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SCUOLA DI SCIENZE
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A UNIQUENESS RESULT FOR THE GROUP OF PERMUTATIONS OF THE NATURAL NUMBERS

Tesi di Laurea in Topologia

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Alla purezza della matematica

Per la credulità il piccolo spirito si persuade siffattamente della verità e certezza de' suoi principi, del suo modo di vedere e giudicare, delle impossibilità ch'egli concepisce, che tutto quello che vi ripugna gli sembra assolutamente falso, qualunque prova v'abbia in contrario; perchè la credulità che immobilmente lo attacca alle precedenti sue idee, lo stacca dalle nuove e lo fa incredulissimo. E così l'eccesso di credulità causa l'eccesso d'incredulità, e impedisce i progressi dello spirito. [...]
Piccolissimo è quello spirito
che non è capace o è difficile al dubbio.

Giacomo Leopardi, Zibaldone

Introduction

Even in non-mathematical environments, a set is well known to be a collection of distinct, well-defined objects, named the elements of the set. A priori, a set can contain any number of elements. If an order can be defined on a set, then a permutation can be seen as any of the various ways in which its elements can be ordered. Now, how a permutation acts on a set can appear quite intuitive as we deal with a finite number of elements. Although the concept of permutation remains the same as we consider an infinite set, some aspects change radically and yield noteworthy results. The purpose of this work is to study the group of infinite permutations from a topological point of view and eventually prove a uniqueness result under defined hypotheses.
More specifically: the combination of a group structure and a "compatible" topology yields what we call a topological group. Our interest is to study the case where a topological group also happens to be completely metrisable and separable, meaning that it has a compatible complete metric and a countable dense subset. When such conditions are fulfilled then a group is said to be Polish, and the relative topology is called a Polish group topology.
In this paper we first go through some foundational theory and at the end we gather all previous knowledge to eventually prove that the group of permutations on the naturals admits a unique Polish group topology.

Contents

Chapter 1

Topological Groups

This first chapter is focused on giving the definition of *topological group* and study what are the implications of giving a group a "good" topological structure. We will see the basic properties of topological groups and prove some relevant theorems on this topic. Ultimately, we will show that if we add the hypothesis of metrisability on a topological group then we obtain some remarkable equivalent conditions. This result goes under the name of Birkhoff Kakutani Theorem, and will be used for following developments in the last chapter. All contents of this chapter refer to [6]

1.1 Topological and metric spaces

In this section we see some preliminary definitions and basic theorems about topological and metric spaces that will be frequently used throughout the paper.

A **topological space** is a set **X** together with a collection $\boldsymbol{\tau}$ of subsets of X, with the property that both $\emptyset$ and X itself are contained in τ and τ is closed under arbitrary unions and finite intersections. Under these conditions, the collection τ is called a **topology** on X, and the relative topological space is denoted by the pair $\boldsymbol{(X, \tau)}$. The elements of the topology τ are called **open** sets, and their complements are called **closed**. Consequently, the collection of closed sets is closed under arbitrary intersection and finite union, and the banal sets $\emptyset$ and X are both open and close. Observe that two immediate topologies on a space X are the **discrete topology**, namely the topology where all subsets are open, and the **indiscrete topology**, namely the one where the only open sets are the empty set and X itself. We say that a topology τ on a set X is **finer** than another τ' if $\tau' \subseteq \tau$, that is, each open set in τ' is also open in τ.

A set that is a countable intersection of open sets is called a $\mathbf{G}_\delta$ set, while one that is a countable union of closed sets is called a $\mathbf{F}_\sigma$ set.

A **subspace** of (X, τ) is a subset Y with the topology $\tau_Y = \{U \cap Y \mid U \in \tau\}$. We call τ_Y the **relative topology**.

A **basis** $\mathcal{B}$ for a topology τ is a subset of τ such that every open set can be written as union of elements of $\mathcal{B}$ (by convention we set the empty union to give the open set $\emptyset$). Notice that a collection $\mathcal{B}$ of subsets of X is a basis for a topology τ if and only if the intersection of any two members of $\mathcal{B}$ is itself obtained by union of elements of $\mathcal{B}$ and the union of all $B \in \mathcal{B}$ gives X. This leads to the following definition: A **subbasis** for a topology τ is a subset $\mathcal{S}$ of τ such that the collection of finite intersections of sets in $\mathcal{S}$ gives a basis for τ. Given a family $\mathcal{S}$ of subsets of X, the **topology generated by** $\mathcal{S}$ is the smallest topology definable on X that contains $\mathcal{S}$, and it is formed by all arbitrary unions of finite intersections of elements of $\mathcal{S}$ (notice that, by definition, $\mathcal{S}$ is obviously a subbasis for such topology).

A topological space is **second countable** if it admits a countable basis. Given any element x of a topological space, an **open neighborhood** of x is any open set that contains x. A **neighborhood basis** for x is a collection $\mathcal{U}$ of open neighborhoods of x with the property that, if V is any open neighborhood of x, then some $U \in \mathcal{U}$ can be found such that $U \in V$. A topological space is **first countable** if each point $x \in X$ admits a countable neighborhood basis.

A topological space (X, τ) is a **Hausdorff** space if, given a pair of arbitrary points $x, y \in X$, there exists a pair of disjoint open sets $U_x, U_y \in \tau$ containing respectively x and y. A Hausdorff space also takes the name of **T2** and **separated**.

A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ from a topological space to another is **continuous** if the inverse image of any open set is open. The map f is **open** if the image of any open set is open, while it is **closed** if the image of any closed set is closed. The map f is a **homeomorphism** if it is a bijective map with the property of being both continuous and open (or, equivalently, if it is a continuous bijection such that its inverse map is also continuous), while it is an **embedding** if X is homeomorphic to its image $f(X)$, where $f(X)$ has the relative topology inherited from (Y, σ). Finally, f is **continuous at** a point $x \in X$ if the inverse image of any open neighborhood of $f(x)$ contains an open neighborhood of x. Hence f is continuous if and only if f is continuous at every point $x \in X$.

A **metric space** is a set X together with a commutative function $d : X \times X \longrightarrow \mathbb{R}^+$ such that the inverse image of 0 is the diagonal $\Delta_{X \times X} = \{(x, x) \in X \times X\}$ and $d(x, y) \leq d(x, z) + d(z, y)$ for each $x, y, z \in X$. Such a function is called a **metric** on X, and the relative metric space is denoted by the pair (X, d). Under these conditions, the **open ball** with center x and radius r is the subset $B(x, r) = \{y \in X \mid d(x, y) < r\}$ of X; the corresponding **closed ball** is $B_{cl}(x, r) = \{y \in X \mid d(x, y) \leq r\}$.

A topological space (X, τ) is **metrisable** if there exists a metric d on X so that τ coincides with the topology generated by the collection of all the open balls in (X, d), named the **topology of the metric space** (X, d). In such case the metric d is said to be **compatible** with τ.

A subset of a topological space is **dense** if it has nonempty intersection with each nonempty open set.

A **separable space** is a topological space admitting a countable dense subset.

By previous definitions it follows that *every second countable space is separable*; in fact, if we consider D to be the union of all sets that constitute a basis for the topology, then D clearly meets each nonempty open set. We will recall this later on for following developments.

The following lemma will be useful for further proofs

Lemma 1.1. *Let X be a topological space and U an open set of X. If D is a dense subset of X, then $U \cap D$ is a dense subset of U.*

Proof. We will show that $U \cap D$ meets every nonempty open subset of U. Let V be an arbitrary nonempty open subset of U, meaning that V is the intersection of U with some open set W of X. As V is a finite intersection of open sets, it is open in X too. Then what happens is that $V \cap (U \cap D) = V \cap D \neq \emptyset$ because D is dense in X by assumption. Hence each nonempty open subset of U has nonempty intersection with $U \cap D$, meaning that $U \cap D$ is dense in U. $\square$

1.2 Topological groups

We have seen that, given any set, we can define a topological structure on it. This process becomes particularly interesting if the set we want to work with has some properties itself. For instance, let $(G, \cdot)$ be a group. Then we are interested in defining on G a topology that preserves the group structure in a continuous way. Such a topology is said to be *compatible* with G.

Definition 1.2. *A **topological group** is a group (G, ·) together with a topology τ on G such that the map $(x, y) \to xy^{-1}$ from $(G \times G, \tau \times \tau)$ into (G, τ) is continuous.*

Observation 1.3. *Given a group G, the discrete topology and the indiscrete topology are both compatible with G. Hence, any group can be seen as a topological group.*

Theorem 1.4. *Given a topological group G, if a,b are fixed elements of G, then each of the following applications is a homeomorphism of G:*

(1) $x \longrightarrow x^{-1}$;

(2) $x \longrightarrow ax$ *(left translation induced by a);*

(3) $x \longrightarrow xa$ *(right translation induced by a);*

(4) $x \longrightarrow axb$*;*

(5) $x \longrightarrow axa^{-1}$ *(inner automorphism of G induced by a);*

Moreover, for fixed $a \in G$*, the mapping:*

(6) $x \longrightarrow xax^{-1}$ *is continuous.*

Proof. (1) The map $x \longrightarrow x^{-1}$ is continuous and self-inverse.
(2),(3) The map $x \longrightarrow (a, x) \longrightarrow ax$ is the composition of two continuous mappings, hence is continuous. Its inverse is given by $x \longrightarrow a^{-1}x$, which also is the composition of two continuous mappings. The case $x \longrightarrow xb$ is analogous.
(4),(5) By the case above we see $x \longrightarrow ax \longrightarrow (ax)b$ as the composition of two homeomorphisms. Similarly for $x \longrightarrow axa^{-1}$.
(6) The map $x \longrightarrow (xa, x^{-1}) \longrightarrow (xa)x^{-1}$ is the composition of two continuous mappings. Continuity of the first one comes from considering the composition of the projections $x \longrightarrow xa$ and $x \longrightarrow x^{-1}$, that are separately continuous by the above properties. The continuity of the second one follows from the previous cases. □

Notation Let G be a group, x a fixed element of G and A, B subsets of G, then we let:

$$Ax = \{ax \mid a \in A\};$$

$$xA = \{xa \mid a \in A\};$$

$$AB = \{ab \mid a \in A, b \in B\};$$

$$A^{-1} = \{a^{-1} \mid a \in A\}.$$

Corollary 1.5. *If G is a topological group and $\mathcal{B}$ is a fundamental system of neighborhoods of the neutral element e, then the sets $\{aV \mid V \in \mathcal{B}\}$ and $\{Va \mid V \in \mathcal{B}\}$ are fundamental systems of neighborhoods of a for each $a \in G$.*

Proof. We know from 1.4 that that the maps $x \longrightarrow ax$ and $x \longrightarrow xa$ are homeomorphisms of G sending e to a, and hence transforming $\mathcal{B}$ into a fundamental system of neighborhoods of a. In a similar way $\mathcal{B}$ is sent to another fundamental system of neighborhoods of e through the homeomorphic map $x \longrightarrow x^{-1}$. □

Corollary 1.6. *If G is a topological group, A is a subset of G and $U \subseteq G$ is open, then the subsets U^{-1}, AU, UA are also open.*

Proof. We can suppose that A and B are both nonempty (otherwise the thesis is automatically verified). Then U^{-1} is open since inversion is a homeomorphism. For the same reason the sets aU and Ua are open for each $a \in A$. Therefore also

$$AU = \bigcup_{a \in A} aU$$

$$UA = \bigcup_{a \in A} Ua$$

are open, for they are countable union of open sets. □

Observation 1.7. *The fact that the translations are homeomorphisms means that for any pair of points $a, b \in G$ the mapping $x \longrightarrow ba^{-1}x$ is a homeomorphism of G that sends a to b. Consequently, topological behaviour at a is reflected at b. For instance, if a point $a \in G$ has a countable neighborhood basis, then each point $b \in G$ has a countable neighborhood basis. Therefore, a topological group is said to be 'topologically homogeneous'.*

Due to the homogeneity of a topological group, the topology is completely determined by the system of neighborhoods of the neutral element e. The following theorem outlines the pertinent properties of this neighborhood system.

Theorem 1.8. *If G is a topological group, then the class $\mathcal{V}$ of all neighborhoods of the neutral element e has the following properties:*

(1) $e \in V$ for all $V \in \mathcal{V}$;

(2) If $V, W \in \mathcal{V}$, then $V \cup W \in \mathcal{V}$;

(3) If $V \in \mathcal{V}$, then there exists $W \in \mathcal{V}$ such that $WW \subset V$;

(4) If $V \in \mathcal{V}$, then $V^{-1} \in \mathcal{V}$;

(5) If $V \in \mathcal{V}$ and $a \in G$, then $aVa^{-1} \in \mathcal{V}$;

(6) If $V \in \mathcal{V}$ and $V \subseteq W$, then $W \subseteq \mathcal{V}$.

Proof. (1),(2),(6) Are more generally properties of the class of neighborhoods of a point in a topological space.
(3) The map $f : (x, y) \longrightarrow xy$ is continuous at (e, e) and $f(e, e) = e$. Hence, given any open neighborhood V of e, its inverse image contains an open neighborhood $A = (W_1 \times W_2) \in G \times G$ of (e, e). If we let $W = V_1 \cap V_2$ then we obtain

$$W \cdot W = f(W \times W) \subseteq f(W_1 \times W_2) \subseteq V.$$

(4),(5) The proof is similar to (3) after recalling that the maps $x \longrightarrow x^{-1}$ and $x \longrightarrow axa^{-1}$ are continuous. □

The latter properties can be considered a characterisation of compatibility of a topology with the group structure.

Proposition 1.9. *Let G be a topological group and B a subset of G. Then $\bar{B} = \cap\{AB \mid A$ is a neighborhood of $e\}$*

Proof. Recall that the closure of a subset B is the intersection of all open sets containing B; then the fact that $\bar{B} \subseteq \cap\{AB \mid A$ is a neighborhood of $e\}$ is obvious.
For the reverse, we will show that if $x \in \cap\{AB \mid A$ is a neighborhood of $e\}$ and U is any neighborhood of x then $U \cap B \neq \emptyset$, for this means that $x \in \overline{B}$.

Suppose $x \in \cap\{AB \mid A$ is a neighborhood of $e\}$. If U is a neighborhood of x then there is a neighborhood A of e such that $Ax \subseteq U$. Now also A^{-1} is a neighborhood of e, so $x \in A^{-1}B$, following that $x = a^{-1}b$ for some $a \in A$ and $b \in B$. Hence $B \ni b = ax \in Ax \subseteq U$, and so $B \cap U \neq \emptyset$. □

Theorem 1.10. *If G is a topological group and $\mathcal{B}$ is any fundamental system of neighborhoods of e, then the following conditions are equivalent:*

(1) G is a Hausdorff space;

(2) $\{e\}$ is a closed subset of G;

(3) $\bigcap_{B\in\mathfrak{B}} B = \{e\}$.

Proof. (1) $\Longrightarrow$ (2) In a Hausdorff space each point is closed, in particular, the singleton e is.
(2) $\Longrightarrow$ (3) If we assume $x \neq e$ then we shall prove that there exists some $B \in \mathcal{B}$ such that $x \notin B$. As $\{e\}$ closed, $\{x\}$ is also closed because by 1.4 the map $x \longrightarrow ax$ is a homeomorphism. Being $\{e\}$ and $\{x\}$ both closed, there exists a neighborhood V of e such that $x \notin V$. Now $\mathcal{B}$ is a fundamental system of neighborhood and we can pick $B \in \mathcal{B}$ such that $B \in V$.
(3) $\Longrightarrow$ (1) Assume $x \neq y$, then we search for a neighborhood V of e such that $xV \cap yV = \emptyset$. Since $xy^{-1} \neq e$, by hypothesis there exists $B \in \mathcal{B}$ such that $xy^{-1} \notin B$. Now we can choose $C \in \mathcal{B}$ so that $C \cdot C \subseteq B$; then $V = C \cap C^{-1}$ is a symmetrical neighborhood of e. If we suppose, by contradiction, that $(xV) \cap (yV) \neq \emptyset$, then it follows that

$$xy^{-1} \in V \cdot V^{-1} = V \cdot V \subseteq C \cdot C \subseteq B.$$

Hence $(xV) \cap (yV) = \emptyset$. □

Theorem 1.11. *If G is a topological group and H is a subgroup of G, then its closure $\overline{H}$ is also a subgroup of G. Moreover, if H is normal, then $\overline{H}$ is also a normal subgroup of G.*

Proof. Suppose H is a subgroup of G and define $f : G \times G \longrightarrow G$ as $f(x, y) = x^{-1}y$; then $f(H \times H) \subseteq H$ from the fact that H is a subgroup. Since f is continuous and $\overline{H} \times \overline{H}$ is the closure of $H \times H$ it follows that $f(\overline{H} \times \overline{H}) \subseteq \overline{H}$, meaning that $\overline{H}$ is a subgroup too.
If, in addiction, H is normal in G, this implies that for each $a \in G$ the homeomorphism $x \longrightarrow axa^{-1}$ sends H into H. Then, by continuity, $x \longrightarrow axa^{-1}$ also sends $\overline{H}$ into $\overline{H}$, yielding that $\overline{H}$ is normal as well. □

Corollary 1.12. *If G is a topological group, then $\overline{\{e\}}$ is a closed normal subgroup of G.*

Theorem 1.13. *If G is a topological group and H is an open subgroup of G, then H is also closed.*

Proof. If H is a subgroup of G, then its complement H^C can be written as union of all left translates of H. In particular:

$$H^C = G \setminus H = \bigcup \{Hx \mid x \notin H\}$$

and this is open according to 1.6, as H is open. □

Proposition 1.14. *If a subgroup H of a topological group G contains a neighborhood of e, then H is open.*

Proof. Let V be a neighborhood of e such that $V \in H$ and let $x \neq e$ be an element of H. Then xV is a neighborhood of x that is contained in H. In fact, let a be any element of xV and write $a = xv$ with $v \in V$. Then $a \in H$ since both x and v belong to the subgroup H. □

Corollary 1.15. *Let G be a topological group and H be a subgroup. If H contains a nonempty open set of G then H is open.*

Proof. Let H be a subgroup of G containing a nonempty open set U. Then choose $x \in U$, so that U is an open neighborhood of x. Consider the left-translate $H' = x^{-1}H$. H' is a subgroup of G that contains $x^{-1}U$, which is a neighborhood of e, hence by Proposition 1.14 H' is open. Therefore H is also open by Corollary 1.6. □

Observation 1.16. *It follows from Corollary 1.15 that a subgroup of a topological group is open if and only if its interior is nonempty.*

Theorem 1.17. *Let G be a topological group and let D be a dense subgroup of G. If H is an open subgroup of D then the closure of H in G is open in G.*

Proof. Assume H nonempty, otherwise there is nothing to prove. Let $\overline{H}$ be the closure of H in G. Since it is the closure of a subgroup of G, $\overline{H}$ is a subgroup as well, according to Theorem 1.11. Then, to see that $\overline{H}$ is open in G it suffices to show that it contains a nonempty open subset of G by Corollary 1.15. Since H is a nonempty open subset of D, there exists a nonempty open subset H' of G such that $H = H' \cap D$. It follows from Lemma 1.1 that $H' \cap D$ is dense in H', hence $\overline{H} = \overline{H' \cap D} = \overline{H'}$. Therefore $\overline{H}$ contains H', which is the open subset of G that we wanted. □

1.3 The Birkhoff-Kakutani Theorem

Recall that a topological group is first countable if each point admits a countable neighborhood basis. On the other hand, we have seen that a topological group has homogeneity properties. For this reason, first countability on a topological group G can be more easily defined by restricting the stated property to any singular point of G. Precisely, *a topological group G is first countable if the neutral element e admits a countable neighborhood basis.*
It is trivial to prove that, in general, a metrisable topological space is both Hausdorff and first countable. A remarkable fact is that, if a topological space has a compatible group structure, then the converse is true: *if the neutral element e of a topological group G has a countable fundamental system of neighborhoods and it is a Hausdorff space, then G is metrisable.*

Definition 1.18. *A **left invariant metric** is a metric d on a metric space X such that $d(x,y)=d(zx,zy)$ for each $x,y,z \in X$.*

Observation 1.19. *If a topological group (G, τ) admits a left-invariant compatible metric d, then the metric $d'(x, y) = d(x^{-1}, y^{-1})$ is clearly right-invariant. Since the mapping $x \longrightarrow x^{-1}$ is a homeomorphism of G, d' generates the same topology of d, the metric d' is also compatible with τ. Thus, in the Birkhoff-Kakutani theorem, it is immaterial whether one says 'left' or 'right.*

Part of the proof can be separated out as the following lemma:

Lemma 1.20. *Let X be a set and suppose that $f : X \times X \longrightarrow \mathbb{R}$ is a function satisfying the following conditions:*

(1) $f(x, y) \geq 0$ *for all* $x, y \in X$

(2) $f(x, x) = 0$ *for all* $x \in X$

(3) For each $\epsilon > 0$ *the relations* $f(w, x) \leq \epsilon$, $f(x, y) \leq \epsilon$, $f(y, z) \leq \epsilon$ *imply that* $f(w, z) \leq 2\epsilon$.

Now define a function d: $X \times X \longrightarrow \mathbb{R}$ *as follows.*
If $(x, y) \in X \times X$ *and* $\mathfrak{p} = \{x = x_0, x_1, ..., x_n = y\}$ *is any finite system of points in* X *that begins at* x *and ends at* y*, then we write*

$$|\mathfrak{p}| = \sum_{k=1}^{n} f(x_{k-1}, x_k)$$

and define

$$d(x, y) = inf(|\mathfrak{p}|),$$

where $\mathfrak{p}$ *varies over all such finite systems. Then* d *has the following properties:*

(a) $\frac{1}{2} f(x, y) \leq d(x, y) \leq f(x, y)$;

(b) $d(x, z) \leq d(x, y) + d(y, z)$;

(c) if $f(x, y) = f(y, x)$ *for all* $x, y \in X$ *then* $d(x, y) = d(y, x)$ *for all* $x, y \in X$.

If $f(x,y)=f(y,x)$ for all $x,y\in X$ and if $f(x,y)>0$ whenever $x\neq y$, then d is a metric on X.

Proof. **(c).** This clearly holds by how d is defined; then if (a) and (b) are verified the last assertion comes consequently. Hence we only have to verify (a) and (b). A preliminary observation derived from condition (3) is that for each $\epsilon>0$, if both $f(x,y)<\epsilon$ and $f(y,z)<\epsilon$, then $f(x,z)<2\epsilon$. If, in particular, $f(x,y)=f(y,z)=0$ then $0<f(x,z)<2\epsilon$ for each ϵ, implying that $f(x,z)=0$. By induction we obtain that, for each system of points $\mathfrak{p}=\{x=x_0,x_1,...,x_n=y\}$ such that $|\mathfrak{p}|=0$, then $f(x,y)=0$.

(a). Consider the system $\mathfrak{q}=\{x=x_0,x_1=y\}$: By definition of d we have that $d(x,y)\leq|\mathfrak{p}|=f(x,y)$, so the second inequality is verified. To prove that $\frac{1}{2}f(x,y)\leq d(x,y)$ we will show that $\frac{1}{2}f(x,y)\leq|\mathfrak{p}|$ for each system $\mathfrak{p}$. We proceed by induction on the number n of points contained in $\mathfrak{p}$. If $n=1$ the only system can be $\mathfrak{p}=\{x=x_0,x_1=y\}$, then $|\mathfrak{p}|=f(x,y)\geq\frac{1}{2}f(x,y)$ since f is non-negative. Now we consider $n\geq2$ and suppose the assertion to be true for systems containing less than n points. We have three cases to consider:

Case 1: $f(x_0,x_1)\geq\frac{1}{2}|\mathfrak{p}|$. Then

$$\frac{1}{2}|\mathfrak{p}|=|\mathfrak{p}|-\frac{1}{2}|\mathfrak{p}|\geq|\mathfrak{p}|-f(x_0,x_1)=\sum_{k=2}^{n}f(x_{k-1},x_k)\geq d(x_1,x_n)\geq\frac{1}{2}f(x_1,x_n),$$

where the last inequality comes by induction on a system of $n-1$ points. We hence obtain that $f(x_1,x_n)\leq|\mathfrak{p}|$, and taking into account the obvious fact that $f(x_0,x_1)\leq|\mathfrak{p}|$ it follows that $f(x_0,x_n)\leq2|\mathfrak{p}|$, which is $\frac{1}{2}f(x,y)\leq|\mathfrak{p}|$.

Case 2: $f(x_{n-1},y)\geq\frac{1}{2}|\mathfrak{p}|$. Then we obtain the same result of Case 1 by a similar argument.

Case 3: $f(x_0,x_1)<\frac{1}{2}|\mathfrak{p}|$ and $f(x_{n-1},x_n)<\frac{1}{2}|\mathfrak{p}|$. We can assume $|\mathfrak{p}|>0$ and $n\geq3$. Let r be the largest natural number such that

$$\sum_{k=1}^{r}f(x_{k-1},x_k)\leq\frac{1}{2}|\mathfrak{p}|.$$

From the fact that $f(x_0, x_1) < \frac{1}{2}|\mathfrak{p}|$ it follows that $r \geq 1$ while, as $f(x_{n-1}, x_n) < \frac{1}{2}|\mathfrak{p}|$, we have that $\sum_{k=1}^{n-1} f(x_{k-1}, x_k) = |\mathfrak{p}| - f(x_{n-1}, x_n) > \frac{1}{2}|\mathfrak{p}|$. This implies that $r < n-1$ and so, by induction and the result above, we can write $\frac{1}{2}f(x_0, x_r) \leq \sum_{k=1}^{r} f(x_{k-1}, x_k) \leq \frac{1}{2}|\mathfrak{p}|$, hence

(i)

$$f(x_0, x_r) \leq |\mathfrak{p}|$$

while clearly

(ii)

$$f(x_r, x_{r+1}) \leq |\mathfrak{p}| .$$

Recall that by maximality of r we had $\sum_{k=1}^{r+1} f(x_{k-1}, x_k) > \frac{1}{2}|\mathfrak{p}|$; it naturally follows that

$$\sum_{k=r+2}^{n} f(x_{k-1}, x_k) < \frac{1}{2}|\mathfrak{p}| .$$

Then by induction we can write $\frac{1}{2}f(x_{r+1}, x_n) \leq \sum_{k=r+2}^{n} f(x_{k-1}, x_k) < \frac{1}{2}|\mathfrak{p}|$, which is

(iii)

$$f(x_{r+1}, x_n) \leq |\mathfrak{p}| .$$

So now we can finally gather (i),(ii) and (iii) together and by (3) obtain that $f(x_0, x_n) \leq 2|\mathfrak{p}|$, which is equivalent to saying that $\frac{1}{2}f(x, y) \leq |\mathfrak{p}|$. So the first inequality is verified as well.

(b). Let x, y, z be elements of X and consider two systems of points

$$\mathfrak{p} = \{x = x_0, x_1, ..., x_n = y\}, \qquad \mathfrak{q} = \{y = y_0, y_1, ..., y_m = z\}$$

and define $\mathfrak{s}$ as the concatenation of $\mathfrak{p}$ and $\mathfrak{q}$, namely

$$\mathfrak{s} = \{x = x_0, x_1, ..., x_n = y = y_0, y_1, ..., y_m = z\}$$

Then by definition of d it is clear that $d(x, z) \leq |\mathfrak{s}| = |\mathfrak{p}| + |\mathfrak{q}|$. Being $\mathfrak{p}$ and $\mathfrak{q}$ two arbitrary independent systems we conclude that $d(x, z) \leq d(x, y) + d(y, z)$.

□

Theorem 1.21 (Birkhoff Kakutani Theorem)**.** *Let G be a topological group. Then G is metrisable if and only if G is a Hausdorff space and the identity e has a countable neighborhood basis. Moreover, if G is metrisable, then G admits a compatible metric d which is left-invariant.*

Proof. We have seen that the first implication is true for any metrisable space, so it is true in particular for topological groups, where first countability coincides with the neutral element having a countable neighborhood basis.

Let now a topological group G be a Hausdorff space with a fundamental sequence of neighborhoods $\{U_n \mid n \in \mathbb{N}\}$ of e. We shall start by constructing an 'improved' fundamental sequence of neighborhoods $\{V_n\}_{n\in\mathbb{N}}$. We first replace U_n with $U_n \cap U_n^{-1}$ so that all U_n are symmetric, meaning that $U_n = U_n^{-1}$, then let $V_1 = U_1$. According to properties listed in 1.8 there exists U_k such that $U_k^3 \subset U_2 \cap V_1$. Let V_2 be the first U_k with this property and, inductively, let V_n be the first U_k such that $U_k^3 \subset U_n \cap V_{n-1}$. By this definition $V_n \subset U_n$ for all n, hence the system of neighborhoods $\{V_n\}_{n\in\mathbb{N}}$ is also fundamental. As G is a Hausdorff space, according to 1.10 we have that

(i)

$$\bigcap_{k\in\mathbb{N}\setminus\{0\}} V_k = \{e\}$$

while by construction we get

(ii)

$$V_{k+1}^3 \subset V_k$$

for each $k = 1, 2, 3,...$ Set V_0=G. Then from (ii) we have

(iii)

$$G = V_0 \supset V_1 \supset V_2 \supset V_3 \supset ...$$

This means that every $x \in G$ must belong to some V_k, and it follows from (i) and (iii) that, if $x \neq e$, then V_k excludes x from some k onward. In other words x belongs to only finitely many V_k. Notice that, if G admits a finite

fundamental system of neighborhoods of e, then it is obviously discrete. In this case the discrete metric, that is $d(x,x) = 0$ for all x and $d(x,y) = 1$ when $x \neq y$, is a left-invariant compatible metric. Thus, let now assume G nondiscrete. Each V_K represents a degree of 'nearness' to the 'origin' e or, equivalently, $x^{-1}y \in V_k$ is a measure of the nearness of x to y. The problem is to express such qualitative statements in terms of a metric, then left-invariance will follow from the fact that the basic relation $x^{-1}y \in V_k$ is itself left-invariant; in fact, $(ax)^{-1}(ay) = x^{-1}y)$.
Suppose $x \neq y$. Two qualitative assertions are that $x^{-1}y \in V_k$ for some k, and that there exists a largest such k. This allows us to define $f(x,y) = \min\{(1/2)^k \mid x^{-1}y \in V_k\}$. On the other hand, if $x = y$ then $x^{-1}y = e \in V_k$ for all k. So, if we set $f(x,x) = 0$ for all x then we can write
(iv)

$$f(x,y) = \inf\{(1/2)^k \mid x^{-1}y \in V_k\}$$

for all $x, y \in G$. The desired metric d will be derived from f via 1.20. Thus, we now show that the hypotheses of the lemma are fulfilled. Clearly $f(x,y) \geq 0$, and $f(x,y) = 0$ if and only if $x = y$. Also, $f(x,y) = f(y,x)$ since the V_n are symmetric by construction. Hence, to apply 1.20 we only need to verify the third condition on f; then the left-invariance of d will follow from the property that $f(ax,ay) = f(x,y)$. With regards to the hypotheses of 1.20, supposing $\epsilon > 0$, $f(w,x) \leq \epsilon$, $f(x,y) \leq \epsilon$, $f(y,z) \leq \epsilon$, it is to be shown that $f(w,z) \leq 2\epsilon$. Notice that this is trivial if $\epsilon > 1/2$, since in any case $f(x,w) < 1$ by definition; hence, suppose $0 < \epsilon < 1/2$. According to (iv) there exist positive integers i, j, k such that

$$w^{-1}x \in V_i \quad and \quad (1/2)^i \leq \epsilon;$$

$$x^{-1}y \in V_j \quad and \quad (1/2)^j \leq \epsilon;$$

$$y^{-1}z \in V_k \quad and \quad (1/2)^k \leq \epsilon.$$

Let $r = \min\{i,j,k\}$; then $(1/2)^r \leq \epsilon$ and it follows from (ii) that

$$w^{-1}z = (w^{-1}x)(x^{-1}y)(y^{-1}z) \in V_1 V_j V_k \subset V_r^3 \subset V_{r-1}$$

hence $f(w,z) \leq (1/2)^{r-1} = 2(1/2)^r \leq 2\epsilon$. We eventually verified the hypotheses in order to apply 1.20 which, as already said, yields a left-invariant metric d; it is left to prove that d generates the given topology. For any $\epsilon > 0$ and any $a \in G$ we define

$$U_\epsilon(a) = \{x \in G \mid f(a,x) < \epsilon\}$$

and we claim that the set $\{U_\epsilon(a) \mid \epsilon > 0\}$ is a fundamental system of neighborhoods for the given topology on G. To begin, every $U_\epsilon(a)$ is a neighborhood of a. In fact, if k is a positive integer such that $(1/2)^k < \epsilon$, then from the chain of implications

$$x \in aV_k \implies a^{-1}x \in V_k \implies f(a,x) \leq (1/2)^k < \epsilon$$

it follows that $aV_k \subset U_\epsilon(a)$, hence $U_\epsilon(a)$ is a neighborhood of a. On the other hand we want to show that if A is any neighborhood of a then $U_\epsilon \subset A$ for some $\epsilon > 0$. According to the fact that $a^{-1}A$ is a neighborhood of e and that V_k is a fundamental system of neighborhoods of e, we let k be a positive integer such that $aV_k \subset A$, then set $\epsilon = (1/2)^k$. If $x \notin aV_K$, then $a^{-1}x$ can belong to V_j only for $j < k$, meaning $f(a,x) > (1/2)^k = \epsilon$. Hence $x \notin U_\epsilon(a)$, yielding to $U_\epsilon(a) \subset aV_k \subset A$. We have shown that $\{U_\epsilon(a) \mid \epsilon > 0\}$ is a fundamental system of neighborhoods for any fixed $a \in G$.
Now recall (1) of 1.20, by which $\frac{1}{2}f(x,y) \leq d(x,y) \leq f(x,y)$; it follows that, for $\epsilon > 0$, $f(x,y) < \epsilon \implies d(x,y) < \epsilon \implies f(x,y) < 2\epsilon$. Thus
(v)

$$U_\epsilon(x) \subset \{y \mid d(x,y) < \epsilon\} \subset U_{2\epsilon}.$$

Now, the $\{U_\epsilon(x) \mid \epsilon > 0\}$ are a fundamental system of neighborhoods of x for the given topology on G, while the open balls $\{y \mid d(x,y) < \epsilon\}$ are a fundamental system of neighborhoods of x for the topology derived from the metric d. An immediate implication of (v) is that the two topologies coincide. Finally, if G is metrisable then it is Hausdorff and first countable by the first inclusion, hence, by the second inclusion, G possesses a left-invariant compatible metric. $\square$

Chapter 2

Polish Groups

A topological space is **Polish** if it is both completely metrisable and separable. Consequently, the topology on such a space is called a **Polish topology**. Throughout the previous chapter we have seen that, given a group G, it is always possible to define a topology in order for it to be a topological group. Then the following definition comes automatically: If a topological group is a Polish space, then it is called a **Polish group**, and the given topology is said a **Polish group topology**.
In this chapter we will see some interesting theorems on Polish groups which will ultimately yield an interesting uniqueness result on the Polish group of our interest: S_∞. The contents in Sections 2.1, 2.2 and 2.3 mostly refer to [1], while Section 2.4 is based on the work in [3] .

2.1 Trees

Let A be a non-empty set and $n \in \mathbb{N}$. The set of finite sequences $s = (s(0), ..., s(n-1))$ of length n from A is denoted by A^n. Concerning the case $n = 0$, we let $A^0 = \{\emptyset\}$, where $\{\emptyset\}$ denotes the **empty sequence**. We now define by $A^{<\mathbb{N}}$ the collection of all finite sequences from A, meaning that

$$A^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} A^n.$$

The length of a finite sequence s is denoted by length(s), by which length($\emptyset$)=0. If $s \in A^n$, then for any $m \leq n$ the subsequence $s|\, m = (s(0), ..., s(m-1))$ ca be defined; hence $s|\, 0 = \emptyset$.

For $m \in \mathbb{N}$, a sequence s from A^m is an **initial segment** of a finite sequence t from A^n, with $n \geq m$, if it happens that $s = t|\, m$. In this case, t is called an **extension** of s, and we denote it by $s \subseteq t$. If such condition is verified, then s and t are **compatible**; otherwise, they are called **incompatible**. The latter case is denoted by $s \perp t$. With this definition, the empty sequence can be seen as an initial segment of any finite sequence from A, and it is thus compatible with any other sequence. Given two sequences s, t in $A^{<\mathbb{N}}$ of length n and m respectively, the **concatenation** of s and t is the the sequence $s * t = (s(0), ..., s(n-1), t(0)..., t(m-1)) \in A^{n+m} \subseteq A^{<\mathbb{N}}$.

Similarly, we define by $A^{\mathbb{N}}$ the set of all infinite sequences $x = (x_n)_{n\in\mathbb{N}}$ from A and we give analogous definitions. If $x \in A^{\mathbb{N}}$ and $n \in \mathbb{N}$, let $x|\, n = (x(0), ..., x(n-1)) \in A^n$. We say that $s \in A^n$ is an **initial segment** of $x \in A^{\mathbb{N}}$ if $s = x|\, n$ and we denote it by $s \subseteq x$. Also, for $s \in A^{<\mathbb{N}}$ and $x \in A^{\mathbb{N}}$ we let the **concatenation** of s, x be the infinite sequence $y = s * x$ where $y(i) = s(i)$ for each $i <$ length(s) and $y(\text{length}(s)+j) = x(j)$. More generally, the **concatenation** $s_0 * s_1 * s_2 * ...$ of $s_i \in A^{<\mathbb{N}}$ is the unique $x \in A^{\mathbb{N}} \cup A^{<\mathbb{N}}$ such that $x(i) = s_0(i)$ if $i <$ length(s_0); $x(\text{length}(s_0) + i) = s_1(i)$ if $i <$ length(s_1); and so on.

A **tree** on a set A is a subset $T \subseteq A^{<\mathbb{N}}$ closed under initial segments, meaning that if $t \in T$ and $s \subseteq t$ then $s \in T$. In particular, $\emptyset \in T$ if T is nonempty. We call the elements of T the **nodes** of T. An **infinite branch** of T is a sequence $x \in A^{\mathbb{N}}$ such that $x|\, n \in T$ for all n. The **body** of T is the set of all infinite branches of T, and we denote it by

$$[T] = \{x \in A^{\mathbb{N}} \mid x|\, n \in T, \forall n\}.$$

Finally, we call a tree T **pruned** if every $s \in T$ has a proper extension $t \supsetneq s, t \in T$.

It is well known that any set A can be viewed as a topological space within the **discrete topology**, that is the topology where every subset of A is open. It is also easy to notice that this can be metrisable with compatible metric $\delta(a,b) = 1$ if $a \neq b$. Therefore $A^{\mathbb{N}}$, viewed as the product space of infinitely many copies of A, is in turn metrisable with compatible metric d defined as follows:

$$d(x,x) = 0$$

for each $x \in X$, while

$$d(x,y) = \frac{1}{2^{n+1}}$$

if $x \neq y$ and n is the least number such that $x_n \neq y_n$.

The **standard basis** for the topology of $A^{\mathbb{N}}$ consists of the sets

$$N_s = \{x \in A^{\mathbb{N}} \mid s \subseteq x\}$$

where $s \in A^{<\mathbb{N}}$. Note that $s \subseteq t \iff N_s \supseteq N_t$, while $s \perp t \iff N_s \cap N_t = \emptyset$.

Proposition 2.1. *The map $T \longrightarrow [T]$ is a bijection between pruned trees on A and closed subsets of $A^{\mathbb{N}}$. Its inverse is given by*

$$F \longrightarrow T_F = \{x|n \mid x \in F, n \in \mathbb{N}\}.$$

We call T_F the tree of F.

Proof. The proof of this fact is quite evident by the fact that, given an arbitrary closed subset of $A^{\mathbb{N}}$ and restricting it to all $n \in \mathbb{N}$, we obtain finite sequences that can be seen as nodes to construct a correspondent pruned tree. Reversely, given a pruned tree T we know that any element of T, that is a subsequence, admits a proper extension. Then the process above is evidently reversible. $\square$

Notation: Let T be a tree on A. Then for any $s \in A^{<\mathbb{N}}$ we define

$$T_s = \{t \in A^{<\mathbb{N}} \mid s * t \in T\};$$

$$T_{[s]} = \{t \in T \mid s \subseteq t\}.$$

Notice that $[T_{[s]}] = [T] \cap N_s$ is a basis for the topology of [T]. Also, while $T_{[s]}$ clearly is a subtree of T, T_s in general is not. For instance, consider

$$T = \{(0,0,1,1),(0,0,2,2),(0,0,3,3)\}$$

and let $s = (0,0) \in A^2$. Then

$$T_s = \{(1,1),(2,2),(3,3)\},$$

which evidently is not even a tree since the condition of being closed under initial segments is not fulfilled.

Definition 2.2. *Let S, T be trees on sets A, B respectively. A map*

$$\phi : S \longrightarrow T$$

*is **monotone** if $s \subseteq t$ implies $\phi(s) \subseteq \phi(t)$. For such ϕ let*

$$D(\phi) = \{x \in [S] \mid lim_{n \longrightarrow \infty} \text{length}(\phi(x|n)) = \infty\}.$$

For $x \in D(\phi)$ define

$$\phi^*(x) = \bigcup_n \phi(x|n) \in [T].$$

*We call ϕ **proper** if $D(\phi) = [S]$.*

Definition 2.3. *Let F be a closed subset of a topological space X. Then F is a **retract** of X if there exists a continuous surjection $f : X \longrightarrow F$ such that $f|_F = id_F$.*

Proposition 2.4. *For each two closed nonempty subsets $F \subseteq H$ of $A^{\mathbb{N}}$, F is a retract of H.*

Proof. By 2.1, each closed subset of $A^{\mathbb{N}}$ is in bijection with a pruned tree; hence we let S, T be pruned trees on A such that $[S]{=}F$ and $[T]{=}H$. Notice that $S \subseteq T$. What we want do is to define a monotone proper $\phi : T \longrightarrow S$

such that $\phi| S = id_S$. With this, we will then extend ϕ to all $[T]$ through ϕ^* as defined in Proposition 2.2 ; this will imply that S is a retract of F.
We now define $\phi(t)$ by induction on length(t). First we let $\phi(\emptyset) = \emptyset$ and we let $\phi(t)$ be already defined. Then, for $a \in A$ with $t * a \in T$ we define $\phi(t * a)$ as follows: If $t * a \in S$, let $\phi(t * a) = t * a$. Otherwise, if $t * a \notin S$, then $\phi(t * a)$ can be any $\phi(t) * b \in S$, which exists since S is pruned. $\square$

Definition 2.5. *A **Lusin scheme** on a set X is a family $\{A_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$ of subsets of X such that*

(1) $A_{s*i} \cap A_{s*j} = \emptyset$, if $s \in \mathbb{N}^{<\mathbb{N}}$ with $i \neq j \in \mathbb{N}$;

(2) $A_{s*i} \subseteq A_s$, if $s \in \mathbb{N}^{<\mathbb{N}}$ with $i \in \mathbb{N}$.

Let (X,d) be a metric space and $\{A_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$ a Lusin scheme on X.
Then if

$$lim_{n \longrightarrow \infty} \text{diam}(A_{x|n}) = 0$$

*for all $x \in \mathcal{N}$ we say that $\{A_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$ has a **vanishing diameter**. In such case we let $D = \{x \in \mathcal{N} \mid \bigcap_n A_{x|n} \neq \emptyset\}$ and define the map from D to X as*

$$\{f(x)\} = \bigcap_n A_{x|n}.$$

*The map f is called the **associated map**.*

Proposition 2.6. *Let $\{A_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$ be a Lusin scheme with a vanishing diameter on a metric space (X,d). Then if $f : D \longrightarrow X$ is the associated map we have:*

(i) f is injective and continuous;

(ii) If (X,d) is complete and each A_s is closed, then D is closed;

(iii) If A_s is open, then f is an embedding.

Proof. (i) This part is straightforward since $\mathcal{N}$ has the standard topology and if $\bigcap_n A_{x|n} = \bigcap_n A_{y|n}$ then $x| n = y| n$ for all n, i.e. $x = y$.
(ii) Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in D such that $x_n \longrightarrow x$. Then:

1. Given $\epsilon > 0$ there exists N such that $\text{diam}(A_{x|N}) < \epsilon$, as $\{A_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ has a vanishing diameter by hypothesis;

2. there exists M such that $x_n|\,N = x|\,N$ for all $n \geq M$, because we set $x_n \longrightarrow x$.

Then it follows that $d(f(x_n), f(x_m)) < \epsilon$ for all $n, m \geq M$, meaning that the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ is Cauchy in (X, d). Hence, there exists $y \in X$ such that $f(x_n) \longrightarrow y$, because (X, d) is complete. By what stated previously, $f(x_n) \in A_{x|n}$ for all $n > M$, and, since each A_s is closed by hypothesis, it follows that also $y \in A_{x|n}$ for all n. Thus $y \in \bigcap_n A_{x|n}$, yielding that $x \in D$. So D is closed.
(iii) This follows from the fact that $f(N_s \cap D) = f(D) \cap A_S$. □

2.2 Topological characterisation of the Baire space

Let X be a topological space. If A is a subset of X, the **interior** of A is the union of all open sets that are contained in A or, in other words, the greater open set contained in A. The **border** of A, indicated with $\partial(A)$, is the set given by the closure of A minus the interior of A. Notice that ∂A is a closed set for it is the intersection of two closed sets, namely $\partial A = \overline{A} \cap \overline{X \setminus A}$. A subset of X is **nowhere dense** if the interior of its closure is empty. A subset of X is **meagre** if it is countable union of nowhere dense sets.
A subset A of X has the **Baire property** if there exists an open set U such that $U \triangle A$ is meagre. Here $U \triangle A$ represents the symmetric difference, namely the set $\{x \in X \mid x \in (U \setminus A) \quad or \quad x \in A \setminus U)\}$.
A **Borel set** of X is any set belonging to the σ-algebra generated by the open sets of X. We denote the class of Borel sets of X by $B(X)$. In other words, the Borel sets are generated from the open sets via the operations of complementation and countable union. This means that the class of Borel sets of a topological space X is the family of all open and closed sets of X.

Let X and Y be topological spaces. A function $f : X \longrightarrow Y$ is **Baire measurable** if $f^{-1}(U)$ has the Baire property for all open sets U in Y. Notice that a continuous function is obviously Baire-measurable.

Lemma 2.7. *If X is a topological space, then every Borel set in X has the Baire property.*

Proof. Let A be a Borel set in a topological space X. Then A is either open or closed. If A is open there is nothing to prove. If A is closed then we consider the open set given by its interior, call it U. Since $\overline{U} = A$, then the set $U \triangle A = A \backslash U = \overline{U} \backslash U$ is the boundary of U. Recall that the boundary of a set is closed and, according to the definition, its interior is empty. Therefore $B = U \triangle A = \partial U$ is nowhere dense, then in particular it is meagre. Hence A has the Baire property. □

Definition 2.8. *The **Baire space** $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ is the set of all mappings from $\mathbb{N}$ to $\mathbb{N}$.*

Theorem 2.9. *Let X be a Polish space. Then there is a closed set $F \subseteq \mathcal{N}$ and a continuous bijection $f : F \longrightarrow X$. In particular, if $X \neq \emptyset$, there is a continuous surjection $g : \mathcal{N} \longrightarrow X$ extending f.*

Proof. To prove the first assertion we begin by using the hypothesis of X being Polish to fix a complete metric $d \leq 1$ on X. Then, recalling Definition 2.5, we construct a Lusin scheme $\{F_s\}_{s\in\mathbb{N}^{\mathbb{N}}}$ on X such that:

(i) $F_\emptyset = X$;

(ii) F_s is an F_σ set;

(iii) $F_s = \bigcup_i F_{s*i} = \bigcup_i \overline{F_{s*i}}$;

(iv) $\operatorname{diam}(F_s) \leq 2^{-\operatorname{length}(s)}$.

We now consider the associate map $f : D \longrightarrow X$. By (iii) we obtain that $f(D) = X$, and according to Proposition 2.6 f is a continuous bijection of D onto X. Hence it is enough to show that D is closed.

If we let $\{x_n\}_{n\in\mathbb{N}}$ be a succession in D such that $x_n \longrightarrow x$, by a similar argument as in Proposition 2.6 it can be proven that $\{f(x_n)\}_{n\in\mathbb{N}}$ is Cauchy. This implies that $f_n \longrightarrow y \in X$, because X is Polish and hence it is complete. Notice that condition(iii) also implies that $y \in \bigcap_n \overline{F_{x|n}} = \bigcap_n F_{x|n}$, thus $x \in D$ and D is close.

Let's now get to the construction of a Lusin scheme with the above-mentioned properties. Let F_σ, $F \subseteq X$ and $\epsilon > 0$ be arbitrary we can write $F = \bigcup_{i\in\mathbb{N}} F_i$, where the F_i are pairwise disjoint F_σ sets of diameter $< \epsilon$ such that $\overline{F_i} \subseteq F$. For that let $F = \bigcup_{i\in\mathbb{N}} C_i$, where C_i is closed and $C_i \subseteq C_{i+1}$. Then $F = \bigcup_{i\in\mathbb{N}} (C_{i+1} \setminus C_i)$. Now write $C_{i+1} \setminus C_i = \bigcup_{j\in\mathbb{N}} E_j^{(i)}$, where $E_j^{(i)}$ are pairwise disjoint F_σ sets of diameter $< \epsilon$. Then $F = \bigcup_{i,j} E_j^{(i)}$ and $\overline{E_j^{(i)}} \subseteq \overline{C_{i+1} \setminus C_i} \subseteq C_{i+1} \subseteq F$.

For what concerns the last assertion, we notice that both F and $\mathcal{N}$ itself are closed subsets of $\mathcal{N}$ such that $F \subseteq \mathcal{N}$. Then according to 2.4 F is a retract of $\mathcal{N}$, meaning that there exists a continuous surjection $g : \mathcal{N} \longrightarrow F$. By the first assertion there exists a continuous bijection $f : F \longrightarrow X$, so $g \circ f : \mathcal{N} \longrightarrow X$ is indeed a continuous surjection that extends f. □

Theorem 2.10. *Let X be a Polish space and $A \subseteq X$ be Borel. Then there exists a closed set $F \subseteq \mathcal{N}$ and a continuous bijection $f : F \longrightarrow A$. In particular, if $A \neq \emptyset$, there is also a continuous surjection $g : \mathcal{N} \longrightarrow A$ extending f.*

Proof. Enlarge the topology τ of X to a Polish topology τ_A in which A is both open and closed. Notice that A is Polish as well because a closed subset of a Polish space is Polish too. So now we can apply Theorem 2.9 to A in order to say that there exists a closed set $F \in \mathcal{N}$ and a bijective function $f : F \longrightarrow A$ that is continuous for $\tau_A|\, A$. Since $\tau \subseteq \tau_A$, then $f : F \longrightarrow A$ is also continuous for the topology τ. Finally we note that $F \subseteq \mathcal{N}$ is an inclusion of closed subsets of $\mathcal{N}$, then we let the last assertion follow from 2.4. □

2.3 Lusin-Souslin Theorem

Generally speaking, the continuous image of a Borel set is not necessarily Borel. However, it is interesting to see that this happens under conditions of injectivity. This result goes under the name of *Lusin-Souslin Theorem* and it will be proven at the end of this section. Towards this objective we will have to go through some more fundational theory first.

Definition 2.11. *A measurable space $(X, \mathcal{S})$ is a* ***standard Borel space*** *if it is isomorphic to $(Y, B(Y))$ for some Polish space Y or, equivalently, if there exists a Polish topology τ on X with $\mathcal{S} = B(\tau)$.*

Definition 2.12. *Let X be a Polish space. A set $A \subseteq X$ is an* ***analytic set*** *if there exist a Polish space Y and a continuous function $f : Y \longrightarrow X$ such that $f(Y) = A$.*

Definition 2.13. *Let X be a standard Borel space. Then $A \subseteq X$ is* ***analytic*** *if there is a Polish space Y and a Borel isomorphism $f : X \longrightarrow Y$ such that $f(A)$ is analytic in Y.*

Definition 2.14. *Let P, Q be two subsets of a Polish space X. Then P and Q are* ***Borel-separable*** *if there exists a Borel set R separating P from Q, meaning that $P \subseteq R$ and $R \cap B = \emptyset$.*

Lemma 2.15. *If $P = \bigcup_m P_m$, $Q = \bigcup_n Q_n$ where P_m, Q_n are Borel-separable for each m, n, then P, Q are Borel-separable.*

Proof. Let $R_{m,n}$ be the Borel-set separating P_m from Q_n for each n, and m and define the set $R = \bigcup_m \bigcap_n R_{m,n}$. R is clearly a Borel-set, for it is generated by unions of intersections of Borel sets, and it separates P from Q since every $R_{m,n}$ does. $\square$

The following result is of fundamental importance.

Theorem 2.16 (The Lusin Separation Theorem)**.** *Let X be a standard Borel space and let $A, B \subseteq X$ be two disjoint analytic sets. Then A and B are Borel-separable.*

Proof. Assume, without loss of generality, that A and B are non-empty and let $f : \mathcal{N} \longrightarrow A$ and $g : \mathcal{N} \longrightarrow B$ be continuous surjections (which exist by Theorem2.10 since A, B are Borel sets). Let also $A_s = f(N_s)$ and $B_s = g(N_s)$, then $A_s = \bigcup_m A_{s*m}$ and $B_s = \bigcup_n B_{s*n}$. If A and B are not Borel-separable, then by iterated use of Lemma 2.15 we can recursively define $x(n), y(n) \in \mathbb{N}$ such that $A_{x|n}, B_{x|n}$ are not Borel-separable for each $n \in \mathbb{N}$. Then $f(x) \in A, g(y) \in B$, so $f(x) \neq g(y)$. Let U, V be disjoint open sets with $f(x) \in U, g(y) \in V$. By continuity of f and g, if n is large enough we have $f(N_{x|n}) \subseteq U, g(N_{y|n}) \subseteq V$. Hence U separates $A_{x|n}$ from $B_{y|n}$, which is a contradiction. Thus A, B must be Borel-separable, as desired. □

The following extension is immediate.

Corollary 2.17. *Let X be a standard Borel space and $\{A_n\}_{n\in\mathbb{N}}$ a pairwise disjoint sequence of analytic sets. Then there is a pairwise disjoint sequence of Borel sets $\{B_n\}_{n\in\mathbb{N}}$ such that $A_n \subseteq B_n$ for each n.*

Theorem 2.18 (Lusin-Souslin Theorem)**.** *Let X, Y be Polish spaces and let $f : X \longrightarrow Y$ be continuous. If $A \subseteq X$ is Borel and $f|_A$ is injective, then f(A) is Borel.*

Proof. In Theorem 2.10 we have seen that there exists a continuous bijection between Borel-sets of Polish spaces and the closed sets of $\mathcal{N}$. Hence we can assume $X = \mathcal{N}$ and that A is a closed subset of $\mathcal{N}$.
Let $B_s = f(A \cap N_s)$ for $s \in \mathbb{N}^{<\mathbb{N}}$. As $f|_A$ is injective, then $\{B_s\}$ is a Lusin scheme such that $B_\emptyset = f(A)$, $B_s = \bigcup_n B_{s*n}$ and B_s is analytic (since is the continuous image of a Polish space). Thus, by Corollary 2.17, there exists a Lusin scheme $\{B'_s\}$ where each B'_s is Borel and such that $B'_\emptyset = Y$ and $B_s \subseteq B'_s$. We will now define inductively on length(s) the Borel sets B^*_s so that each $\{B^*_s\}$ is also a Lusin scheme:

$$B^*_\emptyset = B'_\emptyset$$

$$B^*_{n_0} = B'_{n_0} \cap \overline{B_{n_0}}$$

$$B^*_{(n_0,\dots,n_k)} = B'_{(n_0,\dots,n_k)} \cap B^*_{(n_0,\dots,n_{k-1})} \cap \overline{B_{(n_0,\dots,n_k)}}$$

Then it is easy to prove by induction on k that $B_{(n_0,\dots,n_k)} \subseteq B^*_{(n_0,\dots,n_k)} \subseteq \overline{B_{(n_0,\dots,n_k)}}$. The objective is now to show that

$$f(A) = \bigcap_k \bigcup_{s \in \mathbb{N}^k} B^*_s \quad ,$$

which would obviously yield that $f(A)$ is Borel.
Consider $x \in f(A)$. Then there exists $a \in A$ such that $f(a) = x$: this means that $x \in \cap_k B_{a|k}$, and thus $x \in \cap_k B^*_{a|k}$.
Conversely, consider $x \in \cap_k \cup_{s \in \mathbb{N}^k} B^*_s$. Then there exists unique $a \in \mathcal{N}$ such that $x \in \cap_k B^*_{a|k}$; we will ultimately show that $a \in A$ and that $x = f(A)$. By how x is defined, it is clear that $x \in \cap_k \overline{B_{a|k}}$, meaning in particular that $B_{a|k} \neq \emptyset$ is nonempty for all k. This implies that $A \cap N_{a|k} \neq \emptyset$ for all k, meaning that A contains a succession that converges to a. Then $a \in A$ since A is closed. So $f(a) \in \cap_k B_{a|k}$. We claim that $f(a) = x$. In fact, suppose $f(a) \neq x$. Continuity of f implies that there is an open neighborhood $N_{a|k_0}$ of A with $f(N_{a|k}) \subseteq U$, where U is open such that $x \notin \overline{U}$. It follows that $x \notin \overline{f(N_{a|k_0})} \supseteq \overline{B_{a|k_0}}$, giving a contradiction. Hence it must be $f(a) = x$.
We have shown that the sets $f(A)$ and $\bigcap_k \bigcup_{s \in \mathbb{N}^k} B^*_s$ coincide, then $f(A)$ is Borel for it is generated by operations of intersection and union of Borel-sets. Thus we conclude by saying that **the continuous image of a Borel-set is still a Borel-set under hypothesis of injectivity**. $\square$

2.4 A Uniqueness Result for Polish Group Topologies

This last section is aimed at gathering previous knowledge in order to prove a quite interesting uniqueness fact on Polish group topologies: Any two compatible Polish group topologies on a group G such that one is finer than the other must coincide.

Theorem 2.19 (Petti's Theorem)**.** *Suppose that G is a Polish group and let A,B be subsets of G. If $U(A)$ and $U(B)$ are the largest open sets in G such that A is comeagre in $U(A)$ and B is comeagre in $U(B)$, then $U(A)U(B) \subseteq AB$.*

Proof. The proof is trivial if either $U(A)$ or $U(B)$ are empty, since $U(A)U(B) = \emptyset$ is clearly contained in AB. Suppose now that x is an arbitrary element of $U(A)U(B) = \{uv \mid u \in U(A), v \in U(B)\}$; then there exist $a \in U(A)$ and $b \in U(B)$ such that $x = ab$. It follows that $xb^{-1} = a \in U(A)$, hence $xb^{-1} \in xU(B)^{-1} \cap U(A)$. If we set V=$xU(B)^{-1} \cap U(A)$, then $V \neq \emptyset$. Since $V \subset U(A)$, then $V \setminus A \subset U(A) \setminus A$. Recall that A is comeagre in $U(A)$ by hypothesis, implying that its complement in $U(A)$, namely $U(A) \setminus A$, is a meagre subset of G. Hence, $V \setminus A$ must also be meagre in G. By a similar argument we have that $V \setminus xB^{-1} \subset xU(B)^{-1} \setminus xB^{-1}$. Now we rewrite $xU(B)^{-1} \setminus xB^{-1}$ as $x(U(B)^{-1} \setminus xB^{-1})$ and notice that $U(B)^{-1} = U(B^{-1})$, hence $V \setminus xB^{-1} \subset x(U(B^{-1}) \setminus B^{-1})$. Since B is comeagre in U(B), then $U(B) \setminus B$ is meagre in G, and by the fact that inversion and multiplication by x are homeomorphisms of G, it follows that $x(U(B^{-1}) \setminus B^{-1})$ is meagre too. Hence $V \setminus xB^{-1}$ is meagre in G. Thus, both A and xB^{-1} are comeagre in V, implying that also $A \cap xB^{-1}$ is comeagre in V. But V is nonempty, then also $A \cap xB^{-1}$ is nonempty and there must exist $a \in A$ and $b \in B$ such that $xb^{-1} = a$. This implies that $x = ab \in AB$. Hence we can conclude that if $x \in U(A)U(B)$ then $x \in AB$, which yields the desired subset inclusion. □

Lemma 2.20. *If G is a Polish group and $U \subseteq G$ is open, then G can be covered by countably many left translates of U.*

Proof. Suppose without loss of generality that U is any open neighborhood of e. If G is Polish, then in particular it is metrisable; thus by the Birkhoff-Kakutani theorem there exists a compatible left-invariant metric d on G. But G is also separable, hence there exists a countable dense subset D. Density hypothesis on D implies that the open balls

$$B_d(x, \epsilon) = \{g \in G \mid d(x, g) < \epsilon\}$$

for $x \in D$ and $\epsilon \in \mathbb{Q}$ form a basis for the topology on G. It follows that there exists $\epsilon \in \mathbb{Q}$ such that $B_d(e, \epsilon) \subseteq U$. Now, if x is an arbitrary element of G then $B_d(x, \epsilon) \cap D \neq \emptyset$ because D must meet every nonempty open set. This implies that $d(x, y) < \epsilon$ for some $y \in D$, that is, $x \in B_d(y, \epsilon)$. Bringing all together we obtain the following:

$$G = \bigcup_{y \in D} B_d(y, \epsilon) = \bigcup_{y \in D} yB_d(e, \epsilon) \subseteq \bigcup_{y \in D} yU.$$

Being D a countable set, we have covered G with countably many left translates of U. $\square$

Proposition 2.21. *If G and H are Polish groups, then any Baire-measurable homomorphism $\pi : G \longrightarrow H$ is continuous.*

Proof. It suffices to show that π is continuous at the neutral element of G, namely e_G, as G is a topological group. Therefore it suffices to prove that for every open neighborhood V of the neutral element of H, namely e_H, $\pi^{-1}(V)$ contains an open neighborhood of E_G. Suppose that V is an open neighborhood of $e_H \in H$. Let w be an arbitrary element of H. Then $ww^{-1} = e_H \in V$. Since multiplication is continuous in H, there exist open neighborhoods W_1 and W_2 of w and w^{-1} respectively, such that $gh \in V$ for all $g \in W_1$ and $h \in W_2$. Set $W = W_1 \cap W_2^{-1}$. Then W and W^{-1} are open neighborhoods of w and w^{-1} respectively; if $g \in W \subset W_1$ and $h \in W^{-1} \subset W_2$, then $gh \in V$. Hence $WW^{-1} \subseteq V$, and since $w \in H$ was arbitrary, we may set $w = e_H$ so that $e_H \in W \cap W^{-1}$.
Since W is open, it follows from the previous lemma that there exists a sequence $\{h_i W\}_{i \in \mathbb{N}}$ of left translates of W such that

$$H \subseteq \bigcup_{i \in \mathbb{N}} h_i W$$

Hence

$$G = \bigcup_{i \in \mathbb{N}} \pi^{-1}(h_i W) \subseteq \bigcup_{i \in \mathbb{N}} \pi^{-1}(hi)\pi^{-1}(W)$$

ans so π^{-1}(W) is non-meagre (since a countable union of meagre sets is meagre, but G is not meagre). On the other hand π is Baire-measurable by hypothesis, and so $\pi^{-1}(W)$ has the Baire property. This means that there exists an open set $E \subseteq G$ such that $\pi^{-1}(W) \triangle E$ is meagre. For such choice we have that

$$\pi^{-1}(W) \setminus (\pi^{-1}W \cap E) \subseteq (\pi^{-1}(W) \cup E) \setminus (\pi^{-1}(W) \cup E) = \pi^{-1}(W) \triangle E$$

Notice that if we suppose $\pi^{-1}(W) \cap E$ to be empty, then $\pi^{-1}W$ must be meagre as it is contained in a meagre set. If the intersection is nonempty then E is nonempty as well. Notice that $\pi^{-1}(W)$ is comeagre in E. Therefore, the largest open set $U(\pi^{-1}(W))$ in which $\pi^{-1}(W)$ is comeagre must also be non-empty. By a similar argument, the largest open set $U(\pi^{-1}(W^{-1}))$ in which $\pi^{-1}(W^{-1})$ is comeagre is non-empty too. Hence by Theorem 2.19 we obtain that

$$e_G \in U(\pi^{-1}(W))U(\pi^{-1}(W)^{-1}) \subseteq \pi^{-1}(W)\pi^{-1}(W^{-1}) = \pi^{-1}(WW^{-1}) \subseteq \pi^{-1}(V)$$

But

$$U(\pi^{-1}(W))U(\pi^{-1}(W)^{-1}) = \bigcup_{x \in U(\pi^{-1}(W))} xU(\pi^{-1}(W)^{-1})$$

We have that $U(\pi^{-1}(W))U(\pi^{-1}(W)^{-1})$ is a union of open sets containing e_G (i.e. it is an open neighborhood of e_G) and it is contained in $\pi^{-1}(V)$, hence we can conclude that π is continuous at e_G. $\square$

Corollary 2.22. *If τ and τ' are Polish topologies on a set X such that $\tau \subseteq \tau'$, then every open set in τ' is a Borel set in τ.*

Proof. By definition, since $\tau \subseteq \tau'$, then the identity map $id_X : (X, \tau') \longrightarrow (X, \tau)$ is continuous. Notice that every open set in τ' is Borel in τ', and that the identity map is injective. So we apply Lusin-Souslin Theorem to the identity map, obtaining that for each open set U in (X, τ'), $id_X(U) = U$ is Borel in (X, τ). $\square$

We are now in possession of all the tools needed to prove the following fundamental proposition.

Proposition 2.23. *If τ and τ' are Polish group topologies on a group G such that $\tau \subseteq \tau'$, then $\tau = \tau'$.*

Proof. If τ and τ' are Polish group topologies on a group G and $\tau \subseteq \tau'$, then by Corollary 2.22 it follows that every open set in τ' is a Borel set in τ. By Lemma 2.7, every Borel set in τ' has the Baire Property. Hence if we consider the identity map

$$id_G : (G, \tau) \longrightarrow (G, \tau')$$

its inverse is continuous, for τ' is finer than τ. Then by Theorem 2.18 $id_G^{-1}(U) = U \in \tau$ is Borel in (G, τ) for each open set $U \in (G, \tau')$ and by Corollary 2.22 it has the Baire property. Hence, each open set U in (G, τ') has the Baire property in (G, τ), that is, id_G is Baire measurable. Then it follows from Proposition 2.21 that $id_G : (G, \tau) \longrightarrow (G, \tau')$ is continuous, meaning that each open set $U \in \tau'$ is also open in τ. So the two topologies coincide. $\square$

The meaningful result of this chapter is a uniqueness result and it lies in the latter proposition: **if two Polish group topologies τ and τ' on a group G are such that $\tau \subseteq \tau'$ then $\tau = \tau'$.**

Chapter 3

Infinite Permutation Groups

Throughout this chapter we assume M to be an infinite set, we let $S(M)$ denote the group of all permutations of M and we assume τ to be a topology that is compatible with $S(M)$, so that $(S(M), \circ, \tau)$ is a topological group. Primarily we introduce some remarkable aspects of infinite permutation groups that will be useful in following proofs. Consequently, we prove some meaningful theorems on $S(M)$ with the above-mentioned topology. One of the main facts is that, with the said assumptions, the subgroup that leaves a generic element x fixed is open if and only if it is closed. This, combined with further results, will play a key role in Gaughan's Theorem: If $S(M)$ is a Hausdorff topological group then all stabilisers E_x are open. This will have noteworthy implications on the group of permutations on the natural numbers. All contents in Sections 3.1 and 3.2 are mostly based on [4] and [2]. Section 3.3 refers to [5], while Section 3.4 is based on [3].

3.1 Infinite Permutation Groups

Let M be an infinite set and let $S(M)$ be the group of all permutations of M.

Let n be any natural number. We indicate by S_n the symmetric group on a set of cardinality n, i.e. the group of all permutations of n elements.

The alternating group on n elements will be denoted by A_n, meaning the subgroup of S_n consisting of all permutations that are the composition of an even number of transpositions. Notice that S_n and A_n are subgroups of $S(M)$.

For each $x \in M$ we define $\mathbf{E_x} = \{\sigma \in S(M) \mid \sigma(x) = x\}$. Similarly, for $F \subseteq M$, we define $\mathbf{E_F} = \{\sigma \in S(M) \mid \sigma(x) = x \text{ for each } x \in F\}$. The subgroups of this form take the name of **stabilisers**. Notice that E_F is a subgroup of $S(M)$ for each $F \subseteq M$. In particular, E_x is the subgroup of all permutations that leave the element x fixed.

For each doubleton $\{x, y\}$ we define the subgroup $\widetilde{\mathbf{E}}_{\mathbf{x,y}} = \{\sigma \in S(M) \mid \sigma(\{x, y\}) = \{x, y\}\} = E_{x,y} < (xy) >$. Notice that this is the subgroup of all permutations of $S(M)$ that leave $\{x, y\}$ setwise invariant.

For each $x \in M$ and $A \subseteq S(M)$ we define $\mathbf{A}(\mathbf{x}) = \{\sigma(x) \mid \sigma \in A\}$.

If $\sigma \in S(M)$ then we define $\mathbf{spt}\sigma = \{x \in M \mid \sigma(x) \neq x\}$.

Finally, we say that a subgroup G of $S(M)$ is **n-transitive** on M if, given any pair of sequences $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$ in M^n, there exists a permutation $g \in G$ that sends x_i to y_i for each $i = 1, ..., n$.

Notation: We will indicate by $\mathrm{Alt}(M)$ the subgroup of $S(M)$ generated by all 3-cycles.

Notice that, as A_n is n-transitive, the subgroup $\mathrm{Alt}(M)$ is n-transitive for all $n \in \mathbb{N}$.

Proposition 3.1. *If H is a normal subgroup of $S(M)$ containing a 3-cycle, then H contains* $\mathrm{Alt}(M)$*.*

Proof. For any arbitrary 3-cycle (abc) of $S(M)$ there exists a permutation σ such that $\sigma^{-1} \circ (123) \circ \sigma = (abc)$; sure enough it is sufficient to define σ such that $\sigma(a) = 1, \sigma(b) = 2, \sigma(c) = 3$, so that any element that is not a, b, c is fixed. A direct implication of this is that *if a normal subgroup H of $S(M)$ contains a 3-cycle, then it contains all 3-cycles of $S(M)$*. Then H contains $\mathrm{Alt}(M)$ and it is therefore n-transitive for all $n \in \mathbb{N}$. $\square$

Lemma 3.2. *If H is a normal subgroup of $S(M)$ different from $\{e\}$, then H is n-transitive.*

Proof. First we observe a basic fact: Since a normal subgroup is conjugate-closed, if it contains a 2-cycle (ab) then it contains all other 2-cycles too. It is well known that the set of transpositions generate all finite permutations, then in this case H contains all of $\mathrm{Alt}(M)$ and hence it is n-transitive for all n.
Now suppose that H does not contain a 2-cycle: then H will contain a permutation of $S(M)$ that is not a transposition. Our objective is to show that consequently H contains a 3-cycle, for this implies H containing all 3-cycles and hence all of $\mathrm{Alt}(M)$. Without loss of generality we let $\sigma = (1234...) \in H$. Since H is normal, the permutation

$$\tau = (123)^{-1} \circ \sigma \circ (123) = (321) \circ \sigma \circ (123)$$

belongs to H. If we now consider $\sigma^{-1} \circ \tau \in H$ it is an easy check to see that it is the 3-cycle $(23\sigma^{-1}(1))$, for any number that differs from 1,2,3 only undergoes the action of σ first and σ^{-1} after. So H eventually contains a 3-cycle, then by Proposition 3.1 it contains all of $\mathrm{Alt}(M)$ and it is hence n-transitive. $\square$

Proposition 3.3. *E_x is a maximal subgroup of $S(M)$.*

Proof. Let $\gamma \in S(M)\ E_x$ and $y = \gamma(x)$ with $y \neq x$. We will show that the subgroup generated by E_x and γ, denoted by $< E_x, \gamma >$, coincides with all $S(M)$. Let $\alpha \in S(M)$ be an arbitrary permutation; we want to show that $\alpha \in < E_x, \gamma >$. If $\alpha \in E_x$ then clearly the thesis is satisfied.
Assume $\alpha(x) = z$ with $z \neq x$. We have two cases:
1) If $z = y$, then $(\gamma^{-1} \circ \alpha)(x) = \gamma^{-1}(z) = \gamma^{-1}(y) = x$. Let $\eta = \gamma^{-1} \circ \alpha \in E_x$: If we rewrite $\alpha = \gamma \circ \eta$, then clearly α belongs to $< E_x, \gamma >$.
2) If $z \neq y$, let ψ be the transposition (y, z) so that $\psi \in E_x$. Then $(\gamma^{-1} \circ \psi \circ \alpha)(x) = (\gamma^{-1} \circ \psi)(y) = \gamma^{-1}(z) = x$. Let $\eta = \gamma^{-1} \circ \psi \circ \alpha \in E_x$: If we rewrite $\alpha = \psi^{-1} \circ \gamma \circ \eta$ then, also here, α belongs to $< E_x, \gamma >$. $\square$

Observation 3.4. *By a similar proof we can show that* $E_{x_1,\dots,x_n}$ *is a maximal subgroup of* $E_{x_1,\dots,x_{n-1}}$ *for any set* $\{x_1, \dots x_n\} \subseteq M$ *and any* $n \in \mathbb{N}$*.*

Lemma 3.5. *Suppose that* $A \subseteq S(M)$*,* $e \in A$*, and* $|M \setminus A(x)| = |M|$*. Then there exists* $\sigma \in S(M)$ *such that* $[\sigma A \sigma^{-1} \cap A](x) = \{x\}$*.*

Proof. First of all we shall note two basic facts: for each $x \in M$, $\sigma \in S(M)$, and $A \subseteq M$ we have that $[\sigma A \sigma^{-1}](x) = \sigma[A(\sigma^{-1}(x))]$ and also that $[\sigma A \sigma^{-1} \cap A](x) \subseteq [\sigma A \sigma^{-1}](x) \cap A(X)$.
Since $|M \setminus A(x)| = |M|$, it is possible to define a permutation $\sigma \in S(M)$ such that $\sigma(x) = x$ and $\sigma(A(x) \setminus \{x\}) \subseteq M \setminus A(x)$. With the listed properties it is verified that

$$x \in [\sigma A \sigma^{-1} \cap A](x) \subseteq [\sigma A \sigma^{-1}](x) \cap A(X) = \{x\},$$

hence $[\sigma A \sigma^{-1} \cap A](x) = \{x\}$. □

3.2 Gaughan's Theorem

From, now on we assume τ to be any compatible topology with the group of permutations of an infinite set M, meaning that $(S(M), \circ, \tau)$ is a topological group. Where possible, we will take the liberty to write $S(M)$ instead of $(S(M), \circ, \tau)$ for simplicity reasons.

Theorem 3.6. *Let* M *be an infinite set and let* $S(M)$ *be a topological group. For each* $x \in M$*,* E_x *is closed if and only if* E_x *is open.*

Proof. We have seen in Theorem 1.13 that an open subgroup of a topological group is also closed. Therefore we just have to prove that if E_x is closed then it is also open. If we suppose E_x to be closed then the set $\{\sigma \in S(M) \mid \sigma(x) \neq x\}$ must be open. Hence, by homogeneity of topological groups, for each $y \in M \setminus \{x\}$ the set $V = \{\sigma \in S(M) \mid \sigma(x) \neq y\}$ must be open as well. Since it also contains the identical permutation e, we have that V is a neighborhood of e. Topological group properties linked in 1.8 suggest that there exists a neighborhood W of e such that $W^{-1}W \subseteq V$. In fact, by continuity of the map $f : (x, y) \longrightarrow xy^{-1}$, V being a neighborhood of e implies that $f^{-1}(V)$ contains a neighborhood of (e, e): Let it be (W_1, W_2). If we define $W = W_1 \cap W_2$, then W is a neighborhood of e such that $(W, W) \subseteq (W_1, W_2) \subseteq f^{-1}(V)$. Hence $WW^{-1} = f(W, W) \subseteq f(W_1, W_2) \subseteq V$. Since $W(x)$ and $W(y)$ are disjoint, it must happen that at least one of the following is true:

$$|M \setminus W(x)| = |M|\,;$$

$$|M \setminus W(y)| = |M|\,.$$

Since $e \in W$, it follows from 3.5 that it exists a $\sigma \in S(M)$ and a $u \in M$ such that $[\sigma W \sigma^{-1} \cap W](u)$=$\{u\}$ (where u is either x or y as $M \setminus W(x)$ or $M \setminus W(y)$ has the same cardinality of M).

Hence the subgroup E_u contains $\sigma W \sigma^{-1} \cap W$ which clearly is a neighborhood of e by properties listed in 1.8. By Proposition 1.14 it follows that E_u must be open since it is a subgroup containing a neighborhood of e. We conclude by the fact that all subgroups of permutations that fix exactly one element are conjugate, hence for each $x \in M$ E_x is open if and only if it is closed. $\square$

Observation 3.7. *Notice that the previous theorem can be applied to* $E_x = S(M \setminus \{x\})$ *and any of its subgroups of the form* $E_{x,y}$ *with* $y \in M \setminus \{x\}$ *for these are the stabilisers of* y *in* E_x.

Lemma 3.8. *If* M *is an infinite set and* $S(M)$ *a topological group, then for any doubleton* $\{x, y\} \subset M$ *we have that:*

(a) The subgroup $\widetilde{E}_{x,y}$ *of* $S(M)$ *is maximal;*

(b) Every proper subgroup of $S(M)$ *properly containing* $E_{x,y}$ *coincides with one of the subgroups* E_x, E_y *or* $\widetilde{E}_{x,y}$.

Proof. **(a)** Suppose that H is a subgroup of $S(M)$ properly containing $\widetilde{E}_{x,y}$. Our goal now is to show that both the subgroups E_x and E_y are contained in H, so that by Proposition 3.3 we obtain that $H = S(M)$. Therefore we begin by fixing $\sigma \in E_x$. If $\sigma(y) = y$ then $\sigma \in E_{x,y} \subseteq \widetilde{E}_{x,y} \subseteq H$. Hence we suppose that $\sigma(y) = z \notin \{x, y\}$. As $\widetilde{E}_{x,y}$ is a proper subgroup of H there exists $\overline{\psi} \in H \backslash \widetilde{E}_{x,y}$, then by the fact that $(xy) \in \widetilde{E}_{x,y} \subseteq H$ we have that either $\overline{\psi}(x) \notin \{x, y\}$ or $\overline{\psi}(y) \notin \{x, y\}$. Without loss of generality we assume $\overline{\psi}(y) = w \notin \{x, y\}$. In case $w = z$ we put $\psi = \overline{\psi}$, while if $w \neq z$ we let $\psi = (zw)\overline{\psi}$, so that in any case $\psi \in H$. Now we take $t \in M \setminus \{x, y, \psi^{-1}(x), \psi^{-1}(y)\}$ and let $v = \psi(t) \in M \setminus \{x, y, z\}$. It follows that the cycles (zt) and (zv) both belong to $E_{x,y} \subseteq H$, implying that $(yt) = \psi^{-1}(zv)\psi \in H$. Now if we define $\overline{\overline{\psi}} = (yt)(zt)\sigma$ it is an easy check to see that it belongs to $E_{x,y}$ and hence $\overline{\overline{\psi}} \in H$. This clearly gives that $\sigma \in H$, and as σ varies in E_x we obtain that $E_x \subseteq H$ as wanted.

(b) Suppose that H is a proper subgroup of $S(M)$ properly containing $E_{x,y}$ and that $H \neq E_x$ and $H \neq E_y$. Since our objective is to prove that $H = \widetilde{E}_{x,y}$ and H already contains $E_{x,y}$, then it suffices to show that the cycle (xy) also belongs to H. Notice that $E_{x,y}$ is contained in both H and E_x, hence it belongs to $H \cap E_x$, which clearly is a subgroup of both. According to Observation 3.4, $E_{x,y}$ is a maximal subgroup of E_x and since it is contained in $H \cap E_x$ and we assumed $H \neq E_x$ then it must be $E_{x,y} = H \cap E_x$.
By a similar argument it must be $H \cap E_y = E_{x,y}$. Therefore, if we choose $\rho \in H \setminus E_{x,y}$ then it must be $\rho \notin E_x$ and $\rho \notin E_y$. Hence if we let $z = \rho(x)$ and $t = \rho(y)$ then $z \neq x$ and $t \neq y$. Now we consider the following cases:

Case 1: $\{z, t\} = \{x, y\}$. Then this happens only if $z = y$ and $t = x$, meaning that $(xy)\rho \in E_{x,y} \subseteq H$.

Case 2: $\{z, t\} \cap \{x, y\} = \emptyset$. Then clearly $(zt) \in E_{x,y} \subseteq H$ and $(xy) = \rho^{-1}(zt)\rho \in H$.

Case 3: $\{z, t\} \cap \{x, y\} = z = y$. Then obviously $x \neq t$. Notice that $(tyx)\sigma \in E_{x,y} \subseteq H$ implies $(xyt) = (tyx)^{-1} \in H$. Now take $v \in M \setminus \{x, y, z\}$; then we have that $(tv) \in E_{x,y}$, so $\phi = (xt)(yv) = (xyt)(tv)(xyt)(tv) \in H$. Since $\phi(x) = t \notin \{x, y\}$ and $\phi(y) \notin \{x, y\}$ then we can apply here the same argument as in Case 2 with ϕ in order to obtain $(xy) \in H$.

$\square$

Proposition 3.9. *Let M be an infinite set and let $S(M)$ be a topological group. If $S(M)$ is Hausdorff then there exists a neighborhood of e that is not 2-transitive.*

Proof. Suppose by contradiction that all neighborhoods of e are 2-transitive, then choose distinct $u, v, w \in M$. Our aim is to show that the 3-cycle (uvw) belongs to any arbitrary neighborhood V of e, so that the condition (3) of Theorem 1.10 is not verified and this contradicts Hausdorff hypothesis on $S(M)$. By 1.8 we pick any symmetric neighborhood W of e such that $WW \subseteq V$. If we let $\sigma = (uv)$ then $U = \sigma W \sigma \cap W$ is a symmetric neighborhood of e and $\sigma U \sigma = U$. As U is 2-transitive there exists $\psi \in U$ such that $\psi(u) = u$ and $\psi(v) = w$. Therefore $(uvw) = (uw)(uv) = \psi\sigma\psi^{-1}\sigma \in W(\sigma U \sigma) \subseteq WW \subseteq V$. $\square$

Lemma 3.10. *Let M be an infinite set and $S(M)$ a topological group. Let n be a positive integer and let F be a finite subset of M with n elements. Then E_F is dense in S(M) if and only if each neighborhood of e is n-transitive.*

Proof. Recall that if $B \subseteq S(M)$ then $\bar{B} = \bigcap\{AB \mid A$ is a neighborhood of $e\}$ by Proposition 1.9. Therefore E_F is closed in $S(M)$ if and only if $AE_F = S(M)$ for each neighborhood A of e. But $AE_F = S(M)$ means that, for each $C \subseteq M$ of cardinality n, A must contain all permutations σ such that $\sigma F = C$. Hence each neighborhood A of e must be transitive on F. But if F_1 and F_2 are subsets of M with the same cardinality, then E_{F_1} and E_{F_2} are conjugate subgroups, hence E_{F_1} is dense in $S(M)$ if and only if E_{F_2} is dense in $S(M)$. It follows that E_F is dense in $S(M)$ if and only if each neighborhood of e is n-transitive. $\square$

Notice that Proposition 3.9 together with Lemma 3.10 brings that, if the stabilisers $E_{x,y}$ are dense then every neighborhood of e is 2-transitive.

Corollary 3.11. *Let M be an infinite set and $S(M)$ be a Hausdorff topological group. Then for every pair of distinct elements $x, y \in M$ the stabiliser $E_{x,y}$ is not dense in $S(M)$.*

Proof. As $S(M)$ is Hausdorff, by Proposition 3.9 and Lemma 3.10 there exist distinct $x', y' \in X$ such that $E_{x,y}$ is not dense. Then, from the fact that all stabilisers $E_{x,y}$ for distinct $x, y \in X$ are conjugated, we extend this property to all stabilisers $E_{x,y}$. □

Observation 3.12. *Recalling that E_x is a maximal subgroup of $S(M)$ and that the closure of a subgroup is still a subgroup, it follows that if the closure of E_x is not E_x itself then it must be all $S(M)$. In other words, E_x is either closed or dense in $S(M)$ for each $x \in M$.*

Theorem 3.13 (Gaughan's Theorem)**.** *Let M be an infinite set and let $(S(M), \tau)$ be a topological group. If $S(M)$ is Hausdorff then E_x is open for all $x \in M$*

Proof. Let $S(M)$ be a Hausdorff topological group and suppose, by contradiction, that E_x is not open. Then, as E_x is not open we obtain by Theorem 3.6 that E_x is not closed: hence by Observation 3.12 E_x must be dense in $S(M)$. By assumption $S(M)$ is Hausdorff, then by Corollary 3.11 we have that all stabilisers $E_{x,y}$ are not dense in $S(M)$. Now fix a pair of distinct elements $x, y \in M$ and let $G_{x,y}$ be the closure of $E_{x,y}$: as a matter of fact $G_{x,y}$ is a proper subgroup of $S(M)$ containing $E_{x,y}$, as the closure of a subgroup is itself a subgroup by Theorem 1.11. As we have E_x dense in $S(M)$ and $G_{x,y}$ closed, $G_{x,y}$ does not contain E_x, hence $E_x \cap G_{x,y}$ is a proper subgroup of E_x that contains $E_{x,y}$. Recall that, by Observation 3.4, $E_{x,y}$ is a maximal subgroup of E_x: then it must follow that $E_{x,y} = E_x \cap G_{x,y}$. This shows that $E_{x,y}$ is indeed a closed subgroup of E_x, as it is the intersection of E_x with a closed subgroup of $S(M)$. Then, according to Observation 3.7, $E_{x,y}$ is also open in E_x. Since E_x is dense in $S(M)$ we have that the closure $G_{x,y}$ of the subgroup $E_{x,y}$ is open in $S(M)$, by Theorem 1.17. The fact that E_x is a proper dense subgroup of $S(M)$ implies that E_x cannot contain $G_{x,y}$, and by a similar argument neither E_y can. Hence $G_{x,y} \neq E_{x,y}$, meaning that $G_{x,y}$ is a proper subgroup of $S(M)$ containing $E_{x,y}$ and such that $S_x \neq G_{x,y} \neq E_y$, so we can apply Lemma 3.8 (b) and obtain that $G_{x,y} = \widetilde{E}_{x,y}$. Then $\widetilde{E}_{x,y}$

is open in $S(M)$ and, as all subgroups of the form $\widetilde{E}_{x,y}$ are conjugated, it follows that all subgroups $\widetilde{E}_{x,y}$ are open in $S(M)$. Now we take all stabilisers of finite sets F with $|F| > 2$ and observe that $E_F = \{\widetilde{S}_{x,y} \mid x, y \in F, x \neq y\}$. Hence the subset E_F is open in $S(M)$ and so it is an open neighborhood of e. For all $x \in F$ we have that $E_F \subseteq E_x$, then by Proposition 1.14 E_x is open in $S(M)$. Since all stabilisers of singletons are conjugate, then all E_x are open, a contradiction. □

3.3 The Standard Topology on S_∞

We have defined the **Baire Space** $\mathcal{N}$ as the set of infinitely many copies of $\mathbb{N}$ (see Definition 2.8). We are interested in a specific subset of $\mathcal{N}$, namely the group of permutations on the naturals: we denote it by S_∞.
Our interest is to define a group topology on S_∞. Hence, following the pattern introduced in Section 2.1 for generic $A^{\mathbb{N}}$, we will define a topology on $\mathcal{N}$ and then let the subgroup S_∞ inherit such topology from $\mathcal{N}$.

We begin with considering $\mathbb{N}$ as a topological space with the discrete topology. Then we see the Baire space as the infinite cartesian product of $\mathbb{N}$, and endow it with the product topology. This is what we call the **standard topology** on $\mathcal{N}$. We want to show that this topology has a countable basis.
We see $\mathcal{N}$ as the set of all functions from $\mathbb{N}$ to $\mathbb{N}$. Then a basis for the standard topology on $\mathcal{N}$ is given by the collection $\mathcal{U}$ of all sets of the form

$$U = \{f \in \mathcal{N} \mid f(x_k) = y_k, \quad k = 1, ..., n\}$$

where $x_k < x_{k+1}$ for all $k = 1, ..., n-1$ and $x_k, y_k \in \mathbb{N}$ for all $k = 1, ..., n$. Consider the map ϕ from $\mathcal{U}$ to $\bigcup_{n \in \mathbb{N}} \mathbb{N}^{2n}$ defined by

$$\phi(U) = (x_1, ..., x_n, y_1, ..., y_n).$$

It is easy to see that ϕ is injective, thus

$$|\mathcal{U}| \leq \left| \bigcup_{n \in \mathbb{N}} \mathbb{N}^{2n} \right| \leq \left| \bigcup_{n \in \mathbb{N}} \mathbb{N}^{n} \right| = \aleph_0$$

(for the latter equivalence see [7, Theorem 8.12]). This shows that $\mathcal{N}$ is second countable.

Now we consider S_∞ as the subset of $\mathcal{N}$ consisting of all sequences σ such that $\sigma(i) \neq \sigma(j)$ if $i \neq j$ for all $i, j \in \mathbb{N}$. As a subset of $\mathcal{N}$ we let S_∞ inherit the standard topology from $\mathcal{N}$ and, coherently, name it the **standard topology** on S_∞. Note that, since $\mathcal{N}$ is second countable, S_∞ has the same property. For $\sigma \in S_\infty$, we take a basis for the open neighborhoods of σ consisting of the sets

$$X_{n_1,..,n_k}(\sigma) = \{\tau \in S_\infty \mid \sigma(n_i) = \tau(n_i), i = 1, ...k\}$$

With this definition, it is easy to see that S_∞ is a topological group, since multiplication and inversion are continuous. Hence, the topology can be more simply defined by giving a basis for the open neighborhoods of the neutral element. Since the neutral element in S_∞ is the identical permutation, then the above-mentioned basis consists of all pointwise stabilisers of all finite sets. We denote such stabilisers by

$$E_{n_1,...,n_k} = \{\sigma \in S_\infty \mid \sigma(n_i) = n_i, i = 1, ..., k \quad for \quad k \in \mathbb{N}\}.$$

It follows that the collection of sets

$$E_x = \{\sigma \in S_\infty \mid \sigma(x) = x\}$$

forms a subbasis for the open neighborhoods of the identity as x varies in $\mathbb{N}$. Now that we have given S_∞ a compatible group topology, we will construct a compatible metric d. Also here, we first define a compatible metric in $\mathcal{N}$ and then let S_∞ inherit it. The intuition is that permutations that agree on may points are close. Recall from Section 2.1 that $\mathcal{N}$ can be seen as a metric space by defining compatible metric

$$\delta(f, f) = 0, \forall f \in \mathcal{N}$$

and

$$\delta(f, g) = \frac{1}{2^{n+1}}, \forall f \neq g \in \mathcal{N}$$

where n is the least number such that $f_n \neq g_n$. Being a subspace of $\mathcal{N}$, also (S_∞, δ) is a metric space.

However, S_∞ is not complete in this metric. For instance, let σ_n be the cycle $(0, 1, ..., n)$ for all n; then, for all $m \geq n$, we have that $\delta(\sigma_n, \sigma_m) = \frac{1}{2^n}$. Thus the sequence $\{\sigma_n\}_n \in \mathbb{N}$ is Cauchy, but it does not converge since its pointwise limit is not a permutation: it is the cyclic shift $n \longrightarrow n+1$. We will hence see that a better metric d can be defined on S_∞ such that (S_∞, d) is also **complete**. For each $\sigma, \tau \in S_\infty$ set $d(\sigma, \tau) = \frac{1}{2^n}$ if $\sigma(i) = \tau(i)$ and $\sigma^{-1}(i) = \tau^{-1}(i)$ for $i < n$ but either $\sigma(n) \neq \tau(n)$ or $\sigma^{-1}(n) \neq \tau^{-1}(n)$. In other words,

$$d(\sigma, \tau) = \max\{\delta(\sigma, \tau), \delta(\sigma^{-1}, \tau^{-1})\}.$$

Now (S_∞, d) is a complete metric space. We briefly see this. Let $\{\sigma_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in (S_∞, d). Hence for each $N > 0$ there exists $\overline{N}$ such that $\max\{\delta(\sigma_n, \sigma_m), \delta(\sigma_n^{-1}, \sigma_m^{-1})\} < \frac{1}{2^N}$ for each $n, m > \overline{N}$.

In other words, for each N there exists $\overline{N}$ such that $\delta(\sigma_n, \sigma_m) < \frac{1}{2^N}$ and $\delta(\sigma_n^{-1}, \sigma_m^{-1}) < \frac{1}{2^N}$ for each $n, m > \overline{N}$. By definition of the metric δ, this is equivalent to saying that for all $n, m > \overline{N}$ $\sigma_n(k) = \sigma_m(k)$ and $\sigma_n^{-1}(k) = \sigma_m^{-1}(k)$ for all $k < N$. Then if we fix $k \in \mathbb{N}$ there exists n_k such that $\sigma_n(k) = \sigma_m(k)$ and $\sigma_n^{-1}(k) = \sigma_m^{-1}(k)$ for all $n, m > N_k$. In other words, for fixed k the sequence $\sigma_n(k)$ is the constant sequence from some n_k onward. Then for each $k \in \mathbb{N}$ we define σ as follows:

$$\sigma(k) = \sigma_{n_k+1}(k);$$

$$\sigma^{-1}(k) = \sigma_{n_k+1}^{-1}(k).$$

So obviously $\lim_{n\to\infty} \sigma_n = \sigma$. For each disjoint pair $h, k \in \mathbb{N}$ we have that $\sigma(k) \neq \sigma(h)$ and for each fixed (k) there exists $\sigma^{-1}(k)$. Therefore $\sigma \in S_\infty$ and (S_∞, d) is complete.

It is easy to see that the above-defined metric d is compatible with the standard topology on S_∞. Moreover, it is also a separable space for it is second countable. In fact, as S_∞ is second countable we can find a countable

basis $\{B_n\}_{n\in\mathbb{N}}$ for the open sets. Note that each B_n is nonempty, hence we choose an element x_n for each $B_n \in \mathcal{B}$ and define the set $D = \bigcup_{x_n \in B_n} x_n$. Let now U be any open set. As $\mathcal{B}$ is a basis then U contains B_k for some $k \in \mathbb{N}$, then clearly it must also contain x_k. As U varies in all open sets we obtain that D meets each nonempty subset of S_∞. Therefore we have found that D is a countable dense subset, and we have that S_∞ is separable.
Thus, S_∞ with the standard topology and the metric d previously defined is a Polish group.

3.4 Proof of uniqueness

Consider the group of all permutations on the natural numbers $\mathbb{N}$ and take any Hausdorff compatible topology τ on S_∞. According to Gaughan's Theorem, all subgroups that fix one element $x \in \mathbb{N}$ are both open and closed in such a topology. On the other hand we have seen that the set $\mathcal{S} = \{E_x \mid x \in \mathbb{N}\}$ forms a subbasis for the open neighborhoods of the identity $\mathcal{V}$ in the standard topology. This means that the class of all finite intersections of elements of $\mathcal{S}$, namely

$$\mathcal{B} = \{\bigcap_{j=1}^{n} E_{x_j}, \quad n \in \mathbb{N}, \quad x_j \in \mathbb{N} \quad \text{for} \quad \text{each} \quad \text{j}\}$$

is a basis for the open neighborhoods of e in the standard topology. Then what happens is that all elements of $\mathcal{B}$ are open in any Hausdorff compatible topology on S_∞, implying that *all open neighborhoods of e in the standard topology are open neighborhoods of e also in* (S_∞, τ). We recall one more time the fact that each open set of a topological group can be written as left-translate of an element of $\mathcal{V}$. As translation preserves openness we can say that *if a subset of S_∞ is open in the standard topology then it is open in any arbitrary Hausdorff compatible topology* and conclude that **any Hausdorff topology on S_∞ contains the standard topology**.

Let τ be any Polish compatible group topology on the group of permutations on the naturals, S_∞. If τ is Polish then the topological group (S_∞, τ) is metrisable. This implies in particular that τ is a Hausdorff topology, then by Gaughan's Theorem τ contains the standard topology on S_∞. So τ and the standard topology are both Polish group topologies on S_∞, and one includes the other. The conditions of Proposition 2.23 are hence fulfilled, then we can conclude that **the standard topology is the only Polish group topology on the group of infinite permutations**.

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