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Introduction to black hole thermodynamics

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Abstract

The aim of this thesis is to introduce the reader to the study of black holes, starting from Einstein equations and concluding with the information paradox, along with an overview of the Holographic Principle. The main mathematical and physical properties will be shown by means of the theory of general relativity and quantum field theory. Spherically symmetric space-time and Rindler coordinates will be widely used. Great attention will be paid to thermodynamics and BHs' evaporation process.

The first three chapters are mainly concerned with basic material, which will be later used in chapter four and five for a thermodynamic dissertation.

Sommario

Questa tesi ha l'obiettivo di introdurre il lettore allo studio dei buchi neri, partendo dalle equazioni di campo di Einstein fino ad arrivare a discutere il paradosso dell'informazione di Hawking ed il Principio Olografico. Le principali caratteristiche matematiche e fisiche verrano esposte sulla base delle teorie della relatività generale e dei campi quantistici, facendo ampio uso di uno spazio-tempo a simmetria sferica e delle coordinate di Rindler. Particolare importanza verrà data alle proprietà termodinamiche dei buchi neri ed al loro processo di evaporazione.

I primi tre capitoli contengono materiale di base, che verrà poi utilizzato per esporre la trattazione termodinamica a partire dal quarto capitolo.

Contents

Introduction

The idea of bodies so massive and dense that even light could not escape them was first introduced in the XVIII century relying on Newtonian gravity. Later on, with the general theory of gravity these objects gained more relevance and importance, until they have recently been detected in astronomical observations (see Fig. 1).

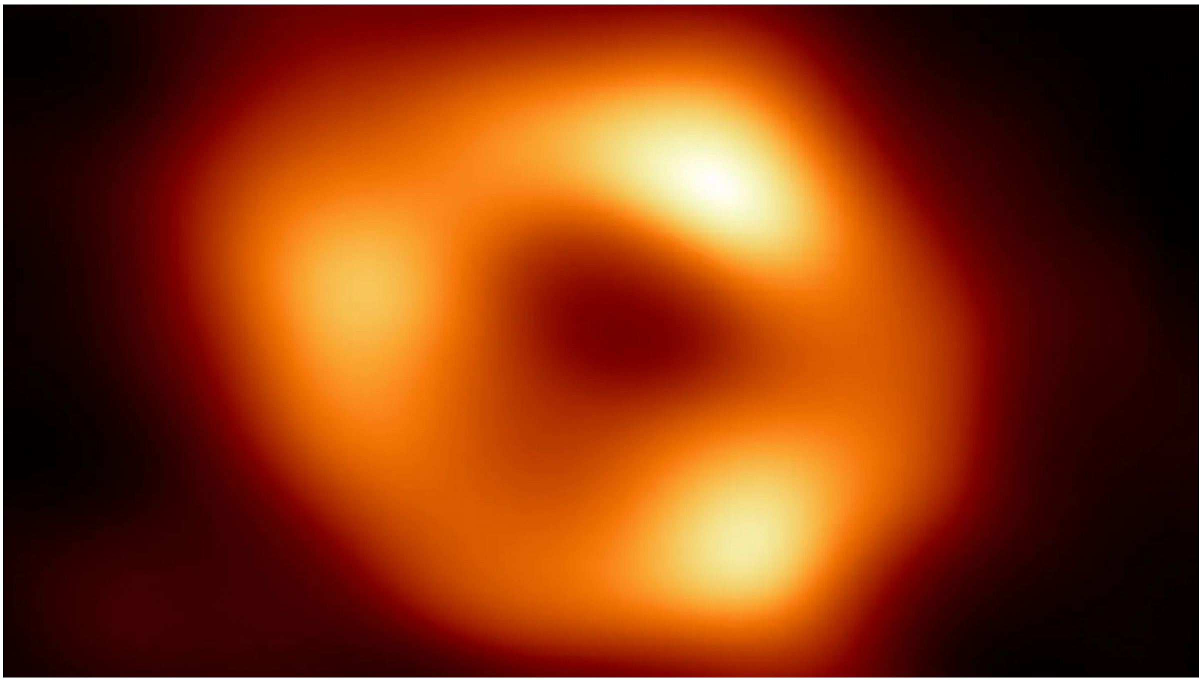


Figure 1: The first photo of superheated material surrounding the supermassive black hole in the Milky Way, Sagittarius A* [4].

Black holes (BHs) are nowadays considered fundamental to understand the history and evolution of the universe; furthermore they appear to be a valid tool for the development of a new theory that combines both gravity and quantum mechanics. Therefore, studying the physics of these peculiar objects is essential, as they seem to be a great opportunity to better understand physics itself.

A BH can be physically defined as an object with a mass and a density for which its escape velocity is greater than c, where $c = 299792458\ ms^{-1}$ indicates the speed of light. Special relativity states that the latter is the velocity limit; in order to accelerate an object at a speed $v = c$ an infinite amount of energy is required. Therefore from this basic definition of a BH it follows that nothing that gets close enough to its centre (light included) can escape it. However it is known that BHs can loose mass when virtual particles generate at the horizon; this mechanism is called Hawking radiation and it is responsible for BHs' temperature. This process will eventually lead to the evaporation of every BH in the Universe. All of these concepts will be better defined and explained during the various sections of this thesis.

The so-called "no hair theorem" tells us that all (stationary) BH are characterised by three independent parameters only: mass, charge and angular momentum. However, for a complete picture of BHs, advanced mathematical tools, like differential geometry, are required to get a complete understanding of their physical properties. Space-time is then considered as a differentiable manifold with a curvature described by the Einstein field equations, which will be introduced in Chapter 1.

Gravitation laws play a crucial role in the derivation of the various metrics which represent different geometries of BHs (see Chapter 2), however, quantum effects must be included to explain their thermodynamic behaviour. The necessary topics regarding quantum fields are presented in Chapter 3, while in Chapter 4 the focus is more on thermodynamics. Lastly, Chapter 5 covers the information conservation laws and the apparent problem of information loss due to BH evaporation. At the end are also introduced the principle of BH complementarity, which is believed to be a valid solution to the information paradox, and a relatively recent theory related with the latter: the Holographic theory.

In the whole thesis the signature of the metric will be $(1, -1, -1, -1)$.

Chapter 1

Useful concepts of general relativity

The principle of general relativity (GR) states that the laws of physics are the same in all reference frames (for all observers), thus the theory does not rely on the concept of inertial observers, unlike special relativity (SR) does, and no preferred reference frames shall exist [3]. However, all equations known in SR can be extended to curved space–time by replacing all partial derivatives with covariant derivatives; this is the so-called *minimal coupling principle* [2].

1.1 Equivalence principle

An important consideration is that gravitational interaction between two bodies cannot be made to vanish, even though gravitational effects can be eliminated by considering a free falling frame of reference.

> "Motion in a uniform gravitational field cannot be distinguished from free fall [3]."

Regarding this, when trying to describe a static BH it is useful to imagine space to be filled with static observers, called fiducial observers (or by abbreviation Fidos) [7]. Each Fido carries a clock which always record the proper time τ. If we consider a particle falling into a BH we can define another type of observers: freely falling observers (or Frefos) who follow the particle as it falls. When a classical particle falls into the BH, according to Fidos' viewpoint, the particle asymptotically approaches the horizon without ever crossing it. On the other hand, according to Frefos, they and the particle cross the horizon after a finite time. However this is not a violation of any GR principle as, once the horizon is crossed, their observations cannot be communicated to any Fido.

We will see more on this later in Chapter 2, Section 2.2.1.

1.2 Einstein's equations

In GR, the gravitational field is characterised by the metric tensor components $g_{\mu\nu}$. Einstein field equations are dynamic equations which can be derived from Hilbert's variation principle [2]

$$\delta S = 0, \quad S = S_m + \int \frac{R - 2\Lambda}{2\varkappa} \sqrt{-g} d^4x \tag{1.1}$$

where $S_m = \int L_m \sqrt{-g} d^4x$ is the action of matter; $R \equiv R^k{}_k$ is called curvature scalar and is the trace of the Ricci tensor R_{ij}, in turn related to the Riemann tensor $R^i{}_{ljk}$ by $R_{ij} = R^k{}_{ikj}$; Λ is the cosmological constant (negligible when considering local configurations); $\varkappa \equiv \frac{8\pi G}{c^4}$ (G is the gravitational constant); $g \equiv \det(g_{\mu\nu})$ and $d^4x \equiv dx^1 dx^2 dx^3 dx^4$. The second term is the action for the field (also called Hilbert action), its variation is

$$\begin{aligned}
\delta S_H &= \frac{1}{2\varkappa} \int \left[\sqrt{-g}\delta R + R\delta\left(\sqrt{-g}\right) - 2\Lambda\delta\left(\sqrt{-g}\right)\right] d^4x \\
&= \frac{1}{2\varkappa} \int \left[\sqrt{-g}\delta R + \frac{R}{2}\sqrt{-g} g^{\mu\nu}\delta g_{\mu\nu} - \Lambda\sqrt{-g} g^{\mu\nu}\delta g_{\mu\nu}\right] d^4x
\end{aligned}$$

the Jacobi's formula $\delta \det(M) = \det(M) \operatorname{Tr}(M^{-1}\delta M)$ has been used to evaluate the variation of the determinant of the metric (for its derivation see appendix A). The variation of $g^{\mu\nu}$ follows from the variation of the identity $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$. In fact we have

$$\begin{aligned}
&\delta g^{\mu\alpha} g_{\alpha\nu} + g^{\mu\alpha}\delta g_{\alpha\nu} = \delta(\delta^\mu_\nu) = 0 \\
&\Rightarrow \quad \delta g^{\mu\alpha}\delta^\beta_\alpha = -g^{\mu\alpha}\delta g_{\alpha\nu} g^{\nu\beta} \quad \Rightarrow \quad \delta g^{\mu\nu} = -g^{\mu\alpha}\delta g_{\alpha\beta} g^{\beta\nu}
\end{aligned}$$

Now we need to evaluate the variation of the curvature scalar

$$\delta R = \delta(g^{\mu\nu} R_{\mu\nu}) = \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} = -g^{\mu\alpha}\delta(g_{\alpha\beta}) g^{\beta\nu} R_{\mu\nu} + g^{\mu\nu}\delta(R^k{}_{\mu k\nu}) \tag{1.2}$$

The Ricci tensor can be expressed in terms of Christoffel symbols as

$$R^k{}_{\alpha k\beta} = \partial_\beta \Gamma^k{}_{\alpha k} - \partial_k \Gamma^k{}_{\alpha\beta} + \Gamma^\sigma{}_{\beta k}\Gamma^k{}_{\sigma\alpha} - \Gamma^k{}_{k\sigma}\Gamma^\sigma{}_{\alpha\beta} \tag{1.3}$$

its variation is then

$$\delta R^k{}_{\alpha k\beta} = \partial_\beta \delta\Gamma^k{}_{\alpha k} - \partial_k \delta\Gamma^k{}_{\alpha\beta} + \delta\Gamma^\sigma{}_{\beta k}\Gamma^k{}_{\sigma\alpha} + \Gamma^\sigma{}_{\beta k}\delta\Gamma^k{}_{\sigma\alpha} - \delta\Gamma^k{}_{k\sigma}\Gamma^\sigma{}_{\alpha\beta} - \Gamma^k{}_{k\sigma}\delta\Gamma^\sigma{}_{\alpha\beta} \tag{1.4}$$

in turn, (using the notation $g_{\mu\nu,\alpha} \equiv \partial_\alpha g_{\mu\nu}$) the variation of the Christoffel symbols can be written as

$$\begin{aligned}
\delta\Gamma^\mu{}_{\alpha\beta} &= \delta\left[\frac{1}{2} g^{\mu\nu}\left(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}\right)\right] \\
&= \frac{1}{2} g^{\mu\nu}\left(\delta g_{\nu\alpha,\beta} + \delta g_{\nu\beta,\alpha} - \delta g_{\alpha\beta,\nu}\right) - g^{\mu\rho}\delta g_{\rho\sigma}\Gamma^\sigma{}_{\alpha\beta}
\end{aligned}$$

Partial derivatives can be replaced by covariant derivatives using

$$\begin{aligned} &\nabla_\alpha \delta g_{\mu\nu} = \partial_\alpha \delta g_{\mu\nu} - \Gamma^\beta{}_{\alpha\mu}\delta g_{\beta\nu} - \Gamma^\beta{}_{\alpha\nu}\delta g_{\mu\beta} \\ \Rightarrow \quad &\delta\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\nabla_\alpha\delta g_{\beta\nu} + \nabla_\beta\delta g_{\alpha\nu} - \nabla_\nu \delta g_{\alpha\beta}\right) \end{aligned} \tag{1.5}$$

Therefore, inserting (1.5) in (1.4), we find

$$\delta R_{\alpha\beta} \equiv \delta R^k_{\alpha k\beta} = \nabla_\beta \Gamma^\mu_{\alpha\mu} - \nabla_\mu \delta\Gamma^\mu_{\alpha\beta}$$

Using this result in (1.2), we obtain

$$\begin{aligned} \delta R &= -g^{\mu\alpha}\delta(g_{\alpha\beta})g^{\beta\nu}R_{\mu\nu} - g^{\mu\nu}\left(\nabla_k\delta\Gamma^k{}_{\mu\nu} - \nabla_\nu\delta\Gamma^k{}_{\mu k}\right) \\ &= -R^{\alpha\beta}\delta(g_{\alpha\beta}) - g^{\alpha\beta}g^{\mu\nu}\left(\nabla_\alpha\nabla_\mu\delta g_{\beta\nu} - \nabla_\alpha\nabla_\beta\delta g_{\mu\nu}\right) \end{aligned}$$

We can finally write the variation of the Hilbert action as

$$\begin{aligned} \delta S_H = &\frac{1}{2\varkappa}\int\left[-\sqrt{-g}R^{\alpha\beta}\delta(g_{\alpha\beta}) + \sqrt{-g}\frac{R}{2}g^{\mu\nu}\delta g_{\mu\nu} - \sqrt{-g}\Lambda g^{\mu\nu}\delta g_{\mu\nu}\right]d^4x \\ &- \frac{1}{2\varkappa}\int\sqrt{-g}g^{\alpha\beta}g^{\mu\nu}\left(\nabla_\alpha\nabla_\mu\delta g_{\beta\nu} - \nabla_\alpha\nabla_\beta\delta g_{\mu\nu}\right)d^4x \end{aligned}$$

The second term forms a total covariant derivative as the metric (and thus also $\sqrt{-g}$) is covariantly conserved.

$$\delta S_H = \frac{1}{2\varkappa}\int\left[\left(-G_{\mu\nu} - g_{\mu\nu}\Lambda\right)\sqrt{-g}\delta g^{\mu\nu} - \nabla_\alpha\left(\sqrt{-g}V^\alpha\right)\right]d^4x \tag{1.6}$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor and $g^{\alpha\beta}g^{\mu\nu}\left(\nabla_\mu\delta g_{\beta\nu} - \nabla_\beta\delta g_{\mu\nu}\right)$ forms a contravariant vector that was named V^α. Notice that

$$\begin{aligned} \nabla_\alpha(\sqrt{-g}V^\alpha) &= \sqrt{-g}\nabla_\alpha V^\alpha = \sqrt{-g}\partial_\alpha V^\alpha + \sqrt{-g}\Gamma^\alpha{}_{\alpha\beta}V^\beta \\ &= \sqrt{-g}\partial_\alpha V^\alpha + \frac{1}{2}\sqrt{-g}g^{\alpha\nu}\partial_\beta g_{\alpha\nu}V^\beta = \partial_\alpha(\sqrt{-g}V^\alpha) \end{aligned}$$

The last term in (1.6) integrates to a boundary term, which can be canceled by placing appropriate boundary conditions. It then follows

$$\begin{aligned} &\delta S_H = \frac{1}{2\varkappa}\int\left[\left(-G_{\mu\nu} - g_{\mu\nu}\Lambda\right)\sqrt{-g}\delta g^{\mu\nu}\right]dx^4 \\ \Rightarrow \quad &\frac{\delta S_H}{\delta g_{\mu\nu}} = \frac{\left(-G_{\mu\nu} - g_{\mu\nu}\Lambda\right)\sqrt{-g}}{2\varkappa} \end{aligned}$$

Considering S_m we have

$$\frac{\delta S_m}{\delta g_{\mu\nu}} \equiv -\frac{1}{2}T_{\mu\nu}\sqrt{-g}$$

where $T_{\mu\nu}$ is the stress energy tensor (SET).

Now, from (1.1), we set $\delta S_H = -\delta S_m$, which results in Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -\varkappa T_{\mu\nu} \tag{1.7}$$

In the weak field limit the local curvature is small and we find the Newtonian potential.

An important feature of the SET is that it is covariantly conserved

$$\nabla_\mu T^{\mu\nu} = 0$$

For the electromagnetic field the SET reads

$$T^{\mu}_{\nu\; e-m} = -\frac{1}{\mu_0}\left[F_{\mu\alpha}F^{\nu\alpha} + \frac{1}{4}\delta^{\nu}_{\mu}F_{\alpha\beta}F^{\alpha\beta}\right] \tag{1.8}$$

where $F_{\mu\nu}$ is the Maxwell tensor, related to the electromagnetic potential A_μ as it follows

$$F_{\mu\nu} \equiv \partial_\nu A_\mu - \partial_\mu A_\nu$$

Before considering the spherically symmetric solution to Einstein's equations, we shall recall that three types of maximally symmetric spaces are possible in a four-dimensional pseudo-Riemannian space–time. The space of zero constant curvature, $\Lambda = 0$, is Minkowski space. The space of constant positive curvature, $\Lambda > 0$, is de Sitter (dS) space. Lastly, the space of constant negative curvature, $\Lambda < 0$, is anti de Sitter (AdS) space [2].

Chapter 2

Spherically symmetric space-time

The simplest solutions to Einstein's equation are in a static, spherically symmetric form, characterizing the gravitational fields in vacuum or in the presence of an electromagnetic field (without charges) and a cosmological constant. With the static assumption we can require the existence of a time-like Killing vector [1], in addition to three space-like Killing vectors corresponding to rotations around axes with origin at the centre of the source of the field

$$\vec{K}_t = \frac{\partial}{\partial t} \quad \text{and} \quad \vec{K}_i = \frac{d}{d\theta_i}, \quad i = 1, 2, 3$$

Spatial vectors must be conserved in time so they commute with $\vec{K}_t$: $\left[\frac{\partial}{\partial t}, \frac{d}{d\theta_i}\right] = 0$. We may therefore assume the metric is such that rotations are orthogonal to $\vec{K}_t$ and that we can use the analogue of polar coordinates on surfaces of constant r. Furthermore, in the planes where only the angular components can vary, the metric is the same as the one of a 2-sphere (see appendix B for a derivation of this metric) [3].

In general, we can write a spherically symmetric metric with exponential coefficients in the form

$$ds^2 = e^{2\gamma} c^2 dt^2 - e^{2\alpha} du^2 - e^{2\beta} d\Omega^2, \quad d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{2.1}$$

where α, β, γ are functions of the radial coordinate u and the time coordinate t. θ and ϕ are, respectively, the usual polar and azimuthal angles. We will use $e^\beta \equiv r$; identifying with r the radius of a coordinate sphere $u = const$, $t = const$. Notice that it does not measure proper spatial distance from the origin, but it is defined so that the area of the 2-sphere at r is $4\pi r^2$. It is important to stress that in curved space the spherical radius r has nothing to do with the distance from the center, as in many spherically symmetric space–times there is no center at all.

[1] A Killing vector $\vec{V}$ is defined as $\mathfrak{L}_{\vec{V}} g = 0$ (g is the metric), where $\mathfrak{L}$ is the Lie derivative [3].

To fix the choice of the radial coordinate we can postulate a relation between the functions α, β, γ or simply choose one of them (or a function of one of them) as the coordinate. In solving different problems, different variants of such coordinate conditions can be convenient.

2.1 Reissner-Nordström-de Sitter solution

In the first place we will look for a static, spherically symmetric solution, considering the gravitational fields in the presence of an electromagnetic field and a cosmological constant; this solution is called Reissner-Nordström-de Sitter solution (if one considers $\Lambda > 0$) [2]. In this configuration $T^{\mu}_{\nu\, matter} = 0$, while $T^{\mu}_{\nu\, e-m}$ is defined as in (1.8). Therefore (1.7) takes the form

$$R^{\nu}_{\mu} - \frac{1}{2} R \delta^{\nu}_{\mu} + \Lambda = -\varkappa T^{\nu}_{\mu\, e-m} \tag{2.2}$$

where the low indices have been contracted once and δ^{ν}_{μ} is the Kronecker delta. In general, the nonzero components of the Ricci tensor are

$$R^0_0 = e^{-2\gamma}\left[2\ddot{\beta} + \ddot{\alpha} + 2\dot{\beta}^2 + \dot{\alpha}^2 - \dot{\gamma}\left(2\dot{\beta} + \dot{\alpha}\right)\right] - e^{-2\alpha}\left[\gamma'' + \gamma'\left(2\beta' + \gamma' - \alpha'\right)\right] \tag{2.3}$$

$$R^1_1 = e^{-2\gamma}\left[\ddot{\alpha} + \dot{\alpha}\left(2\dot{\beta} - \dot{\gamma} + \dot{\alpha}\right)\right] - e^{-2\alpha}\left[2\beta'' + \gamma'' + 2\beta'^2 + \gamma'^2 - \alpha'\left(2\beta' + \gamma'\right)\right] \tag{2.4}$$

$$R^2_2 = e^{-2\beta} + e^{-2\gamma}\left[\ddot{\beta} + \dot{\beta}\left(2\dot{\beta} - \dot{\gamma} + \dot{\alpha}\right)\right] - e^{-2\alpha}\left[\beta'' + \beta'\left(2\beta' + \gamma' - \alpha'\right)\right] = R^3_3 \tag{2.5}$$

$$R_{01} = 2\left[\dot{\beta}' + \dot{\beta}\beta' - \dot{\alpha}\beta' - \dot{\beta}\gamma'\right] \tag{2.6}$$

where dots and primes stand for $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial r}$, respectively. For the derivations see appendix B. We can then write the curvature scalar for a static metric (time derivatives vanish)

$$\begin{aligned} R = R^0_0 + R^1_1 + R^2_2 + R^3_3 = \frac{2}{r^2} - e^{-2\alpha}\Big[&\gamma'' + \gamma'\left(2\beta' + \gamma' - \alpha'\right)2\beta'' + \\ &\gamma'' + 2\beta'^2 + \gamma'^2 - \alpha'\left(2\beta' + \gamma'\right) + 2\beta'' + 2\beta'\left(2\beta' + \gamma' - \alpha'\right)\Big] \\ = \frac{2}{r^2} - e^{-2\alpha}&\left(2\gamma'' + 2\gamma'^2 + \frac{4\gamma'}{r} - 2\gamma'\alpha' - \frac{4\alpha'}{r} + \frac{2}{r^2}\right) \end{aligned}$$

where $e^{-2\beta}$ has been written as $\frac{1}{r^2}$. Note that $e^{\beta} \equiv r \Rightarrow \beta = \ln r \Rightarrow \beta' = \frac{1}{r}, \beta'' = -\frac{1}{r^2}$. Using the so-called curvature coordinates $u = r$, $\gamma = \gamma(r)$, $\alpha = \alpha(r)$, we can write these two components of the Einstein tensor as functions of r

$$\begin{aligned} G^0_0 = R^0_0 - \frac{1}{2}\delta^0_0 R = -e^{-2\alpha}\left[\gamma'' + \gamma'\left(\frac{2}{r} + \gamma' - \alpha'\right)\right] - \frac{1}{r^2} \\ + e^{-2\alpha}\left(\gamma'' + \gamma'^2 + \frac{2\gamma'}{r} - \gamma'\alpha' - \frac{2\alpha'}{r} + \frac{1}{r^2}\right) = e^{-2\alpha}\left(\frac{1}{r^2} - \frac{2\alpha'}{r}\right) - \frac{1}{r^2} \end{aligned} \tag{2.7}$$

$$G^1_1 = R^1_1 - \frac{1}{2}\delta^1_1 R = -e^{-2\alpha}\left[-\frac{2}{r^2} + \gamma'' + \frac{2}{r^2} + \gamma'^2 - \alpha'\left(\frac{2}{r} + \gamma'\right)\right] - \frac{1}{r^2}$$
$$+ e^{-2\alpha}\left(\gamma'' + \gamma'^2 + \frac{2\gamma'}{r} - \gamma'\alpha' - \frac{2\alpha'}{r} + \frac{1}{r^2}\right) = e^{-2\alpha}\left(\frac{1}{r^2} + \frac{2\gamma'}{r}\right) - \frac{1}{r^2} \tag{2.8}$$

Using (2.7) and (2.8), equation (2.2) gives

$$G^0_0 + \Lambda = e^{-2\alpha}\left(\frac{1}{r^2} - \frac{2\alpha'}{r}\right) - \frac{1}{r^2} + \Lambda = -\varkappa T^0_0 \tag{2.9}$$

$$G^1_1 + \Lambda = e^{-2\alpha}\left(\frac{1}{r^2} + \frac{2\gamma'}{r}\right) - \frac{1}{r^2} + \Lambda = -\varkappa T^1_1 \tag{2.10}$$

We shall now write the first two Maxwell equations $\nabla_\alpha F^{\alpha\beta} = -J^\beta = 0$ ($F^{\alpha\beta}$ and J^β are respectively the Maxwell tensor and the four-current) for the spherically symmetric case. If we look at the components of $F_{\mu\nu}$, only the ones describing a radial electric field $F_{01} = -F_{10} = -E_r$ and a radial magnetic field $F_{23} = -F_{32} = B_r$ can be nonzero. For simplicity we will restrict ourselves to an electric field only, hence Maxwell equations become (only the nonzero terms are displayed)

$$\nabla_\alpha F^{\alpha\beta} = \partial_\alpha F^{\alpha\beta} + \Gamma^\alpha{}_{\alpha\mu}F^{\mu\beta} + \Gamma^\alpha{}_{\beta\mu}F^{\alpha\mu} = \partial_1 F^{10} + \Gamma^0{}_{01}F^{10} + \Gamma^0{}_{01}F^{01} + \Gamma^1{}_{11}F^{10}$$
$$= -\partial_r F^{01} - \partial_r\left(\ln\sqrt{-g}\right)F^{01} = -\partial_r F^{01} - \frac{\partial_r\left(\sqrt{-g}\right)}{\sqrt{-g}}F^{01}$$

where the formula $\Gamma^a{}_{ba} = \partial_b\left(\ln\sqrt{-g}\right)$ has been used to replace Christoffel symbols. Now setting the above expression to zero and recognising the composite derivative leads to

$$\partial_r\left(\sqrt{-g}F^{01}\right) = 0 \tag{2.11}$$

The determinant of the metric (2.1) reduces to the product of its diagonal elements:

$$g_{ij} = \begin{pmatrix} e^{2\gamma} & 0 & 0 & 0 \\ 0 & -e^{2\alpha} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \Rightarrow \det g_{ij} = -e^{2\gamma+2\alpha}r^4\sin^2\theta$$

Inserting this result in (2.11) we obtain

$$\partial_r\left[e^{\gamma+\alpha}r^2\sin(\theta)F^{01}\right] = 0 \quad\Rightarrow\quad F^{01} = \frac{\sqrt{\mu_0}Qe^{-\gamma-\alpha}}{4\pi\sqrt{\epsilon_0}r^2} \quad , \quad F_{10} = g_{00}g_{11}F^{01} = \frac{\sqrt{\mu_0}Qe^{\gamma+\alpha}}{4\pi\sqrt{\epsilon_0}r^2}$$

where the constant Q is interpreted as an electric charge and $\left(4\pi\sqrt{\epsilon_0}\right)^{-1}\sqrt{\mu_0}$ is a normalization constant. Using (1.8) we can then write the fist two components of the SET

$$T^0_0 = \frac{1}{4\mu_0}\left[-4\left(F_{01}F^{01}\right) + F_{10}F^{10} + F_{01}F^{01}\right] = \frac{Q^2}{32\pi^2\epsilon_0 r^4} \tag{2.12}$$

$$T^1_1 = \frac{1}{4\mu_0}\left[-4\left(F_{10}F^{10}\right) + F_{10}F^{10} + F_{01}F^{01}\right] = \frac{Q^2}{32\pi^2\epsilon_0 r^4} \tag{2.13}$$

As $T^0_0 = T^1_1$, if we now subtract (2.10) and (2.9) we get

$$e^{-2\alpha}\left(\frac{2\gamma'}{r}+\frac{2\alpha'}{r}\right)=0\Rightarrow\gamma'+\alpha'=0\Rightarrow e^{\alpha+\gamma}=const=1 \tag{2.14}$$

We set the arbitrary constant to 1 so that we asymptotically obtain the Minkowski metric when $r\to\infty$. Now let $A(r)=e^{-2\alpha}$, which plugged in (2.9) gives

$$\begin{aligned}
&A\left(\frac{1}{r^2}-\frac{2\alpha'}{r}\right)-\frac{1}{r^2}=-\left(\Lambda+\varkappa T^0_0\right) \quad\Rightarrow\quad \left[r\left(A-1\right)\right]'=-\left(\Lambda+\varkappa T^0_0\right)r^2\\
\Rightarrow\quad &r\left(A-1\right)=-\frac{\Lambda r^3}{3}-\varkappa\int T^0_0 r^2 dr \quad\Rightarrow\quad A=1-\frac{\Lambda r^2}{3}-\frac{2GM}{c^2}\frac{1}{r}+\frac{GQ^2}{4\pi\epsilon_0 c^4}\frac{1}{r^2}
\end{aligned}$$

where M in the integration constant is interpreted as the active gravitational mass of the gravitational field source. Note that (2.14) implies $e^{2\gamma}=A(r)$.

We can finally write the expression of the Reissner-Nordström-de Sitter metric

$$ds^2=A\left(r\right)c^2dt^2-\frac{dr^2}{A(r)}-r^2d\Omega^2,\quad A(r)=1-\frac{\Lambda r^2}{3}-\frac{2GM}{c^2}\frac{1}{r}+\frac{GQ^2}{4\pi\epsilon_0 c^4}\frac{1}{r^2} \tag{2.15}$$

For $Q=\Lambda=0$, $A(r)=1-\frac{2GM}{c^2}\frac{1}{r}$, and the Schwarzschild metric is obtained.

$$ds^2=\left(1-\frac{2GM}{c^2}\frac{1}{r}\right)c^2dt^2-\left(1-\frac{2GM}{c^2}\frac{1}{r}\right)^{-1}dr^2-r^2d\Omega^2 \tag{2.16}$$

If $M=0$ as well, then $A(r)=1$ and the Minkowski metric is reproduced.

Equation (2.16) reduces to Newton's law of gravity if we consider the geodesic equation

$$\frac{d^2x^i}{ds^2}+\Gamma^i{}_{\mu\nu}u^\mu u^\nu=0 \tag{2.17}$$

where $u^i\equiv\frac{dx^i}{ds}$ (for its derivation see appendix B). For the case of a test particle instantaneously at rest, at a large radius r (if compared to $2GM/c^2$) we have $ds=g_{00}cdt$ and $\frac{2GM}{c^2r}\approx 0$, so that

$$\begin{aligned}
&0=\frac{1}{g_{00}c^2}\frac{d^2x^i}{dt^2}+\Gamma^i{}_{00}\left(u^0\right)^2\approx\frac{\vec{a}}{c^2}+\frac{1}{c}\frac{d^2t}{dt^2}+\frac{1}{2}g^{i\alpha}\left(\partial_0 g_{\alpha 0}+\partial_0 g_{\alpha 0}-\partial_\alpha g_{00}\right)\\
\Rightarrow\quad &0=\vec{a}-\frac{c^2}{2}g^{11}\left(\partial_1 g_{00}\right)\hat{r}\approx\vec{a}-\frac{c^2}{2}\left(\frac{-2GM}{c^2r^2}\right)=\vec{a}+\frac{GM}{r^2}\hat{r}
\end{aligned}$$

Therefore the particle experiences the acceleration $-\frac{GM}{r^2}$ in the direction of smaller radii, in agreement with Newton's law of gravity $\vec{F}=\frac{GM_1M_2}{r^2}\hat{r}$.

To better describe the Reissner–Nordström space-time is useful to introduce "geometrized" mass and charge, having both the dimension of a length

$$m = \frac{GM}{c^2}, \quad q = \frac{\sqrt{G}Q}{\sqrt{4\pi\epsilon_0}c^2}$$

Then, setting $\Lambda = 0$

$$A = 1 - \frac{2m}{r} + \frac{q^2}{r^2} \quad \Rightarrow \quad A = 0 \quad \Leftrightarrow \quad r_\pm = m \pm \sqrt{m^2 - q^2}$$

The values for which $g_{00} = A = 0$ are called horizons [2]. The outer horizon (a sphere with $r = r_+$) is also called *event horizon*, while the inner one (a sphere with $r = r_-$) is the *Cauchy horizon*. At $r_- < r < r_+ \Rightarrow A < 0$, the coordinates r and t exchange their roles: the quantity r now becomes a temporal coordinate and t a spatial coordinate. Such a region is called T-region. On the other hand at $r < r_- \vee r > r_+ \Rightarrow A > 0$ and the region is called an R-region. Under the condition $m^2 = q^2$, so that $r^2 A(r) = (r - m)^2$, the two horizons merge into a single one, $r = r_h = m = |q|$; which is called a double, or extremal horizon, and the region $r \leq m$ (or sometimes the whole configuration) is called an extremal Reissner–Nordström black hole. Lastly, for "large charges," $q^2 > m^2$, the whole space $r > 0$ is occupied by a single R-region and no horizons are present.

At $r \to \infty$, the contribution of the charge can be neglected, and the metric approximately coincides with the Schwarzschild metric. In contrast, at small radii, $r \to 0$, the metric is dominated by the charge.

At $\Lambda \neq 0$, the basic properties of the metric (2.15) are again determined by the function $A(r)$ and, in particular, by the number and the disposition of its zeros, each of them corresponding to a horizon that separates R- and T-regions.

The quasiglobal coordinate $u = x$ (and $\gamma = -\alpha$) is suitable for describing BHs and other similar metrics on both sides of horizons. We can then redefine the static metric (2.1) as

$$ds^2 = A\left(x\right)c^2 dt^2 - \frac{dx^2}{A(x)} - r^2(x) d\Omega^2 \tag{2.18}$$

where $A(x) \equiv e^{2\gamma}$. Note that when we introduced the notion of horizon for the Reissner-Nordström metric, the coordinate $r = x$ was simultaneously a curvature and a quasiglobal coordinate.

It is now possible to define a horizon in any space–time with the metric (2.18) as a regular sphere $x = x_h$, near which

$$A(x) \sim (x - x_h)^n$$

where $n \in \mathbb{N}$ is the order of the horizon. Since $A(x)$ is equal to the norm $\xi^\alpha \xi_\alpha$ of the Killing vector $\xi^\alpha = (1, 0, 0, 0)$, regular surfaces where $A = 0$ are *Killing horizons* (surfaces where the timelike or spacelike Killing vector becomes null).

2.1.1 Rindler approximation

The proper distance from the horizon is measured by the coordinate ρ:

$$\begin{aligned}\rho &= \int_{2m}^{r} \sqrt{g_{rr}(r')}dr' = \int_{2m}^{r} \left(1 - \frac{2m}{r'}\right)^{-\frac{1}{2}} dr' \\ &= \sqrt{r(r-2m)} + 2m \sinh^{-1}\left(\sqrt{\frac{r}{2m-1}}\right)\end{aligned}$$

We can use this coordinate to describe the region near the horizon of a Schwarzschild BH, replacing r in (2.16) with ρ [7].

In terms of ρ and t the metric (2.16) takes the form

$$ds^2 = \left(1 - \frac{2m}{r(\rho)}\right) c^2 dt^2 - d\rho^2 - r(\rho)^2 d\Omega^2$$

Near the horizon,

$$\begin{aligned}&\rho \approx 2\sqrt{2m(r-2m)} \\ \Rightarrow \quad &ds^2 \cong \rho^2 \left(\frac{cdt}{4m}\right)^2 - d\rho^2 - r(\rho)^2 d\Omega^2\end{aligned}$$

Furthermore, in order to study a small angular region of the horizon arbitrarily centred at $\theta = 0$, we can use Cartesian coordinates

$$\begin{aligned}X &= 2m\ \theta \cos\phi = \frac{2GM}{c^2}\theta \cos\phi \\ Y &= 2m\ \theta \sin\phi = \frac{2GM}{c^2}\theta \sin\phi\end{aligned} \tag{2.19}$$

Finally, introducing ω

$$\omega = \frac{t}{4m} = \frac{tc^2}{4GM}$$

the metric then takes the form

$$ds^2 = \rho^2 c^2 d\omega^2 - d\rho^2 - dX^2 - dY^2$$

Clearly, ρ and ω are the radial and hyperbolic angle variables for an ordinary Minkowski space. Lastly, Minkowski coordinates cT, Z can be defined by

$$\begin{aligned}cT &= \rho \sinh(\omega c) \\ Z &= \rho \cosh(\omega c)\end{aligned} \tag{2.20}$$

to get (only near $r = 2m$, and only for a small angular region) the familiar Minkowski metric

$$ds^2 = c^2 dT^2 - dZ^2 - dX^2 - dY^2$$

Therefore, by means of a change of coordinates, we have demonstrated that the horizon is locally nonsingular, and, for a large BH, is almost indistinguishable from flat space-time. This approximation of the near-horizon region by Minkowski space is called the Rindler approximation.

We can now study the case of a particle falling into a BH introduced in Section 1.1 using the concepts of Fidos and Frefos. At different r values Fidos' clocks run at different proper rates. If we take a fixed r, the relation between Schwarzschild time t and the Fidos proper time $\tau = \frac{s}{c}$ is given by

$$\frac{d\tau}{dt} = \sqrt{g_{00}} = \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}}$$

Thus, the Fido near the horizon (where $r = 2m$) sees the Schwarzschild clock running at a very rapid rate. The acceleration of a Fido at proper distance ρ is given by $\frac{c^2}{\rho}$ for $\rho \ll m$.

The motion of the infalling particle near the horizon can be described by the Minkowski coordinates (cT, Z, X, Y) defined in equations (2.19) and (2.20). The dynamic of the particle is free, hence it will have a geodesic motion following a straight line

$$\frac{dZ}{d\tau} = \frac{p^Z}{M} = -\frac{p_Z}{M} \quad \text{and} \quad \frac{cdT}{d\tau} = \frac{p_T}{M}$$

M is the mass of the particle, while p_Z and p_T are the Z and T components of momentum. As the particle goes through the horizon, its momentum can be considered as constant or slowly varying. p_Z and p_T are the components seen by Frefos. The ones seen by Fidos are p_ρ and p_τ which, using (2.20), are given by

$$p_\rho = p_Z \cosh(\omega c) + p_T \sinh(\omega c)$$
$$p_\tau = p_Z \sinh(\omega c) + p_T \cosh(\omega c)$$

For $t \gg 1$ we have

$$p_\rho \approx p_\tau \approx 2p_Z e^{\omega c} = 2p_Z e^{\frac{t}{4m}c}$$

Therefore a Fido registers an exponential growth for the momentum of the infalling particle. We can also say that the proper distance of the particle from the horizon exponentially decreases with time

$$\rho_t \approx \rho_0 e^{-\frac{t}{4m}c}$$

Locally, a time dependent boost along the radial direction links the coordinates of Frefos and Fidos. It follows that Fidos see all matter affacted by Lorentz contraction, appearing arbitrarily thin as the distance from the horizon reduces. More specifically classical physics would describe this as if the particles were stored in “sedimentary” layers of diminishing thickness as they eternally sink toward the horizon. However, in the quantum world we must expect this picture to break down by the time the infalling objects reach a Planck distance from the horizon.

On the other hand for Frefos matter does not show any of this peculiar behaviours.

2.2 General form of geodesic equations

It is now appropriate to have a closer look to the motion of free particles in spherically symmetric Riemannian space-times . Their equation of motion are the geodesic equations (2.17), which have the meaning of trajectory equations, describing the extremum of the world line length between two given points [2]. It is important to remark that s is the interval which coincides either with the proper time of a Frefos, if it is timelike, or with the proper length along the geodesic, if it is spacelike. Geodesics can be spacelike, timelike and null, according to the sign of $v_\mu v^\mu$, and their nature cannot change along their path because $\partial_\alpha(v_\mu v^\mu) = 0$.

Consider the geodesic equations (2.17) as equations relative to the unknowns functions of the canonical parameter λ: $x^0 = t(\lambda), x^1 = u(\lambda), x^2 = \theta(\lambda), x^3 = \phi(\lambda)$; which are coordinates of a point along the trajectory. Thanks to the symmetry of spherical space time, we can assume that the geodesic is located in the equatorial plane $\theta = \pi/2$. Derivatives with respect to λ will be denoted by a dot, whereas the ones with respect to u will be denoted by a prime. For the metric (2.1) the geodesic equations for $x^0 = t(\lambda)$ read

$$\begin{aligned} 0 &= \frac{d^2x^0}{d\lambda^2} + {\Gamma^0}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = c\ddot{t} + c{\Gamma^0}_{01}\dot{t}\dot{u} + c{\Gamma^0}_{10}\dot{u}\dot{t} + 0 + \cdots + 0 \\ &= c\ddot{t} + 2c\left(\frac{1}{2}e^{-2\gamma}\partial_u e^{2\gamma}\right)\dot{t}\dot{u} = c\ddot{t} + 2c\gamma'\dot{t}\dot{u} \\ \Rightarrow \quad \frac{\ddot{t}}{\dot{t}} &= -2\gamma'\dot{u} = -2\dot{\gamma} \quad \Rightarrow \quad \dot{t} = \mathcal{E}e^{-2\gamma} \end{aligned} \tag{2.21}$$

Analogues calculations for $x^1 = u(\lambda)$ and $x^3 = \phi(\lambda)$ lead to

$$\ddot{u} + \gamma' e^{2\gamma-2\alpha}c\dot{t}^2 + \alpha'\dot{u}^2 - \beta' e^{2\beta-2\alpha}\dot{\phi}^2 = 0 \tag{2.22}$$

$$\ddot{\phi} + 2\beta'\dot{\phi}\dot{u} = 0 \quad \Rightarrow \quad |\dot{\phi}| = Le^{-2\beta} \tag{2.23}$$

$\mathcal{E}$ and L are integration constants.

Equations (2.21) and (2.23) are not independent due to the constraint $v_\alpha v^\alpha = k$, where $v^\mu = \frac{dx^\mu}{d\lambda}$; $k = +1$ for timelike geodesics (in this case v^μ is the 4-velocity and the parameter λ corresponds to the proper time), $k = 0$ for null geodesics, and $k = -1$ for spacelike ones. Starting from (2.1) it is possible to write this constraint in its explicit form, which represents an integral of (2.22)

$$\frac{1}{d\lambda^2}ds^2 = \frac{1}{d\lambda^2}\left(e^{2\gamma}c^2dt^2 - e^{2\alpha}du^2 - e^{2\beta}d\Omega^2\right)$$
$$\Rightarrow \quad k = e^{2\gamma}c^2\dot{t}^2 - e^{2\alpha}\dot{u}^2 - e^{2\beta}\dot{\phi}^2$$

Now we can substitute the integrals (2.21) and (2.23). After multiplying the resulting relation by $e^{2\gamma}$ we obtain

$$e^{2\gamma+2\alpha}\dot{u}^2 + ke^{2\gamma} + e^{2\gamma-2\beta}L^2 = c^2\mathcal{E}^2 \equiv E \tag{2.24}$$

The latter reminds to an energy conservation law for a particle moving in a potential field along the u axis: the quantity $E^2 \geq 0$ coincides with the total energy per unit mass, the first term is an analogue of the particle's kinetic energy while the sum of the second and third terms plays the role of a potential energy; moreover, the effective potential

$$V(u) = e^{2\gamma}\left(k + L^2e^{-2\beta}\right) \tag{2.25}$$

in geodesic motion has a similar behaviour as the potential in classical mechanics for one-dimensional motion of a point like particle. Motion is only possible in a region where $E^2 \geq V(u)$, while the values of u at which $E^2 = V$ are turning points. The constant L, related to changes in the azimuthal angle ϕ, can be interpreted as the particle's conserved angular momentum per unit mass.

If we now consider the metric (2.18), remembering the relation $A \equiv e^{2\gamma} = e^{-2\alpha}$, from (2.24) and (2.25) we get

$$\dot{u}^2 = E^2 - V(u) \tag{2.26}$$

We can then study geodesics of the metric (2.18) in the proximity of horizons. From (2.26), in the notation of the metric (where $u \equiv x$),

$$\pm\frac{d\lambda}{dx} = \left[E^2 - A\left(\frac{L^2}{r^2} + k\right)\right]^{-\frac{1}{2}} \tag{2.27}$$

In particular, as $x \to x_h$ such that $A(x_h) = 0$ at $r \neq 0$ (a possible horizon), where the quantity x_h may be finite or infinite, we have

$$d\lambda \approx \frac{dx}{E}$$

If E is finite, then near a possible horizon the coordinate x behaves like a canonical parameter for any geodesic approaching it. We can say that timelike geodesics meet an horizon in a finite proper time $\lambda \equiv \tau$ if and only if this horizon corresponds to a finite value of the quasiglobal coordinate x.

If $A \to 0$ as $x \to \infty$, such a surface may be called a remote horizon because it is reached along a geodesic at infinite proper time. According to (2.27), if $E = 0$, then in the limit $A \to 0$, if $x \to \infty$, then the canonical parameter λ along the geodesics tends to infinity, just as for nonzero values of E. Thus, we obtain the following important result for static, spherically symmetric space–time:
If $x \to \infty$ on a surface where $A \to 0$ (i.e., on a candidate horizon), then this surface (called a remote horizon) is the boundary of the space-time under consideration, and it cannot be reached by any geodesics at finite values of their canonical parameter [2].

So the space–time is complete at a remote horizon: no extension is needed beyond it. The completeness of space-time may be rephrased saying that can follow any path in any direction.

Lastly, we shall return to usual (not remote) horizons and study the time t (which is also the time according to the clocks of a distant observer at rest in the case of an asymptotically flat space–time [2]) for timelike and null geodesics approaching the horizon. Inserting (2.21) in (2.26) we get

$$\dot{u}^2 = c^2\dot{t}^2A^2 - V \quad \Rightarrow \quad \frac{\dot{u}^2}{\dot{t}^2} = A^2 - \frac{V}{\dot{t}^2} = A^2 - \frac{A^2V}{E^2}$$

From which follows that

$$\frac{dx}{dt} = \pm A\sqrt{1 - \frac{V(x)}{E^2}} \tag{2.28}$$

Therefore at the horizon, where $A = 0$, the coordinate time t is infinite for all geodesics that cross it. However, purely spatial geodesics are an exception to this, as for them $E = 0$. In particular t is equal to $+\infty$ for motion to the horizon and $-\infty$ for motion from the horizon; meaning that an horizon is in absolute past or absolute future for any observer located in the static region.

Equation 2.28 for radially moving photons gives $dx/dt = \pm A$, hence the coordinate velocity of photons (and the same for massive particles), from the viewpoint of any static observer, tends to zero as the photon approaches a horizon. It tends to zero so fast that the time t at which a photon reaches or leaves an horizon is infinite (in according with what we have seen in Section 2.1.1). A past horizon and the region beyond it can be called a *white hole*, as their features are opposite to a the ones of a BH: photons or any other particles can appear but in principle it seems to be impossible to get there from the static region, because it is located in the absolute past.

The next question is how to obtain a complete description of space-time, joining the regions separeted by horizons.

[2]As $r \to \infty$, far from gravitational field sources, usually space–time geometry coincides with the Minkowski geometry. Such space–times are called asymptotically flat.

2.3 Kruskal-Szekeres metric

In 1960, M. Kruskal and G. Szekeres independently formulated a transition from the Schwarzschild metric to coordinates providing a complete description of the spherically symmetric space–time [2].

The transformation must involve only the (r, t) subspace while the angular coordinates θ, ϕ will remain the same, so that spherical symmetry is preserved. Let

$$r_* = \int \left(1 - \frac{2m}{r}\right)^{-1} dr = r + 2m \ln\left|\frac{r}{2m} - 1\right|$$

such that $dr_* = \left(1 - \frac{2m}{r}\right)^{-1} dr$ and $dr^2 = \left(1 - \frac{2m}{r}\right)^2 dr_*^2$. Hence (2.16) becomes

$$ds^2 = \left(1 - \frac{2m}{r}\right)(c^2dt^2 - dr_*^2) - r^2 d\Omega^2 \tag{2.29}$$

Next, it is convenient to pass on to the null coordinates V, W:

$$2ct = V + W, \quad 2r_* = V - W$$

such that $cdt = \frac{1}{2}(dV + dW)$ and $dr_* = \frac{1}{2}(dV - dW)$, so $c^2dt^2 - dr_*^2 = dVdW$. Thus (2.29) can be rewritten as

$$ds^2 = \left(1 - \frac{2m}{r}\right) dVdW - r^2 d\Omega^2 \tag{2.30}$$

The last change of coordinates is

$$V = 4m \ln|v|, \quad W = -4m \ln|w|$$

which has the following implications:

$$dV = \frac{4m}{v}dv, \quad dW = -\frac{4m}{w}dw \quad \Rightarrow \quad dVdW = -\frac{16m^2}{vw}dvdw$$

$$2r_* = 4m \ln|v| + 4m \ln|w| = 4m \ln|vw| \quad \Rightarrow \quad vw = -e^{\frac{r_*}{2m}}$$

The sign of $|vw|$ has been chosen to be negative in order to obtain a regular metric.

$$\text{with} \quad e^{\frac{r_*}{2m}} = e^{\frac{r}{2m}}\left|\frac{r}{2m} - 1\right| = e^{\frac{r}{2m}}\left|\frac{r - 2m}{2m}\right| \quad \text{and} \quad \left(1 - \frac{2m}{r}\right) = \frac{r - 2m}{r}$$

It is now easy to see that the metric (2.30) becomes

$$ds^2 = \frac{32m^3}{r}e^{-\frac{r}{2m}}dvdw - r^2 d\Omega^2$$

We can also further transform the null coordinates into spatial (R) and temporal ($\mathcal{T}$) ones by assuming

$$m \cdot v = c\mathcal{T} + R \quad , \quad m \cdot w = c\mathcal{T} - R$$

which implies $dvdw = c^2 d\mathcal{T}^2 - dR^2$, whence

$$ds^2 = \frac{32m}{r} e^{-\frac{r}{2m}} (c^2 d\mathcal{T}^2 - dR^2) - r^2 d\Omega^2 \tag{2.31}$$

This is the Kruskal metric, which allows us to present Schwarzschild space-time in a single, complete picture. Notice that now only one singularity in $r = 0$ has remained, meaning the sphere-shaped singularity of the Schwarzschild metric was only dictated by the choice of coordinates.

An important remark is that Kruskal metric can be misleading when applied to describe observations made by distant observers who have stayed outside the horizon for all the entire history of the BH. For these purposes, Schwarzschild coordinates, which only cover the exterior of the horizon, are in many ways more reliable [7].

2.3.1 Kruskal diagrams

We are now able to represent the whole Schwarzschild space-time in a diagram [2]. Starting from this relations between $r, v, w, \mathcal{T}, R$:

$$-e^{\frac{r}{2m}} \frac{r - 2m}{2m} = vw = \frac{c^2 \mathcal{T}^2 - R^2}{m^2}$$

we can plot every value of r (except $r = \infty$) in a $R\mathcal{T}$ plane. The initial region $r > 2m$ is mapped into the right quadrant $v > 0, w < 0$, or $R > 0$, $-R < c\mathcal{T} < R$ (region I, see Fig. 2.1). The horizon $r = 2m$ is now depicted by the pair of intersecting axes $v = 0$ and $w = 0$, or, equivalently, $c\mathcal{T} = \pm R$. These axes (or these null hypersurfaces if we recall that each point on the diagram corresponds to a 2-sphere) do not belong to any of the R- or $\mathcal{T}$-regions, and the same is true for their intersection point $u = v = 0$. The singularity $r = 0$ is depicted by two hyperbolas: $c\mathcal{T} = -\sqrt{1 + R^2}$ (the past singularity) and $c\mathcal{T} = +\sqrt{1 + R^2}$ (the future singularity). The future singularity is the place where everything goes after crossing the future event horizon. Furthermore, there are two $\mathcal{T}$-regions $\mathcal{T}_-$ and $\mathcal{T}_+$ (the first one is obtained by transition through the future horizon, the second one through the past horizon), and one more R-region, R_-.

For any signal is impossible to ever escape to region I. The only exception is for region IV, whose points can communicate with region I, but the opposite is not allowed. No observer who stays outside $r = 2m$ can ever be influenced by events in region II. For this reason the latter is said to be behind the horizon. These properties are usually described by saying that regions II and III are behind the future horizon while regions III and IV are behind the past horizon [7].

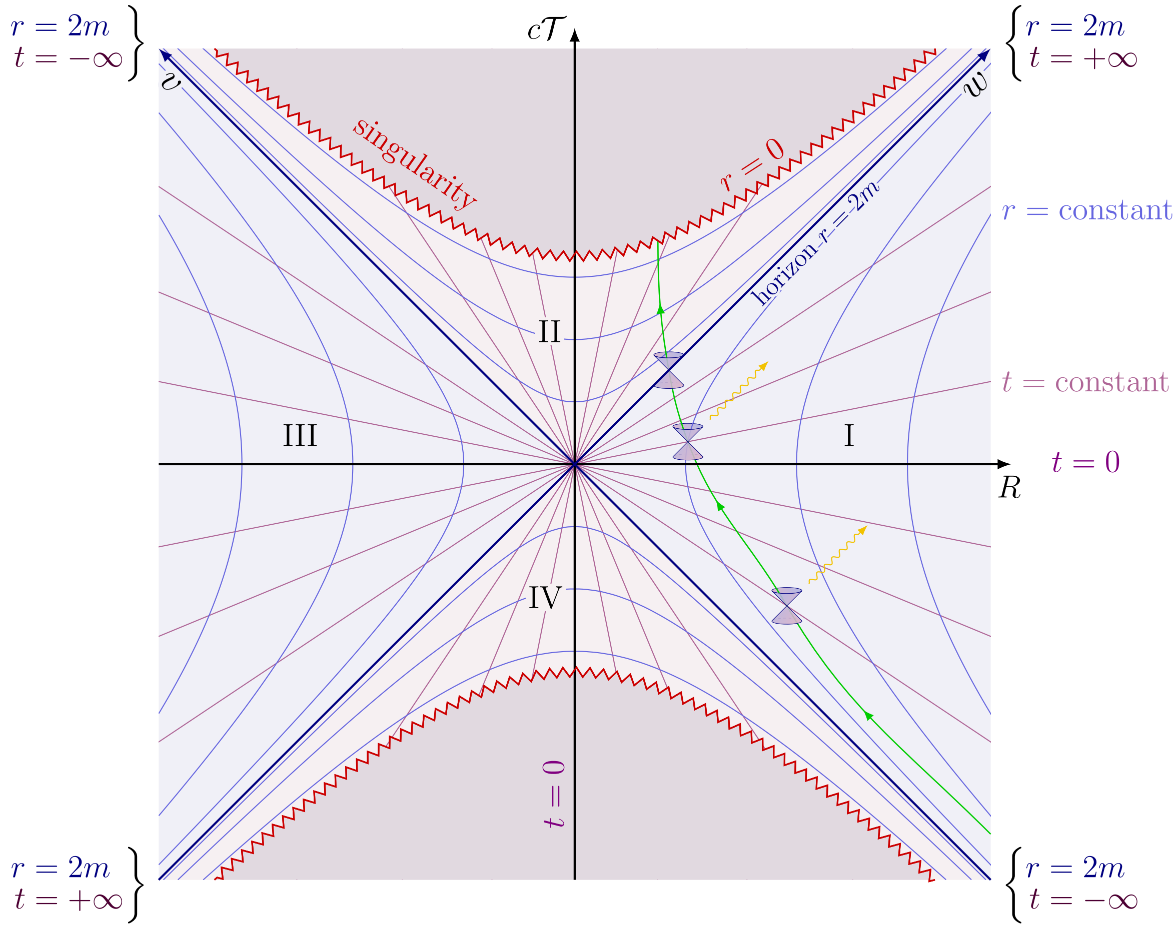


Figure 2.1: Kruskal diagram for a Schwarzschild BH (this plot has been realized based upon the work of Izaak Neutelings on Kruskal diagrams of Schwarzschild BHs [9]). The singularity is represented by a red zigzag line, while the green line in the right side of the picture is a path of a light source falling into the (future) singularity. The future singularity is the upper one, while the other is the past singularity. The yellow lines are photons and the three cones are the light cones of the light source at three different position in space-time. The four regions indicated by roman numbers represent: the outside of the BH (I), the inside of the BH (II), the outside of the BH in a parallel universe (III), the inside of a white hole (IV).

2.3.2 Carter Pensrose diagrams

For the purpose of studying the causal structure of space-time containing Killing horizons, it is often helpful to use Kruskal-like coordinate transformations, providing smooth transitions between horizons [2]. The results are diagrams that, if compared with the usual Kruskal diagram, are more convenient, because each R- or $\mathcal{T}$-region is depicted by a square or triangle of fixed finite size.

We will use the coordinates ξ and η, specified on the segment $(-\pi/2, \pi/2)$,

$$v = \tan\xi, \quad w = \tan\eta$$

The following relation between the variables follows:

$$-e^{\frac{r}{2m}}\frac{r-2m}{2m} = vw = \tan\xi\tan\eta$$

It is easily seen (Fig. 2.2) that the horizon is now represented by the two segments $\xi = 0$, $\eta = 0$ (BE and CF), and the singularity $r = 0$ by other two segments where $\xi+\eta = \pm\pi/2$ (BC and EF). The values $\xi = \pm\pi/2$ and $\eta = \pm\pi/2$ (the line strings FAB and CDE) correspond to an infinite radius r. Therefore the whole two-dimensional (r, t) manifold has been mapped into the interior of the hexagon ABCDEF.

2.4 An outline of rotating black holes

Until now, we have been discussing only spherically symmetric BHs, whose angular momentum is zero. Now, we will very briefly describe the basic properties of rotating BHs [2]. For simplicity we will assume them to be electrically neutral; which is reasonable as in astrophysical conditions their charge may be neglected most of the times. Rotating BHs form as a result of the gravitational collapse of rotating bodies (but practically all celestial bodies are rotating) and at BHs mergers. Stationary rotating BHs in GR are described by the Kerr metric

$$\begin{aligned} ds^2 = &\left(1-\frac{2mr}{\varrho^2}\right)c^2dt^2 + \frac{4cmra^2\sin^2\theta}{\varrho^2}d\phi dt - \frac{\varrho^2}{\Delta}dr^2 \\ &- \varrho^2 d\theta^2 - \left(r^2+a^2+\frac{2mra^2}{\varrho^2}\right)\sin^2\theta d\phi^2 \end{aligned} \tag{2.32}$$

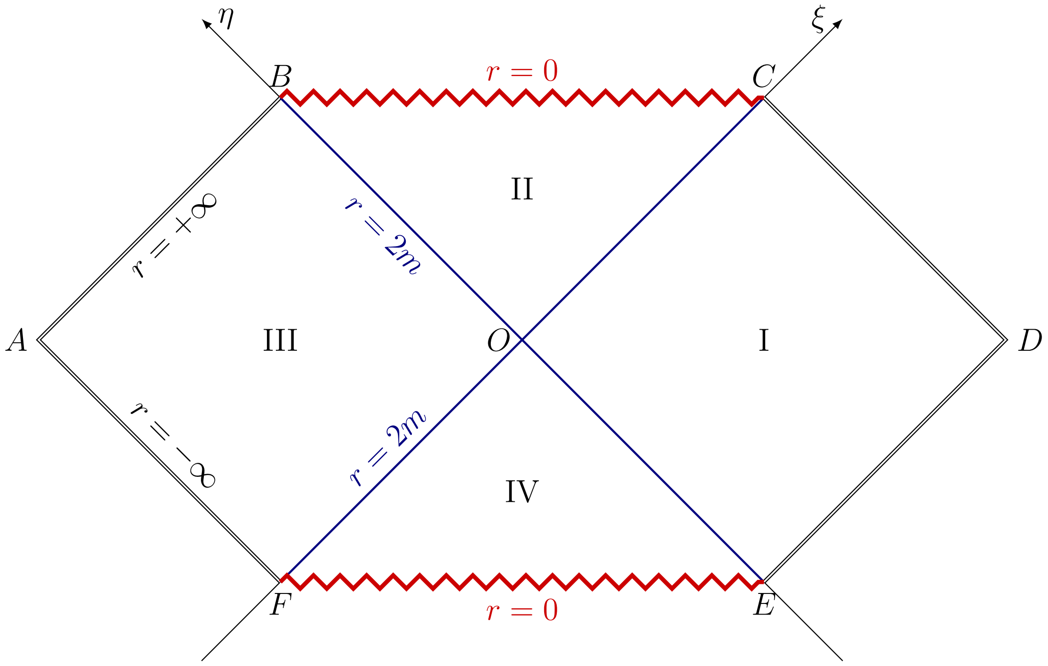
where

$$a = \frac{J}{Mc}, \quad \varrho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2mr + a^2$$

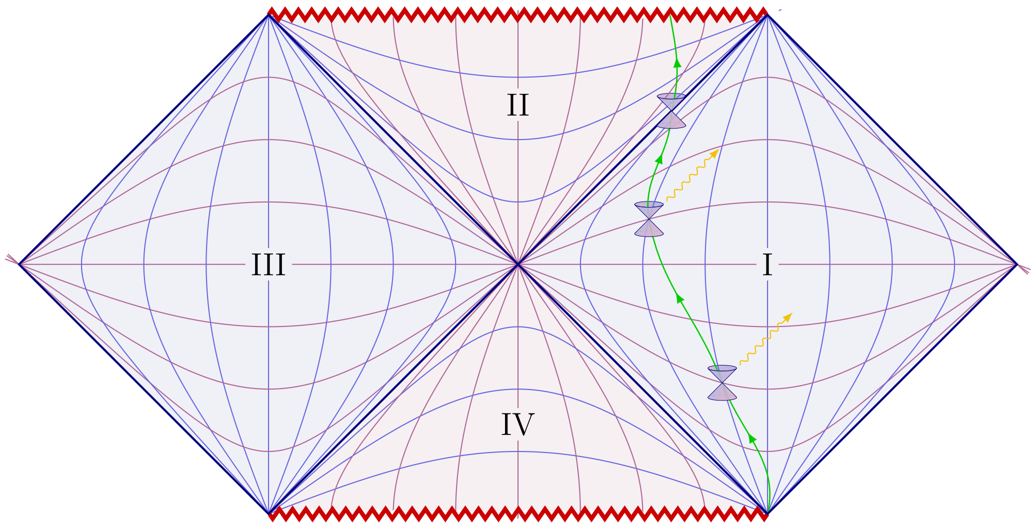
Here, (t, r, θ, ϕ) are the so-called Boyer–Lindquist coordinates. The metric (2.32) represents an asymptotically flat (as $r \to \infty$) stationary, axially symmetric solution to the Einstein equations in vacuum ($T^\mu_\nu \equiv 0$). Note that at $J = 0$ it reduces to the Schwarzschild solution, as J is the angular momentum of the BH. A true curvature singularity occurs at $\varrho = 0, r = 0, \theta = \pi/2$, and an inspection shows that this singularity has the shape of a ring.

The surfaces where $\Delta = 0$, i.e.,

$$r_\pm = m \pm \sqrt{m^2 - a^2}$$



(a) Schematic Carter-Penrose diagram for the Schwarzschild metric.



(b) Carter-Penrose diagram for Schwarzschild BH with lines of constant time and space (this plot has been realized based upon the work of Izaak Neutelings on Carter-Penrose diagrams of Schwarzschild BHs [10]).

Figure 2.2: The Carter–Penrose diagram for the Schwarzschild metric [2]. The objects in the two diagrams are analogous to the ones in Fig. 2.1.

are horizons. In the case of $m \neq a$ there are two simple horizons while for $m = a$ there is only one extremal (double) horizon.

Using (2.32) we can evaluate the norm of the killing vector $\xi^\mu \equiv (1, 0, 0, 0)$

$$\xi_\mu \xi^\mu = g_{00} = 1 - \frac{2mr}{\varrho^2} = \frac{r^2 + a^2 \cos^2\theta - 2mr}{\varrho^2} = \frac{\Delta - a^2 \sin^2\theta}{\varrho^2}$$

and we see it vanishes on the surfaces where

$$\Delta = a^2 \sin^2\theta \tag{2.33}$$

i.e., outside the horizon (with the surface $\Delta = 0$, i.e., the horizon $r = r_+$, it has only two common points: the poles $\theta = 0$ and $\theta = \pi$). The surface (2.33) is called the stationarity limit because for any observer on or inside this surface is impossible to remain stationary. Besides, it is a surface of infinite redshift for photons emitted from it to spatial infinity. The region of space between the horizon $r = r_+$ and the surface (2.33) is called *ergosphere* (see Fig. 2.3).

This region contains orbits of massive particles with a negative total energy (the absolute value of the binding energy exceeds the particle mass). Therefore when an object is launched into the ergosphere, there it will emit particles with negative energy and return to the usual spatial region with a larger energy than it initially had.

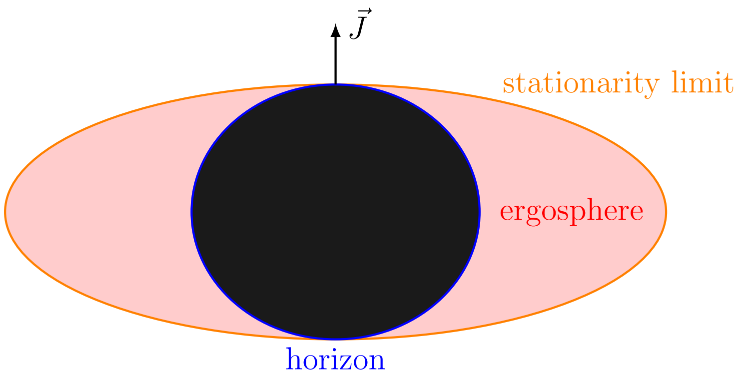


Figure 2.3: A representation of a Kerr BH [7]. The surface in orange is the stationarity limit, while the red volume is the hergosphere. As usual the surface in blue corresponds to the horizon. $\vec{J}$ is the angular momentum of the BH. Notice that the system has now lost its spherical symmetry.

Chapter 3

Quantum fields in Rindler space

According to the equivalence principle, the study of a phenomenon in a gravitational field can be equivalently carried out in an accelerated coordinate system. As we have seen in Section 2.1.2, a uniformly accelerated frame is relativistically analogue to a Rindler space [7].

Recalling equation (2.20) we can visualize the relation between Minkowski and (ρ, ω) coordinates with the diagram shown in Fig. 3.1.

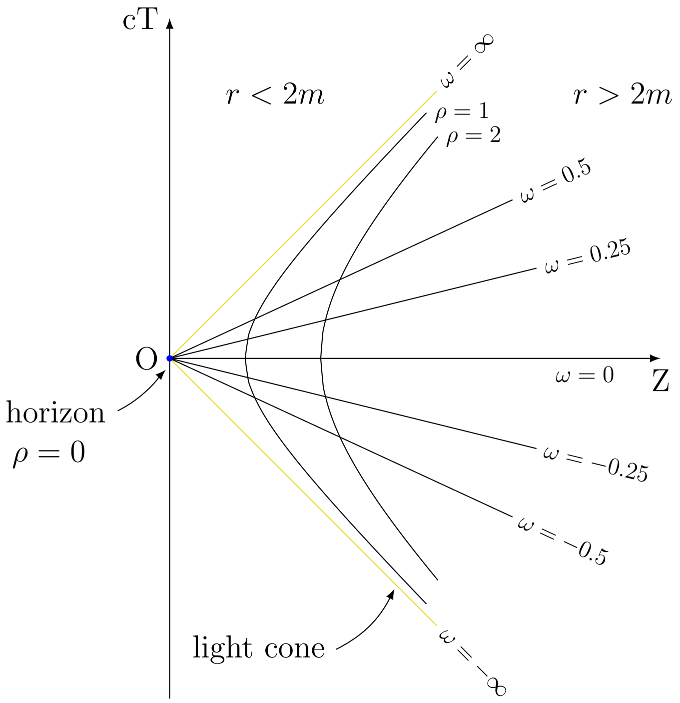


Figure 3.1: Equal time (ω) and proper distance (ρ) surfaces in Rindler space.

As a matter of fact it is easy to show that $T = \pm\sqrt{Z^2 - \rho^2}$ and $T = Z \tanh(\omega c)$.

To evolve between surfaces of constant ω we use the Rindler Hamiltonian. The generator of ω-translations is the usual Hamiltonian density corresponding to the Minkowski observer, which is given by

$$H_R = \int_{\rho=0}^{\infty} d\rho dX dY \rho T^{00}(\rho, X, Y) \tag{3.1}$$

where T^{00} is the $(0,0)$ component of the SET (an energy density). Note that the proper time separation between the surfaces is

$$\delta\tau = \rho\delta\omega$$

Therefore a ρ-dependent time translation is needed in order to transform a ω-surface in time. This why a factor ρ appears attached to T^{00} in (3.1).

Placing ourselves in the viewpoint of the Fidos in Rindler space, we aim to obtain a description of the usual physics of a quantum field in Minkowski space.

First recall that in the usual vacuum state, the correlation between fields at different spatial points does not vanish. For example, in free massless scalar theory the equal time correlator is given by

$$\langle 0|\chi(X, Y, Z)\chi(X', Y', Z')|0\rangle \sim \frac{1}{\Delta^2} \tag{3.2}$$

where Δ is the space-like separation between the points (X, Y, Z) and (X', Y', Z')

$$\Delta^2 = (X - X')^2 + (Y - Y')^2 + (Z - Z')^2$$

When two subsystems become correlated, they are said to be quantum entangled, so that neither can be described in terms of pure states. An appropriate study of pure states and entangled subsystems is carried on by means of the density matrix.

3.1 The density matrix

Usually, a state of maximal information is called a pure state and is identified by the coefficients c_n of the expansion of its state vector ψ into eigenvectors u_n [5]:

$$\psi = \sum_n c_n u_n$$

For a system like this we can say that the mean value of the operator Q, identified by the matrix Q_{nn}, is

$$\langle Q\rangle = \sum_{n'n} Q_{n'n} {c_{n'}}^* c_n$$

Now suppose a nonpure state represented by an incoherent superposition of pure states $\psi^{(i)}$, normalized by statistical weights $p^{(i)}$. In this state, the mean value for the operator is given by the average between pure states:

$$\langle Q\rangle = \sum_i p^{(i)} \langle Q\rangle_i = \sum_{nn'} Q_{n'n} \sum_i p^{(i)} c_{n'}^{(i)*} c_n^{(i)}$$

It is now intuitive to define a matrix, called the density matrix, as

$$\rho_{nn'} \equiv \sum_i p^{(i)} c_{n'}^{(i)*} c_n^{(i)}, \quad \rightarrow \quad \langle Q\rangle = \sum_{n'n} Q_{n'n}\rho_{nn'} = \sum_{n'} (Q\rho)_{n'n'} = \mathrm{Tr}\, Q\rho \tag{3.3}$$

Therefore, *the density matrix represents a minimum set of input data which serves to calculate the mean value of any operator Q for a system* [5].

We can now study a system composed by two subsystems, A and B, which have previously been in contact but are no longer interacting. In these conditions the combined system is described by a wavefunction [7]

$$\Psi = \Psi(\alpha, \beta)$$

where α and β are appropriate commuting variables for the subsystems A and B.

We can then use the density matrix $\rho_A(\alpha, \alpha')$ to describe the measurements of A.

$$\rho_A(\alpha, \alpha') = \sum_\beta \Psi^*(\alpha, \beta)\Psi(\alpha', \beta)$$

Similarly, a complete description of experiments performed on B is provided by $\rho_B(\beta, \beta')$.

$$\rho_B(\beta, \beta') = \sum_\alpha \Psi^*(\alpha, \beta)\Psi(\alpha, \beta')$$

Density matrices have the following properties:

- 1) $\mathrm{Tr}\,\rho = 1$
- 2) $\rho = \rho^\dagger$
- 3) $\rho_j \geq 0$ (all eigenvalues are positive or zero)

In according with (3.3), the expectation value of an operator Q composed of Q degrees of freedom (dof) is given by the formula

$$\langle Q\rangle = \mathrm{Tr}\, Q\rho_A$$

Property 1) can be interpreted as the request that the total probability equals one and it can be demonstrated requiring that the unit operator $\mathbb{1}$ has the mean value 1

$$\mathrm{Tr}\, \mathbb{1}\rho = \mathrm{Tr}\,\rho = 1$$

Hence the eigenvalues ρ_j can be considered to be the probabilities that the system is in the j^{th} state. It is important to focus on the case in which only one eigenvalue ρ_j is nonzero. For this situation to take place we need an uncorrelated product wave function of the form

$$\Psi(\alpha,\beta) = \psi_A(\alpha)\psi_B(\beta) \tag{3.4}$$

This is a special case in which the density matrix describes a pure state. (The eigenvalues of the matrix are invariant under unitary transformations [5]).

An expression for the departure from a pure state is provided by the Von Neumann entropy

$$S = -\operatorname{Tr}\rho\log\rho = -\sum_j \rho_j \log\rho_j$$

For S to be zero all the eigenvalues but one must be zero; the opposite is also true. Furthermore, the only nonzero eigenvalue must be equal to 1 due to condition 1). This entropy is also called the *entropy of entanglement* (or *entanglement entropy*, as we will see in Chapter 5) because it measures the degree of entanglement between A and B.

S has its minimum for a completely incoherent density matrix in which all the eigenvalues are equal to $\frac{1}{N}$, where N is the dimensionality of the Hilbert space. Physically, this matrix describes the opposite of a pure state. In that case

$$S_{max} = -\sum_j \frac{1}{N}\log\frac{1}{N} = \log N$$

In general, when ρ is a projection operator onto a subspace of dimension n, we find

$$S = \log n$$

An example of this is presented at page 46. We therefore identify the Von Neumann entropy (which is profoundly different from the thermal entropy of the second law of thermodynamics) with a measure of the number of states which have an appreciable probability in the statistical ensemble.

From statistical mechanics we know that the state of a system at a temperature T is represented by the incoherent superposition of eigenstates of energy E_m with weights proportional to $e^{-\frac{E_m}{k_B T}}$ (k_B is the Boltzmann constant) [5]. To be sure that the sum of the weights for all eigenstates equals 1, the weight of each state must equal $e^{-\frac{E_m}{k_B T}}$ divided by the "sum of states" $Z(T) = \sum_m e^{-\frac{E_m}{k_B T}}$ (the Maxwell-Boltzmann partition function).

Therefore, a system with Hamiltonian H held in thermal equilibrium at temperature $T = \frac{1}{k_B\beta}$ it is described by a Maxwell–Boltzmann density matrix diagonal in the scheme of energy eigenstates:

$$\rho_{M.B.} = \frac{e^{-\beta H}}{\operatorname{Tr} e^{-\beta H}} \tag{3.5}$$

In this case the thermal entropy is given by

$$S_{thermal} = -\operatorname{Tr}\rho_{M.B}\log\rho_{M.B} \tag{3.6}$$

3.1.1 The Unruh density matrix

Our aim is to find the density matrix describing the Fidos in Region I of Fig. 2.2 [7].

The surface $T = 0$ of Minkowski space can be divided into two halves, one in Region I ($Z > 0$) and one in Region III($Z < 0$). Now consider a scalar field χ, at each point of space a complete set of commuting operators is formed by the fields. These latter may be decomposed into two subsets associated with regions I and III, χ_R and χ_L respectively:

$$\chi(X, Y, Z) = \chi_R(X, Y, Z) \quad \text{for} \quad Z > 0$$

$$\chi(X, Y, Z) = \chi_L(X, Y, Z) \quad \text{for} \quad Z < 0$$

The wave function associated with the system is a functional of χ_L and χ_R

$$\Psi = \Psi(\chi_L, \chi_R)$$

Knowing this, we want to compute the density matrix ρ_R, as it represents the usual Minkowski vacuum to Fidos in Region I.

The state Ψ is invariant for translations with Minkowski coordinates. Therefore for Fidos vacuum must appear to be invariant under translations along the X and Y axes. The Z axis does not enjoy this property due to the act of singling out the origin $Z = 0$ for special consideration. Using this considerations we can conclude that ρ_R commutes with the components of momentum in these directions

$$[p_X, \rho_R] = [p_Y, \rho_R] = 0 \tag{3.7}$$

A very important property of ρ_R is its invariance under Rindler time translations $\omega \to \omega + \textit{constant}$. This follows from the Lorentz boost invariance of Ψ. Thus

$$[H_R, \rho_R] = 0 \tag{3.8}$$

We can proceed further only using the fact that $\Psi = \Psi(\chi_L, \chi_R)$ is the ground state of the Minkowski Hamiltonian. In order for the latter to be computed, we may realy on general path integral methods. The calculation (for the full derivation see [7] page 37) was first carried out by William Unruh in 1976, leading to the following result:

$$\rho_R = \frac{1}{Z} e^{-2\pi(\hbar c)^{-1} H_R} \tag{3.9}$$

Therefore for Fidos the vacuum is a thermal ensemble with a density matrix of the Maxwell–Boltzmann type. From (3.9) we can extract the temperature of the ensemble, which is called *Rindler temperature*:

$$T_R = \frac{\hbar c}{2\pi k_B} \tag{3.10}$$

3.2 Proper temperature

Note that in (3.10) T_R has dimensions of energy multiplied by a length (while ordinarily, temperature has units of energy). However, this should not alarm us because it is due to the dimensionality of ω (time over length), which in turn reflects on the dimensions of H_R as we have already discussed at the beginning of this chapter. Nevertheless we should be able to assign to each Fido a standard temperature that would be recorded by its own standard thermometer [7].

We can consider the thermometer as a system which, when at rest, is characterised by a set of proper energy levels ϵ_i. The interaction between the thermometer and the quantum fields is assumed to be negligible, so that the former will eventually come to thermal equilibrium with the latter. Without loss of generality, we can suppose that the thermometer is at rest with respect to the Fido at position ρ, which means it has a proper acceleration $\frac{c^2}{\rho}$. The Rindler energy in (3.1) is then modified by the thermometer in such a way

$$H_R(thermometer) = \sum_i \rho \left|i\right\rangle \left\langle i\right| \epsilon_i$$

Hence we can say that the Rindler energy level of the ith state of the thermometer is $\rho\epsilon_i$.

When equilibrium is reached between the quantum field at Rindler temperature $\frac{\hbar c}{2\pi k_B}$ and the thermometer, the probability to find the latter excited in the i^{th} level is given by the Boltzmann factor

$$P_i = \frac{e^{-2\pi(\hbar c)^{-1}\rho\epsilon_i}}{\sum_j e^{-2\pi(\hbar c)^{-1}\rho\epsilon_i}}$$

Accordingly, the thermometer registers a proper temperature (sometimes called *Unruh Temperature*)

$$T(\rho) = \frac{\hbar c}{2\pi k_B \rho} = \frac{1}{\rho} T_R \tag{3.11}$$

Therefore each Fido experiences a thermal environment distinguished by a temperature which increases as it moves towards the horizon at $\rho = 0$. It is noteworthy that now the correct energy dimensionality has been recovered.

3.2.1 Virtual vacuum fluctuations

We now want to investigate the origin of these thermal fluctuations felt by the Fidos, which may look ambiguous considering that we are studying the Minkowski space vacuum. This phenomenon is in fact caused by the well known virtual vacuum fluctuations, experienced by accelerated systems.

Fig. 3.2 shows ordinary vacuum fluctuations in a Rindler coordinate frame of reference. The virtual loop (a) in Region I can be thought of as a conventional fluctuation described by the quantum Hamiltonian H_R. The loop (b) contained in Region III obviously has no physical meaning for the Fidos in Region I. Finally, the most interesting are the fluctuations like (c), because they are partially contained both in Region I and III, leading to nontrivial entanglements between the dof χ_L and χ_R, which in turn are responsible for the mixed state of the density matrix in region I.

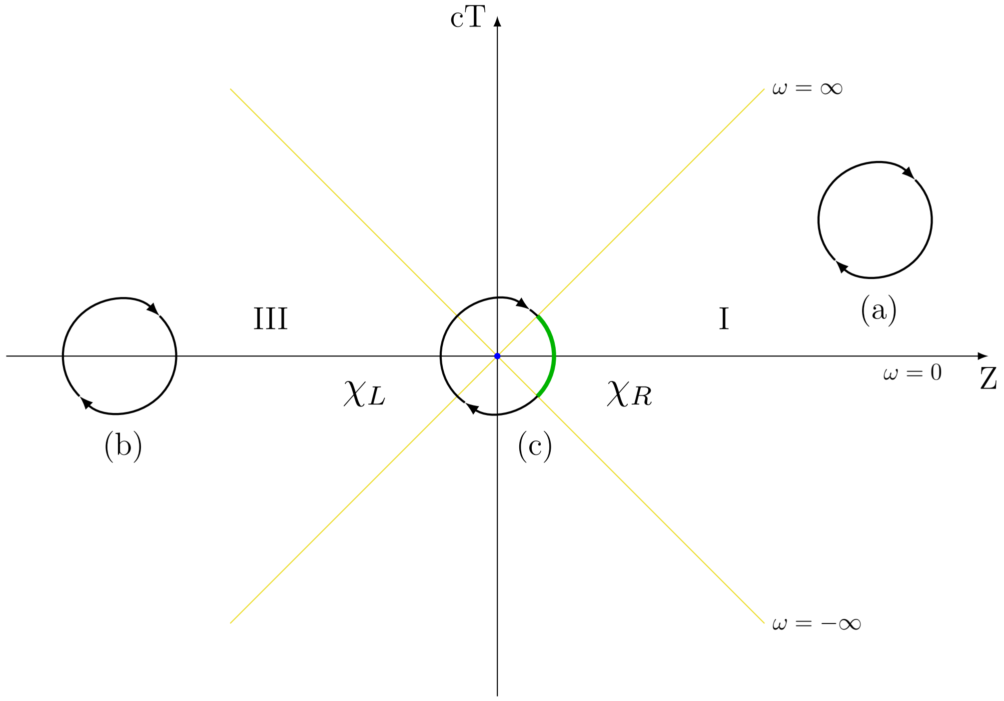


Figure 3.2: Vacuum pair fluctuations near the horizon [7].

Virtual fluctuations usually last for a very short period of time, as they are in some way violating the principle of energy conservation. For instance, a virtual fluctuation that uses an amount E of energy to produce a pair of particles will have a lifetime $\sim E^{-1}$.

Finally, consider the portion of the loop (c) which is found in Region I (highlighted in green in the diagram). Fidos see a particle appearing at $\omega = -\infty$, $\rho = 0$ and then, after travelling inside Region I, they see it disappearing at $\rho = 0$ and $\omega = +\infty$. Therefore Fidos experience a (non) virtual fluctuation that lasts for an infinite amount of time, as if the horizon was a hot membrane radiating and reabsorbing thermal energy.

This being said, we have to underline that a thermometer in the Minkowski vacuum will record zero temperature if held at rest in an inertial frame of reference. We can conclude that every physical phenomena Fidos will ever experience it will be associated with a local proper temperature $T(\rho) = \frac{\hbar c}{2\pi k_B \rho}$. On the other hand, Frefos will see only the zero temperature vacuum state.

The nature of this apparent paradox will be further discussed in the following section.

3.3 Entropy of a Free Quantum Field in Rindler Space

Consider now the region near a horizon described by a Fido. In this background we want to study all the physical phenomena which occur in the range from 0 K to Planck temperature. For the sake of brevity and simplicity, we will limit ourselves to the case in which only a single free field is present in a fixed space-time framework (beware that by doing this we will encounter paradoxes and contradictions regarding BHs, quantum mechanics, and statistical thermodynamics, as a consequence of such simplification) [7].

As we have seen in Section 2.2.1, near the horizon the Rindler metric furnishes a good description of the exterior of a BH

$$ds^2 = \rho^2 c^2 d\omega^2 - d\rho^2 - dX^2 - dY^2$$

For later convenience we will replace ρ by a tortoise-like coordinate which goes to $-\infty$ at the horizon: $u = m \log \frac{\rho}{m}$, so that $d\rho = du \frac{\rho}{m}$, and the metric becomes

$$ds^2 = m^2 e^{\frac{2u}{m}} \left[c^2 d\omega^2 - \frac{1}{m^2} du^2 \right] - dX^2 - dY^2$$

Consider a conventional massless free Klein–Gordon field χ in the Schwarzschild background, the action for χ is

$$I = \frac{1}{2} \int \sqrt{-g} g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi d^4 x$$

$g \equiv \det g_{\mu\nu}$ is just $e^{2u} \cdot (-e^{2u}) \cdot (-1) \cdot (-1) = -e^{4u}$, while $g^{00} = -g^{11} = e^{-2u}$ and $g^{22} = g^{33} = -1$. From now on the constants c and m have been tacitly included inside ω and u (this notation will be adopted until the end of the current chapter). Therefore we obtain

$$I = \frac{1}{2} \int dX dY du d\omega \left[\left(\frac{\partial \chi}{\partial \omega} \right)^2 - \left(\frac{\partial \chi}{\partial u} \right)^2 - e^{2u} \left(\partial_\perp \chi \right)^2 \right]$$

where $\partial_\perp \chi = (\partial_X, \partial_Y)$.

Near the horizon χ can be decomposed into transverse plane waves with transverse wave vector $k_\perp$

$$\chi = \int d^2 k_\perp e^{i k_\perp x_\perp} \chi(k_\perp, u, \omega)$$

the action for a given wave number k is

$$I = \frac{1}{2} \int du d\omega \left[(\partial_\omega \chi)^2 - (\partial_u \chi)^2 - k^2 e^{2u} \chi^2 \right] \tag{3.12}$$

Thus the potential is

$$V(k,u) = k^2 e^{2u} \tag{3.13}$$

Using the principle of least action, from (3.13) we obtain the equation of motion

$$\frac{\partial^2 \chi}{\partial \omega^2} - \frac{\partial^2 \chi}{\partial u^2} + k^2 e^{2u} \chi = 0$$

A solution which behaves like $e^{i\nu t}$ in Schwarzschild time has the form

$$e^{i\nu(4MG\omega)} = e^{i\lambda\omega}$$

We can then evaluate $\frac{\partial^2 \chi}{\partial \omega^2} = -\lambda^2 e^{i\lambda\omega} = -\lambda^2 \chi$, hence the time independent form of the equation of motion is

$$-\frac{\partial^2 \chi}{\partial u^2} + k^2 e^{2u} \chi = \lambda^2 \chi$$

From this equation we see the remarkable fact that, unless $k = 0$, quantum particles experience a potential confining them in a neighbourhood of the horizon. However, for the BH background, the potential barrier is cut off when $\rho = me^u$ is greater than m. On the other hand, in the Rindler approximation V increases as e^u without bound.

In order to quantize the field χ it is necessary to provide a boundary condition when $\rho \to 0$ and consequently $u \to -\infty$. One immediate solution relays on the introduction of a cutoff at $u_o = \log \epsilon$, at which point the field (or its first derivative) is made to vanish. The parameter ϵ can be identified with the proper distance of the cutoff point to the horizon.

It is helpful to think of each transverse Fourier mode χ_k as a free 1+1 dimensional quantum field confined to a box. With this picture in mind, one can imagine one end of the box placed at $u = u_0 = \log \epsilon$, where the reflecting boundary is. The opposite wall of the box is at at $u = u_1 = -\log k$, provided by the repulsive potential (3.13), which becomes large when $u > -\log k$. The total length of the box depends on k and ϵ according to

$$L(k) = -\log(\epsilon k) \tag{3.14}$$

For each value of k the field χ_k can be expanded in mode functions and creation and annihilation operators according to

$$\chi_k = \sum_n \left[a_+(n,k) f_{n,k}(u) + a_-(n,k) f^*_{n,k}(u) \right]$$

where $\lambda(n,k)$ is the frequency of the mode (n,k). The Rindler Hamiltonian is given by

$$H_R = \int d^2k \sum_n \lambda(n,k) a^\dagger(n,k) a(n,k) = \sum_n \lambda(n,k) N(n,k)$$

where $N(n,k) \equiv a^\dagger(n,k)a(n,k)$ is defined as the usual number operator.

Recalling what has been said in Section 3.1.1, we do not identify the vacuum with the state annihilated by $a(n,k)$ (which is not invariant under translations of the Minkowski coordinates), but rather with the thermal density matrix

$$\rho_R = \prod_{n,k} \rho_R(n,k)$$

with

$$\rho_R(n,k) \sim e^{-2\pi\lambda(n,k)a^\dagger(n,k)a(n,k)}$$

Thus the average occupation number of each mode is

$$\langle N(n,k)\rangle = \frac{1}{e^{2\pi\lambda(n,k)} - 1}$$

These particles constitute the *thermal atmosphere*.

Since the relevant density matrix has the Maxwell–Boltzmann form, we can use equations (3.5) and (3.6) to obtain the entropy. Defining

$$\operatorname{Tr} e^{-\beta H} \equiv Z(\beta) \quad \text{and} \quad \rho \log \rho \equiv \left.\frac{\partial}{\partial N}\rho^N\right|_{N=1}$$

we obtain

$$\begin{aligned} S =& -\operatorname{Tr} \left.\frac{\partial}{\partial N}\frac{e^{-N\beta H}}{Z(\beta)^N}\right|_{N=1} \\ =& \operatorname{Tr} \beta H \frac{e^{-\beta H}}{Z} + \ln Z \\ =& \beta \langle H\rangle + \ln Z \end{aligned} \tag{3.15}$$

Defining $E = \langle H\rangle$ and $F = -\frac{1}{\beta}\log Z$ we find the usual thermodynamic identity

$$S = \beta(E - F) \tag{3.16}$$

Another identity follows from using $E = -\frac{\partial \log Z}{\partial \beta}$:

$$S = -\beta^2 \frac{\partial \log Z/\beta}{\partial \beta} \tag{3.17}$$

The entropy S in equations (3.16) and (3.17) can be identified both with entanglement and thermal entropy in the special case of the Rindler space density matrix.

To compute the total entropy we replace the infinite transverse X, Y plane by a finite torus with periodic boundary conditions. In this way we are allowing k to assume only discrete values. Thus

$$k_X = \frac{2n_X\pi}{B} \quad , \quad k_Y = \frac{2n_Y\pi}{B} \tag{3.18}$$

where B is the size of the torus.

The entropy stored in the field χ_k can be estimated from the entropy density of a 1+1 dimensional massless free boson at temperature T. A standard calculation shows that the entropy density $\frac{S}{L}$ is given by

$$\frac{S}{L} \propto \frac{\pi}{3} T$$

Substituting the Rindler temperature and (3.14) for the length L gives the entropy of χ_k

$$S(k) \propto \frac{1}{6} |\log k\epsilon|$$

To sum over the values of k we use (3.17) and let $B \to \infty$

$$S_{Total} \propto \frac{B^2}{24\pi^2} \int d^2k |\log k\epsilon| \tag{3.19}$$

When $k = \frac{1}{\epsilon}$ the potential is already large at $u = u_o$ so that the entire contribution of χ_k is suppressed. Therefore in order to evaluate (3.19) the integral must be cut off when $k > \frac{1}{\epsilon}$. S is then found to be proportional to

$$S_{Total} \propto \frac{1}{96\pi^2} \frac{B^2}{\epsilon^2} \tag{3.20}$$

We immediately notice the remarkable fact that the entropy of Rindler space is proportional to the transverse area of the horizon, B^2. This may come as a surprise as usually someone expects it to diverge as the volume of space. However, in this case the entropy is entirely stored in the proximity of the stretched horizon (because there is where the temperature gets large) and therefore grows proportionally to the area. A second unexpected feature is the $\frac{1}{\epsilon^2}$ divergence of the entropy per unit area. As we will soon delve into, the entropy density of the horizon is a physical quantity whose exact value is known. Nevertheless the divergence in S indicates that its value is sensitive to the ultraviolet physics at very small length scales [7].

We can analyse further the form of S by recalling that the entropy density of a 3+1 dimensional free scalar field is given by

$$S(T) \propto V \frac{2}{\pi^2} \zeta(4) k_B \left(\frac{k_B T}{\hbar c} \right)^3 \propto V \frac{2\pi^2}{45} T^3$$

Now consider the entropy stored in a layer of thickness $\delta\rho$ and area B^2 at a distance ρ from the horizon and substitute (3.11) for the proper temperature

$$\delta S(\rho) \propto \frac{2\pi^2}{45} T^3(\rho) \delta\rho B^2 \propto \frac{2\pi^2}{45} \frac{1}{(2\pi\rho)^3} \delta\rho B^2$$

To find the full entropy we integrate with respect to ρ

$$S \propto \frac{B^2}{(2\pi)^3} \frac{2\pi^2}{45} \int_\epsilon^\infty \frac{d\rho}{\rho^3} \propto \frac{B^2}{(2\pi)^3} \frac{2\pi^2}{45} \cdot \frac{1}{2\epsilon^2} \propto \frac{B^2}{360\pi\epsilon^2} \tag{3.21}$$

Chapter 4

Black hole thermodynamics

In our study of the Rindler space we have enriched the equivalence principle discovering the existence of a temperature experienced by accelerated fiducial observers. It is then natural to expect analogues thermal effects to take place in the vicinity of the horizon of a BH. However, for BHs a new phenomenon must be taken into account: evaporation. Unlike the Rindler case, the thermal atmosphere is not absolutely confined by a centrifugal potential, hence the particles of the thermal atmosphere will gradually leak through the barrier and carry off energy in the form of thermal radiation [7].

Firstly, we shall derive an equation for the temperature T of the BH seen by a distant observer. Recalling the relationship between Rindler time ω and Schwarzschild time t: $\omega = \frac{t}{4m} = \frac{c^2 t}{4GM}$, we can say that a quantum field with Rindler frequency ν_R is seen by the distant Schwarzschild observer to have a red shifted frequency ν

$$\nu = \frac{c^2 \nu_R}{4MG}$$

This means that the red shift affects the temperature of the thermal atmosphere as well, giving the following result

$$T = \frac{\hbar c}{2\pi k_B} \cdot \frac{c^2}{4MG} = \frac{\hbar c^3}{8\pi k_B M G} \tag{4.1}$$

4.1 Four laws of black hole thermodynamics

The Zeroth Law

The surface gravity of a BH does not change from point to point of the event horizon.

The surface gravity is, by definition, the value of the free-fall acceleration, calculated at the event horizon in terms of the particle acceleration [2]. For a Schwarzschild BH, the surface gravity ς is (using $m \equiv GMc^{-2}$)

$$\varsigma = \frac{1}{4m} = \frac{c^2}{4GM}$$

while for a Kerr BH, its expression has been found to be

$$\varsigma = \frac{r_+ - m}{2mr_+} = \frac{\sqrt{m^2 - a^2}}{2m(m + \sqrt{m^2 - a^2})}$$

This equations obey the Zeroth Law as no angular term is present.

The First Law

ΩdJ is the work executed over the BH by adding the angular momentum dJ:

$$dM = \frac{c^2\varsigma}{8\pi G} dA + \Omega dJ \tag{4.2}$$

where A is the horizon area and Ω is the angular velocity at the horizon, respectively

$$A = 4\pi(r_+^2 + a^2) \quad \text{and} \quad \Omega = \frac{a}{r_+^2 + a^2}$$

At $a = 0$ we obtain the expression for the Schwarzschild BH (which can also be obtained integrating (4.2) with the condition $\Omega = a = 0$):

$$A = 4\pi r_+^2 = 16\pi m^2 \tag{4.3}$$

It is important to notice that (4.2) is similar to the first law of conventional thermodynamics: $dU = TdS + dW$. U, the BH energy, is replaced by M, while dS is replaced by dA and T is its coefficient. Recalling (4.1), for a Schwarzschild BH we have

$$T_{BH} = \frac{\hbar c^3}{8\pi MGk_B} \tag{4.4}$$

this means that the additional factor $\frac{ck_B}{4\hbar G}$ lays in the entropy:

$$S_{BH} = \frac{c^3 k_B A}{4\hbar G} \tag{4.5}$$

Note that to recover the first law of traditional thermodynamics we must remember that in replacing E with M inside (4.2) we lost the proportionality factor c^2, which has to come back in the expression of S_{BH}.

The Second Law

There is no physical process that could decrease the horizon area of a BH.

This is similar to the statement that the entropy cannot decrease. However, as we shall in the next section, the quantum process of BH evaporation due to Hawking radiation violates this classical law.

An interesting consequence of the Second Law concerns BHs collisions [6]. The merging between two or more BHs is considered one of the main processes that leads to the formation of supermassive BHs and produces significant gravitational waves (GW). In Fig. 4.1 we can see an example of this; in the simple case of two Schwarzschild BHs (1 and 2) in a head-on collision. Part of the mass of the initial system is converted in GW radiation, implying $M_1 + M_2 < M_3$. However, this do not imply that the entropy has decreased, because, as we have already seen, the latter is stored in the area. Therefore, accordingly with the Second Law, we can write

$$A_1 + A_2 \leq A_3$$

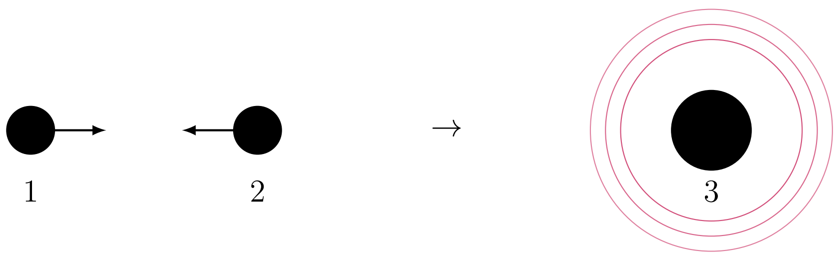


Figure 4.1: The merging of two Schwarzschild BHs (1,2) into a third Schwarzschild BH (3) with the production of GW (which are represented in purple in the figure).

The Third Law

There is no procedure able to bring the BH temperature to zero by a finite sequence of operations.

This law ensures that an absolute zero of temperature is not accessible, and, consequently, asserts the nonexistence of negative temperatures.

If we imagine the BH as a single macroscopic state, then the entropy S_{BH} found in (4.5) is a measure of the number of microscopically distinct quantum states that are "coarse grained" and form the macroscopic state.

In particular, S_{BH} is the famous Bekenstein–Hawking entropy [7]. As we expected from the analysis carried out in Chapter 3, it is proportional to the area of the horizon. This is in fact the place where all the infalling matter accumulates according to external fiducial observers. We have seen with equation (3.20) that the matter fields in proximity of the horizon creates an entropy, which is infinite as $\epsilon \to 0$. This makes it difficult to link this entropy with the total entropy of the BH. However, one way to solve this problem could be to introduce a cut off for the modes which are very close to the horizon. To guess where this cut off should occur we can require that the contribution in (3.20) does not exceed the entropy of the BH S_{BH}.

$$\frac{1}{96\pi^2\epsilon^2} \leq \frac{c^3 k_B A}{4\hbar G}$$

Strictly speaking, the cutoff cannot be much smaller than the Planck length, where the latter is given in terms of the gravitational constant as $l_P = \sqrt{\frac{\hbar G}{c^3}}$. It is widely believed that the nasty divergences of quantum gravity will somehow be cut off by some mechanism when the distance scales become smaller than $\sqrt{G}$ [7].

The direct proportionality between the area and the Bekenstein–Hawking entropy makes the latter an extensive quantity, suggesting that we can study it considering a limiting BH of infinite mass and area and then focus on its local properties. Clearly entropy diverges, but the entropy per unit area remains finite. The local geometry of a limiting BH horizon is of course the Rindler space.

Consider the Rindler energy of the horizon. By definition it is conjugate to the Rindler time ω. Using the geometrized mass $m \equiv \frac{MG}{c^2}$, their commutator is equal to

$$[E_R(M), \omega] = i\frac{c^3 m^2}{G}$$

Here E_R is the Rindler energy which corresponds to the eigenvalue of the Rindler Hamiltonian operator. We have then

$$\left[E_R(M), \frac{t}{4m}\right] = i\frac{c^3 m^2}{G} \quad \to \quad [E_R(M), t] = i\frac{4c^3 m^3}{G} \tag{4.6}$$

We will assume that for a large BH the Rindler energy is a function of the mass of the BH only.

The mass and Schwarzschild time are also conjugate

$$[M, t] = i\frac{m^2 c}{G}$$

It is now convenient to rewrite (4.6) in the following form

$$\frac{\partial E_R}{\partial M} = 4MG \quad \Rightarrow \quad E_R = 2M^2 G$$

Remarkably, Rindler energy is also an extensive quantity. For a Schwarzschild BH, the area density of Rindler energy is

$$\frac{E_R}{A} = \frac{c^4}{8\pi G}$$

It is immediate to verify that Rindler energy and entropy satisfy the first law of BH thermodynamics

$$dE_R = \frac{\hbar c}{2\pi k_B} dS$$

where $\frac{\hbar c}{2\pi k_B}$ is the Rindler temperature. We can then deduce that horizons have universal local properties which resemble the ones of thermal membranes. This latter are also called stretched horizons and will be further studied in Section 4.3.

In conclusion we identify the energy and the mass of the BH respectively with the Rindler energy and the Schwarzschild mass. In particular the latter is the energy as reckoned by observers at infinity using t-clocks, while the former is the energy (rescaled by the proper distance ρ) as defined by observers near the horizon using ω-clocks [7].

4.2 Evaporation process

Until now we have treated Schwarzschild BHs as if they were systems in thermal equilibrium, but of course they are not. They have long lifetimes, but eventually they evaporate [7]. One may try to prevent this by placing them in a thermal heat bath at their Hawking temperature, but the result will be unsuccessful because BHs have a negative specific heat; meaning their temperature decreases as energy or mass is added to the system. Any object with this property is thermodynamically unstable.

For instance, suppose the BH absorbs some energy from the surrounding heat bath due to a fluctuation. If we were dealing with an ordinary system characterised by positive specific heat, this would raise its temperature which in turn would cause it to radiate back into the environment; leading to the conclusion that the fluctuations are self-regulating. However, a system with negative specific heat will lower its temperature when it absorbs energy and will become cooler than the bath. This will cause a chain effect for which the bath conveys additional energy to the BH, making it grow indefinitely. Considering instead the case in which the BH cedes some energy to the environment, it will become hotter than the heath bath and by and by disappear [1].

The evaporation process of a BH will now be presented both with a qualitative and a more quantitative approach. First one can imagine a physical picture of BH evaporation starting with virtual particle-antiparticle pairs, constantly appearing and disappearing

[1]For purposes of completeness we must say that a well known way to make BH stable do exists. The idea is to put it in a box in order to make the environmental heat finite, so that when the BH absorbs some energy it cools but so does the finite heat bath. By choosing a box with the right size, we have that the heat bath will cool more than the BH and the flow of heat will be back to the bath [7].

according to quantum field theory, which are subject to a sufficiently strong gravitational field in the neighbourhood of the horizon (see Fig. 3.2). Then, with a certain probability (which is larger for larger space–time curvatures), the gravitational field can drag apart such virtual pairs. As a result one of the particles falls back towards the horizon while the other escapes to infinity. The BH ends up loosing a particle with nonzero energy, meaning that some of its mass is lost as well [2].

To give a more detailed explanation of this process we can say that the centrifugal barrier described in the Rindler theory by the potential $k^2e^{(2u)}$ is modified at distances $r \approx 3MGc^{-2}$. Any secondary wave (s-wave) quanta with sufficiently high frequencies will easily escape the barrier. In particular, the average energies of massless particles in thermal equilibrium at temperature T is of course of order T, which is enough for the s-wave particles to escape to infinity. Therefore the BH must be kept in equilibrium by a source of incoming radiation, otherwise it will lose energy to the surrounding environment [7].

For particles characterised by an angular momenta higher than s-waves escaping the potential barrier it's harder, because the latter results higher than the thermal scale. The majority of quantum particles in the thermal atmosphere have high angular momenta and are then reflected back towards the horizon. Only a small fraction of quanta is hence able to be radiated in the outer space. In this sense the BH resembles a slightly leaky cavity filled with thermal radiation. Using this interpretation, for the particles that escape the walls are semi-transparent and the cavity slowly radiates its internal energy. This process was first discovered by Stephen Hawking and it is commonly called Hawking radiation.

It is noteworthy that this description of Hawking radiation does not rely on the free field approximation, as this only works if there are interactions capable to keep the system in equilibrium during the course of its evaporation.

For the purpose of estimating the evaporation rate of the BH we will suppose that only s-wave quanta are able to escape. Begin by considering the s-wave quantum particles in terms of a 1+1 dimensional quantum field at Rindler temperature $\frac{\hbar c}{2\pi k_B}$. If one compares this temperature with the potential barrier energy, one obtains that approximately one quantum particle per unit of Rindler time will leave the system. In Schwarzschild time units, the flux of quanta is of order $\frac{c}{MG}$. Each quantum will reach infinity with an energy that is clearly of the same order of the Schwarzschild temperature $\frac{\hbar c^3}{8\pi k_B MG}$. This results in a rate of energy loss $\sim \frac{\hbar c^4}{8\pi M^2 G^2}$, which will be called L, the luminosity. For a Schwarzschild BH, an exact value can be determined by means of Stephan-Boltzmann radiation law:

$$L = A \cdot \sigma T^4 = \frac{16\pi M^2 G^2}{c^4} \cdot \frac{\pi^2 k_B^4}{60\hbar^3 c^4} \left(\frac{\hbar c^3}{8\pi k_B MG}\right)^4 = \frac{\hbar c^4}{15360\pi M^2 G^2}$$

Where A is the surface area and σ is the Stephan-Boltzmann constant. Using energy conservation we can require the BH to lose mass precisely at the rate L [7] [8] (if we assume pure photon emission and we identify the horizon as the radiating surface):

$$\frac{dM}{dt} = -L = -\frac{\hbar c^4 C}{15360\pi M^2 G^2} \tag{4.7}$$

where C is a constant of order unity which depends on the amount of species of particles light enough to be thermally produced. Hence it is actually a variable whose value is fixed case by case. However, if the mass of the BH is large and the temperature low, only a few species of massless particles contribute and C can be traten as an actual constant.

If we neglect the mass dependence of C, (4.7) can be integrated to find the time it takes for a BH to evaporate until it has reached zero mass [7].

$$\int_M^0 \frac{15360\pi M^2 G^2}{\hbar c^4} dM \approx -\int_0^t dt \quad \Rightarrow \quad t_{evaporation} \sim \frac{5120\pi M^3 G^2}{\hbar c^4}$$

A substitution of the corresponding constants leads to the following expressions for the temperature (in Kelvins) and evaporation time of a solar mass Schwarzschild BH:

$$T_{BH} \approx \frac{M_\odot}{M} \cdot 10^{-7}\ K \quad \text{and} \quad t_{evaporation} \approx \left(\frac{M}{M_\odot}\right)^3 \cdot 10^{67}\ years$$

where $M_\odot$ is the solar mass. We can see that the temperatures of BHs of solar masses is very small, and their evaporation occurs very slowly (for instance the Universe is $13.8 \cdot 10^9$ $years$ old): in a realistic scenario, they increase their masses much more rapidly due to accretion of surrounding matter. The evaporation time is even larger for supermassive BH, which are indeed thought to be the very last objects to populate the Universe when everything else will have vanished.

These results also confirm us that BHs have negative heat capacity, as temperature becomes smaller if mass is added. Furthermore they tell us that for small BHs (which have a very large curvature of space near their horizon) the Hawking temperature is high, while the lifetime is low, because they quickly radiate all their internal energy.

When the time instant $t_{evaporation}$ is reached, the evaporation terminates, but it is so far unclear whether the BH must completely disappear or if an object with a mass of Planckian order of magnitude, $M \sim 10^{-5}g$, is left as the end product of the process. Anyway as the mass approaches its Planck value (meaning the horizon radius gets close to the Planck length $\sim 10^{-33}cm$), the so far unknown laws of quantum gravity must come into force [2].

Lastly, it is worthy noting that putting together (4.3) and (4.5) gives an expression relating the entropy and mass for a Schwarzschild BH:

$$S_{BH} = \frac{4\pi k_B G}{\hbar c} M^2$$

Therefore the information paradox (which will be better discussed in Chapter 5) starts to take shape, because we just found that as a BH looses mass through evaporation its entropy will decrease. This seems to be in sharp contrast with the second law of conventional thermodynamics, which states that the entropy of an isolated system can only increase over time [8].

4.3 The stretched horizon

At the end of Chapter 3 we noticed that the entropy per unit area of the horizon diverges as the cutoff ϵ tends to zero (3.21). Furthermore, we later found that BH thermodynamics requires the entropy to be as in (4.5). A contradiction seems to arise because the entropy given by free quantum field theory is too much in the modes very close to the horizon, where the local temperature diverges .

One hypothesis that would fix this is having overestimated the entropy by the assumption of free field theory [7]. The correct entropy density for a general field theory can always be parametrized by

$$S(T) = \gamma(T)T^3$$

where $\gamma(T)$ represents the number of "effective" dof at temperature T.

We are searching for a different theory in which the effective number of dof goes to zero in the proximity of the horizon. Let's start by guessing that ordinary quantum field theory is adequate down to distance scale ϵ. A restriction is needed to make sure that the entropy at distances greater than ϵ does not exceed the Bekenstein–Hawking value:

$$\epsilon^2 \lesssim \frac{\hbar G}{c^3} = l_P^2$$

Clearly at less then one Plank length from the horizon the dof must be very sparse, or even nonexistent. This is another hint concerning the mathematical horizon and its replacement (when the proper length approaches zero) with a physical membrane, or "stretched" horizon. One of the main feature of this different kind of horizon is that real dynamics and evolution can take place there, because a system at the stretched horizon is time-like. Furthermore, the stretched horizon has its own physical properties such as viscosity and electrical conductivity.

To prove this we consider the metric

$$ds^2 = \rho^2 c^2 d\omega^2 - d\rho^2 - dx_\perp^2$$

The stretched horizon is then defined as the surface $\rho = \rho_o$, where ρ_o is a length of order the Planck length.

The action for the electromagnetic field in Rindler space, using Heaviside units (which, from now on, will be used in the whole section), is

$$W = \int \left[\frac{\sqrt{-g}}{16\pi} g^{\mu\nu} g^{\sigma\tau} F_{\mu\sigma} F_{\nu\tau} + j^\mu A_\mu \right] d\omega \; d\rho \; d^2x_\perp$$

or, substituting the form of the metric

$$W = \int \left[\frac{1}{8\pi}\left(\frac{1}{\rho}\left(\dot{\vec{A}} + \vec{\nabla}\phi\right)^2 - \rho\left(\vec{\nabla}\times\vec{A}\right)^2\right) + \vec{j}\cdot\vec{A}\right] d\omega\ d\rho\ d^2x_\perp$$

where $\dot{\vec{A}}$ means $\frac{\partial\vec{A}}{\partial\omega}$ and $\phi = -A_0$, and j is a conserved current: $\partial_\mu j^\mu = 0$. As usual

$$\begin{aligned}\vec{E} &= -\,\vec{\nabla}\phi - \dot{\vec{A}}\\ \vec{B} &= \vec{\nabla}\times\vec{A}\end{aligned}$$

With these definitions, the action becomes

$$W = \int \left[\frac{1}{8\pi}\left(\frac{1}{\rho}|\vec{E}|^2 - \rho|\vec{B}|^2\right) + \vec{j}\cdot\vec{A}\right] d\omega\ d\rho\ d^2x_\perp$$

and the Maxwell equations are

$$\begin{aligned}\frac{1}{\rho}\dot{\vec{E}} - \vec{\nabla}\times(\rho\vec{B}) &= -4\pi\vec{j}\\ \dot{\vec{B}} + \vec{\nabla}\times\vec{E} &= 0\\ \vec{\nabla}\cdot\left(\frac{1}{\rho}\vec{E}\right) &= 4\pi j^0\\ \vec{\nabla}\cdot\vec{B} &= 0\end{aligned} \tag{4.8}$$

First begin by using Rindler coordinates to consider an electrostatic configuration with stationary or slowly moving charges hovering outside the horizon [7]. Since the charge motion is assumed very slow, they are experiencing proper acceleration. In what follows, is also assumed that all length scales related to the charges are much larger than ρ_o.

The surface charge density on the stretched horizon is just the component of the electric field perpendicular to the stretched horizon

$$\sigma = \frac{1}{4\pi\rho}E_\rho\Big|_{\rho=\rho_o} = -\frac{1}{4\pi\rho}\partial_\rho\phi\Big|_{\rho=\rho_o} \tag{4.9}$$

Working in the Coulomb gauge condition (expressed as $\vec{\nabla}\cdot\vec{A} = 0$), we have

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\left(-\vec{\nabla}\phi - \dot{\vec{A}}\right) = \vec{\nabla}\cdot\left(-\vec{\nabla}\phi\right)$$

Therefore the third expression in equation (4.8) becomes

$$\vec{\nabla}\cdot\left(\frac{1}{\rho}\vec{E}\right) = -\vec{\nabla}\cdot\left(\frac{1}{\rho}\vec{\nabla}\phi\right) = 0$$

near the stretched horizon. Thus

$$\partial_\rho^2 \phi - \frac{1}{\rho}\partial_\rho \phi = -\nabla_\perp^2 \phi \tag{4.10}$$

We can attempt a solution for this equation near the horizon via the ansatz $\phi \sim \rho^\alpha$, neglecting the right hand side of (4.10) in virtue of the difference of two powers of ρ with the left hand side.

$$\begin{aligned}
& \partial_\rho^2 \rho^\alpha - \frac{1}{\rho}\partial_\rho \rho^\alpha = -\nabla_\perp^2 \rho^\alpha \\
\rightarrow \quad & \alpha(\alpha-1)\rho^{\alpha-2} - \frac{\alpha}{\rho}\rho^{\alpha-1} = 0 \\
\rightarrow \quad & \alpha(\alpha-1) - \alpha = 0 \\
\Rightarrow \quad & \alpha = 0 \quad \vee \quad \alpha = 2
\end{aligned}$$

Thus we assume

$$\phi = F(x_\perp) + \rho^2 G(x_\perp) + O(\rho^3)$$

Inserting this equation into (4.10) and evaluating at $\rho = \rho_o$ gives

$$\nabla_\perp^2 F + \rho_o \nabla_\perp^2 G = 0 \tag{4.11}$$

If ρ_o is much smaller than all other length scales, then (4.11) is simplified to

$$\nabla_\perp^2 F = 0 \tag{4.12}$$

An analogue equation can also be derived for the finite mass BH. Since the BH horizon is compact, (4.12) proves that $\phi = const$ on the horizon. We have hence demonstrated that the horizon behaves like an electrical conductor.

Let's now try to find an expression for the surface current density. Taking the time derivative of (4.9) and using Maxwell's equations (4.8) we get

$$4\pi\dot{\sigma} = \frac{1}{\rho_o}\dot{E}_\rho = \left(\vec{\nabla} \times \rho\vec{B}\right)_\rho$$

Evidently this is a continuity equation if we define

$$\begin{aligned}
4\pi j_x &= -\rho B_y \\
4\pi j_y &= \rho B_x
\end{aligned} \tag{4.13}$$

Now consider an electromagnetic wave propagating towards the stretched horizon along the ρ axis. From Maxwell's equations we obtain

$$\begin{aligned}
\dot{B}_x &= \partial_\rho E_y \\
\dot{B}_y &= -\partial_\rho E_x \\
\frac{1}{\rho}\dot{E}_x &= -\partial_\rho(\rho B_y) \\
\frac{1}{\rho}\dot{E}_y &= \partial_\rho(\rho B_x)
\end{aligned} \tag{4.14}$$

To make the equation more familiar, we can use these substitutions:

$$\rho\vec{B} = \vec{\beta} \quad \text{and} \quad u = \log\rho$$

Equation (4.14) becomes

$$\begin{aligned}
\dot{\beta}_x &= \partial_u E_y \\
\dot{\beta}_y &= -\partial_u E_x \\
\dot{E}_x &= -\partial_u \beta_y \\
\dot{E}_y &= \partial_u \beta_x
\end{aligned} \tag{4.15}$$

It is noteworthy that these mathematical equations allow solutions characterised by waves propagating in either directions along the u-axis. However it is clear that physics only allows waves propagating towards the horizon from outside the BH. For the latter type of waves, the Maxwell equations (4.15) give

$$\begin{aligned}
\beta_x &= E_y \\
\beta_y &= -E_x
\end{aligned} \tag{4.16}$$

using this in (4.13) we also get

$$\begin{aligned}
j_x &= \frac{1}{4\pi}E_x \\
j_y &= \frac{1}{4\pi}E_y
\end{aligned} \tag{4.17}$$

Hence the horizon appears to be an ohmic conductor with a resistivity of 4π. That is related to a surface resistance of 377 $\Omega/square$ (the resistance of a two-dimensional resistor is scale invariant and only depends on the shape).

Therefore we conclude the horizon has all the traits of a conventional hot conducting membrane. In addition to temperature, entropy, and energy, it exhibits dissipative effects such as electrical resistivity and viscosity; although they are completely unnoticed by a freely falling observer who falls through the horizon [7].

Chapter 5

Information paradox

The idea of a potential loss of information in the process of creation and evaporation of BHs dates back to 1976, when Stephen Hawking introduced it with an argument based on local quantum field theory in the fixed background of a BH. Although Hawking drew undoubtedly wrong conclusions, the questions he raised turned out to be pivotal in the development of a brand new paradigm concerning the concept of locality [7].

As we have already discussed in Section 4.3, quantum field theory struggles in describing systems with horizons, because it leads to an infinite entropy density, instead of the correct Bekenstein–Hawking value of $\frac{c^3 k_B}{4G\hbar}$. This serious problem suggests that a different approach must be taken into account. Before presenting this new theory we shall take a brief detour into information conservation laws.

5.1 Information conservation

Both classical and quantum mechanics agree that a closed isolated system must preserve the information it contains. Classical physics establish this principle in Liouville's theorem: the conservation of phase space volume [7].

Imagine having some limited knowledge about the exact state a specific system is prepared into. In the system's phase space we confine the initial state inside a finite region $\Gamma(0)$. When the system evolves the region $\Gamma(0)$ evolves into $\Gamma(t)$ and the same is for the respective volumes, $V_{\Gamma(0)}$ and $V_{\Gamma(t)}$. Liouville's theorem states that

$$V_{\Gamma(0)} = V_{\Gamma(t)}$$

meaning that the amount of information is conserved.

However, in practice we have a loss of information as, for most cases of interest, the region Γ becomes very complex and this leads to an increment of volume when measured. Indeed if one surrounds every point in the phase space with solid spheres of fixed volume, the union of those spheres is the *coarse grained* volume of phase space, which grows as

Γ becomes more complex (see Fig. 5.1). From this concept it originates the second law of thermodynamics.

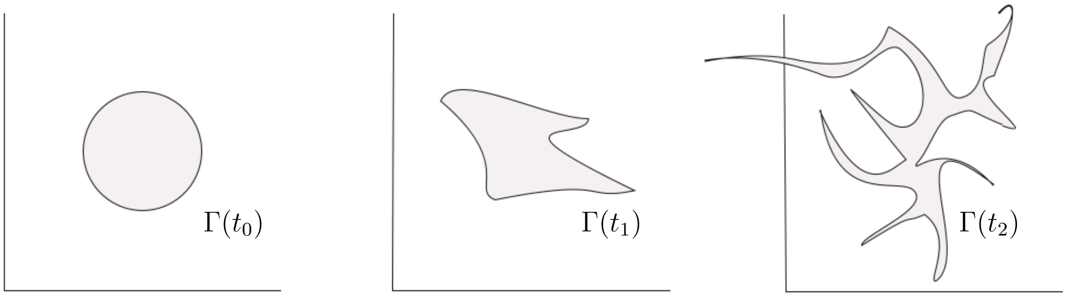


Figure 5.1: Evolution of a fixed volume in phase space ($t_0 < t_1 < t_2$) [7].

In order to make a transition to the quantum realm, consider now a probability density $\rho(p,q)$ in phase space. The quantity which was the volume in the classical case is now generalised by the exponential of the entropy $V_\Gamma \to e^S$, where

$$S = -\int dpdq\rho(p,q)\log\rho(p,q)$$

Entropy may be interpreted as the maximum amount of information that can be hidden in a system. In quantum mechanics information is conserved if and only if the S-matrix is unitary [7]. This time, to study of a system with limited knowledge, we should make use of a projection operator onto a subspace, P, instead of a definite state. Proceeding this way, the phase space volume becomes the dimensionality of the subspace

$$N = \operatorname{Tr} P$$

As the time evolution operator is unitary, N is conserved with time.

The sharp projector P may be replaced by a density matrix ρ, providing the following expression for the fine grained or Von Neumann entropy

$$S = -\operatorname{Tr}\rho\log\rho$$

In the special case $\rho = \frac{P}{\operatorname{Tr} P}$ the entropy reduces to $\log N$ (see appendix C for an explicit derivation). We can then infer that the entropy is an estimate of the logarithm of the number of quantum states that make up the initial ensemble. As far as isolated systems are concerned, the equations of motion entail the exact conservation of S both in quantum and classical mechanics.

5.1.1 Entanglement entropy

In the quantum world different subsystems belonging to the same composite system enjoy a property called entanglement. The word *entanglement* refers to quantum correlations between the system under investigation and a second system [7].

For instance, imagine to study a system made of 2 subsystems A and B. Let $\alpha(\beta)$ be a complete set of commuting observables describing the subsystem $A(B)$. We will now assume the composite system to be in a pure state with wave function $\Psi(\alpha, \beta)$. If one considers A and B as separate systems, then all measurements performed on $A(B)$ are describable by means of a density matrix $\rho_A(\rho_B)$.

$$
\begin{aligned}
(\rho_A)_{\alpha\alpha'} &= \sum_\beta \Psi^*(\alpha, \beta)\Psi(\alpha', \beta) \\
(\rho_B)_{\beta\beta'} &= \sum_\alpha \Psi^*(\alpha, \beta)\Psi(\alpha, \beta')
\end{aligned}
$$

It is crucial to underline that subsystems are not described by pure states even if the composite system is. This is not concerned with a possible lack of knowledge of the state of the latter; in fact the reason for this regards the concept of entanglement and consequently results in an "entanglement entropy" for the subsystems.

We have seen (p. 24) that if the subsystem A is in a pure state, then one of the eigenvalues of ρ_A is equal to 1 and all the others must vanish (the same holds for B). Furthermore, we can prove that the nonzero eigenvalues of ρ_A and ρ_B are equal if the composite system is in a pure state. If ϕ is the eigenvector of ρ_A we have

$$
\sum_{\beta\alpha'} \Psi^*(\alpha, \beta)\Psi(\alpha', \beta)\phi(\alpha') = \lambda\phi(\alpha)
$$

Assume $\lambda \neq 0$. Then suppose $\chi(\beta')$ is an eigenvector of ρ_B

$$
\chi(\beta') \equiv \sum_{\alpha'} \Psi^*(\alpha', \beta')\phi^*(\alpha')
$$

This allows us to write

$$
\begin{aligned}
\sum_{\beta'} (\rho_B)_{\beta\beta'}\ \chi(\beta') &= \sum_{\alpha\beta'} \Psi^*(\alpha, \beta)\Psi(\alpha, \beta')\chi(\beta') \\
&= \sum_{\alpha\alpha'\beta'} \Psi^*(\alpha, \beta)\Psi(\alpha, \beta')\Psi^*(\alpha', \beta')\phi^*(\alpha') \\
&= \lambda \sum_\alpha \Psi^*(\alpha, \beta)\phi^*(\alpha) = \lambda\chi(\beta)
\end{aligned}
$$

(A more rigorous poof is in appendix C). The equality between non-vanishing eigenvalues of ρ_A and ρ_B has an important repercussion:

$$S_A = -\operatorname{Tr}\rho_A \log \rho_A = -\sum \rho_{Aj} \log \rho_{Aj} = -\sum \rho_{Bj} \log \rho_{Bj} = S_B \tag{5.1}$$

which holds if the combined state is pure (ρ_{Aj} and ρ_{Bj} are, respectively, the eigenvalues of ρ_A and ρ_B). In this case the entropy of the composite system vanishes; meaning that entropy, in general, is not additive. However from (5.1) we see that the entropy of the subsystem does not vanish; we refer to this entanglement entropy as $S_{entanglement}$.

We now want to study a large system Σ, made of a large number of small similar subsystems σ_i. Consider these latter to be weakly interacting and suppose further that Σ is in a pure state with total energy E. Each subsystem will be characterised by an average energy ϵ. A small subsystem which is part of a complex interacting structure will have, in general, a thermal density matrix

$$\rho_i = \frac{e^{-\beta H_i}}{Z_i}$$

where H_i is the energy of the subsystem. ρ_i maximizes the entropy for a certain average energy ϵ. We now assume the entire system Σ to be in a pure state, having as a consequence zero entropy. The sum of the entropies of every small subsystem gives the so called *coarse grained* or *thermal entropy*

$$S_{thermal} = \sum_i S_i$$

This kind of entropy is what we usually think of in the context of thermodynamics. By definition is additive and we will now see that it is also not conserved.

Consider all the subsystems prepared in a product state with no correlations. The entropy of each subsystem S_i as well as the entropy σ_Σ of the whole system and the coarse grained entropy of Σ all vanish. Now let the subsystems interact: the wave function develops correlations, meaning that it cannot be factorized anymore. In this situation, the entropies of the subsystems become nonzero

$$S_i \neq 0$$

Consequently, the coarse grained entropy also becomes nonzero

$$S_{thermal} = \sum_i S_i \neq 0$$

However, the "fine grained" entropy of Σ is exactly conserved and therefore remains zero.

Usually the fine grained entropy of an arbitrary subsystem Σ_1 of Σ is defined as the entanglement entropy $S(\Sigma_1)$ of Σ_1 with the remaining subsystem $\Sigma - \Sigma_1$ (Σ_1 may be composed by several subsystems of Σ, or even all of them). The coarse grained entropy of Σ_1 will always be greater than its fine grained entropy

$$S_{thermal}(\Sigma_1) > S(\Sigma_1)$$

It is clear that as Σ_1 approaches Σ, the fine grained entropy $S(\Sigma_1)$ will tend to zero; because $S(\Sigma_1) \to S(\Sigma) = S(\Sigma - \Sigma) = 0$.

We can now the define information as

$$I \equiv S_{thermal} - S$$

Commonly, the coarse grained entropy coincides with the thermal entropy of the system, so that the information is the difference between coarse grained and fine grained entropy. In this case, if one considers small subsystems, one finds the latter do not contain any information at all.

Both entropy and information are measured in *bits*. A bit is the entropy of a two state system if nothing is known. The numerical value of a bit is $\log 2$ [7].

It is counterintuitive but the amount of information in a moderately sized subsystem does not vary smoothly from zero (for the σ_i) to $S_{CoarseGrained}$ (for Σ). In fact, for subsystems smaller than about $\frac{1}{2}$ of the total system, the amount of information they contain is smaller than 1 bit and therefore negligible.

The subsystem $\frac{1}{2}\Sigma$ has about 1 bit of information. Thus for $\Sigma_1 < \frac{1}{2}\Sigma$, using the definition of bit, we can say

$$S(\Sigma_1) \cong S_{thermal}(\Sigma_1) \quad \Rightarrow \quad I(\Sigma_1) \approx 0$$

Next consider the amount of information in a subsystem with $\Sigma_1 > \frac{1}{2}\Sigma$. To compute it, we use two facts:

$$S(\Sigma - \Sigma_1) = S(\Sigma_1) \quad \text{and} \quad S(\Sigma - \Sigma_1) \cong S_{thermal}(\Sigma - \Sigma_1)$$

it then follows

$$S(\Sigma_1) \cong S_{thermal}(\Sigma - \Sigma_1)$$

The coarse grained entropy of $\Sigma - \Sigma_1$ will be of order $(1 - f)S_{thermal}(\Sigma)$, where f is the fraction of the total dof contained in Σ_1. Hence we obtain

$$\begin{aligned} I(\Sigma_1) &= S_{thermal}(\Sigma_1) - S(\Sigma_1) \\ &= f S_{thermal}(\Sigma) - (1 - f)S_{thermal}(\Sigma) = (2f - 1)S_{thermal}(\Sigma) \end{aligned}$$

Before applying these considerations to BHs, we shall first focus on a less exotic system so that we can better understand the implications of the concepts we just derived. Imagine an experiment in which a bomb explodes inside a box with perfectly reflecting walls [7]. The thermal radiation emitted inside slowly leaks out through a small hole. In this case Σ consists of the whole subsystem B that includes everything in the box. On the other hand, the subsystem A consists of everything outside of the box, in this case, outgoing photons.

Initially the bomb is in its ground state, clearly both B and A have vanishing entropy. When the explosion occurs, thermal radiation makes the thermal entropy of the box increasing, but its fine grained entropy stays constant. This is because no photons have yet escaped, so $S(A) = 0$.

Now the first photons start leaking out. As a result, the systems A and B become entangled. The entanglement entropy, which is equal for the two subsystems, begins to increase. Conversely, the thermal entropy in the box decreases:

$$S_{entanglement} \neq 0, \quad S_{thermal}(A) \neq 0, \quad S_{thermal}(B) \neq 0$$

Eventually, all of the photons have left the box. The latter is now empty and the thermal or coarse grained entropy as well as the fine grained entropy of B all tend to zero. The box has came back in a pure state; its ground state. At this time, the thermal entropy of A has reached its final value.

The evolution over time of the various entropies, along with the one of information, are shown in Fig. 5.2.

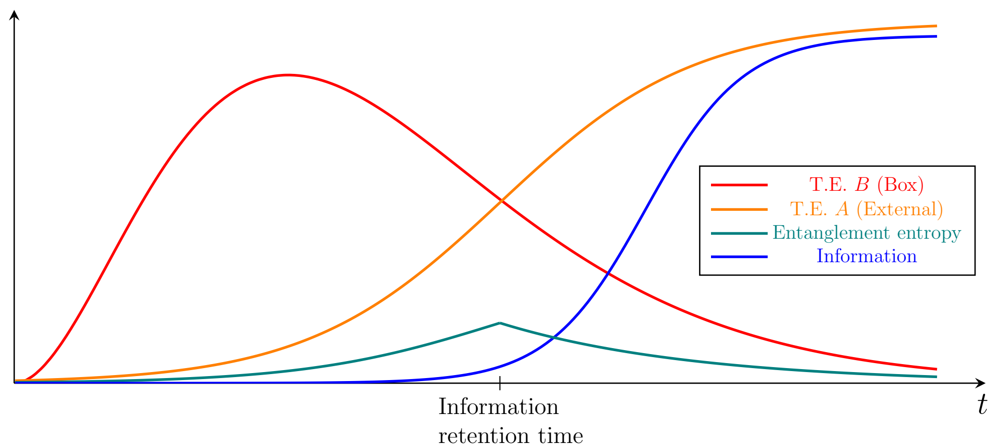


Figure 5.2: Evolution of thermal entropy (T.E.) of box and exterior, entanglement entropy and information [7].

Due to the second law of thermodynamics, $S_{thermal}(A)$ must be larger than $S_{thermal}(B)$ just after the explosion. However, the fine grained entropy of A vanishes as the entanglement between the two subsystems has gone to zero. The actual entanglement entropy must be less than the thermal entropy of A or B.

It is remarkable that when $S_{thermal}(A) = S_{thermal}(B)$ we have the time at which the information in the outside region begins to grow (see Fig. 5.2). Even if at that time a good deal of energy has already escaped, no information was carried along with it. Roughly information appears outside of the box when half of the final entropy of the photons has emerged. This time is called the *information retention time* and corresponds to the amount of time needed to retrieve a single bit of information about the initial state of the box [7].

From the analysis above, we can conclude that the final radiation field outside the box must be in a pure state. However, locally, regions containing a small fraction of the photons may be extremely thermal. The key point is that they usually carry negligible information.

This is how information conservation works for a conventional quantum system. The description of the evolution of different kinds of entropy has been derived from very general principles. Therefore we expect BHs evaporation to obey this fundamental laws of nature.

5.1.2 Quantum Xerox principle

Consider a machine (usually called *Quantum Xerox Machine*) having an input terminal, which takes a generic system, and an output terminal, which will copy the input system, producing the original and a duplicate. This principle, often called the *no-cloning principle*, states the impossibility of creating such a machine [7].

A simple gedankenexperiment demonstrates this clearly. Imagine inserting a single spin particle (such as an electron or a proton) in the input port of the machine (Fig. 5.3).

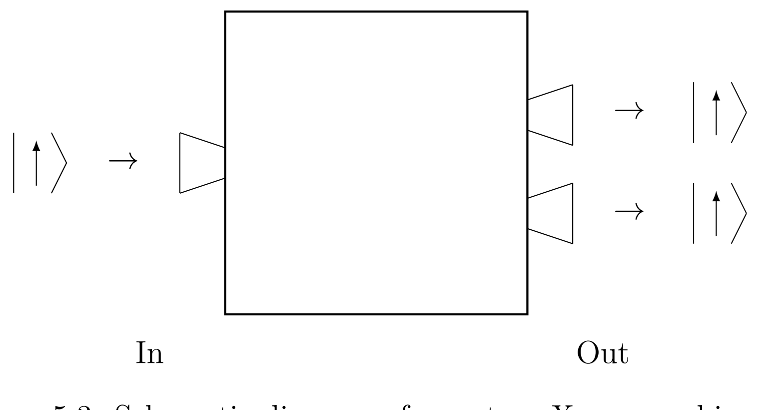


Figure 5.3: Schematic diagram of quantum Xerox machine [7].

If this is in the up state with respect to the z-axis then it duplicates

$$|\uparrow\rangle \rightarrow |\uparrow\rangle\,|\uparrow\rangle \tag{5.2}$$

Similarly, if it is in the down state, it is duplicated

$$|\downarrow\rangle \rightarrow |\downarrow\rangle\,|\downarrow\rangle \tag{5.3}$$

Where $|\uparrow\rangle \equiv |1/2, 1/2\rangle$ and $|\downarrow\rangle \equiv |1/2, -1/2\rangle$.

Now suppose that the spin is inserted with its polarization along the x-axis, i.e.

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \tag{5.4}$$

The general principles of quantum mechanics require the state to evolve linearly. Thus from equations (5.2) and (5.3)

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \rightarrow \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle\right) \tag{5.5}$$

On the other hand, a true quantum Xerox machine is required to duplicate the spin in (5.4)

$$\begin{aligned} &\frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \rightarrow \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle\right) \\ &= \frac{1}{2}|\uparrow\rangle|\uparrow\rangle + \frac{1}{2}|\uparrow\rangle|\downarrow\rangle + \frac{1}{2}|\downarrow\rangle|\uparrow\rangle + \frac{1}{2}|\downarrow\rangle|\downarrow\rangle \end{aligned} \tag{5.6}$$

The two states in equations (5.5) and (5.6) are obviously not the same. Therefore quantum Xerox machines cannot exist precisely due to the linearity principle. Furthermore, if we were able to build Xeroxed quantum states, we would also be able to violate the Heisenberg uncertainty principle by a set of measurements on those states [7].

5.2 The black hole information problem

We have now the tools in order to better understand the argument made by Hawking about the loss of information during BHs creation and evaporation.

To start with, recall that an initial state $|\psi_{in}\rangle$ evolves according to the S-matrix in a final state given by [7]

$$|\psi_{out}\rangle = S\,|\psi_{in}\rangle$$

Information conservation require S to be unitary, hence is possible to recover the initial state from the final state

$$|\psi_{in}\rangle = S^{\dagger}\,|\psi_{out}\rangle$$

Now consider two correlated particles falling towards a BH. Assume one is able to escape to infinity while the other is captured and crosses the horizon. The Hilbert space of the final state will be a tensor product between the state at spatial infinity and that at the singularity:

$$\mathcal{H}_{out} = \mathcal{H}_{\infty} \otimes \mathcal{H}_{S}$$

In virtue of the entanglement between the two particles, the state of the one which escaped to infinity becomes mixed. Indeed if an external observer (for whom the singularity is clearly precluded) performs an experiment on the outgoing particle, this will be described by

$$\rho_{out} = \mathrm{Tr}_{singularity} \left|\psi_{out}\right\rangle \left\langle\psi_{out}\right|$$

where $\mathrm{Tr}_{singularity}$ stands for a trace over the states on the singularity. The problem relies in the fact that the purity of the initial state is not recovered if the BH evaporates. This is a violation of the information conservation principle because S would not result to be a unitary matrix. Therefore this reasoning leads to think that information is somewhere lost during the process of BH formation and evaporation.

Applying the concepts presented in the previous section to BHs evaporation, we can say that once a BH has lost more than a half of its original entropy, information must begin to come out in the emitted radiation. When the majority of mass and entropy has been lost, we expect the entanglement entropy of radiation to be smaller than the remaining entropy of the BH (see Fig. 5.2). Thus, even if all information were emitted at the very end of the evaporation process, a law of nature would be violated from the viewpoint of the external observer. The situation is even worse if one hypotheses information not being emitted at all.

Another possibility is that black holes never completely evaporate. Instead at the end of their lives they remain a stable Planck-mass containing all the lost information. However, such remnants should have an enormous, or even infinite entropy, making it hard to imagine them as real physical systems.

Many theories have been proposed in order to solve the paradox, here we will focus on two of them in particular. The first is based on the idea that the horizon is not penetrable. If this holds, from the viewpoints of an infalling system, the horizon would behave like an infinite potential barrier. This assumption was never seriously taken into account, especially by relativists, because the near horizon region of a Schwarzschild BH is essentially flat space-time, hence any violent disturbance to a freely falling system would violate the equivalence principle.

Finally, quantum Xerox principle discards the other theory, which hypnotise all information to be returned to the outside in Hawking radiation. As we already discussed, the equivalence principle requires information to freely pass through the horizon. On account of Xerox principle, the horizon cannot duplicate information, so it results impossible for a particle inside the BH to escape via Hawking radiation. It is clear that fundamental laws of nature seem to be defied, at least for some observers [7].

5.2.1 Black hole complementarity

Fortunately, BH complementarity comes into help telling us that no observer ever witnesses a violation of a law of nature. For an external observer it can be formulated in the following way

> “A BH is a complex system whose entropy is a measure of its capacity to store information. Entropy is the logarithm of the number of microstates of the dof that make up the BH. The micro-dof can absorb, thermalize, and eventually re-emit all information in the form of Hawking radiation [7].”

The principle does not say what those micro-dof are, but one can estimate their number to find it is about the area of the horizon in Planck units.

For Frefos, one can use the principle above to say that as long as the infalling system is much smaller than the BH, the horizon is seen as flat featureless space-time, no high temperatures or other anomalies are encountered and the equivalence principle is satisfied.

All of this seems to be reassuring, but its validity may not be obvious in view of the contradictions we met in the previous paragraph. To challenge BH complementarity, we can perform the following theoretical experiment.

Consider system A which is freely falling inside a BH. We know that A, along with the information it contains, will cross the horizon without incident. Next, consider an observer B who hovers above the same BH detecting Hawking radiation. The complementarity principle implies that the information carried in by A will be contained in the photons recorded by B. In this way B is able to collect some information about A from outside the BH. Now if B decides to enter the BH a serious problem may occur. Indeed if A sends a signal to B after it passed through the horizon, B would posses two exact copies of the same information.

If this experiment was possible, then BH complementarity would not be self-consistent as the quantum xerox principle would be broken.

However, the restriction to the time when information starts being emitted (half the entropy of the BH must evaporate first) implies that the experiment cannot take place in the real world. From what we have seen, half of the evaporation time t^* is of order M^3G^2. Assume that, at the beginning, the distance between B and the stretched horizon is at least one the Planck length l_P. As a consequence, observer B jumps into the BH from a point in space-time defined by

$$\begin{aligned} \omega^* &\geq \frac{t^* c^2}{4MG} \sim M^2 G \\ \rho^* &\geq l_P \end{aligned}$$

Using the light cone coordinates $x^{\pm} = \rho e^{\pm \omega c}$ we have

$$\begin{aligned} x_*^+ x_*^- &> l_P^2 \\ x_*^+ &\gtrapprox l_P e^{\omega^* c} \end{aligned}$$

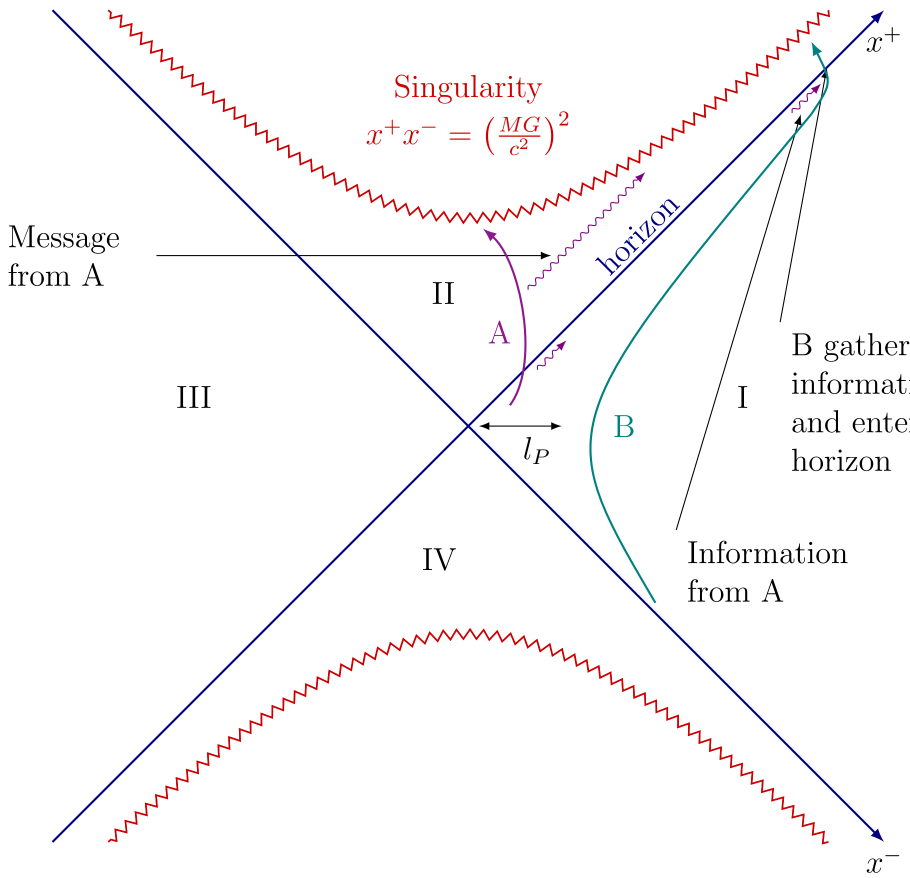


Figure 5.4: Kruskal diagram of a Schwarzschild BH showing the resolution of Xerox paradox for observers within horizon [7].

As shown in Fig. 5.4, observer B hits the singularity before the arrival of the message. The singularity is the surface given by

$$x^+x^- = \left(\frac{MG}{c^2}\right)^2$$

Observer B will reach the singularity at a point with

$$x^- \lesssim \frac{MG}{c^2}e^{-\omega^* c}$$

Therefore if A sends a signal that B is able to receive, the collection must occur at $x^- < \frac{MG}{c^2}e^{-\omega^* c}$, meaning A has a time of order $\Delta t \sim (MG)e^{-\omega^*}$ to send the message.

Classical physics does not put any limit on how much information can be sent in an arbitrarily small period of time. On the other hand, quantum mechanics require at least one quantum in order to send a single bit. In our experiment we can assume the information A wants to send to B consists only of one bit. A quantum needed to convey the message must be emitted between $x^- = 0$ and $x^- = \frac{MG}{c^2} e^{-\omega^* c}$, with an energy bounded by the uncertainty principle, for which

$$E \Delta x^- > \hbar c \quad \Rightarrow \quad E > \frac{e^{\omega^* c} \hbar c^3}{MG} \gg \frac{e^{M^2 G}}{MG}$$

This analysis reveals that the quantum Xerox paradox takes place only if the energy carried by A is many orders of magnitude larger than the BH mass. Obviously this is impossible, otherwise the system A wouldn't have fit inside the horizon.

This example, as many others, manifests how the constraints of quantum mechanics, combined with those of relativity, forbid violations of the complementarity principle [7].

5.3 An overview of the Holographic theory

BH Complementarity had a great impact in the world of physics, leading to brand new interpretations of the nature of space, time and locality. One of the key elements of these new paradigms is the Holographic Principle [7].

As indicated by the divergent entropy density found in equation (3.20), quantum field theory (QFT) possesses too many dof to consistently describe a gravitational theory. This overabundance can be quantitatively measured by means of the Holographic Principle, which can be summarised saying that there are vastly fewer dof in quantum gravity than in any QFT.

To better understand the Holographic Principle we can consider a wide region of space Γ. Assume it to be a sphere for the sake of simplicity. We aim to determine the dimensionality of a specific state-space, the one made by all the states describing arbitrary systems capable to fit into Γ in a way that the region outside is empty space. When counting the states of a system it is helpful to define the concept of *maximum entropy* as the logarithm of the total number of states. Clearly, it is an intrinsic property of the system. The name comes from the fact that it can be viewed as the entropy given that we know nothing about the state of the system.

To give an example, we can take a lattice of discrete spins. If the lattice spacing is a and Γ has a volume V, then the number of spins is Va^{-3} and the number of orthogonal states supported in Γ is

$$N_{states} = 2^{V/a^3}$$

In this case the maximum entropy is

$$S_{max} = \frac{V}{a^3} \log 2$$

Whenever the latter exists, it is proportional to the number of simple dof needed to get a complete description of the system. Usually this means it is proportional to the volume as in the example above. Note that a simple dof represents a single bit of information.

Now we want to include gravity in our description. Consider the spherical region Γ with a boundary $\partial\Gamma$ of area A in a four-dimensional space-time. Suppose Γ contains a thermodynamic system with entropy S. It is safe to say that the total mass of this system cannot exceed the mass of a BH of area A. Now imagine a spherically symmetric light-like shell of matter collapses and with the original mass it forms a BH. Using the right amount of energy the area of the horizon of the BH will result equal to A. In this way we known the entropy of the end product: $S = \frac{c^3 k_B A}{4\hbar G}$. Next we use the second law of thermodynamics to place an upper bound to the original entropy inside Γ, which has to be $\leq S$. In other words, this remarkable results is achieved

> "The maximum entropy of a region of space is proportional to its area measured in Planck units [7]."

A fundamental property of gravitational systems has therefore emerged: they have the maximum number of non-redundant dof proportional to their area.

For a large macroscopic region like a BH this is an enormous reduction in the required dof. Indeed for a system with linear dimensions of order L, a factor $\frac{1}{L}$ in Planck units scales the number of dof per unit volume. Furthermore, the latter can be made arbitrarily sparse in space by increasing the value of L. Nevertheless it is important to keep trace of the microscopic processes happening anywhere in the region. In order to do this, we can imagine that the dof of Γ belong to $\partial\Gamma$, with an area density of no more than ~ 1 dof per Planck area.

We can then conclude with the statement of the Holographic Principle

> "Three-dimensional space is described by a two-dimensional hologram at its boundary [7]."

Appendix

Appendix A

Derivation of Jacobi's formula

Starting from the definition of $\delta \det(M)$:

$$\delta \det(M) \equiv \det(M + \delta M) - \det(M) = \det\left(M\left(I + M^{-1}\right)\right) - \det(M)$$

Using Binet's theorem for the determinant of a product

$$\det\left(M\left(I + M^{-1}\right)\right) = \det(M)\det\left(I + M^{-1}\right)$$

Therefore

$$\begin{aligned}
\delta \det(M) &= \det(M)\det\left(I + M^{-1}\right) - \det(M) \\
&= \det(M)\left[1 + \mathrm{Tr}(M^{-1}\delta M) + \ldots\right] - \det(M) \\
&\approx \det(M)\,\mathrm{Tr}\left(M^{-1}\delta M\right)
\end{aligned}$$

$\square$

Note that the formula holds only if δM becomes infinitesimally small.

Appendix B

Metric of a 2-sphere

Start with the parametrization of a 2-sphere with a radius R and define the metric η_{ij}

$$\begin{cases} x = R\sin(\theta)\cos(\phi) \\ y = R\sin(\theta)\sin(\phi) \\ z = R\cos(\theta) \end{cases} \qquad \text{and} \quad \eta_{ij} = \begin{pmatrix} \frac{\partial \vec{x}}{\partial \theta}\frac{\partial \vec{x}}{\partial \theta} & \frac{\partial \vec{x}}{\partial \theta}\frac{\partial \vec{x}}{\partial \phi} \\ \frac{\partial \vec{x}}{\partial \phi}\frac{\partial \vec{x}}{\partial \theta} & \frac{\partial \vec{x}}{\partial \phi}\frac{\partial \vec{x}}{\partial \phi} \end{pmatrix}$$

$$\Rightarrow \quad \eta_{ij} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2(\theta) \end{pmatrix}$$

where $\vec{x} = (x, y, z)$. Now we can write the metric as

$$dl^2 = \eta_{ij}\left(du^i, du^j\right) = R^2\,(d\theta, d\phi)\begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix}\begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = R^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)$$

where $\vec{u} = (\theta, \phi)$. □

Components of the Ricci tensor for a spherically symmetric metric

The explicit derivation will be shown only for the first component R^0_0, as the procedure is analogous for the others.

Starting from the definition of the Ricci tensor in terms of Christoffel symbols (1.3) we have

$$\begin{aligned} R^k{}_{\alpha k\beta} &= \partial_\beta \Gamma^k{}_{\alpha k} - \partial_k \Gamma^k{}_{\alpha\beta} + \Gamma^\sigma{}_{\beta k}\Gamma^k{}_{\sigma\alpha} - \Gamma^k{}_{k\sigma}\Gamma^\sigma{}_{\alpha\beta} \\ \to \quad R_{00} \equiv R^k{}_{0k0} &= \partial_0 \Gamma^k{}_{0k} - \partial_k \Gamma^k{}_{00} + \Gamma^\sigma{}_{0k}\Gamma^k{}_{\sigma 0} - \Gamma^k{}_{k\sigma}\Gamma^\sigma{}_{00} \end{aligned}$$

In order to evaluate each Christoffel symbol for the metric (2.1) we can use these important relationships between the two:

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}\right) \quad \text{and} \quad \Gamma^\alpha{}_{\mu\alpha} = \Gamma^\alpha{}_{\alpha\mu} = \partial_\mu\left(\ln\sqrt{-g}\right)$$

We shall first evaluate the term $\ln\sqrt{-g} \equiv \ln\sqrt{-\det g_{\mu\nu}}$:

$$\begin{aligned} g &= -e^{2\gamma+2\alpha+4\beta}\sin^2\theta \\ \sqrt{-g} &= e^{\gamma+\alpha+2\beta}\sin\theta \\ \ln\sqrt{-g} &= \gamma + \alpha + 2\beta + \ln\sin\theta \end{aligned}$$

The terms that define R_{00} are then expressed as

$$
\begin{aligned}
\bullet\ \Gamma^k{}_{0k} =& \partial_0(\gamma+\alpha+2\beta+\ln\sin\theta) = \dot\gamma+\dot\alpha+2\dot\beta \\
\rightarrow \quad \partial_0\Gamma^k{}_{0k} =& \partial_0(\dot\gamma+\dot\alpha+2\dot\beta) = \ddot\gamma+\ddot\alpha+2\ddot\beta\ ;
\end{aligned}
$$

$$
\begin{aligned}
\bullet\ \Gamma^k{}_{00} =& \frac{1}{2}g^{k\nu}\left(g_{\nu 0,0}+g_{\nu 0,0}-g_{00,\nu}\right) = \frac{1}{2}g^{k\nu}\left(2g_{\nu 0,0}-g_{00,\nu}\right) \\
=& \frac{1}{2}\left[g^{k0}g_{00,0}+g^{k1}(2g_{01,0}-g_{00,1})\right] = \frac{1}{2}\left[g^{k0}g_{00,0}-g^{k1}g_{00,1}\right] \\
\rightarrow \quad \partial_k\Gamma^k{}_{00} =& \partial_0\frac{1}{2}\left(e^{-2\gamma}2\dot\gamma e^{2\gamma}\right)-\partial_1\frac{1}{2}\left(-e^{-2\alpha}2\gamma' e^{2\gamma}\right) \\
=& \ddot\gamma+e^{2\gamma-2\alpha}\left(\gamma''+2\gamma'^2-2\gamma'\alpha'\right);
\end{aligned}
$$

$$
\begin{aligned}
\bullet\ \Gamma^\sigma{}_{0k} =& \frac{1}{2}g^{\sigma\nu}\left(g_{\nu 0,k}+g_{\nu k,0}-g_{0k,\nu}\right) \\
=& \frac{1}{2}\left[g^{\sigma 0}\left(g_{00,k}+g_{0k,0}-g_{0k,0}\right)+g^{\sigma 1}\left(g_{10,k}+g_{1k,0}-g_{0k,1}\right)\right] \\
\Gamma^k{}_{\sigma 0} =& \frac{1}{2}g^{k\nu}\left(g_{\nu\sigma,0}+g_{\nu 0,\sigma}-g_{\sigma 0,\nu}\right) \\
=& \frac{1}{2}\left[g^{k0}\left(g_{0\sigma,0}+g_{00,\sigma}+-g_{\sigma 0,0}\right)+g^{k1}\left(g_{01,\sigma}+g_{\sigma 1,0}-g_{\sigma 0,1}\right)\right] \\
\rightarrow \quad \Gamma^\sigma{}_{0k}\Gamma^k{}_{\sigma 0} =& \Gamma^0{}_{00}\Gamma^0{}_{00}+\Gamma^0{}_{01}\Gamma^1{}_{00}+\cancel{\Gamma^0{}_{02}\Gamma^2{}_{00}}+\cancel{\Gamma^0{}_{03}\Gamma^3{}_{00}}+\Gamma^1{}_{00}\Gamma^0{}_{10}+\Gamma^1{}_{01}\Gamma^1{}_{10} \\
&+\cancel{\Gamma^1{}_{02}\Gamma^2{}_{10}}+\cancel{\Gamma^1{}_{03}\Gamma^3{}_{10}}+\cancel{\Gamma^2{}_{00}\Gamma^0{}_{20}}+\cancel{\Gamma^2{}_{01}\Gamma^1{}_{20}}+\Gamma^2{}_{02}\Gamma^2{}_{20}+\cancel{\Gamma^2{}_{03}\Gamma^3{}_{20}} \\
&+\cancel{\Gamma^3{}_{00}\Gamma^0{}_{30}}+\cancel{\Gamma^3{}_{01}\Gamma^1{}_{30}}+\cancel{\Gamma^3{}_{02}\Gamma^2{}_{30}}+\Gamma^3{}_{03}\Gamma^3{}_{30} \\
=& \dot\gamma^2+\gamma'^2e^{2\gamma-2\alpha}+\gamma'^2e^{2\gamma-2\alpha}+\dot\alpha^2+\dot\beta^2+\dot\beta^2 \\
=& \dot\gamma^2+\dot\alpha^2+2\dot\beta^2+\gamma'^2e^{2\gamma-2\alpha}\ ;
\end{aligned}
$$

$$
\begin{aligned}
\bullet\ \Gamma^\sigma{}_{00} =& \frac{1}{2}g^{\sigma\nu}(2g_{\nu 0,0}-g_{00,\nu}) = \frac{1}{2}\left[g^{\sigma 0}g_{00,0}-g^{\sigma 1}g_{00,1}\right] \\
\Gamma^k{}_{k\sigma} =& \partial_\sigma(\gamma+\alpha+2\beta+\ln\sin\theta) \\
\rightarrow \quad \Gamma^k{}_{k\sigma}\Gamma^\sigma{}_{00} =& \frac{1}{2}e^{-2\gamma}\left(2\dot\gamma e^{2\gamma}\right)\left(\dot\gamma+\dot\alpha+2\dot\beta\right)-\frac{1}{2}e^{-2\alpha}\left(-2\gamma' e^{2\gamma}\right)\left(\gamma'+\alpha'+2\beta'\right) \\
=& \dot\gamma^2+\dot\gamma\dot\alpha+2\dot\gamma\dot\beta+e^{2\gamma-2\alpha}\left(\gamma'^2+\gamma'\alpha'+2\gamma'\beta'\right)
\end{aligned}
$$

where dots and primes stand for $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial r}$, respectively. In the evaluation of the second to last term the sums containing Christoffel symbols equal to zero have been canceled out.

We are now able to write

$$\begin{aligned}R_{00} &= \ddot{\gamma} + \ddot{\alpha} + 2\ddot{\beta} - \ddot{\gamma} - e^{2\gamma-2\alpha}\left(\gamma'' + 2\gamma'^2 - 2\gamma'\alpha'\right) + \dot{\gamma}^2 + \dot{\alpha}^2 + 2\dot{\beta}^2 + \gamma'^2 e^{2\gamma-2\alpha} \\ &\quad -\dot{\gamma}^2 - \dot{\gamma}\dot{\alpha} - 2\dot{\gamma}\dot{\beta} - e^{2\gamma-2\alpha}\left(\gamma'^2 + \gamma'\alpha' + 2\gamma'\beta'\right) \\ &= \ddot{\alpha} + 2\ddot{\beta} + 2\dot{\beta}^2 - \dot{\gamma}\left(\dot{\alpha} + 2\dot{\beta}\right) + \dot{\alpha}^2 - e^{2\gamma-2\alpha}\left(\gamma'' - \gamma'\alpha' + \gamma'^2 + 2\gamma'\beta'\right)\end{aligned}$$

To obtain R^0_0 we use the relationship $R^0_0 = g^{00}R_{00}$, where $g^{00} = e^{-2\gamma}$.

$$R^0_0 = e^{-2\gamma}\left[2\ddot{\beta} + \ddot{\alpha} + 2\dot{\beta}^2 + \dot{\alpha}^2 - \dot{\gamma}\left(2\dot{\beta} + \dot{\alpha}\right)\right] - e^{-2\alpha}\left[\gamma'' + \gamma'\left(2\beta' + \gamma' - \alpha'\right)\right]$$

□

Derivation of the geodesic equation

Starting from Euler-Lagrange equations of motion

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0$$

$L(x^i, \dot{x}^j) \equiv \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j$ is the Lagrangian function, hence

$$\frac{\partial L}{\partial x^\mu} = \frac{1}{2}\frac{\partial}{\partial x^\mu}\left(g_{ij}\dot{x}^i\dot{x}^j\right) = \frac{1}{2}g_{ij,\mu}\dot{x}^i\dot{x}^j + 0 + 0$$

and

$$\frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{2}\frac{\partial}{\partial \dot{x}^\mu}\left(g_{ij}\dot{x}^i\dot{x}^j\right) = \frac{1}{2}\left(g_{\mu j}\dot{x}^i + g_{\mu\nu}\dot{x}^j\right) = g_{\mu j}\dot{x}^j$$

Taking the derivative with respect to the canonical parameter s

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = g_{ij}\ddot{x}^j + g_{ij,\nu}\dot{x}^j\dot{x}^\nu = g_{ij}\ddot{x}^j + \frac{1}{2}\left(g_{\mu j,\nu} + g_{\mu\nu,j}\right)\dot{x}^j\dot{x}^\nu$$

Putting now the two parts together and multiplying by $g^{i\mu}$ we obtain

$$\ddot{x}^i + \frac{1}{2}g^{il}\left(g_{l\nu,j} + g_{lj,\nu} - g_{j\nu,l}\right)\dot{x}^j\dot{x}^\nu = 0$$

Expressing the second term with a Christoffel symbol we find the geodesic equation

$$\ddot{x}^i + \Gamma^i{}_{j\nu}\dot{x}^j\dot{x}^\nu = 0$$

□

Appendix C

Fine grained entropy

For $\rho = \frac{P}{\operatorname{Tr} P}$, the entropy is

$$S = -\operatorname{Tr} \rho \log \rho = -\operatorname{Tr} \frac{P}{\operatorname{Tr} P} \log \left(\frac{P}{\operatorname{Tr} P} \right) = -\frac{1}{\operatorname{Tr} P} \operatorname{Tr} P(\log P - \log \operatorname{Tr} P)$$

The projection operator P has a diagonal form with $N = \operatorname{Tr} P$ eigenvalues equal to 1 and all the remaining are zeros. Therefore we have that

$$\operatorname{Tr} P \log P = 1 \cdot \log 1 + \cdots + 1 \cdot \log 1 = 0$$

Using this result we obtain

$$S = -\frac{1}{\operatorname{Tr} P} (\operatorname{Tr} P \log P - \operatorname{Tr} P \log \operatorname{Tr} P) = 0 + \frac{N}{N} \log N = \log N$$

□

Entanglement entropy

The choice of a composite system A+B leads to the expansion of its wavefunction $|\Psi\rangle$ over a chosen orthonormal complete basis $|n\rangle_O$ in the Hilbert space of B:

$$|\Psi\rangle = \sum_n |n\rangle_O |\phi_n\rangle_S$$

where $|\phi_n\rangle_S = {}_O \langle n|\Psi\rangle$ are the states of the observable subsystem A relative to $|n\rangle_O$ [1].

From the symmetry of subsystems, in the sense that no one is preferred, we can choose both sets of basis vectors $|n\rangle_A$ and their relative states $|n\rangle_B$ to be orthonormal

$${}_A \langle n|m\rangle_A = \delta_{nm} \quad {}_B \langle n|m\rangle_B = \delta_{nm}$$

and the preferred-basis decomposition transforms into

$$|\Psi\rangle = \sum_n c_n |n\rangle_A |n\rangle_B$$

Now consider the density matrix of the system $\rho = |\psi\rangle \langle\psi|$ and the density matrix of A, obtained by tracing out the dof of B: $\rho_A = \operatorname{Tr}_B |\Psi\rangle \langle\Psi|$ (analogously $\rho_B = \operatorname{Tr}_A |\Psi\rangle \langle\Psi|$). Substituting the expressions above we can rewrite ρ_A (and ρ_B) as

$$\rho_A = \sum_n \rho_n |n\rangle_A {}_A\langle n| \quad , \quad \rho_B = \sum_n \rho_n |n\rangle_B {}_B\langle n| ; \quad \text{with} \quad \rho_n \equiv |c|^2$$

$$\Rightarrow \quad \rho_A |n\rangle_A = \rho_n |n\rangle_A \quad \text{and} \quad \rho_B |n\rangle_B = \rho_n |n\rangle_B$$

□

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