Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE
Corso di Laurea Magistrale in Matematica

Existence and uniqueness of degenerate SDEs with Hölder diffusion and measurable drift

Tesi di Laurea in Analisi Stocastica

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"Mathematics can explore the fourth dimension and the world of what is possible, but the tsar can only be overthrown in the third dimension."

Introduction

In the following paper, we will address the existence and uniqueness of the solution of a stochastic differential equation, known as an SDE. We shall therefore start by defining what an SDE is:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t.$$

We will then illustrate the basic notions, such as solution in the strong sense, in the weak sense and uniqueness, also examining the first theorems on uniqueness and the existence of the solution.In these proofs, we will make use of very strong hypotheses such as Lipschitz and linear growth, also known as the standard hypothesis. The results are of great importance for the theory of SDEs, and are analogous to the results seen in deterministic differential equations.

In the final chapter we will discuss an extremely important result, namely the uniqueness of the weak solution of an SDE with a Hölder diffusive parameter and a measurable drift parameter. This result was initially demonstrated by Ito and Watanabe for one-dimensional SDEs, and they also showed that uniqueness could only be obtained for a Hölder parameter $\alpha \geq \dfrac{1}{2}$ by providing a counterexample. In this work, we will extend this result to d-dimensional SDEs and prove the existence and uniqueness of the solution with Hölder parameter

In order to achieve this, we will utilise the estimates formulated by Krylov, which compare the expected value conditional on a filtration:

$$\mathbb{E}\left(\int_{t_0\wedge\tau_R}^{t_1\wedge\tau_R} (\det(\sigma_s\sigma_s^*))^{\frac{1}{d+1}} f(s,\xi_s) \middle| \mathscr{F}_{t_0\wedge\tau_R} \right)$$

with functions in L^d.

Our objective will then be to search for a non-negative functional that satisfies the following properties:

* The functional must satisfy the following

$$\int_0^t \int_{B_R} F(s,x)^p \det(\sigma\sigma^*)^{-1}(s,x)dxds < \infty$$

** The functional must provide an estimate on the diffusive part and the drift parameter of our stochastic equation.

These assumptions serve to obtain the uniqueness of the solution.

Finally, we have provided examples on which to apply the theorem. To show the validity of the theorem on these examples, we have introduced the Hardy-Littlewood operator:

$$\mathcal{M}_R\phi(x) = \sup_{0<r<R} \frac{1}{\mid B_r \mid} \int_{B_r} \phi(x+y)dy$$

which plays the non-negative operetor role described above. We have also stated a corollary that allows us to apply the theorem in a simpler and more intuitive manner. By applying this corollary, we can verify the theorem for such examples.

Existence and uniqueness of degenerate SDEs with Hölder diffusion and measurable drift

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20/06/2023

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Chapter 1

Stochastic analysis

1.1 Preliminary concepts

In the first chapter, we see the first definitions of what a stochastic process is, uniqueness, existence and where solutions live.

Definition 1.1 (SDE)**.** Given the interval $I = [t_0, T]$ and a W Brownian motion m-dimensional, we define Borels measurable functions :

$$b : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$$

and

$$\sigma : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$$

Let us consider the following equation:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \quad X_0 = 0 \tag{1.1}$$

We define a space where we search for solutions

Definition 1.2 (set-up)**.** We define set-up as a triple $(Z, W, \mathscr{F})$, where:

Z is a random variable and $\mathscr{F}_{t_0}$-measurable,

W is a d-dimensional Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$

$\mathbb{F} = (\mathscr{F}_t)_{t \geq 0}$ is a filtration where the usual assumptions hold (completeness and continuity on the right).

We indicate that X is a solution of an SDE we use:

$$X \in SDE(b, \sigma, Z, W, \mathbb{F})$$

Let us give the generic definition of the solution of an SDE

Definition 1.3 (solution of an SDE)**.** A stochastic process is solution of (1.1) relative to our set-up if:

(i) X is continuous process on $(\Omega, \mathscr{F}, \mathbb{P})$;

(ii) X is a process adapted to $\mathbb{F}$.

(iii)
$$\int_{t_0}^{T} \mid b(t, X_t) \mid dt + \int_{t_0}^{T} \|\sigma(t, X_t)\|^2 dt < +\infty \quad \mathbb{P} - a.s$$
where $\|\cdot\|$ stands for Hilbert-Schmidt norm of a matrix.

(iv)
$$\big(X_t = Z + \int_{t_0}^{T} b(s, X_s)ds + \int_{t_0}^{T} \sigma(s, X_s)dW_s \quad \forall t \in I\big) \quad a.s$$
we will say X_t that solution of our SDE,

In the study of solutions of the weak type it is useful to refer to the canonical set-up. Let's introduce it gradually, making some considerations:

- Let μ be a law on $\mathbb{R}^N$, let's take:

$$Z : (\mathbb{R}^N, \mathbb{B}, \mu) \to \mathbb{R}^N$$

$$x \mapsto Z(x) = x$$

  then $Z \sim \mu$.

- Let W be a Brownian motion:

$$W : (\Omega, \mathscr{F}, \mathbb{P}) \to C(I, \mathbb{R})$$

$$\omega \mapsto W(\omega)$$

We now denote by μ_W the Wiener measure of W. This is a probability measure on the space $(C(I,\mathbb{R}), \mathbb{B}(\mathbb{C}))$, where $\mathbb{B}(\mathbb{C})$ is a Borel algebra, hence

$$\mu_W(H) = \mathbb{P}(W \in H).$$

We can actually observe that the Wiener measure μ_W is uniquely determined by the finite-dimensional distributions, ie by the laws of $(W_{t_1}, ..., W_{t_n})$.

Example 1.1.1. We can build the canonical version of a Browian motion:

Let us take the interval $I = [t_0, T]$, Borel's σ-algebra on the space of continuous functions $\mathbb{B}(C) = \sigma(w(t) \in H, H \in \mathbb{B}, t \in I)$(1-dimensional cylinder) and a filtration, the one-dimensional cylinder: $B_t(C) = \sigma(w(s) \in H, s \leq t)$.

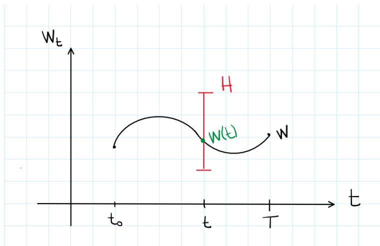

Figure 1.1: the canonical version of a Browian motion

So if we take:

$$(C(I,\mathbb{R}), \mathbb{B}(C), \mu_W, \mathbb{B}_t(C))$$

($\mathbb{B}_t(C)$ then we will replace it with the completed filtration $\mathscr{F}_t^W$)

$$W : C(I,\mathbb{R}) \to C(I,\mathbb{R})$$

$$w \mapsto W(w) = w$$

whene W is a browian motion.

We have just described the canonical version of a Browian motion with the outcomes coinciding with the trajectories of W, i.e. $W_t(w) = w(t)$.

Let us give the formal definition of canonical set-up

Definition 1.4 (canonincal setup)**.** Taking the interval $I = [t_0, T]$ and the space of continuous functions $C(I, \mathbb{R}^d) = \Omega_d$, we define the canonical set-up for μ the following quantity:

$$(\mathbb{R}^N \times \Omega_d, \mathscr{F}_T^{Z,W}, \mu \otimes \mu_W, \mathscr{F}_t^{Z,W})$$

where:

$$Z : \mathbb{R}^N \times \Omega_d \to \mathbb{R}^N$$
$$(x, w) \longmapsto x$$

$$W : \mathbb{R}^N \times \Omega_d \to \Omega_d$$
$$(x, w) \longmapsto w$$

- $Z \sim \mu$
- W is a d-dimenional Brownian motion
- Z and W are independent
- $(\mathbb{R}^N \times \Omega_d)$ is a Polish space, that is, it is a Banach space and it is separable.

Let us now look at the definition of a solution in the strong and weak sense of an SDE

Definition 1.5 (solution of the SDE of coefficients b and σ)**.**

(I) weak type: for every μ distribution on $\mathbb{B}_N$ (Borelian of $\mathbb{R}^N$) there exists a set-up $(Z, W, \mathscr{F})$ such that $Z \backsim \mu$ and there exists $X \in SDE(b, \sigma, Z, W, \mathbb{F})$

(II) strong type: for every set-up $(Z, W, \mathscr{F})$ there exists $X \in SDE(b, \sigma, Z, W, \mathscr{F}^{Z,W})$, where $\mathscr{F}^{Z,W} = (\mathscr{F}_t^{Z,W})_{t \geq 0})$ with standard filtration, which is the completion of $\sigma(Z, W_s \quad with \quad s \leq t)$

Remark 1.1.2. In general the solution of an SDE is unique when the coefficients b and σ are uniformly Lipschitz continuous to the variable x uniform in t. However, when b and σ are non-Lipschitz continuous, the pathwise uniqueness would not hold as in the case of ordinary differential equations.

In one dimensional case, the famous Yamada and Watanabe theorem provides a sufficient condition for pathwise uniqueness.

In this thesis we see some new conditions to ensure the pathwise uniqueness and also the existence of weak solutions for multidimensional SDEs without assuming uniform ellipticity and Lipschitz continuity.

Definition 1.6 (uniqueness)**.** We say that for the SDE of coefficients b, σ we have uniqueness:

- in a strong sense (pathwise uniqueness) , if $X \in SDE(b, \sigma, W, \mathscr{F}_t)$ and $Y \in SDE(b, \sigma, W, \mathscr{G}_t)$ with $X_{t_0} = Y_{t_0}$ a.s implies that X and Y are indistinguishable processes;
- in a weak sense (in law), if $X \in SDE(b, \sigma, W, \mathscr{F}_t)$ and $Y \in SDE(b, \sigma, B, \mathscr{G}_t)$ with $X_{t_0} \stackrel{\mathrm{d}}{=} Y_{t_0}$, implies that $(X, W) \stackrel{\mathrm{d}}{=} (Y, B)$, or, equivalently, (X, W) and (Y, B), have the same finite-dimensional distributions.

Remark 1.1.3. In the definition of strong uniqueness the two processes X and Y are defined on the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and are solutions of the SDE related to the setups $(W, \mathscr{F}_t)$and $(W, \mathscr{G}_t)$ where W is a Brownian motion with respect to both filtrations $\mathscr{F}_t$ and $\mathscr{G}_t$ which may be different. In the definition of uniqueness in law, the processes X and Y can be solutions related to set-up $(W, \mathscr{F}_t)$and $(B, \mathscr{G}_t)$ distinct, also defined on different probability spaces.

we observe the standard assumptions with which the first results on the existence and uniqueness of the solution were provided.

Definition 1.7 (standard hypothesis)**.**

- b and σ have a linear growth, i.e $\exists c_1$ such that

$$|b(t,x)| + |\sigma(t,x)| \leq c_1(1+|x|) \quad \forall t \in I, \quad \forall x \in \mathbb{R}^N;$$

- b and σ are Lipschitz, i.e $\exists c_2$ such that

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq c_2(|x-y|) \quad \forall t \in I, \quad \forall x,y \in \mathbb{R}^N;$$

Let us now look at two theorems that we will need for the third chapter

Lemma 1.1.4 (Gronwall lemma)**.** We assume that $u \in \mathbf{C}([0,T);R), T \in (0,+\infty)$, satisfies the differential inequality:

$$u^{'} \leq a(t)u + b(t) \quad \text{on} \quad \forall t \in [0,T) \tag{1.2}$$

for some $a, b \in L^1(0,T)$ Then, u satisfies the pointwise estimate

$$u(t) \leq \mathrm{e}^{A(t)}u(0) + \int_0^t b(s)\mathrm{e}^{A(t)-A(s)}ds, \quad \forall t \in (0,T) \tag{1.3}$$

where we have defined the primitive function:

$$A(t) := \int_0^t a(s)ds$$

Proof. From inequality (1.2), for every function test $\varphi \in \mathcal{D}(0,T)$ with $\varphi \geq 0$ we get:

$$-\langle u, \varphi^{'} \rangle = \langle u^{'}, \varphi \rangle \leq \langle a(t)u + b(t), \varphi \rangle \tag{1.4}$$

and for every $0 \leq \varphi \in C_c(0,T) \cap W^{1,\infty}(0,T)$.

If we take

$$v(t) = u(t)\mathrm{e}^{-A(t)} - \int_0^t b(s)\mathrm{e}^{-A(t)}ds$$

we can observe:

$$v^{'}(t) \leq 0, \quad v \in \mathcal{D}^{'}(0,T) \quad v \in C([0,T])$$

We can note that:

- if $v \in C^1(or \quad even \quad v \in W^{1,1})$ we immediately conclude:

$$v(t) = v(0) + \int_0^t v^{'}(s)ds \leq v(0) = u(0)$$

  from the (1.2) follows.

- In the general case when $v \in C([0,T])$, we proceed as follows.

  We fix $\varepsilon > 0$ and $\varrho \in C_c^1(0,\varepsilon)$ such that:

$$\varrho \geq 0 \quad \int \varrho = 1$$

.

For any function $w \geq 0 \quad w \in C_c^1(\varepsilon, T)$, the function

$$\psi := -w + \left(\int_0^T w\right)\varrho$$

belongs to $C_c([0,T])$ and

$$\int_0^T \psi = 0$$

.

As a consequence ψ has a primitive φ such that $\varphi(0) = \varphi(T) = 0$ The function φ thus enjoys the following properties:

$$\varphi \in C_c^1([0,T]), \quad \varphi \geq 0 \quad \text{and} \quad \varphi^{'} = \psi$$

.

We deduce:

$$0 \geq \langle u^{'}, \varphi \rangle = \int_0^T v\left\{w - \left(\int_0^T w\right)\varrho\right\}dt = \int_0^T w\left\{v - \int_0^T w\varrho\right\}$$

Because the above inequality is true for any $w \in C_c^1([\varepsilon, T])$,with $w \geq 0$ than:

$$v \leq \int_0^T v\varrho \quad \text{on} \quad (\varepsilon, T)$$

Taking $\varrho = \varrho_\alpha$ a mollifier sequence,(i.e. $\varrho \to \delta_0$)and letting $\alpha \to 0$ we deduce again $v(t) \leq v(0)$ on $(0,T)$.

$\square$

Lemma 1.1.5 (Stochastic Gronwall's inequality)**.** Let $\xi(t)$ and $\mu(t)$ be two nonnegative càdlàg $\mathscr{F}_t$-adapted processes, A_t a continuous nondecreasing $\mathscr{F}_t$-adapted process with $A_0 = 0$, M_t a local martingale with $M_0 = 0$. Suppose that:

$$\xi(t) \leq \mu(t) + \int_0^t \xi(s)dA_s + M_t \quad \forall t \geq 0 \tag{1.5}$$

Then for any $0 < q < p < 1$ and stopping time τ , we have:

$$[\mathbb{E}(\xi(\tau)^*)^q]^{\frac{1}{q}} \leq \left(\frac{p}{p-q}\right)^{\frac{1}{q}} \left(\mathbb{E}\mathrm{e}^{\frac{pA_r}{(1-p)}}\right)^{\frac{(1-p)}{p}} \mathbb{E}(\mu(\tau)^*), \tag{1.6}$$

where

$$\xi(t)^* := \sup_{s\in[0,t]} \xi(s)$$

Proof. We fix a stopping time τ . Without loss of generality, we may assume that the right hand side of (1.6) is finite and $\mu(t)$ is nondecreasing. Otherwise, we may replace $\mu(t)$ with $\mu(t)^*$.

We define $\overline{\xi}(t)$ the right side of (1.5) and

$$\overline{A}_t := \int_0^t \frac{\xi(s)}{\overline{\xi}(s)} dA_s$$

.

Then

$$\xi(t) \leq \overline{\xi}(t) = \mu(t) + \int_0^t \overline{\xi}(s)d\overline{A}_s + M_t$$

we use Ito's formula to get:

$$\mathrm{e}^{-\overline{A}_t}\overline{\xi}(t) = \mu(0) + \int_0^t -\overline{\mathrm{A}}_s d\mu(s) + \int_0^t \mathrm{e}^{\overline{A}_s} dM_s.$$

We take $(\tau_n)_{n\in\mathbb{N}}$ localization sequence stopping times of the local martingale M, that is, for every $n \in \mathbb{N}$,

$$t \longmapsto M_{t\bigwedge\tau_n} \quad \text{is a martingale.}$$

since we have that $\mathrm{e}^{-\overline{A}_t} \leq 1$, we have:

$$\mathbb{E}\Big(\mathrm{e}^{-\overline{A}_{t\wedge\tau\wedge\tau_n}}\overline{\xi}(t\wedge\tau\wedge\tau_n)\Big) \leq \mathbb{E}\Big(\mu(t\wedge\tau\wedge\tau_n)\Big) \leq \mathbb{E}\Big(\mu(t\wedge\tau)\Big).$$

Going to limit for $n \to \infty$ a.s. by Fatou's lemma, we get:

$$\mathbb{E}\Big(\mathrm{e}^{-\overline{A}_\tau}\overline{\xi}(\tau)\Big) \le \mathbb{E}\Big(\mu(\tau))\Big).$$

Because thanks to the Holder inequality of , $\xi(t) \le \overline{\xi}(t)$ and $\overline{A}_t \le A_t$, that for any $p \in (0,1)$,

$$\mathbb{E}\Big(\xi(\tau)^p\Big) \le \mathbb{E}\Big(\overline{\xi}(\tau)^p\Big) \le \Big(\mathbb{E}\Big(\mathrm{e}^{(\dfrac{pA_\tau}{1-p})}\Big)\Big)^{(1-p)}\Big(\mathbb{E}\Big(\mu(\tau)\Big)\Big)^p.$$

Now we define for each $\lambda \ge 0$ a stopping time:

$$\tau_\lambda := \inf\{s \ge 0 \quad s.t. \quad \xi(s) \ge \lambda\}.$$

Since ξ is càdlàg, we have $\xi_{\tau_\lambda} \ge \lambda$ and

$$\lambda^p\mathbb{P}\Big(\xi(\tau)^* > \lambda\Big) \le \lambda^p\mathbb{P}(\tau_\lambda \le \tau) \le \mathbb{E}\Big(\xi(\tau \wedge \tau_\lambda)^p\Big) \le \Big(\mathbb{E}\Big(\mathrm{e}^{(\dfrac{pA_\tau}{1-p})}\Big)\Big)^{(1-p)}\Big(\mathbb{E}\Big(\mu(\tau)\Big)\Big)^p =: \delta$$

and for any $q \in (0,p)$,

$$\mathbb{E}\Big(|\xi(\tau)^*|^p\Big) = q\int_0^\infty \lambda^{q-1}\mathbb{P}(\xi(\tau)^* > \lambda)d\lambda \le q\int_0^\infty \lambda^{q-1}((\lambda^{-p}\delta)\wedge 1)d\lambda = \frac{p\delta^{(\frac{q}{p})}}{(p-q)}.$$

the proof is complete. $\square$

Chapter 2

Uniqueness in the classical sense

Now let's see uniqueness in the classical sense, ie with very strong hypotheses on the diffusion and drift parameter. We see an estimate derived from standard hypotheses

Lemma 2.0.1. Let X,Y be adapted and continuous processes a.s. and $p \leq 2$. Then

- if b, σ verify the condition of linear growth, there exists a positive constant $\overline{c_1} = \overline{c_1}(T, d, N, p, c_1)$, such that

$$E[\sup_{t_o \leq t \leq t_1} |\int_{t_0}^{t} b(s, X_s)ds + \int_{t_0}^{t} \sigma(s, X_s)dW_s|^p]$$
$$\leq \overline{c_1}(t_1 - t_0)^{\frac{p-2}{2}} \int_{t_0}^{t_1} (1 + E[\sup_{t_o \leq r \leq s} |X_r|^p])ds$$

  for each $t_1 \in (t_0, T)$;

- if b,σ verify the global lipschitz condition, there exists a positive constant $\overline{c_2} = \overline{c_2}(T, d, N, p, c2)$ such that

$$E[\sup_{t_o \leq t \leq t_1} |\int_{t_0}^{t} b(s, X_s) - b(s, Y_s)ds + \int_{t_0}^{t} \sigma(s, X_s) - \sigma(s, Y_s)dW_s|^p]$$
$$\leq \overline{c_2}(t_1 - t_0)^{\frac{p-2}{2}} \int_{t_0}^{t_1} (E[\sup_{t_o \leq r \leq s} |X_r - Y_r|^p])ds$$

We now see theorems that give existence and uniqueness of the solution of an SDE, obviously under standard assumptions

Theorem 2.0.2. Suppose that the coefficient

$$b : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d$$

is a bounded Borel-measurable function. Then the SDE

$$dX_t = b(t, Xt)dt + dWt$$

and solvable in the weak sense with unique solution in law.

Proof. We divide the proof into 2 parts, existence and uniqueness.

- **Existence**
  Let μ_0 be a distribution on $\mathbb{R}^d$ and let X be a d-dimensional Brownian motion with initial value $X_0 \sim \mu_0$ defined on the space $(\Omega, \mathscr{F}, \mathbb{P}, \mathscr{F}_t)$. For the borderness assumption on b we have:

  $$M_t := \exp\Big(\int_0^t b(s, X_s)dX_S - \frac{1}{2}\int_0^t |b(s, X_s)|^2 ds\Big) \quad t \in [0, T]$$

  Is a martingale. So, we know that the process:

  $$W_t := X_t - X_0 - \int_0^t b(s, X_s)ds$$

  Is a standard Brownian motion under the measure Q defined by $\dfrac{dQ}{dP} = M_T$. Thus, the equation just seen shows that X is the weak solution of the SDE under Q.In addition:

  $$Q(X_0 \in H) = E^P[\mathbb{1}_{\mathbb{H}}(\mathbb{X}_0 E^0[M_T|\mathscr{F}]] = P(X_0 \in H)$$

  by the martingale property of the process M, and thus $X_0 \sim \mu_0$ in Q.

- **Uniqueness**
  Let $X^{(i)}$, $i = 1, 2$ be solutions of the SDE for the setups respectively $(W^{(i)}, \mathscr{F}_t^{(i)}$ defined on the spaces $(\Omega_i, \mathscr{F}^{(i)}, \mathbb{P}_i)$ such that $X_0^{(1)}$ and $X_0^{(2)}$ are equal in law. Still for the borderness b we have the process :

  $$M_i^{(i)} \exp\Big(-\int_0^t b(s, X_s^{(i)})dW_s^{(i)} - \frac{1}{2}\int_0^t |b(s, X_s^{(i)})|^2 ds\Big) \quad t \in [0, T]$$

are martingale.

From which we obtain that:

$$X_t^{(i)} := X_0^{(i)} - \int_0^t b(s, X_s^{(i)})ds + W_t^{(i)}$$

are Brownian motions on spaces respectively $(\Omega_i, \mathscr{F}^{(i)}, Q_i, \mathbb{P}_i)$ where $\frac{dQ_i}{dP_i} = M_T^{(i)}$. Thus, the law of $X^{(1)}$ of Q_1 is equal to the law of $X^{(2)}$ of Q_2 So we have that the law of $(X^{(1)}, W^{(1)}, M^{(1)})$ in Q_1 is egual to the law of $(X^{(2)}, W^{(2)}, M^{(2)})$ in Q_2.Finally, for every $0 \leq t_1 < \cdots < t_n \leq T$ and $H \in \mathscr{B}_{2nd}$ we have:

$$P_1((X_{t_1}^{(1)}, (W_{t_1}^{(1)}, \cdots, (X_{t_n}^{(1)}, (W_{t_n}^{(1)}) \in H) = \int_{\Omega_1} \mathbb{1}_H(X_{t_1}^{(1)}, (W_{t_1}^{(1)}, \cdots, (X_{t_n}^{(1)}, (W_{t_n}^{(1)}) \frac{dQ_1}{M_T^{(1)}} =$$
$$\int_{\Omega_2} \mathbb{1}_H(X_{t_1}^{(2)}, (W_{t_1}^{(2)}, \cdots, (X_{t_n}^{(2)}, (W_{t_n}^{(2)}) \frac{dQ_2}{M_T^{(2)}} =$$
$$P_2((X_{t_1}^{(2)}, (W_{t_1}^{(2)}, \cdots, (X_{t_n}^{(2)}, (W_{t_n}^{(2)}) \in H)$$

□

The following result establishes the relationship between solvability and uniqueness for an SDE in a weak and strong sense

Theorem 2.0.3. (Yamada e Watanabe)

i) If an SDE is solvable in the strong sense then it is also solvable in the weak sense;

ii) If for an SDE there is uniqueness in the strong sense then there is also uniqueness in the weak sense;

iii) If for an SDE there is solvability in the weak sense and uniqueness in the strong sense then there is solvability in the ‘ strong sense;

Sketch of proof. i) It is sufficient to construct a set-up to deduce the weak resolubility from the strong one.

More precisely, having assigned a distribution μ_0 on $\mathbb{R}^N$, we consider the canonical space $\mathbb{R}^N \times \Omega_d$ equipped with the filtration $(\mathscr{G}_t)_{t\in[0,T]}$ generated by the identity process

$$(\mathbf{Z},\mathbf{W}) : \mathbb{R}^N \times \Omega_d \longrightarrow \mathbb{R}^N \times \Omega_d, \qquad \mathbf{Z}(z,w) = z, \quad \mathbf{W}_t(z,w) = w(t), \quad t \in [0,T]$$

and the product measure $\mu_0 \otimes \mu_W$, where μ_W is the law of Brownian motion d-dimensional. Then $Z \sim \mu_0$ is $\mathscr{G}_0 - \textit{measurable}$ and $\mathbf{W}$ is a Brownian motion (withrespect to $\mathscr{G}_t$). Therefore, by the strong solvability hypothesis, there exists a solution X relative to the set-up $(\mathbf{W}, \mathscr{G}_t)$ and such that $\mathbf{X}_0 = \mathbf{Z} \sim \mu_0$.

ii) Let us consider two solutions $X^i = SDE(b,\sigma,W^i,\mathscr{F}_t^i)$ such that $X_0^i = x \in \mathbb{R}^N$ almost surely, for $i = 1,2$. We prove that the strong uniqueness hypothesis implies that (X^1,W^1) and (X^2,W^2) are equal in law. The problem is that the solutions X^1 and X^2 are in general defined on different sample spaces: so the idea is to construct versions of X^1 and X^2 that are solutions of the SDE on the same space and relatively the same Brownian. To this end, we construct a canonical space on which three processes are defined: a Brownian motion and versions of X^1 and X^2. We know that there is a regular version of law X^i conditional on W^i, for each $w \in \Omega_d$ i.e $\mu_{X^i|W^i} = (\mu_{X^i|W^i}(\cdot;w))_{w\in\Omega_d}$ is a distribution over the Borelians $\mathscr{G}_T^N$ of Ω_N and applies:

$$\int_A \mu_{X^i|W^i}(H;w)\mu_W(dw) = \mathbb{E}[\mathbb{E}[\mathbb{1}_H(X^i)|W^i]\mathbb{1}_A(W^i)] = \mu_{X^i,W^i}(H\times A) \quad (H,A) \in \mathscr{G}_T^N \times \mathscr{G}_T^d.$$

Now on the space of trajectories $\Omega_N \times \Omega_N \times \Omega_d$ we define the probability measure

$$P(H\times K\times A) := \int_A \mu_{X^1|W^2}(H;w)\mu_{X^2|W^2}(K;w)\mu_W(dw), \quad (H,K,A) \in \mathscr{G}_T^N \times \mathscr{G}_T^N \times \mathscr{G}_T^d.$$

and we denote by $(\mathbf{X}^1,\mathbf{X}^2,\mathbf{W})$ the canonical process on that space. Given respectively H $=\Omega_N$ or $K = \Omega_N$ in the equation above,we obtain:

$$(\mathbf{X}^i,\mathbf{W}) \overset{\mathrm{d}}{=} (X^i,W^i), \quad i = 1,2$$

from which it follows in particular that $\mathbf{W}$ is a Brownian motion in the measure P, $(\mathbf{X}^1, \mathbf{W})$ and $(\mathbf{X}^2, \mathbf{W})$ are both solutions of the SDE of coefficients b, σ and with initial datum x. For uniqueness in the strong sense we have that $\mathbf{X}^1$ and $\mathbf{X}^2$ are indistinguishable in the measure P and therefore

$$(X^1, W^1) \stackrel{\mathrm{d}}{=} (\mathbf{X}^1, \mathbf{W}) = (\mathbf{X}^2, \mathbf{W}) \stackrel{\mathrm{d}}{=} (X^2, W^2).$$

iii) Also for this point, let us only consider the case of a deterministic initial datum. Let therefore $X \in SDE(b, \sigma, W, \mathscr{F}_t)$ be a solution with initial datum $X_0 = x \in \mathbb{R}^N$ a.s.

We apply the construction of the previous point (ii) with $X^1 = X^2 = X$ i.e. we construct a space $\Omega_N \times \Omega_N \times \Omega_d$ the measure P and the canonical process $(\mathbf{X}^1, \mathbf{X}^2, \mathbf{W})$ in which $\mathbf{X}^1$,$\mathbf{X}^2$ are equal in law to X and are solutions of the SDE with respect to Brownian motion $\mathbf{W}$.

Consider the conditional probability $P(\cdot|\mathbf{W}) = (P_w(\cdot|\mathbf{W}))_{w \in \Omega_d}$and related conditional laws

$$\mu_{X^i|\mathbf{W}}(H) = P(\mathbf{X}^i \in H|\mathbf{W}), \quad H \in \Omega_N, \quad i = 1, 2,$$

we observe that $\mu_{X^i|\mathbf{W}} = \mu_{X|W}$. It is verified that the random variables $\mathbf{X}^1$ and $\mathbf{X}^2$ are simultaneously equal a.s. and independent in $P_w(\cdot|\mathbf{W})$ for almost any $w \in \Omega_d$ and therefore $\mathbf{X}^1$ and $\mathbf{X}^2$ have as law in $P_w(\cdot|\mathbf{W})$ a Dirac delta. In other words, for almost every $w \in \Omega_d$ we have $\mu_{X|W}(H; w) = \mu_{X^i|\mathbf{W}}(H; w) = \delta_{F(w)}$ for a certain measurable map F from Ω_d in Ω_N and thus $X = F(W)$ a.s. To conclude, it is necessary to show that X is adapted to the standard Brownian filtration $\mathscr{F}^W$.

$\square$

Theorem 2.0.4 (Uniqueness in a strong sense)**.** Let us assume that the following assumption of local smoothness is valid in x, uniform in t: for each $n \in \mathbb{N}$ there exists a constant k_n such that

$$|b(t, x) - b(t, y)| \leq k_n |x - y|$$

for each t and $x, y \in \mathbb{R}^N$ such that $|x|, |y| \leq n$ Then for the SDE (1.1) there is uniqueness in the strong sense according to Definition (1.6)

Proof. Let X,Y be two solutions of the SDE $X \in SDE(b, \sigma, W, \mathscr{F}_t)$ and $Y \in SDE(b, \sigma, W, \mathscr{G}_t)$.

Let us use a localization argument: we pose

$$\tau_n = \inf\{t \in [t_0, T] \quad | \quad |X_t| \vee |Y_t| \geq n\}, \quad n \in \mathbb{N},$$

with the convention $\min \emptyset = T$. We note that $\tau_n = t_0$ on $(|Z| > n)$. Being by hypothesis X,Y adapted and continuous a.s., τ_n and an increasing succession of stopping times at values in $[t_0, T]$, such that $\tau_n \nearrow T$ a.s. let us assume

$$b_n(t, x) = b(t, x)\mathbb{1}_{[t_0, \tau_n]}(t), \quad \sigma_n(t, x) = \sigma(t, x)\mathbb{1}_{[t_0, \tau_n]}(t) \quad n \in \mathbb{N}$$

The processes $X_{t\vee\tau_n}$, $Y_{t\vee\tau_n}$ almost surely satisfy the equation

$$\begin{aligned} X_{t\wedge\tau_n} - Y_{t\vee\tau_n} &= \int_{t_0}^{t\wedge\tau_n} (b(s, X_s) - b(s, Y_s))ds + \int_{t_0}^{t\wedge\tau_n} (\sigma(s, X_s) - \sigma(s, Y_s))dW_s \\ &= \int_{t_0}^{t} (b(s, X_{s\wedge\tau_n}) - b(s, Y_{s\wedge\tau_n})ds + \int_{t_0}^{t} (\sigma(s, X_{s\wedge\tau_n}) - \sigma(s, Y_{s\wedge\tau_n})dW_s \end{aligned}$$

we also have:

$$|b_n(s, X_{s\wedge\tau_n}) - b_n(s,_Y s \vee \tau_n)| = |b_n(s, X_{s\wedge\tau_n}) - b_n(s,_Y s \wedge \tau_n)|\mathbb{1}_{(|Z|\leq n)} \leq k_n|X_{s\wedge\tau_n} - X_{s\vee\tau_n}|$$

because $|X_{s\wedge\tau_n}|, |X_{s\wedge\tau_n}| \leq n$ on $(|Z|) \leq n$ for $s \in [t_0, T]$

and a similar estimation occurs with σ_n instead of b_n.Now let us assume

$$v_n(t) = E[\sup_{t_0\leq s\leq t} |X_{s\wedge\tau_n} - Y_{s\wedge\tau_n}|^2], \quad t \in [t_0, T].$$

thanks to the earlier findings we obtain:

$$v_n(t) \leq \bar{c}\int_{t_0}^{t} v(s)ds, \quad t \in [t_0, T]$$

for a positive constant $\bar{c} = \bar{c}(T, d, N, k_n)$.

Since X and Y are continuous processes a.s. and adapted (and therefore progressively measurable), Fubini's theorem ensures that v is a measurable function on $[t_0, T]$, i.e $v_n \in m\mathscr{B}$. Moreover, v_n is bounded, precisely $|v_n| \leq 4n$ by construction. From Gronwall's lemma we obtain that $v_n \equiv 0$ and therefore

$$E[\sup_{t_0 \le t \le T} |X_{s\wedge\tau_n} - Y_{s\wedge\tau_n}|^2] = v_n(T) = 0$$

Turning to the limit as $n \to \infty$, by the Beppo-Levi theorem, we have that X and Y are indistinguishable on $[t_0, T]$. □

In the one-dimensional case, the following stronger result applies, which we report without proof.

Theorem 2.0.5 (Watanabe-Yamada)**.** Let be given a one-dimensional SDE (1.1),$N = d = 1$, with the following conditions: for $x, y \in \mathbb{R}, t \ge 0$ we have:

- $|b(t,x) - b(t,y)| \le k(|x-y|)$

- $|\sigma(t,x) - \sigma(t,y)| \le h(|x-y|)$

where

- h and a strictly increasing function such that $h(0) = 0$ and for every $\varepsilon \ge 0$:
$$\int_0^\varepsilon \frac{1}{h^2(t)} dt = \infty;$$

- k and a strictly increasing function such that $k(0) = 0$ and for every $\varepsilon \ge 0$:
$$\int_0^\varepsilon \frac{1}{k(t)} dt = \infty;$$

then we have unique strong solution for the SDE.

Example 2.0.6. Ito-Watanabe

The following SDE,
$$dX_t = 3X_t^{(1/3)} dt + 3X_t^{(2/3)} dW_t$$
has infinitely many solutions of the following form:

$$X_t^{(a)} = \begin{cases} 0 \quad for \quad 0 \le t < \tau_a \\ W_t^3 \quad for \quad \tau_a \le t \end{cases}$$

Where $a \in [0, \infty]$ and $\tau_a = \inf\{t \geq a \quad such \quad that \quad W_t = 0\}$.

In fact by applying Ito's formula, in one dimension, we verify that:

$$dW_t^3 = 3W_t dt + \frac{6}{2} W_t^2 dW_t = 3W_t dt + 3W_t^2 dW_t$$

substitute $W_t = X_t^{1/3}$ we obtain:

$$dX_t = 3X_t^{(1/3)} dt + 3X_t^{(2/3)} dW_t$$

To give an idea, we set $a = 1$ as we see in the graph τ_a (stopping time) is immediately after, therefore the solution is zero before τ_a and W_t after.

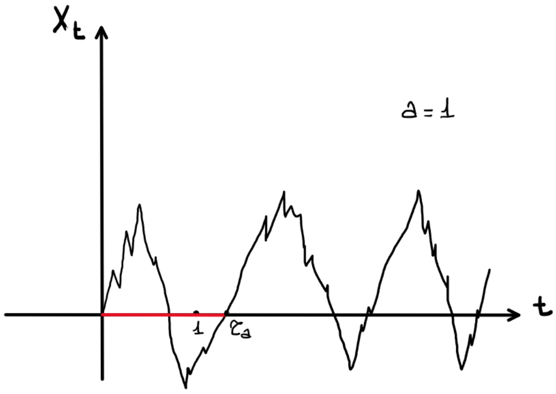

Figure 2.1: the stocastic prosses

Remark 2.0.7. We note that if the term $a = \infty$ the solution is the null one, instead with $a = 0$ we have the solution W_t^3.

Theorem 2.0.8 (Solvability in the strong sense and flow properties)**.** Let us assume that the coefficients b, σ satisfy the standard assumptions(1.7) on $(t_0, T) \times \mathbb{R}^N$ Given a set-up $(W, \mathscr{F}_t)$, we have:

i) for each $x \in \mathbb{R}^N$, there exists the strong solution $X^{t_0,x} \in SDE(b, \sigma, W, \mathscr{F}^W)$ with initial data $X_{t_0}^{t_0,x} = x$. In addition, for each $t \in [t_0, T]$ we have:

$$(x, \omega) \longmapsto \psi_{t_0,t} := X_t^{t_0,x}(\omega) \in m(\mathscr{B}_N \otimes \mathscr{F}_t^W);$$

ii) for each $Z \in m\mathscr{F}_{t_0}$ the process $X^{t_0,Z}$ defined by

$$X_t^{t_0,Z}(\omega) := \psi_{t_0,t}(Z(\omega),\omega), \quad \omega \in \Omega, t \in [t_0,T],$$

is a strong solution of the SDE (1.1),$X^{t_0,Z} \in SDE(b,\sigma,W,\mathscr{F}^{Z,W})$ with initial data $X_{t_0}^{t_0,Z}$;

iii) The flow property holds for each $t \in [t_0,T]$ the process $X^{t_0,Z}$ and $X^{t,X_t^{t_0,Z}}$ are indistinguishable on $[t,T]$ i.e. almost surely we have

$$X_s^{t_0,Z} = X_s^{t,X_t^{t_0,Z}} \quad \forall s \in [t,T]$$

Proof. We divide the demonstration into steps:

step one We prove the existence of the solution of (1.1) on $[t_0,T]$ with the deterministic initial datum $X_{t_0} = x \in \mathbb{R}^N$. We use the method of successive approximations and define recursively the sequence of Ito processes:

$$X_t^{(0)} \equiv x,$$

$$X_t^{(n)} = x + \int_{t_0}^t b(s, X_s^{(n-1)})ds + \int_{t_0}^t \sigma(s, X_s^{(n-1)})DW_s, \quad n \in \mathbb{N},$$

with $t \in [t_0,T]$. The sequence is well defined and $X^{(n)}$ is adapted to $\mathscr{F}^W$ and continuous a.s. for every n. Moreover, with an inductive argument in n we prove that $X_t^{(n)} = X_t^{(n)}(x,\omega) \in m(\mathscr{B}_N \otimes \mathscr{F}^{W_t})$ for every $n \leq 0$ and $t \in [t_0,T]$. We now prove by induction the following estimate:

$$E = [\sup_{t_0 \leq t \leq t_1} |X_t^{(n)} - X_t^{(n-1)}|^2] \leq \frac{c^n(t_1-t_0)^n}{n!}, \quad t_1 \in (t_0,T), n \in \mathbb{N}$$

with $c = c(T,d,N,x,c_1,c_2) > 0$ where c_1 and c_2 are the constants of the standard hypotheses. Let n = 1: by (2.0.1) we have:

$$E[\sup_{t_0 \leq t \leq t_1} |X_t^{(1)} - X_t^{(0)}|^2] = E[\sup_{t_0 \leq t \leq t_1} |\int_{t_0}^t b(s,x)ds + \int_{t_0}^t \sigma(s,x)DW_s|^2 \leq \overline{c_1}(1+|x|^2)(t_1-t_0)$$

for $n = 1$ the estimate is verified, let us assume it is valid for n and prove it for $n+1$: we have

$$E = [\sup_{t_0 \le t \le t_1} |X_t^{(n+1)} - X_t^{(n)}|^2] =$$

$$= E[\sup_{t_0 \le t \le t_1} |\int_{t_0}^{t} (b(s, X_s^{(n)}) - b(s, X_s^{(n-1)}))ds + \int_{t_0}^{t} (\sigma(s, X_s^{n}) - \sigma(s, X_s^{n-1}))DW_s|^2] \le$$

for (2.0.1)

$$\le c^{n+1} \int_{t_0}^{t_1} \frac{(s-t_0)^n}{n!} ds$$

this proves the estimate. Combining Markov's inequality with the estimate just proved, we obtain:

$$P(\sup_{t_0 \le t \le T} |X_t^{(n)} - X_t^{(n-1)}| \ge \frac{1}{2^n}) \le 2^{2n} E[\sup_{t_0 \le t \le T} |X_t^{(n)} - X_t^{(n-1)}|^2] \le \frac{(4cT)^n}{n!}, \quad n \in \mathbb{N}$$

Then, by the Borel-Cantelli Lemma we have

$$P(\sup_{t_0 \le t \le T} |X_t^{(n)} - X_t^{(n-1)}| \ge \frac{1}{2^n} \quad i.o) = 0$$

that is, for almost every $\omega \in \Omega$ there exists $n_\omega \in \mathbb{N}$ such that

$$\sup_{t_0 \le t \le T} |X_t^{(n)}(\omega) - X_t^{(n-1)}(\omega)| \le \frac{1}{2^n}, \quad n \ge n_\omega.$$

Being

$$X_t^{(n)} = x + \sum_{k=1}^{n} (X_t^{(k)} - X_t^{(k-1)})$$

we have that, almost surely, $X_t^{(n)}$ converges uniformly in $t \in [t_0, T]$ as $n \to \infty$ to a limit that we denote by X_t: as notation we will use : $X_t^n \rightrightarrows X_t$ a.s. We note that $X = (X_t)_{t \in [t_0, T]}$ is a continuous a.s. process (due to uniform convergence) and adapted to $\mathscr{F}^W$: moreover, $X_t = X_t(x, \omega) \in m(\mathscr{B}_N \otimes \mathbb{F}_t^W)$ for each $t \in [t_0, T]$ because this measurability property holds for $X_t^{(n)}$ for each $n \in \mathbb{N}$. By the standard assumptions and X being continuous a.s. it is clear that the condition

$$\int_{t_0}^{T} |b(s, X_s)|ds + \int_{t_0}^{T} |\sigma(s, X_s)|^2 dW_s < \infty$$

is satisfied. To verify that, almost surely

$$X_t = x + \int_{t_0}^{t} b(s, X_s)ds + \int_{t_0}^{t} \sigma(s, X_s)dW_s, \quad t \in [t_0, T],$$

it is sufficient to observe:

- by the Lipschitz property of b and uniform σ in t, we have that

  $$b(t, X_t^{(n)}) \rightrightarrows b(t, X_t) \quad a.s$$

  and

  $$\sigma(t, X_t^{(n)}) \rightrightarrows \sigma(t, X_t) \quad a.s$$

  hence

  $$\lim_{x \longrightarrow \infty} \int_{t_0}^{t} b(t, X_t^{(n)})ds = \int_{t_0}^{t} b(t, X_t)ds \quad a.s$$

  and

  $$\lim_{x \longrightarrow \infty} \int_{t_0}^{t} |\sigma(t, X_t^{(n)}) - \sigma(t, X_t)|^2 ds \quad a.s$$

- $$\lim_{x \longrightarrow \infty} \int_{t_0}^{t} \sigma(t, X_t^{(n)})dW_s = \int_{t_0}^{t} \sigma(t, X_t)dW_s$$

  This concludes the proof of existence in the case of the deterministic initial datum.

step two Let us now consider the case of a random initial $Z \in m\mathscr{F}_{t_0}$. Let $f = f(x, \omega)$ be the function on $\mathbb{R}^N \times \Omega$ defined by

$$f(x, \cdot) := \sup_{t_0 \le t \le T} |X_t^{t_0,x} - x - \int_{t_0}^{t} b(s, X_s^{t_0,x})ds - \int_{t_0}^{t} \sigma(s, X_s^{t_0,x})dW_s|.$$

We note that $f \in m(\mathscr{B}_N \otimes \mathscr{F}_T^W)$ because $X_t^{t_0,\cdot} \in m(\mathscr{B}_N \otimes \mathscr{F}_t^W)$ for each $t \in [t_0, T]$. Moreover, for each $x \in \mathbb{R}^N$ we have $f(x, \cdot) = 0$ a.s. and thus also $F(x) := E[f(x, \cdot)] = 0$. Then we have

$$0 = F(Z) = E[f(x, \cdot)]|_{x=Z} = E[f(Z, \cdot)|\mathscr{F}_{t_0}]$$

since $Z \in m\mathscr{F}_{t_0}, f \in m(\mathscr{B}_N \otimes \mathscr{F}_T^W)$ with $\mathscr{F}_{t_0}$ and $mathscrF_t^W$ σ-algebras independent and $f \geq 0$.

Applying the expected value we also have

$$E[f(Z,\cdot)] = 0$$

and therefore $X^{t_0,Z}$ is a solution of the SDE (1.1) and is also a solution in the strong sense because it is clearly adapted to $\mathscr{F}^{Z,W}$.

step three For $t_0 \le t \le s \le T$, with equalities that almost surely hold, we have

$$\begin{aligned} X_s^{t_0,Z} &= Z + \int_{t_0}^{s} b(r, X_r^{t_0,Z})dr + \int_{t_0}^{s} \sigma(r, X_r^{t_0,Z})dW_r \\ &= Z + \int_{t_0}^{t} b(r, X_r^{t_0,Z})dr + \int_{t_0}^{t} \sigma(r, X_r^{t_0,Z})dW_r \\ &\quad + \int_{t}^{s} b(r, X_r^{t_0,Z})dr + \int_{t}^{s} \sigma(r, X_r^{t_0,Z})dW_r \\ &= X_t^{t_0,Z} \int_{t}^{s} b(r, X_r^{t_0,Z})dr + \int_{t}^{s} \sigma(r, X_r^{t_0,Z})dW_r \end{aligned}$$

i.e. $X^{t_0,Z}$ is solution on $[t,T]$ of the SDE (1.1) with initial datum $X_t^{t_0,Z}$. On the other hand, as far as proved in the second step, $X^{t,X_t^{t_0,Z}}$ is also a solution of the same SDE. By uniqueness, the processes $X^{t_0,Z}$ and $X^{t,X_t^{t_0,Z}}$ are indistinguishable on $[t,T]$. This proves $X_s^{t_0,Z} = X_s^{t,X_t^{t_0,Z}}$ and concludes the proof of the theorem.

$\square$

Chapter 3

Existence and uniqueness for SDE

In the last chapter we analyse a very important result on the existence and uniqueness under weaker than standard assumptions, i.e. with Hölder diffusion parameter and measurable drift. This fact is very counterintuitive in the sense that, for ODS, the uniqueness of the solution occurs with Lipschitz parameters, whereas, as we shall see in SDEs, Hölder is sufficient. We shall also see an important corollary with the Hardy-Littlewood operator, and some examples of SDEs.

Before we look at the theorem, let us see some estimates prove by Krylov

Theorem 3.0.1 (Krylov's estimate)**.** We consider the d-dimensional Ito process:

$$\xi_t = \xi_0 + \int_0^t b_s ds + \int_0^t \sigma_s dWs$$

where $\xi_0 \in \mathscr{F}_0$

For $R > 0$ define a stopping time:

$$\tau_R = \left\{ t \geq 0 \quad such \quad that \mid \xi_t \mid \vee \int_0^t \mid b_s \mid ds \vee \int_0^t \|\sigma_s\|^2 ds > R \right\}$$

For any $R, T > 0$ there exist constants C_1, C_2 such that for all $0 \leq t_0 < t_1 < T$ and all $f \in L^{d+1}_{loc}(\mathbb{R}^{d+1})$ and $g \in L^d_{loc}(\mathbb{R}^d)$

then the following inequalities:

- $$\mathbb{E}\left(\int_{t_0 \wedge \tau_R}^{t_1 \wedge \tau_R} (\det(\sigma_s \sigma_s^*))^{\frac{1}{d+1}} f(s, \xi_s) \middle| \mathscr{F}_{t_0 \wedge \tau_R} \right) \leq C_1 \|f\|_{L^{d+1}([t_0, t_1 \times B_R])} \qquad (3.1)$$

- $$\mathbb{E}\left(\int_{t_0\wedge\tau_R}^{t_1\wedge\tau_R} (\det(\sigma_s\sigma_s^*))^{\frac{1}{d}} g(\xi_s) \middle| \mathscr{F}_{t_0\wedge\tau_R}\right) \leq C_1(t_1-t_0)^{\frac{d}{2p}} \|g\|_{L^d([t_0,t_1\times B_R])} \tag{3.2}$$

Theorem 3.0.2. Let $R > 0$ with $B_R := \{x \in \mathbb{R}^d \quad s.t. \quad |x| < R\}$ suppose that there exist a nonnegative measurable function F with

$$\int_0^t \int_{B_R} F(s,x)^p \det(\sigma\sigma^*)^{-1}(s,x)dxds < \infty, \tag{3.3}$$

for some $p \geq d+1$ and such that for Lebesgue-almost all s, x, y :

$$2\langle x-y, b(s,x)-b(s,y)\rangle + \|\sigma(s,x)-\sigma(s,y)\|^2 \leq |x-y|^2(F(s,x)-F(s,y)).$$

Then the local pathwise uniqueness holds def the SDE (1.1). Moreover, is b and σ are time-independent, then the above requirement $p \geq d+1$ can be replaced with $p \geq d$.

Proof. Let X_t and Y_t, two solutions for SDE (1.1) with starting point x and y, respectively.

For $R > 0$ define, a stopping time for process X:

$$\tau_R^X := \inf\left\{t > 0 \quad s.t. \quad |X_t| \vee \int_0^t |b(s,X_s)|ds \vee \int_0^t \|\sigma(s,X_s)\|^2 ds > R\right\}$$

and a stopping time for process Y:

$$\tau_R^Y := \inf\left\{t > 0 \quad s.t. \quad |Y_t| \vee \int_0^t |b(s,Y_s)|ds \vee \int_0^t \|\sigma(s,Y_s)\|^2 ds > R\right\}.$$

Now define a new stocastic process as a combination of X and Y in this way:

$$Z_t := X_t - Y_t, \quad \tau_R := \tau_R^X \wedge \tau_R^Y.$$

By Itô's formula and the assumption, we have:

$$
\begin{aligned}
|Z_{t\wedge\tau_R}|^2 = |x-y|^2 + 2\int_0^{t\wedge\tau_R} \langle Z_s, b(s,X_s) - b(s,Y_s)\rangle ds \\
+2\int_0^{t\wedge\tau_R} \langle Z_s, \sigma(s,X_s) - \sigma(s,Y_s)\rangle dW_s \\
+ \int_0^{t\wedge\tau_R} \|\sigma(s,X_s) - \sigma(s,Y_s)\|^2 ds \\
\leq |x-y|^2 + \int_0^{t\wedge\tau_R} |Z_s|^2 (F(s,X_s) + F(s,Y_s)) ds \\
+2\int_0^{t\wedge\tau_R} \langle Z_s, \sigma(s,X_s) - \sigma(s,Y_s)\rangle dW_s.
\end{aligned}
$$

By stochastic Gronwall's inequality (1.6), for any $0 < q < p < 1$ and a stopping time $\tau^{'}$, we have:

$$
\mathbb{E}\bigg(\sup_{t\in[0,T\wedge\tau^{'}]} |Z_{t\wedge\tau_R}|^{2q}] \bigg) \leq C|x-y|^2 \bigg(\mathbb{E}e^{\dfrac{pA_{T\wedge\tau^{'}}}{(1-p)}} \bigg)^{\dfrac{(1-p)}{p}}, \tag{3.4}
$$

where

$$
A_t := \int_0^{t\wedge\tau_R} (F(s,X_s) + F(s,Y_s)) ds.
$$

Now,we multiply and divide by $\det(\sigma\sigma^*)^{(1/p)}(s,X_s) > 0$ and calculate the expected value of A_t:

$$
\mathbb{E}A_t = \mathbb{E}\bigg(\int_0^{t\wedge\tau_R} \det(\sigma\sigma^*)^{(1/p)}(s,X_s)(F\det(\sigma\sigma^*)^{(-1/p)}(s,X_s) ds \bigg)
$$

by Krylov's estimate (3.1) and the assumption, we have:

$$
\mathbb{E}A_t \leq \bigg(\int_0^t \int_{B_R} |F(s,x)|^p \det(\sigma\sigma^*)^{(-1)}(s,x) \bigg)^p < \infty.
$$

In particular, $t \to A_t$ in a continuous adapted process and if we define a new stopping time:

$$
\tau^{'}_N := \inf\{t > 0 \quad s.t. \quad A_t > N\}
$$

then

$$\mathbb{P}\bigg(\lim_{N\to\infty\tau'_N=\infty}\bigg) = 1.$$

In (3.4), we replace $\tau\prime$ by τ'_N and let $x = y$, then

$$\mathbb{E}\Bigg(\sup_{t\in[0,T\wedge\tau'_N]} |Z_{t\wedge\tau_R}|^{2q}]\Bigg) = 0$$

Letting $N, R \to \infty$, by Fatou's lemma we get the pathwise uniqueness.

If b and σ are time-independent, we can use the Krylov's estimate (3.2) instead of (3.1) to derive the same result. □

Now let's see a corollary that follows from the theorem that uses the Hardy-Littlewood one as an operator Let us first give the definition of the Hardy-Littlewood operator:

Definition 3.1 (Local Hardy-Littlewood maximal function)**.** The operator takes a locally integrable function:

$$\phi : \mathbb{R}^d \longrightarrow \mathbb{R}$$

and return a function $M_R\phi$.

For each $x \in \mathbb{R}^d$, it returns the maximum value by considering the average of the values of the function ϕ over all balls of radius r, regardless of their size. Defined:

$$\mathcal{M}_R\phi(x) = \sup_{0<r<R} \frac{1}{\mid B_r \mid} \int_{B_r} \phi(x+y)dy$$

Corollary 3.0.3. Suppose that $\sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^m$ satisfies that for some $p \geq d$,

$$\int_{B_R} (\mathcal{M}_R \mid \nabla\sigma \mid (x))^p \det(\sigma\sigma^*)^{-1}(x) < \infty \tag{3.5}$$

where $\nabla\sigma$ stands for the generalized gradient, and $\mathcal{M}_R \mid \nabla\sigma \mid$ is the local Hardy-Littlewood maximal function.

Then the local pathwise uniqueness holds for SDE

Let's also look at some examples

Example 3.0.4. Let $d > 2n$ with $n \in N$ and $\alpha \in [0,1]$, $\beta \in [\alpha,1]$. Consider the following diffusion matrix $\sigma(x)$:

$$\sigma(x) := \begin{pmatrix} |x|^\alpha & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & |x|^\alpha & 0 & \cdots & 0 \\ 0 & \cdots & 0 & |x|^\beta + 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & |x|^\beta + 1 \end{pmatrix}$$

we can then see matrix σ as a block matrix:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where A is a $n \times n$ matrix and B is a $(d-n) \times (d-n)$ matrix. One can check that (3.5) holds.

We now compute the $\det(\sigma\sigma^*)$:

$$\det(\sigma\sigma^*)(x) = |x|^{2n\alpha}(|x|^\beta + 1)^{2n(d-n)} \geq |x|^{2n\alpha}.$$

Remark 3.0.5. We can see for $\gamma \in (0,1)$ that the (3.5) operator can be estimated as follows:

$$\frac{1}{|B_s|}\int_{B_s} |x+y|^{-\gamma} dy \leq \begin{cases} \frac{1}{|B_s|}\int_{B_s} ||x|-|y||^{-\gamma} dy \leq 2^\gamma |x|^{-\gamma}, & s < \frac{|x|}{2} \\ \frac{1}{|B_s|}\int_{4B_s} |y|^{-\gamma} dy \leq cs^{-\gamma} \leq c|x|^{-\gamma}, & s \geq \frac{|x|}{2} \end{cases} \tag{3.6}$$

where $c = c(d,\gamma)$ is a costant.

So, we can see the following fact:

$$\sup_{s>0} \frac{1}{|B_s|}\int_{B_s} |x+y|^{-\gamma} dy \leq c|x|^{-\gamma}$$

Let's calculate:

$$\nabla\sigma(x) = (\alpha|x|^{\alpha-1} sgn(x), \beta|x|^{\beta-1} sgn(x))$$

Now:

$$|\nabla\sigma|(x) = |\alpha|x|^{\alpha-1} + \beta|x|^{\beta-1}|$$

So substituting inside the integral (3.5) and using the above estimate with $-\gamma = \alpha - 1$ and with $d > 2n$ and for any $R > 0$ we get:

$$\int_{B_R} (\mathcal{M}_R \mid \nabla\sigma \mid (x))^d \det(\sigma\sigma^*)^{-1}(x) \le C \int_{B_R} |x|^{(\alpha-1)d-2n\alpha} dx < \infty.$$

Notice that the function $x \longmapsto |x|^\alpha$ is only α-Hölder continuous at point 0.

Example 3.0.6. Let $d = 3$ and $m_1, m_2, m_3 \ge 2$. Let $\alpha \in (0,1)$ and $y_{ij} \in \mathbb{R}^d$, $i = 1, , m_j$, $j = 1, 2, 3$. Consider the following $\sigma(x)$:

$$\sigma(x) := \begin{pmatrix} \sum_{i=1}^{m_1} |x - y_{i1}|^\alpha & 0 & 0 \\ 0 & \sum_{i=1}^{m_2} |x - y_{i2}|^\alpha & 0 \\ 0 & 0 & \sum_{i=1}^{m_3} |x - y_{i3}|^\alpha \end{pmatrix}$$

As above, one can check that (3.5) holds. Notice that σ is Hölder continuous at points y_{ij}.

We show that the integral

$$\int_{B_R} (\mathcal{M}_R \mid \nabla\sigma \mid (x))^p \det(\sigma\sigma^*)^{-1}(x) < \infty$$

We have

$$\nabla\sigma(x) = \big(\alpha \sum_{i=1}^{m_1} \operatorname{sign}(x-y_{i1})|x-y_{i1}|^{\alpha-1}, \alpha \sum_{i=1}^{m_2} \operatorname{sign}(x-y_{i2})|x-y_{i2}|^{\alpha-1}, \alpha \sum_{i=1}^{m_3} \operatorname{sign}(x-y_{i3})|x-y_{i3}|^{\alpha-1}\big)$$

Next, we can compute the norm of the gradient:

$$|\nabla\sigma(x)| = \alpha \sum_{j=1}^{3} \left(\sum_{i=1}^{m_j} |x - y_{ij}|^{\alpha-1} \right).$$

We can see for $\gamma \in (0,1)$ that the (3.5) operator can be estimated as follows:

$$\frac{1}{|B_s|} \sum_{j=1}^{3} \sum_{i=1}^{m_j} \int_{B_s} |x - y_{ij} + y|^{-\gamma} dy \le$$

$$\leq \begin{cases} \frac{1}{|B_s|}\sum_{j=1}^3\sum_{i=1}^{m_j}\int_{B_s}||x-y_{ij}|-|y||^{-\gamma}dy \leq C\sum_{j=1}^3\sum_{i=1}^{m_j}|x-y_{ij}|^{-\gamma}, & s < \frac{|x-y_{ij}|}{2} \\ \frac{1}{|B_s|}\sum_{j=1}^3\sum_{i=1}^{m_j}\int_{4B_s}|y|^{-\gamma}dy \leq c\sum_{j=1}^3\sum_{i=1}^{m_j}|x-y_{ij}|^{-\gamma}, & s \geq \frac{|x-y_{ij}|}{2} \end{cases}$$

We now calculate the determinant of $\sigma\sigma^*$:

$$\det(\sigma\sigma^*)(x) \leq (\prod_{j=1}^{3}\sum_{i=1}^{m_j}|x-y_{ij}|^{2\alpha})$$

As in the previous example we analyze:

$$\int_{B_R}(\mathcal{M}_R \mid \nabla\sigma \mid (x))^d \det(\sigma\sigma^*)^{-1}(x)$$

Now imposing $\gamma = \alpha - 1$,$R > 0$ and substituting the we obtain:

$$\int_{B_R}(\mathcal{M}_R \mid \nabla\sigma \mid (x))^3 \det(\sigma\sigma^*)^{-1}(x) \leq K\sum_{j=1}^{3}\sum_{i=1}^{m_j}\int_{B_R}|x-y_{ij}|^{3(\alpha-1)}\prod_{j=1}^{3}|x-y_{ij}|^{-2\alpha}dx =$$

$$= K\sum_{j=1}^{3}\sum_{i=1}^{m_j}\int_{B_R}\prod_{j=1}^{3}|x-y_{ij}|^{-3+3\alpha-2\alpha}dx < \infty$$

because $\alpha - 3 \leq -2$.

Notice that the function $x \longmapsto |x-y_{ij}|^\alpha$ is only α-Hölder continuous at point y_{ij}.

Example 3.0.7. Let $\alpha \in (0,1)$. Consider the following one-dimensional equation:

$$dZ_t = (|Z_t|^\alpha + |W_t^{(2)}|^\alpha + |W_t^{(3)}|^\alpha)dW_t^{(1)}$$

$$Z_0 = z \quad z \in \mathbb{R}$$

where $(W_t^{(1)}, W_t^{(2)}, W_t^{(3)})$ is a three dimensional standard Brownian motion. The above SDE can be written as a three dimensional SDE with $X_t = (Z_t, W_t^{(2)}, W_t^{(3)})$) and

$$\sigma(x) := \begin{pmatrix} (|x_1|^\alpha + |x_2|^\alpha + |x_3|^\alpha) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

One can verify that (3.5) holds for the above σ. It should be observed that the Hölder continuity index can be less than $1/2$ compared with Yamada-Watanabe's classical result due to the regularization effect of Brownian noises.

To verify the integrability condition (3.5) we need to compute the matrix $\nabla\sigma$ and the function $\det(\sigma\sigma^*)^{-1}$, and then check that the resulting integral is finite.

The matrix $\nabla\sigma$ is the Jacobian matrix of the function $\sigma : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ with entries $\sigma_{i,j}(x)$, $1 \leq i, j \leq 3$, given by

$$\begin{cases} |x_1|^\alpha + |x_2|^\alpha + |x_3|^\alpha, & i = j = 1, \\ 1, & i = j = 2 \text{ or } i = j = 3, \\ 0, & \text{otherwise.} \end{cases}$$

The Jacobian matrix is therefore

$$\nabla\sigma(x) = \begin{pmatrix} \alpha|x_1|^{\alpha-1}\text{sign}(x_1) & \alpha|x_2|^{\alpha-1}\text{sign}(x_2) & \alpha|x_3|^{\alpha-1}\text{sign}(x_3) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix $\sigma\sigma^*$ is the product of σ and its transpose, and it's determinant is given by

$$\det(\sigma\sigma^*)(x) = \det \begin{pmatrix} (|x_1|^\alpha + |x_2|^\alpha + |x_3|^\alpha)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= (|x_1|^\alpha + |x_2|^\alpha + |x_3|^\alpha)^2 \geq |x_1|^{2\alpha} + |x_2|^{2\alpha} + |x_3|^{2\alpha} \geq |x|^{2\alpha}$$

$$|\nabla\sigma(x)| = \alpha|x_1|^{\alpha-1} + \alpha|x_2|^{\alpha-1} + \alpha|x_3|^{\alpha-1}$$

Similarly to inequality (3.6) we can estimate our operator (3.5) and obtain with $-\gamma = \alpha - 1$:

$$\int_{B_R} (\mathcal{M}_R \mid \nabla\sigma \mid (x))^3 \det(\sigma\sigma^*)^{-1}(x) \leq K \int_{B_R} (|x_1|^{\alpha-1} + |x_2|^{\alpha-1} + |x_3|^{\alpha-1})^3 |x_1^{2\alpha} + x_2^{2\alpha} + x_3^{2\alpha}|^{-1} dx$$

$$\leq K \int_{B_R} |x|^{3(\alpha-1)} |x|^{-2\alpha} < \infty$$

We have just verified that this integral is finite, so thanks to corollary 3 we have a unique solution for each Holder parameter.

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Ringraziamenti

I would like to thank Professor Andrea Pascucci for his willingness to constantly follow me along this path. I thank my love, Maria Grazia, for her continuous support and being always beside me on this path. I thank my father, mother, Sara and Alessandro for the sacrifices they made to allow me to continue my studies. A big thank you goes to my grandmother Paola for their support on this path. I thank Grandma Marisa, Rosa, Marco and Paolino. I especially thank Anna, Tonino, Nicola, Sveva little Samuel and Gabriele. I also thank my friends Carlo, Filippo, Riccardo, Hassan, Davide, Pasquale, Giovanni, Stefano, Michele, Miriana, Filomena, Giulia, Jack, Enrico, Valentina and Alby for listening to me and supporting me in all my craziness. I thank my water polo teammates, who have kept me entertained and relaxed this year. Finally, I thank Simone for last (I will teach you how to drive one of these days!) for being there for me during the difficult times.