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Heat kernel methods in perturbative quantum gravity

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Abstract

The heat kernel method is a powerful technique in mathematical physics, with applications ranging from black hole entropy to mathematical finance. It consists in a variety of perturbation methods applied to elliptic second–order differential operators on manifolds, which allow to study asymptotic expansions and singularities of Green functions. Perturbative quantum gravity is grounded on the background field method, where the metric tensor is split into a fixed background and quantum perturbations. By moving to euclidean time, the kinetic operators of BRST–quantised gravity become elliptic operators of second–order in partial derivatives, which can be studied via heat kernel techniques. The heat kernel coefficients obtained in this expansion correspond to the counterterms needed to renormalise the one–loop effective action; once computed on–shell, i.e. by using Einstein equations, they become gauge invariant. Up to now, only the first three coefficients for perturbative quantum gravity were known, so the main goal of this thesis is to compute the fourth one, which allows to study renormalisation theory for $D = 6$ gravity at one–loop, behaving similarly to $D = 4$ gravity at two–loops. Both theories are known to be non–renormalisable, and our calculation shows the precise coefficient of the one–loop term that gives the logarithmic divergences in $D = 6$ extended to arbitrary dimensions (for $D > 6$ these divergences are not anymore logarithmic). Our result is in accordance with the one–loop calculation performed independently through the $\mathcal{N} = 4$ spinning particle in the worldline formalism. The computations are then extended to the case of matter fields coupled to the graviton, in the vacuum approximation; a suitable extension of these results to the general matter case might have a role in evaluating quantum corrections to the entropy of Kerr–Newmann black holes.

Contents

Introduction

In a well–known and over quoted paper [1], Eugene Wigner described a peculiar despite recurrent phenomenon in theoretical physics: studying the mathematical structure of a theory sometimes leads to truly physical developments of the theory itself. At the same time, the need to model unintelligible physical phenomena has traditionally led to develop new mathematical structures. This *unreasonable effectiveness* of mathematics, and, more broadly, the continuous interplay between mathematical formulations and their corresponding physical theories, appear especially apparent in mathematical physics, where the same model finds unexpected applications in different disciplines, completely independent one from the other.

The heat kernel method, object of the present thesis, is a powerful technique in mathematical physics, with applications ranging from black hole entropy to mathematical finance; its name comes from the heat equation, which was studied by Fourier in his *Théorie analytique de la chaleur* (1822). His mathematical treatment of the heat propagation led to significant developments in mathematical analysis and differential equation theory [2], but the advent of Riemannian geometry has shown that the technique can be generalised further to study curvature perturbations on manifolds. This idea, however, did not emerge until DeWitt pioneered the computation of the first counterterms for the one–loop effective action for perturbative quantum gravity in four dimensions [3; 4]. His computational techniques were not efficient, and the results contained errors [5], presumably transcription errors or typos; nevertheless, his deep intuition allowed to find more refined computational methods [6; 7], which form now a full coherent mathematical theory [8]. Nowadays, these methods are commonly employed for studying second–order elliptic partial differential operators; since most systems can be described through second–order PDEs, their appereance is ubiquitous in theoretical physics and mathematical modeling.

In this thesis, we apply the heat kernel method to perturbative quantum gravity, in order to investigate deeper the issue of renormalisation in quantum gravity, and in particular the calculation of the counterterms necessary to make the theory finite at one–loop. We shall see that perturbative quantum gravity at two–loops is non–renormalisable, so it can be seen as an effective theory of a more general and up to now unknown quantum theory of gravity. This is already well–known in the literature, and the novelty here is the determination of the gauge–invariant structure of the fourth coefficient in arbitrary dimensions, which reduces to the logarithmic divergence in six dimensions.

In the first chapter, we present a general mathematical description of the heat kernel method, following the most recent treatment of the subject by Avramidi [8]. After having described the connection between second–order elliptic differential operators and Riemannian manifolds, the heat kernel is defined in full generality. The focus is then restricted to the particular case of perturbation theory: the heat kernel can be written as an expansion around small curvature perturbations, whose form is discussed in detail, introducing the necessary mathematical tools. The coefficients of this expansion, defined through a Mellin transform, are the heat kernel coefficients needed for solving the partial differential equation at small times. In the second part of the chapter, we review the effective action formalism for a generic bosonic or fermionic theory, compute the effective action at one–loop and show how this quantity, upon moving to euclidean time, is related to the heat kernel coefficients found above. The chapter ends by sketching the most efficient method that allows to compute these coefficients, and providing the generic values for the first four of them.

In the second chapter, we describe the BRST quantisation of quantum gravity at one–loop: after expanding the Einstein–Hilbert action at one–loop level, gauge fixing of the quantum gauge symmetry is accomplished by introducing ghosts and auxiliary fields. The kinetic operator related to the graviton and ghost fields are found to be second–order elliptic differential operators, which can thereby be treated with heat kernel techniques.

In the third chapter, the computation of the first four heat kernel coefficients for perturbative quantum gravity is described in full detail. After having introduced the curvature monomial expansion for Einstein spaces, which allows to express any quantity defined on these manifolds as a function of invariants up to a fixed order in curvature, the kinetic operators for the ghost and graviton are simplified accordingly. The heat kernel coefficients are then evaluated on–shell, i.e. on a background satisfying Einstein equations, which ensures them to be gauge independent. Thus, they can serve as a benchmark for verifying alternative approaches to perturbative quantum gravity. The main contribution of this work is the evaluation of the fourth coefficient, which has never been computed before in its full generality. The value of this new coefficient is shown to enforce non–renormalisability of one–loop quantum gravity in $D = 6$, but the result, given in a generic dimension D, could be useful in other contexts. The final results are then compared with a different computation, which exploits the $\mathcal{N} = 4$ spinning particle treated within the worldline formalism, providing a strong consistency check on their correctness.

In the fourth and last chapter, the heat kernel method is applied to a slightly more general theory, including both the graviton and matter fields. By adding to the gravity action a scalar, spinor or vector field, the heat kernel coefficients are modified; the computations are performed here only in the vacuum approximation, where matter fields do not contribute to the background metric, which is therefore still a solution of vacuum Einstein equations. Generalisation of these results are found to be useful in computing quantum corrections to Kerr–Newmann black holes entropy, even though a general computation is still missing.

Chapter 1

Heat kernel method

In this chapter we briefly review the mathematical foundations of the heat kernel method, and show its connection with the effective action defined in the background field method for quantum field theories. We also sketch the fundamental ideas of the most recent technique for computing the HDMS coefficients. Our main references are [9; 8].

1.1 Elliptic second–order differential operators

Let us write the coordinate vector in n spatial dimensions as $(t, x^i) \in \mathbb{R}^{n+1}$ with $i = 1, \ldots, n$. The most general form for a *second–order differential operator* is

$$L(t, x, \partial_x) = -\sum_{i,j=1}^{n} \alpha^{ij}(t,x)\partial_i\partial_j + \sum_{i=1}^{n} \beta^i(t,x)\partial_i + \gamma(t,x)\,, \tag{1.1}$$

where $\alpha^{ij}(t,x)$ are real functions, while $\beta^i(t,x)$ and $\gamma(t,x)$ can be complex; we will assume that they are all $C^\infty(M)$–functions, with $M \subseteq \mathbb{R}^{n+1}$. Here and in the following we assume also that M has no boundary, $\partial M = \varnothing$. By introducing the *dual* variables p_i and the inner product $\langle p, x\rangle \equiv \sum_i p_i x^i$, we define the *symbol* of the operator (1.1) as the quantity

$$\sigma(t,x,p) \equiv \langle p,\ A(t,x)\,p\rangle + i\,\langle p, \beta(t,x)\rangle + \gamma \tag{1.2}$$

and the *leading symbol* as

$$\sigma_L(t,x,p) \equiv \langle p,\ A(t,x)\,p\rangle\ , \tag{1.3}$$

where $A \equiv (\alpha^{ij})$ and $\beta \equiv (\beta^i)$, and all derivatives have been replaced as $\partial_j \to i p_j$. An operator is said to be *elliptic* if $\forall x \in M$ and $p \neq 0$ the leading symbol of the operator is positive, $\sigma_L(t,x,p) > 0$.[1]

[1] This is the case, for instance, of the Laplacian $\Delta \equiv \sum_i \partial_i^2$, which has $\sigma_L = |p|^2 > 0$.

The operator (1.1) can be seen as acting on $L^2(M,\mu)$, the space of square–integrable functions on M with weight function $\mu(x)$, and its (formal) *adjoint* L^* is defined by $(\varphi, L\psi) = (L^*\varphi, \psi)$.[2] It turns out to be really useful to decompose the quantities β^j and γ in terms of real–valued functions A_i, B^j and Q:

$$\begin{aligned}\beta^j &\equiv B^j - \sum_i \mu^{-1}\partial_i\left(\mu\alpha^{ij}\right) - 2i\sum_k \alpha^{jk}A_k \\ \gamma &\equiv Q + \sum_{ij} A_i\alpha^{ij}A_j - i\sum_{ij}\mu^{-1}\partial_i\left(\mu\alpha^{ij}A_j\right),\end{aligned} \tag{1.4}$$

since this allows to write (1.1) as

$$L = -\sum_{kj}\mu^{-1}\mathcal{D}_j\mu\alpha^{jk}\mathcal{D}_k + B^j\mathcal{D}_j + Q\,, \tag{1.5}$$

where we introduced the *covariant derivative* $\mathcal{D}_j \equiv \partial_j + iA_j$. Equation (1.5) enlightens that the theory of second–order differential operators is deeply intertwined with Riemannian geometry, since if L is elliptic, α^{ij} is a real, symmetric and non–degenerate tensor, which can be naturally identified with a metric tensor.

1.1.1 Extension to Riemannian manifolds

Riemannian geometry shall be really helpful in studying elliptic second–order differential operators, since they can be defined in terms of the intrinsic geometry of the manifold. Consider a Riemannian manifold $(\mathcal{M}, \boldsymbol{g})$, equipped with a metric tensor $\boldsymbol{g} = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$; the operator (1.1) can likewise be written as

$$L = -\alpha^{ij}(x)\partial_i\partial_j + \beta^j(x)\partial_j + \gamma(x)\,, \tag{1.6}$$

where the symmetric and positive–definite tensor α^{ij} is to be identified with the inverse metric $g^{ij} = (g_{ij})^{-1}$ and $x \in \mathcal{U} \subseteq \mathcal{M}$, $\partial\mathcal{U} = \varnothing$. The decomposition (1.5) can be applied again, leading this time to

$$L = -g^{-\frac{1}{2}}\left(\partial_i + \mathcal{A}_i\right) g^{\frac{1}{2}} g^{ij}\left(\partial_j + \mathcal{A}_j\right) + Q\,, \tag{1.7}$$

since now $\mu(x) = g^{\frac{1}{2}}(x)$ in the Hilbert space $L^2(\mathcal{M}, g^{\frac{1}{2}})$, where $g \equiv |\det \boldsymbol{g}|$. The quantities $\mathcal{A}_i$ and Q appearing in (1.7) are defined as

$$\begin{aligned}\mathcal{A}_i &\equiv -\frac{1}{2}g_{ij}\left(\beta^j + \Gamma^j\right) \\ Q &\equiv \gamma + g^{ij}\mathcal{A}_i\mathcal{A}_j + g^{-\frac{1}{2}}\partial_i\left(g^{\frac{1}{2}}g^{ij}\mathcal{A}_j\right),\end{aligned} \tag{1.8}$$

[2] The inner product $(\cdot,\cdot)$ in the space of square–integrable functions $L^2(M,\mu)$ should not be confused with the one $\langle\cdot,\cdot\rangle$ between coordinates x^i and dual variables p_i.

where $\mathcal{A}_i$ is known as *generalised connection*, while Γ^j are the contracted Christoffel symbols.[3] Using the covariant derivatives defined above and a simple identity,[4] (1.7) can be written as

$$L = -\left(\nabla_i + \mathcal{A}_i\right) g^{ij} \left(\nabla_j + \mathcal{A}_j\right) + Q \tag{1.9}$$

$$Q = \gamma + g^{ij}\mathcal{A}_i\mathcal{A}_j + g^{ij}\nabla_i\mathcal{A}_j \, . \tag{1.10}$$

This allows us to interpret the quantity $\mathcal{R}_{ij} \equiv \partial_i\mathcal{A}_j - \partial_j\mathcal{A}_i$ as a *generalised curvature*, since the covariant derivative can be extended to a *generalised covariant derivative* containing both a gravitational (spacetime) term and a purely gauge (internal) term,

$$\nabla_i^{\mathcal{A}} \equiv \nabla_i + \mathcal{A}_i \, , \tag{1.11}$$

allowing to write the *Laplace–type operator* (1.9) more compactly as

$$L = -g^{ij}\nabla_i^{\mathcal{A}}\nabla_j^{\mathcal{A}} + Q \, . \tag{1.12}$$

1.2 The heat kernel

Consider now a positive–definite operator A in a Hilbert space, $(A, A) > 0$. We define the *heat semigroup* as the operator

$$U(t) = \exp\left(-tA\right) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}(tA)^j \, , \tag{1.13}$$

which is well defined for $t > 0$, and satisfies the *operator heat equation*

$$\left(\partial_t + A\right) U(t) = 0 \qquad \text{with} \qquad U(0) = \mathbb{1} \, . \tag{1.14}$$

The operator $U(t)$ enjoys the semigroup defining properties, since for any $t_1, t_2 > 0$ one has $U(t_1 + t_2) = U(t_1)\, U(t_2)$, but since in general it is not invertible, as $U^{-1}(t) = \exp\left(tA\right)$ is not well defined for $t > 0$, it cannot be considered as an element of a group.

[3] Starting from $\Gamma^i_{jk} = \frac{1}{2}g^{im}\left(\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{jk}\right)$ and contracting with δ^k_i we get the expression

$$\Gamma_i \equiv \Gamma^j_{ij} = \frac{1}{2}g^{jk}\partial_i g_{jk} = \frac{1}{2}g^{-1}\partial_i g = g^{-\frac{1}{2}}\partial_i g^{\frac{1}{2}} \, .$$

[4] As a consequence of the previous result, given a vector field K^i defined on the manifold, we have the useful formula

$$\nabla_i K^i = \partial_i K^i + \Gamma_i K^i = g^{-\frac{1}{2}}\partial_i\left(g^{\frac{1}{2}}K^i\right) .$$

The *heat kernel* is then defined as the *integral kernel*[5] of the heat semi-group for the operator L in (1.1), which in spectral decomposition reads

$$U(t;x,x') \equiv \sum_j e^{-t\lambda_j} \varphi_j(x)\overline{\varphi_j(x')}\,, \tag{1.15}$$

where $L\varphi_j(x) = \lambda_j \varphi_j(x)$ are the eigenfunctions and eigenvalues of the differential operator L. According to (1.14), $U(t;x,x')$ satisfies the *differential heat equation*[6]

$$[\partial_t + L(x,\partial_x)]\, U(t;x,x') = 0\,, \qquad \text{with} \qquad U(0;x,x') = \delta(x,x')\,, \tag{1.16}$$

where the δ–function, in the case of curved space, is to be interpreted as a biscalar density.[7]

The usefulness of this operator appears once we consider how it can be exploited to solve the initial value problem for the "heat equation"

$$\begin{cases} [\partial_t + L(x,\partial_x)]\, V(t,x) = 0 \\ V(0,x) = f(x)\,, \end{cases} \tag{1.17}$$

which is solved indeed by

$$V(t,x) = \int_M \mathrm{d}x'\;\mu(x')\, U(t;x,x')\, f(x')\,. \tag{1.18}$$

Similarly, for the non–homogeneous case, where a generic (smooth) function $h \in C^\infty(M)$ appears on the right–hand side, we have that

$$\begin{cases} [\partial_t + L(x,\partial_x)]\, V(t,x) = h(t,x) \\ V(0,x) = f(x) \end{cases} \tag{1.19}$$

[5]The integral kernel of an integral operator K acting on $L^2(M,\mu)$, $M \subseteq \mathbb{R}^n$, is a two–point function $K(x,x')$ such that $\forall f \in L^2(M,\mu)$ one has

$$(Kf)(x) = \int_M \mathrm{d}x'\;\mu(x')K(x,x')f(x')\,.$$

[6]There is also an equivalent equation for the adjoint operator L^* [8, § 2.4.1] which for sake of brevity we will omit in the following discussion.

[7]In detail, $\delta(x,x') \equiv g^{-\frac{1}{4}}(x)\delta(x-x')g^{-\frac{1}{4}}(x')$, where the factors of $g(x)$ are inserted in a symmetric way to satisfy the definition

$$\int_{\mathcal{M}} \mathrm{d}x\;\sqrt{g}\;\delta(x,x')f(x) \equiv f(x')\,,$$

so that the heat kernel is a biscalar (and not a biscalar density, as it would have been, had we chosen the standard definition for the δ–function).

is solved by

$$V(t,x) = \int_M \mathrm{d}x'\ \mu(x')\,U(t;x,x')\,f(x') + \int_0^t \mathrm{d}t' \int_M \mathrm{d}x'\ \mu(x')\,U(t;x,x')\,h(t',x')\,. \tag{1.20}$$

The forcing term, according to *Duhamel's principle*, is adding a new initial condition at each instant of time [10, § 1.1]. On a Riemannian manifold $\mathcal{M}$, we can formulate the same problem, upon replacing $L(x,\partial_x)$ by the Laplace–type operator (1.12). It can be shown that for $t > 0$ the heat kernel is a smooth function of the time t and of x, x' [8, § 5.1]; however, it is impossible to analytically evaluate it in general; in the following, we will therefore focus on the approximation at small times, $t \to 0$.

1.2.1 Singular perturbation theory

Consider the operator L written in the form (1.6), and rescale all derivatives by a constant parameter $\varepsilon > 0$, to get a *singularly perturbed* operator

$$L(x,\varepsilon\partial) = -\varepsilon^2\alpha^{jk}(x)\partial_j\partial_k + \varepsilon\beta^j(x)\partial_j + \gamma\,. \tag{1.21}$$

The corresponding heat equation becomes

$$[\varepsilon\partial_t + L(x,\varepsilon\partial_x)]\,U(t;x,x') = 0\,, \qquad \text{with} \qquad U(0;x,x') = \delta(x,x')\,. \tag{1.22}$$

We now look for a solution of (1.22) by introducing the following *semi–classical ansatz* for the heat kernel:

$$U(t;x,x') \sim \varepsilon^{-\frac{n}{2}} g^{-\frac{1}{4}}(x) g^{-\frac{1}{4}}(x') \exp\left\{-\frac{1}{\varepsilon}S(t;x,x')\right\} \sum_{j=0}^{\infty} \varepsilon^j b_j(t;x,x')\,. \tag{1.23}$$

This choice is motivated as follows: the solution of equation (1.22) for operators (1.21) with costant coefficients is a sort of "plane wave" in euclidean space [8, §§ 4.2.1, 4.3], so the idea is to replace in the limit $\varepsilon \to 0$ *and* $t \to 0$ this euclidean plane wave by a *distorted* one which depends on a function $S(t;x,x')$ and on an expansion around $\varepsilon = 0$, with coefficients $b_j(t;x,x')$. The factor $\varepsilon^{-\frac{n}{2}} g^{-\frac{1}{4}}(x) g^{-\frac{1}{4}}(x')$ is introduced to satisfy the initial condition (1.22), that is

$$\begin{aligned}\lim_{\varepsilon,\,t\to 0} U(t;x,x') &\sim \lim_{\varepsilon,\,t\to 0} \varepsilon^{-\frac{n}{2}} g^{-\frac{1}{4}}(x) g^{-\frac{1}{4}}(x') \exp\left\{-\frac{1}{\varepsilon}S(t;x,x')\right\} b_0(t;x,x') \\ &= g^{-\frac{1}{4}}(x)\delta(x-x')g^{-\frac{1}{4}}(x') = \delta(x,x')\,.\end{aligned} \tag{1.24}$$

The algorithm for determining the function $S(t;x,x')$ and the coefficients $b_j(t;x,x')$ is rather simple: we plug the ansatz (1.23) in the differential equation (1.22) and equate to zero the coefficients at order ε^j. For $j = 0$ we get a non–linear first–order partial differential equation for S, known as *Hamilton–Jacobi equation*. To solve this equation, one has to introduce the Hamiltonian system of equations, whose solution is the action S. For $j \geq 1$, we get a system of differential recursion relations for b_j, known as *transport equations*, which allow to find as many coefficients b_j as needed.

Another form for the expansion

Introduce now a real two–point function $\Phi = \Phi(x, x')$ such that $\Phi(x, x') > 0$ for $x \neq x'$ and $\Phi(x, x') = 0$ iff $x = x'$. We also assume that it is an analytic function of x, so that it can be expanded in Taylor series around x', with

$$\partial_i \Phi(x, x')\big|_{x=x'} = \partial_{j'} \Phi(x, x')\big|_{x=x'} = 0\,, \tag{1.25}$$

and that it has a non degenerate Hessian $\det\left[-\partial_i\partial_{j'}\Phi(x, x')\right]\big|_{x=x'} \neq 0$. This means that $\Phi(x, x')$ has a non–degenerate absolute minimum at x' equal to zero.[8] Then the limit derived in appendix 1.A, equation (1.108), can be generalised as

$$\lim_{t\to 0} (4\pi t)^{-\frac{n}{2}} \det\left[-\partial_i\partial_{j'}\Phi(x, x')\right]^{\frac{1}{2}} \exp\left\{-\frac{1}{2t}\Phi(x, x')\right\} = \delta(x, x')\,, \tag{1.26}$$

and this equation still holds if $\Phi(t; x, x')$ is an analytic function of t as well, such that $\Phi_0(x, x') = \Phi(0; x, x')$ satisfies the conditions above and $\partial_t\Phi(0; x, x')|_{x=x'} = 0$. By comparison between (1.24) and (1.26) the functions S and b_0 have the following asymptotics at $t \to 0$:

$$S(t; x, x') = \frac{1}{2t}\Phi(x, x') + \mathcal{O}(1) \tag{1.27}$$

$$b_0(t; x, x') = (4\pi t)^{-\frac{n}{2}} \det\left[-\partial_i\partial_{j'}\Phi(x, x')\right]^{\frac{1}{2}} + \mathcal{O}\left(t^{1-\frac{n}{2}}\right). \tag{1.28}$$

These functions can then be written in terms of expansions around $t \to 0$, by introducing the unknown functions $S_j(x, x')$ and $b_{jk}(x, x')$,

$$S(t; x, x') = \frac{1}{2t}\Phi(x, x') + \sum_{j=0}^{\infty} t^j S_j(x, x') \tag{1.29}$$

$$b_j(t; x, x') = (4\pi t)^{-\frac{n}{2}} \det\left[-\partial_i\partial_{j'}\Phi(x, x')\right]^{\frac{1}{2}} \sum_{k=0}^{\infty} t^k b_{jk}(x, x')\,, \tag{1.30}$$

[8] We denote by primed indices, i.e. $\partial_{i'}$, the derivatives with respect to x'.

and by plugging (1.29) and (1.30) inside our ansatz (1.23), upon a suitable redefinition of the generic functions $b_{jk}(x,x')$, we find

$$U(t;x,x') \sim \varepsilon^{-\frac{n}{2}}(4\pi t)^{-\frac{n}{2}} g^{-\frac{1}{4}}(x) \det\left[-\partial_i\partial_{j'}\Phi(x,x')\right]^{\frac{1}{2}} g^{-\frac{1}{4}}(x') \cdot \exp\left\{-\frac{1}{2\varepsilon t}\Phi(x,x')\right\}\sum_{j=0}^{\infty}(\varepsilon t)^j c_j(x,x')\,, \tag{1.31}$$

where $\Phi(x,x')$ and $c_j(x,x')$ are still undetermined functions.

1.2.2 Hamilton–Jacobi equation

At this point, it is important to note that the ansatz (1.31) is not written in a covariant way, but can be easily made so. Consider the form (1.12) of the operator L, and singularly perturb it as

$$L_\varepsilon \equiv \varepsilon^2 L = -\varepsilon^2 g^{ij}\nabla_i^{\mathcal{A}}\nabla_j^{\mathcal{A}} + \varepsilon^2 Q\,. \tag{1.32}$$

Assuming that the coefficients of L do not depend on ε, the following *commutation formula* [8, § 5.1] is easily seen to hold:

$$\exp\left(\frac{1}{\varepsilon}S\right)\left(\varepsilon\partial_t + \varepsilon^2 L\right)\exp\left(-\frac{1}{\varepsilon}S\right) = T_0 + \varepsilon T_1 + \varepsilon^2 L\,, \tag{1.33}$$

with

$$\begin{cases} T_0 = -\dot{S} - g^{ij}S_iS_j \\ T_1 = \partial_t + 2g^{ij}S_j\left(\nabla_i + \mathcal{A}_i\right) + g^{ij}S_{ij} \end{cases} \tag{1.34}$$

by denoting $S_i \equiv \nabla_i S$ and $S_{ij} \equiv \nabla_i\nabla_j S$. In order to cancel the leading term in (1.33), we require that $T_0 = 0$ in (1.34), that is,

$$\partial_t S + g^{ij}\nabla_i S\nabla_j S = 0\,, \tag{1.35}$$

which is our Hamilton–Jacobi equation. Moreover, due to (1.29) we can assume that $S(t;x,x')$ takes the form

$$S(t;x,x') \equiv \frac{1}{2t}\sigma(x,x')\,, \tag{1.36}$$

hence (1.35) becomes

$$\sigma = \frac{1}{2}g^{ij}(\nabla_i\sigma)(\nabla_j\sigma)\,, \tag{1.37}$$

which is the definition of the *Synge function* $\sigma(x,x')$, see appendix 1.B, and gives the exact form of the action S. The ansatz (1.31) can now be written in an explicitly covariant manner,

$$U(t;x,x') \sim \varepsilon^{-\frac{n}{2}}(4\pi t)^{-\frac{n}{2}}\Delta^{\frac{1}{2}}(x,x')\exp\left\{-\frac{1}{2\varepsilon t}\sigma(x,x')\right\}\sum_{j=0}^{\infty}(\varepsilon t)^j c_j(x,x')\,, \tag{1.38}$$

where $\Delta(x,x')$ is *Van Vleck–Morette determinant*, defined in appendix 1.C.

With the asymptotic ansatz (1.38) in mind, the second line of (1.34), together with (1.36), gives the following *transport operator*

$$T_1 = \partial_t + \frac{1}{t}\left(D + \sigma^i \mathcal{A}_i + \frac{1}{2}\sigma^i{}_i\right), \tag{1.39}$$

where $\sigma_i \equiv \nabla_i \sigma$, $\sigma_{ij} \equiv \nabla_i \nabla_j \sigma$ and $D \equiv \sigma^i \nabla_i$. Using (1.129), proved in appendix 1.C, (1.39) can equivalently be written as

$$T_1 = t^{-\frac{n}{2}} \Delta^{\frac{1}{2}} \left[\partial_t + \frac{1}{t}\left(D + \sigma^i \mathcal{A}_i\right)\right] t^{\frac{n}{2}} \Delta^{-\frac{1}{2}}\,; \tag{1.40}$$

moreover, by introducing of the *generalised operator of parallel transport* $\mathcal{P}(x,x')$, defined in appendix 1.D,

$$T_1 = t^{-\frac{n}{2}} \mathcal{P} \Delta^{\frac{1}{2}} \left(\partial_t + \frac{1}{t} D\right) t^{\frac{n}{2}} \mathcal{P}^{-1} \Delta^{-\frac{1}{2}}\,. \tag{1.41}$$

Since the asymptotic ansatz (1.38) depends on ε and t only through their product εt, by replacing $\varepsilon t \to t$ and using (1.41), it becomes

$$U(t;x,x') \sim (4\pi t)^{-\frac{n}{2}} \mathcal{P}(x,x') \Delta^{\frac{1}{2}}(x,x') \exp\left\{-\frac{1}{2t}\sigma(x,x')\right\} \Omega(t;x,x') \tag{1.42}$$

where we assume that x and x' are sufficiently close, so that all two–point functions are well defined, i.e. the geodesic connecting the two points is unique. The *transport function* $\Omega(t;x,x')$ is arbitrary, besides satisfying a *transport equation*

$$\left(\partial_t + \frac{1}{t} D + \hat{L}\right) \Omega(t;x,x') = 0 \qquad \text{with} \qquad \Omega(0;x,x') = 1\,, \tag{1.43}$$

where $\hat{L} \equiv \mathcal{P}^{-1} \Delta^{-\frac{1}{2}} L \Delta^{\frac{1}{2}} \mathcal{P}$. The condition (1.43) comes from imposing the commutation formula (1.33).

1.3 Minakshisundaram–Pleijel equation

We assume that the potential $Q(x)$ is bounded from below by a positive parameter m^2, so that $Q(x) \geq m^2$. Then the operator L is positive–definite and the heat kernel $U(t;x,x')$ and the function $\Omega(t;x,x')$ decrease at the infinity $t \to \infty$ more rapidly than any power of t, whereas around $t \to 0$ we can expand $\Omega(t;x,x')$ in positive integer powers of t. Therefore, for any $\alpha, N \geq 0$:

$$\lim_{t \to \infty,\,0} t^{\alpha} \left(\frac{\partial}{\partial t}\right)^N \Omega(t;x,x') = 0\,. \tag{1.44}$$

Consider now the Mellin transform of the transport function $\Omega(t;x,x')$,

$$b_q(x,x') \equiv \frac{1}{\Gamma(-q)} \int_0^\infty \mathrm{d}t\; t^{-q-1} \Omega(t;x,x') \,, \tag{1.45}$$

which converges in the region $\mathrm{Re}\,(q) < 0$, $q \in \mathbb{C}$. The function $b_q(x,x')$ can be extended for $\mathrm{Re}\,(q) \geq 0$ via analytic continuation [7, § 2]. By integrating by parts, for $\mathrm{Re}\,(q) < N \in \mathbb{N}$,

$$b_q(x,x') = \frac{(-1)^N}{\Gamma(-q+N)} \int_0^\infty \mathrm{d}t\; t^{-q-1+N} \left(\frac{\partial}{\partial t}\right)^N \Omega(t;x,x') \,, \tag{1.46}$$

and the asymptotic property (1.44), together with integration by parts, allows to compute $b_q(x,x')$ at the positive integers $q = k$,

$$b_k(x,x') = \left(-\frac{\partial}{\partial t}\right)^k \Omega(t;x,x')\bigg|_{t=0} \tag{1.47}$$

with asymptotic behaviour

$$\lim_{\substack{|q|\to\infty \\ \mathrm{Re}(q)<N}} \Gamma(-q+N)\; b_q(x,x') = 0 \,. \tag{1.48}$$

The functions $b_q(x,x')$ can therefore be seen as an analytical continuation of the coefficients b_k on the whole complex plane of q from positive integer values, with the asymptotic condition (1.48). By inverting the Mellin transform,

$$\Omega(t;x,x') = \int_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}q}{2\pi i}\; t^q\; \Gamma(-q)\; b_q(x,x') \,, \qquad c < 0 \,, \tag{1.49}$$

which gives the *heat kernel diagonal* as an integral of the *coincidence limit* $x' \to x$ of the function b_q,

$$U(t;x,x) = (4\pi t)^{-\frac{n}{2}} \int_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}q}{2\pi i}\; t^q\; \Gamma(-q)\; b_q(x,x) \,. \tag{1.50}$$

Shifting to the right the contour of integration in (1.49) and taking into account the properties (1.47) and (1.48), it is possible to prove that

$$\begin{aligned}\Omega(t;x,x') &= \sum_{j=0}^{N-1} \frac{t^j}{j!} b_j(x,x') + \int_{c_N-i\infty}^{c_N+i\infty} \frac{\mathrm{d}q}{2\pi i}\; t^q\; \Gamma(-q)\; b_q(x,x') \\ &\equiv \sum_{j=0}^{N-1} \frac{t^j}{j!} b_j(x,x') + R_N(t;x,x') \,, \end{aligned} \tag{1.51}$$

with $N-1 < c_N < N$. Here, $R_N(t;x,x') \sim \mathcal{O}(t^N)$ as $t \to 0$ and is smaller than the last term in the sum, so (1.51) gives the asymptotic expansion of $\Omega(t;x,x')$ as $t \to 0$ in the form

$$\Omega(t;x,x') \sim \sum_{j=0}^{\infty} \frac{t^j}{j!} b_j(x,x')\,, \tag{1.52}$$

leading to the asymptotic expansion of the heat kernel diagonal,

$$U(t;x,x) \sim (4\pi t)^{-\frac{n}{2}} \sum_{j=0}^{\infty} \frac{t^j}{j!} b_j(x,x)\,. \tag{1.53}$$

The coefficients $b_j(x,x')$ are smooth functions known as *Hadamard–DeWitt–Minakshisundaram–Seeley (HDMS) coefficients*, and the asymptotic expansion (1.53) is known as *Minakshisundaram–Pleijel equation.*

Note that the expansion (1.52) is convergent only if the remainder term $R_N(t;x,x')$ vanishes as $N \to \infty$ in a neighborhood of $t=0$, in which case $\Omega(t;x,x')$ is analytic at $t=0$. In general, for any fixed $t>0$, $R_N(t;x,x')$ does not vanish as $N \to \infty$ and (1.52) diverges for any $t>0$. Thus the asymptotic ansatz (1.52) makes sense only when its lowest–order terms are essential.

1.4 One–loop effective action

Consider a generic field $\varphi(x)$ on a n–dimensional space–time, with components $\varphi^A(x)$ which transform with respect to some representation of the diffeomorphism group, and which can be either bosonic or fermionic:

$$\varphi^A \varphi^B = (-1)^{AB} \varphi^B \varphi^A\,, \tag{1.54}$$

with indices A, B equal to zero in the first case, and equal to one in the latter. In order to construct a local action functional $S[\varphi]$ we also need to introduce a metric for the configuration space E_{AB}, which allows to define a scalar product

$$(\varphi_1, \varphi_2) = \varphi_1^A E_{AB} \varphi_2^B \tag{1.55}$$

and to higher and lower the indices, $\varphi_A = \varphi^B E_{BA}$ and $\varphi^B = \varphi_A (E^{-1})^{AB}$, with $(E^{-1})^{AB} E_{BC} = \delta^A_C$. If the theory is gauge–redundant — which is the case of quantum gravity — we assume that the ghost fields are included in the set of fields $\varphi(x)$, and that the action is modified accordingly. We also employ *DeWitt hyper–condensed notation.*[9]

[9] This notation combines summation with integration, allowing to use a unique discrete–continuous index $i \equiv (A,x)$ to denote the fields, $\varphi^i \equiv \varphi^A(x)$. Summation–integration is then combined in a brief contraction of indices, $\varphi_i \psi^i \equiv \int \mathrm{d}^n x\ \varphi_A(x) \psi^A(x)$.

Consider two causally connected in– and out–regions in spacetime, that lie in the past and in the future with respect to the region containing the physical system we are looking at, and define two vacuum states $|\text{in}\rangle$, $\langle\text{out}|$ in the two regions. The *transition amplitude* between these two states, in presence of background classical sources J_i, is given by the path integral

$$\langle\text{out}|\text{in}\rangle = \int \mathrm{d}\varphi\, \mathcal{M}[\varphi]\ \exp\left\{\frac{i}{\hbar}\left(S[\varphi] + J_i\varphi^i\right)\right\} \equiv \exp\left\{\frac{i}{\hbar}W[J]\right\}. \tag{1.56}$$

The quantity $\mathcal{M}[\varphi]$ is a measure functional, fixed by canonical quantisation of the theory, while $W[J]$ is the generating functional for connected correlations functions,

$$\langle\varphi^{i_1}\dots\varphi^{i_k}\rangle_C = \left(\frac{\hbar}{i}\right)^{k-1}\frac{\delta_L^k}{\delta J_{i_1}\dots\delta J_{i_k}}W[J]\,, \tag{1.57}$$

where δ_L is the left functional derivative. In particular, the first functional derivative of $W[J]$ gives the *background field*

$$\langle\varphi^i\rangle \equiv \Phi^i[J] = \frac{\delta_L}{\delta J_i}W[J]\,, \tag{1.58}$$

in terms of which we can define the key concept of *effective action* $\Gamma[\Phi]$, through the functional Legendre transform

$$\Gamma[\Phi] = W[J] - J_i\Phi^i\,. \tag{1.59}$$

The right functional derivative of (1.59) gives the sources,

$$\frac{\delta_R}{\delta\Phi^i}\Gamma[\Phi] = -J_i[\Phi]\,, \tag{1.60}$$

and (1.60) shows that, by taking $J_i[\Phi] \equiv 0$, the effective action generates *effective equations of motion* in the background field Φ. The second derivative of (1.59) determines the propagator, while higher derivatives give all vertex functions. By using the definition (1.59), as well as (1.56), one finds that $\Gamma[\Phi]$ satisfies the integro–differential equation

$$\exp\left\{\frac{i}{\hbar}\Gamma[\Phi]\right\} = \int \mathrm{d}\varphi\, \mathcal{M}[\varphi]\ \exp\left\{\frac{i}{\hbar}\left[S[\varphi] - \frac{\delta_R\Gamma[\Phi]}{\delta\Phi^i}\left(\varphi^i - \Phi^i\right)\right]\right\}. \tag{1.61}$$

This shows that, when using the effective action functional for the construction of the S–matrix, one needs only tree–level diagrams, since all quantum corrections determined by the loops are already included in the full propagator and vertex functions. In other words, the fundamental entities of the theory (effective action and Green's function) become functionals of an external classical (background) field and in principle contain the entire information of quantum field theory [11]. The effective equations of motion

(1.60), with $J_i = 0$, describe the dynamics of the background fields with regard to all quantum corrections.

At this point, a problem arises: the off–shell effective action depends on the choice of the gauge, so it is not gauge–invariant. The solution of this problem has been proposed by Vilkovisky [12] and later extended to curved spacetimes by DeWitt [13], and goes under the name of *Vilkovisky–DeWitt formalism*. The key idea is to introduce a connection in configuration space, which allows to keep gauge invariance safe. Very recent developments [14], however, show that a more refined definition of background independence, rooted on the extensive studies on this subject made by Anderson [15; 16; 17], appears to conflict with gauge invariance. We will not tackle this issue here,[10] even though it is important, from a more general–theoretical point of view, to be aware of its existence.

Going back to our formalism, we can now expand the effective action by orders of $\hbar$, which correspond to the number of loops,[11]

$$\Gamma[\Phi] \equiv S[\Phi] + \sum_{k\geq 1} \hbar^k \Gamma_{(k)}[\Phi]\,, \tag{1.62}$$

and substitute (1.62) in (1.61), with the shift of variable $\varphi^i \equiv \Phi^i + \sqrt{\hbar}\,\phi^i$. After having expanded the action $S[\varphi]$ and the measure $\mathcal{M}[\varphi]$ in quantum fields ϕ^i, by equating coefficients at equal powers of $\hbar$ we get recurrence relations that uniquely define the coefficients $\Gamma_{(k)}[\Phi]$. The fundamental elements of this computation are the bare propagators, i.e. the Green functions of the differential operator

$$\Delta_{ij}[\varphi] \equiv \frac{\delta_L \delta_R}{\delta\varphi^i \delta\varphi^j} S[\varphi]\,. \tag{1.63}$$

In particular, the one–loop effective action can be obtained from the computation of a Gaussian–like integral, and has the well–known form

$$\Gamma_{(1)}[\Phi] = -\frac{1}{2i} \log \frac{\operatorname{sdet} \Delta[\Phi]}{\mathcal{M}^2[\Phi]}\,, \tag{1.64}$$

where we have introduced the functional *Berezin superdeterminant*

$$\operatorname{sdet}\Delta \equiv \exp\left(\operatorname{sTr}\,\log\Delta\right) \tag{1.65}$$

and the *supertrace*

$$\operatorname{sTr} F_{ij} \equiv (-1)^i F^i{}_i\,. \tag{1.66}$$

[10] Our computations in the background field method will be carried out only *on–shell*, that is, by assuming that the background solves Einstein equations, so covariance will not be a problem in our case.

[11] At L loops we will have an amplitude proportional to $\hbar^{L-1}$.

The local functional measure $\mathcal{M}[\varphi]$ can be taken to be the square root of the superdeterminant of the metric in configuration space,

$$\mathcal{M} = [\text{sdet}\, E_{ij}(\varphi)]^{\frac{1}{2}} \;, \tag{1.67}$$

where we assumed *ultralocality* [18, § 3]

$$E_{ij}[\varphi] = E_{AB}(\varphi(x))\, \delta(x, x')\,. \tag{1.68}$$

Note that the measure $\mathrm{d}\varphi\, \mathcal{M}[\varphi]$ is invariant under point–wise transformations of the fields, $\varphi(x) \to F[\varphi(x)]$. Multiplicativity of the superdeterminant [19, § I.2] allows then to write (1.64) as

$$\Gamma_{(1)}[\Phi] = -\frac{1}{2i} \log \text{sdet}\, \hat{\Delta} \qquad \text{with} \qquad \hat{\Delta} \equiv E^{-1} \Delta\,. \tag{1.69}$$

1.4.1 Effective action and heat kernel expansion

The most general form of the operator $\hat{\Delta}$ (1.69) is covariantly given by (1.12); by explicitly writing down the configuration space indices A, B, as well as factoring out the lower bound of $Q(x) \geq m^2 > 0$ as

$$Q^A{}_B(x) \equiv \widetilde{Q}^A{}_B - m^2 \delta^A_B\,, \tag{1.70}$$

we have

$$\hat{\Delta}^A{}_B(x, x') = \left[\delta^A_B \left(\nabla^2 - m^2\right) + \widetilde{Q}^A{}_B(x)\right] g^{\frac{1}{2}}(x)\; \delta(x, x')\,. \tag{1.71}$$

By means of the *Fock–Schwinger–DeWitt proper time* method, we can write down the Green function $G^A{}_{B'}(x, x')$ for the operator (1.71), that is,[12]

$$\left[\delta^A_C \left(\nabla^2 - m^2\right) + \widetilde{Q}^A{}_C\right] G^C{}_{B'}(x, x') = -\delta^A{}_{B'}\, g^{-\frac{1}{2}}(x)\; \delta(x, x')\,, \tag{1.72}$$

as a contour integral over an auxiliary "time" variable s [9, § 1.2]. Then, as the heat kernel is the integral kernel of the heat semigroup,

$$G(x, x') \equiv i \int_C \mathrm{d}s\;\; \exp\left(-ism^2\right)\, U(s; x, x')\,. \tag{1.73}$$

In the following we will choose the Feynman causal propagator by integrating s from 0 to ∞, and shifting $m^2 \to m^2 + i\varepsilon$. Using (1.52) and (1.53), at coinciding points $x = x'$ we then have

$$G(x, x) = i \int_0^\infty \mathrm{d}s\;\; (4\pi s)^{-\frac{n}{2}} \exp\left(-ism^2 + \varepsilon s\right)\, \Omega(s; x, x)\,. \tag{1.74}$$

[12] Note that this Green function is a two–point quantity, transforming as the field $\varphi^A(x)$ under the transformation of coordinates in x, and as the current $J_{B'}(x')$ under the transformation at x'.

We now move for further convenience to *euclidean time* $\beta \equiv is$, and exploit the operatorial identity [20, § 1]

$$\log A = -\int_0^\infty \frac{\mathrm{d}\beta}{\beta}\, \exp(-\beta A)\,, \tag{1.75}$$

which can be seen to hold by differentiating both sides with respect to A, and neglecting an infinite additive constant which does not depend on A itself. By recalling that $\log \mathrm{sdet}\,\hat{\Delta} = \mathrm{sTr}\,\log\hat{\Delta}$ and setting $A \equiv \hat{\Delta}$, (1.69) becomes[13]

$$\begin{aligned}\Gamma_{(1)} &= \frac{1}{2}\log \mathrm{sdet}\,\hat{\Delta} = -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}\beta}{\beta}\, \mathrm{sTr}\,\exp(-\beta\hat{\Delta}) \\ &= -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}\beta}{\beta}\, \exp(-\beta m^2)\int \mathrm{d}^n x\,\sqrt{g}\,\mathrm{sTr}\,U(\beta;x,x)\,, \end{aligned}\tag{1.76}$$

and the integral (1.76) admits two different kinds of divergences:

- for $\beta \to \infty$, *infrared divergences* can be due to negative or zero eigenvalues of $\hat{\Delta}$ (they can be removed by assuming that m is sufficiently large, which is not the case of the graviton);
- for $\beta \to 0$, we have instead *ultraviolet divergences*; indeed, by introducing a cutoff at $\beta = \Lambda^{-1}$,

$$\Gamma_\Lambda = -\frac{1}{2}\int_{\Lambda^{-1}}^\infty \frac{\mathrm{d}\beta}{\beta}\, \exp(-\beta m^2)\int \mathrm{d}^n x\,\sqrt{g}\,\mathrm{sTr}\,U(\beta;x,x)\,, \tag{1.77}$$

  it is easy to see that the divergent part of Γ_Λ, for $\Lambda \to \infty$, is a subgroup of the heat kernel coefficients.

By using the HDMS expansion (1.53) in euclidean time β,[14] the action (1.76) becomes

$$\Gamma_{(1)} = -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}\beta}{\beta}\, \exp(-\beta m^2)\int \frac{\mathrm{d}^n x\,\sqrt{g}}{(4\pi\beta)^{\frac{n}{2}}}\,\mathrm{sTr}\sum_{j=0}^\infty \frac{\beta^j}{j!} b_j(x,x)\,, \tag{1.78}$$

which is the general form for the effective action in terms of HDMS coefficients. From a physical point of view, the expansion in series in proper time corresponds to the expansion in the dimensionless parameter $\alpha \equiv \lambda/L$, where $\lambda = \hbar/mc$ is the Compton wavelength, and L is the characteristic scale of variation of the background fields. When $\lambda \ll L$, this approximation is nothing else than the *semi–classical approximation* of quantum mechanics, since $\alpha \sim \hbar$ in usual units.

[13] When going to euclidean time $\beta = is$, the factor $-(2i)^{-1}$ in front of the effective action (1.69) becomes 2^{-1}.

[14] Note that (1.53) is written for a Riemannian manifold; since, when shifting to euclidean time, we move from a pseudo-Riemannian manifold to a Riemaniann one, the "time" coordinate t introduced in (1.53) is to be identified with β, and not with s.

1.5 Heat kernel coefficients computation

The computation of the HDMS coefficients was carried out by DeWitt [3, § 17; 4, §§ 9.7–9.9], but the method employed by him, which exploits a recursive relation between the coefficients, although being the simplest in principle, turns out to be cumbersome when looking for higher order terms (indeed, he did not compute a_3). The fourth heat kernel coefficient a_3 was computed for the first time by Gilkey [21, §§ 3–4], with all terms spelled out explicitly, and to which we refer as a starting point in our computations. However, the method due to Gilkey is still formulated in a very complicated form, which cannot be immediately applied to the physical problems: it is not covariant nor can be extended to computer calculations [6, § 1]. A fully covariant and algorithmic method has been more recently developed by Avramidi [11; 6; 7; 9, §§ 2.1–2.4; 8, §§ 5.5–5.7].

To briefly sketch this latter procedure, we rewrite the transport equation (1.43) for the differential operator (1.71) with $t = is$, to get

$$\left(\frac{\mathrm{d}}{\mathrm{d}is} + \frac{1}{is}D\right)\Omega(s) = \mathcal{P}^{-1}\left(\mathbb{1}\Delta^{-\frac{1}{2}}\nabla^2\Delta^{\frac{1}{2}} + \widetilde{Q}\right)\mathcal{P}\Omega(s)\,, \tag{1.79}$$

where $\Omega(s) \equiv \Omega(s; x, x)$ in the coincidence limit, satisfying the boundary condition $\Omega^{A'}_{\ B'}(0; x, x) = \delta^{A'}_{\ B'}$. Then, from (1.52) we get

$$\Omega(s) = \sum_{j=0}^{\infty} \frac{(is)^j}{j!} b_j\,, \tag{1.80}$$

and hence, by plugging (1.80) into (1.79) we find

$$a_0 = \mathbb{1} \tag{1.81}$$

$$\left(1 + \frac{1}{j}D\right) b_j = \mathcal{P}^{-1}\left(\mathbb{1}\Delta^{-\frac{1}{2}}\nabla^2\Delta^{\frac{1}{2}} + \widetilde{Q}\right)\mathcal{P} b_{j-1}\,. \tag{1.82}$$

While equation (1.81) gives immediately the first heat kernel coefficient as reported in (1.96), we can rewrite the recursion relation (1.82) as

$$b_j = \left(1 + \frac{1}{j}D\right)^{-1} F \left(1 + \frac{1}{j-1}D\right)^{-1} F \ldots (1 + D)^{-1} F\,, \tag{1.83}$$

where

$$F \equiv \mathcal{P}^{-1}\left(\mathbb{1}\Delta^{-\frac{1}{2}}\nabla^2\Delta^{\frac{1}{2}} + \widetilde{Q}\right)\mathcal{P}\,. \tag{1.84}$$

Assuming that HDMS coefficients admit a coincidence limit for $x \to x'$, we can introduce the covariant Taylor series described in appendix 1.E:

$$b_j = \sum_{n \geq 0} |n\rangle \langle n|b_j\rangle \tag{1.85}$$

and hence

$$\left(1+\frac{1}{j}D\right)^{-1}=\sum_{n\geq 0}\frac{j}{j+n}\left|n\right\rangle\left\langle n\right| . \tag{1.86}$$

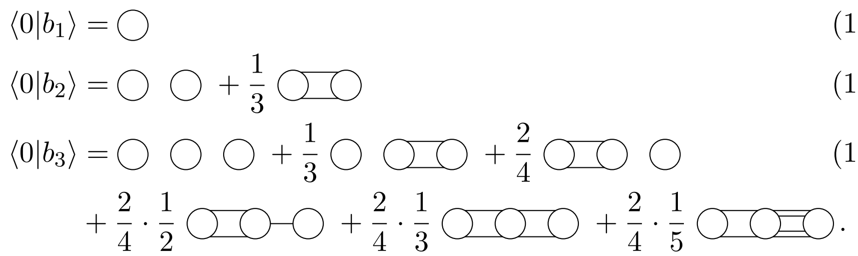
The recursive relation (1.83) can therefore be rewritten as

$$\left\langle n|b_j\right\rangle=\sum_{n_1,\dots,n_{j-1}\geq 0}\frac{j}{j+n}\cdot\frac{j-1}{j-1+n_{j-1}}\cdots\frac{1}{1+n_j} \cdot\left\langle n|F|n_{j-1}\right\rangle\left\langle n_{j-1}|F|n_{j-2}\right\rangle\dots\left\langle n_1|F|0\right\rangle , \tag{1.87}$$

where the matrix elements of the operator F are

$$\left\langle m|F|n\right\rangle=\nabla_{(\mu_1}\dots\nabla_{\mu_m)}F\frac{(-1)^n}{n!}\sigma^{\nu_1'}\dots\sigma^{\nu_n'} . \tag{1.88}$$

As the operator F is a differential operator of second order, the matrix elements (1.88) do not vanish only for $n\leq m+2$, so the summation (1.87) is always finite, an in particular $n_1\geq 0$, $n_j\leq n_{j+1}+2$. The problem of computing HDMS coefficients is therefore reduced by (1.87) to compute the matrix elements (1.88). This last procedure can be simplified even more by introducing a *diagrammatic technique*, which consists in writing a generic element $\langle m|F|n\rangle$ as a "block" of m lines coming from the left and n lines going out to the right, whereas contractions between matrix elements $\langle m|F|k\rangle\langle k|F|n\rangle$ are represented by two blocks connected by k intermediate lines. To get the contraction (1.87), we must then draw all possible diagrams with j blocks connected in all possible ways by any number of intermediate lines k, which does not exceed the number of incoming lines by more than two nor by exactly one. The different diagrams are then summed with the weights provided by (1.87); for the first coefficients these diagrams are:



(1.89)

(1.90)

(1.91)

Explicit results

The exact computation of the matrix elements is quite involved, and will not be carried out here; the results obtained by Avramidi [11; 6], however, are in accordance with the earlier ones by Gilkey [21], which is a strong cross–

check for the correctness of both procedures.[15] More recently, the same coefficients have been written in [23] with all curvature monomials spelled out.

To show the outcome of these involved computations, it is useful to write the operator (1.71) in the following simplified form

$$H \equiv -\nabla^2 - V\,, \tag{1.92}$$

where we keep implicit configuration space indices, and define $V \equiv -Q$ [20, § 2.1]. We will also denote by $n = D$ the dimension of our spacetime manifold. The trace of the heat kernel coefficients (1.53) can be written as

$$\frac{1}{(4\pi t)^{\frac{D}{2}}}\mathrm{Tr}\left[\sum_{j=0}^{\infty} t^j a_j(x,x)\right] \equiv \frac{1}{(4\pi t)^{\frac{D}{2}}}\mathrm{Tr}\left[\exp\left(\sum_{j=1}^{\infty} t^j \alpha_j(x,x)\right)\right]\,, \tag{1.93}$$

whereas the standard coefficients, in the normalisation previously employed, are the b_j's defined as

$$a_j \equiv \frac{1}{j!}\, b_j \qquad \text{with} \qquad a_j \equiv \alpha_j + \beta_j\,. \tag{1.94}$$

We separated the contribution α_j, coming from connected diagrams, from β_j, due to disconnected ones, as it is commonly performed in quantum field theory [22, § B]. By comparing the two terms in (1.93) we find

$$\begin{cases} \beta_0 = \beta_1 = 0 \\ \beta_2 = \dfrac{1}{2}\alpha_1^2 \\ \beta_3 = \dfrac{1}{6}\alpha_1^3 + \alpha_1\alpha_2\,. \end{cases} \tag{1.95}$$

Proof. Neglecting the prefactor in (1.93), we expand the exponential as

$$\begin{aligned} \mathrm{Tr}\left[\exp\left(\sum_{j=1}^{\infty} t^j \alpha_j(x,x)\right)\right] &= \mathrm{Tr}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{j=1}^{\infty} t^j \alpha_j(x,x)\right)^k \\ &= \mathrm{Tr}\left[\mathbb{1} + \alpha_1 t + \left(\alpha_2 + \frac{1}{2}\alpha_1^2\right)t^2 + \left(\alpha_3 + \frac{1}{6}\alpha_1^3 + \alpha_1\alpha_2\right)t^3 + \mathcal{O}(t^4)\right] \end{aligned}$$

and direct comparison between the two terms identifies the coefficients $\beta_2 = \frac{1}{2}\alpha_1^2$ and $\beta_3 = \frac{1}{6}\alpha_1^3 + \alpha_1\alpha_2$, as reported in (1.95). ∎

[15] It is important to note that the notation we employ here for the Riemann tensor, Ricci tensor and scalar is the same of [22; 5],

$$R^\mu{}_{\nu\rho\sigma} \equiv \partial_\rho\Gamma^\mu_{\sigma\nu} - \partial_\sigma\Gamma^\mu_{\rho\nu} - \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\rho\nu}\,, \qquad R_{\mu\nu} \equiv R_{\alpha\mu}{}^\alpha{}_\nu\,, \qquad R \equiv R^\mu{}_\mu\,,$$

while [21; 20] adopt the opposite sign in the Riemann tensor and in its contractions, $R_{\mu\nu} \equiv R_{\alpha\mu\nu}{}^\alpha$, so the heat kernel coefficients (1.96)–(1.99) have been changed accordingly.

By using the form (1.92) for the differential operator and by denoting the gauge field strength tensor as $\Omega_{\mu\nu} \equiv [\nabla_\mu, \nabla_\nu]$, the first four HDMS coefficients[16] are given by:

$$\alpha_0(x) = \mathbb{1} \tag{1.96}$$

$$\alpha_1(x) = \frac{1}{6}R\mathbb{1} + V \tag{1.97}$$

$$\alpha_2(x) = \frac{1}{6}\nabla^2\left(\frac{1}{5}R\mathbb{1} + V\right) + \frac{1}{180}\left(R^2_{\mu\nu\rho\sigma} - R^2_{\mu\nu}\right)\mathbb{1} + \frac{1}{12}\Omega^2_{\mu\nu} \tag{1.98}$$

$$\begin{aligned}\alpha_3(x) = {} & \frac{1}{7!}\Big[18\nabla^4 R + 17(\nabla_\mu R)^2 - 2(\nabla_\mu R_{\nu\sigma})^2 - 4\nabla_\mu R_{\nu\sigma}\nabla^\nu R^{\mu\sigma} \\ & + 9(\nabla_\alpha R_{\mu\nu\rho\sigma})^2 - 8R_{\mu\nu}\nabla^2 R^{\mu\nu} + 24 R_{\mu\nu}\nabla^\nu\nabla_\sigma R^{\mu\sigma} \\ & + 12 R_{\mu\nu\rho\sigma}\nabla^2 R^{\mu\nu\rho\sigma} - \frac{208}{9}R_\mu{}^\nu R_\nu{}^\sigma R_\sigma{}^\mu + \frac{64}{3}R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma} \\ & - \frac{16}{3}R_{\mu\nu}R^\mu{}_{\rho\sigma\tau}R^{\nu\rho\sigma\tau} + \frac{44}{9}R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu} \\ & + \frac{80}{9}R_{\mu\nu\rho\sigma}R^{\mu\alpha\rho\beta}R^\nu{}_\alpha{}^\sigma{}_\beta\Big]\mathbb{1} \\ & + \frac{2}{6!}\Big[8(\nabla_\mu\Omega_{\nu\sigma})^2 + 2(\nabla^\mu\Omega_{\mu\nu})^2 + 12\Omega_{\mu\nu}\nabla^2\Omega^{\mu\nu} - 12\Omega_\mu{}^\nu\Omega_\nu{}^\sigma\Omega_\sigma{}^\mu \\ & + 6R_{\mu\nu\rho\sigma}\Omega^{\mu\nu}\Omega^{\rho\sigma} - 4R_{\mu\nu}\Omega^{\mu\sigma}\Omega^\nu{}_\sigma + 6\nabla^4 V + 30(\nabla_\mu V)^2 \\ & + 4R_{\mu\nu}\nabla^\mu\nabla^\nu V + 12\nabla_\mu R\nabla^\mu V\Big].\end{aligned} \tag{1.99}$$

These results can be equivalently employed for gauge fields and potentials expressed in matrix form, as it will be apparent in the following.

[16] In the following, when referring to the HDMS coefficients for perturbative quantum gravity, we will simply speak of *Seeley–DeWitt coefficients*.

Appendix

In this appendix we review some mathematical tools which were needed to define the heat kernel expansion in the previous chapter.

1.A Elliptic operators with constant coefficients

One of the very few simple cases in which computations can be carried out explicitly is that of elliptic differential operators (1.1) with constant coefficients (A, β, γ). Let $\lambda \in \mathbb{C}$ and $h = h(x) \in C^\infty(M)$ be a given smooth function, and let $f(x)$ be an unknown function satisfying the equation

$$(L - \lambda)\, f(x) = h(x) \qquad \text{with} \qquad \lim_{x\to\pm\infty} f(x) = 0\,. \tag{1.100}$$

By Fourier transforming, (1.100) is solved by

$$\begin{aligned} f(x) = \left(\mathcal{F}^{-1}\hat{f}\right)(x) &= \int_{\mathbb{R}^n} \frac{\mathrm{d}^n p}{(2\pi)^n}\, e^{i\langle p,x\rangle}\, \frac{\hat{h}(p)}{\sigma(p)-\lambda} \\ &\equiv \int_{\mathbb{R}^n} \mathrm{d}^n x'\; G(\lambda; x, x')\; h(x')\,, \end{aligned} \tag{1.101}$$

having introduced the *resolvent* of the operator L,

$$G(\lambda; x, x') \equiv \int_{\mathbb{R}^n} \frac{\mathrm{d}^n p}{(2\pi)^n} e^{i\langle p,(x-x')\rangle} \frac{1}{\sigma(p)-\lambda}\,, \tag{1.102}$$

which satisfies the condition

$$(L-\lambda)\, G(\lambda; x, x') = \delta(x - x')\,. \tag{1.103}$$

Let us assume that $\mathrm{Re}\,(\lambda)$ is sufficiently large and negative, so that for any $p \in \mathbb{R}$, $\mathrm{Re}\,(\sigma(p) - \lambda) > 0$. Then, we can parametrise

$$\frac{1}{\sigma(p)-\lambda} = \int_0^\infty \mathrm{d}t\; \exp\left\{-t[\sigma(p)-\lambda]\right\}, \tag{1.104}$$

and therefore (1.102) admits an integral representation

$$G(\lambda; x, x') \equiv \int_0^\infty \mathrm{d}t\; e^{\lambda t} U(t; x, x')\,, \tag{1.105}$$

in terms of the *heat kernel operator*

$$U(t;x,x') \equiv \int_{\mathbb{R}^n} \frac{\mathrm{d}^n p}{(2\pi)^n} \exp\left\{-t\sigma(p) + i\left\langle p,(x-x')\right\rangle\right\}. \qquad (1.106)$$

Since in this case the symbol $\sigma(p) = \langle p, Ap\rangle + i\langle p, \beta\rangle + \gamma$ (1.2) is a quadratic polynomial, (1.106) is a Gaussian integral and can be computed exactly, to give [8, § 2.5.3]

$$U(t;x,x') = (4\pi t)^{-\frac{n}{2}}(\det A)^{-\frac{1}{2}} \exp\left\{\frac{1}{2}\left\langle (x-x'), A^{-1}\beta\right\rangle\right\}$$
$$\cdot \exp\left\{-t\left[\gamma + \frac{1}{4}\left\langle \beta, A^{-1}\beta\right\rangle\right]\right\} \exp\left\{-\frac{1}{4t}\left\langle (x-x'), A^{-1}(x-x')\right\rangle\right\} \qquad (1.107)$$

and by using the Fourier integral representation of the δ–function we find

$$\lim_{t\to 0^+} (4\pi t)^{-\frac{n}{2}}(\det A)^{-\frac{1}{2}} \exp\left\{-\frac{1}{4t}\left\langle (x-x'), A^{-1}(x-x')\right\rangle\right\} = \delta(x-x'). \qquad (1.108)$$

This implies that $U(t;x,x')$ satisfies the initial condition (1.16), because in the limit $t \to 0^+$ the other two exponential terms give the identity.

1.B Synge function

Consider a fixed point $x' \in \mathcal{M}$ and assume that there is a sufficiently small neighborhood of x' such that each point x therein can be connected to x' by a single geodesic $x = x(\tau)$, with $\tau \in [0,t]$ and $x(0) = x'$, $x(t) = x$. The *Synge function* is defined as half of the square of the geodesic distance

$$\sigma(x,x') \equiv \frac{1}{2}d^2(x,x') = \frac{1}{2}t\int_0^t \mathrm{d}\tau\; g_{ij}\dot{x}^i\dot{x}^j. \qquad (1.109)$$

It is a biscalar function that fully determines the local geometry of the manifold. Consider now the one–parameter family of geodesics $x^i = x^i(\tau,\varepsilon)$, such that the initial point $x^i(0,\varepsilon) = x'^i$ is fixed, and let us denote

$$h^i(\tau) = \left.\frac{\partial x^i(\tau,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0}. \qquad (1.110)$$

Since x' is fixed, $h^i(0) = 0$, and we can compute $\nabla_i\sigma$ as

$$\left.\frac{\partial\sigma}{\partial\varepsilon}\right|_{\varepsilon=0} = h^i(t)\nabla_i\sigma. \qquad (1.111)$$

By plugging (1.111) into the integral definition (1.109) we find

$$\begin{aligned}\left.\frac{\partial\sigma}{\partial\varepsilon}\right|_{\varepsilon=0} &= \frac{1}{2}t\int_0^t \mathrm{d}\tau\left[2g_{ij}\dot{x}^i\frac{\partial\dot{x}^j}{\partial\varepsilon}+\frac{\partial g_{ij}}{\partial\varepsilon}\dot{x}^i\dot{x}^j\right]_{\varepsilon=0}\\ &= \frac{1}{2}t\int_0^t \mathrm{d}\tau\left[2g_{ij}\dot{x}^i\frac{\partial h^j}{\partial\tau}+h^k\partial_k g_{ij}\dot{x}^i\dot{x}^j\right]_{\varepsilon=0}\end{aligned}\tag{1.112}$$

and integrating by parts the first term returns

$$\begin{aligned}\left.\frac{\partial\sigma}{\partial\varepsilon}\right|_{\varepsilon=0} &= t\left[g_{ij}\dot{x}^i h^j\big|_0^t\right]_{\varepsilon=0}+t\int_0^t \mathrm{d}\tau\left[-g_{ij}\ddot{x}^i h^j-\partial_k g_{ij}\dot{x}^k\dot{x}^i h^j\right.\\ &\qquad\left.+\frac{1}{2}\partial_k g_{ij}\dot{x}^i\dot{x}^j h^k\right]_{\varepsilon=0}\\ &= t\left[g_{ij}\dot{x}^i h^j\big|_0^t\right]_{\varepsilon=0}-t\int_0^t \mathrm{d}\tau\left[\ddot{x}^i+\Gamma^i_{jl}\dot{x}^j\dot{x}^l\right]g_{ik}h^k\Big|_{\varepsilon=0}\,.\end{aligned}\tag{1.113}$$

The second term in (1.113) vanishes, since $x = x(\tau,\varepsilon)$ is a geodesic, and as $h(0) = 0$ we are left with

$$\left.\frac{\partial\sigma}{\partial\varepsilon}\right|_{\varepsilon=0} = t g_{ij}\dot{x}^i(t)h^j(t) \qquad\longrightarrow\qquad \nabla_j\sigma = t g_{ij}\dot{x}^i(t)\,,\tag{1.114}$$

where we recalled (1.110). Equation (1.114) shows that first derivatives of the Synge function are proportional to the tangent vectors, and hence

$$g^{ij}(\nabla_i\sigma)(\nabla_j\sigma) = t^2||\dot{x}||^2 = d^2\,,\tag{1.115}$$

from which, by comparing once more with (1.109), we find

$$\sigma = \frac{1}{2}g^{ij}(\nabla_i\sigma)(\nabla_j\sigma)\,.\tag{1.116}$$

Equation (1.114) for the first covariant derivative of Synge function can be equivalently derived for primed indices — that is, for derivatives with respect to x' — in which case it reads as

$$\sigma^{\mu'} = -t\dot{x}^{\mu'}(0)\,,\tag{1.117}$$

and then

$$\sigma = \frac{1}{2}g^{i'j'}(\nabla_{i'}\sigma)(\nabla_{j'}\sigma)\,.\tag{1.118}$$

1.C Van Vleck–Morette determinant

The *Van Vleck–Morette determinant* is a two–point quantity $\Delta(x,x')$, defined in terms of the Synge function as [8, § 3.6.3]

$$\Delta(x,x') \equiv g^{-\frac{1}{2}}(x)\det\left[-\sigma_{ij'}(x,x')\right]g^{-\frac{1}{2}}(x')\,,\tag{1.119}$$

where we denote by pedices the covariant derivatives of Synge function, i.e. $\sigma_{ij'} \equiv \nabla_i \nabla_{j'} \sigma$. In the following, we will also denote $D \equiv \sigma^i \nabla_i$, so that (1.116) takes the form $D\sigma = 2\sigma$. By differentiating (1.116) and (1.120),

$$\sigma_j = \sigma_{ji}\sigma^i \qquad \text{and} \qquad \sigma_{j'} = \sigma_{j'i}\sigma^i \,, \tag{1.120}$$

or, equivalently,

$$D\sigma^i = \sigma^i \qquad \text{and} \qquad D\sigma^{i'} = \sigma^{i'} \,. \tag{1.121}$$

By derivating again the second equation in (1.120), and using the fact that derivatives with respect to x always commute with derivatives with respect to x', we find

$$\sigma_{j'k} = \sigma_{j'ik}\sigma^i + \sigma_{j'i}\sigma^i{}_k \quad \longrightarrow \quad \sigma^i \nabla_i \sigma^{j'}{}_k + \sigma^{j'}{}_i \sigma^i{}_k - \sigma^{j'}{}_k = 0\,. \tag{1.122}$$

Define now the matrices $\xi = (\xi^i{}_j) \equiv \sigma^i{}_j$ and $\eta = (\eta^{i'}{}_j) \equiv \sigma^{i'}{}_j$. Then (1.122) can be written in a more compact form:

$$D\eta + \eta(\xi - \mathbb{1}) = 0\,. \tag{1.123}$$

By setting $\gamma \equiv \eta^{-1}$, one has $\sigma_{ij'} = g_{j'k'}\eta^{k'}_i$ and

$$\det(-\sigma_{ij'}) = g(x')\det(-\eta) = g(x')\det(-\gamma)^{-1}\,; \tag{1.124}$$

therefore, the Van Vleck–Morette determinant can also be written as

$$\Delta(x,x') = g^{-\frac{1}{2}}(x)\det(-\eta) g^{\frac{1}{2}}(x') = g^{-\frac{1}{2}}(x)\det(-\gamma)^{-1} g^{\frac{1}{2}}(x')\,. \tag{1.125}$$

We may now compute

$$\begin{aligned}\Delta^{-1}\nabla_i\Delta &= \partial_i \log\left[g^{-\frac{1}{2}}\det(-\eta)\right] = \mathrm{Tr}\left[\gamma\partial_i\eta\right] - g^{-\frac{1}{2}}\partial_i g^{\frac{1}{2}} \\ &= \gamma^k{}_{j'}\partial_i\eta^{j'}{}_k - \Gamma^k_{ik} = \gamma^k{}_{j'}\nabla_i\eta^{j'}{}_k = \mathrm{Tr}\left[\gamma\nabla_i\eta\right]\end{aligned} \tag{1.126}$$

and by contracting with σ^i we get

$$\Delta^{-1}D\Delta = \mathrm{Tr}\left[\gamma D\eta\right]\,. \tag{1.127}$$

The trace of (1.123), contracted with γ on the left, gives $\mathrm{Tr}\left[\gamma D\eta\right] = n - \sigma^i{}_i$, therefore the Van Vleck–Morette determinant satisfies the linear differential equation

$$D\Delta = (n - \sigma^i{}_i)\Delta\,, \tag{1.128}$$

and its square root satisfies the equation

$$D\Delta^{\frac{1}{2}} = \frac{1}{2}(n - \sigma^i{}_i)\Delta^{\frac{1}{2}}\,. \tag{1.129}$$

1.D Operator of parallel transport

With the same conventions as in appendix 1.C, we define the *operator of parallel transport* $\mathcal{P}(x,x')$ as the two–point scalar function which solves the equation $D^{\mathcal{A}}\mathcal{P}(x,x') = 0$ with initial condition $\mathcal{P}(x,x) = 1$. Here, we denote by $D^{\mathcal{A}} \equiv \sigma^i \nabla_i^{\mathcal{A}}$ the operator of differentiation along the geodesic connecting x and x'. Formally, the solution is given by [8, § 3.7.3]

$$\mathcal{P}(x,x') = \exp\left\{-\int_0^t \mathrm{d}\tau\ \dot{x}^i \mathcal{A}_i[x(\tau)]\right\}, \tag{1.130}$$

where τ is the affine parameter along the geodesic, and it satisfies the symmetry property $\mathcal{P}(x',x) = \mathcal{P}^{-1}(x,x')$. Note that (1.130) is unitary and preserves the norm only if $\mathcal{A}_i$ is purely imaginary.

1.E Covariant Taylor series

The subject of covariant Taylor series is quite involved, and here we will just scratch the surface of this mathematical technique [9, § 2.1]. Consider the usual collective field φ, and transport it parallel along the geodesic connecting x to x', to obtain:

$$\bar{\varphi} = \bar{\varphi}^{C'}(x') = \mathcal{P}^{C'}_{A}(x',x)\varphi^A(x) = \mathcal{P}^{-1}\varphi\,, \tag{1.131}$$

where $\mathcal{P}^{-1} = \mathcal{P}^{C'}_{A}(x',x)$ is the parallel transport operator along the opposite path, satisfying $\mathcal{P}\mathcal{P}^{-1} = \mathbb{1}$. The function $\bar{\varphi}$ is a scalar under coordinate transformations at point x, since it does not have non–primed indices; therefore, it can be expanded in Taylor series as a function of the geodesic affine parameter τ:

$$\bar{\varphi} = \sum_{k\geq 0} \frac{\tau^k}{k!}\, \frac{\mathrm{d}^k}{\mathrm{d}\tau^k}\bar{\varphi}\bigg|_{\tau=0}\,. \tag{1.132}$$

Now, exploiting again the fact that $\bar{\varphi}$ is a scalar,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\bar{\varphi} = \dot{x}^\mu \partial_\mu \bar{\varphi} = \dot{x}^\mu \nabla_\mu \bar{\varphi}\,. \tag{1.133}$$

Moreover, equation (1.117) in this context reads as

$$\sigma^{\mu'} = -\tau \dot{x}^{\mu'}(0)\,. \tag{1.134}$$

Combining these results with the geodesic equation $\dot{x}^\mu \nabla_\mu \dot{x}^\nu = 0$ we eventually find

$$\varphi = \mathcal{P}\bar{\varphi} = \mathcal{P}\sum_{k\geq 0} \frac{(-1)^k}{k!}\sigma^{\mu_1'}\ldots\sigma^{\mu_k'}\nabla_{(\mu_1}\ldots\nabla_{\mu_k)}\varphi\,, \tag{1.135}$$

which is know as *generalised covariant Taylor series* for an arbitrary field and connection in curved space. The key point is that the series (1.135) is a complete set of eigenfunctions for the operator $D \equiv \sigma^{\mu}\nabla_{\mu}$. Indeed, as the vectors σ^{μ} and $\sigma^{\mu'}$ are eigenfunctions of D with eigenvalue one, we can construct the basis

$$|0\rangle \equiv 1 \qquad \text{and} \qquad |n\rangle \equiv |\nu_1' \dots \nu_n'\rangle = \frac{(-1)^n}{n!}\sigma^{\nu_1'} \dots \sigma^{\nu_n'} \,, \tag{1.136}$$

which, due to (1.120), satisfies

$$D\,|n\rangle = n\,|n\rangle \;, \tag{1.137}$$

and which has dual elements

$$\langle m| \equiv \langle \mu_1' \dots \mu_m'| = (-1)^m g^{\mu_1}_{\mu_1'} \dots g^{\mu_m}_{\mu_m'} \nabla_{(\mu_1} \dots \nabla_{\mu_m)} \delta(x, x') \,, \tag{1.138}$$

which satisfy

$$\langle m|n\rangle = \int \mathrm{d}^n x \, \langle \mu_1' \dots \mu_m' | \nu_1' \dots \nu_n' \rangle = \delta_{mn} \delta^{\nu_1}_{(\mu_1} \dots \delta^{\nu_n}_{\mu_n)} \,. \tag{1.139}$$

The scalar product (1.139) allows to write the covariant series (1.135) in a much more compact form, since

$$\langle m|\varphi\rangle = \nabla_{(\mu_1} \dots \nabla_{\mu_m)} \varphi \qquad \text{and} \qquad |\varphi\rangle = \mathcal{P} \sum_{n \geq 0} |n\rangle \langle n|\varphi\rangle \;. \tag{1.140}$$

It is now easy to see that a relation of completeness holds,

$$\mathbb{1} = \sum_{n \geq 0} |n\rangle \langle n| \;, \tag{1.141}$$

and that a generic differential operator $F[\varphi]$ can be written in the form

$$F = \sum_{m,n \geq 0} \mathcal{P}\,|m\rangle \langle m|\mathcal{P}^{-1} F \mathcal{P}|n\rangle \langle n|\,\mathcal{P}^{-1} \,, \tag{1.142}$$

with

$$\langle m|\mathcal{P}^{-1} F \mathcal{P}|n\rangle \equiv \nabla_{(\mu_1} \dots \nabla_{\mu_m)} \mathcal{P}^{-1} F \mathcal{P} \frac{(-1)^n}{n!} \sigma^{\nu_1'} \dots \sigma^{\nu_n'} \,. \tag{1.143}$$

Chapter 2

Gauge–fixing the graviton

In this chapter we apply the BRST quantisation procedure to find a quantum action principle for the graviton in the background field method [5].

2.1 Perturbative quantum gravity action

Consider a D-dimensional Riemannian manifold $(\mathcal{M}, \boldsymbol{G})$ equipped with a metric tensor $\boldsymbol{G}$ with euclidean signature. The line element is given by $ds^2 = G_{\mu\nu}dx^\mu dx^\nu$, where $G_{\mu\nu}$ are the components of $\boldsymbol{G}$, and the metric enjoys the local gauge symmetry[1]

$$G_{\mu\nu}(x) \quad \longrightarrow \quad G'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}G_{\alpha\beta}(x)\,, \tag{2.1}$$

leaving the Einstein–Hilbert action

$$S_{EH}[\boldsymbol{G}] = -\frac{1}{k^2}\int \mathrm{d}^D x\, \sqrt{G}\,[R(\boldsymbol{G}) - \Lambda] \tag{2.2}$$

invariant, where $k^2 \equiv 16\pi G_N$, being G_N the Newtonian universal gravitational constant, $R(\boldsymbol{G})$ is the Ricci scalar computed from $\boldsymbol{G}$, $G \equiv |\det G_{\mu\nu}|$ and a cosmological constant $\Lambda \neq 0$ has been included. As well known, Einstein field equations in presence of a cosmological constant may be obtained from (2.2) through the principle of least action.

In perturbative quantum gravity we employ the *background field method* described above, by splitting the metric tensor $\boldsymbol{G}$ into a fixed (but generic) classical background $\boldsymbol{g}$ and a "small" perturbation $\boldsymbol{h}$, that is, $|\boldsymbol{h}| \ll |\boldsymbol{g}|$:

$$G_{\mu\nu}(x) = g_{\mu\nu}(x) + h_{\mu\nu}(x)\,. \tag{2.3}$$

[1] In general, gravity has a global gauge symmetry leaving $(\mathcal{M}, \boldsymbol{G}, \boldsymbol{T})$ invariant, where $\boldsymbol{T}$ are all tensor structures on $\mathcal{M}$; this wide symmetry is described by the non–Lie type Bergmann–Komar algebra, and is larger than local diffeomorphism invariance (2.1), which describes only *local* transformations that are *connected to the identity.*

Quanta of the field $\boldsymbol{h}$ are usually called *gravitons*. As a consequence of the splitting (2.3), the action (2.2) can be expanded in power series in the fluctuations $\boldsymbol{h}$,

$$S_{EH}[\boldsymbol{g}+\boldsymbol{h}] \equiv \frac{1}{k^2}\sum_{n=0}^{\infty} S_n[\boldsymbol{g},\boldsymbol{h}^n]\,. \tag{2.4}$$

Since we are to consider perturbative quantum gravity at one–loop, we will be concerned with the terms up to S_2 in (2.4). This expansion can be explicitly computed, as described in appendix 2.A. The result is:

$$S_0 = -\int \mathrm{d}^D x\sqrt{g}[R-2\Lambda] \tag{2.5}$$

$$S_1 = \int \mathrm{d}^D x\sqrt{g}\left[h^{\mu\nu}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+g_{\mu\nu}\Lambda\right)\right] \tag{2.6}$$

$$\begin{aligned} S_2 = \int \mathrm{d}^D x\sqrt{g}\Bigg[&-\frac{1}{4}h^{\mu\nu}\left(\nabla^2+2\Lambda-R\right)h_{\mu\nu}+\frac{1}{8}h\left(\nabla^2+2\Lambda-R\right)h \\ &-\frac{1}{2}\left(\nabla^\nu h_{\mu\nu}-\frac{1}{2}\nabla_\mu h\right)^2-\frac{1}{2}\left(h^{\mu\lambda}h_\lambda^{\ \nu}-hh^{\mu\nu}\right)R_{\mu\nu} \\ &-\frac{1}{2}h^{\mu\lambda}h^{\nu\rho}R_{\mu\nu\lambda\rho}\Bigg]\,. \end{aligned} \tag{2.7}$$

It is important to note that in (2.5)–(2.7) the Ricci tensor $R_{\mu\nu}=R_{\mu\nu}(\boldsymbol{g})$ and scalar $R=R(\boldsymbol{g})$, as well as covariant derivatives $\nabla_\mu=\nabla_\mu(\boldsymbol{g})$, are computed with respect to the background metric $\boldsymbol{g}$. As expected, the principle of least action on the tree–level action S_1 (2.6) leads to the graviton equations of motion, which are the Einsten equations for the metric $\boldsymbol{g}$.

2.2 Gauge fixing

To overcome gauge redundancy (2.1), the computation of Seeley–DeWitt coefficients requires to fix a specific gauge. Consider an infinitesimal change of coordinates

$$x^\mu \;\longrightarrow\; x'^\mu = x^\mu-\xi^\mu(x)\,, \qquad |\xi^\mu(x)| \ll |x^\mu|\,, \tag{2.8}$$

under which the components of the metric tensor $\boldsymbol{G}$ transform as

$$\delta G_{\mu\nu}(x) = G'_{\mu\nu}(x)-G_{\mu\nu}(x) = \xi^\alpha\partial_\alpha G_{\mu\nu}+G_{\alpha\nu}\partial_\mu\xi^\alpha+G_{\mu\alpha}\partial_\nu\xi^\alpha \tag{2.9}$$

$$= \nabla_\mu\xi_\nu+\nabla_\nu\xi_\mu = \left(\mathcal{L}_\xi\boldsymbol{G}\right)_{\mu\nu}\,, \tag{2.10}$$

where we denote by $\mathcal{L}_\xi$ the Lie derivative along the vector field ξ^μ, and the last equality (2.10) comes from its definition when applied to tensor fields.

Proof. By inverting equation (2.8) we get $x^\mu = x'^\mu + \xi^\mu$, which plugged into the transformation law (2.1) gives

$$\begin{aligned} G'_{\mu\nu}(x') &= \frac{\partial(x'^\alpha + \xi^\alpha)}{\partial x'^\mu}\frac{\partial(x'^\beta + \xi^\beta)}{\partial x'^\nu} G_{\alpha\beta}(x) = (\delta^\alpha_\mu + \partial'_\mu \xi^\alpha)(\delta^\beta_\nu + \partial'_\nu \xi^\beta) G_{\alpha\beta}(x) \\ &= G_{\mu\nu} + G_{\alpha\nu}\partial_\mu \xi^\alpha + G_{\mu\alpha}\partial_\nu \xi^\alpha \,, \end{aligned}$$

where we used the fact that, at first order in ξ, we can approximate $\partial'\xi \sim \partial\xi$, with ∂' denoting derivatives w.r.t. x'. By Taylor expanding and keeping only first order terms,

$$G'_{\mu\nu}(x') = G'_{\mu\nu}(x-\xi) = G'_{\mu\nu}(x) - \xi^\alpha \partial_\alpha G'_{\mu\nu}(x) = G'_{\mu\nu}(x) - \xi^\alpha \partial_\alpha G_{\mu\nu}(x) \,,$$

hence

$$\begin{aligned} G'_{\mu\nu}(x) &= G'_{\mu\nu}(x') + \xi^\alpha \partial_\alpha G_{\mu\nu}(x) \\ &= G_{\mu\nu} + G_{\alpha\nu}\partial_\mu \xi^\alpha + G_{\mu\alpha}\partial_\nu \xi^\alpha + \xi^\alpha \partial_\alpha G_{\mu\nu} \,, \end{aligned}$$

from which (2.9) follows. To get (2.10), recall that the covariant derivative is defined as $\nabla_\mu \xi_\nu \equiv \partial_\mu \xi_\nu - \Gamma^\alpha_{\mu\nu}\xi_\alpha$, therefore

$$\begin{aligned} \nabla_\mu \xi_\nu &= \partial_\mu \xi_\nu - \frac{1}{2} G^{\alpha\beta} \left(\partial_\mu G_{\beta\nu} + \partial_\nu G_{\beta\mu} - \partial_\beta G_{\mu\nu}\right) \xi_\alpha \\ &= \left[G_{\nu\alpha}\partial_\mu - \frac{1}{2}\left(\partial_\mu G_{\alpha\nu} + \partial_\nu G_{\alpha\mu} - \partial_\alpha G_{\mu\nu}\right)\right] \xi^\alpha + \xi^\alpha \partial_\mu G_{\nu\alpha} \\ \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu &= \xi^\alpha \partial_\alpha G_{\mu\nu} + G_{\alpha\nu}\partial_\mu \xi^\alpha + G_{\mu\alpha}\partial_\nu \xi^\alpha \,, \end{aligned}$$

where we used $\partial_\mu \xi_\nu = \partial_\mu (G_{\nu\alpha}\xi^\alpha) = G_{\nu\alpha}\partial_\mu \xi^\alpha + \xi^\alpha \partial_\mu G_{\nu\alpha}$ to remove antisymmetric terms under the exchange of indices $\{\mu \leftrightarrow \nu\}$. ■

2.2.1 Two gauge symmetries

The splitting (2.3) we introduced before gives rise to *two* different gauge symmetries:

1. the truly *quantum gauge symmetry*, which transforms $\boldsymbol{h}$, leaving the background $\boldsymbol{g}$ unchanged:

$$\begin{cases} \delta_1 \boldsymbol{g} = 0 \\ \delta_1 \boldsymbol{h} = \mathcal{L}_\xi \boldsymbol{G} = \mathcal{L}_\xi(\boldsymbol{g} + \boldsymbol{h}) \,; \end{cases} \tag{2.11}$$

2. a *background gauge symmetry*, which transforms $\boldsymbol{g}$ as a dynamical field as well, so that $\boldsymbol{g}$ and $\boldsymbol{h}$ transform as tensor fields:

$$\begin{cases} \delta_2 \boldsymbol{g} = \mathcal{L}_\xi \boldsymbol{g} \\ \delta_2 \boldsymbol{h} = \mathcal{L}_\xi \boldsymbol{h} \,. \end{cases} \tag{2.12}$$

Only (2.11) is a true dynamical symmetry, and is to be gauge–fixed. The gauge–fixing procedure will remove $2D$ unphysical degrees of freedom from the metric tensor (they are canceled by the ghost fields), leaving us with

$$n = \frac{D(D+1)}{2} - 2D = \frac{D(D-3)}{2} \tag{2.13}$$

degrees of freedom for the graviton. Note that the above expression for $D = 4$ gives the familiar result of $n = 2$ degrees of freedom of a massless spin–2 gauge theory, while for $D = 3$ the number of degrees of freedom is $n = 0$ and Einstein gravity has no dynamics.

2.2.2 BRST quantisation

Following the BRST quantisation procedure, we introduce the BRST variation of the field $\boldsymbol{h}$ by performing the substitution $\xi^\mu \longrightarrow \Lambda c^\mu$ in (2.11), with the aid of (2.10):

$$\delta_B h_{\mu\nu} \equiv \Lambda(\nabla_\mu c_\nu + \nabla_\nu c_\mu) = \Lambda(s h_{\mu\nu})\,, \tag{2.14}$$

where c^μ is the *ghost* field, Λ is a constant Grassmann number and s is the *Slavnov variation* operator, defined by $\delta_B \equiv \Lambda s$. Moreover, two non–minimal fields are needed: the *antighost* $\bar{c}^\mu$, satisfying fermionic statistics, and the *Nakanishi–Lautrup auxiliary field* B_μ, which instead is a bosonic field, with the following BRST variations:

$$\begin{cases} \delta_B \bar{c}^\mu \equiv \Lambda B^\mu = \Lambda(s\bar{c}^\mu) \\ \delta_B B^\mu \equiv 0\,. \end{cases} \tag{2.15}$$

The BRST variation of the ghost c^μ can be found by requiring that it is nilpotent (i.e. $s^2 = 0$) when acting on a generic scalar field ϕ,

$$\delta_B c^\mu = \Lambda c^\nu \partial_\nu c^\mu = \Lambda(s c^\mu)\,. \tag{2.16}$$

Proof. Since the Lie derivative on a scalar field ϕ acts as the usual directional derivative, $\mathcal{L}_\xi \phi = \xi^\mu \partial_\mu \phi$, we have by the usual substitution $\xi^\mu \longrightarrow \Lambda c^\mu$ that $\delta_B \phi = \Lambda c^\mu \partial_\mu \phi$, and requiring nilpotency:

$$\begin{aligned} s^2\phi &= 0 \\ &= s(c^\mu \partial_\mu \phi) = (s c^\mu)\partial_\mu \phi - c^\mu \partial_\mu (c^\nu \partial_\nu \phi) \\ &= (s c^\mu)\partial_\mu \phi - c^\mu c^\nu \partial_\mu \partial_\nu \phi - c^\mu (\partial_\mu c^\nu)\partial_\nu \phi \\ &= (s c^\mu - c^\nu \partial_\nu c^\mu)\partial_\mu \phi\,, \end{aligned}$$

where in the last line we removed the vanishing term $c^\mu c^\nu \partial_\mu \partial_\nu \phi$ and relabeled the indices μ, ν in the last term of the sum. By comparing the first with the last line, we end up with (2.16). ∎

The most suitable gauge–fixing choice for our problem is de Donder gauge, which is defined by the *gauge–fixing function*

$$f_\mu(x) \equiv \nabla^\nu h_{\mu\nu} - \frac{1}{2}\nabla_\mu h\,, \tag{2.17}$$

with $h \equiv \mathrm{Tr}\,[\boldsymbol{h}]$. Using (2.17), we define the following *gauge fermion*:

$$\Psi \equiv \bar{c}^\mu\,(f_\mu + \alpha B_\mu)\,, \tag{2.18}$$

where $\alpha \in \mathbb{R}$ is a constant. The action S_2 computed in (2.7) can be made BRST–invariant by adding the Slavnov variation of the gauge fermion,

$$S_2^{tot}[\boldsymbol{h}, c, \bar{c}] = S_2[\boldsymbol{h}] + s\int \mathrm{d}^D x\sqrt{g}\;\Psi\,. \tag{2.19}$$

The actions S_2^{tot} and S_2 belong to the same cohomology class, as their difference is an exact element (a BRST variation), therefore they give rise to the same physical observables.

The Slavnov variation in (2.19) can be computed by the aid of the BRST variations (2.14) and (2.15),

$$\begin{aligned} s\int \mathrm{d}^D x\sqrt{g}\;\Psi &= \int \mathrm{d}^D x\sqrt{g}\;\left[(s\bar{c}^\mu)\,(f_\mu + \alpha B_\mu) - \bar{c}^\mu(sf_\mu)\right] \\ &= \int \mathrm{d}^D x\sqrt{g}\;\left[B^\mu f_\mu + \alpha B^2 - \bar{c}^\mu(sf_\mu)\right]\,. \end{aligned} \tag{2.20}$$

From the first two terms in (2.20), $\mathcal{L}_B \equiv B^\mu f_\mu + \alpha B^2$, we can compute the equations of motion for the auxiliary field B^μ,

$$0 = \frac{\partial \mathcal{L}_B}{\partial B^\mu} = f_\mu + 2\alpha B_\mu \qquad \longrightarrow \qquad B_\mu = -\frac{1}{2\alpha}f_\mu\,, \tag{2.21}$$

which, plugged back in (2.20), returns

$$\mathcal{L}_B = -\frac{1}{2\alpha}f^2 + \alpha\frac{1}{4\alpha^2}f^2 = \frac{1}{4\alpha}f^2\,. \tag{2.22}$$

With the gauge choice (2.17), the Lagrangian (2.22) cancels off in the total action (2.19) by hitting the term

$$-\frac{1}{2}\left(\nabla^\nu h_{\mu\nu} - \frac{1}{2}\nabla_\mu h\right)^2 \subset \mathcal{L}_2\,, \tag{2.23}$$

with

$$S_2 \equiv \int \mathrm{d}^D x\;\mathcal{L}_2$$

identified from (2.7), *provided that* we set the arbitrary parameter $\alpha \equiv \frac{1}{2}$. What remains in S_2^{tot} (2.19) is then the action (2.7), stripped off of the term (2.23), which in the following will be denoted by

$$S_h[\boldsymbol{h}] \equiv S_2[\boldsymbol{h}] + \frac{1}{2}\int \mathrm{d}^D x \left(\nabla^\nu h_{\mu\nu} - \frac{1}{2}\nabla_\mu h\right)^2 , \tag{2.24}$$

and the ghost action, which can be proved to be

$$S_{gh} \equiv -\int \mathrm{d}^D x \sqrt{g}\; \bar{c}^\mu (s f_\mu) = -\int \mathrm{d}^D x \sqrt{g}\; \bar{c}^\mu (\nabla^2 c_\mu + R_{\mu\nu} c^\nu) \, . \tag{2.25}$$

Proof. Using the expression for the Slavnov variation of the metric (2.14) we get

$$\begin{aligned} s f_\mu = s\left(\nabla^\nu h_{\mu\nu} - \frac{1}{2}\nabla_\mu h\right) &= \nabla^\nu(\nabla_\mu c_\nu + \nabla_\nu c_\mu) - \frac{1}{2} s(\nabla_\mu {h^\nu}_\nu) \\ &= \nabla^2 c_\mu + \nabla^\nu \nabla_\mu c_\nu - \frac{1}{2}(2\nabla_\mu \nabla^\nu c_\nu) = \nabla^2 c_\mu + [\nabla_\nu, \nabla_\mu] c^\nu \\ &= \nabla^2 c_\mu + {R_{\nu\mu}}^\nu{}_\sigma c^\sigma = \nabla^2 c_\mu + R_{\mu\nu} c^\nu \, , \end{aligned}$$

where we used the definition of the Riemann tensor as a commutator of covariant derivatives, $[\nabla_\mu, \nabla_\nu] V^\rho = {R_{\mu\nu}}^\rho{}_\sigma V^\sigma$, which allows to reproduce (2.25). ∎

2.2.3 Kinetic operators

We are now able to identify the invertible (thanks to gauge–fixing) kinetic operators from the actions S_{gh} and S_h. The ghost action (2.25) can be immediately rewritten as

$$S_{gh} = -\int \mathrm{d}^D x \sqrt{g}\; \bar{c}_\mu (\delta^\mu_\nu \nabla^2 + {R^\mu}_\nu) c^\nu \equiv \int \mathrm{d}^D x \sqrt{g}\; \bar{c}_\mu {\mathcal{F}^\mu}_\nu c^\nu \, , \tag{2.26}$$

allowing us to identify the ghost kinetic operator

$${\mathcal{F}^\mu}_\nu \equiv -(\delta^\mu_\nu \nabla^2 + {R^\mu}_\nu) \, . \tag{2.27}$$

Similarly, the graviton kinetic operator can be defined by writing the action S_h (2.24) as

$$S_h \equiv \int \mathrm{d}^D x \, \sqrt{g} \, \frac{1}{2} h_{\mu\nu} F^{\mu\nu\alpha\beta} h_{\alpha\beta} \, , \tag{2.28}$$

where

$$\begin{aligned} F^{\mu\nu\alpha\beta} \equiv &-\frac{1}{4}\left(g^{\mu\alpha}g^{\nu\beta} + g^{\nu\alpha}g^{\mu\beta} - g^{\mu\nu}g^{\alpha\beta}\right)\left(\nabla^2 + 2\Lambda - R\right) \\ &-\frac{1}{2}\left(R^{\mu\alpha\nu\beta} + R^{\mu\beta\nu\alpha} - g^{\mu\nu}R^{\alpha\beta} - g^{\alpha\beta}R^{\mu\nu}\right) \\ &-\frac{1}{4}\left(g^{\mu\alpha}R^{\nu\beta} + g^{\mu\beta}R^{\nu\alpha} + g^{\nu\alpha}R^{\mu\beta} + g^{\nu\beta}R^{\mu\alpha}\right) . \end{aligned} \tag{2.29}$$

Proof. The explicit form of S_h, see (2.7) and (2.24), is

$$S_h = \int \mathrm{d}^D x \sqrt{g} \left[-\frac{1}{4} h^{\mu\nu} \left(\nabla^2 + 2\Lambda - R\right) h_{\mu\nu} + \frac{1}{8} h \left(\nabla^2 + 2\Lambda - R\right) h \right.$$
$$\left. + \frac{1}{2} \left(h^{\mu\lambda} h_\lambda^{\ \nu} - h h^{\mu\nu}\right) R_{\mu\nu} - \frac{1}{2} h^{\mu\lambda} h^{\nu\rho} R_{\mu\nu\lambda\rho} \right] .$$

The first line in (2.29), when plugged into (2.28), gives

$$-\frac{1}{4} h^{\alpha\beta} \left(\nabla^2 + 2\Lambda - R\right) h_{\alpha\beta} + \frac{1}{8} h \left(\nabla^2 + 2\Lambda - R\right) h \,,$$

which corresponds to the first line of S_h; note that indices are highered and lowered by the background metric $\boldsymbol{g}$. The second line gives instead

$$- \frac{1}{4} \left(h^{\mu\nu} h^{\alpha\beta} R_{\mu\alpha\nu\beta} + h^{\mu\nu} h^{\alpha\beta} R_{\mu\beta\nu\alpha} - 2 h h^{\alpha\beta} R_{\alpha\beta}\right)$$
$$= -\frac{1}{2} \left(h h^{\alpha\beta} R_{\alpha\beta} + h^{\mu\nu} h^{\alpha\beta} R_{\mu\alpha\nu\beta}\right) ,$$

where the symmetry of $h_{\alpha\beta}$ allows to relabel the indices α, β and obtain twice the first term. This corresponds to the last two terms in S_h; to conclude, the third line gives

$$-\frac{1}{8} \left(h^\alpha_{\ \nu} h_{\alpha\beta} R^{\nu\beta} + h^\beta_{\ \nu} h_{\alpha\beta} R^{\nu\alpha} + h_\mu^{\ \alpha} h_{\alpha\beta} R^{\mu\beta} + h_\mu^{\ \beta} h_{\alpha\beta} R^{\mu\alpha}\right) = -\frac{1}{2} h_{\alpha\beta} h^\alpha_{\ \nu} R^{\nu\beta} \,,$$

which corresponds to the third to last term in S_h. ■

To fit properly in the following computations, the graviton kinetic operator (2.29) needs to be slightly transformed. To lower its first two indices, we introduce the *DeWitt super–metric*[2]

$$\gamma^{\mu\nu\alpha\beta} \equiv \frac{1}{4} \left(g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} - g^{\mu\nu} g^{\alpha\beta}\right) \tag{2.30}$$

$$\gamma_{\mu\nu\alpha\beta} = g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha} - \frac{2}{D-2} g_{\mu\nu} g_{\alpha\beta} \,, \tag{2.31}$$

which is symmetric under the exchange of the two couples of indices and is normalised by

$$\gamma_{\mu\nu\alpha\beta}\, \gamma^{\alpha\beta\rho\sigma} = \frac{1}{2} \left(\delta^\rho_\mu \delta^\sigma_\nu + \delta^\sigma_\mu \delta^\rho_\nu\right) . \tag{2.32}$$

[2] In general, the form of the graviton operator is compatible with a more general family of metrics,

$$\gamma^{\mu\nu\alpha\beta}_{(k)} \equiv \frac{1}{4} \left(g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} - k g^{\mu\nu} g^{\alpha\beta}\right)$$
$$\gamma^{(k)}_{\mu\nu\alpha\beta} = g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha} - \frac{2k}{kD-2} g_{\mu\nu} g_{\alpha\beta} \,,$$

where the parameter k is arbitrary. Note that $k = \frac{2}{D}$ is not allowed, since it brings a singularity in the inverse metric: indeed, for $k = \frac{2}{D}$ the metric $\boldsymbol{\gamma}$ is the projector on the traceless subspace [5, § 2]. In this thesis we will always keep $k \equiv 1$.

Proof. By direct computation,

$$\begin{aligned}\gamma_{\mu\nu\alpha\beta}\,\gamma^{\alpha\beta\rho\sigma} &= \frac{1}{4}\left(g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha} - \frac{2}{D-2}g_{\mu\nu}g_{\alpha\beta}\right)\left(g^{\alpha\rho}g^{\beta\sigma} + g^{\alpha\sigma}g^{\beta\rho} - g^{\alpha\beta}g^{\rho\sigma}\right)\\ &= \frac{1}{4}\left[2\left(\delta^\rho_\mu\delta^\sigma_\nu + \delta^\sigma_\mu\delta^\rho_\nu\right) - 2g_{\mu\nu}g^{\rho\sigma} - \frac{2}{D-2}\,(2-D)g_{\mu\nu}g^{\rho\sigma}\right]\\ &= \frac{1}{2}\left(\delta^\rho_\mu\delta^\sigma_\nu + \delta^\sigma_\mu\delta^\rho_\nu\right),\end{aligned}$$

where we used the fact that, in D dimensions, $g_{\alpha\beta}g^{\alpha\beta} = D$. ■

Under the action of the super–metric, the graviton kinetic operator (2.29) assumes the form

$$\begin{aligned}F_{\mu\nu}{}^{\alpha\beta} =& -\frac{1}{2}\left(\delta^\alpha_\mu\delta^\beta_\nu + \delta^\beta_\mu\delta^\alpha_\nu\right)\left(\nabla^2 + 2\Lambda - R\right) - \frac{1}{D-2}g_{\mu\nu}g^{\alpha\beta}R\\ &+ \frac{2}{D-2}g_{\mu\nu}R^{\alpha\beta} + g^{\alpha\beta}R_{\mu\nu} - R_\mu{}^\alpha{}_\nu{}^\beta - R_\mu{}^\beta{}_\nu{}^\alpha\\ &- \frac{1}{2}\left(\delta^\alpha_\mu R^\beta_\nu + \delta^\beta_\mu R^\alpha_\nu + \delta^\alpha_\nu R^\beta_\mu + \delta^\beta_\nu R^\alpha_\mu\right).\end{aligned} \tag{2.33}$$

Proof. The first line in (2.29), when contracted as $F_{\rho\sigma}{}^{\alpha\beta} = \gamma_{\rho\sigma\mu\nu}F^{\mu\nu\alpha\beta}$, gives the usual factor $\left(\nabla^2 + 2\Lambda - R\right)$ multiplied by

$$\begin{aligned}&-\frac{1}{4}\left(g^{\mu\alpha}g^{\nu\beta} + g^{\nu\alpha}g^{\mu\beta} - g^{\mu\nu}g^{\alpha\beta}\right)\left(g_{\rho\mu}g_{\sigma\nu} + g_{\rho\nu}g_{\sigma\mu} - \frac{2}{D-2}g_{\rho\sigma}g_{\mu\nu}\right)\\ &\qquad = \frac{1}{2}\left(\delta^\alpha_\rho\delta^\beta_\sigma + \delta^\beta_\rho\delta^\alpha_\sigma\right),\end{aligned}$$

which is exactly the same computation performed in the contraction $\gamma_{\mu\nu\alpha\beta}\,\gamma^{\alpha\beta\rho\sigma}$ above; this reproduces the first term of $F_{\rho\sigma}{}^{\alpha\beta}$. The second line, instead, gives:

$$\begin{aligned}&-\frac{1}{2}\left(R^{\mu\alpha\nu\beta} + R^{\mu\beta\nu\alpha} - g^{\mu\nu}R^{\alpha\beta} - g^{\alpha\beta}R^{\mu\nu}\right)\left(g_{\rho\mu}g_{\sigma\nu} + g_{\rho\nu}g_{\sigma\mu} - \frac{2}{D-2}g_{\rho\sigma}g_{\mu\nu}\right)\\ &\quad = -\frac{1}{2}\Big(R_\rho{}^\alpha{}_\sigma{}^\beta + R_\rho{}^\beta{}_\sigma{}^\alpha + R_\sigma{}^\alpha{}_\rho{}^\beta + R_\sigma{}^\beta{}_\rho{}^\alpha - \frac{2}{D-2}\,2g_{\rho\sigma}R^{\alpha\beta}\\ &\qquad\qquad -2g_{\rho\sigma}R^{\alpha\beta} - 2g^{\alpha\beta}R_{\rho\sigma} + \frac{2D}{D-2}g_{\rho\sigma}R^{\alpha\beta} + \frac{2}{D-2}g_{\rho\sigma}g^{\alpha\beta}R\Big)\\ &\quad = -R_\rho{}^\alpha{}_\sigma{}^\beta - R_\rho{}^\beta{}_\sigma{}^\alpha + g^{\alpha\beta}R_{\rho\sigma} - \frac{1}{D-2}g_{\rho\sigma}g^{\alpha\beta}R\\ &\qquad\qquad + \left(\frac{2}{D-2} + 1 - \frac{D}{D-2}\right)g_{\rho\sigma}R^{\alpha\beta}\\ &\quad = -R_\rho{}^\alpha{}_\sigma{}^\beta - R_\rho{}^\beta{}_\sigma{}^\alpha + g^{\alpha\beta}R_{\rho\sigma} - \frac{1}{D-2}g_{\rho\sigma}g^{\alpha\beta}R,\end{aligned}$$

where we used the symmetries of the Riemann tensor to enforce

$$R_\sigma{}^\beta{}_\rho{}^\alpha = R^{\mu\beta\nu\alpha}g_{\rho\nu}g_{\sigma\mu} = R^{\nu\alpha\mu\beta}g_{\rho\nu}g_{\sigma\mu} = R_\rho{}^\alpha{}_\sigma{}^\beta\,,$$

and similarly for $R_{\sigma\ \rho}^{\ \alpha\ \beta}$. This corresponds to the last term in the first line and the three last terms in the second one of (2.33); to conclude, the third line gives

$$
\begin{aligned}
-\frac{1}{4}\left(g^{\mu\alpha}R^{\nu\beta}+g^{\mu\beta}R^{\nu\alpha}+g^{\nu\alpha}R^{\mu\beta}+g^{\nu\beta}R^{\mu\alpha}\right)&\left(g_{\rho\mu}g_{\sigma\nu}+g_{\rho\nu}g_{\sigma\mu}-\frac{2}{D-2}g_{\rho\sigma}g_{\mu\nu}\right)\\
&=-\frac{1}{2}\left(\delta^\alpha_\mu R^\beta_\nu+\delta^\beta_\mu R^\alpha_\nu+\delta^\alpha_\nu R^\beta_\mu+\delta^\beta_\nu R^\alpha_\mu\right)+\frac{1}{2(D-2)}\,4g_{\rho\sigma}R^{\alpha\beta}\\
&=-\frac{1}{2}\left(\delta^\alpha_\mu R^\beta_\nu+\delta^\beta_\mu R^\alpha_\nu+\delta^\alpha_\nu R^\beta_\mu+\delta^\beta_\nu R^\alpha_\mu\right)+\frac{2}{(D-2)}g_{\rho\sigma}R^{\alpha\beta}\,,
\end{aligned}
$$

corresponding to the last line and to the first term of the second line of (2.33). ■

Appendix

In this appendix we prove the expansions for the Einstein–Hilbert action up to one–loop, as given in equations (2.5)–(2.7), following [24, § 4].

2.A Expansion of the action at one–loop

In order to expand the Einstein–Hilbert action (2.2), which we here rewrite by expanding $R(\boldsymbol{G}) = G^{\mu\nu}R_{\mu\nu}(\boldsymbol{G})$,

$$S_{EH}[\boldsymbol{G}] = -\frac{1}{k^2}\int \mathrm{d}^D x\, \sqrt{G}\,[G^{\mu\nu}R_{\mu\nu}(\boldsymbol{G}) - \Lambda]\,, \tag{2.34}$$

we need to consider three different terms separately, namely: the square root of the metric determinant $\sqrt{G}$ (2.A.1), the inverse metric $G^{\mu\nu}$ (2.A.2) and the Ricci tensor $R_{\mu\nu}(\boldsymbol{G})$ (2.A.3).

2.A.1 Square root of the metric determinant

By using some well know properties of the exponential and logarithmic functions of operators, we find:

$$\begin{aligned}
\sqrt{G} &= \sqrt{|\det \boldsymbol{g} + \boldsymbol{h}|} = \sqrt{|\det \boldsymbol{g}|}\;\sqrt{\det|\mathbb{1} + \boldsymbol{g}^{-1}\boldsymbol{h}|} \\
&= \sqrt{|\det \boldsymbol{g}|}\;\exp\left[\frac{1}{2}\log\det\left|\mathbb{1} + \boldsymbol{g}^{-1}\boldsymbol{h}\right|\right] \\
&= \sqrt{|\det \boldsymbol{g}|}\;\exp\left[\frac{1}{2}\mathrm{Tr}\log\left|\mathbb{1} + \boldsymbol{g}^{-1}\boldsymbol{h}\right|\right] \\
&= \sqrt{|\det \boldsymbol{g}|}\;\exp\left[\frac{1}{2}\mathrm{Tr}\left(\boldsymbol{g}^{-1}\boldsymbol{h} - \frac{1}{2}(\boldsymbol{g}^{-1}\boldsymbol{h})^2 + \mathcal{O}(\boldsymbol{h}^3)\right)\right] \\
&= \sqrt{|\det \boldsymbol{g}|}\left[1 + \frac{1}{2}\mathrm{Tr}\left(\boldsymbol{g}^{-1}\boldsymbol{h}\right) - \frac{1}{4}\mathrm{Tr}\left(\boldsymbol{g}^{-1}\boldsymbol{h}\right)^2 + \frac{1}{8}\mathrm{Tr}^2\left(\boldsymbol{g}^{-1}\boldsymbol{h}\right) + \mathcal{O}(\boldsymbol{h}^3)\right] \\
&= \sqrt{g}\left[1 + \frac{1}{2}h^{\mu}{}_{\mu} - \frac{1}{4}h^{\mu\nu}h_{\mu\nu} + \frac{1}{8}\left(h^{\mu}{}_{\mu}\right)^2\right] + \mathcal{O}(\boldsymbol{h}^3)\,.
\end{aligned} \tag{2.35}$$

2.A.2 Inverse metric

To invert the metric $\boldsymbol{G}$ we observe that

$$\begin{aligned}\boldsymbol{G}^{-1} &= (\boldsymbol{g}+\boldsymbol{h})^{-1} = \boldsymbol{g}^{-1}(\mathbb{1}+\boldsymbol{g}^{-1}\boldsymbol{h})^{-1}\\ &= \boldsymbol{g}^{-1} - \boldsymbol{g}^{-2}\boldsymbol{h} + \boldsymbol{g}^{-2}\boldsymbol{h}\boldsymbol{g}^{-1}\boldsymbol{h} + \mathcal{O}(\boldsymbol{h}^3)\,. \end{aligned} \tag{2.36}$$

Since $\left(\boldsymbol{g}^{-2}\boldsymbol{h}\right)_{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}h_{\alpha\beta} = h^{\mu\nu}$ and $\left(\boldsymbol{g}^{-2}\boldsymbol{h}\boldsymbol{g}^{-1}\boldsymbol{h}\right)_{\mu\nu} = h^{\mu\alpha}h_{\alpha}^{\ \nu}$ in components (2.36) becomes

$$G^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu} + h^{\mu\alpha}h_{\alpha}^{\ \nu} + \mathcal{O}(\boldsymbol{h}^3)\,. \tag{2.37}$$

2.A.3 Ricci tensor

The most involved computation is the expansion of the Ricci tensor $R_{\mu\nu}(\boldsymbol{G})$, since we need to start from the full Riemann tensor and contract it. The starting point is therefore the evaluation of Christoffel symbols:

$$\begin{aligned}\Gamma^{\lambda}_{\mu\nu}(\boldsymbol{G}) &= \frac{1}{2}G^{\lambda\alpha}\left(\partial_\mu G_{\nu\alpha} + \partial_\nu G_{\mu\alpha} - \partial_\alpha G_{\mu\nu}\right)\\ &= \frac{1}{2}\left(g^{\lambda\alpha} - h^{\lambda\alpha} + h^{\lambda\beta}h_{\beta}^{\ \alpha}\right)\\ &\qquad \cdot\left[\partial_\mu(g_{\nu\alpha}+h_{\nu\alpha}) + \partial_\nu(g_{\mu\alpha}+h_{\mu\alpha}) - \partial_\alpha(g_{\mu\nu}+h_{\mu\nu})\right] + \mathcal{O}(\boldsymbol{h}^3)\\ &= \Gamma^{\lambda}_{\mu\nu}(0) + \Gamma^{\lambda}_{\mu\nu}(1) + \Gamma^{\lambda}_{\mu\nu}(2) + \mathcal{O}(\boldsymbol{h}^3)\,,\end{aligned} \tag{2.38}$$

where we used (2.37) and denoted by $\Gamma^{\lambda}_{\mu\nu}(n)$ the Christoffel symbol at n-th order in $\boldsymbol{h}$. It is then clear that

$$\Gamma^{\lambda}_{\mu\nu}(0) = \frac{1}{2}g^{\lambda\alpha}\left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right), \tag{2.39}$$

while

$$\begin{aligned}\Gamma^{\lambda}_{\mu\nu}(1) &= \frac{1}{2}g^{\lambda\alpha}\left(\partial_\mu h_{\nu\alpha} + \partial_\nu h_{\mu\alpha} - \partial_\alpha h_{\mu\nu}\right) - \frac{1}{2}h^{\lambda\alpha}\left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right)\\ &= \frac{1}{2}g^{\lambda\alpha}\left(\partial_\mu h_{\nu\alpha} + \partial_\nu h_{\mu\alpha} - \partial_\alpha h_{\mu\nu}\right) - g_{\rho\alpha}h^{\lambda\alpha}\Gamma^{\rho}_{\mu\nu}(0)\\ &= \frac{1}{2}g^{\lambda\alpha}\left[\partial_\mu h_{\nu\alpha} + \partial_\nu h_{\mu\alpha} - \partial_\alpha h_{\mu\nu} - 2h_{\alpha\rho}\Gamma^{\rho}_{\mu\nu}(0)\right]\\ &= \frac{1}{2}g^{\lambda\alpha}\left(\nabla_\mu h_{\nu\alpha} + \nabla_\nu h_{\mu\alpha} - \nabla_\alpha h_{\mu\nu}\right),\end{aligned} \tag{2.40}$$

where we used (2.39) and in the last step we added and subtracted the term $(h_{\mu\rho}\Gamma^{\rho}_{\alpha\nu} + h_{\nu\rho}\Gamma^{\rho}_{\alpha\mu})$, in order to reproduce $\nabla_\mu h_{\nu\alpha} = \partial_\mu h_{\nu\alpha} - h_{\nu\rho}\Gamma^{\rho}_{\alpha\mu} - h_{\rho\alpha}\Gamma^{\rho}_{\mu\nu}$ and so on. Note that all covariant derivatives in (2.40) and in the following equations are computed with respect to the background metric $\boldsymbol{g}$. When not specified, moreover, $\Gamma^\lambda_{\mu\nu} \equiv \Gamma^\lambda_{\mu\nu}(0) = \Gamma^\lambda_{\mu\nu}(\boldsymbol{g})$. In the same manner,

$$\begin{aligned}\Gamma^\lambda_{\mu\nu}(2) &= \frac{1}{2} h^\lambda_\beta h^{\beta\alpha} \left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right) - \frac{1}{2} h^{\lambda\alpha} \left(\partial_\mu h_{\nu\alpha} + \partial_\nu h_{\mu\alpha} - \partial_\alpha h_{\mu\nu}\right) \\ &= -h^\lambda_\alpha \Gamma^\alpha_{\mu\nu}(1)\,. \end{aligned} \tag{2.41}$$

The results obtained in (2.40) and (2.41) show that both $\Gamma^\lambda_{\mu\nu}(1)$ and $\Gamma^\lambda_{\mu\nu}(2)$ are tensors, so it is meaningful to apply covariant derivatives to them, as it will be done in the following.

With the aid of equations (2.39)–(2.41) we can now expand the Riemann tensor around perturbations:

$$R^\mu{}_{\nu\rho\sigma}(\boldsymbol{G}) \equiv R^\mu{}_{\nu\rho\sigma}(0) + R^\mu{}_{\nu\rho\sigma}(1) + R^\mu{}_{\nu\rho\sigma}(2) + \mathcal{O}(\boldsymbol{h}^3)\,. \tag{2.42}$$

The term of order zero is just given by the definition of Riemann tensor,

$$R^\mu{}_{\nu\rho\sigma}(0) = \nabla_\rho \Gamma^\mu_{\nu\sigma}(0) - \nabla_\sigma \Gamma^\mu_{\rho\nu}(0)\,, \tag{2.43}$$

while the first order term will be of the form

$$\begin{aligned} R^\mu{}_{\nu\rho\sigma}(1) = \partial_\rho \Gamma^\mu_{\sigma\nu}(1) - \partial_\sigma \Gamma^\mu_{\rho\nu}(1) + \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\sigma\nu}(1) + \Gamma^\mu_{\rho\lambda}(1)\Gamma^\lambda_{\sigma\nu} \\ - \Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\rho\nu}(1) - \Gamma^\mu_{\sigma\lambda}(1)\Gamma^\lambda_{\rho\nu}\,. \end{aligned} \tag{2.44}$$

This rather involved expression can be simplified by adding and subtracting the quantity $\Gamma^\lambda_{\rho\sigma}\Gamma^\mu_{\lambda\nu}(1)$, to reproduce covariant derivatives as in (2.43),

$$R^\mu{}_{\nu\rho\sigma}(1) = \nabla_\rho \Gamma^\mu_{\nu\sigma}(1) - \nabla_\sigma \Gamma^\mu_{\rho\nu}(1)\,. \tag{2.45}$$

The second order expression is

$$\begin{aligned} R^\mu{}_{\nu\rho\sigma}(2) = \partial_\rho \Gamma^\mu_{\sigma\nu}(2) - \partial_\sigma \Gamma^\mu_{\rho\nu}(2) + \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\sigma\nu}(2) + \Gamma^\mu_{\rho\lambda}(2)\Gamma^\lambda_{\sigma\nu} \\ - \Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\rho\nu}(2) - \Gamma^\mu_{\sigma\lambda}(2)\Gamma^\lambda_{\rho\nu} + \Gamma^\mu_{\rho\lambda}(1)\Gamma^\lambda_{\sigma\nu}(1) - \Gamma^\mu_{\sigma\lambda}(1)\Gamma^\lambda_{\rho\nu}(1)\,, \end{aligned} \tag{2.46}$$

which can be likewise simplified to give

$$R^\mu{}_{\nu\rho\sigma}(2) = \nabla_\rho \Gamma^\mu_{\nu\sigma}(2) - \nabla_\sigma \Gamma^\mu_{\rho\nu}(2) + \Gamma^\mu_{\rho\lambda}(1)\Gamma^\lambda_{\sigma\nu}(1) - \Gamma^\mu_{\sigma\lambda}(1)\Gamma^\lambda_{\rho\nu}(1)\,. \tag{2.47}$$

Another equivalent way of writing (2.47), by exploiting (2.41), is

$$R^\mu{}_{\nu\rho\sigma}(2) = -h^\mu_\beta R^\beta{}_{\nu\rho\sigma}(1) - g^{\mu\alpha} g_{\beta\gamma}\left[\Gamma^\gamma_{\rho\alpha}(1)\Gamma^\beta_{\sigma\nu}(1) - \Gamma^\gamma_{\sigma\alpha}(1)\Gamma^\beta_{\rho\nu}(1)\right]. \tag{2.48}$$

Using again (2.37), the Ricci tensor is then given by

$$R_{\nu\sigma}(\boldsymbol{G}) \equiv R_{\nu\sigma}(0) + R_{\nu\sigma}(1) + R_{\nu\sigma}(2) + \mathcal{O}(\boldsymbol{h}^3)\,, \tag{2.49}$$

where quite trivially $R_{\nu\sigma}(0) = R^{\mu}{}_{\nu\mu\sigma}(0)$, while

$$\begin{aligned} R_{\nu\sigma}(1) = R^{\mu}{}_{\nu\mu\sigma}(1) &= \nabla_{\mu}\Gamma^{\mu}_{\nu\sigma}(1) - \nabla_{\sigma}\Gamma^{\mu}_{\mu\nu}(1) \\ &= \frac{1}{2}\left(\nabla_{\mu}\nabla_{\nu}h^{\mu}{}_{\sigma} + \nabla_{\mu}\nabla_{\sigma}h^{\mu}{}_{\nu} - \nabla_{\sigma}\nabla_{\nu}h - \nabla^2 h_{\sigma\nu}\right), \end{aligned} \tag{2.50}$$

where we used the explicit form (2.40). At the next order, thanks to the alternative form (2.48), we have

$$\begin{aligned} R_{\nu\sigma}(2) &= R^{\mu}{}_{\nu\mu\sigma}(2) \\ &= -h^{\mu}{}_{\beta}R^{\beta}{}_{\nu\mu\sigma}(1) - g^{\mu\alpha}g_{\beta\gamma}\left[\Gamma^{\gamma}_{\mu\alpha}(1)\Gamma^{\beta}_{\sigma\nu}(1) - \Gamma^{\gamma}_{\sigma\alpha}(1)\Gamma^{\beta}_{\mu\nu}(1)\right]. \end{aligned} \tag{2.51}$$

The Ricci scalar curvature can likewise be decomposed as

$$R(\boldsymbol{G}) = R(0) + R(1) + R(2) = \left(g^{\nu\sigma} - h^{\nu\sigma} + h^{\nu}{}_{\lambda}h^{\lambda\sigma}\right)R_{\nu\sigma}(\boldsymbol{G}), \tag{2.52}$$

where $R(0) = g^{\nu\sigma}R_{\nu\sigma}(0)$ as before, while from (2.50)

$$\begin{aligned} R(1) &= -h^{\nu\sigma}R_{\nu\sigma}(0) + g^{\nu\sigma}R_{\nu\sigma}(1) \\ &= -h^{\nu\sigma}R_{\nu\sigma}(0) + \nabla_{\nu}\nabla_{\sigma}h^{\nu\sigma} - \nabla^2 h, \end{aligned} \tag{2.53}$$

and from (2.51)

$$\begin{aligned} R(2) = &- h^{\nu\sigma}R_{\nu\sigma}(1) - g^{\nu\sigma}h^{\mu}{}_{\beta}R^{\beta}{}_{\nu\mu\sigma}(1) + h^{\nu}{}_{\lambda}h^{\lambda\sigma}R_{\nu\sigma}(0) \\ &- g^{\nu\sigma}g^{\mu\alpha}g_{\beta\gamma}\left[\Gamma^{\gamma}_{\mu\alpha}(1)\Gamma^{\beta}_{\sigma\nu}(1) - \Gamma^{\gamma}_{\sigma\alpha}(1)\Gamma^{\beta}_{\mu\nu}(1)\right]. \end{aligned} \tag{2.54}$$

We can expand some of the terms in (2.54) by using (2.45) and (2.40), to obtain

$$\begin{aligned} -g^{\nu\sigma}h^{\mu}{}_{\beta}R^{\beta}{}_{\nu\mu\sigma}(1) &= -g^{\nu\sigma}h^{\mu}{}_{\beta}\left[\nabla_{\mu}\Gamma^{\beta}_{\nu\sigma}(1) - \nabla_{\sigma}\Gamma^{\beta}_{\mu\nu}(1)\right] \\ &= -\Big(h^{\mu}{}_{\beta}\nabla_{\mu}\nabla_{\nu}h^{\beta\nu} + \frac{1}{2}h^{\mu}{}_{\beta}\nabla^{\nu}\nabla^{\beta}h_{\mu\nu} - \frac{1}{2}h_{\mu\nu}\nabla^2 h^{\mu\nu} \\ &\qquad -\frac{1}{2}h^{\mu}{}_{\beta}\nabla^{\nu}\nabla_{\mu}h^{\beta}{}_{\nu} - \frac{1}{2}h^{\mu}{}_{\beta}\nabla_{\mu}\nabla^{\beta}h\Big) \\ &= -h^{\mu}{}_{\beta}\nabla_{\mu}\nabla_{\nu}h^{\beta\nu} + \frac{1}{2}h_{\mu\nu}\nabla^2 h^{\mu\nu} + \frac{1}{2}h^{\mu}{}_{\beta}\nabla_{\mu}\nabla^{\beta}h, \end{aligned} \tag{2.55}$$

and similarly

$$\begin{aligned} &- g^{\nu\sigma}g^{\mu\alpha}g_{\beta\gamma}\left[\Gamma^{\gamma}_{\mu\alpha}(1)\Gamma^{\beta}_{\sigma\nu}(1) - \Gamma^{\gamma}_{\sigma\alpha}(1)\Gamma^{\beta}_{\mu\nu}(1)\right] = \\ &-\frac{1}{4}\Big(4\nabla^{\mu}h_{\mu\beta}\nabla_{\nu}h^{\beta\nu} + \nabla_{\beta}h\nabla^{\beta}h - 4\nabla^{\mu}h_{\beta\mu}\nabla^{\beta}h \\ &\qquad -3\nabla^{\mu}h^{\nu\beta}\nabla_{\mu}h_{\beta\nu} + 2\nabla^{\nu}h^{\mu}{}_{\beta}\nabla^{\beta}h_{\mu\nu}\Big), \end{aligned} \tag{2.56}$$

which plugged back into (2.54) gives

$$\begin{aligned}R(2) = {} & h^{\nu}{}_{\lambda} h^{\lambda\sigma} R_{\nu\sigma}(0) - h^{\nu\sigma}\nabla_\mu \nabla_\sigma h^{\mu}{}_{\nu} - h^{\nu\sigma}\nabla_\mu\nabla_\nu h^{\mu}{}_{\sigma} + h^{\nu\sigma}\nabla^2 h_{\nu\sigma} \\ & + h^{\nu\sigma}\nabla_\sigma\nabla_\nu h - \nabla^\mu h_{\mu\beta}\nabla_\nu h^{\beta\nu} - \frac{1}{4}\nabla_\beta h \nabla^\beta h + \nabla^\mu h_{\beta\mu}\nabla^\beta h \\ & + \frac{3}{4}\nabla^\mu h^{\nu\beta}\nabla_\mu h_{\beta\nu} - \frac{1}{2}\nabla^\nu h^{\mu}{}_{\beta}\nabla^\beta h_{\mu\nu} \,. \end{aligned} \tag{2.57}$$

Collecting all terms, the action (2.2) becomes

$$\begin{aligned} S_{EH}[\boldsymbol{G}] = -\frac{1}{k^2}\int \mathrm{d}^D x\, \sqrt{g}\, \Bigg\{ & \left(1 + \frac{1}{2}h - \frac{1}{4}h^{\mu\nu}h_{\mu\nu} + \frac{1}{8}h^2\right) \\ & \cdot \Big[R(0) + \left(-h^{\nu\sigma}R_{\nu\sigma}(0) + \nabla_\nu\nabla_\sigma h^{\nu\sigma} - \nabla^2 h\right) \\ & + \Big(h^{\nu}{}_{\lambda}h^{\lambda\sigma}R_{\nu\sigma}(0) - h^{\nu\sigma}\nabla_\mu\nabla_\sigma h^{\mu}{}_{\nu} - h^{\nu\sigma}\nabla_\mu\nabla_\nu h^{\mu}{}_{\sigma} + h^{\nu\sigma}\nabla^2 h_{\nu\sigma} \\ & + h^{\nu\sigma}\nabla_\sigma\nabla_\nu h - \nabla^\mu h_{\mu\beta}\nabla_\nu h^{\beta\nu} - \frac{1}{4}\nabla_\beta h\nabla^\beta h + \nabla^\mu h_{\beta\mu}\nabla^\beta h \\ & + \frac{3}{4}\nabla^\mu h^{\nu\beta}\nabla_\mu h_{\beta\nu} - \frac{1}{2}\nabla^\nu h^{\mu}{}_{\beta}\nabla^\beta h_{\mu\nu}\Big) - 2\Lambda\Big] + \mathcal{O}(\boldsymbol{h}^3)\Bigg\} \,, \end{aligned} \tag{2.58}$$

whence S_0 (2.5) follows immediately by taking only zeroth order terms, and S_1 (2.6) from terms first order in $\boldsymbol{h}$, recalling that $h \equiv g_{\mu\nu}h^{\mu\nu}$ and neglecting the total derivative $\nabla_\nu\nabla_\sigma h^{\nu\sigma} - \nabla^2 h$. The one–loop result S_2 (2.7) is instead given by

$$\begin{aligned} S_2 = -\frac{1}{k^2}\int \mathrm{d}^D x\,\sqrt{g}\,\Bigg[& R(2) + \frac{1}{2}hR(1) + \frac{1}{8}\left(h^2 - 2h^{\mu\nu}h_{\mu\nu}\right)R(0) \\ & + 2\Lambda\left(\frac{1}{4}h_{\mu\nu}h^{\mu\nu} - \frac{1}{8}h^2\right)\Bigg] \,, \end{aligned} \tag{2.59}$$

where integration by parts, neglecting total derivatives as before, allows to simplify the expression in the form (2.7). In detail,

$$\begin{aligned} \frac{1}{2}hR(1) & = -\frac{1}{2}hh^{\mu\nu}R_{\mu\nu}(0) + \frac{1}{2}h\nabla_\mu\nabla_\nu h^{\mu\nu} - \frac{1}{2}h\nabla^2 h \\ & \to -\frac{1}{2}hh^{\mu\nu}R_{\mu\nu}(0) - \frac{1}{2}\nabla_\mu h\nabla_\nu h^{\mu\nu} - \frac{1}{2}h\nabla^2 h \end{aligned} \tag{2.60}$$

while for the terms contained in $R(2)$ we have $h^{\nu\sigma}\nabla_\sigma\nabla_\nu h = -\nabla_\sigma h^{\nu\sigma}\nabla_\nu h$, which cancels off with $\nabla^\mu h_{\beta\mu}\nabla^\beta h$; $\frac{1}{4}\nabla_\beta h\nabla^\beta h = -\frac{1}{4}h\nabla^2 h$, which sums with the similar term in (2.60); $\frac{3}{4}\nabla^\mu h^{\nu\beta}\nabla_\mu h_{\beta\nu} = -\frac{3}{4}h^{\nu\beta}\nabla^2 h_{\beta\nu}$, which can be summed with $h^{\nu\sigma}\nabla^2 h_{\nu\sigma}$. The other five terms can be related one to the other by using the explicit form of the commutator between covariant derivatives,

$$[\nabla_\mu, \nabla_\nu]h^{\gamma}{}_{\alpha} = R^{\gamma}{}_{\lambda\mu\beta}h^{\lambda}{}_{\alpha} + R^{\lambda}{}_{\alpha\beta\mu}h^{\gamma}{}_{\lambda} \,. \tag{2.61}$$

Indeed:

$$
\begin{aligned}
-h^{\nu\sigma}\nabla_\mu\nabla_\sigma h^{\mu\nu} &= \nabla_\sigma h^{\nu\sigma}\nabla_\mu h^\mu{}_\nu - h^{\nu\sigma}\left(R^\mu{}_{\lambda\mu\sigma}h^\lambda{}_\nu + R^\lambda{}_{\nu\mu\sigma}h^\mu{}_\lambda\right) \\
&= \nabla_\sigma h^{\nu\sigma}\nabla_\mu h^\mu{}_\nu - h^{\nu\sigma}h^\lambda{}_\nu R_{\lambda\sigma} - h^{\nu\sigma}h^\mu{}_\lambda R^\lambda{}_{\nu\mu\sigma}\,, \qquad (2.62)
\end{aligned}
$$

and the same for $-h^{\nu\sigma}\nabla_\mu\nabla_\nu h^\mu{}_\sigma$, while

$$
\begin{aligned}
-\frac{1}{2}\nabla^\nu h^\mu{}_\beta\nabla^\beta h_{\mu\nu} &= \frac{1}{2}\nabla^\beta\nabla^\nu h^\mu{}_\beta h_{\mu\nu} \\
&= -\frac{1}{2}\nabla^\beta h^\mu{}_\beta\nabla^\nu h_{\mu\nu} + \left(R^\mu{}_\lambda{}^{\beta\nu} + R^\lambda{}_\beta{}^{\beta\nu}h_\lambda{}^\mu\right)h_{\mu\nu} \\
&= -\frac{1}{2}\nabla^\beta h^\mu{}_\beta\nabla^\nu h_{\mu\nu} + R^\mu{}_\lambda{}^{\beta\nu}h^\lambda{}_\beta h_{\mu\nu} - h^\mu{}_\lambda h_{\mu\nu}R^{\lambda\nu}\,, \qquad (2.63)
\end{aligned}
$$

so that what remains is really the expansion (2.7). It is important to note that a more general form of the action can be guessed from first principles: by requiring gauge–invariance in Einstein spaces it is possible to fix three of the five arbitrary parameters therein [25, § 3.3]. However, the direct computation performed here shows in a more explicit form the terms to be considered in expanding the action at one–loop.

Chapter 3

Seeley–DeWitt coefficients

In this section we apply the heat kernel method to euclidean perturbative quantum gravity. The computation is performed assuming that the background satisfies Einstein equations, and the results are extended up to the fourth coefficient, which was unknown up to now [5]. At the end of the chapter, a new form for the effective action is provided.

3.1 Einstein manifolds

A D-dimensional Riemannian manifold without boundary can be described through an (infinite) basis of *curvature monomials* $\mathcal{R}_i^{n,k}$. These are geometric invariants of order n in the Riemann tensor, Ricci tensor and scalar, and have been introduced precisely to deal with heat kernel computations [26], and more recently reviewed by [27; 28]. The *order* n of such monomials, that is, the number of differentiations of the metric tensor implicitly appearing in the monomial, should not be confused with his *rank* k, defined as the number of free indices [27; 28]. Since we consider only scalar invariants, we will always take $k = 0$. Up to order $n = 2$, there are only five,

$$\mathcal{R}^0 = 1 \tag{3.1}$$

$$\mathcal{R}^1 = R \tag{3.2}$$

$$\mathcal{R}^2_1 = R^2 \qquad \mathcal{R}^2_2 = R_{\mu\nu}R^{\mu\nu} \qquad \mathcal{R}^2_3 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\,, \tag{3.3}$$

while moving to third order we encounter ten more invariants,

$$\begin{aligned}
&\mathcal{R}^3_1 = R\nabla^2 R && \mathcal{R}^3_2 = R_{\mu\nu}\nabla^2 R^{\mu\nu} && \mathcal{R}^3_3 = R^3 \\
&\mathcal{R}^3_4 = RR_{\mu\nu}R^{\mu\nu} && \mathcal{R}^3_5 = R_\mu{}^\nu R_\nu{}^\rho R_\rho{}^\mu && \mathcal{R}^3_6 = R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma} \\
&\mathcal{R}^3_7 = RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} && \mathcal{R}^3_8 = R_{\mu\nu}R^{\mu\rho\sigma\tau}R^\nu{}_{\rho\sigma\tau} && \mathcal{R}^3_9 = R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu} \\
&\mathcal{R}^3_{10} = R^\alpha{}_\mu{}^\beta{}_\nu R^\mu{}_\rho{}^\nu{}_\sigma R^\rho{}_\alpha{}^\sigma{}_\beta\,,
\end{aligned} \tag{3.4}$$

where $\nabla^2 \equiv G^{\mu\nu}\nabla_\mu\nabla_\nu$ and we used the assumption that the manifold has no boundary to exclude two invariants, which are total derivatives.[1] In this thesis, however, we will restrict ourselves to the case of *Einstein manifolds*, which are a special class of Riemannian manifolds whose metric satisfies Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R + \Lambda G_{\mu\nu} = 0\,. \tag{3.5}$$

By contracting (3.5) with $G^{\mu\nu}$ we get

$$R - \frac{1}{2}DR + D\Lambda = 0 \qquad \longrightarrow \qquad \Lambda = \frac{D-2}{2D}R\,, \tag{3.6}$$

which allows to express the cosmological constant Λ as a function of R; moreover, by plugging (3.6) back into (3.5) we find

$$R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R + \frac{D-2}{2D}RG_{\mu\nu} = 0 \qquad \longrightarrow \qquad R_{\mu\nu} = \frac{1}{D}G_{\mu\nu}R\,, \tag{3.7}$$

so the Ricci curvature $R_{\mu\nu}$ is proportional to the Ricci scalar R on Einstein manifolds. Another important result comes from the *second Bianchi identity*,

$$\nabla_\mu R_{\alpha\beta\nu\rho} + \nabla_\nu R_{\alpha\beta\rho\mu} + \nabla_\rho R_{\alpha\beta\mu\nu} = 0\,, \tag{3.8}$$

that through multiple contractions leads to

$$\nabla^\rho R_{\mu\nu\alpha\rho} = \nabla_\nu R_{\mu\alpha} - \nabla_\mu R_{\nu\alpha} \tag{3.9}$$

$$\nabla^\nu R_{\mu\nu} = \frac{1}{2}\nabla_\mu R\,, \tag{3.10}$$

which are known as *contracted Bianchi identities*. Plugging equation (3.7) in (3.10), as $R_{\mu\nu} \propto R$, we find that $\nabla_\mu R = \nabla^\nu R_{\mu\nu} = 0$. Moreover, by taking the covariant derivative of (3.7), $\nabla_\alpha R_{\mu\nu} = 0$, so that equation (3.9) together with the last result implies that $\nabla^\rho R_{\mu\nu\alpha\rho} = 0$ as well. Therefore, *on Einstein manifolds all covariant derivatives of the form $\nabla_\mu R$, $\nabla_\alpha R_{\mu\nu}$ and $\nabla^\sigma R_{\mu\nu\rho\sigma}$ vanish identically.*

As a consequence, the basis given by (3.1)–(3.4) reduces to

$$\mathcal{E}^0 = 1 \tag{3.11}$$

$$\mathcal{E}^1 = R \tag{3.12}$$

$$\mathcal{E}^2_1 = R^2 \qquad \mathcal{E}^2_2 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \tag{3.13}$$

$$\begin{aligned} &\mathcal{E}^3_1 = R^3 && \mathcal{E}^3_2 = RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\ &\mathcal{E}^3_3 = R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu} && \mathcal{E}^3_4 = R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}R^{\rho}{}_{\alpha}{}^{\sigma}{}_{\beta}\,, \end{aligned} \tag{3.14}$$

[1] These are $\mathcal{R}^2_4 = \nabla^2 R$ and $\mathcal{R}^3_{11} = \nabla^4 R$ [28, § A].

since we can substitute (3.6) and (3.7) repeatedly in (3.3)–(3.4) to find the equations relating the new basis elements to the old ones,

$$\mathcal{R}_2^2 = \frac{1}{D}\mathcal{E}_1^2 \tag{3.15}$$

$$\begin{aligned} &\mathcal{R}_1^3 = \mathcal{R}_2^3 = 0 && \mathcal{R}_4^3 = \frac{1}{D}\mathcal{E}_1^3 \\ &\mathcal{R}_5^3 = \mathcal{R}_6^3 = \frac{1}{D^2}\mathcal{E}_1^3 && \mathcal{R}_8^3 = \frac{1}{D}\mathcal{E}_2^3 \,. \end{aligned} \tag{3.16}$$

Proof. Equation (3.15) comes from direct substitution,

$$\mathcal{R}_2^2 = R_{\mu\nu}R^{\mu\nu} = \frac{1}{D}RG_{\mu\nu}R^{\mu\nu} = \frac{1}{D}R^2 = \frac{1}{D}\mathcal{E}_1^2\,,$$

and the second equation in (3.16) follows immediately; to get the last two conditions of (3.16) we have to manipulate indices,

$$\begin{aligned} \mathcal{R}_5^3 &= R_\mu{}^\nu R_\nu{}^\rho R_\rho{}^\mu = G_{\mu\alpha}G_{\nu\beta}G_{\rho\gamma}R^{\alpha\nu}R^{\beta\rho}R^{\gamma\mu} = \frac{1}{D^3}R^3 G_{\mu\alpha}G_{\nu\beta}G_{\rho\gamma}G^{\alpha\nu}G^{\beta\rho}G^{\gamma\mu} \\ &= \frac{1}{D^3}R^3\delta^\mu_\nu\delta^\nu_\rho G_{\rho\gamma}G^{\gamma\mu} = \frac{1}{D^3}R^3 G_{\mu\gamma}G^{\mu\gamma} = \frac{1}{D^2}R^3 = \frac{1}{D^2}\mathcal{E}_1^3\,, \end{aligned}$$

and similarly

$$\begin{aligned} \mathcal{R}_6^3 &= R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma} = \frac{1}{D^2}R^2 G_{\mu\nu}G_{\rho\sigma}R^{\mu\rho\nu\sigma} = \frac{1}{D^2}R^3 = \frac{1}{D^2}\mathcal{E}_1^3 \\ \mathcal{R}_8^3 &= R_{\mu\nu}R^{\mu\rho\sigma\tau}R^\nu{}_{\rho\sigma\tau} = \frac{1}{D}RG_{\mu\nu}R^{\mu\rho\sigma\tau}R^\nu{}_{\rho\sigma\tau} = \frac{1}{D}RR^{\mu\rho\sigma\tau}R_{\mu\rho\sigma\tau} = \frac{1}{D}\mathcal{E}_2^3\,, \end{aligned}$$

which completes the proof of (3.16). ∎

Some useful identites

The basis constructed above, when combined with the second Bianchi identity, leads to useful identities involving contractions of the Riemann tensor. At order $\mathcal{R}^2$ we just have

$$R_{\mu\nu\alpha\beta}R^{\mu\alpha\nu\beta} = \frac{1}{2}R^2_{\mu\nu\alpha\beta} = \frac{1}{2}\mathcal{E}_2^2\,, \tag{3.17}$$

while at order $\mathcal{R}^3$ there are three independent conditions,

$$R^{\mu\alpha\nu\beta}R_{\mu\nu\rho\sigma}R_{\alpha\beta}{}^{\rho\sigma} = \frac{1}{2}\mathcal{E}_3^3 \tag{3.18}$$

$$R_{\alpha\beta}{}^{\rho\sigma}R^{\alpha\mu\beta\nu}R_{\rho\mu\sigma\nu} = \frac{1}{4}\mathcal{E}_3^3 \tag{3.19}$$

$$R_{\mu\alpha\nu\beta}R^{\mu\rho\nu\sigma}R^\alpha{}_\sigma{}^\beta{}_\rho = -\frac{1}{4}\mathcal{E}_3^3 + \mathcal{E}_4^3\,. \tag{3.20}$$

Proof. Since these identities are crucial in the following computations, we derive them in full detail. The first identity (3.17) comes from

$$\begin{aligned} R_{\mu\nu\alpha\beta}R^{\mu\alpha\nu\beta} &= -R_{\mu\nu\beta\alpha}R^{\mu\alpha\nu\beta} = R_{\mu\nu\beta\alpha}\left(R^{\mu\beta\alpha\nu} + R^{\mu\nu\beta\alpha}\right) = R^2_{\mu\nu\alpha\beta} + R_{\mu\nu\beta\alpha}R^{\mu\beta\alpha\nu} \\ &= \mathcal{E}^2_2 - R_{\mu\nu\alpha\beta}R^{\mu\alpha\nu\beta}\,, \end{aligned}$$

having exploited the first Bianchi identity and the symmetry properties of the Riemann tensor, as well as the relabeling of indices $\alpha \leftrightarrow \beta$. Equation (3.17) follows from comparing the first with the last term in the above equality. For what concerns (3.18) we can write

$$\begin{aligned} R^{\mu\alpha\nu\beta}R_{\mu\nu\rho\sigma}R_{\alpha\beta}{}^{\rho\sigma} &= -\left(R^{\mu\nu\beta\alpha} + R^{\mu\beta\alpha\nu}\right)R_{\mu\nu\rho\sigma}R_{\alpha\beta}{}^{\rho\sigma} \\ &= R^{\mu\nu\alpha\beta}R_{\mu\nu\rho\sigma}R_{\alpha\beta}{}^{\rho\sigma} - R^{\mu\beta\alpha\nu}R_{\mu\nu\rho\sigma}R_{\alpha\beta}{}^{\rho\sigma} \\ &= \mathcal{E}^3_3 - R^{\mu\alpha\nu\beta}R_{\mu\nu\rho\sigma}R_{\alpha\beta}{}^{\rho\sigma}\,, \end{aligned}$$

where in the last line we again relabeled $\alpha \leftrightarrow \beta$. Similarly, (3.19) is proven by

$$\begin{aligned} R_{\alpha\beta}{}^{\rho\sigma}R^{\alpha\mu\beta\nu}R_{\rho\mu\sigma\nu} &= -R_{\alpha\beta}{}^{\rho\sigma}R^{\alpha\mu\beta\nu}\left(R_{\rho\sigma\nu\mu} + R_{\rho\nu\mu\sigma}\right) \\ &= R_{\alpha\beta}{}^{\rho\sigma}R^{\alpha\mu\beta\nu}R_{\rho\sigma\mu\nu} - R_{\alpha\beta}{}^{\rho\sigma}R^{\alpha\mu\beta\nu}R_{\rho\nu\mu\sigma} \\ &= \frac{1}{2}\mathcal{E}^3_3 - R_{\alpha\beta}{}^{\rho\sigma}R^{\alpha\mu\beta\nu}R_{\rho\mu\sigma\nu}\,, \end{aligned}$$

where we used (3.18) as well. The last one can be deduced by exploiting the previous identities in a similar manner,

$$\begin{aligned} R_{\mu\alpha\nu\beta}R^{\mu\rho\nu\sigma}R^{\alpha}{}_{\sigma}{}^{\beta}{}_{\rho} &= -R_{\mu\alpha\nu\beta}R^{\mu\rho\nu\sigma}\left(R^{\alpha\beta}{}_{\rho\sigma} + R^{\alpha}{}_{\rho\sigma}{}^{\beta}\right) \\ &= -R_{\mu\alpha\nu\beta}R^{\mu\rho\nu\sigma}R^{\alpha\beta}{}_{\rho\sigma} + R_{\mu\alpha\nu\beta}R^{\mu\rho\nu\sigma}R^{\alpha}{}_{\rho}{}^{\beta}{}_{\sigma} = -\frac{1}{4}\mathcal{E}^3_3 + \mathcal{E}^3_4\,, \end{aligned}$$

which corresponds to (3.20). These identities are stated without proof in [27, § 5.1], equations (5.5 *a*–*c*); identity (3.17) is proved in [26, § 2]. ∎

There is also a useful identity which involves covariant derivatives of the Riemann tensor, not contracted with the tensor itself:

$$R_{\mu\nu\alpha\beta}\nabla^2 R^{\mu\nu\alpha\beta} = \frac{2}{D}\mathcal{E}^3_2 - \mathcal{E}^3_3 - 4\mathcal{E}^3_4\,. \tag{3.21}$$

Proof. By using the second Bianchi identity (3.8) and the commutator of covariant derivatives acting on the Riemann tensor,

$$[\nabla_\alpha, \nabla_\beta]R_{\mu\nu\rho\sigma} = R_{\alpha\beta\mu}{}^{\lambda}R_{\lambda\nu\rho\sigma} + R_{\alpha\beta\nu}{}^{\lambda}R_{\mu\lambda\rho\sigma} + R_{\alpha\beta\rho}{}^{\lambda}R_{\mu\nu\lambda\sigma} + R_{\alpha\beta\sigma}{}^{\lambda}R_{\mu\nu\rho\lambda}\,,$$

we find

$$\begin{aligned} R_{\mu\nu\alpha\beta}\nabla^2 R^{\mu\nu\alpha\beta} &= -R_{\mu\nu\alpha\beta}\nabla_\rho\left(\nabla^\beta R^{\mu\nu\rho\alpha} + \nabla^\alpha R^{\mu\nu\beta\rho}\right) = -2R_{\mu\nu\alpha\beta}\nabla_\rho\nabla^\beta R^{\mu\nu\rho\alpha} \\ &= -2R_{\mu\nu\alpha\beta}\Big(\nabla^\beta\nabla_\rho R^{\rho\alpha\mu\nu} + R_{\rho}{}^{\beta\mu}{}_{\lambda}R^{\lambda\nu\rho\alpha} + R_{\rho}{}^{\beta\nu}{}_{\lambda}R^{\mu\lambda\rho\alpha} \\ &\qquad + R^{\beta}{}_{\lambda}R^{\mu\nu\lambda\alpha} + R_{\rho}{}^{\beta\alpha}{}_{\lambda}R^{\mu\nu\rho\lambda}\Big) \\ &= \frac{2}{D}RR^2_{\mu\nu\alpha\lambda} - 2R_{\mu\nu\alpha\beta}R^{\alpha\lambda\beta\rho}R^{\mu\nu}{}_{\lambda\rho} - 4R_{\mu\nu\alpha\beta}R_{\rho}{}^{\beta\mu}{}_{\lambda}R^{\rho\alpha\lambda\nu} \\ &= \frac{2}{D}\mathcal{E}^3_2 - \mathcal{E}^3_3 - 4R_{\mu\nu\beta\alpha}R^{\nu\lambda\alpha\rho}R_{\lambda}{}^{\mu}{}_{\rho}{}^{\beta} = \frac{2}{D}\mathcal{E}^3_2 - \mathcal{E}^3_3 - 4\mathcal{E}^3_4\,, \end{aligned}$$

where we used $\nabla_\rho R^{\rho\alpha\mu\nu} = 0$ and the antisymmetry properties of the Riemann tensor; in the second–to–last step, we also employed the identity (3.18).[2] A similar identity, stated without proof, appears in [27, § 5.1], equation (5.4). ■

3.1.1 Kinetic operators on Einstein manifolds

The expression for the kinetic operators for the ghost (2.27) and for the graviton (2.33) can be greatly simplified in the context of Einstein manifolds. In a previous work [5], the operators are not reduced to this simpler form, and the first three coefficients are computed off–shell. Without employing Vilkovisky–DeWitt formalism [12; 13], however, these results are not gauge invariant, and have to be hereafter reduced to Einstein manifolds, as it occours indeed in [5]. Here, to simplify computations, we immediately reduce the differential operators to the case of Einstein manifolds; the outcoming coefficients will then be automatically gauge invariant. Therefore, from now on we assume that the background field $\boldsymbol{g}$ satisfies Einstein equations (3.5).

The ghost kinetic operator (2.27), with the aid of (3.7), becomes

$$\mathcal{F}^{\mu}{}_{\nu} = -(\delta^\mu_\nu \nabla^2 + R^{\mu}{}_{\nu}) = -\delta^\mu_\nu \left(\nabla^2 + \frac{1}{D}R\right), \tag{3.22}$$

while the graviton kinetic operator (2.33) can be reduced to

$$F_{\mu\nu}{}^{\alpha\beta} = -\frac{1}{2}\left(\delta^\alpha_\mu\delta^\beta_\nu + \delta^\beta_\mu\delta^\alpha_\nu\right)\nabla^2 - R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta} - R_{\mu}{}^{\beta}{}_{\nu}{}^{\alpha}\,, \tag{3.23}$$

by repeated application of (3.6) and (3.7).

Proof. The starting point is equation (2.33): it is simple to see that Einstein manifold constraints do not modify the terms appearing in (3.23). We now prove that all the remaining terms vanish identically:

$$\begin{aligned}
&-\frac{1}{2}\left(\delta^\alpha_\mu\delta^\beta_\nu + \delta^\beta_\mu\delta^\alpha_\nu\right)(2\Lambda - R) - \frac{1}{D-2}g_{\mu\nu}g^{\alpha\beta}R + \frac{2}{D-2}g_{\mu\nu}R^{\alpha\beta} + g^{\alpha\beta}R_{\mu\nu}\\
&\qquad -\frac{1}{2}\left(\delta^\alpha_\mu R^\beta_\nu + \delta^\beta_\mu R^\alpha_\nu + \delta^\alpha_\nu R^\beta_\mu + \delta^\beta_\nu R^\alpha_\mu\right)\\
&= -\frac{1}{2}\left(\delta^\alpha_\mu\delta^\beta_\nu + \delta^\beta_\mu\delta^\alpha_\nu\right)\left(\frac{D-2}{D} - 1\right)R + \left[-\frac{1}{D-2} + \frac{2}{D(D-2)} + \frac{1}{D}\right]g_{\mu\nu}g^{\alpha\beta}\\
&\qquad -\frac{1}{D}\left(\delta^\alpha_\mu\delta^\beta_\nu + \delta^\beta_\mu\delta^\alpha_\nu\right)R\\
&= -\left(\frac{D-2}{2D} - \frac{1}{2} + \frac{1}{D}\right)\left(\delta^\alpha_\mu\delta^\beta_\nu + \delta^\beta_\mu\delta^\alpha_\nu\right)R = 0\,.
\end{aligned}$$

■

[2] I thank Francesco Comberiati and Filippo Fecit for having drawn my attention on this identity. The proof given here was suggested by them.

3.1.2 Maximally symmetric spaces

The basis (3.11)–(3.14) simplifies even more in the case of *maximally symmetric spaces*, where the Riemann tensor is proportional to the Ricci scalar,

$$R_{\mu\nu\rho\sigma} = \frac{1}{D(D-1)} \left(G_{\mu\rho}G_{\nu\sigma} - G_{\mu\sigma}G_{\nu\rho}\right) R \,, \tag{3.24}$$

and the number of linearly independent Killing vectors can be proved to be $\frac{1}{2}D(D+1)$, which is the highest possible one. In euclidean time, these spaces are given by the D–spheres S^D [29, § A.2]. From condition (3.24), it is possible to prove that the basis of curvature monomials, in the case of maximally symmetric spaces, contains only one element at each order $\mathcal{R}^n$, proportional to the Ricci scalar n–th power R^n, since

$$\mathcal{E}_2^2 = \frac{2}{D(D-1)}R^2 \qquad \mathcal{E}_2^3 = \frac{2}{D(D-1)}R^3 \tag{3.25}$$

$$\mathcal{E}_3^3 = \frac{4}{D^2(D-1)^2}R^3 \qquad \mathcal{E}_4^3 = \frac{D-2}{D^2(D-1)^2}R^3 \,. \tag{3.26}$$

Proof. The second equation in (3.25) is trivially deduced from the first one, which can be proved by direct computation using (3.24):

$$\begin{aligned}\mathcal{E}_2^2 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} &= \frac{1}{D^2(D-1)^2} \left(G_{\mu\rho}G_{\nu\sigma} - G_{\mu\sigma}G_{\nu\rho}\right)\left(G^{\mu\rho}G^{\nu\sigma} - G^{\mu\sigma}G^{\nu\rho}\right) R^2 \\ &= \frac{1}{D^2(D-1)^2}\left(2D^2 - 2D\right) R^2 = \frac{2}{D(D-1)}R^2 \,.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{E}_3^3 &= R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu} \\ &= \frac{1}{D^3(D-1)^3}\left(\delta^\rho_\mu\delta^\sigma_\nu - \delta^\sigma_\mu\delta^\rho_\nu\right)\left(\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\beta_\rho\delta^\alpha_\sigma\right)\left(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\nu_\alpha\delta^\mu_\beta\right) R^3 \\ &= \frac{1}{D^3(D-1)^3}2\left(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\beta_\mu\delta^\alpha_\nu\right)\left(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\nu_\alpha\delta^\mu_\beta\right) R^3 \\ &= \frac{2}{D^3(D-1)^3}2\left(D^2 - D\right) R^3 = \frac{4}{D^2(D-1)^2}R^3 \,,\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}_4^3 &= R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}R^{\rho}{}_{\alpha}{}^{\sigma}{}_{\beta} \\ &= \frac{1}{D^3(D-1)^3}\left(G^{\alpha\beta}G_{\mu\nu} - \delta^\alpha_\nu\delta^\beta_\mu\right)\left(G^{\mu\nu}G_{\rho\sigma} - \delta^\mu_\sigma\delta^\nu_\rho\right)\left(G^{\rho\sigma}G_{\alpha\beta} - \delta^\rho_\beta\delta^\sigma_\alpha\right) R^3 \\ &= \frac{1}{D^3(D-1)^3}\left[(D-2)G^{\alpha\beta}G_{\rho\sigma} + \delta^\alpha_\rho\delta^\beta_\sigma\right]\left(G^{\rho\sigma}G_{\alpha\beta} - \delta^\rho_\beta\delta^\sigma_\alpha\right) R^3 \\ &= \frac{1}{D^3(D-1)^3}(D-2)(D^2-D)R^3 = \frac{D-2}{D^2(D-1)^2}R^3 \,,\end{aligned}$$

which proves (3.26). ■

The ghost kinetic operator (3.22) cannot be simplified more, while the graviton operator (3.23) can be cast in the even simpler form

$$F_{\mu\nu}{}^{\alpha\beta} = -\frac{1}{2}\left[\delta^\alpha_\mu\delta^\beta_\nu + \delta^\beta_\mu\delta^\alpha_\nu\right]\left(\nabla^2 + \frac{2}{D(D-1)}R\right) - \frac{2}{D(D-1)}G_{\mu\nu}G^{\alpha\beta}R\,. \tag{3.27}$$

Proof. Starting from (3.23) we get

$$\begin{aligned} F_{\mu\nu}{}^{\alpha\beta} &= -\frac{1}{2}\left(\delta^\alpha_\mu\delta^\beta_\nu + \delta^\beta_\mu\delta^\alpha_\nu\right)\nabla^2 - \frac{1}{D(D-1)}\left(G_{\mu\nu}G^{\alpha\beta} - \delta^\beta_\mu\delta^\alpha_\nu + G_{\mu\nu}G^{\beta\alpha} - \delta^\alpha_\mu\delta^\beta_\nu\right)R \\ &= -\frac{1}{2}\left(\delta^\alpha_\mu\delta^\beta_\nu + \delta^\beta_\mu\delta^\alpha_\nu\right)\left[\nabla^2 + \frac{2}{D(D-1)}R\right] - \frac{2}{D(D-1)}G_{\mu\nu}G^{\alpha\beta}R\,. \end{aligned}$$

■

3.2 Seeley-DeWitt coefficients computation

Everything is now ready to start a detailed computation of Seeley–DeWitt coefficients for perturbative quantum gravity. In the following, we consider the ghost, graviton and total coefficients — obtained as a combination of the previous two — up to the fourth one, $a_3(x)$. The result will allow us to write a more general form for the effective action in D dimensions.

3.2.1 Ghost coefficients

To compute the ghost heat kernel coefficients we have to relate the ghost kinetic operator in its simplified form (3.22) to the general form for an elliptic operator (1.92). This implies to replace $\mathbb{1} \leftrightarrow \delta^\mu_\nu$ and $V \leftrightarrow \frac{1}{D}R\delta^\mu_\nu$. Moreover, the contravariant ghost field c^μ satisfies the commutation relation

$$[\nabla_\mu, \nabla_\nu]c^\rho = R_{\mu\nu}{}^\rho{}_\sigma c^\sigma\,, \tag{3.28}$$

which stems from the definition of the Riemann tensor. Since in non–abelian gauge theories $\Omega_{\mu\nu} \equiv [\nabla_\mu, \nabla_\nu]$, we conclude that $(\Omega_{\mu\nu})^\rho{}_\sigma = R_{\mu\nu}{}^\rho{}_\sigma$. Note that in this expression the indices μ, ν label the different elements of the gauge field strength $\Omega_{\mu\nu}$, which are $D \times D$ matrices whose components are given by the (spacetime) indices ρ, σ. Therefore, the substitutions to be performed in the heat kernel coefficients (1.96)–(1.99) are

$$\begin{cases} \mathbb{1} \;\leftrightarrow\; \delta^\mu_\nu \\ V \;\leftrightarrow\; \dfrac{1}{D}R\delta^\mu_\nu \\ (\Omega_{\mu\nu})^\rho{}_\sigma \;\leftrightarrow\; R_{\mu\nu}{}^\rho{}_\sigma\,. \end{cases} \tag{3.29}$$

To compute the first two coefficients we note that $\mathrm{Tr}\,[\delta^\mu_\nu] = D$, and with the aid of (1.94) and the first line of (1.95) we can transform (1.96) and (1.97) in

$$\mathrm{Tr}\left[a_0^{gh}(x)\right] = \mathrm{Tr}\left[\alpha_0^{gh}(x)\right] = D \tag{3.30}$$

$$\mathrm{Tr}\left[a_1^{gh}(x)\right] = \mathrm{Tr}\left[\alpha_1^{gh}(x)\right] = \left(\frac{1}{6}D + 1\right) R\,. \tag{3.31}$$

Similar computations are to be performed in the case of (1.98), where the term containing ∇^2 identically vanish because of Einstein manifolds properties, and we are left with

$$\begin{aligned}\mathrm{Tr}\left[\alpha_2^{gh}(x)\right] &= \frac{D}{180}\left(R^2_{\mu\nu\rho\sigma} - R^2_{\mu\nu}\right) + \frac{1}{12}\mathrm{Tr}\left[\Omega^2_{\mu\nu}\right] \\ &= \frac{D}{180}\left(R^2_{\mu\nu\rho\sigma} - \frac{1}{D}R^2\right) - \frac{1}{12}R^2_{\mu\nu\rho\sigma}\,.\end{aligned} \tag{3.32}$$

Proof. By the last of (3.29),

$$\begin{aligned}\mathrm{Tr}\left[\Omega^2_{\mu\nu}\right] &= \mathrm{Tr}\left[(\Omega_{\mu\nu})^\rho{}_\sigma\,(\Omega^{\mu\nu})^\sigma{}_\tau\right] = (\Omega_{\mu\nu})^\rho{}_\sigma\,(\Omega^{\mu\nu})^\sigma{}_\rho \\ &= R_{\mu\nu\rho\sigma}R^{\mu\nu\sigma\rho} = -R^2_{\mu\nu\rho\sigma}\,.\end{aligned}$$

■

According to (1.95), α_2^{gh} is to be summed to

$$\beta_2^{gh} = \frac{1}{2}\left(\alpha_1^{gh}\right)^2 = \frac{1}{2}\left[\delta^\mu_\nu\left(\frac{1}{6}+\frac{1}{D}\right)R\right]^2 = \frac{1}{2}\delta^\mu_\nu\left(\frac{1}{6}+\frac{1}{D}\right)^2 R^2\,, \tag{3.33}$$

hence, by taking the trace of (3.33),

$$\mathrm{Tr}\left[\beta_2^{gh}(x)\right] = \frac{D}{2}\left(\frac{1}{6}+\frac{1}{D}\right)^2 R^2\,, \tag{3.34}$$

which combined with (3.32) gives in the end

$$\mathrm{Tr}\left[a_2^{gh}(x)\right] = \frac{5D^2+58D+180}{360D}\,R^2 + \frac{D-15}{180}R^2_{\mu\nu\rho\sigma}\,. \tag{3.35}$$

In $D = 4$ these coefficients reduce to

$$\mathrm{Tr}\left[a_0^{gh}(x)\right]_{D=4} = 4 \tag{3.36}$$

$$\mathrm{Tr}\left[a_1^{gh}(x)\right]_{D=4} = \frac{5}{3}R \tag{3.37}$$

$$\mathrm{Tr}\left[a_2^{gh}(x)\right]_{D=4} = \frac{41}{120}R^2 - \frac{11}{180}R^2_{\mu\nu\rho\sigma}\,, \tag{3.38}$$

where (3.36) reproduces the correct number of degrees freedom for the ghost field. On maximally symmetric spaces the third coefficient (3.35) can be further simplified, using the first equation in (3.25), to[3]

$$\mathrm{Tr}\left[\widetilde{a}_2^{gh}(x)\right]_{D=4} = \left.\frac{5D^3+53D^2+126D-240}{360D(D-1)}\right|_{D=4} R^2 = \frac{179}{540}R^2\,. \tag{3.39}$$

Fourth heat kernel coefficient for the ghost

We are now ready to compute the fourth heat kernel coefficient for the ghost field, starting from the general formula (1.99) and performing the substitutions (3.29). It is convenient to write

$$\alpha_3^{gh}(x) \equiv \frac{1}{7!}\boldsymbol{A}_{gh}[R,R_{\mu\nu},R_{\mu\nu\rho\sigma}] + \frac{2}{6!}\boldsymbol{B}_{gh}[R,R_{\mu\nu},R_{\mu\nu\rho\sigma},\Omega_{\mu\nu},V]\,, \tag{3.40}$$

where $\boldsymbol{A}_{gh}$ and $\boldsymbol{B}_{gh}$ are two (involved) functions of the metric invariants and of the gauge field strength, as reported in (1.99).

Let us start from $\boldsymbol{A}_{gh}$: six of the first eight terms vanish identically, since they are all proportional to covariant derivatives of R, $R_{\mu\nu}$ or $R_{\mu\nu\rho\sigma}$, except from the two proportional to $(\nabla_\alpha R_{\mu\nu\rho\sigma})^2$ and $R_{\mu\nu\rho\sigma}\nabla^2 R^{\mu\nu\rho\sigma}$, as the covariant derivatives are not contracted with the Riemann tensor there. Recalling again that $\mathrm{Tr}[\delta^\mu_\nu]=D$, we are therefore left with

$$\begin{aligned}\mathrm{Tr}\left[\boldsymbol{A}_{gh}\right] = D\Big(&3R_{\mu\nu\rho\sigma}\nabla^2 R^{\mu\nu\rho\sigma} - \frac{208}{9}R_\mu^{\ \nu}R_\nu^{\ \sigma}R_\sigma^{\ \mu} + \frac{64}{3}R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma}\\ &-\frac{16}{3}R_{\mu\nu}R^\mu_{\ \rho\sigma\tau}R^{\nu\rho\sigma\tau} + \frac{44}{9}R_{\mu\nu}^{\ \ \ \rho\sigma}R_{\rho\sigma}^{\ \ \ \alpha\beta}R_{\alpha\beta}^{\ \ \ \mu\nu}\\ &+\frac{80}{9}R_{\mu\nu\rho\sigma}R^{\mu\alpha\rho\beta}R^\nu_{\ \alpha\ \beta}^{\ \ \sigma}\Big)\\ = -&\frac{16}{9D}\mathcal{E}_1^3 + \frac{2}{3}\mathcal{E}_2^3 + \frac{17D}{9}\mathcal{E}_3^3 - \frac{28D}{9}\mathcal{E}_4^3\,,\end{aligned} \tag{3.41}$$

which, on maximally symmetric spaces, reduces to

$$\mathrm{Tr}\left[\widetilde{\boldsymbol{A}}_{gh}\right] = -\frac{16\left(D^2-D-6\right)}{9D(D-1)^2}\,R^3\,. \tag{3.42}$$

Proof. Let us start from the vanishing terms: $\nabla^4 R = (\nabla_\mu R)^2 = (\nabla_\mu R_{\nu\sigma})^2 = 0$ by Einstein manifolds properties. Similarly,

$$\nabla_\mu R_{\nu\sigma}\nabla^\nu R^{\mu\sigma} = 0 \qquad \text{and} \qquad R_{\mu\nu}\nabla^\nu\nabla_\sigma R^{\mu\sigma} = 0\,.$$

[3] Here and in the following equations, a tilde indicates that the quantity is computed in maximally symmetric spaces.

Moreover, it is useful to integrate by parts, and use the assumption that $\partial\mathcal{M} = \varnothing$ to neglect boundary terms, so that

$$R_{\mu\nu}\nabla^2 R^{\mu\nu} = -\nabla_\alpha R_{\mu\nu}\nabla^\alpha R^{\mu\nu} = 0\,.$$

What remains are then two terms, proportional to $(\nabla_\alpha R_{\mu\nu\rho\sigma})^2$ and $R_{\mu\nu\rho\sigma}\nabla^2 R^{\mu\nu\rho\sigma}$. By integrating by parts,

$$(\nabla_\alpha R_{\mu\nu\rho\sigma})^2 = -R_{\mu\nu\rho\sigma}\nabla^2 R^{\mu\nu\rho\sigma}\,,$$

where we neglected again boundary terms, so that (1.99), by identity (3.21), will contain only

$$3R_{\mu\nu\rho\sigma}\nabla^2 R^{\mu\nu\rho\sigma} = 3\left(\frac{2}{D}\mathcal{E}_2^3 - \mathcal{E}_3^3 - 4\mathcal{E}_4^3\right).$$

For what concerns the terms in (1.99) which do not contain covariant derivatives, using (3.16) we have

$$\begin{aligned}
R_\mu{}^\nu R_\nu{}^\sigma R_\sigma{}^\mu &= \mathcal{R}_5^3 = \frac{1}{D^2}\mathcal{E}_1^3 \\
R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma} &= \mathcal{R}_6^3 = \frac{1}{D^2}\mathcal{E}_1^3 \\
R_{\mu\nu}R^\mu{}_{\rho\sigma\tau}R^{\nu\rho\sigma\tau} &= \mathcal{R}_8^3 = \frac{1}{D}\mathcal{E}_2^3\,,
\end{aligned}$$

while the other two factors cannot be simplified, being already equal to the elements of the Einstein manifold basis $\mathcal{E}_3^3$ and $\mathcal{E}_4^3$, respectively. On maximally symmetric spaces, instead, these reduce to

$$\begin{aligned}
R_{\mu\nu}R^\mu{}_{\rho\sigma\tau}R^{\nu\rho\sigma\tau} &= \mathcal{R}_8^3 = \frac{1}{D}\mathcal{E}_2^3 = \frac{2}{D^2(D-1)}R^3 \\
R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu} &= \mathcal{E}_3^3 = \frac{4}{D^2(D-1)^2}R^3 \\
R_{\mu\nu\rho\sigma}R^{\mu\alpha\rho\beta}R^\nu{}_\alpha{}^\sigma{}_\beta &= \mathcal{E}_4^3 = \frac{D-2}{D^2(D-1)^2}R^3\,,
\end{aligned}$$

by using (3.25) and (3.26). ∎

We now compute $\boldsymbol{B}_{gh}$. Since $(\Omega_{\mu\nu})^\rho{}_\sigma = R_{\mu\nu}{}^\rho{}_\sigma$, the second term of $\boldsymbol{B}_{gh}$ in (1.99) vanishes identically; moreover, as $V \propto R$, the same occours for the last four, leaving us with

$$\begin{aligned}
\mathrm{Tr}\left[\boldsymbol{B}_{gh}\right] &= 4\mathrm{Tr}\left[\Omega_{\mu\nu}\nabla^2\Omega^{\mu\nu}\right] - 12\mathrm{Tr}\left[\Omega_\mu{}^\nu\Omega_\nu{}^\sigma\Omega_\sigma{}^\mu\right] \\
&\quad + 6\mathrm{Tr}\left[R_{\mu\nu\rho\sigma}\Omega^{\mu\nu}\Omega^{\rho\sigma}\right] - 4\mathrm{Tr}\left[R_{\mu\nu}\Omega^{\mu\sigma}\Omega^\nu{}_\sigma\right] \\
&= -\frac{4}{D}\mathcal{E}_2^3 - 2\mathcal{E}_3^3 + 4\mathcal{E}_4^3\,,
\end{aligned} \tag{3.43}$$

and on maximally symmetric spaces

$$\mathrm{Tr}\left[\widetilde{\boldsymbol{B}}_{gh}\right] = -\frac{4(D+2)}{D^2(D-1)^2}\,R^3\,. \tag{3.44}$$

Proof. The vanishing terms are $(\nabla^\mu \Omega_{\mu\nu})^2 = (\nabla^\mu R_{\mu\nu\rho\sigma})^2 = 0$, as well as $\nabla^4 V = (\nabla_\mu V)^2 = R_{\mu\nu}\nabla^\mu\nabla^\nu V = \nabla_\mu R \nabla^\mu V = 0$, since $V \propto R$. The only terms containing covariant derivatives which do not vanish are $8(\nabla_\mu \Omega_{\nu\sigma})^2 + 12\Omega_{\mu\nu}\nabla^2\Omega^{\mu\nu}$; by integrating by parts they sum up to

$$\begin{aligned} 4\mathrm{Tr}\left[\Omega_{\mu\nu}\nabla^2\Omega^{\mu\nu}\right] &= 4\mathrm{Tr}\left[R_{\mu\nu\rho\sigma}\nabla^2 R^{\mu\nu\sigma\alpha}\right] = -4R_{\mu\nu\rho\sigma}\nabla^2 R^{\mu\nu\rho\sigma} \\ &= -4\left(\frac{2}{D}\mathcal{E}_2^3 - \mathcal{E}_3^3 - 4\mathcal{E}_4^3\right), \end{aligned}$$

where we used the identity (3.21). The traces appearing in the remaining three terms can be computed by explicitly writing all the matrix indices:

$$\begin{aligned} \mathrm{Tr}\left[\Omega_\mu{}^\nu\Omega_\nu{}^\sigma\Omega_\sigma{}^\mu\right] &= \mathrm{Tr}\left[R^\alpha{}_{\beta\mu}{}^\nu R^\beta{}_{\gamma\nu}{}^\sigma R^\gamma{}_{\delta\sigma}{}^\mu\right] = R^\alpha{}_{\beta\mu}{}^\nu R^\beta{}_{\gamma\nu}{}^\sigma R^\gamma{}_{\alpha\sigma}{}^\mu = \mathcal{E}_4^3 \\ \mathrm{Tr}\left[R_{\mu\nu\rho\sigma}\Omega^{\mu\nu}\Omega^{\rho\sigma}\right] &= R_{\mu\nu\rho\sigma}\mathrm{Tr}\left[R^\alpha{}_\beta{}^{\mu\nu}R^\beta{}_\gamma{}^{\rho\sigma}\right] = R_{\mu\nu\rho\sigma}R^\alpha{}_\beta{}^{\mu\nu}R^\beta{}_\alpha{}^{\rho\sigma} \\ &= -R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu} = -\mathcal{E}_3^3 \\ \mathrm{Tr}\left[R_{\mu\nu}\Omega^{\mu\sigma}\Omega^\nu{}_\sigma\right] &= R_{\mu\nu}\mathrm{Tr}\left[R^\alpha{}_\beta{}^{\mu\sigma}R^\beta{}_\gamma{}^\nu{}_\sigma\right] = R_{\mu\nu}R^\alpha{}_\beta{}^{\mu\sigma}R^\beta{}_\alpha{}^\nu{}_\sigma \\ &= -R_{\mu\nu}R^{\mu\sigma\alpha\beta}R^\nu{}_{\sigma\alpha\beta} = -\frac{1}{D}RR_{\mu\sigma\alpha\beta}R^{\mu\sigma\alpha\beta} = -\frac{1}{D}\mathcal{E}_2^3\,, \end{aligned}$$

and the result for maximally symmetric spaces can be obtained by the same identities recalled in the previous proof. ■

Going back to (3.40) we conclude that

$$\begin{aligned} \mathrm{Tr}\left[\alpha_3^{gh}(x)\right] &= -\frac{1}{2835D}\mathcal{E}_1^3 + \frac{D-84}{7560D}\mathcal{E}_2^3 + \frac{17D-252}{45360}\mathcal{E}_3^3 - \frac{D-18}{1620}\mathcal{E}_4^3 \\ \mathrm{Tr}\left[\widetilde{\alpha}_3^{gh}(x)\right] &= -\frac{2D^3-2D^2+51D+126}{5670D^2(D-1)^2}\,R^3\,, \end{aligned} \tag{3.45}$$

which however does not provide the full coefficient for the ghost, since we still have to add the term β_3 defined in (1.95). This is evaluated as

$$\mathrm{Tr}\left[\beta_3^{gh}(x)\right] = \frac{(5D^2+54D+180)(D+6)}{6480D^2}\mathcal{E}_1^3 + \frac{(D+6)(D-15)}{1080D}\mathcal{E}_2^3\,, \tag{3.46}$$

while on maximally symmetric spaces

$$\mathrm{Tr}\left[\widetilde{\beta}_3^{gh}(x)\right] = \frac{(D+6)(5D^3+49D^2+138D-360)}{6480D^2(D-1)}\,R^3\,. \tag{3.47}$$

Proof. The computation to get (3.46) is made a bit more complicated than before, since we cannot take the trace at first:

$$\begin{aligned}
\beta_3^{gh} &= \frac{1}{6}\left(\alpha_1^{gh}\right)^3 + \alpha_1^{gh}\alpha_2^{gh} \\
&= \frac{1}{6}\left[\delta^\tau_\alpha\left(\frac{1}{6}+\frac{1}{D}\right)R\right]^3 + \left[\delta^\tau_\alpha\left(\frac{1}{6}+\frac{1}{D}\right)R\right]\left[\frac{1}{180}\left(R^2_{\mu\nu\rho\sigma}-R^2_{\mu\nu}\right)\delta^\alpha_\gamma + \frac{1}{12}\Omega^2_{\mu\nu}\right] \\
&= \frac{1}{6}\delta^\tau_\gamma\left(\frac{1}{6}+\frac{1}{D}\right)^3 R^3 + \left[\delta^\tau_\gamma\left(\frac{1}{6}+\frac{1}{D}\right)R\right]\frac{1}{180}\left(R^2_{\mu\nu\rho\sigma}-\frac{1}{D}R^2\right) \\
&\qquad + \frac{1}{12}\left(\frac{1}{6}+\frac{1}{D}\right)RR^\tau{}_{\beta\mu\nu}R^\beta{}_\gamma{}^{\mu\nu}\,,
\end{aligned}$$

where it is important to note that δ^τ_α is contracted with the first index of $(\Omega_{\mu\nu})^\alpha{}_\beta = R^\alpha{}_{\beta\mu\nu}$, to give $R^\tau{}_{\beta\mu\nu}$. Taking the trace,

$$\begin{aligned}
\mathrm{Tr}\left[\beta_3^{gh}\right] &= \frac{D}{6}\left(\frac{1}{6}+\frac{1}{D}\right)^3 R^3 + \frac{D}{180}\left(\frac{1}{6}+\frac{1}{D}\right)R\left(R^2_{\mu\nu\rho\sigma}-\frac{1}{D}R^2\right) \\
&\qquad + \frac{1}{12}\left(\frac{1}{6}+\frac{1}{D}\right)RR^\tau{}_{\beta\mu\nu}R^\beta{}_\tau{}^{\mu\nu} \\
&= \frac{(5D^2+54D+180)(D+6)}{6480D^2}\mathcal{E}_1^3 + \frac{(D+6)(D-15)}{1080D}\mathcal{E}_2^3\,,
\end{aligned}$$

where the permutation of indices β, τ brought a negative sign in the last term. The maximally symmetric case (3.47) is obtained by replacing $\mathcal{E}_2^3$, as usual. ■

The ghost coefficient is then computed by summing up the two results (3.45) and (3.46), to get

$$\begin{aligned}
\mathrm{Tr}\left[a_3^{gh}(x)\right] &= \frac{35D^3+588D^2+3512D+7560}{45360D^2}\,\mathcal{E}_1^3 \\
&\qquad + \frac{7D^2-62D-714}{7560D}\mathcal{E}_2^3 + \frac{17D-252}{45360}\mathcal{E}_3^3 - \frac{D-18}{1620}\mathcal{E}_4^3\,, \qquad (3.48)
\end{aligned}$$

while on maximally symmetric spaces

$$\mathrm{Tr}\left[\tilde{a}_3^{gh}(x)\right] = \frac{35D^5+518D^4+2455D^3+268D^2-18804D+14112}{45360(D-1)^2D^2}\,R^3\,. \qquad (3.49)$$

Due to the rather complicated final expressions (3.48) and (3.49), it might be useful to evaluate them at $D=4$ as well:

$$\mathrm{Tr}\left[a_3^{gh}(x)\right]_{D=4} = \frac{4157}{90720}\mathcal{E}_1^3 - \frac{85}{3024}\mathcal{E}_2^3 - \frac{23}{5670}\mathcal{E}_3^3 + \frac{7}{810}\mathcal{E}_4^3 \qquad (3.50)$$

$$\mathrm{Tr}\left[\tilde{a}_3^{gh}(x)\right]_{D=4} = \frac{5599}{136080}\,R^3\,. \qquad (3.51)$$

3.2.2 Graviton coefficients

To compute the graviton heat kernel coefficients we again relate the ghost kinetic operator in its simplified form (3.23) to the general form (1.92). This implies to replace $\mathbb{1} \leftrightarrow \delta_{\mu\nu}{}^{\alpha\beta}$ and $V \leftrightarrow \mathcal{V}_{\mu\nu}{}^{\alpha\beta}$ where

$$\delta_{\mu\nu}{}^{\alpha\beta} \equiv \frac{1}{2}\left(\delta_\mu^\alpha \delta_\nu^\beta + \delta_\mu^\beta \delta_\nu^\alpha\right) \tag{3.52}$$

$$\mathcal{V}_{\mu\nu}{}^{\alpha\beta} \equiv R_\mu{}^\alpha{}_\nu{}^\beta + R_\mu{}^\beta{}_\nu{}^\alpha \,. \tag{3.53}$$

$\delta_{\mu\nu}{}^{\alpha\beta}$ is a symmetrised version of the usual Kronecker delta, acting on a space of dimension $\frac{1}{2}D(D+1)$, as it can be seen from the fact that

$$\mathrm{Tr}\left[\delta_{\mu\nu}{}^{\alpha\beta}\right] = \delta_{\mu\nu}{}^{\mu\nu} = \frac{1}{2}(\delta_\mu^\mu \delta_\nu^\nu + \delta_\mu^\nu \delta_\nu^\mu) = \frac{1}{2}D(D+1)\,. \tag{3.54}$$

Note that both $\delta_{\mu\nu}{}^{\alpha\beta}$ and $\mathcal{V}_{\mu\nu}{}^{\alpha\beta}$ are symmetric under the exchange of the lower or upper indices among themselves. Moreover, as the graviton kinetic operator (3.23) acts on covariant symmetric tensors, the commutation relation of covariant derivatives is now given by

$$[\nabla_\mu, \nabla_\nu] h_{\rho\sigma} = R_{\rho\sigma}{}^{\alpha\beta}{}_{\mu\nu} h_{\alpha\beta}\,, \tag{3.55}$$

having defined the correspondingly symmetrised version of the Riemann tensor,

$$R_{\rho\sigma}{}^{\alpha\beta}{}_{\mu\nu} \equiv \frac{1}{2}\left(\delta_\rho^\alpha R_\sigma{}^\beta{}_{\mu\nu} + \delta_\rho^\beta R_\sigma{}^\alpha{}_{\mu\nu} + \delta_\sigma^\alpha R_\rho{}^\beta{}_{\mu\nu} + \delta_\sigma^\beta R_\rho{}^\alpha{}_{\mu\nu}\right). \tag{3.56}$$

From the general definition $\Omega_{\mu\nu} \equiv [\nabla_\mu, \nabla_\nu]$, we find $(\Omega_{\mu\nu})_{\rho\sigma}{}^{\alpha\beta} = R_{\rho\sigma}{}^{\alpha\beta}{}_{\mu\nu}$. Again, the indices μ, ν label the different elements of the gauge field strength $\Omega_{\mu\nu}$, which are $\frac{1}{2}D(D+1) \times \frac{1}{2}D(D+1)$ matrices whose components are given by the indices α, β and ρ, σ. Therefore, the substitutions to be performed in the heat kernel coefficients (1.96)–(1.99) are

$$\begin{cases} \mathbb{1} \;\leftrightarrow\; \delta_{\mu\nu}{}^{\alpha\beta} \\ V \;\leftrightarrow\; \mathcal{V}_{\mu\nu}{}^{\alpha\beta} \\ (\Omega_{\mu\nu})_{\rho\sigma}{}^{\alpha\beta} \;\leftrightarrow\; R_{\rho\sigma}{}^{\alpha\beta}{}_{\mu\nu}\,. \end{cases} \tag{3.57}$$

The first coefficient is computed directly from (1.96) and (1.94)–(1.95), corresponding to the trace (3.54),

$$\mathrm{Tr}\,[a_0^{gr}(x)] = \mathrm{Tr}\,[\alpha_0^{gr}(x)] = \frac{1}{2}D(D+1)\,. \tag{3.58}$$

For the second coefficient, we have to evaluate (1.97) with

$$\begin{aligned} \mathrm{Tr}\,[a_1^{gr}(x)] = \mathrm{Tr}\,[\alpha_1^{gr}(x)] &= \frac{1}{12}D(D+1)R + \mathrm{Tr}\left[\mathcal{V}_{\mu\nu}{}^{\alpha\beta}\right] \\ &= \frac{(D+4)(D-3)}{12}\,R\,. \end{aligned} \tag{3.59}$$

Proof. The trace is

$$\mathrm{Tr}\left[\mathcal{V}_{\mu\nu}{}^{\alpha\beta}\right] = \mathcal{V}_{\mu\nu}{}^{\mu\nu} = R_{\mu}{}^{\mu}{}_{\nu}{}^{\nu} + R_{\mu}{}^{\nu}{}_{\nu}{}^{\mu} = -R_{\mu\nu}{}^{\mu\nu} = -R$$

where the first term vanishes by antisymmetry of the Riemann tensor. ■

The third coefficient (1.98) has no terms containing ∇^2, since $\mathcal{V}_{\mu\nu}{}^{\alpha\beta}$ is a function of the Riemann tensor only, whose covariant derivatives vanish on Einstein manifolds. We are left with

$$\begin{aligned}\mathrm{Tr}\left[\alpha_2^{gr}(x)\right] &= \frac{D(D+1)}{360}\left(R^2_{\mu\nu\rho\sigma} - \frac{1}{D}R^2\right) + \frac{1}{12}\mathrm{Tr}\left[\Omega^2_{\mu\nu}\right] \\ &= -\frac{D+1}{360}R^2 + \frac{D^2-29D-60}{360}R^2_{\mu\nu\rho\sigma}\,. \end{aligned} \tag{3.60}$$

Proof. The trace in (3.60) is

$$\begin{aligned}\mathrm{Tr}\left[\Omega^2_{\mu\nu}\right] &= \mathrm{Tr}\left[(\Omega_{\mu\nu})_{\rho\sigma}{}^{\alpha\beta}(\Omega^{\mu\nu})_{\alpha\beta}{}^{\gamma\delta}\right] = (\Omega_{\mu\nu})_{\rho\sigma}{}^{\alpha\beta}(\Omega^{\mu\nu})_{\alpha\beta}{}^{\rho\sigma} = R_{\rho\sigma}{}^{\alpha\beta}{}_{\mu\nu}R_{\alpha\beta}{}^{\rho\sigma\mu\nu} \\ &= \frac{1}{4}\left(\delta^{\alpha}_{\rho}R_{\sigma}{}^{\beta}{}_{\mu\nu} + \delta^{\beta}_{\rho}R_{\sigma}{}^{\alpha}{}_{\mu\nu} + \delta^{\alpha}_{\sigma}R_{\rho}{}^{\beta}{}_{\mu\nu} + \delta^{\beta}_{\sigma}R_{\rho}{}^{\alpha}{}_{\mu\nu}\right) \\ &\qquad \cdot\left(\delta^{\rho}_{\alpha}R_{\beta}{}^{\sigma}{}_{\mu\nu} + \delta^{\sigma}_{\alpha}R_{\beta}{}^{\rho}{}_{\mu\nu} + \delta^{\rho}_{\beta}R_{\alpha}{}^{\sigma}{}_{\mu\nu} + \delta^{\sigma}_{\beta}R_{\alpha}{}^{\rho}{}_{\mu\nu}\right) \\ &= \frac{1}{4}\left(-DR^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu\rho\sigma} + \text{permutations}\right) \\ &= -(D+2)R^2_{\mu\nu\rho\sigma}\,,\end{aligned}$$

where we used the fact that the other three terms are just like the first one, upon relabeling contracted indices. ■

According to (1.95), α_2^{gr} is to be added to

$$\begin{aligned}\beta_2^{gr} &= \frac{1}{2}\left(\alpha_1^{gr}\right)^2 = \frac{1}{2}\left(\frac{1}{6}R\delta_{\mu\nu}{}^{\alpha\beta} + \mathcal{V}_{\mu\nu}{}^{\alpha\beta}\right)^2 \\ &= \frac{1}{72}R^2\delta_{\mu\nu}{}^{\alpha\beta} + \frac{1}{2}\mathcal{V}_{\mu\nu}{}^{\rho\sigma}\mathcal{V}_{\rho\sigma}{}^{\alpha\beta} + \frac{1}{6}R\mathcal{V}_{\mu\nu}{}^{\alpha\beta}\,;\end{aligned} \tag{3.61}$$

hence, by taking the trace of (3.61),

$$\mathrm{Tr}\left[\beta_2^{gr}(x)\right] = \frac{D^2+D-24}{144}R^2 + \frac{3}{2}R^2_{\mu\nu\rho\sigma}\,, \tag{3.62}$$

which combined with (3.60) gives in the end

$$\mathrm{Tr}\left[a_2^{gr}(x)\right] = \frac{5D^2+3D-122}{720}R^2 + \frac{D^2-29D+480}{360}R^2_{\mu\nu\rho\sigma}\,. \tag{3.63}$$

Proof. The trace of (3.61) can be computed by recalling that

$$\mathrm{Tr}\left[\delta_{\mu\nu}{}^{\alpha\beta}\right] = \frac{1}{2}D(D+1) \qquad \text{and} \qquad \mathrm{Tr}\left[\mathcal{V}_{\mu\nu}{}^{\alpha\beta}\right] = -R\,,$$

were we used the result coming from the proof of (3.59). What remains to be computed is then

$$\begin{aligned}\mathrm{Tr}\left[\mathcal{V}_{\mu\nu}{}^{\rho\sigma}\mathcal{V}_{\rho\sigma}{}^{\alpha\beta}\right] &= \mathcal{V}_{\mu\nu}{}^{\rho\sigma}\mathcal{V}_{\rho\sigma}{}^{\mu\nu} = \left(R_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma} + R_{\mu}{}^{\sigma}{}_{\nu}{}^{\rho}\right)\left(R_{\rho}{}^{\mu}{}_{\sigma}{}^{\nu} + R_{\rho}{}^{\nu}{}_{\sigma}{}^{\mu}\right) \\ &= 2\left(R^2_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho}R^{\rho\mu\sigma\nu}\right) = 3R^2_{\mu\nu\rho\sigma}\,,\end{aligned}$$

where, by identity (3.17),

$$R_{\mu\sigma\nu\rho}R^{\rho\mu\sigma\nu} = R_{\mu\sigma\nu\rho}R^{\mu\nu\sigma\rho} = \frac{1}{2}\mathcal{E}_2^2\,.$$ ■

In $D = 4$ these coefficients reduce to

$$\mathrm{Tr}\left[a_0^{gr}(x)\right]_{D=4} = 10 \tag{3.64}$$

$$\mathrm{Tr}\left[a_1^{gr}(x)\right]_{D=4} = \frac{2}{3}R \tag{3.65}$$

$$\mathrm{Tr}\left[a_2^{gr}(x)\right]_{D=4} = -\frac{1}{24}R^2 + \frac{19}{18}R^2_{\mu\nu\rho\sigma}\,, \tag{3.66}$$

where (3.64) reproduces the number of degrees freedom of a symmetric tensor of rank 2 in dimension $D = 4$.[4] On maximally symmetric spaces, (3.63) reduces to

$$\mathrm{Tr}\left[\tilde{a}_2^{gr}(x)\right] = \frac{5D^4 - 2D^3 - 121D^2 + 6D + 1920}{720(D-1)D}\,R^2 \tag{3.67}$$

$$\mathrm{Tr}\left[\tilde{a}_2^{gr}(x)\right]_{D=4} = \frac{29}{216}R^2\,. \tag{3.68}$$

Fourth heat kernel coefficient for the graviton

To compute the fourth heat kernel coefficient for the graviton, we start once more from the general formula (1.99) and perform the substitutions (3.57). It is again convenient to split

$$\alpha_3^{gr}(x) \equiv \frac{1}{7!}\boldsymbol{A}_{gr}[R, R_{\mu\nu}, R_{\mu\nu\rho\sigma}] + \frac{2}{6!}\boldsymbol{B}_{gr}[R, R_{\mu\nu}, R_{\mu\nu\rho\sigma}, \Omega_{\mu\nu}, V]\,, \tag{3.69}$$

as we did in (3.40) for the ghost. By inspecting more closely the substitution rules (3.29) and (3.57), though, it is clear that $\boldsymbol{A}_{gr}$ and $\boldsymbol{A}_{gh}$ differ only by the trace of the identity operator $\mathbb{1}$. Using (3.41), we find in particular

$$\begin{aligned}\mathrm{Tr}\left[\boldsymbol{A}_{gr}\right] &= \frac{D+1}{2}\,\mathrm{Tr}\left[\boldsymbol{A}_{gh}\right] \\ &= (D+1)\left(-\frac{8}{9D}\mathcal{E}_1^3 + \frac{1}{3}\mathcal{E}_2^3 + \frac{17D}{18}\mathcal{E}_3^3 - \frac{14D}{9}\mathcal{E}_4^3\right),\end{aligned} \tag{3.70}$$

[4] To find the correct number of degrees of freedom for the graviton, which correspond to its physical polarisations, we have to sum these coefficients to the ghost ones.

which, on maximally symmetric spaces, reduces to

$$\mathrm{Tr}\left[\widetilde{\boldsymbol{A}}_{gr}\right] = -\frac{8\left(D^3-7D-6\right)}{9D(D-1)^2}\,R^3\,. \tag{3.71}$$

For the computation of $\boldsymbol{B}_{gr}$ we can repeat the previous observations, and consider the only non–vanishing terms

$$\begin{aligned}\mathrm{Tr}\left[\boldsymbol{B}_{gr}\right] &= 4\mathrm{Tr}\left[\Omega_{\mu\nu}\nabla^2\Omega^{\mu\nu}\right] - 12\mathrm{Tr}\left[\Omega_{\mu}{}^{\nu}\Omega_{\nu}{}^{\sigma}\Omega_{\sigma}{}^{\mu}\right]\\ &\quad + 6\mathrm{Tr}\left[R_{\mu\nu\rho\sigma}\Omega^{\mu\nu}\Omega^{\rho\sigma}\right] - 4\mathrm{Tr}\left[R_{\mu\nu}\Omega^{\mu\sigma}\Omega^{\nu}{}_{\sigma}\right] + 30\mathrm{Tr}\left[(\nabla_\mu V)^2\right]\\ &= -\frac{4(D+47)}{D}\,\mathcal{E}_2^3 - 2(D-43)\,\mathcal{E}_3^3 + 4(D+92)\,\mathcal{E}_4^3\,,\end{aligned} \tag{3.72}$$

which on maximally symmetric spaces reduce to

$$\mathrm{Tr}\left[\widetilde{\boldsymbol{B}}_{gr}\right] = -\frac{4\left(D+2\right)^2}{D^2(D-1)^2}\,R^3\,. \tag{3.73}$$

Proof. The traces containing no covariant derivatives are computed as follows:

$$\begin{aligned}\mathrm{Tr}\left[\Omega_{\mu}{}^{\nu}\Omega_{\nu}{}^{\sigma}\Omega_{\sigma}{}^{\mu}\right] &= \mathrm{Tr}\left[(\Omega_{\mu}{}^{\nu})_{\rho\lambda}{}^{\alpha\beta}(\Omega_{\nu}{}^{\sigma})_{\alpha\beta}{}^{\gamma\delta}(\Omega_{\sigma}{}^{\mu})_{\gamma\delta}{}^{\eta\tau}\right]\\ &= (\Omega_{\mu}{}^{\nu})_{\rho\lambda}{}^{\alpha\beta}(\Omega_{\nu}{}^{\sigma})_{\alpha\beta}{}^{\gamma\delta}(\Omega_{\sigma}{}^{\mu})_{\gamma\delta}{}^{\rho\lambda} = R_{\rho\lambda}{}^{\alpha\beta}{}_{\mu}{}^{\nu}R_{\alpha\beta}{}^{\gamma\delta}{}_{\nu}{}^{\sigma}R_{\gamma\delta}{}^{\rho\lambda}{}_{\sigma}{}^{\mu}\\ &= R_{\rho\lambda}{}^{\alpha\beta}{}_{\mu}{}^{\nu}R_{\alpha\beta}{}^{\gamma\delta}{}_{\nu\sigma}R_{\gamma\delta}{}^{\rho\lambda\sigma\mu}\\ &= \frac{1}{8}\left(\delta^\alpha_\rho R_\lambda{}^{\beta}{}_{\mu}{}^{\nu} + \delta^\beta_\rho R_\lambda{}^{\alpha}{}_{\mu}{}^{\nu} + \delta^\alpha_\lambda R_\rho{}^{\beta}{}_{\mu}{}^{\nu} + \delta^\beta_\lambda R_\rho{}^{\alpha}{}_{\mu}{}^{\nu}\right)\\ &\qquad\qquad \cdot\left(\delta^\gamma_\alpha R_\beta{}^{\delta}{}_{\nu\sigma} + \delta^\delta_\alpha R_\beta{}^{\gamma}{}_{\nu\sigma} + \delta^\gamma_\beta R_\alpha{}^{\delta}{}_{\nu\sigma} + \delta^\delta_\beta R_\alpha{}^{\gamma}{}_{\nu\sigma}\right)\\ &\qquad\qquad \cdot\left(\delta^\rho_\gamma R_\delta{}^{\lambda\sigma\mu} + \delta^\lambda_\gamma R_\delta{}^{\rho\sigma\mu} + \delta^\rho_\delta R_\gamma{}^{\lambda\sigma\mu} + \delta^\lambda_\delta R_\gamma{}^{\rho\sigma\mu}\right),\end{aligned}$$

where the last two factors are

$$\begin{aligned}&\delta^\rho_\alpha R_\beta{}^{\delta}{}_{\nu\sigma}R_\delta{}^{\lambda\sigma\mu} + \delta^\lambda_\alpha R_\beta{}^{\delta}{}_{\nu\sigma}R_\delta{}^{\rho\sigma\mu} + R_\alpha{}^{\lambda\sigma\mu}R_\beta{}^{\rho}{}_{\nu\sigma} + R_\alpha{}^{\rho\sigma\mu}R_\beta{}^{\lambda}{}_{\nu\sigma}\\ &+ \delta^\rho_\alpha R_\beta{}^{\gamma}{}_{\nu\sigma}R_\gamma{}^{\lambda\sigma\mu} + \delta^\lambda_\alpha R_\beta{}^{\gamma}{}_{\nu\sigma}R_\gamma{}^{\rho\sigma\mu} + R_\alpha{}^{\lambda\sigma\mu}R_\beta{}^{\rho}{}_{\nu\sigma} + R_\alpha{}^{\rho\sigma\mu}R_\beta{}^{\lambda}{}_{\nu\sigma}\\ &+ \delta^\rho_\beta R_\alpha{}^{\delta}{}_{\nu\sigma}R_\delta{}^{\lambda\sigma\mu} + \delta^\lambda_\beta R_\alpha{}^{\delta}{}_{\nu\sigma}R_\delta{}^{\rho\sigma\mu} + R_\beta{}^{\lambda\sigma\mu}R_\alpha{}^{\rho}{}_{\nu\sigma} + R_\beta{}^{\rho\sigma\mu}R_\alpha{}^{\lambda}{}_{\nu\sigma}\\ &+ \delta^\rho_\beta R_\alpha{}^{\gamma}{}_{\nu\sigma}R_\gamma{}^{\lambda\sigma\mu} + \delta^\lambda_\beta R_\alpha{}^{\gamma}{}_{\nu\sigma}R_\gamma{}^{\rho\sigma\mu} + R_\beta{}^{\lambda\sigma\mu}R_\alpha{}^{\rho}{}_{\nu\sigma} + R_\beta{}^{\rho\sigma\mu}R_\alpha{}^{\lambda}{}_{\nu\sigma}\\ &= 2\Big(\delta^\rho_\alpha R_\beta{}^{\delta}{}_{\nu\sigma}R_\delta{}^{\lambda\sigma\mu} + \delta^\lambda_\alpha R_\beta{}^{\delta}{}_{\nu\sigma}R_\delta{}^{\rho\sigma\mu} + R_\alpha{}^{\lambda\sigma\mu}R_\beta{}^{\rho}{}_{\nu\sigma} + R_\alpha{}^{\rho\sigma\mu}R_\beta{}^{\lambda}{}_{\nu\sigma}\\ &\qquad + \delta^\rho_\beta R_\alpha{}^{\delta}{}_{\nu\sigma}R_\delta{}^{\lambda\sigma\mu} + \delta^\lambda_\beta R_\alpha{}^{\delta}{}_{\nu\sigma}R_\delta{}^{\rho\sigma\mu} + R_\beta{}^{\lambda\sigma\mu}R_\alpha{}^{\rho}{}_{\nu\sigma} + R_\beta{}^{\rho\sigma\mu}R_\alpha{}^{\lambda}{}_{\nu\sigma}\Big)\,,\end{aligned}$$

since the first two rows are equal, and so the last two: notice how the switching of the indices α, β does change the value of the expression, while the one of δ, γ does not, since they are contracted. The product with the first factor in the full expression for the trace will be, by a similar symmetry, given by four times the product of the above expression by $\delta^\alpha_\rho R_\lambda{}^{\beta\mu\nu}$. Using the symmetry properties of the Riemann tensor we then find

$$
\begin{aligned}
&\mathrm{Tr}\left[\Omega_\mu{}^\nu\Omega_\nu{}^\sigma\Omega_\sigma{}^\mu\right] = \\
&= \delta^\alpha_\rho R_\lambda{}^\beta{}_\mu{}^\nu \Big(\delta^\rho_\alpha R_\beta{}^\delta{}_{\nu\sigma}R_\delta{}^{\lambda\sigma\mu} + \delta^\lambda_\alpha R_\beta{}^\delta{}_{\nu\sigma}R_\delta{}^{\rho\sigma\mu} + R_\alpha{}^{\lambda\sigma\mu}R_\beta{}^\rho{}_{\nu\sigma} + R_\alpha{}^{\rho\sigma\mu}R_\beta{}^\lambda{}_{\nu\sigma} \\
&\qquad + \delta^\rho_\beta R_\alpha{}^\delta{}_{\nu\sigma}R_\delta{}^{\lambda\sigma\mu} + \delta^\lambda_\beta R_\alpha{}^\delta{}_{\nu\sigma}R_\delta{}^{\rho\sigma\mu} + R_\beta{}^{\lambda\sigma\mu}R_\alpha{}^\rho{}_{\nu\sigma} + R_\beta{}^{\rho\sigma\mu}R_\alpha{}^\lambda{}_{\nu\sigma}\Big) \\
&= (D+3)R_\lambda{}^\beta{}_\mu{}^\nu R_\beta{}^\delta{}_\nu{}^\sigma R_\delta{}^\lambda{}_\sigma{}^\mu + R_\lambda{}^\beta{}_\mu{}^\nu R_\beta{}^{\rho\sigma\mu}R_\rho{}^\lambda{}_{\nu\sigma} = (D+2)\mathcal{E}^3_4\,,
\end{aligned}
$$

since

$$
R_\lambda{}^\beta{}_\mu{}^\nu R_\beta{}^{\rho\sigma\mu}R_\rho{}^\lambda{}_{\nu\sigma} = R_{\lambda\beta\mu\nu}R^{\beta\rho\sigma\mu}R_\rho{}^{\lambda\nu}{}_\sigma = -R_{\beta\lambda\mu\nu}R^{\beta\rho\mu\sigma}R^\lambda{}_\rho{}^\nu{}_\sigma = -\mathcal{E}^3_4\,.
$$

The second trace is similar to computations already performed,

$$
\begin{aligned}
\mathrm{Tr}\left[R_{\mu\nu\rho\sigma}\Omega^{\mu\nu}\Omega^{\rho\sigma}\right] &= R_{\mu\nu\rho\sigma}\mathrm{Tr}\left[(\Omega^{\mu\nu})_{\tau\lambda}{}^{\alpha\beta}(\Omega^{\rho\sigma})_{\alpha\beta}{}^{\gamma\delta}\right] \\
&= R_{\mu\nu\rho\sigma}(\Omega^{\mu\nu})_{\tau\lambda}{}^{\alpha\beta}(\Omega^{\rho\sigma})_{\alpha\beta}{}^{\tau\lambda} \\
&= \frac{1}{4}R_{\mu\nu\rho\sigma}\left(\delta^\alpha_\tau R_\lambda{}^{\beta\mu\nu} + \delta^\beta_\tau R_\lambda{}^{\alpha\mu\nu} + \delta^\alpha_\lambda R_\tau{}^{\beta\mu\nu} + \delta^\beta_\lambda R_\tau{}^{\alpha\mu\nu}\right) \\
&\qquad \cdot\left(\delta^\tau_\alpha R_\beta{}^{\lambda\rho\sigma} + \delta^\lambda_\alpha R_\beta{}^{\tau\rho\sigma} + \delta^\tau_\beta R_\alpha{}^{\lambda\rho\sigma} + \delta^\lambda_\beta R_\alpha{}^{\tau\rho\sigma}\right) \\
&= -\frac{1}{4}\left[R_{\mu\nu\rho\sigma}(D+2)R^{\lambda\beta\mu\nu}R_{\lambda\beta}{}^{\rho\sigma} + \text{permutations}\right] \\
&= -(D+2)R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\lambda\beta}R_{\lambda\beta}{}^{\mu\nu} = -(D+2)\ \mathcal{E}^3_3\,,
\end{aligned}
$$

while the third one can be easily computed exploiting the result found in the proof of (3.60), since on Einstein manifolds

$$
\mathrm{Tr}\left[R_{\mu\nu}\Omega^{\mu\sigma}\Omega^\nu{}_\sigma\right] = \frac{1}{D}Rg_{\mu\nu}\mathrm{Tr}\left[\Omega^{\mu\sigma}\Omega^\nu{}_\sigma\right] = \frac{1}{D}R\ \mathrm{Tr}\left[\Omega^2_{\mu\nu}\right] = -\frac{D+2}{D}\ \mathcal{E}^3_2\,.
$$

We are left to compute the trace which contains the covariant derivative,

$$
\begin{aligned}
\mathrm{Tr}\left[\Omega_{\mu\nu}\nabla^2\Omega^{\mu\nu}\right] &= (\Omega_{\mu\nu})_{\rho\sigma}{}^{\alpha\beta}\nabla^2(\Omega^{\mu\nu})_{\alpha\beta}{}^{\rho\sigma} = R_{\rho\sigma}{}^{\alpha\beta}{}_{\mu\nu}\nabla^2 R_{\alpha\beta}{}^{\rho\sigma\mu\nu} \\
&= \frac{1}{4}\left(\delta^\alpha_\rho R_\sigma{}^\beta{}_{\mu\nu} + \delta^\beta_\rho R_\sigma{}^\alpha{}_{\mu\nu} + \delta^\alpha_\sigma R_\rho{}^\beta{}_{\mu\nu} + \delta^\beta_\sigma R_\rho{}^\alpha{}_{\mu\nu}\right) \\
&\qquad \cdot\nabla^2\left(\delta^\rho_\alpha R_\beta{}^{\sigma\mu\nu} + \delta^\sigma_\alpha R_\beta{}^{\rho\mu\nu} + \delta^\rho_\beta R_\alpha{}^{\sigma\mu\nu} + \delta^\sigma_\beta R_\alpha{}^{\rho\mu\nu}\right) \\
&= \delta^\alpha_\rho R_\sigma{}^\beta{}_{\mu\nu}\nabla^2\left(\delta^\rho_\alpha R_\beta{}^{\sigma\mu\nu} + \delta^\sigma_\alpha R_\beta{}^{\rho\mu\nu} + \delta^\rho_\beta R_\alpha{}^{\sigma\mu\nu} + \delta^\sigma_\beta R_\alpha{}^{\rho\mu\nu}\right) \\
&= -(D+2)R_{\mu\nu\rho\sigma}\nabla^2 R^{\mu\nu\rho\sigma} = -(D+2)\left(\frac{2}{D}\mathcal{E}^3_2 - \mathcal{E}^3_3 - 4\mathcal{E}^3_4\right).
\end{aligned}
$$

Three of the last four terms vanish identically. Indeed, quite trivially $\nabla_\mu R\nabla^\mu V = 0$, and by integrating by parts we see that

$$
R_{\mu\nu}\nabla^\mu V\nabla^\nu V = -\nabla^\mu R_{\mu\nu}\nabla^\nu V = 0\,.
$$

Notice that for these computations we do not need the explicit form of V. The term $\nabla^4 V$ is not identically zero, but has vanishing trace, as

$$
\mathrm{Tr}\left[\nabla^4\mathcal{V}_{\alpha\beta}{}^{\mu\nu}\right] = \nabla^4\mathrm{Tr}\left[\mathcal{V}_{\alpha\beta}{}^{\mu\nu}\right] = -\nabla^4 R = 0\,,
$$

while for the term proportional to $(\nabla_\mu V)^2$ we can integrate by parts and neglect the boundary term, $(\nabla_\mu V)^2 = -V\nabla^2 V$, to compute

$$\begin{aligned}\mathrm{Tr}\left[(\nabla_\alpha \mathcal{V}_{\mu\nu}{}^{\rho\sigma})^2\right] &= -\mathcal{V}_{\mu\nu}{}^{\rho\sigma}\nabla^2\mathcal{V}_{\rho\sigma}{}^{\mu\nu} = -\left(R_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma} + R_{\mu}{}^{\sigma}{}_{\nu}{}^{\rho}\right)\nabla^2\left(R_{\rho}{}^{\mu}{}_{\sigma}{}^{\nu} + R_{\rho}{}^{\nu}{}_{\sigma}{}^{\mu}\right)\\ &= -2\left(R_{\mu\nu\rho\sigma}\nabla^2 R^{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho}\nabla^2 R^{\rho\mu\sigma\nu}\right)\\ &= -3R_{\mu\nu\rho\sigma}\nabla^2 R^{\mu\nu\rho\sigma} = -3\left(\frac{2}{D}\mathcal{E}_2^3 - \mathcal{E}_3^3 - 4\mathcal{E}_4^3\right),\end{aligned}$$

since we still can use the identity (3.17), by keeping the covariant derivative between the two Riemann tensors:

$$R_{\mu\sigma\nu\rho}\nabla^2 R^{\rho\mu\sigma\nu} = R_{\mu\sigma\rho\nu}\nabla^2 R^{\mu\rho\sigma\nu} = \frac{1}{2}R_{\mu\nu\rho\sigma}\nabla^2 R^{\mu\nu\rho\sigma}\,. \qquad \blacksquare$$

Going back to (3.69) we conclude that

$$\begin{aligned}\mathrm{Tr}\left[\alpha_3^{gr}(x)\right] = &-\frac{D+1}{5670D}\,\mathcal{E}_1^3 + \frac{D^2-167D-7896}{15120D}\,\mathcal{E}_2^3\\ &+\frac{17D^2-487D+21672}{90720}\,\mathcal{E}_3^3 - \frac{D^2-35D-3312}{3240}\,\mathcal{E}_4^3\,,\end{aligned} \tag{3.74}$$

which on maximally symmetric spaces reduces to

$$\mathrm{Tr}\left[\widetilde{\alpha}_3^{gr}(x)\right] = -\frac{D^4+56D^2+246D+252}{5670D^2(D-1)^2}\,R^3\,. \tag{3.75}$$

The last step is to compute the term β_3 defined in (1.95), which turns out to be given by

$$\begin{aligned}\mathrm{Tr}\left[\beta_3^{gr}(x)\right] = &\frac{5D^3-D^2-186D+72}{12960D}\,\mathcal{E}_1^3\\ &+\frac{D^3-29D^2+468D+2520}{2160D}\,\mathcal{E}_2^3 - \frac{5}{12}\mathcal{E}_3^3 - \frac{2}{3}\mathcal{E}_4^3\,,\end{aligned} \tag{3.76}$$

while on maximally symmetric spaces

$$\mathrm{Tr}\left[\widetilde{\beta}_3^{gr}(x)\right] = \frac{\begin{pmatrix}5D^6-11D^5-167D^4+83D^3\\ +5634D^2+16056D-34560\end{pmatrix}}{12960D^2(D-1)^2}\,R^3\,. \tag{3.77}$$

Proof. We start again from the last of (1.95), which reads

$$\beta_3^{gr} = \frac{1}{6}\left(\alpha_1^{gr}\right)^3 + \alpha_1^{gr}\alpha_2^{gr}\,.$$

The first term, before computing the trace, is

$$\begin{aligned}\frac{1}{6}\left(\alpha_1^{gr}\right)^3 &= \frac{1}{6}\left(\frac{1}{6}R\delta_{\mu\nu}{}^{\alpha\beta} + \mathcal{V}_{\mu\nu}{}^{\alpha\beta}\right)^3\\ &= \frac{1}{6}\left(\frac{1}{216}R^3\delta_{\mu\nu}{}^{\alpha\beta} + \frac{1}{12}R^2\mathcal{V}_{\mu\nu}{}^{\alpha\beta} + \frac{1}{2}R\mathcal{V}_{\mu\nu}{}^{\rho\sigma}\mathcal{V}_{\rho\sigma}{}^{\alpha\beta} + \mathcal{V}_{\mu\nu}{}^{\rho\sigma}\mathcal{V}_{\rho\sigma}{}^{\lambda\tau}\mathcal{V}_{\lambda\tau}{}^{\alpha\beta}\right)\end{aligned}$$

and we already know that

$$\mathrm{Tr}\left[\delta_{\mu\nu}{}^{\alpha\beta}\right] = \frac{1}{2}D(D+1)\,, \qquad \mathrm{Tr}\left[\mathcal{V}_{\mu\nu}{}^{\alpha\beta}\right] = -R\,, \qquad \mathrm{Tr}\left[\mathcal{V}_{\mu\nu}{}^{\rho\sigma}\mathcal{V}_{\rho\sigma}{}^{\alpha\beta}\right] = 3R^2_{\mu\nu\rho\sigma}\,,$$

so the only trace we are left to compute is

$$\begin{aligned}
\mathrm{Tr}\left[\mathcal{V}_{\mu\nu}{}^{\rho\sigma}\mathcal{V}_{\rho\sigma}{}^{\lambda\tau}\mathcal{V}_{\lambda\tau}{}^{\alpha\beta}\right] &= \mathcal{V}_{\mu\nu}{}^{\rho\sigma}\mathcal{V}_{\rho\sigma}{}^{\lambda\tau}\mathcal{V}_{\lambda\tau}{}^{\mu\nu} \\
&= \left(R_{\mu\ \nu}^{\ \rho\ \sigma} + R_{\mu\ \nu}^{\ \sigma\ \rho}\right)\left(R_{\rho\ \sigma}^{\ \lambda\ \tau} + R_{\rho\ \sigma}^{\ \tau\ \lambda}\right)\left(R_{\lambda\ \tau}^{\ \mu\ \nu} + R_{\lambda\ \tau}^{\ \nu\ \mu}\right) \\
&= \left(R_{\mu\ \nu}^{\ \rho\ \sigma} + R_{\mu\ \nu}^{\ \sigma\ \rho}\right)\left(R_{\rho\ \sigma}^{\ \lambda\ \tau}R_{\lambda\ \tau}^{\ \mu\ \nu} + R_{\rho\ \sigma}^{\ \lambda\ \tau}R_{\lambda\ \tau}^{\ \nu\ \mu}\right. \\
&\qquad\qquad \left. + R_{\rho\ \sigma}^{\ \tau\ \lambda}R_{\lambda\ \tau}^{\ \mu\ \nu} + R_{\rho\ \sigma}^{\ \tau\ \lambda}R_{\lambda\ \tau}^{\ \nu\ \mu}\right) \\
&= 4\left(R_{\mu\ \nu}^{\ \rho\ \sigma}R_{\rho\ \sigma}^{\ \lambda\ \tau}R_{\lambda\ \tau}^{\ \mu\ \nu} + R_{\mu\ \nu}^{\ \rho\ \sigma}R_{\rho\ \sigma}^{\ \lambda\ \tau}R_{\lambda\ \tau}^{\ \nu\ \mu}\right) \\
&= 4\left(\mathcal{E}_4^3 - \frac{1}{4}\mathcal{E}_3^3 + \mathcal{E}_4^3\right) = 8\mathcal{E}_4^3 - \mathcal{E}_3^3\,,
\end{aligned}$$

where we used the identity (3.20). Putting everything together,

$$\frac{1}{6}\,\mathrm{Tr}\left[(\alpha_1^{gr})^3\right] = \frac{D^2+D-36}{2592}\,\mathcal{E}_1^3 + \frac{1}{4}\mathcal{E}_2^3 - \frac{1}{6}\mathcal{E}_3^3 + \frac{4}{3}\mathcal{E}_4^3\,.$$

The second term, instead, is

$$\begin{aligned}
\alpha_1^{gr}\alpha_2^{gr} &= \left(\frac{1}{6}R\delta_{\mu\nu}{}^{\alpha\beta} + \mathcal{V}_{\mu\nu}{}^{\alpha\beta}\right)\left[\frac{1}{180}\left(R^2_{\mu\nu\rho\sigma} - \frac{1}{D}R^2\right)\delta_{\alpha\beta}{}^{\lambda\tau} + \frac{1}{12}\left(\Omega^2_{\rho\sigma}\right)_{\alpha\beta}{}^{\lambda\tau}\right. \\
&\qquad\qquad \left. + \frac{1}{6}\mathcal{V}_{\mu\nu}{}^{\alpha\beta}\nabla^2\mathcal{V}_{\alpha\beta}{}^{\lambda\tau}\right] \\
&= \frac{1}{1080}\left(RR^2_{\mu\nu\rho\sigma} - \frac{1}{D}R^3\right)\delta_{\mu\nu}{}^{\lambda\tau} + \frac{1}{72}R\left(\Omega^2_{\rho\sigma}\right)_{\mu\nu}{}^{\lambda\tau} \\
&\quad + \frac{1}{180}\left(R^2_{\mu\nu\rho\sigma} - \frac{1}{D}R^2\right)\mathcal{V}_{\mu\nu}{}^{\lambda\tau} + \frac{1}{12}\mathcal{V}_{\mu\nu}{}^{\alpha\beta}\left(\Omega^2_{\rho\sigma}\right)_{\alpha\beta}{}^{\lambda\tau} + \frac{1}{6}\mathcal{V}_{\mu\nu}{}^{\alpha\beta}\nabla^2\mathcal{V}_{\alpha\beta}{}^{\lambda\tau}\,,
\end{aligned}$$

where all the traces are known, except from

$$\begin{aligned}
\mathrm{Tr}\left[\mathcal{V}_{\mu\nu}{}^{\alpha\beta}\left(\Omega^2_{\rho\sigma}\right)_{\alpha\beta}{}^{\lambda\tau}\right] &= \mathcal{V}_{\mu\nu}{}^{\alpha\beta}\left(\Omega^{\rho\sigma}\right)_{\alpha\beta}{}^{\lambda\tau}\left(\Omega_{\rho\sigma}\right)_{\lambda\tau}{}^{\mu\nu} \\
&= \frac{1}{4}\left(R_{\mu\ \nu}^{\ \alpha\ \beta} + R_{\mu\ \nu}^{\ \beta\ \alpha}\right) \\
&\qquad \cdot\left(\delta_\alpha^\lambda R_\beta{}^{\tau\rho\sigma} + \delta_\alpha^\tau R_\beta{}^{\lambda\rho\sigma} + \delta_\beta^\lambda R_\alpha{}^{\tau\rho\sigma} + \delta_\beta^\tau R_\alpha{}^{\lambda\rho\sigma}\right) \\
&\qquad \cdot\left(\delta_\lambda^\mu R_\tau{}^\nu{}_{\rho\sigma} + \delta_\lambda^\nu R_\tau{}^\mu{}_{\rho\sigma} + \delta_\tau^\mu R_\lambda{}^\nu{}_{\rho\sigma} + \delta_\tau^\nu R_\lambda{}^\mu{}_{\rho\sigma}\right) \\
&= \left(R_{\mu\ \nu}^{\ \alpha\ \beta} + R_{\mu\ \nu}^{\ \beta\ \alpha}\right)\left(\delta_\alpha^\mu R_\beta{}^{\tau\rho\sigma}R_\tau{}^\nu{}_{\rho\sigma} + \delta_\alpha^\nu R_\beta{}^{\tau\rho\sigma}R_\tau{}^\mu{}_{\rho\sigma}\right. \\
&\qquad \left. + R_\beta{}^{\mu\rho\sigma}R_\alpha{}^\nu{}_{\rho\sigma} + R_\beta{}^{\nu\rho\sigma}R_\alpha{}^\mu{}_{\rho\sigma}\right) \\
&= 2\left(R_{\mu\beta}R^{\beta\tau\rho\sigma}R^\mu{}_{\tau\rho\sigma} + R_{\mu\alpha}{}^{\nu\beta}R_{\nu\beta}{}^{\rho\sigma}R_{\rho\sigma}{}^{\mu\alpha}\right. \\
&\qquad \left. + R^{\mu\alpha\nu\beta}R_{\mu\beta}{}^{\rho\sigma}R_{\rho\sigma\nu\alpha}\right) \\
&= \frac{2}{D}\mathcal{E}_2^3 + 2\mathcal{E}_3^3 + 2R^{\mu\alpha\nu\beta}R_{\mu\beta}{}^{\rho\sigma}R_{\rho\sigma\nu\alpha}\,,
\end{aligned}$$

where by identity (3.18)

$$R^{\mu\alpha\nu\beta}R_{\mu\beta}{}^{\rho\sigma}R_{\rho\sigma\nu\alpha}=R^{\mu\alpha\beta\nu}R_{\mu\beta}{}^{\rho\sigma}R_{\alpha\nu\rho\sigma}=\frac{1}{2}\mathcal{E}_3^3\,,$$

so that

$$\mathrm{Tr}\left[\mathcal{V}_{\mu\nu}{}^{\alpha\beta}\left(\Omega^2_{\rho\sigma}\right)_{\alpha\beta}{}^{\lambda\tau}\right]=\frac{2}{D}\mathcal{E}_2^3+3\mathcal{E}_3^3\,,$$

and

$$\begin{aligned}\mathrm{Tr}\left[\mathcal{V}_{\mu\nu}{}^{\alpha\beta}\nabla^2\mathcal{V}_{\alpha\beta}{}^{\lambda\tau}\right]&=\mathcal{V}_{\mu\nu}{}^{\alpha\beta}\nabla^2\mathcal{V}_{\alpha\beta}{}^{\mu\nu}=\left(R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}+R_{\mu}{}^{\beta}{}_{\nu}{}^{\alpha}\right)\nabla^2\left(R_{\alpha}{}^{\mu}{}_{\beta}{}^{\nu}+R_{\alpha}{}^{\nu}{}_{\beta}{}^{\mu}\right)\\&=2\left(R_{\mu\alpha\nu\beta}\nabla^2R^{\mu\alpha\nu\beta}+R_{\mu\beta\nu\alpha}\nabla^2R^{\mu\alpha\nu\beta}\right)\\&=2\left(R_{\mu\alpha\nu\beta}\nabla^2R^{\mu\alpha\nu\beta}+\frac{1}{2}R_{\mu\alpha\nu\beta}\nabla^2R^{\mu\alpha\nu\beta}\right)\\&=3R_{\mu\alpha\nu\beta}\nabla^2R^{\mu\alpha\nu\beta}=3\left(\frac{2}{D}\mathcal{E}_2^3-\mathcal{E}_3^3-4\mathcal{E}_4^3\right),\end{aligned}$$

where we used (3.17) and (3.21). In the end

$$\mathrm{Tr}\left[\alpha_1^{gr}\alpha_2^{gr}\right]=-\frac{D^2+D-12}{2160D}\,\mathcal{E}_1^3+\frac{D^3-29D^2-72D+2520}{2160D}\,\mathcal{E}_2^3-\frac{1}{4}\mathcal{E}_3^3-2\mathcal{E}_4^3\,,$$

which, when summed with the first term, gives (3.76). ■

By summing (3.74) and (3.76) we conclude that

$$\begin{aligned}\mathrm{Tr}\left[a_3^{gr}(x)\right]=&\frac{35D^3-7D^2-1318D+488}{90720D}\,\mathcal{E}_1^3\\&+\frac{7D^3-202D^2+3109D+9744}{15120D}\,\mathcal{E}_2^3\\&+\frac{17D^2-487D-16128}{90720}\,\mathcal{E}_3^3-\frac{D^2-35D-1152}{3240}\,\mathcal{E}_4^3\end{aligned}\tag{3.78}$$

and

$$\mathrm{Tr}\left[\widetilde{a}_3^{gr}(x)\right]=\frac{\begin{pmatrix}35D^6-77D^5-1185D^4+581D^3\\+38542D^2+108456D-245952\end{pmatrix}}{90720D^2(D-1)^2}\,R^3\,.\tag{3.79}$$

In dimension $D=4$ they reduce to

$$\mathrm{Tr}\left[a_3^{gr}(x)\right]_{D=4}=-\frac{83}{11340}\mathcal{E}_1^3+\frac{4849}{15120}\mathcal{E}_2^3-\frac{4451}{22680}\mathcal{E}_3^3+\frac{319}{810}\mathcal{E}_4^3\tag{3.80}$$

$$\mathrm{Tr}\left[\widetilde{a}_3^{gr}(x)\right]_{D=4}=\frac{157}{3402}\,R^3\,.\tag{3.81}$$

Before proceeding, we notice that (3.78) is also useful as it stands, as it can be interpreted as the total coefficient of a system composed of gravity coupled to a complex spin–1 field and D real scalars [5, § 2].

3.2.3 Total coefficients and effective action

Let us now go back to the effective action (1.78), which we rewrite here in terms of the Seeley–DeWitt coefficients a_j,

$$\Gamma_{(1)} = -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}\beta}{\beta}\,\exp\left(-\beta m^2\right)\int \frac{\mathrm{d}^D x\,\sqrt{g}}{(4\pi\beta)^{\frac{D}{2}}}\,\mathrm{sTr}\sum_{j=0}^{\infty}\beta^j a_j(x,x)\,. \tag{3.82}$$

By using the definition of the supertrace (1.66), we find that the ghost field, being a complex fermion, brings a factor of -2, while the graviton, a real boson, a factor of $+1$. Using the ghost kinetic operator $\mathcal{F}$ (3.22) and the graviton one F (3.23), and the fact that these fields are massless ($m=0$) we can therefore write (3.82) more compactly as

$$\begin{aligned}\Gamma_{(1)} &= -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}\beta}{\beta}\int \frac{\mathrm{d}^D x\,\sqrt{g}}{(4\pi\beta)^{\frac{D}{2}}}\,\mathrm{sTr}\sum_{j=0}^{\infty}\beta^j a_j(x,x)\\ &= -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}\beta}{\beta}\left\{\mathrm{Tr}\left[e^{-\beta F}\right] - 2\mathrm{Tr}\left[e^{-\beta\mathcal{F}}\right]\right\}.\end{aligned} \tag{3.83}$$

The heat kernel expansion (1.53) can be then plugged in (3.83), to give

$$\mathrm{Tr}\left[a_i\right] = \mathrm{Tr}\left[a_i^{gr}\right] - 2\mathrm{Tr}\left[a_i^{gh}\right]. \tag{3.84}$$

Inserting the results found above in (3.84) returns

$$\mathrm{Tr}\left[a_0(x)\right] = \frac{D(D-3)}{2} \tag{3.85}$$

$$\mathrm{Tr}\left[a_1(x)\right] = \frac{D^2-3D-36}{12}\,R \tag{3.86}$$

$$\mathrm{Tr}\left[a_2(x)\right] = \frac{5D^3-17D^2-354D-720}{720D}\,\mathcal{E}_1^2 + \frac{D^2-33D+540}{360}\,\mathcal{E}_2^2 \tag{3.87}$$

$$\begin{aligned}\mathrm{Tr}\left[a_3(x)\right] &= \frac{35D^4-147D^3-3670D^2-13560D-30240}{90720D^2}\,\mathcal{E}_1^3\\ &+ \frac{7D^3-230D^2+3357D+12600}{15120D}\,\mathcal{E}_2^3\\ &+ \frac{17D^2-555D-15120}{90720}\,\mathcal{E}_3^3 - \frac{D^2-39D-1080}{3240}\,\mathcal{E}_4^3\,,\end{aligned} \tag{3.88}$$

which in $D=4$ reduce to

$$\mathrm{Tr}\left[a_0(x)\right]_{D=4} = 2 \tag{3.89}$$

$$\mathrm{Tr}\left[a_1(x)\right]_{D=4} = -\frac{8}{3}R \tag{3.90}$$

$$\mathrm{Tr}\left[a_2(x)\right]_{D=4} = -\frac{29}{40}\mathcal{E}_1^2 + \frac{53}{45}\mathcal{E}_2^2 \tag{3.91}$$

$$\mathrm{Tr}\left[a_3(x)\right]_{D=4} = -\frac{4489}{45360}\mathcal{E}_1^3 + \frac{5699}{15120}\mathcal{E}_2^3 + \frac{4267}{22680}\mathcal{E}_3^3 + \frac{61}{162}\mathcal{E}_4^3 \,. \tag{3.92}$$

On maximally symmetric spaces:

$$\mathrm{Tr}\left[\widetilde{a}_2(x)\right] = \frac{5D^4 - 22D^3 - 333D^2 - 498D + 2880}{720(D-1)D}\, R^2 \tag{3.93}$$

$$\mathrm{Tr}\left[\widetilde{a}_3(x)\right] = \frac{\begin{pmatrix} 35D^6 - 217D^5 - 3257D^4 - 9239D^3 \\ + 37470D^2 + 183672D - 302400 \end{pmatrix}}{90720D^2(D-1)^2}\, R^3 \,, \tag{3.94}$$

and in $D=4$,

$$\mathrm{Tr}\left[\widetilde{a}_2(x)\right]_{D=4} = -\frac{571}{1080}R^2 \tag{3.95}$$

$$\mathrm{Tr}\left[\widetilde{a}_3(x)\right]_{D=4} = -\frac{2459}{68040}R^3 \,. \tag{3.96}$$

The effective action for perturbative quantum gravity can now be written, up to third order in euclidean time, as

$$\Gamma_{(1)} = -\frac{1}{2}\int_0^\infty \mathrm{d}\beta \int \frac{\mathrm{d}^D x\,\sqrt{g}}{(4\pi)^{\frac{D}{2}}} \Big[a_0(x)\beta^{-1-\frac{D}{2}} + a_1(x)\beta^{-\frac{D}{2}} + a_2(x)\beta^{1-\frac{D}{2}} + a_3(x)\beta^{2-\frac{D}{2}} + \mathcal{O}\left(\beta^{3-\frac{D}{2}}\right)\Big] \,, \tag{3.97}$$

from which it is apparent that in $D=4$ divergences occur up to $a_2(x)$, so that the first three coefficients are the counterterms needed to renormalise the one–loop effective action. For $4 < D \leq 6$, instead, $a_3(x)$ has to be included as well to perform renormalisation.

3.3 Discussion

Before ending this chapter, we comment on some details which emerge from the result obtained, and compare it with the literature on this subject.

3.3.1 Divergences and topological invariants

A well–known result from 't Hooft and Veltman states that $D = 4$ pure gravity (that is, with no cosmological constant $\Lambda = 0$) is finite at one loop [24]. To be more precise, it is free of logarithmic divergences [5]: indeed, we see from (3.97) that for $D = 4$ these divergences would come from $a_2(x)$. By setting $\Lambda = 0$, however, we have $R = 0$ as well, due to (3.6). This allows to drop the term proportional to $\mathcal{E}_1^2$ in (3.91), and therefore to conclude that $\mathrm{Tr}\,[a_2(x)] \propto \mathcal{E}_2^2 = R^2_{\mu\nu\rho\sigma}$. The connection between this result and [24] will be apparent after having introduced the *Gauss–Bonnet theorem*, which allows to compute the *Euler character* of the manifold $\chi_E(\mathcal{M})$ as a volume integral of the 2–form $\mathcal{R}^{\mu\nu} \equiv R^{\mu\nu}{}_{\alpha\beta}\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta$:

$$\chi_E(\mathcal{M}) = \frac{1}{(4\pi)^d} \int_{\mathcal{M}} \varepsilon_{\mu_1\nu_1\ldots\mu_d\nu_d}\mathcal{R}^{\mu_1\nu_1} \wedge \cdots \wedge \mathcal{R}^{\mu_d\nu_d}\,, \tag{3.98}$$

where $D \equiv 2d$ is the dimension of the manifold, assumed here to be even.[5] In local coordinates, (3.98) becomes

$$\chi_E(\mathcal{M}) = \frac{1}{2(4\pi)^d} \int \mathrm{d}^D x\sqrt{g}\; \frac{D!}{D}\delta^{\alpha_1}_{[\mu_1}\delta^{\beta_1}_{\nu_1}\ldots\delta^{\alpha_d}_{\mu_d}\delta^{\beta_d}_{\nu_d]}R^{\mu_1\nu_1}{}_{\alpha_1\beta_1}\ldots R^{\mu_d\nu_d}{}_{\alpha_d\beta_d}\,. \tag{3.99}$$

It is possible to prove that $\chi_E(\mathcal{M})$ defined in this way does not depend on the metric settled upon $\mathcal{M}$, and is fixed only by the global topology of the manifold. For $D = 2$, $d = 1$, (3.99) becomes

$$\chi_E(\mathcal{M})\bigg|_{D=2} = \frac{1}{16\pi}\int \mathrm{d}^2x\sqrt{g}\; R\,, \tag{3.100}$$

which is proportional to the Einstein–Hilbert action, while at $D = 4$, $d = 2$ the Euler character reads

$$\chi_E(\mathcal{M})\bigg|_{D=4} = \frac{1}{32\pi^2}\int \mathrm{d}^4x\sqrt{g}\;\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right). \tag{3.101}$$

On Einstein spaces the first two terms in the integrand of (3.101) cancel off, and therefore we are left with

$$\chi_E(\mathcal{M})\bigg|_{D=4} = \frac{1}{32\pi^2}\int \mathrm{d}^4x\sqrt{g}\; R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \equiv \int \mathrm{d}^4x\sqrt{g}\; E_4\,. \tag{3.102}$$

Therefore, the third heat kernel coefficient $\mathrm{Tr}\,[a_2(x)] \propto R^2_{\mu\nu\rho\sigma}$ is proportional to the *Euler density* E_4 and hence is a total derivative, which can be neglected in the effective action. This result is no more true when $\Lambda \neq 0$, even if we drop the total derivative term corresponding to Euler density [5].

[5] If D is odd, the integral (3.98) vanishes, and the theorem does not provide a useful way of computing $\chi_E(\mathcal{M})$.

3.3.2 Pure gravity in six dimensions

The newly computed coefficient (3.88), which contains terms proportional both to $\mathcal{E}_3^3$ and $\mathcal{E}_4^3$, allows to see what happens in dimension $D = 6$ when $\Lambda = 0$. The Euler character is now [29, § A.3]

$$\chi_E(\mathcal{M})\Big|_{D=6} = \frac{1}{384\pi^3} \int d^6x\sqrt{g}\ \big(4\mathcal{R}_3^3 - 48\mathcal{R}_4^3 + 64\mathcal{R}_5^3 + 96\mathcal{R}_6^3 + 12\mathcal{R}_7^3 - 96\mathcal{R}_8^3 + 16\mathcal{R}_9^3 - 32\mathcal{R}_{10}^3\big)\,, \quad (3.103)$$

which on Einstein spaces, according to (3.16), reduces to

$$\chi_E(\mathcal{M})\Big|_{D=6} = \frac{1}{384\pi^3} \int d^6x\sqrt{g}\ \left(\frac{4}{9}\mathcal{E}_1^3 - 4\mathcal{E}_2^3 + 16\mathcal{E}_3^3 - 32\mathcal{E}_4^3\right). \quad (3.104)$$

The condition $\Lambda = 0$, that is $R = 0$, forces $\mathcal{E}_1^3 = \mathcal{E}_2^3 = 0$, so that (3.104) eventually becomes

$$\chi_E(\mathcal{M})\Big|_{D=6} = \frac{1}{384\pi^3} \int d^6x\sqrt{g}\ \big(16\mathcal{E}_3^3 - 32\mathcal{E}_4^3\big) \equiv \int d^6x\sqrt{g}\ E_6\,. \quad (3.105)$$

The fourth coefficient (3.88), when evaluated at $D = 6$, reduces to

$$\mathrm{Tr}\,[a_3(x)]_{D=6} = -\frac{799}{11340}\mathcal{E}_1^3 + \frac{481}{1680}\mathcal{E}_2^3 - \frac{991}{5040}\mathcal{E}_3^3 + \frac{71}{180}\mathcal{E}_4^3 \quad (3.106)$$

and we see that $\mathrm{Tr}\,[a_3(x)]$ is not proportional to the Euler density E_6. Thus, at dimension $D = 6$, even with $\Lambda = 0$ (and therefore $\mathcal{E}_1^3 = \mathcal{E}_2^3 = 0$), the perturbative quantum gravity effective action is not free of logarithmic divergences. The result found by 't Hooft and Veltman appears to be a specific property of four–dimensional spacetime, as it was already conjectured by van Nieuwenhuizen in 1977 [30] and shown again more recently [31]. From (3.105) and (3.106), in particular, we have

$$\mathcal{E}_4^3 = \frac{1}{2}\mathcal{E}_3^3 - 12\pi^3 E_6 \qquad \longrightarrow \qquad \mathrm{Tr}\,[a_3(x)]_{D=6} = \frac{1}{1680}\mathcal{E}_3^3\,, \quad (3.107)$$

and the divergence in (3.97) becomes

$$-\frac{1}{2(4\pi)^3}\mathrm{Tr}\,[a_3(x)]_{D=6} = -\frac{1}{2(4\pi)^3}\frac{1}{1680}\mathcal{E}_3^3\,. \quad (3.108)$$

This result (3.108) is in agreement with van Nieuwenhuizen's pioneering calculation [30], besides a computational error in the numerical factor in his equation (81), already noted a year later by Critchley [32, § 5]. Confirmation of the validity of the result is also found in [31, § 5.2] or [33].

3.3.3 Comparison with the $\mathcal{N}=4$ spinning particle

The heat kernel coefficients can be computed in a completely independent manner by studying the $\mathcal{N}=4$ spinning particle in the so–called *worldline formalism*, where scattering amplitudes are computed as the correlators or n–point functions of a 1–dimensional QFT that lives on the Feynman graphs, namely the *worldline theory* (usually, a sigma–model into the given target spacetime). Intuitively, the edges in a Feynman diagram correspond to worldlines of virtual particles. The worldline formalism is equivalent to the traditional formulation, but it has the conceptual advantage that it expresses the Feynman perturbation series of QFT manifestly as a second quantisation of its particle content, given explicitly as the superposition of all one–particle processes, and the calculational advantage of automatically summing over subsets of Feynman diagrams related by exchange of external legs, thus maintaining permutation symmetry and explicit gauge invariance of on–shell scattering amplitudes.

In this section, we briefly describe how to obtain the one-loop effective action for Einstein-Hilbert gravity in the worldline formalism; however, we do not perform any explicit computation, referring to our outcoming paper [34]. The starting point is the so-called $O(\mathcal{N})$ spinning particle, a relativistic particle with $\mathcal{N}$–extended local supersymmetries on the worldline, which has been shown to produce the spectrum of a particle of spin $s=\frac{1}{2}\mathcal{N}$ in four dimensions. In our case, we are interested in reproducing the graviton, which corresponds to setting $\mathcal{N}=4$. It is important to note that the model is consistent only on background metrics satisfying Einstein equations [35; 36]. The one–loop effective action $\Gamma[g_{\mu\nu}]$ for Einstein-Hilbert gravity corresponds to the circle path integral of the worldline $\mathcal{N}=4$ spinning particle action $S[X,G;g_{\mu\nu}]$,

$$\Gamma[g_{\mu\nu}] = \int_{S^1} \frac{\mathcal{D}G\,\mathcal{D}X}{\mathrm{Vol(Gauge)}}\, e^{-S[X,G;g_{\mu\nu}]}\,, \tag{3.109}$$

where the action depends on the worldline gauge fields $G=(e,\chi,\bar{\chi},a)$ and on the coordinates with supersymmetric partners $X=\big(x,\psi,\bar{\psi}\big)$. Explicitly, the effective action (3.109) is related to the $\mathcal{N}=4$ spinning particle path integral $Z(T)$ through its Schwinger representation, which, in euclidean configuration space, is given by

$$\Gamma[g_{\mu\nu}] = -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}T}{T} Z(T)\,. \tag{3.110}$$

The partition function $Z(T)$, upon gauge fixing, becomes

$$Z(T) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi}\, P(\theta,\phi) \int_{\mathrm{PBC}} DxDaDbDc \int_{\mathrm{ABC}} D\bar{\psi}D\psi\, e^{-S[X;g_{\mu\nu}]}\,, \tag{3.111}$$

where $P(\theta, \phi)$ is the measure on the moduli space (θ, ϕ) generated by the gauge fixing on the graviton degrees of freedom, and which implements the correct projection on the physical graviton Hilbert space. The worldline variables $X = \big(x, \psi, \bar{\psi}, a, b, c\big)$ now include the ghosts (a, b, c). The path integral over bosonic variables is evaluated by fixing periodic boundary conditions (PBC), while the fermionic path integral is performed by choosing anti–periodic boundary conditions (ABC) on each flavour of fermionic fields ψ_i, with the internal index i taking values $i = 1, 2$. The nonlinear sigma model action reads

$$S[X; g_{\mu\nu}] = \int \mathrm{d}\tau \left[\frac{1}{4T} g_{\mu\nu} \left(\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu \right) + \overline{\psi}^{ai} \left(\delta_i^j \partial_\tau - \Lambda_i^j \right) \psi_{aj} \right.$$
$$\left. + \omega_{\mu ab} \dot{x}^\mu \overline{\psi}^a \cdot \psi^b - T R_{abcd} \overline{\psi}^a \cdot \psi^b \overline{\psi}^c \cdot \psi^d - T \mathcal{V} \right] , \quad (3.112)$$

where we used flat indices on the worldline complex fermions ψ_i^a, a dot denotes contraction on the internal indices and $D_\tau \equiv \delta_i^j \partial_\tau - \Lambda_i^j$ is the covariant derivative with the spin connection, having defined

$$\Lambda_i^j = \begin{pmatrix} \theta & 0 \\ 0 & \phi \end{pmatrix} . \quad (3.113)$$

The scalar potential term $\mathcal{V}$ is needed even at the classical level, since it contains the counterterm required by the regularization scheme, and at the quantum level to achieve nilpotency of the BRST charge [5]. The latter condition requires $V_{\mathrm{BRST}} = -\frac{2}{D} R$, while the former, adopting dimensional regularization on the worldline [37], introduces a counterterm $V_{\mathrm{CT}} = -\frac{1}{4} R$ [38], producing an effective potential

$$\mathcal{V} = V_{\mathrm{BRST}} + V_{\mathrm{CT}} = \left(\frac{2}{D} - \frac{1}{4} \right) R . \quad (3.114)$$

It is convenient to rewrite the angular integration in the complex plane. This can be achieved by introducing the Wilson variables $z \equiv e^{i\theta}$ and $\omega \equiv e^{i\phi}$,

$$Z(T) = \oint \frac{dz}{2\pi i} \frac{d\omega}{2\pi i} \, P(z, \omega) \int_{\mathrm{PBC}} DxDaDbDc \int_{\mathrm{ABC}} D\bar{\psi} D\psi \, e^{-S[X; g_{\mu\nu}]} , \quad (3.115)$$

where the measure on the moduli space reads

$$P(z, \omega) = \frac{1}{2} \frac{(z+1)^{D-2}}{z^3} \frac{(\omega+1)^{D-2}}{\omega^3} (z - \omega)^2 (z\omega - 1) . \quad (3.116)$$

Perturbative expansion

The perturbative expansion around the free theory, which is to be expressed in terms of the parameter T, requires to:

- factorise out the zero mode in the kinetic operator, that is, parametrise the bosonic coordinates of the circle as $x^\mu(\tau) \equiv x_0^\mu + q^\mu(\tau)$, and set all loops in spacetime with a fixed base point x_0^μ plus quantum fluctuations with Dirichlet boundary conditions, represented by $q^\mu(\tau)$; note that fermions coordinates have no zero modes, due to their antiperiodic boundary conditions on the circle;

- use Riemann normal coordinates [39; 36] centered around x_0^μ, so to expand the metric tensor and the spin connection at order T^3. The Riemann tensor appearing in the four–fermions Weyl vertex in (3.109) has to be Taylor expanded around x_0^μ, as well.

With these prescriptions (3.115) can be written as

$$Z(T) = \oint \frac{\mathrm{d}z}{2\pi i}\frac{\mathrm{d}\omega}{2\pi i}\, P(z,\omega) \int \mathrm{d}^D x_0 \frac{\sqrt{g(x_0)}}{(4\pi T)^{\frac{D}{2}}} \left\langle e^{-S_{\text{int}}} \right\rangle , \tag{3.117}$$

factorising out for convenience the $\sqrt{g(x_0)}$ arising from the free ghost part of the action when functional integrating, together with the numerical factor arising each time one evaluates vacuum expectation values. The expectation value in (3.117) is to be evaluated using the Wick theorem on the free path integral, with the free action $S_0[X]$ being the first line of (3.109), while higher order terms form the interacting action S_{int}. Introducing the double expectation value of the interacting action $\langle\!\langle \cdots \rangle\!\rangle$, defined as the average over the path integral *and* over the moduli space,

$$\left\langle\!\!\left\langle e^{-S_{\text{int}}} \right\rangle\!\!\right\rangle = \oint \frac{\mathrm{d}z}{2\pi i}\frac{\mathrm{d}\omega}{2\pi i}\, P(z,\omega) \left\langle e^{-S_{\text{int}}} \right\rangle , \tag{3.118}$$

we can rewrite (3.109) in a more compact form:

$$Z(T) = \int \frac{\mathrm{d}^D x_0}{(4\pi T)^{\frac{D}{2}}} \sqrt{g(x_0)} \left\langle\!\!\left\langle e^{-S_{\text{int}}} \right\rangle\!\!\right\rangle . \tag{3.119}$$

Now, since the perturbative expansion of the path integral is a Taylor expansion in the proper time T, we can rearrange (3.119) so to make explicit the Seeley–DeWitt coefficients arising from the perturbative expansion,

$$\begin{aligned} Z(T) &= \int \frac{\mathrm{d}^D x_0}{(4\pi T)^{\frac{D}{2}}} \sqrt{g(x_0)} \sum_{n=0}^{\infty} a_n(D) T^n \qquad (3.120) \\ &= \int \frac{\mathrm{d}^D x_0}{(4\pi T)^{\frac{D}{2}}} \sqrt{g(x_0)} \oint \frac{\mathrm{d}z}{2\pi i}\frac{\mathrm{d}\omega}{2\pi i}\, P(z,\omega) \sum_{n=0}^{\infty} a_n(D,z,\omega) T^n , \end{aligned}$$

thus identifying

$$\left\langle e^{-S_{\text{int}}} \right\rangle = \sum_{n=0}^{\infty} a_n(D, z, \omega) T^n \tag{3.121}$$

$$\left\langle\!\left\langle e^{-S_{\text{int}}} \right\rangle\!\right\rangle = \sum_{n=0}^{\infty} a_n(D) T^n = a_0 + a_1 T + a_2 T^2 + a_3 T^3 + \mathcal{O}(T^4)\,. \tag{3.122}$$

By computing (3.119) we are then able to extract the values of Seeley–DeWitt coefficients. This computation has been carried out by Filippo Fecit, and the comparison with the same result obtained through heat kernel techniques led to an improvement of both methods, producing the same results (3.85)–(3.88), which are to be published soon [34].

Chapter 4

Extended models

In this last chapter we broaden the discussion by adding matter fields that couple to the graviton; the computations are performed in the matter vacuum, but could be extended to a more general background. From the point of view of renormalisation theory, only pure gravity in $D = 4$ is renormalisable [24], as the inclusion of matter fields introduces terms which cannot be absorbed into the effective Lagrangian as counterterms [30].

4.1 Scalar field

Consider the most general[1] action principle for a free real scalar field $\phi(x)$, $\phi : \mathcal{M} \to \mathbb{R}^D$ in curved spacetime and euclidean time,

$$S_\phi = \frac{1}{2} \int \mathrm{d}^D x \, \sqrt{G} \left[G^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^i + (m^2 + R\xi)\phi^2 \right] , \tag{4.1}$$

and assume that the scalar field itself can be decomposed into a classical background φ and quantum fluctuations $\tilde{\phi}$, so that $\phi = \varphi + \tilde{\phi}$.[2] The classical equations of motion stemming from (4.1) are

$$\left(-\Box + m^2 + R\xi \right) \varphi = 0 \, . \tag{4.2}$$

[1] Actually, the most general action principle would contain a generic potential $V(\phi)$. However, by splitting the field as described above, we can write

$$V(\phi) = V(\varphi + \tilde{\phi}) = V(\varphi) + V'(\varphi)\tilde{\phi} + \frac{1}{2} V''(\varphi)\tilde{\phi}^2 + \mathcal{O}(\tilde{\phi}^3) \, ,$$

and we can argue that $V(\varphi) = V'(\varphi) = 0$. Indeed, the first condition comes just from a rescaling of the potential, while the second one can be justified by observing that the background (classical) configuration should be a stationary one, i.e. $V'(\varphi) = 0$.

[2] The components of the fields are denoted by ϕ^i, $i = 1, \ldots, D$; when indices do not appear, we look at the field as a vector in $\mathbb{R}^D$.

4.1.1 The vacuum case

We start from the simplest case of *vacuum Einstein equations*: if the classical field $\varphi \equiv 0$, the momentum–energy tensor of the background is

$$T_{\mu\nu}(\varphi) = \partial_\mu \varphi^i \partial_\nu \varphi^i - g_{\mu\nu}\left[\frac{1}{2}\partial_\alpha \varphi^i \partial^\alpha \varphi^i + V(\varphi)\right] = 0\,, \tag{4.3}$$

and vacuum Einstein equations (3.5) are still satisfied. We now expand $G^{\mu\nu}$ in the metric perturbations $h^{\mu\nu}$ according to (2.37), and the square root of the determinant $\sqrt{G}$ following (2.35), while R should be expanded according to (2.52)–(2.57),

$$\begin{aligned} S_\phi = \frac{1}{2}\int \mathrm{d}^D x\, \sqrt{g}\left[1 + \frac{1}{2}h - \frac{1}{4}h^{\mu\nu}h_{\mu\nu} + \frac{1}{8}h^2\right] \\ \cdot \Bigg[\left(g^{\mu\nu} - h^{\mu\nu} + h^{\mu\alpha}h_\alpha^{\ \nu}\right)\partial_\mu\left(\varphi + \tilde{\phi}\right)^i \partial_\nu \left(\varphi + \tilde{\phi}\right)^i \\ + \left[m^2 + \left(R(0) + R(1) + R(2)\right)\xi\right](\varphi + \tilde{\phi})^2\Bigg]\,. \end{aligned} \tag{4.4}$$

However, with $\varphi = 0$, the only one–loop contributions in (4.4) come from the leading order terms in the metric tensor,

$$S_{\phi,2} = \frac{1}{2}\int \mathrm{d}^D x\, \sqrt{g}\left[g^{\mu\nu}\partial_\mu \tilde{\phi}^i \partial_\nu \tilde{\phi}^i + \left(m^2 + R(0)\xi\right)\tilde{\phi}^2\right], \tag{4.5}$$

which has the same form as (4.1), except from the fact that now the background metric involved in the action $S_{\phi,2}$ is purely classical, while the field $\tilde{\phi}$ is quantum. Note that, however, the action (4.5) is to be summed with the graviton one–loop action S_2 (2.7), so that the system described by $S_2^{g+\phi} \equiv S_2 + S_{\phi,2}$ is not just a quantum scalar field in a classical curved spacetime, as instead is the case in [9, § 2.5]. Indeed, spacetime contains itself quantum perturbations, which do not interact directly with the field at one–loop level, but do contribute to the evaluation of heat kernel coefficients.

Heat kernel coefficients

The evaluation of the heat kernel coefficients for the action $S_2^{g+\phi}$ in the above setting is really simple, as one needs to add to the coefficients already computed for the graviton the term corresponding to the scalar field. By inspecting the action (4.5), we easily identify the kinetic operator $\mathcal{F}_\phi = \delta_{ij}\nabla^2$ and the potential $V_\phi = R(0)\xi\delta_{ij}$, as (4.5) can be cast into the form

$$S_{\phi,2} = -\frac{1}{2}\int \mathrm{d}^D x\, \sqrt{g}\left[\tilde{\phi}^i \delta_{ij}\partial^\mu \partial_\mu \tilde{\phi}^j - \tilde{\phi}^i\left(m^2 + R(0)\xi\right)\delta_{ij}\tilde{\phi}^j\right], \tag{4.6}$$

while $[\nabla_\mu, \nabla_\nu]\phi = [\partial_\mu, \partial_\nu]\phi = 0$, so that $\Omega_{\mu\nu} = 0$ as well. Recall that the mass term, as in (1.71) and (1.78), is singled out in the Schwinger–DeWitt parametrisation, and transformed into an exponential term in front of the effective action: for this reason, it does not appear in the definition of V_ϕ. By comparison with (1.92) we get the following substitution rules:

$$\begin{cases} \mathbb{1} \ \leftrightarrow \ \delta_{ij} \\ V \ \leftrightarrow \ -R(0)\xi\delta_{ij} \\ \Omega_{\mu\nu} \ \leftrightarrow \ 0\,. \end{cases} \tag{4.7}$$

Since $\mathrm{Tr}\,[\delta_{ij}] = D$, and denoting from now on $R(0) \equiv R$, the first two heat kernel coefficients for the scalar field are

$$\mathrm{Tr}\left[a_0^\phi(x)\right] = D \tag{4.8}$$

$$\mathrm{Tr}\left[a_1^\phi(x)\right] = D\left(\frac{1}{6} - \xi\right) R\,. \tag{4.9}$$

For the third one, we have

$$\mathrm{Tr}\left[\alpha_2^\phi(x)\right] = \frac{D}{180}\left(R^2_{\mu\nu\rho\sigma} - \frac{1}{D}R^2\right) \tag{4.10}$$

$$\mathrm{Tr}\left[\beta_2^\phi(x)\right] = \frac{1}{2}\mathrm{Tr}\left[\left(\frac{1}{6} - \xi\right)\delta_{ij}R\right]^2 = \frac{D}{2}\left(\frac{1}{6} - \xi\right)^2 R^2\,, \tag{4.11}$$

therefore

$$\mathrm{Tr}\left[a_2^\phi(x)\right] = \frac{5D(1-6\xi)^2 - 2}{360}\, R^2 + \frac{D}{180}R^2_{\mu\nu\rho\sigma}\,. \tag{4.12}$$

The fourth coefficient is even simpler: $V \propto R$ and $\Omega_{\mu\nu} = 0$, in the notation of (3.40), imply that $\boldsymbol{B}_\phi = 0$; so from (3.41) we get

$$\begin{aligned} \mathrm{Tr}\left[\alpha_3^\phi(x)\right] &= \frac{1}{7!}\mathrm{Tr}\left[\boldsymbol{A}_{gh}\right] \\ &= \frac{1}{7!}\left(-\frac{16}{9D}\mathcal{E}_1^3 + \frac{2}{3}\mathcal{E}_2^3 + \frac{17D}{9}\mathcal{E}_3^3 - \frac{28D}{9}\mathcal{E}_4^3\right), \end{aligned} \tag{4.13}$$

while

$$\begin{aligned} \mathrm{Tr}\left[\beta_3^\phi(x)\right] &= \mathrm{Tr}\left[\frac{1}{6}\left(\alpha_1^\phi\right)^3 + \alpha_1^\phi\alpha_2^\phi\right] \\ &= \frac{D}{6}\left(\frac{1}{6} - \xi\right)^3 R^3 + \frac{D}{180}\left(\frac{1}{6} - \xi\right)\left(RR^2_{\mu\nu\rho\sigma} - \frac{1}{D}R^3\right) \\ &= \frac{1}{180}\left[30D\left(\frac{1}{6} - \xi\right)^3 + \xi - \frac{1}{6}\right]\mathcal{E}_1^3 - \frac{D(6\xi - 1)}{1080}\,\mathcal{E}_2^3\,, \end{aligned} \tag{4.14}$$

resulting in

$$\mathrm{Tr}\left[a_3^{\phi}(x)\right] = \left[\frac{1}{6}D\left(\frac{1}{6}-\xi\right)^3 - \frac{1}{2835D} + \frac{1}{180}\left(\xi - \frac{1}{6}\right)\right]\mathcal{E}_1^3 + \frac{D(7-42\xi)+1}{7560}\,\mathcal{E}_2^3 + \frac{17D}{45360}\mathcal{E}_3^3 - \frac{D}{1620}\mathcal{E}_4^3\,. \qquad (4.15)$$

Effective action

The effective action (1.78) for the scalar field ϕ becomes then

$$\Gamma_{(1)}^{\phi} = -\frac{1}{2}\int_0^{\infty}\frac{\mathrm{d}\beta}{\beta}\;\exp\left(-\beta m^2\right)\int\frac{\mathrm{d}^D x\,\sqrt{g}}{(4\pi\beta)^{\frac{D}{2}}}\sum_{j=0}^{\infty}\frac{\beta^j}{j!}\mathrm{Tr}\left[a_j^{\phi}(x)\right], \qquad (4.16)$$

which cannot be directly summed with the gravity one, since in general $m \neq 0$, giving

$$\Gamma_{(1)}^{g+\phi} = -\frac{1}{2}\int_0^{\infty}\frac{\mathrm{d}\beta}{\beta}\left\{\int\frac{\mathrm{d}^D x\,\sqrt{g}}{(4\pi\beta)^{\frac{D}{2}}}\sum_{j=0}^{\infty}\frac{\beta^j}{j!}\mathrm{Tr}\left[a_j(x)\right] + \exp\left(-\beta m^2\right)\int\frac{\mathrm{d}^D x\,\sqrt{g}}{(4\pi\beta)^{\frac{D}{2}}}\sum_{j=0}^{\infty}\frac{\beta^j}{j!}\mathrm{Tr}\left[a_j^{\phi}(x)\right]\right\}. \qquad (4.17)$$

If we assume that the scalar field mass is small enough to expand in series the exponential in (4.17), this introduces for each heat kernel coefficient a term of the form

$$a_0^{\phi,m}(x) = a_0^{\phi}(x) \qquad (4.18)$$

$$a_1^{\phi,m}(x) = a_1^{\phi}(x) - m^2 a_0^{\phi}(x) \qquad (4.19)$$

$$a_2^{\phi,m}(x) = a_2^{\phi}(x) - m^2 a_1^{\phi}(x) + \frac{1}{2!}m^4 a_0^{\phi}(x) \qquad (4.20)$$

$$a_3^{\phi,m}(x) = a_3^{\phi}(x) - m^2 a_2^{\phi}(x) + \frac{1}{2!}m^4 a_1^{\phi}(x) - \frac{1}{3!}m^6 a_0^{\phi}(x)\,, \qquad (4.21)$$

and in general

$$a_j^{\phi,m}(x) = \sum_{k=0}^{j}\frac{(-1)^k}{k!}m^{2k}\,a_{j-k}^{\phi}(x)\,, \qquad (4.22)$$

so that $a_j^{g+\phi}(x) = a_j(x) + a_j^{\phi,m}(x)$ represent the total coefficients for the theory containing an almost massless scalar field coupled to a graviton.

4.1.2 Outlook of the general case

More interesting, though, is the general case of $\varphi \neq 0$, where the background metric is influenced by the presence of the scalar field. The one–loop expansion of the action (4.4) is much more involved; it has already been carried out in the slightly particular case $\xi = 0$, at $D = 4$ [24]; indeed, any additional term proportional to R in the action does not change the UV behaviour of the theory in four dimensions [24, § 7]. In the generic D–dimensional case, however, one should consider the general form of the action (4.4) with $\xi \neq 0$, and repeat the gauge fixing procedure discussed for the case of gravity with no background matter.

This computation will not be carried out here, but the heat kernel method, combined with an algorithmic evaluation of tensor contractions, can provide a reliable technique to solve this problem. The expansion could also be extended to the two–loop level; very recently, a result at two–loops has been obtained for $D = 4$ and with a *flat* gravitational background, while the scalar background is left arbitrary [40].

4.2 Spinor field

The action for a D–dimensional spinor field $\psi : \mathcal{M} \to V$ in a gravitational field[3] is written by exploiting the *vierbein* or *tetrad basis* $\mathbf{e}_a \equiv e_a^\mu \mathbf{e}_\mu$, with $\mathbf{e}_a \cdot \mathbf{e}_b = e_a^\mu e_b^\nu G_{\mu\nu} = \eta_{ab}$:

$$S_\psi = \int \mathrm{d}^D x \; e \; \overline{\psi} \left(\nabla\!\!\!\!/ + m \right) \psi \,, \tag{4.23}$$

where $e \equiv \det e_\mu^a = \sqrt{G}$ is the vierbein determinant,[4] and the covariant derivative is defined as

$$\nabla\!\!\!\!/ = \gamma^a e_a^\mu \nabla_\mu \,, \qquad \nabla_\mu \equiv \partial_\mu + \frac{1}{4} \omega_{\mu ab} \gamma^a \gamma^b \,, \tag{4.24}$$

where $\gamma^\mu \equiv \gamma^a e_a^\mu$ are the *curved γ–matrices*, and $\omega_{\mu ab}$ is the *spin connection*

$$\omega_\mu^{ab} \equiv e_\nu^a \Gamma^\nu_{\mu\sigma} e^{\sigma b} + e_\nu^a \partial_\mu e^{\nu b} \,, \tag{4.25}$$

whose expression in terms of the vierbein coefficients is found by requiring that a vector $\mathbf{X} \in T_P\mathcal{M}$ satisfies $X^\mu = e_a^\mu X^a$ [41, § 2]. In order to have Hermitian γ–matrices, we assume that our euclidean spacetime manifold

[3] Here, V is a vector space in which the spin group Spin (D) is suitably represented.

[4] Indeed,

$$g = \left| \det \left(e_a^\mu e_b^\nu \eta^{ab} \right) \right| = \left(\det e_a^\mu \right)^2 \left| \det \eta^{ab} \right| = \left(\det e_a^\mu \right)^2 .$$

has *even* dimension D: in this way, the kinetic operator $\not\nabla$ is Hermitian as well. The equations of motions corresponding to (4.23) are

$$\left(\not\nabla + m\right)\psi = 0\,. \tag{4.26}$$

By decomposing the metric as usual, $G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$, we correspondingly assume that the spinor field can be written as $\psi = \Psi + \delta\psi$, where Ψ is the background field and $\delta\psi$ are quantum fluctuations. We can use again the Einstein background hypothesis by taking the solution of (4.26) corresponding to $\Psi \equiv 0$, which gives $T_{\mu\nu}(\Psi) = \overline{\Psi}\gamma_\mu\nabla_\nu\Psi = 0$, hence preserving Einstein equations in vacuum (3.5). Then (4.23) becomes

$$\begin{aligned} S_\psi = \int \mathrm{d}^D x\sqrt{g}\left[1 + \frac{1}{2}h - \frac{1}{4}h^{\mu\nu}h_{\mu\nu} + \frac{1}{8}h^2\right]\left(\overline{\Psi} + \delta\overline{\psi}\right) \qquad (4.27)\\ \cdot\left[\gamma^c e^\mu_c\left(\partial_\mu + \frac{1}{4}\left[\omega_{\mu ab}(0) + \omega_{\mu ab}(1) + \omega_{\mu ab}(2)\right]\gamma^a\gamma^b\right) + m\right](\Psi + \delta\psi)\,, \end{aligned}$$

where, according to (4.25) and (2.39)–(2.41),

$$\omega_{\mu ab}(n) \equiv e^a_\nu \Gamma^\nu_{\mu\sigma}(n) e^{\sigma b} + e^a_\nu \partial_\mu e^{\nu b}\,\delta_{n0}\,. \tag{4.28}$$

However, since $\Psi \equiv 0$, the action principle (4.27) at one–loop reduces to the much simpler expression

$$\begin{aligned} S_{\psi,2} &= \int \mathrm{d}^D x\sqrt{g}\;\delta\overline{\psi}\left[\gamma^c e^\mu_c\left(\partial_\mu + \frac{1}{4}\omega_{\mu ab}(0)\gamma^a\gamma^b\right) + m\right]\delta\psi \\ &= \int \mathrm{d}^D x\;\sqrt{g}\;\delta\overline{\psi}\left(\not\nabla(0) + m\right)\delta\psi\,, \end{aligned} \tag{4.29}$$

where $\not\nabla(0)$ corresponds to the covariant derivative computed at order zero in quantum perturbations.

The differential operator in (4.23) is not in the form (1.92), but we can observe that the effective action can also be written as

$$\begin{aligned} \Gamma_\psi &= -\log\det\left(\not\nabla + m\right) = -\frac{1}{2}\log\left[\det\left(\not\nabla + m\right)\det\left(-\not\nabla + m\right)\right] \\ &= -\frac{1}{2}\mathrm{Tr}\log\left(-\not\nabla^2 + m^2\right) = -\frac{1}{2}\mathrm{Tr}\log\left(-\nabla^2 + \frac{R}{4} + m^2\right), \end{aligned} \tag{4.30}$$

where we have used *Lichnerowicz identity*

$$\not\nabla^2 = \gamma^\mu\gamma^\nu\nabla_\mu\nabla_\nu = \mathbb{1}_S\left(\nabla^2 - \frac{R}{4}\right), \tag{4.31}$$

in which $\mathbb{1}_S$ denotes the identity operator in spinor space, and $R \equiv R(0)$ as before. This procedure, known as *bosonisation* of the fermionic differential operator, consists in a change of basis of γ–matrices [42], and can be applied to operators whose potential does not depend on γ–matrices [43, § 2.2].

Proof. To prove that (4.31) works in any even dimensional spacetime D we observe that the Clifford fundamental relation $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_S$ is true in any even dimension D, and that exploiting this relation we have

$$\begin{aligned}\nabla\!\!\!/^2 &= \gamma^\mu\gamma^\nu\nabla_\mu\nabla_\nu = \gamma^\mu\gamma^\nu[\nabla_\mu, \nabla_\nu] + \gamma^\mu\gamma^\nu\nabla_\nu\nabla_\mu \\ &= \gamma^\mu\gamma^\nu[\nabla_\mu, \nabla_\nu] + 2\mathbb{1}_S\nabla^2 - \gamma^\nu\gamma^\mu\nabla_\nu\nabla_\mu\,,\end{aligned}$$

and by comparing the second term with the last one, as well as using the commutation relation (4.32),

$$\gamma^\mu\gamma^\nu\nabla_\mu\nabla_\nu = \mathbb{1}_S\nabla^2 + \frac{1}{8}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta R_{\mu\nu\alpha\beta}\,.$$

At this point we can employ the identity

$$\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta R_{\mu\nu\alpha\beta} = -2R\mathbb{1}_S\,,$$

which can be proved by using the symmetry properties of the Riemann tensor and the identity

$$R_{\alpha\beta}\gamma^\beta = -\frac{1}{2}R_{\alpha\mu\nu\beta}\gamma^\mu\gamma^\nu\gamma^\beta\,,$$

that, in turn, comes from the Clifford algebra definition [41, § 4], which is independent on the dimension D. Indeed,

$$\begin{aligned}R_{\alpha\beta}\gamma^\beta &= -\frac{1}{2}R_{\alpha\mu\nu\beta}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right)\gamma^\beta \\ &= -\frac{1}{2}R_{\alpha\mu\nu\beta}\gamma^\mu\gamma^\nu\gamma^\beta + \frac{1}{2}\left(R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu}\right)\gamma^\nu\gamma^\mu\gamma^\beta \\ &= -R_{\alpha\mu\nu\beta}\gamma^\mu\gamma^\nu\gamma^\beta + \frac{1}{2}R_{\alpha\beta\mu\nu}\gamma^\nu\gamma^\mu\gamma^\beta\,,\end{aligned}$$

where

$$\begin{aligned}\gamma^\nu\gamma^\mu\gamma^\beta &= 2g^{\mu\beta}\gamma^\nu - \gamma^\nu\gamma^\beta\gamma^\mu \\ &= 2g^{\mu\beta}\gamma^\nu - 2g^{\nu\beta}\gamma^\mu + \gamma^\beta\gamma^\nu\gamma^\mu\,,\end{aligned}$$

and therefore

$$\begin{aligned}R_{\alpha\beta}\gamma^\beta &= -R_{\alpha\mu\nu\beta}\gamma^\mu\gamma^\nu\gamma^\beta + \frac{1}{2}\left(-2R_{\alpha\nu}\gamma^\nu - 2R_{\alpha\mu}\gamma^\mu + R_{\alpha\beta\mu\nu}\gamma^\beta\gamma^\nu\gamma^\mu\right) \\ &= -\frac{3}{2}R_{\alpha\mu\nu\beta}\gamma^\mu\gamma^\nu\gamma^\beta - 2R_{\alpha\beta}\gamma^\beta\,,\end{aligned}$$

from which

$$R_{\alpha\beta}\gamma^\beta = -\frac{1}{2}R_{\alpha\mu\nu\beta}\gamma^\mu\gamma^\nu\gamma^\beta\,,$$

that is, the identity above. By contracting this result with γ^α on the left we end up with the desired result:

$$\begin{aligned}R_{\alpha\beta}\gamma^\alpha\gamma^\beta &= \frac{1}{2}R_{\alpha\beta}\left\{\gamma^\alpha, \gamma^\beta\right\} = R\mathbb{1}_S \\ &= -\frac{1}{2}R_{\alpha\mu\nu\beta}\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\beta\,.\end{aligned}$$

■

4.2.1 Heat kernel coefficients

By inspecting (4.30), we easily identify the kinetic operator $\mathcal{F}_\psi = -\mathbb{1}_S \nabla^2$ and the potential $V_\psi = \frac{1}{4} R \mathbb{1}_S$, while in any dimension D the commutator of covariant derivatives has the form

$$\Omega_{\mu\nu} = [\nabla_\mu, \nabla_\nu] = \frac{1}{4} \gamma^\alpha \gamma^\beta R_{\alpha\beta\mu\nu} \,. \tag{4.32}$$

Proof. Indeed,

$$\begin{aligned}
\nabla_\nu \nabla_\mu \psi &= \partial_\nu (\nabla_\mu \psi) + \frac{1}{4} \omega_{\nu ab} \gamma^a \gamma^b \nabla_\mu \psi - \Gamma^\lambda_{\mu\nu} \nabla_\lambda \psi \\
&= \partial_\nu \partial_\mu \psi + \frac{1}{4} \partial_\nu \omega_{\mu ab} \gamma^a \gamma^b \psi + \frac{1}{4} \omega_{\mu ab} \gamma^a \gamma^b \partial_\nu \psi + \frac{1}{4} \omega_{\nu ab} \gamma^a \gamma^b \partial_\mu \psi \\
&\quad + \frac{1}{16} \omega_{\nu ab} \omega_{\mu cd} \gamma^a \gamma^b \gamma^c \gamma^d \psi - \Gamma^\lambda_{\mu\nu} \partial_\lambda \psi - \frac{1}{4} \Gamma^\lambda_{\mu\nu} \omega_{\lambda ab} \gamma^a \gamma^b \psi \,,
\end{aligned}$$

and, by neglecting all terms that are symmetric under the exchange $\mu \leftrightarrow \nu$, we are left with

$$[\nabla_\mu, \nabla_\nu] \, \psi = \frac{1}{4} \partial_\mu \omega_{\nu ab} \gamma^a \gamma^b \psi + \frac{1}{16} \omega_{\mu ab} \omega_{\nu cd} \gamma^a \gamma^b \gamma^c \gamma^d \psi - \{\mu \leftrightarrow \nu\} \,.$$

By means of the same identity exploited in the previous proof [41, § 4] we can rewrite the four γ–matrices product as

$$\begin{aligned}
\omega_{\mu ab} \omega_{\nu cd} \gamma^a \gamma^b \gamma^c \gamma^d &= 4 \left(\omega_{\mu a}{}^b \omega_{\nu bd} \gamma^a \gamma^d - \omega_{\mu a}{}^b \omega_{\nu bd} \gamma^d \gamma^a \right) - \omega_{\nu ab} \omega_{\mu cd} \gamma^a \gamma^b \gamma^c \gamma^d \\
&= 8 \omega_{\mu a}{}^b \omega_{\nu bd} \gamma^a \gamma^d - \omega_{\nu ab} \omega_{\mu cd} \gamma^a \gamma^b \gamma^c \gamma^d \,,
\end{aligned}$$

where we also used the fact that the spin connection is antisymmetric when swapping the last two indices. Therefore,

$$\omega_{\mu ab} \omega_{\nu cd} \gamma^a \gamma^b \gamma^c \gamma^d = 4 \omega_{\mu a}{}^b \omega_{\nu bd} \gamma^a \gamma^d \,,$$

and going back to the commutator we find

$$[\nabla_\mu, \nabla_\nu] = \frac{1}{4} \left(\partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu a}{}^c \omega_{\nu cb} - \omega_{\nu a}{}^c \omega_{\mu cb} \right) \gamma^a \gamma^b \equiv \frac{1}{4} R_{\mu\nu ab} \gamma^a \gamma^b \,,$$

having defined the Riemann tensor with mixed indices

$$R_{\mu\nu ab} \equiv \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu a}{}^c \omega_{\nu cb} - \omega_{\nu a}{}^c \omega_{\mu cb} \,.$$

The fully covariant form (4.32) can be immediately obtained by introducing the curved γ–matrices as $\gamma^a = e^a_\alpha \gamma^\alpha$. ■

Again, the mass term is singled out in the Schwinger–DeWitt parametrisation, and transformed into an exponential term in front of the effective action. By comparison with (1.92) we get the following substitution rules:

$$\begin{cases}
\mathbb{1} \;\leftrightarrow\; -\mathbb{1}_S \\
V \;\leftrightarrow\; \frac{1}{4} R \mathbb{1}_S \\
\Omega_{\mu\nu} \;\leftrightarrow\; \frac{1}{4} \gamma^\alpha \gamma^\beta R_{\alpha\beta\mu\nu} \,.
\end{cases} \tag{4.33}$$

The heat kernel coefficients for a massive spinor field have been already computed up to a_2, see for instance the most recent work [44]. Here, we repeat the computation in generic dimension D and we extend the computations to a_3, introducing the coupling to the graviton as well. The first two coefficients can be easily computed by recalling that

$$\mathrm{Tr}\left[\mathbb{1}_S\right] = 2^{\frac{D}{2}}\,, \tag{4.34}$$

since the group $\mathrm{Spin}\,(D)$ is represented in a vector space of dimension $2^{\frac{D}{2}}$ (assuming that D is even). From (1.96)–(1.97) we then obtain

$$\mathrm{Tr}\left[a_0^{\psi}(x)\right] = -2^{\frac{D}{2}} \tag{4.35}$$

$$\mathrm{Tr}\left[a_1^{\psi}(x)\right] = 2^{\frac{D}{2}}\frac{1}{12}R\,. \tag{4.36}$$

For the third coefficient we have to compute

$$\mathrm{Tr}\left[\alpha_2^{\psi}(x)\right] = 2^{\frac{D}{2}}\frac{1}{180}\left(R^2_{\mu\nu\rho\sigma} - \frac{1}{D}R^2\right) - 2^{\frac{D}{2}}\frac{1}{96}R^2_{\mu\nu\rho\sigma} \tag{4.37}$$

$$\mathrm{Tr}\left[\beta_2^{\psi}(x)\right] = \frac{1}{2}\mathrm{Tr}\left[\left(\frac{1}{12}R\mathbb{1}_S\right)^2\right] = 2^{\frac{D}{2}}\frac{1}{288}R^2\,, \tag{4.38}$$

therefore

$$\mathrm{Tr}\left[a_2^{\psi}(x)\right] = 2^{\frac{D}{2}}\left(\frac{5D-8}{1440D}\,\mathcal{E}_1^2 - \frac{7}{1440}\,\mathcal{E}_2^2\right). \tag{4.39}$$

Proof. The only term we have to compute more carefully in (4.37) is the one containing the gauge field strength, which turns out to be

$$\mathrm{Tr}\left[\Omega_{\mu\nu}^2\right] = \frac{1}{16}R_{\mu\nu\alpha\beta}R^{\mu\nu}{}_{\rho\sigma}\mathrm{Tr}\left[\gamma^\alpha\gamma^\beta\gamma^\rho\gamma^\sigma\right],$$

and since

$$\mathrm{Tr}\left[\gamma^\alpha\gamma^\beta\gamma^\rho\gamma^\sigma\right] = 2^{\frac{D}{2}}\left(g^{\alpha\beta}g^{\rho\sigma} - g^{\alpha\rho}g^{\beta\sigma} + g^{\alpha\sigma}g^{\beta\rho}\right),$$

we conclude that

$$\mathrm{Tr}\left[\Omega_{\mu\nu}^2\right] = -2^{\frac{D}{2}}\frac{1}{8}\mathcal{E}_2^2\,.$$ ■

Regarding the fourth coefficient, consider again that $V \propto R$, as in the scalar field case, but now $\Omega_{\mu\nu} \neq 0$. Therefore, we still have

$$\begin{aligned}\mathrm{Tr}\left[\boldsymbol{A}_\psi\right] &= 2^{\frac{D}{2}}\frac{1}{D}\mathrm{Tr}\left[\boldsymbol{A}_{gh}\right] \\ &= 2^{\frac{D}{2}}\left(-\frac{16}{9D^2}\mathcal{E}_1^3 + \frac{2}{3D}\mathcal{E}_2^3 + \frac{17}{9}\mathcal{E}_3^3 - \frac{28}{9}\mathcal{E}_4^3\right),\end{aligned} \tag{4.40}$$

while $\boldsymbol{B}_\psi \neq 0$, and in particular:

$$\mathrm{Tr}\left[\boldsymbol{B}_\psi\right] = 2^{\frac{D}{2}}\left(-\frac{1}{2D}\mathcal{E}_2^3 - \frac{1}{4}\mathcal{E}_3^3 + \frac{1}{2}\mathcal{E}_4^3\right). \tag{4.41}$$

Proof. Indeed, using the rather involved identity

$$\begin{aligned}\mathrm{Tr}\left[\gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\nu\gamma^\rho\gamma^\sigma\right] &= \eta^{\mu\alpha}\mathrm{Tr}\left[\gamma^\beta\gamma^\nu\gamma^\rho\gamma^\sigma\right] - \eta^{\mu\beta}\mathrm{Tr}\left[\gamma^\alpha\gamma^\nu\gamma^\rho\gamma^\sigma\right] + \eta^{\mu\nu}\mathrm{Tr}\left[\gamma^\alpha\gamma^\beta\gamma^\rho\gamma^\sigma\right] \\ &\quad - \eta^{\mu\rho}\mathrm{Tr}\left[\gamma^\alpha\gamma^\beta\gamma^\nu\gamma^\sigma\right] + \eta^{\mu\sigma}\mathrm{Tr}\left[\gamma^\alpha\gamma^\beta\gamma^\nu\gamma^\rho\right]\end{aligned}$$

we have

$$\begin{aligned}\mathrm{Tr}\left[\Omega_\mu{}^\nu\Omega_\nu{}^\sigma\Omega_\sigma{}^\mu\right] &= \frac{1}{64}\mathrm{Tr}\left[\gamma^\alpha\gamma^\beta\gamma^\delta\gamma^\epsilon\gamma^\lambda\gamma^\tau\right] R_\mu{}^\nu{}_{\alpha\beta}R_\nu{}^\sigma{}_{\delta\epsilon}R_\sigma{}^\mu{}_{\lambda\tau} \\ &= \frac{1}{64}\left(-\mathrm{Tr}\left[\gamma^\beta\gamma^\epsilon\gamma^\lambda\gamma^\tau\right] R_\mu{}^{\nu\alpha}{}_\beta R_\nu{}^\sigma{}_{\alpha\epsilon}R_\sigma{}^\mu{}_{\lambda\tau}\right. \\ &\qquad + \mathrm{Tr}\left[\gamma^\beta\gamma^\delta\gamma^\lambda\gamma^\tau\right] R_\mu{}^{\nu\alpha}{}_\beta R_\nu{}^\sigma{}_{\delta\alpha}R_\sigma{}^\mu{}_{\lambda\tau} \\ &\qquad - \mathrm{Tr}\left[\gamma^\beta\gamma^\delta\gamma^\epsilon\gamma^\tau\right] R_\mu{}^{\nu\alpha}{}_\beta R_\nu{}^\sigma{}_{\delta\epsilon}R_\sigma{}^\mu{}_{\alpha\tau} \\ &\qquad \left.+ \mathrm{Tr}\left[\gamma^\beta\gamma^\delta\gamma^\epsilon\gamma^\lambda\right] R_\mu{}^{\nu\alpha}{}_\beta R_\nu{}^\sigma{}_{\delta\epsilon}R_\sigma{}^\mu{}_{\lambda\alpha}\right) \\ &= 2^{\frac{D}{2}}\frac{1}{64}R_\mu{}^{\nu\alpha}{}_\beta\left(R_\nu{}^\sigma{}_\alpha{}^\tau R_\sigma{}^{\mu\beta}{}_\tau - R_\nu{}^\sigma{}_\alpha{}^\lambda R_\sigma{}^\mu{}_\lambda{}^\beta\right. \\ &\qquad - R_\nu{}^{\sigma\tau}{}_\alpha R_\sigma{}^{\mu\beta}{}_\tau + R_\nu{}^{\sigma\lambda}{}_\alpha R_\sigma{}^\mu{}_\lambda{}^\beta \\ &\qquad - R_\nu{}^{\sigma\beta\tau}R_\sigma{}^\mu{}_{\alpha\tau} + R_\nu{}^{\sigma\tau\beta}R_\sigma{}^\mu{}_{\alpha\tau} \\ &\qquad \left.+ R_\nu{}^{\sigma\beta\lambda}R_\sigma{}^\mu{}_{\lambda\alpha} - R_\nu{}^{\sigma\lambda\beta}R_\sigma{}^\mu{}_{\lambda\alpha}\right) \\ &= 2^{\frac{D}{2}}\frac{1}{16}R_\mu{}^{\nu\alpha}{}_\beta\left(R_\nu{}^\sigma{}_\alpha{}^\tau R_\sigma{}^{\mu\beta}{}_\tau - R_\nu{}^{\sigma\beta\tau}R_\sigma{}^\mu{}_{\alpha\tau}\right) = 2^{\frac{D}{2}}\frac{1}{8}\mathcal{E}_4^3\,,\end{aligned}$$

while, in a slightly simpler way,

$$\begin{aligned}\mathrm{Tr}\left[\Omega_{\mu\nu}\nabla^2\Omega^{\mu\nu}\right] &= \frac{1}{16}\mathrm{Tr}\left[\gamma^\alpha\gamma^\beta\gamma^\rho\gamma^\sigma\right] R_{\mu\nu\alpha\beta}\nabla^2R^{\mu\nu}{}_{\rho\sigma} \\ &= 2^{\frac{D}{2}}\frac{1}{16}\left(g^{\alpha\sigma}g^{\beta\rho} - g^{\alpha\rho}g^{\beta\sigma}\right)R_{\mu\nu\alpha\beta}\nabla^2R^{\mu\nu}{}_{\rho\sigma} \\ &= -2^{\frac{D}{2}}\frac{1}{8}\left(\frac{2}{D}\mathcal{E}_2^3 - \mathcal{E}_3^3 - 4\mathcal{E}_4^3\right)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Tr}\left[R_{\mu\nu\rho\sigma}\Omega^{\mu\nu}\Omega^{\rho\sigma}\right] &= \frac{1}{16}\mathrm{Tr}\left[\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\tau\right] R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\alpha\beta}R^{\rho\sigma}{}_{\lambda\tau} \\ &= 2^{\frac{D}{2}}\frac{1}{16}\left(g^{\alpha\tau}g^{\beta\lambda} - g^{\alpha\lambda}g^{\beta\tau}\right)R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\alpha\beta}R^{\rho\sigma}{}_{\lambda\tau} = -2^{\frac{D}{2}}\frac{1}{8}\mathcal{E}_3^3\,,\end{aligned}$$

and exploiting the results of the previous proof,

$$\mathrm{Tr}\left[R_{\mu\nu}\Omega^{\mu\sigma}\Omega^\nu{}_\sigma\right] = \frac{1}{D}R\,\mathrm{Tr}\left[\Omega_{\mu\nu}^2\right] = -\frac{1}{8D}2^{\frac{D}{2}}\mathcal{E}_2^3\,. \qquad \blacksquare$$

We then conclude that

$$\begin{aligned}\mathrm{Tr}\left[\alpha_3^\psi(x)\right] &= \frac{1}{7!}\mathrm{Tr}\left[\boldsymbol{A}_\psi\right] + \frac{2}{6!}\mathrm{Tr}\left[\boldsymbol{B}_\psi\right] \\ &= 2^{\frac{D}{2}}\left(-\frac{1}{2835D^2}\,\mathcal{E}_1^3 - \frac{19}{15120D}\,\mathcal{E}_2^3 - \frac{29}{90720}\,\mathcal{E}_3^3 + \frac{1}{1296}\,\mathcal{E}_4^3\right).\end{aligned} \tag{4.42}$$

Moreover,

$$\begin{aligned}\mathrm{Tr}\left[\beta_3^{\psi}(x)\right] &= \mathrm{Tr}\left[\frac{1}{6}\left(\alpha_1^{\psi}\right)^3 + \alpha_1^{\psi}\alpha_2^{\psi}\right]\\ &= \frac{R^3}{10368} 2^{\frac{D}{2}} + \frac{R}{12}\left\{\frac{1}{180}\left(R_{\mu\nu\rho\sigma}^2 - \frac{1}{D}R^2\right) 2^{\frac{D}{2}} + \frac{1}{12}\mathrm{Tr}\left[\Omega_{\mu\nu}^2\right]\right\}\\ &= 2^{\frac{D}{2}}\left(\frac{5D-24}{51840D}\,\mathcal{E}_1^3 - \frac{7}{17280}\,\mathcal{E}_2^3\right),\end{aligned} \tag{4.43}$$

resulting in

$$\begin{aligned}\mathrm{Tr}\left[a_3^{\psi}(x)\right] = 2^{\frac{D}{2}}\Bigg(&\frac{35D^2-168D-128}{362880D^2}\,\mathcal{E}_1^3\\ &- \frac{49D+152}{120960D}\,\mathcal{E}_2^3 - \frac{29}{90720}\,\mathcal{E}_3^3 + \frac{1}{1296}\,\mathcal{E}_4^3\Bigg).\end{aligned} \tag{4.44}$$

4.2.2 Effective action

The effective action (1.78) for the spinor field ψ becomes then

$$\Gamma_{(1)}^{\psi} = \frac{1}{2}\int_0^{\infty}\frac{\mathrm{d}\beta}{\beta}\ \exp\left(-\beta m^2\right)\int\frac{\mathrm{d}^D x\,\sqrt{g}}{(4\pi\beta)^{\frac{D}{2}}}\sum_{j=0}^{\infty}\frac{\beta^j}{j!}\mathrm{Tr}\left[a_j^{\psi}(x)\right], \tag{4.45}$$

where the factor in front has been changed, according to the definition of the supertrace (1.66). Again, when summed to the gravity action, since $m \neq 0$, we have

$$\begin{aligned}\Gamma_{(1)}^{g+\psi} = \frac{1}{2}\int_0^{\infty}\frac{\mathrm{d}\beta}{\beta}\Bigg\{&\int\frac{\mathrm{d}^D x\,\sqrt{g}}{(4\pi\beta)^{\frac{D}{2}}}\sum_{j=0}^{\infty}\frac{\beta^j}{j!}\mathrm{Tr}\left[a_j(x)\right]\\ &+\exp\left(-\beta m^2\right)\int\frac{\mathrm{d}^D x\,\sqrt{g}}{(4\pi\beta)^{\frac{D}{2}}}\sum_{j=0}^{\infty}\frac{\beta^j}{j!}\mathrm{Tr}\left[a_j^{\psi}(x)\right]\Bigg\},\end{aligned} \tag{4.46}$$

and the same analysis carried out for the scalar field can be repeated here, including relation (4.22).

Again, the results could be extended to the more general case of matter contributions to the background. Already in 1974, Deser and van Nieuwenhuizen [45] proved that this theory is non–renormalisable at one–loop in $D = 4$; a general computation could allow to extend this result to $D > 4$. Moreover, it would be interesting to consider a model composed of both massive spinors and a $U(1)$ gauge field (the *Einstein–Maxwell–Dirac* system). If the gauge field is not considered as a dynamical one — which we will do in the next section — this amounts to introduce a gauge connection A_μ in the covariant derivative (4.24), which becomes

$$\nabla\!\!\!/ = \gamma^a e_a^\mu \nabla_\mu \,, \qquad \nabla_\mu \equiv \partial_\mu + A_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^a\gamma^b \,. \tag{4.47}$$

This situation has already been considered in the special case of a quantised fermionic field, with a classical background [44].

4.3 Vector field

The last model we consider is the minimal coupling between the graviton and a massless vector field $A_\mu(x)$, described by the action principle

$$S[G, A] = \int \mathrm{d}^D x \, \sqrt{G} \left(-\frac{1}{k^2} R + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) , \tag{4.48}$$

where $F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the gauge field strength. (4.48) is known as *Einstein–Maxwell model* and its effective action has already been studied in detail through worldline methods up to order RF^4 [46], as these terms contain information on the modifications of light propagation by weak gravitational fields in the limit of zero photon energies [47]. The effect of non–dynamical scalar and spinor fields propagating on Einstein–Maxwell background has been studied as well [48].

Here, we shall consider a different limit, in which curvature dominates over the electromagnetic field strength, which is expected to occur in the vicinity of a black hole. Indeed, the Einstein–Maxwell theory has been used, together with the heat kernel method, to compute the quantum corrections to the entropy of Kerr–Newmann black holes in Einstein gravity [49] and, more recently, in $\mathcal{N} = 2$ supergravity [50; 51] and low–energy string theory models [52]. In these computations, however, the cosmological constant Λ is set to zero, and due to the tracelessness of the electromagnetic energy–momentum tensor, $R = 0$ is a solution of Einstein equations: all terms proportional to R are then disregarded [49; 50; 51; 52]. Here, instead, we consider the case of $\Lambda \neq 0$, and keep all terms proportional to R, assuming as usual that the background electromagnetic field vanishes.

The action (4.48) can then be expanded by setting $G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ and $A_{\mu\nu} = \overline{A}_{\mu\nu} + a_{\mu\nu}$ (correspondingly, $F_{\mu\nu} = \overline{F}_{\mu\nu} + f_{\mu\nu}$). Since the vector field brings another term to the gauge connection and to the covariant derivative, $\nabla^A_\mu \equiv \nabla_\mu + A_\mu$, the expansion of the action at one–loop performed in appendix 2.A should be computed again from the beginning. In general, the gravitational term of the action (4.48) contains couplings between the background vector field and the gravitational one [53, § 4.A];[5] however, if we

[5]Note that in this paper the author keeps $\Lambda \neq 0$ and employs Vilkovisky–DeWitt formalism, but restricts himself to a flat background field $g_{\mu\nu} = \eta_{\mu\nu}$.

again consider only vacuum solutions $\overline{F}_{\mu\nu} \equiv 0$, all these terms vanish and we are left with the action (2.7), *plus* the new term

$$S_A = \int \mathrm{d}^D x \, \sqrt{g} \left(\frac{1}{4} f_{\mu\nu} f^{\mu\nu} \right) = \frac{1}{2} \int \mathrm{d}^D x \, \sqrt{g} \, a_\mu \left(-g^{\mu\nu} \nabla^2 + \nabla^\mu \nabla^\nu \right) a_\nu$$
$$\equiv \int \mathrm{d}^D x \, \sqrt{g} \, a^\mu \left(\mathcal{F}^A_{\mu\nu} + \nabla_\mu \nabla_\nu \right) a^\nu \, . \tag{4.49}$$

The term proportional to $\nabla^\mu \nabla^\nu$ can be removed by fixing the gauge symmetry associated to the vector field, as we did for the graviton. On the quantum level, this requires to introduce the additional ghost and antighost fields c and $\overline{c}$, which are described by the Lagrangian

$$S_A^{gh} = \int \mathrm{d}^D x \, \sqrt{g} \, \overline{c} \, \nabla^2 c \, . \tag{4.50}$$

Additional interaction terms between the newly introduced ghost field c and the fields c_μ associated to the graviton are removed by the requirement that $\overline{A}_\mu = 0$, as it can be seen from equation (2.11) in [49, § 2].

4.3.1 Heat kernel coefficients

From (4.49) and (4.50) we easily identify the substitution rules for the gauge field A_μ and for the associated ghost c:

$$\begin{cases} \mathbb{1} \leftrightarrow \delta^\mu_\nu \\ V \leftrightarrow 0 \\ \Omega_{\mu\nu} \leftrightarrow F_{\mu\nu} = f_{\mu\nu} \end{cases} \qquad \text{and} \qquad \begin{cases} \mathbb{1} \leftrightarrow 1 \\ V \leftrightarrow 0 \\ \Omega_{\mu\nu} \leftrightarrow 0 \end{cases} \tag{4.51}$$

where for the ghost field $[\nabla_\mu, \nabla_\nu] \, c = 0$, since c behaves as a scalar under covariant derivatives containing only spacetime components. Therefore, for the ghost we immediately have

$$\mathrm{Tr} \left[a_0^{gh,A}(x) \right] = 1 \tag{4.52}$$

$$\mathrm{Tr} \left[a_1^{gh,A}(x) \right] = \frac{1}{6} R \tag{4.53}$$

$$\mathrm{Tr} \left[a_2^{gh,A}(x) \right] = \frac{5D-2}{360D} \mathcal{E}_1^2 + \frac{1}{180} \mathcal{E}_2^2 \, , \tag{4.54}$$

while for the fourth coefficient, from (3.41),

$$\mathrm{Tr} \left[\alpha_3^{gh,A}(x) \right] = \frac{1}{7! \, D} \mathrm{Tr} \left[\boldsymbol{A}_{gh} \right]$$
$$= \frac{1}{7!} \left(-\frac{16}{9D^2} \mathcal{E}_1^3 + \frac{2}{3D} \mathcal{E}_2^3 + \frac{17}{9} \mathcal{E}_3^3 - \frac{28}{9} \mathcal{E}_4^3 \right) \tag{4.55}$$

and

$$\mathrm{Tr}\left[\beta_3^{gh,A}(x)\right] = \frac{1}{6}\left(\frac{1}{6}R\right)^3 + \frac{1}{6}R\cdot\frac{1}{180}\left(R^2_{\mu\nu\rho\sigma} - \frac{1}{D}R^2\right)$$
$$= \frac{5D-6}{6480}\mathcal{E}_1^3 + \frac{1}{1080}\mathcal{E}_2^3\,, \tag{4.56}$$

resulting in

$$\mathrm{Tr}\left[a_3^{gh,A}(x)\right] = \frac{35D^2-42D-16}{45360D^2}\mathcal{E}_1^3 + \frac{7D+1}{7560D}\mathcal{E}_2^3 + \frac{17}{45360}\mathcal{E}_3^3 - \frac{1}{1620}\mathcal{E}_3^4\,. \tag{4.57}$$

Similarly, for the gauge field,

$$\mathrm{Tr}\left[a_0^{g,A}(x)\right] = D \tag{4.58}$$

$$\mathrm{Tr}\left[a_1^{g,A}(x)\right] = \frac{D}{6}R \tag{4.59}$$

$$\mathrm{Tr}\left[a_2^{g,A}(x)\right] = \frac{5D-2}{360}\mathcal{E}_1^2 + \frac{D}{180}\mathcal{E}_2^2 + \frac{1}{12}f_{\mu\nu}f^{\mu\nu}\,, \tag{4.60}$$

where we see the appereance of the new invariant $\mathcal{F}_1^2 \equiv f_{\mu\nu}f^{\mu\nu}$. Since the tensor $f_{\mu\nu}$ is traceless, $f^{\mu}_{\ \mu}=0$, we do not have invariants of order smaller than two. Before computing the fourth coefficient, it is useful to list all invariants of order three in curvature *and* gauge field strength:

$$\mathcal{F}_1^3 = f_\mu^{\ \nu}f_\nu^{\ \sigma}f_\sigma^{\ \nu}\,,\qquad \mathcal{F}_2^3 = R_{\mu\nu\rho\sigma}f^{\mu\nu}f^{\rho\sigma}\,,\qquad \mathcal{F}_3^3 = Rf^{\mu\nu}f_{\mu\nu}\,, \tag{4.61}$$

where we already neglected the invariants containing the Ricci scalar, since on Einstein manifolds they can be simplified further. Note that we do not have any invariant of the form $f_{\mu\nu}\nabla^2 f^{\mu\nu}$, since it vanishes identically.

Proof. From Maxwell equations in vacuum we have $\nabla_\mu F^{\mu\nu}=0$, and by using the second Bianchi identity, with $[\nabla_\mu,\nabla_\nu]=f_{\mu\nu}$, we find

$$\begin{aligned}\mathcal{F}_4^3 &= f_{\mu\nu}\nabla_\alpha\nabla^\alpha f^{\mu\nu} = -f_{\mu\nu}\nabla_\alpha\left(\nabla^\mu f^{\nu\alpha} + \nabla^\nu f^{\alpha\mu}\right)\\ &= -f_{\mu\nu}\left\{\left([\nabla_\alpha,\nabla^\mu] + \nabla^\mu\nabla_\alpha\right)f^{\nu\alpha} - \left([\nabla_\alpha,\nabla^\nu]+\nabla^\nu\nabla_\alpha\right)f^{\alpha\mu}\right\}\\ &= -f_{\mu\nu}f_\alpha^{\ \mu}f^{\nu\alpha} - f_{\mu\nu}\nabla^\mu\nabla_\alpha f^{\nu\alpha} - f_{\mu\nu}f_\alpha^{\ \nu}f^{\alpha\mu} - f_{\mu\nu}\nabla^\nu\nabla_\alpha f^{\alpha\mu}\\ &= -f_\mu^{\ \nu}f_\nu^{\ \alpha}f_\alpha^{\ \mu} + f_\mu^{\ \nu}f_\nu^{\ \alpha}f_\alpha^{\ \mu} = 0\,.\end{aligned}$$

■

On this basis, the computations give

$$\mathrm{Tr}\left[\boldsymbol{A}_{g,A}(x)\right] = \mathrm{Tr}\left[\boldsymbol{A}_{gh}\right] = -\frac{16}{9D}\mathcal{E}_1^3 + \frac{2}{3}\mathcal{E}_2^3 + \frac{17D}{9}\mathcal{E}_3^3 - \frac{28D}{9}\mathcal{E}_4^3 \tag{4.62}$$

$$\mathrm{Tr}\left[\boldsymbol{B}_{g,A}(x)\right] = -12\mathcal{F}_1^3 + 6\mathcal{F}_2^3 - \frac{4}{D}\mathcal{F}_3^3 \tag{4.63}$$

and

$$\mathrm{Tr}\left[\alpha_3^{g,A}(x)\right] = -\frac{1}{2835D}\mathcal{E}_1^3 + \frac{1}{7560}\mathcal{E}_2^3 + \frac{17D}{45360}\mathcal{E}_3^3 - \frac{D}{1620}\mathcal{E}_4^3 - \frac{1}{30}\mathcal{F}_1^3 + \frac{1}{60}\mathcal{F}_2^3 - \frac{1}{90D}\mathcal{F}_3^3 \,, \quad (4.64)$$

while

$$\mathrm{Tr}\left[\beta_3^{g,A}(x)\right] = \frac{5D-6}{6480}\mathcal{E}_1^3 + \frac{D}{1080}\mathcal{E}_2^3 + \frac{1}{72}\mathcal{F}_3^3 \,, \quad (4.65)$$

to get

$$\mathrm{Tr}\left[a_3^{g,A}(x)\right] = \frac{35D^2-42D-16}{45360D}\mathcal{E}_1^3 + \frac{7D+1}{7560}\mathcal{E}_2^3 + \frac{17D}{45360}\mathcal{E}_3^3 - \frac{D}{1620}\mathcal{E}_4^3 - \frac{1}{30}\mathcal{F}_1^3 + \frac{1}{60}\mathcal{F}_2^3 + \frac{5D-4}{360}\mathcal{F}_3^3 \,. \quad (4.66)$$

The total heat kernel coefficient, according to (3.84) (the coefficients for the ghost and vector field are the same), is then given by

$$\mathrm{Tr}\left[a_3^{A}(x)\right] = \frac{35D^3-112D^2+68D+32}{45360D^2}\mathcal{E}_1^3 + \frac{(D-2)(7D+1)}{7560D}\mathcal{E}_2^3 + \frac{17(D-2)}{45360}\mathcal{E}_3^3 + \frac{2-D}{1620}\mathcal{E}_4^3 - \frac{1}{30}\mathcal{F}_1^3 + \frac{1}{60}\mathcal{F}_2^3 + \frac{5D-4}{360}\mathcal{F}_3^3 \,. \quad (4.67)$$

The results of this chapter are just a first example of the kind of computations that can be performed by means of the heat kernel method, when considering more complex systems. The most apparent drawback of the approach described here is that it works only in the vacuum approximation; to include matter background fields we would have to consider Einstein equations with non–zero momentum–energy tensor, and the computations quickly become daunting. A possible solution would be to implement a symbolic calculus program that allows to perform easily all the tensor contractions and evaluations required. In any case, we leave this topic to further research.

Conclusions and outlook

The main results obtained in this thesis, together with possible directions that could be undertaken to proceed further in research, are summarised for sake of clearness in the list below.

- The main result is the evaluation of the first four heat kernel coefficients for pure gravity with nonzero cosmological constant, in generic dimension D and on Einstein manifolds, so that the computation outcome is gauge invariant:
  - the fourth complete and gauge–invariant heat kernel coefficient for perturbative quantum gravity was previously unknown [5];
  - this result allows to prove in a more general framework *non–finiteness of quantum gravity* in $D = 6$ at one–loop, in contrast to what happens in $D = 4$ at one–loop [24]. It would be interesting to see if the precise one–loop coefficient that we have found could have a relation with the coefficient at two–loops in $D = 4$, going beyond the formal analogies observed in [30].
- The result is then compared with the same coefficients coming from the $\mathcal{N} = 4$ spinning particle in the worldline formalism:
  - the fourth coefficient has never been computed before [5];
  - the comparison led to improvement of both methods, as it already occurred for the previous coefficients [5];
  - further useful comparisons could be described by applying other perturbative techniques, like the *covariant derivative expansion*, which has recently been extended to all fermionic operators [43];
  - moreover, further important developments include implementation of *Vilkovisky–DeWitt formalism* [12; 13], which allows to write a background independent and covariant effective action,[6] and of *Borel resummation*, that after having been applied to heat

[6] As we already noted while describing the background field method, this appears not to be the case. A comment on this will be given at the end of this section.

kernel computations by Avramidi [9], has no more been considered in this context.

- In the last part of the thesis, we tried to extend the procedure to systems containing *matter fields* which couple to the graviton:
  - the computations have been performed in the vacuum case, in which matter fields do not give contribution to spacetime curvature;
  - an extension to the *Einstein–Maxwell* general case, with non–vacuum background, would lead to improvements in computing *quantum corrections to Kerr–Newmann black holes entropy*, according to a well–established research line [49; 50; 51; 52].
- A more theoretical issue that could be explored is the tension which has been very recently found to exist between covariance and background independence in the background field method [14]. This result would imply that, even when taking advantage of Vilkovisky–DeWitt formalism, there is no way to define a fully background independent and covariant effective action. Computations perfomed within the heat kernel method might shed more light on this result, providing explicit examples of the construction described formally in [14].

In conclusion, this thesis has investigated the computation of the fourth heat kernel coefficient for perturbative quantum gravity, which plays a crucial role in understanding the behavior of quantum gravitational interactions at one–loop in $D = 6$ and could provide insight for two–loops computations in $D = 4$. These results try to strengthen our theoretical understanding of quantum gravity and to shed light on the higher–order effects that arise in gravitational interactions.

The findings presented in this thesis could be the starting point to advance our understanding of the perturbative approach for quantum gravity. The derived expressions for the fourth heat kernel coefficient provide valuable tools for future investigations and pave the way for more accurate and precise computations in perturbative quantum gravity, as the ones that can be performed with matter fields in the non–vacuum case.

The insights gained from the computation of the fourth heat kernel coefficient provide a small contribution to the ongoing efforts to develop a consistent and comprehensive theory of quantum gravity. We hope that further research building upon the results of this thesis will continue to refine our understanding of the fundamental nature of spacetime and the quantum world.

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E poi ci sono tutti gli altri, quelli che sanno dell'esistenza di questa tesi, ma inevitabilmente non ne conoscono i dettagli; ma importa davvero? Inutile scrivere qui un arido elenco di nomi, o squadernare goffamente, dopo pagine di calcoli, frammenti di una vita che poco hanno a che vedere con questi. Fortunatamente, *verba volant*: preferisco affidarmi alla voce, e spero di avere già ringraziato molti di questi vivendo; nell'egoistica ipotesi che qualcuno si creda escluso, si domandi: escluso da cosa? e si sentirà immediatamente rincuorato.

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