Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE
Corso di Laurea Magistrale in Matematica

INDUCED AND PRODUCED REPRESENTATIONS OF LIE ALGEBRAS: A REALIZATION THEOREM

Tesi di Laurea in Algebra

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Introduction

In mathematics, it is useful to represent algebraic structures as linear maps between vector spaces. This branch of algebra is called representation theory and here is where this thesis is located. The concepts of induced and produced representations of rings were introduced by D. G. Higman in [2] and then further explored by R. J. Blattner in [1] in the context of Lie algebras (which are closely related to rings through their universal enveloping algebras).

There is a Realization Theorem proved by Guillemin and Sternberg (and later extended by Rim) which embeds any transitive Lie algebra into the Lie algebra of continuous derivations of the dual of a certain symmetric algebra (which is isomorphic to an algebra of power series if the field has characteristic 0). Blattner's paper gives a stronger statement for the Realization Theorem and it showcases analogous results to the ones already exposed by G. W. Mackey on group representations. His work isn't aimed at students and is rather concise compared to the complexity of the subject, which results in many details being skimmed over or given for granted. Because of this, directly approaching Blattner's study might prove itself challenging even for readers with reasonable knowledge of the concepts of Lie algebras and their representations. This thesis tries to cover (most of) [1] as clearly and thoroughly as possible, so that the reader's effort may be reduced to the minimum. The original work also includes results regarding Lie groups and further expands on the topological structure of produced representations. We will not cover these parts.

The first chapter is dedicated to introducing the notions of tensor products between modules over rings and universal enveloping algebras of Lie algebras. This is not a very in-depth introduction and will only cover what is needed for understanding the later chapters. In §2, we show important properties and structures of induced and produced Lie modules. The next two chapters are focused on produced representations: we showcase the Guillemin-Sternberg-Rim theorem in §3 and give some results on systems of imprimitivity in §4. The module produced from the field (regarded as a trivial Lie module) is of particular importance here. Lastly, in §5 we look at induced representations and prove a theorem concerning irreducibility criteria for certain Lie modules.

Despite the thesis not being targeted at experts only, readers are still required to be familiar with the basic concepts of linear algebra and topology, as well as the notions of rings, algebras, Lie algebras and modules over these algebraic structures. We refer to [4] and [5] for these concepts.

Contents

Chapter 1

The universal enveloping algebra

The universal enveloping algebra of a Lie algebra is a central tool in this work. Here we will only recall the most important results that will be needed later on; many theorems and proofs will therefore be skipped and may be found in [4] and [5].

1.1 Tensor products

Let R be a ring. Let V be a right R-module and let W be a left R-module.

Definition 1.1 (Tensor product of R-modules)**.** A *tensor product* of V and W is a pair (U, φ) where U is an R-module, φ is an R-bilinear map $\varphi : V \times W \to U$ and the following universal property holds: for any other pair (M, σ) where M is an R-module and $\sigma : V \times W \to M$ is an R-bilinear map then there exists a unique R-linear map $\theta : U \to M$ such that $\sigma = \theta \circ \varphi$.

$$\begin{array}{ccc} V \times W & \xrightarrow{\varphi} & U \\ & \searrow^{\sigma} & \downarrow{\scriptstyle\theta} \\ & & M \end{array}$$

The tensor product of two R-modules V and W exists and is unique up to isomorphism (see [5]). Here is how to construct it: define the free R-module

$$R^{(V \times W)} = \{ \sum_{(v,w) \in I} r_{(v,w)}(v, w) \mid r_{(v,w)} \in R, v \in V, w \in W, I \subseteq V \times W, |I| < \infty \}$$

as the direct sum of R over the set $V \times W$ and let Z be its submodule generated by the bilinear conditions, i.e. all elements of the form

$$\begin{aligned}&(v, w_1 + w_2) - (v, w_1) - (v, w_2)\\&(v_1 + v_2, w) - (v_1, w) - (v_2, w)\\&(vr, w) - r(v, w)\\&(v, rw) - r(v, w)\end{aligned}$$

for all $r \in R$, $v, v_i \in V$ and $w, w_i \in W$, $i = 1, 2$. Set

$$V \otimes_R W = R^{V \times W}/Z$$

and let $\pi : R^{(V \times W)} \to V \otimes_R W$ be the canonical projection. Write $\pi(v, w) = v \otimes w$ so that we have the relations

$$\begin{aligned}v \otimes (w_1 + w_2) &= v \otimes w_1 + v \otimes w_2\\(v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w\\r(v \otimes w) &= (vr) \otimes w = v \otimes (rw)\end{aligned}$$

for all $r \in R$, $v, v_i \in V$ and $w, w_i \in W$, $i = 1, 2$. Define $\otimes_R$ to be the restriction of π to $V \times W \subseteq R^{(V \times W)}$, which is bilinear by the above formulas. Then the pair $(V \otimes_R W, \otimes_R)$ is the tensor product of V and W. The elements mapped into $V \otimes_R W$ by $\otimes_R$ are called *simple tensors*.

1.1.1 Properties of tensor products

Tensor products are associative, i.e. if V_i, $i = 1, 2, 3$ are R-modules then there is a canonical isomorphism

$$\begin{aligned}\sigma : (V_1 \otimes_R V_2) \otimes_R V_3 &\to V_1 \otimes_R (V_2 \otimes_R V_3)\\(v_1 \otimes v_2) \otimes v_3 &\mapsto v_1 \otimes (v_2 \otimes v_3)\end{aligned}$$

where V_1, $V_1 \otimes_R V_2$ are right R-modules, $V_2 \otimes_R V_3$, V_3 are left R-modules and V_2 is both a left and right R-module. Other properties are commutativity

$$V_1 \otimes_R V_2 \cong V_2 \otimes_R V_1$$

and distributivity with respect to the direct sum

$$(V_1 \oplus V_2) \otimes_R V_3 \cong (V_1 \otimes_R V_3) \oplus (V_2 \otimes_R V_3).$$

If R is a unitary ring and V is a left unitary R-module, i.e. $1v = v$ for each $v \in V$, we also have the isomorphism

$$V \otimes_R R \cong V.$$

See [5] for proofs and details.

1.1.2 The tensor algebra

If $R = \mathbb{K}$ is a field, the R-modules V and W are $\mathbb{K}$-vector spaces. Let $\{v_i\}_{i\in I}$ and $\{w_j\}_{i\in J}$ be bases for V and W respectively: then the tensor product of V and W is a $\mathbb{K}$-vector space with the basis $\{v_i \otimes w_j\}_{i\in I, j\in J}$. We will denote it by $V \otimes_{\mathbb{K}} W$ or just $V \otimes W$ equivalently.

For each $i \geq 0$ we define $T^i(V) = V \otimes \ldots \otimes V$ to be the tensor product of i copies of V. If $i = 0$, set $T^0(V) = \mathbb{K}$. The vector space

$$T(V) = \bigoplus_{i=0}^{\infty} T^i(V)$$

is called the *tensor algebra* of V. This is a unitary associative algebra with the product

$$(x_1 \otimes \ldots \otimes x_p)(y_1 \otimes \ldots \otimes y_q) = x_1 \otimes \ldots \otimes x_p \otimes y_1 \otimes \ldots \otimes y_q$$

defined on all $x_i, y_j \in V$, $i \in \{1, \ldots, p\}$, $j \in \{1, \ldots, q\}$ and extended linearly on $T(\mathfrak{g})$. Now define J to be the (two sided) ideal in $T(V)$ generated by all elements of the form $x \otimes y - y \otimes x$ for all $x, y \in V$. The quotient algebra

$$S(V) = T(V)/J$$

is called the *symmetric algebra* of V. It is a unitary associative and commutative algebra by the definition of J. Moreover, if $\{x_i\}_{i\in I}$ is a basis for V then we have $S(V) \cong \mathbb{K}[x_i]_{i\in I}$ (see [4]).

1.1.3 Tensor product of modules over algebras

Let A be an associative $\mathbb{K}$-algebra. Modules over A are $\mathbb{K}$-vector spaces with a module structure over the ring A. If V is a right A-module and W is a left A-module, then we have two different tensor products: the $\mathbb{K}$-vector space $V \otimes W$ and the A-module $V \otimes_A W$. Moreover, we may also define the $\mathbb{K}$-vector space $A \otimes A$ and give it the obvious product

$$(a \otimes b)(c \otimes d) = (ac) \otimes (bd)$$

for all $a, b, c, d \in A$ so that $A \otimes A$ also becomes an associative algebra. Then we may regard $V \otimes W$ as an $(A \otimes A)$-module with the action

$$(a \otimes b)(v \otimes w) = (va) \otimes (bw)$$

where $a, b \in A$, $v \in V$ and $w \in W$. This might be referred to as the *external tensor product* of V and W. It will be denoted by $V \boxtimes W$ and its simple tensors will also be written as $v \boxtimes w$.

For what concerns Lie algebras, things are obviously different as there is no ring structure. If $\mathfrak{g}$ is a $\mathbb{K}$-Lie algebra, we recall that V is a (left) $\mathfrak{g}$-module if it is a $\mathbb{K}$-vector space and there is a bilinear action $\mathfrak{g} \times V \to V$ which satisfies

$$[x, y]v = x(yv) - y(xv)$$

for each $x, y \in \mathfrak{g}$ and $v \in V$. If V and W are $\mathfrak{g}$-modules then $V \otimes W$ is well-defined as a vector space. We may give it a structure of $\mathfrak{g}$-module by defining

$$x(v \otimes w) = (xv) \otimes w + v \otimes (xw)$$

for each $x \in \mathfrak{g}$, $v \in V$, $w \in W$ and extending this action linearly on $V \otimes W$. We quickly show that this formula defines a module structure on $V \otimes W$. We have

$$\begin{aligned}
[x, y]\big(v \otimes w\big) &= \big([x, y]v\big) \otimes w + v \otimes \big([x, y]w\big) \\
&= \big(x(yv)\big) \otimes w - \big(y(xv)\big) \otimes w + v \otimes \big(x(yw)\big) - v \otimes \big(y(xw)\big) \\
&= \big(x(yv)\big) \otimes w + (yv) \otimes (xw) - \big(y(xv)\big) \otimes w - (xv) \otimes (yw) \\
&\quad + (xv) \otimes (yw) + v \otimes \big(x(yw)\big) - (yv) \otimes (xw) - v \otimes \big(y(xw)\big) \\
&= x\big((yv) \otimes w\big) - y\big((xv) \otimes w\big) + x\big(v \otimes (yw)\big) - y\big(v \otimes (xw)\big) \\
&= x\big(y(v \otimes w)\big) - y\big(x(v \otimes w)\big)
\end{aligned}$$

for each $x, y \in \mathfrak{g}$, $v \in V$ and $w \in W$. The $\mathfrak{g}$-module $V \otimes W$ may be called the *internal tensor product* of V and W.

1.2 Enveloping algebras

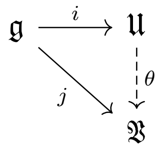
Recall that for any associative algebra A over a field $\mathbb{K}$ there exists a corresponding Lie algebra A_{Lie} which is the same vector space as A equipped with the bracket

$$[a, b] = ab - ba$$

for each $a, b \in A$. If $\varphi : A \to B$ is an associative algebra homomorphism, then $\varphi : A_{\text{Lie}} \to B_{\text{Lie}}$ is a Lie algebra homomorphism.

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$.

Definition 1.2 (Universal enveloping algebra)**.** A *universal enveloping algebra* of $\mathfrak{g}$ is a pair $(\mathfrak{U}, i)$ where $\mathfrak{U}$ is an associative unitary algebra, $i : \mathfrak{g} \to \mathfrak{U}_{\text{Lie}}$ is a Lie algebra homomorphism and the following universal property holds: for any other pair $(\mathfrak{V}, j)$ where $\mathfrak{V}$ is an associative unitary algebra and $j : \mathfrak{g} \to \mathfrak{V}_{\text{Lie}}$ is a Lie algebra homomorphism then there exists a unique associative unitary algebra homomorphism $\theta : \mathfrak{U} \to \mathfrak{V}$ such that $j = \theta \circ i$.

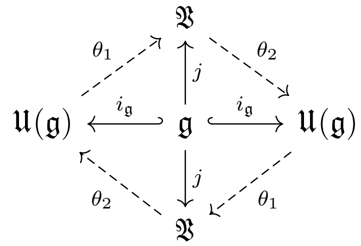


We may construct an enveloping algebra of $\mathfrak{g}$ as follows: let $T(\mathfrak{g})$ be the tensor algebra of $\mathfrak{g}$. Define $\mathfrak{J}$ as the (two sided) ideal in $T(\mathfrak{g})$ generated by all elements of the form $x \otimes y - y \otimes x - [x, y]$ for all $x, y \in \mathfrak{g}$ and set

$$\mathfrak{U}(\mathfrak{g}) = T(\mathfrak{g})/\mathfrak{J}.$$

Let $\pi : T(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$ be the canonical projection map. Write $\pi(x_1 \otimes \ldots \otimes x_p) = x_1 \ldots x_p$ for all $x_i \in \mathfrak{g}$, $i \in \{1, \ldots, p\}$ so that the associative product in $\mathfrak{U}(\mathfrak{g})$ between any two elements a and b is the juxtaposition ab. Define $i_\mathfrak{g}$ to be the restriction of π to $T^1(\mathfrak{g}) = \mathfrak{g}$. Then the pair $\big(\mathfrak{U}(\mathfrak{g}), i_\mathfrak{g}\big)$ is an universal enveloping algebra of $\mathfrak{g}$ and $i_\mathfrak{g} : \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$ is injective (see [4]).

The universal property that comes from the definition ensures that $\mathfrak{U}(\mathfrak{g})$ is unique up to isomorphism. Indeed, let $(\mathfrak{V}, j)$ be another universal enveloping algebra of $\mathfrak{g}$. We obtain unique homomorphisms $\theta_1 : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{V}$ and $\theta_2 : \mathfrak{V} \to \mathfrak{U}(\mathfrak{g})$ such that $j = \theta_1 \circ i_\mathfrak{g}$ and $i_\mathfrak{g} = \theta_2 \circ j$.



Then $i_\mathfrak{g} = (\theta_2 \circ \theta_1) \circ i_\mathfrak{g}$ and obviously also $i_\mathfrak{g} = \mathrm{id}_{\mathfrak{U}(\mathfrak{g})} \circ i_\mathfrak{g}$, thus $\theta_2 \circ \theta_1 = \mathrm{id}_{\mathfrak{U}(\mathfrak{g})}$ by the uniqueness of θ_1 and θ_2. Similarly, $\theta_1 \circ \theta_2 = \mathrm{id}_\mathfrak{V}$ and therefore θ_1 and θ_2 are algebra isomorphisms. We may then refer to the pair $\big(\mathfrak{U}(\mathfrak{g}), i_\mathfrak{g}\big)$ as the universal enveloping algebra of $\mathfrak{g}$.

Remark 1. If $\mathfrak{g}$ is abelian, then the ideal $\mathfrak{J}$ is generated by elements of the form $x \otimes y - y \otimes x$ with $x, y \in \mathfrak{g}$. Therefore $\mathfrak{U}(\mathfrak{g})$ is the symmetric algebra $S(\mathfrak{g})$.

1.2.1 Representation equivalence

Thanks to the existence of the universal enveloping algebra, we may study $\mathfrak{g}$- and $\mathfrak{U}(\mathfrak{g})$-modules equivalently. Indeed, if $\rho : \mathfrak{g} \to \mathrm{gl}(V)$ is the representation map for the $\mathfrak{g}$-module V, then by universal property there exists a unique unitary associative algebra homomorphism $\bar{\rho} : \mathfrak{U}(\mathfrak{g}) \to \mathrm{End}(V)$ such that $\rho = \bar{\rho} \circ i_\mathfrak{g}$.

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{i_{\mathfrak{g}}} & \mathfrak{U}(\mathfrak{g}) \\ & \searrow^{\rho} & \downarrow{\scriptstyle \bar{\rho}} \\ & & \mathrm{End}(V) \end{array}$$

Therefore $\bar{\rho}$ defines a $\mathfrak{U}(\mathfrak{g})$-module structure on V. Conversely, if V is a $\mathfrak{U}(\mathfrak{g})$-module and $\bar{\rho}$ is its representation map then the map $\rho = \bar{\rho} \circ i_{\mathfrak{g}}$ is a Lie algebra homomorphism which defines a $\mathfrak{g}$-module structure on V.

Let V_1, V_2 be $\mathfrak{g}$-modules and let ρ_1, ρ_2 be their representation maps.

Definition 1.3 (Representation equivalence)**.** We write $\rho_1 \cong \rho_2$ (ρ_1 is *equivalent* to ρ_2) if there exists a linear isomorphism $\varphi : V_1 \to V_2$ such that $\varphi \circ \rho_1(x) = \rho_2(x) \circ \varphi$ for each $x \in \mathfrak{g}$.

$$\begin{array}{ccc} V_1 & \xrightarrow{\rho_1(x)} & V_1 \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ V_2 & \xrightarrow{\rho_2(x)} & V_2 \end{array}$$

We may define equivalence of $\mathfrak{U}(\mathfrak{g})$-representations in the same way. Therefore, if ρ_1, ρ_2 are Lie representation maps and $\bar{\rho}_1, \bar{\rho}_2$ are their associative correspondents then we have $\rho_1 \cong \rho_2$ if and only if $\bar{\rho}_1 \cong \bar{\rho}_2$. Equivalently, if V_1 and V_2 are the $\mathfrak{g}$-modules associated to ρ_1 and ρ_2 then $V_1 \cong V_2$ as $\mathfrak{g}$-modules if and only if $V_1 \cong V_2$ as $\mathfrak{U}(\mathfrak{g})$-modules.

Remark 2. A Lie representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ is irreducible if and only if its correspondent associative representation $\bar{\rho} : \mathfrak{U}(\mathfrak{g}) \to \mathrm{gl}(V)$ is irreducible.

Indeed, assume ρ irreducible and $\bar{\rho}$ reducible. Then there exists $\{0\} \neq W \subsetneqq V$ such that $\bar{\rho}(x)W \subseteq W$ for each $x \in \mathfrak{U}(\mathfrak{g})$. But $\mathfrak{g}$ is canonically immersed into $\mathfrak{U}(\mathfrak{g})$ through $i_{\mathfrak{g}}$, therefore W is a proper $\mathfrak{g}$-submodule of V which is absurd. Conversely, assume $\bar{\rho}$ irreducible and ρ reducible. Again, there exists $\{0\} \neq W \subsetneqq V$ such that $\rho(x)W \subseteq W$ for each $x \in \mathfrak{g}$. By the construction of the enveloping algebra, each element of $\mathfrak{U}(\mathfrak{g})$ is a finite sum of elements of the form $x_1 \dots x_p$ where $x_i \in \mathfrak{g}$ for $i \in \{1, \dots, p\}$. For such monomials we may write $\rho(x_i) = \bar{\rho}(x_i)$ for each i due to the immersion of $\mathfrak{g}$ into $\mathfrak{U}(\mathfrak{g})$. Then by virtue of $\bar{\rho}$ being an associative algebra isomorphism we have

$$\bar{\rho}(x_1 \dots x_p)W = \rho(x_1) \dots \rho(x_p)W \subseteq W$$

and this is true for all monomials, which implies the absurd consequence that W is a $\mathfrak{U}(\mathfrak{g})$-submodule of V.

1.2.2 Filtered structure of $\mathfrak{U}(\mathfrak{g})$

Let A be an algebra over the field $\mathbb{K}$ and let $\mathfrak{A} = \{A_p\}_{p\in\mathbb{Z}}$ be a collection of vector subspaces of A such that $\bigcup_{p\in\mathbb{Z}} A_p = A$ and $A_p \subseteq A_{p+1}$ (or $A_p \supseteq A_{p+1}$) for each $p \in \mathbb{Z}$.

Definition 1.4 (Filtered algebra, upward and downward filtration)**.** The pair $(A, \mathfrak{A})$ is called a *filtered algebra* if $A_pA_q \subseteq A_{p+q}$ for each $p, q \in \mathbb{Z}$.

If $(A, \mathfrak{A})$ is a filtered algebra, $\mathfrak{A}$ is called an *upward filtration* if $A_p \subseteq A_{p+1}$ for each $p \in \mathbb{Z}$ and it is called a *downward filtration* if $A_p \supseteq A_{p+1}$ for each $p \in \mathbb{Z}$.

The enveloping algebra $\mathfrak{U}(\mathfrak{g})$ has a natural upward filtration given by $\mathfrak{U}_p(\mathfrak{g}) = \mathrm{span}\{y_1 \dots y_q \mid y_i \in \mathfrak{g}\ \forall i \in \{1, \dots, q\}\ ,\ q \le p\}$ for all $p \in \mathbb{Z}$. Note that $\mathfrak{U}_0(\mathfrak{g}) = \mathrm{Span}\{1\} \cong \mathbb{K}$ and $\mathfrak{U}_p(\mathfrak{g}) = \{0\}$ for $p < 0$. Set $\mathfrak{U} = \{\mathfrak{U}_p(\mathfrak{g})\}_{p\in\mathbb{Z}}$. It is easy to see that $\big(\mathfrak{U}(\mathfrak{g}), \mathfrak{U}\big)$ is indeed a filtered algebra as the condition $\mathfrak{U}_p(\mathfrak{g})\mathfrak{U}_q(\mathfrak{g}) \subseteq \mathfrak{U}_{p+q}(\mathfrak{g})$ is obvious due to the product of monomials being their juxtaposition.

Remark 3. For each $a \in \mathfrak{U}_p(\mathfrak{g})$ and $b \in \mathfrak{U}_p(\mathfrak{g})$, we have that

$$ab = ba \quad \big(\text{modulo } \mathfrak{U}_{p+q-1}(\mathfrak{g})\big).$$

Indeed, assume $a = a_1 \dots a_p$ and $b = b_1 \dots b_q$ where $a_i, b_j \in \mathfrak{g}$ for $i \in \{1, \dots, p\}$, $j \in \{1, \dots, q\}$. Then

$$\begin{aligned} ab &= a_1 \dots a_p b_1 \dots b_q = a_1 \dots a_{p-1}(a_p b_1) b_2 \dots b_q \\ &= a_1 \dots a_{p-1}(b_1 a_p) b_2 \dots b_q + a_1 \dots a_{p-1}[a_p, b_1] b_2 \dots b_q \end{aligned}$$

where clearly $a_1 \dots a_{p-1}[a_p, b_1] b_2 \dots b_q \in \mathfrak{U}_{p+q-1}(\mathfrak{g})$, hence

$$ab = a_1 \dots a_{p-1} b_1 a_p b_2 \dots b_q \quad \big(\text{modulo } \mathfrak{U}_{p+q-1}(\mathfrak{g})\big).$$

This process can be repeated any number of times in order to rearrange the terms a_i and b_j until we reach the desired conclusion.

1.2.3 PBW Theorem

The following theorem (usually abbreviated with PBW Theorem) comes in multiple forms. We give the one that is most useful in our work.

Theorem 1.2.1 (Poincaré-Birkhoff-Witt)**.** *Let I be a totally ordered set and let $\{x_i\}_{i\in I}$ be an ordered basis for $\mathfrak{g}$. Then*

$$\{x_{i_1} \dots x_{i_p} \mid i_1 \le \dots \le i_p, i_j \in I\ \forall j \in \{1, \dots, p\}, p \in \mathbb{Z}_{\ge 0}\}$$

is a basis for $\mathfrak{U}(\mathfrak{g})$.

Proof. See [4]. □

Note that the unit 1 is an element of the basis given in the above theorem, as it corresponds to the empty product obtained when $p = 0$.

Let I be a totally ordered index set such that $|I| = \dim \mathfrak{g}$. Choose $\{x_i\}_{i\in I}$ as a basis for $\mathfrak{g}$. For every $m \in (\mathbb{Z}_{\geq 0})^I$ multi-index, we can define its height as

$$|m| = \sum_{i\in I} m_i.$$

Set $|m| = \infty$ if the sum diverges. Define $M = \{m \in (\mathbb{Z}_{\geq 0})^I \mid |m| < \infty\}$, i.e. the set of multi-indexes such that $m_i = 0$ for all but a finite number of $i \in I$. If $m \in M$, we will write

$$x^m = \prod_{i\in I} {x_i}^{m_i}.$$

Since the above is a finite product and I is totally ordered, x^m is a well-defined element of $\mathfrak{U}(\mathfrak{g})$. By the PBW Theorem, $\{x^m\}_{m\in M}$ is a basis for $\mathfrak{U}(\mathfrak{g})$.

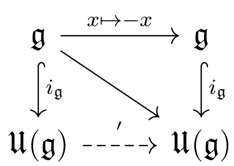
1.2.4 The main antiautomorphism

Definition 1.5 (Antihomomorphism and antiautomorphism)**.** Let A and B be algebras over the field $\mathbb{K}$. A linear map $\varphi : A \to B$ is called an *antihomomorphism* of the algebra A into the algebra B if $\varphi(xy) = \varphi(y)\varphi(x)$ for each $x, y \in A$. If $A = B$ and φ is bijective, we will say that φ is an *antiautomorphism* of the algebra A.

The above definition can be applied to any Lie algebra $\mathfrak{g}$, in which case the condition for φ will be that $\varphi([x, y]) = [\varphi(y), \varphi(x)]$. Now consider the mapping $x \mapsto -x$ of $\mathfrak{g}$ into itself. It is clearly bijective and it is also a Lie algebra antiautomorphism because

$$[x, y] \mapsto -[x, y] = [y, x] = [-y, -x]$$

for all $x, y \in \mathfrak{g}$. Note that the fundamental property of the enveloping algebra can be formulated analogously for antihomomorphisms so that this mapping extends into an associative algebra antiautomorphism of $\mathfrak{U}(\mathfrak{g})$. We will call this the *main antiautomorphism* of $\mathfrak{U}(\mathfrak{g})$ and it will be denoted by $'$.



Note that $'$ has period two, i.e. $' \circ ' = \mathrm{id}_{\mathfrak{U}(\mathfrak{g})}$. This is because $x \mapsto -x$ also has period two.

1.3 The coproduct Δ

If $\mathfrak{g}$ is a Lie algebra, let $\Delta : \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$ be the diagonal map $\Delta(x) = x \oplus x$ for each $x \in \mathfrak{g}$.

Proposition 1.3.1. *The diagonal map Δ is a Lie homomorphism.*

Proof. The bracket in $\mathfrak{g}$ is extended naturally to $\mathfrak{g} \oplus \mathfrak{g}$ by defining

$$[x_1 \oplus y_1, x_2 \oplus y_2] = [x_1, x_2] \oplus [y_1, y_2]$$

for each $x_i, y_i \in \mathfrak{g}$, $i = 1, 2$. We have

$$\begin{aligned} \Delta([x, y]) &= [x, y] \oplus [x, y] \\ [\Delta(x), \Delta(y)] &= [x \oplus x, y \oplus y] = [x, y] \oplus [x, y] \end{aligned}$$

for each $x, y \in \mathfrak{g}$. □

Thanks to the universal property of the enveloping algebra, we may extend Δ to a homomorphism $\mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g} \oplus \mathfrak{g})$ as shown below.

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\Delta} & \mathfrak{g} \oplus \mathfrak{g} \\ {\scriptstyle i_{\mathfrak{g}}}\downarrow & \searrow^{i_{\mathfrak{g}\oplus\mathfrak{g}} \circ \Delta} & \downarrow{\scriptstyle i_{\mathfrak{g}\oplus\mathfrak{g}}} \\ \mathfrak{U}(\mathfrak{g}) & \dashrightarrow & \mathfrak{U}(\mathfrak{g} \oplus \mathfrak{g}) \end{array}$$

We will still use Δ to refer to the extended map.

Proposition 1.3.2. *The enveloping algebra $\mathfrak{U}(\mathfrak{g} \oplus \mathfrak{g})$ is canonically isomorphic to $\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$.*

Proof. Let $\varphi : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$ be the linear map $\varphi(x \oplus y) = x \otimes 1 + 1 \otimes y$ for all $x, y \in \mathfrak{g}$. We have seen in §1.1.3 that the tensor product $\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$ is an associative algebra and thus it inherits the Lie algebra structure of

$\big(\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})\big)_{\mathrm{Lie}}$. We show that φ is a Lie algebra homomorphism:

$$\begin{aligned}
\varphi([x_1 \oplus y_1, x_2 \oplus y_2]) &= \varphi([x_1, x_2] \oplus [y_1, y_2]) = [x_1, x_2] \otimes 1 + 1 \otimes [y_1, y_2] \\
[\varphi(x_1 \oplus y_1), \varphi(x_2 \oplus y_2)] &= [x_1 \otimes 1 + 1 \otimes y_1, x_2 \otimes 1 + 1 \otimes y_2] \\
&= [x_1 \otimes 1, x_2 \otimes 1] + [x_1 \otimes 1, 1 \otimes y_2] + [1 \otimes y_1, x_2 \otimes 1] \\
&+ [1 \otimes y_1, 1 \otimes y_2] \\
&= x_1x_2 \otimes 1 - x_2x_1 \otimes 1 + x_1 \otimes y_2 - x_1 \otimes y_2 \\
&+ x_2 \otimes y_1 - x_2 \otimes y_1 + 1 \otimes y_1y_2 - 1 \otimes y_2y_1 \\
&= (x_1x_2 - x_2x_1) \otimes 1 + 1 \otimes (y_1y_2 - y_2y_1) \\
&= [x_1, x_2] \otimes 1 + 1 \otimes [y_1, y_2].
\end{aligned}$$

Therefore, by the universal property of the enveloping algebra φ extends to an associative algebra homomorphism θ which sends $\mathfrak{U}(\mathfrak{g}\oplus\mathfrak{g})$ into $\mathfrak{U}(\mathfrak{g})\otimes\mathfrak{U}(\mathfrak{g})$.

$$\begin{array}{ccc}
\mathfrak{g} \oplus \mathfrak{g} & \xrightarrow{i_{\mathfrak{g}\oplus\mathfrak{g}}} & U(\mathfrak{g} \oplus \mathfrak{g}) \\
& {}_{\varphi}\searrow & \downarrow \theta \\
& & U(\mathfrak{g}) \otimes U(\mathfrak{g})
\end{array}$$

If $x \otimes 1 + 1 \otimes y = 0$ then $x = y = 0$, hence θ is injective. Any element in $\mathfrak{U}(\mathfrak{g})\otimes\mathfrak{U}(\mathfrak{g})$ is a sum of elements of the form $x_1 \dots x_p \otimes y_1 \dots y_q$ with $x_i, y_j \in \mathfrak{g}$ for $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$. By observing that $\theta(x \oplus 0) = x \otimes 1$ we have

$$\begin{aligned}
\theta\big((x_1 \oplus 0) \dots (x_p \oplus 0)(0 \oplus y_1) \dots (0 \oplus y_q)\big) &= (x_1 \otimes 1) \dots (x_p \otimes 1)(1 \otimes y_1) \dots (1 \otimes y_q) \\
&= x_1 \dots x_p \otimes y_1 \dots y_q
\end{aligned}$$

because θ is an algebra homomorphism. Therefore θ is surjective and thus it is an algebra isomorphism. It is canonical by the independence a basis choice for $\mathfrak{g}$. $\square$

Through the above isomorphism, we will refer to the map

$$\Delta : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$$

as the *coproduct* of $\mathfrak{U}(\mathfrak{g})$. Note that $\Delta(1) = 1 \otimes 1$ as the extension must be a unitary associative algebra homomorphism.

Let $y_1, \dots, y_p \in \mathfrak{g}$ and define $y^m = y_1^{m_1} \dots y_p^{m_p} \in \mathfrak{U}(\mathfrak{g})$, m being the multi-index $(m_1, \dots, m_p) \in (\mathbb{Z}_{\geq 0})^p$. For each other $k = (k_1, \dots, k_p)$ multi-index, define

$$\binom{m}{k} = \binom{m_1}{k_1} \dots \binom{m_p}{k_p}.$$

Multi-indices are partially ordered in $(\mathbb{Z}_{\geq 0})^p$ in the obvious way: $k \leq m$ if $k_1 \leq m_1, \ldots, k_p \leq m_p$.

Lemma 1.3.3.
$$\Delta(y^m) = \sum_{0 \leq k \leq m} \binom{m}{k} y^k \otimes y^{m-k}.$$

Proof. First, assume $m_i = 1$ for each $i \in \{1, \ldots, p\}$. Then all binomial terms are 1. Remember that

$$\Delta(y_1 \ldots y_p) = \Delta(y_1) \ldots \Delta(y_p) = (y_1 \otimes 1 + 1 \otimes y_1) \ldots (y_p \otimes 1 + 1 \otimes y_p).$$

We show this case by induction on p. If $p = 1$, the lemma is clearly true. Now assume our result to be true for a product of p elements of $\mathfrak{g}$.

$$\begin{aligned}
\Delta(y_1 \ldots y_{p+1}) = (y_1 \otimes 1 + 1 \otimes y_1) &\sum_{\substack{0 \leq k_i \leq 1 \\ i \in \{2,\ldots,p+1\}}} {y_2}^{k_2} \ldots {y_{p+1}}^{k_{p+1}} \otimes {y_2}^{1-k_2} \ldots {y_{p+1}}^{1-k_{p+1}} \\
&= \sum_{\substack{0 \leq k_i \leq 1 \\ i \in \{2,\ldots,p+1\}}} y_1 {y_2}^{k_2} \ldots {y_{p+1}}^{k_{p+1}} \otimes {y_2}^{1-k_2} \ldots {y_{p+1}}^{1-k_{p+1}} \\
&+ \sum_{\substack{0 \leq k_i \leq 1 \\ i \in \{2,\ldots,p+1\}}} {y_2}^{k_2} \ldots {y_{p+1}}^{k_{p+1}} \otimes y_1 {y_2}^{1-k_2} \ldots {y_{p+1}}^{1-k_{p+1}} \\
&= \sum_{\substack{0 \leq k_i \leq 1 \\ i \in \{1,\ldots,p+1\}}} {y_1}^{k_1} {y_2}^{k_2} \ldots {y_{p+1}}^{k_{p+1}} \otimes {y_1}^{1-k_1} {y_2}^{1-k_2} \ldots {y_{p+1}}^{1-k_{p+1}}.
\end{aligned}$$

This proves the lemma for a product of $p+1$ elements of $\mathfrak{g}$. To get the general case, set $z_{m_1+\ldots+m_{i-1}+1} = \ldots = z_{m_1+\ldots+m_i} = y_i$ for each $i \in \{1, \ldots, p\}$ and apply the previous result to $z_1 \ldots z_{|m|}$:

$$\begin{aligned}
\Delta(z_1 \ldots z_{|m|}) &= \sum_{\substack{0 \leq k_i \leq 1 \\ i \in \{1,\ldots,|m|\}}} {z_1}^{k_1} \ldots {z_{|m|}}^{k_{|m|}} \otimes {z_1}^{1-k_1} \ldots {z_{|m|}}^{1-k_{|m|}} \\
&= \sum_{\substack{0 \leq k_i \leq 1 \\ i \in \{1,\ldots,|m|\}}} {y_1}^{k_1} \ldots {y_1}^{k_{m_1}} \ldots {y_p}^{k_{(m_1+\ldots+m_{p-1}+1)}} \ldots {y_p}^{k_{|m|}} \\
&\otimes {y_1}^{1-k_1} \ldots {y_1}^{1-k_{m_1}} \ldots {y_p}^{1-k_{(m_1+\ldots+m_{p-1}+1)}} \ldots {y_p}^{1-k_{|m|}} \\
&= \sum_{\substack{0 \leq k_i \leq 1 \\ i \in \{1,\ldots,|m|\}}} {y_1}^{k_1+\ldots+k_{m_1}} \ldots {y_p}^{k_{(m_1+\ldots+m_{p-1}+1)}+\ldots+k_{|m|}} \\
&\otimes {y_1}^{1-(k_1+\ldots+k_{m_1})} \ldots {y_p}^{1-\left(k_{(m_1+\ldots+m_{p-1}+1)}+\ldots+k_{|m|}\right)}.
\end{aligned}$$

Now let $h_i = k_{(m_1+\ldots+m_{i-1}+1)} + \ldots + k_{(m_1+\ldots+m_i)}$ for each $i \in \{1,\ldots,p\}$. Note that $h_i \in \{0,\ldots,m_i\}$. Counting the terms in the sum that contain ${y_i}^{h_i}$, we observe that they are $\binom{m_i}{h_i}$ in number. Collecting them in the above formula proves the lemma:

$$\begin{aligned}\Delta({y_1}^{m_1}\ldots{y_p}^{m_p}) &= \sum_{\substack{0\le k_i\le 1\\ i\in\{1,\ldots,|m|\}}} {y_1}^{h_1}\ldots{y_p}^{h_p}\otimes{y_1}^{1-h_1}\ldots{y_p}^{1-h_p}\\ &= \sum_{\substack{0\le h_i\le m_i\\ i\in\{1,\ldots,p\}}} \binom{m_1}{h_1}\cdots\binom{m_p}{h_p}{y_1}^{h_1}\ldots{y_p}^{h_p}\otimes{y_1}^{1-h_1}\ldots{y_p}^{1-h_p}.\end{aligned}$$

□

Remark 4. By setting $h = m - k$, we may reformulate this identity as

$$\Delta(y^m) = \sum_{\substack{0\le h,k\le m\\ h+k=m}} \frac{m!}{h!\,k!}\, y^h \otimes y^k$$

where $m! = m_1!\ldots m_p!$ and similarly for $h!$ and $k!$, consistently with the previous definition of the binomial product for multi-indexes.

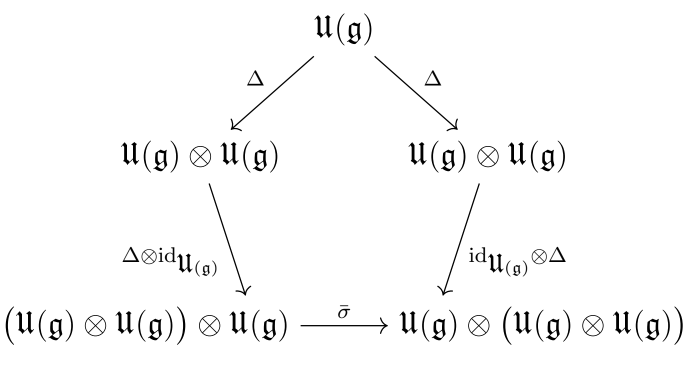
Proposition 1.3.4. *The coproduct Δ is coassociative.*

Proof. Coassociativity for Δ means that

$$\left(\mathrm{id}_{\mathfrak{U}(\mathfrak{g})}\otimes\Delta\right)\circ\Delta = \bar\sigma\circ\left(\Delta\otimes\mathrm{id}_{\mathfrak{U}(\mathfrak{g})}\right)\circ\Delta$$

where $\bar\sigma$ denotes the canonical isomorphism

$$\bar\sigma : \left(\mathfrak{U}(\mathfrak{g})\otimes\mathfrak{U}(\mathfrak{g})\right)\otimes\mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})\otimes\left(\mathfrak{U}(\mathfrak{g})\otimes\mathfrak{U}(\mathfrak{g})\right).$$



We can use the barycentric formula written in Remark 4 to show this fact. Choose $\{x_i\}_{i\in I}$ basis for $\mathfrak{g}$ and define M, x^m for $m \in M$ as in §1.2.3 so that $\{x^m\}_{m\in M}$ is a basis for $\mathfrak{U}(\mathfrak{g})$. Then

$$\begin{aligned}
\big(\bar\sigma \circ (\Delta \otimes \mathrm{id}_{\mathfrak{U}(\mathfrak{g})}) \circ \Delta\big)(x^m) &= \big(\bar\sigma \circ (\Delta \otimes \mathrm{id}_{\mathfrak{U}(\mathfrak{g})})\big) \left(\sum_{\substack{0\le h,k\le m\\ h+k=m}} \frac{m!}{h!\ k!} x^h \otimes x^k \right)\\
&= \bar\sigma \left(\sum_{\substack{0\le h,k\le m\\ h+k=m}} \frac{m!}{h!\ k!} \Delta(x^h) \otimes x^k \right)\\
&= \bar\sigma \left(\sum_{\substack{0\le h,k\le m\\ h+k=m}} \frac{m!}{h!\ k!} \left(\sum_{\substack{0\le i,j\le h\\ i+j=h}} \frac{h!}{i!\ j!} x^i \otimes x^j \right) \otimes x^k \right)\\
&= \bar\sigma \left(\sum_{\substack{0\le i,j,k\le m\\ i+h+k=m}} \frac{m!}{i!\ j!\ k!} (x^i \otimes x^j) \otimes x^k \right)\\
&= \sum_{\substack{0\le i,j,k\le m\\ i+j+k=m}} \frac{m!}{i!\ j!\ k!} x^i \otimes (x^j \otimes x^k)
\end{aligned}$$

and

$$\begin{aligned}
\big((\mathrm{id}_{\mathfrak{U}(\mathfrak{g})} \otimes \Delta) \circ \Delta\big)(x^m) &= (\mathrm{id}_{\mathfrak{U}(\mathfrak{g})} \otimes \Delta) \left(\sum_{\substack{0\le h,k\le m\\ h+k=m}} \frac{m!}{h!\ k!} x^h \otimes x^k \right) =\\
&= \sum_{\substack{0\le h,k\le m\\ h+k=m}} \frac{m!}{h!\ k!} x^h \otimes \Delta(x^k)\\
&= \sum_{\substack{0\le h,k\le m\\ h+k=m}} \frac{m!}{h!\ k!} x^h \otimes \left(\sum_{\substack{0\le i,j\le k\\ i+j=k}} \frac{k!}{i!\ j!} x^i \otimes x^j \right)\\
&= \sum_{\substack{0\le i,j,h\le m\\ i+j+h=m}} \frac{m!}{i!\ j!\ h!} x^i \otimes (x^j \otimes x^h).
\end{aligned}$$

□

One may find in literature that $\big(\mathfrak{U}(\mathfrak{g}), \Delta\big)$ is called a *coalgebra* if Δ is coassociative and that $\big(\mathfrak{U}(\mathfrak{g}), \cdot, \Delta\big)$ is called a *bialgebra* if $\big(\mathfrak{U}(\mathfrak{g}), \cdot\big)$ is an algebra, $\big(\mathfrak{U}(\mathfrak{g}), \Delta\big)$ is a coalgebra and Δ is an algebra homomorphism.

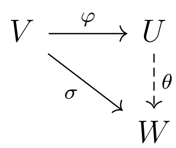
Chapter 2

Induction and production

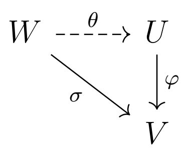
We introduce the concepts of induced and produced representations. From now on, the notation for modules will be left-sided unless otherwise specified. Everything said in §1.1 still holds regardless.

Let $\mathbb{K}$ be a field and let A be an associative, unitary $\mathbb{K}$-algebra. Let B be a unitary subalgebra of A and let V be a unitary B-module.

Definition 2.1 (Induced pair)**.** A pair (U, φ) where U is a unitary A-module and $\varphi : V \to U$ is a B-homomorphism is said to be *induced* from V if the following universal property holds: for every other pair (W, σ) as above then there exists a unique A-homomorphism $\theta : U \to W$ such that $\sigma = \theta \circ \varphi$.



Definition 2.2 (Produced pair)**.** A pair (U, φ) where U is a unitary A-module and $\varphi : U \to V$ is a B-homomorphism is said to be *produced* from V if the following universal property holds: for every other pair (W, σ) as above then there exists a unique A-homomorphism $\theta : W \to U$ such that $\sigma = \varphi \circ \theta$.



Induced (respectively produced) pairs exist and are unique up to isomorphism (see [2]), thus we may refer to them as the pair induced (respectively produced) from V. To construct the induced pair (U, φ), regard A as a right

B-module: we have $U = A \otimes_B V$ and $\varphi(v) = 1 \otimes v$ for all $v \in V$ as our pair. U is an A-module with the action

$$a(b \otimes v) = (ab) \otimes v$$

for every $a, b \in A$ and $v \in V$, extended linearly on all $u \in U$. For another pair (W, σ), θ is defined by $\theta(b \otimes v) = b\,\sigma(v)$ for every $b \in A, v \in V$. For produced pairs, we regard A as a left B-module instead and thus $U = \mathrm{Hom}_B(A, V)$, $\varphi(u) = u(1)$ for all $u \in U$ is our pair. The action on U is given by

$$(au)(b) = u(ba)$$

for all $b \in A$ and, for a pair (W, σ), we have $\big(\theta(w)\big)(b) = \sigma(bw)$ for all $w \in W$ and $b \in A$. See [2] for proofs and details on the construction of these modules.

We can extend these definitions to a $\mathbb{K}$-Lie algebra $\mathfrak{g}$ with a subalgebra $\mathfrak{h}$, obtaining equivalent notions for Lie-modules induced and produced from an $\mathfrak{h}$-module V. Moreover, let $\mathfrak{U}(\mathfrak{g})$ and $\mathfrak{U}(\mathfrak{h})$ be the universal enveloping algebras of $\mathfrak{g}$ and $\mathfrak{h}$ respectively, and regard $\mathfrak{U}(\mathfrak{h})$ as a unitary subalgebra of $\mathfrak{U}(\mathfrak{g})$: since V is an $\mathfrak{h}$-module, it is also a $\mathfrak{U}(\mathfrak{h})$-module by the universal property of the enveloping algebra, hence we know that the representations induced and produced from V are $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{h})} V$ and $\mathrm{Hom}_{\mathfrak{U}(\mathfrak{h})}(\mathfrak{U}(\mathfrak{g}), V)$ respectively, as seen above. We will denote them by $\mathcal{I}_{\mathfrak{h}}^{\mathfrak{g}}(V)$ and $\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$.

Induced and produced Lie representations are intertwined through duality. For any $\mathfrak{g}$-module W, let W^* be its contragradient $\mathfrak{g}$-module. Recall that $W^* = \mathrm{Hom}_{\mathbb{K}}(W, \mathbb{K})$ is given the action

$$(x\psi)(w) = -\psi(xw)$$

for all $x \in \mathfrak{g}$, $\psi \in W^*$ and $w \in W$. We may reformulate the above as $(x\psi)(w) = \psi(x'w)$, where $'$ is the main antiautomorphism introduced in §1.2.4 and it coincides with the map $x \mapsto -x$ on $\mathfrak{g}$. Then by extending this action on $\mathfrak{U}(\mathfrak{g})$ we obtain that

$$(a\psi)(w) = \psi(a'w)$$

for all $a \in \mathfrak{U}(\mathfrak{g})$, $\psi \in W^*$ and $w \in W$.

Proposition 2.0.1. *Let V be an h-module. Then $\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V^*) \cong \big(\mathcal{I}_{\mathfrak{h}}^{\mathfrak{g}}(V)\big)^*$.*

Proof. Realize $\mathcal{I}_{\mathfrak{h}}^{\mathfrak{g}}(V)$ as $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{h})} V$ and $\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V^*)$ as $\mathrm{Hom}_{\mathfrak{U}(\mathfrak{h})}(\mathfrak{U}(\mathfrak{g}), V^*)$. Let $\psi \in \big(\mathcal{I}_{\mathfrak{h}}^{\mathfrak{g}}(V)\big)^* = \big(\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{h})} V\big)^*$. For each $a \in \mathfrak{U}(\mathfrak{g})$ define a map $\hat{\psi}(a) : V \to \mathbb{K}$ by

$$\big(\hat{\psi}(a)\big)(v) = \psi(a' \otimes v)$$

for all $v \in V$. By the linearity of ψ and the bilinear properties of the tensor product, $\hat{\psi}(a) \in V^*$ and $\hat{\psi} \in \mathrm{Hom}_{\mathbb{K}}(\mathfrak{U}(\mathfrak{g}), V^*)$. Now let $z \in \mathfrak{U}(\mathfrak{h})$. The restriction of $'$ to $\mathfrak{U}(\mathfrak{h})$ is the main antiautomorphism of $\mathfrak{U}(\mathfrak{h})$, therefore

$$\begin{aligned}\big(\hat{\psi}(za)\big)(v) &= \psi\big((za)' \otimes v\big) = \psi(a'z' \otimes v) = \psi(a' \otimes z'v) = \big(\hat{\psi}(a)\big)(z'v)\\ &= \big(z\,\hat{\psi}(a)\big)(v)\end{aligned}$$

because $\hat{\psi}(a) \in V^*$. Hence $\hat{\psi} \in \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V^*)$. The map

$$\begin{aligned}\widehat{\ }: \big(\mathcal{I}^{\mathfrak{g}}_{\mathfrak{h}}(V)\big)^* &\to \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V^*)\\ \psi &\mapsto \hat{\psi}\end{aligned}$$

is clearly linear by definition of sum and scalar multiplication in the Hom spaces. If $b \in \mathfrak{U}(\mathfrak{g})$ and $\psi \in \big(\mathcal{I}^{\mathfrak{g}}_{\mathfrak{h}}(V)\big)^*$, we have

$$\begin{aligned}\big((\widehat{b\psi})(a)\big)(v) &= (b\psi)(a' \otimes v) = \psi\big(b'(a' \otimes v)\big) = \psi\big((ab)' \otimes v\big)\\ &= \big(\hat{\psi}(ab)\big)(v) = \big((b\,\hat{\psi})(a)\big)(v)\end{aligned}$$

for each $a \in \mathfrak{U}(\mathfrak{g})$ and $v \in V$. This implies that $\widehat{\ }$ is a $\mathfrak{U}(\mathfrak{g})$-homomorphism.

If $\hat{\psi} = 0$, then ψ vanishes on a set of generators for $\mathcal{I}^{\mathfrak{g}}_{\mathfrak{h}}(V)$ and therefore $\psi = 0$. So $\widehat{\ }$ is injective. Now let $\zeta \in \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V^*)$. Define $\xi : \mathfrak{U}(\mathfrak{g}) \times V \to \mathbb{K}$ by $\xi(a, v) = \big(\zeta(a')\big)(v)$ for all $a \in \mathfrak{U}(\mathfrak{g})$ and $v \in V$. The map ξ is bilinear by the linearity of ζ and $\zeta(a')$. Moreover, for all $z \in \mathfrak{U}(\mathfrak{h})$ it is

$$\begin{aligned}\xi(az, v) &= \big(\zeta(z'a')\big)(v) = \big(z'\zeta(a')\big)(v) = \big(\zeta(a')\big)(zv)\\ &= \xi(a, zv)\end{aligned}$$

because $'$ has period two. This implies that ξ is $\mathfrak{U}(\mathfrak{h})$-bilinear. By the universal property of the tensor product, there exists a unique linear map $\psi : \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{h})} V \to \mathbb{K}$ such that $\psi(a \otimes v) = \xi(a, v) = \big(\zeta(a')\big)(v)$ for all $a \in \mathfrak{U}(\mathfrak{g})$ and $v \in V$.

$$\begin{array}{ccc}\mathfrak{U}(\mathfrak{g}) \times V & \xrightarrow{\ \otimes\ } & \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{h})} V\\ & \searrow^{\xi} & \downarrow{\scriptstyle \psi}\\ & & \mathbb{K}\end{array}$$

Since $\psi(a \otimes v) = \big(\hat{\psi}(a')\big)(v)$ then $\hat{\psi} = \zeta$, therefore $\widehat{\ }$ is surjective. This proves that $\widehat{\ }$ is an $\mathfrak{U}(\mathfrak{g})$-isomorphism. $\square$

2.1 Filtered structure of $\mathcal{I}_{\mathfrak{h}}^{\mathfrak{g}}(V)$ and $\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$

In §1.2.2, we have seen that the enveloping algebra $\mathfrak{U}(\mathfrak{g})$ has a natural upward filtration $\mathfrak{U} = \{\mathfrak{U}_p(\mathfrak{g})\}_{p\in\mathbb{Z}}$. This filtration gives rise to an upward filtration for $\mathcal{I}_{\mathfrak{h}}^{\mathfrak{g}}(V)$ and a downward one for $\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$: we have

$$\mathcal{I}_{\mathfrak{h}}^{\mathfrak{g}}(V)_p = \mathrm{Span}\{a \otimes v \mid a \in \mathfrak{U}_p(\mathfrak{g}),\ v \in V\}$$
$$\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)_p = \{u \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V) \mid u|_{\mathfrak{U}_{p-1}(\mathfrak{g})} = 0\}.$$

Now consider the quotient space $\mathfrak{g}/\mathfrak{h}$ and regard it as an abelian Lie algebra, so that $\mathfrak{U}(\mathfrak{g}/\mathfrak{h})$ is the symmetric algebra $S(\mathfrak{g}/\mathfrak{h})$. The singleton $\{0\}$ is obviously a subalgebra of $\mathfrak{g}/\mathfrak{h}$ and its action on V is trivial, thus $\mathfrak{U}(\{0\})$ acts on V like the scalar multiplication (the action is unitary). Replacing A with $S(\mathfrak{g}/\mathfrak{h})$ and B with $\mathfrak{U}(\{0\}) = \mathbb{K}$ in our definitions, we obtain that $S(\mathfrak{g}/\mathfrak{h}) \otimes_{\mathbb{K}} V$ and $\mathrm{Hom}_{\mathbb{K}}(S(\mathfrak{g}/\mathfrak{h}), V)$ are, respectively, the modules induced and produced by the $\{0\}$-module V, so we have

$$\mathcal{I}_{\{0\}}^{\mathfrak{g}/\mathfrak{h}}(V) = S(\mathfrak{g}/\mathfrak{h}) \otimes_{\mathbb{K}} V$$
$$\mathcal{P}_{\{0\}}^{\mathfrak{g}/\mathfrak{h}}(V) = \mathrm{Hom}_{\mathbb{K}}(S(\mathfrak{g}/\mathfrak{h}), V).$$

We may give them the filtrations seen above.

Let I be a totally ordered index set such that $|I| = \dim(\mathfrak{g}/\mathfrak{h})$. Choose $\{\bar{x}_i\}_{i\in I}$ basis for $\mathfrak{g}/\mathfrak{h}$ and, for every $i \in I$, choose a class representative $x_i \in \bar{x}_i \subseteq \mathfrak{g}$. Define M, x^m for $m \in M$ as in §1.2.3 and define $\bar{x}^m \in S(\mathfrak{g}/\mathfrak{h})$ analogously to x^m. By the PBW Theorem, $\{\bar{x}^m\}_{m\in M}$ is a basis for $S(\mathfrak{g}/\mathfrak{h})$ and thus we can define $\tau : S(\mathfrak{g}/\mathfrak{h}) \to \mathfrak{U}(\mathfrak{g})$ as the linear map $\bar{x}^m \mapsto x^m$. Let's consider the following maps

$$\iota : S(\mathfrak{g}/\mathfrak{h}) \otimes_{\mathbb{K}} V \to \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{h})} V$$
$$a \otimes v \mapsto \tau(a) \otimes v$$
$$\pi : \mathrm{Hom}_{\mathfrak{U}(\mathfrak{h})}(\mathfrak{U}(\mathfrak{g}), V) \to \mathrm{Hom}_{\mathbb{K}}(S(\mathfrak{g}/\mathfrak{h}), V)$$
$$(\pi u)(a) = u\big(\tau(a)\big)$$

where $a \in S(\mathfrak{g}/\mathfrak{h})$, $v \in V$ and $u \in \mathrm{Hom}_{\mathfrak{U}(\mathfrak{h})}(\mathfrak{U}(\mathfrak{g}), V)$. Note that

$$\iota : \mathcal{I}_{\{0\}}^{\mathfrak{g}/\mathfrak{h}}(V) \to \mathcal{I}_{\mathfrak{h}}^{\mathfrak{g}}(V)$$
$$\pi : \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V) \to \mathcal{P}_{\{0\}}^{\mathfrak{g}/\mathfrak{h}}(V)$$

hence these are maps between filtered spaces as seen earlier.

Proposition 2.1.1. *The maps ι and π are filtration preserving linear isomorphisms.*

Proof. The map ι is defined by linear extension on $S(\mathfrak{g}/\mathfrak{h}) \otimes_{\mathbb{K}} V$ and π is clearly linear from the notion of sum and scalar multiplication in the produced module $\mathrm{Hom}_{\mathfrak{U}(\mathfrak{h})}(\mathfrak{U}(\mathfrak{g}), V)$.

The injectivity of ι comes from the obvious injectivity of τ. For the surjectivity, consider $\mathfrak{g} \cong \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{h}$. Complete $\{x_i\}_{i\in I}$ to a basis of $\mathfrak{g}$ and call this completion $\{x_i, y_j\}_{i\in I, j\in J}$, where $\{y_j\}_{j\in J}$ is a basis for $\mathfrak{h}$ and J is a totally ordered index set. We order $I \cup J$ by saying that $i < j$ for all $i \in I$, $j \in J$. Thanks to the PBW Theorem, we know that a basis for $\mathfrak{U}(\mathfrak{g})$ is given by $\{x^m y^n\}_{m\in M, n\in N}$ where $N = \{n \in (\mathbb{Z}_{\geq 0})^J \mid |n| < \infty\}$, $|n|$ and y^n being defined analogously to $|m|$ and x^m. $\{y^n\}_{n\in N}$ is a basis for $\mathfrak{U}(\mathfrak{h})$, thus $\{x^m\}_{m\in M}$ is a basis for $\mathfrak{U}(\mathfrak{g})$ as a $\mathfrak{U}(\mathfrak{h})$-module. This proves that ι is surjective, as $x^m y^n \otimes v = x^m \otimes y^n v = \iota(\bar{x}^m \otimes y^n v)$. Thus, ι is an isomorphism.

To prove the same for π, assume $\pi u = 0$: then $(\pi u)(a) = u(\tau(a)) = 0$ for all $a \in S(\mathfrak{g}/\mathfrak{h})$, i.e. $\tau(S(\mathfrak{g}/\mathfrak{h})) \subseteq \ker u$. Since $\{x^m\}_{m\in M}$ is a basis for the left $\mathfrak{U}(\mathfrak{h})$-module $\mathfrak{U}(\mathfrak{g})$, for all $z \in \mathfrak{U}(\mathfrak{g})$ we may write $z = \sum_{m\in M} y_m x^m$, $y_m \in \mathfrak{U}(\mathfrak{h})$ and only a finite amount of them being non-zero. Since u is a $\mathfrak{U}(\mathfrak{h})$-homomorphism, we have

$$u(z) = \sum_{m\in M} y_m u(x^m) = \sum_{m\in M} y_m u(\tau(\bar{x}^m)) = 0$$

which means $u = 0$ and π is injective.
Let $\varphi \in \mathrm{Hom}_{\mathbb{K}}(S(\mathfrak{g}/\mathfrak{h}), V)$ be a linear map such that $\bar{x}^m \mapsto v_m$. Let $u \in \mathrm{Hom}_{\mathfrak{U}(\mathfrak{h})}(\mathfrak{U}(\mathfrak{g}), V)$ be the map $x^m \mapsto v_m$, extended to $\mathfrak{U}(\mathfrak{g})$ as an $\mathfrak{U}(\mathfrak{h})$-homomorphism. Then $(\pi u)(\bar{x}^m) = u(\tau(\bar{x}^m)) = u(x^m) = v_m$, which means $\pi u = \varphi$. Therefore, π is an isomorphism.

We will now prove that ι preserves the filtration. It is clear that $\iota\big(\mathcal{I}_{\{0\}}^{\mathfrak{g}/\mathfrak{h}}(V)_p\big) \subseteq \mathcal{I}_{\mathfrak{h}}^{\mathfrak{g}}(V)_p$. Let $u \in \mathcal{I}_{\mathfrak{h}}^{\mathfrak{g}}(V)_p$. From the PBW Theorem, it follows that

$$\begin{aligned} u &= \sum_{\substack{m\in M,\ n\in N \\ |m|+|n|\leq p}} x^m y^n \otimes v_{m,n} = \sum_{\substack{m\in M,\ n\in N \\ |m|+|n|\leq p}} x^m \otimes y^n v_{m,n} \\ &= \sum_{\substack{m\in M \\ |m|\leq p}} x^m \otimes \sum_{\substack{n\in N \\ |n|\leq p-|m|}} y^n v_{m,n} \\ &= \sum_{\substack{m\in M \\ |m|\leq p}} \tau(\bar{x}^m) \otimes \sum_{\substack{n\in N \\ |n|\leq p-|m|}} y^n v_{m,n} \in \iota(\mathcal{I}_{\{0\}}^{\mathfrak{g}/\mathfrak{h}}(V)_p) \end{aligned}$$

where $v_{m,n} \in V$ and all zero except for a finite amount. Hence $\mathcal{I}_{\mathfrak{h}}^{\mathfrak{g}}(V)_p \subseteq \iota\big(\mathcal{I}_{\{0\}}^{\mathfrak{g}/\mathfrak{h}}(V)_p\big)$.

As for π, let $u \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)_p$ and $a \in S_{p-1}(\mathfrak{g}/\mathfrak{h})$. Clearly $\tau(a) \in \mathfrak{U}_{p-1}(\mathfrak{g})$, so we have $(\pi u)(a) = u(\tau(a)) = 0$ which means $\pi\big(\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)_p\big) \subseteq \mathcal{P}_{\{0\}}^{\mathfrak{g}/\mathfrak{h}}(V)_p$.
Let $\varphi \in \mathcal{P}_{\{0\}}^{\mathfrak{g}/\mathfrak{h}}(V)_p$ be a linear map $\varphi(\bar{x}^m) = v_m \in V$. We know that $v_m = 0$ if $|m| \leq p-1$. By setting $u(x^m) = v_m$ for each $m \in M$ and extending u as an $\mathfrak{U}(\mathfrak{h})$-homomorphism of $\mathfrak{U}(\mathfrak{g})$ into V, we showed that $\pi u = \varphi$. For $|m| \leq p-1$, $u(x^m) = 0$; since $\{x^m \mid |m| \leq p-1\}_{m\in M}$ is a $\mathfrak{U}(\mathfrak{h})$-basis for $\mathfrak{U}_{p-1}(\mathfrak{g})$, it follows that $u \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)_p$. This implies $\mathcal{P}_{\{0\}}^{\mathfrak{g}/\mathfrak{h}}(V)_p \subseteq \pi\big(\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)_p\big)$. □

2.1.1 Filtration topology in $\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$

The filtration $\{\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)_p\}_{p\in\mathbb{Z}}$ induces a topology on the vector space $\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$ by providing a basis of open neighbourhoods of 0; neighbourhoods of any $u \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$ are defined through translation. Similarly, the enveloping algebra also has the filtration topology where the neighbourhoods of 0 are $\mathfrak{U}_p(\mathfrak{g})$, but since $\mathfrak{U}_p(\mathfrak{g}) = \{0\}$ for all $p < 0$ this coincides with the discrete topology. If V is given the structure of a topological vector space, then we may also consider the finite-open topology on $\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$ defined through the basis $\{\mathcal{U}(F,A) \neq \emptyset \mid F \subseteq \mathfrak{U}(\mathfrak{g}), |F| < \infty, A \subseteq V \text{ open}\}$ where $\mathcal{U}(F,A) = \{u \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V) \mid u(F) \subseteq A\}$. Regard V as a discrete topological space.

Proposition 2.1.2. *The finite-open topology is weaker than the filtration topology on $\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$. If $\dim(\mathfrak{g}/\mathfrak{h}) < \infty$ then the filtration and finite-open topologies coincide.*

Proof. To prove that the finite-open topology is weaker than the filtration topology, we show that for any F finite subset of $\mathfrak{U}(\mathfrak{g})$ and any $A \subseteq V$ such that $\mathcal{U}(F,A) \neq \emptyset$ there exists an open neighbourhood in the filtration topology contained into $\mathcal{U}(F,A)$. Let $p = \min\{m \geq -1 \mid F \subset \mathfrak{U}_m(\mathfrak{g})\}$ and let $u \in \mathcal{U}(F,A)$. Then $u + \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)_{p+1} \subseteq \mathcal{U}(F,A)$. Now assume $\dim(\mathfrak{g}/\mathfrak{h}) < \infty$ and let $p \in \mathbb{Z}$. Let F be a basis for $\mathfrak{U}_{p-1}(\mathfrak{g})$ as an $\mathfrak{U}(\mathfrak{h})$-module (choose $F = \{0\}$ if $p \leq 0$) and let $A = \{0\}$. Since $\dim(\mathfrak{g}/\mathfrak{h}) < \infty$, the basis F is finite. Then we have $\mathcal{U}(F,A) \subseteq \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)_p$ and therefore the filtration topology is weaker than the finite-open topology. □

Proposition 2.1.3. *The $\mathfrak{g}$-module $\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$ is filtration complete.*

Proof. Convergence for a sequence $\{u_n\}_{n\geq 0}$ in the filtration topology of $\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$ is obtained by saying that $\{u_n\}_{n\geq 0}$ converges to 0 if for every $p \in \mathbb{Z}$ there exists $N_p \geq 0$ such that $u_n \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)_p$ for all $n \geq N_p$. Convergence to any element $u \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$ is defined through the sequence $\{u - u_n\}_{n\geq 0}$ and the notion of a Cauchy sequence is also given analogously. We may write $u_n \xrightarrow{n\to\infty} u$ as is standard to denote convergence. Now we show that $\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$ is complete with respect to this convergence. Let $\{u_n\}_{n\geq 0}$ be a Cauchy sequence in $\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$. Then for every $p \geq 0$ there exists $N_p \geq 0$ such that $u_n - u_m \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)_{p+1}$ for each $n, m \geq N_p$. This implies $u_n(a) = u_m(a)$ for all $a \in \mathfrak{U}_p(\mathfrak{g})$ and $n, m \geq N_p$; in other words, for each $a \in \mathfrak{U}_p(\mathfrak{g})$ the sequence $\{u_n(a)\}_{n\geq N_p}$ is stationary in V. Therefore we can define $u(a) = u_{N_p}(a)$ for all $a \in \mathfrak{U}_p(\mathfrak{g})$. Note that if $p_1 \leq p_2$ we may choose N_{p_2} to satisfy $N_{p_1} \leq N_{p_2}$, so that if $a \in \mathfrak{U}_{p_1}(\mathfrak{g}) \subseteq \mathfrak{U}_{p_2}(\mathfrak{g})$ we have $u_{N_{p_1}}(a) = u_{N_{p_2}}(a)$. We may then repeat the process for all p to obtain a well-defined function $u : \mathfrak{U}(\mathfrak{g}) \to V$. Since u is defined in terms of u_n it also holds all of its properties, hence u is a $\mathfrak{U}(\mathfrak{h})$-homomorphism which means $u \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$. Now consider the sequence $\{u - u_n\}_{n\geq 0}$: for any $p \geq 0$, there exists $N_p \geq 0$ such that $u_n(a) = u(a)$ for all $a \in \mathfrak{U}_p(\mathfrak{g})$ and $n \geq N_p$, which implies $u_n - u \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)_{p+1}$ for all $n \geq N_p$. Therefore $\{u_n\}_{n\in\mathbb{Z}_{>0}}$ converges to $u \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$ and $\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$ is filtration complete. □

2.2 Multiplicative structure of produced representations

By 1.3.2, we can treat $\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$ and $\mathfrak{U}(\mathfrak{g} \oplus \mathfrak{g})$ representations interchangeably. Let V_1 and V_2 be $\mathfrak{h}$-modules and thus $\mathfrak{U}(\mathfrak{h})$-modules. Now consider the tensor product of these vector spaces: as seen in §1.1.3, it has multiple module structures. Indeed, we may consider the external tensor product $V_1 \boxtimes V_2$, which is a $\mathfrak{U}(\mathfrak{h}) \otimes \mathfrak{U}(\mathfrak{h})$-module with the action

$$(a \otimes b)(v \boxtimes w) = (av) \boxtimes (bw)$$

for all $a, b \in \mathfrak{U}(\mathfrak{h})$, $v \in V_1$ and $w \in V_2$. However, there is also the internal tensor product $V_1 \otimes V_2$, which is an $\mathfrak{h}$-module with the action

$$h(v \otimes w) = (hv) \otimes w + v \otimes (hw)$$

for all $h \in \mathfrak{h}$, $v \in V_1$ and $w \in V_2$. It is possible to establish a relation between these representations as follows.

Proposition 2.2.1. $h(v \otimes w) = \Delta(h)\big(v \boxtimes w\big)$ *for each* $h \in \mathfrak{h},\ v \in V_1,\ w \in V_2$.

Proof. By Lemma 1.3.3 we immediately see that

$$\Delta(h) = h \otimes 1 + 1 \otimes h \in \mathfrak{U}(\mathfrak{h}) \otimes \mathfrak{U}(\mathfrak{h})$$

for all $h \in \mathfrak{h}$. Hence

$$\begin{aligned}\Delta(h)\big(v \boxtimes w\big) &= (h \otimes 1 + 1 \otimes h)\big(v \boxtimes w\big) = (hv) \boxtimes w + v \boxtimes (hw)\\ &= (hv) \otimes w + v \otimes (hw) = h(v \otimes w)\end{aligned}$$

remembering that $V_1 \boxtimes V_2$ and $V_1 \otimes V_2$ are the same vector space. □

2.2.1 The external multiplication $\boxtimes$

Let $u_i \in \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_i)$ for $i = 1, 2$. $\mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_1)$ and $P^{\mathfrak{g}}_{\mathfrak{h}}(V_2)$ are produced by V_1 and V_2 respectively, hence they are $\mathfrak{U}(\mathfrak{g})$-modules and $\mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_1) \boxtimes \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_2)$ is a $\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$-module. $u_1 \boxtimes u_2$ is an element of $P^{\mathfrak{g}}_{\mathfrak{h}}(V_1) \boxtimes P^{\mathfrak{g}}_{\mathfrak{h}}(V_2)$. Now consider

$$\mathcal{P}^{\mathfrak{g}\oplus\mathfrak{g}}_{\mathfrak{h}\oplus\mathfrak{h}}(V_1 \boxtimes V_2) = \mathrm{Hom}_{\mathfrak{U}(\mathfrak{h}\oplus\mathfrak{h})}(\mathfrak{U}(\mathfrak{g} \oplus \mathfrak{g}), V_1 \boxtimes V_2).$$

This is the $(\mathfrak{g}\oplus\mathfrak{g})$-module produced by $V_1 \boxtimes V_2$, hence it is also a $\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$-module. Define

$$\overline{u_1 \boxtimes u_2} : \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g}) \to V_1 \boxtimes V_2$$

as the linear map $(\overline{u_1 \boxtimes u_2})(a \otimes b) = u_1(a) \boxtimes u_2(b)$ for each $a, b \in \mathfrak{U}(\mathfrak{g})$. This is clearly a $\mathfrak{U}(\mathfrak{h}) \otimes \mathfrak{U}(\mathfrak{h})$-homomorphism, hence $\overline{u_1 \boxtimes u_2} \in \mathcal{P}^{\mathfrak{g}\oplus\mathfrak{g}}_{\mathfrak{h}\oplus\mathfrak{h}}(V_1 \boxtimes V_2)$. By the fundamental property of the tensor product, the bilinear map $(u_1, u_2) \mapsto \overline{u_1 \boxtimes u_2}$ gives rise to a linear map

$$\overline{} : \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_1) \boxtimes \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_2) \to \mathcal{P}^{\mathfrak{g}\oplus\mathfrak{g}}_{\mathfrak{h}\oplus\mathfrak{h}}(V_1 \boxtimes V_2)$$

which sends $u_1 \boxtimes u_2 \mapsto \overline{u_1 \boxtimes u_2}$, as shown in the diagram below.

$$\begin{array}{ccc} \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_1) \times \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_2) & \longrightarrow & \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_1) \boxtimes \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_2) \\ & \searrow & \Big\downarrow \overline{} \\ & & \mathcal{P}^{\mathfrak{g}\oplus\mathfrak{g}}_{\mathfrak{h}\oplus\mathfrak{h}}(V_1 \boxtimes V_2) \end{array}$$

Proposition 2.2.2. *The map* $\overline{}$ *is a* $\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$*-homomorphism.*

Proof. For each $a, b, \alpha, \beta \in \mathfrak{U}(\mathfrak{g})$ and $u_i \in \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_i)$ with $i = 1, 2$ we have

$$\begin{aligned}\big(\overline{(a \otimes b)(u_1 \boxtimes u_2)}\big)(\alpha \otimes \beta) &= (\overline{au_1 \boxtimes bu_2})(\alpha \otimes \beta) = \big((au_1)(\alpha)\big) \boxtimes \big((bu_2)(\beta)\big)\\ &= u_1(\alpha a) \boxtimes u_2(\beta b) = (\overline{u_1 \boxtimes u_2})(\alpha a \otimes \beta b)\\ &= (\overline{u_1 \boxtimes u_2})\big((\alpha \otimes \beta)(a \otimes b)\big)\\ &= \big((a \otimes b)(\overline{u_1 \boxtimes u_2})\big)(\alpha \otimes \beta).\end{aligned}$$

□

It may be useful to think of $\overline{}$ as the map which sends a formal object in $\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_1) \boxtimes \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_2)$ to its realization in $\mathcal{P}_{\mathfrak{h}\oplus\mathfrak{h}}^{\mathfrak{g}\oplus\mathfrak{g}}(V_1 \boxtimes V_2)$ while preserving the action of $\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$. Abusing the notation, from now on we shall omit writing $\overline{}$ and we will directly write $u_1 \boxtimes u_2$ to denote the realized homomorphism, unless otherwise specified.

2.2.2 The internal multiplication $\otimes$

Define $u_1 \otimes u_2 : \mathfrak{U}(\mathfrak{g}) \to V_1 \boxtimes V_2$ by

$$(u_1 \otimes u_2)(a) = (u_1 \boxtimes u_2)(\Delta(a)) \tag{2.1}$$

for each $a \in \mathfrak{U}(\mathfrak{g})$ and $u_i \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_i)$, $i = 1, 2$.

Lemma 2.2.3. *$u_1 \otimes u_2 \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_1 \otimes V_2)$.*

Proof. Note that

$$\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_1 \otimes V_2) = \mathrm{Hom}_{\mathfrak{U}(\mathfrak{h})}(\mathfrak{U}(\mathfrak{g}), V_1 \otimes V_2).$$

Let $a \in \mathfrak{U}(\mathfrak{g}), z \in \mathfrak{U}(\mathfrak{h})$. Then

$$\begin{aligned}(u_1 \otimes u_2)(za) &= (u_1 \boxtimes u_2)\big(\Delta(za)\big) = (u_1 \boxtimes u_2)\big(\Delta(z)\Delta(a)\big) = \\ &= \Delta(z)\big((u_1 \boxtimes u_2)(\Delta(a))\big) = \Delta(z)\big((u_1 \otimes u_2)(a)\big) = \\ &= z\big((u_1 \otimes u_2)(a)\big).\end{aligned}$$

The last identity comes from 2.2.1, considering $(u_1 \otimes u_2)(a)$ as an element of $V_1 \otimes V_2$. □

Let $\mathscr{P}(\mathfrak{g}, \mathfrak{h})$ be the class consisting of all $\mathfrak{U}(\mathfrak{h})$-modules $\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$, V running over every h-module. Equation (2.1) defines a multiplication $\otimes$ on $\bigcup \mathscr{P}(\mathfrak{g}, \mathfrak{h})$.

Proposition 2.2.4. *The multiplication $\otimes$ is associative.*

Proof. Let σ be the canonical isomorphism $(V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3)$. By saying that $\otimes$ is associative, we mean that the identity

$$\sigma \circ \big((u_1 \otimes u_2) \otimes u_3\big) = u_1 \otimes (u_2 \otimes u_3)$$

must hold for each $u_i \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_i)$, $i = 1, 2, 3$.

$$\begin{array}{ccc} \mathfrak{U}(\mathfrak{g}) & \xrightarrow{(u_1 \otimes u_2) \otimes u_3} & (V_1 \otimes V_2) \otimes V_3 \\ & \searrow^{u_1 \otimes (u_2 \otimes u_3)} & \downarrow \sigma \\ & & V_1 \otimes (V_2 \otimes V_3) \end{array}$$

By 1.3.4, the coproduct Δ is coassociative. If $\bar{\sigma}$ is the canonical isomorphism $\big(\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})\big) \otimes \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g}) \otimes \big(\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})\big)$, then it is evident that

$$\big(u_1 \boxtimes (u_2 \boxtimes u_3)\big) \circ \bar{\sigma} = \sigma \circ \big((u_1 \boxtimes u_2) \boxtimes u_3\big)$$

for each $u_i \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_i)$, $i = 1, 2, 3$.

$$\begin{array}{ccc} \big(\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})\big) \otimes \mathfrak{U}(\mathfrak{g}) & \xrightarrow{(u_1 \boxtimes u_2) \boxtimes u_3} & (V_1 \otimes V_2) \otimes V_3 \\ \bar{\sigma} \big\downarrow & & \big\downarrow \sigma \\ \mathfrak{U}(\mathfrak{g}) \otimes \big(\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})\big) & \xrightarrow[u_1 \boxtimes (u_2 \boxtimes u_3)]{} & V_1 \otimes (V_2 \otimes V_3) \end{array}$$

Let $a \in \mathfrak{U}(\mathfrak{g})$. We obtain

$$\begin{aligned} \big(u_1 \otimes (u_2 \otimes u_3)\big)(a) &= \big(u_1 \boxtimes (u_2 \otimes u_3)\big)\big(\Delta(a)\big) \\ &= \big(u_1 \boxtimes (u_2 \boxtimes u_3)\big)\big(((\mathrm{id}_{\mathfrak{U}(\mathfrak{g})} \otimes \Delta) \circ \Delta)(a)\big) \\ &= \big(u_1 \boxtimes (u_2 \boxtimes u_3)\big)\big((\bar{\sigma} \circ (\Delta \otimes \mathrm{id}_{\mathfrak{U}(\mathfrak{g})}) \circ \Delta)(a)\big) \\ &= \big(\sigma \circ ((u_1 \boxtimes u_2) \boxtimes u_3)\big)\big(((\Delta \otimes \mathrm{id}_{\mathfrak{U}(\mathfrak{g})}) \circ \Delta)(a)\big) \\ &= \big(\sigma \circ ((u_1 \otimes u_2 \boxtimes u_3)\big)\big(\Delta(a)\big) \\ &= \big(\sigma \circ ((u_1 \otimes u_2) \otimes u_3)\big)(a). \end{aligned}$$

□

Proposition 2.2.5. *Let $x \in \mathfrak{g}$. x acts as a derivation on $\mathscr{P}(\mathfrak{g}, \mathfrak{h})$, i.e. $x(u_1 \otimes u_2) = (xu_1) \otimes u_2 + u_1 \otimes (xu_2)$ for each $u_i \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_i)$, $i = 1, 2$.*

Proof. Let $a \in \mathfrak{U}(\mathfrak{g})$ and $u_i \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_i)$, $i = 1, 2$. Then

$$\begin{aligned} \big(x(u_1 \otimes u_2)\big)(a) &= (u_1 \otimes u_2)(ax) = (u_1 \boxtimes u_2)\big(\Delta(ax)\big) \\ &= (u_1 \boxtimes u_2)\big(\Delta(a)\Delta(x)\big) = \big(\Delta(x)(u_1 \boxtimes u_2)\big)\big(\Delta(a)\big) \\ &= \big((x \otimes 1 + 1 \otimes x)(u_1 \boxtimes u_2)\big)\big(\Delta(a)\big) \\ &= \big((xu_1) \boxtimes u_2 + u_1 \boxtimes (xu_2)\big)\big(\Delta(a)\big) \\ &= \big((xu_1) \otimes u_2 + u_1 \otimes (xu_2)\big)(a). \end{aligned}$$

□

Remark 5. A useful consideration involves the dual coproduct

$$\begin{aligned} \Delta^* : \mathrm{Hom}_{\mathfrak{U}(\mathfrak{h} \oplus \mathfrak{h})}(\mathfrak{U}(\mathfrak{g} \oplus \mathfrak{g}), V_1 \boxtimes V_2) &\to \mathrm{Hom}_{\mathfrak{U}(\mathfrak{h})}(\mathfrak{U}(\mathfrak{g}), V_1 \otimes V_2) \\ u &\mapsto u \circ \Delta \end{aligned}$$

which maps $\mathcal{P}_{\mathfrak{h} \oplus \mathfrak{h}}^{\mathfrak{g} \oplus \mathfrak{g}}(V_1 \boxtimes V_2)$ into $\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_1 \otimes V_2)$. Indeed, thanks to Lemma 2.2.3, it now follows that $u_1 \otimes u_2 = \Delta^*(u_1 \boxtimes u_2)$ for each $u_i \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_i)$, $i = 1, 2$.

It is interesting to note that, while Δ^* is only a linear map at first glance, one may define an action of $\mathfrak{U}(\mathfrak{g})$ on each $u \in \mathcal{P}_{\mathfrak{h}\oplus\mathfrak{h}}^{\mathfrak{g}\oplus\mathfrak{g}}(V_1 \boxtimes V_2)$ by $au = \Delta(a)u$, where $a \in \mathfrak{U}(\mathfrak{g})$. Due to Δ being an algebra homomorphism, this definition turns $\mathcal{P}_{\mathfrak{h}\oplus\mathfrak{h}}^{\mathfrak{g}\oplus\mathfrak{g}}(V_1 \boxtimes V_2)$ into a $\mathfrak{U}(\mathfrak{g})$-module, here denoted by $\Delta\mathcal{P}_{\mathfrak{h}\oplus\mathfrak{h}}^{\mathfrak{g}\oplus\mathfrak{g}}(V_1 \boxtimes V_2)$. In general, this process may be done for any $(\mathfrak{g} \oplus \mathfrak{g})$-module U in order to turn it into the $\mathfrak{g}$-module ΔU. It follows that, for the $\mathfrak{h}$-modules V_1 and V_2, we will have $V_1 \otimes V_2 = \Delta(V_1 \boxtimes V_2)$. The dual map

$$\Delta^* : \Delta\mathcal{P}_{\mathfrak{h}\oplus\mathfrak{h}}^{\mathfrak{g}\oplus\mathfrak{g}}(V_1 \boxtimes V_2) \to \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_1 \otimes V_2)$$

becomes then a $\mathfrak{U}(\mathfrak{g})$-homomorphism, since for each $a, \alpha \in \mathfrak{U}(\mathfrak{g})$ we have

$$\begin{aligned}\big(\Delta^*(au)\big)(\alpha) &= (au)\big(\Delta(\alpha)\big) = \big(\Delta(a)u\big)(\Delta(\alpha)) = u\big(\Delta(\alpha)\Delta(a)\big) = u\big(\Delta(\alpha a)\big)\\ &= \big(\Delta^*(u)\big)(\alpha a) = \big(a(\Delta^*(u))\big)(\alpha).\end{aligned}$$

Moreover, this action also turns the realization map

$$\overline{} : \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_1) \otimes \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_2) \to \Delta\mathcal{P}_{\mathfrak{h}\oplus\mathfrak{h}}^{\mathfrak{g}\oplus\mathfrak{g}}(V_1 \boxtimes V_2)$$

into a $\mathfrak{U}(\mathfrak{g})$-homomorphism. Thanks to the fundamental property of the tensor product, the multiplication $\otimes$ gives rise to a linear map

$$\begin{aligned}\nu : \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_1) \otimes \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_2) &\to \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_1 \otimes V_2)\\ u_1 \boxtimes u_2 &\mapsto u_1 \otimes u_2\end{aligned}$$

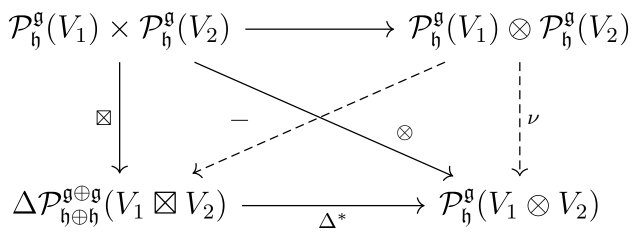
where $u_1 \boxtimes u_2$ is here regarded as a formal object. This argument assumes that the operation $\otimes$ is bilinear, which directly comes from the obvious bilinearity of the multiplication $\boxtimes$.

$$\begin{array}{ccc} \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_1) \times \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_2) & \longrightarrow & \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_1) \otimes \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_2) \\ & \searrow^{\otimes} & \downarrow \nu \\ & & \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_1 \otimes V_2) \end{array}$$

We obtain a result analogous to 2.2.2.

Proposition 2.2.6. *The map ν is a $\mathfrak{U}(\mathfrak{g})$-homomorphism.*

Proof. As seen in Remark 5, we have $\otimes = \Delta^* \circ \boxtimes$. The diagram below shows that $\nu = \Delta^* \circ \overline{}$ for every simple tensor, which implies the identity holds true for all elements of $\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_1) \otimes \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_2)$ by linearity.



Both Δ^* and $\overline{}$ are $\mathfrak{U}(\mathfrak{g})$-homomorphisms, therefore ν also is. $\square$

2.2.3 Filtered structure of $\boxtimes$ and $\otimes$

The multiplications $\boxtimes$ and $\otimes$ can be put in relation with the filtrations introduced in §2.1. We note that, for each $p \in \mathbb{Z}$, Lemma 1.3.3 directly implies the following.

Corollary 2.2.7.

$$\Delta\big(\mathfrak{U}_p(\mathfrak{g})\big) \subseteq \sum_{k=0}^{p} \mathfrak{U}_k(\mathfrak{g}) \otimes \mathfrak{U}_{p-k}(\mathfrak{g}).$$

Proposition 2.2.8. *Let $u_1 \in \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_1)_p$ and $u_2 \in \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_2)_q$. Then $u_1 \boxtimes u_2 \in \mathcal{P}^{\mathfrak{g}\oplus\mathfrak{g}}_{\mathfrak{h}\oplus\mathfrak{h}}(V_1 \boxtimes V_2)_{p+q}$, $u_1 \otimes u_2 \in \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_1 \otimes V_2)_{p+q}$ and $xu_1 \in \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_1)_{p-1}$ for each $x \in \mathfrak{g}$.*

Proof. By the isomorphism $\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g}) \cong \mathfrak{U}(\mathfrak{g} \oplus \mathfrak{g})$, we have

$$\mathfrak{U}_p(\mathfrak{g} \oplus \mathfrak{g}) = \sum_{k=0}^{p} \mathfrak{U}_k(\mathfrak{g}) \otimes \mathfrak{U}_{p-k}(\mathfrak{g}).$$

Remember that $\mathcal{P}^{\mathfrak{g}\oplus\mathfrak{g}}_{\mathfrak{h}\oplus\mathfrak{h}}(V_1 \boxtimes V_2)_{p+q} = \{u \in \mathcal{P}^{\mathfrak{g}\oplus\mathfrak{g}}_{\mathfrak{h}\oplus\mathfrak{h}}(V_1 \boxtimes V_2) \mid u|_{\mathfrak{U}_{p+q-1}(\mathfrak{g}\oplus\mathfrak{g})} = 0\}$. Then from 2.2.7 it is

$$\begin{aligned}(u_1 \boxtimes u_2)\big(\mathfrak{U}_{p+q-1}(\mathfrak{g} \oplus \mathfrak{g})\big) &= (u_1 \boxtimes u_2)\left(\sum_{k=0}^{p+q-1} \mathfrak{U}_k(\mathfrak{g}) \otimes \mathfrak{U}_{p+q-1-k}(\mathfrak{g})\right) \\ &= \sum_{k=0}^{p+q-1} u_1\big(\mathfrak{U}_k(\mathfrak{g})\big) \otimes u_2\big(\mathfrak{U}_{p+q-1-k}(\mathfrak{g})\big) = 0\end{aligned}$$

because $u_1\big(\mathfrak{U}_k(\mathfrak{g})\big) = 0$ for $k < p$ and $u_2\big(\mathfrak{U}_{p+q-1-k}(\mathfrak{g})\big) = 0$ for $k \geq p$. This shows that $u_1 \boxtimes u_2 \in \mathcal{P}^{\mathfrak{g}\oplus\mathfrak{g}}_{\mathfrak{h}\oplus\mathfrak{h}}(V_1 \boxtimes V_2)_{p+q}$. By 2.2.7 we get

$$\begin{aligned}(u_1 \otimes u_2)\big(\mathfrak{U}_{p+q-1}(\mathfrak{g})\big) &= (u_1 \boxtimes u_2)\big(\Delta(\mathfrak{U}_{p+q-1}(\mathfrak{g}))\big) \\ &\subseteq (u_1 \boxtimes u_2)\left(\sum_{k=0}^{p+q-1} \mathfrak{U}_k(\mathfrak{g}) \otimes \mathfrak{U}_{p+q-1-k}(\mathfrak{g})\right) \\ &= \sum_{k=0}^{p+q-1} u_1\big(\mathfrak{U}_k(\mathfrak{g})\big) \otimes u_2\big(\mathfrak{U}_{p+q-1-k}(\mathfrak{g})\big) = 0\end{aligned}$$

from the same argument as before, which means $u_1 \otimes u_2 \in \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_1 \otimes V_2)_{p+q}$. Finally, let $a \in \mathfrak{U}_{p-2}(\mathfrak{g})$; then $ax \in \mathfrak{U}_{p-1}(\mathfrak{g})$, hence $(xu_1)(a) = u_1(ax) = 0$. This proves that $xu_1 \in \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_1)_{p-1}$. □

Remark 6. If U and V are filtered $\mathfrak{g}$-modules then $U \otimes V$ is the $\mathfrak{g}$-module filtered by

$$(U \otimes V)_n = \sum_{\substack{p+q=n \\ p,q\in\mathbb{Z}}} U_p \otimes V_q.$$

Clearly the same holds when regarding $U \otimes V$ as the external tensor product $U \boxtimes V$. By 2.2.8, the maps

$$\begin{aligned}\overline{} &: \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_1) \boxtimes \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_2) \to \mathcal{P}^{\mathfrak{g}\oplus\mathfrak{g}}_{\mathfrak{h}\oplus\mathfrak{h}}(V_1 \boxtimes V_2) \\ \nu &: \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_1) \otimes \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_2) \to \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_1 \otimes V_2)\end{aligned}$$

become filtered homomorphisms between $\mathfrak{g} \oplus \mathfrak{g}$ and $\mathfrak{g}$-modules respectively. Moreover, $x\mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_1)_p \subseteq \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(V_1)_{p-1}$ for each $x \in \mathfrak{g}$.

Chapter 3

The Guillemin-Sternberg-Rim realization theorem

In the previous chapter, we have defined a multiplication $u_1 \otimes u_2$ between elements of produced $\mathfrak{h}$-modules. Now regard $\mathbb{K}$ as a trivial $\mathfrak{h}$-module and set $F = \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(\mathbb{K})$. Give F the filtration $\mathfrak{F} = \{F_m\}_{m \in \mathbb{Z}}$ introduced in §2.1. Let V be any $\mathfrak{h}$-module and assume that $u_1 \in F$, $u_2 \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$. By identifying $\mathbb{K} \otimes V$ with V through the natural isomorphism $k \otimes v \mapsto kv$, it follows that $u_1 \otimes u_2 \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$. In this case we will denote the multiplication by $u_1 u_2$. If $V = \mathbb{K}$, this operation is internal in F.

Proposition 3.0.1. *The pair $(F, \mathfrak{F})$ is a filtered unitary associative algebra.*

Proof. By 2.2.4, $F = \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(\mathbb{K})$ is an associative algebra with the multiplication mentioned above. Choose $\{x_i\}_{i \in I}$ basis for $\mathfrak{g}$ so that $\{x^m\}_{m \in M}$ is a basis for $\mathfrak{U}(\mathfrak{g})$, M and x^m defined as in §1.2.3. Define e as the linear map $\mathfrak{U}(\mathfrak{g}) \to \mathbb{K}$ such that

$$e(x^m) = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m \neq 0. \end{cases}$$

The trivial action of $\mathfrak{h}$ on $\mathbb{K}$ ensures this is a well-defined element of F. Let $u \in F$. From Lemma 1.3.3 we have

$$\begin{aligned} (eu)(x^m) = (e \boxtimes u)(\Delta(x^m)) &= \sum_{0 \leq k \leq m} \binom{m}{k} (e \boxtimes u)(x^k \otimes x^{m-k}) \\ &= \sum_{0 \leq k \leq m} \binom{m}{k} e(x^k) u(x^{m-k}) = e(1) u(x^m) = u(x^m) \end{aligned}$$

and

$$(ue)(x^m) = (u \boxtimes e)(\Delta(x^m)) = \sum_{0 \le k \le m} \binom{m}{k} (u \boxtimes e)(x^k \otimes x^{m-k})$$
$$= \sum_{0 \le k \le m} \binom{m}{k} u(x^k)e(x^{m-k}) = u(x^m)e(1) = u(x^m)$$

which means that e is the identity of F. Finally, let $u_1 \in F_p, u_2 \in F_q$ and $a \in \mathfrak{U}_{p+q-1}(\mathfrak{g})$. Then

$$(u_1 u_2)(a) = (u_1 \boxtimes u_2)(\Delta a) = 0$$

because $\Delta a \in \sum_{k=0}^{p+q-1} \mathfrak{U}_k(\mathfrak{g}) \otimes \mathfrak{U}_{p+q-k-1}(\mathfrak{g})$ by 2.2.7 but $k \ge p$ implies

$$p + q - k - 1 \le p + q - p - 1 \le q - 1.$$

Therefore $u_1 u_2 \in F_{p+q}$, i.e. $F_p F_q \subseteq F_{p+q}$. □

Proposition 3.0.2. *For each $\mathfrak{h}$-module V, the produced module $\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$ is a unitary F-module.*

Proof. Let $u_1 \in F$ and $u_2 \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$. As stated earlier, $u_1 u_2 \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$. The associativity that comes from 2.2.4 guarantees that this multiplication defines a module structure on $\mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$. By repeating the previous proof and supposing that $u \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$, we obtain $eu = u$, hence the action is unitary. □

3.1 Separated filtrations

Definition 3.1 (Separated filtration)**.** Let A be an algebra and let $\{A_p\}_{p \in \mathbb{Z}}$ be a (downward) filtration on A. $\{A_p\}_{p \in \mathbb{Z}}$ is said to be *separated* if $\bigcap_{p \in \mathbb{Z}} A_p = \{0\}$.

Proposition 3.1.1. *The filtration $\mathfrak{F}$ is separated.*

Proof. Let $u \in \bigcap_{p \in \mathbb{Z}} F_p = \{0\}$. Then $u = 0$ on $\mathfrak{U}_p(\mathfrak{g})$ for all $p \in \mathbb{Z}$ and therefore $u = 0$ on $\bigcup_{p \in \mathbb{Z}} \mathfrak{U}_p(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g})$. □

We introduce a filtration on the Lie algebra $\mathfrak{g}$ as follows:

$$\mathfrak{g}_p = \begin{cases} \mathfrak{g} & \text{if } p < 0 \\ \mathfrak{h} & \text{if } p = 0 \\ \{x \in \mathfrak{g}_{\mathfrak{p}-1} \mid [y, x] \in \mathfrak{g}_{p-1} \ \forall y \in \mathfrak{g}\} & \text{if } p > 0. \end{cases} \tag{3.1}$$

Let $\mathfrak{G} = \{\mathfrak{g}_p\}_{p \in \mathbb{Z}}$.

Proposition 3.1.2. *The pair* $(\mathfrak{g}, \mathfrak{G})$ *is a filtered Lie algebra.*

Proof. Clearly $\mathfrak{g}_p \subseteq \mathfrak{g}_{p-1}$ by definition. We will prove that $[\mathfrak{g}_p, \mathfrak{g}_q] \subseteq \mathfrak{g}_{p+q}$ for each $p, q \in \mathbb{Z}$. Set $n = p + q$: the result is trivial for $n < 0$, so let's assume $n \geq 0$. If $p \leq -1$ then $q \geq 1$, hence we have $[\mathfrak{g}_p, \mathfrak{g}_q] = [\mathfrak{g}, \mathfrak{g}_q] \subseteq \mathfrak{g}_{q-1} \subseteq \mathfrak{g}_{p+q}$ by definition of $\mathfrak{g}_q$. We can thus suppose $p, q \geq 0$ and proceed by induction on n. If $n = 0$ then it must be $p = q = 0$, therefore $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$ is trivially true. Now suppose inductively that $[\mathfrak{g}_p, \mathfrak{g}_q] \subseteq \mathfrak{g}_n$ for a certain $n > 0$ and all $p, q \in \{0, \ldots, n\}$ such that $p + q = n$; this is equivalent to assuming that $[\mathfrak{g}_p, \mathfrak{g}_{n-p}] \subseteq \mathfrak{g}_n$ for each $p \in \{0, \ldots, n\}$. Let $p \in \{0, \ldots, n+1\}$, $x \in \mathfrak{g}_{n+1-p}$ and $y \in \mathfrak{g}_p$: we will show that $[y, x] \in \mathfrak{g}_{n+1}$, i.e. that $[z, [y, x]] \in \mathfrak{g}_n$ for each $z \in \mathfrak{g}$. Since $x \in \mathfrak{g}_{n+1-p} \subseteq \mathfrak{g}_{n-p}$, by induction hypothesis it is $[y, x] \in \mathfrak{g}_n$. Moreover, from the Jacobi identity we have

$$[z, [y, x]] = [[x, y], z] = [x, [y, z]] - [y, [x, z]].$$

By definition of $\mathfrak{g}_p$ and $\mathfrak{g}_{n+1-p}$ respectively, we see that $[y, z] \in \mathfrak{g}_{p-1}$ and $[x, z] \in \mathfrak{g}_{n-p}$, therefore both $[x, [y, z]]$ and $[y, [x, z]]$ are in $\mathfrak{g}_n$ thanks to the induction hypothesis. Hence $[z, [y, x]] \in \mathfrak{g}_n$ and $[\mathfrak{g}_p, \mathfrak{g}_{n+1-p}] \subseteq \mathfrak{g}_{n+1}$. □

Proposition 3.1.3. *The filtration* $\mathfrak{G}$ *is separated if and only if* $\mathfrak{h}$ *contains no nontrivial* $\mathfrak{g}$*-ideals.*

Proof. Let $\mathfrak{i} \subseteq \mathfrak{h}$ be a non-trivial $\mathfrak{g}$-ideal. We shall show the inclusion $\mathfrak{i} \subseteq \mathfrak{g}_p$ for each $p \in \mathbb{Z}$ by induction on p. This is trivial for $p \leq 0$, so let $p > 0$ and assume $\mathfrak{i} \subseteq \mathfrak{g}_p$; then $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i} \subseteq \mathfrak{g}_p$, which means $\mathfrak{i} \subseteq \mathfrak{g}_{p+1}$. This concludes the proof by induction, which gives us $\mathfrak{i} \subseteq \bigcap_{p \in \mathbb{Z}} \mathfrak{g}_p$, i.e. $\mathfrak{F}$ is not separated. Conversely, let $\{0\} \neq \mathfrak{i} = \bigcap_{p \in \mathbb{Z}} \mathfrak{g}_p$. Then $\mathfrak{i} \subseteq \mathfrak{g}_p$ for each $p \in \mathbb{Z}$ (in particular we have $\mathfrak{i} \subseteq \mathfrak{h}$), hence $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{g}_{p-1}$; this is also true for every $p \in \mathbb{Z}$, implying $[\mathfrak{g}, \mathfrak{i}] \subseteq \bigcap_{p \in \mathbb{Z}} \mathfrak{g}_p = \mathfrak{i}$. □

For the rest of this chapter, $\mathfrak{G}$ will be assumed to be separated. Let D_p be the subspace of $\mathrm{der}(F)$ containing all derivations d such that $dF_m \subseteq F_{m+p}$ for each $m \in \mathbb{Z}$. Let $D = \bigcup_{p \in \mathbb{Z}} D_p$ and $\mathfrak{D} = \{D_p\}_{p \in \mathbb{Z}}$.

Proposition 3.1.4. *The pair* $(D, \mathfrak{D})$ *is a filtered Lie algebra.*

Proof. Let $d \in D_p$. Then $dF_m \subseteq F_{m+p} \subseteq F_{m+p-1}$ for each $m \in \mathbb{Z}$, i.e. $d \in D_{p-1}$ which implies $D_p \subseteq D_{p-1}$ for each $p \in \mathbb{Z}$. We will show that $[D_p, D_q] \subseteq D_{p+q}$ for every $p, q \in \mathbb{Z}$. Let $d_1 \in D_p$, $d_2 \in D_q$ and $u \in F_m$: we have $[d_1, d_2]u = d_1(d_2 u) - d_2(d_1 u)$. Both terms on the right are in F_{m+p+q}, hence $[d_1, d_2] \in D_{p+q}$. □

Proposition 3.1.5. *The filtration* $\mathfrak{D}$ *is separated.*

Proof. Let $d \in \cap_{p\in\mathbb{Z}} D_p$. Then we have $dF = dF_0 \subseteq F_p$ for all $p \in \mathbb{Z}$, i.e. $dF \subseteq \cap_{p\in\mathbb{Z}} F_p = \{0\}$ by 3.1.1. It follows that $d = 0$. □

If L is a Lie subalgebra of D, we will filter L in the obvious way, i.e. $L_p = L \cap D_p$ for each $p \in \mathbb{Z}$. This filtration on L is also separated.

Remark 7. Note that $F = \mathbb{K}e + F_1$ and $D_0 F \subseteq F_1$. Moreover, $u = u(1)\,e$ (modulo F_1) for all $u \in F$.
The identity $F = \mathbb{K}e + F_1$ clearly comes by the fact that e is nonzero only on the constant terms while F_1 contains $\mathfrak{U}(\mathfrak{h})$-homomorphisms that are zero on $\mathbb{K} \subseteq \mathfrak{U}(\mathfrak{g})$. To show that $D_0 F \subseteq F_1$, let $d \in D_0$. Then $dF = d(\mathbb{K}e) + dF_1 \subseteq \mathbb{K}de + F_1$. Since e is the identity element of F, we have

$$d(e) = d(e^2) = d(e)\,e + e\,d(e) = 2d(e)$$

which implies $d(e) = 0$. Hence $dF \subseteq F_1$. Finally, since $F = \mathbb{K}e + F_1$ it is $u = ke$ (modulo F_1) for some $k \in \mathbb{K}$, but

$$u(1) = (ke)(1) = e(k) = k\,e(1) = k.$$

3.2 Embedding theorem

Let γ be the representation map for the g-module F, i.e.

$$\begin{aligned} \gamma : \mathfrak{g} &\to \mathrm{gl}(F) \\ x &\mapsto \gamma(x) \end{aligned}$$

where $\gamma(x)u = xu$ for each $u \in F$. We remind that γ can be extended to an associative algebra homomorphism of $\mathfrak{U}(\mathfrak{g})$ into $\mathrm{Hom}_{\mathbb{K}}(F, F)$ through the fundamental property of the enveloping algebra and thus $(xu)(a) = u(ax)$ for all $a \in \mathfrak{U}(\mathfrak{g})$.

We show that the representation γ is faithful (i.e.: injective) and essentially unique up to isomorphism in the hypothesis of a separated filtration for the Lie algebra $\mathfrak{g}$.

Theorem 3.2.1. *The map γ is a filtered Lie algebra isomorphism of $\mathfrak{g}$ with a subalgebra of D. For every Lie algebra homomorphism $\gamma_1 : \mathfrak{g} \to D$ such that $\gamma_1(x) - \gamma(x) \in D_0$ for each $x \in \mathfrak{g}$, there exists a unique filtered algebra automorphism $\theta : F \to F$ such that $\theta \circ \gamma_1(x) = \gamma(x) \circ \theta$ for all $x \in \mathfrak{g}$.*

$$\begin{array}{ccc} F & \xrightarrow{\gamma_1(x)} & F \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta} \\ F & \xrightarrow{\gamma(x)} & F \end{array}$$

Proof. By 2.2.5 we have $\gamma(x) \in \mathrm{der}(F)$. Moreover, it is $\gamma(x) \in D_{-1}$: if $u \in F_m$, then $xu \in F_{m-1}$ directly from 2.2.8. In particular, this shows that $\gamma : \mathfrak{g} \to D$. We shall split the rest of the proof in some steps.

Step 1. We prove that $\gamma(h) \in D_0$ if and only if $h \in \mathfrak{h}$.

Let $h \in \mathfrak{h}$, $u \in F_m$ and $a \in \mathfrak{U}_{m-1}(\mathfrak{g})$. Then $(\gamma(h)u)(a) = (hu)(a) = u(ah)$. As seen in Remark 3, we have $ah = ha + b$ where $b \in \mathfrak{U}_{m-1}(\mathfrak{g})$. Since u is a $\mathfrak{U}(\mathfrak{h})$-homomorphism we obtain that $u(ha) = h\,u(a) = 0$, also $u(b) = 0$ hence $\gamma(h)u \in F_m$ and $\gamma(h) \in D_0$. Now suppose $h \notin \mathfrak{h}$ and consider its equivalence class $\bar{h} \in \mathfrak{g}/\mathfrak{h} \subseteq S(\mathfrak{g}/\mathfrak{h})$. Obviously $\bar{h} \neq 0$, thus we may regard it as an element of a basis for $\mathfrak{g}/\mathfrak{h}$. Consider the maps τ and π defined in §2.1: we can say that $\tau(\bar{h}) = h$. Choose $w \in \mathrm{Hom}_{\mathbb{K}}(S(\mathfrak{g}/\mathfrak{h}), \mathbb{K})$ such that $w(\bar{h}) \neq 0$ and set $u = \pi^{-1}(w)$. Remember that π is bijective by 2.1.1. Then

$$(\gamma(h)u)(1) = (hu)(1) = u(h) = u(\tau(\bar{h})) = (\pi u)(\bar{h}) = w(\bar{h}) \neq 0$$

which means $\gamma(h)u \notin F_1$. From Remark 7, it follows immediately that $\gamma(h) \notin D_0$.

Step 2. The map γ is a Lie algebra isomorphism.

We show the injectivity of γ. Let $\mathfrak{t} = \ker\gamma$ and let $x \in \mathfrak{t}$. Then $\gamma(x)F_m = \{0\} \subseteq F_m$, hence $\gamma(x) \in D_0$ and thus $x \in \mathfrak{h}$ as seen above, which means that $\mathfrak{t} \subseteq \mathfrak{h}$. Since the filtration is separated and $\mathfrak{t}$ is a $\mathfrak{g}$-ideal, it must be $\mathfrak{t} = \{0\}$, i.e. γ is injective. This is enough to show that γ is isomorphic to a subalgebra of D, as $\gamma : \mathfrak{g} \to \gamma(\mathfrak{g}) \subseteq D$ and $[\gamma(\mathfrak{g}), \gamma(\mathfrak{g})] = \gamma([\mathfrak{g}, \mathfrak{g}])$.

Step 3. The map γ is filtered.

We shall prove that $\gamma(\mathfrak{g}_p) = \gamma(\mathfrak{g}) \cap D_p$ for each $p \in \mathbb{Z}$. Remember that $(\gamma(\mathfrak{g}))_p = \gamma(\mathfrak{g}) \cap D_p$ by definition. If $p < 0$ then $\mathfrak{g}_p = \mathfrak{g}$, also we have already shown that $\gamma(\mathfrak{g}) \subseteq D_{-1}$ which implies

$$\gamma(\mathfrak{g}_\mathfrak{p}) = \gamma(\mathfrak{g}) = \gamma(\mathfrak{g}) \cap D_{-1} = \gamma(\mathfrak{g}) \cap D_p.$$

If $p \geq 0$, we proceed by induction on p. If $p = 0$ then $\mathfrak{g}_0 = \mathfrak{h}$ and we have seen that $\gamma(h) \in D_0$ if and only if $h \in \mathfrak{h}$, which proves that $\gamma(\mathfrak{h}) = \gamma(\mathfrak{g}) \cap D_0$. Suppose that $\gamma(\mathfrak{g}_p) = \gamma(\mathfrak{g}) \cap D_p$ for some $p > 0$: we show that $\gamma(\mathfrak{g}_{p+1}) = \gamma(\mathfrak{g}) \cap D_{p+1}$.

Let $x \in \mathfrak{g}_{p+1}$; then we have $x, [y, x] \in \mathfrak{g}_p$ for each $y \in \mathfrak{g}$ and also $\gamma(x), \gamma([y, x]) \in D_p$ by induction hypothesis. To prove that $\gamma(x) \in D_{p+1}$, we need to show $\gamma(x)F_{m-1} \subseteq F_{m+p}$ for all $m \in \mathbb{Z}$: we proceed with a secondary induction on m. If $m \leq -p$, then $F_{m-1} \supseteq F_{-p-1} \supseteq F_0 = F$ and

$F_{m+p} \supseteq F_{-p+p} = F_0 = F$, so the inclusion becomes just $\gamma(x)F \subseteq F$, which is trivially true. Now assume $\gamma(x)F_{m-1} \subseteq F_{m+p}$ for some $m > -p$ and let $u \in F_m$, $a \in \mathfrak{U}_{m+p-1}(\mathfrak{g})$ and $y \in \mathfrak{g}$. Then

$$\begin{aligned}\big(\gamma(x)u\big)(ay) &= (xu)(ay) = u(ayx) = u(a[y,x]) + u(axy)\\ &= ([y,x]u)(a) + (xyu)(a) = \big(\gamma([y,x])u\big)(a) + \big(\gamma(xy)u\big)(a)\\ &= \big(\gamma([y,x])u\big)(a) + \big(\gamma(x)\gamma(y)u\big)(a).\end{aligned}$$

Since $\gamma([y,x]) \in D_p$, we have $\gamma([y,x])u \in F_{m+p}$ and $\big(\gamma([y,x])u\big)(a) = 0$; analogously, $\gamma(y) \in D_{-1}$ implies $\gamma(y)u \in F_{m-1}$ and $\gamma(x)\gamma(y)u \in F_{m+p}$ by induction hypothesis (on m), thus it is also $\big(\gamma(x)\gamma(y)u\big)(a) = 0$. Therefore $\gamma(x)u = 0$ on $\mathfrak{U}_{m+p}(\mathfrak{g})$ and $\gamma(x)u \in F_{m+p+1}$; this proves the secondary induction so that $\gamma(x)F_{m-1} \subseteq F_{m+p}$ for all $m \in \mathbb{Z}$, i.e. $\gamma(x) \in D_{p+1}$. We have shown that $\gamma(\mathfrak{g}_{p+1}) \subseteq D_{p+1}$.

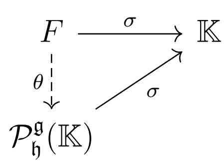
Conversely, suppose $\gamma(x) \in D_{p+1}$ for some $x \in \mathfrak{g}$ and let $u \in F_m$, $y \in \mathfrak{g}$. Naturally $\gamma([y,x]) = [\gamma(y), \gamma(x)]$, but $\gamma(x) \in D_{p+1}$ and $\gamma(y) \in D_{-1}$ imply that $\gamma([y,x]) \in D_p$ thanks to 3.1.4. By induction hypothesis both $x, [y,x] \in \mathfrak{g}_p$, thus $x \in \mathfrak{g}_{p+1}$ and $\gamma(\mathfrak{g}) \cap D_{p+1} \subseteq \gamma(\mathfrak{g}_{p+1})$. This proves that $\gamma : \mathfrak{g} \to \gamma(\mathfrak{g}) \cap D$ is a filtered Lie algebra isomorphism.

Step 4. Existence of θ.

Let $\gamma_1 : \mathfrak{g} \to D$ be a Lie algebra homomorphism such that $\gamma_1(x) - \gamma(x) \in D_0$ for each $x \in \mathfrak{g}$. Observe that γ_1 is a representation map and as such it defines another action on F as a (left) $\mathfrak{g}$-module by $(x,u) \mapsto \gamma_1(x)u$ for each $x \in \mathfrak{g}$ and $u \in F$. From the results obtained earlier for γ, we have that $\gamma_1(\mathfrak{g}) \subseteq D_{-1}$ and $\gamma_1(\mathfrak{h}) \subseteq D_0$. Regard F as a $\mathfrak{g}$-module with the action given by γ_1 and define $\sigma : F \to \mathbb{K}$ by $\sigma(u) = u(1)$ for each $u \in F$. We obtain $\sigma \circ \gamma(h) = \sigma \circ \gamma_1(h) = 0$ for $h \in \mathfrak{h}$: indeed, Remark 7 tells us that $\gamma(h)u, \gamma_1(h)u \in F_1$ for all $u \in F$, however clearly $\sigma F_1 = \{0\}$. This implies that $\sigma : F \to \mathbb{K}$ is a $\mathfrak{h}$-homomorphism, as for every $u \in F$ and $h \in \mathfrak{h}$ it is

$$\sigma(\gamma_1(h)u) = \big(\sigma \circ \gamma_1(h)\big)(u) = 0 = h\,\sigma(u)$$

due to the trivial $\mathfrak{h}$-module structure of $\mathbb{K}$. Since $F = \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(\mathbb{K})$, we know that the map $\sigma : \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(\mathbb{K}) \to \mathbb{K}$ defined as above is also the $\mathfrak{h}$-homomorphism produced from $\mathbb{K}$, as seen at the start of §2. Therefore, there exists a unique $\mathfrak{g}$-homomorphism $\theta : F \to \mathcal{P}^{\mathfrak{g}}_{\mathfrak{h}}(\mathbb{K})$ such that $\sigma = \sigma \circ \theta$, as shown in the diagram below.



Then for each $x \in \mathfrak{g}$, $u \in F$ we have $\theta(\gamma_1(x)u) = x(\theta u) = \gamma(x)(\theta u)$, i.e. $\theta \circ \gamma_1(x) = \gamma(x) \circ \theta$.

Step 5. The map θ is filtered.

The Lie algebra homomorphism γ_1 can be extended to an associative algebra homomorphism of $\mathfrak{U}(\mathfrak{g})$ into $\mathrm{Hom}_{\mathbb{K}}(F, F)$, just like γ. From §2 we also know that θ is defined by $(\theta u)(a) = \sigma(\gamma_1(a)u) = (\gamma_1(a)u)(1)$ for each $u \in F$ and $a \in \mathfrak{U}(\mathfrak{g})$. Since $\gamma_1(\mathfrak{g}) \subseteq D_{-1}$ then $\gamma_1(\mathfrak{g})F_m \subseteq F_{m-1}$ for all $m \in \mathbb{Z}$; for any product $y_1 \dots y_p$ of p elements of $\mathfrak{g}$, we have

$$\begin{aligned}\gamma(y_1 \dots y_p)F_m = \gamma(y_1) \dots \gamma(y_p)F_m &\subseteq \gamma_1(y_1) \dots \gamma_1(y_{p-1})F_{m-1} \\ &\subseteq \dots \subseteq F_{m-p}\end{aligned}$$

hence $\gamma_1\big(\mathfrak{U}_p(\mathfrak{g})\big)F_m \subseteq F_{m-p}$ for all m and p. Let $u \in F_m$ and $a \in \mathfrak{U}_{m-1}(\mathfrak{g})$. Then $\gamma_1(a)u \in F_{m-(m-1)} = F_1$ and therefore $(\theta u)(a) = (\gamma_1(a)u)(1) = 0$. It follows that $\theta u \in F_m$ which implies $\theta F_m \subseteq F_m$, i.e. $\theta \in \mathrm{Hom}_{\mathbb{K}}(F, F)$ is filtration preserving.

Step 6. The map θ is bijective.

Let $u \in F_m$ and $y \in \mathfrak{g}$. Since $\gamma_1(y) - \gamma(y) \in D_0$ we have $\big(\gamma_1(y) - \gamma(y)\big)u \in F_m$, i.e. $\gamma_1(y)u = \gamma(y)u$ (modulo F_m). Now let $y_1, \dots, y_p \in \mathfrak{g}$. We show that $\gamma_1(y_1 \dots y_p)u = \gamma(y_1 \dots y_p)u$ (modulo F_{m-p+1}) by induction on p. We have already proven the case $p = 1$, so assume our result to be true for a product of $p > 1$ elements of $\mathfrak{g}$. Then

$$\begin{aligned}\gamma_1(y_1 \dots y_{p+1})u &= \gamma_1(y_1)\gamma_1(y_2 \dots y_{p+1})u \\ &= \gamma_1(y_1)\gamma(y_2 \dots y_{p+1})u && (\text{modulo } \gamma_1(y_1)F_{m-p+1}) \\ &= \gamma_1(y_1)\gamma(y_2 \dots y_{p+1})u && (\text{modulo } F_{m-p}) \\ &= \gamma(y_1)\gamma(y_2 \dots y_{p+1})u && (\text{modulo } F_{m-p}) \quad (3.2) \\ &= \gamma(y_1 \dots y_{p+1})u && (\text{modulo } F_{m-p})\end{aligned}$$

where identity (3.2) holds because $\gamma(y_2 \dots y_{p+1})u \in F_{m-p}$ and hence $\big(\gamma_1(y) - \gamma(y)\big)\big(\gamma(y_2 \dots y_{p+1})u\big) \in F_{m-p}$. Therefore it is also $\gamma_1(a)u = \gamma(a)u$ (modulo F_{m-p+1}) for each $a \in \mathfrak{U}_p(\mathfrak{g})$. Moreover, the identity $\theta \circ \gamma_1(y) = \gamma(y) \circ \theta$ for $y \in \mathfrak{g}$ shown in step 4 gives us

$$\begin{aligned}\theta\big(\gamma_1(y_1 \dots y_p)u\big) &= \theta\big(\gamma_1(y_1) \dots \gamma_1(y_p)u\big) = \big(\theta \circ \gamma_1(y_1)\big)\big(\gamma_1(y_2) \dots \gamma_1(y_p)u\big) \\ &= \big(\gamma(y_1) \circ \theta\big)\big(\gamma_1(y_2) \dots \gamma_1(y_p)u\big) = \gamma(y_1)\,\theta\big(\gamma_1(y_2) \dots \gamma(y_p)u\big) \\ &= \dots = \gamma(y_1) \dots \gamma(y_p)(\theta u) \\ &= \gamma(y_1 \dots y_p)(\theta u)\end{aligned}$$

for any product $y_1 \dots y_p$ of p elements of $\mathfrak{g}$. This clearly implies $\theta \circ \gamma_1(a) = \gamma(a) \circ \theta$ for each $a \in \mathfrak{U}(\mathfrak{g})$. Now let $a \in \mathfrak{U}_m(\mathfrak{g})$ so that $\gamma_1(a)u = \gamma(a)u + v$ with $v \in F_1$. Then by definition of θ we have

$$\begin{aligned}(\theta u)(a) &= \sigma(\gamma_1(a)u) = \big(\gamma_1(a)u\big)(1) = \big(\gamma(a)u + v\big)(1) \\ &= \big(\gamma(a)u\big)(1) = u(a)\end{aligned}$$

which means $\theta u = u$ on $\mathfrak{U}_m(\mathfrak{g})$. Therefore, for all $u \in F_m$ it is $\theta u - u \in F_{m+1}$ or, in other words, $\theta u = u$ (modulo F_{m+1}).

It follows that θ is injective: indeed, assume that $\theta u = 0$ for some $u \in F$. If $u \in F_m$ for a certain $m \in \mathbb{Z}$, then $u = u - \theta u \in F_{m+1}$, therefore $u \in \bigcap_{m \in \mathbb{Z}} F_m = \{0\}$ because the filtration is separated, hence $u = 0$ and θ is injective.

To prove that θ is surjective, let $u \in F$. Set $u_0 = u$ and define $u_{n+1} = u_n - \theta u_n \in F_{n+1}$ inductively for all $n \geq 0$. From the notion of convergence seen in 2.1.3, we have $u_n \xrightarrow{n\to\infty} 0$. Let $v_n = \sum_{i=0}^{n} u_i$ for each $n \geq 0$. For any $p \geq 0$ choose $N_p = p$ and let $n, m \geq N_p$ while also assuming $n \geq m$ without losing of generality. Then

$$v_n - v_m = \sum_{i=0}^{n} u_i - \sum_{i=0}^{m} u_i = \sum_{i=m+1}^{n} u_i \in F_{m+1} \subseteq F_{p+1}$$

which implies that $\{v_n\}_{n\geq 0}$ is a Cauchy sequence in F: by 2.1.3 there exists $v \in F$ such that $v_n \xrightarrow{n\to\infty} v$. Using the same argument and remembering that θ is filtration preserving, the sequence $\{\theta v_n\}_{n\geq 0}$ is also a Cauchy sequence, therefore it must be $\theta v_n \xrightarrow{n\to\infty} \theta v$. By definition of u_n we see that

$$\begin{aligned}u = u_0 = u_1 + \theta u_0 &= u_2 + \theta u_1 + \theta u_0 \\ &= \dots = u_{n+1} + \sum_{i=0}^{n} \theta u_i \\ &= u_{n+1} + \theta v_n\end{aligned}$$

for any $n \geq 0$. Therefore $u - \theta v_n = u_{n+1} + \theta v_n - \theta v_n = u_{n+1} \xrightarrow{n\to\infty} 0$, i.e. $\theta v_n \xrightarrow{n\to\infty} u$. Hence $u = \theta v$.

Step 7. The map θ is an automorphism.

We finally show that $\theta(u_1 u_2) = \theta(u_1)\theta(u_2)$ for each $u_1, u_2 \in F$. Note that F is a D-module with the obvious action $du = d(u)$ for all $d \in D$ and $u \in F$, therefore $F \boxtimes F$ and $F \otimes F$ are $\mathfrak{U}(D) \otimes \mathfrak{U}(D)$ and $\mathfrak{U}(D)$-modules respectively as reminded in §2.2. By the fundamental property of the tensor product, the multiplication in F gives rise to a linear map $\nu : F \otimes F \to F$.

$$\begin{array}{ccc} F \times F & \longrightarrow & F \otimes F \\ & \searrow^{\cdot} & \downarrow{\scriptstyle \nu} \\ & & F \end{array}$$

Observe that the identification $F = \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(\mathbb{K} \otimes \mathbb{K})$ implies ν is indeed the map seen in 2.2.6, which is a $\mathfrak{U}(\mathfrak{g})$-homomorphism. Since $D \subseteq \operatorname{der}(F)$, for each $d \in D$ and $u_1, u_2 \in F$ we have

$$\begin{aligned} d\big(\nu(u_1 \otimes u_2)\big) &= d(u_1 u_2) = (du_1)\, u_2 + u_1\,(du_2) \\ &= \nu\big((du_1) \otimes u_2\big) + \nu\big(u_1 \otimes (du_2)\big) \\ &= \nu\big((du_1) \otimes u_2 + u_1 \otimes (du_2)\big) \\ &= \nu\big(d(u_1 \otimes u_2)\big) \end{aligned}$$

therefore ν is also a $\mathfrak{U}(D)$-homomorphism. To avoid confusion with the previous notations, we remark that $u_1 \otimes u_2$ is here regarded as a formal element of $F \otimes F$, as the multiplication in F has been denoted with $u_1 u_2$ instead. The same holds true for $u_1 \boxtimes u_2 \in F \boxtimes F$ through the rest of the proof. Put in terms of ν mapping $F \boxtimes F \to F$, the map ν intertwines the action of Δl on $F \boxtimes F$ with the action of l on F for each $l \in \mathfrak{U}(D)$ as follows:

$$\begin{aligned} l(u_1 u_2) = l\big(\nu(u_1 \otimes u_2)\big) &= \nu\big(l(u_1 \otimes u_2)\big) \\ &= \nu\big(\Delta(l)(u_1 \boxtimes u_2)\big). \end{aligned} \tag{3.3}$$

We will extend the Lie algebra homomorphisms $\gamma, \gamma_1 : \mathfrak{g} \to D$ into associative algebra homomorphisms $\mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(D)$ through the fundamental property of the enveloping algebra.

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\gamma} & D \\ {\scriptstyle i_{\mathfrak{g}}}\downarrow & \searrow^{i_D \circ \gamma} & \downarrow{\scriptstyle i_D} \\ \mathfrak{U}(\mathfrak{g}) & \dashrightarrow & \mathfrak{U}(D) \end{array} \qquad \begin{array}{ccc} \mathfrak{g} & \xrightarrow{\gamma_1} & D \\ {\scriptstyle i_{\mathfrak{g}}}\downarrow & \searrow^{i_D \circ \gamma_1} & \downarrow{\scriptstyle i_D} \\ \mathfrak{U}(\mathfrak{g}) & \dashrightarrow & \mathfrak{U}(D) \end{array}$$

Then $\gamma \boxtimes \gamma$ and $\gamma_1 \boxtimes \gamma_1$ are homomorphisms $\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(D) \otimes \mathfrak{U}(D)$. For each $x \in \mathfrak{g}$, we have

$$\begin{aligned} \Delta\big(\gamma(x)\big) &= \gamma(x) \otimes 1_{\mathfrak{U}(D)} + 1_{\mathfrak{U}(D)} \otimes \gamma(x) \\ (\gamma \boxtimes \gamma)(\Delta x) &= (\gamma \boxtimes \gamma)(x \otimes 1_{\mathfrak{U}(\mathfrak{g})} + 1_{\mathfrak{U}(\mathfrak{g})} \otimes x) = \gamma(x) \otimes \gamma(1_{\mathfrak{U}(\mathfrak{g})}) + \gamma(1_{\mathfrak{U}(\mathfrak{g})}) \otimes \gamma(x) \\ &= \gamma(x) \otimes 1_{\mathfrak{U}(D)} + 1_{\mathfrak{U}(D)} \otimes \gamma(x) \end{aligned}$$

because γ must map $1_{\mathfrak{U}(\mathfrak{g})} \mapsto 1_{\mathfrak{U}(D)}$. The same holds for γ_1 and we know that Δ also extends to an associative algebra homomorphism, therefore the identities

$$
\begin{aligned}
\Delta \circ \gamma &= (\gamma \boxtimes \gamma) \circ \Delta \\
\Delta \circ \gamma_1 &= (\gamma_1 \boxtimes \gamma_1) \circ \Delta
\end{aligned}
\tag{3.4}
$$

hold on $\mathfrak{U}(\mathfrak{g})$. Be mindful that we are (ab)using the symbol Δ to denote both maps $\mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$ and $\mathfrak{U}(D) \to \mathfrak{U}(D) \otimes \mathfrak{U}(D)$ as their action on any enveloping algebra is the same.

$$
\begin{array}{ccccccc}
\mathfrak{U}(\mathfrak{g}) & \xrightarrow{\Delta} & \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g}) & \quad & \mathfrak{U}(\mathfrak{g}) & \xrightarrow{\Delta} & \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g}) \\
{\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \gamma \boxtimes \gamma} & & {\scriptstyle \gamma_1}\downarrow & & \downarrow{\scriptstyle \gamma_1 \boxtimes \gamma_1} \\
\mathfrak{U}(D) & \xrightarrow{\Delta} & \mathfrak{U}(D) \otimes \mathfrak{U}(D) & & \mathfrak{U}(D) & \xrightarrow{\Delta} & \mathfrak{U}(D) \otimes \mathfrak{U}(D)
\end{array}
$$

For each $a \in \mathfrak{U}(\mathfrak{g})$ identities (3.3) and (3.4) imply

$$
\gamma(a) \circ \nu = \nu \circ \Delta(\gamma(a)) = \nu \circ \big((\gamma \boxtimes \gamma)(\Delta a)\big)
$$

and the same is also true swapping γ with γ_1. Since θ intertwines γ_1 with γ through the identity $\theta \circ \gamma_1(a) = \gamma(a) \circ \theta$ for each $a \in \mathfrak{U}(\mathfrak{g})$, then the map $\theta \boxtimes \theta : F \boxtimes F \to F \boxtimes F$ intertwines $\gamma_1 \boxtimes \gamma_1$ with $\gamma \boxtimes \gamma$ through the identity

$$
(\theta \boxtimes \theta) \circ \big(\gamma_1(a) \otimes \gamma_1(b)\big) = \big(\gamma(a) \otimes \gamma(b)\big) \circ (\theta \boxtimes \theta)
$$

for each $a, b \in \mathfrak{U}(\mathfrak{g})$. Let $\sigma : F \to \mathbb{K}$ be defined as previously. By identifying $\mathbb{K} \otimes \mathbb{K}$ with $\mathbb{K}$ through field multiplication, the map $\sigma \boxtimes \sigma$ acts as follows:

$$
\begin{aligned}
\sigma \boxtimes \sigma : F \boxtimes F &\to \mathbb{K} \\
u_1 \boxtimes u_2 &\mapsto u_1(1)\, u_2(1)
\end{aligned}
$$

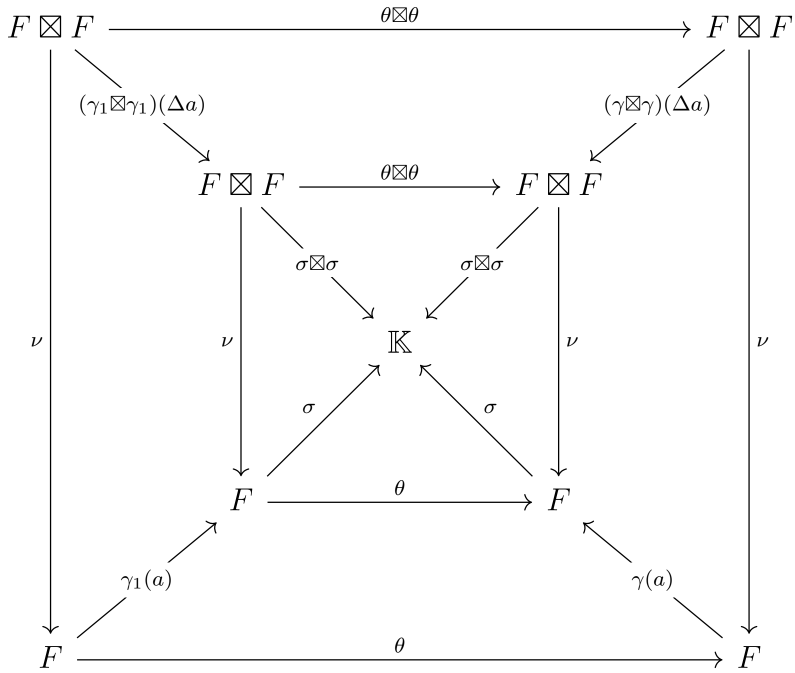
for each $u_1, u_2 \in F$. Note that $\sigma \boxtimes \sigma = \sigma \circ \nu$ because

$$
\begin{aligned}
\sigma\big(\nu(u_1 \otimes u_2)\big) &= \sigma(u_1 u_2) = (u_1 u_2)(1) = (u_1 \boxtimes u_2)(\Delta(1)) = (u_1 \boxtimes u_2)(1 \otimes 1) \\
&= u_1(1)\, u_2(1)
\end{aligned}
$$

for each $u_1, u_2 \in F$. Remembering that $\sigma = \sigma \circ \theta$ and therefore $\sigma \boxtimes \sigma = (\sigma \boxtimes \sigma) \circ (\theta \boxtimes \theta)$, we have

$$
\begin{aligned}
\sigma \circ \gamma(a) \circ \theta \circ \nu &= \sigma \circ \theta \circ \gamma_1(a) \circ \nu = \sigma \circ \gamma_1(a) \circ \nu \\
&= \sigma \circ \nu \circ \big((\gamma_1 \boxtimes \gamma_1)(\Delta a)\big) = (\sigma \boxtimes \sigma) \circ \big((\gamma_1 \boxtimes \gamma_1)(\Delta a)\big) \\
&= (\sigma \boxtimes \sigma) \circ (\theta \boxtimes \theta) \circ \big((\gamma_1 \boxtimes \gamma_1)(\Delta a)\big) \\
&= (\sigma \boxtimes \sigma) \circ \big((\gamma \boxtimes \gamma)(\Delta a)\big) \circ (\theta \boxtimes \theta) \\
&= \sigma \circ \nu \circ \big((\gamma \boxtimes \gamma)(\Delta a)\big) \circ (\theta \boxtimes \theta) \\
&= \sigma \circ \gamma(a) \circ \nu \circ (\theta \boxtimes \theta)
\end{aligned}
$$

for every $a \in \mathfrak{U}(\mathfrak{g})$, as shown in the diagram below.



Note that if $u \in F$ and $a \in \mathfrak{U}(\mathfrak{g})$ then

$$\big(\sigma \circ \gamma(a)\big)(u) = \big(\sigma(au)\big) = (au)(1) = u(a)$$

therefore if the identity $\big(\sigma\circ\gamma(a)\big)(u_1) = \big(\sigma\circ\gamma(a)\big)(u_2)$ holds for all $a \in \mathfrak{U}(\mathfrak{g})$, then it must be $u_1 = u_2 \in F$. This implies $\theta \circ \nu = \nu \circ (\theta \boxtimes \theta)$, i.e. for each $u_1, u_2 \in F$ we have $\theta(u_1 u_2) = \theta(u_1)\theta(u_2)$. □

Corollary 3.2.2. *Let γ_1 be as in 3.2.1. Then $\gamma_1 : \mathfrak{g} \to \gamma_1(\mathfrak{g})$ is a filtered Lie algebra isomorphism.*

Proof. It is sufficient to prove that γ_1 is injective and filtered. Let $\gamma_1(x) = 0$. From the theorem, $\theta^{-1} \circ \gamma(x) \circ \theta = 0$ which implies $\gamma(x) = 0$. Since γ is an isomorphism, $x = 0$.
Let $x \in \mathfrak{g}_p$. Then $\gamma(x) \in D_p$, also

$$\gamma_1(x)F_m = (\theta^{-1} \circ \gamma(x) \circ \theta)F_m \subseteq (\theta^{-1} \circ \gamma(x))F_m \subseteq \theta^{-1}F_{m+p} \subseteq F_{m+p}$$

for each $m \in \mathbb{Z}$ due to 3.2.1. Hence $\gamma_1(x) \in D_p$. □

3.3 The dual symmetric algebra $S(\mathfrak{g}/\mathfrak{h})^*$

Theorem 3.2.1 gives a canonical embedding of $\mathfrak{g}$ into the filtered Lie algebra D of derivations of the produced $\mathfrak{g}$-module F. We will use this result to realize $\mathfrak{g}$ into the algebra of derivations of $S(\mathfrak{g}/\mathfrak{h})^*$. In the hypothesis that $\operatorname{char}\mathbb{K} = 0$, the dual symmetric algebra $S(\mathfrak{g}/\mathfrak{h})^*$ is isomorphic to the algebra of formal power series on $\dim \mathfrak{g}/\mathfrak{h}$ variables.

Proposition 3.3.1. *Let $u_i \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V_i)$, $i = 1, 2$. Then*

$$\pi(u_1 \otimes u_2) = \pi u_1 \otimes \pi u_2.$$

Proof. Choose an ordered index set I, a basis $\{\bar{x}_i\}_{i\in I}$ for $\mathfrak{g}/\mathfrak{h}$, a set of class representatives $\{x_i\}_{i\in I}$, define the multi-indices set M and the monomials x^m, $\bar{x}^m$ for $m \in M$ as in §2.1. Remember that the map $\tau : S(\mathfrak{g}/\mathfrak{h}) \to \mathfrak{U}(\mathfrak{g})$ is defined by $\tau(\bar{x}^m) = x^m$ for each $m \in M$. By Lemma 1.3.3, we have

$$\Delta\big(\tau(\bar{x}^m)\big) = \Delta(x^m) = \sum_{0\le k\le m} \binom{m}{k} x^k \otimes x^{m-k}$$

$$(\tau \otimes \tau)(\Delta\bar{x}^m) = (\tau \otimes \tau)\left(\sum_{0\le k\le m} \binom{m}{k} \bar{x}^k \otimes \bar{x}^{m-k}\right) = \sum_{0\le k\le m} \binom{m}{k} x^k \otimes x^{m-k}$$

for each $m \in M$, i.e. $\Delta \circ \tau = (\tau \otimes \tau) \circ \Delta$.

$$\begin{array}{ccc} S(\mathfrak{g}/\mathfrak{h}) & \xrightarrow{\ \tau\ } & \mathfrak{U}(\mathfrak{g}) \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta} \\ S(\mathfrak{g}/\mathfrak{h}) \otimes S(\mathfrak{g}/\mathfrak{h}) & \xrightarrow[\tau\otimes\tau]{} & \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g}) \end{array}$$

Remember that for any $\mathfrak{h}$-module V, the map $\pi : \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V) \to \mathcal{P}_{\{0\}}^{\mathfrak{g}/\mathfrak{h}}(V)$ is defined by $(\pi u)(a) = u(\tau a)$ for each $u \in \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$ and $a \in S(\mathfrak{g}/\mathfrak{h})$. Then

$$\begin{aligned} (\pi u_1 \boxtimes \pi u_2)(a \otimes b) &= (\pi u_1)(a) \otimes (\pi u_2)(b) = u_1(\tau a) \otimes u_2(\tau b) \\ &= (u_1 \boxtimes u_2)(\tau a \otimes \tau b) \\ &= (u_1 \boxtimes u_2)\big((\tau \otimes \tau)(a \otimes b)\big) \end{aligned}$$

for each $a, b \in S(\mathfrak{g}/\mathfrak{h})$, which means $\pi u_1 \boxtimes \pi u_2 = (u_1 \boxtimes u_2) \circ (\tau \otimes \tau)$.

$$\begin{array}{ccc} S(\mathfrak{g}/\mathfrak{h}) \otimes S(\mathfrak{g}/\mathfrak{h}) & \xrightarrow{\ \tau\otimes\tau\ } & \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g}) \\ & {\scriptstyle \pi u_1 \boxtimes \pi u_2}\searrow & \downarrow{\scriptstyle u_1 \boxtimes u_2} \\ & & V_1 \otimes V_2 \end{array}$$

Therefore

$$\begin{aligned}(\pi(u_1 \otimes u_2))(a) &= (u_1 \otimes u_2)(\tau a) = (u_1 \boxtimes u_2)(\Delta(\tau a)) \\ &= (u_1 \boxtimes u_2)((\tau \otimes \tau)(\Delta a)) \\ &= (\pi u_1 \boxtimes \pi u_2)(\Delta a) \\ &= (\pi u_1 \otimes \pi u_2)(a)\end{aligned}$$

for each $a \in S(\mathfrak{g}/\mathfrak{h})$. □

Corollary 3.3.2. *The filtered associative algebras F and $S(\mathfrak{g}/\mathfrak{h})^*$ are isomorphic. If* $\operatorname{char} \mathbb{K} = 0$*, then F is isomorphic to the filtered algebra $\mathbb{K}[[\bar{x}_i]]_{i\in I}$, where $\{\bar{x}_i\}_{i\in I}$ is a basis for $\mathfrak{g}/\mathfrak{h}$.*

Proof. By considering $V_1 = V_2 = \mathbb{K}$ in 3.3.1 we have that π maps $F \to S(\mathfrak{g}/\mathfrak{h})^*$ and it satisfies $\pi(u_1 u_2) = (\pi u_1)(\pi u_2)$ for each $u_1, u_2 \in F$, i.e. the filtered linear isomorphism π is also an algebra isomorphism.

Choose an ordered index set I, a basis $\{\bar{x}_i\}_{i\in I}$ for $\mathfrak{g}/\mathfrak{h}$, define the multi-indices set M and the monomials $\bar{x}^m$ for $m \in M$ as in §2.1. For each $u \in S(\mathfrak{g}/\mathfrak{h})^*$, set $u_m = u(\bar{x}^m)$ for all $m \in M$. Let Φ be the map

$$\begin{aligned}\Phi : S(\mathfrak{g}/\mathfrak{h})^* &\to \mathbb{K}[[\bar{x}_i]]_{i\in I} \\ u &\mapsto \sum_{m\in M} \frac{u_m}{m!}\bar{x}^m.\end{aligned}$$

We show that Φ is an associative algebra isomorphism. The map Φ is clearly linear by the definition of the linear operations on $S(\mathfrak{g}/\mathfrak{h})^*$ and $\mathbb{K}[[\bar{x}_i]]_{i\in I}$. Since $\operatorname{char} \mathbb{K} = 0$, if $\Phi(u) = 0$ then it must be $u_m = 0$ for all $m \in M$, which implies $u = 0$. Now let $a \in \mathbb{K}[[\bar{x}_i]]_{i\in I}$: since a is a formal series in the variables $\{\bar{x}_i\}_{i\in I}$ with coefficients in $\mathbb{K}$, then it is $a = \sum_{m\in M} a_m \bar{x}^m$ where $a_m \in \mathbb{K}$ for all $m \in M$. If $u \in S(\mathfrak{g}/\mathfrak{h})^*$ is the linear map $u(\bar{x}^m) = m!\, a_m$ for each $m \in M$, then $\Phi(u) = a$. Therefore Φ is bijective. Now let $u, v \in S(\mathfrak{g}/\mathfrak{h})^*$. By Lemma 1.3.3, we have

$$\begin{aligned}(uv)_m = (uv)(\bar{x}^m) = (u \boxtimes v)(\Delta \bar{x}^m) &= (u \boxtimes v)\left(\sum_{0\le k\le m} \binom{m}{k} \bar{x}^k \otimes \bar{x}^{m-k}\right) \\ &= \sum_{0\le k\le m} \binom{m}{k} u(\bar{x}^k)\, v(\bar{x}^{m-k}) = \sum_{0\le k\le m} \binom{m}{k} u_k\, v_{m-k}.\end{aligned}$$

Then

$$\begin{aligned}
\Phi(uv) &= \sum_{m\in M} \frac{(uv)_m}{m!}\bar{x}^m = \sum_{m\in M}\left(\sum_{0\le k\le m} \frac{1}{m!}\binom{m}{k} u_k\, v_{m-k}\right)\bar{x}^m \\
&= \sum_{m\in M}\left(\sum_{0\le k\le m} \frac{u_k\, v_{m-k}}{k!\,(m-k)!}\right)\bar{x}^m \\
\Phi(u)\Phi(v) &= \left(\sum_{m\in M}\frac{u_m}{m!}\bar{x}^m\right)\left(\sum_{m\in M}\frac{v_m}{m!}\bar{x}^m\right) = \sum_{m\in M}\left(\sum_{0\le k\le m} \frac{u_k\, v_{m-k}}{k!\,(m-k)!}\right)\bar{x}^m
\end{aligned}$$

which means that Φ is an algebra isomorphism.

Lastly, we show that Φ is filtered. If $a = \sum_{m\in M} a_m \bar{x}^m$ define $\deg a = \min\{|m| \mid a_m \neq 0, m \in M\}$ and consider the downward filtration $\{K_m\}_{m\in\mathbb{Z}}$ on $\mathbb{K}[[\bar{x}_i]]_{i\in I}$ where $K_m = \{a \in \mathbb{K}[[\bar{x}_i]]_{i\in I} \mid \deg a \geq m\}$. Since $\deg(ab) = \deg a + \deg b$ then $\mathbb{K}[[\bar{x}_i]]_{i\in I}$ is a filtered associative algebra. If $u \in S(\mathfrak{g}/\mathfrak{h})^*_p$ then $u(a) = 0$ for each $a \in S_{p-1}(\mathfrak{g}/\mathfrak{h})$, therefore $u_m = 0$ for each $m \in M$ such that $|m| \leq p-1$. Hence $\deg \Phi(u) \geq p$ and $\Phi(u) \in K_p$, i.e $\Phi\big(S(\mathfrak{g}/\mathfrak{h})^*_p\big) \subseteq K_p$. Conversely, if $\Phi(u) \in K_p$ then $u_m = 0$ for each $m \leq p-1$ which immediately implies that $u(a) = 0$ for each $a \in S_{p-1}(\mathfrak{g}/\mathfrak{h})$. Therefore $u \in S(\mathfrak{g}/\mathfrak{h})^*_p$ and $K_p \subseteq \Phi\big(S(\mathfrak{g}/\mathfrak{h})^*_p\big)$. □

Thanks to 3.3.2, we may transfer the action of γ to $E = S(\mathfrak{g}/\mathfrak{h})^*$ through π. Rename the Lie subalgebra $D \subseteq \mathrm{der}(F)$ to $D(F)$ and define $D(E)$ analogously for derivations of E such that $E_m \to E_{m+p}$ for some p and all m. By the same arguments seen in 3.1.4, $D(E)$ is a filtered Lie algebra.

Remark 8. Since π is a filtered map, we have that $d \in D_p(F)$ if and only if $\pi \circ d \circ \pi^{-1} \in D_p(E)$ for any $p \in \mathbb{Z}$ as is shown below.

$$\begin{array}{ccc}
F_m & \xrightarrow{\ \pi\ } & E_m \\
{\scriptstyle d}\downarrow & & \downarrow{\scriptstyle \pi\circ d\circ\pi^{-1}} \\
F_{m+p} & \xrightarrow{\ \pi\ } & E_{m+p}
\end{array}$$

For each $x \in \mathfrak{g}$, define $\bar{\gamma}(x) = \pi \circ \gamma(x) \circ \pi^{-1}$.

Proposition 3.3.3. *The map* $\bar{\gamma} : \mathfrak{g} \to D(E) \cap \bar{\gamma}(\mathfrak{g})$ *is a filtered algebra isomorphism.*

Proof. For each $x, y \in \mathfrak{g}$ and $u \in E$ it is

$$\begin{aligned}
[\bar\gamma(x), \bar\gamma(y)]u &= [\pi \circ \gamma(x) \circ \pi^{-1}, \pi \circ \gamma(y) \circ \pi^{-1}]u \\
&= \pi\gamma(x)\pi^{-1}\pi\gamma(y)\pi^{-1}u - \pi\gamma(y)\pi^{-1}\pi\gamma(x)\pi^{-1}u \\
&= \pi\gamma(x)\gamma(y)\pi^{-1}u - \pi\gamma(y)\gamma(x)\pi^{-1}u \\
&= \pi\big(\gamma(x)\gamma(y) - \gamma(y)\gamma(x)\big)\pi^{-1}u \qquad (3.5)\\
&= \pi\big([\gamma(x), \gamma(y)]\big)\pi^{-1}u \\
&= \pi\big(\gamma([x, y])\big)\pi^{-1}u \\
&= \bar\gamma([x, y])u
\end{aligned}$$

so $\bar\gamma$ is a Lie algebra homomorphism. We have $\bar\gamma(x) = 0$ if and only if $\gamma(x) = 0$, which by 3.2.1 only happens if $x = 0$ so $\bar\gamma$ is injective. Finally, Remark 8 shows that $\gamma(x) \in D_p(F)$ if and only if $\bar\gamma(x) \in D_p(E)$ for any $p \in \mathbb{Z}$ and $x \in \mathfrak{g}$. Therefore $\bar\gamma$ is filtered because γ is. □

3.3.1 Realization theorem

For each $y \in \mathfrak{g}$, let $\bar y$ be its equivalence class in $\mathfrak{g}/\mathfrak{h}$. Remember that $E = \mathcal{P}^{\mathfrak{g}/\mathfrak{h}}_{\{0\}}(\mathbb{K})$ is a $\mathfrak{g}/\mathfrak{h}$-module with the action $(\bar y u)(a) = u(a\bar y)$ for each $y \in \mathfrak{g}$, $u \in E$ and $a \in S(\mathfrak{g}/\mathfrak{h})$. Let δ be the corresponding representation, i.e. $\delta(\bar y)u = \bar y u$ for all $y \in \mathfrak{g}$, $u \in E$. We can substitute γ with δ in Theorem 3.2.1 to find that δ is a filtered Lie algebra isomorphism of $\mathfrak{g}/\mathfrak{h}$ into a subalgebra of $D(E)$. Note that $\mathfrak{g}/\mathfrak{h}$ is abelian, hence $\delta(\bar x)\delta(\bar y) = \delta(\bar y)\delta(\bar x)$ for each $x, y \in \mathfrak{g}$.

Proposition 3.3.4. *Let $y \in \mathfrak{g}$. Then $\bar\gamma(y) - \delta(\bar y) \in D_0(E)$.*

Proof. Choose an ordered index set I, a basis $\{\bar x_i\}_{i\in I}$ for $\mathfrak{g}/\mathfrak{h}$, a set of class representatives $\{x_i\}_{i\in I}$, define the multi-indices set M and the monomials x^m, $\bar x^m$ for $m \in M$ as in §2.1. Let $i \in I$ and define $r \in M$ by $r_j = \delta_{ij}$ where δ_{ij} is the Kronecker delta. Let $u \in E_k$ and $w = \pi^{-1}(u)$. If $m \in M$ is such that $|m| \le k - 1$, then

$$\begin{aligned}
\big(\bar\gamma(x_i)u\big)(\bar x^m) &= \big((\pi \circ \gamma(x_i) \circ \pi^{-1})u\big)(\bar x^m) = \big((\pi \circ \gamma(x_i))w\big)(\bar x^m) \\
&= \big(\gamma(x_i)w\big)(\tau\bar x^m) = \big(\gamma(x_i)w\big)(x^m) = (x_i w)(x^m) \\
&= w(x^m x_i) \\
\big(\delta(\bar x_i)u\big)(\bar x^m) &= (\bar x_i u)(\bar x^m) = u(\bar x^m \bar x_i) = u(\bar x^{m+r}) = u(\tau x^{m+r}) \\
&= w(x^{m+r})
\end{aligned}$$

because $\mathfrak{g}/\mathfrak{h}$ is abelian. By the PBW Theorem, we have

$$x^m x_i = x^{m+r} \quad \big(\text{modulo } \mathfrak{U}_{|m|}(\mathfrak{g})\big)$$

and therefore $\big(\bar{\gamma}(x_i)u\big)(\bar{x}^m) = \big(\delta(\bar{x}_i)u\big)(\bar{x}^m)$, as $w = 0$ on $\mathfrak{U}_{|m|}(\mathfrak{g})$. This implies that $\bar{\gamma}(x_i)u = \delta(\bar{x}_i)u$ on $S_{k-1}(\mathfrak{g}/\mathfrak{h})$, i.e. $\big(\bar{\gamma}(x_i) - \delta(\bar{x}_i)\big)u \in E_k$ and thus $\bar{\gamma}(x_i) - \delta(\bar{x}_i) \in D_0(E)$. Since this is true for any $i \in I$, it must be $\bar{\gamma}(x) - \delta(\bar{x}) \in D_0(E)$ for all $x \in \mathfrak{g}/\mathfrak{h}$. But $\bar{\gamma}$ is a filtered Lie algebra isomorphism, so we have $\bar{\gamma}(h) - \delta(\bar{h}) = \bar{\gamma}(h) \in D_0(E)$ for all $h \in \mathfrak{h}$ and therefore $\bar{\gamma}(y) - \delta(\bar{y}) \in D_0(E)$ for all $y \in \mathfrak{g}$. □

Corollary 3.3.5 (Guillemin-Sternberg-Rim)**.** *There exists a Lie algebra homomorphism $\bar{\beta} : \mathfrak{g} \to D(E)$ such that $\bar{\beta}(y) - \delta(\bar{y}) \in D_0(E)$ for all $y \in \mathfrak{g}$. If $\bar{\alpha}, \bar{\beta} : \mathfrak{g} \to D(E)$ satisfy the previous conditions then there exists a unique filtered algebra automorphism $\theta : E \to E$ such that $\theta \circ \bar{\alpha}(y) = \bar{\beta}(y) \circ \theta$ for all $y \in \mathfrak{g}$, also $\bar{\alpha}$ and $\bar{\beta}$ are filtered Lie algebra isomorphisms of $\mathfrak{g}$ with a subalgebra of $D(E)$.*

Proof. By 3.3.4, choosing $\bar{\beta} = \bar{\gamma} : \mathfrak{g} \to D(E)$ satisfies the requirement of a Lie algebra homomorphism such that $\bar{\beta}(y) = \delta(\bar{y})$ $\big($modulo $D_0(E)\big)$ for all $y \in \mathfrak{g}$. Moreover, if $\bar{\alpha}, \bar{\beta}$ are as such, then

$$\bar{\alpha}(y) = \bar{\beta}(y) = \delta(\bar{y}) = \bar{\gamma}(y) \quad \big(\text{modulo } D_0(E)\big)$$

for all $y \in \mathfrak{g}$. Define $\alpha(y) = \pi^{-1} \circ \bar{\alpha}(y) \circ \pi$ and $\beta(y) = \pi^{-1} \circ \bar{\beta}(y) \circ \pi$ for each $y \in \mathfrak{g}$. By computations equivalent to those seen in (3.5), the maps $\alpha, \beta : \mathfrak{g} \to D(F)$ are Lie algebra homomorphisms.

$$\begin{array}{ccc} F & \xrightarrow{\pi} & E \\ {\scriptstyle \alpha(y)}\downarrow & & \downarrow{\scriptstyle \bar{\alpha}(y)} \\ F & \xrightarrow{\pi} & E \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{\pi} & E \\ {\scriptstyle \beta(y)}\downarrow & & \downarrow{\scriptstyle \bar{\beta}(y)} \\ F & \xrightarrow{\pi} & E \end{array}$$

Thanks to Remark 8, it follows that $\alpha(y) = \beta(y) = \gamma(y)$ $\big($modulo $D_0(F)\big)$ for all $y \in \mathfrak{g}$. By Theorem 3.2.1, there exist unique θ_1, θ_2 filtered algebra automorphisms of F such that $\theta_1 \circ \alpha(y) = \gamma(y) \circ \theta_1$ and $\theta_2 \circ \beta(y) = \gamma(y) \circ \theta_2$ for each $y \in \mathfrak{g}$. Therefore $\gamma(y) = \theta_1 \circ \alpha(y) \circ \theta_1^{-1}$ and $\gamma(y) = \theta_2 \circ \beta(y) \circ \theta_2^{-1}$, which imply the following identities:

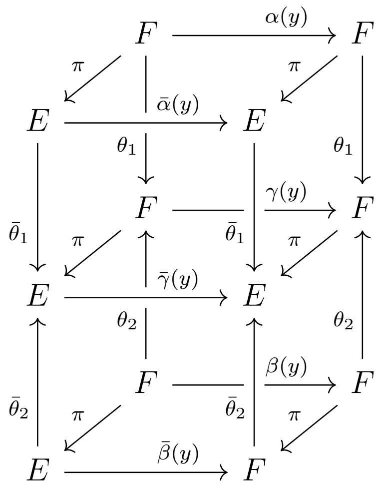
$$\begin{aligned} \theta_1 \circ \alpha(y) \circ \theta_1^{-1} &= \theta_2 \circ \beta(y) \circ \theta_2^{-1} \\ \theta_1 \circ \pi^{-1} \circ \bar{\alpha}(y) \circ \pi \circ \theta_1^{-1} &= \theta_2 \circ \pi^{-1} \circ \bar{\beta}(y) \circ \pi \circ \theta_2^{-1} \\ \pi \circ \theta_2^{-1} \circ \theta_1 \circ \pi^{-1} \circ \bar{\alpha}(y) &= \bar{\beta}(y) \circ \pi \circ \theta_2^{-1} \circ \theta_1 \circ \pi^{-1}. \end{aligned}$$

Define $\bar{\theta}_1 = \pi \circ \theta_1 \circ \pi^{-1}$ and $\bar{\theta}_2 = \pi \circ \theta_2 \circ \pi^{-1}$. Then

$$\bar{\theta}_2^{-1} \circ \bar{\theta}_1 \circ \bar{\alpha}(y) = \bar{\beta}(y) \circ \bar{\theta}_2^{-1} \circ \bar{\theta}_1$$

for all $y \in \mathfrak{g}$. Since π is also multiplicative by 3.3.1, the maps $\bar{\theta}_1, \bar{\theta}_2 : E \to E$ are filtered algebra automorphisms. Define $\theta = \bar{\theta}_2^{-1} \circ \bar{\theta}_1$. We obtain that

$\theta : E \to E$ is the unique filtered algebra automorphism such that $\theta \circ \bar{\alpha}(y) = \bar{\beta}(y) \circ \theta$ for each $y \in \mathfrak{g}$.



Moreover, by 3.2.2 it follows that α and β are filtered Lie algebra isomorphisms of $\mathfrak{g}$ with a subalgebra of $D(F)$, therefore we may substitute $\bar{\gamma}$ with $\bar{\alpha}$ and $\bar{\beta}$ in 3.3.3 to obtain that $\bar{\alpha}$ and $\bar{\beta}$ are filtered Lie algebra isomorphisms of $\mathfrak{g}$ with a subalgebra of $D(E)$. □

Chapter 4

Systems of imprimitivity

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$ and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Let $F = \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(\mathbb{K})$ and let W be a $\mathfrak{g}$-module.

Definition 4.1 (System of imprimitivity). A *system of imprimitivity* based on $\mathfrak{g}/\mathfrak{h}$ for W is an F-module structure on W such that

$$x(fw) = (xf)w + f(xw)$$

for each $x \in \mathfrak{g}$, $f \in F$ and $w \in W$.

Proposition 4.0.1. *Let V be an $\mathfrak{h}$-module and let $W = \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$. Let F act on W as shown in 3.0.2. This F-module structure on W is a system of imprimitivity based on $\mathfrak{g}/\mathfrak{h}$ for W.*

Proof. The proposition is immediate by 2.2.5. □

Therefore systems of imprimitivity based on $\mathfrak{g}/\mathfrak{h}$ exist for any Lie algebra $\mathfrak{g}$ with a subalgebra $\mathfrak{h}$.

4.1 Filtered structure of imprimitivity

Let W be a $\mathfrak{g}$-module with a system of imprimitivity based on $\mathfrak{g}/\mathfrak{h}$. Set $W_p = F_pW$ for each $p \in \mathbb{Z}$ and let $\mathfrak{W} = \{W_p\}_{p\in\mathbb{Z}}$. Since $F_p = F$ for all $p \leq 0$ and obviously $FW = W$, then $\mathfrak{W}$ is a filtration on W such that $W_p = W$ for all $p \leq 0$. Moreover, we have $F_pW_q \subseteq W_{p+q}$ directly by 3.0.1. Theorem 3.2.1 implies $\mathfrak{g}F_p \subseteq F_{p-1}$ and $\mathfrak{h}F_p \subseteq F_p$ (γ is a filtered map and $\mathfrak{g} = \mathfrak{g}_{-1}$) therefore $\mathfrak{g}W_p \subseteq W_{p-1}$ and $\mathfrak{h}W_p \subseteq W_p$.

Lemma 4.1.1. *Let $w \in W_p$ for some $p \geq 1$. If $\mathfrak{g}w \in W_p$ then $w \in W_{p+1}$.*

Proof. Since $w \in W_p$ we may write

$$w = \sum_{j=1}^{k} f_j w_j \quad (\text{modulo } W_{p+1})$$

where $f_j \in F_p$ for all $j \in \{1, \ldots, k\}$ and $\{w_j\}_{j\in\{1,\ldots,k\}}$ is linearly independent in W (modulo W_1), i.e. $\{\bar{w}_j\}_{j\in\{1,\ldots,k\}}$ is linearly independent in W_0/W_1 with $w_j \in \bar{w}_j$ for each j. To obtain such a form for w, simply replace elements $\bar{w}_j$ as linear combinations of the others until it is no longer possible or the result is null (in which case it is $w \in W_{p+1}$ and the proof concluded), then collect the coefficients f_j. As seen earlier, $\mathfrak{g}W_p \subseteq W_{p-1}$ and thus

$$yw = \sum_{j=1}^{k} (yf_j) w_j \quad (\text{modulo } W_p)$$

for any $y \in \mathfrak{g}$. However, by hypothesis we know that $yw \in W_p$, hence

$$\sum_{j=1}^{k} (yf_j) w_j = 0 \quad (\text{modulo } W_p).$$

By iterating, we have

$$0 = y_1 \ldots y_q w = \sum_{j=1}^{k} (y_1 \ldots y_q f_j) w_j \quad (\text{modulo } W_{p-q+1})$$

for any product of q elements of $\mathfrak{g}$. Therefore

$$0 = aw = \sum_{j=1}^{k} (af_j) w_j \quad (\text{modulo } W_1)$$

for any $a \in \mathfrak{U}_p(\mathfrak{g})$. Note that $af_j = f_j(a)\, e$ (modulo F_1) for all $j \in \{1, \ldots, k\}$ by Remark 7, as $(af_j)(1) = f_j(a)$. Then $(af_j)w_j = f_j(a)w_j$ (modulo W_1), hence

$$0 = \sum_{j=1}^{k} (af_j) w_j = \sum_{j=1}^{k} f_j(a) w_j \quad (\text{modulo } W_1)$$

which imply $f_j(a) = 0$ for all $j \in \{1, \ldots, k\}$ by the linear independence (modulo W_1) of $\{w_j\}$. Since this holds for all $a \in \mathfrak{U}_p(\mathfrak{g})$, then $f_j \in F_{p+1}$ for all $j \in \{1, \ldots, k\}$ and so $w \in W_{p+1}$. □

Assume the filtration $\mathfrak{W}$ on W is separated. Define $V = W_0/W_1$. Ignoring the trivial case $W = \{0\}$, then $V \neq \{0\}$. Indeed, if $W_0 = W_1$ then $\mathfrak{g}W_1 \subseteq W_0 = W_1$ and therefore $W_0 = W_1 \subseteq W_2$ by Lemma 4.1.1, which would imply $W = W_p$ for all $p \in \mathbb{Z}$ by iteration. But this is absurd if $\mathfrak{W}$ is separated.

Proposition 4.1.2. *The quotient space V is an $\mathfrak{h}$-module and the projection $\sigma : W \to V$ is an $\mathfrak{h}$-homomorphism.*

Proof. For each $h \in \mathfrak{h}$ and $w \in W$, define $h\sigma(w) = \sigma(hw)$. If $w_1, w_2 \in W$ are such that $w_1 - w_2 \in W_1$, then $h(w_1 - w_2) \in W_1$ and therefore $\sigma(hw_1) = \sigma(hw_2)$. Moreover, if $h, k \in \mathfrak{h}$ and $w \in W$ then

$$\begin{aligned}[h,k]\sigma(w) &= \sigma([h,k]w) = \sigma(hkw - khw) = \sigma(hkw) - \sigma(khw) \\ &= h\sigma(kw) - k\sigma(hw) = (hk - kh)\sigma(w)\end{aligned}$$

which means that the Lie module action of $\mathfrak{h}$ on V is well-defined. Then the projection σ is an $\mathfrak{h}$-homomorphism by its own definition. □

4.2 An imprimitivity embedding theorem

For the rest of this section, we will set $U = \mathcal{P}_{\mathfrak{h}}^{\mathfrak{g}}(V)$. By denoting with φ the $\mathfrak{h}$-homomorphism produced by V (which maps U into V and is defined by $\varphi(u) = u(1)$ for each $u \in U$), there exists a unique $\mathfrak{g}$-homomorphism θ such that $\sigma = \theta \circ \varphi$, as seen in §2. We also know that θ is defined by $(\theta w)(a) = \sigma(aw)$ for all $w \in W$ and $a \in \mathfrak{U}(\mathfrak{g})$.

Lemma 4.2.1. *For each $w \in W$, $w \in W_p$ if and only if $\theta w \in U_p$.*

Proof. Let $w \in W_p$. For each $a \in \mathfrak{U}_{p-1}(\mathfrak{g})$ we have $aw \in W_1$ so that $(\theta w)(a) = \sigma(aw) = 0$, therefore $\theta w \in U_p$. This implies that θ is a filtered $\mathfrak{g}$-homomorphism. Now let $\theta w \in U_p$. We show by induction on p that $w \in W_p$ for all $p \in \mathbb{Z}$. This is trivial for $p \leq 0$ and also true for $p = 1$, as $\theta w \in U_1$ implies $(\theta w)(1) = \sigma(w) = 0$, i.e. $w \in W_1$. Suppose our result to be true for some $p > 1$ and let $\theta w \in U_{p+1}$. Then $\theta w \in U_p$ and $w \in W_p$ by induction hypothesis. Since θ is a $\mathfrak{g}$-homomorphism, we have $\theta(xw) = x(\theta w) \in U_p$ for all $x \in \mathfrak{g}$. By induction hypothesis again, $xw \in W_p$ for any $x \in \mathfrak{g}$, thus $w \in W_{p+1}$ by Lemma 4.1.1. □

Corollary 4.2.2. *The map θ is injective.*

Proof. If $\theta w = 0$, then clearly $(\theta w)(a) = 0$ for all $a \in \mathfrak{U}(\mathfrak{g})$, i.e. $\theta w \in U_p$ for all $p \in \mathbb{Z}$. By Lemma 4.2.1, $w \in W_p$ for all $p \in \mathbb{Z}$, which means $w = 0$ because the filtration $\mathfrak{W}$ is separated. □

Lemma 4.2.3. *The map θ is an F-homomorphism.*

Proof. Similarly to Step 7 of the proof for Theorem 3.2.1, consider the $\mathfrak{U}(\mathfrak{g})\otimes\mathfrak{U}(\mathfrak{g})$-modules $F\boxtimes W$ and $F\boxtimes U$, also define $\nu : F\boxtimes W \to W$ by $\nu(f\boxtimes w) = fw$ for each $f \in F, w \in W$. Since the F-module structure on W is a system of imprimitivity based on $\mathfrak{g}/\mathfrak{h}$, we have $x(fw) = (xf)w + f(xw)$ for any $x \in \mathfrak{g}$, $f \in F$ and $w \in W$. This relation might be written in terms of ν and the coproduct Δ as follows:

$$\begin{aligned} x\nu(f\boxtimes w) &= x(fw) = (xf)w + f(xw) = \nu\big((xf)\boxtimes w\big) + \nu\big(f\boxtimes(xw)\big) \\ &= \nu\big((xf)\boxtimes w + f\boxtimes(xw)\big) = \nu\big((x\otimes 1 + 1\otimes x)(f\boxtimes w)\big) \\ &= \nu\big(\Delta(x)(f\boxtimes w)\big) \end{aligned}$$

for each $x \in \mathfrak{g}$, $f \in F$ and $w \in W$. Then $x\nu(g) = \nu\big((\Delta x)g\big)$ for all $x \in \mathfrak{g}$ and $g \in F\boxtimes W$. Moreover, we have

$$\begin{aligned} xy\nu(g) &= x\nu\big(\Delta(y)g\big) = \nu\big(\Delta(x)\Delta(y)g\big) \\ &= \nu\big(\Delta(xy)g\big) \end{aligned}$$

for all $x, y \in \mathfrak{g}$ and $g \in F\boxtimes W$, therefore $a\nu(g) = \nu(\Delta(a)g)$ for any $a \in \mathfrak{U}(\mathfrak{g})$ and $g \in F\boxtimes W$. By Remark 7, for any $f \in F$ it is $f = f(1)e$ (modulo F_1), which means that $fw = f(1)w$ (modulo W_1) and thus $\sigma(fw) = \sigma(f(1)w)$ for any $w \in W$. Then

$$\begin{aligned} \big(f\boxtimes(\theta w)\big)(a\otimes b) &= f(a)\big((\theta w)(b)\big) = f(a)\sigma(bw) = \sigma\big(f(a)(bw)\big) \\ &= \sigma\big((af)(1)\,(bw)\big) = \sigma\big((af)(bw)\big) = \sigma\big(\nu(af\boxtimes bw)\big) \\ &= \sigma\big(\nu((a\otimes b)(f\boxtimes w))\big) \end{aligned}$$

for all $a, b \in \mathfrak{U}(\mathfrak{g})$, $f \in F$ and $w \in W$. Hence $(f\boxtimes\theta w)(c) = \sigma\big(\nu\big(c(f\boxtimes w)\big)\big)$ for any $c \in \mathfrak{U}(\mathfrak{g})\otimes\mathfrak{U}(\mathfrak{g})$ and therefore

$$\begin{aligned} \big(f(\theta w)\big)(a) &= (f\boxtimes\theta w)(\Delta(a)) = \sigma\big(\nu\big(\Delta(a)(f\boxtimes w)\big)\big) \\ &= \sigma\big(a\nu(f\boxtimes w)\big) = \sigma\big(a(fw)\big) \\ &= \big(\theta(fw)\big)(a) \end{aligned}$$

for all $a \in \mathfrak{U}(\mathfrak{g})$, $f \in F$ and $w \in W$. □

Lemma 4.2.4. *Let $X \subseteq U$ be such that $FX \subseteq X$ and $\varphi(X) = V$. Suppose one of the following hypotheses:*
(1) X is closed in the finite-open topology of U;
(2) $\dim V < \infty$.
Then $X = U$.

Proof. Choose an ordered index set I, a basis $\{\bar{x}_i\}_{i\in I}$ for $\mathfrak{g}/\mathfrak{h}$, a set of class representatives $\{x_i\}_{i\in I}$, define the multi-indices set M and the monomials x^m for $m \in M$ as in §2.1. Remember that $\{x^m\}_{m\in M}$ is a basis for $\mathfrak{U}(\mathfrak{g})$ as a $\mathfrak{U}(\mathfrak{h})$-module, as seen in 2.1.1. For any $m \in M$, define $f_m \in F$ by $f_m(x^l) = \delta_{lm}$ for all $l \in M$, where δ_{lm} is the Kronecker delta. If $u \in U$, we have

$$\begin{aligned}(f_m u)(x^l) = (f_m \boxtimes u)(\Delta(x^l)) &= (f_m \boxtimes u)\left(\sum_{0\le k\le l} \binom{l}{k} x^k \otimes x^{l-k}\right)\\ &= \sum_{0\le k\le l} \binom{l}{k} f_m(x^k) u(x^{l-k}) = \sum_{0\le k\le l} \binom{l}{k} \delta_{km} u(x^{l-k})\end{aligned}$$

for all $l, m \in M$. The above formula imply that $(f_m u)(x^l) = 0$ unless $m \le l$, in which case it is $(f_m u)(x^l) = \binom{l}{m} u(x^{l-m})$. In particular, $(f_m u)(x^m) = u(1)$.

Now let $\{u_m\}_{m\in M} \subseteq U$. Be mindful that M is partially ordered and I has any cardinality, therefore $\{u_m\}_{m\in M}$ is not a sequence. Consider the unordered sum $\sum_{m\in M} f_m u_m$: for every x^l with $l \in M$, we have

$$\sum_{m\in M} (f_m u_m)(x^l) = \sum_{m\le l} (f_m u_m)(x^l)$$

where the sum on the right is finite. We can thus define the element $u^* \in U$ by $u^*(x^l) = \sum_{m\le l} (f_m u_m)(x^l)$ for all $l \in M$. We will show that the unordered sum $\sum_{m\in M} f_m u_m$ converges to u^*. For an unordered sum to converge in a topological vector space, it must happen that for any neighbourhood of 0 there exists a finite index subset $L \subseteq M$ such that for any other finite index subset H where $L \subseteq H \subseteq M$ we have that

$$u^* - \sum_{m\in H} f_m u_m$$

belongs in the neighbourhood. Any neighbourhood of 0 in the finite-open topology is of the form $\mathcal{U}(K, A) = \{u \in U \mid u(K) \subseteq A\}$ where $K \subseteq \mathfrak{U}(\mathfrak{g})$ is finite and $\{0\} \subseteq A \subseteq V$, so we may assume that $K = \{y_1, \dots, y_p\}$. For each $j \in \{1, \dots, p\}$ we can write $y_j = \sum_{m\in M_j} a_m^{(j)} x^m$ where $a_m^{(j)} \in \mathfrak{U}(\mathfrak{h})$ and the M_j are finite subsets of M. Let $M^* = \bigcup_{j=1}^{p} M_j$ and let $L = \{m \in M \mid \exists\, l \in M^* : m \le l\}$. Clearly M^* and therefore L are both finite. For any H finite such that $L \subseteq H \subseteq M$ and for any $l \in L$ we have

$$\begin{aligned} u^*(x^l) - \sum_{m\in H}(f_m u_m)(x^l) &= \sum_{m\leq l}(f_m u_m)(x^l) - \sum_{m\in H}(f_m u_m)(x^l) \\ &= \sum_{m\leq l}(f_m u_m)(x^l) - \sum_{\substack{m\in H\\ m\leq l}}(f_m u_m)(x^l) \\ &= \sum_{\substack{m\in M\smallsetminus H\\ m\leq l}}(f_m u_m)(x^l) = 0 \end{aligned}$$

because the last sum is empty. Therefore

$$u^*(y_j) - \sum_{m\in H}(f_m u_m)(y_j) = 0$$

for all $j \in \{1, \ldots, p\}$, which implies that

$$u^* - \sum_{m\in H} f_m u_m \in \mathcal{U}(K, A)$$

for all H finite where $L \subseteq H \subseteq M$. Hence the unordered sum $\sum_{m\in M} f_m u_m$ converges in the finite-open topology to some $u^* \in U$ for any $\{u_m\}_{m\in M} \subseteq U$.

Choose a linear map $\alpha : V \to X$ where $\varphi \circ \alpha = \mathrm{id}_V$. Such a map always exists, since $\varphi(X) = V$ by hypothesis implies that for any $v \in V$ there exists a linear map in X that sends $1 \mapsto v$ and thus we may choose $\alpha(v) \in X$ to be said map. Clearly we will have $(\varphi \circ \alpha)(v) = \varphi(\alpha(v)) = \big(\alpha(v)\big)(1) = v$. Let $u \in U$. Proceeding inductively on $p \geq 0$, we will define a family $\{w_m\}_{m\in M} \subseteq \alpha V$ and a sequence $\{u_p\}_{p\geq 0}$ in U such that

$$u_p = u - \sum_{|m|\leq p} f_m w_m \in U_{p+1}$$

for all $p \geq 0$. As shown earlier, the unordered sum in the expression converges and therefore this is always a well-defined element of U. If $p = 0$, set $w_0 = \alpha(u(1))$ so that $w_0(1) = \varphi(w_0) = \varphi\big(\alpha(u(1))\big) = u(1)$. Then

$$u_0(1) = u(1) - (f_0 w_0)(1) = u(1) - w_0(1) = 0$$

hence $u_0 \in U_1$. Now suppose that $\{w_m\}_{m\in M} \subseteq \alpha V$ have been defined for $|m| \leq p$ in such a way that $u_p \in U_{p+1}$. For $|m| = p + 1$, set $w_m = \alpha(u_p(x^m))$ so that $w_m(1) = u_p(x^m)$. Then for any $k \in M$ such that $|k| \leq p+1$ it must be

$(f_m w_m)(x^k) = \delta_{mk}\, w_m(1) = \delta_{mk}\, u_p(x^m)$ because $m \leq k$ if and only if $m = k$. Therefore

$$\begin{aligned} u_{p+1}(x^k) &= u(x^k) - \sum_{|m| \leq p+1} (f_m w_m)(x^k) \\ &= u_p(x^k) + \sum_{|m| \leq p} (f_m w_m)(x^k) - \sum_{|m| \leq p+1} (f_m w_m)(x^k) \\ &= u_p(x^k) - \sum_{|m| = p+1} (f_m w_m)(x^k) = u_p(x^k) - \sum_{|m|=p+1} \delta_{mk} u_p(x^m) \\ &= u_p(x^k) - u_p(x^k) = 0 \end{aligned}$$

for any $k \leq p+1$, which implies $u_{p+1} \in U_{p+2}$. It follows that $u_p \xrightarrow{p\to\infty} 0$ in the filtration topology and therefore also in the finite-open topology by 2.1.2. Hence if $v_p = \sum_{|m|\leq p} f_m w_m$ then the sequence $\{v_p\}_{p\geq 0}$ converges to u in the filtration and finite-open topologies. Moreover, the unordered sum $\sum_{m\in M} f_m w_m$ also converges to u as

$$\sum_{m\in M} (f_m w_m)(x^k) = v_p(x^k) = u(x^k) - u_p(x^k) = u(x^k)$$

for any $k \in M$ with $|k| = p$.

Suppose hypothesis (1) is true. Then X is closed in the finite-open topology of U, which implies that the limit of any convergent sequence in X belongs to X. Since $w_m \in \alpha V \subseteq X$ and $FX \subseteq X$, we have that $\{f_m w_m\}_{m\in M} \subseteq X$ and therefore $\{v_p\}_{p\geq 0}$ is a sequence in X. It follows that $u \in X$ because $v_p \xrightarrow{p\to\infty} u$.

Now suppose hypothesis (2). Then $\dim V < \infty$, which implies that $\dim \alpha V \leq \dim V < \infty$. Choose a basis $\{w_{(1)}, \ldots, w_{(q)}\}$ for αV and write

$$w_m = \sum_{j=1}^{q} \lambda_{jm} w_{(j)}$$

for all $m \in M$, where $\lambda_{jm} \in \mathbb{K}$ for any j and m. Define $f_{(1)}, \ldots, f_{(q)} \in F$ by

$f_{(j)}(x^m) = \lambda_{jm}$ for each $j \in \{1, \ldots, q\}$ and $m \in M$. It follows that

$$\begin{aligned}\sum_{j=1}^{q}(f_{(j)}w_{(j)})(x^m) &= \sum_{j=1}^{q}\sum_{k\le m}\binom{m}{k}f_{(j)}(x^k)w_{(j)}(x^{m-k})\\ &= \sum_{k\le m}\binom{m}{k}\sum_{j=1}^{q}\lambda_{jk}w_{(j)}(x^{m-k})\\ &= \sum_{k\le m}\binom{m}{k}w_k(x^{m-k}) = \sum_{k\le m}(f_k w_k)(x^m)\\ &= \sum_{k\in M}(f_k w_k)(x^m) = u(x^m)\end{aligned}$$

for each $m \in M$. Therefore $u = \sum_{j=1}^{q} f_{(j)}w_{(j)}$, but $w_{(j)} \in \alpha V \subseteq X$ and $FX \subseteq X$, hence $u \in X$. □

Thanks to this lemma, we reach the following main result on imprimitivity.

Theorem 4.2.5. *The map $\theta : W \to U$ is an injective $\mathfrak{g}$- and F-homomorphism. Moreover, suppose one of the following hypotheses:*
(1) $\dim(\mathfrak{g}/\mathfrak{h}) < \infty$ *and W is filtration complete;*
(2) $\dim V < \infty$.
Then θ is bijective.

Proof. By 4.2.2 and 4.2.3, θ is an injective $\mathfrak{g}$- and F-homomorphism. Let $X = \theta W$. We have $FX = F(\theta W) = \theta(FW) = \theta W = X$, also $\varphi X = \varphi(\theta W) = \sigma W = V$.

Suppose hypothesis (1) is true. Then $\dim(\mathfrak{g}/\mathfrak{h}) < \infty$ and W is filtration complete. This means that each Cauchy sequence $\{w_n\}_{n\ge 0}$ in W must converge to some $w \in W$ in the filtration topology. Since θ is filtration preserving by Lemma 4.2.1, this implies that $\theta w_n \xrightarrow{n\to\infty} \theta w$ in the filtration topology of U. In other words, $X = \theta W$ is closed in the filtration topology of U and therefore also in the finite-open topology by 2.1.2. Hypothesis (1) of Lemma 4.2.4 is satisfied and thus $\theta W = X = U$, i.e. θ is surjective.

Now suppose hypothesis (2). Then $X = U$ by case (2) of Lemma 4.2.4 and θ is surjective. □

Chapter 5

An irreducibility theorem for induced representations

In this chapter we will reach a result concerning irreducibility criteria for certain induced representations. Theorem 5.2.4 given at the end may be dualized thanks to 2.0.1 to obtain an analogous proposition for produced representations, however the result obtained is primarily of topological nature and therefore it will not be included in this work. The dual version may be found in [1].

5.1 Absolute irreducibility

Let $\mathfrak{g}$ be a Lie algebra.

Definition 5.1 (Absolute irreducibility)**.** A $\mathfrak{g}$-module V is called *absolutely irreducible* if it is irreducible under arbitrary extensions of the field $\mathbb{K}$, i.e. if the $(\mathbb{L} \otimes_{\mathbb{K}} \mathfrak{g})$-module $\mathbb{L} \otimes_{\mathbb{K}} V$ is irreducible for any field extension $\mathbb{K} \subseteq \mathbb{L}$.

Note that for any field extension $\mathbb{K} \subseteq \mathbb{L}$, the $\mathbb{K}$-vector space $\mathbb{L} \otimes_{\mathbb{K}} \mathfrak{g}$ is an $\mathbb{L}$-Lie algebra with the following operations of scalar multiplication and bracket:

$$l(k \otimes x) = (lk) \otimes x$$
$$[k \otimes x, l \otimes y] = (kl) \otimes [x, y]$$

for any $k, l \in \mathbb{L}$ and $x, y \in \mathfrak{g}$. The extended module $\mathbb{L} \otimes_{\mathbb{K}} V$ is defined by regarding $\mathbb{L}$ as a right $\mathbb{K}$-module and it is a $(\mathbb{L} \otimes_{\mathbb{K}} \mathfrak{g})$-module with the action

$$(k \otimes x)(l \otimes v) = (kl) \otimes (xv)$$

for all $k, l \in \mathbb{L}$, $x \in \mathfrak{g}$ and $v \in V$.

Remark 9. For the enveloping algebra, we have $\mathfrak{U}(\mathbb{L}\otimes_{\mathbb{K}}\mathfrak{g}) \cong \mathbb{L}\otimes_{\mathbb{K}}\mathfrak{U}(\mathfrak{g})$ where the multiplication on $\mathbb{L}\otimes_{\mathbb{K}}\mathfrak{U}(\mathfrak{g})$ is the obvious one defined by $(k\otimes x)(l\otimes y) = (kl)\otimes(xy)$ for all $k, l \in \mathbb{L}$ and $x, y \in \mathfrak{g}$. The canonical isomorphism is given by $(l_1 \otimes x_1)\ldots(l_p \otimes x_p) \mapsto (l_1 \ldots l_p) \otimes (x_1 \ldots x_p)$ for all $l_j \in \mathbb{L}$ and $x_j \in \mathfrak{g}$, $j \in \{1, \ldots, p\}$.

Example 5.1. Let $\mathbb{K} = \mathbb{R}$ and $\mathfrak{g} = \mathrm{gl}_n(\mathbb{R})$ where $n \in \mathbb{Z}_{\leq 0}$. Let $\mathbb{L} = \mathbb{C}$ and $V = \mathbb{R}^n$. Then since $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ we have

$$\begin{aligned}
\mathbb{C}\otimes_{\mathbb{R}} \mathrm{gl}_n(\mathbb{R}) &= \big(\mathbb{R}\oplus i\mathbb{R}\big)\otimes_{\mathbb{R}} \mathrm{gl}_n(\mathbb{R}) \cong \big(\mathbb{R}\otimes_{\mathbb{R}} \mathrm{gl}_n(\mathbb{R})\big) \oplus \big(i\mathbb{R}\otimes_{\mathbb{R}} \mathrm{gl}_n(\mathbb{R})\big)\\
&\cong \mathrm{gl}_n(\mathbb{R}) \oplus i\mathrm{gl}_n(\mathbb{R}) = \mathrm{gl}_n(\mathbb{C})\\
\mathbb{C}\otimes_{\mathbb{R}} \mathbb{R}^n &= \big(\mathbb{R}\oplus i\mathbb{R}\big)\otimes_{\mathbb{R}} \mathbb{R}^n \cong \big(\mathbb{R}\otimes_{\mathbb{R}} \mathbb{R}^n\big) \oplus \big(i\mathbb{R}\otimes_{\mathbb{R}} \mathbb{R}^n\big)\\
&\cong \mathbb{R}^n \otimes_{\mathbb{R}} i\mathbb{R}^n = \mathbb{C}^n
\end{aligned}$$

congruently with the intuitive idea of field extension for a Lie algebra and module.

Example 5.2. We give an example of a $\mathfrak{g}$-module which is irreducible but not absolutely irreducible. Let $\mathbb{K} = \mathbb{R}$ and $\mathfrak{g} = \mathrm{span}_{\mathbb{R}}\{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\}$. This is clearly an abelian subalgebra of $\mathrm{gl}_2(\mathbb{R})$. Let $V = \mathbb{R}^2$ be a $\mathfrak{g}$-module with the inherited natural action. To show that $\mathbb{R}^2$ is irreducible, we must prove that there is no $W \subseteq \mathbb{R}^2$ such that $\{0\} \neq W \neq \mathbb{R}^2$ and $\mathfrak{g}W \subseteq W$. Nontrivial submodules of $\mathbb{R}^2$ must be 1-dimensional because $\dim_{\mathbb{R}} \mathbb{R}^2 = 2$, therefore there must be a nonzero $v \in \mathbb{R}^2$ such that for each $x \in \mathfrak{g}$ there exists $\lambda_x \in \mathbb{R}$ which satisfies $xv = \lambda_x v$. This implies that v is an eigenvector of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. But

$$\det\begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1$$

which is a polynomial with no roots for $\lambda \in \mathbb{R}$, hence $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ has no eigenvalues and therefore no eigenvectors. So $\mathbb{R}^2$ is an irreducible $\mathfrak{g}$-module. Now let $\mathbb{L} = \mathbb{C}$. As seen earlier in Example 5.1 we have $\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{g} \cong \mathrm{span}_{\mathbb{C}}\{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\}$ and $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}^2 \cong \mathbb{C}^2$. The polynomial $\lambda^2 + 1$ has roots $\lambda = \pm i$ in $\mathbb{C}$, therefore there exists a nonzero $v \in \mathbb{C}^2$ such that $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)v = iv$ (choose $v = \left(\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right)$ for example). Hence $W = \mathrm{span}_{\mathbb{C}}\{v\}$ is a subspace of $\mathbb{C}^2$ such that $\dim_{\mathbb{C}} W = 1$ and $(\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{g})W \subseteq W$, so $\mathbb{C}^2$ is not an irreducible $(\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{g})$-module.

5.1.1 Schur's Lemma and Chevalley-Jacobson density theorem

We give a couple of results on irreducible representations that will be needed later in this chapter.

Lemma 5.1.1 (Schur). *Let A be an algebra over an algebraically closed field $\mathbb{K}$ and let V be an irreducible A-module. If $\varphi \in \mathrm{End}_A(V)$ then $\varphi = \lambda \,\mathrm{Id_V}$ for some $\lambda \in \mathbb{K}$.*

Proof. Since $\mathbb{K}$ is algebraically closed, there exists a nonzero $v \in V$ eigenvector for φ of eigenvalue λ. Denote with $V_\lambda = \{v \in V \mid \varphi(v) = \lambda v\}$ the eigenspace for λ. We show that V_λ is an A-submodule of V. For each $a \in A$ and $v \in V_\lambda$ we have

$$\varphi(av) = a\varphi(v) = a\lambda v = \lambda av$$

which implies $av \in V_\lambda$. Since V is irreducible, it must be $V = V_\lambda$ and therefore $\varphi = \lambda \,\mathrm{Id}_V$. □

Theorem 5.1.2 (Chevalley-Jacobson). *Let R be a ring and let V be an irreducible R-module. Set $D = \mathrm{End}_R(V)$ and let $v_1, \dots, v_k \in V$ be D-linearly independent. Then for any $w_1, \dots, w_k \in V$ there exists $z \in R$ such that $zv_j = w_j$ for all $j \in \{1, \dots, k\}$.*

Proof. See [3]. □

Remark 10. If $\mathbb{K}$ is algebraically closed and V is a module over the $\mathbb{K}$-algebra A, by Schur's Lemma we may replace the hypothesis of $x_1, \dots, x_k$ being $\mathrm{End}_A(V)$-linearly independent in 5.1.2 with $x_1, \dots, x_k$ being linearly independent.

Corollary 5.1.3. *Let $\mathfrak{g}$ be a Lie algebra and let V be an absolutely irreducible $\mathfrak{g}$-module. Let $v_1, \dots, v_k \in V$ be linearly independent. Then for any $w_1, \dots, w_k \in V$ there exists $z \in \mathfrak{U}(\mathfrak{g})$ such that $zv_j = w_j$ for all $j \in \{1, \dots, k\}$.*

Proof. The $\mathfrak{g}$-module V is absolutely irreducible, therefore by considering the algebraic closure $\bar{\mathbb{K}}$ of $\mathbb{K}$ we have that $\bar{\mathbb{K}} \otimes_{\mathbb{K}} V$ is an irreducible $\big(\bar{\mathbb{K}} \otimes_{\mathbb{K}} \mathfrak{U}(\mathfrak{g})\big)$-module. Clearly $v_1, \dots, v_k$ linearly independent in V implies $1 \otimes v_1, \dots, 1 \otimes v_k$ linearly independent in $\bar{\mathbb{K}} \otimes_{\mathbb{K}} V$. By Theorem 5.1.2 and Remark 10 there exists $\bar{z} \in \bar{\mathbb{K}} \otimes_{\mathbb{K}} \mathfrak{U}(\mathfrak{g})$ such that $\bar{z}(1 \otimes v_j) = 1 \otimes w_j$ for all $j \in \{1, \dots, k\}$. Choose $\{k_i\}_{i \in I}$ basis for $\bar{\mathbb{K}}$ as a $\mathbb{K}$-vector space and let $i_0 \in I$ be the index such that $k_{i_0} = 1$. Then $\bar{z} = \sum_{i \in I} k_i \otimes a_i$ where only a finite number of $a_i \in \mathfrak{U}(\mathfrak{g})$ are nonzero. This implies that

$$\left(\sum_{i \in I} k_i \otimes a_i \right) (1 \otimes v_j) = \sum_{i \in I} k_i \otimes a_i v_j = 1 \otimes w_j$$

for all $j \in \{1, \dots, k\}$ and therefore it must be $a_i = 0$ for $i \neq i_0$. Hence if $z = a_{i_0}$ then $\bar{z} = 1 \otimes z$ so that $1 \otimes zv_j = 1 \otimes w_j$ for all $j \in \{1, \dots, k\}$, which implies our desired result due to the isomorphism $1 \otimes V \cong V$. □

5.2 Extended representations

In this section, $\mathfrak{g}$ will be a Lie algebra with an ideal $\mathfrak{t}$ and V will be a $\mathfrak{t}$-module. Define $\mathfrak{h} = \{y \in \mathfrak{g} \mid \exists\ s \in \mathrm{Hom}_{\mathbb{K}}(V,V) : [y,k]v = skv - ksv\ \forall v \in V, k \in \mathfrak{t}\}$.

Proposition 5.2.1. *The set $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ such that $\mathfrak{t} \subseteq \mathfrak{h}$.*

Proof. Since V is a $\mathfrak{t}$-module, for any $z \in \mathfrak{t}$ we have $[z,k]v = zkv - kzv$ for all $k \in \mathfrak{t}$ and $v \in V$. If $\rho : \mathfrak{t} \to \mathrm{Hom}_{\mathbb{K}}(V,V)$ is the representation map of V, then $zv = \rho(z)v$ for any $v \in V$, hence the previous formula implies that $\mathfrak{t} \subseteq \mathfrak{h}$. Now let $x, y \in \mathfrak{h}$ and let $\lambda \in \mathbb{K}$. Then there exist $s, t \in \mathrm{Hom}_{\mathbb{K}}(V,V)$ such that

$$\begin{aligned}[x,k]v &= skv - ksv\\ [y,k]v &= tkv - ktv\end{aligned}$$

for all $v \in V$ and $k \in \mathfrak{t}$. Since $\mathfrak{t}$ is an ideal, $[x,k], [y,k] \in \mathfrak{t}$. Therefore

$$\begin{aligned}[x+y,k]v &= [x,k]v + [y,k]v = skv - ksv + tkv - ktv\\ &= (s+t)kv - k(s+t)v\\ [\lambda x,k]v &= \lambda[x,k]v = \lambda skv - \lambda ksv\\ &= (\lambda s)kv - k(\lambda s)v\\ [[x,y],k]v &= [x,[y,k]]v - [y,[x,k]]v = s[y,k]v - [y,k]sv - (t[x,k]v - [x,k]tv)\\ &= (stkv - sktv) - (tksv - ktsv) - (tskv - tksv) + (sktv - kstv)\\ &= stkv + ktsv - tskv - kstv = (st - ts)kv - k(st - ts)v\\ &= [s,t]kv - k[s,t]v\end{aligned}$$

for all $v \in V$ and $k \in \mathfrak{t}$. Hence $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. □

For each $x \in \mathfrak{g}$ and $k \in \mathfrak{t}$, define $\delta(x) \in \mathrm{Hom}_{\mathbb{K}}(\mathfrak{t}, \mathfrak{t})$ by $\delta(x)k = [k,x]$.

Proposition 5.2.2. *The map $\delta : \mathfrak{g} \to \mathrm{der}(\mathfrak{t})$ is a Lie algebra antihomomorphism.*

Proof. The map δ is linear by the bracket's bilinearity. We show that δ maps $\mathfrak{g}$ into $\mathrm{der}(\mathfrak{t})$. For each $x \in \mathfrak{g}$ and $h, k \in \mathfrak{t}$ we have

$$\begin{aligned}\delta(x)[h,k] &= [[h,k],x] = [h,[k,x]] - [k,[h,x]] = [h,\delta(x)k] - [k,\delta(x)h]\\ &= [h,\delta(x)k] + [\delta(x)h,k]\end{aligned}$$

by the Jacobi identity, which proves that $\delta(x)$ is a derivation of $\mathfrak{t}$. For each $x, y \in \mathfrak{g}$ and $k \in \mathfrak{t}$ we have

$$\begin{aligned}
\delta([x,y])k &= [k,[x,y]] \\
[\delta(x),\delta(y)]k &= \delta(x)\delta(y)k - \delta(y)\delta(x)k = \delta(x)[k,y] - \delta(y)[k,x] \\
&= [[k,y],x] - [[k,x],y] = [x,[y,k]] - [y,[x,k]] \\
&= [[x,y],k]
\end{aligned}$$

by the Jacobi identity again. Since $[\delta(y),\delta(x)] = -[\delta(x),\delta(y)] = -[[x,y],k] = [k,[x,y]]$, we obtain that $\delta([x,y])k = [\delta(x),\delta(y)]k$ for any $x,y \in \mathfrak{g}$ and $k \in \mathfrak{t}$. Therefore δ is a Lie algebra antihomomorphism. □

For each $x \in \mathfrak{g}$ we may extend $\delta(x)$ to $\mathfrak{U}(\mathfrak{t})$ by means of derivation: for any $k_1, k_2 \in \mathfrak{t}$ define

$$\delta(x)\big(k_1k_2\big) = \big(\delta(x)k_1\big)k_2 + k_1\big(\delta(x)k_2\big)$$

and do the same iteratively for any product $k_1 \dots k_p$ of p elements of $\mathfrak{t}$, so that $\delta(x) \in \mathrm{der}\big(\mathfrak{U}(\mathfrak{t})\big)$. The degenerate case is $\delta(x)1 = 0$ as constants must be null under derivation. We can then extend δ to be an associative unitary algebra antihomomorphism $\mathfrak{U}(\mathfrak{g}) \to \mathrm{Hom}_{\mathbb{K}}\big(\mathfrak{U}(\mathfrak{t}), \mathfrak{U}(\mathfrak{t})\big)$ by defining

$$\delta(y_1 \dots y_p) = \delta(y_p) \dots \delta(y_1)$$

for all products $y_1 \dots y_p$ of p elements of $\mathfrak{g}$ (and obviously $\delta(1) = \mathrm{Id}_{\mathfrak{U}(\mathfrak{t})}$).

Remark 11. Note that $zx = xz + \delta(x)z$ for all $z \in \mathfrak{U}(\mathfrak{t})$ and $x \in \mathfrak{g}$.
Indeed, for any $z_1, \dots, z_q \in \mathfrak{t}$ and $x \in \mathfrak{g}$ then

$$\begin{aligned}
z_1 \dots z_q x &= z_1 \dots z_{q-1} x z_q + z_1 \dots z_{q-1}[z_q, x] \\
&= z_1 \dots z_{q-1} x z_q + z_1 \dots z_{q-1}\big(\delta(x) z_q\big) \\
&= \dots = x z_1 \dots z_q + \sum_{i=1}^{q} z_1 \dots z_{i-1}\big(\delta(x) z_i\big) z_{i+1} \dots z_q \\
&= x z_1 \dots z_q + \delta(x)(z_1 \dots z_q)
\end{aligned}$$

which implies the desired result.

Let $z \in \mathfrak{U}(\mathfrak{t})$ and $y_1, \dots, y_p \in \mathfrak{g}$. For any multi-index $m = (m_1, \dots, m_p) \in (\mathbb{Z}_{\geq 0})^p$ define $y^m = y^{m_1} \dots y^{m_p} \in \mathfrak{U}(\mathfrak{g})$ like in Lemma 1.3.3.

Lemma 5.2.3.

$$zy^m = \sum_{0 \leq k \leq m} \binom{m}{k} y^k \big(\delta(y^{m-k})z\big).$$

Proof. We show the identity holds for the case $m_1 = \ldots = m_p = 1$. The general result will follow by identifying the elements y_i and collecting terms exactly as in Lemma 1.3.3. Proceed by induction on p. If $p = 0$ the identity holds trivially as $z = \delta(1)z$, however this is a degenerate case. If $p = 1$ we have

$$\begin{aligned} zy_1 &= y_1 z + \delta(y_1)z = y_1\delta(1)z + \delta(y_1)z \\ &= \sum_{k=0}^{1} y^k \big(\delta(y^{1-k})z\big) \end{aligned}$$

by Remark 11. Now suppose the result holds for some $p > 1$. Then

$$\begin{aligned} zy_1 \ldots y_{p+1} &= y_1 z y_2 \ldots y_{p+1} + \big(\delta(y_1)z\big)y_2 \ldots y_{p+1} \\ &= y_1 \sum_{\substack{0\le k_i\le 1 \\ i\in\{2,\ldots,p+1\}}} y_2^{k_2} \ldots y_{p+1}^{k_{p+1}} \big(\delta(y_2^{1-k_2} \ldots y_{p+1}^{1-k_{p+1}})z\big) \\ &+ \sum_{\substack{0\le k_i\le 1 \\ i\in\{2,\ldots,p+1\}}} y_2^{k_2} \ldots y_{p+1}^{k_{p+1}} \big(\delta(y_2^{1-k_2} \ldots y_{p+1}^{1-k_{p+1}})\delta(y_1)z\big) \\ &= \sum_{\substack{0\le k_i\le 1 \\ i\in\{2,\ldots,p+1\}}} y_1 y_2^{k_2} \ldots y_{p+1}^{k_{p+1}} \big(\delta(y_1^{1-1} y_2^{1-k_2} \ldots y_{p+1}^{1-k_{p+1}})z\big) \\ &+ \sum_{\substack{0\le k_i\le 1 \\ i\in\{2,\ldots,p+1\}}} y_1^0 y_2^{k_2} \ldots y_{p+1}^{k_{p+1}} \big(\delta(y_1 y_2^{1-k_2} \ldots y_{p+1}^{1-k_{p+1}})z\big) \\ &= \sum_{\substack{0\le k_i\le 1 \\ i\in\{1,\ldots,p+1\}}} y_1^{k_1} y_2^{k_2} \ldots y_{p+1}^{k_{p+1}} \big(\delta(y_1^{1-k_1} y_2^{1-k_2} \ldots y_{p+1}^{1-k_{p+1}})z\big) \end{aligned}$$

because $\delta(y_1)z \in \mathfrak{U}(\mathfrak{t})$ and δ is an antihomomorphism. □

5.2.1 Irreducibility theorem

Theorem 5.2.4. (1) *If* $\mathfrak{h} \neq \mathfrak{t}$ *then* $\mathcal{I}_{\mathfrak{t}}^{\mathfrak{g}}(V)$ *is reducible.*
(2) *Suppose* $\operatorname{char}\mathbb{K} = 0$ *and* V *is absolutely irreducible. Let* W *be an irreducible (respectively absolutely irreducible)* $\mathfrak{h}$*-module such that* $W = \bigoplus_{\alpha\in A} V_\alpha$ *as a* $\mathfrak{t}$*-module, where* $V_\alpha \cong V$ *for all* α*. Then* $\mathcal{I}_{\mathfrak{h}}^{\mathfrak{g}}(W)$ *is irreducible (respectively absolutely irreducible).*

Proof. (1) Assume $\mathfrak{h} \neq \mathfrak{t}$ and set $U = \mathcal{I}_{\mathfrak{t}}^{\mathfrak{g}}(V) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{t})} V$. Choose $y \in \mathfrak{h} \smallsetminus \mathfrak{t}$ so that its equivalence class $\bar{y} \in \mathfrak{g}/\mathfrak{t}$ is nonzero. Choose an ordered index set I, a basis $\{\bar{x}_i\}_{i\in I}$ for $\mathfrak{g}/\mathfrak{t}$ and $\{x_i\}_{i\in I} \subseteq \mathfrak{g}$ such that $x_i \in \bar{x}_i$ for each $i \in I$.

We may reorder I so that it has an highest element i_0 and since $\bar{y} \neq 0$ we may also assume $y = x_{i_0}$. Define the multi-indices set M, the monomials x^m for $m \in M$ and filter U as in §2.1. Let $s \in \mathrm{Hom}_{\mathbb{K}}(V, V)$ be such that $[y, k]v = skv - ksv$ for all $k \in \mathfrak{t}$ and $v \in V$, as in the definition of $\mathfrak{h}$. Define $r \in M$ by $r_i = \delta_{ii_0}$ for each $i \in I$, where δ_{ii_0} is the Kronecker delta. Let $T = \mathrm{span}\{x^{m+r} \otimes v - x^m \otimes sv \mid m \in M, v \in V\}$. We show that T is a non-$\{0\}$ subspace of U such that $T \cap U_0 = \{0\}$. The map $\iota : S(\mathfrak{g}/\mathfrak{t}) \otimes V \to U$ is a filtration preserving linear isomorphism by 2.1.1 and $\{\bar{x}^m\}_{m \in M}$ is a basis for $S(\mathfrak{g}/\mathfrak{t})$, therefore if $\{v_j\}_{j \in J}$ is a basis for V then $\{x^{m+r} \otimes v_j - x^m \otimes sv_j\}_{m \in M, j \in J}$ is a basis for T. Clearly $\bar{x}^{m+r} \otimes v_j \neq \bar{x}^m \otimes sv_j$ for any m and j, hence $x^{m+r} \otimes v_j - x^m \otimes sv_j \neq 0$ so that $T \neq \{0\}$. Remember that

$$U_0 = \mathfrak{U}_0(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{t})} V = \{1 \otimes v \mid v \in V\}.$$

We may write $1 \otimes V = \{1 \otimes v \mid v \in V\}$ for convenience, so that $1 \otimes V$ coincides with the internal tensor product between $\mathbb{K}$ (regarded as a trivial $\mathfrak{t}$-module) and V. If $w \in T \cap U_0$ then

$$\begin{aligned} w &= \sum_{j=1}^{k} \lambda_j (x^{m_{(j)}+r} \otimes v_{i_j} - x^{m_{(j)}} \otimes sv_{i_j}) \\ w &= 1 \otimes v \end{aligned}$$

where $\lambda_j \in \mathbb{K}$, $m_{(j)} \in M$, $v_{i_j} \in V$ and $v \in V$. Both writings are unique by the argument above and since $m_{(j)} + r \neq 0$ for any $m_{(j)} \in M$ then it must be $w = 0$. Therefore $T \cap U_0 = \{0\}$, which also implies that T is a proper subspace of U.

We finish by showing that $\mathfrak{U}(\mathfrak{g})T \subseteq T$. If $k \in \mathfrak{t}$, we have

$$\begin{aligned} k(x_{i_0} \otimes v - 1 \otimes sv) &= kx_{i_0} \otimes v - k \otimes sv = x_{i_0}k \otimes v + [k, x_{i_0}] \otimes v - k \otimes sv \\ &= x_{i_0} \otimes kv - 1 \otimes (ksv + [x_{i_0}, k]v) \\ &= x_{i_0} \otimes kv - 1 \otimes skv \end{aligned}$$

because $y = x_{i_0}$ and $[y, k]v = skv - ksv$. Thus $S = \{x_{i_0} \otimes v - 1 \otimes sv \mid v \in V\}$ is $\mathfrak{t}$-invariant and hence it is $\mathfrak{U}(\mathfrak{t})$-invariant. But $\{x_i\}_{i \in I}$ is a basis for $\mathfrak{U}(\mathfrak{g})$ as an $\mathfrak{U}(\mathfrak{t})$-module, therefore $\mathfrak{U}(\mathfrak{g}) = \sum_{m \in M} x^m \mathfrak{U}(\mathfrak{t})$. It follows that

$$\mathfrak{U}(\mathfrak{g})S = \sum_{m \in M} x^m \mathfrak{U}(\mathfrak{t})S = \sum_{m \in M} x^m S = T$$

because $x^m x_{i_0} = x^{m+r}$ for all $m \in M$. We then have

$$\mathfrak{U}(\mathfrak{g})T = \mathfrak{U}(\mathfrak{g}) \sum_{m \in M} x^m S = \sum_{m \in M} x^m \mathfrak{U}(\mathfrak{g})S = \sum_{m \in M} x^m T = T$$

and therefore T is a nontrivial $\mathfrak{U}(\mathfrak{g})$-submodule of U. So U is reducible.

(2) Assume that $\operatorname{char}\mathbb{K} = 0$, V is an absolutely irreducible $\mathfrak{t}$-module, W is an irreducible $\mathfrak{h}$-module and $W = \bigoplus_{\alpha\in A} V_\alpha$ as a $\mathfrak{t}$-module with $V_\alpha \cong V$. Set $U = \mathcal{I}_{\mathfrak{h}}^{\mathfrak{g}}(W) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{h})} W$. Choose an ordered index set I, a basis $\{\bar{x}_i\}_{i\in I}$ for $\mathfrak{g}/\mathfrak{h}$, a set of class representatives $\{x_i\}_{i\in I}$, define the multi-indices set M, the monomials x^m for $m \in M$ and filter U as in §2.1. Let T be a non-$\{0\}$ $\mathfrak{U}(\mathfrak{g})$-invariant subspace of U: we show that $T = U$, which implies that U is irreducible.

Case I. Suppose $T \cap U_0 \neq \{0\}$. Remember that $U_0 = 1 \otimes W$. We have $h(1 \otimes w) = h \otimes w = 1 \otimes hw$ for all $h \in \mathfrak{h}$ and $w \in W$, therefore U_0 is $\mathfrak{U}(\mathfrak{h})$-invariant. Since T is $\mathfrak{U}(\mathfrak{g})$-invariant it is also $\mathfrak{U}(\mathfrak{h})$-invariant, hence $T \cap U_0$ is a non-$\{0\}$ $\mathfrak{U}(\mathfrak{h})$-submodule of $1 \otimes W \cong W$. By the irreducibility of W, it is $T \cap U_0 = 1 \otimes W$. Therefore

$$\begin{aligned} T = \mathfrak{U}(\mathfrak{g})T \supseteq \mathfrak{U}(\mathfrak{g})(1 \otimes W) &= \sum_{m\in M} x^m \mathfrak{U}(\mathfrak{h})(1 \otimes W) = \sum_{m\in M} x^m \otimes W \\ &= U \end{aligned}$$

thanks to the isomorphism $\iota : S(\mathfrak{g}/\mathfrak{h}) \otimes_{\mathbb{K}} W \to U$ seen in 2.1.1.

Case II. Now suppose $T \cap U_0 = \{0\}$. We will show that this case leads to an absurd. Let $p \geq 1$ be the smallest integer such that $T \cap U_p \neq \{0\}$, which must exists as $\{0\} \neq T \subseteq U$. Let u be a nonzero element of $T \cap U_p$. By 2.1.1, the map ι is a filtration preserving isomorphism and therefore we can write

$$u = \sum_{|m|\leq p} x^m \otimes w_m$$

with $w_m \in W$. By hypothesis, $W = \bigoplus_{\alpha\in A} V_\alpha = \{\sum_{j=1}^k v_j \mid v_j \in V_{\alpha_j}, \alpha_j \in A, k \in \mathbb{Z}_{\geq 0}\}$. Therefore only a finite number of the w_m in the expression of u are nonzero and the w_m have components only in a finite number of the V_α. Let's denote these with $V_{\alpha_1}, \ldots, V_{\alpha_q}$. For each $n \in \{1, \ldots, q\}$ let $\zeta^n : W \to V$ be a $\mathfrak{t}$-homomorphism which vanishes on $\bigoplus_{\alpha\neq\alpha_n} V_\alpha$ and is an isomorphism on V_α, i.e. ζ^n is the composition between an isomorphism $V_{\alpha_n} \to V$ and the projection of W into its α_n-th coordinate.

$$\begin{array}{ccc} W & \xrightarrow{\pi_{\alpha_n}} & V_{\alpha_n} \\ & {\scriptstyle \zeta^n}\searrow & \downarrow{\scriptstyle \cong} \\ & & V \end{array}$$

If $z \in \mathfrak{U}(\mathfrak{t})$ then $zw_m = 0$ if and only if $z\zeta^n w_m = 0$ for all $n \in \{1, \dots, q\}$.

Let v be a nonzero element of V. Without loss of generality, we may assume that $\zeta^n w_m = \lambda_m^{(n)} v$ for all $n \in \{1, \dots, q\}$ and $|m| = p$, where $\lambda_m^{(n)} \in \mathbb{K}$. In fact, let $\mathfrak{Z}$ be a maximal linearly independent subset of $\{\zeta^n w_m \mid |m| = p, n \in \{1, \dots, q\}\}$. By 5.1.3 there exists $z_0 \in \mathfrak{U}(\mathfrak{t})$ such that $z_0 \zeta^n w_m = \lambda_m^{(n)} v$ for all $\zeta^n w_m \in \mathfrak{Z}$, these coefficients $\lambda_m^{(n)}$ being chosen freely in $\mathbb{K}$. Writing every $\zeta^n w_m$ as a linear combination of elements in $\mathfrak{Z}$ gives the above result for some $\lambda_m^{(n)} \in \mathbb{K}$ and all $n \in \{1, \dots, q\}$, $|m| = p$. We can assume that the $\lambda_m^{(n)}$ are not all zero, thus the $z_0 w_m$ are not all zero either and so it is $z_0 u \neq 0$. From Lemma 5.2.3 we have

$$
\begin{aligned}
z_0 x^m \otimes w_m &= \sum_{0 \le k \le m} \binom{m}{k} x^k \big(\delta(x^{m-k}) z_0\big) \otimes w_m = \sum_{0 \le k \le m} \binom{m}{k} x^k \otimes \big(\delta(x^{m-k}) z_0\big) w_m \\
&= x^m \otimes z_0 w_m + \sum_{0 \le k < m} \binom{m}{k} x^k \otimes \big(\delta(x^{m-k}) z_0\big) w_m
\end{aligned}
$$

where the sum on the right always belongs to U_{p-1}. It follows that

$$
z_0 u = \sum_{|m| = p} x^m \otimes z_0 w_m \quad \big(\text{modulo } U_{p-1}\big).
$$

Since T is $\mathfrak{U}(\mathfrak{g})$-invariant we obtain that $z_0 u$ is a nonzero element of $\in T \cap U_p$, therefore we may choose $z_0 u$ instead of our original u to find an element that satisfies the above assumptions.

For all $z \in \mathfrak{U}(\mathfrak{t})$ and $|m| = p$ we have $\zeta^n(zw_m) = z\zeta^n w_m = \lambda_m^{(n)} zv$. Therefore $w_m = 0$, $|m| = p$ if and only if $\lambda_m^{(n)} = 0$ for each $n \in \{1, \dots, q\}$. Moreover, if $w_m \neq 0$, then $zw_m = 0$ if and only if $zv = 0$. Now choose m such that $|m| = p$ and $w_m \neq 0$. For each $i \in I$, define the multi-index $r(i) = \delta_{ij}$ for all $j \in I$, where δ_{ij} is the Kronecker delta. Let $i_0 \in I$ be an index such that $m_{i_0} > 0$ and set $l = m - r(i_0)$. Observing that $|l| = p - 1$ we may write

$$
\begin{aligned}
u &= \sum_{|k| = p} x^k \otimes w_k + \sum_{|k| < p} x^k \otimes w_k \\
&= \sum_{|k| = p} x^k \otimes w_k + x^l \otimes w_l + \sum_{\substack{|k| < p \\ k \neq l}} x^k \otimes w_k \\
&= \sum_{i \in I} x^{l + r(i)} \otimes w_{l + r(i)} + \sum_{\substack{|k| = p \\ k \ngeq l}} x^k \otimes w_k + x^l \otimes w_l + \sum_{\substack{|k| < p \\ k \neq l}} x^k \otimes w_k.
\end{aligned}
$$

Our objective is to find the coefficient of x^l in $\iota^{-1}(zu)$. We apply Lemma 5.2.3 to zu written as below:

$$zu = \sum_{i\in I} zx^{l+r(i)} \otimes w_{l+r(i)} + \sum_{\substack{|k|=p\\ k\ngeq l}} zx^k \otimes w_k + zx^l \otimes w_l + \sum_{\substack{|k|<p\\ k\neq l}} zx^k \otimes w_k.$$

Clearly the sums over $k \ngeq l$ and $k \neq l$ will not contain the term x^l, therefore they can be discarded. For each $i \in I$, we have

$$\begin{aligned} zx^{l+r(i)} &= \binom{l+r(i)}{l} x^l \big(\delta(x^{l+r(i)-l})z\big) + \sum_{\substack{0\le k\le l+r(i)\\ k\neq l}} \binom{l+r(i)}{k} x^k \big(\delta(x^{l+r(i)-k})z\big) \\ &= (l_i+1)x^l\big(\delta(x_i)z\big) + \sum_{\substack{0\le k\le l+r(i)\\ k\neq l}} \binom{l+r(i)}{k} x^k \big(\delta(x^{l+r(i)-k})z\big) \end{aligned}$$

so we may discard the sum over $k \neq l$ written above. Similarly, since it is

$$zx^l = x^l z + \sum_{0\le k<l} \binom{l}{k} x^k \big(\delta(x^{l-k})z\big)$$

we will also discard the sum over $k < l$. Hence we have

$$\begin{aligned} zu &= \sum_{i\in I} (l_i+1)x^l\big(\delta(x_i)z\big) \otimes w_{l+r(i)} + x^l z \otimes w_l + \sum_{\substack{|k|\le p\\ k\neq l}} x^k \otimes \bar{w}_k \\ &= x^l \otimes \left(zw_l + \sum_{i\in I} (l_i+1)\big(\delta(x_i)z\big) w_{l+r(i)} \right) + \sum_{\substack{|k|\le p\\ k\neq l}} x^k \otimes \bar{w}_k \end{aligned}$$

for some $\bar{w}_k \in W$ and therefore the coefficient of x^l in zu is

$$zw_l + \sum_{i\in I} (l_i+1)\big(\delta(x_i)z\big) w_{l+r(i)}. \tag{5.1}$$

Choose $n_0 \in \{1,\ldots,q\}$ such that $\lambda_m^{(n_0)} \neq 0$. As explained earlier, this is always possible because we assumed $w_m \neq 0$. We apply ζ^{n_0} to (5.1) to obtain

$$\begin{aligned} z\zeta^{n_0} w_l + \sum_{i\in I} (l_i+1)\big(\delta(x_i)z\big)\zeta^{n_0} w_{l+r(i)} &= z\zeta^{n_0} w_l + \sum_{i\in I} (l_i+1)\big(\delta(x_i)z\big)\lambda_{l+r(i)}^{(n_0)} v \\ &= zv_0 + \big(\delta(y)z\big)v \end{aligned}$$

where $v_0 = \zeta^{n_0} w_l$ and

$$y = \sum_{i \in I} (l_i + 1) \lambda^{(n_0)}_{l+r(i)} x_i \in \mathfrak{g}.$$

Since $\operatorname{char} \mathbb{K} = 0$ and $\lambda^{(n_0)}_m \neq 0$, the coefficient of x_{i_0} in y is not zero. Therefore the equivalence class of y in $\mathfrak{g}/\mathfrak{h}$ is not zero, i.e. $y \notin \mathfrak{h}$.

At last we prove that $y \in \mathfrak{h}$, which contradicts the above argument and thus implies that Case II is void. By 5.1.3, we know that every element of V is of the form zv for some $z \in \mathfrak{U}(\mathfrak{t})$. Suppose $zv = 0$: as seen earlier, this means $zw_k = 0$ for all $|k| = p$. Since we have already proven

$$zu = \sum_{|k|=p} x^k \otimes zw_k \quad (\text{modulo } U_{p-1})$$

then $zu \in T \cap U_{p-1}$, but due to the minimal choice of p it must be $zu = 0$. Hence (5.1) vanishes and therefore $zv_0 + \big(\delta(y)z\big)v = 0$. So we may define an operator $s \in \operatorname{Hom}_{\mathbb{K}}(V, V)$ by

$$szv = zv_0 + \big(\delta(y)z\big)v$$

for each $z \in \mathfrak{U}(\mathfrak{t})$. This expression is linear in z and so s is well-defined as a linear operator. Let $k \in \mathfrak{t}$. Because $\delta(y) \in \operatorname{der}\big(\mathfrak{U}(\mathfrak{t})\big)$, then

$$\begin{aligned} sk(zv) - ks(zv) &= kzv_0 + \big(\delta(y)(kz)\big)v - \Big(kzv_0 + k\big(\delta(y)z\big)v\Big) \\ &= \Big(\big(\delta(y)k\big)z + k\big(\delta(y)z\big)\Big)v - k\big(\delta(y)z\big)v \\ &= \big(\delta(y)k\big)zv = [k, y]zv \end{aligned}$$

for any $z \in \mathfrak{U}(\mathfrak{t})$. It follows that $y \in \mathfrak{h}$, which leads to an absurd.

To show absolute irreducibility for $\mathcal{I}^{\mathfrak{g}}_{\mathfrak{h}}(W)$ in (2) we just need to repeat the above proof for any field extension $\mathbb{K} \subseteq \mathbb{L}$. We obtain that the $\mathfrak{U}(\mathbb{L} \otimes_{\mathbb{K}} \mathfrak{g})$-module $\mathcal{I}^{\mathbb{L}\otimes\mathfrak{g}}_{\mathbb{L}\otimes\mathfrak{h}}(\mathbb{L} \otimes_{\mathbb{K}} W)$ is irreducible, but

$$\begin{aligned} \mathcal{I}^{\mathbb{L}\otimes\mathfrak{g}}_{\mathbb{L}\otimes\mathfrak{h}}(\mathbb{L} \otimes_{\mathbb{K}} W) &= \mathfrak{U}(\mathbb{L} \otimes_{\mathbb{K}} \mathfrak{g}) \otimes_{\mathfrak{U}(\mathbb{L}\otimes_{\mathbb{K}}\mathfrak{h})} (\mathbb{L} \otimes_{\mathbb{K}} W) \\ &\cong \big(\mathbb{L} \otimes_{\mathbb{K}} \mathfrak{U}(\mathfrak{g})\big) \otimes_{\mathbb{L}\otimes_{\mathbb{K}}\mathfrak{U}(\mathfrak{h})} \big(\mathbb{L} \otimes_{\mathbb{K}} W\big) \\ &\cong \mathbb{L} \otimes_{\mathbb{K}} \big(\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{h})} W\big) = \mathbb{L} \otimes_{\mathbb{K}} \mathcal{I}^{\mathfrak{g}}_{\mathfrak{h}}(W) \end{aligned}$$

by Remark 9 and the isomorphism $(l_1 \otimes a) \otimes (l_2 \otimes w) \mapsto (l_1 l_2) \otimes (a \otimes w)$ for each $l_1, l_2 \in \mathbb{L}$, $a \in \mathfrak{U}(\mathfrak{g})$ and $w \in W$. That the above map defines an isomorphism comes from the fact that the tensor product is associative and $\mathbb{L} \otimes_{\mathbb{L}} \mathbb{L} \cong \mathbb{L}$. □

Example 5.3. We give an example for case (1) of the theorem. Let $\mathbb{K} = \mathbb{C}$, $\mathfrak{g} = \mathrm{gl}_{\mathbb{C}}(n)$ and $\mathfrak{t} = \mathrm{sl}_{\mathbb{C}}(n)$. Let V be an $\mathrm{sl}_{\mathbb{C}}(n)$-module. We know that $\mathrm{gl}_{\mathbb{C}}(n) = \mathrm{sl}_{\mathbb{C}}(n) \oplus \mathbb{C}\mathrm{Id}_n$ and $\mathbb{C}\mathrm{Id}_n$ commutates with any $k \in \mathfrak{t}$. Therefore, for any $y = \lambda \mathrm{Id}_n$ with $\lambda \in \mathbb{C}$ the condition $[y,k]v = skv - ksv$ is trivially satisfied for all $k \in \mathrm{sl}_{\mathbb{C}}(n)$ and $v \in V$ by setting $s = 0$. As $\mathbb{C}\mathrm{Id}_n$ is 1-dimensional, this must imply $\mathfrak{h} = \mathrm{gl}_{\mathbb{C}}(n)$ so that $\mathfrak{h} \neq \mathrm{sl}_{\mathbb{C}}(n)$. Then the assumptions for (1) are satisfied and therefore the induced representation $\mathcal{I}_{\mathrm{sl}(n)}^{\mathrm{gl}(n)}(V)$ is reducible. Indeed, we have

$$\mathfrak{g}/\mathfrak{t} = \frac{\mathrm{gl}_{\mathbb{C}}(n)}{\mathrm{sl}_{\mathbb{C}}(n)} \cong \mathbb{C}\mathrm{Id}_n$$

and therefore by the isomorphism $\iota : \mathcal{I}_{\mathrm{sl}(n)}^{\mathrm{gl}(n)}(V) \to S(\mathbb{C}\mathrm{Id}_n) \otimes_{\mathbb{C}} V$ we obtain

$$\mathcal{I}_{\mathrm{sl}(n)}^{\mathrm{gl}(n)}(V) \cong S(\mathbb{C}\mathrm{Id}_n) \otimes V \cong \bigoplus_{i=0}^{\infty} \big(S_i(\mathbb{C}\mathrm{Id}_n) \otimes V \big)$$

because $S(\mathbb{C}\mathrm{Id}_n) = \bigoplus_{i=0}^{\infty} S_i(\mathbb{C}\mathrm{Id}_n)$ (remember that $S(\mathbb{C}\mathrm{Id}_n) \cong \mathbb{C}[\mathrm{Id}_n]$).

Bibliography

[1] R. J. Blattner, *Induced and produced representations of Lie algebras*, Trans. Amer. Math. Soc. 144 (1969), 457-474.

[2] D. G. Highman, *Induced and produced modules*, Canad. J. Math. 7 (1955), 490-508.

[3] N. Jacobson, *Structure theory of simple rings without finiteness assumptions*, Trans. Amer. Math. Soc. 57 (1945), 228-245.

[4] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer, New York, 1972.

[5] N. Bourbaki, *Algebra I*, Hermann, Paris, 1971.