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LSZ Reduction Formula In The Worldline Formalism

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*“On ne trouvera point de Figures
dans cet Ouvrage.”*

*Mécanique analytique [1]
Joseph-Luois Lagrange*

Abstract

The LSZ reduction formula is one of the key equations of QFT as it is used to reduce S-matrix elements to scattering amplitudes, from which one obtains observables such as cross sections. The reduction formula was originally obtained in the second quantized quantum field theory, in which time dependent creation/annihilation operators acting on the distant past/far future vacuum state are time evolved all the way to the far future/distant past. Recently the reduction formula has been applied to the worldline formalism, a first quantized approach to the calculation of Feynman diagrams and integrals. Then, derived, in the context of worldline path integrals, in a recent reformulation of the binary inspiraling problem, the so-called Worldline Quantum Field Theory (WQFT). In particular, the LSZ reduction formula has been applied, in momentum space, to the worldline representation of the Green function for a scalar particle propagating in a gravitational field, resumming the irreducible part of the related Feynman diagrams, arising from the perturbative expansion in the gravitational coupling constant. Then, based on such relation, a configuration space path integral has been written to compute amplitudes, which, however, are dressed with coherent wave-functions of the Poincarè group. This is needed since this analysis aims at classical applications.

In this thesis we propose a method to LSZ reduce the worldline Green function directly in position space, obtaining a position space path integral, without any dressing with classical wave-functions, thus generating on-shell scattering amplitudes with external asymptotic states. In particular, a position space representation of the Green function is obtained from the mixed position and momentum space representation and path integrating out the momentum perturbation degrees of freedom. Our proposal, thought shown to be equivalent to the usual one, allows to write down, directly in configuration space, a worldline representation for the half reduced dressed propagator, thus making explicit the pole structure of the diagrams with respect to the external lines, otherwise hidden in the standard phase space formulation. As an application we present the Compton scattering amplitude obtained from the reduced dressed propagator in the cases of ϕ^3 and scalar electrodynamics, and in addition, we study the classical limit of such amplitude, reproducing known results in the literature.

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Chapter 1

Introduction

The original Reduction Formula was proposed by Lehmann, Symanzik and Zimmermann (LSZ) in 1955 [2] with the purpose to obtain a consistent Quantum Field Theory (QFT) free from infinities. This celebrated LSZ formula is still used nowadays to reduce S-matrix elements to amplitudes, see e.g. [3–6], from which one can assemble predictions such as cross sections that can be measured. The final formula for the reduction of a correlation function involving only real scalars reads[1]

$$\langle f|i\rangle = i^{n+n'} \prod_{l'} \int d^D x_{l'}\ e^{-ik_{l'}x_{l'}}(-\Box_{l'} + m^2)\ \prod_{l} \int d^D x_l e^{ik_l x_l}(-\Box_l + m^2) \\ \times\ \langle\Omega|\mathrm{T}\phi(x_{l'})\phi(x_l)|\Omega\rangle \quad (1.1)$$

which doesn't show any presence of the creation/annihilation operators that are the backbone of the second quantized QFT, only the inverse free propagator and the time ordered product of fields. The latter is the Green functions of the interacting theory, often represented by path integrals in the space of fields. One interesting consequence of the absence of explicit second quantized creation/annihilation operators suggests that the Green functions may be instead represented by a worldline path integral. This is shown in the main reference article [7], and applied, e.g. in [8, 9]. The worldline formalism is a first quantized quantum-mechanical theory on $0 + 1$ spacetime dimensions embedded into some target space. First developed by Feynman as a mathematical tool for perturbative QED in the early 50's [10, 11] was shown to coincide with the infinite string tension limit of a first quantized *worldsheet* theory, see e.g. [12–15]. More recently, string inspired vertex operators where adopted as a method to recast worldline path integrals into Gaussian integrals [16, 17], see [18–20], in particular [21] for a comprehensive review.

[1] throughout this thesis the Minkowski metric signature is the one with mostly + signs.

1.1 Worldline formalism and scattering amplitudes

Worldline path integrals are representations of the Green function or one-loop effective action of interacting field theories. One way to introduce such path integrals relies on the Schwinger exponentiation of differential operators, represented by the interacting kinetic terms in some field theory. Then, the Schwinger parameter [22], (see, e.g. Chapter 33 in [6]) is interpreted as the proper time of a relativistic particle moving along the worldline, while the differential operator is interpreted as a point particle Hamiltonian. Here is where the worldline enters the picture, since the related transition amplitude, appearing in the Schwinger representation, can be realized as a quantum-mechanical path integral. Thus, worldline path integrals are sums over all possible paths, given specific boundary conditions for the trajectories. There are two main topologies of boundary conditions, the circle, with which one computes effective actions (see e.g. [21, 23–27]) and the topology of the line. In this thesis we will focus on the latter, whose path integral produce *un-reduced* scattering amplitudes ([28–32]). The goal of this thesis is to propose a worldline path integral in position space which evaluates sums of 1PI Feynman diagrams, with external propagators already on-shell and amputated, for the cases of scalars moving through background fields. More specifically, such reduced dressed propagators have been proposed in momentum space in [33, 34]. In particular, the article [7] exemplifies the procedure in momentum space, highlighting the presence of the correct poles structure. It constitutes the main reference.

Inspired by the mixed-position and momentum space representation of the Green functions [35, 36], we build up a path integral representation of the Green function fully in configuration space, which delivers directly the massive external lines propagator, once using the procedure proposed in [7] −namely, a change of coordinates−. We explicitly recover the same poles structure, which is then amputated by the LSZ reduction. This is obtained firstly in the context of a ϕ^3 theory and later on in the case of scalar Electrodynamics (sQED). The reduced, on-shell dressed propagators are then tested, computing the vertices and Compton scattering of the two theories. The sQED Compton amplitude is then studied in the classical limit, employing the Kosover-Maybeee-O'Connel procedure [37]. The Compton amplitude in ϕ^3 is also studied in the classical limit, this time using the appropriate Worldline Quantum Field Theory (WQFT) Feynman Rules, proposed in [38], and applied in the context of Hard Thermal Loops (HTL)([39–42]). Finally, employing the Kawai-Lewellen-Tye (KLT)([43]) like relation, straight at the classical level [44], we obtain the classical gravitational Compton amplitude as the double copy of the sQED Compton amplitude.

1.2 Organization of the thesis

The scope of this thesis is to review the LSZ reduction for momentum space dressed propagators and propose a method to LSZ reduce the position space worldline path integral. In Chapter 2 the reference paper [7] is reviewed, from action of the theory, the derivation of the graviton dressed scalar propagator, to the reduction and the derivation of Feynman rules. In Chapter 3 the same argument is applied to a ϕ^3 toy model. In Chapter 4 the mixed position and momentum representation of the Green function is introduced. We will see how out of this representation one can obtain a Green function representation expressed in configuration space, for the case of ϕ^3. The LSZ reduction of this version of the position space representation of the Green function is remarkably similar to the procedure reviewed in Chapters 2 and 3. The validity of the Green function introduced in Chapter 4 is tested in Chapter 5, obtaining some simple amplitudes in ϕ^3. In the same Chapter we introduce the worldline path integral for a complex scalar propagating in an Abelian background gauge field, first in the phase space representation and then in configuration space. This path integral can be reduced in the same fashion as the ϕ^3 case. It is then tested, obtaining simple sQED amplitudes. Finally, the KLT double copy is used to obtain the classical limit of gravitational Compton scattering.

Chapter 2

Review

In this Chapter the reduction *procedure* found in [7] is reviewed. The paper itself is mainly concerned with the computation of graviton mediated scattering amplitudes of two black holes, represented as differently flavoured scalars, in classical limit. The worldline formalism is used to describe black holes as classical objects [45], treating gravity as an EFT [46, 47]. Of particular interest for this review is how the worldline action is used to obtain a positions space path integral, how the dressed propagator looks like in momentum space, and how the latter is LSZ reduced.

2.1 Worldline action and position space propagator

Let the action S describing the motion of a black hole coupled to gravity be

$$S = S_{EH} + S_{gf} + S_{pm} \tag{2.1}$$

where S_{EH} is the Einstein-Hilbert action

$$S_{EH} = -2m_{Pl}^{D-2} \int d^D x \, \sqrt{g} R. \tag{2.2}$$

The metric is taken as small fluctuations about Minkowski metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}. \tag{2.3}$$

The gauge fix is the De Donder gauge

$$S_{gf} = \int d^D x \, \left(\partial_\nu h^{\mu\nu} - \frac{1}{2} \partial^\mu h \right)^2 . \tag{2.4}$$

S_{pm} is the worldline action, obtained as an EFT by expanding in powers of the curvature tensor, for some Wilson coefficients $\{c_R,\ c_V,\ \dots\}$ necessary to describe an extended object [48], see [45, 49] for a review.

$$S_{pm} = m\int d\tau + c_R\int d\tau R(x) + c_V\int d\tau R_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + \dots \tag{2.5}$$

where $d\tau = \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$ is the proper time. The first term corresponds with geodesics motion with respect to the metric $g_{\mu\nu}$ on the geodesic x^μ. Usually, up to 4^{th} order in the Post Newtonian expansion, terms proportional to c_R and c_V are removed by a field redefinition, as they don't carry consequences for physical observables [50]. Introducing the einbein $e(\tau)$ we can rewrite Equation (2.5) into a Polyakov form

$$S_{pm} = \frac{m}{2}\int_{-\infty}^{+\infty} d\tau\left(e^{-1}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + e\right). \tag{2.6}$$

Integrating out the einbein using its equation of motion

$$e^2 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \tag{2.7}$$

recovers Equation (2.5). We choose instead to gauge fix $e = 1$, which sets $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 1$ and τ as the proper time. The action then reads:

$$S_{pm} = \frac{m}{2}\int_{-\infty}^{+\infty} d\tau\left(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + 1\right). \tag{2.8}$$

In the sequel we will use the action in Equation (2.8), as there are no square roots.

Let us now sketch the derivation of the Polyakov action to describe the black hole scattering, starting from QFT arguments. We picture the black holes as two different flavoured massive scalar fields. The QFT action for a scalar field coupled to gravity is

$$\begin{aligned} S' &= S_{EH} + S_{gf} + \sum_i S_i \\ &= S_{EH} + S_{gf} + \sum_i \int d^Dx\sqrt{g}\left(g^{\mu\nu}\partial_\mu\phi_i^\dagger\partial_\nu\phi_i + m^2\phi_i^\dagger\phi_i - \xi R\phi_i^\dagger\phi_i\right) \end{aligned} \tag{2.9}$$

where the index i refers to the flavours and ξ is a non minimal, dimensionless coupling with the Ricci scalar. Let us now introduce the graviton dressed propagator $G(x, x')$ for the scalar field. It is defined to be the inverse of the interacting kinetic term for the scalar, namely

$$\left(\nabla_\mu\nabla^\mu + m^2 + \xi R\right)G(x, x') = \sqrt{g}\delta^{(D)}(x - x') \tag{2.10}$$

where ∇^μ is the covariant derivative. We represent this Green function as a path integral in field space, namely:

$$G_i(x,x') = \mathcal{Z}_i^{-1} \int \mathcal{D}\phi_i \; \phi_i(x)\phi_i^\dagger(x')e^{iS_i} \tag{2.11}$$

where the index i refers again to the flavour and $\mathcal{Z}_i$ is a normalization constant. Having introduced the dressed propagator, let us now go the the full four scalar amplitude $\phi_1\phi_2 \to \phi_1\phi_2$. As know, from QFT, amplitudes are Fourier transform of amputated Green functions, thus, studying the amplitude in the classical limit meas studying how the related correlator behaves in the classical limit. Let us focus on the following time ordered correlation function

$$\begin{aligned}
&\langle\Omega|\mathrm{T}\phi_1(x_1)\phi_1^\dagger(x_1')\phi_2(x_2)\phi_2^\dagger(x_2')|\Omega\rangle = \\
&= \tilde{\mathcal{Z}}^{-1} \int \mathcal{D}[\phi_1,\phi_2,h_{\mu\nu}] \; \phi_1(x_1)\phi_1^\dagger(x_1')\phi_2(x_2)\phi_2^\dagger(x_2')e^{iS'} \\
&= \mathcal{Z}^{-1} \int \mathcal{D}h_{\mu\nu} \; G_1(x_1,x_1')G_2(x_2,x_2')e^{iS_{EH}+iS_{gf}}
\end{aligned} \tag{2.12}$$

where in the last step the scalar degrees of freedom have been integrated, suppressing all virtual loops including scalars, obtaining the product of G_1G_2. It should be noticed that here, the dressed propagator must be amputated because we are interested in the scattering amplitude. We then represent the Green functions using worldline path integrals. A first guess for the Green function is

$$G(x,x') \sim \int_0^{+\infty} dT \; e^{im^2T} \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x(\tau) \; \exp\left\{ i\int_0^T d\tau \left[\frac{1}{4}g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \tilde{\xi}R(x)\right]\right\}. \tag{2.13}$$

It must be noticed that in a gravitational background the trajectory-space measure depends on the metric [51–53], namely

$$\begin{aligned}
\mathcal{D}x &= Dx \prod_{0\le\tau\le T} \sqrt{g[x(\tau)]} \\
&= \int D[\mathfrak{a},\mathfrak{b},\mathfrak{c}] \exp\left\{ -i\int_0^T d\tau \frac{1}{4}g_{\mu\nu}\left(\mathfrak{a}^\mu\mathfrak{a}^\nu + \mathfrak{b}^\mu\mathfrak{c}^\nu\right)\right\}
\end{aligned} \tag{2.14}$$

where Dx is the flat spacetime path integral measure

$$Dx = \prod_\tau d^Dx(\tau) \tag{2.15}$$

and $\mathfrak{a}^\mu$, $\mathfrak{b}^\mu$ and $\mathfrak{c}^\mu$ are the Lee-Yang ghost fields [54]; the first is Grassmann even, the latter two are Grassmann odd. Reassembling the Green function:

$$\begin{aligned} G(x,x') = \int_0^{+\infty} dT\ e^{im^2T} \int_{x(0)=x}^{x(T)=x'} Dx \int D[\mathfrak{a},\mathfrak{b},\mathfrak{c}] \\ \exp\left\{ i\int_0^T d\tau \left[\frac{1}{4} g_{\mu\nu}\left(\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \mathfrak{a}^\mu\mathfrak{a}^\nu + \mathfrak{b}^\mu\mathfrak{c}^\nu\right) + \left(\xi - \frac{1}{4}\right) R(x)\right]\right\} \end{aligned} \tag{2.16}$$

Equation (2.16) is a prime example to showcase why τ, T are rescaled to $\tau/2m$ and $T/2m$ specifically:

$$\begin{aligned} G(x,x')\Big|_{rescaled} = \int_0^{+\infty} \frac{dT}{2m} \int_{x(0)=x}^{x(T)=x'} Dx \int D[\mathfrak{a},\mathfrak{b},\mathfrak{c}] \exp\left\{ i\int_0^T d\tau \frac{1}{8m} g_{\mu\nu}\left(\mathfrak{a}^\mu\mathfrak{a}^\nu + \mathfrak{b}^\mu\mathfrak{c}^\nu\right)\right\} \\ \exp\left\{ i\int_0^T d\tau \left[\frac{m}{2} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{m}{2} + \frac{1}{2m}\left(\xi - \frac{1}{4}\right) R(x)\right]\right\} \end{aligned} \tag{2.17}$$

where $\dot{x}$ now does indicate the derivative with respect to proper time, in fact, the factor $1/2$ recovers the gauge fixed worldline action in Equation (2.8), up to the not propagating ghosts fields.

2.2 Momentum space dressed propagator

Now that a position space representation of the propagator is available we can Fourier transform to a momentum space representation $D(p,p';\{p_l\})$ so to amputate the external legs. We denote the initial momentum p and the final momentum p', both taken as ingoing into the worldline. Then

$$D(p,p';\{\varepsilon_l,p_l\}) = \int d^D[x,x']\ e^{ip\cdot x + ip'\cdot x'} G(x,x'). \tag{2.18}$$

Let us introduce an expansion in plane waves for the off-shell graviton field:

$$h_{\mu\nu} = \sum_{l=1}^{N} \varepsilon_{\mu\nu}^{(l)} e^{ip_l\cdot x(\tau_l)}. \tag{2.19}$$

We set the boundary conditions for the path integral in Equation (2.18):

$$x^\mu(\tau) = x^\mu + \frac{\tau}{T}\Delta x^\mu + q^\mu(\tau), \quad \Delta x^\mu = x'^\mu - x^\mu \tag{2.20}$$

where q^μ is a fluctuation around the straight line trajectory, obeying Dirichlet Boundary Conditions (DBC). Inserting both the parameterization Equation (2.20) and the field expansion, Equation (2.19), into the Fourier transform of the propagator in Equation (2.16) we obtain

$$D(p,p';\{\varepsilon_l,p_l\}) = \left(\frac{i\kappa}{4}\right)^N \int d^D[x,x']\; e^{ip\cdot x+ip'\cdot x'} \int_0^\infty dT\; e^{im^2T} e^{i\frac{\Delta x^2}{4T}}$$

$$\int_{DBC} D[q] \int D[\mathfrak{a},\mathfrak{b},\mathfrak{c}] \prod_{l=1}^{N} \int_0^T d\tau_l\; \varepsilon^{(l)}_{\mu\nu}\Big(\dot{x}^\mu(\tau_l)\dot{x}^\nu(\tau_l) + \mathfrak{a}^\mu(\tau_l)\mathfrak{a}^\nu(\tau_l) + \mathfrak{b}^\mu(\tau_l)\mathfrak{c}^\nu(\tau_l)\Big) e^{ip_l\cdot x(\tau_l)}$$

$$\times \exp\left\{ i\int_0^T d\tau \frac{1}{4}(\dot{q}^2 + \mathfrak{a}^2 + \mathfrak{b}\mathfrak{c})\right\}. \tag{2.21}$$

The integral can be cast in Gaussian form, by completing the square. To do so, a propagator on the worldline is required, such as:

$$\Delta(\tau,\tau') = \frac{|\tau-\tau'|}{2} - \frac{\tau+\tau'}{2} + \frac{\tau\tau'}{T} \tag{2.22}$$

with coincidence limit

$$\Delta(\tau,\tau) = \frac{\tau^2}{T} - \tau. \tag{2.23}$$

Then the Wick contractions needed to compute the path integrals are:

$$\begin{aligned} \langle q^\mu(\tau) q^\nu(\tau')\rangle &= 2i\eta^{\mu\nu}\Delta(\tau,\tau') \\ \langle \mathfrak{a}^\mu(\tau)\mathfrak{a}^\nu(\tau')\rangle &= -2i\eta^{\mu\nu}\delta(\tau-\tau') \\ \langle \mathfrak{b}^\mu(\tau)\mathfrak{c}^\nu(\tau')\rangle &= 4i\eta^{\mu\nu}\delta(\tau-\tau'). \end{aligned} \tag{2.24}$$

To proceed further let us introduce some more notation. Let $\mathcal{O}(q,\mathfrak{a},\mathfrak{b},\mathfrak{c})$ be some operator valued function of the worldline fluctuation and the ghosts, then we can define the following unnormalized expectation value:

$$\langle \mathcal{O}(q,\mathfrak{a},\mathfrak{b},\mathfrak{c})\rangle = \int_{DBC} Dq \int D[\mathfrak{a},\mathfrak{b},\mathfrak{c}]\; \mathcal{O}(q,\mathfrak{a},\mathfrak{b},\mathfrak{c}) e^{i\int_0^T d\tau \frac{1}{4}(\dot{q}^2+\mathfrak{a}^2+\mathfrak{b}\mathfrak{c})} \tag{2.25}$$

Let us also define

$$\tilde{D}^{(l)}_{\mu\nu}(x,x') = \left[\left(\frac{\Delta x^\mu}{T} + \dot{q}^\mu(\tau_l)\right)\left(\frac{\Delta x^\nu}{T} + \dot{q}^\nu(\tau_l)\right) + \mathfrak{a}^\mu(\tau_l)\mathfrak{a}^\nu(\tau_l) + \mathfrak{b}^\mu(\tau_l)\mathfrak{c}^\nu(\tau_l)\right] e^{ip_l\cdot q(\tau_l)} \tag{2.26}$$

then, using the parameterization in Equation (2.20) the relevant part of the integral in Equation (2.21) can be rewritten as

$$\begin{aligned}\left\langle \prod_{l=1}^{N} \int_0^T d\tau_l\ \varepsilon^{(l)}_{\mu\nu}\Big(\dot{x}^\mu(\tau_l)\dot{x}^\nu(\tau_l) + \mathfrak{a}^\mu(\tau_l)\mathfrak{a}^\nu(\tau_l) + \mathfrak{b}^\mu(\tau_l)\mathfrak{c}^\nu(\tau_l)\Big) e^{ip_l\cdot x(\tau_l)} \right\rangle \\ = \left\langle \prod_{l=1}^{N} \int_0^T d\tau_l\ e^{ip_l\cdot\left(x+\frac{\tau_l}{T}\Delta x\right)} \varepsilon^{(l)\mu\nu}\tilde{D}_{\mu\nu}(x,x') \right\rangle\end{aligned} \tag{2.27}$$

The integral on the fluctuation is performed by:

$$\int_{DBC} Dq\ e^{i\int_0^T d\tau \frac{\dot{q}^2}{4}} = (4\pi i T)^{-D/2}. \tag{2.28}$$

Let us introduce an auxiliary scalar function $F(\varepsilon, \alpha, \beta, \gamma)$, for the "polarization" vectors ε^μ, α^μ which are Grassmann even and β^μ and γ^μ which are Grassmann odd;

$$F(\varepsilon, \alpha, \beta, \gamma) = \left\langle \exp\left\{ \sum_{l=1}^{N} \varepsilon_l\cdot\dot{q}(\tau_l) + \alpha_l\cdot\mathfrak{a}(\tau_l) + \beta_l\cdot\mathfrak{b}(\tau_l) + \gamma_l\cdot\mathfrak{c}(\tau_l) \right\} \right\rangle \tag{2.29}$$

which lets us cast Equation (2.27) as

$$\begin{aligned}&\left\langle \prod_{l=1}^{N} \int_0^T d\tau_l\ \varepsilon^{(l)}_{\mu\nu}\Big(\dot{x}^\mu(\tau_l)\dot{x}^\nu(\tau_l) + \mathfrak{a}^\mu(\tau_l)\mathfrak{a}^\nu(\tau_l) + \mathfrak{b}^\mu(\tau_l)\mathfrak{c}^\nu(\tau_l)\Big) e^{ip_l\cdot x(\tau_l)} \right\rangle = \\ &\prod_{l=1}^{N} \int_0^T d\tau_l \left[\left(\frac{\Delta x_\mu}{T} + \partial_{\varepsilon_l^\mu}\right)\left(\frac{\Delta x_\nu}{T} + \partial_{\varepsilon_l^\nu}\right) + \partial_{\alpha_l^\mu}\partial_{\alpha_l^\nu} + \partial_{\beta_l^\mu}\partial_{\gamma_l^\nu} \right] F(\varepsilon, \alpha, \beta, \gamma)\Bigg|_{\varepsilon=\alpha=\beta=\gamma=0}.\end{aligned} \tag{2.30}$$

Using the expectation value identity

$$e^{\langle\mathcal{O}\rangle} = (4\pi i T)^{-D/2} \exp\left\{ \frac{1}{2}\langle\mathcal{O}\mathcal{O}\rangle \right\} \tag{2.31}$$

which in our case returns

$$F(\varepsilon, \alpha, \beta, \gamma) = (4\pi i T)^{-D/2} \exp\left\{ \frac{1}{2} \left\langle \sum_{l,l'=1}^{N} \mathcal{O}_l\mathcal{O}_{l'} \right\rangle \right\}. \tag{2.32}$$

In our case we will set

$$\mathcal{O}_l = \varepsilon_l\cdot\dot{q}(\tau_l) + \alpha_l\cdot\mathfrak{a}(\tau_l) + \beta_l\cdot\mathfrak{b}(\tau_l) + \gamma_l\cdot\mathfrak{c}(\tau_l) + ip_l\cdot q(\tau_l). \tag{2.33}$$

Using the Wick contractions in Equation (2.24), the expression of the worldline Green function in Equation (2.22) and noticing the derivatives

$$\begin{aligned}
{}^\bullet\Delta(\tau,\tau') &= \frac{1}{2}\,\mathrm{sign}(\tau-\tau') + \frac{\tau'}{T} - \frac{1}{2} \\
\Delta^\bullet(\tau,\tau') &= -\frac{1}{2}\,\mathrm{sign}(\tau-\tau') + \frac{\tau}{T} - \frac{1}{2} \\
{}^\bullet\Delta^\bullet &= \frac{1}{T} - \delta(\tau-\tau') \\
{}^{\bullet\bullet}\Delta(\tau,\tau') &= \delta(\tau-\tau')
\end{aligned} \tag{2.34}$$

we can compute

$$\begin{aligned}
\langle \mathcal{O}_l \mathcal{O}_{l'} \rangle = {}& 2\delta(\tau-\tau')\left(\varepsilon_l\cdot\varepsilon_{l'} + \alpha_l\cdot\alpha_{l'} - 2\gamma_l\cdot\beta_{l'} - 2\gamma_{l'}\cdot\beta_l\right) \\
& - \frac{2}{T}\left(ip_l\tau_l+\varepsilon_l\right)\cdot\left(ip_{l'}\tau_{l'}+\varepsilon_{l'}\right) + ip_l\cdot\left(ip_{l'}\tau_{l'}+\varepsilon_{l'}\right) + ip_{l'}\cdot\left(ip_l\tau_l+\varepsilon_l\right) \\
& - i\,\mathrm{sign}(\tau_l-\tau_{l'})\left(\varepsilon_l\cdot p_{l'} - \varepsilon_{l'}\cdot p_l\right) + p_l\cdot p_{l'}|\tau_l-\tau_{l'}|.
\end{aligned} \tag{2.35}$$

At this point one can promote the $\Delta x^\mu/T$ to the exponent of Equation (2.26) by manually inserting $\sum_{l=1}^N \varepsilon_l\cdot\frac{\Delta x}{T}$ on the r.h.s. in Equation (2.31). At this point the integral in dx and dx' can be taken, yielding a total momentum conservation δ function and the Gaussian integral. Finally, the expression for the graviton dressed scalar propagator in momentum space is:

$$\begin{aligned}
D(p,p';\{\varepsilon_l,p_l\}) = {}& \left(\frac{i\kappa}{4}\right)^N \delta^{(D)}\left(p+p'+\sum_{l=1}^N p_l\right)\int_0^\infty dT\ e^{i(p'^2+m^2)T} \\
& \times\prod_{l=1}^N\int_0^T d\tau_l\ \varepsilon^{(l),\mu\nu}\left[\partial_{\varepsilon_l^\mu}\partial_{\varepsilon_l^\nu} + \partial_{\alpha_l^\mu}\partial_{\alpha_l^\nu} + \partial_{\beta_l^\mu}\partial_{\gamma_l^\nu}\right] \\
& \times\exp\left\{-(p-p')\cdot\sum_{l=1}^N(ip_l\tau_l+\varepsilon_l) - 2i\sum_{l,l'=1}^N\left[\frac{|\tau_l-\tau_{l'}|}{2}p_l\cdot p_{l'} - i\,\mathrm{sign}(\tau_l-\tau_{l'})\varepsilon_l\cdot p_{l'}\right]\right\} \\
& \times\exp\{\delta(\tau_l-\tau_{l'})\left(\varepsilon_l\cdot\varepsilon_{l'}+\alpha_l\cdot\alpha_{l'}-\gamma_l\cdot\beta_{l'}\right)\}\Big|_{\varepsilon_l=\alpha_l=\beta_l=\gamma_l=0}.
\end{aligned} \tag{2.36}$$

2.3 Cutting external legs

Now that we have a momentum space representation of the graviton dressed scalar propagator, Equation (2.36) we may move on to the LSZ reduction. Notice how the graviton legs in Equation (2.36) are *already* cut, in the sense that no poles in $p_l^2 = 0$ arise. Thus the only external scalar legs have to be put on-shell and to cut. Let us begin with the outgoing external scalar leg, carrying momentum p':

$$\begin{aligned}
&\lim_{on-shell} -i(p'^2+m^2)D(p,p';\{\varepsilon_l,p_l\}) = \\
&\qquad\qquad = \lim_{on-shell} -i(p'^2+m^2)\int_0^\infty dT\; e^{i(p'^2+m^2)T}\Omega_N(T) \\
&\qquad\qquad = -\int_0^\infty dT\frac{d}{dT}\left(e^{i(p'^2+m^2)T}\right)\Omega_N(T)\Bigg|_{on-shell}
\end{aligned} \tag{2.37}$$

where Ω_N is a radiative correction[36], defined as

$$\begin{aligned}
\Omega_N(T) = &\left(\frac{i\kappa}{4}\right)^N \delta^{(D)}\left(p+p'+\sum_{l=1}^N\right)\prod_{l=1}^N\int_0^T d\tau_l\varepsilon^{(l),\mu\nu}\left[\partial_{\varepsilon_l^\mu}\partial_{\varepsilon_l^\nu}+\partial_{\alpha_l^\mu}\partial_{\alpha_l^\nu}+\partial_{\beta_l^\mu}\partial_{\gamma_l^\nu}\right] \\
&\times\exp\left\{-(p-p')\cdot\sum_{l=1}^N(ip_l\tau_l+\varepsilon_l)-i\sum_{l,l'=1}^N\left[\frac{|\tau_l-\tau_{l'}|}{2}p_l\cdot p_{l'}-i\,\mathrm{sign}(\tau_l-\tau_{l'})\varepsilon_l\cdot p_{l'}\right]\right\} \\
&\times\exp\{\delta(\tau_l-\tau_{l'})\left(\varepsilon_l\cdot\varepsilon_{l'}+\alpha_l\cdot\alpha_{l'}-\gamma_l\cdot\beta_{l'}\right)\}\Bigg|_{\varepsilon_l=\alpha_l=\beta_l=\gamma_l=0}.
\end{aligned} \tag{2.38}$$

The integral in the last line of Equation (2.37) can be done by parts, and using the on-shell limit:

$$\begin{aligned}
\lim_{on-shell} -i(p'^2+m^2)D(p,p';\{\varepsilon_l,p_l\}) &= \lim_{on-shell}\left(\Omega_N(\infty)-\Omega_N(0)\right) \\
&= \lim_{on-shell}\Omega_N(\infty)
\end{aligned} \tag{2.39}$$

where we used the fact that $\Omega_N(0)$ vanishes.

Now, $\Omega_N(\infty)$ must contain the incoming scalar external propagator, that is, it must develop a simple pole in $p^2 = -m^2$. To show this let us perform a change of coordinates for the proper times τ_l: defining the "center of mass" proper time τ_+

and the "relative" proper times $\tilde{\tau}_l$ as

$$\begin{aligned}
\tau_+ &= \frac{1}{N}\sum_{l=1}^{N}\tau_l, \qquad && \tilde{\tau}_l = \tau_l - \tau_+ \\
\sum_{l=1}^{N}\tilde{\tau}_l &= 0 && \tilde{\tau}_l - \tilde{\tau}_{l'} = \tau_l - \tau_{l'}
\end{aligned} \tag{2.40}$$

with this reparameterization the τ_l integrals may be rewritten as

$$\prod_{l=1}^{N}\int_0^{+\infty} d\tau_l \ldots = \prod_{l=1}^{N}\int_{-\infty}^{+\infty} d\tilde{\tau}_{l'} \int_0^{+\infty} d\tau_+ \ldots \tag{2.41}$$

The radiative function $\Omega_N(T)$, Equation (2.38) only contains the difference $\tau_l - \tau_{l'}$ which is unchanged by the reparameterization, except for $-i(p-p')\cdot\sum_{l=1}^{N} p_l\tau_l$. Hence the effect of the τ_+ integration can be computed as

$$\begin{aligned}
\int_0^{+\infty} d\tau_+ e^{-i(p-p')\cdot\sum_{l=1}^{N} p_l\tau_l} &= e^{-i(p-p')\cdot\sum_{l=1}^{N} p_l\tilde{\tau}_l}\int_0^{+\infty} d\tau_+ e^{i(p-p')\cdot(p+p')\tau_+} \\
&= \frac{ie^{-i(p-p')\cdot\sum_{l=1}^{N} p_l\tilde{\tau}_l}}{p^2+m^2}
\end{aligned} \tag{2.42}$$

where the total momentum conservation δ function has been used, as well as the fact that the outgoing propagator with p' is already on-shell.
The LSZ formula reduces the dressed propagator to a form factor $F(p,p'|\{\varepsilon_l,p_l\})$.

Renaming $\tilde{\tau}_l \to \tau_l$:

$$\begin{aligned}
F(p,p'|\{\varepsilon_l,p_l\}) &= \lim_{on-shell} i(p^2+m^2-i\varepsilon)i(p'^2+m^2-i\varepsilon)D(p,p';\{\varepsilon_l,p_l\}) \\
&= \left(\frac{i\kappa}{4}\right)^N \delta^{(D)}\left(p+p'+\sum_{l=1}^{N}p_l\right)\prod_{l=1}^{N}\int_{-\infty}^{+\infty} d\tau_l\varepsilon^{(l),\mu\nu}\left[\partial_{\varepsilon_l^\mu}\partial_{\varepsilon_l^\nu}+\partial_{\alpha_l^\mu}\partial_{\alpha_l^\nu}+\partial_{\beta_l^\mu}\partial_{\gamma_l^\nu}\right]\delta\left(\sum_{l=1}^{N}\tau_l\right) \\
&\times \exp\left\{-(p-p')\cdot\sum_{l=1}^{N}(ip_l\tau_l+\varepsilon_l) - i\sum_{l,l'=1}^{N}\left[\frac{|\tau_l-\tau_{l'}|}{2}p_l\cdot p_{l'} - i\,\mathrm{sign}(\tau_l-\tau_{l'})\varepsilon_l\cdot p_{l'}\right]\right\} \\
&\times \exp\{\delta(\tau_l-\tau_{l'})\left(\varepsilon_l\cdot\varepsilon_{l'}+\alpha_l\cdot\alpha_{l'}-\gamma_l\cdot\beta_{l'}\right)\}\Big|_{\substack{\varepsilon_l=\alpha_l=\beta_l=\gamma_l=0 \\ p^2=p'^2=-m^2+i\varepsilon}}.
\end{aligned} \tag{2.43}$$

The authors of [7] point out that this reduction can be performed by hand: start with the momentum space representation of the dressed propagator, Equation

(2.36), then drop the overall T integral, insert a total proper time δ function and let the proper time integrals run on $\mathbb{R}$. Similar computations will be carried out in Chapter 3, for the case of a scalar particle moving through a scalar background in ϕ^3 theory, obtaining a comparable result.
For the moment we will continue the review of [7].

2.4 Position space

Let us construct a normalized WQFT partition function, starting with the position space representation of the propagator in Equation (2.16), except we drop the dT integral, let the action in the exponential run on $\mathbb{R}$, setting $\xi = 1/4$.

$$\Xi(b, v; \{\varepsilon_l, p_l\}) = \int_x^{x'} D[x] \int D[\mathfrak{a}, \mathfrak{b}, \mathfrak{c}] \exp\left\{ i \int_{-\infty}^{+\infty} d\tau \left[\frac{1}{4} g_{\mu\nu}(\dot{x}^\mu \dot{x}^\nu + \mathfrak{a}^\mu \mathfrak{a}^\nu + \mathfrak{b}^\mu \mathfrak{c}^\nu) \right] \right\}. \tag{2.44}$$

We expand the graviton field as a collection of plane waves, Equation (2.19), allowing us to construct again the vertex operator,

$$\begin{aligned}\Xi(b, v;\{\varepsilon_l, p_l\}) = &\left(\frac{i\kappa}{4}\right)^N \int_x^{x'} Dx \int D[\mathfrak{a}, \mathfrak{b}, \mathfrak{c}] \\ &\prod_{l=1}^N \int_{-\infty}^{+\infty} d\tau_l \varepsilon_{\mu\nu}^{(l)} \Big(\dot{x}^\mu(\tau_l)\dot{x}^\nu(\tau_l) + \mathfrak{a}^\mu(\tau_l)\mathfrak{a}^\nu(\tau_l) + \mathfrak{b}^\mu(\tau_l)\mathfrak{c}^\nu(\tau_l) \Big) e^{ip_l x(\tau_l)} \\ &\times \exp\left\{ i \int_{-\infty}^{+\infty} d\tau \frac{1}{4} \left[\dot{x}^2(\tau) + \mathfrak{a}^2(\tau) + \mathfrak{b}(\tau)\mathfrak{c}(\tau) \right] \right\}.\end{aligned} \tag{2.45}$$

Let the worldline trajectory be expressed as perturbations z^μ about the straight line trajectory

$$x^\mu(\tau) = b^\mu + v^\mu \tau + z^\mu(\tau) \tag{2.46}$$

where b^μ is the impact parameter for the black holes scattering (assumed to be much larger than the radii of the black holes), v^μ is the 4-velocity of a particle moving along the worldline, $z^\mu(\tau)$ is the usual perturbation.

In order to evaluate the integral over the fluctuations and the ghosts we impose Wick contractions in a similar form of those in Equation (2.24). The propagator on the worldline is chosen to be time symmetric:

$$\Delta(\tau, \tau') = \frac{|\tau - \tau'|}{2} \tag{2.47}$$

which satisfies

$${}^\bullet\Delta^\bullet(\tau - \tau') = -\delta(\tau - \tau'). \tag{2.48}$$

This choice of the propagator on the worldline is equivalent to the previous, Equation (2.22) and amounts to a shift of the background trajectory parameters b^μ and v^μ. Imposing again the change of coordinates for the proper time, as in Equation (2.40) we find that

$$\begin{aligned}
&\Xi(b, v; \{\varepsilon_l, p_l\}) = \Xi_0 \left(\frac{i\kappa}{4}\right)^N \delta^{(D)}\left(v \cdot \sum_{l_1}^{N} p_l\right) e^{ib\cdot\sum_{l=1}^N p_l} \\
&\prod_{l=1}^{N} \int_{-\infty}^{+\infty} d\tau_l \varepsilon^{(l),\mu\nu} \left[\partial_{\varepsilon_l^\mu}\partial_{\varepsilon_l^\nu} + \partial_{\alpha_l^\mu}\partial_{\alpha_l^\nu} + \partial_{\beta_l^\mu}\partial_l^\nu\right] \delta\left(\sum_{l=1}^{N} \tau_l\right) \\
&\times \exp\left\{ v \cdot \sum_{l=1}^{N} (ip_l\tau_l + \varepsilon_l) - i \sum_{l,l'=1}^{N} \Delta(\tau_l - \tau_{l'}) p_l \cdot p_{l'} - i \,{}^\bullet\Delta(\tau_l - \tau_{l'}) \varepsilon_l \cdot p_{l'} \Big]\right\} \\
&\times \exp\{- \,{}^\bullet\Delta^\bullet(\tau_l - \tau_{l'}) \left(\varepsilon_l \cdot \varepsilon_{l'} + \alpha_l \cdot \alpha_{l'} - \gamma_l \cdot \beta_{l'}\right)\}\Big|_{\varepsilon_l = \alpha_l = \beta_l = \gamma_l = 0}
\end{aligned} \tag{2.49}$$

where Ξ_0 is a measure factor.

By denoting q^μ as the total momentum transferred then we can rewrite the boundary conditions on x^μ in terms of the momenta p^μ and p'^μ, recall that $p^\mu = -\frac{1}{2}\dot{x}^{\mu 2}$

$$p^\mu = -\frac{1}{2}\dot{x}^\mu(-\infty) = -\frac{v^\mu}{2} + \frac{q^\mu}{2} \qquad - p'^\mu = -\frac{1}{2}\dot{x}^\mu(+\infty) = -\frac{v^\mu}{2} - \frac{q^\mu}{2}. \tag{2.50}$$

Substituting $p^\mu - p'^\mu = -v^\mu$ in Equation (2.49) then we recover the form factor:

$$\frac{\Xi(b, v; \{\varepsilon_l, p_l\})}{\Xi_0} = \delta^{(D)}\left(v \cdot \sum_{l=1}^{N} p_l\right) e^{ib\cdot\sum_{l=1}^N p_l} \; F(p, p'|\{\varepsilon_l, p_l\}). \tag{2.51}$$

The authors of [7] have shown classical observables can be recovered as WQFT expectation values. The δ function and the plane waves multiplying the amplitude show that the partition function Ξ is not resumming standard Feynman diagrams. In [38] it is shown that those factors arise from having weighted the external states in the Feynman diagrams with coherent wave-functions [55]. Based on such result, the our aim is to find a configuration space path integral representation, which might be similar to the above one, able to generate on-shell scattering amplitudes (or 1PI Feynman diagrams) without any dressing with coherent wave-functions.

[2] to recover the canonical momentum simply rescale $\tau \to \tau/2m$

2.5 WQFT Feynman rules

We conclude this review of [7] by showcasing how one can work out the WQFT Feynman rules for the emission of gravitons from the worldline. We introduce the notation:

$$\int_k = \int \frac{d^4k}{(2\pi)^4} \qquad \int_\omega = \int \frac{d\omega}{(2\pi)} \tag{2.52}$$

having set $D = 4$. We start by going in momentum space for the graviton and in energy space for the position-space fluctuations

$$h_{\mu\nu}(x) = \int_k e^{-ik\cdot x} h_{\mu\nu}(k) \qquad z^\mu(\tau) = \int_\omega e^{-i\omega\tau} z^\mu(\omega). \tag{2.53}$$

The Einstein-Hilbert action, Equation (2.2) is integrated over all of spacetime, imposing momentum conservation on all vertices. The point mass action on the worldline, Equation (2.8) is integrated only in time, hence it imposes only energy ω conservation on vertices. The graviton propagator in De-Donder gauge is the usual:

$$\mu\nu \overset{k}{\rightsquigarrow} \rho\sigma \;= iP_{\mu\nu;\rho\sigma} \int_k \frac{e^{ik\cdot(x-y)}}{k^2 \pm i\varepsilon} \tag{2.54}$$

where the tensor $P_{\mu\nu;\rho\sigma}$ is the usual, in 4 dimensions

$$P_{\mu\nu;\rho\sigma} = \frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\rho\nu} - \eta_{\mu\nu}\eta_{\rho\sigma}) \tag{2.55}$$

Next we find the propagator for z^μ. Starting from the worldline action, Equation (2.8), we insert the trajectory expansion Equation (2.46) and we set

$$S_{pm}\Big|_{h_{\mu\nu}=0} = \int_{-\infty}^{+\infty} d\tau \left(m + mv\cdot\dot{z} + \frac{m}{2}\dot{z}^2\right) \tag{2.56}$$

where the spacetime indices are contracted using the Minkowski metric $\eta_{\mu\nu}$. We can ignore the constant m and the boundary term $mv\dot{z}$, then the third term gives the propagator:

$$\mu \overset{\omega}{\longrightarrow} \nu \;= -i\frac{\eta^{\mu\nu}}{m}\int_\omega \frac{e^{i\omega(\tau_1-\tau_2)}}{(\omega \pm i\varepsilon)^2} = \frac{i\eta^{\mu\nu}}{2m}[|\tau_1-\tau_2| \pm (\tau_1-\tau_2)] \tag{2.57}$$

Where in both propagators, Equations (2.54) and (2.57) we remain agnostic on the choice of Feynman prescription which implements the causality. We can proceed to the interactions. Evaluating the graviton on the worldline, we get

$$\begin{aligned} h_{\mu\nu}(x(\tau)) &= \int_k e^{ik\cdot(b+v\tau+z(\tau))} h_{\mu\nu}(-k) = \sum_{n=0}^{\infty} \frac{i^n}{n!}\int_k e^{ik(b+v\tau)}(kz)^n h_{\mu\nu}(-k) \\ &= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{k,\omega_1,\ldots,\omega_n} e^{ik\cdot b} e^{i(k\cdot v + \sum_{l=1}^n \omega_l)\tau}\left(\prod_{l=1}^n k\cdot z(-\omega_l)\right) h_{\mu\nu}(-k). \end{aligned} \tag{2.58}$$

then, plugging it back into the worldline action we get

$$
\begin{aligned}
S_{pm}^{int} = S_{pm} - S_{pm}\Big|_{h_{\mu\nu}=0} &= \frac{m}{2m_{Pl}} \int_{-\infty}^{+\infty} d\tau h_{\mu\nu}(x(\tau))\ \dot{x}^{\mu}(\tau)\dot{x}^{\nu}(\tau) \\
&= \frac{m}{2m_{Pl}} \int_{-\infty}^{+\infty} h_{\mu\nu}(x(\tau))\ \left(v^{\mu}v^{\nu} + 2v^{(\mu}\dot{z}^{\nu)}(\tau) + \dot{z}^{\mu}(\tau)\dot{z}^{\nu}(\tau)\right) \\
&= \frac{m}{2m_{Pl}} \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{k,\omega_1,\dots,\omega_n} e^{ik\cdot b}\hat{\delta}\left(k\cdot v + \sum_{l=1}^{n}\omega_l\right) h_{\mu\nu}(-k) \left(\prod_{l=1}^{n} z^{\rho_l}(-\omega_l)\right) \\
&\times \left[\left(\prod_{l=1}^{n} k_{\rho_l}\right) v^{\mu}v^{\nu} + \sum_{l=1}^{n}\omega_l \left(\prod_{l'}^{n} k_{\rho'_l}\right) v^{(\mu}\delta^{\nu)}_{\rho_l} + \sum_{l<l'}^{n} \omega_l\omega_{l'} \left(\prod_{j\neq l,l'} k_{\rho_j}\right) \delta^{(\mu}_{\rho_l}\delta^{\nu)}_{\rho_{l'}}\right]
\end{aligned}
\tag{2.59}
$$

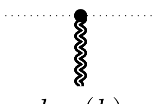
where, in order to avoid the proliferation of powers of (2π) the reduced $\hat{\delta}$ is defined as

$$
\hat{\delta} = (2\pi)^D \delta^{(D)}
\tag{2.60}
$$

from which we can read the n^{th} order in z^{μ}, linear in $h_{\mu\nu}$ vertex

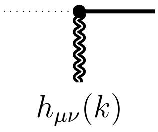
$$
\begin{aligned}
V^{WL,\mu\nu}_{\rho_1,\dots,\rho_n}(k,\omega_1,\dots,\omega_n) &= i^{n-1}\frac{m}{2m_{Pl}} e^{ik\cdot b}\hat{\delta}\left(k\cdot v + \sum_{l=1}^{n}\omega_l\right) \times \\
&\left[\left(\prod_{l=1}^{n} k_{\rho_l}\right) v^{\mu}v^{\nu} + \sum_{l=1}^{n}\omega_l \left(\prod_{l'}^{n} k_{\rho'_l}\right) v^{(\mu}\delta^{\nu)}_{\rho_l} + \sum_{l<l'}^{n} \omega_l\omega_{l'} \left(\prod_{j\neq l,l'} k_{\rho_j}\right) \delta^{(\mu}_{\rho_l}\delta^{\nu)}_{\rho_{l'}}\right].
\end{aligned}
\tag{2.61}
$$

As an example one can compute, at 0^{th} order in z^{μ}:


$h_{\mu\nu}(k)$

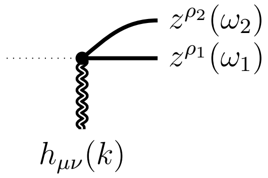
$$
= i\frac{m}{2m_{Pl}} e^{ik\cdot b}\hat{\delta}\,(k\cdot v)\, v^{\mu}v^{\nu}.
\tag{2.62}
$$

which allows to read out the scalar point particle stress-energy tensor, thus representing the classical limit of the three-point amplitude with two scalars sourcing a graviton. Next, at 1^{st} order in z^{μ} one obtains the two point vertex, taking ω outgoing:


$z^{\rho}(\omega)$
$h_{\mu\nu}(k)$

$$
= -\frac{m}{2m_{Pl}} e^{ik\cdot b}\hat{\delta}\,(k\cdot v + \omega)\left[2\omega v^{(\mu}\delta^{\nu)}_{\rho} + v^{\mu}v^{\nu}k_{\rho}\right].
\tag{2.63}
$$

Finally, at 2^{nd} order in z^μ, the three-point vertex reads:



$$= -\, i\frac{m}{m_{\mathrm{Pl}}}e^{ik\cdot b}\hat{\delta}(k\cdot v+\omega_1+\omega_2)\times$$

$$\left(\frac{1}{2}k_{\rho_1}k_{\rho_2}v^\mu v^\nu+\omega_1 k_{\rho_2}v^{(\mu}\delta^{\nu)}_{\rho_1}+\omega_2 k_{\rho_1}v^{(\mu}\delta^{\nu)}_{\rho_2}+\omega_1\omega_2\delta^{(\mu}_{\rho_1}\delta^{\nu)}_{\rho_2}\right). \tag{2.64}$$

Of course, one can have vertices with more fluctuations, but this is enough for our purposes.

2.6 Remarks

Let us recap the results of the last Chapter. Starting from the worldline action it was possible to construct a position space representation of the Green function for the scalars propagating in a graviton background. Then we inserted the latter in the LSZ reduction formula in lieu of the QFT path integral. The reduction itself is preformed on the momentum space representation of the dressed propagator.

It is worth noting in fact that gravitons emitted by the dressed propagators are *already* reduced, at least in the sense that no massless propagator with poles in $p_l^2 = 0$ arise from the dressed propagator of the scalar. The reduction has a net effect on the dressed propagator in momentum space that can be implemented as a simple procedure: remove the dT integration, insert a total proper time δ function and let the integrals on proper time(s) run on $\mathbb{R}$.

One can compute the dressed propagator with only one vertex operator, this result can be found in [7], obtaining a reduced amplitude that may be compared against the scalar-scalar-graviton vertex in the non-minimal coupling $\xi = \frac{1}{4}$.

Equation (2.51) is of central importance. On the l.h.s. the reduced dressed propagator in positions space was reassembled as a partition function with which to compute WQFT expectation values. On the r.h.s. we have the form factor −the amplitude− dressed with coherent wave-functions. In the next Chapter we will walk again those steps in a simpler theory, namely ϕ^3, recovering the nice properties discussed so far.

The WQFT Feynman rules that were obtained can be used in computing classical observables such as the momentum deflection and the eikonal phase [56–58], see [7] itself, [8, 9] in which the same argument is applied to the graviton dressed photon propagator and the photon dressed scalar propagator.

Chapter 3

Application to ϕ^3 theory

In this Chapter we will walk again the steps of the previous Chapter. In particular the QFT and correspondent worldline actions are presented, then the position space and momentum space representations of the dressed propagator. The latter can be reduced by cutting the external scalar propagators. After the reduction, the WQFT Feynman rules of the theory are discussed.

3.1 Worldline action and path integral

Let the QFT action for a real scalar field involving a three point self interaction be:

$$S_{QFT}[\phi] = \int d^D x \left[\frac{1}{2}(\partial^\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{3!}\phi^3 \right] \tag{3.1}$$

then, employing background field methods, the corresponding worldline action reads:

$$S[x, \phi] = \int_0^T d\tau \left[\frac{\dot{x}^2}{4} + m^2 + \lambda \phi(x(\tau)) \right] \tag{3.2}$$

from which we can obtain the Feynman-Schwinger representation of the propagator reads:

$$G(x, x') = \int_0^\infty dT e^{im^2 T} \int_{x(0)=x}^{x(T)=x'} Dx \; e^{i \int_0^T d\tau \left[\frac{\dot{x}^2}{4} + \lambda \phi(x(\tau)) \right]}. \tag{3.3}$$

The background scalar field ϕ, appearing in the above path integral, can be expanded as a sum of plane waves:

$$\phi(x) = \sum_{l=1}^{N} e^{ip_l \cdot x} \tag{3.4}$$

then the interaction potential in the exponential of Equation (3.3) is expanded to order λ^N, while keeping each plane wave exactly once. This defines the vertex operator:

$$V[p_l] = \int_0^T d\tau_l \; e^{ip_l \cdot x(\tau_l)}. \tag{3.5}$$

Hence the position space dressed propagator can be written as

$$\begin{aligned} G(x, x') &= (i\lambda)^N \int_0^\infty dT \; e^{im^2 T} \int_{x(0)=x}^{x(T)=x'} Dx \; e^{i\int_0^T d\tau \frac{\dot{x}^2}{4}} \prod_{l=1}^N V[p_l] \\ &= (i\lambda)^N \int_0^\infty dT \; e^{im^2 T} \int_{x(0)=x}^{x(T)=x'} Dx \; e^{i\int_0^T d\tau \frac{\dot{x}^2}{4}} \prod_{l=1}^N \int_0^T d\tau_l \; e^{ip_l \cdot x(\tau_l)}. \end{aligned} \tag{3.6}$$

The path integral can be computed, imposing the boundary conditions of a straight line:

$$x^\mu(\tau) = x^\mu + \frac{\tau}{T}\Delta x^\mu + q^\mu(\tau), \quad \Delta x^\mu = x'^\mu - x^\mu \tag{3.7}$$

then the propagator reads:

$$\begin{aligned} G(x, x') &= (i\lambda)^N \int_0^\infty dT \; e^{im^2 T} e^{i\frac{\Delta x^2}{4T}} \int_{DBC} Dq \; e^{i\int_0^T d\tau \frac{\dot{q}^2}{4}} \\ &\times \prod_{l=1}^N \int_0^T d\tau_l \; \exp\left\{ \sum_{l=1}^N \left[ip_l \cdot \Delta x \frac{\tau_l}{T} + ip_l \cdot x + ip_l \cdot q(\tau_l) \right] \right\}. \end{aligned} \tag{3.8}$$

In order to perform the integration in Equation (3.8) only one Wick contraction is required. Using the finite Schwinger time T propagator on the worldline:

$$\begin{aligned} \langle q^\mu(\tau) q^\nu(\tau') \rangle &= 2i\eta^{\mu\nu} \Delta(\tau, \tau') \\ \Delta(\tau, \tau') &= \frac{|\tau - \tau'|}{2} - \frac{\tau + \tau'}{2} + \frac{\tau\tau'}{T}. \end{aligned} \tag{3.9}$$

We can cast the path integral into a Gaussian integral. Noticing that

$$\begin{aligned} &\int_0^T d\tau \frac{\dot{x}^2}{4} = \frac{(\Delta x)^2}{4T} + \int_0^T d\tau \frac{\dot{q}^2}{4} \\ &\int_{DBC} Dq \; e^{i\int_0^T d\tau \frac{\dot{q}^2}{4}} = (4\pi i T)^{-D/2} \end{aligned} \tag{3.10}$$

then we can write the path integral as:

$$
\begin{aligned}
G(x,x') = (i\lambda)^N \int_0^\infty dT\ e^{im^2T} e^{i\frac{\Delta x^2}{4T}} (4\pi i T)^{-D/2} \\
\times \prod_{l=1}^{N} \int_0^T d\tau_l\ \exp\left\{ \sum_{l=1}^{N} \left[ip_l \cdot \Delta x \frac{\tau_l}{T} + ip_l \cdot x \right] \right\} \\
\times \exp\left\{ i \sum_{l,l'=1}^{N} \Delta(\tau_l, \tau_{l'}) p_l \cdot p_{l'} \right\}.
\end{aligned}
\tag{3.11}
$$

Now we can take the Fourier transform of the position space dressed propagator, that is

$$
D(p,p';\{p_l\}) = \int d^D[x,x']\ e^{ip\cdot x + ip'\cdot x'} G(x,x'). \tag{3.12}
$$

Let

$$
2x_+^\mu = x'^\mu + x^\mu \tag{3.13}
$$

be the center of mass x coordinate. The integration can then be performed with respect to $d[x_+, \Delta x]$, noticing that this coordinate transformation has Jacobian $= 1$. The integration over dx_+ yields a total momentum δ function:

$$
\begin{aligned}
D(p,p';\{p_l\}) = (i\lambda)^N (2\pi)^D \delta^{(D)} \left(p + p' + \sum_{l=1}^{N} p_l \right) \\
\int_0^\infty dT\ e^{im^2T} (4\pi i T)^{-D/2} \int d^D \Delta x\ e^{i\frac{\Delta x^2}{4T}} \\
\prod_{l=1}^{N} \int_0^T d\tau_l \exp\left\{ i\Delta x \cdot \left(p' + \sum_{l=1}^{N} \frac{\tau_l}{T} p_l \right) + i \sum_{l.l'=1}^{N} \Delta(\tau_l, \tau_{l'}) p_l \cdot p_{l'} \right\}.
\end{aligned}
\tag{3.14}
$$

Performing the $d\Delta x$ integral and inserting the worldline propagator $\Delta(\tau,\tau')$ in Equation (3.9), we obtain a compact result:

$$D(p,p';\{p_l\}) = (i\lambda)^N (2\pi)^D \delta^{(D)}\left(p+p'+\sum_{l=1}^N p_l\right)\int_0^\infty dT\ \prod_{l=1}^N \int_0^T d\tau_l$$

$$\exp\left\{ i\left[m^2 + \left(p' + \frac{1}{T}\sum_{l=1}^N p_l\tau_l\right)^2\right]T + i\sum_{l,l'=1}^N \Delta(\tau_l,\tau_{l'})p_l\cdot p_{l'}\right\}$$

$$= (i\lambda)^N (2\pi)^D \delta^{(D)}\left(p+p'+\sum_{l=1}^N p_l\right)\int_0^\infty dT\ \ \exp\big\{i(m^2+p'^2)T\big\}$$

$$\times \prod_{l=1}^N \int_0^T d\tau_l \exp\left\{ 2ip'\cdot \sum_{l=1}^N p_l\tau_l - i\sum_{l,l'=1}^N \left(\frac{|\tau_l-\tau_{l'}|}{2} - \frac{\tau+\tau_{l'}}{2}\right)p_l\cdot p_{l'}\right\}. \tag{3.15}$$

The last version is the one that will be reduced.

3.2 Cutting external legs

Given the momentum space representation of the dressed propagator, Equation (3.15) the reduction of the *external legs* proceeds as done before. As seen in the graviton dressed propagator, Chapter 2 the background field lines, with momenta $\{p_l\}$ are already reduced, in the sense that no $p_l^2 = -M^2$ poles arise from the dressed propagator.

Starting from the outgoing external line, with momentum p':

$$\lim_{on-shell} -i(p'^2+m^2)D(p,p';\{p_l\}) = \lim_{on-shell} -i(p'^2+m^2)\int_0^\infty dT\ e^{i(p'^2+m^2)T}\Omega_N(T)$$

$$= -\int_0^\infty dT\frac{d}{dT}\left(e^{i(p'^2+m^2)T}\right)\Omega_N(T)\Bigg|_{on-shell}$$

$$= \Omega_N(\infty)\Bigg|_{on-shell} \tag{3.16}$$

where, this time the radiative function $\Omega_N(\infty)$ is defined as

$$\begin{aligned}\Omega_N(\infty) = (i\lambda)^N (2\pi)^D \delta^{(D)}\left(p + p' + \sum_{l=1}^{N} p_l\right) \\ \times \prod_{l=1}^{N} \int_0^\infty d\tau_l \exp\left\{2ip' \cdot \sum_{l=1}^{N} p_l \tau_l - i \sum_{l,l'=1}^{N} \left(\frac{|\tau_l - \tau_{l'}|}{2} - \frac{\tau_l + \tau_{l'}}{2}\right) p_l \cdot p_{l'}\right\}.\end{aligned} \tag{3.17}$$

We introduce the same center of mass proper time coordinates as in Equation (2.40). Focusing on the exponent in Equation (3.17) we can single out from the double sum the case where $l = l'$. Then the effect of this coordinate transformation is:

$$\begin{aligned}&\sum_{l=1}^{N} \left(2p' \cdot p_l + p_l^2\right) \tau_l - \sum_{l\neq l'}^{N} \left(\frac{|\tau_l - \tau_{l'}|}{2} - \frac{\tau_l + \tau_{l'}}{2}\right) p_l \cdot p_{l'} \\ &= \sum_{l=1}^{N} \left(2p' \cdot p_l + p_l^2\right) \left(\tilde{\tau}_l + \tau_+\right) - \sum_{l\neq l'}^{N} \left(\frac{|\tilde{\tau}_l - \tilde{\tau}_{l'}|}{2} - \frac{\tilde{\tau}_l + \tilde{\tau}_{l'}}{2} - \tau_+\right) p_l \cdot p_{l'} \\ &= \sum_{l=1}^{N} \left(2p' \cdot p_l + p_l^2\right) \tilde{\tau}_l - \sum_{l\neq l'}^{N} \left(\frac{|\tilde{\tau}_l - \tilde{\tau}_{l'}|}{2} - \frac{\tilde{\tau}_l + \tilde{\tau}_{l'}}{2}\right) p_l \cdot p_{l'} \\ &+ \sum_{l=1}^{N} \left(2p' \cdot p_l + p_l^2 + \sum_{l\neq l'}^{N} p_l \cdot p_{l'}\right) \tau_+\end{aligned} \tag{3.18}$$

thus, the effect of this change of coordinates is to create a duplicate of the original exponent plus a term proportional only to τ_+. The latter can be recognized to be:

$$\begin{aligned}\sum_{l=1}^{N} \left(2p' \cdot p_l + p_l^2 + \sum_{l\neq l'}^{N} p_l \cdot p_{l'}\right) &= \left(p' + \sum_{l=1}^{N} p_l\right)^2 - p'^2 \\ &= p^2 - p'^2 \\ &= p^2 + m^2\end{aligned} \tag{3.19}$$

where in the first line the conservation of total momentum δ function has been used and in the second line the on-shell condition for $p'^2 = -m^2$ has been used.

Reassembling the reduction formula as was done in Equation (2.43), reintroducing the Feynman prescription and dropping the tildes

$$\begin{aligned} F(p,p'|\{p_l\}) &= \lim_{on-shell} i(p^2+m^2-i\varepsilon)i(p'^2+m^2-i\varepsilon)D(p,p';\{p_l\}) \\ &= (i\lambda)^N(2\pi)^D\delta^{(D)}\left(p+p'+\sum_{l=1}^N p_l\right)\prod_{l=1}^N\int_{-\infty}^{+\infty}d\tau_l\ \delta\left(\sum_{l=1}^N\tau_l\right) \\ &\exp\left\{2ip'\cdot\sum_{l=1}^N p_l\tau_l - i\sum_{l,l'=1}^N\left(\frac{|\tau_l-\tau_{l'}|}{2}-\frac{\tau_l+\tau_{l'}}{2}\right)p_l\cdot p_{l'}\right\} \end{aligned} \tag{3.20}$$

where we notice that the total proper time δ function cancels the contribution from the double sum of $\tau_l+\tau_{l'}$.

Moreover the reduction can be carried by hand, at the level of the dressed propagator: remove the dT integration, insert a total proper time δ function and let the $d\tau_l$ integrals to run on $\mathbb{R}$.

We can perform similar operations on the position space path integral, obtaining the partition function:

$$\Xi(b,v;\{p_l\}) = \int Dx\ e^{i\int_{-\infty}^{+\infty}d\tau\left(\frac{\dot{x}^2}{4}+\lambda\phi(x(\tau))\right)}. \tag{3.21}$$

We can insert N vertex operators, the boundary condition in Equation (2.46). To perform the Wick contraction, Equation (3.9), we choose the time symmetric worldline propagator, $\Delta(\tau,\tau')=\frac{|\tau-\tau'|}{2}$. After some computations we arrive to

$$\frac{\Xi(b,v;\{p_l\})}{\Xi_0} = \delta^{(D)}\left(v\cdot\sum_{l=1}^N p_l\right)e^{ib\cdot\sum_{l=1}^N p_l}\ F(p,p'|\{p_l\}) \tag{3.22}$$

which has the same structure as for the result obtained in the previous Chapter. The partition function Ξ generates the dressed propagator, with in addition the usual factors arising when dressing the external scattering states with coherent wave-functions.

3.3 ϕ^3 WQFT Feynman Rules

Let us end this Chapter with a presentation of the WQFT Feynman Rules for a scalar coupled to a background field ϕ. Recalling the notation introduced in Equation (2.52), the position space and time space representation of the external field ϕ and the worldline perturbation z^μ read:

$$\phi(x) = \int_k e^{-ik\cdot x}\phi(k) \qquad z^\mu(\tau) = \int_\omega e^{-i\omega\tau}z^\mu(\omega). \tag{3.23}$$

The propagator for the background field is the usual

$$\xrightarrow{\;\;k\;\;} = i \int_k \frac{e^{ik\cdot(x-y)}}{k^2 + M^2 \pm i\varepsilon} \tag{3.24}$$

where M is the mass of the external scalar field. Starting from the worldline action in Equation (3.2), rescaling $\tau \to \tau/2m$. Extending the integration domain to $\mathbb{R}$ we obtain:

$$\begin{aligned} S[x,\phi] &= S_{free} + S_{int} \\ &= \frac{m}{2}\int_{-\infty}^{+\infty} d\tau \left(\dot{x}^2 + 1 + \frac{2\lambda}{m}\phi(x(\tau))\right). \end{aligned} \tag{3.25}$$

Expanding the trajectory around the straight line, Equation (2.46) we obtain the same action in Equation (2.56), out of which the propagator for the perturbation z^μ is the same as that in Equation (2.57), here reported for convenience:

$$\mu \overset{\omega}{\rule[0.5ex]{3em}{0.4pt}} \nu = -i\frac{\eta^{\mu\nu}}{m}\int_\omega \frac{e^{i\omega(\tau_1-\tau_2)}}{(\omega \pm i\varepsilon)^2} = i\frac{\eta^{\mu\nu}}{2m}[|\tau_1 - \tau_2| \pm (\tau_1 - \tau_2)]. \tag{3.26}$$

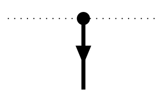
We can insert the trajectory from Equation (2.46) in the representation of the field in Equation (3.23), obtaining:

$$\begin{aligned} \phi(x(\tau)) &= \int_k e^{ik\cdot(b+v\tau+z)}\phi(-k) = \sum_{n=0}^{\infty}\frac{i^n}{n!}\int_k e^{ik\cdot(b+v\tau)}(k\cdot z)^n\phi(-k) \\ &= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{k,\omega_1,\dots\omega_n} e^{ik\cdot b}e^{i(k\cdot v+\sum_{l=1}^{n}\omega_l)\tau}\left(\prod_{l=1}^{n} k\cdot z(-\omega_l)\right)\phi(-k). \end{aligned} \tag{3.27}$$

Feeding this expression back in the interaction action we obtain:

$$\begin{aligned} S_{int} = S - S_{free} &= \lambda\int_{-\infty}^{+\infty} d\tau\ \phi(x(\tau)) \\ &= \lambda\sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{k,\omega_1,\dots\omega_n} e^{ik\cdot b}\hat{\delta}\left(k\cdot v + \sum_{l=1}^{n}\omega_l\right)\phi(-k)\left(\prod_{l=1}^{n}k\cdot z(-\omega_l)\right). \end{aligned} \tag{3.28}$$

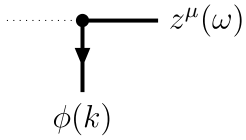
Out of this action we can read infinitely many Feynman rules, all linear in the background field ϕ. At order 0 in the perturbation z^μ we have:



$\phi(k)$

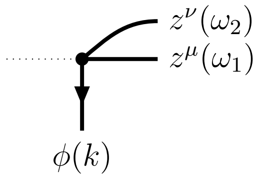
$$= i\lambda e^{ik\cdot b}\hat{\delta}\,(k\cdot v) \tag{3.29}$$

at first order in the perturbation z^μ we obtain:



$$= -\lambda e^{ik\cdot b}\hat{\delta}\left(k\cdot v+\omega\right)k_\mu. \tag{3.30}$$

The last example, the 2^{nd} order in the perturbation z^μ vertex, linear in the background field reads:



$$= -\,i\lambda e^{ik\cdot b}\hat{\delta}(k\cdot v+\omega_1+\omega_2)k_\mu k_\nu. \tag{3.31}$$

Such Feynman rules are in agreement with the ones derived in [38] for the bi-adjoint scalar, once stripping off the color factors.

3.4 Remarks

It is worth to remark that, despite the differences between ϕ^3 and the theory for a scalar non-minimally coupled to gravity, the reduction of the momentum space dressed propagator is performed the same way in both cases. Indeed this is expected, as the LSZ reduction formula, cuts the external lines independently of the interactions of the theory.

The WQFT Feynman rules are a novelty, they where proposed for the first time in [7], where expanded upon and used to compute classical observables in [59–64] in gravity. See also [9] for rules in the case of Scalar QED, [38] where a connection between WQFT and the classical limit is established.

Chapter 4

LSZ reduction in position space

The Green functions in Equation (3.11) is the position space representation of the worldline propagator, already path integrated. Its dependence on the initial and final coordinate x_i^μ and x_f^μ is explicit, hence,hence, one could act with $(\Box_f - m^2)$ and $(\Box_i - m^2)$ on the dressed propagator to achieve an on-shell formulation. However we found this to be hard to pursuit thus we choose instead a different path. We begin by introducing the *mixed* position and momentum representation of the path integral, see [36], although we will follow the derivation in [35].

4.1 Position-momentum representation of the Green function

Let the Green function of the KG operator, $G(x, y)$ satisfy:

$$(\Box_x - m^2)G(x, y) = -i\delta^D(x - y) \tag{4.1}$$

where the $i\varepsilon$ prescription has been ignored. Writing the Green function using the Schwinger representation:

$$G(x_i, x_f) = \int_0^\infty d\beta \, \langle x_f | e^{-i\beta \hat{H}} | x_i \rangle \tag{4.2}$$

where β is the Schwinger proper time, and

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + m^2 - \lambda\phi(x)) \tag{4.3}$$

is the ϕ^3 Hamiltonian on the worldline. The operator $e^{-i\beta\hat{H}}$ is the time evolution operator. It satisfies Schrödinger's equation:

$$i\frac{d}{d\beta}e^{-i\beta\hat{H}} = \hat{H}e^{-i\beta\hat{H}} \tag{4.4}$$

and coincides with the identity at $\beta = 0$. Let p_f^μ be the 4-momentum on the outgoing worldline, we can insert the resolution of the identity in Equation (4.2):

$$G(x_i, x_f; \phi) = \int_0^\infty d\beta \int \frac{d^D p_f}{(2\pi)^D} \langle x_f | p_f \rangle \, \langle p_f | e^{-i\beta \hat{H}} | x_i \rangle \, . \tag{4.5}$$

The scalar product reads:

$$\langle x_f | p_f \rangle = \frac{e^{i p_f \cdot x_f}}{(2\pi)^D}. \tag{4.6}$$

In order to evaluate the matrix element we introduce states in the Hilbert space $|x\rangle$ and $|p\rangle$, which are continuous eigenstates of the operators $\hat{x}$ and $\hat{p}$, then we cast the Hamiltonian in the form:

$$\hat{H}(\hat{x}, \hat{p}) = \sum_{n=0}^{\infty} \hat{p}_{\mu_1} \dots \hat{p}_{\mu_n} H^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_n} \hat{x}^{\nu_1} \dots \hat{x}^{\nu_n} \tag{4.7}$$

then, for a small time parameter Δt the matrix element

$$\langle p | e^{-i \hat{H} \Delta t} | x \rangle = e^{-i H \Delta t + \mathcal{O}[(\Delta) t]^2} \langle p | x \rangle \tag{4.8}$$

where H is a c-number. We can slice the matrix element corresponding to the full time variable into N steps of duration Δt, inserting the resolution of the identity for both momentum and position states at each step:

$$\begin{aligned} &\langle p | e^{-i\beta \hat{H}} | x \rangle = \\ &= \int d^D x_1 \dots d^D x_N \int d^D p_0 \dots d^D p_{N-1} e^{-i \sum_{k=0}^{N-1} H(p_k, x_k) \Delta t} \prod_{k=0}^{N} \langle p_k | x_k \rangle \prod_{k=0}^{N-1} \langle x_{k+1} | p_k \rangle \end{aligned} \tag{4.9}$$

where powers of 2π have been absorbed in the measure. We can take the continuous limit, obtaining the path integral:

$$\langle p_f | e^{-i\beta \hat{H}} | x_i \rangle = \int_{x(0)=x_i}^{p(\beta)=p_f} D[x, p] \; \exp\left\{ -i p(\beta) \cdot x(\beta) + i \int_0^\beta d\tau (p \cdot \dot{x} - H(p, x)) \right\}. \tag{4.10}$$

Reassembling the Green function from Equation (4.5):

$$\begin{aligned} G(x_i, x_f; \phi) = &\int_0^\infty d\beta \int d^D p_f \; e^{i p_f \cdot x_f} \\ &\int_{x(0)=x_i}^{p(\beta)=p_f} D[x, p] \; \exp\left\{ -i p(\beta) \cdot x(\beta) + i \int_0^\beta d\tau (p \cdot \dot{x} - H(p, x)) \right\} \end{aligned} \tag{4.11}$$

where powers of 2π have been absorbed in the measure. Notice the presence of $p(\beta)\cdot x(\beta)$, it is a rather unusual term which arises from the fact that we evaluated the time evolution matrix element between a position eigenstate and a momentum eigenstate. For further details on quantum-mechanical path integrals see e.g. [4]. The sum on the trajectories can be performed around the straight line background:

$$\begin{aligned} x^\mu(\tau) &= x_i^\mu + p_f^\mu \tau + \xi^\mu(\tau) \\ p^\mu(\tau) &= p_f^\mu + \pi^\mu(\tau) \end{aligned} \tag{4.12}$$

where $\xi^\mu(\tau)$ and $\pi^\mu(\tau)$ are perturbations on the trajectory and on the momentum respectively. Then the path integral may be recast into:

$$\int_{x(0)=x_i}^{p(\beta)=p_f} D[x,p] = \int_{\xi(0)=0}^{\pi(\beta)=0} D[\xi,\pi]. \tag{4.13}$$

Under this expansion the relevant exponent in Equation (4.11) reads:

$$\begin{aligned} &p_f\cdot x_f - p(\beta)\cdot x(\beta) + \int_0^\beta d\tau [p\cdot\dot{x} - H(p,x)] = \\ =&p_f\cdot x_f - p_f\cdot(x_i + p_f\beta + \xi(\beta)) + \int_0^\beta d\tau\left[(p_f+\pi)\cdot(p_f+\dot{\xi}) - \frac{1}{2}(p_f+\pi)^2 - \frac{m^2}{2} + \frac{\lambda\phi}{2}\right] \\ =&p_f\cdot(x_f-x_i) - \frac{1}{2}\beta(p_f^2+m^2) + \int_0^\beta d\tau\left(\pi\cdot\dot{\xi} - \frac{\pi^2}{2} + \frac{1}{2}\lambda\phi(x_i+p_f\tau+\xi(\tau))\right). \end{aligned} \tag{4.14}$$

Finally, reassembling into the path integral from Equation (4.11) we obtain:

$$\begin{aligned} G(x_i,x_f;\phi) = &\int_0^\infty d\beta \int d^D p_f\; e^{ip_f\cdot(x_f-x_i)} e^{-\frac{i}{2}\beta(p_f^2+m^2)} \\ &\int_{\xi(0)=0}^{\pi(\beta)=0} D[\xi,\pi]\exp\left\{i\int_0^\beta d\tau\left[\pi\cdot\dot{\xi} - \frac{\pi^2}{2} + \frac{1}{2}\lambda\phi(x_i+p_f\tau+\xi(\tau))\right]\right\}. \end{aligned} \tag{4.15}$$

So far the path integral representation of the Green function is still in the mixed position ξ^μ and momentum π^μ representation, however the integration in $D\pi$ can be performed as a Gaussian integral, obtaining a path integral representation in position space. Rescaling $\beta \to 2T$ we obtain the path integral:

$$
\begin{aligned}
G(x_i, x_f; \phi) = &\int_0^\infty dT \int d^D p_f \; e^{ip_f\cdot(x_f - x_i)} e^{-iT(p_f^2+m^2)} \\
&\int_{\xi(0)=0} D\xi \exp\left\{ i \int_0^T d\tau \left[\frac{\dot{\xi}^2}{2} + \lambda\phi(x_i + p_f\tau + \xi(\tau)) \right] \right\} \\
= &\int d^D p_f \, e^{ip_f(x_f-x_i)} f(x_i, p_f)
\end{aligned}
\tag{4.16}
$$

where, in the last line, we emphasize that the Green function in position space is an inverse Fourier transform of the off-shell current $f(x_i, p_f)$. This version of the path integral can be compared with that in Equation (3.2).

4.2 Cutting external legs

The position space representation of the path integral in Equation (4.16) shows an explicit dependence on x_f^μ, out of which we can take derivatives and eventually the KG operator.

$$
\frac{\partial}{\partial x_f^\mu} G(x_i, x_f; \phi) = i \left\langle p_f^\mu \right\rangle_G \tag{4.17}
$$

$$
\Box_f G(x_i, x_f; \phi) = - \left\langle p_f^2 \right\rangle_G \tag{4.18}
$$

where the subscript G refers to the fact that expectation values are taken with respect to the path integral in Equation (4.16).

The full KG operator then reads:

$$
\begin{aligned}
-i(\Box_f - m^2) G(x_i, x_f; \phi) &= i \left\langle p_f^2 + m^2 \right\rangle_G \\
&= \int d^D p_f \; e^{ip_f\cdot(x_f-x_i)} \left(- \int_0^\infty dT \frac{d}{dT} \left(e^{-iT(p_f^2+m^2)} \right) \Omega(T) \right) .
\end{aligned}
\tag{4.19}
$$

The term in brackets is a one leg amputated off-shell current i.e. $(p_f^2+m^2) f(x_i, p_f)$. So, we continue by performing the LSZ reduction on the outgoing leg, directly at the level of such current, which is really the quantity we are interested in

$$
\begin{aligned}
(p_f^2+m^2)f(x_i,p_f) =& -\left[e^{-iT(p_f^2+m^2)}\Omega(T)\Big|_0^\infty - e^{-iT(p_f^2+m^2)}(\Omega(\infty)-\Omega(0))\right] \\
=& -\left[-\Omega(0) - e^{-iT(p_f^2+m^2)}(\Omega(\infty)-\Omega(0))\right] \\
\xrightarrow[on-shell]{}& \ \Omega(\infty)
\end{aligned}
\tag{4.20}
$$

where in the last line the on-shell condition $p_f^2 = -m^2$ has been taken and $\Omega(T)$ is the radiative function in position space:

$$
\Omega(T) = \int_{\xi(0)=0} D\xi \ \exp\left\{ i\int_0^T d\tau \left[\frac{\dot{\xi}^2}{2} + \lambda\phi(x_i + p_f\tau + \xi(\tau))\right]\right\}. \tag{4.21}
$$

Let us know move on to the amputation of the incoming leg directly in position space. To achieve this, we propose here a path integral representation for the half reduced Green function and later on, we show that it reproduces the correct result for the scattering amplitude. Such representation we propose is based on the above amputation. We start by sending p_f^μ to an arbitrary momentum k^μ, and we redefine the background split point particle trajectory as

$$
x^\mu(\tau) = x_i^\mu - (k+p_i)^\mu\tau + \xi^\mu(\tau). \tag{4.22}
$$

In this way, we propose the following path integral representation, for the half reduced Green function in position space

$$
\begin{aligned}
\bar{G}(x_i,x_f;\phi) =& \int d^Dk \ e^{ik\cdot(x_f-x_i)} \\
& \int_{\xi(0)=0} D\xi \ \exp\left\{ i\int_0^\infty d\tau \left[\frac{\dot{\xi}^2}{2} + \lambda\phi(x_i - (k+p_i)\tau + \xi(\tau))\right]\right\}
\end{aligned}
\tag{4.23}
$$

where the meaning of this redefinition of the trajectory coupled to the external field will become clear in a moment. Let us now show that the above path integral generates the half-reduced current. We start by plane wave expanding the external background field. At order N in the coupling constant the half reduced Green function reads:

$$
\begin{aligned}
\bar{G}(x_i,x_f;\phi) =& (i\lambda)^N \int d^Dk \ e^{ik\cdot(x_f-x_i)} \int_{\xi(0)=0} D\xi \ e^{i\int_0^\infty d\tau \frac{\dot{\xi}^2}{2}} \\
& \prod_{l=1}^N \int_0^\infty d\tau_l \ \exp\{ip_l\cdot[x_i - (k+p_i)\tau_l + \xi(\tau_l)]\}.
\end{aligned}
\tag{4.24}
$$

Now we move to momentum space for the Green function in Equation (4.24):

$$
\begin{aligned}
\bar{D}(p_i, p_f; \phi) = & \int d^D[x_i, x_f] e^{ip_i \cdot x_x + ip_f \cdot x_f} \bar{G}(x_i, x_f; \phi) \\
= & (i\lambda)^N \int d^D k \, \bar{\delta} \left(p_i - k + \sum_{l=1}^{N} p_l \right) \bar{\delta} \left(p_f + k \right) \\
& \times \prod_{l=1}^{N} \int_0^{\infty} d\tau_l \; \exp\left\{ -i(p_i + k) \cdot \sum_{l=1}^{N} p_l \tau_l + i \sum_{l,l'=1}^{N} \Delta_{ll'} p_l \cdot p_{l'} \right\}
\end{aligned}
\tag{4.25}
$$

where $\Delta_{ll'}$ is the time symmetric propagator on the worldline from Equation (2.47). Here, as a consequence of having considered $\tau \in [0, \infty]$, we are implicitly assuming p_f to be on-shell. At this point the integration in $d^D k$ can be performed:

$$
\begin{aligned}
\bar{D}(p_i, p_f; \phi) = & (i\lambda)^N \bar{\delta} \left(p_i + p_f + \sum_{l=1}^{N} p_l \right) \\
& \times \prod_{l=1}^{N} \int_0^{\infty} d\tau_l \; \exp\left\{ i(p_f - p_i) \cdot \sum_{l=1}^{N} p_l \tau_l + i \sum_{l,l'=1}^{N} \Delta_{ll'} p_l \cdot p_{l'} \right\}.
\end{aligned}
\tag{4.26}
$$

We can introduce the same change of coordinates as in Equations (2.40) and (2.41):

$$
\begin{aligned}
\bar{D}(p_i, p_f; \phi) = & (i\lambda)^N \bar{\delta} \left(p_i + p_f + \sum_{l=1}^{N} p_l \right) \\
\times \prod_{l=1}^{N} \int_{-\infty}^{+\infty} d\tilde{\tau}_l \int_0^{+\infty} d\tau_+ \delta \left(\sum_{l=1}^{N} \tilde{\tau}_l \right) & \exp\left\{ i(p_f - p_i) \cdot \sum_{l=1}^{N} p_l (\tilde{\tau}_l + \tau_+) + i \sum_{l,l'=1}^{N} \Delta_{ll'} p_l \cdot p_{l'} \right\}.
\end{aligned}
\tag{4.27}
$$

Again, the only exponential that couples to $d\tau_+$ is

$$
\begin{aligned}
\int_0^{\infty} d\tau_+ e^{i(p_f - p_i) \cdot \sum_{l=1}^{N} p_l \tau_+} &= \frac{-1}{i(p_f - p_i) \cdot \sum_{l=1}^{N} p_l} \\
&= \frac{1}{i(p_f - p_i) \cdot (p_f + p_i)} \\
&= \frac{-1}{i(p_i^2 + m^2)}
\end{aligned}
\tag{4.28}
$$

where the fact that p_f^μ was set on-shell in Equation (4.19) was used. This explicitly shows that our proposal generates exactly the half reduced current, because of the appearance of the above propagator in the current, related to the incoming line. Finally, simply amputating such propagator leads to the momentum space representation of the fully reduced current

$$\begin{aligned} D^c(p_i, p_f; \phi) =& (i\lambda)^N \bar{\delta}\left(p_i + p_f + \sum_{l=1}^N p_l\right) \\ &\times \prod_{l=1}^N \int_{-\infty}^{+\infty} d\tau_l\ \delta\left(\sum_{l=1}^N \tau_l\right) \exp\left\{ i(p_f - p_i)\cdot \sum_{l=1}^N p_l\tau_l + i\sum_{l,l'=1}^N \Delta_{ll'} p_l \cdot p_{l'} \right\}. \end{aligned} \tag{4.29}$$

which is in agreement with (3.20), but obtained from a path integral representation written directly in position space.

4.3 Remarks

As seen in this Chapter a position space representation of the Green function, Equation (4.16) can be obtained by path integrating the momentum perturbation out of the mixed position-momentum space representation, Equation (4.15). In particular, the reduction of the incoming external line requires to fix the worldline trajectory of the point particle, Equation (4.22), tuning the momentum term so that the KG operator generates the correct inverse propagator. In the next Chapter we will test the reduced, *partially* on-shell, Equation (4.29) and the boundary conditions in Equation (4.12). The dressed propagator in Equation (4.29) is partially on-shell in the sense that the external scalars which couple to the worldline are off-shell.

Chapter 5

Applications

In this Chapter some straightforward tests are shown for the ϕ^3 theory, namely the three and four point 1PI diagrams. The latter coincides with the result obtained by summing Feynman diagrams, up to a overall constant factor 1/2 which can be absorbed in the definition of the dressed propagator.

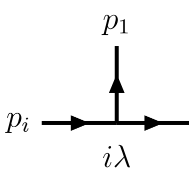
Later in this Chapter the dressed propagator for a complex scalar coupled to a $U(1)$ background gauge field (sQED) is derived, reduced and put on-shell using the same procedure as in the previous Chapter. The reduced sQED propagator is tested against the 3-point vertex and the Compton amplitude. Out of the latter, we take the classical limit via the KMOC method [37].

Finally, we use the KLT relations to obtain the gravitational Compton amplitude as the double copy of sQED amplitudes.

5.1 ϕ^3 Amplitudes

Equipped with the dressed propagator in Equation (4.29) we can begin the testing with the simplest possible amplitude: the ϕ^3 vertex:

$\mathcal{A}_{\phi^3} =$



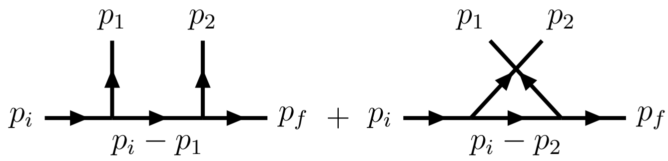
$= i\lambda. \qquad (5.1)$

The reduced dressed propagator recovers this result, in fact, inserting one vertex operator in Equation (4.29), it is straightforward to verify:

$$
\begin{aligned}
D^c(p_i, -p_f; -p_1) &= -i\lambda\, \bar{\delta}\,(p_i - p_f - p_1) \int_{-\infty}^{+\infty} d\tau_1\ \delta(\tau_1) e^{-i(p_f+p_i)\cdot p_1 \tau_1} \\
&= -i\lambda\, \bar{\delta}\,(p_i - p_f - p_1) \\
&= \bar{\delta}(p_i - p_f - p_1)\mathcal{A}_{\phi^3}.
\end{aligned} \tag{5.2}
$$

which is the correct vertex, coupled to the total momentum conservation δ function. We can move on to the 4-point amplitude. Diagrammatically the amplitude is the sum of the t and u channels:

$\mathcal{A}_{\phi^3} =$



(5.3)

inserting two vertex operators in the reduced on-shell dressed propagator in Equation (4.29) and assigning the signs of the momenta according to the momentum flow in Equation (5.3) one obtains:

$$
\begin{aligned}
&D^c(p_i, -p_f; -p_1, -p_2) = (i\lambda)^2 \bar{\delta}(p_i - p_f - p_1 - p_2) \\
&\quad \int_{-\infty}^{+\infty} d[\tau_1, \tau_2]\delta(\tau_1 + \tau_2) \exp\left\{ i(p_f + p_i)\cdot(p_1\tau_1 + p_2\tau_2) - i\sum_{l,l'=1}^{2} \Delta_{ll'} p_l \cdot p_{l'} \right\}.
\end{aligned} \tag{5.4}
$$

The integral in $d\tau_2$ can be performed immediately using the total proper time δ function:

$$
\begin{aligned}
D^c(p_i, -p_f; -p_1, -p_2) &= (-i\lambda)^2 \bar{\delta}(p_i - p_f - p_1 - p_2) \\
&\int_{-\infty}^{+\infty} d\tau_1\ e^{i(p_f+p_i)\cdot(p_1-p_2)\tau_1 - 2i|\tau_1|p_1\cdot p_2}.
\end{aligned} \tag{5.5}
$$

In order to evaluate $|\tau_1|$ we can perform a branch cut on the integral:

$$
\begin{aligned}
D^c(p_i, -p_f; -p_1, -p_2) &= (i\lambda)^2 \bar{\delta}(p_i - p_f - p_1 - p_2) \\
&\times \Bigg[\int_{-\infty}^{0} d\tau_1\ e^{i(p_f+p_i)\cdot(p_1-p_2)\tau_1 + 2ip_1\cdot p_2\tau_1} \\
&+ \int_{0}^{+\infty} d\tau_1\ e^{i(p_f+p_i)\cdot(p_1-p_2)\tau_1 - 2ip_1\cdot p_2\tau_1} \Bigg]
\end{aligned} \tag{5.6}
$$

both integrals can be performed immediately, each producing its respective channel shown in Equation (5.3):

$$D^c(p_i,-p_f;-p_1,-p_2) = (i\lambda)^2\tilde{\delta}(p_i-p_f-p_1-p_2) \times\left[\frac{-i}{(p_f+p_i)\cdot(p_1-p_2)+2p_1\cdot p_2}+\frac{i}{(p_f+p_i)\cdot(p_1-p_2)-2p_1\cdot p_2}\right]. \tag{5.7}$$

Recall that τ_1 was originally the proper time corresponding to the emission of the scalar with momentum p_1^μ, if this emission happens in the past with respect to the center of mass proper time coordinates (the negative proper time domain of the integral in Equation (5.6)) means that it happened *before* the emission of the other scalar, with momentum p_2^μ. Following this reasoning we can interpret the first integral and the corresponding fraction to the propagator in the t channel, while the other integral and corresponding fraction must yield the u channel. With this insight, using the total momentum conservation δ function we can remove p_2^μ from the the first fraction and p_1^μ from the second:

$$\begin{aligned} D^c(p_i,-p_f;-p_1,-p_2) &= (i\lambda)^2\tilde{\delta}(p_i-p_f-p_1-p_2) \\ &\times\left[\frac{-i}{(p_f+p_i)\cdot(p_i-p_f)+2(p_f+p_i)\cdot p_1+2p_1\cdot p_i-2p_1\cdot p_f-2p_1^2}\right. \\ &\left.+\frac{i}{(p_f+p_i)\cdot(p_i-p_f)-2(p_f+p_i)\cdot p_2-2p_2\cdot p_i+2p_2\cdot p_f+2p_2^2}\right]. \end{aligned} \tag{5.8}$$

Since both p_i^μ and p_f^ν are on-shell, the difference of the squares $p_f^2-p_i^2$ vanish in each case. What is left is:

$$\begin{aligned} D^c(p_i,-p_f;-p_1,-p_2) &= (i\lambda)^2\tilde{\delta}(p_i-p_f-p_1-p_2)\left[\frac{-i}{4p_1\cdot p_i-2p_1^2}+\frac{i}{-4p_2\cdot p_i+2p_2^2}\right] \\ &= (i\lambda)^2\tilde{\delta}(p_i-p_f-p_1-p_2)\ \frac{1}{2}\left[\frac{i}{(p_i-p_1)^2+m^2}+\frac{i}{(p_i-p_2)^2+m^2}\right] \\ &= \frac{1}{2}\tilde{\delta}(p_i-p_f-p_1-p_2)\mathcal{A}_{\phi^3}. \end{aligned} \tag{5.9}$$

The factor $1/2$ multiplying the scattering amplitude can be reabsorbed in the path integral.

5.2 Scalar electrodynamics Amplitudes

We begin by obtaining the mixed position and momentum representation of the Green function as done in the previous Chapter. In particular, up to the path integral in Equation (4.11) and the boundary conditions in Equation (4.12) the Hamiltonian was never specified. Let us set the Hamiltonian to

$$2H(p,x) = (p - A(x))^2 + m^2. \tag{5.10}$$

The relevant exponent in the path integral reads:

$$\begin{aligned}
&p_f \cdot x_f - p(\beta)\cdot x(\beta) + \int_0^\beta d\tau [p\cdot \dot{x} - H(p,x)] = \\
=&p_f \cdot x_f - p_f\cdot (x_i + p_f\beta + \xi(\beta)) + \int_0^\beta d\tau \left[(p_f+\pi)\cdot(p_f+\dot{\xi}) - \frac{1}{2}(p_f+\pi-A)^2 - \frac{1}{2}m^2\right] \\
=&p_f\cdot(x_f - x_i) - \frac{\beta}{2}(p_f^2+m^2) + \int_0^\beta d\tau \left[\pi\cdot\dot{\xi} - \frac{1}{2}\pi^2 - \frac{1}{2}A^2 + \pi\cdot A + p_f\cdot A\right] \\
=&p_f\cdot(x_f - x_i) - \frac{\beta}{2}(p_f^2+m^2) + \int_0^\beta d\tau \left[\pi\cdot\dot{\xi} - \frac{1}{2}(\pi - A)^2 + p_f\cdot A\right] \\
=&p_f\cdot(x_f - x_i) - \frac{\beta}{2}(p_f^2+m^2) + \int_0^\beta d\tau \left[(\pi - A)\cdot\dot{\xi} - \frac{1}{2}(\pi - A)^2 + p_f\cdot A + \dot{\xi}\right] \\
=&p_f\cdot(x_f - x_i) - \frac{\beta}{2}(p_f^2+m^2) + \int_0^\beta d\tau \left[(\pi - A)\cdot\dot{\xi} - \frac{1}{2}(\pi - A)^2 + \dot{x}\cdot A\right]
\end{aligned} \tag{5.11}$$

where the minimal coupling between π^μ and A^μ has been put in evidence. The last line follows from the boundary conditions in Equation (4.12). Since the measure is invariant under translation

$$D\pi = D(\pi - A) \tag{5.12}$$

we can use the Gaussian integration to path integrate over the momentum perturbation, obtaining

$$\begin{aligned}
G(x_i, x_f; A) = &\int_0^\infty d\beta \int d^D p_f \; e^{ip_f\cdot(x_f - x_i)} e^{-i\frac{\beta}{2}(p_f^2+m^2)} \\
&\int_{\xi(0)=0} D\xi \exp\left\{ i\int_0^\beta d\tau \left[\frac{\dot{\xi}^2}{2} + \dot{x}\cdot A(x(\tau))\right]\right\},
\end{aligned} \tag{5.13}$$

rescaling $\beta \to 2T$ and restoring the electric charge e we arrive to

$$\begin{aligned} G(x_i, x_f; A) = &\int_0^\infty dT \int d^D p_f \; e^{ip_f\cdot(x_f - x_i)} e^{-iT(p_f^2+m^2)} \\ &\int_{\xi(0)=0} D\xi \exp\left\{ i \int_0^T d\tau \left[\frac{\dot{\xi}^2}{2} + e\dot{x}\cdot A(x(\tau)) \right] \right\}. \end{aligned} \tag{5.14}$$

The reduction of the external legs proceeds *exactly* as seen in the previous Chapter (see e.g. Equations (4.19), (4.28)). We write a representation similar to (4.23), by introducing a momentum k^μ, and allowing $\tau \in [0, \infty]$. Then, going in momentum space in Equation (5.14) we arrive to the momentum space representation of the dressed propagator, which, when LSZ reduced as before reads as

$$\begin{aligned} D^c(p_i, p_f; \{\varepsilon_l, p_l\}) = &(ie)^N \tilde{\delta}\left(p_i + p_f + \sum_{l=1}^N p_l \right) e^{(p_f - p_i)\cdot \sum_{l=1}^N \varepsilon_l} \\ &\times \prod_{l=1}^N \int_{-\infty}^{+\infty} d\tau_l \exp\left\{ i(p_i - p_f)\cdot \sum_{l=1}^N p_l \tau_l - i \sum_{l,l'=1}^N \Delta_{ll'} p_l \cdot p_{l'} \right\} \\ &\times \exp\left\{ -2 \sum_{l,l'=1}^N {}^\bullet\Delta_{ll'} \varepsilon_l \cdot p_{l'} + i \sum_{l,l'=1}^N {}^\bullet\Delta^\bullet_{ll'} \varepsilon_l \cdot \varepsilon_{l'} \right\} \delta\left(\sum_{l=1}^N \tau_l \right) \Bigg|_{m.l.} \end{aligned} \tag{5.15}$$

where the propagator on the worldline is chosen to be time symmetric:

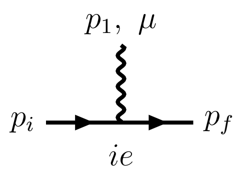
$$\begin{aligned} \Delta(\tau, \tau') &= \frac{|\tau - \tau'|}{2} \\ {}^\bullet\Delta(\tau, \tau') &= \frac{1}{2}\,\mathrm{sign}(\tau - \tau') \\ -{}^\bullet\Delta^\bullet(\tau, \tau') &= \delta(\tau - \tau') \end{aligned} \tag{5.16}$$

while the $|_{m.l.}$ prescription means

$$D^{c,\mu_1\ldots\mu_N}(p_i, p_f; \{p_l\})\Big|_{m.l.} = \prod_{k=1}^N \partial_{\varepsilon_k^{\mu_k}} D^c(p_i, p_f; \{\varepsilon_l, p_l\})\Big|_{\varepsilon=0}. \tag{5.17}$$

Again, we can start testing the validity of the reduced dressed propagator in Equation (5.15) by comparing with QFT amplitudes. The first test is the 3-point vertex:

$$\mathcal{A}^\mu_{sQED} =$$



$$= ie(p_i + p_f)^\mu \tag{5.18}$$

with p_1 taken outgoing. Specializing the reduced dressed propagator to one vertex operator, with momenta assigned as in the amplitude, Equation (5.18) we obtain:

$$
\begin{aligned}
D^c(p_i, -p_f; \varepsilon_1, -p_1) =& ie\, \bar{\delta}(p_i - p_f - p_1)\ e^{-(p_f+p_i)\cdot\varepsilon_1} \\
&\times \int_{-\infty}^{+\infty} d\tau_1\ \exp\{i(p_f + p_i)\cdot p_1\tau_1\}\delta(\tau_1)\Big|_{m.l.}
\end{aligned}
\tag{5.19}
$$

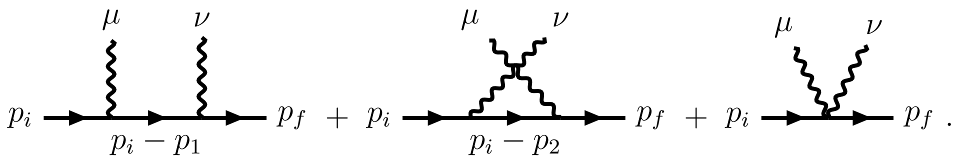
where the singular term proportional to ε_1^2 has been eliminated by the $|_{m.l.}$ prescription. The integral is straightforward since the proper time δ function sets the integral to 1. Finally, taking the $|_{m.l.}$ prescription as in Equation (5.17) one finds:

$$
D^c(p_i, -p_f; \varepsilon_1, -p_1) = ie\ \bar{\delta}(p_i - p_f - p_1)\ (p_i + p_f)^\mu = \bar{\delta}(p_i - p_f - p_1)\mathcal{A}^\mu_{sQED} \tag{5.20}
$$

which is the expected result, up to a redefinition of the charge $e \to -e$.

The amplitude at the next order in the charge, $\mathcal{O}(e^2)$ can be expressed in terms of Feynman diagrams as

$\mathcal{A}^{\mu\nu}_{sQED} =$



(5.21)

At this point we specialize the reduced dressed propagator in Equation (5.15) to the case of two vertex operators, assigning the momenta in the fashion of Equation (5.21) (the photon momenta are taken outgoing):

$$
\begin{aligned}
D^c(p_i, -p_f|\varepsilon_1, \varepsilon_2; -p_1, -p_2) =& (ie)^2\bar{\delta}(p_i - p_f - p_1 - p_2)e^{-(p_i+p_f)\cdot(\varepsilon_1+\varepsilon_2)} \\
&\times \int_{-\infty}^{+\infty} d[\tau_1, \tau_2] \exp\left\{ i(p_f + p_i)\cdot(p_1\tau_1 + p_2\tau_2) - i\sum_{l,l'=1}^{2} \Delta_{ll'} p_l \cdot p_{l'} \right\} \\
&\times \exp\left\{ 2\sum_{l,l'=1}^{N} {}^\bullet\Delta_{ll'}\varepsilon_l \cdot p_{l'} + i\sum_{l,l'=1}^{N} {}^\bullet\Delta^{\bullet}_{ll'}\varepsilon_l\cdot\varepsilon_{l'} \right\}\Bigg|_{m.l.} .
\end{aligned}
\tag{5.22}
$$

Inserting the worldline propagator from Equation (5.16) in the reduced dressed propagator in Equation (5.22) and performing the $d\tau_2$ integration using the total proper time δ function we obtain:

$$\begin{aligned}D^c(p_i,-p_f|\varepsilon_1,\varepsilon_2;-p_1,-p_2) &= (ie)^2\bar{\delta}(p_i-p_f-p_1-p_2)e^{-(p_i+p_f)\cdot(\varepsilon_1+\varepsilon_2)}\\ &\times\int_{-\infty}^{+\infty} d\tau_1\ \exp\{i(p_f+p_i)\cdot(p_1-p_2)\,\tau_1-2i|\tau_1|ip_1\cdot p_2\}\\ &\times\exp\{\mathrm{sign}(\tau_1)\cdot(\varepsilon_1 p_2-\varepsilon_2 p_1)-2i\delta(\tau_1)\varepsilon_1\cdot\varepsilon_2\}\Big|_{m.l.}.\end{aligned} \tag{5.23}$$

At this point the $|_{m.l.}$ prescription can be taken as in Equation (5.17), which, at order $\mathcal{O}(e^2)$ reads:

$$D^{c,\mu\nu}(p_i,-p_f;-p_1,-p_2)\Big|_{m.l.}=\partial_{\varepsilon_1^\mu}\partial_{\varepsilon_2^\nu}D^c(p_i,-p_f|\varepsilon_1,\varepsilon_2;-p_1,-p_2)\Big|_{\varepsilon_1=\varepsilon_2=0}. \tag{5.24}$$

In flat space the partial derivatives commune, hence without loss of generality the explicit $|_{m.l.}$ prescription reads:

$$\begin{aligned}D^{c,\ \mu\nu}(p_i,-p_f;-p_1,-p_2) &= (ie)^2\bar{\delta}(p_i-p_f-p_1-p_2)\\ &\times\int_{-\infty}^{+\infty} d\tau_1\ \exp\{i(p_f+p_i)\cdot(p_1-p_2)\,\tau_1-2i|\tau_1|ip_1p_2\}\\ &\times\partial_{\varepsilon_1^\mu}\partial_{\varepsilon_2^\nu}\ e^{-(p_i+p_f)\cdot(\varepsilon_1+\varepsilon_2)+\mathrm{sign}(\tau_1)(\varepsilon_1\cdot p_2-\varepsilon_2\cdot p_1)-2i\delta(\tau_1)\varepsilon_1\cdot\varepsilon_2}\Big|_{\varepsilon_1=\varepsilon_2=0}\\ &= (ie)^2\bar{\delta}(p_i-p_f-p_1-p_2)\left\langle\partial_{\varepsilon_1^\mu}\partial_{\varepsilon_2^\nu}\right\rangle_{D^c}\end{aligned} \tag{5.25}$$

where the expectation value $\langle(\dots)\rangle_{D^c}$ in the last line of Equation (5.25) is taken with respect to the reduced dressed propagator in the same Equation. Ignoring for now the charge and the total momentum conservation δ function, the following equalities hold:

$$\begin{aligned}D^{c,\mu\nu}(p_i,-p_f;-p_1,-p_2) &= \left\langle\partial_{\varepsilon_1^\mu}\partial_{\varepsilon_2^\nu}\right\rangle_{D^c}\\ &= \left\langle\partial_{\varepsilon_1^\mu}\left[-(p_f+p_i)^\nu+\mathrm{sign}(\tau_1)p_1^\nu-2i\varepsilon_{1,\mu}\eta^{\mu\nu}\delta(\tau_1)\right]\right\rangle_{D^c}\\ &= \langle-2i\eta^{\mu\nu}\delta(\tau_1)+[-(p_f+p_i)^\nu-\mathrm{sign}(\tau_1)p_1^\nu][-(p_f+p_i)^\mu+\mathrm{sign}(\tau_1)p_2^\mu]\rangle_{D^c}\\ &= \langle-2i\eta^{\mu\nu}\delta(\tau_1)+(p_f+p_i)^\mu(p_f+p_i)^\nu+\mathrm{sign}(\tau_1)[(p_f+p_i)^\mu p_1^\nu-p_2^\mu(p_f+p_i)^\nu]\\ &\quad-\mathrm{sign}^2(\tau_1)p_2^\mu p_1^\nu\rangle_{D^c}.\end{aligned} \tag{5.26}$$

The proper time δ function immediately yields the correct 4-point vertex shown in Equation (5.21). In order to evaluate the absolute value $|\tau_1|$ and the sign functions one can perform the branch cut. Explicitly, the reduced dressed propagator reads:

$$
\begin{aligned}
D^{c,\mu\nu}(p_i, -p_f; -p_1, -p_2) &= (ie)^2 \bar{\delta}(p_i - p_f - p_1 - p_2)\Big\{ -2i\eta^{\mu\nu} \\
&+ \int_{-\infty}^{0} d\tau_1 \; e^{i[(p_i+p_f)\cdot(p_1-p_2)+2p_1\cdot p_2]\tau_1} \\
&\quad \times \left[(p_f+p_i)^\mu (p_f+p_i)^\nu + (p_f+p_i)^\mu p_1^\nu - p_2^\mu (p_f+p_i)^\nu - p_2^\mu p_1^\nu\right] \\
&+ \int_{0}^{+\infty} d\tau_1 \; e^{i[(p_i+p_f)\cdot(p_1-p_2)-2p_1\cdot p_2]\tau_1} \\
&\quad \times \left[(p_f+p_i)^\mu (p_f+p_i)^\nu - (p_f+p_i)^\mu p_1^\nu + p_2^\mu (p_f+p_i)^\nu - p_2^\mu p_1^\nu\right] \Big\}.
\end{aligned}
\tag{5.27}
$$

On top of the 4-point vertex −the seagull− which we can immediately recognize that the remaining integrals produce the correct propagators as done in Equation (5.9). All which is left is to check whether the numerators match the correct vertices from sQED.
Let us discuss them separately, starting with the t channel (the first integral in Equation (5.27)). We can use the total momentum conservation δ function to remove p_f^μ:

$$
\begin{aligned}
&(p_f+p_i)^\mu (p_f+p_i)^\nu + (p_f+p_i)^\mu p_1^\nu - p_2^\mu (p_f+p_i)^\nu - p_2^\mu p_1^\nu = \\
&(2p_i - p_1 - p_1)^\mu (2p_i - p_1 - p_2)^\nu + p_2^\mu (2p_i - p_1 - p_2)^\nu - (2p_i - p_1 - p_2)^\mu p_1^\nu - p_2^\mu p_1^\nu = \\
&(2p_i - p_1 - p_2)^\mu (2p_i - p_1 - p_2)^\nu + p_2^\mu (2p_i - 2p_1 - p_2)^\nu = \\
&(2p_i - p_1)^\mu (2p_i - 2p_1 - p_2)^\nu.
\end{aligned}
\tag{5.28}
$$

The u channel numerator is studied similarly, removing again p_f^μ:

$$
\begin{aligned}
&(p_f+p_i)^\mu (p_f+p_i)^\nu - (p_f+p_i)^\mu p_1^\nu + p_2^\mu (p_f+p_i)^\nu - p_2^\mu p_1^\nu = \\
&(2p_i - p_1 - p_2)^\mu (2p_i - p_1 - p_2)^\nu - p_2^\mu (2p_i - p_1 - p_2)^\nu + (2p_i - p_1 - p_2)^\mu p_1^\nu - p_2^\mu p_1^\nu = \\
&(2p_i - p_1 - p_2)^\mu (2p_i - p_2)^\nu - p_2^\mu (2p_i - p_2)^\nu = \\
&(2p_i - 2p_2 - p_1)^\mu (2p_i - p_2)^\nu.
\end{aligned}
\tag{5.29}
$$

Finally, the reduced dressed propagator can be reassembled:

$$
\begin{aligned}
D^{c,\mu\nu}(p_i,-p_f;-p_1,-p_2) &= ie^2\hat{\delta}(p_i-p_f-p_1-p_2)\Bigg\{2\eta^{\mu\nu} \\
&- \frac{(2p_i-p_1)^\mu(2p_i-2p_1-p_2)^\nu}{(p_i-p_1)^2+m^2} - \frac{(2p_i-2p_2-p_1)^\mu(2p_i-p_2)^\nu}{(p_i-p_2)^2+m^2}\Bigg\} \\
&= \hat{\delta}(p_i-p_f-p_1-p_2)\mathcal{A}^{\mu\nu}_{sQED}.
\end{aligned} \tag{5.30}
$$

5.3 Classical limit à la KMOC

As a further application we can apply the KMOC procedure [37] to obtain the classical limit of the sQED Compton amplitude, (5.30). The procedure is essentially a power counting for the Planck constant $\hbar$. The constant $\hbar$ can be reintroduced in the amplitude by means of dimensional analysis[3], expressing the photon momenta p_l in terms of *wavenumbers*

$$p_l = \hbar q_l \tag{5.31}$$

which have dimensions of $[L]^{-1}$. We can revisit the total momentum conservation δ function which appears in Equation (5.30), making the wavenumbers explicit:

$$\hat{\delta}(p_i - p_f - \hbar q_1 - \hbar q_2) \tag{5.32}$$

then, denoting $p_i^\mu = p^\mu$ we get that $p_f = p+\hbar q_1+\hbar q_2$. Then squaring such relations and neglecting subleading $\hbar$ terms, we get that

$$p\cdot q_2 = -p\cdot q_1, \tag{5.33}$$

which, as we will see, will be automatically produced when using the WQFT to evaluate such classical result. In addition, we take the photon momenta to be off-shell. Performing a Laurent expansion in $\hbar$ we get that

$$
\begin{aligned}
\frac{1}{-2\hbar p\cdot q_l+\hbar^2 q_l^2} &= \frac{1}{-2\hbar p\cdot q_l\left(1-\frac{\hbar q_l^2}{2p\cdot q_l}\right)} \\
&= \frac{1}{-2\hbar p\cdot q_l}\sum_{n=0}^{\infty}\left(\frac{\hbar q_l^2}{2p\cdot q_l}\right)^n \\
&= \frac{1}{-2\hbar p\cdot q_l}\left[1+\frac{\hbar q_l^2}{2p\cdot q_l}+\mathcal{O}(\hbar^2)\right].
\end{aligned} \tag{5.34}
$$

[3] restoring $\hbar$ the correct dimensionless electric charge is $e/\sqrt{\hbar}$ but since we are interested only in the channels terms of the amplitude this is ignored

Dropping for now the prefactor and the seagull term, $\eta^{\mu\nu}$ which are trivial the remaining t and u channels can be expanded to

$$\begin{aligned}&\frac{4p^\mu p^\nu - 4\hbar p^\mu q_1^\nu - 2\hbar p^\mu q_2^\nu - 2\hbar q_1^\mu p^\nu}{2\hbar p\cdot q_1}\left(1+\frac{\hbar q_1^2}{2p\cdot q_1}\right)\\ +&\frac{4p^\mu p^\nu - 2\hbar p^\mu q_2^\nu - 4\hbar q_2^\mu p^\nu - 2\hbar q_1^\mu p^\nu}{2\hbar p\cdot q_2}\left(1+\frac{\hbar q_2^2}{2p\cdot q_2}\right)+\mathcal{O}(\hbar).\end{aligned} \tag{5.35}$$

On the support of the δ function in Equation (5.33) we can write

$$\begin{aligned}&\frac{4p^\mu p^\nu - 4\hbar p^\mu q_1^\nu - 2\hbar p^\mu q_2^\nu - 2\hbar q_1^\mu p^\nu}{2\hbar p\cdot q_1}\left(1+\frac{\hbar q_1^2}{2p\cdot q_1}\right)\\ -&\frac{4p^\mu p^\nu - 2\hbar p^\mu q_2^\nu - 4\hbar q_2^\mu p^\nu - 2\hbar q_1^\mu p^\nu}{2\hbar p\cdot q_1}\left(1-\frac{\hbar q_2^2}{2p\cdot q_1}\right)+\mathcal{O}(\hbar)\end{aligned} \tag{5.36}$$

from which we immediately see that the superclassical terms $p^\mu p^\nu$ cancel. After some algebraic manipulations and reintroducing the seagull term $\eta^{\mu\nu}$ we arrive to

$$A^{\mu\nu}_{sQED} = 2ie^2\hat{\delta}(p\cdot(q_1+q_2))\left[\eta^{\mu\nu} - \frac{q_1^\mu p^\nu}{p\cdot q_1} + \frac{q_2^\mu p^\nu}{p\cdot q_1} + \frac{1}{2}\frac{p^\mu p^\nu (q_1^2+q_2^2)}{(p\cdot q_1)^2}\right] + \mathcal{O}(\hbar). \tag{5.37}$$

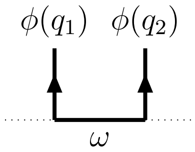
Finally, using momentum conservation we arrive to

$$A^{\mu\nu}_{sQED} = 2ie^2\hat{\delta}(p\cdot(q_1+q_2))\left[\eta^{\mu\nu} - \frac{q_1^\mu p^\nu}{p\cdot q_1} + \frac{q_2^\mu p^\nu}{p\cdot q_1} - \frac{p^\mu p^\nu q_1\cdot q_2}{(p\cdot q_1)^2}\right] \tag{5.38}$$

which reproduces the result found in [38], computed by setting up the WQFT for scalar electrodynamics. The same calculation can be performed by using the results from ϕ^3 cube. In such a case we present the calculation of the 2-point HTL-current ([38]) both from Feynman diagrams and from a WQFT perspective. Using the same procedure as above, to take the classical limit of the 1PI sum of Feynman diagrams in (5.3), we get the following answer

$$\bar{A}_{\phi^3} = \lim_{\hbar\to 0}\mathcal{A}_{\phi^3} = -i\lambda^2\frac{q_1\cdot q_2}{2\,(p\cdot q_1)^2} \tag{5.39}$$

while, on the WQFT side, we just have one diagram, namely



$$\begin{aligned}&= -\lambda^2\int_{-\infty}^{\infty}\mathrm{d}\omega\,\delta(q_1\cdot p+\omega)\delta(q_2\cdot p-\omega)\frac{i}{\omega^2}q_1\cdot q_2\\ &= \delta(p\cdot(q_1+q_2))\left(-i\lambda^2\frac{q_1\cdot q_2}{2\,(p\cdot q_1)^2}\right) = \delta(p\cdot(q_1+q_2))\bar{A}_{\phi^3}\end{aligned} \tag{5.40}$$

which matches the result obtained above using Feynman diagrams.

5.4 Gravitational Compton amplitude from double copy

In addition to the above applications, here, using the KLT relation showed in [38], at the classical level, we generate the classical limit of the gravitational Compton amplitude, describing the scattering of linearized gravitational waves off massive scalar particles. Such a scenario holds when the wave length of the gravitational waves is much bigger than the Schwarschild radius of the black holes, thus described as massive scalar particles.

To generate the Compton current, we use the double copy relation

$$\mathcal{M}^{\mu_1\nu_1,\mu_2\nu_2} = \frac{(q_1\cdot p)^2}{q_1\cdot q_2} A^{\mu_1\mu_2}_{sQED} A^{\nu_1\nu_2}_{sQED} \tag{5.41}$$

then, contracting with trace-less and transverse polarization tensors, written as a copy of two null photon polarizations i.e. $\epsilon^h_{\mu\nu}(q) = \epsilon^h_\mu(q)\epsilon^h_\nu(q)$, we get the on-shell gravitational Compton as

$$\begin{aligned}
\mathcal{M}^{h_1h_2}(q_1,q_2) &= \mathcal{M}^{\mu\nu,\alpha\beta}(q_1,q_2)\epsilon^{h_1}_\mu(q_1)\epsilon^{h_1}_\nu(q_1)\epsilon^{h_2}_\alpha(q_2)\epsilon^{h_2}_\beta(q_2)\\
&= -\frac{\kappa^4 q_1\cdot q_2\,(p\cdot\epsilon_1)^2\,(p\cdot\epsilon_2)^2}{16\omega^2} - \frac{\kappa^4\,(p\cdot\epsilon_1)^2 p\cdot\epsilon_2 q_1\cdot\epsilon_2}{8\omega} + \frac{\kappa^4 p\cdot\epsilon_1\,(p\cdot\epsilon_2)^2 q_2\cdot\epsilon_1}{8\omega}\\
&+ \frac{\kappa^4\omega\epsilon_1\cdot\epsilon_2 p\cdot\epsilon_1 q_1\cdot\epsilon_2}{8q_1\cdot q_2} - \frac{\kappa^4\omega\epsilon_1\cdot\epsilon_2 p\cdot\epsilon_2 q_2\cdot\epsilon_1}{8q_1\cdot q_2} - \frac{\kappa^4\,(p\cdot\epsilon_1)^2\,(q_1\cdot\epsilon_2)^2}{16q_1\cdot q_2}\\
&- \frac{\kappa^4\,(p\cdot\epsilon_2)^2\,(q_2\cdot\epsilon_1)^2}{16q_1\cdot q_2} + \frac{\kappa^4 p\cdot\epsilon_1 p\cdot\epsilon_2 q_1\cdot\epsilon_2 q_2\cdot\epsilon_1}{8q_1\cdot q_2}\\
&+ \frac{1}{8}\kappa^4\epsilon_1\cdot\epsilon_2 p\cdot\epsilon_1 p\cdot\epsilon_2 - \frac{\kappa^4\omega^2\,(\epsilon_1\cdot\epsilon_2)^2}{16q_1\cdot q_2}
\end{aligned} \tag{5.42}$$

where we defined $\omega = p\cdot q_1$ being the classical limit of the $t-$channel propagator in the QFT calculations. The above amplitude is gauge invariant by construction, given that the sQED classical amplitude is gauge invariant by itself, and correctly reproduces the known results from the literature.

Conclusion

The Worldline formalism is a powerful tool to compute Green functions and effective actions. In Chapters 2 and 3 we have seen how momentum space representations of dressed propagators can be reduced and put on-shell. In particular, in [7] the net effects of the reduction on the momentum space dressed propagator is derived. Then, an extension was performed in configuration space for classical applications. However it does not evaluates on-shell Green functions in vacuum, but, such correlators are dressed with coherent wave-functions of the Poincarè group. This has an effect on the worldline action, now integrated over all real values of the proper time. Out of this infinitely extended worldline action one can compute WQFT Feynman Rules, which are a novelty in the literature of scattering amplitudes and thus, quickly became influential. The main goal of this thesis is to show that the reduction can be performed directly at the level of the path integral in configuration space, without dressing the amplitude with such wave-functions. So to be able to evaluate on-shell Feynman diagrams, with external asymptotic states, from a worldline formulation.

To do so the mixed position and momentum space representation of the Green function is presented, and out of the latter a path integral fully in position space is obtained, see Equations (4.16) and (5.14), and in general Chapters 4 and 5. This position space Green function includes and integration in the outgoing momentum, which turns $\Box$ operators into squares of momenta: it is a position space representation of the Green function compatible with the LSZ reduction proposed in [7].

One can investigate whether this procedure of obtaining a fully position space representation of the Green function out of the mixed position and momentum representation and its subsequent reduction, can be extended to other theories, e.g. the bi-adjoint scalar, Scalar Chromodynamics, gravity theories; as well as having fermions propagating on the worldline, e.g. fermion QED [65, 66], fermion QCD and fermion-gravity; having vectors propagating on the worldline, recovering e.g. the dressed propagator in [8]. Our representation allows to generate on-shell Feynman diagrams very easily, since it only boils down to the calculation of unbounded Schwinger integrals which are easier to perform with respect to the purely off-shell case. In addition our formulation gives a very compact expression for a Feynman diagram re-summation, which is quite remarkable. In addition such amplitudes can also be used to study on-shell features of the double copy construction for matter lines [67, 68], recently extended, straight at the classical level by [44]. Such double copy constructions might boost the efficiency in the generation of classical integrands needed for applications to classical black hole scattering since they allows to generates integrands for spinning particles, reproducing a multipole expansion of Kerr black holes observables [69].

Bibliography

[1] J.L. Lagrange. *Mécanique analytique*. Mécanique analytique v. 1. Ve Courcier, 1811. URL: https://books.google.it/books?id=Q8MKAAAAYAAJ.

[2] H. Lehmann, Kurt Symanzik, and Wolf Zimmermann. "Zur Formulierung quantisierter Feldtheorien". In: *Il Nuovo Cimento (1955-1965)* 1 (1955), pp. 205–225.

[3] M. Srednicki. *Quantum field theory*. Cambridge University Press, Jan. 2007. ISBN: 978-0-521-86449-7, 978-0-511-26720-8.

[4] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to quantum field theory*. Reading, USA: Addison-Wesley, 1995. ISBN: 978-0-201-50397-5.

[5] Claude Itzykson and Jean-Bernard Zuber. *Quantum field theory*. Courier Corporation, 2012.

[6] Matthew D. Schwartz. *Quantum Field Theory and the Standard Model*. Cambridge University Press, Mar. 2014. ISBN: 978-1-107-03473-0, 978-1-107-03473-0.

[7] Gustav Mogull, Jan Plefka, and Jan Steinhoff. "Classical black hole scattering from a worldline quantum field theory". In: *Journal of High Energy Physics* 2021.2 (Feb. 2021). DOI: 10.1007/jhep02(2021)048. URL: https://doi.org/10.1007%2Fjhep02%282021%29048.

[8] Fiorenzo Bastianelli, Francesco Comberiati, and Leonardo de la Cruz. "Light bending from eikonal in worldline quantum field theory". In: *Journal of High Energy Physics* 2022.2 (Feb. 2022). DOI: 10.1007/jhep02(2022)209. URL: https://doi.org/10.1007%2Fjhep02%282022%29209.

[9] Tianheng Wang. *Binary Dynamics from Worldline QFT for Scalar-QED*. 2022. DOI: 10.48550/ARXIV.2205.15753. URL: https://arxiv.org/abs/2205.15753.

[10] R. P. Feynman. "Mathematical Formulation of the Quantum Theory of Electromagnetic Interaction". In: *Phys. Rev.* 80 (3 Nov. 1950), pp. 440–457. DOI: 10.1103/PhysRev.80.440. URL: https://link.aps.org/doi/10.1103/PhysRev.80.440.

[11] Richard P. Feynman. “An Operator Calculus Having Applications in Quantum Electrodynamics”. In: *Phys. Rev.* 84 (1 Oct. 1951), pp. 108–128. DOI: 10.1103/PhysRev.84.108. URL: https://link.aps.org/doi/10.1103/PhysRev.84.108.

[12] J. Scherk. “Zero-slope limit of the dual resonance model”. In: *Nuclear Physics B* 31.2 (1971), pp. 222–234. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(71)90227-6. URL: https://www.sciencedirect.com/science/article/pii/0550321371902276.

[13] J.L. Gervais and A. Neveu. “Ghost-free dual model with leading intercept minus one”. In: *Nuclear Physics B* 47.2 (1972), pp. 422–435. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(72)90443-9. URL: https://www.sciencedirect.com/science/article/pii/0550321372904439.

[14] T. Yoneya. “Quantum gravity and the zero slope limit of the generalized Virasoro model”. In: *Lett. Nuovo Cim.* 8 (1973), pp. 951–955. DOI: 10.1007/BF02727806.

[15] J. Scherk and J.H. Schwarz. “Dual models and the geometry of space-time”. In: *Physics Letters B* 52.3 (1974), pp. 347–350. ISSN: 0370-2693. DOI: https://doi.org/10.1016/0370-2693(74)90059-8. URL: https://www.sciencedirect.com/science/article/pii/0370269374900598.

[16] Zvi Bern and David A. Kosower. “Color decomposition of one-loop amplitudes in gauge theories”. In: *Nuclear Physics B* 362.1 (1991), pp. 389–448. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(91)90567-H. URL: https://www.sciencedirect.com/science/article/pii/055032139190567H.

[17] Zvi Bern and David A. Kosower. “The computation of loop amplitudes in gauge theories”. In: *Nuclear Physics B* 379.3 (1992), pp. 451–561. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(92)90134-W. URL: https://www.sciencedirect.com/science/article/pii/055032139290134W.

[18] M. G. Schmidt and C. Schubert. *The Worldline Path Integral Approach to Feynman Graphs*. 1994. DOI: 10.48550/ARXIV.HEP-PH/9412358. URL: https://arxiv.org/abs/hep-ph/9412358.

[19] Olindo Corradini et al. *Spinning Particles in Quantum Mechanics and Quantum Field Theory*. 2015. DOI: 10.48550/ARXIV.1512.08694. URL: https://arxiv.org/abs/1512.08694.

[20] James P. Edwards and Christian Schubert. *Quantum mechanical path integrals in the first quantised approach to quantum field theory*. 2019. DOI: 10.48550/ARXIV.1912.10004. URL: https://arxiv.org/abs/1912.10004.

[21] Christian Schubert. “Perturbative quantum field theory in the string-inspired formalism”. In: *Physics Reports* 355.2-3 (Dec. 2001), pp. 73–234. DOI: 10.1016/s0370-1573(01)00013-8. URL: https://doi.org/10.1016%2Fs0370-1573%2801%2900013-8.

[22] Julian Schwinger. “On Gauge Invariance and Vacuum Polarization”. In: *Phys. Rev.* 82 (5 June 1951), pp. 664–679. DOI: 10.1103/PhysRev.82.664. URL: https://link.aps.org/doi/10.1103/PhysRev.82.664.

[23] Naser Ahmadiniaz et al. “Tadpole contribution to magnetic photon-graviton conversion”. In: *16th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Astrophysics and Relativistic Field Theories.* Nov. 2021. DOI: 10.1142/9789811269776_0370. arXiv: 2111.01980 [hep-ph].

[24] Fiorenzo Bastianelli and Francesco Comberiati. “Path integral calculation of heat kernel traces with first order operator insertions”. In: *Nucl. Phys. B* 960 (2020), p. 115183. DOI: 10.1016/j.nuclphysb.2020.115183. arXiv: 2005.08737 [hep-th].

[25] Fiorenzo Bastianelli et al. “One-loop quantum gravity from the $\mathcal{N} = 4$ spinning particle”. In: *JHEP* 11 (2019), p. 124. DOI: 10.1007/JHEP11(2019)124. arXiv: 1909.05750 [hep-th].

[26] Naser Ahmadiniaz et al. “One-particle reducible contribution to the one-loop spinor propagator in a constant field”. In: *Nucl. Phys. B* 924 (2017), pp. 377–386. DOI: 10.1016/j.nuclphysb.2017.09.012. arXiv: 1704.05040 [hep-th].

[27] Fiorenzo Bastianelli, Francesco Comberiati, and Leonardo de la Cruz. “Worldline description of a bi-adjoint scalar and the zeroth copy”. In: *JHEP* 12 (2021), p. 023. DOI: 10.1007/JHEP12(2021)023. arXiv: 2107.10130 [hep-th].

[28] Naser Ahmadiniaz, Fiorenzo Bastianelli, and Olindo Corradini. “Dressed scalar propagator in a non-Abelian background from the worldline formalism”. In: *Phys. Rev. D* 93.2 (2016). [Addendum: Phys.Rev.D 93, 049904 (2016)], p. 025035. DOI: 10.1103/PhysRevD.93.025035. arXiv: 1508.05144 [hep-th].

[29] Fiorenzo Bastianelli et al. “Integral representations combining ladders and crossed-ladders”. In: *Journal of High Energy Physics* 2014 (May 2014). DOI: 10.1007/JHEP07(2014)066.

[30] Naser Ahmadiniaz et al. “Worldline master formulas for the dressed electron propagator. Part 2. On-shell amplitudes”. In: *JHEP* 01 (2022), p. 050. DOI: 10.1007/JHEP01(2022)050. arXiv: 2107.00199 [hep-th].

[31] N. Ahmadiniaz et al. "Worldline master formulas for the dressed electron propagator. Part I. Off-shell amplitudes". In: *JHEP* 08.08 (2020), p. 049. DOI: 10.1007/JHEP08(2020)018. arXiv: 2004.01391 [hep-th].

[32] Olindo Corradini and Gianluca Degli Esposti. "Dressed Dirac propagator from a locally supersymmetric N=1 spinning particle". In: *Nucl. Phys. B* 970 (2021), p. 115498. DOI: 10.1016/j.nuclphysb.2021.115498. arXiv: 2008.03114 [hep-th].

[33] E. Fradkin. "Application of functional methods in quantum field theory and quantum statistics (II)". In: *Nucl. Phys.* 76.3 (1966), pp. 588–624. DOI: 10.1016/0029-5582(66)90200-8.

[34] M. Fabbrichesi et al. "Planckian energy scattering and surface terms in the gravitational action". In: *Nuclear Physics B* 419.1 (May 1994), pp. 147–188. DOI: 10.1016/0550-3213(94)90361-1. URL: https://doi.org/10.1016%2F0550-3213%2894%2990361-1.

[35] Eric Laenen, Gerben Stavenga, and Chris D White. "Path integral approach to eikonal and next-to-eikonal exponentiation". In: *Journal of High Energy Physics* 2009.03 (Mar. 2009), pp. 054–054. DOI: 10.1088/1126-6708/2009/03/054. URL: https://doi.org/10.1088%2F1126-6708%2F2009%2F03%2F054.

[36] Domenico Bonocore. "Asymptotic dynamics on the worldline for spinning particles". In: *Journal of High Energy Physics* 2021.2 (Feb. 2021). DOI: 10.1007/jhep02(2021)007. URL: https://doi.org/10.1007%2Fjhep02%282021%29007.

[37] David A. Kosower, Ben Maybee, and Donal O'Connell. "Amplitudes, observables, and classical scattering". In: *Journal of High Energy Physics* 2019.2 (Feb. 2019). DOI: 10.1007/jhep02(2019)137. URL: https://doi.org/10.1007%2Fjhep02%282019%29137.

[38] Francesco Comberiati and Leonardo de la Cruz. "Classical off-shell currents". In: (Dec. 2022). arXiv: 2212.09259 [hep-th].

[39] Eric Braaten and Robert D. Pisarski. "Soft Amplitudes in Hot Gauge Theories: A General Analysis". In: *Nucl. Phys. B* 337 (1990), pp. 569–634. DOI: 10.1016/0550-3213(90)90508-B.

[40] J. Frenkel and J. C. Taylor. "High Temperature Limit of Thermal QCD". In: *Nucl. Phys. B* 334 (1990), pp. 199–216. DOI: 10.1016/0550-3213(90)90661-V.

[41] Eric Braaten and Robert D. Pisarski. "Deducing Hard Thermal Loops From Ward Identities". In: *Nucl. Phys. B* 339 (1990), pp. 310–324. DOI: 10.1016/0550-3213(90)90351-D.

[42] R. R. Francisco, J. Frenkel, and J. C. Taylor. "Effective Action for Hard Thermal Loops in Gravitational Fields". In: *Phys. Lett. B* 756 (2016), pp. 205–207. DOI: 10.1016/j.physletb.2016.03.011. arXiv: 1603.03074 [hep-th].

[43] H. Kawai, D. C. Lewellen, and S. H. H. Tye. "A Relation Between Tree Amplitudes of Closed and Open Strings". In: *Nucl. Phys. B* 269 (1986), pp. 1–23. DOI: 10.1016/0550-3213(86)90362-7.

[44] Francesco Comberiati and Canxin Shi. *Classical Double Copy of Spinning Worldline Quantum Field Theory.* 2022. DOI: 10.48550/ARXIV.2212.13855. URL: https://arxiv.org/abs/2212.13855.

[45] Walter D. Goldberger and Ira Z. Rothstein. "Effective field theory of gravity for extended objects". In: *Physical Review D* 73.10 (May 2006). DOI: 10.1103/physrevd.73.104029. URL: https://doi.org/10.1103%2Fphysrevd.73.104029.

[46] Walter D Goldberger and Ira Z Rothstein. "Towers of gravitational theories". In: *International Journal of Modern Physics D* 15.12 (2006), pp. 2293–2302.

[47] Walter D. Goldberger and Andreas Ross. "Gravitational radiative corrections from effective field theory". In: *Physical Review D* 81.12 (June 2010). DOI: 10.1103/physrevd.81.124015. URL: https://doi.org/10.1103%2Fphysrevd.81.124015.

[48] Gregor Kälin, Zhengwen Liu, and Rafael A. Porto. "Conservative tidal effects in compact binary systems to next-to-leading post-Minkowskian order". In: *Physical Review D* 102.12 (Dec. 2020). DOI: 10.1103/physrevd.102.124025. URL: https://doi.org/10.1103%2Fphysrevd.102.124025.

[49] Rafael A. Porto. "The effective field theorist's approach to gravitational dynamics". In: *Physics Reports* 633 (May 2016), pp. 1–104. DOI: 10.1016/j.physrep.2016.04.003. URL: https://doi.org/10.1016%2Fj.physrep.2016.04.003.

[50] Stefano Foffa et al. "Conservative dynamics of binary systems to fourth post-Newtonian order in the EFT approach. II. Renormalized Lagrangian". In: *Physical Review D* 100.2 (July 2019). DOI: 10.1103/physrevd.100.024048. URL: https://doi.org/10.1103%2Fphysrevd.100.024048.

[51] Bryce S. DeWitt. "Dynamical theory in curved spaces. 1. A Review of the classical and quantum action principles". In: *Rev. Mod. Phys.* 29 (1957), pp. 377–397. DOI: 10.1103/RevModPhys.29.377.

[52] Leonard Parker. "Path integrals for a particle in curved space". In: *Phys. Rev. D* 19 (2 Jan. 1979), pp. 438–441. DOI: 10.1103/PhysRevD.19.438. URL: https://link.aps.org/doi/10.1103/PhysRevD.19.438.

[53] Jacob D. Bekenstein and Leonard Parker. "Path-integral evaluation of Feynman propagator in curved spacetime". In: *Phys. Rev. D* 23 (12 June 1981), pp. 2850–2869. DOI: 10.1103/PhysRevD.23.2850. URL: https://link.aps.org/doi/10.1103/PhysRevD.23.2850.

[54] Fiorenzo Bastianelli and Peter van Nieuwenhuizen. "Trace anomalies from quantum mechanics". In: *Nuclear Physics B* 389.1 (Jan. 1993), pp. 53–80. DOI: 10.1016/0550-3213(93)90285-w. URL: https://doi.org/10.1016%2F0550-3213%2893%2990285-w.

[55] A. M. Perelomov. "Coherent states for arbitrary Lie group". In: (2002). DOI: 10.48550/ARXIV.MATH-PH/0203002. URL: https://arxiv.org/abs/math-ph/0203002.

[56] Paolo Di Vecchia et al. "The eikonal approach to gravitational scattering and radiation at \mathcal{O} (G3)". In: *Journal of High Energy Physics* 2021.7 (July 2021). DOI: 10.1007/jhep07(2021)169. URL: https://doi.org/10.1007%2Fjhep07%282021%29169.

[57] Carlo Heissenberg. "Infrared divergences and the eikonal exponentiation". In: *Physical Review D* 104.4 (Aug. 2021). DOI: 10.1103/physrevd.104.046016. URL: https://doi.org/10.1103%2Fphysrevd.104.046016.

[58] Paolo Di Vecchia et al. *The eikonal operator at arbitrary velocities I: the soft-radiation limit.* 2022. DOI: 10.48550/ARXIV.2204.02378. URL: https://arxiv.org/abs/2204.02378.

[59] Gustav Uhre Jakobsen et al. "Classical Gravitational Bremsstrahlung from a Worldline Quantum Field Theory". In: *Physical Review Letters* 126.20 (May 2021). DOI: 10.1103/physrevlett.126.201103. URL: https://doi.org/10.1103%2Fphysrevlett.126.201103.

[60] Gustav Uhre Jakobsen et al. "Gravitational Bremsstrahlung and Hidden Supersymmetry of Spinning Bodies". In: *Physical Review Letters* 128.1 (Jan. 2022). DOI: 10.1103/physrevlett.128.011101. URL: https://doi.org/10.1103%2Fphysrevlett.128.011101.

[61] Gustav Uhre Jakobsen et al. "SUSY in the sky with gravitons". In: *Journal of High Energy Physics* 2022.1 (Jan. 2022). DOI: 10.1007/jhep01(2022)027. URL: https://doi.org/10.1007%2Fjhep01%282022%29027.

[62] Gustav Uhre Jakobsen and Gustav Mogull. "Conservative and Radiative Dynamics of Spinning Bodies at Third Post-Minkowskian Order Using Worldline Quantum Field Theory". In: *Physical Review Letters* 128.14 (Apr. 2022). DOI: 10.1103/physrevlett.128.141102. URL: https://doi.org/10.1103%2Fphysrevlett.128.141102.

[63] Gustav Uhre Jakobsen et al. *All Things Retarded: Radiation-Reaction in Worldline Quantum Field Theory*. 2022. DOI: 10.48550/ARXIV.2207.00569. URL: https://arxiv.org/abs/2207.00569.

[64] Gustav Uhre Jakobsen and Gustav Mogull. *Linear Response, Hamiltonian and Radiative Spinning Two-Body Dynamics*. 2022. DOI: 10.48550/ARXIV.2210.06451. URL: https://arxiv.org/abs/2210.06451.

[65] N. Ahmadiniaz et al. "Worldline master formulas for the dressed electron propagator. Part I. Off-shell amplitudes". In: *Journal of High Energy Physics* 2020.8 (Aug. 2020). DOI: 10.1007/jhep08(2020)018. URL: https://doi.org/10.1007%2Fjhep08%282020%29018.

[66] N. Ahmadiniaz et al. "Worldline master formulas for the dressed electron propagator. Part 2. On-shell amplitudes". In: *Journal of High Energy Physics* 2022.1 (Jan. 2022). DOI: 10.1007/jhep01(2022)050. URL: https://doi.org/10.1007%2Fjhep01%282022%29050.

[67] Henrik Johansson and Alexander Ochirov. "Double copy for massive quantum particles with spin". In: *JHEP* 09 (2019), p. 040. DOI: 10.1007/JHEP09(2019)040. arXiv: 1906.12292 [hep-th].

[68] Yilber Fabian Bautista and Alfredo Guevara. "On the double copy for spinning matter". In: *JHEP* 11 (2021), p. 184. DOI: 10.1007/JHEP11(2021)184. arXiv: 1908.11349 [hep-th].

[69] Alfredo Guevara, Alexander Ochirov, and Justin Vines. "Scattering of Spinning Black Holes from Exponentiated Soft Factors". In: *JHEP* 09 (2019), p. 056. DOI: 10.1007/JHEP09(2019)056. arXiv: 1812.06895 [hep-th].