Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE

Corso di Laurea Magistrale in Matematica

STRONG EXISTENCE AND PATHWISE UNIQUENESS FOR SDE WITH ROUGH COEFFICIENTS

Tesi di Laurea in Analisi Stocastica

Relatore:
Chiar.mo Prof.
ANDREA PASCUCCI

Presentata da:
SASCHA PORTARO

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To my parents, relatives, and friends
who have wholeheartedly believed in me,
with deep gratitude and eternal love.

Introduzione

In questa tesi studiamo l'esistenza e l'unicità forte della soluzione dell'equazione differenziale stocastica in $\mathbb{R}^d$, $d \geq 1$,

$$dX_t = F(t, X_t)dt + \sigma(t, X_t)dW_t, \qquad X_0 = \xi,$$

dove $F : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ e $\sigma : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{d\times r}$ sono funzioni Borel-misurabili, $(W_t, t \geq 0)$ è un moto Browniano standard r-dimensionale su uno spazio di probabilità completo con filtrazione $(\Omega, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, e ξ è una variabile aleatoria $\mathscr{F}_0$-misurabile.

Quando σ e F sono limitati, la legge $u(t, dx)$ di X_t appartiene all'insieme M_1 delle funzioni da $\mathbb{R}^+$ con valori nell'insieme P_1 delle misure di probabilità su $\mathbb{R}^d$ tali che, per ogni insieme Borel-misurabile $A \subseteq \mathbb{R}^d$ la funzione $t \mapsto u(t, A)$ è misurabile. É facile dedurre dalla formula di Itô che $u(t, dx)$ è una soluzione debole dell'equazione differenziale di Fokker-Planck su $\mathbb{R}^+ \times \mathbb{R}^d$

$$\partial_t u + \nabla_x \cdot (Fu) = \nabla_x^2(au), \qquad u(t = 0, dx) = u^0,$$

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$$\partial_t u + \sum_{i=1}^{d} \frac{\partial(F_i u)}{\partial x_i} = \sum_{1\leq i,j\leq d} \frac{\partial^2(a_{ij}u)}{\partial x_i \partial x_j},$$

dove $a = \frac{1}{2}\sigma\sigma^*$ e u^0 è la legge della variabile aleatoria iniziale ξ.
Per l'esistenza, consideriamo una succcessione di approssimazioni dell'equazione stocastica precedente

$$dX_t^n = F_n(t, X_t^n)dt + \sigma_n(t, X_t^n)dW_t, \qquad X_0^n = \xi,$$

con lo stesso moto Browniano W per ogni $n \in \mathbb{N}$. E introduciamo la corrispondente approssimazione dell'equazione di Fokker-Planck

$$\partial_t u_n + \nabla_x \cdot (F_n u_n) = \nabla_x^2(a^n u_n), \qquad u_n(t = 0, dx) = u^0,$$

dove $a^n = \frac{1}{2}\sigma_n\sigma_n^*$ e $u_n \in M_1$.

Per questo lavoro ci siamo basati su [6], il metodo descritto usa stime su funzionali della differenza tra due soluzioni della regolarizzazione dell'equazione differenziale stocastica considerata. Il vantaggio principale di questo approccio è la sua flessibilità poiché assumiamo bound in spazi di Sobolev per i coefficienti di drift e diffusione, e bound in spazi L^p per la soluzione della corrispondente equazione di Fokker–Planck, che possono essere provati separatamente. Ciò ci dà una certa libertà nello scegliere il metodo migliore per trattare l'equazione di Fokker–Planck tenendo conto di qualsiasi struttura aggiuntiva. Pertanto tali risultati possono essere applicati in vari casi, incluso quello uniformemente ellittico in ogni dimensione e quello dell'equazione di Kolmogorv dove non è richiesta nessuna ipotesi di ellitticità sulla matrice di diffusione.

Contents

Introduction

In this thesis we study strong existence and pathwise uniqueness of the solution to the stochastic differential equation (SDE) in $\mathbb{R}^d$, $d \geq 1$,

$$dX_t = F(t, X_t)dt + \sigma(t, X_t)dW_t, \qquad X_0 = \xi, \tag{1}$$

where $F : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{d\times r}$ are Borel measurable function, $(W_t, t \geq 0)$ is a r-dimensional standard Brownian motion on some given complete filtered probability space $(\Omega, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, and ξ is a $\mathscr{F}_0$-measurable random variable.

When σ and F are bounded, the law $u(t, dx)$ of X_t belongs to the set M_1 of functions from $\mathbb{R}^+$ with value in the set P_1 of probability measure on $\mathbb{R}^d$ such that, for all Borel set $A \subseteq \mathbb{R}^d$ the function $t \mapsto u(t, A)$ is measurable. It is standard to deduce from Itô's formula that $u(t, dx)$ is a weak solution to the Fokker-Planck PDE on $\mathbb{R}^+ \times \mathbb{R}^d$

$$\partial_t u + \nabla_x \cdot (Fu) = \nabla_x^2(au), \qquad u(t = 0, dx) = u^0, \tag{2}$$

that is

$$\partial_t u + \sum_{i=1}^{d} \frac{\partial(F_i u)}{\partial x_i} = \sum_{1\leq i,j\leq d} \frac{\partial^2(a_{ij}u)}{\partial x_i \partial x_j},$$

where $a = \frac{1}{2}\sigma\sigma^*$ and u^0 is the law of the initial random variable ξ.
For existence, we consider a sequence of approximations to (1)

$$dX_t^n = F_n(t, X_t^n)dt + \sigma_n(t, X_t^n)dW_t, \qquad X_0^n = \xi, \tag{3}$$

with the same Brownian motion W for any n. And we introduce the corresponding approximation to (2)

$$\partial_t u_n + \nabla_x \cdot (F_n u_n) = \nabla_x^2(a^n u_n), \qquad u_n(t = 0, dx) = u^0, \tag{4}$$

where $a^n = \frac{1}{2}\sigma_n\sigma_n^*$ and $u_n \in M_1$.

We based our work on [6], the method described uses estimates on functionals of the difference between two solutions of the regularization of (1). The main advantage of this approach is its flexibility since we assume Sobolev bounds on the drift and diffusion coefficients, and L^p bounds for the solution of the corresponding Fokker–Planck PDE (2), which can be proved separately. This allows us to have a certain freedom in choosing the best method to deal with (2) according to any additional structure. Hence the results can be applied in various cases, including the uniformly elliptic case in any dimension and the Kolmogorv case where no assumption of ellipticity on the diffusion matrix is required.

Chapter 1

Stochastic differential equations and Fokker-Planck equation

We first introduce the notion of solution to a SDE, the problem of existence and uniqueness (in both weak and strong formulation) and we will state some classical results about this topic. Then in the last section of this chapter we will explain how the Fokker-Planck equation (2) is obtained starting from the SDE (1).

1.1 Weak and strong solutions: Yamada and Watanabe Theorem

Let us start by giving the following definitions.

Definition 1.1.1 (Weak existence). *We say that weak existence holds for* (1) *if there exist a filtered probability space* $(\Omega, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$, *an adapted* r*-dimensional Brownian motion* W *and an adapted process* X *on this space such that satisfy* (1).

Note that to show weak existence for a SDE one needs to build not only the process X but also the filtered probability space and the Brownian motion W: for this reason it is generally said that the solution is not only X but (X, W).

Definition 1.1.2 (Strong existence). *We say that strong existence holds for* (1) *if for* *any given filtered probability space* $(\Omega, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ *equipped with any given adapted* r*-dimensional Brownian motion* W*, there exists a process* X *solution to* (1)*.*

Definition 1.1.3 (Uniqueness in law)**.** *We say that uniqueness in law holds if every solution* X *to* (1)*, possibly on different probability spaces, has the same law.*

Definition 1.1.4 (Pathwise uniqueness)**.** *We say that pathwise uniqueness holds for* (1) *if, on any given filtered probability space* $(\Omega, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ *equipped with any given adapted* r*-dimensional Brownian motion* W*, any two solutions to* (1) *with the same given* $\mathscr{F}_0$*-measurable initial condition* ξ *coincide.*

The following is a well-known result which establishes a link between the previous notions.

Theorem 1.1.5 (Yamada and Watanabe)**.** *The following statements hold:*

i) If strong existence holds for a SDE then also weak existence holds;

ii) if pathwise uniqueness holds for a SDE then also uniqueness in law holds;

iii) if weak existence and pathwise uniqueness holds for a SDE then also strong existence holds;

Proof. See [21]. □

Note that, although it is quite simple to prove i) of the previous Theorem by building the right filtered probability space equipped with the right standard Brownian motion, it is not the case of ii). Indeed one wants to show that if X and Y are two solution of (1), possibly on different probability spaces, then they have the same law, so one needs to build a version of X and Y on the same probability space equipped with the same Brownian motion in order to apply the assumption of pathwise uniqueness.

1.2 Classical results on existence and uniqueness

In this section we recall some well-known theorems without proving them. These results illustrate what sort of assumptions one generally makes on the coefficients F and

σ in order to obtain existence and/or uniqueness.
One of the main hypotheses that one usually requires is the uniform ellipticity of the diffusion coefficient σ.

Definition 1.2.1 (Uniform ellipticity). *We call bC^α_T the space of the functions $f \in C((0,T)\times\mathbb{R}^d)$ such that f is bounded and uniformly α-Hölder in $x \in \mathbb{R}^d$ where $\alpha \in (0,1]$, with the norm*

$$[f]_\alpha := \sup_{(0,T)\times\mathbb{R}^d} |f| + \sup_{t\in(0,T)} \frac{\left|f(t,x)-f(t,y)\right|}{|x-y|^\alpha}.$$

Assume $F_i, \sigma_{i,j} \in bC^\alpha_t$ for an $\alpha \in (0,1]$ and for all $i=1,\dots,d$, $j=1,\dots,r$, we say that σ is uniformly elliptic if there exists a positive constant λ such that

$$\frac{1}{\lambda} I \leq a(t,x) \leq \lambda I, \qquad \forall (t,x) \in (0,T)\times\mathbb{R}^d, \tag{1.1}$$

where $a = \frac{1}{2}\sigma\sigma^*$.

Theorem 1.2.2 (Skorokhod, Stroock and Varadhan, Krylov). *Assume that F and σ are bounded measurable functions such that at least one of the following assumptions holds:*

1. *$F(t,\cdot), \sigma(t,\cdot) \in C(\mathbb{R}^d)$, for all $t \in [0,T]$;*

2. *condition (1.1) of uniform ellipticity.*

Then weak existence holds for (1). Moreover, if both the conditions hold one also has uniqueness in law.

Proof. See [18],[21] and [14]. □

Theorem 1.2.3. *Assume that F and σ satisfy the hypothesis of local Lipschitzianity, i.e. for all $n \in \mathbb{N}$ there exists a constant c_n such that*

$$\left|F(t,x)-F(t,y)\right| + \left|\sigma(t,x)-\sigma(t,y)\right| \leq c_n |x-y|,$$

for all $t \in [0,T]$ and $x,y \in \mathbb{R}^d$ such that $|x|, |y| \leq n$. Then pathwise uniqueness holds for (1).

Proof. See [16] and [17]. □

Theorem 1.2.4 (Yamada and Watanabe). *Assume $r = d = 1$, then pathwise uniqueness holds for* (1) *under the following assumptions:*

$$\left|F(t,x) - F(t,y)\right| \leq h\left(|x-y|\right), \qquad \left|\sigma(t,x) - \sigma(t,y)\right| \leq k\left(|x-y|\right), \qquad \forall t \geq 0, x, y \in \mathbb{R},$$

where:

- *h is a strictly increasing function such that $h(0) = 0$ and for all $\epsilon > 0$*

$$\int_0^\epsilon \frac{1}{h^2(s)} ds = +\infty;$$

- *k is a strictly increasing function, concave such that $k(0) = 0$ and for all $\epsilon > 0$*

$$\int_0^\epsilon \frac{1}{k(s)} ds = +\infty.$$

Proof. See [7] or [13]. □

Now we assume additional hypotheses on the coefficients F and σ and state some results on strong solutions.

Definition 1.2.5 (Standard hypotheses). *We say that F and σ satisfy the standard hypotheses if there exist two constant c_1 and c_2 such that:*

- *Linear growth: $\left|F(t,x)\right| + \left|\sigma(t,x)\right| \leq c_1\left(1 + |x|\right)$, for all $t \geq 0$ and $x \in \mathbb{R}^d$;*
- *Lipschitzianity: $\left|F(t,x) - F(t,y)\right| + \left|\sigma(t,x) - \sigma(t,y)\right| \leq c_2|x-y|$, for all $t \geq 0$ and $x, y \in \mathbb{R}^d$;*

Theorem 1.2.6. *If F and σ satisfy the standard hypotheses then strong existence holds for* (1).

Proof. See [16] and [17]. □

The following theorem illustrates the regularizing effect of the noise, that is the diffusive part of the SDE.

Theorem 1.2.7 (Zvonkin,Veretennikov). *Let us assume the following:*

- *there exists* $\alpha \in (0,1]$ *such that* $F_i \in bC_T^{\alpha}$ *for all* $i = 1, \dots, d$;
- $\sigma_{i,j} \in bC_T^1$ *for all* $i = 1, \dots, d$, $j = 1, \dots, r$, *that is* σ *is bounded and Lipschitz;*
- *condition* (1.1) *of uniform ellipticity holds.*

Then strong existence and pathwise uniqueness hold for (1).

Proof. See [22] and [28]. □

1.3 Derivation of the Fokker-Planck equation

Since we are interested in studying strong existence and pathwise uniqueness for an SDE through suitable estimates in Sobolev spaces for the solution of the associated Fokker-Planck equation, now we give an overview of the link between (1) and (2).

Definition 1.3.1 (Infinitesimal generator)**.** *Let* p *be a transition law on* $\mathbb{R}^d$, $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$ *and let's suppose that there exists the limit*

$$\mathscr{A}_t\varphi(x) := \lim_{T-t\to 0^+} \int_{\mathbb{R}^d} \frac{p(t,x;T,dy) - p(t,x;t,dy)}{T-t}\varphi(y)$$

for all $\varphi \in C_c^{\infty}(\mathbb{R}^d)$. *Then we say that* $\mathscr{A}_t$ *is the infinitesimal generator of* p. *If* p *is the transition law of a Markov process* X *we say that* $\mathscr{A}_t$ *is the infinitesimal generator of* X.

Let p be the transition law of a Markov process X, then it satisfies the Chapman-Kolmogorov equation (Theorem B.0.8). By the definition of infinitesimal generator and assuming that there exists the derivative $\partial_T p(t,x;T,dz)$, for all $\varphi \in C_c^{\infty}(\mathbb{R}^d)$ we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \partial_T p(t,x;T,dz)\varphi(z) &= \int_{\mathbb{R}^d} \lim_{h\to 0^+} \frac{p(t,x;T+h,dz) - p(t,x;T,dz)}{h}\varphi(z) \\
&\underset{\substack{\uparrow \\ \text{(Chapman-Kolmogorov)}}}{=} \int_{\mathbb{R}^d} p(t,x;T,dy) \lim_{h\to 0^+} \int_{\mathbb{R}^d} \frac{p(T,y;T+h,dz) - p(T,y;T,dz)}{h}\varphi(z) \\
&= \int_{\mathbb{R}^d} p(t,x;T,dy)\mathscr{A}_T\varphi(y).
\end{aligned}$$

So we have

$$\int_{\mathbb{R}^d} \partial_T p(t,x;T,dz)\varphi(z) = \int_{\mathbb{R}^d} p(t,x;T,dy)\mathscr{A}_T\varphi(y), \qquad \forall \varphi \in C_c^{\infty}(\mathbb{R}^d), \tag{1.2}$$

that is the *Fokker-Planck equation* (or *Kolmogorov forward equation*). Since φ is a test function (1.2) can be written in a distributional sense as

$$\partial_T p(t,x;T,\cdot) = \mathscr{A}_T^* p(t,x;T,\cdot), \tag{1.3}$$

where $\mathscr{A}_T^*$ is the adjoint operator of $\mathscr{A}_T$.

Finally, we can associate to (1) the following operator

$$\mathscr{A}_t := \sum_{i,j=1}^d a_{ij} \frac{\partial^2 \cdot}{\partial x_i \partial x_j} + \sum_{i=1}^d F_i \frac{\partial \cdot}{\partial x_i}. \tag{1.4}$$

Indeed, if we suppose that F and σ satisfy the standard hypotheses then X solution to the SDE is such that

$$\begin{aligned}\mathbb{E}\left[\frac{\varphi(X_t)-\varphi(x)}{t}\right] \underset{\text{(Itô formula)}}{\underset{\uparrow}{=}} & \ \mathbb{E}\left[\frac{1}{t}\int_0^t \mathscr{A}_s\varphi(X_s)ds + \frac{1}{t}\int_0^t \nabla\varphi(X_s)\sigma(s,X_s)dW_s\right] \\ &= \mathbb{E}\left[\frac{1}{t}\int_0^t \mathscr{A}_s\varphi(X_s)ds\right] \xrightarrow[t\to 0^+]{} \mathscr{A}_0\varphi(x) \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^d),\end{aligned}$$

where the passage to the limit is justify by the dominated convergence Theorem.
In other words we have

$$\frac{d}{dt}\mathbb{E}\left[\varphi(X_t)\right]\Big|_{t=0} = \mathscr{A}_0\varphi(x) \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^d),$$

and in general if X starts from $s>0$ we have

$$\frac{d}{dt}\mathbb{E}\left[\varphi(X_t)\right]\Big|_{t=s} = \mathscr{A}_s\varphi(x) \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^d),$$

which reminds us the definition of infinitesimal generator of a Markov process. So we give the following

Definition 1.3.2 (Characteristic operator of a SDE)**.** *The operator* $\mathscr{A}_t$ *in* (1.4) *is called characteristic operator of the SDE* (1)*.*

Lastly we observe that the adjoint operator of $\mathscr{A}_t$ is

$$\mathscr{A}_t^* = \sum_{i,j=1}^d \frac{\partial^2(a_{ij}\cdot)}{\partial x_i \partial x_j} - \sum_{i=1}^d \frac{\partial(F_i\cdot)}{\partial x_i},$$

then by (1.3) we obtain

$$\partial_t p + \sum_{i=1}^{d} \frac{\partial (F_i p)}{\partial x_i} = \sum_{1 \le i,j \le d} \frac{\partial^2 (a_{ij} p)}{\partial x_i \partial x_j},$$

that is (2).

Chapter 2

General results on strong solutions

The goal of this chapter is to give the statement of the main results on strong existence and pathwise uniqueness to (1) where F and σ are rough.
First of all we want to state a result which follows from the main Theorem 2.2.1. It does not require any additional definition, in fact it only deals with the classical Sobolev spaces L^p, but it illustrates the type of assumptions we need.

Theorem 2.0.1. *Assume $d \geq 2$.*

- *Existence: Assume that there exists two sequences $F_n, \sigma_n \in C^\infty \cap L^\infty$ converging in the sense of distributions to F and σ, respectively, such that the solution $u_n \in M_1$ to (4) satisfies for $1 \leq p, q \leq +\infty$, with $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$*

  1. $\sigma_n - \sigma \xrightarrow[n\to+\infty]{} 0$ *in* $L^q_{t,loc}(L^p_x)$
  2. $F_n - F \xrightarrow[n\to+\infty]{} 0$ *in* $L^q_{t,loc}(L^p_x)$*;*
  3. $\sup_n \left(\|\sigma_n\|_{L^{2q}_{t,loc}(W^{1,2p}_x)} + \|F_n\|_{L^q_{t,loc}(W^{1,p}_x)} + \|\sigma_n\|_{L^\infty} + \|F_n\|_{L^\infty} \right) < +\infty$*;*
  4. $\sup_n \left(\|u_n\|_{L^{q'}_{t,loc}(L^{p'}_x)} \right) < +\infty$*;*
  5. $u_n \xrightarrow[n\to+\infty]{} u$ *in the weak topology of* M_1.

  Then there exists a strong solution X_t to (1) and $(X^n_t - \xi, t \in [0,T])_n$ converges in $L^p(\Omega, L^\infty([0,T]))$ for all $p > 1$ and $T > 0$ to $(X_t - \xi, t \in [0,T])$, with X^n_t the solutions to (3). In addition, $u(t, dx)$ is the law of X_t for almost all $t \geq 0$.

- *Uniqueness: Let* X *and* Y *be two solutions to* (1) *with one-dimensional time marginals* $u_X(t,x)dx$ *and* $u_Y(t,x)dx$ *both in* $L^{q'}_{t,loc}(L^{p'}_x)$. *Assume that* $F, \sigma \in L^\infty$, $X_0 = Y_0$ *a.s. and that*

  6. $\|F\|_{L^q_{t,loc}(W^{1,p}_x)} + \|\sigma\|_{L^{2q}_{t,loc}(W^{1,2p}_x)} < +\infty$;

  with $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$. *Then one has pathwise uniqueness:* $\sup_{t\geq 0}|X_t - Y_t| = 0$ *a.s.*

Proof. See Appendix A. □

In the one-dimensional case we obtain better results.

Theorem 2.0.2. *Assume* $d = 1$.

- *The existence result of Theorem 2.0.1 holds under the same assumptions on* F_n, σ_n, u_n, *except that the assumption* $\sup_n \|\sigma_n\|_{L^{2q}_{t,loc}(W^{1,2p}_x)} < +\infty$ *can be replaced by*

$$\sup_n \|\sigma_n\|_{L^{2q}_{t,loc}(W^{\frac{1}{2},2p}_x)} < +\infty,$$

  and in the case $p = 1$, *the assumption* $\sup_n \|F_n\|_{L^q_{t,loc}(W^{1,p}_x)} < +\infty$ *must be replaced by*

$$\sup_n \|F_n\|_{L^q_{t,loc}(W^{1,1+\epsilon}_x)} < +\infty$$

  for some $\epsilon > 0$.

- *The uniqueness result of Theorem 2.0.1 holds under the same assumptions on* F, σ, u_X, u_Y, *except that the assumption* $\|\sigma\|_{L^{2q}_{t,loc}(W^{1,2p}_x)} < +\infty$ *can be replaced by*

$$\|\sigma\|_{L^{2q}_{t,loc}(W^{\frac{1}{2},2p}_x)} < +\infty,$$

  and in the case $p = 1$, *the assumption* $\|F\|_{L^q_{t,loc}(W^{1,p}_x)} < +\infty$ *must be replaced by*

$$\|F\|_{L^q_{t,loc}(W^{1,1+\epsilon}_x)} < +\infty$$

  for some $\epsilon > 0$.

However, in the most general case the conditions that we impose on F and σ can be simplified as follows. We want σ to be L^2 in time and H^1 (in dimension $d \geq 2$) or $H^{1/2}$ (in dimension $d = 1$) in space with respect to the measure u solution to (2), and F to be L^1 in time and $W^{1,1}$ in space with respect to the measure u.
Moreover, since no regularity is known on u one must be careful and this is why maximal functions are required.

2.1 The spaces $H^1_T(u)$, $H^{\frac{1}{2}}_T(u)$ and $W^{\phi,weak}_T(u)$

In this section we give the definitions of the spaces needed to state and prove the two main theorems (in dimension $d \geq 2$ and $d = 1$).
Since our spaces depends on the measure u, we need the following

Definition 2.1.1. *Let P_1 be the set of probability measure on $\mathbb{R}^d$, M_1 is defined as the set of functions from $\mathbb{R}^+$ with value in the set P_1 such that, for all Borel set $A \subseteq \mathbb{R}^d$ the function $t \mapsto u(t, A)$ is measurable.*

2.1.1 The space $H^1_T(u)$

First of all we recall that if $f \in L^1_{loc}(\mathbb{R}^d)$, the usual maximal function Mf is defined as follows

$$Mf(x) := \sup_{r>0} \frac{1}{\left|B(x,r)\right|} \int_{B(x,r)} \left|f(y)\right| dy \qquad \forall x \in R^d$$

and M is called maximal operator.
Moreover following [19], if $f \in BV_{loc}(\mathbb{R}^d)$ then $|\nabla f|$ is a locally finite measure. This allows us to define

$$M|\nabla f|\,(x) = \sup_{r>0} \frac{1}{\left|B(x,r)\right|} \int_{B(x,r)} |\nabla f|\,(dz) \qquad \forall x \in R^d.$$

In that case $M|\nabla f|$ is a Borel function with value in $\mathbb{R}^+ \cup \{+\infty\}$. In fact it locally belongs to the weak L^1 space, that is, for any $R > 0$ there exists $C_R > 0$, such that

$$\left|\{x \in B(0,R); M|\nabla f|\,(x) > L\}\right| \leq \frac{C_R}{L}.$$

Hence, the integral of $(M|f|\,(x))^2$ against v is well defined with value in $\mathbb{R}^+ \cup \{+\infty\}$.
This justify the following definition.

Definition 2.1.2. *Fixing $v \in P_1$, the space $H^1(v)$ is the subspace of function $f \in BV_{loc}(\mathbb{R}^d)$, such that*

$$\|f\|_{H^1(v)}^2 := \int_{\mathbb{R}^d} ((M|f|(x))^2 + (M|\nabla f|(x))^2) v(dx) < \infty. \tag{2.1}$$

The space $H^1(v)$ defined above is well-behaving independently of the regularity of v.

Theorem 2.1.3. *If $v \in P_1$ then $H^1(v)$ is a Banach space with norm (2.1). Moreover, the norm is lower semicontinuous with respect to convergence in the sense of distribution: if $f_n \xrightarrow[n\to+\infty]{} f$ in the sense of distribution*[1]*, then*

$$\|f\|_{H^1(v)} \leq \liminf_n \|f_n\|_{H^1(v)}. \tag{2.2}$$

And if for a given $f \in BV_{loc}(\mathbb{R}^d)$, v_n converges to v in the tight topology of probability measures, then

$$\|f\|_{H^1(v)} \leq \liminf_n \|f\|_{H^1(v_n)}.$$

Proof. First of all $\|\cdot\|_{H^1(v)}$ is a norm on $H^1(v)$.
By definition it is non-negative and finite on $H^1(v)$. Moreover, if $\lambda > 0$ then $M(|\lambda f|) = \lambda M(|f|)$, and thus $\|\lambda f\|_{H^1(v)} = |\lambda| \, \|f\|_{H^1(v)}$. The triangle inequality is also trivially since $M(f+g) \leq Mf + Mg$. Finally, if $\|f\|_{H^1(v)} = 0$ then $M|f| = 0$ on the support of v which contains at least one point x_0 since v is a probability measure. But now $M|f|(x_0) = 0$ implies $f = 0$ by the definition of the maximal function.
Now we prove that (2.2) holds.
Consider a sequence f_n in $H^1(v)$ such that converges to some f in $\mathcal{D}'$ and assume (eventually restrictiong to a subsequence) that

$$\sup_n \|f_n\|_{H^1(v)} < +\infty.$$

Otherwise, there is nothing to prove.
Note that f_n is uniformly bounded in BV_{loc}. Indeed for any $R > 0$ and any $x \in B(0,R)$

$$|\nabla f_n| \big(B(0,R)\big) \leq c_d R^d M|\nabla f_n|(x),$$

[1]If f, f_n with $n \in \mathbb{N}$ are distributions on $\mathbb{R}^d$, we say that $f_n \xrightarrow[n\to+\infty]{} f$ if for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ we have that $\langle f_n|\varphi\rangle \xrightarrow[n\to+\infty]{} \langle f|\varphi\rangle$. In that case we write $f_n \xrightarrow[n\to+\infty]{} f$ in $\mathcal{D}'$.

so that by Cauchy-Schwarz,

$$|\nabla f_n|\big(B(0,R)\big) \leq \frac{c_d R^d}{\Big(\int_{B(0,R)} v(dx)\Big)^{\frac{1}{2}}} \|f_n\|_{H^1(v)}. \tag{2.3}$$

As $f_n \xrightarrow[n\to+\infty]{} f$ in $\mathcal{D}'$ then $f \in BV_{loc}$ as well. Therefore, $M|\nabla f|$ is well defined.
On the other hand $\nabla f_n \xrightarrow[n\to+\infty]{} \nabla f$ in $\mathcal{D}'$ too, and so by the Theorem on the lower semicontinuity of variation measure we have

$$\int \varphi|\nabla f|\,(dx) \leq \liminf_n \int \varphi|\nabla f_n|\,(dx), \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^d).$$

Now fix $c > 1$ and any $r > 0$ and note that the previous inequality implies that

$$\frac{1}{\big|B(x,r)\big|}\int_{B(x,r)} |\nabla f|\,(dz) \leq \frac{1}{\big|B(x,r)\big|}\liminf_n \int_{B(x,cr)} |\nabla f_n|\,(dz) \leq c^d \liminf_n M|\nabla f_n|\,(x).$$

Taking now the supremum in r, we deduce that for any $c > 1$

$$M|\nabla f|\,(x) \leq c^d \liminf_n M|\nabla f_n|\,(x).$$

By Fatou's lemma and sending $c \to 1$ we get

$$\int \big(M|\nabla f|\,(x)\big)^2 v(dx) \leq \liminf_n \int \big(M|\nabla f_n|\,(x)\big)^2 v(dx).$$

The same steps with $M|f_n|$ and $M|f|$ prove that $f \in H^1(v)$ and that (2.2) holds.
To conclude the proof that $H^1(v)$ is a Banach space we have to show that it is complete.
Accordingly, consider any Cauchy sequence f_n in $H^1(v)$.
The sequence is the also Cauchy in BV_{loc}. Indeed, for any $R > 0$ using (2.3) for $f_n - f_m$ we obtain

$$\big|\nabla(f_n - f_m)\big|\big(B(0,R)\big) \leq \frac{c_d R^d}{\Big(\int_{B(0,R)} v(dx)\Big)^{\frac{1}{4}}} \|f_n - f_m\|_{H^1(v)}.$$

Therefore, there exists $f \in BV_{loc}$ such that f_n converges toward f in BV_{loc}. In particular, f_n converges to f in $\mathcal{D}'$ and by (2.2) we deduce that $f \in H^1(v)$.
It remains to show that $\|f_n - f_m\|_{H^1(v)} \to 0$. Fixing n we consider the sequence $f_n - f_m$ in m which converges in the sense of distribution to $f_n - f$. So now we can conclude again using (2.2) that

$$\|f_n - f\|_{H^1(v)} \leq \liminf_m \|f_n - f_m\|_{H^1(v)} = 0,$$

whence $\|f_n - f_m\|_{H^1(v)} \to 0$.

Let us now prove the last part of the Theorem. We first recall that if μ is a finite, non-negative Radon measure, then $M\mu$ is lower semicontinuous. This can be shown using similar arguments to the ones above: consider any $x_n \to x$, then for any $c > 1$ and $r > 0$ we have

$$\frac{1}{\big|B(0,r)\big|}\int_{B(0,r)} \mu(x+dz) \le \frac{1}{\big|B(0,r)\big|}\liminf_n \int_{B(0,cr)} \big|\mu(x_n+dz)\big| \le c^d \liminf_n M|\mu|\,(x_n).$$

The lower semicontinuity of $M\mu$ then follows taking the supremum in r and then the infimum in c.
Denote now $g := (M|\nabla f|)^2 + (M|f|)^2$, g is a non-negative Borel function with values in $\mathbb{R}^+ \cup \{+\infty\}$. By the previous remark it is also lower semicontinuous.
Note also that for any positive measure μ

$$\int g d\mu = \int_0^{+\infty}\int \mathbb{1}_{g(x)>\xi}\mu(dx)d\xi.$$

Now assume $v_n \to v$ in the tight topology of P_1. For any open set $O \subseteq \mathbb{R}^d$

$$\int_O dv \le \liminf_n \int_O dv_n.$$

Taking $O := \{g(x) > \xi\}$ which is open by the lower semicontinuity of g, by Fatou's lemma we obtain

$$\begin{aligned}\int g dv = \int_0^{+\infty}\int_O dv d\xi &\le \int_0^{+\infty} \liminf_n \int_O dv_n \xi\\ &\le \liminf_n \int_0^{+\infty}\int_O dv_n\xi = \liminf_n \int g dv_n,\end{aligned}$$

which concludes the proof of the Theorem. □

Note that we need the maximal operator in the definition of the norm (2.1) because, for example, v could vanish where ∇f is too large. In particular, without the maximal function the lower semicontinuity (2.2) does not hold in general.
Finally we can give the definition of $H^1_T(u)$.

Definition 2.1.4. *Fix $u \in M_1$. For all $T > 0$, the space $H^1_T(u)$ is defined as the subspace of the set of the measurable functions on $[0,T] \times \mathbb{R}^d$ such that, for almost $t \in [0,T]$, $f(t,\cdot) \in H^1(u(t,\cdot))$ and*

$$\|f\|^2_{H^1_T(u)} := \int_0^T \|f\|^2_{H^1(u(t,\cdot))} dt < \infty. \tag{2.4}$$

We note that if $u(t,\cdot)$ is the distribution of X_t solution to (1), then, for all $T > 0$ and $\sigma \in H^1_T(u)$,

$$\|\sigma\|^2_{H^1_T(u)} = \mathbb{E}\left[\int_0^T M|\sigma|^2(t,X_t)dt\right] + \mathbb{E}\left[\int_0^T (M|\nabla\sigma|(t,X_t))^2 dt\right]. \tag{2.5}$$

As a consequence of Theorem 2.1.3 we have the following

Corollary 2.1.5. *Fix $T > 0$ and assume $u \in M_1$, then $H^1_T(u)$ is a Banach space with norm (2.4). Moreover, the norm is lower semicontinuous with respect to convergence in the sense of distribution: if $f_n \xrightarrow[n\to+\infty]{} f$ in the sense of distribution, then*

$$\|f\|_{H^1_T(u)} \leq \liminf_n \|f_n\|_{H^1_T(u)}. \tag{2.6}$$

And if for a given f measurable on $\mathbb{R}^+ \times \mathbb{R}^d$ with $f(t,\cdot) \in BV_{loc}(\mathbb{R}^d)$ for almost all $t \geq 0$, u_n converges to u for the weak topology in M_1, then

$$\|f\|_{H^1_T(u)} \leq \liminf_n \|f\|_{H^1_T(u_n)}.$$

2.1.2 The space $H^{\frac{1}{2}}_T(u)$

In the one-dimensional case we require $H^{\frac{1}{2}}$ assumptions on the coefficient σ. The definitions and the properties of the spaces $H^{\frac{1}{2}}_T(u)$ follow exactly the same steps as before.
We first need the following

Definition 2.1.6. *Let $f \in L^1_{loc}(\mathbb{R}^d)$, we define*

$$\partial^{\frac{1}{2}} f = \mathcal{F}^{-1}|\xi|^{\frac{1}{2}}\mathcal{F}f,$$

where $\mathcal{F}$ is the Fourier trasform in $\mathbb{R}^d$.

Now following the same steps of the previous subsection we can define the desired spaces.
Note that as for $H^1(v)$ with $v \in P_1$, the maximal function can be extended to measures by

$$M|\partial^{\frac{1}{2}} f|(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |\partial^{\frac{1}{2}} f(dz)| \qquad \forall x \in R^d.$$

We have again that $M|\partial^{\frac{1}{2}} f|$ is a Borel function with value in $\mathbb{R}^+ \cup \{+\infty\}$ belonging to the local weak L^1 space. The integral against the Borelian measure v is hence well defined in $\mathbb{R}^+ \cup \{+\infty\}$, independently of the regularity of v.
This justify the following definition.

Definition 2.1.7. *Fixing $v \in P_1$, the space $H^{\frac{1}{2}}(v)$ is the subspace of function $f \in L^1_{loc}(\mathbb{R}^d)$, such that $\partial^{\frac{1}{2}} f$ is a locally finite Radon measure and*

$$\|f\|^2_{H^{\frac{1}{2}}(v)} := \int_{\mathbb{R}^d} ((M|f|(x))^2 + (M|\partial^{\frac{1}{2}} f|(x))^2) v(dx) < \infty. \tag{2.7}$$

Now we can state the properties of the space $H^{\frac{1}{2}}(v)$.

Theorem 2.1.8. *If $v \in P_1$ then $H^{\frac{1}{2}}(v)$ is a Banach space with norm (2.7). Moreover, the norm is lower semicontinuous with respect to convergence in the sense of distribution: if $f_n \xrightarrow[n\to+\infty]{} f$ in the sense of distribution, then*

$$\|f\|_{H^{\frac{1}{2}}(v)} \leq \liminf_n \|f_n\|_{H^{\frac{1}{2}}(v)}. \tag{2.8}$$

And if for a given $f \in L^1_{loc}(\mathbb{R}^d)$ such that $\partial^{\frac{1}{2}} f$ is a locally finite Radon measure, v_n converges to v in the tight topology of probability measures, then

$$\|f\|_{H^{\frac{1}{2}}(v)} \leq \liminf_n \|f\|_{H^{\frac{1}{2}}(v_n)}.$$

Proof. We omit the proof since it is similar to that of Theorem 2.1.3. The only difference is that the space BV_{loc} is replaced by the space of L^1_{loc} functions f such that $\partial^{\frac{1}{2}} f$ is a locally finite measure. □

Finally we can give the definition of $H^{\frac{1}{2}}_T(u)$.

Definition 2.1.9. *Fix $u \in M_1$. For all $T > 0$, the space $H^{\frac{1}{2}}_T(u)$ is defined as the subspace of the set of the measurable functions on $[0,T] \times \mathbb{R}^d$ such that, for almost $t \in [0,T]$, $f(t,\cdot) \in H^{\frac{1}{2}}_T(u(t,\cdot))$ and*

$$\|f\|^2_{H^{\frac{1}{2}}_T(u)} := \int_0^T \|f\|^2_{H^{\frac{1}{2}}_T(u(t,\cdot))} dt < \infty. \tag{2.9}$$

Note that if $u(t,\cdot)$ is the distribution of X_t solution to (1), then, for all $T > 0$ and $\sigma \in H^{\frac{1}{2}}_T(u)$,

$$\|\sigma\|^2_{H^{\frac{1}{2}}_T(u)} = \mathbb{E}\left[\int_0^T M|\sigma|^2(t,X_t)dt\right] + \mathbb{E}\left[\int_0^T (M|\partial_x^{\frac{1}{2}}\sigma|(t,X_t))^2 dt\right]. \tag{2.10}$$

Again, as a consequence of Theorem 2.1.8 we have the following

Corollary 2.1.10. *Fix $T > 0$ and assume $u \in M_1$, then $H^{\frac{1}{2}}_T(u)$ is a Banach space with norm (2.9). Moreover, the norm is lower semicontinuous with respect to convergence in the sense of distribution: if $f_n \xrightarrow[n\to+\infty]{} f$ in the sense of distribution, then*

$$\|f\|_{H^{\frac{1}{2}}_T(u)} \le \liminf_n \|f_n\|_{H^{\frac{1}{2}}_T(u)}. \tag{2.11}$$

And if for a given $f \in L^1(\mathbb{R}^+ \times \mathbb{R}^d)$ such that $\partial^{\frac{1}{2}} f(t,\cdot)$ is a locally finite Radon measure for almost all $t \ge 0$, u_n converges to u for the weak topology in M_1, then

$$\|f\|_{H^{\frac{1}{2}}_T(u)} \le \liminf_n \|f\|_{H^{\frac{1}{2}}_T(u_n)}.$$

2.1.3 The space $W^{\phi,weak}_T(u)$

We also need some $W^{1,1}$ assumptions on the coefficient F and following the previous definition of H^1, it seems natural to define

$$\|F\|_{W^{1,1}_T(u)} := \int_0^T \int_{\mathbb{R}^d} \big(M|F|(t,x) + M|\nabla F|(t,x)\big)\, u(t,dx)dt. \tag{2.12}$$

Unfortunately, while this definition would work, it is too strong in some cases. This is due to the fact that the maximal operator M is bounded on L^p, $p > 1$, but not on L^1. So, if for example $u \in L^\infty$, then the norm defined in (2.5) would automatically be finite

if σ is in the usual H^1 space but the norm (2.12) would not be finite if $F \in W^{1,1}$ in general.
Therefore, in order to obtain better assumptions we will work with a more complicated space. So we need further definitions.

Definition 2.1.11. *A function $\phi : [1, +\infty) \to \mathbb{R}$ is a superlinear function if $\frac{\phi(\xi)}{\xi}$ is non-decreasing and converges to $+\infty$ as $\xi \to +\infty$.*

Definition 2.1.12. *For any locally finite Radon measure μ, let μ_a and μ_s be respectively the absolutely continuous part and the singular part of the decomposition of μ with respect to the Lebesgue measure, one defines for $L \geq 1$*

$$M_L\mu := \sqrt{\log L} + \int_{\mathbb{R}^d} \frac{|\mu_a|(z)\mathbb{1}_{|\mu_a(z)|>\sqrt{\log L}}dz + |\mu_s|(dz)}{(L^{-1} + |x-z|)|x-z|^{d-1}}.$$

Note that the very reason why we need to define M_L as above becomes clear in the proof of Lemma B.0.2 in Appendix B.

Now, for any $f \in BV_{loc}(\mathbb{R}^d)$, the decomposition of ∇f into a part absolutely continuous with respect to the Lebesgue measure $(\nabla f)_a$ and the singular part $(\nabla f)_s$, makes $M_L\nabla f$ well defined. So it makes sense to give the following definition.

Definition 2.1.13. *Fix $v \in P_1$ and let ϕ be a superlinear function. The space $W^{\phi,weak}(v)$ is the subspace of function $f \in BV_{loc}(\mathbb{R}^d)$ such that*

$$\|f\|_{W^{\phi,weak}(v)} := \int_{\mathbb{R}^d} M|f|(x)v(dx) + \sup_{L\geq 1} \frac{\phi(L)}{L\log L}\int_{\mathbb{R}^d} M_L\nabla f v(dx) < +\infty.$$

Note that the space heavily depends on the choice of ϕ. In particular $M_L\nabla f \geq \sqrt{\log L}$, then $\|f\|_{W^{\phi,weak}(v)} \geq \sup_{L\geq 1} \frac{\phi(L)}{L\sqrt{\log L}}$. So $\|f\|_{W^{\phi,weak}(v)} = +\infty$ for all f if $\phi(L) \gg L\sqrt{\log L}$ asymptotically as $L \to +\infty$. On the other hand, we want to choose ϕ superlinear as we need to control the integrability of $|\nabla f|$. This leads to the assumptions:

$$\frac{\phi(L)}{L} \to +\infty, \qquad \frac{\phi(L)}{L\sqrt{\log L}} \to 0 \qquad \text{as } L \to +\infty. \tag{2.13}$$

Even with this assumption, $W^{\phi,weak}(v)$ is not a Banach space and $\|\cdot\|_{W^{\phi,weak}(v)}$ is not a norm. Of course, $\|0\|_{W^{\phi,weak}(v)} \neq 0$ but this could easily be fixed by considering $\|\cdot\|_{W^{\phi,weak}(v)} - \alpha_\phi$ instead, for the right constant α_ϕ. The main problem is that

$\|\lambda f\|_{W^{\phi,weak}(v)} \neq |\lambda| \, \|f\|_{W^{\phi,weak}(v)}$ and this cannot easily be corrected.
However, the space $W^{\phi,weak}(v)$ as defined above is still interesting for us because it satisfies the following two requirements that we need:

- The estimates that we perform later would not work for instance with the simple requirement that $\int_{\mathbb{R}^d} \left(|f| + |\nabla f|\right) v(dx) < +\infty$, so the maximal operator is needed;

- We want to recover the classical assumption if v is bounded from below and above. That means that if $\frac{1}{C} \leq v \leq C$, then any $f \in W^{1,1}$ must be in $W^{\phi,weak}(v)$ for some well chosen ϕ (depending on f). This is in particular why we do not use the direct extension $W^{1,1}(v)$ of the space $H^1(v)$, given by (2.12).

To be more precise we have the following

Theorem 2.1.14. *Assume that $v \in P_1$, ϕ is superlinear, continuous and that satisfies* (2.13)*. Then $W^{\phi,weak}(v)$ is well defined and $\|\cdot\|_{W^{\phi,weak}(v)}$ is lower semicontinuous with respect to convergence in the sense of distribution: if $f_n \xrightarrow[n\to+\infty]{} f$ in the sense of distribution, then*

$$\|f\|_{W^{\phi,weak}(v)} \leq \liminf_n \|f_n\|_{W^{\phi,weak}(v)}. \tag{2.14}$$

And if for a given $f \in BV_{loc}(\mathbb{R}^d)$, v_n converges to v in the tight topology of probability measure then

$$\|f\|_{W^{\phi,weak}(v)} \leq \liminf_n \|f\|_{W^{\phi,weak}(v_n)}.$$

Moreover, if $v \geq \frac{1}{C}$ over a smooth open set Ω and $f \in W^{\phi,weak}(v)$ then $f \in W^{1,1}(K)$ for any compact set $K \subset \Omega$. Reciprocally, if $v \leq C$ over Ω and $f \in W^{1,1}(\Omega)$ with compact support in Ω, then there exists a superlinear ϕ satisfying (2.13) *such that $f \in W^{\phi,weak}(v)$.*

Proof. The first part of the proof concerning the lower semicontinuity follows exactly the same steps as the proof of Thoerem 2.1.3. Indeed we have the following control through the BV norm

$$\begin{aligned} M_L|\nabla f|(x) &\geq \sqrt{\log L} + \frac{1}{R^{d-1}(R+L^{-1})} \int_{B(0,R)} \left(|\nabla f|_a \, \mathbb{1}_{|\nabla f|_a(z) > \sqrt{\log L}} dz + |\nabla f|_s (dz) \right) \\ &\geq \frac{1}{C\sqrt{L}(1+R^d)} \int_{B(0,R)} |\nabla f| (dz), \end{aligned}$$

for all $R, L \geq 1$.
Now one could obtain the same type of lower semicontinuity properties.
For istance if $f_n \xrightarrow[n\to+\infty]{} f$ in $\mathcal{D}'$ for a sequence f_n uniformly bounded in BV_{loc}, then

$$\begin{aligned} M_L \nabla f(x) &= \sqrt{\log L} + \int_{\mathbb{R}^d} \frac{|\nabla f|_a \mathbb{1}_{|\nabla f|_a(z)>\sqrt{\log L}} dz + |\nabla f|_s(dz)}{(L^{-1} + |x-z|)|x-z|^{d-1}} \\ &\leq \sqrt{\log L} + \liminf_n \int_{\mathbb{R}^d} \frac{|\nabla f_n|_a \mathbb{1}_{|\nabla f_n|_a(z)>\sqrt{\log L}} dz + |\nabla f_n|_s(dz)}{(L^{-1} + |x-z|)|x-z|^{d-1}} \\ &= \liminf_n M_L \nabla f_n(x). \end{aligned}$$

Therefore, taking the supremum over $L \geq 1$ we obtain (2.14) as ϕ is continuous.

We skip the details for the first part of the Theorem and instead we focus on the connection with $W^{1,1}$.
By contradiction, assume $f \in W^{\phi,weak}(v)$ and $v \geq \frac{1}{C}$ over Ω but that $f \notin W^{1,1}(B(x_0, 2r))$ for some ball such that $B(x_0, r) \subset \Omega$. We have

$$\begin{aligned} \int_{\mathbb{R}^d} M_L \nabla f(x) &\geq \int_{B(x_0,2r)} \int_{B(x_0,2r)} \frac{|\nabla f|_s(dz)}{(L^{-1} + |x-z|)|x-z|^{d-1}} v(dx) \\ &\geq \frac{1}{C} \int_{B(x_0,2r)} \int_{B(x_0,2r)} \frac{|\nabla f|_s(dz)}{(L^{-1} + |x-z|)|x-z|^{d-1}} dx. \end{aligned}$$

Define the kernel

$$K_L = C_L \frac{\mathbb{1}_{|x|\leq 2r}}{(L^{-1} + |x|)|x|^{d-1}},$$

with C_L such that $\|K_L\|_{L^1} = 1$. Observe that K_L is a standard approximation by convolution so in particular

$$\liminf_{L\to+\infty} \int_{B(x_0,2r)} K_L * \left(|\nabla f|_s\right) dx \geq \int_{B(x_0,r)} |\nabla f|_s (B(x_0, 2r)) > 0.$$

Note that $C_L \sim \frac{1}{\log L}$ as $L \to +\infty$. Indeed,

$$\begin{aligned} 1 &= \int_{\mathbb{R}^d} K_L(x) dx = C_L c_d \int_0^{2r} \frac{1}{(L^{-1} + s)s^{d-1}} s^{d-1} ds \\ &= C_L c_d \left(\log(L^{-1} + 2r) - \log L^{-1}\right) \\ &= C_L c_d \left(\log(L^{-1} + 2r) + \log L\right) \sim C_L \log L, \qquad \text{as } L \to +\infty. \end{aligned}$$

So there exists a constant $C > 0$ such that for L large enough

$$\int_{B(x_0,2r)} \int_{B(x_0,2r)} \frac{|\nabla f|_s(dz)}{(L^{-1} + |x-z|)|x-z|^{d-1}} dx \geq \frac{\log L}{C}.$$

Therefore,

$$\|f\|_{W^{\phi,weak}(v)} \geq \frac{1}{C} \sup_{L \geq 1} \frac{\phi(L)}{L} = +\infty,$$

giving the desired contradiction.

Reciprocally, assume that $v \leq C$ on Ω and that $f \in W^{1,1}(K)$ compactly supported in $K \subset \Omega$.

First, by Sobolev embedding, f and hence Mf belong to L^p for some $p > 1$ and $Mf \in L^\infty(\Omega^c)$. Therefore,

$$\begin{aligned}\int M|f|(x)v(dx) &= \int_\Omega M|f|(x)v(dx) + \int_{\Omega^c} M|f|(x)v(dx) \\ &\leq C \int_\Omega M|f|(x)dx + C \\ &\leq C\|Mf\|_{L^p} + C < \infty.\end{aligned}$$

Then for $x \notin \Omega$

$$M_L \nabla f(x) \leq \sqrt{\log L} + \frac{1}{d(x,K)^d} \int_K |\nabla f(z)|\, dz.$$

As a consequence for any ϕ satisfying (2.13), there exists some finite constant C_ϕ such that

$$\|f\|_{W^{\phi,weak}(v)} \leq C_\phi + C \sup_{L \geq 1} \frac{\phi(L)}{L \log L} \int_\Omega M_L \nabla f(x) dx.$$

Now decompose ∇f in level sets by defining for all $n \in \mathbb{Z}$

$$\omega_n := \{z \in K; 2^n \leq |\nabla f(z)| \leq 2^{n+1}\}.$$

Then

$$\begin{aligned}\int_\Omega M_L \nabla f(x) dx &\leq |\Omega| \sqrt{\log L} + \sum_{n \geq \log_2 L - 1} \int_\Omega \int_K \frac{2^{n+1} \mathbb{1}_{z \in \omega_n}}{(L^{-1} + |x-z|)|x-z|^{d-1}} dz dx \\ &\leq |\Omega| \sqrt{\log L} + \sum_{n \geq \log_2 L - 1} 2^{n+1} |\omega_n| \log L.\end{aligned}$$

Since $\nabla f \in L^1(K)$, one has $\sum_n 2^n|\omega_n| < +\infty$, and thus

$$S_N = \sum_{n\geq N} 2^n|\omega_n| \to 0, \text{as } N \to +\infty.$$

We can now define an appropriate ϕ. Choose any smooth function such that $\frac{\phi(x)}{x}$ is non decreasing and

$$\phi(2^{N+1}) = 2^{N+1} \min\{N^{\frac{1}{4}}, S_N^{-1}\}.$$

Then ϕ satisfies (2.13) and

$$\sup_{L\geq 1} \frac{\phi(L)}{L\log L} \int_\Omega M_L \nabla f(x)dx \leq 2\sup_N \frac{\phi(2^{N+1})}{2^N} S_N \leq 4.$$

Therefore, we can conclude that $f \in W^{\phi,weak}(v)$. □

We can now define the space $W_T^{\phi,weak}(u)$.

Definition 2.1.15. *Fix ϕ to be a superlinear function and $u \in M_1$. For all $T > 0$, the space $W_T^{\phi,weak}(u)$ is defined as the set of the measurable functions on $[0,T] \times \mathbb{R}^d$ such that, for almost $t \in [0,T]$, $f(t,\cdot) \in W^{\phi,weak}(u(t,\cdot))$ and*

$$\|f\|_{W_T^{\phi,weak}(u)} := \int_0^T \|f\|_{W^{\phi,weak}(u(t,\cdot))}dt < +\infty. \tag{2.15}$$

We note that if $u(t,\cdot)$ is the distribution of X_t solution to (1), then, for all $T > 0$ and $F \in W_T^{\phi,weak}(u)$,

$$\|F\|_{W_T^{\phi,weak}(u)} \geq C \sup_{L\geq 1} \frac{\phi(L)}{L\log L} \mathbb{E}\left[\int_0^T (M|F|(t,X_t)dt + M_L|\nabla F|(t,X_t))dt\right]. \tag{2.16}$$

As a consequence of Theorem 2.1.14 we have the following

Corollary 2.1.16. *Fix $T > 0$ and assume $u \in M_1$ and that ϕ is superlinear, continuous and satisfies (2.13). Then $W_T^{\phi,weak}(u)$ is well defined and $\|\cdot\|_{W_T^{\phi,weak}(u)}$ is lower semi-continuous with respect to convergence in the sense of distribution: if $f_n \xrightarrow[n\to+\infty]{} f$ in the sense of distribution, then*

$$\|f\|_{W_T^{\phi,weak}(u)} \leq \liminf_n \|f_n\|_{W_T^{\phi,weak}(u)}. \tag{2.17}$$

And if for a given f measurable on $\mathbb{R}^+ \times \mathbb{R}^d$ with $f(t,\cdot) \in BV_{loc}(\mathbb{R}^d)$ for almost all $t \geq 0$, u_n converges to u for the weak topology in M_1, then

$$\|f\|_{W_T^{\phi,weak}(u)} \leq \liminf_n \|f\|_{W_T^{\phi,weak}(u_n)}.$$

Moreover, if $u \geq \frac{1}{C}$ over $[0,T] \times \Omega$ where Ω is a smooth open set Ω and $f \in W_T^{\phi,weak}(u)$ then $f \in L_t^1([0,T], W^{1,1}(K))$ for any compact set $K \subset \Omega$. Reciprocally, if $u \leq C$ over $[0,T] \times \Omega$ and $f \in L_t^1([0,T], W^{1,1}(\Omega))$ with compact support in $[0,T] \times \Omega$, then there exists a superlinear ϕ satisfying (2.13) such that $f \in W_T^{\phi,weak}(u)$.

2.2 Statement of the results

Now we can state the two most general results. In the multidimensional case we have the following

Theorem 2.2.1. *Assume $d \geq 2$.*

1. *Existence: Fix $T > 0$ and assume that there exists two sequences $F_n, \sigma_n \in C^\infty \cap L^\infty$ converging in the sense of distributions to F and σ respectively, such that the solution $u_n \in M_1$ to (4) satisfies for some superlinear ϕ,*

   (a) $\int_0^T \int_{\mathbb{R}^d} \left(|\sigma_n - \sigma| + |F_n - F|\right) du_n dt \xrightarrow[n\to+\infty]{} 0$;

   (b) $\sup_n \left(\|F\|_{W_T^{\phi,weak}(u_n)} + \|\sigma\|_{H_T^1(u_n)} + \|F_n\|_{L^\infty} + \|\sigma_n\|_{L^\infty}\right) < \infty$;

   (c) $u_n \xrightarrow[n\to+\infty]{} u$ *in the weak topology of M_1.*

   Then there exists a strong solution X_t to (1) such that $(X_t^n - \xi, t \in [0,T])_n$ converges in $L^p(\Omega, L^\infty([0,T]))$ for all $p > 1$ to $(X_t - \xi, t \in [0,T])$, with X_t^n the solutions to (3). In addition, $u(t,dx)$ is the law of X_t for almost all $t \in [0,T]$.

2. *Uniqueness: Let X and Y be two solutions to (1) with one-dimensional time marginals $u_X(t,\cdot)$ and $u_Y(t,\cdot)$ on $[0,T]$. Assume that $F, \sigma \in L^\infty$, $X_0 = Y_0$ a.s. and that*

   (a) $\|F\|_{W_T^{\phi,weak}(u_X)} + \|F\|_{W_T^{\phi,weak}(u_Y)} + \|\sigma\|_{H_T^1(u_X)} + \|\sigma\|_{H_T^1(u_Y)} < \infty$

for some superlinear function ϕ. Then one has pathwise uniqueness on $[0,T]$, that is, $\sup_{t\in[0,T]}|X_t - Y_t| = 0$ *a.s.*

Proof. Existence. Let us define the quantities

$$Q_{nm}^{(\epsilon)}(t) := \log\left(1 + \frac{|X_t^n - X_t^m|^2}{\epsilon^2}\right), \qquad \epsilon \in (0,1], n, m \geq 1. \tag{2.18}$$

We have the following estimates on the expectation of $Q_{nm}^{(\epsilon)}(t)$.

Lemma 2.2.2. *There exist a constant C such that for all $0 < \epsilon \leq 1$ and $n, m \geq 1$,*

$$\sup_{t\in[0,T]} \mathbb{E}[Q_{nm}^{(\epsilon)}(t)] \leq C|\log\epsilon|\,\tilde{\eta}(\epsilon) + C\frac{\eta(n,m)}{\epsilon^2}, \tag{2.19}$$

where $\eta(n,m) \to 0$ when $n, m \to +\infty$ and $\tilde{\eta}(\epsilon) := (\epsilon\phi(\epsilon^{-1}))^{-1} \to 0$ when $\epsilon \to 0$.

Proof. Note that

$$\left|\nabla\left(\log\left(1 + \frac{|x|^2}{\epsilon^2}\right)\right)\right| = \left|\frac{2x}{\epsilon^2 + |x^2|}\right| \leq \frac{C}{\epsilon + |x|} \tag{2.20}$$

and

$$\left|\nabla^2\left(\log\left(1 + \frac{|x|^2}{\epsilon^2}\right)\right)\right| = \left|\nabla\left(\frac{2x}{\epsilon^2 + |x^2|}\right)\right| \leq \frac{C}{\epsilon^2 + |x|^2} \tag{2.21}$$

By Itô's formula, for any $f \in bC^2$,

$$f(X_t^n - X_t^m) = f(0) + \int_0^t \nabla f(X_s^n - X_s^m)d(X_s^n - X_s^m) + \frac{1}{2}\int_0^t \nabla^2 f(X_s^n - X_s^m)d\langle X^n - X^m\rangle_s.$$

Now,

$$dX_t^n = F_n(t, X_t^n)dt + \sigma_n(t, X_t^n)dW_t \qquad \text{and} \qquad d\langle X^n\rangle_t = \sigma_n(t, X_t^n)\sigma_n^*(t, X_t^n)dt,$$

whence

$$d(X_t^n - X_t^m) = (F_n(t, X_t^n) - F_m(t, X_t^m))dt + (\sigma_n(t, X_t^n) - \sigma_m(t, X_t^m))dW_t$$

and

$$\begin{aligned} d\langle X^n - X^m\rangle_t = (&\sigma_n(t, X_t^n)\sigma_n^*(t, X_t^n) + \sigma_m(t, X_t^m)\sigma_m^*(t, X_t^m) \\ &- \sigma_m(t, X_t^m)\sigma_n^*(t, X_t^n) - \sigma_n(t, X_t^n)\sigma_m^*(t, X_t^m))dt. \end{aligned}$$

Thus, taking the expectation we have

$$\begin{aligned}\mathbb{E}\left[f(X_t^n - X_t^m)\right] &= f(0) + \int_0^t \mathbb{E}[\nabla f(X_s^n - X_s^m)(F_n(s,X_s^n) - F_m(s,X_s^m))]ds \\ &+ \frac{1}{2}\int_0^t \mathbb{E}[\nabla^2 f(X_s^n - X_s^m)(\sigma_n(s,X_s^n)\sigma_n^*(s,X_s^n) + \sigma_m(s,X_s^m)\sigma_m^*(s,X_s^m) \\ &- \sigma_m(s,X_s^m)\sigma_n^*(s,X_s^n) - \sigma_n(s,X_s^n)\sigma_m^*(s,X_s^m))]ds.\end{aligned}$$

Now, since $\sup_n \|\sigma_n\|_{L^\infty} < +\infty$,

$$\begin{aligned}&\left|\sigma_n(t,X_t^n)\sigma_n^*(t,X_t^n) + \sigma_m(t,X_t^m)\sigma_m^*(t,X_t^m) - \sigma_m(t,X_t^m)\sigma_n^*(t,X_t^n) - \sigma_n(t,X_t^n)\sigma_m^*(t,X_t^m)\right| \\ &\leq \left|\sigma_n(t,X_t^n)\sigma_n^*(t,X_t^n) - \sigma(t,X_t^n)\sigma_n^*(t,X_t^n)\right| + \left|\sigma_m(t,X_t^m)\sigma_m^*(t,X_t^m) - \sigma(t,X_t^m)\sigma_m^*(t,X_t^m)\right| \\ &+ \left|\sigma(t,X_t^n)\sigma_n^*(t,X_t^n) - \sigma_m(t,X_t^m)\sigma_n^*(t,X_t^n)\right| + \left|\sigma(t,X_t^m)\sigma_m^*(t,X_t^m) - \sigma_n(t,X_t^n)\sigma_m^*(t,X_t^m)\right| \\ &\leq 2\sup_k \|\sigma_k\|_{L^\infty}\left(\left|\sigma_n(t,X_t^n) - \sigma(t,X_t^n)\right| + \left|\sigma_m(t,X_t^m) - \sigma(t,X_t^m)\right|\right)\end{aligned}$$

hence

$$\begin{aligned}&\mathbb{E}\left[f(X_t^n - X_t^m)\right] \\ &\leq f(0) + \int_0^t \mathbb{E}\Big[\left|\nabla^2 f(X_s^n - X_s^m)\right| \sup_k \|\sigma_k\|_{L^\infty}\Big(\left|\sigma_n(s,X_s^n) - \sigma(s,X_s^n)\right| \\ &+ \left|\sigma_m(s,X_s^m) - \sigma(s,X_s^m)\right|\Big)\Big]ds + \int_0^t \mathbb{E}\left[\left|\nabla f(X_s^n - X_s^m)\right|\left|F_n(s,X_s^n) - F_m(s,X_s^m)\right|\right]ds.\end{aligned}$$

Hence, from (2.20) and (2.21)

$$\mathbb{E}[Q_{nm}^{(\epsilon)}(t)] \leq C\int_0^t \mathbb{E}\left[\frac{\left|F(s,X_s^n) - F(s,X_s^m)\right|}{\epsilon + |X_s^n - X_s^m|}\right]ds + C\frac{\eta(n,m)}{\epsilon^2},$$

where C is a constant independent of n and ϵ and

$$\begin{aligned}\eta(n,m) := \int_0^t \mathbb{E}\Big[&\left|\sigma_n(s,X_s^n) - \sigma(s,X_s^n)\right| + \left|\sigma_m(s,X_s^m) - \sigma(s,X_s^m)\right| \\ &+ \left|F_n(s,X_s^n) - F(s,X_s^n)\right| + \left|F_m(s,X_s^m) - F(s,X_s^m)\right|\Big]ds\end{aligned}$$

which is such that $\eta(n,m) \to 0$ as $n,m \to +\infty$ by assumption (1a).

Now, let $h := |F| + M_{\frac{1}{\epsilon}} \nabla F$. Then by Lemma B.0.2,

$$\begin{aligned}
&\int_0^t \mathbb{E}\left[\frac{\left|F(s,X_s^n) - F(s,X_s^m)\right|}{\epsilon + |X_s^n - X_s^m|}\right] ds \\
&\leq C \int_0^t \mathbb{E}\left[\frac{\left|h(s,X_s^n) - h(s,X_s^m)\right|}{\epsilon + |X_s^n - X_s^m|} \left(|X_s^n - X_s^m| + \epsilon\right)\right] ds \\
&= C \int_0^t \int h(s,x) \left(u_n(s,x) + u_m(s,x)\right) dxds.
\end{aligned}$$

Note that since $BV_{loc}(\mathbb{R}^d) \subseteq L^1_{loc}(\mathbb{R}^d)$, Theorem B.0.7 follows that $|F| \leq M|F|$ a.e. So by (2.16) and assumption (1b),

$$\begin{aligned}
&\int_0^t \int h(s,x) \left(u_n(s,x) + u_m(s,x)\right) dxds \\
&\leq C \frac{\log\left(\frac{1}{\epsilon}\right)}{\epsilon \phi(\frac{1}{\epsilon})} \left(\|F\|_{W_T^{\phi,weak}(u_n)} + \|F\|_{W_T^{\phi,weak}(u_m)}\right) \leq C \frac{|\log \epsilon|}{\epsilon \phi(\epsilon^{-1})}.
\end{aligned}$$

Combining the previous inequalities we obtain (2.19), where $\tilde{\eta}(\epsilon) := (\epsilon \phi(\epsilon^{-1}))^{-1} \to 0$ as $\epsilon \to 0$ since ϕ is superlinear. □

Fix $p > 1$. We want to deduce from Lemma 2.2.2 that $(X_t^n - \xi)$ is a Cauchy sequence in $L^p(\Omega, L^\infty([0,T]))$.

Since F_n and σ_n are uniformly bounded, it is standard to deduce from the Burkholder-Davis-Gundy inequality (Theorem B.0.9) that for all $p > 1$, $X_t^n - \xi$ are uniformly bounded in $L^p(\Omega, L^\infty([0,T]))$. Indeed,

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} |X_t^n - \xi|^p\right] &\leq C_p \mathbb{E}\left[\langle X^n - \xi\rangle_T^{\frac{p}{2}}\right] = \mathbb{E}\left[\left(\int_0^T \sigma_n(s,X_s^n)\sigma_n^*(s,X_s^n) ds\right)^{\frac{p}{2}}\right] \\
&\leq \mathbb{E}\left[\left(\int_0^T \|\sigma_n\|_{L^\infty}^2 ds\right)^{\frac{p}{2}}\right] = T^{\frac{p}{2}} \|\sigma_n\|_{L^\infty}^p
\end{aligned}$$

and since from hypothesis (1b) $\sup_n \|\sigma_n\|_{L^\infty} = C < \infty$, we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t^n - \xi|^p\right] \leq C < \infty, \qquad \forall n \in \mathbb{N}^*. \tag{2.22}$$

So now we can prove that $(X_t^n - \xi)$ is a Cauchy sequence in $L^p(\Omega, L^\infty([0,T]))$.

Lemma 2.2.3. *For all $p > 1$,*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t^n - X_t^m|^p\right] \to 0 \qquad \text{as } n, m \to +\infty. \tag{2.23}$$

Proof. Fix $t \geq 0$, for any ϵ and L such that $0 \leq \epsilon \leq L$,

$$\begin{aligned}
&\mathbb{E}\left[|X_t^n - X_t^m|^p\right] \leq \mathbb{E}\left[|X_t^n - X_t^m|^p \,||X_t^n - X_t^m| \geq L\right] \\
&+ \mathbb{E}\left[|X_t^n - X_t^m|^p \,||X_t^n - X_t^m| \leq \sqrt{\epsilon}\right] + \mathbb{E}\left[|X_t^n - X_t^m|^p \,|\sqrt{\epsilon} \leq |X_t^n - X_t^m| \leq L\right] \\
&\leq \mathbb{E}\left[|X_t^n - X_t^m|^p \,||X_t^n - X_t^m| \geq L\right] + \epsilon^{\frac{p}{2}} + \mathbb{E}\left[L^p|\sqrt{\epsilon} \leq |X_t^n - X_t^m| \leq L\right] \\
&\leq \mathbb{E}\left[|X_t^n - X_t^m|^p \,||X_t^n - X_t^m| \geq L\right] + \epsilon^{\frac{p}{2}} + L^p\mathbb{P}\left(|X_t^n - X_t^m| \geq \sqrt{\epsilon}\right).
\end{aligned} \tag{2.24}$$

Note that

$$\begin{aligned}
\mathbb{E}\left[|X_t^n - X_t^m|^p \,||X_t^n - X_t^m| \geq L\right] &\leq \mathbb{E}\left[|X_t^n - \xi + \xi - X_t^m|^{p+1} \frac{1}{|X_t^n - X_t^m|} ||X_t^n - X_t^m| \geq L\right] \\
&\leq \frac{1}{L}\left(\mathbb{E}\left[|X_t^n - \xi|^{p+1}\right] + \mathbb{E}\left[|X_t^m - \xi|^{p+1}\right]\right).
\end{aligned} \tag{2.25}$$

By (2.22),

$$\sup_{n\geq 1, t\in[0,T]} \mathbb{E}\left[|X_t^n - \xi|^{p+1}\right] \leq \sup_{n\geq 1, t\in[0,T]} \mathbb{E}\left[\sup_{t\in[0,T]} |X_t^n - \xi|^{p+1}\right] \leq C < +\infty. \tag{2.26}$$

By the generalised Markov inequality,

$$\begin{aligned}
\mathbb{P}\left(|X_t^n - X_t^m| \geq \sqrt{\epsilon}\right) = \mathbb{P}\left(|X_t^n - X_t^m|^2 \geq \epsilon\right) &\leq \frac{1}{|\log \epsilon|}\mathbb{E}\left[\left|\log\left(|X_t^n - X_t^m|^2\right)\right|\right] \\
&\leq \frac{1}{|\log \epsilon|}\mathbb{E}\left[\left|\log\left(1 + \frac{|X_t^n - X_t^m|^2}{\epsilon^2}\right)\right|\right] = \frac{\mathbb{E}[Q_{nm}^{(\epsilon)}(t)]}{|\log \epsilon|}.
\end{aligned} \tag{2.27}$$

Now, by the previous inequalities and Lemma 2.2.2 we have,

$$\mathbb{E}\left[|X_t^n - X_t^m|^p\right] \leq C\left(\frac{1}{L} + \epsilon^{\frac{p}{2}} + \frac{L^p}{|\log \epsilon|}\left(|\log \epsilon|\,\tilde{\eta}(\epsilon) + \frac{\eta(n,m)}{\epsilon^2}\right)\right). \tag{2.28}$$

Taking, for example, $\epsilon^2 = \eta(n,m)$ and $L = \left(\frac{1}{|\log \epsilon|} + \tilde{\eta}(\epsilon)\right)^{-\frac{1}{2p}}$, we can conclude that

$$\sup_{t\in[0,T]} \mathbb{E}\left[|X_t^n - X_t^m|^p\right] \to 0 \qquad \text{as } n, m \to +\infty.$$

Lastly, we have to pass the supremum inside the expectation. It suffices to repeat the computation in the proof of Lemma 2.2.2 applied to $|A^n_{t\wedge\tau} - A^m_{t\wedge\tau}|^2 \vee |M^n_{t\wedge\tau} - M^m_{t\wedge\tau}|^2$, where τ is any stopping time and $X^n_t = \xi + A^n_t + M^n_t$ is Doob's decomposition of the semimartingale X^n_t, that is,

$$A^n_t = \int_0^t F_n(s, X^n_s)ds \quad \text{and} \quad M^n_t = \int_0^t \sigma_n(s, X^n_s)dW_s.$$

Indeed,

$$\begin{aligned}&\mathbb{E}\left[\log\left(1 + \frac{|A^n_{t\wedge\tau} - A^m_{t\wedge\tau}|^2 \vee |M^n_{t\wedge\tau} - M^m_{t\wedge\tau}|^2}{\epsilon^2}\right)\right]\\ &\leq C\int_0^t \mathbb{E}\left[\frac{\left|F(s, X^n_s) - F(s, X^m_s)\right|}{\epsilon + |A^n_{t\wedge\tau} - A^m_{t\wedge\tau}| \vee |M^n_{t\wedge\tau} - M^m_{t\wedge\tau}|}\right] ds + C\frac{\eta(n,m)}{\epsilon^2},\end{aligned} \tag{2.29}$$

from which

$$\begin{aligned}&\mathbb{E}\left[\log\left(1 + \frac{|A^n_{t\wedge\tau} - A^m_{t\wedge\tau}|^2 \vee |M^n_{t\wedge\tau} - M^m_{t\wedge\tau}|^2}{\epsilon^2}\right)\right]\\ &\leq C\int_0^t \mathbb{E}\left[\frac{\left|F(s, X^n_s) - F(s, X^m_s)\right|}{\epsilon + \frac{1}{2}|X^n_t - X^m_t|}\right] ds + C\frac{\eta(n,m)}{\epsilon^2}.\end{aligned} \tag{2.30}$$

Now, the same computation as in(2.24)-(2.28) gives

$$\sup_{\substack{t\in[0,T],\\ \tau \text{ stopping time}}} \mathbb{E}\left[|A^n_{t\wedge\tau} - A^m_{t\wedge\tau}|^p \vee |M^n_{t\wedge\tau} - M^m_{t\wedge\tau}|^p\right] \to 0 \qquad \text{as } n, m \to +\infty.$$

Since $p > 1$, from Doob's inequality

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]} |M^n_t - M^m_t|^p\right] &\leq C\mathbb{E}\left[|M^n_T - M^m_T|^p\right]\\ &\leq C\sup_{t\in[0,T]} \mathbb{E}\left[|M^n_t - M^m_t|^p\right] \to 0 \qquad \text{as } n, m \to +\infty.\end{aligned}$$

Fix $\eta > 0$ and $n_0 \in \mathbb{N}^*$ such that

$$\sup_{\substack{t\in[0,T],\\ \tau \text{ stopping time}}} \mathbb{E}\left[|A^n_{t\wedge\tau} - A^m_{t\wedge\tau}|^p\right] \leq \eta$$

for all $n, m \geq n_0$. For all $M > 0$, let $\tau := \inf\{t \geq 0 : |A^n_t - A^m_t| \geq M\}$. Then, by Markov inequality

$$\mathbb{P}\left(\sup_{t\in[0,T]} |A^n_t - A^m_t| \geq M\right) = \mathbb{P}(\tau \leq T) \leq \frac{\mathbb{E}\left[\sup_{t\in[0,T]}|A^n_{t\wedge\tau} - A^m_{t\wedge\tau}|^p\right]}{M^p} \leq \frac{\eta}{M^p}.$$

Now, for all $1 < q < p$,

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]} |A^n_t - A^m_t|^q\right] &= q\int_0^{+\infty} x^{q-1}\mathbb{P}\left(\sup_{t\in[0,T]} |A^n_t - A^m_t| \geq x\right) dx \\ &\leq q\int_0^{+\infty} x^{q-1}\left(\frac{\eta}{x^p}\wedge 1\right) dx = \frac{p\eta^{\frac{q}{p}}}{p-q}.\end{aligned}$$

Therefore,

$$\mathbb{E}\left[\sup_{t\in[0,T]} |A^n_t - A^m_t|^p\right] \to 0 \qquad \text{as } n, m \to \infty,$$

which concludes the proof of (2.23).

□

From the fact that $(X^n_t - \xi)$ is a Cauchy sequence in $L^p(\Omega, L^\infty([0,T]))$, we deduce that exists a subsequence of $(X^n_t, t \in [0,T])_n$ which converges almost sure to a process $(X_t, t \in [0,T])$ in the L^∞ norm and $(X_t - \xi, t \in [0,T]) \in L^p(\Omega, L^\infty([0,T]))$ for all $p > 1$. Since the convergence holds in the L^∞ norm, the process X is continuous a.s. and adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$.
Since u_n converges to u in the weak topology of M_1, we have for all $f \in bC([0,T]\times\mathbb{R}^d)$

$$\mathbb{E}\left[\int_0^T f(t, X_t)dt\right] = \int_{\mathbb{R}^d}\int_0^t f(t,x)u(dt,dx),$$

so $u(t,dx)$ is the law of X_t for almost t.
Defining for all $t \in [0,T]$,

$$Y_t := \int_0^t F(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s,$$

it only remains to check that $Y_t = X_t - \xi$ for all $t \in [0,T]$ a.s. As

$$X^n_t - \xi = \int_0^t F_n(s, X^n_s)ds + \int_0^t \sigma_n(s, X^n_s)dW_s$$

we have $Y_t = X_t - \xi$ provided that

$$\int_0^t \mathbb{E}\left[\left|F_n(s,X_s^n) - F(s,X_s)\right| + \left|\sigma_n(s,X_s^n) - \sigma(s,X_s)\right|^2\right] ds \xrightarrow[n\to+\infty]{} 0. \tag{2.31}$$

Indeed,

$$\mathbb{E}\left[\int_0^t \left|F_n(s,X_s^n) - F(s,X_s)\right| ds\right] \xrightarrow[n\to+\infty]{} 0 \implies \int_0^t \left|F_n(s,X_s^n) - F(s,X_s)\right| ds \xrightarrow[n\to+\infty]{\mathbb{P}} 0$$

and

$$\mathbb{E}\left[\int_0^t \left|\sigma_n(s,X_s^n) - \sigma(s,X_s)\right|^2 ds\right] \xrightarrow[n\to+\infty]{} 0 \implies \int_0^t \left|\sigma_n(s,X_s^n) - \sigma(s,X_s)\right|^2 ds \xrightarrow[n\to+\infty]{\mathbb{P}} 0.$$

Now, by Lemma B.0.6

$$\int_0^t \left|\sigma_n(s,X_s^n) - \sigma(s,X_s)\right|^2 ds \xrightarrow[n\to+\infty]{\mathbb{P}} 0 \implies \int_0^t \big(\sigma_n(s,X_s^n) - \sigma(s,X_s)\big)\, dW_s \xrightarrow[n\to+\infty]{\mathbb{P}} 0.$$

Hence, up to subsequences, for all $t \in [0,T]$

$$|X_t^n - \xi - Y_t| \le \int_0^t \left|F_n(s,X_s^n) - F(s,X_s)\right| ds + \left|\int_0^t \big(\sigma_n(s,X_s^n) - \sigma(s,X_s)\big)\, dW_s\right| \xrightarrow[n\to+\infty]{a.s.} 0,$$

that is

$$X_t^n - \xi \xrightarrow[n\to+\infty]{a.s.} Y_t, \qquad \forall t \in [0,T].$$

But we know that

$$X_t^n \xrightarrow[n\to+\infty]{a.s.} X_t \qquad \forall t \in [0,T],$$

so $X_t - \xi = Y_t$ a.s. for all $t \in [0,T]$.

It only remains to prove (2.31) which is implied by:

For any fixed $\epsilon > 0$,

$$\int_0^T \left(\mathbb{P}\left(\left|F(s,X_s^n) - F(s,X_s)\right| > \epsilon\right) + \mathbb{P}\left(\left|\sigma(s,X_s^n) - \sigma(s,X_s)\right| > \epsilon\right)\right) ds \xrightarrow[n\to+\infty]{} 0. \tag{2.32}$$

Indeed,

$$\begin{aligned}
&\int_0^t \mathbb{E}\left[\left|F_n(s,X_s^n) - F(s,X_s)\right| + \left|\sigma_n(s,X_s^n) - \sigma(s,X_s)\right|^2\right] ds \\
&\le \int_0^T \mathbb{E}\Big[\left|F_n(s,X_s^n) - F(s,X_s^n)\right| + \left|F(s,X_s^n) - F(s,X_s)\right| \\
&+ 3\left(\left|\sigma_n(s,X_s^n) - \sigma(s,X_s^n)\right|^2 + \left|\sigma(s,X_s^n) - \sigma(s,X_s)\right|^2\right)\Big] ds
\end{aligned}$$

Now, by assumption (1a) we have

$$\int_0^T \mathbb{E}\left[\left|F_n(s,X_s^n)-F(s,X_s^n)\right|\right] ds \xrightarrow[n\to+\infty]{} 0,$$

and

$$\int_0^T \mathbb{E}\left[\left|\sigma_n(s,X_s^n)-\sigma(s,X_s^n)\right|\right] ds \xrightarrow[n\to+\infty]{} 0,$$

so

$$\left|\sigma_n(s,X_s^n)-\sigma(s,X_s^n)\right| \xrightarrow[n\to+\infty]{} 0, \qquad \text{a.s. for almost } s\in[0,T]$$

and in particular

$$\left|\sigma_n(s,X_s^n)-\sigma(s,X_s^n)\right|^2 \xrightarrow[n\to+\infty]{} 0, \qquad \text{a.s. for almost } s\in[0,T].$$

In addition, by assumption (1b)

$$\left|\sigma_n(s,X_s^n)-\sigma(s,X_s^n)\right|^2 \le 3\left(\sup_{n\in\mathbb{N}^*}\|\sigma_n\|_{L^\infty}+\left|\sigma(s,X_s^n)\right|^2\right) \le C+\left|\sigma(s,X_s^n)\right|^2 \in L^2[0,T]\times\Omega,$$

in fact, again by assumption (1b)

$$\int_0^T \mathbb{E}\left[\left|\sigma(s,X_s^n)\right|^2\right] ds \le \|\sigma_n\|^2_{H^1_T(u_n)} \le \sup_{n\in\mathbb{N}^*}\|\sigma_n\|^2_{H^1_T(u_n)} \le C.$$

Then by dominated convergence Theorem

$$\int_0^T \mathbb{E}\left[\left|\sigma_n(s,X_s^n)-\sigma(s,X_s^n)\right|^2\right] ds \xrightarrow[n\to+\infty]{} 0.$$

Moreover, by Theorem 2.1.14, Theorem 2.1.3 and assumption (1b) we have

$$\begin{aligned}&\mathbb{E}\left[\left|F(s,X_s^n)-F(s,X_s)\right|\right] \le \int_{\mathbb{R}^d}\left|F(s,x)\right|\big(u_n(s,dx)+u(s,dx)\big)\\ &\le \|F(s,\cdot)\|_{W^{\phi,weak}(u_n(s,\cdot))}+\liminf_n \|F(s,\cdot)\|_{W^{\phi,weak}(u_n(s,\cdot))} \le C_1(s)\in L^1(\Omega,\mathbb{P})\end{aligned} \tag{2.33}$$

and

$$\begin{aligned}&\mathbb{E}\left[\left|\sigma(s,X_s^n)-\sigma(s,X_s)\right|^2\right] \le 3\int_{\mathbb{R}^d}\left|\sigma(s,x)\right|\big(u_n(s,dx)+u(s,dx)\big)\\ &\le 3\left(\|\sigma(s,\cdot)\|^2_{H^1(u_n(s,\cdot))}+\liminf_n \|F(s,\cdot)\|_{H^1(u_n(s,\cdot))}{}^2\right) \le C_2(s)\in L^2(\Omega,\mathbb{P}).\end{aligned} \tag{2.34}$$

Now, by (2.32) we have for almost $s \in [0,T]$

$$\mathbb{P}\left(\left|F(s,X_s^n) - F(s,X_s)\right| > \epsilon\right) + \mathbb{P}\left(\left|\sigma(s,X_s^n) - \sigma(s,X_s)\right| > \epsilon\right) \xrightarrow[n\to+\infty]{} 0,$$

which gives, together with (2.33) and (2.34)

$$\mathbb{E}\left[\left|F(s,X_s^n) - F(s,X_s)\right|\right] \xrightarrow[n\to+\infty]{} 0$$

and

$$\mathbb{E}\left[\left|\sigma(s,X_s^n) - \sigma(s,X_s)\right|^2\right] \xrightarrow[n\to+\infty]{} 0.$$

In addition, again by assumption (1b)

$$\mathbb{E}\left[\left|F(s,X_s^n) - F(s,X_s)\right|\right] \leq C_1(s) \in L^1([0,T])$$

and

$$\mathbb{E}\left[\left|\sigma(s,X_s^n) - \sigma(s,X_s)\right|^2\right] \leq C_2(s) \in L^1([0,T]).$$

Then by dominated convergence Theorem

$$\int_0^T \mathbb{E}\left[\left|F(s,X_s^n) - F(s,X_s)\right|\right] ds \xrightarrow[n\to+\infty]{} 0$$

and

$$\int_0^T \mathbb{E}\left[\left|\sigma(s,X_s^n) - \sigma(s,X_s)\right|^2\right] ds \xrightarrow[n\to+\infty]{} 0.$$

Lastly, we prove (2.32) for σ, the argument for F is similar.
By Corollary 2.1.5 and assumption (1b),

$$\int_0^T \int_{\mathbb{R}^d} \left(M\left|\nabla\sigma(t,x)\right|\right)^2 \left(u(t,dx) + u_n(t,dx)\right) dt \leq \|\sigma\|_{H^1_T(u_n)} + \liminf_n \|\sigma\|_{H^1_T(u_n)} \leq C. \tag{2.35}$$

Now, by Lemma B.0.1

$$\mathbb{P}\left(\left|\sigma(s,X_s^n) - \sigma(s,X_s)\right| > \epsilon\right) \leq \mathbb{P}\left(\left(M\left|\nabla\sigma(s,X_s^n)\right| + M\left|\nabla\sigma(s,X_s)\right|\right) > \frac{\epsilon}{|X_s^n - X_s|}\right)$$
$$\leq \mathbb{P}\left(|X_s^n - X_s| > \epsilon^2\right) + \mathbb{P}\left(M\left|\nabla\sigma(s,X_s^n)\right| \geq \frac{1}{2\epsilon}\right) + \mathbb{P}\left(M\left|\nabla\sigma(s,X_s)\right| \geq \frac{1}{2\epsilon}\right),$$

where the last inequality is justify by the fact that $M\big|\nabla\sigma(s,X_s^n)\big| + M\big|\nabla\sigma(s,X_s)\big| > \frac{\epsilon}{\big|X_s^n - X_s\big|}$ implies $|X_s^n - X_s| > \frac{\epsilon}{M\big|\nabla\sigma(s,X_s^n)\big|+M\big|\nabla\sigma(s,X_s)\big|} > \epsilon^2$, if $M\big|\nabla\sigma(s,X_s^n)\big| < \frac{1}{2\epsilon}$ and $M\big|\nabla\sigma(s,X_s)\big| < \frac{1}{2\epsilon}$, so

$$\begin{aligned}
&\left(\left(M\big|\nabla\sigma(s,X_s^n)\big| + M\big|\nabla\sigma(s,X_s)\big|\right) > \frac{\epsilon}{|X_s^n - X_s|}\right)\\
&\subseteq \left(|X_s^n - X_s| > \epsilon^2\right) \cup \left(M\big|\nabla\sigma(s,X_s^n)\big| \geq \frac{1}{2\epsilon}\right) \cup \left(M\big|\nabla\sigma(s,X_s)\big| \geq \frac{1}{2\epsilon}\right).
\end{aligned}$$

Therefore,by Markov inequality and (2.35),

$$\begin{aligned}
&\int_0^T \mathbb{P}\left(\big|\sigma(s,X_s^n) - \sigma(s,X_s)\big| > \epsilon\right) ds\\
&\leq \int_0^T \left(\mathbb{P}\left(|X_s^n - X_s| > \epsilon^2\right) + \mathbb{P}\left(M\big|\nabla\sigma(s,X_s^n)\big| \geq \frac{1}{2\epsilon}\right) + \mathbb{P}\left(M\big|\nabla\sigma(s,X_s)\big| \geq \frac{1}{2\epsilon}\right)\right) ds\\
&\leq \int_0^T \left(\mathbb{P}\left(|X_s^n - X_s| > \epsilon^2\right) + \mathbb{E}\left[4\epsilon^2(M\big|\nabla\sigma(s,X_s^n)\big|)^2\right] + \mathbb{E}\left[4\epsilon^2(M\big|\nabla\sigma(s,X_s)\big|)^2\right]\right) ds\\
&\leq \int_0^T \mathbb{P}\left(|X_s^n - X_s| > \epsilon^2\right) ds + 4C\epsilon^2 \qquad \forall\epsilon > 0,
\end{aligned}$$

so we conclude from the fact that $|X_s^n - X_s| \xrightarrow[n\to+\infty]{} 0$ almost surely.

Uniqueness. Consider two solutions X and Y satisfying the assumptions in point (2). Define a family of functions $(L_\epsilon)_{\epsilon>0} \subset C^\infty(\mathbb{R}^d)$ satisfying

$$\begin{aligned}
L_\epsilon(x) = 1 \quad &\text{if} \quad |x| \geq \epsilon, \qquad L_\epsilon(x) = 0 \quad \text{if} \quad |x| \leq \frac{\epsilon}{2},\\
&\epsilon\|\nabla L_\epsilon\|_{L^\infty} + \epsilon^2\|\nabla^2 L_\epsilon\|_{L^\infty} \leq C,
\end{aligned}$$

with C independent of ϵ, and $L_\epsilon(x) \geq L_{\epsilon'}(x)$ for all $\epsilon \leq \epsilon'$ and $x \in \mathbb{R}^d$. Use Itô's formula,

$$\begin{aligned}
\mathbb{E}\left[L_\epsilon(X_t - Y_t)\right] = L(0) &+ \int_0^t \mathbb{E}\left[\nabla L_\epsilon(X_s - Y_s)\left(F(s,X_s) - F(s,Y_s)\right)\right] ds\\
&+ \int_0^t \mathbb{E}\Big[\nabla^2 L_\epsilon(X_s - Y_s)\big(\sigma(s,X_s)\sigma^*(s,X_s) + \sigma(s,Y_s)\sigma^*(s,Y_s)\\
&- \sigma(s,X_s)\sigma^*(s,Y_s) - \sigma(s,Y_s)\sigma^*(s,X_s)\big)\Big] ds.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E}\left[L_\epsilon(X_t - Y_t)\right] \leq C\int_0^t \mathbb{E}\Bigg[\mathbb{1}_{\frac{\epsilon}{2}\leq|X_s - Y_s|\leq\epsilon}\Bigg(&\frac{\big|\sigma(s,X_s) - \sigma(s,Y_s)\big|^2}{\epsilon^2}\\
&+ \frac{\big|F(s,X_s) - F(s,Y_s)\big|}{\epsilon}\Bigg)\Bigg] ds.
\end{aligned}$$

Now denote $h := M|\nabla\sigma|$ so that

$$\int_0^T \int_{\mathbb{R}^d} \left|h(t,x)\right|^2 \big(u_X(t,dx) + u_Y(t,dx)\big)dt \le C < +\infty.$$

Define as well $\tilde{h}_\epsilon := |F| + M_{\frac{1}{\epsilon}} \nabla F$, so we have

$$\int_0^T \int_{\mathbb{R}^d} \tilde{h}_\epsilon(t,x) \big(u_X(t,dx) + u_Y(t,dx)\big)dt \le \frac{C|\log\epsilon|}{\epsilon\phi(\epsilon^{-1})}.$$

Note that we can always assume that ϕ satisfies (2.13), and so $\frac{\phi(\xi)}{\xi}$ is a non-increasing function which grows not faster than $\log\xi$. In particular, there exists a constant $C > 0$ such that

$$\frac{1}{C}\epsilon\phi(\epsilon^{-1}) \le \frac{\phi(\xi)}{\xi} \le C\epsilon\phi(\epsilon^{-1}) \qquad \forall\xi \in [\epsilon^{-\frac{1}{2}}, \epsilon^{-1}].$$

Consider the partition of $(0,1) = \cup_{i\in\mathbb{N}} I_i$ where $I_0 = [\frac{1}{2}, 1)$, $I_i = [a_i, b_i)$ for $i \in \mathbb{N}^*$ are disjoint with $b_i = \sqrt{a_i}$, so $|I_i| = b_i - a_i$ and $|I_i| \sim \sqrt{a_i}$ when $i \to +\infty$.
Now for any $\epsilon \in I_i$, choose $\bar{h}_\epsilon = \tilde{h}_{a_i}$. We have

$$\int_0^T \int_{\mathbb{R}^d} \bar{h}_\epsilon(t,x) \big(u_X(t,dx) + u_Y(t,dx)\big)dt \le C\frac{|\log b_i|}{b_i\phi(b_i^{-1})}.$$

By Lemma B.0.1 and Lemma B.0.2,

$$\begin{aligned}\mathbb{E}\left[L_\epsilon(X_t - Y_t)\right] \le C&\int_0^t \mathbb{E}\left[\big(h^2(s,X_s) + h^2(s,Y_s)\big)\,\mathbb{1}_{\frac{\epsilon}{2}\le|X_s - Y_s|\le\epsilon}\right] ds \\ + C&\int_0^t \mathbb{E}\left[\Big(\tilde{h}_\epsilon(s,X_s) + \tilde{h}_\epsilon(s,Y_s)\Big)\,\mathbb{1}_{\frac{\epsilon}{2}\le|X_s - Y_s|\le\epsilon}\right] ds.\end{aligned}$$

Denote for $k \in \mathbb{N}$

$$\alpha_k := \int_0^t \mathbb{E}\left[\big(h^2(s,X_s) + h^2(s,Y_s)\big)\,\mathbb{1}_{2^{-k-1}\le|X_s - Y_s|\le 2^{-k}}\right] ds.$$

Note that

$$\begin{aligned}\sum_{k\in\mathbb{N}} \alpha_k &\le \int_0^t \mathbb{E}\left[\big(h^2(s,X_s) + h^2(s,Y_s)\big)\right] ds \\ &= \int_0^t \int_{\mathbb{R}^d} h^2(s,x) \big(u_X(s,dx) + u_Y(s,dx)\big)ds \le C.\end{aligned}$$

Therefore, $\alpha_k \to 0$ as $k \to +\infty$.
Denote similarly for $k \in \mathbb{N}$

$$\beta_k := \int_0^t \mathbb{E}\left[\left(\bar{h}_{2^{-k}}(s, X_s) + \bar{h}_{2^{-k}}(s, Y_s)\right) \mathbb{1}_{2^{-k-1} \leq |X_s - Y_s| \leq 2^{-k}}\right] ds.$$

Denote $J_i := \{k : [2^{-k-1}, 2^{-k}) \subseteq I_i\}$. Note that $|J_i| \geq \frac{1}{C}|\log b_i|$, in fact

$$[2^{-k-1}, 2^{-k}) \subseteq [b_i^2, b_i) \iff 2^{-k-1} \geq b_i^2 \quad \text{and} \quad 2^{-k} < b_i,$$

whence

$$-k - 1 \geq \log_2 b_i^2 = -2|\log_2 b_i| \quad \text{and} \quad -k < \log_2 b_i = -|\log_2 b_i|\,,$$

that is

$$k < 2|\log_2 b_i| = 2\frac{|\log b_i|}{\log 2} \quad \text{and} \quad k > |\log_2 b_i| = \frac{|\log b_i|}{\log 2},$$

so $|J_i| = \frac{|\log b_i|}{\log 2}$.
Since $\bar{h}_\epsilon$ is fixed on $\epsilon \in I_i$,

$$\begin{aligned}\frac{1}{|J_i|}\sum_{k \in J_i} \beta_k &\leq \frac{1}{|J_i|}\int_0^t \int_{\mathbb{R}^d} \bar{h}_{b_i}(s, x)\left(u_X(s, dx) + u_Y(s, dx)\right) ds \\ &\leq \frac{C}{b_i \phi(b_i^{-1})} \to 0 \qquad \text{as } i \to +\infty.\end{aligned}$$

Therefore, there exists a subsequence β_{n_k} such that $\beta_{n_k} \to 0$ as $k \to +\infty$.
Consequently, since the sequence of functions L_ϵ is non-increasing,

$$\mathbb{E}\left[L_\epsilon(X_t - Y_t)\right] \to 0 \qquad \text{as } \epsilon \to 0.$$

On the other hand, by Markov inequality we have

$$\mathbb{P}\left(|X_t - Y_t| > \epsilon\right) \leq \mathbb{P}\left(L_\epsilon(X_t - Y_t) \geq 1\right) \leq \mathbb{E}\left[L_\epsilon(X_t - Y_t)\right],$$

and by taking the limit as $\epsilon \to 0$, we deduce that for any $t \in [0, T]$

$$\mathbb{P}\left(|X_t - Y_t| > 0\right) = 0.$$

Since X_t and Y_t have a.s. continuous path, we finally conclude that

$$\mathbb{P}\left(\sup_{t \in [0,T]} |X_t - Y_t| = 0\right) = 1.$$

□

In dimension 1, the result is slightly better: we require $H^{\frac{1}{2}}$ assumptions on the coefficient σ, but we lose a little bit on F (we have to use (2.12) instead of (2.16)).

Theorem 2.2.4. *Assume $d = 1$.*

1. *Existence: Fix $T > 0$ and assume that there exists two sequences $F_n, \sigma_n \in C^\infty \cap L^\infty$ converging in the sense of distributions to F and σ respectively, such that the solution $u_n \in M_1$ to (4) satisfies*

   (a) $\int_0^T \int_{\mathbb{R}^d} \left(|\sigma_n - \sigma| + |F_n - F|\right) du_n dt \xrightarrow[n\to+\infty]{} 0$;

   (b) $\sup_n \left(\|F\|_{W_T^{1,1}(u_n)} + \|\sigma\|_{H_T^{\frac{1}{2}}(u_n)} + \|F_n\|_{L^\infty} + \|\sigma_n\|_{L^\infty}\right) < \infty$;

   (c) $u_n \xrightarrow[n\to+\infty]{} u$ *in the weak topology of M_1.*

   Then there exists a strong solution X_t to (1) such that $(X_t^n - \xi, t \in [0,T])_n$ converges in $L^p(\Omega, L^\infty([0,T]))$ for all $p > 1$ to $(X_t - \xi, t \in [0,T])$, with X_t^n the solutions to (3). In addition, $u(t, dx)$ is the law of X_t for almost all $t \in [0,T]$.

2. *Uniqueness: Let X and Y be two solutions to (1) with one-dimensional time marginals $u_X(t,\cdot)$ and $u_Y(t,\cdot)$ on $[0,T]$. Assume that $F, \sigma \in L^\infty$, $X_0 = Y_0$ a.s. and that*

   (a) $\|F\|_{W_T^{1,1}(u_X)} + \|F\|_{W_T^{1,1}(u_Y)} + \|\sigma\|_{H_T^{\frac{1}{2}}(u_X)} + \|\sigma\|_{H_T^{\frac{1}{2}}(u_Y)} < \infty$.

   Then one has pathwise uniqueness on $[0,T]$, that is, $\sup_{t\in[0,T]}|X_t - Y_t| = 0$ a.s.

Proof. The proof follows exactly the same steps as the one of the multidimensional case. The only differences are the functionals used so we will skip the parts of the proof which are identical.

Existence. Let us define the quantities

$$Q_{n,m}^{(\epsilon)}(t) := e^{-U_t^{n,m}} |X_t^n - X_t^m| \log\left(1 + \frac{|X_t^n - X_t^m|^2}{\epsilon^2}\right), \qquad \epsilon \in (0,1], n, m \geq 1,$$

where $U_t^{n,m}$ is a non-negative stochastic process with bounded variation satisfying $dU_t^{n,m} = \lambda_t^{n,m} dt$ with $\lambda_t^{n,m}$ an adapted process (measurable function of a continuous, adapted process) to be chosen later.

Note that $f(x) = |x| \log\left(1+\frac{|x|^2}{\epsilon^2}\right)$ satisfies

$$\left|f'(x)\right| \leq 4\log\left(1+\frac{|x|^2}{\epsilon^2}\right) \qquad \text{and} \qquad \left|f''(x)\right| \leq \frac{C}{\epsilon+|x|}.$$

Therefore, by Itô's formula with $g(t,x,u) = e^{-u}f(x)$, we have

$$\begin{aligned} g(t, X_t^n - X_t^m, U_t^{n,m}) &= \int_0^t \partial_x g(s, X_s^n - X_s^m, U_s^{n,m}) d(X_s^n - X_s^m) \\ &+ \int_0^t \partial_u g(s, X_s^n - X_s^m, U_s^{n,m}) dU_s^{n,m} \\ &+ \frac{1}{2}\int_0^t \partial_x^2 g(s, X_s^n - X_s^m, U_s^{n,m}) d\langle X_s^n - X_s^m\rangle_s, \end{aligned}$$

whence, since $U_t^{n,m}$ is non-negative,

$$\begin{aligned} &\mathbb{E}\left[Q_{n,m}^{(\epsilon)}(t)\right] \leq \int_0^t \mathbb{E}\left[e^{-U_s^{n,m}} 4\log\left(1+\frac{|X_s^n - X_s^m|^2}{\epsilon^2}\right)\left|F(s,X_s^n) - F(s,X_s^m)\right|\right] ds \\ &+ \int_0^t \mathbb{E}\left[-e^{-U_s^{n,m}}|X_s^n - X_s^m| \log\left(1+\frac{|X_s^n - X_s^m|^2}{\epsilon^2}\right)\lambda_s^{n,m}\right] ds + C\frac{\eta(n,m)}{\epsilon} \\ &\leq \int_0^t \mathbb{E}\left[|X_s^n - X_s^m| \log\left(1+\frac{|X_s^n - X_s^m|^2}{\epsilon^2}\right)\left(4\frac{\left|F(s,X_s^n) - F(s,X_s^m)\right|}{|X_s^n - X_s^m|} - \lambda_s^{n,m}\right)\right] ds \\ &+ C\frac{\eta(n,m)}{\epsilon}. \end{aligned}$$

The term with F must be dealt differently than we did in the multidimensional case. We introduce $\tilde{h} := M|\nabla f|$ such that

$$\int_0^T \int_{\mathbb{R}} \tilde{h}(t,x)\left(u_n(t,dx) + u_m(t,dx)\right) dt \leq C.$$

We choose

$$\lambda_t^{n,m} := 4\left(\tilde{h}(t,X_t^n) + \tilde{h}(t,X_t^m)\right).$$

Therefore, we deduce that

$$\sup_{t\leq T} \mathbb{E}\left[Q_{n,m}^{(\epsilon)}(t)\right] \leq C + C\frac{\eta(n,m)}{\epsilon}.$$

Using a similar method as in Theorem 2.2.1, we write for $p < 1$ and for constants L and K to be chosen later

$$\begin{aligned}\mathbb{E}\left[|X_t^n - X_t^m|^p\right] &\leq \mathbb{E}\left[|X_t^n - X_t^m|^p \,||X_t^n - X_t^m| \geq L\right] \\ &+ E\left[|X_t^n - X_t^m|^p \,||X_t^n - X_t^m| \leq \frac{1}{\sqrt{|\log\epsilon|}}\right] \\ &+ \mathbb{E}\left[|X_t^n - X_t^m|^p \,|\frac{1}{\sqrt{|\log\epsilon|}} \leq |X_t^n - X_t^m| \leq L\right] \\ &\leq \mathbb{E}\left[|X_t^n - X_t^m|^p \,||X_t^n - X_t^m| \geq L\right] + \frac{1}{|\log\epsilon|^{\frac{p}{2}}} \\ &+ \mathbb{E}\left[|X_t^n - X_t^m|^p \,|\frac{1}{\sqrt{|\log\epsilon|}} \leq |X_t^n - X_t^m| \leq L\right].\end{aligned}$$

Now

$$\begin{aligned}&\mathbb{E}\left[|X_t^n - X_t^m|^p \,|\frac{1}{\sqrt{|\log\epsilon|}} \leq |X_t^n - X_t^m| \leq L\right] \\ &= \mathbb{E}\left[|X_t^n - X_t^m|^p \,|\frac{1}{\sqrt{|\log\epsilon|}} \leq |X_t^n - X_t^m| \leq L; U_t^{n,m} \geq \log K\right] \\ &+ \mathbb{E}\left[|X_t^n - X_t^m|^p \,|\frac{1}{\sqrt{|\log\epsilon|}} \leq |X_t^n - X_t^m| \leq L; U_t^{n,m} \leq \log K\right] \\ &\leq L^p\mathbb{P}\left(U_t^{n,m} \geq \log K\right) + L^p\mathbb{P}\left(|X_t^n - X_t^m| \geq \frac{1}{\sqrt{|\log\epsilon|}}; U_t^{n,m} \leq \log K\right).\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}\left[|X_t^n - X_t^m|^p\right] &\leq \mathbb{E}\left[|X_t^n - X_t^m|^p \,||X_t^n - X_t^m| \geq L\right] + \frac{1}{|\log\epsilon|^{\frac{p}{2}}} \\ &+ L^p\mathbb{P}\left(U_t^{n,m} \geq \log K\right) + L^p\mathbb{P}\left(|X_t^n - X_t^m| \geq \frac{1}{\sqrt{|\log\epsilon|}}; U_t^{n,m} \leq \log K\right).\end{aligned}$$

Note that

$$\mathbb{E}\left[U_t^{n,m}\right] = \mathbb{E}\left[\int_0^t \lambda_s^{n,m} ds\right] \leq 4\int_0^t \tilde{h}(s,x)\left(u_n(s,dx) + u_m(s,dx)\right) ds \leq C.$$

Consequently, by Markov inequality

$$\mathbb{P}\left(U_t^{n,m} \geq \log K\right) \leq \frac{C}{\log K}.$$

In addition, for ϵ small enough, again by Markov inequality

$$\mathbb{P}\left(|X_t^n - X_t^m| \geq \frac{1}{\sqrt{|\log\epsilon|}}; U_t^{n,m} \leq \log K\right) \leq \sqrt{|\log\epsilon|}\mathbb{E}\left[|X_t^n - X_t^m|\,|U_t^{n,m} \leq \log K\right]$$

$$= \sqrt{|\log\epsilon|}\mathbb{E}\left[Q_{n,m}^{(\epsilon)}(t)e^{U_t^{n,m}}\log\left(1+\frac{|X_s^n - X_s^m|^2}{\epsilon^2}\right)^{-1}|U_t^{n,m} \leq \log K\right]$$

$$\leq \frac{K\mathbb{E}\left[Q_{n,m}^{(\epsilon)}(t)\right]}{2\sqrt{|\log\epsilon|}},$$

since $\log\left(1+\frac{|X_s^n - X_s^m|^2}{\epsilon^2}\right) \geq 2|\log\epsilon|$, for ϵ small enough.
Therefore, using (2.25) as in the proof of Lemma 2.2.3, we have

$$\mathbb{E}\left[|X_t^n - X_t^m|^p\right] \leq C\left(\frac{1}{L} + \frac{1}{|\log\epsilon|^{\frac{p}{2}}} + \frac{L^p}{\log K} + \frac{L^pK\left(1+\frac{\eta(n,m)}{\epsilon}\right)}{\sqrt{|\log\epsilon|}}\right).$$

Taking, for example, $\epsilon := \eta(n,m)$, $K := |\log\epsilon|^{\frac{1}{4}}$ and $L := \log\left(|\log\epsilon|\right)^{\frac{1}{2p}}$, we deduce that

$$\sup_{t\in[0,T]}\mathbb{E}\left[|X_t^n - X_t^m|^p\right] \to 0, \qquad \text{as } n,m \to +\infty.$$

The rest of the proof is similar.

Uniqueness. For simplicity, we assume here that $F = 0$. Otherwise it is necessary to introduce U_t as in the previous subsection but it is handled in exactly the same way.

We proceed as in the proof of Theorem 2.2.1. We similarly change the definition of L_ϵ in

$$L_\epsilon(x) = |x| \quad \text{if} \quad |x| \geq \epsilon, \qquad L_\epsilon(x) = 0 \quad \text{if} \quad |x| \leq \frac{\epsilon}{2},$$

$$\|\nabla L_\epsilon\|_{L^\infty} + \epsilon\|\nabla^2 L_\epsilon\|_{L^\infty} \leq C,$$

with C independent of ϵ.
Applying Itô's formula,

$$\mathbb{E}\left[L_\epsilon(X_t - Y_t)\right] \leq C\int_0^t \mathbb{E}\left[\mathbb{1}_{\frac{\epsilon}{2}\leq|X_s - Y_s|\leq\epsilon}\frac{|\sigma(s,X_s) - \sigma(s,Y_s)|^2}{\epsilon^2}\right]ds.$$

By using as before the assumptions, Lemma B.0.4 and the corresponding definition of $H_T^{\frac{1}{2}}(u_X)$ and $H_T^{\frac{1}{2}}(u_Y)$, we deduce that

$$\mathbb{E}\left[L_\epsilon(X_t - Y_t)\right] \to 0 \qquad \text{as } \epsilon \to 0.$$

This is slightly less strong than before but still enough. In particular, if $\alpha \geq \epsilon$ by Markov inequality we have

$$\mathbb{P}\left(|X_t - Y_t| \geq \alpha\right) \leq \frac{1}{\alpha}\mathbb{E}\left[|X_t - Y_t|\right] = \frac{1}{\alpha}\mathbb{E}\left[L_\epsilon(X_t - Y_t)\right],$$

since $|X_t - Y_t| = L_\epsilon(X_t - Y_t)$ when $|X_t - Y_t| \geq \alpha \geq \epsilon$.
Therefore, by taking $\epsilon \to 0$, we still obtain that for any $t \in [0, T]$,

$$\mathbb{P}\left(|X_t - Y_t| \geq 0\right) = 0,$$

which allows us to conclude as before. □

Chapter 3

Consequences and applications

3.1 The uniformly elliptic case

In this section we want to recover classical results if the coefficient σ satisfies an ellipticity condition. To be more precise we will consider the case where σ is uniformly elliptic: for all t, x,

$$\frac{1}{2}\sigma(t,x)\sigma^*(t,x) = a(t,x) \geq cI, \tag{3.1}$$

for some $c > 0$.

A first useful result is the following inequality due to Krylov.

Theorem 3.1.1. *Assume that F and σ are bounded and σ satisfies* (3.1). *Then for all solution X of* (1) *with any initial distribution, for all $T > 0$ and $p, q > 1$ such that $\frac{d}{p} + \frac{2}{q} < 2$, there exists a constant C such that for all f* inf $L_t^q(L_x^p)$

$$\mathbb{E}\left[\int_0^T f(t, X_t)dt\right] \leq C\|f\|_{L_t^q(L_x^p)}.$$

Proof. See [26]. □

This Theorem means that

$$u \in L_t^{q'}(L_x^{p'}), \tag{3.2}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$, and we obtain the following corollary.

Corollary 3.1.2. *1. Assume that $d \geq 2$, $F, \sigma \in L^\infty$, σ satisfies (3.1), $F \in L^{\frac{q}{2}}_{t,loc}(W_x^{-1,\frac{p}{2}})$ and $\sigma \in L^q_{t,loc}(W_x^{1,p})$ with $\frac{d}{p} + \frac{2}{q} < 1$. Then one has both existence of a strong solution to (1) and pathwise uniqueness for any initial condition ξ.*

2. Assume that $d = 1$, $F, \sigma \in L^\infty$, σ satisfies (3.1), $\sigma \in L^q_{t,loc}(W_x^{\frac{1}{2},p})$ with $\frac{d}{p} + \frac{2}{q} < 1$ and $F \in L^{\frac{q}{2}}_{t,loc}(W_x^{-1,\frac{p}{2}})$ if $p > 2$, $F \in L^{\frac{q}{2}}_{t,loc}(W_x^{1,1+\epsilon})$ for some $\epsilon > 0$ if $p \leq 2$. Then one has both existence of a strong solution to (1) and pathwise uniqueness for any initial condition ξ.

Proof. The conclusion follows from (3.2), Theorem 2.0.1 and 2.0.2. □

Note that in this case, since $u \in L_t^{q'}(L_x^{p'})$ for all solution to (1) we do not need any additional assumption to obtain pathwise uniqueness.

Now in our setting, we can state a priori estimates on u solution to (2) which leads to results on existence and uniqueness. For instance, we have the following

Proposition 3.1.3. *For any $d \geq 1$, assume $u^0 \in L^1 \cap L^\infty$, $F, \sigma \in L^\infty$, σ satisfies (3.1) and $\nabla\sigma \in L^q_{t,loc}(L^p_x)$ satisfying $\frac{d}{p} + \frac{2}{q} = 1$ with $p > d$. Then any u solution to (2), limit for the weak topology in M_1 of smooth solutions, belongs to $L_t^\infty(L_x^r)$ for any $1 \leq r \leq +\infty$.*

Proof. We use the energy estimates to prove the result. The computations below are formal but could easily be made rigorous by taking a regularization of σ, F, and hence a and then pass to the limit.

$$\begin{aligned}\frac{d}{dt}\int u^\alpha(t,x)dx = (-1)^{d+1}\alpha(\alpha-1)&\int u^{\alpha-1}(t,x)\nabla u(t,x)\cdot F(t,x)dx \\ +(-1)^d\alpha(\alpha-1)&\int u^{\alpha-2}(t,x)\nabla u^*(t,x)a(t,x)\nabla u(t,x)dx \\ +(-1)^d\alpha(\alpha-1)&\int u^{\alpha-1}(t,x)\nabla u(t,x)\cdot\nabla a(t,x)dx.\end{aligned}$$

Note that

$$\left|\nabla u^{\frac{\alpha}{2}}\right|^2 = \left|\frac{\alpha}{2}u^{\frac{\alpha}{2}-1}\nabla u\right|^2 = \left(\frac{\alpha}{2}\right)^2 u^{\alpha-2}|\nabla u|^2,$$

so by (3.1) we get

$$\int u^{\alpha-2}(t,x)\nabla u^*(t,x)a(t,x)\nabla u(t,x)dx \geq C\int u^{\alpha-2}(t,x)\left|\nabla u(t,x)\right|^2 dx \geq C\|\nabla u^{\frac{\alpha}{2}}\|_{L^2}^2.$$

Note also that

$$\left(\nabla u^{\frac{\alpha}{2}}\right) u^{\frac{\alpha}{2}} = \left(\frac{\alpha}{2} u^{\frac{\alpha}{2}-1} \nabla u\right) u^{\frac{\alpha}{2}} = \frac{\alpha}{2} u^{\alpha-1} \nabla u,$$

so by Hölder and Young inequalities

$$\begin{aligned}\int u^{\alpha-1}(t,x)\nabla u(t,x)\cdot F(t,x)dx &\leq C\|\nabla u^{\frac{\alpha}{2}}\|_{L^2}\|u^{\frac{\alpha}{2}}\|_{L^2}\|F\|_{L^\infty} \\ &\leq \frac{C}{4}\|\nabla u^{\frac{\alpha}{2}}\|_{L^2}^2 + C'\int u^{\alpha}(t,x)dx.\end{aligned}$$

Similarly,

$$\begin{aligned}\int u^{\alpha-1}(t,x)\nabla u(t,x)\cdot \nabla a(t,x)dx &\leq C\|\nabla u^{\frac{\alpha}{2}}\|_{L^2}\|u^{\frac{\alpha}{2}}\nabla a\|_{L^2} \\ &\leq C\|\nabla u^{\frac{\alpha}{2}}\|_{L^2}\|\nabla a\|_{L^p}\|u^{\frac{\alpha}{2}}\nabla a\|_{L^r},\end{aligned}$$

with $\frac{1}{2} = \frac{1}{p} + \frac{1}{r}$. Now by Sobolev embedding

$$\|u^{\frac{\alpha}{2}}\nabla a\|_{L^r} \leq \left(\int u^{\alpha}(t,x)dx\right)^{\frac{\theta}{2}} \|\nabla u^{\frac{\alpha}{2}}\|_{L^2}^{1-\theta},$$

for $\theta \in (0,1]$, such that $\frac{1}{r} = \frac{1}{2} - \frac{1-\theta}{d}$ or $\frac{1-\theta}{d} = \frac{1}{p}$, provided that $p > d$. Combining with the previous inequality, using Young inequality we get

$$\int u^{\alpha-1}(t,x)\nabla u(t,x)\cdot\nabla a(t,x)dx \leq \frac{C}{4}\|\nabla u^{\frac{\alpha}{2}}\|_{L^2}^2 + C''\|\nabla a\|_{L^p}^{\frac{2}{\theta}}\int u^{\alpha}(t,x)dx.$$

Therefore, we have

$$\frac{d}{dt}\int u^{\alpha}(t,x)dx + \frac{C}{2}\|\nabla u^{\frac{\alpha}{2}}\|_{L^2}^2 \leq C'''\left(1 + \|\nabla a\|_{L^p}^{\frac{2}{\theta}}\right)\int u^{\alpha}(t,x)dx.$$

Now note that

$$\int_0^T \|\nabla a\|_{L^p}^{\frac{2}{\theta}} dt < +\infty,$$

since $\nabla a \in L^q_{t,loc}(L^p_x)$ with $\frac{1}{q} = \frac{\theta}{2} = \frac{1}{2} - \frac{d}{2p}$, which corresponds exactly to the condition $\frac{2}{q} + \frac{d}{p} = 1$ with $p > d$. Therefore, we can finally conclude by Grönwall inequality that for any t and $\alpha < +\infty$,

$$\|u(t,\cdot)\|_{L^\alpha} \leq C\|u(t=0,\cdot)\|_{L^\alpha} \leq C,$$

with C independent of α since $u^0 \in L^1 \cap L^\infty$. This implies that $\|u(t,\cdot)\|_{L^\infty} \leq C$ and completes the proof. □

This, combined with Theorem 2.0.1, gives slightly better conditions for σ and much better conditions for F, assuming additional conditions on the initial distribution. We obtain the following

Corollary 3.1.4. *Assume that $d \geq 2$, $u^0 \in L^1 \cap L^\infty$, $F, \sigma \in L^\infty$, $F \in L^1_{t,loc}(W^{1,1}_x)$, σ satisfies* (3.1) *and $\nabla\sigma \in L^q_{t,loc}(L^p_x)$ satisfying $\frac{d}{p} + \frac{2}{q} = 1$ with $p > d$. Then one has existence of a strong solution to* (1) *with marginal distributions $u(t, dx)$ in $L^\infty_{t,loc}(L^\infty_x)$. In addition, pathwise uniqueness holds among all solutions with marginal distributions in $L^\infty_{t,loc}(L^\infty_x)$.*

3.2 The Langevin kinetic case

One of the main advantages of the exposed method is its flexibility. In fact, in any dimension, depending on the precise structure of (1) one can have strong solutions without requiring any ellipticity condition. It is the case of the following classical problem in the phase space $\mathbb{R}^{2d}$:

$$dX_t = V_t dt, \qquad dV_t = F(t, X_t)dt + \sigma(t, X_t)dW_t, \qquad (X_0, V_0) = \xi. \tag{3.3}$$

The joint law $u(t, x, v)$ of the process $(X_t, V_t)_{t\geq 0}$ solves the kinetic equation

$$\partial_t u(t, x, v) + v \cdot \nabla_x u(t, x, v) + F(t, x) \cdot \nabla_v u(t, x, v) = \sum_{1\leq i,j\leq d} a_{i,j}(t, x) \frac{\partial^2 u(t, x, v)}{\partial v_i \partial v_j}. \tag{3.4}$$

Since the previous equation (3.4) is better behaved than (2) we have the following

Corollary 3.2.1. *Assume $\sigma \in L^\infty \cap L^2_{t,loc}(H^1_x)$, $F \in L^1_{t,loc}(W^{1,1}_x)$, and $u^0 \in L^\infty$. Then there is both existence of a strong solution to* (3.3) *and pathwise uniqueness among all solutions with marginal distributions in $L^\infty_{t,loc}(L^\infty_x)$.*

Proof. In order to obtain the existence of a strong solution and the pathwise uniqueness we want to use Theorem 2.0.1. So we need to recover the hypothesis (1)-(6).
Fix $T > 0$, we define σ_n, F_n as follows:

$$\sigma_n(t, x) := (\sigma * \phi_n)(t, x); \qquad F_n(t, x) := ((F \wedge n) * \phi_n)(t, x),$$

where $\phi_n(t,x) := n^{2d+1}\phi(n(t,x))$, and $\phi \in C_c^\infty([0,T] \times \mathbb{R}^d)$ such that $\int \phi = 1$.
Then the sequence of σ_n and F_n satisfy the assumptions (1)-(3) of Theorem 2.0.1 with $p = q = 1$. To be precise we can not say that $\sup_n \|F_n\|_{L^\infty} < +\infty$, but in the proof of Theorem 2.2.1 (and hence Theorem 2.0.1) we do not use this hypothesis.

Moreover since $p' = q' = +\infty$, the hypothesis that $u^0 \in L^\infty$ guarantees that $\|u_n\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^{2d})} \leq \|u^0\|_{L^\infty}$ (see [2]), and hence we obtain the assumptions (4) and (5).

Finally note that (6) holds by hypothesis. This concludes the proof. $\square$

Conclusions

In the conclusions of this thesis, it is essential to emphasize the significant advantage of the proposed method that is its remarkable flexibility. The approach adopted in this study relies on direct quantitative estimates of solutions to the stochastic differential equation (1), assuming Sobolev bounds on the drift and diffusion coefficients, as well as L^p bounds for the solution of the corresponding Fokker–Planck PDE (2). Importantly, these latter bounds can be proven independently, further enhancing the robustness of the approach.

By employing this methodology, we are granted a certain degree of freedom in choosing the most suitable approach to address the Fokker–Planck equation, based on any additional structural considerations present in the system. This adaptability is particularly advantageous, as it allows us to tailor our methodology to different scenarios, accommodating diverse sets of assumptions and constraints. Consequently, the results obtained in this study can be effectively applied to a wide range of cases.

This approach excels in handling systems characterized by uniformly elliptic behavior in any dimension, making it a notable application of the method. It effectively accounts for this condition while considering appropriate assumptions on the drift and diffusion coefficients, ensuring an accurate and reliable analysis of such systems. This capability is of great practical importance, as uniformly elliptic behavior is commonly encountered in various scientific and engineering domains.

Additionally, the described method proves effective in handling the Kolmogorov case, where assumptions of ellipticity on the diffusion matrix are not required. This is particularly noteworthy, as the Kolmogorov case represents a distinct class of systems that often exhibit complex and non-standard behavior. By removing the need for ellipticity assumptions, this approach expands the applicability of the findings to a broader range of systems, enabling us to gain insights and draw conclusions that were previously challenging to obtain.

In summary, the flexibility of the proposed method is a key strength, as it allows for the selection of the most appropriate technique to address the Fokker–Planck equation, considering the unique characteristics of each system. This flexibility empowers us to apply the results to a diverse array of scenarios, encompassing the two mentioned above. The broad scope of applicability enhances the practical significance of this approach and opens up new possibilities for analyzing and understanding complex systems.

Appendix A

Proof of Theorem 2.0.1

Proof. The only thing left to prove after Theorem 2.2.1 is that if $u \in L^{q'}_{t,loc}(L^{p'}_x(\mathbb{R}^d))$ then there exists a superlinear function ϕ such that

$$\|\sigma\|_{H^1_T(u)} \le C\|\sigma\|_{L^{2q}_t([0,T],W^{1,2p}_x)}, \qquad \|F\|_{W^{\phi,weak}_T(u)} \le C\|F\|_{L^q_t([0,T],W^{1,p}_x)}.$$

Since the maximal operator M is bounded on L^p for $p > 1$, this is straightforward for σ as $2p \ge 2 > 1$.
Therefore, the key point is how to prove that for F when $p \ge 1$. We give the proof for $p = 1$, the case $p > 1$ can be treated following the same lines.

Fix $L \ge 1$ and denote

$$h(t,x) := M_L \nabla F = \sqrt{\log L} + \int_{\mathbb{R}^d} \frac{\left|\nabla F(t,z)\right| \mathbb{1}_{|\nabla F| \ge \sqrt{\log L}} dz}{(L^{-1} + |x-z|)|x-z|^{d-1}}.$$

$p = 1$ implies $p' = \infty$, then for almost any fixed t, $u(t,\cdot) \in L^{q'} \cap L^\infty$, and hence

$$\begin{aligned}
\int_{\mathbb{R}^d} h(t,x)u(t,x)dx &\le \sqrt{\log L} + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\left|\nabla F(t,z)\right| \mathbb{1}_{|\nabla F| \ge \sqrt{\log L}}}{(L^{-1} + |x-z|)|x-z|^{d-1}} u(t,x)dzdx \\
&= \sqrt{\log L} + \iint_{|x-z|\le 1} \frac{\left|\nabla F(t,z)\right| \mathbb{1}_{|\nabla F| \ge \sqrt{\log L}}}{(L^{-1} + |x-z|)|x-z|^{d-1}} u(t,x)dzdx \\
&+ \iint_{|x-z|>1} \frac{\left|\nabla F(t,z)\right| \mathbb{1}_{|\nabla F| \ge \sqrt{\log L}}}{(L^{-1} + |x-z|)|x-z|^{d-1}} u(t,x)dzdx \\
&\le \sqrt{\log L} + C \log L \|u(t,\cdot)\|_{L^\infty} \|\nabla F(t,\cdot)\mathbb{1}_{|\nabla F| \ge \sqrt{\log L}}\|_{L^1} \\
&+ \|u(t,\cdot)\|_{L^1} \|\nabla F(t,\cdot)\mathbb{1}_{|\nabla F| \ge \sqrt{\log L}}\|_{L^1}
\end{aligned}$$

Whence

$$\int_{\mathbb{R}^d} h(t,x)u(t,x)dx \leq \sqrt{\log L} + C\log L(\|u(t,\cdot)\|_{L^\infty} + \|u(t,\cdot)\|_{L^1})\|\nabla F(t,\cdot)\mathbb{1}_{|\nabla F|\geq\sqrt{\log L}}\|_{L^1}.$$

Therefore, integrating now in time, by Hölder's estimates we have

$$\int_0^T \int_{\mathbb{R}^d} h(t,x)u(t,x)dxdt \leq T\sqrt{\log L} + C\log L\|\nabla F\mathbb{1}_{|\nabla F|\geq\sqrt{\log L}}\|_{L^q_t([0,T],L^1_x)}.$$

Now, if $\nabla F \in L^q_t([0,T],L^1_x)$, then there exists a superlinear ψ such that

$$\|\psi(\nabla F)\|_{L^q_t([0,T],L^1_x)} \leq +\infty.$$

Consequently, if $|\nabla F| \geq \sqrt{\log L}$ then

$$\frac{\psi(|\nabla F|)}{|\nabla F|} \leq \frac{\psi(\sqrt{\log L}}{\sqrt{\log L}},$$

and so

$$|\nabla F| \leq \psi(|\nabla F|)\frac{\sqrt{\log L}}{\psi(\sqrt{\log L})}.$$

Hence

$$\int_0^T \int_{\mathbb{R}^d} h(t,x)u(t,x)dxdt \leq T\sqrt{\log L} + C\frac{(\log L)^{\frac{3}{2}}}{\psi(\sqrt{\log L})}.$$

We conclude that $\|\nabla F\|_{W^{\phi,weak}_T(u)}$ is bounded for ϕ defined by

$$\frac{L}{\phi(L)} = \frac{C\sqrt{\log L}}{\log L} + \frac{C\sqrt{\log L}}{\psi(\sqrt{\log L})},$$

which is hence superlinear. □

Appendix B

Technical results

Lemma B.0.1. *Fix $t \geq 0$ and assume that $\sigma(t,\cdot) \in BV(\mathbb{R}^d)$. Then for any $x, y \in \mathbb{R}^d$*

$$\left|\sigma(t,x) - \sigma(t,y)\right| \leq C_d \left(M|\nabla_x \sigma|(t,x) + M|\nabla_x \sigma|(t,y)\right)|x-y|, \tag{B.1}$$

for some constant C_d that depends only on d.

The next lemma provides an extension of (B.1).

Lemma B.0.2. *Fix $t \geq 0$ and assume that $F(t,\cdot) \in BV(\mathbb{R}^d)$. For any $x \in \mathbb{R}^d$, if $h(t,x) < +\infty$ with $h(t,x) := \left|F(t,x)\right| + M_L \nabla F(t,x)$, then x is a Lebesgue point of F. Then for any $x, y \in \mathbb{R}^d$*

$$\left|F(t,x) - F(t,y)\right| \leq C_d \left(h(t,x) + h(t,y)\right)\left(|x-y| + \frac{1}{L}\right), \tag{B.2}$$

for some constant C_d that depends only on d.

Proof. First observe that if $|x-y| \geq 1$ the conclusion is obvious, so it is not restrictive assuming $|x-y| < 1$.
We recall the following lemma.

Lemma B.0.3. *Assume $F \in C^1(\mathbb{R}^d)$. There exists a constant C_d depending only on d such that for any $x, y \in \mathbb{R}^d$,*

$$\left|F(x) - F(y)\right| \leq C_d \int_{B(x,y)} \left(\frac{1}{|x-z|^{d-1}} + \frac{1}{|y-z|^{d-1}}\right)|\nabla F|(dz), \tag{B.3}$$

where $B(x,y)$ denotes the ball of center $\frac{x+y}{2}$ and diameter $|x-y|$.

Proof. See [12]. □

The first step is to extend inequality (B.3) to any $F \in BV_{loc}$. Consider the classic convolution kernel $K \geq 0$ with $K(-x) = K(x)$ and $supp(K) \subseteq B(0,1)$, so the sequence $K_\epsilon * F \in C^\infty(\mathbb{R}^d)$. At every x Lebesgue point of F, one has $(K_\epsilon * F)(x) \to F(x)$ as $\epsilon \to 0$ and, therefore, if x, y are distinct Lebesgue points of F applying (B.3) to $K_\epsilon * F$ and taking the limit as $\epsilon \to 0$ one obtains

$$\begin{aligned}\left|F(x) - F(y)\right| &= \lim \epsilon \to 0 \left|(K_\epsilon * F)(x) - (K_\epsilon * F)(y)\right| \\ &\leq C_d \lim \epsilon \to 0 \int_{B(x,y)} \left(\frac{1}{|x-z|^{d-1}} + \frac{1}{|y-z|^{d-1}} \right) |\nabla K_\epsilon * F| \, (dz).\end{aligned}$$

Now notice that

$$|\nabla K_\epsilon * F| \, (z) = \left| \int k_\epsilon(z-w) \nabla F(dw) \right| \leq \int k_\epsilon(z-w) \left|\nabla F(dw)\right| \leq K_\epsilon * |\nabla F| \, (z).$$

Therefore, since we are considering $\epsilon \to 0$ we can suppose that ϵ is small with respect to $|x-y|$, and then we have

$$\int_{B(x,y)} \left(\frac{1}{|x-z|^{d-1}} + \frac{1}{|y-z|^{d-1}} \right) |\nabla K_\epsilon * F| \, (dz) \leq \int_{\tilde{B}(x,y)} K_\epsilon * \phi_{x,y}(z) |\nabla F| \, (dz),$$

where $\phi_{x,y}(z) := \frac{1}{|x-z|^{d-1}} + \frac{1}{|y-z|^{d-1}}$ and $\tilde{B}(x,y)$ denotes the ball of center $\frac{x+y}{2}$ and diameter $2|x-y|$.
Now observe that, since w^{-d+1} is integrable, one has for all $z \in \mathbb{R}^d$

$$\int K_\epsilon(z-w) w^{-d+1} dw \leq \frac{C}{(|z| + \epsilon)^{d-1}} \leq \frac{C}{|z|^{d-1}}.$$

Whence,

$$K_\epsilon * \phi_{x,y}(z) \leq C \phi_{x,y}(z).$$

Since $|\nabla F|$ is a positive measure we obtain

$$\int_{\tilde{B}(x,y)} K_\epsilon * \phi_{x,y}(z) |\nabla F| \, (dz) \leq C \int_{\tilde{B}(x,y)} \phi_{x,y}(z) |\nabla F| \, (dz).$$

We have proved that for any x, y Lebesgue points of F,

$$\left|F(x) - F(y)\right| \leq C \int_{\tilde{B}(x,y)} \left(\frac{1}{|x-z|^{d-1}} + \frac{1}{|y-z|^{d-1}} \right) |\nabla F| \, (dz). \tag{B.4}$$

On the other hand, following [1], if $F \in BV_{loc}$ the set of non-Lebesgue points of F can be defined as the set of x such that

$$\liminf r^{-d+1} \int_{B(x,r)} \left|\nabla F(dz)\right| > 0.$$

At such point x, one has

$$\int_{B(x,r)} \frac{|\nabla F|(dz)}{|x-z|^{d-1}} \geq \sum_{n \geq \lfloor \log_2 r \rfloor} 2^{n(-d+1)} \int_{2^{-n-1} \leq |x-z| < 2^{-n}} |\nabla F|(dz) = +\infty,$$

and inequality (B.4) is trivial.
Moreover, if

$$\int_{B(x,r)} \frac{|\nabla F|(dz)}{|x-z|^{d-1}} = +\infty,$$

for some $r > 0$, then $h(x,y) = +\infty$. This implies that x is necessarily a Lebesgue point of F if $h(t,x) < +\infty$.
Now we notice that $|\nabla F| \leq |\nabla F|_s + \sqrt{\log L}\mathcal{L} + |\nabla F|_a \mathbb{1}_{|\nabla F|_a \geq \sqrt{\log L}}$ where $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}^d$, $|\nabla F|_a$ and $|\nabla F|_s$ are the absolutely continuous and singular parts of the measure $|\nabla F|$, and where we identified $|\nabla F|_a$ with its density with respect to $\mathcal{L}$ in the indicator function.
Thus, if $\frac{1}{L} \leq |x-y| \leq 1$ then $|x-z| + \frac{1}{L} \leq 3|x-y|$ for all $z \in \tilde{B}(x,y)$, and so we obtain

$$\frac{1}{|x-y|} \int_{\tilde{B}(x,r)} \frac{|\nabla F|(dz)}{|x-z|^{d-1}} \leq C \left(\sqrt{\log L} + \int_{B(x,2)} \frac{|\nabla F|_a(z) \mathbb{1}_{|\nabla F|_a \geq \sqrt{\log L}} dz + |\nabla F|_s(dz)}{\left(\frac{1}{L} + |x-z|\right)|x-z|^{d-1}} \right),$$

where $B(x,2)$ is the ball of radius 2 centered at x.
Similarly, if $|x-y| \leq \frac{1}{L}$ then $|x-z| + \frac{1}{L} \leq \frac{2}{L}$ for all $z \in B(x,y)$, and so

$$\int_{\tilde{B}(x,r)} \frac{|\nabla F|(dz)}{|x-z|^{d-1}} \leq \frac{C}{L} \left(\sqrt{\log L} + \int_{B(x,2)} \frac{|\nabla F|_a(z) \mathbb{1}_{|\nabla F|_a \geq \sqrt{\log L}} dz + |\nabla F|_s(dz)}{\left(\frac{1}{L} + |x-z|\right)|x-z|^{d-1}} \right).$$

Finally, by definition of M_L, using (B.4) and the last two inequalities we can conclude the proof. □

Proof of Lemma B.0.1. Recall that we have proved that the estimate (B.4) holds for any $F \in BV$ at any points x and y. Applying this inequality to σ, we obtain that for any

x, y

$$\left|\sigma(x)-\sigma(y)\right| \le C\int_{\tilde{B}(x,y)}\left(\frac{1}{|x-z|^{d-1}}+\frac{1}{|y-z|^{d-1}}\right)|\nabla\sigma|(dz).$$

Now for any given x we have

$$\begin{aligned}\int_{\tilde{B}(x,r)}\frac{|\nabla\sigma|(dz)}{|x-z|^{d-1}} &= \sum_{k=0}^{+\infty}\int_{2^{-k}\le\frac{|x-z|}{|x-y|}\le 2^{-k+1}}\frac{|\nabla\sigma|(dz)}{|x-z|^{d-1}}\\ &\le \sum_{k=0}^{+\infty}2^{k(d-1)}|x-y|^{1-d}\int_{\frac{|x-z|}{|x-y|}\le 2^{-k+1}}|\nabla\sigma|(dz)\\ &\le \sum_{k=0}^{+\infty}2^{-k+d)}|x-y|\,M|\nabla\sigma|(x)\\ &\le 2^{d+1}|x-y|\,M|\nabla\sigma|(x),\end{aligned}$$

by definition of the maximal function. This concludes the proof. □

Lemma B.0.4. *Fix $t \ge 0$ and assume that $\sigma(t,\cdot) \in L^1_{loc}$ and $\partial_x^{\frac{1}{2}}\sigma(t,\cdot)$ is a locally finite Radon measure. Then for any $x, y \in \mathbb{R}^d$,*

$$\left|\sigma(t,x)-\sigma(t,y)\right| \le \left(M|\partial_x^{\frac{1}{2}}\sigma|(t,x)+M|\partial_x^{\frac{1}{2}}\sigma|(t,y)\right)|x-y|^{\frac{1}{2}}.$$

Proof. By definition of $\partial_x^{\frac{1}{2}}\sigma$, we have that

$$\sigma = K * \partial_x^{\frac{1}{2}}\sigma, \tag{B.5}$$

for the convolution kernel $K := \mathcal{F}^{-1}|\xi|^{-\frac{1}{2}}$. Indeed,

$$\mathcal{F}(K*\partial_x^{\frac{1}{2}}\sigma) = \mathcal{F}K\mathcal{F}\partial_x^{\frac{1}{2}}\sigma = \mathcal{F}\mathcal{F}^{-1}|\xi|^{-\frac{1}{2}}\,\mathcal{F}\mathcal{F}^{-1}|\xi|^{\frac{1}{2}}\,\mathcal{F}\sigma = \mathcal{F}\sigma,$$

which implies (B.5).

Moreover, by definition of K we have (see [9])

$$\left|K(x)\right| \le C\frac{1}{|x|^{d-\frac{1}{2}}}, \qquad \left|\nabla K(x)\right| \le C\frac{1}{|x|^{d+\frac{1}{2}}}. \tag{B.6}$$

Now we have

$$\begin{aligned}\left|\sigma(x)-\sigma(y)\right| \le &\int_{|z-x|\ge 2|x-y|}\left|K(x-z)-K(y-z)\right|\,|\partial_x^{\frac{1}{2}}\sigma|(dz)\\ &+\int_{|z-x|\le 2|x-y|}\left(\left|K(x-z)\right|+\left|K(y-z)\right|\right)|\partial_x^{\frac{1}{2}}\sigma|(dz).\end{aligned}$$

Denote $|x-y|=r$. By (B.6) we obtain

$$\begin{aligned}\int_{|z-x|\le 2r} \big|K(x-z)\big|\,|\partial_x^{\frac{1}{2}}\sigma|(dz) &\le \sum_{n\ge -1}\int_{|z-x|\le 2^{-n}r}\big|K(x-z)\big|\,|\partial_x^{\frac{1}{2}}\sigma|(dz)\\ &\le \sum_{n\ge -1}\int_{|z-x|\le 2^{-n}r}\frac{1}{|z-x|^{d-\frac{1}{2}}}|\partial_x^{\frac{1}{2}}\sigma|(dz)\\ &\le C\sum_{n\ge -1}\int_{|z-x|\le 2^{-n}r}\frac{2^{n(d-\frac{1}{2})}}{r^{d-\frac{1}{2}}}|\partial_x^{\frac{1}{2}}\sigma|(dz)\\ &\le C\sum_{n\ge -1}2^{-\frac{n}{2}}r^{\frac{1}{2}}M|\partial_x^{\frac{1}{2}}\sigma|(x) = Cr^{\frac{1}{2}}M|\partial_x^{\frac{1}{2}}\sigma|(x).\end{aligned}$$

Since $|z-x|\le 2r$ implies that $|z-y|\le 3r$, we obtain the same inequality:

$$\int_{|z-x|\le 2r}\big|K(y-z)\big|\,|\partial_x^{\frac{1}{2}}\sigma|(dz)\le Cr^{\frac{1}{2}}M|\partial_x^{\frac{1}{2}}\sigma|(y).$$

For the last term, first note that if $|x-z|\ge 2|x-y|$ then $|y-z|\ge \frac{|x-z|}{2}$, so

$$\sup_{\zeta\in U}|\nabla K|\,(\zeta)\le C\frac{1}{\min\{|x-z|\,,|y-z|\}^{d+\frac{1}{2}}} = C\frac{1}{|x-z|^{d+\frac{1}{2}}},$$

where U denotes the segment in $\mathbb{R}^d$ from $x-z$ to $y-z$.
Hence by (B.6) if $|x-z|\ge 2|x-y|$ we have

$$\big|K(x-z)-K(y-z)\big|\le \sup_{\zeta\in U}|\nabla K|\,(\zeta)\big|x-z-(y-z)\big|\le C\frac{|x-y|}{|x-z|^{d+\frac{1}{2}}}.$$

Therefore,

$$\begin{aligned}\int_{|z-x|\ge 2r}\big|K(x-z)-K(y-z)\big|\,|\partial_x^{\frac{1}{2}}\sigma|(dz) &\le C\sum_{n\ge 1}\int_{|z-x|\ge 2^n r}\frac{r}{|x-z|^{d+\frac{1}{2}}}|\partial_x^{\frac{1}{2}}\sigma|(dz)\\ &\le C\sum_{n\ge 1}\int_{|z-x|\ge 2^n r}\frac{r}{(2^n r)^{d+\frac{1}{2}}}|\partial_x^{\frac{1}{2}}\sigma|(dz)\\ &\le C\sum_{n\ge 1}2^{-\frac{n}{2}}r^{\frac{1}{2}}M|\partial_x^{\frac{1}{2}}\sigma|(x) = Cr^{\frac{1}{2}}M|\partial_x^{\frac{1}{2}}\sigma|(x).\end{aligned}$$

Summing up the three estimates we can conclude the proof. □

Definition B.0.5. *A stochastic process $X=(X_t)_{t\ge 0}$ is progressively measurable if, for all $t\ge 0$, the function $[0,t]\times\Omega\ni(s,\omega)\mapsto X_s(\omega)\in\mathbb{R}^d$ is measurable with respect to the product σ-algebra $\mathscr{B}\otimes\mathscr{F}_t$.*

Lemma B.0.6. *Let b, b_n, $n \in \mathbb{N}$, such that*

- *b, b_n are progressively measurable;*
- *for all $t \geq 0$, $\mathbb{E}\left[\int_0^t \left|b(s)\right|^2 ds\right] < +\infty$ and $\mathbb{E}\left[\int_0^t \left|b_n(s)\right|^2 ds\right] < +\infty$.*

Then

$$\int_0^t \left|b_n(s) - b(s)\right|^2 ds \xrightarrow[n\to+\infty]{\mathbb{P}} 0 \implies \int_0^t \big(b_n(s) - b(s)\big)\, dW_s \xrightarrow[n\to+\infty]{\mathbb{P}} 0.$$

Proof. See [16] and [17]. □

Theorem B.0.7 (Lebesgue's differentiation theorem)**.** *For any $f \in L^1_{loc}(\mathbb{R}^d)$ we have*

$$\lim_{r\to 0} \frac{1}{\left|B(x,r)\right|} \int_{B(x,r)} f(y)dy = f(x)$$

for almost all $x \in \mathbb{R}^d$. Consequently we have $|f| \leq M|f|$ a.e.

Proof. See [10]. □

Theorem B.0.8 (Chapman-Kolmogorov equation)**.** *Let X be a Markov process with transition law p. For all $0 \leq t_1 < t_2 < t_3$ and for all Borel set $H \subset \mathbb{R}^d$, we have*

$$p(t_1, X_{t_1}; t_3, H) = \int_{\mathbb{R}^d} p(t_1, X_{t_1}; t_2, dx_2) p(t_2, x_2; t_3, H).$$

Proof. See [16] and [17]. □

Theorem B.0.9 (Burkholder-Davis-Gundy inequality)**.** *For all $p > 0$ there exist two constants $c_p, C_p > 0$ such that*

$$c_p \mathbb{E}\left[\langle X\rangle_\tau^{\frac{p}{2}}\right] \leq \mathbb{E}\left[\sup_{t\in[0,\tau]} |X_t|^p\right] \leq C_p \mathbb{E}\left[\langle X\rangle_\tau^{\frac{p}{2}}\right],$$

for all continuous local martingale X such that $X_0 = 0$ a.s. and for all finite stopping time τ ($\tau < +\infty$ a.s.).

Proof. See [16] and [17]. □

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