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Some new results on the matching distance in biparameter persistent homology

Tesi di Laurea in Topological Data Analysis

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A mis padres.

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Abstract

When studyng monoparameter persistent homology, the bottleneck distance is the most standard choice in literature. The advantage of this choice is that persistence diagrams in the monoparameter setting are a complete invariant as shown by the Isometry Theorem. However, they are not a complete invariant in multiparameter persistent homology. This makes the multiparameter case much more harder to study.

In this work we study some properties of the biparametric matching distance, which is an analogue of the bottleneck distance for higher dimensions, between persistence diagrams, along with the set of special values (a, b) in which the matching realising this distance may change abruptly. We prove a result that drastically reduces the cost of the computation of the matching distance, under suitable regularity conditions on the filtered space (X, φ) and on the special set. Moreover, we give a proof for an extension of the Position Theorem, which is a central step in the proof of our main theorem.

Chapter 1

Introduzione

L'analisi topologica dei dati (TDA) svolge un ruolo importante in uno dei più grandi problemi della società contemporanea: l'organizzazione e l'analisi di grandi quantità di dati. L'approccio di questo ramo della topologia applicata consiste nell'analizzare questi grandi insiemi di dati utilizzando strumenti di topologia e dell'algebra omologica. Nell'ambito della TDA, un modo abituale per rappresentare i dati è quello di considerarli come una nuvola di punti, cioè come un insieme finito. Questo insieme viene ulteriormente arricchito con delle strutture simpliciali e metriche. I diversi invarianti topologici e geometrici di questo spazio corrispondono a certe *features*, ossia a particolari informazioni intrinseche ai dati, che sono di interesse per la loro comprensione. In particolare, l'omologia persistente cerca di costruire un collegamento tra la topologia e la geometria utilizzando i gruppi di omologia, che sono un invariante centrale nella topologia algebrica, per identificare le proprietà topologiche che persistono nei diversi livelli di analisi dell'insieme rappresentante i dati e risultano contemporaneamente compatti e discriminanti.

L'approccio classico all'omologia persistente si basa sullo studio dei cambiamenti omologici degli insiemi di sottolivello X_u^φ di uno spazio topologico filtrato (X, φ), con $\varphi\colon X \to \mathbb{R}^n$ una funzione continua, al variare di u in $\mathbb{R}^n$. Quando il codominio della funzione filtrante φ ha dimensione 1 parleremo di omologia persistente monodimensionale o monoparametrica, e per $n > 1$ parleremo di omologia persistente multidimensionale o multiparametrica. Gli oggetti di studio dell'omologia persistente sono i diagrammi di persistenza, e il nostro lavoro si incentra sullo studio di una delle pseudometriche più comunemente usate tra questi oggetti, la *bottleneck distance*.

La persistenza monoparametrica fornisce un riassunto dei dati attraverso una filtrazione unidimensionale, fornendo una panoramica dei dati a molte scale diverse. L'omologia persistente monoparametrica è stata oggetto di numerosi studi ed è risultata utile in molte applicazioni, tra cui [11], [14]. Inoltre la persistenza monoparametrica rende disponibile una descrizione completa delle features osservate dall'omologia, nel seguente senso: i diagrammi di persistenza offrono un invariante completo dell'oggeto geometrico in analisi. Inoltre, nel caso monoparametrico basato su un approccio categoriale alla TDA, il cosiddetto *Isometry theorem* fornisce un'uguaglianza tra la distance di bottleneck e la distanza di interleaving [9]. La rilevanza di questo risultato e una motivazione per lo studio della distanza di bottleneck è il fatto che per il calcolo di questa distanza esistono algoritmi con costi computazionali bassi. Al contrario, è stato dimostrato in [15] che il calcolo dell'interleaving distance è NP-hard. Il lettore interessato agli algoritmi per il calcolo della bottleneck distance può consultare [5], [17].

Nonostante la persistenza monoparametrica sia stata ampiamente studiata in ambito teorico e a lungo utiliizzata nelle applicazioni, alcuni dati richiedono una filtrazione lungo più parametri per catturare multiple e più complesse informazioni: questo è il ruolo dell'omologia persistente multiparametrica [7]. In alcuni contesti, può, infatti, essere utile utilizzare più parametri per catturare diverse misurazioni dei dati **miller2020data**. Purtroppo, comprendere, visualizzare e calcolare gli invarianti nell'omologia persistente multiparametrica rimane un compito difficile sia dal punto di vista matematico che computazionale. Questa difficoltà si applica anche al calcolo delle distanze tra tali invarianti. Per esempio, non vale in generale un equivalente dell'Isometry Theorem.

L'assenza di invarianti analoghi ai diagrammi di persistenza nel caso biparametrico, lo rende più difficile da studiare rispetto al caso monoparametrico e richiedi lo sviluppo di nuove idee e metodi matematici. Uno di questi metodi consiste nel ridurre una filtrazione bidimensionale associata a una certa funzione $\varphi\colon X \to \mathbb{R}^2$ a una famiglia di filtrazioni unidimensionali associate a funzioni $\varphi^*_{(a,b)}\colon X \to \mathbb{R}$, con $a \in]0,1[, b \in \mathbb{R}$, definite come

$$\varphi^*_{(a,b)}(x) = \min\{a, 1-a\} \max\left\{ \frac{\varphi_1(x) - b}{a}, \frac{\varphi_2(x) + b}{1-a} \right\}.$$

Ogni coppia (a, b) identifica la retta con pendenza positiva $r_{(a,b)}$ in $\mathbb{R}^2$ definita dall'equazione parametrica $(u, v) = (at + b, (1 - a)t - b)$. In altre parole, la precedente filtrazione unidimensionale associata alla funzione $\varphi^*_{(a,b)}$ viene ottenuta proiettando X sul piano $\mathbb{R}^2$ mediante φ e considerando, per ogni $p \in r_{(a,b)}$, il sottoinsieme $X_p \subset X$ formato dai punti che si trovano in basso a sinistra di p. È noto che, per ogni grado k, la conoscenza di tutte le funzioni dei numeri di Betti persistenti associate alle filtrazioni di X definite dalle funzioni $\varphi^*_{(a,b)}$ al variare di a e b è equivalente alla conoscenza della funzione dei numeri di Betti persistenti associata alla filtrazione di X data da φ [7].

In questo lavoro affrontiamo lo studio di una metrica ampiamente studiata in persistenza biparametrica, la *matching distance*. Questa metrica è definita per due funzioni $\varphi, \psi\colon X \to \mathbb{R}^2$ come il supremo delle distanze di bottleneck tra i diagrammi di persistenza $\varphi^*_{(a,b)}, \psi^*_{(a,b)}$ al variare di (a, b).

In particolare, affrontiamo una congettura ampiamente studiata nella letteratura (si vedano, per esempio, [13], [19]) che riguarda la persistenza biparametrica: è possibile calcolare la distanza di matching studiando soltanto le rette filtranti $r_{(a,b)}$ di pendenza 1, ovvero, fissando il parametro $a = \frac{1}{2}$? Un tale risultato sarebbe molto utile dal punto di vista applicativo e computazionale, giacché consentirebbe di passare dello studio dei valori su una striscia allo studio dei valori su una retta, riducendo nettamente il costo computazionale del calcolo.

Purtroppo questo risultato non sussiste in generale nel caso discreto ed è probabilmente falso anche nel caso topologico.. Alcuni controesempi possono essere trovati in [19]. Tuttavia alcune versioni deboli, ancora computazionalmente interessanti, della congettura possono essere dimostrate sotto opportune ipotesi di stabilità. In questa tesi imponiamo delle condizioni sull'insieme dei valori (a, b) che sono "speciali" nel senso che in tali punti il matching il cui costo realizza la distanza di matching potrebbe cambiare in modo discontinuo. In questo modo riusciamo a dimostrare che la distanza viene realizzata su almeno una delle tre rette, $a = 0, a = \frac{1}{2}$ e $a = 1$ oppure su almeno un punto di un insieme costituito da un numero finito di valori (a, b) speciali. Questo studio è il contenuto dei capitoli 5 e 6 di questo lavoro.

È interessante osservare che, in un setting più generale di quello presentato in questa tesi, l'omologia persistente appartiene a una classe di operatori adatti a modelizzare il ruolo degli osservatori che trasformano e interpretano i dati, i recentemente introdotti *Group Equivariant Non-Expansive Operators, GENEOs.* Questi costituiscono un recente approccio al collegamento tra topologia e analisi dei dati, e sono attualmente oggeto di studio, p.e. in [12], [20], [22].

Il contenuto della tesi è il seguente. Nel capitolo 2 richiamiamo la definizione di diagramma di persistenza insieme ad altri concetti e teoremi di base in omologia persistente monoparametrica e multiparametrica che utilizzeremo nello sviluppo del resto dei capitoli. Inoltre introduciamo l'extended Pareto grid, che sarà lo strumento centrale per la dimostrazione del Position Theorem 6. Nel capitolo 3 introduciamo le distanze di bottleneck e di matching per lo studio dei diagrammi di persistenza monoparametrici e multiparametrici, rispettivamente. Il contenuto del capitolo 4 è l'estensione del Position Theorem presentato in [13]. Nel capitolo 5 ci concentriamo sullo studio dei valori (a, b) in cui i matching che realizzano la distanza di matching bidimensionale potrebbero cambiare in modo discontinuo. Finalmente, nel capitolo 6 introduciamo le nostre assunzioni di regolarità sui valori (a, b) e dopo dimostriamo il nostro risultato principale.

Chapter 1

Introduction

Topological Data Analysis (TDA) plays a crucial role in one of the most significant challenges of contemporary society: organizing and analyzing large quantities of data. The approach of this branch of applied topology consists in the analysis of these large datasets by using tools from topology and homological algebra. In the field of TDA, a common way to represent data is by considering them as a point cloud, which is a finite set. This set is further enriched with simplicial and metric structures. The different topological and geometric invariants of this space correspond to particular *features*, which are specific intrinsic information about the data that are of interest for their understanding. Persistent homology is a method in TDA that aims to establish a connection between topology and geometry by using homology groups, which are a central invariant in algebraic topology, to identify the topological properties that persist across different levels of analysis of the data and are relevant for their interpretation.

The classical approach to persistent homology is based on studying homological changes of the sublevel sets X_u^φ of a filtered topological space (X, φ), where $\varphi\colon X \to \mathbb{R}^n$ is a continuous function, as u varies in $\mathbb{R}^n$. When the codomain of the filtering function φ has dimension 1, we will speak of one-parameter or monodimensional persistent homology, and for $n > 1$, we will speak of multi-parameter or multidimensional persistent homology. The objects of study in persistent homology are persistence diagrams, and our work focuses on studying one of the most standard pseudometrics on these objects, the *bottleneck distance*.

Monodimensional persistence provides a summary of data through a one-dimensional filtration, offering an overview of the data at multiple scales.

Monodimensional persistent homology has been extensively studied and proven useful in various applications, including [11], [14]. Furthermore, monodimensional persistence provides a complete description of the features observed by homology. Persistence diagrams provide a complete invariant of the geometric object under analysis. Furthermore, in the monoparametric case based on a categorical approach to TDA, the *Isometry theorem* provides an equality between the bottleneck distance and the interleaving distance [9]. This result highlights the relevance of the bottleneck distance and its computationally efficient algorithms, as opposed to the NP-hardness [15] of effectively calculating the interleaving distance. The reader interested in the algorithms for the calculation of the bottleneck distance can refer to [5], [17].

Despite monoparametric persistence having been extensively studied in theory and widely used in applications, some data require filtrations along multiple parameters to fully capture their information: this is the role of multiparameter persistent homology [7]. In certain contexts, using multiple parameters can be helpful in capturing data details. Unfortunately, understanding, visualizing, and computing invariants in multiparameter persistent homology remains a challenging task from both the mathematical and computational viewpoints. This difficulty also applies to the computation of distances between such invariants. For example, the Isometry Theorem does not generally hold in the multiparameter setting.

The absence of invariants analogous to persistence diagrams in the biparametric case makes it more challenging to study compared to the monoparametric case and requires the development of new mathematical ideas and methods. One such method consists on the reduction from the two-dimensional case to the one-dimensional case by using a family of functions $\varphi^*_{(a,b)} \colon X \to \mathbb{R}^2$, where $a \in]0,1[$ and $b \in \mathbb{R}$, defined as follows:

$$\varphi^*_{(a,b)}(x) = \min\{a, 1-a\} \max\left\{ \frac{\varphi_1(x) - b}{a}, \frac{\varphi_2(x) + b}{1-a} \right\}.$$

Each pair (a,b) corresponds to a line with positive slope $r_{(a,b)}$ in $\mathbb{R}^2$, defined by the parametric equation $(u,v) = (at+b, (1-a)t-b)$. The function $\varphi^*_{(a,b)}$ allows us to pass from a two-dimensional filtration to a one-dimensional filtration. In simple terms, the aforementioned one-dimensional filtration associated with the function $\varphi^*_{(a,b)}$ is obtained by projecting X onto the plane $\mathbb{R}^2$ through φ and considering, for each $p \in r(a,b)$, the subset $X_p \subset X$ consisting of points that lie below and to the left of p. It is known that, for each degree k, the knowledge of the persistent Betti numbers functions associated with the filtrations defined by the filtering functions $\varphi^*_{(a,b)}$ for varying a and b is equivalent to the knowledge of the persistent Betti numbers function associated with the filtration defined by φ.

In this work, we tackle the study of a widely studied metric in biparametric persistence, the matching distance. This metric is defined for two continuous functions $\varphi, \psi \colon X \to \mathbb{R}^2$ as the supremum of bottleneck distances between the persistence diagrams $\varphi^*_{(a,b)}, \psi^*_{(a,b)}$ over varying (a, b).

In particular, we address a conjecture that has been extensively studied in the literature (see, for example, [13], [19]) regarding biparameter persistent homology: Can we effectively compute the matching distance by considering only the filtering lines $r_{(a,b)}$ with slope 1, that is, by fixing the parameter $a = \frac{1}{2}$? Such a result would be highly useful from both an applicational and computational perspective, as it would allow us to transition from studying values on a strip to studying values on a line, significantly reducing the computational cost of calculations.

Unfortunately, this result does not generally hold in the discrete case, and it is likely false in the topological case too. Counterexamples can be found in [19] for the discrete case. However, weakened versions of the conjecture, which are still computationally interesting, can be proven under suitable stability assumptions. In this thesis, we impose conditions on the set of values (a, b) that are "special" in the sense that at these points, the matching that realizes the matching distance may change discontinuously. In this way, we can prove that the distance is realized either on at least one of the three lines $a = 0, a = \frac{1}{2}$, and $a = 1$, or on at least one point of a set consisting of a finite number of exceptional (a, b) values. This is the content of Chapters 5 and 6 of this work.

It is interesting to note that, in a more general setting, persistent homology belongs to a class of operators suitable for modeling the role of observers that transform and interpret data, namely the recently introduced Group Equivariant Non-Expansive Operators (GENEOs). GENEOs represent a novel approach to the connection between topology and data analysis, and they are currently being studied in works such as [12], [20], [22].

The content of the thesis is as follows:

In Chapter 2, we review the definition of persistence diagrams along with other basic concepts and theorems in monodimensional and multidimensional persistent homology that we will use in the development of the remaining chapters. We also introduce the extended Pareto grid, which will be the central tool for proving the Position Theorem 6. In Chapter 3, we introduce the bottleneck and matching distances for the study of monodimensional and multidimensional persistence diagrams, respectively. The content of Chapter 4 is the extension of the Position Theorem presented in [13]. In Chapter 5, we focus on studying the values (a, b) where the matchings that realize the two-dimensional matching distance may change discontinuously. Finally, in Chapter 6, we introduce our assumptions of regularity on the values (p_a, q_b) and then prove our main result.

Chapter 2

Preliminaries

Throughout this chapter let $\mathbb{F}$ be a field and let X be a finitely triangulable topological space. Also, fix the following notations:

$$\begin{aligned}
\Delta &= \{(u,u) \in \mathbb{R}^2\} \quad \text{that we refer to as the diagonal,}\\
\Delta^+ &= \{(u,v) \in \mathbb{R}^2 \text{ such that } u < v\} \sqcup \{\Delta\},\\
\Delta^* &= \Delta^+ \sqcup \{(u,\infty)\}_{u \in \mathbb{R}}.
\end{aligned}$$

Most of the definitions and results in this chapter and the following one come from [7].

2.1 One-parameter Persistent Homology

We start with a general definition:

Definition 2.1 (Persistence module)**.** A (n-parameter) *persistence module* is a covariant functor $\mathbf{M}\colon \mathbf{R}^n \to \mathbf{Vect}_{\mathbb{F}}$ where $\mathbf{Vect}_{\mathbb{F}}$ is the category of $\mathbb{F}-$vector spaces and linear maps, and $\mathbf{R}^n$ is the category induced by the poset $(\mathbb{R}^n, \preceq)$, with

$$x, y \in \mathbb{R}^n,\ x \preceq y \text{ if and only if } x_i \leqslant y_i \text{ for every } 0 \leqslant i \leqslant n.$$

Persistence modules can be defined in a more general setting, which we present briefly in appendix A.

In TDA, persistence modules codify topological information about a topological space X which represents the data, and the homology of X represents "features" of the data we wish to study. Persistence modules encode the variation in the homology of a filtered topological space. They are usually obtained by applying the homology functor to a filtration of topological spaces, where an inclusion of topological spaces is associated to the linear maps between their homology vector spaces.

Definition 2.2. A *filtration* is a functor $\mathcal{F}\colon \mathbf{R}^n \to \mathbf{Top}$ from the poset category $\mathbf{R}^n$ induced by the partially ordered set $(\mathbb{R}^n, \preceq)$ to the category of topological spaces and continuous functions, such that for each morphism $u \leqslant v$, $\mathcal{F}(u \leqslant v)$ is an inclusion.

Any continuous map $\varphi\colon X \to \mathbb{R}$ induces a filtration $\mathcal{F}\varphi\colon \mathbf{R} \to \mathbf{Top}$, with $\mathcal{F}\varphi(t) = X_t^\varphi$ for each $t \in \mathbb{R}$, and $\mathcal{F}\varphi(s \leqslant t) = \iota_{s,t}^\varphi$ for each $s \leqslant t$ in the following way: for any $t \in \mathbb{R}$ define

$$X_t^\varphi = \{x \in X\colon \varphi(x) \leqslant t\}.$$

For each $(s \leqslant t)$ there is a canonical inclusion

$$\iota_{s,t}^\varphi\colon X_s^\varphi \hookrightarrow X_t^\varphi.$$

Notice that, since X is finitely triangulable, each continuous function $\varphi\colon X \to \mathbb{R}$ is bounded and thus there exists $T = \max_{t\in\mathbb{R}} \varphi(t) \in \mathbb{R}$ such that for every $t' \geqslant t \geqslant T$

$$X_T^\varphi = X_{t'}^\varphi = X \quad \text{and} \quad \iota_{t,t'}^\varphi = \mathrm{id}_X.$$

We will write $X_\infty^\varphi = X$, and this allows us to naturally extend the indexing poset of our filtration $\mathcal{F}\varphi$ to $(\mathbb{R} \cup \{\infty\}, \leqslant)$, by putting

$$\mathcal{F}\varphi(\infty) = X, \quad \text{and} \quad \mathcal{F}\varphi(t \leqslant \infty)\colon X_t^\varphi \hookrightarrow X,$$

extending the order relation with $t < \infty$ for each $t \in \mathbb{R}$.

Fix a degree $k \in \mathbb{Z}$. Let $H_k(-;\mathbb{F})$ denote the Čech homology functor in degree k. To avoid some encumbering notation, in the following we will omit the coefficient field $\mathbb{F}$ and we will write $\iota_{s,t}^{\varphi *} = H_k(\iota_{s,t}^\varphi)$ when the degree of homology is not specified or clear from the context.

Remark 2.1. The collection of objects $\{H_k(X_t^\varphi)\}_{t\in\mathbb{R}\cup\infty}$ and morphisms $\{\iota_{s,t}^{\varphi *}\}_{(s\leqslant t)\in\mathbb{R}\cup\infty}$ parametrised by the ordered set $(\mathbb{R}, \leqslant)$ is a persistence module.

Definition 2.3 (Persistent homology group)**.** For any $(u \leqslant v)$ define

$$H_k^\varphi(u, v; X) = H_k^\varphi(u, v) = \mathrm{Im}\iota_{u,v}^{\varphi *} < H_k(X_v^\varphi)$$

the *k-th persistent homology group* of X at (u, v).

Remark 2.2. Since we are taking Čech homology with coefficients in a field, the Čech homology groups are $\mathbb{F}$-vector spaces fully described by $\dim_{\mathbb{F}} H_k(X_v^{\varphi})$, hence the persistent homology groups are determined up to isomorphism of vector spaces by their dimension as subspaces of $H_k(X_v^{\varphi})$.

Definition 2.4 (PBNF)**.** For any continuous function $\varphi\colon X \to \mathbb{R}$ and for any $k \in \mathbb{Z}$ define the PBNF, standing for *persistent Betti numbers function*, of φ in degree p as follows:

$$\begin{aligned} \beta_k^{\varphi}\colon \Delta^* &\to \mathbb{N} \cup \{\infty\} \\ (u, v) &\mapsto \dim_{\mathbb{F}} H_k^{\varphi}(u, v) \end{aligned}$$

One could wonder why working with Čech homology and why working with coefficients in a field $\mathbb{F}$, therefore losing the topological information about the torsion of the space X. The reason for the latter choice is that there is a decomposition theorem for persistence modules with codomain $\mathbf{Vect}_{\mathbb{F}}$ which is central in persistent homology. The result appeared initially in [2] and we present a modern generalization from [16] in Appendix A. The motivation for using Čech is to exploit some properties proved in [7] for persistent Betti numbers functions that we are about to introduce. In particular the authors of [7] showed that Proposition 2.1 is false in general for singular homology. This is Proposition 2.9 from [7]:

Proposition 2.1. *$\beta_k^{\varphi}(u, v)$ is right-continuous in both its variables.*

The next result is Theorem 2.3 in [7] and guarantees that persistent homology groups are finite-dimensional. The authors of [7] showed it in general for multiparameter persistent homology.

Theorem 1. *Let $\varphi\colon X \to \mathbb{R}$ be a continuous function, X a finitely triangulable topological space and $k \in \mathbb{Z}$. Then $\beta_k^{\varphi}(u, v) < \infty$ for every $(u, v) \in \Delta^*$.*

Definition 2.5 (Multiplicity)**.** Let $\varphi\colon X \to \mathbb{R}$ and $k \in \mathbb{Z}$. Define, for every $p = (u, v) \in \Delta^+$, the *multiplicity of p* as

$$\mu_k^{\varphi}(p) = \lim_{\varepsilon \to 0} \beta_p^{\varphi}(u+\varepsilon, v-\varepsilon) - \beta_p^{\varphi}(u-\varepsilon, v-\varepsilon) - \beta_p^{\varphi}(u+\varepsilon, v+\varepsilon) + \beta_p^{\varphi}(u-\varepsilon, v+\varepsilon),$$

and, for $q = (u, \infty) \in \Delta^* \backslash \Delta^+$, define the *multiplicity of q* as

$$\mu_k^{\varphi}(q) = \lim_{\varepsilon \to 0} \beta_p^{\varphi}(u + \varepsilon, \infty) - \beta_p^{\varphi}(u - \varepsilon, \infty).$$

We will omit the degree k and the filtering function φ for the sake of clarity when it is not ambiguous. Let us justify the terminology "multiplicity" for the number $\mu(p)$:

Proposition 2.2. *For every $p = (u, v) \in \Delta^*$ $\mu(p) \in \mathbb{N} \cup \{0\}$.*

Proof. It is enough to show that $\beta(u_2, v_1) - \beta(u_1, v_1) \geqslant \beta(u_2, v_2) - \beta(u_1, v_2)$ and $\beta(u_2, \infty) - \beta(u_1, \infty) \geqslant 0$ for any real numbers $u_1 \leqslant u_2 < v_1 \leqslant v_2$. Indeed, note that if

$$\iota^*_{u_2,v_2} = \iota^*_{v_1,v_2} \circ \iota^*_{u_2,v_1}$$

then the Rank-nullity theorem - see Appendix C - implies

$$\beta(u_2, v_2) = \dim \mathrm{Im}\iota^*_{u_2,v_2} = \dim \mathrm{Im}\iota^*_{u_2,v_1} - \dim \ker \left(\iota^*_{v_1,v_2}|_{\mathrm{Im}\iota^*_{u_2,v_1}} \right).$$

Conversely,

$$\iota^*_{u_1,v_2} = \iota^*_{v_1,v_2} \circ \iota^*_{u_1,v_1}$$

implies

$$\beta(u_1, v_2) = \dim \mathrm{Im}\iota^*_{u_1,v_2} = \dim \mathrm{Im}\iota^*_{u_1,v_1} - \dim \ker \left(\iota^*_{v_1,v_2}|_{\mathrm{Im}\iota^*_{u_1,v_1}} \right).$$

Now observe that $\mathrm{Im}\iota^*_{u_1,v_1} = \mathrm{Im}\left(\iota^*_{u_2,v_1} \circ \iota^*_{u_1,u_2}\right) \subset \mathrm{Im}\iota^*_{u_2,v_1}$; and from this follows

$$\dim \ker \left(\iota^*_{v_1,v_2}|_{\mathrm{Im}\iota^*_{u_1,v_1}} \right) \subset \dim \ker \left(\iota^*_{v_1,v_2}|_{\mathrm{Im}\iota^*_{u_2,v_1}} \right).$$

All together yields

$$\begin{aligned}
\beta(u_2, v_2) - \beta(u_1, v_2) &= \dim \mathrm{Im}\iota^*_{u_2,v_1} - \dim \mathrm{Im}\iota^*_{u_1,v_1} \\
&\quad - \dim \ker \left(\iota^*_{v_1,v_2}|_{\mathrm{Im}\iota^*_{u_2,v_1}} \right) + \dim \ker \left(\iota^*_{v_1,v_2}|_{\mathrm{Im}\iota^*_{u_1,v_1}} \right) \\
&\leqslant \dim \mathrm{Im}\iota^*_{u_2,v_1} - \dim \mathrm{Im}\iota^*_{u_1,v_1} \\
&= \beta(u_2, v_1) - \beta(u_1, v_1).
\end{aligned}$$

To prove $\beta(u_2, \infty) - \beta(u_1, \infty) \geqslant 0$ just observe

$$\mathrm{Im}\iota^*_{u_1,\infty} = \mathrm{Im}\iota^*_{u_1,\max\varphi} = \mathrm{Im}\left(\iota^*_{u_2,\max\varphi} \circ \iota^*_{u_1,u_2}\right) \subset \mathrm{Im}\iota^*_{u_2,\max\varphi} = \mathrm{Im}\iota^*_{u_2,\infty}.$$

□

Definition 2.6 (Multi-set)**.** A *multi-set* is a pair $(\mathcal{S}, f)$ of a set $\mathcal{S}$ and a function $f\colon \mathcal{S} \to \mathbb{N} \cup \{\infty\}$. If $S \in \mathcal{S}$ and $f(S) > 0$, we say that S is an element of the multi-set and its *multiplicity* is $f(S)$. We may refer to the multi-set $(\mathcal{S}, f)$ as the collection $\mathcal{S}_f = \{(S, n) \in \mathcal{S} \times (\mathbb{N} \cup \{\infty\}) : 0 < n \leqslant f(S)\}$.

Definition 2.7 (Map between multi-sets)**.** Let $(\mathcal{S}, f)$ and $(\mathcal{S}', f')$ be two multi-sets. Any map from $\mathcal{S}f$ to $\mathcal{S}'f'$ is called a *multi-set map* from the multi-set $(\mathcal{S}, f)$ to the multi-set $(\mathcal{S}', f')$.

Now we are ready to define the geometrical invariant for persistence modules announced in 2.1. Persistence diagrams play a central role in the computational aspects of TDA, in particular in multiparameter persistence, when comparing multiparameter persistence modules directly can be a computationally infeasible task, see Appendix A.

Definition 2.8 (Persistence diagram)**.** Let $\varphi \colon X \to \mathbb{R}$ be a continuous function. The *persistence diagram* $\mathrm{Dgm}(\varphi)$ of φ is the multiset consisting of points $(u, v) \in \Delta^* \backslash \{\Delta\}$ with multiplicity $\mu^{\varphi}(u, v)$ union the singleton $\{\Delta\}$ with infinite multiplicity.

With a small abuse of notation, throughout this thesis we will also use the symbol $\mathrm{Dgm}(\varphi)$ to indicate the realisation of the multiset $\mathrm{Dgm}(\varphi)$. The elements of $\mathrm{Dgm}(\varphi)$ are called *cornerpoints*. A cornerpoint $(u, v) \in \Delta^{\backslash}\Delta$ will be called a *proper cornerpoint* and a cornerpoint (u, ∞) will be called an *essential cornerpoint*. We will denote by $\mathrm{Prp}(\varphi)$ the set of proper cornerpoints of $\mathrm{Dgm}(\varphi)$ union the diagonal $\{\Delta\}$ and $\mathrm{Ess}(\varphi)$ the set of essential cornerpoints of $\mathrm{Dgm}(\varphi)$.

Now we will endow the set Δ^* with a metric structure. We introduce the notion of intrinsic metric in a metric space, and then we define the metric d in Δ^* as the intrinsic metric induced by the maximum norm distance. This definition, that may seem artificial and complicated, is at the base of the proof of the Stability Theorem 3, which will be fundamental in the construction that will take place in the following chapters.

The next definitions are from [4].

Definition 2.9. Let (M, d) be a metric space. For any $x, y \in M$, a *path* from x to y is a continuous map $\gamma \colon [0, 1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$. We will use the notation $\gamma \colon x \mapsto y$.

The union $\mathcal{P}_{[0,1]} = \bigcup_{n\in\mathbb{N}}\{(0 = t_0, t_1, \ldots, t_n, t_{n+1} = 1)\}_{0<t_1,\ldots<t_n<1}$ parametrises the finite partitions $\{[0, t_1), \ldots, [t_{n-1}, t_n), [t_n, 1]\}$ of $[0, 1]$.

Define the *length of a path* γ in M as the quantity

$$\ell(\gamma) = \sup_{(0,t_1,\ldots,t_n,1)\in\mathcal{P}_{[0,1]}} \sum_{i=0}^{n} d(\gamma(t_i), \gamma(t_{i+1})) \geqslant 0.$$

Define a new (extended) metric d_i on M, the *intrinsic metric induced by* d, as follows: for each $x, y \in M$ set $d_i(x, y) = \infty$, if there are no paths of finite length from x to y. Otherwise,

$$d_i(x, y) = \inf_{\gamma\colon x \mapsto y} \ell(\gamma).$$

Remark 2.3. (M, d_i) is a metric space and, in general, $d_i \leqslant d$, meaning that the topology on M induced by d_i is always finer than the one induced by d. For more details see [4].

Definition 2.10. Consider the maximum norm distance, or Chebyshev distance in Δ^*. For each $p = (u, v), p' = (u', v') \in \Delta^*$ it is defined as

$$d_\infty(p, p') = \begin{cases} \max\{|u - u'|, |v - v'|\} \text{ when } p, p' \neq \Delta, \\ \frac{v-u}{2} \text{ when } p' = \Delta, \\ \frac{v'-u'}{2} \text{ when } p = \Delta, \\ 0 \text{ otherwise.} \end{cases}$$

with the convention that for each $r \in \mathbb{R}$, $\infty - r = r - \infty = \infty, \infty - \infty = 0, |\infty| = \infty, \frac{\infty}{2} = \infty, \min\{\infty, r\} = r$ and $\max\{\infty, r\} = \infty$.

We will denote with d the intrinsic metric induced by d_∞.

Remark 2.4. Explicitly, the extended metric d is given by

$$d(p, p') = \begin{cases} \min\{\max\{|u - u'|, |v - v'|\}, \max\{\frac{v-u}{2}, \frac{v'-u'}{2}\}\} & \text{if } p,\ p' \in \Delta^+ \\ |u - u'| & \text{if } v = \infty,\ v' = \infty \\ \frac{v-u}{2} & \text{if } p \in \Delta^+,\ p' = \Delta \\ \frac{v'-u'}{2} & \text{if } p' \in \Delta^+,\ p = \Delta \\ 0 & \text{if } p = p' = \Delta \\ \infty & \text{otherwise} \end{cases}$$

for each $p = (u, v), p' = (u', v')$.

The meaning of the term $\min\{\max\{|u - u'|, |v - v'|\}, \max\{\frac{v-u}{2}, \frac{v'-u'}{2}\}\}$ is the following. For each pair of points $p, p' \in \Delta^*$ there are two paths to consider when computing the intrinsic metric: the path $\gamma\colon p \mapsto p'$ such that $\mathrm{Im}\gamma$ coincides with the segment $[p, p']$ and the path $\gamma'\colon p \mapsto p'$ passing through the point $\Delta \in \Delta^*$ such that $\mathrm{Im}\gamma$ is the disjoint union of segments $[p, \left(\frac{v-u}{2}, \frac{v-u}{2}\right)]$ and $[\left(\frac{v'-u'}{2}, \frac{v'-u'}{2}\right), p']$. Indeed, $\left(\frac{v-u}{2}, \frac{v-u}{2}\right), \left(\frac{v'-u'}{2}, \frac{v'-u'}{2}\right)$ are the points in Δ nearest from p, p', respectively.

The next proposition is Proposition 3.9 together with Remark 3.10 in [7], and will be used very frequently in the proofs that will follow.

Proposition 2.3 (Local finiteness of cornerpoints)**.** *Let $\varphi\colon X \to \mathbb{R}$ be a continuous function and for any $\varepsilon > 0$ let U_ε denote the open ball of radius ε and center Δ with respect to the metric d. Then*

- $\mathrm{Prp}(\varphi)\backslash U_\varepsilon$ *is a finite set.*
- $\mathrm{Ess}(\varphi)$ *is a finite set.*

Remark 2.5. The previous proposition implies that the set of accumulation points of any persistence diagram is $\{\Delta\}$.

Proposition 2.4. *For every continuous function $\varphi\colon X \to \mathbb{R}$, $\mathrm{Dgm}(\varphi)$ is a compact subset of (Δ^*, d).*

Proof. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of $\mathrm{Dgm}(\varphi)$ in (Δ^*, d). Then there exists $i_0 \in I$ such that $\Delta \in U_{i_0}$. Because of Proposition 2.3, $\mathrm{Dgm}(\varphi)\backslash U_{i_0}$ is a finite set. Put $\mathrm{Dgm}(\varphi)\backslash U_{i_0} = \{p_1, p_2, \ldots, p_N\}$. Therefore it is enough to choose open sets U_{i_j} containing the cornerpoints p_j, for $1 \leqslant j \leqslant N$. Then, $\{U_{i_0}, U_{i_1}, \ldots, U_{i_N}\}$ is a finite subcovering of $\mathcal{U}$. $\square$

2.2 Biparameter Persistent Homology

All of the definitions of the previous section can be extended to multiparameter persistence. We are particularly interested in the biparameter case, and in the next section we will introduce a useful tool from differential geometry, the extended Pareto grid, which the authors of [13] used prove a powerful result characterising the coordinates of cornerpoints in a persistence diagram. This result, referred as Position Theorem, will be discused and extended in Chapter 4.

Any continuous map $\varphi\colon X \to \mathbb{R}^n$ induces a filtration $\mathcal{F}\varphi\colon \mathbf{R}^n \to \mathbf{Top}$, with $\mathcal{F}\varphi(u) = X_u^\varphi$ for each $u \in \mathbb{R}^n$, and $\mathcal{F}\varphi(u \preceq v) = \iota_{u,v}^\varphi$ for each $u \preceq v$ in the following way: for any $u \in \mathbb{R}^n$ put

$$X_u^\varphi = \{x \in X : \varphi(x) \preceq u\},$$

and for each $(u \preceq v)$ there is a canonical inclusion

$$\iota_{u,v}^\varphi\colon X_u^\varphi \hookrightarrow X_v^\varphi.$$

With the same notation, define multiparater persistent homology groups and multiparameter PBNFs exactly as in the previous section. The collection of objects $\{H_k(X_u^{\varphi})\}_{u\in\mathbb{R}^n}$ and morphisms $\{\iota_{u,v}^{\varphi *}\}_{(v\preceq v)\in\mathbb{R}^n}$ parametrised by the partially ordered set $(\mathbb{R}^n, \preceq)$ is a persistence module. Theorem 1 still holds for multiparameter persistence, allowing us to give the definitions and results that will follow.

From now on we will focus on the biparameter case. We introduce a way of studying a biparameter PBNF through a family of one-parameter PBNFs associated with filtrations induced by lines of positive slope. This technique is commonly referred as *foliation method.* The reader interested in the generalisation to n-parameter persistence of this technique can refer to [7].

Definition 2.11. Consider the set of strictly positive slope lines in $\mathbb{R}^2$ parametrised by the set $(0,1)\times\mathbb{R}$ through

$$(0,1)\times\mathbb{R} \ni (a,b) \mapsto r_{(a,b)} = \{(at+b, (1-a)t-b)\}_{t\in\mathbb{R}}.$$

We will call $r_{(a,b)}$ the *filtering line* associated with the pair (a,b).

Remark 2.6. In chapters 5, 6 we will use the cartesian equation of the filtering line $r_{(a,b)}$. It follows from a trivial calculation from the parametrisation:

$$y = \frac{1-a}{a}x - \frac{b}{a}$$

Now we construct a filtration $\mathcal{F}_{(a,b)}\colon \mathbf{R} \to \mathbf{Top}$ induced by $r_{(a,b)}$. Each point $(u(t), v(t)) \in r_{(a,b)}$ can be associated with the subspace $X^{\varphi}_{(u(t),(v(t))} = \{x \in X\colon \varphi_1(x) \leqslant u(t),\ \varphi_2(x) \leqslant v(t)\} \subset X$. The filtration given by

$$\mathcal{F}_{(a,b)}(t) = X^{\varphi}_{(u(t),v(t))}, \quad \mathcal{F}_{(a,b)}(t) = \iota^{\varphi}_{(u(s),v(s)),(u(t),v(t))}$$

for each $s \leqslant t \in \mathbb{R}$ is a one-parameter filtration of X.

The following is a useful result charaterising the filtration we just defined as the one-parameter filtration for a real valued function $\varphi^*_{(a,b)}$. This is theorem 4.2 in [7].

Theorem 2. *Let* $(a,b) \in (0,1)\times\mathbb{R}$ *The filtration* $\mathcal{F}_{(a,b)}$ *just introduced is the one-parameter filtration on* X *induced by the real-valued function*

$$\varphi^*_{(a,b)}(x) = \min\{a, 1-a\}\max\left\{\frac{\varphi_1(x)-b}{a}, \frac{\varphi_2(x)+b}{1-a}\right\}$$

Definition 2.12 (Order relation on a filtering line)**.** For each filtering line $r_{(a,b)}$ we will consider the following binary relation on $r_{(a,b)}$:

$$X \leqslant_{(a,b)} Y \iff x_X \leqslant x_Y \text{ or, equivalently } y_X \leqslant y_Y, \quad \forall X, Y \in r_{(a,b)}.$$

When $X \leqslant_{(a,b)} Y$ and $X \neq Y$ we will write $X <_{(a,b)} Y$. Observe that $x_X \leqslant x_Y$ is equivalent to $y_X \leqslant y_Y$ since $r_{(a,b)}$ is a line with strictly positive slope.

We will extend the definition of $\leqslant_{(a,b)}$ to a broader set of filtering lines in Chapter 4. Showing that $\leqslant_{(a,b)}$ is indeed a total order is an application of the definition:

Proposition 2.5. *For each filtering line, $\leqslant_{(a,b)}$ is a total order relation and $<_{(a,b)}$ is a strict order relation on that line.*

Proof. Fix $(a, b) \in]0, 1[\times\mathbb{R}$. Let $P = (x_P, y_P), Q = (x_Q, y_Q), R = (x_R, y_R) \in r_{(a,b)}$. Let us check $\leqslant_{(a,b)}$ first.

- Reflexive and antisymmetric properties follow directly from reflexive and antisymmetric properties for the order $\leqslant$ in $\mathbb{R}$.
- Transitivity: let $P \leqslant_{(a,b)} Q$ and $Q \leqslant_{(a,b)} R$. Then $x_P \leqslant x_Q \leqslant x_R$, and it follows $P \leqslant_{(a,b)} R$.
- The order relation $\leqslant_{(a,b)}$ is total: by contradiction suppose that neither $P \leqslant_{(a,b)} Q$ nor $Q \leqslant_{(a,b)} P$. Equivalently, neither $x_P \leqslant x_Q$ nor $x_P \geqslant x_Q$, but that is an absurd since $(\mathbb{R}, \leqslant)$ is totally ordered.

As for $<_{(a,b)}$:

- Irreflexive and asymmetric properties follow directly from irreflexive and asymmetric properties for the strict order $<$ in $\mathbb{R}$.
- Transitivity: let $P <_{(a,b)} Q$ and $Q <_{(a,b)} R$. Then $x_P < x_Q < x_R$, and it follows $P <_{(a,b)} R$.
- Connectedness: by contradiction suppose that $P \neq Q$ and neither $P <_{(a,b)} Q$ nor $Q <_{(a,b)} P$. Equivalently, neither $x_P < x_Q$ nor $x_P < x_Q$, but that is an absurd when $x_P \neq x_Q$ since $<$ is a strict total order on $\mathbb{R}$.

□

Now we present the natural pseudo-distance. The natural-pseudo distance is a dissimilarity measure for topological spaces endowed with vector-valued continuous functions that is intrinsically hard to compute, motivating the interest in methods for its estimation, which will be treated in Section 3. The following is definition 5.1 from [7]:

Definition 2.13 (Natural pseudo-distance)**.** Let $\varphi, \psi \colon X \to \mathbb{R}^n$ be two continuous functions. The *natural pseudo-distance* between the pairs φ and ψ is

$$\delta(\varphi, \psi) = \inf_{h \in \mathrm{Homeo}(X)} \sup_{x \in X} \|\varphi(x) - \psi(h(x))\|_\infty$$

2.3 Extended Pareto grid

Let M be a closed smooth manifold paired with a Riemannian structure. In this chapter we recall the relation between a differential construction associated with a smooth function $\varphi \colon M \to \mathbb{R}^2$, called the extended Pareto grid, and the points of the persistence diagrams $\mathrm{Dgm}(\varphi^*_{(a,b)})$. This connection is established in the Position Theorem proved in [13].

Definition 2.14. The *Jacobi set* of φ is the collection

$$\mathbb{J}(\varphi) = \{p \in M \mid \nabla\varphi_1 = \lambda\nabla\varphi_2 \text{ or } \nabla\varphi_2 = \lambda\nabla\varphi_1, \text{ for some } \lambda \in \mathbb{R}\}.$$

The *Pareto critical set* of φ is the subset of $\mathbb{J}(\varphi)$ given by

$$\mathbb{J}_P(\varphi) = \{p \in \mathbb{J}(\varphi) \mid \nabla\varphi_1 = \lambda\nabla\varphi_2 \text{ or } \nabla\varphi_2 = \lambda\nabla\varphi_1, \text{ for some } \lambda \leqslant 0\}.$$

Assume now that φ is not only smooth, but it also satisfies the following properties:

(i) No point p exists in M at which both $\nabla\varphi_1$ and $\nabla\varphi_2$ vanish.

(ii) $\mathbb{J}(\varphi)$ is a 1-manifold smoothly embedded in M consisting of finitely many components, each one diffeomorphic to a circle.

(iii) $\mathbb{J}_P(\varphi)$ is a 1-dimensional closed submanifold of M, with boundary in $\mathbb{J}(\varphi)$.

(iv) If we denote by $\mathbb{J}_C(\varphi)$ the subset of $\mathbb{J}(\varphi)$ where $\nabla\varphi_1$ and $\nabla\varphi_2$ are orthogonal to $\mathbb{J}(\varphi)$, then the connected components of $\mathbb{J}_P(\varphi)\backslash\mathbb{J}_C(\varphi)$ are finite in number, each one being diffeomorphic to an interval. With respect to any parameterisation of each component, one of φ_1 and φ_2 is strictly increasing and the other is strictly decreasing. Each component can meet critical points for φ_1, φ_2 only at its endpoints.

These properties are generic on the set of smooth maps $M \to \mathbb{R}^2$ (see [1]).

Definition 2.15 (Extended Pareto grid)**.** Denote by $\{p_1, \ldots, p_h\}$ and $\{q_1, \ldots, q_k\}$, respectively, the critical points of φ_1 and φ_2. Since the function φ satisfies (i), then $\{p_1, \ldots, p_h\} \cap \{q_1, \ldots, q_k\} = \varnothing$. The *extended Pareto grid* of φ is defined as the union

$$\Gamma(\varphi) = f\left(\mathbb{J}_P(\varphi)\right) \cup \left(\bigcup_i v_i\right) \cup \left(\bigcup_j h_j\right)$$

where v_i is the vertical half-line $\{(x, y) \in \mathbb{R}^2 \mid x = \varphi_1(p_i), y \geqslant \varphi_2(p_i)\}$ and h_j is the horizontal half-line $\{(x, y) \in \mathbb{R}^2 \mid x \geqslant \varphi_1(q_j), y = \varphi_2(q_j)\}$. We refer to these half-lines as *improper contours* and to the closure of the image of the connected components of $\mathbb{J}_P(\varphi)\backslash\mathbb{J}_C(\varphi)$ as *proper contours* of $\Gamma(\varphi)$. For any improper contour extending from the point $(x_0, y_0) \in \mathbb{R}^2$, we will call (x_0, y_0) the *basepoint* of that improper contour.

We will denote $\mathrm{Ctr}(\varphi)$ the set of contours in $\Gamma(\varphi)$.

Remark 2.7. Observe that, because of property (ii), $\mathrm{Ctr}(\varphi)$ is a finite set. Moreover, property (iv) ensures that every contour can be parametrised as a curve whose two coordinates are respectively non-increasing and non-decreasing. In particular, this implies that the slope of the tangent to any contour at any point cannot have strictly positive slope. This fact will be useful in the proof of our main theorem.

Remark 2.8. Properties (ii) and (iv) guarantee that proper contours are bounded in $\mathbb{R}^2$. For any proper contour $\alpha \subset \Gamma(\varphi)$ its closure in $\mathbb{J}(\varphi)$, which is just α union one or both its endpoints, is diffeomorphic to a closed interval in $\mathbb{R}$.

We can consider the embedding $\iota\colon \Gamma(\varphi) \hookrightarrow \overline{\mathbb{R}}^2$, with $\overline{\mathbb{R}}$ the extended half-line. The closure of the image of this embedding

$$\hat{\Gamma}(\varphi) = \overline{\mathrm{Im}\iota}^{\overline{\mathbb{R}}^2}.$$

is homeomorphic to the union of $\Gamma(\varphi)$ with the endpoints $(\varphi_1(p_i), +\infty)$ and $(+\infty, \varphi_2(q_j))$ for each improper contour v_i, h_j,respectively. Call $\beta \subset \hat{\Gamma}(\varphi)$ a (proper or improper) contour in $\hat{\Gamma}(\varphi)$ if there exists a (proper or improper) α in $\Gamma(\varphi)$ such that $\beta = \overline{\alpha}^{\overline{\mathbb{R}}^2}$.

The topology on $\hat{\Gamma}(\varphi)$ is the topology induced by the product topology in $\overline{\mathbb{R}}^2$, where $\overline{\mathbb{R}}$ is equipped with the usual topology on the extended real line.

Then every contour in $\hat{\Gamma}(\varphi)$ is bounded and homeomorphic to a closed interval. Moreover, proper contours are diffeomorphic to a closed interval. With a small abuse of notation, we will use the symbol $\Gamma(\varphi)$ to denote both spaces in the remaining pages of this work.

Proposition 2.6. *Let $\alpha \subset \Gamma(\varphi)$ be a contour. Then $r_{(a,b)} \cap \alpha$ is either the empty set or a single point, for each $(a, b) \in]0, 1[\times\mathbb{R}$.*

Proof. By contradiction, suppose $r_{(a,b)} \cap \alpha = \{P, Q\}$ two different points. Because of property (iv) in Section 2.3, there is a diffeomorphism

$$\begin{aligned} \xi\colon [0, 1] &\to \alpha \\ t &\mapsto (\xi_1(t), \xi_2(t)) \end{aligned}$$

such that when x_1 or x_2 is strictly increasing, the other one is strictly decreasing. Let $T_1, T_2 \in [0, 1]$ such that $P = \xi(T_1)$ and $Q = \xi(T_2)$, and without loss of generality assume $T_1 < T_2$.

The vector $(P - Q)$ is a multiple of the director vector of the filtering line $r_{(a,b)}$, which is $(a, 1 - a)$, and since $a \in]0, 1[$, both its coordinates are positive. Therefore, there exists a point $Q' \in \alpha$ such that the line tangent to α in X has director vector $(a, 1 - a)$, that is, slope equal to $\frac{1-a}{a} > 0$, This is against (iv). $\square$

Chapter 3

Matching distance

In this chapter we introduce the bottleneck distance and the matching distance. The first was firstly introduced as a lower bound for the pseudo-natural distance 2.13, making it relevant and computable in the applications. It is also deeply connected with a certain notion of distance between persistence modules, see Appendix A. The bottleneck distance is a readily computable mean of comparing persistence modules and is widely used in TDA and shape comparison problems. The latter is its natural generalisation in n-parameter persistence. For more details about these distances, see [3].

Definition 3.1 (Matching between multi-sets)**.** Let $(\mathcal{S}, f)$ and $(\mathcal{S}', f')$ be two multi-sets. Any bijection from $\mathcal{S}f$ to $\mathcal{S}'f'$ is called a *matching* from the multi-set $(\mathcal{S}, f)$ to the multi-set $(\mathcal{S}', f')$.

With a slight abuse of notation, we will identify the multiset $(\mathcal{S}, f)$ with the support of f, $\{S_i\}_{i\in I}$, where its elements S_i are taken with the multiplicity $m_i = f(S_i)$.

Definition 3.2 (Bottleneck distance)**.** For any pair of continuous functions $\varphi, \psi \colon X \to \mathbb{R}$ we define the *bottleneck distance*:

$$d_{\mathrm{B}}(\mathrm{Dgm}(\varphi), \mathrm{Dgm}(\psi)) = \inf_{\sigma\in S(\varphi,\psi)} \mathrm{cost}\sigma = \inf_{\sigma\in S(\varphi,\psi)} \sup_{p\in\mathrm{Dgm}(\varphi)} d(p, \sigma(p))$$

where $S(\varphi, \psi)$ is the set of matchings between $\mathrm{Dgm}(\varphi)$ and $\mathrm{Dgm}(\psi)$.

If there exists a matching $\sigma \in S(\varphi, \psi)$ such that

$$d_{\mathrm{B}}(\mathrm{Dgm}(\varphi), \mathrm{Dgm}(\psi)) = \mathrm{cost}\sigma$$

we will say that the matching σ *realises* the bottleneck distance between $\mathrm{Dgm}(\varphi)$ and $\mathrm{Dgm}(\psi)$. Such a matching will be called an *optimal matching.*

With a slight abuse of notation, if σ is an optimal matching for $\mathrm{Dgm}(\varphi)$ and $\mathrm{Dgm}(\psi)$ and there exist $p \in \mathrm{Dgm}(\varphi)$, $q = \sigma(p) \in \mathrm{Dgm}(\psi)$ such that

$$\mathrm{cost}\sigma = d(p, q)$$

we will also say that the points p, q *realise* the bottleneck distance.

Remark 3.1. Let $\varphi, \psi \colon M \to \mathbb{R}$ be any two continuous functions. For any matching $\sigma \colon \mathrm{Dgm}(\varphi) \to \mathrm{Dgm}(\psi)$ the cost of σ is ∞ if and only if there is one of the two following situations:

1. There exists $p \in \mathrm{Prp}(\varphi)$ with $\sigma(p) \in \mathrm{Ess}(\psi)$.

2. There exists $p \in \mathrm{Ess}(\varphi)$ with $\sigma(p) \in \mathrm{Prp}(\psi)$.

Moreover, $d_{\mathrm{B}}(\mathrm{Dgm}(\varphi), \mathrm{Dgm}(\psi))$ is infinite if and only if $\mathrm{Ess}(\varphi)$ and $\mathrm{Ess}(\psi)$ have different cardinality. In this case, any matching $\sigma \in S(\varphi, \psi)$ necessarily sends a proper cornerpoint in $\mathrm{Dgm}(\varphi)$ to an essential one in $\mathrm{Dgm}(\psi)$ or viceversa.

Proposition 3.1. *d_{B} is an extended metric between persistence diagrams.*

Proof. For any continuous function φ, $d_{\mathrm{B}}(\mathrm{Dgm}(\varphi), \mathrm{Dgm}(\varphi)) = 0$ since $\mathrm{id}_{\mathrm{Dgm}(\varphi)} \in S(\varphi, \varphi)$ is a matching with cost zero. Symmetric property is also trivial since every matching is a bijection on its domain. Let us check triangular inequality for $\mathrm{Dgm}(\varphi), \mathrm{Dgm}(\varphi'), \mathrm{Dgm}(\varphi'')$. Fixing $\sigma \in S(\varphi, \varphi')$, for any $\tau \in S(\varphi, \varphi'')$, there exists $\sigma' \in S(\varphi', \varphi'')$ such that $\sigma' \circ \sigma = \tau$. Using this fact on the definition:

$$\begin{aligned}
d_{\mathrm{B}}(\mathrm{Dgm}(\varphi), \mathrm{Dgm}(\varphi'')) &= \inf_{\tau \in S(\varphi, \varphi'')} \sup_{p \in \mathrm{Dgm}(\varphi)} d(p, \tau(p))) \\
&= \inf_{\sigma' \in S(\varphi', \varphi'')} \sup_{p \in \mathrm{Dgm}(\varphi)} d(p, \sigma'(\sigma(p))) \\
&\leqslant \inf_{\sigma' \in S(\varphi', \varphi'')} \sup_{p \in \mathrm{Dgm}(\varphi)} \Big(d(p, (\sigma(p)) + d(\sigma(p), \sigma'(\sigma(p)) \Big) \\
&\leqslant \sup_{p \in \mathrm{Dgm}(\varphi)} d(p, \sigma(p)) + \inf_{\sigma' \in S(\varphi', \varphi'')} \sup_{q \in \mathrm{Dgm}(\varphi')} d(q, \sigma'(q)) \\
&= \sup_{p \in \mathrm{Dgm}(\varphi)} d(p, \sigma(p)) + d_{\mathrm{B}}(\mathrm{Dgm}(\varphi'), \mathrm{Dgm}(\varphi'')).
\end{aligned}$$

Taking the infimum varying $\sigma \in S(\varphi, \varphi')$ yields the triangle inequality

$$d_{\mathrm{B}}(\mathrm{Dgm}(\varphi), \mathrm{Dgm}(\varphi'')) \leqslant d_{\mathrm{B}}(\mathrm{Dgm}(\varphi), \mathrm{Dgm}(\varphi')) + d_{\mathrm{B}}(\mathrm{Dgm}(\varphi'), \mathrm{Dgm}(\varphi'')).$$

Now we have to check that d_{B} is actually an extended metric. By contradiction, we assume that $d_{\mathrm{B}}(\mathrm{Dgm}(\varphi), \mathrm{Dgm}(\psi)) = 0$ and $\mathrm{Dgm}(\varphi) \neq \mathrm{Dgm}(\psi)$. By definition of infimum, the distance $d_{\mathrm{B}}(\mathrm{Dgm}(\varphi), \mathrm{Dgm}(\psi))$ equals zero if and only if for every $\varepsilon > 0$ a matching $\sigma_\varepsilon \in S(\varphi, \psi)$ exists such that $d(p, \sigma_\varepsilon(p)) < \varepsilon$ for every $p \in \mathrm{Dgm}(\varphi)$. Assuming $\mathrm{Dgm}(\varphi) \neq \mathrm{Dgm}(\psi)$ implies that there exists at least a point $p = (u, v) \in \Delta^* \backslash \{\Delta\}$ with $\mu^\varphi(p) \neq \mu^\psi(p)$. Without loss of generality assume $\mu^\varphi(p) > \mu^\psi(p)$. Let us consider

$$c = \inf_{\substack{q \in \mathrm{Dgm}(\psi) \\ q \neq p}} d(p, q) > 0.$$

Then for every matching $\sigma \colon \mathrm{Dgm}(\varphi) \to \mathrm{Dgm}(\psi)$, $\mathrm{cost}\sigma \geqslant d(p, \sigma(p)) \geqslant c$. Hence, choosing $\varepsilon < c$ contradicts our assumption.

□

Lemma 3.1. *For any matching* $\sigma \colon \mathrm{Dgm}(\varphi) \to \mathrm{Dgm}(\psi)$ *there exists* $\overline{p} \in \mathrm{Dgm}(\varphi)$ *such that* $\mathrm{cost}\sigma = \sup_{p \in \mathrm{Dgm}(\varphi)} d(p, \sigma(p)) = d(\overline{p}, \sigma(\overline{p}))$.

Proof. Fixing a matching σ, assume that $\mathrm{cost}\sigma = C$. If $C = 0$, then σ can be assumed to be the identity of $\mathrm{Dgm}(\varphi)$ and every cornerpoint p realises the cost of the matching, $d(p, \sigma(p)) = 0$.

Now, we suppose that $C > 0$. By contradiction, assume that $d(p, \sigma(p)) < C$ for each $p \in \mathrm{Dgm}(\varphi)$. Then for every $n \in \mathbb{N}$ there exists a cornerpoint $p_n \in \mathrm{Dgm}(\varphi)$ such that $0 < C - d(p_n, \sigma(p_n)) < \frac{1}{n}$ and the sequence of real numbers $d(p_n, \sigma(p_n))_{n \in \mathbb{N}}$ is increasing and bounded by C. By using monotone convergence lemma, see Appendix C:

$$\lim_{n \to \infty} d(p_n, \sigma(p_n)) = \sup d(p_n, \sigma(p_n)) = C.$$

Because of the compactness of persistence diagrams 2.4, there exists a converging subsequence $(p_{n_k})_{k \in \mathbb{N}}$ of $(p_n)_{n \in \mathbb{N}}$. Local finiteness 2.3 of cornerpoints implies that the only accumulation point in a persistence diagram is Δ. Therefore,

$$\lim_{k \to \infty} p_{n_k} = \Delta.$$

Consider now the sequence $(\sigma(p_n))_{n\in\mathbb{N}}$. As before, there exists a subsequence $(\sigma(p_{n_j}))_{j\in\mathbb{N}}$ converging to Δ. Therefore, there are subsequences $(p_{n_{k_i}})_{i\in\mathbb{N}}$, $(\sigma(p_{n_{j_i}}))_{i\in\mathbb{N}}$ with

$$\lim_{i\to\infty} d(p_{n_{k_i}}, \sigma(p_{n_{j_i}})) = 0.$$

But this contradicts $C > 0$.

$\square$

Proposition 3.2 (Existence of an optimal matching)**.** *For every $\varphi, \psi\colon X \to \mathbb{R}$ continuous functions there exists an optimal matching σ such that*

$$d_{\mathrm{B}}(\mathrm{Dgm}(\varphi), \mathrm{Dgm}(\psi)) = \mathrm{cost}\sigma.$$

Proof. Let $D = d_{\mathrm{B}}(\mathrm{Dgm}(\varphi), \mathrm{Dgm}(\psi))$. If $D = \infty$ the statement follows from the fact that there exists a matching $\sigma\colon \mathrm{Dgm}(\varphi) \to \mathrm{Dgm}(\psi)$ and a point $p \in \mathrm{Dgm}(\varphi)$ with $p \in \mathrm{Prp}(\varphi)$ and $\sigma(p) \in \mathrm{Ess}(\psi)$ or viceversa. Then $\mathrm{cost}\sigma = \infty$ and σ realises the bottleneck distance. Hence, assume $D = d_{\mathrm{B}}(\mathrm{Dgm}(\varphi), \mathrm{Dgm}(\psi)) < \infty$.

By contradiction suppose $\mathrm{cost}\sigma > D$ for every matching $\sigma\colon \mathrm{Dgm}(\varphi) \to \mathrm{Dgm}(\psi)$. Then there is a sequence of matchings $(\sigma_n)_{n\in\mathbb{N}}$ such that

$$0 < \mathrm{cost}\sigma_n - D < \frac{1}{n}.$$

Let U_D be the metric open ball of radius D centered in Δ. For each matching $\sigma \in S(\varphi, \psi)$ consider the new matching

$$\begin{aligned} \overline{\sigma}\colon \mathrm{Dgm}(\varphi) &\to \mathrm{Dgm}(\psi) \\ p &\mapsto \Delta \quad \text{if } p \in U_D \cap \mathrm{Dgm}(\varphi) \\ q &\mapsto \sigma(q) \quad \text{otherwise.} \end{aligned}$$

Observe that $\mathrm{cost}\overline{\sigma} \leqslant \mathrm{cost}\sigma$, for each $\sigma \in S(\varphi, \psi)$. Let $\overline{S}(\varphi, \psi) = \{\overline{\sigma} \mid \sigma \in S(\varphi, \psi)\} \subset S(\varphi, \psi)$. The definition of D yields:

$$D = d_{\mathrm{B}}(\mathrm{Dgm}(\varphi), \mathrm{Dgm}(\psi)) = \inf_{\sigma\in S(\varphi,\psi)} \mathrm{cost}\sigma = \inf_{\sigma\in \overline{S}(\varphi,\psi)} \mathrm{cost}\overline{\sigma}.$$

Moreover, recall from Lemma 3.1 that for each matching σ there is a cornerpoint p that realises the cost of the matching. Observe that if $\mathrm{cost}\overline{\sigma} =$

$d(p, \Delta)$, for some $p \in U_D \cap \mathrm{Dgm}(\varphi)$, then $\mathrm{cost}\overline{\sigma} < D$, against the assumption that $D = \inf_{\sigma \in S(\varphi,\psi)} \mathrm{cost}\sigma$. Thus, for each $\overline{\sigma}$, $\mathrm{cost}\overline{\sigma}$ belong to the set

$$\{d(p, \sigma(p))\}_{p \in \mathrm{Dgm}(\varphi) \setminus U_D}.$$

But this set is finite because of Proposition 2.3.

In particular, this implies that the set $\{\mathrm{cost}\overline{\sigma}_n\}_{n \in \mathbb{N}}$ is finite. But then there exists $N \in \mathbb{N}$ sufficiently big such that for each $n > N$

$$\mathrm{cost}\sigma_n = D.$$

We have reached a contradiction and we can conclude.

□

Remark 3.2. Since we just proved that a matching realising the bottleneck distance always exists, by combining this result with the previous lemma the definition of bottleneck distance can be rewritten as

$$d_{\mathrm{B}}(\mathrm{Dgm}(\varphi), \mathrm{Dgm}(\psi)) = \min_{\sigma \in S(\varphi,\psi)} \mathrm{cost}\sigma = \min_{\sigma \in S(\varphi,\psi)} \max_{p \in \mathrm{Dgm}(\varphi)} d(p, \sigma(p))$$

where $S(\varphi, \psi)$ is the set of all matchings between $\mathrm{Dgm}(\varphi)$ and $\mathrm{Dgm}(\psi)$.

The following is Theorem 3.13 from [7], which is a generalisation to continuous functions of the original Stability Theorem for tame functions in [3]:

Theorem 3 (Stability theorem)**.** *Let $\varphi, \psi \colon M \to \mathbb{R}$ be two continuous functions. Then*

$$d_{\mathrm{B}}(\mathrm{Dgm}\varphi, \mathrm{Dgm}\psi) \leqslant \|\varphi - \psi\|_\infty.$$

Now we are ready to estabilish a metric between persistence diagrams in biparameter persistence.

Definition 3.3 (Biparameter matching distance)**.** Let $\varphi, \psi \colon M \to \mathbb{R}^2$ be two continuous functions. Define

$$D_{\mathrm{match}}(\varphi, \psi) = \sup_{(a,b) \in]0,1[\times \mathbb{R}} d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(a,b)}), \mathrm{Dgm}(\psi^*_{(a,b)}))$$

If there exists a pair $(a, b) \in]0, 1[\times [-C, C]$ such that

$$D_{\mathrm{match}}(\varphi, \psi) = d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(a,b)}), \mathrm{Dgm}(\psi^*_{(a,b)}))$$

we will say that the pair (a, b) *realises* the matching distance between φ and ψ.

There is an analogous theorem to 3 for the biparameter matching distance. The following is Corollary 2.5 from [13]:

Theorem 4 (Stability theorem for biparameter matching distance)**.** *Let* $\varphi, \psi\colon M \to \mathbb{R}^2$ *be two continuous functions. Then*

$$D_{\mathrm{match}}(\varphi, \psi) \leqslant \|\varphi - \psi\|_{\infty}.$$

Chapter 4

Position Theorem

In this chapter we state the Position Theorem proved in [13], which benefits from the extended Pareto grid defined in the previous chapter to give a characterisation of the coordinates of cornerpoints in the persistence diagram. We also prove an extension of this theorem for vertical and horizontal filtering lines. To do this, we first recall some results from [21] and introduce a new operator $\overset{\infty}{\cap}$ that formalises the informal idea of "intersection at infinity" between parallel filtering lines and contours of the extended Pareto grid.

For this chapter, let M be a closed smooth manifold with a Riemannian structure on it, and let $\varphi, \psi\colon M \to \mathbb{R}^2$ be two smooth functions satisfying properties (i)-(iv) from Section 2.3 and $\Gamma(\varphi), \Gamma(\psi)$ are the corresponding extended Pareto grids. Moreover, let us define the following constant

$$C = \max\{\|\varphi\|_\infty, \|\psi\|_\infty\}.$$

Remark 4.1. Since φ, ψ are continuous and M is compact, the constant C is finite.

4.1 Horizontal and vertical filtering lines

In [21, Theorem 4.3], the authors observed that:

Theorem 5. *If $|b| \leqslant C$, then for every $a, a' \in]0, 1[$ and $b \in \mathbb{R}$ the following inequality holds:*

$$\|\varphi^*_{(a,b)} - \varphi^*_{(a',b')}\|_\infty \leqslant 4|a - a'|(\|\varphi\|_\infty + C) + 3|b - b'|.$$

In particular, the previous theorem implies that $\varphi^*_{(a,b)}$ is locally Lipschitz on the variables a, b, and so it can be uniquely extended to $[0,1] \times \mathbb{R}$ as

$$\varphi^*_{(0,b)}(x) = \lim_{(a',b') \to (0,b)} \varphi^*_{(a',b')},$$
$$\varphi^*_{(1,b)}(x) = \lim_{(a',b') \to (1,b)} \varphi^*_{(a',b')}.$$

Explicitly, we can state the following result characterising $\varphi^*_{(0,b)}$ and $\varphi^*_{(1,b)}$:

Proposition 4.1. *Let $\varphi \colon M \to \mathbb{R}^2$ be a continuous function and let $b \in \mathbb{R}$. Then*

$$\varphi^*_{(0,b)}(x) = \max\{\varphi_1(x) - b, 0\},$$
$$\varphi^*_{(1,b)}(x) = \max\{0, \varphi_2(x) + b\}.$$

Proof. We will omit the dependency on x for the sake of clarity. For $a = 0$,

$$\begin{aligned}
\lim_{(a',b') \to (0,b)} \varphi^*_{(a',b')} &= \lim_{(a',b') \to (0,b)} \min\{a', 1-a'\} \max\{\frac{\varphi_1 - b'}{a'}, \frac{\varphi_2 + b'}{1-a'}\} \\
&= \max\{(\varphi_1 - b) \lim_{a' \to 0} \frac{a'}{a'}, (\varphi_2 + b) \lim_{a' \to 0} \frac{a'}{1-a'}\} \\
&= \max\{\varphi_1 - b, 0\}.
\end{aligned}$$

For $a = 1$, an analogous calculation gives the result. □

Thus, the persistence diagrams $\mathrm{Dgm}(\varphi^*_{(a,b)})$ and $\mathrm{Dgm}(\psi^*_{(a,b)})$ are defined for every (a,b) in $[0,1] \times \mathbb{R}$, and the function

$$(a,b) \mapsto d_B(\mathrm{Dgm}(\varphi^*_{(a,b)}), \mathrm{Dgm}(\psi^*_{(a,b)}))$$

can also be extended naturally to a continuous function defined on $[0,1] \times \mathbb{R}$, since the limit commutes with the operator $\mathrm{Dgm}(\cdot)$ because of the Stability Theorem from [7], see Lemma 4.2.

Remark 4.2. From Theorem 4 and the continuity of $(a,b) \mapsto d_B(\mathrm{Dgm}(\varphi^*_{(a,b)}), \mathrm{Dgm}(\psi^*_{(a,b)}))$, it follows that:

$$\lim_{a \to 0} d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(a,b)}), \mathrm{Dgm}(\varphi^*_{(0,b)})) = 0.$$

Let us now define the lines $r_{(0,b)}$ and $r_{(1,b)}$ as the lines of equations $x = b$ and $y = -b$, respectively. We observe that this definition agrees with 2.11, since for both $r_{(0,b)}$ and $r_{(1,b)}$ the director vector is $(a, 1-a)$ with $a = 0$ and $a = 1$.

The total ordering we introduced in 2.12 extends naturally to horizontal and vertical filtering lines:

Definition 4.1. Fixing (a,b) in $[0,1]\times\mathbb{R}$, for each $P=(x_P,y_P), Q=(x_Q,y_Q)\in r_{(a,b)}$ define

$$P \leqslant_{(a,b)} Q \iff \begin{cases} x_P \leqslant x_Q \text{ and } y_P \leqslant y_Q \text{ when } a \in]0,1[, \\ y_P \leqslant y_Q \text{ when } a=0, \\ x_P \leqslant x_Q \text{ when } a=1. \end{cases}$$

When $P \leqslant_{(a,b)} Q$ and $P \neq Q$ we will write $P <_{(a,b)} Q$.

Remark 4.3. For $a \in]0,1[$, $x_P \leqslant x_Q$ is equivalent to $y_P \leqslant y_Q$ and the order relation $P <_{(a,b)}$ can be read as "Q is above and on the right with respect to P". If $a=0$ then $P <_{(a,b)} Q \iff y_P < y_Q$ and if $a=1$ then $P <_{(a,b)} Q \iff x_P < x_Q$. Thus in these cases the strict order relation can be read as "Q is above P" and "Q on the right with respect to P", respectively.

Furthermore, in [21, Proposition 4.4] the authors showed that the matching distance can be realised by parameter values lying in a bounded region of $[0,1]\times\mathbb{R}$:

Proposition 4.2. *There exists* $(\overline{a},\overline{b})$ *in* $[0,1]\times[-C,C]$, *such that*

$$\begin{aligned} D_{\text{Match}}(\varphi,\psi) &= \max_{[0,1]\times[-C,C]} d_{\text{B}}\left(\text{Dgm}(\varphi^*_{(a,b)}), \text{Dgm}(\psi^*_{(a,b)})\right) \\ &= d_{\text{B}}\left(\text{Dgm}(\varphi^*_{(\overline{a},\overline{b})}), \text{Dgm}(\psi^*_{(\overline{a},\overline{b})})\right). \end{aligned}$$

The previous results allow us to work with a compact space of parameters, $[0,1]\times[-C,C]$.

4.2 The intersection operator $\overset{\infty}{\cap}$

In order to proceed, we need a notion of intersection between vertical ($a=0$) and horizontal ($a=1$) lines with the vertical and horizontal half-lines in the extended Pareto grid. Also, later in the exposition - see Chapter 6 - we will need a notion of intersection between two vertical (respectively, horizontal) parallel lines. For this purpose we define a new operator $\overset{\infty}{\cap}$ between lines and contours in $\overline{\mathbb{R}}^2$, where $\overline{\mathbb{R}}$ denotes the extended real line endowed with its usual topology.

Definition 4.2 (Intersection operator)**.** Let $\varphi\colon M \to \mathbb{R}^2$ be a smooth function satisfying properties (i)-(iv) in Section 2.3 and let $\Gamma(\varphi)$ be the extended Pareto grid of φ.

For every $(a, b) \in [0, 1] \times [-C, C]$ and every contour $\alpha \subset \Gamma(\varphi)$ put

$$r_{(a,b)} \overset{\infty}{\cap} \alpha = \begin{cases} r_{(a,b)} \cap \alpha & \text{if } a \neq 0, 1 \\ \{P = \lim_{n\to\infty} P_n \mid P_n \in r_{(a_n,b_n)} \cap \alpha\} & \text{if } a \in \{0, 1\}. \end{cases}$$

where $((a_n, b_n))_{n\in\mathbb{N}}$ is a sequence, whenever it exists, such that $\lim_{n\to\infty} b_n = b$ and
$\lim_{n\to\infty} a_n = a$.

Remark 4.4. With a small abuse of notation we will write, for any $(a, b) \in [0, 1] \times [-C, C]$ and for any filtering function φ

$$r_{(a,b)} \overset{\infty}{\cap} \Gamma(\varphi) = \bigcup_{\alpha\in\mathrm{Ctr}(\varphi)} r_{(a,b)} \overset{\infty}{\cap} \alpha.$$

Example 4.1. Notice that the operator $\overset{\infty}{\cap}$ can differ from the regular intersection, and therefore $\overset{\infty}{\cap}$ is not an extension of the intersection.

Fix some $b \in [-C, C]$. Now consider there exists a horizontal improper contour h_{x_0,y_0} in $\Gamma(\varphi)$, with $x_0 > b, y_0 > -b$. There exists a strictly positive real value $0 < A < 1$ such that for every $a \in [A, 1[$ the filtering line $r_{(a,b)}$ intersects the contour. In particular, A is uniquely determined by the condition $(x_0, y_0) \in r_{(A,b)}$:

$$\begin{aligned} r_{(A,b)} \ni (x_0, y_0) &\iff \begin{cases} x_0 = At + b \\ y_0 = (1 - A)t - b \end{cases} \\ &\iff \text{There exists } t = \frac{x_0 - b}{A} = \frac{y_0 + b}{1 - A} \\ &\iff A = \frac{x_0 - b}{x_0 + y_0}. \end{aligned}$$

Note that $x_0 + y_0 > 0$ because of $x_0 > b, y_0 > -b$, and it is trivial that any line with smaller slope than $r_{(A,b)}$ intersects the horizontal half-line with basepoint (x_0, y_0).

Therefore the sequence $\left(\left(\frac{n-1+A}{n}, b\right)\right)_{n\in\mathbb{N}}$ converges to $(1, b)$ and yields the sequence $(P_n)_{n\in\mathbb{N}}$ of intersection points given by

$$\begin{aligned}
\{P_n\} &= r_{\left(\frac{n-1+A}{n},b\right)} \cap h_{x_0,y_0} \\
\text{with ordinate } y_0 &= \left(1-\frac{n-1+A}{n}\right)t-b = \frac{1-A}{n}t-b \\
&\Longrightarrow t = n\frac{b+y_0}{1-A}. \\
\text{Therefore, its abscissa is } x_n &= \left(\frac{n-1+A}{n}\right)t+b \\
&= (n-1+A)\frac{b+y_0}{1-A}+b = \frac{nb+(n-1+A)y_0}{1-A}.
\end{aligned}$$

Since $\lim_{n\to\infty} x_n = \infty$, we conclude

$$r_{(1,b)} \overset{\infty}{\cap} h_{x_0,y_0} = \{(+\infty, y_0)\}.$$

More in general, there is the following characterisation of the operator $\overset{\infty}{\cap}$:

Proposition 4.3. *Let $h_{x_h,y_h} = \{x \geqslant x_h,\ y = y_h\}$, $v_{x_v,y_v} = \{x = x_v,\ y \geqslant y_v\} \subset \Gamma(\varphi)$ denote the improper horizontal and vertical contours with base-points $(x_h, y_h), (x_v, y_v) \in \mathbb{R}^2$, respectively. Then*

1. *If both the filtering line and the contour are vertical then:*

$$r_{(0,b)} \overset{\infty}{\cap} v_{x_v,y_v} = \begin{cases} \varnothing & \text{if } x_v < b \\ \{(x_v, \infty)\} & \text{if } x_v > b \\ v_{x_v,y_v} & \text{if } x_v = b \end{cases}$$

2. *If both the filtering line and the contour are horizontal then:*

$$r_{(1,b)} \overset{\infty}{\cap} h_{x_h,y_h} = \begin{cases} \varnothing & \text{if } y_h < -b \\ \{(\infty, y_h)\} & \text{if } y_h > -b \\ h_{x_h,y_h} & \text{if } y_h = -b \end{cases}$$

In any other case, the operator $\overset{\infty}{\cap}$ is just the regular intersection in $\mathbb{R}^2$.

Remark 4.5. Recall from section 2.3 that we are considering the closure of $\Gamma(\varphi)$ in $\overline{\mathbb{R}}^2$, *Proof.* Case 1. We distinguish the following subcases:

- If $x_v < b$, then
$$r_{(0,b)} \cap v_{x_v,y_v} = \varnothing$$
and then there is a right neighbourhood $[0, \varepsilon[, \varepsilon > 0$ of 0 such that
$$r_{(a',b)} \cap v_{x_v,y_v} = \varnothing \quad \forall a' \in [0, \varepsilon[.$$
We will show the existence of this neighbourhood explicitly. Indeed, for any $a' \in]0, \varepsilon[$ the parametrisation of $r_{(a',b)}$ allows us to write a rational (in particular is smooth on $]0, \varepsilon[$) expression for the ordinate $y(a')$ of the point in $r_{(a',b)}$ having abscissa x_v:
$$x_v = a't + b \Longrightarrow t = \frac{x_v - b}{a'}$$
$$y(a') = (1 - a')t - b = (1 - a')\frac{x_v - b}{a'} - b = \frac{x_v - b}{a'} - x_v.$$
Notice that $\frac{x_v - b}{a'}$ is negative for every $a' \in (0, \varepsilon)$, and consequently the sequence $y\left(\frac{1}{n}\right)$ tends to $-\infty$. So there exists some $\hat{a} > 0$ such that $y(\hat{a}) < y_v$ and thus $(x_v, y(\hat{a})) \notin v_{x_v,y_v}$. In other words:
$$r_{(a',b)} \cap v_{x_v,y_v} = \varnothing \quad \forall a' < \hat{a}.$$
Hence there are no sequences
$$((a_n, b_n))_{n \in \mathbb{N}}, \quad \text{with } r_{(a_n,b_n)} \cap v_{x_v,y_v} \neq \varnothing, \ a_n > 0 \ \forall n \in \mathbb{N}$$
such that $\lim_{n\to\infty} a_n = 0$ and therefore by definition $r_{(0,b)} \overset{\infty}{\cap} v_{x_v,y_v} = \varnothing$.

- If $x_v > b$, then
$$r_{(0,b)} \cap v_{x_v,y_v} = \varnothing$$
but there exist $\varepsilon > 0$ such that
$$r_{(a',b)} \cap v_{x_v,y_v} \neq \varnothing \quad \forall a' \in]0, \varepsilon[.$$
Indeed, reasoning in a similar fashion as before, when we compute the abscissa $y(a)$ of a point in $r_{(a',b)}$ having abscissa x_v we obtain

$$y(a') = \frac{x_v - b}{a'} - x_v$$

but this time the quantity $\frac{x_v - b}{a'}$ is positive for any $a' \in]0, \varepsilon[$, hence the sequence $y\left(\frac{1}{n}\right)$ tends to ∞. This amounts to say that there exists an $\hat{a} > 0$ such that $y(\hat{a}) > y_v$ and thus $(x_v, y(a')) \in v_{x_v,y_v}$ for every $a' \in]0, \hat{a}[$.

Equivalently, there exist sequences

$$((a_n, b_n))_{n\in\mathbb{N}}, \quad (P_n)_{n\in\mathbb{N}} \quad \text{with } \{P_n\} = r_{(a_n,b_n)} \cap v_{x_v,y_v} = \{(x_v, y(a_n))\}$$

with $a_n > 0$ for each $n \in \mathbb{N}$ such that $\lim_{n\to\infty} a_n = 0$. It follows

$$\lim_{n\in\mathbb{N}} P_n = \lim_{n\in\mathbb{N}} \left(x_v, \frac{x_v - b}{a_n} - x_v \right) = (x_v, \infty) \in r_{(0,b)} \overset{\infty}{\cap} v_{x_v,y_v}.$$

Let us check that (x_v, ∞) is the only point in $r_{(0,b)} \overset{\infty}{\cap} v_{x_v,y_v}$. By contradiction suppose there was another point $Q \neq (x_v, \infty)$ in $r_{(0,b)} \overset{\infty}{\cap} v_{x_v,y_v}$. Since $Q \in v_{x_v,y_v} \backslash \{(x_v, \infty)\}$,

$$Q = (x_v, y_Q) \text{ for some } y_Q \in [y_v, \infty[.$$

Then there are sequences

$$((a'_n, b'_n))_{n\in\mathbb{N}}, \quad (Q_n)_{n\in\mathbb{N}} \quad \text{with } \{Q_n\} = r_{(a'_n,b'_n)} \cap v_{x_v,y_v} = \{(x_v, y(a'_n))\}$$

such that $\lim_{(a'_n,b'_n)} = (0, b)$ and $\lim_{y(a'_n)} = y_Q < \infty$. But

$$\lim_{n\to\infty} y(\frac{1}{n} = \lim_{n\to\infty} n(x_v - b) - x_v = \infty.$$

Hence $r_{(0,b)} \overset{\infty}{\cap} v_{x_v,y_v} = \{(x_v, \infty\}$.

- Lastly, if $x_v = b$, we will show that any point (b, y) with $y \geqslant y_v$ belongs to $r_{(0,b)} \overset{\infty}{\cap} v_{x_v,y_v}$.

  In general, the sheaf of filtering lines passing through a fixed point $(x_0, y_0) \in \mathbb{R}^2$ corresponds to a segment in the space of parameters $[0, 1] \times [-C, C]$. The equation of this segment in a, b can be deduced from the cartesian equation in 2.11. Indeed, (x_0, y_0) belongs to $r_{(a,b)}$ if there exists $t \in \mathbb{R}$ such that

$$\begin{cases} x_0 = at + b \\ y_0 = (1-a)t - b. \end{cases}$$

Putting both equations together gives a grade 1 polynomial on a, b which describes a segment in $[0,1] \times [-C, C]$:

$$t = \frac{x_0 - b}{a} = \frac{y_0 + b}{1-a},$$
$$a(x_0 + y_0) - x_0 + b = 0.$$

Therefore, every rotation around a point $(x_v, y) \in v_{x_v,y_v}$ is contained in the segment

$$a(x_v + y) - x_v + b = 0$$

Applying Definition 4.2, (x_v, y) belongs to $r_{(0,b)} \overset{\infty}{\cap} v_{x_v,y_v}$ if and only if there exist suitable sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ converging to 0 and b, respectively. We can choose such sequences with

$$P_n = (x_v, y) \in r_{(a_n,b_n} \cap v_{x_v,y_v}$$

constant for each $n \in \mathbb{N}$. It is enough to put $a_n = \frac{1}{n}$ and, using the equation of the segment:

$$b_n = x_0 - \frac{x_0 + y_0}{n}.$$

Then $\lim_{n\to\infty}(a_n, b_n) = (0, b)$ and for each $n \in \mathbb{N}$ the line $r_{(a_n,b_n)}$ passes through the point $(x_v, y) \in v_{x_v,y_v}$. Hence $r_{(a_n,b_n)} \cap v_{x_v,y_v} = \{(x_v, y)\}$ for each $n \in \mathbb{N}$. Similar sequences can be constructed for each $y \geqslant y_v$. This proves $r_{(0,b)} \overset{\infty}{\cap} v_{x_v,y_v} = v_{x_v,y_v}$.

Case 2 is completely analogous to case 1 swapping abscissa with ordinates and $a = 0$ with $a = 1$. This concludes the proof.

□

Now we make an exhaustive example exploring all possible cases in 4.3.

Example 4.2. Consider the improper contour $v_{(2,2)} = \{x = 2, y \geqslant 2\}$ and the following filtering lines:

$r_{(0,0)}, r_{\left(\frac{1}{5},0\right)}, r_{\left(\frac{1}{3},0\right)}$ centered in the point$(0,0)$,

$r_{(0,2)}, r_{\left(\frac{1}{9},2\right)}, r_{\left(\frac{1}{5},2\right)}$ centered in the point$(2,-2)$.

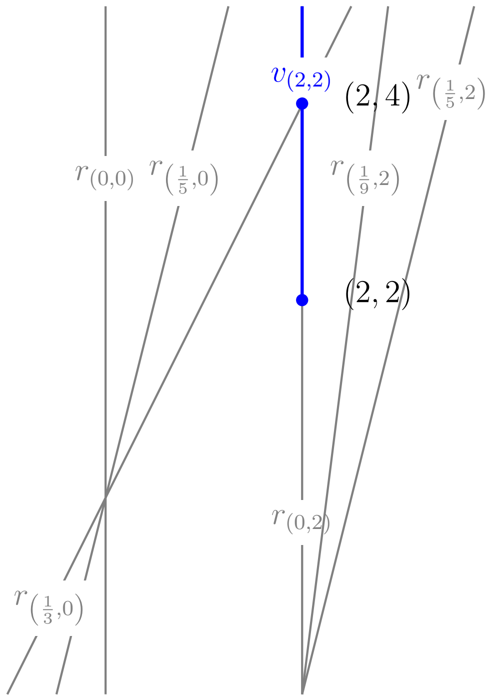

FIGURE 4.1: A depiction of the intersections between these filtering lines and the improper vertical contour $v_{(2,2)}$

We will show that $r_{(0,0)} \overset{\infty}{\cap} v_{(2,2)} = \{(2,\infty)\}$, but instead $r_{(0,2)} \overset{\infty}{\cap} v_{(2,2)} = v_{(2,2)}$. The first three lines are parametrised by

$$(at,(1-a)t) \quad \text{with } a \in \{0, \frac{1}{3}, \frac{1}{5}\}$$

and the latter by

$$(at+2,(1-a)t-2) \quad \text{with } a \in \{0, \frac{1}{9}, \frac{1}{5}\}.$$

Substituting these parametrisations in the inequalities describing $v_{(2,2)}$ it is trivial to compute the regular intersections in $\mathbb{R}^2$. We exhibit this computation for the first two intersections:

- $$(x,y) \in r_{(0,0)} \cap v_{(2,2)} \iff \begin{cases} x = at = 0 \text{ and } x = 2 \\ y = (1-a)t = t \text{ and } y \geqslant 2 \end{cases}$$

  The first line $0 = x = 2$ is a contradiction, so the point (x,y) does not exist and the intersection is empty.

- $$(x,y) \in r_{\left(\frac{1}{5},0\right)} \cap v_{(2,2)} \iff \begin{cases} x = at = \frac{t}{5} \text{ and } x = 2 \\ y = (1-a)t = \frac{4t}{5} \text{ and } y \geqslant 2 \end{cases}$$

  Substituting $t = 5x = 10$ gives $(x,y) = (2,8)$.

The complete list of intersections is the following:

$$\begin{aligned} r_{(0,0)} \cap v_{(2,2)} &= \varnothing \\ r_{\left(\frac{1}{5},0\right)} \cap v_{(2,2)} &= \{(2,8)\}, \\ r_{\left(\frac{1}{3},0\right)} \cap v_{(2,2)} &= \{(2,4)\}, \\ r_{(0,2)} \cap v_{(2,2)} &= v_{(2,2)}, \\ r_{\left(\frac{1}{9},2\right)} \cap v_{(2,2)} &= \varnothing, \\ r_{\left(\frac{1}{5},2\right)} \cap v_{(2,2)} &= \varnothing. \end{aligned}$$

In general, observe that for $a \neq 0$ any filtering line $r_{(a,0)}$ centered in $(0,0)$ has slope equal to $\frac{1-a}{a}$, and the intersection with $v_{(2,2)}$ will be the point $(2, 2\frac{1-a}{a})$ when $a \leqslant \frac{1}{2}$ (in particular for $r_{\frac{1}{2},0} \cap v_{(2,2)}$ is the basepoint $(2,2)$) and the empty set otherwise. Instead for $a = 0$ the filtering line is parallel to $v_{(2,2)}$ and they do not intersect in $\mathbb{R}^2$. All of this implies that

$$\{a \in]0,1[\text{ such that } r_{(a,b)} \cap v_{(2,2)} \neq \varnothing\} = \left(0, \frac{1}{2}\right].$$

Conversely, for $a \neq 0$ no filtering line $r_{(a,2)}$ centered in $(2,-2)$ meets $v_{(2,2)}$, since the only point with abscissa 2 in $r_{(a,2)}$ is the common point $(2,-2)$, which is below the improper contour. Therefore,

$$\{a \in]0,1[\text{ such that } r_{(a,b)} \cap v_{(2,2)} \neq \varnothing\} = \varnothing.$$

However, $v_{(2,2)}$ is contained on the vertical line $(x = 2) = r_{(0,2)}$, so the regular intersection in this case is the whole contour (unlike the one given by our operator).

For $a \neq 0$, $r_{(a,b)} \overset{\infty}{\cap} v_{(2,2)} = r_{(a,b)} \cap v_{(2,2)}$. As for $a = 0$, since we just remarked that $r_{(a,2)} \cap v_{(2,2)} = \varnothing$ for every $a \neq 0$ and for the lines $r_{(a,0)}$ the intersection point is given by the function

$$\left(0, \frac{1}{2}\right] \ni a \mapsto r_{(a,0)} \cap v_{(2,2)} = \begin{cases} \{\, (2, 2\frac{1-a}{a}) \,\} & \text{when } a \leqslant \frac{1}{2} \\ \varnothing & \text{otherwise.} \end{cases}$$

Let us check the definition and the characterisation agree for $r_{(0,0)}$ and $r_{(0,2)}$. In the first case the characterisation gives us

$$r_{(0,0)} \overset{\infty}{\cap} v_{(2,2)} = \{(2,\infty)\}$$

since $b = 0 < 2$. Indeed, consider the sequence

$$\left(\left(\frac{1}{n}, 0\right)\right)_{n\in\mathbb{N}} \quad \text{with } \{P_n\} = r_{\left(\frac{1}{n},0\right)} \overset{\infty}{\cap} v_{(2,2)},$$
$$\text{with } \left(2, 2\frac{1-\frac{1}{n}}{\frac{1}{n}}\right) = (2, 2(n-1)).$$

The sequence satisfies the Definition 4.2 and therefore $r_{(0,0)} \overset{\infty}{\cap} v_{(2,2)} = \lim_{n\to\infty} P_n = (2,\infty)$.

Instead, in the second case, the characterisation yields

$$r_{(0,2)} \overset{\infty}{\cap} v_{(2,2)} = v_{(2,2)}$$

which again agrees with the definition, since for each point $(2, y) \in v_{(2,2)}$, $y \geqslant 2$, we can find suitable sequences $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$ that satisfy the conditions on 4.2.

The sheaf of lines passing through $(2, y)$ corresponds to the segment of equation

$$(2 + y)a + b = 2.$$

Then, for each $n \in \mathbb{N}$ we can choose

$$a_n = \frac{1}{n}, \quad b_n = 2 - \frac{2+y}{n}.$$

That way, $\lim_{n \to \infty}(a_n, b_n) = (0, b)$ and each line $r_{(a_n,b_n)}$ meets $v_{(2,2)}$ at the point $(2, y)$, for every $n \in \mathbb{N}$.

Hence $(2, y) \in r_{(0,2)} \overset{\infty}{\cap} v_{(2,2)}$ for every $y \geqslant 2$. This concludes the example.

4.3 An extension of the Position Theorem

Now, we will give an extension of the Position Theorem proved in [13, Theorem 2] for horizontal and vertical filtering lines.

Theorem 6 (Extended Position Theorem)**.** *Let $\varphi, \psi \colon M \to \mathbb{R}^2$ be two continuous functions satisfying properties (i)-(iv) from Section 2.3. Let $(a, b) \in [0, 1] \times \mathbb{R}$ and $p \in \mathrm{Dgm}(\varphi^*_{(a,b)}) \backslash \{\Delta\}$. Then: For each $a \in [0, 1]$ and for each finite coordinate w of p, a point $P = (x, y) \in r_{(a,b)} \overset{\infty}{\cap} \Gamma(\varphi)$ exists such that*

$$w = \begin{cases} \min\{1, \frac{1-a}{a}\}(x - b) & \text{if } a \in [0, 1), \\ \min\{1, \frac{a}{1-a}\}(y + b) & \text{if } a \in (0, 1]. \end{cases}$$

with the conventions $\frac{1}{0} = \infty$, $\min\{1, \infty\} = 1$.

Proof. For $a \in]0, 1[$, the proof is the original Position Theorem as it appears in [13]. Let $a = 0, b \in \mathbb{R}$ and let $p \in \mathrm{Dgm}(\varphi^*_{(0,b)})$. Let w be a finite coordinate of p.

Remark 4.2 guarantees that for each sequence $(a_n, b_n)_{n\in\mathbb{N}}$ converging to $(0, b)$ in $[0,1]\times[-C, C]$, possibly extracting a subsequence $(a_{n_k}, b_{n_k})_{k\in\mathbb{N}}$, there exists a sequence of cornerpoints

$$p_k \in \mathrm{Dgm}(\varphi^*_{(a_{n_k}, b_{n_k})}), \text{ with } d(p_k, p) < \frac{1}{k}$$

with $\lim_{k\to\infty} p_k = p$.

Fixing $k \in \mathbb{N}$ such that $a_{n_k} < \frac{1}{2}$, let w_k be the finite coordinate of $p_k \in \mathrm{Dgm}(\varphi^*_{(a_{n_k}, b_{n_k})})$ corresponding to the coordinate w of $p \in \mathrm{Dgm}(\varphi^*_{(0,b)})$. Apply the classical Position Theorem to w_k. There exist a point P_k and a contour α_k such that $P_k \in \alpha_k \subset \Gamma(\varphi)$ and

$$w_k = x_{P_k} - b_{n_k} = \frac{a_{n_k}}{1 - a_{n_k}}(y_{P_k} + b_{n_k})$$

Recall from Section 2.3 that we are considering the completion of the extended Pareto grid $\Gamma(\varphi)$ in $\overline{\mathbb{R}}^2$ and each contour is homeomorphic to a closed interval. By compactness of the contour α_k, the sequence $(P_k)_{k\in\mathbb{N}}$ has a converging subsequence, that is there exists $P \in \alpha_k$ with

$$P = \lim_{l\to\infty} P_{k_l}.$$

By continuity of $(a, b) \mapsto$, taking the limit as l approaches ∞ on the first equality yields

$$w = \lim_{l\to\infty} x_{P_{k_l}} - b_{n_{k_l}} = x_P - b$$

as we wanted to show.

The proof for the case $a = 1$ is analogous by considering sequences $(a'_n, b'_n)_{n\in\mathbb{N}}$ converging to $(1, b)$ and $q_k \in \mathrm{Dgm}(\varphi^*_{(a'_{n_k}, b'_{n_k})})$ for opportune subsequences, with $d(q_k, p) < \frac{1}{n}$ and $\lim_{k\to\infty} q_k = p$. Then when applying Position Theorem to the finite coordinates w'_k of q_k it is enough to take the limit as k approaches to ∞ on the second equality.

□

Remark 4.6. In the following chapters, whenever we apply the Position Theorem to the coordinates w_p, w_q of points p, q realising the bottleneck distance $d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(a,b)}), \mathrm{Dgm}(\psi^*_{(a,b)}))$, we will say that the points $P, Q \in \Gamma(\varphi) \cup \Gamma(\psi)$ satisfying

$$w_p = \min\{1, \frac{1-a}{a}\}(x_P - b) = \min\{1, \frac{a}{1-a}\}(y_P + b)$$

and
$$w_q = \min\{1, \frac{1-a}{a}\}(x_Q - b) = \min\{1, \frac{a}{1-a}\}(y_Q + b)$$
realise the bottleneck distance between $\mathrm{Dgm}(\varphi^*_{(a,b)})$ and $\mathrm{Dgm}(\psi^*_{(a,b)})$.

Chapter 5

Special values

Fix (φ, ψ) a pair of filtering functions and let $\Gamma(\varphi), \Gamma(\psi)$ be their respective extended Pareto grids. There are pairs $(a, b) \in]0, 1[\times[-C, C]$ in which the optimal matching between $\mathrm{Dgm}(\varphi^*_{(a,b)})$ and $\mathrm{Dgm}(\psi^*_{(a,b)})$ may change abruptly. In this chapter we introduce the special set associated with (φ, ψ), which contains such pairs, and prove some new results relating this set with the bottleneck distance.

5.1 The special set

Definition 5.1. Let $\mathrm{Ctr}(\varphi, \psi) = \mathrm{Ctr}(\varphi) \cup \mathrm{Ctr}(\psi)$ be the set of all curves that are contours of φ or ψ. From our assumptions in Section 2.3 on the functions φ, ψ, it follows that $\mathrm{Ctr}(\varphi, \psi)$ is a finite set. The *special set* of (φ, ψ), denoted by $\mathrm{Sp}(\varphi, \psi)$, is the collection of all (a, b) at $]0, 1[\times[-C, C]$ for which two distinct pairs $\{\alpha_P, \alpha_Q\}$, $\{\alpha_S, \alpha_T\}$ of contours in $\mathrm{Ctr}(\varphi, \psi)$ intersecting $r_{(a,b)}$ exist, such that $\{\alpha_P, \alpha_Q\} \neq \{\alpha_R, \alpha_S\}$ and

- $c_1|x_P - x_Q| = c_2|x_R - x_S|$, with $c_1, c_2 \in \{1, 2\}$, if $a \leqslant \frac{1}{2}$,
- $c_1|y_P - y_Q| = c_2|y_R - y_S|$, with $c_1, c_2 \in \{1, 2\}$, if $a \geqslant \frac{1}{2}$,

where $P = P_{(a,b)} = r_{(a,b)} \cap \alpha_P$, $Q = Q_{(a,b)} = r_{(a,b)} \cap \alpha_Q$, $R = R_{(a,b)} = r_{(a,b)} \cap \alpha_R$ and $S = S_{(a,b)} = r_{(a,b)} \cap \alpha_S$, and x_*, y_* denote abscissas and ordinates of these points, respectively. We will say that two different pairs of contours $\{\alpha_P, \alpha_Q\}, \{\alpha_R, \alpha_S\}$ as above *satisfy the condition of speciality.* An element of the special set $\mathrm{Sp}(\varphi, \psi)$ is called a *special value* of the pair (φ, ψ).

Remark 5.1. Note that in the above definition, the two contours of the two distinct pairs $\{\alpha_P, \alpha_Q\}, \{\alpha_R, \alpha_S\}$ do not necessarily need to differ. For example, the definition allows $\alpha_P = \alpha_R$, $\alpha_P = \alpha_S$, $\alpha_Q = \alpha_R$, $\alpha_Q = \alpha_S$, but not two of these conditions simultaneously. Moreover, the contours may all belong to the same extended Pareto grid, as long as the pairs $\{\alpha_P, \alpha_Q\}, \{\alpha_R, \alpha_S\}$ are different.

Notice that, in general, the special set is not finite. The next proposition shows that for any point belonging to two different contours, there are segments in $]0,1[\times[-C,C]$ entirely contained in the special set.

Proposition 5.1. *Let $r_{(a,b)}$, $a \in]0,1[$ be a filtering line intersecting $\Gamma(\varphi) \cup \Gamma(\psi)$ in at least two different points. If there exists X in $r_{(a,b)} \cap \alpha_X \cap \beta_X$, where $\alpha_X \neq \beta_X$ are in $\mathrm{Ctr}(\varphi,\psi)$, then (a,b) is a special value.*

Proof. Let $Y \in \alpha_Y$ be a point in $r_{(a,b)} \cap (\Gamma(\varphi) \cup \Gamma(\psi))$ different from X. Then the pairs of contours $\{\alpha_X, \alpha_Y\}, \{\beta_X, \alpha_Y\}$ are different and satisfy the condition of speciality for (a,b). $\square$

There is a relation between the condition of speciality and the pairs of cornerpoints realising the cost of an optimal matching between the corresponding persistence diagrams. This relation is a consequence of the Position Theorem.

Proposition 5.2. *Let $\sigma\colon \mathrm{Dgm}(\varphi^*_{(a,b)}) \to \mathrm{Dgm}(\psi^*_{(a,b)})$ be an optimal matching, $\sigma \neq id_{\mathrm{Dgm}(\varphi^*_{(a,b)})}$. If the cost of σ is realised by two different pairs of points $\{p_1, q_1\} \neq \{p_2, q_2\}$ then (a,b) is a special value.*

Proof. Let us first examine the case $a \leqslant \frac{1}{2}$. Let w_i, z_i be the coordinates of p_i, q_i, respectively, realising the distance $d(p_i, q_i)$, for $i = 1, 2$. With this notation the hypothesis becomes

$$\mathrm{cost}\sigma = d(p_1, q_1) = c_1|w_1 - z_1| = d(p_2, q_2) = c_2|w_2 - z_2| > 0,$$

for some $c_1, c_2 \in \{1, 2\}$.

Note that cost $\sigma > 0$ because σ is not the identity. Apply Position Theorem. There are points $P, Q, R, S \in r_{(a,b)} \cap (\Gamma(\varphi) \cup \Gamma(\psi))$ such that

$$w_1 = x_P - b, \quad z_1 = x_Q - b, \quad w_2 = x_R - b, \quad z_2 = x_S - b,$$
$$|x_P - x_Q| = |x_R - x_S| > 0.$$

From the previous equalities we get $x_P \neq x_Q$ and $x_R \neq x_S$. In particular $P \neq Q$ and $R \neq S$. This implies that there exist some contours $\alpha_P \ni P, \alpha_Q \ni Q, \alpha_R \ni R$ and $\alpha_S \ni S$ such that $\{\alpha_P, \alpha_R\} \neq \{\alpha_Q, \alpha_S\}$ and satisfy the condition of speciality.

The case $a \geqslant \frac{1}{2}$ is obtained analogously by substituting abscissas with ordinates. □

We recall Proposition 5.2 from [21] which shows an important property of the special set fundamental in the proof of our main theorem in chapter 6.

Proposition 5.3. $\mathrm{Sp}(\varphi, \psi)$ *is a closed subset in* $]0,1[\times[-C, C]$.

5.2 The ultraspecial set

Definition 5.2. The *ultraspecial set* of (φ, ψ), denoted by $\mathrm{USp}(\varphi, \psi)$, is the collection of all (a, b) in $[0,1] \times [-C, C]$ for which there are three distinct pairs $\{\alpha_P, \alpha_Q\}, \{\alpha_R, \alpha_S\}, \{\alpha_T, \alpha_U\}$ intersecting $r_{(a,b)}$ and such that every two of them satisfy the speciality condition; that is

- $c_1|x_P - x_Q| = c_2|x_R - x_S| = c_3|x_T - x_U|$, with $c_1, c_2, c_3 \in \{1, 2\}$, if $a \leqslant \frac{1}{2}$,
- $c_1|y_P - y_Q| = c_2|y_R - y_S| = c_3|y_T - y_U|$, with $c_1, c_2, c_3 \in \{1, 2\}$, if $a \geqslant \frac{1}{2}$,

where $P = P_{(a,b)} = r_{(a,b)} \cap \alpha_P$, $Q = Q_{(a,b)} = r_{(a,b)} \cap \alpha_Q$, $R = R_{(a,b)} = r_{(a,b)} \cap \alpha_R$, $S = S_{(a,b)} = r_{(a,b)} \cap \alpha_S$, $T = T_{(a,b)} = r_{(a,b)} \cap \alpha_T$, and $U = U_{(a,b)} = r_{(a,b)} \cap \alpha_U$ and x_*, y_* denote abscissas and ordinates of these points, respectively. An element of the ultraspecial set $\mathrm{USp}(\varphi, \psi)$ is called an *ultraspecial value* of the pair (φ, ψ).

By definition, $\mathrm{USp}(\varphi, \psi) \subset \mathrm{Sp}(\varphi, \psi)$.

Proposition 5.4. *Consider* X *and* Y *such that* $X \neq Y$*. Let* $\alpha_X, \beta_X, \alpha_Y, \beta_Y \in \mathrm{Ctr}(\varphi, \psi)$*, with* $\alpha_X \neq \beta_X$ *and* $\alpha_Y \neq \beta_Y$*. If* X *is in* $r_{a,b} \cap \alpha_X \cap \beta_X$ *and* Y *is in* $r_{a,b} \cap \alpha_Y \cap \beta_Y$*, then* (a, b) *is an ultraspecial value.*

Proof. The claim follows from observing that any three of the four different pairs

$\{\alpha_X, \alpha_Y\}, \{\alpha_X, \beta_Y\}, \{\beta_X, \alpha_Y\}, \{\beta_X, \beta_Y\}$ satisfy the condition of speciality between them. □

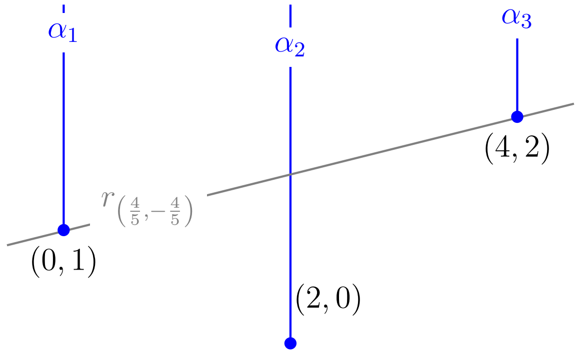

FIGURE 5.1: The contours α_1, α_2 and α_3 with basepoints $(0,1),(2,0)$ and $(4,2)$ from Example 5.1; with the filtering line $r_{\left(\frac{4}{5},-\frac{4}{5}\right)}$, which is the only admissible one passing through 2 basepoints and intersects all three contours.

We now give some examples where the ultraspecial set is not finite.

Example 5.1. [An example where $\mathrm{USp}(\varphi,\psi)$ are not a measure zero subset.]

Let $\alpha_1, \alpha_2, \alpha_3$ be three different parallel improper contours in $\mathrm{Ctr}(\varphi,\psi)$ such that $\mathrm{dist}(\alpha_1,\alpha_2) = \mathrm{dist}(\alpha_2,\alpha_3)$, where dist denotes the infimum of the distance between any two points of the contours. Then any (a,b) such that the filtering line $r_{(a,b)}$ meets all three contours is ultraspecial. Indeed, if $P_i = r_{(a,b)} \cap \alpha_i$, $i = 1,2,3$, then

$$|x_{P_1} - x_{P_3}| = 2|x_{P_1} - x_{P_2}| = 2|x_{P_2} - x_{P_3}|,$$
$$|y_{P_1} - y_{P_3}| = 2|y_{P_1} - y_{P_2}| = 2|y_{P_2} - y_{P_3}|,$$

so the condition of speciality is fulfilled for $\{\alpha_1,\alpha_2\},\{\alpha_2,\alpha_3\}$ and $\{\alpha_3,\alpha_1\}$ both when $a \leqslant \frac{1}{2}$ and $a \geqslant \frac{1}{2}$.

In particular, according to Defintion 2.11, when $a \in]0,1[$ the filtering line is described by the cartesian equation

$$y = \frac{1-a}{a}x - \frac{b}{a}$$

which means that if the improper contours $\alpha_1,\alpha_2,\alpha_3$ have basepoints $(x_1,y_1),(x_2,y_2)$ and (x_3,y_3) then $r_{(a,b)}$ intersects all three contours when

$$b \leqslant (1-a)x_i - ay_i = -(x_i+y_i)a + x_i \quad \text{for } i = 1,2,3$$

which, when non-empty, is a subset of $]0,1[\times[-C,C]$ limited by three lines with different slope $-(x_i+y_i)$.

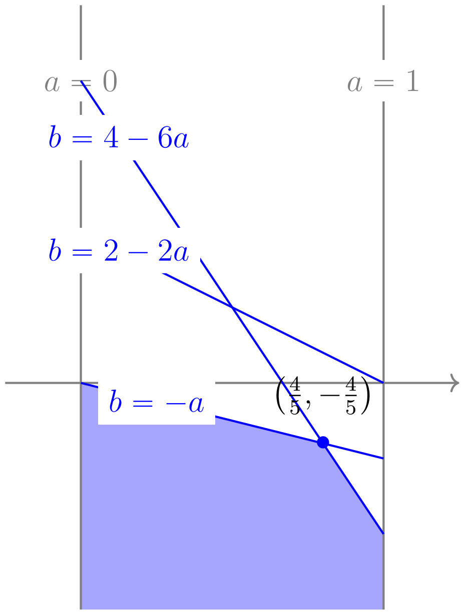

FIGURE 5.2: In example 5.1, with the choice of contours from figure 5.1, the pairs (a, b) in the blue region of the rectangle $]0, 1[\times[-C, C]$ are ultraspecial.

Example 5.2. Let α_P, α_Q be two different parallel improper contours in $\mathrm{Ctr}(\varphi, \psi)$ and let X be a point belonging to two different contours α_X, β_X such that $\mathrm{dist}(X, \alpha_P) = \mathrm{dist}(X, \alpha_Q)$. In this case, any line $r_{(a,b)}$ passing through X and meeting α_P and α_Q corresponds to an ultraspecial value, since the same way as before if $P = r_{(a,b)} \cap \alpha_P$ and $Q = r_{(a,b)} \cap \alpha_Q$ then

$$\begin{aligned}|x_P - x_Q| &= 2|x_P - x_X| = 2|x_X - x_Q|,\\ |y_P - y_Q| &= 2|y_P - y_X| = 2|y_X - y_Q|,\end{aligned}$$

so any three of the pairs $\{\alpha_P, \alpha_Q\}, \{\alpha_P, \alpha_X\}, \{\alpha_Q, \alpha_X\}, \{\alpha_P, \beta_X\}$ and $\{\alpha_Q, \beta_X\}$ satisfy the condition of speciality.

In particular, if the improper contours α_P, α_Q have respective basepoints $(x_0, y_0), (x_1, y_1)$; then the filtering lines passing through X and above those basepoints:

$$b = -(x_X + y_X)a + x_X; \quad b \geqslant \max\{-(x_P + y_P)a + x_P, -(x_Q + y_Q)a + x_Q\}$$

are either the empty set or a closed segment in the parameter space.

Chapter 6

Results

In this chapter we prove our main theorem. Before doing so, we prove two lemmas, Lemma 6.2 and Lemma 6.3 which will allow us to find, for any fixed smooth functions φ, ψ satisfying the assumptions in Section 2.3 and for any $(a, b) \in]0, 1[\times[-C, C]$, a pair (a', b') such that the bottleneck distance between the corresponding diagrams increases. We will be able to do this outside a set of pairs (a, b) that we conjecture to be finite. The proof of Lemma 6.3 will require the study of the gradients of two auxiliar functions f, g. For the sake of clarity, some of the calculations for this matter will be left to the Appendix C.

6.1 Statement of our main theorem

Definition 6.1. Given a contour α, consider the set

$$U_\alpha = \left\{(a, b) \in \left]0, \frac{1}{2}\right] \times [-C, C] \mid \alpha \cap r_{(a,b)} \neq \varnothing\right\}.$$

For any four contours $\alpha_P, \alpha_Q, \alpha_R, \alpha_S \in \mathrm{Ctr}(\varphi, \psi)$ denote

$$\mathcal{Q}_{P,Q,R,S} = \bigcap_{X \in \{P,Q,R,S\}} U_{\alpha_X} \subset \left]0, \frac{1}{2}\right] \times [-C, C]$$

and

$$\begin{aligned} P(a,b) &= (x_P(a,b), y_P(a,b)) = r_{(a,b)} \cap \alpha_P, \\ Q(a,b) &= (x_Q(a,b), y_Q(a,b)) = r_{(a,b)} \cap \alpha_Q, \\ R(a,b) &= (x_R(a,b), y_R(a,b)) = r_{(a,b)} \cap \alpha_R, \\ S(a,b) &= (x_S(a,b), y_S(a,b)) = r_{(a,b)} \cap \alpha_S \end{aligned}$$

the intersection points when varying (a,b).

In this setting, define the functions

$$\begin{aligned} f_{P,Q,\alpha_P,\alpha_Q} = f \colon \mathcal{Q}_{P,Q,R,S} &\to [0,\infty[\\ (a,b) &\mapsto (x_P(a,b) - x_Q(a,b))^2 \\ g_{R,S,\alpha_R,\alpha_S} = g \colon \mathcal{Q}_{P,Q,R,S} &\to [0,\infty[\\ (a,b) &\mapsto (x_R(a,b) - x_S(a,b))^2 \end{aligned}$$

We will prove in 6.1 that the functions f, g defined above are differentiable in $\mathcal{Q}^{\circ}_{P,Q,R,S}$, the interior of $\mathcal{Q}_{P,Q,R,S}$. Let

$$\mathcal{U}(\varphi,\psi) = US_p(\varphi,\psi) \cup \left\{ (a,b) \in \mathrm{Sp}(\varphi,\psi) \;\middle|\; \begin{array}{l} \nabla f_{P,Q,\alpha_P,\alpha_Q}, \nabla g_{R,S,\alpha_R,\alpha_S} \\ \text{are parallel and } \alpha_P, \alpha_Q, \alpha_R, \alpha_S \\ \text{are the contours in} \\ \text{condition of speciality for } (a,b) \end{array} \right\}$$

We will assume the following hypothesis before stating our main result:

$\dagger$: $\mathcal{U}(\varphi,\psi)$ is a finite set.

$\ddagger$: $\mathrm{Sp}(\varphi,\psi)$ is a finite union of curves.

Now we state our main theorem, which we shall prove at the end of this chapter.

Theorem 7. *[Main theorem] Let $\varphi, \psi \colon M \to \mathbb{R}^2$ be smooth functions satisfying the properties (i)-(iv) in Section 2.3. Under the hypothesis in $\dagger$, $\ddagger$, $D_{\mathrm{match}}(\varphi,\psi)$ is realised either for $a \in \{0, \frac{1}{2}, 1\}$ or in the finite set of parameters $\mathcal{U}(\varphi,\psi)$.*

In other words,

$$D_{\mathrm{match}}(\varphi,\psi) = \max_{\{0,\frac{1}{2},1\}\times[-C,C]\cup\mathcal{U}(\varphi,\psi)} d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(a,b)}), \mathrm{Dgm}(\psi^*_{(a,b)}))$$

6.2 Preliminary lemmas

Before proving our main theorem, we will write explicit conditions characterizing the set $\mathcal{U}(\varphi, \psi)$ and prove auxiliary lemmas.

Firstly we will show that for $(a, b) \notin \mathrm{Sp}(\varphi, \psi)$ there exists a clockwise (when $a \leqslant \frac{1}{2}$) or counter-clockwise (when $a \geqslant \frac{1}{2}$) rotation of the filtering line for which the bottleneck distance increases.

Definition 6.2. For any $(a, b) \neq (a', b') \in]0, 1[\times[-C, C]$, the symbol $(a, b) \to (a', b')$ represents the rotation taking the line $r_{(a,b)}$ to the line $r_{(a',b')}$. We will say $(a, b) \to (a', b')$ is *clockwise* if $a < a'$, and *counter-clockwise* if $a' < a$.

Remark 6.1. Observe that any rotation or translation corresponds to a closed segment $[(a, b), (a', b')] \subset [0, 1] \times [-C, C]$.

Remark 6.2. Let $(a, b) \to (a', b')$ be a clockwise rotation around a point $A = (x_A, y_A)$. Then $x_A + y_A > 0$ if and only if $b' < b$. Indeed, the lines $r_{(a,b)}, r_{(a',b')}$ have cartesian equations

$$ay = (1 - a)x - b \quad a'y = (1 - a')x - b'.$$

Since both lines contain the point A, $b' > b$ if and only if

$$\begin{aligned}(1 - a')x_A - a'y_A > (1 - a)x_A - ay_A &\iff (a - a')x_A > (a' - a)y_A \\ &\iff x_A < -y_A.\end{aligned}$$

The total order $\leqslant_{(a,b)}$ induced on the intersection points of $r_{(a,b)} \cap (\Gamma(\varphi) \cup \Gamma(\psi))$ determines in which way the corresponding points in the persistence diagram move when rotating the filtering line.

Lemma 6.1. *Let $A, B, C \in r_{(a,b)} \cap \Gamma(\varphi)$ belonging to different contours $\alpha_A, \alpha_B, \alpha_C$ such that $C <_{(a,b)} A <_{(a,b)} B$. Let $(a, b) \to (a', b')$, $(a, b) \to (a'', b'')$ be, respectively, a clockwise and a counter-clockwise rotation fixing the point A; and make the further assumption that the lines $r_{(a',b')}, r_{(a'',b'')}$ meet α_B, α_C. Let*

$$B' = r_{(a',b')} \cap \alpha_B, \quad B'' = r_{(a'',b'')} \cap \alpha_B, \quad C' = r_{(a',b')} \cap \alpha_C, \quad C'' = r_{(a'',b'')} \cap \alpha_C.$$

Then

$$\begin{gathered}x_{B'} \geqslant x_B \geqslant x_A, \quad y_{B''} \geqslant y_B \geqslant y_A, \quad x_{C'} \leqslant x_C \leqslant x_A, \quad y_{C''} \leqslant y_C \leqslant y_A \\ B' >_{(a',b')} A, \quad B'' >_{(a'',b'')} A, \quad C' <_{(a',b')} A, \quad C'' <_{(a'',b'')} A.\end{gathered}$$

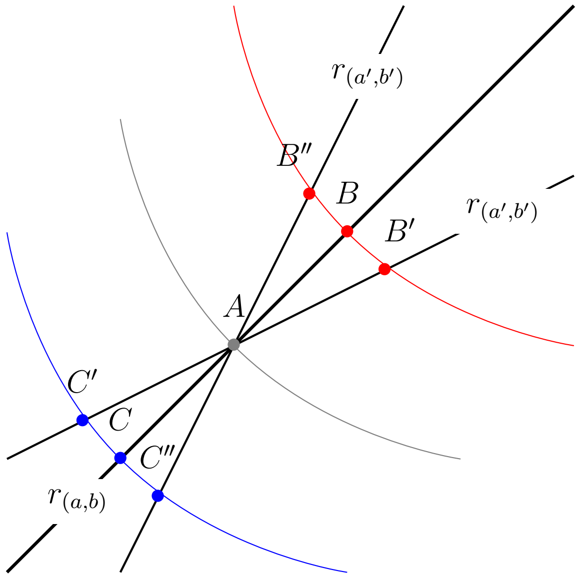

FIGURE 6.1: The points, lines and contours described in Lemma 6.1.

Proof. We will show the proof for the clockwise rotation $(a,b) \to (a',b')$ and the inequalities $x_{B'} \geqslant x_B \geqslant x_A$ and $B' >_{(a',b')} A$, since the proof for all the other cases are completely analogous. Let us prove $B' >_{(a',b')} A$ first. By 2.3 (iv), every proper contour is diffeomorphic to an interval. Hence, there exists a smooth parametrisation $\xi = (\xi_1, \xi_2)\colon [0,1] \to \mathbb{R}^2$ of α_B.

By contradiction suppose $B' \leqslant_{(a',b')} A$. This implies $x_{B'} \leqslant x_A \leqslant x_B$ and $y_{B'} \leqslant y_A \leqslant y_B$. Thus, $\xi_1(s) \leqslant \xi_1(t)$ and $\xi_2(s) \leqslant \xi(t)$ for any $s \leqslant t$, which contradicts property (iv) in Section 2.3. This proves $B' >_{(a',b')} A$.

Now we prove $x_{B'} \geqslant x_B \geqslant x_A$. By definition of $\leqslant_{(a,b)}$, $x_B \geqslant x_A$ follows. In order to prove $x_{B'} \geqslant x_B$, suppose, by contradiction, that $x_{B'} < x_B$. That would imply, because of point (iv) in Section 2.3 that $y_{B'} > y_B$. We will study two cases separately:

1. Case $x_{B'} - x_A, x_B - x_A > 0$.

   The inequality $y_B' \geqslant y_B$ implies that

$$\frac{y_{B'} - y_A}{x_{B'} - x_A} > \frac{y_B - y_A}{x_B - x_A},$$

   being the slopes of $r_{(a,b)}$, on the right, and the one of $r_{(a',b')}$ on the left.

   Note that $y_{B'} - y_A, x_{B'} - x_A$ are non-negative quantities since we just

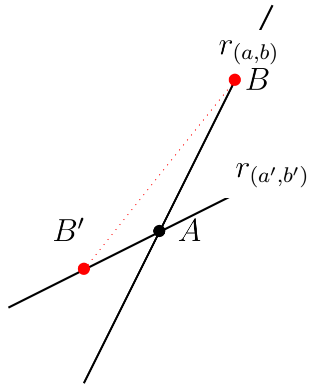

FIGURE 6.2: A graphical representation of the contradiction in the first part of the proof of Lemma 6.1. Any C^1 path from B' to B is strictly increasing in both its coordinates in some open subset. Recall from Section 2.3 that contours are either horizontal or vertical half-lines or curves with negative local slope.

proved $B' >_{(a',b')} A$. Thus, the rotation sending $r_{(a,b)}$ to $r_{(a',b')}$ is counter-clockwise, which contradicts the hypothesis

2. Case $x_{B'} - x_A = 0$ or $x_B - x_A = 0$.

   The hypothesis $\alpha_A \neq \alpha_B$ and $B >_{(a,b)} A, B' >_{(a',b')} A$ imply that one of the filtering lines $r_{(a,b)}, r_{(a',b')}$ is vertical, that is, $a = 0$ or $a' = 0$ (otherwise for every abscissa there would be a unique point in $r_{(a,b)}$ and $r_{(a','b)}$). In particular, since $(a, b) \to (a', b')$ is clockwise, we can study directly $a = 0$ since it must be $a < a'$.

   For $a = 0$, supposing $x_{B'} < x_B = x_A$ implies $B' <_{(a',b')} A$, that is, B' belongs to the part of $r_{(a',b')}$ which is to the left of $r_{(a,b)}$. But we saw before $B' >_{(a',b')} A$, which is a contradiction.

□

By Proposition 3.2, for any (a, b) there exists $\overline{p} \in \mathrm{Dgm}(\varphi^*_{(a,b)})$, $\overline{q} \in \mathrm{Dgm}(\psi^*_{(a,b)})$ such that

$$d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(a,b)}), \mathrm{Dgm}(\psi^*_{(a,b)})) = d(\overline{p}, \overline{q});$$

and by the definition of the metric d 2.10 and the Position Theorem 6, there exist A and C in $r_{(a,b)} \overset{\infty}{\cap} (\Gamma(\varphi) \cup (\Gamma(\psi))$ and $c \in \{\frac{1}{2}, 1\}$ such that

$$d(\overline{p}, \overline{q}) = c|(x_A - b) - (x_C - b)| = c|x_A - x_C|$$

when $a \leqslant \frac{1}{2}$; otherwise there exist B and D in $r_{(a,b)} \overset{\infty}{\cap} (\Gamma(\varphi) \cup (\Gamma(\psi))$ and $c' \in \{\frac{1}{2}, 1\}$ with

$$d(\overline{p}, \overline{q}) = c'|(y_B + b) - (y_D + b)| = c'|y_B - y_D|$$

Consider $(a, b) \notin \mathrm{Sp}(\varphi, \psi)$. Assume that the bottleneck distance between $\mathrm{Dgm}(\varphi^*_{(a,b)})$ and $\mathrm{Dgm}(\psi^*_{(a,b)})$ is equal to $|x_A - x_C|$. It is always possible to find a rotation $(a, b) \to (a', b')$ around either A or C corresponding to a segment in the parameter space $[0, 1] \times [-C, C]$ such that the bottleneck distance between the new diagrams $\mathrm{Dgm}(\varphi^*_{(a',b')})$ and $\mathrm{Dgm}(\psi^*_{(a','b)})$ is bigger or equal than the initial one. This is the content of the following lemma.

Lemma 6.2. *Let $\varphi, \psi \colon M \to \mathbb{R}^2$ be two smooth functions satisfying conditions (i)-(iv) on Section 2.3. Let $(a, b) \in]0, 1[\times[-C, C]$. If $(a, b) \notin \mathrm{Sp}(\varphi, \psi)$ then there exists a rotation $((a, b) \to (a', b'))$ around a point $A \in r_{(a,b)} \cap (\Gamma(\varphi) \cup \Gamma(\psi))$ corresponding to one of the coordinates realising the bottleneck distance between the corresponding persistence diagrams and for which:*

$$d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(a,b)}), \mathrm{Dgm}(\psi^*_{(a,b)})) \leqslant d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(a',b')}), \mathrm{Dgm}(\psi^*_{(a',b')})).$$

Proof. From the compactness of the persistence diagrams and Proposition 2.4, there is an optimal matching $\sigma \colon \mathrm{Dgm}(\varphi^*_{(a,b)}) \to \mathrm{Dgm}(\psi^*_{(a,b)})$ and there exists a unique - since (a, b) is not special - pair of cornerpoints $\overline{p} \in \mathrm{Dgm}(\varphi^*_{(a,b)}), \overline{q} \in \mathrm{Dgm}(\psi^*_{(a,b)})$ such that $\mathrm{cost}(\sigma) = d(\overline{p}, \overline{q})$. If $d(\overline{p}, \overline{q}) = \infty$ the result is clear, so let us assume $d(\overline{p}, \overline{q}) < \infty$. We have that $d(\overline{p}, \overline{q}) = c|w_1 - w_2|$ for some finite coordinates $w_1, w_2 \in \{x_{\overline{p}}, x_{\overline{q}}, y_{\overline{p}}, y_{\overline{q}}\}$ and for $c \in \{\frac{1}{2}, 1\}$. In particular, by the definition of d 2.10, $c = \frac{1}{2}$ if $d(\overline{p}, \overline{q}) = d(\overline{p}, \Delta)$ or $d(\overline{p}, \overline{q}) = d(\overline{q}, \Delta)$, and $c = 1$ otherwise.

Consider the case $a \leqslant \frac{1}{2}$ and assume that $\overline{p}, \overline{q}$ are proper cornerpoints. Then, by Theorem 6, there are $A, B, C, D \in r_{(a,b)} \overset{\infty}{\cap} (\Gamma(\varphi) \cup \Gamma(\psi)), A \leqslant_{(a,b)} B, C \leqslant_{(a,b)} D$ such that $\overline{p} = (x_A - b, x_B - b), \overline{q} = (x_C - b, x_D - b)$. Let $\alpha_A, \alpha_B, \alpha_C, \alpha_D$ be the contours in $\Gamma(\varphi) \cup \Gamma(\psi)$ containing A, B, C, D respectively. These are univoquely determined because $(a, b) \notin \mathrm{Sp}(\varphi, \psi)$, so by Proposition 5.1 each point belongs to a unique contour. If $\overline{p}, \overline{q}$ were improper cornerpoints, then there would still exist some $A, C \in r_{(a,b)} \overset{\infty}{\cap} (\Gamma(\varphi) \cup \Gamma(\psi))$ such that $\overline{p} = (x_A - b, \infty), \overline{q} = (x_C - b, \infty)$ and the following holds.

Let $(a,b) \to (a',b')$ be a clockwise rotation around the point A such that the segment $[(a,b),(a',b')]$ (which is not a singleton set, because by definition $a' > a$) does not meet $\mathrm{Sp}(\varphi,\psi)$. This is because from 5.3 we know $\mathrm{Sp}(\varphi,\psi)$ is closed, so for any $(a,b) \notin \mathrm{Sp}(\varphi,\psi)$ there exists such a pair (a',b'). This implies in particular that for any $(a'',b'') \in [(a,b),(a',b')]$, the filtering line $r_{(a'',b'')}$ does not encounter any points from $\Gamma(\varphi) \cup \Gamma(\psi)$ incident to more than one contour. In particular, no $r_{(a'',b'')}$ encounters the endpoint of any contour, for any $(a'',b'') \in [(a,b),(a',b')]$. Thus, the rotation $(a,b) \to (a',b')$ induces a bijection:

$$\Psi\colon r_{(a,b)} \overset{\infty}{\cap} (\Gamma(\varphi) \cup \Gamma(\psi)) \longleftrightarrow r_{(a',b')} \overset{\infty}{\cap} (\Gamma(\varphi) \cup \Gamma(\psi))$$
$$r_{(a,b)} \overset{\infty}{\cap} \alpha_X = X \longleftrightarrow \Psi(X) = r_{(a',b')} \overset{\infty}{\cap} \alpha_X$$

With this terminology, the rotation $(a,b) \to (a',b')$ fixes $A = \Psi(A)$ and sends C to $\Psi(C)$. Applying Lemma 6.1, if $C \leqslant_{(a,b)} A$ then $x_{\Psi(C)} \leqslant x_C \leqslant x_A$ and, conversely, if $C \geqslant_{(a,b)} A$ then $x_{\Psi(C)} \geqslant x_C \geqslant x_A$. In any case

$$|x_{\Psi(C)} - x_A| \geqslant |x_C - x_A| = d(\overline{p},\overline{q}) = d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(a,b)}), \mathrm{Dgm}(\psi^*_{(a,b)})).$$

Since outside the special set the optimal matching cannot change abruptly, the cost of the optimal matching between $\mathrm{Dgm}(\varphi^*_{(a',b')})$ and $\mathrm{Dgm}(\psi^*_{(a',b')})$ is the distance between the points $(x_{\Psi(A)}-b, x_{\Psi(B)}-b)$ and $(x_{\Psi(C)}-b, x_{\Psi(D)}-b)$, which is precisely $|x_{\Psi(C)} - x_{\Psi}(A)| = |x_{\Psi(C)} - x_A|$. Combining this with the previous inequality gives the result:

$$d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(a',b')}), \mathrm{Dgm}(\psi^*_{(a',b')})) \geqslant d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(a,b)}), \mathrm{Dgm}(\psi^*_{(a,b)})).$$

The proof for the case $a \geqslant \frac{1}{2}$ is completely analogous by looking at the ordinates of the points in the persistence diagram, considering a counter-clockwise rotation and applying Lemma 6.1. □

Before stating the technical lemma to give proof for our main theorem, we will study the gradients of the functions f, g from Definition 6.1.

It is possible to see that U_α is a closed area of $]0,1[\times[-C,C]$ bounded by one or two lines, depending on whether α is an improper or proper contour, respectively. Below we see an example of this fact.

Example 6.1. We give an example of the computation of U_α as defined in 6.1. Let α be a proper contour with endpoints $(1,0),(0,1)$. Let L be the segment

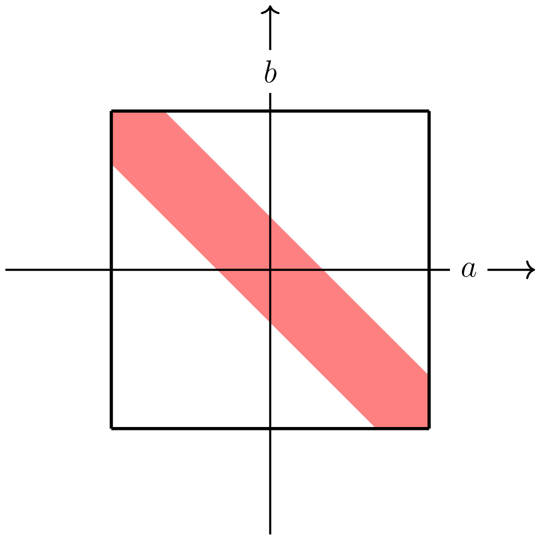

FIGURE 6.3: The domain U_α computed in the example (it is a strip between the lines $b = 1-a$ and $b = -a$). The black square represents the space of parameters $[0,1] \times [-C, C]$

with the same endpoints. Then,

$$\begin{aligned} r_{(a,b)} \cap \alpha \neq \varnothing &\iff r_{(a,b)} \cap L \neq \varnothing \\ &\iff \text{There exists } \lambda \in [0,1] \text{ such that} \\ &\qquad \lambda(0,1) + (1-\lambda)(1,0) \text{ belongs to } r_{(a,b)} \\ &\iff \text{There exists } \lambda \in [0,1] \text{ and } t \in \mathbb{R} \text{ such that} \\ &\qquad \begin{cases} 1 - \lambda = at + b \\ \lambda = (1-a)t - b. \end{cases} \end{aligned}$$

The computation becomes a linear system of two equations on the variables λ, t. In this particular case, substituting λ yields

$$1 - (1-a)t = at \Longleftarrow t = 1,$$

which gives the solution $(\lambda, t) = (1-a-b, 1)$. Therefore, the filtering line $r_{(a,b)}$ intersects α if and only if $0 \leqslant 1 - a - b \leqslant 1$, which corresponds to a "strip" in the space of parameters.

Proposition 6.1. *The functions f, g as in Definition 6.1 are of class C^1 on the interior of $\mathcal{Q}_{P,Q,R,S}$.*

Proof. Since f and g are polynomials of degree 2 on the functions $x_P(a,b), x_Q(a,b), x_R(a,b)$ and $x_S(a,b)$ it is enough to show that $x_P(a,b)$ is of class C^1 on $\mathcal{Q}^{\mathrm{o}}_{P,Q,R,S}$, the interior of $\mathcal{Q}_{P,Q,R,S}$. Let α be a contour containing P

- Assume α is a proper contour. Recall from Section 2.3 that α is diffeomorphic to an interval and $\alpha \subset \mathbb{J}(\varphi) \hookrightarrow M$ is a smooth embedding. Thus, there exists a differentiable function $h\colon \mathbb{R} \to \mathbb{R}$ such that $y = h(x)$ is the equation of α.

  By Propositions 2.6 and 4.3, there exists a unique point (x, y) in $(x, y) = (x(a,b), y(a,b))$ in $r_{(a,b)} \cap \alpha$ which is determined by the system:

  $$\begin{cases} y = \frac{1-a}{a}x - \frac{b}{a} \\ y = h(x) \end{cases}$$

  Consider the set $X = \{(a, b, x(a,b))\}$, such that $(x(a,b), y(a,b))$ is a solution of the previous equation system. The set X is the set of zeroes of a function F on $a, b, x(a,b)$, with

  $$F(a,b,x) = h(x) - \frac{1-a}{a}x + \frac{b}{a}.$$

  The last entries x of the solutions (a,b,x) of $F = 0$ are the abscissas of the points in $r_{(a,b)} \cap \alpha$.

  The function F is differentiable in x, because h is. Furthermore, $\frac{\partial F}{\partial x} = \frac{\partial h}{\partial x} - \frac{1-a}{a} < 0$ because the derivative of h is strictly negative due to point (iv) in Section 2.3 when α is a proper contour. In particular, $\frac{\partial F}{\partial x} \neq 0$ in its domain $\mathcal{Q}^{\mathrm{o}}_{P,Q,R,S}$, and the implicit function theorem can be applied to F. This yields the differentiability of $x(a,b)$ over $\mathcal{Q}^{\mathrm{o}}_{P,Q,R,S}$.

- Assume now that α is a vertical improper contour lying on the line $x = x_0$. Then $x(a,b) = x_0$ is a constant function, and therefore of class C^∞ in $\mathcal{Q}^{\mathrm{o}}_{P,Q,R,S}$.

- Lastly, suppose α is a horizontal vertical contour lying on the line $y = y_0$.

Then, for each $(a,b) \in]0,1[\times[-C,C]$ the only point $(x(a,b), y(a,b))$ in $r_{(a,b)} \cap \alpha_P$ is the solution to the system:

$$\begin{cases} y = \frac{1-a}{a}x - \frac{b}{a} \\ y = y_0 \end{cases}$$

But then $x(a,b)$ can be calculated explicitly:

$$x(a,b) = \frac{a}{1-a}y_0 + \frac{b}{1-a}.$$

Recall from Definition 6.1 that $0 < a \leqslant \frac{1}{2}$ for each $(a,b) \in \mathcal{Q}^{\circ}_{P,Q,R,S}$. Hence $x(a,b)$ as above is of class C^{∞} in its domain.

□

The functions f, g measure the distance between the abscissa of the points P, Q and R, S.

We are interested in understanding when the gradients $\nabla f, \nabla g$, defined for $(a,b) \in \mathcal{Q}^{\circ}_{P,Q,R,S}$, are parallel, that is, we wish to study the condition

$$\nabla f \text{ is parallel to } \nabla g. \tag{6.1}$$

When $\frac{\partial g}{\partial a}, \frac{\partial g}{\partial b} \neq 0$ (we show in full detail when does this condition hold in Appendix B), condition (6.1) is equivalent to

$$\frac{\frac{\partial f}{\partial a}}{\frac{\partial g}{\partial a}} = \frac{\frac{\partial f}{\partial b}}{\frac{\partial g}{\partial b}} \tag{6.2}$$

Let $X \in \{P, Q, R, S\}$ and let $T_X\alpha_X$ denote the line tangent to α at the point X. An approximation of the first order of the position of a point $X(a,b)$ when varying a, b is given by the intersection between the contour α_X and the tangent line:

$T_X\alpha_X \colon y - y_X = m_X(x - x_X)$	when X does not belong to an improper contour
$y = y_X$	when X belongs to a horizontal improper contour, $\alpha_X = h_{x_0,y_X} = \{x \geqslant x_0,\ y = y_X\}$
$x = x_X$	when X belongs to a vertical improper contour, $\alpha_X = v_{x_X,y_0} = \{x = x_X,\ y \geqslant y_0\}$.

where m_X is the local slope of α_X at X, when α_X is a proper contour . Explicitly, recall from Section 2.3 that each proper contour α_X comes with a diffeomorphism

$$\begin{aligned}\omega\colon [0,1] &\to \alpha_X\\ t &\mapsto (\omega_1(t), \omega_2(t))\end{aligned}$$

such that $\omega_1'(t) \cdot \omega_2'(t) \leqslant 0$ - this is property (iv) in Section 2.3 - and $\omega_1'(t), \omega_2'(t) \neq 0$ for all $t \in]0,1[$.

Let $t_X \in [0,1]$ be the value such that $\omega(t_X) = X$. If X is not an endpoint of α_X, then the local slope of α_X at X is

$$m_X = \frac{\omega_2'(t_X)}{\omega_1'(t_X)} < 0$$

Moreover, recall the parameterisation of the filtering line $r_{(a,b)}$, from Definition 2.11:

$$(u(t), v(t)) = (at + b, (1-a)t - b), \quad \text{for } (a,b) \in \mathcal{Q}^\circ_{P,Q,R,S} \subset \left]0, \frac{1}{2}\right] \times [-C, C].$$

Substituting the parametrisation of $r_{(a,b)}$ on the equation of $T_X\alpha_X$ gives

$$\begin{aligned}(1-a)t_X - b - y_X &= m_X(at_X + b - x_X)\\ t_X^{(a,b)} = t_X &= \frac{y_X + b - m_X(x_X - b)}{1 - a - m_X a}\end{aligned}$$

when X does not belong to an improper contour and

$$\begin{aligned}(1-a)t_X - b &= y_X & \qquad at_X + b &= x_X\\ t_X &= \frac{y_X + b}{1-a} & t_X &= \frac{x_X - b}{a}\end{aligned}$$

in the cases X belongs to a horizontal half-line and a vertical half-line, respectively. Remember that we are supposing $a \in \left]0, \frac{1}{2}\right]$, hence t_X is well defined in any case and substituting t_X in the parametrisation gives the intersection point in $r_{(a,b)} \cap T_X\alpha_X$.

In order to avoid some encumbering notation, for each point $X \in \alpha_X$ and for every $(a,b) \in \mathcal{Q}_{P,Q,R,S} \subset \left]0, \frac{1}{2}\right] \times [-C, C]$ consider the following degree 1 polynomials:

$$\xi_X(b) = \begin{cases} y_X + b - m_X(x_X - b) & \text{when } \alpha_X \text{ is a proper contour,} \\ y_X + b & \text{when } \alpha_X \text{ is an improper horizontal contour,} \\ x_X - b & \text{when } \alpha_X \text{ is an improper vertical contour,} \end{cases}$$

$$\eta_X(a) = \begin{cases} 1 - a - m_X a & \text{when } \alpha_X \text{ is a proper contour,} \\ 1 - a & \text{when } \alpha_X \text{ is an improper horizontal contour,} \\ a & \text{when } \alpha_X \text{ is an improper vertical contour.} \end{cases}$$

Notice that, for any $X \in \{P, Q, R, S\}$:

$$\begin{aligned} &\frac{\partial \xi_X}{\partial a} = \frac{\partial \eta_X}{\partial b} = 0, \\ &\frac{\partial \xi_X}{\partial b} = -\frac{\partial \eta_X}{\partial a} = \begin{cases} (1 + m_X) & \text{when } \alpha_X \text{ is a proper contour,} \\ 1 & \text{when } \alpha_X \text{ is an improper horizontal contour,} \\ -1 & \text{when } \alpha_X \text{ is an improper vertical contour.} \end{cases} \end{aligned} \tag{6.3}$$

Remark 6.3. Notice that $\eta_X(a) > 0 \ \forall a \in \left]0, \frac{1}{2}\right]$ since the tangent lines $T_X\alpha_X$ are either horizontal, vertical or with negative slope because of assumption (iv) in Section 2.3, and $a, 1 - a > 0$. Therefore for $a \in \left]0, \frac{1}{2}\right]$ we can write

$$t_X = \frac{\xi_X(b)}{\eta_X(a)}$$

and this value uniquely determines the point X via the (injective) parametrisation of $r_{(a,b)}$. Because of this, for each fixed (a, b), with $a \in \left]0, \frac{1}{2}\right]$:

$$\frac{\xi_X(b)}{\eta_X(a)} - \frac{\xi_{X'}(b)}{\eta_{X'}(a)} = 0 \Longleftrightarrow X = X' \in r_{(a,b)}. \tag{6.4}$$

Combining the polynomials ξ_X, η_X with the parametrisation gives an expression for the intersection point in $r_{(a,b)} \cap T_X\alpha_X$, for each $(a, b) \in \left]0, \frac{1}{2}\right] \times [-C, C]$,

$$X(a, b) = \left(a\frac{\xi_X(b)}{\eta_X(a)} + b,\ (1 - a)\frac{\xi_X(b)}{\eta_X(a)} - b \right).$$

Therefore, for $X \in \alpha_X$, $Y \in \alpha_Y$ and $(a,b) \in \mathcal{Q}_{P,Q,R,S} \cap \left]0, \frac{1}{2}\right] \times [C, -C]$:

$$\begin{aligned} x_X(a,b) - x_Y(a,b) &= a\frac{\xi_X(b)}{\eta_X(a)} + b - a\frac{\xi_Y(b)}{\eta_Y(a)} - b \\ &= a\left(\frac{\xi_X(b)}{\eta_X(a)} - \frac{\xi_Y(b)}{\eta_Y(a)}\right). \end{aligned}$$

We can rewrite our distance functions as

$$\begin{aligned} f(a,b) &= a^2\left(\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)}\right)^2, \\ g(a,b) &= a^2\left(\frac{\xi_R(b)}{\eta_R(a)} - \frac{\xi_S(b)}{\eta_S(a)}\right)^2, \\ &\text{for } (a,b) \in \mathcal{Q}_{P,Q,R,S} \cap \left]0, \frac{1}{2}\right] \times [C, -C]. \end{aligned}$$

With this notation, a simple computation gives

$$\begin{aligned} \frac{\partial f}{\partial a} &= 2a\left(\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)}\right)^2 + 2a^2\left(\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)}\right)\left(\frac{\xi_Q(b)\frac{\partial \eta_Q}{\partial a}}{\eta_Q(a)^2} - \frac{\xi_P(b)\frac{\partial \eta_P}{\partial a}}{\eta_P(a)^2}\right) \\ &= 2a\left(\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)}\right)\left(\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)} - \frac{a\xi_P(b)\frac{\partial \eta_P}{\partial a}}{\eta_P(a)^2} + \frac{a\xi_Q(b)\frac{\partial \eta_Q}{\partial a}}{\eta_Q(a)^2}\right), \end{aligned} \tag{6.5}$$

$$\frac{\partial f}{\partial b} = 2a^2\left(\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)}\right)\left(\frac{\frac{\partial \xi_P}{\partial b}}{\eta_P(a)} - \frac{\frac{\partial \xi_Q}{\partial b}}{\eta_Q(a)}\right),$$

and an analogous expression for $\frac{\partial g}{\partial a}$, $\frac{\partial g}{\partial b}$ just substituting P, Q with R, S. This expression is well-defined , since as we pointed out before, $\eta_X(a) > 0$ for every $a \in \left]0, \frac{1}{2}\right]$, $X \in \{P, Q, R, S\}$.

We rewrite the condition (6.1) we want to study using (6.5). For this step we are using $a \in \left]0, \frac{1}{2}\right]$, which implies $\eta_X \neq 0$:

$$\begin{aligned} &\frac{\left(\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)}\right)\left(\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)} - \frac{a\xi_P(b)\frac{\partial \eta_P}{\partial a}}{\eta_P(a)^2} + \frac{a\xi_Q(b)\frac{\partial \eta_Q}{\partial a}}{\eta_Q(a)^2}\right)}{\left(\frac{\xi_R(b)}{\eta_R(a)} - \frac{\xi_S(b)}{\eta_S(a)}\right)\left(\frac{\xi_R(b)}{\eta_R(a)} - \frac{\xi_S(b)}{\eta_S(a)} - \frac{a\xi_R(b)\frac{\partial \eta_R}{\partial a}}{\eta_R(a)^2} + \frac{a\xi_S(b)\frac{\partial \eta_S}{\partial a}}{\eta_S(a)^2}\right)} \\ &\qquad = \frac{\left(\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)}\right)\left(\frac{\frac{\partial \xi_P}{\partial b}}{\eta_P(a)} - \frac{\frac{\partial \xi_Q}{\partial b}}{\eta_Q(a)}\right)}{\left(\frac{\xi_R(b)}{\eta_R(a)} - \frac{\xi_S(b)}{\eta_S(a)}\right)\left(\frac{\frac{\partial \xi_R}{\partial b}}{\eta_R(a)} - \frac{\frac{\partial \xi_S}{\partial b}}{\eta_S(a)}\right)}. \end{aligned} \tag{6.6}$$

The further assumption that $P \neq Q$, $R \neq S$ implies $\left(\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)}\right) \neq 0$ and $\left(\frac{\xi_R(b)}{\eta_R(a)} - \frac{\xi_S(b)}{\eta_S(a)}\right) \neq 0$ by (6.4). Observe that the definition of special pair can be misleading: note that this assumption is not equivalent to $\{\alpha_P, \alpha_R\}, \{\alpha_Q, \alpha_S\}$ satisfying the condition of speciality: even if $\alpha_P \neq \alpha_Q$ and $\alpha_R \neq \alpha_S$, nor $P \neq Q$ nor $R \neq S$ is implied.

Under the assumption that $P \neq Q$ and $R \neq S$ the condition (6.1) becomes

$$\frac{\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)} - \frac{a\xi_P(b)\frac{\partial \eta_P}{\partial a}}{\eta_P(a)^2} + \frac{a\xi_Q(b)\frac{\partial \eta_Q}{\partial a}}{\eta_Q(a)^2}}{\frac{\xi_R(b)}{\eta_R(a)} - \frac{\xi_S(b)}{\eta_S(a)} - \frac{a\xi_R(b)\frac{\partial \eta_R}{\partial a}}{\eta_R(a)^2} + \frac{a\xi_S(b)\frac{\partial \eta_S}{\partial a}}{\eta_S(a)^2}} = \frac{\frac{\frac{\partial \xi_P}{\partial b}}{\eta_P(a)} - \frac{\frac{\partial \xi_Q}{\partial b}}{\eta_Q(a)}}{\frac{\frac{\partial \xi_R}{\partial b}}{\eta_R(a)} - \frac{\frac{\partial \xi_S}{\partial b}}{\eta_S(a)}}. \tag{6.7}$$

Now that we have fully characterised $f, g, \frac{\partial f}{\partial a}, \frac{\partial f}{\partial b}, \frac{\partial g}{\partial a}$, and $\frac{\partial g}{\partial b}$ in their respective domains, we provide the necessary and sufficient conditions on the points (a, b) for the gradient ∇f or ∇g to vanish.

Proposition 6.2. *Using the preceding notation, for any $(a, b) \in \mathcal{Q}^{\circ}_{P,Q,R,S}$ and for any $X, Y \in r_{(a,b)}$ consider the functions*

$$\Theta_{X,Y}(a, b) = \frac{\partial \xi_X}{\partial b}\eta_Y(a) - \eta_X(a)\frac{\partial \xi_Y}{\partial b},$$
$$\Xi_{X,Y}(a, b) = t_X\left(1 - \frac{a}{\eta_X(a)}\frac{\partial \eta_X}{\partial a}\right) - t_Y\left(1 - \frac{a}{\eta_Y(a)}\frac{\partial \eta_Y}{\partial a}\right)$$

Then,

$$\nabla f(a, b) = 0 \iff \begin{cases} P = Q, & \text{or} \\ \Theta_{P,Q}(a, b) = 0 \text{ and } P = Q, & \text{or} \\ \Xi_{P,Q}(a, b) = 0 \text{ and } P = Q, & \text{or} \\ \Theta_{P,Q}(a, b) = 0 \text{ and } \Xi_{P,Q}(a, b) = 0. \end{cases}$$

$$\nabla g(a, b) = 0 \iff \begin{cases} R = S & \text{or} \\ \Theta_{R,S}(a, b) = 0 \text{ and } R = S, & \text{or} \\ \Theta'_{R,S}(a, b) = 0 \text{ and } R = S, & \text{or} \\ \Theta_{R,S}(a, b) = 0 \text{ and } \Theta'_{R,S}(a, b) = 0. \end{cases}$$

Proof. We will give the proof for ∇f since the one for ∇g is completely identical substituting the pair $\{P, Q\}$ with $\{R, S\}$. Fix $(a, b) \in \mathcal{Q}^{\circ}_{P,Q,R,S}$ and

$P \in r_{(a,b)} \cap \alpha_P$, $Q \in r_{(a,b)} \cap \alpha_Q$. From the expression (6.5) for the partial derivatives it follows

$$\frac{\partial f}{\partial a} = 0 \Longleftrightarrow \underbrace{\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)}}_{\mathbf{C1}} = 0$$

$$\text{or } \underbrace{\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)} - \frac{a\xi_P(b)\frac{\partial \eta_P}{\partial a}}{\eta_P(a)^2} + \frac{a\xi_Q(b)\frac{\partial \eta_Q}{\partial a}}{\eta_Q(a)^2} = 0,}_{\mathbf{C2}}$$

$$\frac{\partial f}{\partial b} = 0 \Longleftrightarrow \frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)} = 0 \text{ or } \underbrace{\frac{\frac{\partial \xi_P}{\partial b}}{\eta_P(a)} - \frac{\frac{\partial \xi_Q}{\partial b}}{\eta_Q(a)} = 0.}_{\mathbf{C3}}$$

Therefore,

$$\nabla f = 0 \Longleftrightarrow \begin{cases} \mathbf{C1}, \\ \text{ or } \mathbf{C1} \text{ and } \mathbf{C2}, \\ \text{ or } \mathbf{C1} \text{ and } \mathbf{C3}, \\ \text{ or } \mathbf{C2} \text{ and } \mathbf{C3}. \end{cases}$$

The conditions **C1**, **C2** and **C3** are well defined when $a \in \left]0, \frac{1}{2}\right]$. We pointed out in 6.3 that **C1** is equivalent to $P = Q$. Conditions **C2** and **C3** are equivalent to $\Theta_{P,Q}(a,b) = 0$ and $\Xi_{P,Q}(a,b) = 0$, respectively. □

The previous discussion was made under the assumption $a \leqslant \frac{1}{2}$, but we will also need to know how to increase the Bottleneck distance when the filtering line $r_{(a,b)}$ is rotating with $a \geqslant \frac{1}{2}$. All the previous definitions and formulas have an analogous for $a \geqslant \frac{1}{2}$. For the sake of brevity, we report here only the essential steps.

Definition 6.3. For any four fixed contours $\alpha_P, \alpha_Q, \alpha_R, \alpha_S \in \mathrm{Ctr}(\varphi, \psi)$, $X \in \{P, Q, R, S\}$ denote

$$U'_{\alpha_X} = \{(a,b) \in \left[\frac{1}{2}, 1\right[\times [-C, C] \text{ such that } \alpha_X \cap r_{(a,b)} \neq \varnothing\},$$

and

$$\mathcal{Q}'_{P,Q,R,S} = \bigcap_{X \in \{P,Q,R,S\}} U'_{\alpha_X} \subset \left[\frac{1}{2}, 1\right[\times [-C, C],$$

and

$$\begin{aligned}P(a,b) &= (x_P(a,b), y_P(a,b)) = r_{(a,b)} \cap \alpha_P,\\ Q(a,b) &= (x_Q(a,b), y_Q(a,b)) = r_{(a,b)} \cap \alpha_Q,\\ R(a,b) &= (x_R(a,b), y_R(a,b)) = r_{(a,b)} \cap \alpha_R,\\ S(a,b) &= (x_S(a,b), y_S(a,b)) = r_{(a,b)} \cap \alpha_S\end{aligned}$$

the intersection points when varying (a,b).

In this setting, define the functions

$$\begin{aligned}f'_{P,Q,\alpha_P,\alpha_Q} = f\colon \mathcal{Q}'_{P,Q,R,S} &\to [0,\infty[\\ (a,b) &\mapsto (y_P(a,b) - y_Q(a,b))^2\\ g'_{R,S,\alpha_R,\alpha_S} = g\colon \mathcal{Q}'_{P,Q,R,S} &\to [0,\infty[\\ (a,b) &\mapsto (y_R(a,b) - y_S(a,b))^2\end{aligned} \tag{6.8}$$

An analogous to Proposition 6.1 holds for f' and g':

Proposition 6.3. *The functions f', g' are of class C^1 on $\mathcal{Q}'^{\circ}_{P,Q,R,S}$.*

The definitions we gave for ξ_X, η_X and the first order approximation of $X(a,b)$ extend to $(a,b) \in \mathcal{Q}'_{P,Q,R,S} \subset \left[\frac{1}{2}, 1\right[\times [-C, C]$:

$$\xi_X(b) = \begin{cases} y_X + b - m_X(x_X - b) & \text{when } \alpha_X \text{ is a proper contour,}\\ y_X + b & \text{when } \alpha_X \text{ is an improper horizontal contour,}\\ x_X - b & \text{when } \alpha_X \text{ is an improper vertical contour,}\end{cases}$$

$$\eta_X(a) = \begin{cases} 1 - a - m_X a & \text{when } \alpha_X \text{ is a proper contour,}\\ 1 - a & \text{when } \alpha_X \text{ is an improper horizontal contour,}\\ a & \text{when } \alpha_X \text{ is an improper vertical contour,}\end{cases}$$

$$X(a,b) = \left(a\frac{\xi_X(b)}{\eta_X(a)} + b,\ (1-a)\frac{\xi_X(b)}{\eta_X(a)} - b\right).$$

This time we want to look at the second coordinate of $X(a,b)$, for $X \in \{P, Q, R, S\}$. The functions f', g' can be rewritten as

$$\begin{aligned}f'(a,b) &= (1-a)^2\left(\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)}\right)^2 = \left(\frac{1-a}{a}\right)^2 f(a,b),\\ g'(a,b) &= (1-a)^2\left(\frac{\xi_R(b)}{\eta_R(a)} - \frac{\xi_S(b)}{\eta_S(a)}\right)^2 = \left(\frac{1-a}{a}\right)^2 g(a,b),\\ &\text{for } (a,b) \in \mathcal{Q}_{P,Q,R,S} \cap \left[\frac{1}{2}, 1\right[\times [C, -C].\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial f'}{\partial a} &= 2(a-1)\left(\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)}\right)^2 \\
&\quad + 2(1-a)^2\left(\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)}\right)\left(\frac{\xi_Q(b)\frac{\partial \eta_Q}{\partial a}}{\eta_Q(a)^2} - \frac{\xi_P(b)\frac{\partial \eta_P}{\partial a}}{\eta_P(a)^2}\right) \\
&= 2(1-a)\left(\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)}\right) \\
&\quad \left(\frac{\xi_Q(b)}{\eta_Q(a)} - \frac{\xi_P(b)}{\eta_P(a)} - (1-a)\frac{\xi_P(b)\frac{\partial \eta_P}{\partial a}}{\eta_P(a)^2} + (1-a)\frac{\xi_Q(b)\frac{\partial \eta_Q}{\partial a}}{\eta_Q(a)^2}\right), \\
\frac{\partial f'}{\partial b} &= 2(1-a)^2\left(\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)}\right)\left(\frac{\frac{\partial \xi_P}{\partial b}}{\eta_P(a)} - \frac{\frac{\partial \xi_Q}{\partial b}}{\eta_Q(a)}\right) = \left(\frac{1-a}{a}\right)^2 \frac{\partial f}{\partial b},
\end{aligned}$$

for $(a,b) \in \mathcal{Q}'^{\circ}_{P,Q,R,S} \subset \left[\frac{1}{2}, 1\right[\times [-C, C]$.

Hence, assuming $\frac{\partial g'}{\partial a}, \frac{\partial g'}{\partial b} \neq 0$ (see Appendix B), the parallelism condition between $\nabla f'$ and $\nabla g'$ can be written as

$$\begin{aligned}
&\frac{\left(\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)}\right)\left(\frac{\xi_Q(b)}{\eta_Q(a)} - \frac{\xi_P(b)}{\eta_P(a)} - (1-a)\frac{\xi_P(b)\frac{\partial \eta_P}{\partial a}}{\eta_P(a)^2} + (1-a)\frac{\xi_Q(b)\frac{\partial \eta_Q}{\partial a}}{\eta_Q(a)^2}\right)}{\left(\frac{\xi_R(b)}{\eta_R(a)} - \frac{\xi_S(b)}{\eta_S(a)}\right)\left(\frac{\xi_S(b)}{\eta_S(a)} - \frac{\xi_R(b)}{\eta_R(a)} - (1-a)\frac{\xi_R(b)\frac{\partial \eta_R}{\partial a}}{\eta_R(a)^2} + (1-a)\frac{\xi_S(b)\frac{\partial \eta_S}{\partial a}}{\eta_S(a)^2}\right)} \\
&\qquad = \frac{\left(\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)}\right)\left(\frac{\frac{\partial \xi_P}{\partial b}}{\eta_P(a)} - \frac{\frac{\partial \xi_Q}{\partial b}}{\eta_Q(a)}\right)}{\left(\frac{\xi_R(b)}{\eta_R(a)} - \frac{\xi_S(b)}{\eta_S(a)}\right)\left(\frac{\frac{\partial \xi_R}{\partial b}}{\eta_R(a)} - \frac{\frac{\partial \xi_S}{\partial b}}{\eta_S(a)}\right)},
\end{aligned}$$

and, under the further assumption that $P \neq Q$ and $R \neq S$,

$$\frac{\frac{\xi_Q(b)}{\eta_Q(a)} - \frac{\xi_P(b)}{\eta_P(a)} - (1-a)\frac{\xi_P(b)\frac{\partial \eta_P}{\partial a}}{\eta_P(a)^2} + (1-a)\frac{\xi_Q(b)\frac{\partial \eta_Q}{\partial a}}{\eta_Q(a)^2}}{\frac{\xi_S(b)}{\eta_S(a)} - \frac{\xi_R(b)}{\eta_R(a)} - (1-a)\frac{\xi_R(b)\frac{\partial \eta_R}{\partial a}}{\eta_R(a)^2} + (1-a)\frac{\xi_S(b)\frac{\partial \eta_S}{\partial a}}{\eta_S(a)^2}} = \frac{\frac{\frac{\partial \xi_P}{\partial b}}{\eta_P(a)} - \frac{\frac{\partial \xi_Q}{\partial b}}{\eta_Q(a)}}{\frac{\frac{\partial \xi_R}{\partial b}}{\eta_R(a)} - \frac{\frac{\partial \xi_S}{\partial b}}{\eta_S(a)}}.$$

Proposition 6.2 has an equivalent for f', g':

Proposition 6.4. *Using the preceding notation, for any $(a,b) \in \mathcal{Q}'^{\circ}_{P,Q,R,S}$ and for any $X, Y \in r_{(a,b)}$ consider the functions*

$$\begin{aligned}
\Theta'_{X,Y}(a,b) &= \frac{\partial \xi_X}{\partial b}\eta_Y(a) - \eta_X(a)\frac{\partial \xi_Y}{\partial b}, \\
\Xi'_{X,Y}(a,b) &= t_X\left(1 + \frac{1-a}{\eta_X(a)}\frac{\partial \eta_X}{\partial a}\right) - t_Y\left(1 + \frac{1-a}{\eta_Y(a)}\frac{\partial \eta_Y}{\partial a}\right)
\end{aligned}$$

Then,

$$\nabla f'(a,b) = 0 \iff \begin{cases} P = Q, & \text{or} \\ \Theta'_{P,Q}(a,b) = 0 \text{ and } P = Q, & \text{or} \\ \Xi'_{P,Q}(a,b) = 0 \text{ and } P = Q, & \text{or} \\ \Theta'_{P,Q}(a,b) = 0 \text{ and } \Xi'_{P,Q}(a,b) = 0. & \end{cases}$$

$$\nabla g'(a,b) = 0 \iff \begin{cases} R = S & \text{or} \\ \Theta'_{R,S}(a,b) = 0 \text{ and } R = S, & \text{or} \\ \Xi'_{R,S}(a,b) = 0 \text{ and } R = S, & \text{or} \\ \Theta'_{R,S}(a,b) = 0 \text{ and } \Xi'_{R,S}(a,b) = 0. & \end{cases}$$

Proof. We will give the proof for $\nabla f'$ since the one for $\nabla g'$ is completely identical substituting the pair $\{P, Q\}$ with $\{R, S\}$.

Fix $(a,b) \in \mathcal{Q}^{\circ}_{P,Q,R,S}$ and $P \in r_{(a,b)} \cap \alpha_P$, $Q \in r_{(a,b)} \cap \alpha_Q$. From the expression for the partial derivatives it follows

$$\frac{\partial f'}{\partial a} = 0 \iff \underbrace{\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)} = 0}_{\mathbf{C1}}$$

$$\text{or } \underbrace{\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)} - \frac{1 - a\xi_P(b)\frac{\partial \eta_P}{\partial a}}{\eta_P(a)^2} + \frac{1 - a\xi_Q(b)\frac{\partial \eta_Q}{\partial a}}{\eta_Q(a)^2} = 0,}_{\mathbf{C2}}$$

$$\frac{\partial f'}{\partial b} = 0 \iff \frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)} = 0 \text{ or } \underbrace{\frac{\frac{\partial \xi_P}{\partial b}}{\eta_P(a)} - \frac{\frac{\partial \xi_Q}{\partial b}}{\eta_Q(a)} = 0.}_{\mathbf{C3}}$$

Therefore,

$$\nabla f' = 0 \iff \begin{cases} \mathbf{C1}, \\ \text{or } \mathbf{C1} \text{ and } \mathbf{C2}, \\ \text{or } \mathbf{C1} \text{ and } \mathbf{C3}, \\ \text{or } \mathbf{C2} \text{ and } \mathbf{C3}. \end{cases}$$

The conditions **C1**, **C2** and **C3** are well defined when $a \in \left]0, \frac{1}{2}\right]$. Condition **C1** is equivalent to $P = Q$. Conditions **C2** and **C3** are equivalent to $\Theta'_{P,Q}(a,b) = 0$ and $\Xi'_{P,Q}(a,b) = 0$, respectively. $\square$

Without loss of generality, put $a \leqslant \frac{1}{2}$. If the matching distance is realised by the distance between two different pairs of points $\{p, q\} \neq \{p', q'\}$ simultaneously, i.e.

$$d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(a,b)}), \mathrm{Dgm}(\psi^*_{(a,b)})) = d(p,q) = d(p',q'),$$
$$\text{such that } \begin{cases} d(p,q) & = \lambda|w_0 - w_1| = |x_P - x_Q|, \\ d(p',q') & = \lambda|w'_0 - w'_1| = |x_R - x_S|, \end{cases}$$

where w_0, w_1, w'_0, w'_1 are the opportune coordinates of p, q, p', q', then by Proposition 5.2 (a, b) is a special pair. We can no longer apply the reasoning from Lemma 6.2 because by choosing a rotation that increases the first distance $d(p, q)$ the same rotation may simultaneously diminish the second distance $d(p', q')$, therefore decreasing the Bottleneck distance.

The functions f, g (or f', g', depending on whether $a \leqslant \frac{1}{2}$ or $a \geqslant \frac{1}{2}$) measure the distance between two pairs of points $\{P, Q\}, \{R, S\}$ in $\Gamma(\varphi)$. We now have a complete - see Appendix B - characterisation for the conditions under which $\nabla f, \nabla g$ are parallel or vanish. When none of these happen, a vector $v \in \mathbb{R}^2$ can be chosen such that $v \cdot \nabla f, v \cdot \nabla g > 0$; thus the vector v applied to a pair (a, b) represents a direction in which both functions $f(a, b), g(a, b)$ strictly increase, and hence the bottleneck distance cannot decrease. This is the main idea in the proof of the following lemma:

Lemma 6.3. *Let $(a, b) \notin \mathrm{USp}(\varphi, \psi)$. Assume hypothesis ‡. Let $\alpha_P, \alpha_Q, \alpha_R, \alpha_S$ be contours in $\Gamma(\varphi) \cup \Gamma(\psi)$ and let $X = r_{(a,b)} \cap \alpha_X$ for X in $\{P, Q, R, S\}$. For X, Y in $\{P, Q, R, S\}$ consider the functions $\Theta_{X,Y}, \Xi_{X,Y}, \Theta'_{X,Y}$ and $\Xi'_{X,Y}$ from Propositions 6.2 and 6.4.*

When $P \neq Q$, $R \neq S$, the function $\Theta_{X,Y}, \Xi_{X,Y}, \Theta'_{X,Y}$ and $\Xi'_{X,Y}$ do not vanish on (a, b) and the condition 6.7 is not satisfied, there exists a rotation $(a, b) \to (a', b')$, with $(a', b') \notin \mathrm{Sp}(\varphi, \psi)$ such that

$$d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(a,b)}), \mathrm{Dgm}(\psi^*_{(a,b)})) < d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(a',b')}), \mathrm{Dgm}(\psi^*_{(a',b')})).$$

Proof. If (a, b) is not a special value, the result follows directly from Lemma 6.2. Assume now that (a, b) is special.

Then there exist two pairs of contours $\{\alpha_A, \alpha_B\} \neq \{\alpha_C, \alpha_D\}$ in $\mathrm{Ctr}(\varphi, \psi)$ fulfilling the conditions of speciality. That is,

$$c_1|x_A - x_B| = c_2|x_C - x_D|, \text{ with } c_1, c_2 \in \{1, 2\},$$

where $A = \alpha_A \cap r_{(\overline{a},\overline{b})}, B = \alpha_B \cap r_{(\overline{a},\overline{b})}, C = \alpha_C \cap r_{(\overline{a},\overline{b})}$ and $D = \alpha_D \cap r_{(\overline{a},\overline{b})}$.

Note that the two pairs $\{\alpha_A, \alpha_B\} \neq \{\alpha_C, \alpha_D\}$ in condition of speciality may not be unique.

But, at the same time, (a, b) is not ultraspecial, so no three different paits of contours $\{\alpha_A, \alpha_B\} \neq \{\alpha_C, \alpha_D\} \neq \{\alpha_E, \alpha_F\} \neq \{\alpha_A, \alpha_B\}$ in $\mathrm{Ctr}(\varphi, \psi)$ satisfy

$$c_1|x_A - x_B| = c_2|x_C - x_D| = c_3|x_E - x_F|, \text{ with } c_1, c_2, c_3 \in \{1, 2\},$$

where $E = \alpha_E \cap r_{(\overline{a},\overline{b})}, F = \alpha_F \cap r_{(\overline{a},\overline{b})}$.

On the other hand, from 3.2 we know that there exist a matching $\sigma\colon \mathrm{Dgm}(\varphi^*_{(\overline{a},\overline{b})}) \to \mathrm{Dgm}(\psi^*_{(\overline{a},\overline{b})})$ and two cornerpoints $p \in \mathrm{Dgm}(\varphi^*_{(\overline{a},\overline{b})})$, $q \in \mathrm{Dgm}(\psi^*_{(\overline{a},\overline{b})})$ such that

$$d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(\overline{a},\overline{b})}), \mathrm{Dgm}(\psi^*_{(\overline{a},\overline{b})})) = \mathrm{cost}\sigma = d(p, q) = \lambda|w_0 - w_1|,$$

where w_0, w_1 are some coordinates of p and q and $\lambda \in \{\frac{1}{2}, 1\}$.

Assume $a \leqslant \frac{1}{2}$. Applying Position Theorem, there are points $P, Q, R, S \in \Gamma(\varphi)$ belonging to contours $\alpha_P, \alpha_Q, \alpha_R, \alpha_S$ respectively; and $p = (x_P - b, x_R - b), q = (x_Q - b, x_S - b)$. Without loss of generality, suppose from now on that the coordinates realising the cost of the optimal matching σ are $w_0 = x_p = x_P - b$ and $w_1 = x_q = x_Q - b$. Hence, the computation of the Bottleneck distance becomes just

$$d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(\overline{a},\overline{b})}), \mathrm{Dgm}(\psi^*_{(\overline{a},\overline{b})})) = \lambda|x_P - b - x_Q + b| = \lambda|x_P - x_Q|.$$

The pair $\{\alpha_P, \alpha_Q\}$ may coincide with $\{\alpha_A, \alpha_B\}$ or $\{\alpha_C, \alpha_D\}$, which are in speciality condition, so we cannot choose a point to rotate around as we did in previous Lemma 6.2.

Instead, recall from 6.1 the functions

$$\begin{aligned} f_{P,Q,\alpha_P,\alpha_Q} = f\colon \mathcal{Q}_{P,Q,R,S} &\to [0, \infty[\\ (a, b) &\mapsto (x_P(a, b) - x_Q(a, b))^2 \\ g_{R,S,\alpha_R,\alpha_S} = g\colon \mathcal{Q}_{P,Q,R,S} &\to [0, \infty[\\ (a, b) &\mapsto (x_R(a, b) - x_S(a, b))^2 \end{aligned}$$

$\mathcal{Q}_{P,Q,R,S}$ is not empty because $r_{(a,b)}$ intersects the contours $\alpha_P, \alpha_Q, \alpha_R, \alpha_S$. f, g are of class C^1 on its domain due to 6.1, so we can consider its gradients. Because of Preposition 6.2, our hypothesis guarantee that $\nabla f(a, b), \nabla g(a, b) \neq$

0 and they are not parallel. For any vector $v = \lambda_f \nabla f + \lambda_g \nabla g$ with $\lambda_f, \lambda_g > 0$ the following scalar products are strictly positive:

$$\nabla f \cdot v, \quad \nabla g \cdot v > 0.$$

That is, the following directional derivatives are strictly positive:

$$\frac{\partial f}{\partial v} = \lim_{\varepsilon \to 0} \frac{f((a,b)+\varepsilon v) - f(a,b))}{\varepsilon}, \quad \frac{\partial g}{\partial v} = \lim_{\varepsilon \to 0} \frac{g((a,b)+\varepsilon v) - g(a,b))}{\varepsilon} > 0.$$

In particular, since both limits are strictly positive, it is implied that there exist $\varepsilon_v > 0$ such that $f((a,b)+\varepsilon v) > f(a,b)$ and $g((a,b)+\varepsilon v) > g(a,b)$ for every $\varepsilon < \varepsilon_v$. Let $(a',b') = (a,b)+\varepsilon v \neq (a'.b')$. Since $\mathrm{Sp}(\varphi,\psi)$ is closed, and since it is a finite union of curves because of ‡, v and $\varepsilon < \varepsilon_v$ can be chosen such that $(a',b') \notin \mathrm{Sp}(\varphi,\psi)$ and $a \neq a'$. Then the rotation $(a,b) \to (a',b')$ strictly increases the Bottleneck distance:

$$d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(a,b)}), \mathrm{Dgm}(\psi^*_{(a,b)})) < d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(a',b')}), \mathrm{Dgm}(\psi^*_{(a',b')})).$$

If $a \geqslant \frac{1}{2}$, applying Position Theorem yields points $P', Q', R', S' \in \Gamma(\varphi)$ belonging to contours $\alpha_{P'}, \alpha_{Q'}, \alpha_{R'}, \alpha_{S'}$ respectively; and $p' = (y_{P'}+b, y_{R'}+b), q' = (y_{Q'}+b, y_{S'}+b)$. As we did in the case $a \leqslant \frac{1}{2}$, without loss of generality, suppose that the coordinates realising the Bottleneck distance are $y_{P'}$ and $y_{Q'}$;

$$d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(\bar{a},\bar{b})}), \mathrm{Dgm}(\psi^*_{(\bar{a},\bar{b})})) = \lambda |y_{P'} - y_{Q'}|,$$

and suppose the pair $\{\alpha_{P'}, \alpha_{Q'}\}$ is in condition of speciality with some other pair of contours $\{\beta_0, \beta_1\} \subset \mathrm{Ctr}(\varphi,\psi)$.

Then, recall from 6.8 the functions

$$\begin{aligned} f'_{P',Q',\alpha_{P'},\alpha_{Q'}} = f' \colon \mathcal{Q}'_{P',Q',R',S'} &\to [0,\infty[\\ (a,b) &\mapsto (y_{P'}(a,b) - y_{Q'}(a,b))^2 \\ g'_{R',S',\alpha_{R'},\alpha_{S'}} = g' \colon \mathcal{Q}'_{P',Q',R',S'} &\to [0,\infty[\\ (a,b) &\mapsto (y_{R'}(a,b) - y_{S'}(a,b))^2 \end{aligned}$$

which are of class C^1 in $\mathcal{Q}'^{\mathrm{o}}_{P',Q',R',S'}$ because of 6.3; and Proposition 6.2 implies that the gradients $\nabla f', \nabla g'$ do not vanish in $\mathcal{Q}'^{\mathrm{o}}_{P',Q',R',S'}$ and they are not parallel under our hypothesis.

Therefore, for any $\lambda_{f'}, \lambda_{g'} > 0$ the scalar products $\nabla f' \cdot v', \nabla g' \cdot v'$ are strictly positive, for $v' = \lambda_{f'}\nabla f' + \lambda_{g'}\nabla g'$. Hence, there exists $\varepsilon_{v'} > 0$ such that for any $0 < \varepsilon < \varepsilon_{v'}$,

$$\frac{f'((a,b)+\varepsilon v') - f'(a,b))}{\varepsilon}, \quad \frac{g'((a,b)+\varepsilon v') - g'(a,b))}{\varepsilon} > 0.$$

Let $(a'', b'') = (a,b) + \varepsilon v'$. As in the previous case $a \leqslant \frac{1}{2}$, our hypothesis guarantee $0 < \varepsilon < \varepsilon_{v'}$ can be chosen such that $(a'', b'') \notin \mathrm{Sp}(\varphi, \psi)$ and $a \neq a''$. Then:

$$d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(a,b)}), \mathrm{Dgm}(\psi^*_{(a,b)})) < d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(a',b')}), \mathrm{Dgm}(\psi^*_{(a'',b'')})),$$

and we can conclude. □

6.3 Proof of our main theorem

Now we are finally ready to prove our main theorem 7.

Proof. By contradiction, let $(\overline{a}, \overline{b}) \notin \mathcal{U}(\varphi, \psi)$, $\overline{a} \notin \{0, \frac{1}{2}, 1\}$ such that

$$D_{\mathrm{match}}(\varphi, \psi) = d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(\overline{a},\overline{b})}), \mathrm{Dgm}(\psi^*_{(\overline{a},\overline{b})})).$$

Without loss of generality, choose $(\overline{a}, \overline{b})$ minimizing $|a - \frac{1}{2}|$. We are allowed to do so since $[0,1] \times [-C, C]$ is compact. This will lead to a contradiction. For now assume $\overline{a} < \frac{1}{2}$. Let us study separate cases.

1. Case $(\overline{a}, \overline{b}) \notin \mathrm{Sp}(\varphi, \psi)$.

   Because of Lemma 6.2 there is a clockwise rotation $(\overline{a}, \overline{b}) \to (a', b')$, with $(a', b') \notin \mathrm{Sp}(\varphi, \psi)$, such that

   $$d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(\overline{a},\overline{b})}), \mathrm{Dgm}(\psi^*_{(\overline{a},\overline{b})})) \leqslant d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(a',b')}), \mathrm{Dgm}(\psi^*_{(a',b')}))$$

   Since $(\overline{a}, \overline{b}) \to (a', b')$ is clockwise, $a' > \overline{a}$ and so $|a' - \frac{1}{2}| < |\overline{a} - \frac{1}{2}|$, against the minimality of $\overline{a}$.

2. Case $(\overline{a}, \overline{b}) \in \mathrm{Sp}(\varphi, \psi) \backslash \mathcal{U}(\varphi, \psi)$.

   Apply Lemma 6.3. Then there is a (not necessarily clockwise or counter-clockwise) rotation $(\overline{a}, \overline{b}) \to (a'', b'')$ which strictly increases the bottleneck distance; and such that (a'', b'') is not a special value.

But this contradicts the hypothesis that $(\overline{a}, \overline{b})$ realises the matching distance:

$$\begin{aligned} d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(a'',b'')}), \mathrm{Dgm}(\psi^*_{(a'',b'')})) &> d_{\mathrm{B}}(\mathrm{Dgm}(\varphi^*_{(\overline{a},\overline{b})}), \mathrm{Dgm}(\psi^*_{(\overline{a},\overline{b})})) \\ &= D_{\mathrm{match}}(\varphi, \psi). \end{aligned}$$

We began by supposing $(\overline{a}, \overline{b}) \notin \mathcal{U}(\varphi, \psi)$, hence we have reached an absurd.

The proof for $\overline{a} > \frac{1}{2}$ is completely analogous by considering a counter-clockwise rotation in case (1).

□

Chapter 7

Conclusion

In this thesis we have presented a geometric approach to biparameter persistent homology. At the end of Chapter 6 we have shown a result that consents to qualitatively improve the computation cost for the calculation of the biparameter matching distance. Such a result might be important to the effective application of persistent homology to data analysis, machine learning, bioinformatics and many other fields.

Moreover, we have extended the Position Theorem from [13], which is the key step in the proof of our main result.

Regarding future works, we believe that our extended Position Theorem could be used to prove a further generalisation of our main result, reducing the computation of the matching distance to the study of lines of slope one, and maybe a finite quantity of exceptional values (a, b). Such a generalisation will require extending the functions f, g used in Chapter 6 to study the changes in the distance between intersection points for fixed contours, and a deeper understanding of the special and ultraspecial set introduced in Chapter 5.

We believe that these techniques can be easily generalised to general n-parameter persistent homology, virtually permitting to cut the amount of parameters needed for the calculation of the n-dimensional matching distance by one half.

Appendix A

The categorical viewpoint

A.1 Grothendieck categories and decomposition theorem

In this appendix we review the definitions and basic results from category theory underlying the theory of persistence modules. We also present the interleaving distance between persistence modules and give a motivation for the study of discrete invariants such as persistence diagrams. Most of the notation and results in this appendix come from [8], [9], [18].

Let $(P, \leqslant)$ be a partially ordered set, which can also be seen as a category $\mathbf{P}$ where $\mathrm{Mor}_P(a, b)$ is empty, if $a \nleqslant b$, and it has a unique element, if $a \leqslant b$. Let R be a unital ring and ${}_R\mathbf{Mod}$, $\mathbf{Mod}_R$ the categories of left and right (respectively) R-modules with R-module homomorphisms. Let X be a finitely triangulable topological space and $\varphi\colon X \to \mathbb{R}$ a continuous function. Consider the filtration $\{X_t^{\varphi}\}_{t \in \mathbb{R}}$ as in Section 2.

Definition A.1. An *abelian category* is a category $\mathbf{C}$ with a zero object, that is, an object that is both initial and terminal, with the following additional properties:

- $\mathbf{C}$ is *preadditive*: for each pair of objects $A, B \in \mathrm{ob}(\mathbf{C})$ $Hom_{\mathbf{C}}(A, B)$ is an abelian group and the composition of morphisms is bilinear in the sense that it can be seen as a morphism:

$$Hom_{\mathbf{C}}(B, C) \otimes Hom_{\mathbf{C}}(A, B) \to Hom_{\mathbf{C}}(A, C).$$

- $\mathbf{C}$ is *additive*: It admits finite products and coproducts.

- $\mathbf{C}$ is *preabelian*: It admits kernels and cokernels for each morphism.
- Every monomorphism is the kernel of some morphism.
- Every epimorphism is the cokernel of some morphism.

Definition A.2. A *Grothendieck category* is an abelian category $\mathbf{C}$ that

- admits a generator $G \in \mathrm{ob}(\mathbf{C})$, that is, an object G such that for every pair of distinct morphisms $f, g\colon A \to B$ in $\mathbf{C}$, there exist a morphism $h\colon G \to A$ such that $f \circ h \neq g \circ h$;
- admits coproducts of arbitrary families of objects.
- direct limits are exact: for every directed system of short exact sequences $\{0 \to A_i \to B_i \to C_i \to 0\}_{i \in I}$ the short exact sequence

$$0 \to \varinjlim A_i \to \varinjlim B_i \to \varinjlim C_i \to 0$$

  is exact.

With this terminology, a general definition of persistence module (2.1) can be given:

Definition A.3 (Persistence module)**.** A (n-parameter) *persistence module* is a covariant functor $\mathbf{M}\colon \mathbf{P} \to \mathbf{C}$ where $\mathbf{P}$ is a poset category and $\mathbf{C}$ is a Grothendieck category.

Persistence modules $\mathbf{M}\colon \mathbf{P} \to \mathbf{C}$ with natural transformations between them form themselves a category that we shall denote with $\mathbf{C}^{\mathbf{P}}$. In particular in section 2.3 in [18] the authors showed that when $(P, \leqslant; +, 0)$ admits a structure of abelian group compatible with the order $\leqslant$ and $\mathbf{C} = {}_R\mathbf{Mod}$ or $\mathbf{C} = \mathbf{Mod}_R$, there is an isomorphism of categories between the category of persistence modules and the category of P-graded $R[U_0]$-modules, with $U_0 = \{x \in P \mid 0 \leqslant x\}$, the *principal up-set* of $0 \in P$.

An immediate consequence of this is that when the poset $(P, \leqslant)$ is finite and R is a field then $R[U_0]$ is a PID and thus the structure theorem for finitely-generated modules over PIDs - see Appendix C- can be applied: there exists a unique decreasing sequence of proper ideals $(d_1) \supseteq (d_2) \supseteq \ldots$ such that

$$\mathbf{M} \cong \bigoplus_i R/(d_i).$$

However in general, P is not finite. Recall from Section 2 that in this work we considered the persistence module

$$H_k\colon \mathbf{R}^n \to \mathbf{Vect}_{\mathbb{F}}$$

associating to each $u \in \mathbb{R}^n$ the $\mathbb{F}$-vector space $H_k(X^{\varphi}_u; \mathbb{F})$, where H_k denotes the Čech homology functor in degree $k \in \mathbb{Z}$, and to each morphism $u \leqslant v$ the linear application induced in homology by the inclusion $X^{\varphi}_u \hookrightarrow X^{\varphi}_u$.

Nevertheless, there is a much more general decomposition theorem. The following is Theorem 1.1 from [16], which generalises the classical result from Carlsson, Zomorodian, Collins and Guibas (theorem 5.2 in [2]).

Definition A.4. A persistence module $\mathbf{M}\colon \mathbf{P} \to \mathbf{Vect}_{\mathbb{F}}$ is *pointwise finite-dimensional* if $\dim_{\mathbb{F}} M(p) < \infty$ for every $p \in P$.

Remark A.1. Theorem 1 guarantees that the persistence module $H_k\colon \mathbf{R}^n \to \mathbf{Vect}_{\mathbb{F}}$ is pointwise finite-dimensional.

Theorem 8 (Botnan, Crawley-Boevey). *Let* $\mathbf{M}\colon \mathbf{P} \to \mathbf{Vect}_{\mathbb{F}}$ *be pointwise finite-dimensional persistence module. Then there is a family of indecomposable persistence modules* $\{M_i\}_{i\in I}$ *such that* $\mathrm{End}(M_i)$ *is local for each* $i \in I$ *and*

$$M \cong \bigoplus_{i\in I} M_i.$$

We now illustrate briefly the categorical construction corresponding to the foliation method used in this work to study multiparameter persistence along a filtering line. Consider a line $L \subset \mathbb{R}^n$ of positive slope parametrised by $ut + v$ when $t \in \mathbb{R}$ for some $u, v \in \mathbb{R}^n$ fixed. Consider the functor

$$\mathbf{L}\colon \mathbf{R} \to \mathbf{R}^n$$

taking each $t \in \mathbb{R}$ to $ut + v \in \mathbb{R}^n$ and each $s \leqslant t$ to $us + v \preceq ut + v$. Then the composition functor $\mathbf{LM}$ is a one-parameter persistence module, corresponding to the filtering line L.

Also, the multiparameter matching distance we defined in Chapter 3 has an analogue on persistence modules. The following is definition 2.7 from [19] and for $n = 2$ agrees with our definition of matching distance:

Definition A.5. Let M, N be n-parameter persistence modules. Denote by $\ell(\mathbb{R}^n)$ the set of lines of positive slope in $\mathbb{R}^n$ and, for each $L \in \ell(\mathbb{R}^n)$, let

$\hat{u}(L) > 0$ be the minimal coordinate of the director vector $u(L)$ of L. Then the matching distance between M and N is defined as

$$D_{\mathrm{match}}(M,N) = \sup_{L \in \ell(\mathbb{R}^n)} \hat{u}(L) \cdot d_{\mathrm{B}}(\mathrm{Dgm}(\mathbf{L}M), \mathrm{Dgm}(\mathbf{L}N))$$

where $\mathrm{Dgm}(\mathbf{L}M), \mathrm{Dgm}(\mathbf{L}N)$ are the persistence diagrams corresponding to the persistence modules $\mathbf{L}M, \mathbf{L}N$.

A.2 Interleaving distance and isometry theorem

The following is definition 3.1 from [8].

Definition A.6 (Interleaving distance)**.** Let $\mathbf{C}$ be a category. For any $\varepsilon \geqslant 0$, let $\vec{\varepsilon} = (\varepsilon, \ldots, \varepsilon) \in \mathbb{R}^n$. Consider the shift functor

$$\begin{aligned} T_{\vec{\varepsilon}} \colon \mathbf{R}^n &\to \mathbf{R}^n \\ a &\mapsto a + \vec{\varepsilon} \\ a \preceq b &\mapsto a + \vec{\varepsilon} \preceq b + \vec{\varepsilon}. \end{aligned}$$

Note that $T_{\vec{\varepsilon}_1} T_{\vec{\varepsilon}_2} = T_{\vec{\varepsilon}_1 + \vec{\varepsilon}_2}$.

Then, an ε-*interleaving* between functors $F, G \colon \mathbf{R}^n \to \mathbf{C}$ is a pair of natural transformations (α, β) defined by the commutative diagram:

$$\begin{tikzcd} \mathbf{R}^n \arrow[r, "T_{\vec{\varepsilon}}"] \arrow[d, "F"] & \mathbf{R}^n \arrow[r, "T_{\vec{\varepsilon}}"] \arrow[d, "G"] & \mathbf{R}^n \arrow[d, "F"] \\ \mathbf{C} \arrow[r] \arrow[ur, phantom, "\overset{\alpha}{\Longrightarrow}"] & \mathbf{C} \arrow[r] \arrow[ur, phantom, "\overset{\beta}{\Longrightarrow}"] & \mathbf{C} \end{tikzcd}$$

and such that
the following diagrams commute:

$$\begin{tikzcd} F(a) \arrow[rr, "F(a \preceq a+2\vec{\varepsilon})"] \arrow[dr, "\alpha(a)"'] & & F(a+2\vec{\varepsilon}) \\ & G(a+\vec{\varepsilon}) \arrow[ur, "\beta(a+\vec{\varepsilon})"'] & \end{tikzcd} \qquad \begin{tikzcd} & F(a+\vec{\varepsilon}) \arrow[dr, "\alpha(a+\vec{\varepsilon})"] & \\ G(a) \arrow[ur, "\beta(a)"] \arrow[rr, "G(a \preceq a+2\vec{\varepsilon})"'] & & G(a+2\vec{\varepsilon}) \end{tikzcd}$$

We say that F, G are ε-*interleaved* if there is an ε-interleaving between them.

The *interleaving distance* between two persistence modules M, N is defined as

$$d_I(M, N) = \inf\{\varepsilon \in [0, \infty) \text{ such that } M, N \text{ are } \varepsilon\text{-interleaved}\}.$$

For $n = 1$ there is an important result relating the interleaving distance and the Bottleneck distance. This result can be found in [10] and [9]:

Theorem 9 (Isometry theorem)**.** *Let $\varphi, \psi \colon X \to \mathbb{R}$ two continous, real-valued functions. Fix a degree $k \in \mathbb{Z}$. Let $M_\varphi, M_\psi \colon \mathbf{R} \to \mathbf{Vect}_{\mathbb{F}}$ be the persistence modules given by*

$$M_\varphi(t) = H_k(X_t^\varphi; \mathbb{F}), \quad M_\varphi(s \leqslant t) = H_k(X_s^\varphi \hookrightarrow X_t^\varphi),$$
$$M_\psi(t) = H_k(X_t^\psi; \mathbb{F}), \quad M_\psi(s \leqslant t) = H_k(X_s^\psi \hookrightarrow X_t^\psi).$$

Then

$$d_I(M_\varphi, M_\psi) = d_{\mathrm{B}}(\mathrm{Dgm}(\varphi), \mathrm{Dgm}(\psi))$$

where $\mathrm{Dgm}(\varphi)$, $\mathrm{Dgm}(\psi)$ *are the persistence diagrams of φ and ψ in degrees k_φ, k_ψ, respectively.*

The isometry theorem allows the computation of the interleaving distance in one-parameter persistence. However, for $n > 1$, the computation of the interleaving distance is a NP-hard problem, as shown in [15].

This obstruction gives a motivation for the study of other metrics, such as D_{match}, to study persistence in TDA. The reader interested in other metrics studied in the same spirit should refer to [6], [13].

Appendix B

Necessary and sufficient conditions for $\frac{\partial f}{\partial a}, \frac{\partial f}{\partial b}, \frac{\partial g}{\partial a}, \frac{\partial g}{\partial b}$ to vanish

We study here the conditions under which the partial derivatives of the functions f, g studied in Chapter 6 vanish. Let $\varphi, \psi \colon M \to \mathbb{R}^2$ be smooth functions satisfying conditions (i)-(iv) in Section 2.3. Let $\alpha_P, \alpha_Q, \alpha_R, \alpha_S \in \operatorname{Ctr}(\varphi, \psi)$ be contours in $\Gamma(\varphi) \cup \Gamma(\psi)$ and set

$$\begin{aligned} P(a,b) &= (x_P(a,b), y_P(a,b)) = r_{(a,b)} \cap \alpha_P, \\ Q(a,b) &= (x_Q(a,b), y_Q(a,b)) = r_{(a,b)} \cap \alpha_Q, \\ R(a,b) &= (x_R(a,b), y_R(a,b)) = r_{(a,b)} \cap \alpha_R, \\ S(a,b) &= (x_S(a,b), y_S(a,b)) = r_{(a,b)} \cap \alpha_S \end{aligned}$$

the intersection points of the contours with the filtering line $r_{(a,b)}$ when varying $(a,b) \in \mathcal{Q}_{P,Q,R,S} \subset \left]0, \frac{1}{2}\right] \times [-C, C]$.

Recall Definition 6.1

$$\begin{aligned} f_{P,Q,\alpha_P,\alpha_Q} = f \colon \mathcal{Q}_{P,Q,R,S} &\to [0, \infty[\\ (a,b) &\mapsto (x_P(a,b) - x_Q(a,b))^2 \\ g_{R,S,\alpha_R,\alpha_S} = g \colon \mathcal{Q}_{P,Q,R,S} &\to [0, \infty[\\ (a,b) &\mapsto (x_R(a,b) - x_S(a,b))^2 \end{aligned}$$

and the auxiliary functions

$$\xi_X(b) = \begin{cases} y_X + b - m_X(x_X - b) & \text{when } X \text{ does not belong to an improper contour,} \\ y_X + b & \text{when } X \text{ belongs to an improper horizontal contour,} \\ x_X - b & \text{when } X \text{ belongs to an improper vertical contour,} \end{cases}$$

$$\eta_X(a) = \begin{cases} 1 - a - m_X a & \text{when } X \text{ does not belong to an improper contour,} \\ 1 - a & \text{when } X \text{ belongs to an improper horizontal contour,} \\ a & \text{when } X \text{ belongs to an improper vertical contour,} \end{cases}$$

for $X \in \{P, Q, R, S\}$.

A crucial step in the proof of Lemma 6.3 and our main theorem 7 is to study the values $(a, b) \in \left]0, \frac{1}{2}\right] \times [-C, C]$ for which the gradients $\nabla f, \nabla g$ are parallel. When $\frac{\partial g}{\partial a} \neq 0$ and $\frac{\partial g}{\partial b} \neq 0$ or $\frac{\partial f}{\partial a} \neq 0$ and $\frac{\partial f}{\partial b} \neq 0$ the parallelism condition is equivalent, respectively, to

$$\frac{\frac{\partial f}{\partial a}}{\frac{\partial g}{\partial a}} = \frac{\frac{\partial f}{\partial b}}{\frac{\partial g}{\partial b}} \quad \text{or} \quad \frac{\frac{\partial g}{\partial a}}{\frac{\partial f}{\partial a}} = \frac{\frac{\partial g}{\partial b}}{\frac{\partial f}{\partial b}} \tag{B.1}$$

If the above conditions on the partial derivatives are not satisfied, then there are four possibilities regarding the parallelism of the gradients $\nabla f, \nabla g$:

Proposition B.1. *Let $(a, b) \in \left]0, \frac{1}{2}\right] \times [-C, C]$. If $\frac{\partial g}{\partial a} = 0$ or $\frac{\partial g}{\partial b} = 0$, and $\frac{\partial f}{\partial a} = 0$ or $\frac{\partial f}{\partial b} = 0$, then one of the following is true:*

1. *$\nabla f = 0$ and $\nabla g \neq 0$,*
2. *$\nabla g = 0$ and $\nabla f \neq 0$,*
3. *∇f and ∇g are parallel and non-vanishing on (a, b),*
4. *∇f and ∇g are orthogonal on (a, b).*

In particular, under these hypotheses, the gradients of f and g are parallel if and only if

$$\begin{aligned} &\frac{\partial f}{\partial a} = \frac{\partial g}{\partial a} = 0, \text{ and } \frac{\partial f}{\partial b}, \frac{\partial g}{\partial b} \neq 0 \\ \text{or} \quad &\frac{\partial f}{\partial b} = \frac{\partial g}{\partial b} = 0, \text{ and } \frac{\partial f}{\partial a}, \frac{\partial g}{\partial a} \neq 0 \end{aligned}$$

Proof. There are four possibilities:

- $\frac{\partial f}{\partial a} = \frac{\partial f}{\partial b} = 0$. Then $\nabla f = 0$.
- $\frac{\partial g}{\partial a} = \frac{\partial g}{\partial b} = 0$. Then $\nabla g = 0$.
- $\frac{\partial f}{\partial a} = 0$ and $\frac{\partial f}{\partial b} \neq 0$.
  - $\frac{\partial g}{\partial a} = 0$ and $\frac{\partial g}{\partial b} \neq 0$. Then $\nabla f, \nabla g$ are parallel and non-vanishing.
  - $\frac{\partial g}{\partial a} \neq 0$ and $\frac{\partial g}{\partial b} = 0$. Then $\nabla f, \nabla g$ are orthogonal and non-vanishing.
- $\frac{\partial g}{\partial a} = 0$ and $\frac{\partial g}{\partial b} \neq 0$.
  - $\frac{\partial f}{\partial a} = 0$ and $\frac{\partial f}{\partial b} \neq 0$. Then $\nabla f, \nabla g$ are parallel and non-vanishing.
  - $\frac{\partial f}{\partial a} \neq 0$ and $\frac{\partial f}{\partial b} = 0$. Then $\nabla f, \nabla g$ are orthogonal and non-vanishing.

□

In Chapter 6 we reduced our study, for the sake of brevity, to the first equation in B.1. With the previous proposition, all the cases for $\frac{\partial f}{\partial a}, \frac{\partial f}{\partial b}, \frac{\partial g}{\partial a}, \frac{\partial g}{\partial b}$ when the form B.1 of the parallelism condition is not available have been covered. But we want to show explicit conditions under which we restrict our study to condition B.1.

We will now study when $\frac{\partial f}{\partial a}$ and $\frac{\partial f}{\partial b}$ do not vanish. The reasoning is completely analogous for the study of the partial derivatives of g. In 6.5 we found an expression for the partial derivatives as a function of ξ_X, η_X, $X \in \{P, Q, R, S\}$, $(a, b) \in \left]0, \frac{1}{2}\right] \times [-C, C]$:

$$\frac{\partial f}{\partial a} = 2a \left(\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)} \right) \left(\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)} - \frac{a\xi_P(b)\frac{\partial \eta_P}{\partial a}}{\eta_P(a)^2} + \frac{a\xi_Q(b)\frac{\partial \eta_Q}{\partial a}}{\eta_Q(a)^2} \right),$$

$$\frac{\partial f}{\partial b} = 2a^2 \left(\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)} \right) \left(\frac{\frac{\partial \xi_P}{\partial b}}{\eta_P(a)} - \frac{\frac{\partial \xi_Q}{\partial b}}{\eta_Q(a)} \right)$$

and an analogous expression for $\frac{\partial g}{\partial a}, \frac{\partial g}{\partial b}$ just substituting P, Q with R, S.

Recall that η_X is strictly positive when $a \in \left]0, \frac{1}{2}\right]$; in particular, it is strictly positive where the gradients $\nabla f, \nabla g$ are defined. Moreover, recall from 6.3 that for each $(a,b) \in]0,1[\times[-C,C]$:

$$\frac{\partial \xi_X}{\partial a} = \frac{\partial \eta_X}{\partial b} = 0,$$

$$\frac{\partial \xi_X}{\partial b} = -\frac{\partial \eta_X}{\partial a} = \begin{cases} (1+m_X) & \text{when } X \text{ does not belong to an improper} \\ & \quad \text{contour,} \\ 1 & \text{when } X \text{ belongs to an improper horizontal} \\ & \quad \text{contour,} \\ -1 & \text{when } X \text{ belongs to an improper vertical} \\ & \quad \text{contour.} \end{cases}$$

Proposition B.2. *Let $(a,b) \in \left]0, \frac{1}{2}\right] \times [-C, C]$ and $P \neq Q$. With the notations of Chapter 6, the conditions on table B.1 are necessary and sufficient conditions for $\frac{\partial f}{\partial a} \neq 0$ and $\frac{\partial f}{\partial b} \neq 0$ depending on the contours α_P, α_Q*

TABLE B.1: Table for the partial derivatives of f in Proposition B.2

	$\frac{\partial f}{\partial a} \neq 0$	$\frac{\partial f}{\partial b} \neq 0$
α_P, α_Q proper	$\xi_P(b)\eta_Q(a)^2 \neq \xi_Q(b)\eta_P(a)^2$	$m_P \neq m_Q$
α_P horizontal, α_Q proper	$\xi_P(b)\eta_Q(a)^2 \neq \xi_Q(b)(1-a)^2$	Q is an endpoint of α_Q and the basepoint of β_Q a horizontal contour belonging to the same EPG
α_P vertical, α_Q proper	$\xi_Q(b) \neq 0$	Always
α_P, α_Q horizontal	$y_P \neq y_Q$	Never
α_P vertical, α_Q horizontal	$y_Q \neq -b$	Always
α_P, α_Q vertical	Never	Never

Remark B.1. In the previous table we skipped the cases α_P proper and α_Q horizontal, α_P proper and α_Q vertical, and α_P horizontal and α_Q vertical since they are symmetric to the cases presented in the second, third and fifth rows, respectively.

Proof. We will first show the first column in the table. We pointed out in 6.3 that for $P \neq Q$

$$\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)} \neq 0$$

Therefore, since for $a \in \left]0, \frac{1}{2}\right]$ $\eta_P(a), \eta_Q(a) > 0$, the equation $\frac{\partial f}{\partial a} = 0$ is equivalent to

$$\begin{aligned}
&\frac{\xi_P(b)}{\eta_P(a)} - \frac{a\xi_P(b)\frac{\partial \eta_P}{\partial a}}{\eta_P(a)^2} = \frac{\xi_Q(b)}{\eta_Q(a)} - \frac{a\xi_Q(b)\frac{\partial \eta_Q}{\partial a}}{\eta_Q(a)^2} \\
\iff &\xi_P(b)\eta_P(a)\eta_Q(a)^2 - a\xi_P(b)\eta_Q(a)^2\frac{\partial \eta_P}{\partial a} \\
&\quad = \xi_Q(b)\eta_P(a)^2\eta_Q(a) - a\xi_Q(b)\eta_P(a)^2\frac{\partial \eta_Q}{\partial a} \\
\iff &\xi_P(b)\eta_Q(a)^2\left(\eta_P(a) - a\frac{\partial \eta_P}{\partial a}\right) = \xi_Q(b)\eta_P(a)^2\left(\eta_Q(a) - a\frac{\partial \eta_Q}{\partial a}\right).
\end{aligned}$$

Let us denote this condition on P, Q with the letter $\mathbf{C}$.

From the expression for the partial derivatives of the auxiliary equations ξ_X, η_X we can refine the condition $\mathbf{C}$ using that

$$\eta_X(a) - a\frac{\partial \eta_X}{\partial a} = \begin{cases} 1 & \text{if } X \text{ does not belong to a vertical contour,} \\ 0 & \text{if } X \text{ belongs to a vertical contour,} \end{cases}$$

for $X \in \{P, Q\}$, $a \in v$. Indeed, if $X \in \alpha_X$ proper contour, then

$$\eta_X(a) - a\frac{\partial \eta_X}{\partial a} = 1 - a - m_X a + a + m_X a = 1.$$

Similarly, if α_X is an improper horizontal contour,

$$\eta_X(a) - a\frac{\partial \eta_X}{\partial a} = 1 - a + a = 1.$$

But when α_X is vertical

$$\eta_X(a) - a\frac{\partial \eta_X}{\partial a} = a - a = 0.$$

Now we distinguish cases for α_P, α_Q:

1. α_P, α_Q proper.

   In this case, the condition **C** becomes $\xi_P(b)\eta_Q(a)^2 = \xi_Q(b)\eta_P(a)^2$.

2. α_P horizontal and α_Q proper.

   The condition **C** is the same than in the previous case but can be specialised using that $\eta_P(a) = 1 - a$. Therefore **C** is equivalent to

   $$\xi_P(b)\eta_Q(a)^2 = \xi_Q(b)(1-a)^2$$

3. α_P vertical and α_Q proper.

   Then $\eta_P(a) - a\frac{\partial \eta_P}{\partial a} = 0$, and since $a \in \left]0, \frac{1}{2}\right]$ implies $\eta_Q(a) > 0$, condition **C** can be written as

   $$0 = \xi_P(b)\eta_Q(a)^2 = \xi_P(b).$$

4. α_P, α_Q horizontal.

   For X belonging to improper horizontal contours, $\eta_X(a) = 1 - a > 0$, hence condition **C** is

   $$\xi_P(b) = y_P + b = \xi_Q(b) = y_Q + b,$$

   which is equivalent to $y_P = y_Q$.

5. α_P vertical and α_Q horizontal.

   In this case $\eta_Q(a) - a\frac{\partial \eta_Q}{\partial a} = 1$ but $\eta_P(a) - a\frac{\partial \eta_P}{\partial a} = 0$. Therefore condition **C** becomes

   $$0 = \xi_Q(b)\eta_P(a)^2 = a^2(y_Q + b).$$

   Equivalently, $y_Q = -b$

6. α_P, α_Q vertical.

   In this situation $\eta_Q(a) - a\frac{\partial \eta_Q}{\partial a} = \eta_P(a) - a\frac{\partial \eta_P}{\partial a} = 0$ and hence condition **C** is tautological. But that means that $\frac{\partial f}{\partial a}$ always vanishes when $P \neq Q$.

Now let us check the second column of the table.
Using again

$$\frac{\xi_P(b)}{\eta_P(a)} - \frac{\xi_Q(b)}{\eta_Q(a)} \neq 0$$

for $P \neq Q$, the equation $\frac{\partial f}{\partial b}$ is equivalent to

$$\frac{\frac{\partial \xi_P}{\partial b}}{\eta_P(a)} = \frac{\frac{\partial \xi_Q}{\partial b}}{\eta_Q(a)}.$$

Since $a \in \left]0, \frac{1}{2}\right]$ implies $\eta_P(a), \eta_Q(a) > 0$ we are reduced to studying the equation

$$\frac{\partial \xi_P}{\partial b}\eta_Q(a) = \frac{\partial \xi_Q}{\partial b}\eta_P(a).$$

As we did before, now we study different possibilities for the contours α_P, α_Q:

1. α_P, α_Q proper.

$$\begin{aligned}\frac{\partial \xi_P}{\partial b}\eta_Q(a) = \frac{\partial \xi_Q}{\partial b}\eta_P(a) &\iff (1+m_P)(1-a-m_Qa)\\ &\quad = (1+m_Q)(1-a-m_Pa)\\ &\iff 1-a-m_Qa+m_P-m_Pa-m_Pm_Qa\\ &\quad = 1-a-m_Pa+m_Q-m_Qa-m_Pm_Qa\\ &\iff m_Q = m_P\end{aligned}$$

2. α_P horizontal, α_Q proper.

$$\begin{aligned}\frac{\partial \xi_P}{\partial b}\eta_Q(a) = \frac{\partial \xi_Q}{\partial b}\eta_P(a) &\iff 1-a-m_Qa\\ &\quad = (1+m_Q)(1-a) = 1+m_Q-a-m_Qa\\ &\iff m_Q = 0\end{aligned}$$

   Notice that if α_Q is proper, then $m_Q = 0$ if and only if m_Q is simultaneously an endpoint of α_Q and the basepoint of some horizontal contour $\beta_Q \neq \alpha_Q$ belonging to the same extended Pareto grid as α_Q.

3. α_P vertical, α_Q proper.

$$\begin{aligned}\frac{\partial \xi_P}{\partial b}\eta_Q(a) = \frac{\partial \xi_Q}{\partial b}\eta_P(a) &\iff m_Qa+a-1 = (1+m_Q)a = a+m_Qa\\ &\iff -1 = 0\end{aligned}$$

   In this case we reached a contradiction, so $\frac{\partial \xi_P}{\partial b}\eta_Q(a)$ and $\frac{\partial \xi_Q}{\partial b}\eta_P(a)$ cannot be equal and therefore $\frac{\partial f}{\partial b}$ never vanishes in this situation.

4. α_P, α_Q horizontal.

$$\frac{\partial \xi_P}{\partial b}\eta_Q(a) = \frac{\partial \xi_Q}{\partial b}\eta_P(a) \Longleftrightarrow 1 - a = 1 - a$$

   A tautology means that $\frac{\partial f}{\partial b} = 0$ for each $(a, b) \in \mathcal{Q}^{o}_{P,Q,R,S}$ in this situation.

5. α_P vertical, α_Q horizontal.

   This case yields the equation $a - 1 = a$, which is contradictory for $a \in \left]0, \frac{1}{2}\right]$. Therefore $\frac{\partial f}{\partial b} \neq 0$.

6. α_P, α_Q vertical.

   This last case yields $a = a$; so once again, a tautology which implies $\frac{\partial f}{\partial b} = 0$ for each $(a, b) \in \mathcal{Q}^{o}_{P,Q,R,S}$.

This concludes the proof. □

An analogous result can be proven to characterise $\frac{\partial g}{\partial a} \neq 0$, $\frac{\partial g}{\partial b} \neq 0$ depending on the contours α_R, α_S.

Of course, when $a \in \left[\frac{1}{2}, 1\right[$, the proof of Proposition B.1 still holds for f', g' and the necessary and sufficient conditions for $\frac{\partial f'}{\partial a}, \frac{\partial f'}{\partial b}, \frac{\partial g'}{\partial a}, \frac{\partial g'}{\partial b}$ to be non-vanishing on $(a, b) \in \mathcal{Q}'^{o}_{P,Q,R,S}$ are proven following the same reasoning as in the proof of Proposition B.2.

Hence we skip the proofs and directly report the tables B.2, B.3, and B.4 containing the conditions under which the partial derivatives of g, f', g' are non-zero in (a, b).

Propositions B.1 and B.2 completely classify the relative position of the gradients $\nabla f, \nabla g, \nabla f', \nabla g'$ depending on the partial derivatives $\frac{\partial f}{\partial a}, \frac{\partial f}{\partial b}, \frac{\partial g}{\partial a}, \frac{\partial g}{\partial b}, \frac{\partial f'}{\partial a}, \frac{\partial f'}{\partial b}, \frac{\partial g'}{\partial a}, \frac{\partial g'}{\partial b}$ and the contours $\alpha_P, \alpha_Q, \alpha_R, \alpha_S$.

Table B.2: Table for the partial derivatives of g

	$\frac{\partial g}{\partial a} \neq 0$	$\frac{\partial g}{\partial b} \neq 0$
α_R, α_S proper	$\xi_R(b)\eta_S(a)^2 \neq \xi_S(b)(1-a)^2$	$m_R \neq m_S$
α_R horizontal, α_S proper	$\xi_R(b)\eta_S(a)^2 \neq \xi_S(b)\eta_R(a)^2$	S is an endpoint of α_S and the basepoint of β_S a horizontal contour belonging to the same EPG
α_R vertical, α_S proper	$\xi_S(b) \neq 0$	Always
α_R, α_S horizontal	$y_R \neq y_S$	Never
α_R vertical, α_S horizontal	$y_S \neq -b$	Always
α_R, α_S vertical	Never	Never

Table B.3: Table for the partial derivatives of f'

	$\frac{\partial f'}{\partial a} \neq 0$	$\frac{\partial f'}{\partial b} \neq 0$
α_P, α_Q proper	$m_P\xi_P(b)\eta_Q(a)^2 \neq m_Q\xi_Q(b)\eta_P(a)^2$	$m_P \neq m_Q$
α_P horizontal, α_Q proper	$\xi_Q(b) \neq 0$	Q is an endpoint of α_Q and the basepoint of β_Q a horizontal contour belonging to the same EPG
α_P vertical, α_Q proper	$\xi_P(b)\eta_Q(a)^2 \neq -m_Q a^2 \xi_Q(b)$	Always
α_P, α_Q horizontal	Never	Never
α_P vertical, α_Q horizontal	$x_P \neq b$	Always
α_P, α_Q vertical	$x_P \neq x_Q$	Never

Table B.4: Table for the partial derivatives of g'

	$\frac{\partial g'}{\partial a} \neq 0$	$\frac{\partial g'}{\partial b} \neq 0$
α_R, α_S proper	$m_R \xi_R(b) \eta_S(a)^2 \neq m_S \xi_S(b) \eta_R(a)^2$	$m_R \neq m_S$
α_R horizontal, α_S proper	$\xi_S(b) \neq 0$	S is an endpoint of α_S and the basepoint of β_S a horizontal contour belonging to the same EPG
α_R vertical, α_S proper	$\xi_R(b) \eta_S(a)^2 \neq -m_S a^2 \xi_S(b)$	Always
α_R, α_S horizontal	Never	Never
α_R vertical, α_S horizontal	$x_R \neq b$	Always
α_R, α_S vertical	$x_R \neq x_S$	Never

Appendix C

Auxiliary elementary results

We report here the statements of some well-known results from linear algebra classical real analysis we used in our exposition.

Theorem 10 (Rank-nullity theorem)**.** *Let $\mathbb{F}$ be a field and V, W be vector spaces over $\mathbb{F}$. For each $\mathbb{F}$-linear map $T\colon V \to W$,*

$$\dim V = \dim \mathrm{Im} T + \dim \ker T.$$

Proposition C.1 (Monotone convergence for sequences)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a monotonic sequence of real numbers such that there exists an upper bound $B \in \mathbb{R}$ such that $|a_N| < B$ for each $n \in \mathbb{N}$. Then the sequence $(a_n)_{n\in\mathbb{N}}$ is convergent and:*

$$\lim_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} a_n \quad \text{if } (a_n)_{n\in\mathbb{N}} \text{ is increasing,}$$
$$\lim_{n\to\infty} a_n = \inf_{n\in\mathbb{N}} a_n \quad \text{if } (a_n)_{n\in\mathbb{N}} \text{ is decreasing.}$$

Theorem 11 (Implicit function theorem)**.** *Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ be an open set, whose points are written in the form $(x_1, \ldots, x_n, y_1, \ldots, y_m)$.. Let $F\colon \Omega \to \mathbb{R}^m$ be a function of class C^1 on Ω and $(a, b) \in \Omega$ such that $F(a, b) = 0$ and the matrix $\frac{\partial F}{\partial y}(a, b)$ is invertible, where*

$$\frac{\partial F}{\partial y} = \left(\frac{\partial F_i}{\partial y_j}\right)_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant m}} = \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}.$$

Then, there exists open sets $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ with $a \in X$, $b \in Y$ and such that

- *For each* $x \in X$ *there is a unique* $y = f(x) \in Y$ *such that* $F(x, f(x)) = 0$.
- $f(a) = b$ *and* $f\colon X \to Y$ *is of class* C^1 *on* X.

Theorem 12 (Structure theorem for finitely-generated modules over PIDs)**.** *For every finitely generated module* M *over a principal ideal domain* R*, there is a unique decreasing sequence of proper ideals* $(d_1) \supseteq (d_2) \supseteq \cdots \supseteq (d_n)$ *such that* M *is isomorphic to the sum of cyclic modules:*

$M \cong \bigoplus_i R/(d_i) = R/(d_1) \oplus R/(d_2) \oplus \cdots \oplus R/(d_n)$.

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