{(x,\ell) \in \mathbb{A}^n \times \mathbb{P}^{n-1} \mid x \in \ell\}$$

together with the natural projection

$$\begin{aligned} B &\xrightarrow{\pi} \mathbb{A}^n \\ (x,\ell) &\longmapsto x \end{aligned}$$

(the same set, together with the other projection to $\mathbb{P}^{n-1}$ defines the tautological bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$). B is a closed subvariety of $\mathbb{A}^n \times \mathbb{P}^{n-1}$: indeed, if we use coordinates $x = (x_1, \dots, x_n)$ for $\mathbb{A}^n$, $\ell = [y_1 : \dots : y_n]$ for $\mathbb{P}^{n-1}$, then $x \in \ell$ if and only if the matrix

$$\begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix}$$

has rank less than or equal to one. This holds precisely if all the 2×2 minors of the matrix vanish:

$$B = \mathbb{V}(x_i y_j - x_j y_i \mid 0 \leq i < j \leq n) \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}.$$

The fibre of π over any point x other than the origin is just the single point (x,ℓ), where ℓ is the unique line passing through x and the origin. However, the origin lies on all lines through the origin, so the preimage of 0 is an entire copy of $\mathbb{P}^{n-1}$, namely $\{0\} \times \mathbb{P}^{n-1} \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$. π is a birational map, with inverse

$$\begin{aligned} \mathbb{A}^n \setminus \{0\} &\longrightarrow \mathbb{A}^n \times \mathbb{P}^{n-1} \\ (x_1, \dots, x_n) &\longmapsto (x_1, \dots, x_n; x_1 : \dots : x_n). \end{aligned}$$

We explicitly compute the canonical bundle of B for $n = 2$: in this case B is defined by a single polynomial $f = x_1 y_2 - x_2 y_1$. Consider the affine open cover of $B = B_p(\mathbb{A}^2)$

$$\{V_i\} = \{(\mathbb{A}^2 \times U_i) \cap B\} \qquad i = 1, 2.$$

We have

$$V_1 = \{(x, y, [1 : s]) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid xs - y = 0\} \cong \{(x, y, s) \in \mathbb{A}^3 \mid xs - y = 0\}$$

$$\begin{aligned} V_1 \cong \{(x, y, s) \in \mathbb{A}^3 \mid xs - y = 0\} &\xleftrightarrow{\sim} \mathbb{A}^2_{x,s} \\ (x, y, s) &\longmapsto (x, s) \\ (x, xs, s) &\longleftarrow\!\shortmid (x, s) \end{aligned}$$

$$V_2 = \{(x, y, [t : 1]) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid x - yt = 0\} \cong \mathbb{A}^2_{y,t}$$

where $x = x_1$, $y = x_2$, $s = \frac{y_2}{y_1}$, $t = \frac{y_1}{y_2}$ and in the intersection V_{ij} we have the relations

$$\begin{aligned} & x = yt & & s = t^{-1} \\ \implies\ & dx = tdy + ydt & & ds = -\frac{1}{t^2}dt \end{aligned}$$

so

$$dx \wedge ds = (tdy + ydt) \wedge \left(-\frac{1}{t^2}dt\right) = -\frac{1}{t}dy \wedge dt.$$

The transition map g_{12} is thus $t^{-1} = \frac{y_2}{y_1}$. These are the same transition maps as for $\pi_2^*\mathcal{O}_{\mathbb{P}^1}(-1)$, where $\pi_2 : B \to \mathbb{P}^1$ is the projection to the second factor. Let $h_{12} = \frac{z_2}{z_1}$ be the transition function $h_{12} : \mathbb{P}^1_{[z_1:z_2]} \to GL_1(k)$ for $\mathcal{O}_{\mathbb{P}^1}(-1)$. We have $V_i = \pi_2^{-1}(U_i)$, so by Remark 1.3.3 the transition functions for $\pi_2^*\mathcal{O}_{\mathbb{P}^1}(-1)$ are

$$\pi_2^*(h_{12})(x_1, x_2; y_1, y_2) = h_{12} \circ \pi_2(x_1, x_2; y_1, y_2) = h_{12}([y_1 : y_2]) = \frac{y_2}{y_1}.$$

In conclusion, $\omega_B \cong \pi_2^*\mathcal{O}_{\mathbb{P}^1}(-1)$.

3.3 Divisors

In this section we introduce the terminology of divisors, which are a generalisation of codim 1 subvarieties of an irreducible smooth variety X. In a sense, divisors are zero sets "counted with multiplicity".

Definition 3.3.1. Let X be a smooth irreducible variety. A *prime divisor* on X is an irreducible closed subvariety $D \subset X$ of codimension 1.

Let D be a prime divisor of a smooth irreducible variety X. Then there exist an affine open cover of X $\{U_i\}_{i\in I}$ and functions $f_i \in \mathcal{O}_X(U_i)$ such that

$$D \cap U_i = \{f_i = 0\} \subseteq U_i.$$

More precisely, we require that each f_i is a generator of the ideal of the closed embedding $D \cap U_i \hookrightarrow U_i$. On the intersections U_{ij} necessarily $\{f_i = 0\} = \{f_j = 0\}$, so

$$\exists\, g_{ij} \in \mathcal{O}_X(U_{ij}) \mid f_i = g_{ij} f_j.$$

Definition 3.3.2. The *line bundle associated to a prime divisor D $\mathcal{O}_X(D)$* is the line bundle on X with as transition maps the maps g_{ij}. It has a global section s (given locally by the f_i) such that $\{s = 0\} = D$.

Remark 3.3.3. Beware of the notation! $\mathcal{O}_X(U)$ (regular functions on U) and $\mathcal{O}_X(D)$ (a line bundle) have very different meanings for an open U of X and respectively D a closed irreducible subvariety of codimension 1 of X.

Proposition 3.3.4 (Adjunction Formula)**.** *[2, II.8.20] Let X be a smooth irreducible variety, $D \subseteq X$ be a smooth prime divisor. Then*

$$\omega_D \cong \left(\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)\right)\big|_D.$$

Remark 3.3.5. The adjunction formula can be used to compute the canonical bundle for the cases we already examined. Let $X = \mathbb{A}^n$, $D = \mathbb{V}(f)$ be a smooth irreducible hypersurface. Since D is described by a single polynomial in the affine $\mathbb{A}^n$, the transition maps for $\mathcal{O}_{\mathbb{A}^n}(D)$ are just the identity: $\mathcal{O}_{\mathbb{A}^n}(D) \cong \mathcal{O}_{\mathbb{A}^n}$. Thus

$$\omega_D \cong \left(\mathcal{O}_{\mathbb{A}^n} \otimes \mathcal{O}_{\mathbb{A}^n}\right)\big|_D \cong \mathcal{O}_D.$$

If $X = \mathbb{P}^n$, $D = \mathbb{V}(F)$ a smooth irreducible hypersurface where F is homogenous of degree d, then the polynomial F is a global section of $\mathcal{O}_X(d)$, so the adjunction formula yields

$$\omega_D \cong \left(\mathcal{O}_{\mathbb{P}^n}(-n-1) \otimes \mathcal{O}_{\mathbb{P}^n}(d)\right)\big|_D \cong \mathcal{O}_D(-n-1+d).$$

Consider an arbitrary codimension 1 closed (reduced, but not necessarily irreducible) subvariety Y (whose irreducible components we require to all have codimension 1) of an irreducible smooth variety X. Let $D_1, \dots, D_m$ denote its irreducible components, which are prime divisors of X. Then on a suitable affine open cover $\{U_i\}$ such that $D_k \cap U_i = \{f_i^{(k)} = 0\}$, we get that Y is described locally by

$$Y \cap U_i = \{f_i^{(1)} \cdot \ldots \cdot f_i^{(m)} = 0\}$$

and on the intersections we get

$$f_i^{(1)} \cdot \ldots \cdot f_i^{(m)} = g_{ij}^{(1)} \cdot \ldots \cdot g_{ij}^{(m)} f_j^{(1)} \cdot \ldots \cdot f_j^{(m)}$$

where $g_{ij}^{(k)}$ are the transition functions of $\mathcal{O}_X(D_k)$ with respect to the cover $\{U_i\}$. We denote Y by

$$\sum_{k=1}^{m} D_k.$$

From Definition 3.3.2 we see the importance of taking generators of the ideal of D_k in X: a different choice of functions would give a different line bundle. For example, if instead of the f_i we took f_i^2, then we would still get $D \cap U_i = \{f_i^2 = 0\}$, but on the intersections we get transition functions $f_i^2 = g_{ij}^2 f_j^2$. These transition functions still induce a line bundle, namely $\mathcal{O}_X(D)^{\otimes 2}$, which we denote by $\mathcal{O}_X(2D)$.

Definition 3.3.6. Let X be an irreducible smooth variety. Let $\mathrm{Div}(X)$ denote the abelian free group with

$$\{D \subseteq X \text{ irreducible subvariety of } \operatorname{codim} 1\}$$

as $\mathbb{Z}$-basis. The elements of $\mathrm{Div}(X)$ are called the *divisors* of X. For $\sum n_k D_k \in \mathrm{Div}(X)$, we call

$$\bigotimes \mathcal{O}_X(D_k)^{\otimes n_k}$$

the *line bundle associated to the divisor* (for negative n_k consider $\mathcal{O}_X(D)^\vee$). Furthermore:

- If the coefficients of a divisor are either 0 or 1, then the divisor is called a *reduced effective divisor.* For instance (if not all coefficients are 0), $\sum_k D_k$ corresponds to the subvariety $Y \subseteq X$ of codim 1 whose irreducible components are D_k; $\bigotimes \mathcal{O}_X(D_k)$ has a global section s, given locally by the product of the $f_i^{(k)}$ defining the D_k (as discussed above) such that $\{s = 0\} = Y = \sum D_k$.

- If the coefficients of a divisor are non-negative, then the divisor is called an *effective divisor.* For instance, $\sum_k n_k D_k$, with $n_k \geq 1$. Its local equations still describe the same subvariety Y as the reduced divisor $\sum D_k$, but the multiplicity of the prime divisors may be more than one. Indeed, $\bigotimes \mathcal{O}_X(D_k)^{\otimes n_k}$ has a global section s whose zero set is Y, but it is given locally by the product of the $(f_i^{(k)})^{n_k}$.

- *Non-effective divisors* do not correspond to any subvariety of X.

Definition 3.3.7. Let L be a line bundle on a smooth, irreducible variety X. $\sum n_k D_k \in \mathrm{Div}(X)$ is a *divisor of* L if $L \cong \bigotimes \mathcal{O}_X(D_k)^{\otimes n_k}$. In particular, if L admits a non-zero global section s, then its zero set, counted with the right multiplicity given by the transition functions of L, is an *effective divisor of* L. Different non-zero global sections of L might have different zero sets, so the (effective) divisor of a line bundle is not unique. Two divisors giving the same line bundle are said to be *linearly equivalent.*

Proposition 3.3.8. *[3, 1.1.5] Let X be a smooth irreducible variety. The mapping*

$$\begin{aligned} \mathrm{Div}(X) &\xrightarrow{\phi} \mathrm{Pic}(X) \\ \sum n_i D_i &\longmapsto \bigotimes \mathcal{O}_X(D_i)^{\otimes n_i}. \end{aligned}$$

is a surjective group homomorphism. If $\dim X = 1$ *the following statement also holds:*

$$\sum n_i D_i \in \ker \phi \implies \sum n_i = 0.$$

3.4 Kodaira Dimension of Curves

Definition 3.4.1. Let X be a smooth, irreducible projective curve, L a line bundle over X. The *degree of a L* $\deg L$ is $\sum n_i \in \mathbb{Z}$, where $\sum n_i D_i$ is any preimage of L under ϕ. The previous proposition ensures that the degree of a bundle is well defined.

Remark 3.4.2. deg is a group morphism $\mathrm{Pic}(X) \to \mathbb{Z}$: since $\phi : \mathrm{Div}(X) \to \mathrm{Pic}(X)$ is surjective, then

$$\mathrm{Pic}(X) \cong \frac{\mathrm{Div}(X)}{\ker \phi}.$$

Let ψ denote the morphism

$$\begin{aligned} \psi : \mathrm{Div}(X) &\longrightarrow \mathbb{Z} \\ \sum n_i D_i &\longmapsto \sum n_i. \end{aligned}$$

Now since $ker\phi \subseteq \ker \psi$, there is a unique map $\overline{\psi}$ such that the diagram

$$\begin{tikzcd} \mathrm{Div}(X) \arrow[rr, "\psi"] \arrow[d, "\phi"'] & & \mathbb{Z} \\ \mathrm{Pic}(X) \cong \frac{\mathrm{Div}(X)}{\ker \phi} \arrow[urr, "\overline{\psi}"'] & & \end{tikzcd}$$

commutes, and by definition $\deg = \overline{\psi}$.

Lemma 3.4.3. *Let X be a smooth, irreducible projective curve, L a line bundle on X of degree $\deg L < 0$. Then $h^0(X, L) = 0$.*

Proof. By contradiction suppose that there exists $s \in H^0(X, L) \setminus \{0\}$. Then $\sum n_i D_i = \{s = 0\}$ is an effective divisor of L. So we get

$$\deg L = \sum n_i \geq 0$$

so we have reached a contradiction. $\square$

Theorem 3.4.4 (Riemann-Roch for curves)**.** *Let X be a smooth irreducible projective curve of genus g. For any $L \in \mathrm{Pic}(X)$*

$$h^0(X, L) - h^0(X, L^\vee \otimes \omega_X) = \deg L + 1 - g.$$

Proof. We refer the reader to [2, IV.1.3]. □

Corollary 3.4.5. *Let X be a smooth irreducible projective curve of genus g. Then*

$$\deg \omega_X = 2g - 2.$$

Theorem 3.4.6 (Kodaira Dimension Classification of Curves)**.** *Let X be a smooth, irreducible projective curve of genus $g = h^0(X, \omega_X)$. Then*

$$\begin{aligned} &\text{if } g = 0 && \kappa(X) = -\infty, \\ &\text{if } g = 1 && \kappa(X) = 0, \\ &\text{if } g \geq 2 && \kappa(X) = 1. \end{aligned}$$

Proof. By Corollary 3.4.5 $\deg \omega_X = 2g - 2$, so

$$\deg \omega_X^{\otimes m} = m(2g-2) \qquad \forall m \in \mathbb{Z}.$$

Suppose $g = 0$. Then $\deg \omega_X^{\otimes m} = -2m$ for all $m \in \mathbb{Z}$. By Lemma 3.4.3 $h^0(X, \omega_X^{\otimes m}) = 0$ for all $m > 0$, so

$$\kappa(X) = \kappa(\omega_X) = -\infty$$

by definition of Iitaka dimension of a bundle.

Now suppose $g = 1$. Then $\deg \omega_X^{\otimes m} = 0$ for all $m \in \mathbb{Z}$. In particular, ω_X is trivial. Since there exists $0 \neq s \in h^0(X, \omega_X)$, ω_X admits an effective divisor $\sum n_i D_i$. However, $\deg \omega_X = 0$ implies $\sum n_i = 0$ and since $n_i \geq 0$ for all i, s must have empty zero set. Thus ω_X and consequently all of its powers of are trivial, so

$$\kappa(X) = 0.$$

Finally, suppose $g \geq 2$. Then $\deg \omega_X^{\otimes m} = m(2g-2)$ for all $m \in \mathbb{Z}$. In particular $h^0(X, \omega_X^{\otimes m}) = 0$ for all $m < 0$ by Lemma 3.4.3. By the Riemann-Roch theorem for curves, for $m \geq 2$ we get

$$h^0(X, \omega_X^{\otimes m}) - h^0(X, \omega_X^{-m+1}) = h^0(X, \omega_X^{\otimes m}) - 0 = m(2g-2) + 1 - g.$$

Since the plurigenera grow linearly in m, by Proposition 3.1.14 we get

$$\kappa(X) = 1.$$

$\square$

Remark 3.4.7. The geometric genus of a *singular* (non-smooth) irreducible curve is defined as the geometric genus of any of its smooth models. Proposition 3.1.18 ensures that it is well-defined. Thus, if $\operatorname{char} k = 0$, by the birationality of the Kodaira dimension smoothness is not required in the Kodaira classification of curves.

Remark 3.4.8. The product of two irreducible smooth projective varieties X, Y is also smooth, projective and irreducible.

Proposition 3.4.9 (Additivity of the Kodaira dimension)**.** *Let X, Y be irreducible smooth projective varieties. Then*

$$H^0(\omega_{X\times Y}^{\otimes m}) \cong H^0(\omega_X^{\otimes m}) \otimes_k H^0(\omega_Y^{\otimes m}).$$

In particular,

$$\kappa(X \times Y) = \kappa(X) + \kappa(Y).$$

Proof. This follows from Remark 2.3.7. $\square$

After fixing a dimension n, we now have a way to construct n-dimensional varieties of each possible Kodaira dimension by considering the product of smooth, irreducible, projective curves:

- For $\kappa = -\infty$, we have $\mathbb{P}^n$, alternatively consider the product of a curve of genus 0 with $n-1$ other curves of arbitrary genus;
- For $\kappa = i$, $0 \leq i \leq n$, we consider the product of i curves of genus ≥ 2 with $n-i$ curves of genus 1;
- Alternatively for $\kappa = n$ we may consider an irreducible smooth hypersurface in $\mathbb{P}^{n+1}$ defined by a polynomial of degree $d > n+2$.

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