ALMA MATER STUDIORUM · UNIVERSITÀ DI BOLOGNA

Scuola di Scienze
Dipartimento di Fisica e Astronomia
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MODIFIED GRAVITY WITH TENSOR COMPUTER ALGEBRA: A CASE STUDY

Relatore:
Prof. Marco Baldi

Correlatore:
Dott. Alessandro Casalino

Presentata da:
Alessandro Borys

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...vidi l'Aleph, da tutti i punti, vidi nell'Aleph la terra e nella terra di nuovo l'Aleph e nell'Aleph la terra, vidi il mio volto e le mie viscere, vidi il tuo volto, e provai vertigini e piansi, poiché i miei occhi avevano visto l'oggetto segreto e supposto, il cui nome usurpano gli uomini, ma che nessun uomo ha contemplato: l'inconcepibile universo.

J.L.Borges, "L'Aleph"

Sommario

Il modello ΛCDM è il modello cosmologico più semplice, ma finora più efficace, per descrivere l'evoluzione dell'universo. Esso si basa sulla teoria della Relatività Generale di Einstein e fornisce una spiegazione dell'espansione accelerata dell'universo introducendo la costante cosmologica Λ, che rappresenta il contributo della cosiddetta energia oscura, un'entità di cui ben poco si sa con certezza. Sono stati tuttavia proposti modelli teorici alternativi che descrivono gli effetti di questa quantità misteriosa, introducendo ad esempio gradi di libertà aggiuntivi, come nella teoria di Horndeski. L'obiettivo principale di questa testi è quello di studiare questi modelli tramite il tensor computer algebra `xAct`. In particolare, il nostro scopo sarà quello di implementare una procedura universale che permette di derivare, a partire dall'azione, le equazioni del moto e l'evoluzione temporale di qualunque modello generico.

Abstract

The ΛCDM model is the simplest, yet most efficient, model that describes the evolution of the Universe. It is well embedded in the context of General Relativity and explains the accelerated expansion of the Universe by introducing the cosmological constant Λ, which entails the so-called dark energy contribution. However, alternative theoretical models have been proposed to describe the effects of this unknown quantity by including additional degrees of freedom, such as in Horndeski's theory. The focus of this thesis is to study these models within the framework of the tensor computer algebra `xAct`. In particular, we will implement a consistent and universal workflow which allows us to derive from the action the equations of motion and the time evolution for any arbitrary model.

Contents

Introduction

General Relativity is one of the most beautiful theories ever discovered. It allows us to describe the gravitational interaction among objects in a neat and elegant way, and provides an interpretation of gravity as the geometry of space-time. Modern cosmology lies its foundations within the framework of General Relativity, which explains almost all the phenomenology observed, e.g. the recent detection of gravitational waves [1], or the CMB experiment Planck [2]. However, there are still some unresolved mysteries of the cosmos, such as the nature of dark matter (non-baryonic matter, only detectable because of its gravitational interaction with ordinary matter) and dark energy. In both cases, the attribute "dark" emphasizes our ignorance about the fundamental nature of these entities. We hold dark energy accountable for the accelerated expansion of the Universe, which cannot be explained, in General Relativity, considering exclusively standard matter and radiation or even another spatial curvature. The simplest description of this dark energy consists in introducing a cosmological constant Λ in the General Relativity action, and thus enters in the equations of motion: the model is the so-called ΛCDM model, the standard model of cosmology. Alternative theoretical models have been proposed to solve some issues with the cosmological constant (see for instance [3]) by including additional dynamical quantities or degrees of freedom, for instance a scalar field, whose evolution could explain the accelerated expansion and the dark matter effects. All these models are usually called *Modified Gravity* theories. See for instance [4] for a review of these models.

This thesis is divided into four chapters. The first one presents a brief summary of General Relativity, with special attention to the most important equations of the theory, which is followed by an overview of standard cosmology and the ΛCDM model. The second chapter is more technical; in fact, it contains an introduction to `xAct` [5, 6], the tensor computer algebra used to derive most of the results presented in the next chapters. In particular, we will discuss a simple change of coordinates, very useful to get started with the general setup and commands of `xAct`. In the third chapter, we derive the equations of motion and the time evolution for the ΛCDM model using the theoretical and numerical methods explained in the previous chapter and we present the graphics thus obtained, which correspond accurately to the observational data. Finally, the fourth chapter deals with two modified gravity models, belonging to the Horndeski class of models [7]: one is *Extended Quintessence* and the other is an original model, whose validity is tested using the workflow previously implemented on `xAct`. Both models show a behaviour which reproduces in a very satisfying way the time evolution of the ΛCDM model.

Chapter 1

A brief review of General Relativity and Cosmology

General Relativity is the standard theory of gravity, which allows us to describe gravitational interactions among objects, from the smallest length scales to cosmology. Being embedded in the framework of differential geometry, we can write its equations in any coordinate system, inertial or not. Laws of physics in general can be thus formulated in a way that is totally independent of the reference frame, a fact well stated in the **Principle of General Relativity**: *The laws of physics are the same in all reference frames for all observers.* This means that all physical laws must contain only tensors and tensorial operations in the sense of differential geometry.

1.1 Gravity and geometry

General Relativity is a classical field theory, in which the dynamical field is the metric tensor describing the curvature of space-time itself and not some additional field propagating through space-time. The physical principle that led Einstein to formulate his theory is the **Equivalence Principle**: *Motion in an uniform gravitational field cannot be distinguished from free fall.* This statement implies that free-falling observers are the true inertial observers, and the local metric in their own inertial frames is the canonical Minkowski metric. The equations of motion for a freely-falling particle in space-time are called *geodesic equations.* If u^μ is the four-velocity of an inertial observer subject only to gravity, its motion in the freely falling frame will occur along a straight line, or *geodesic* of the metric

$$u^\mu \nabla_\mu u^\alpha = \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} = 0\,. \tag{1.1.1}$$

In a different (non inertial) frame, the Christoffel symbols will not be zero: this suggests that the metric tensor $g_{\mu\nu}$ can be viewed as a potential for the gravitational interaction,

$\Gamma^{\alpha}_{\mu\nu} \sim g_{\mu\nu}$. In fact, the connection between the Christoffel symbols and the metric is given by

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\alpha} \left[g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha} \right] , \tag{1.1.2}$$

where the "," subscript followed by a tensor index denotes a partial derivative ($V^{\mu}_{,\nu} \equiv \partial_{\nu} V^{\mu}$). If geodesics are the extension of straight line trajectories in Euclidean space-time, the mathematical quantity which allows us to define the *curvature* of a manifold is called the Riemann tensor, identified as [8]

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\alpha}_{\mu\alpha} . \tag{1.1.3}$$

The Riemann tensor will vanish if and only if the metric is perfectly flat. A crucial property of this tensor is that it is "block symmetric" (invariant under interchange of the first pair of indices with the second):

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma} . \tag{1.1.4}$$

In fact, it allows us to define the following tensors

- the Ricci-Curbastro tensor

$$R^{\lambda}_{\mu\lambda\nu} = R_{\mu\nu}, \text{ with } R_{\mu\nu} = R_{\nu\mu} , \tag{1.1.5}$$

- the Ricci scalar

$$R = R^{\mu}_{\nu} = g^{\mu\nu} R_{\mu\nu} , \tag{1.1.6}$$

- the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} . \tag{1.1.7}$$

The latter, in particular, satisfies the very important *Bianchi identity*

$$\nabla_{\mu} G^{\mu\nu} = 0 . \tag{1.1.8}$$

We will see how these quantities play a fundamental role in General Relativity.

1.1.1 Einstein Equations

The theory of General Relativity is governed by the Einstein-Hilbert action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} [R(g_{\mu\nu}, \partial_{\rho} g_{\mu\nu}, \partial_{\sigma}\partial_{\rho} g_{\mu\nu}) - 2\Lambda] + S_M , \tag{1.1.9}$$

where G is the Newton constant, Λ represents the cosmological constant and $\sqrt{-g} \equiv \sqrt{-det(g_{\mu\nu})}$ (since the signature of the metric is Sign[$g_{\mu\nu}$]=$(-,+,+,+)$, det[$g_{\mu\nu} < 0$]).

The S_M part of the action contains all the matter sources, which in the standard model of cosmology come down to baryons, dark matter, photons and neutrinos. S_M is usually written from a Lagrangian

$$S_M = \int d^4x\sqrt{-g}\mathcal{L}_M(g_{\mu\nu}) \,. \tag{1.1.10}$$

Applying the variational principle on this action with respect to the metric $g_{\mu\nu}$, we find the *Einstein equations*

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \,, \tag{1.1.11}$$

which are ten (with only six independent) partial differential equations. These equations are covariant, meaning all quantities in the equations are invariant under local coordinate transformations; in other words, if we change local coordinates at a given point in space-time x^μ, the result of an experiment doesn't change.
$T_{\mu\nu}$ is the *energy-momentum* tensor, which comes from the functional derivative of the S_M part of the Einstein-Hilbert action with respect to the metric $g_{\mu\nu}$

$$T^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g_{\mu\nu}} \,. \tag{1.1.12}$$

Since Eq.(1.1.8) tells us that $G_{\mu\nu}$ is divergenceless, it must be that $T_{\mu\nu}$ is divergenceless too. For a perfect fluid with four-velocity u^μ, $T_{\mu\nu}$ is given by

$$T^{\mu\nu} = \rho u^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu) = (\rho + p)u^\mu u^\nu + pg^{\mu\nu} \,, \tag{1.1.13}$$

where ρ and P are, respectively, the density and the pressure of the fluid. Moreover, in the rest frame of the fluid the four-velocity u^μ = (1,0,0,0) and dτ=dt, which means that

$$T^{\mu\nu} = \begin{bmatrix} \rho & 0 \\ 0 & pg^{ij} \end{bmatrix} \,, \tag{1.1.14}$$

and

$$T^\mu_\nu = diag(-\rho, p, p, p) \,. \tag{1.1.15}$$

Einstein equations contain an interpretation of gravity as the geometry of space-time: the energy-momentum tensor (mass) determines the curvature of space-time (the Ricci tensor), which in turn affects the matter motion.
Finally, the cosmological constant Λ that enters in the Einstein equations entails the contribution of *vacuum energy*, an energy density characteristic of empty space. There is a perfect equivalence between the cosmological constant and vacuum energy

$$\rho_{vac} = \frac{\Lambda}{8\pi G} \,. \tag{1.1.16}$$

We will see in the next chapters how to replace the definition of a cosmological constant with some other quantities, for instance a scalar potential.

1.2 Cosmology

Contemporary cosmological models are based on the **Cosmological Principle**, which states that the Universe is homogeneous and isotropic [9]. Isotropy is an observational statement, while homogeneity follows from assuming isotropy is independent of the observation point.

Isotropy states that the space looks the same no no matter in what direction we look. Obviously, looking at the night sky, it doesn't look isotropic at all: we can see scattered stars forming constellations and clusters, not to mention the Milky Way, which appears as a bright strip across the sky. In fact, the Cosmological Principle applies only on very large scales ($\geq 100Mpc$), where local variations of density are averaged over: if we could detect all the matter in the Universe, it's reasonable to think the average distribution would be the same in all directions. Homogeneity is the statement that the metric is the same throughout a manifold. If a space is isotropic everywhere, then it is also homogeneous. Since there is large observational evidence for isotropy, and we believe observers in another part of the Universe should also observe isotropy, we are allowed to assume both homogeneity and isotropy for space.

In the following, we will deal with an idealized model of the Universe, in which galaxies form a homogeneous fluid filling the entire space.

1.2.1 FLRW metric

The form of the cosmological metric can be partly fixed if we apply the Cosmological Principle only to space, but not to space-time. In fact, whereas the Universe is *spatially* homogeneous and isotropic, it is constantly evolving in time (in General relativity, this translates into the statement that the Universe can be foliated into spacelike slices Σ such that each three-dimensional slice is *maximally symmetric*). Observations show that the Universe is expanding and, the farther away galaxies are from us, the faster they are receding.

The metric which describe an isotropic and homogeneous Universe in expansion is identified by the *Friedmann-Lemaître-Robertson-Walker metric*

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + sin^2(\theta) d\phi^2 \right] \tag{1.2.1}$$

$$\equiv -dt^2 + a(t)^2 d\sigma^2 \,, \tag{1.2.2}$$

where the origin r=0 is totally arbitrary and t is the proper time of an observer moving along with the cosmic fluid at constant r, θ and ϕ. The coordinates used here are known as *comoving coordinates*. Only a comoving observer will think that the Universe looks isotropic (on Earth, we are not comoving at all, that's why we see a dipole anisotropy in the Cosmic Microwave Background as a result of the conventional Doppler effect).

The function $a(t)$ is called the *cosmic scale factor*, and describes the expansion of the Universe starting from the initial value $a_0 = 0$ to a conventional value of $a(t_0) = 1$, corresponding to the present day (t_0 is the age of the Universe). Since $a(t)$ is an increasing monotonic function of the proper time t, we often use it instead of the latter to study the evolution of the Universe.
The parameter k is called the *curvature constant* and describes the curvature, and therefore the size, of the spatial surfaces. k could indeed take any real value, but it is common to normalize it to

$$k = 0, \pm 1\,, \tag{1.2.3}$$

and absorb the physical size of the manifold into the scale factor $a(t)$. Depending on the value of the curvature scalar, it is possible to introduce new coordinates such that the topology of the hypersurface Σ is apparent from the line element $d\sigma^2$

- The $k = 0$ case corresponds to no curvature on Σ

$$d\sigma^2 = dr^2 + r^2 d\Omega^2 = dx^2 + dy^2 + dz^2\,, \tag{1.2.4}$$

  and Σ is called **flat** (euclidean geometry).

- The $k = +1$ case corresponds to constant positive curvature on Σ

$$r = sin(X) \Rightarrow d\sigma^2 = dX^2 + sin^2(X) d\Omega^2\,, \tag{1.2.5}$$

  and Σ is a **three-dimensional sphere** (spherical geometry).

- The $k = -1$ case corresponds to constant negative curvature on Σ

$$r = \sinh(\varphi) \Rightarrow d\sigma^2 = d\varphi^2 + \sinh^2(\varphi) d\Omega^2\,, \tag{1.2.6}$$

  and Σ is a **three-dimensional hyperboloid** (hyperbolic geometry).

The proper distance between two points depends on time, in fact

$$dR = a(t) \frac{dr}{\sqrt{1 - kr^2}} = a(t) \mathrm{d}R_{(3)} \tag{1.2.7}$$

where $R_{(3)}$ is the rescaled proper distance on Σ. Observations show that the distance among galaxies increases in time, whereas their typical size remains constant. We can therefore state that the universe is expanding, with the farther galaxies moving away faster from us, like dots on an inflating balloon. The Earth-Sun distance, for instance, doesn't scale like the intergalactic distance: at a local level, the gravitational interaction is dominant.

1.2.2 Hubble law

When we observe the spectrum of a galaxy, we typically see absorption lines. Suppose we know the wavelength λ_{em} of an atomic transition observed on Earth; the wavelength λ_{obs} we measure in the galaxy spectrum won't be in general the same because, due to the Doppler effect, the emitted radiation will be shifted towards red or blue, depending on the velocity of the galaxy along our line of observation. We can define the *redshift* z of a galaxy as (in the non-relativistic case)

$$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} = \frac{v_r}{c} \,. \tag{1.2.8}$$

In 1929, plotting z as a function of distance, Hubble discovered a linear law, known as the *Hubble law*

$$v = H_0 D \,, \tag{1.2.9}$$

which states that the observed recession velocity is directly proportional to the distance (for galaxies that are not too far away). H_0 is the *Hubble constant*, whose present value is $H_0 \simeq 70$ km/s · Mpc. Albeit H_0 is called constant, it is not at all constant in general. We can in fact define the *Hubble parameter* H as

$$H = \frac{\dot{a}}{a} \,, \tag{1.2.10}$$

where $\dot{a} = \frac{da}{dt}$. H contains the information relative to the expansion rate of the Universe, which, as we will see, has changed over different epochs.

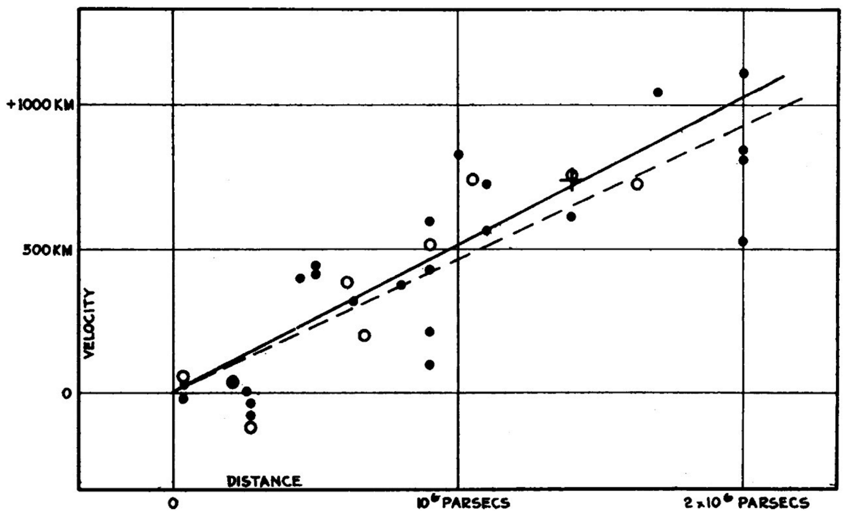

Figure 1.1: *Original plot made by Edwin Hubble, which shows a linear dependence of velocity on distance.*

1.2.3 Cosmic fluids

As we discussed above, we will assume that the Universe is permeated with an ideal cosmic fluid of matter and energy. In the rest frame of the fluid, its energy-momentum tensor becomes

$$T^{\mu}_{\nu} = diag(-\rho, p, p, p)\,, \tag{1.2.11}$$

where $\rho = \rho(t)$ e $p = p(t)$. Moreover, the following *continuity equation* holds:

$$\nabla_{\mu} T^{\mu}_{\nu} = 0\,. \tag{1.2.12}$$

The zero-component of this equation yields

$$\begin{aligned} 0 &= \nabla_{\mu} T^{\mu}_{\nu} && (1.2.13)\\ &= \partial_{\mu} T^{\mu}_{0} + \Gamma^{\mu}_{\mu\lambda} T^{\lambda}_{0} - \Gamma^{\lambda}_{\mu 0} T^{\mu}_{\lambda} && (1.2.14)\\ &= -\dot{\rho} - 3H(\rho + p)\,. && (1.2.15) \end{aligned}$$

We can assume our fluid is barotropic, meaning it obeys the following *equation of state*

$$p = \omega \rho\,, \tag{1.2.16}$$

where the parameter ω is called *state parameter* and is a constant value independent of time. Energy conservation then reads

$$\frac{\dot{\rho}}{\rho} = -3(1+\omega)\frac{\dot{a}}{a}\,, \tag{1.2.17}$$

which means

$$\frac{\rho(t)}{\rho_0} = \left[\frac{a_0}{a(t)}\right]^{3(1+\omega)} \quad \Rightarrow \quad \rho \propto a^{-3(1+\omega)}\,. \tag{1.2.18}$$

Until not too long ago, the present Universe was believed to be dominated by ordinary matter (dust), while the primordial one by radiation (the density of the latter, indeed, rises as we go back in time). However, the expansion of the Universe is accelerating ($\ddot{a} > 0$), a fact which is incompatible with the gravitational effect of matter. Among the possible sources responsible of this acceleration, it has been hypothesised the so-called *dark energy*, encompassed in the cosmological constant Λ.

The components of the cosmic fluid therefore are

- **Dust**: pressureless matter, or non-relativistic matter almost exactly at rest with the cosmic frame. In this case other forces beside gravity are absent, and $\omega = 0$ (so that $p = 0$). Eq.(1.2.18) therefore yields

$$\rho_m = \frac{E}{V} \propto a^{-3}\,, \tag{1.2.19}$$

which is interpreted as the decrease in the number density of particles as the Universe expands

- **Radiation**: pure electromagnetic radiation or highly-relativistic matter. Since the mass is totally negligible, so is the energy-momentum trace

$$T = -\rho + 3p = 0\,, \tag{1.2.20}$$

which implies

$$p = \frac{1}{3}\rho \quad \Rightarrow \quad \omega = \frac{1}{3}\,. \tag{1.2.21}$$

Eq. (1.2.18) yields

$$\rho_r = \frac{E}{V} \propto a^{-4}\,. \tag{1.2.22}$$

This is because the number density of photons decreases in the same way as the number density of non-relativistic particeles, but individual photons lose energy as $E \propto a^{-1}$ as they redshift.

- **Vacuum energy**: a fluid with equation of state

$$\rho = -p = \frac{\Lambda}{8\pi G}, \quad \omega = -1\,. \tag{1.2.23}$$

The energy density of this fluid is therefore constant

$$\rho_\Lambda \propto 1\,. \tag{1.2.24}$$

1.2.4 Friedmann equations

The Einstein equations

$$T_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.2.25}$$

evaluated on the FLRW metric are the so-called *Friedmann equations*

$$3\left[(\frac{\dot{a}}{a})^2 + \frac{k}{a^2}\right] = 8\pi G_N \rho \quad \Rightarrow \quad H^2 = \frac{8\pi G_N}{3}\rho - \frac{k}{a^2}\,, \tag{1.2.26}$$

$$3\frac{\ddot{a}}{a} = -4\pi G_N(\rho + 3P) \quad \Rightarrow \quad \dot{H} + H^2 = -\frac{4\pi G_N}{3}(\rho + 3P)\,, \tag{1.2.27}$$

where $\rho = \rho(t)$ is the *total* energy density of all components of the cosmic fluid and $P = P(t)$ the total pressure. The Friedmann equations should be solved to find $a(t)$, which depends on the components considered through ρ and P. The first one, Eq. (1.2.26), is technically a constraint which selects all possible combinations of the initial conditions $a(t_0) = a_0$ and $\dot{a}(t_0) = \dot{a}_0$ for the truly dynamical (second order) Eq. (1.2.27). Furthermore, it can be shown that Eq. (1.2.27) can be derived from Eq. (1.2.26); for a fluid which satisfies the continuity equation (1.2.12), thus, it is easier to just solve for the constraint (1.2.26) at all times $t \geq t_0$. We further define the *deceleration parameter*

$$q = -\frac{a\ddot{a}}{\dot{a}^2}\,, \tag{1.2.28}$$

which measures the rate of change of the rate of expansion, and the *density parameter*

$$\Omega = \frac{8\pi G_N}{3H^2}\rho = \frac{\rho}{\rho_{critical}}\,, \tag{1.2.29}$$

where the *critical density* is defined as

$$\rho_{critical} = \frac{3H^2}{8\pi G_N}\,. \tag{1.2.30}$$

The Friedmann equation 1.2.26 can thus be written as

$$\Omega - 1 = \frac{k}{H^2 a^2}\,, \tag{1.2.31}$$

where the density parameter Ω contains the information on the topology of the Universe. In fact

- $\rho < \rho_{critical} \iff \Omega < 1 \iff k = -1 \iff$ Open Universe
- $\rho = \rho_{critical} \iff \Omega = 1 \iff k = 0 \iff$ Flat Universe
- $\rho > \rho_{critical} \iff \Omega > 1 \iff k = 1 \iff$ Closed Universe .

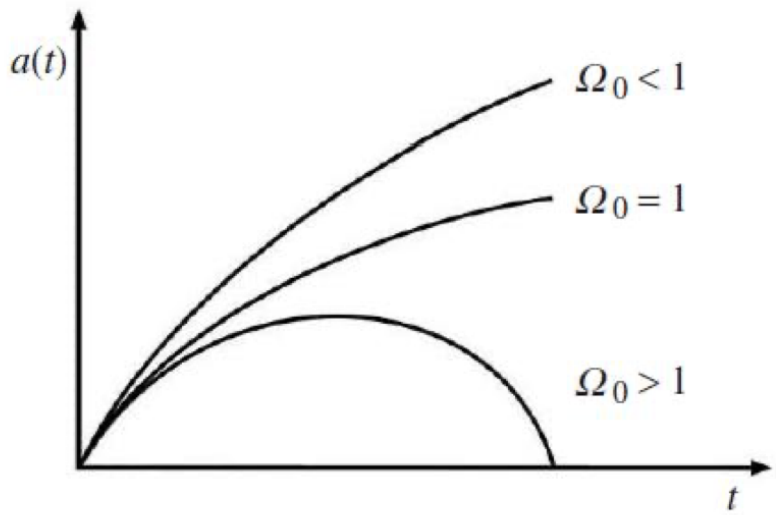

Figure 1.2: *Evolution of the scale factor a(t) for different values of Ω. If $\Omega > 1$, the universe will eventually contract to a singularity; if $\Omega = 1$, the Universe will continue expanding at a constant rate; if $\Omega < 1$, the expansion will be accelerated.*

The density parameter, then, tells us which of the three FLRW geometries describes our Universe. Determining it observationally is crucial: recent measurements of the CMB anisotropy suggest that Ω is very close to 1.
We can from now on assume flatness ($k = 0$). For a flat, matter dominated Universe, we have

$$\rho_m \sim \frac{1}{a^3} \;\;\Rightarrow\;\; \frac{\dot{a}^2}{a^2} \sim \frac{1}{a^3} \;\;\Rightarrow\;\; \sqrt{a}da \sim dt \;\;\Rightarrow\;\; t \sim a^{3/2}\,, \tag{1.2.32}$$

thus $a \sim t^{2/3}$.
For a flat, radiation dominated Universe, instead

$$\rho_r \sim \frac{1}{a^4} \;\;\Rightarrow\;\; \frac{\dot{a}^2}{a^2} \sim \frac{1}{a^4} \;\;\Rightarrow\;\; ada \sim dt \;\;\Rightarrow\;\; t \sim a^2\,, \tag{1.2.33}$$

thus $a \sim t^{1/2}$.
Finally, for a flat and empty Universe, with only a positive vacuum energy present, we obtain the extact solution

$$\rho_\Lambda \sim \Lambda \;\;\Rightarrow\;\; \frac{\dot{a}^2}{a^2} \sim \frac{\Lambda}{3} \;\;\Rightarrow\;\; \sqrt{\frac{\Lambda}{3}} \sim \frac{\dot{a}}{a} \equiv H_0 \;\;\Rightarrow\;\; a \sim e^{H_0 t}\,, \tag{1.2.34}$$

where H_0 is now a true cosmological constant. The latter case is also known as *de Sitter Universe*, a Universe subject to a constant exponential expansion.
As $a \to 0$ in the past, the vacuum energy is negligible, while the radiation is dominant; as $a \to \infty$ in the future, matter will be negligible and the Universe will be asymptotic to de Sitter.

It is very useful to write the Friedmann equations in another way. In particular, we can write Eq. (1.2.26) in the form

$$H(t)^2 = \frac{8\pi G}{3}\rho(t)\,, \tag{1.2.35}$$

where the density is given by the sum of the single components densities

$$\rho(t) = \rho_m(t) + \rho_r(t) + \rho_\Lambda(t)\,. \tag{1.2.36}$$

If we divide Eq. (1.2.35) by H_0^2, we obtain the equation

$$1 = \Omega_m(t) + \Omega_r(t) + \Omega_\Lambda(t)\,, \tag{1.2.37}$$

where Ω_i is called the *fractional energy density* of the i-th matter species, defined as

$$\Omega_i(t) \equiv \frac{8\pi G}{3}\frac{\rho(t)}{H(t)^2}\,. \tag{1.2.38}$$

We can take some other steps. At time t_0, Eq. (1.2.35) reads

$$H_0^2 \equiv H(t_0)^2 = \frac{8\pi G}{3}\rho(t_0) \equiv \frac{8\pi G}{3}\rho_{cr}\,, \tag{1.2.39}$$

where we define the *critical energy density* $\rho_{cr} \equiv \rho(t_0)$ to be the energy density today. So, the fractional energy densities evaluated today, at $t = t_0$, are

$$\Omega_i^0 \equiv \Omega_i(t_0) = \frac{\rho_i(t_0)}{\rho_{cr}}\,. \tag{1.2.40}$$

Using Eq. (1.2.18) for $\rho(t)$, we obtain the explicit equation for the energy density

$$\rho_m(t) = \frac{\rho_{cr}\Omega_m^0}{a(t)^3}, \quad \rho_r(t) = \frac{\rho_{cr}\Omega_r^0}{a(t)^4}, \quad \rho_\Lambda(t) = \rho_{cr}\Omega_\Lambda^0\,, \tag{1.2.41}$$

and therefore

$$\Omega_m(t) \equiv \frac{H_0^2}{H(t)^2}\frac{\Omega_m^0}{a(t)^3}, \quad \Omega_r(t) \equiv \frac{H_0^2}{H(t)^2}\frac{\Omega_r^0}{a(t)^4}, \quad \Omega_\Lambda(t) \equiv \frac{H_0^2}{H(t)^2}\Omega_\Lambda^0\,. \tag{1.2.42}$$

In general, at any time t we can write

$$1 = \sum_i \Omega_i(t)\,. \tag{1.2.43}$$

The Friedmann equation with this new notation reads

$$H(t)^2 = \frac{8\pi G}{3}\left[\frac{\Omega_m^0}{a(t)^3} + \frac{\Omega_r^0}{a(t)^4} + \Omega_\Lambda^0\right]\,. \tag{1.2.44}$$

Therefore, once we the know the value of H_0 and the value of all fractional energy densities today, we can obtain the value of $H(t)$ for any value of the scale factor $a(t)$. However, to obtain the value of the scale factor at a given t, we still need to solve the differential equation.

1.3 The ΛCDM model

The solutions of the Friedmann equations for dust and radiation show a common behaviour. According to the inflation theory, the Universe started from a *singularity* and is expanding ever since (at least until a maximum scale factor). The standard model of cosmology is divided into two parts

- a primordial stage, behind the Last Scattering Surface, very well described by the theory of *inflation*,

- a later stage during which large scale structures such as clusters and galaxies came into being, described by the *ΛCDM model*. This model can be summarized as following: the dark matter is cold and the dark energy has a constant energy density, which means $\omega = -1$.

The observations collected so far denote the fact that the present Universe is spatially flat with $\Omega \simeq 1$, which corresponds to an average density

$$\rho_0 = \rho_{critical} \simeq 10^{-29} g/cm^3 \,, \tag{1.3.1}$$

equivalent to about 6 protons per square cubic meter. Three sources that contribute to ρ_0 have been identified

- **Regular baryonic matter**, approximated by a fluid made of dust and estimated through the luminosity of galaxies,

$$\frac{\rho_{matter}}{\rho_0} \simeq 5\% \,, \tag{1.3.2}$$

- **Nonbaryonic dark matter**,which behaves like dust (gravitationally), but can't be directly detected

$$\frac{\rho_{DM}}{\rho_0} \simeq 25\% \,. \tag{1.3.3}$$

  The existence of dark matter is necessary to explain several cosmological effect. Among these, the first signature of dark matter came from the motion of stars inside galaxies [10]. However, no direct detection of dark matter has been performed so far.

- **Dark energy**, with the equation of state of the vacuum,

$$\frac{\rho_{DE}}{\rho_0} \simeq 70\% \,. \tag{1.3.4}$$

  It is necessary to explain the current value of the deceleration parameter $q_0 < 0$ $(\ddot{a} > 0)$.

In terms of the fractional densities, we can estimate that $\Omega^0_m = (0.3 \pm 0.1)$, $\Omega^0_r = 9 \cdot 10^{-5}$ and $\Omega^0_\Lambda = (0.7 \pm 0.1)$. Note that the value of Ω^0_m is given by the sum of the baryonic matter and the dark matter contribution, which accounts for most of the mass in the Universe

$$\Omega^0_m = \Omega^0_b + \Omega^0_{CDM} = 0.0486 + 0.2589 \,. \tag{1.3.5}$$

The standard ΛCDM model therefore describes a spatially flat Universe composed by radiation, matter (baryonic and non-baryonic) and dark energy.
The Universe has gone through three different epochs, each one governed by a different component

- **Radiation dominated epoch**: the radiation is prevalent up to redshift $z \simeq 10^4$; given the high speed with which it decreases, its density becomes soon negligible.
- **Matter dominated epoch**: baryonic matter and dark matter are the dominant components from $z \simeq 10^4$ until $z \simeq 0.7$.
- **Dark energy epoch**: the dark energy density is very small but fixed, as it is constantly generated by the vacuum itself, and becomes dominant when the expansion of the Universe has decreased the density of the other components.

Chapter 2

Tensor computer algebra: `xAct`

In order to study our models of the Universe, we need a tool to solve some crucial equations and define the most suitable setup for a correct description of our systems (we will see that solving Einstein and Klein-Gordon equations leads us to a complete understanding of the evolution of the Universe and thus of its components). `Mathematica` is a symbolic mathematical computation program, sometimes called a computer algebra program, used in many scientific, engineering, mathematical, and computing fields. In particular, `xAct` is a very powerful tensor computer algebra `Mathematica` add-on, which can be used to study General Relativity and cosmology with a computer analytically, allowing us to use all of the mathematical tools of tensor calculus. For instance, on `xAct` we can work with 4-dimensional manifolds, metric tensors and their perturbations and much more.
Before entering the core of our thesis, we will first focus on some of the most important features of `xAct` by briefly studying two examples, i.e. a change from spherical/cylindrical coordinates to the Cartesian ones.

2.1 Change to spherical coordinates

The change between spherical and Cartesian coordinates can be parametrized with the three coordinates ρ, θ and ϕ, as in the following figure.

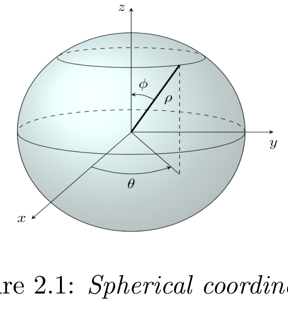

Figure 2.1: *Spherical coordinates.*

The change of coordinates is given by the following transformation

$$\varphi : \begin{cases} x = \rho \sin(\theta) \cos(\phi) \\ y = \rho \sin(\theta) \sin(\phi) \\ z = \rho \cos(\theta) \end{cases} , \tag{2.1.1}$$

with $(\rho, \theta, \phi) \in [0, +\infty[\times [0, \pi[\times [0, 2\pi[$. The determinant of the Jacobian matrix of φ equals

$$\det J_\varphi(\rho, \theta, \phi) = \det \begin{pmatrix} \sin(\theta)\cos(\phi) & \rho\cos(\theta)\cos(\phi) & -\rho\sin(\theta)\sin(\phi) \\ \sin(\theta)\sin(\phi) & \rho\cos(\theta)\sin(\phi) & \rho\sin(\theta)\cos(\phi) \\ \cos(\theta) & -\rho\sin(\theta)\cos(\phi) & 0 \end{pmatrix} = \rho^2 \sin(\theta) . \tag{2.1.2}$$

We start by opening a new file in `Mathematica` and digiting the command << `xAct`‘`xCoba`‘, which brings us in the framework of `xAct` with the package `xCoba`. The first step is the definition of a differentiable n dimensional manifold M of dimension dim and tensor abstract indices with the command `DefManifold[name, dim, indices]`

```
DefManifold[R3, 3, {a, b, c, d, e, f, i, j, k}]
** DefManifold: Defining manifold R3.
** DefVBundle: Defining vbundle TangentR3.
```

Then, we need to define the spherical and Cartesian charts. The command `DefChart[chart, manifold, Cnumbers, scalars]` defines a chart as a coordinate system on the manifold, with coordinate fields given by the list scalars and the integer coordinate numbers (`Cnumbers`), identifying those coordinates and the associated vectors/1-forms

```
DefChart[sph, R3, {1, 2, 3}, {r[], θ[], ϕ[]}, FormatBasis → {"Partials", "Differentials"}]
DefChart[cart, R3, {1, 2, 3}, {x[], y[], z[]}, FormatBasis → {"Partials", "Differentials"}]
```

The transformation relations between Cartesian to spherical coordinates and the converse are respectively

```
cart2sph = {Sqrt[x[] ^ 2 + y[] ^ 2 + z[] ^ 2], ArcCos[z[] / Sqrt[x[] ^ 2 + y[] ^ 2 + z[] ^ 2]],
  ArcTan[x[], y[]]}
sph2cart = r[] {Sin[θ[]] Cos[ϕ[]], Sin[θ[]] Sin[ϕ[]], Cos[θ[]]}
```

$$\left\{\sqrt{x^2 + y^2 + z^2}, \text{ArcCos}\left[\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right], \text{ArcTan}[x, y]\right\}$$

We also need to create the Mathematica rules for a change of coordinates

```
cart2sphRules = Thread[{x[], y[], z[]} → sph2cart]
```

$$\{x \to \text{Cos}[\phi]\, r\, \text{Sin}[\theta],\ y \to r\, \text{Sin}[\theta]\, \text{Sin}[\phi],\ z \to \text{Cos}[\theta]\, r\}$$

```
sph2cartRules = Thread[{r[], θ[], ϕ[]} → cart2sph]
```

$$\left\{r \to \sqrt{x^2+y^2+z^2},\ \theta \to \text{ArcCos}\left[\frac{z}{\sqrt{x^2+y^2+z^2}}\right],\ \phi \to \text{ArcTan}[x, y]\right\}$$

To check the validity of our transformations, we can use the command `Simplify` (or `FullSimplify`), verifying that going forth and back from one coordinate system to another leads to the identity

```
cart2sphRules /. sph2cartRules // Simplify
```

$$\{x \to x,\ y \to y,\ z \to z\}$$

```
sph2cartRules /. cart2sphRules // FullSimplify
```

$$\{r \to r,\ \theta \to \theta,\ \phi \to \text{ArcTan}[\text{Cos}[\phi], \text{Sin}[\phi]]\}$$

It is now necessary to compute the Jacobian matrices of the transformations in order to specify the basis change. To do so, we use the composition of the commands `Simplify` and `Outer[`f`, `$list_1$`, `$list_2$`, ...]`. The latter gives the generalized outer product of the $list_i$, forming all possible combinations of lowest-level elements in each of them and feeding them as arguments to f

```
jc2s = Simplify@Outer[D, cart2sph, {x[], y[], z[]}]
js2c = Simplify@Outer[D, sph2cart, {r[], θ[], ϕ[]}]
```

$$\left\{\left\{\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right\},\right.$$
$$\left.\left\{\frac{x\,z}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)}, \frac{y\,z}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)}, -\frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2}\right\}, \left\{-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right\}\right\}$$

$$\{\{\text{Cos}[\phi]\,\text{Sin}[\theta],\ \text{Cos}[\theta]\,\text{Cos}[\phi]\,r,\ -r\,\text{Sin}[\theta]\,\text{Sin}[\phi]\},$$
$$\{\text{Sin}[\theta]\,\text{Sin}[\phi],\ \text{Cos}[\theta]\,r\,\text{Sin}[\phi],\ \text{Cos}[\phi]\,r\,\text{Sin}[\theta]\},\ \{\text{Cos}[\theta],\ -r\,\text{Sin}[\theta],\ 0\}\}$$

Note that the letter `D` stands for 'partial derivative', in fact we are differentiating each change of coordinates with respect to the local coordinate basis. We can thus build the Jacobians in the following way

```
cart2sphJacobian = CTensor[jc2s, {sph, -cart}]
sph2cartJacobian = CTensor[js2c, {cart, -sph}]
```

$$\text{CTensor}\left[\left\{\left\{\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right\}, \left\{\frac{x\,z}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)}, \frac{y\,z}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)}, -\frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2}\right\}, \left\{-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right\}\right\}, \{sph, -cart\}, 0\right]$$

```
CTensor[{{Cos[ϕ] Sin[θ], Cos[θ] Cos[ϕ] r, -r Sin[θ] Sin[ϕ]},
  {Sin[θ] Sin[ϕ], Cos[θ] r Sin[ϕ], Cos[ϕ] r Sin[θ]},
  {Cos[θ], -r Sin[θ], 0}}, {cart, -sph}, 0]
```

```
{cart2sphJacobian[a, -b], sph2cartJacobian[a, -b]}
```

$$\left\{ \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \\ \bullet & \bullet & -\frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix}^{a}{}_{b}, \begin{pmatrix} \text{Cos}[\phi]\ \text{Sin}[\theta] & \text{Cos}[\theta]\ \text{Cos}[\phi]\ r & -r\ \text{Sin}[\theta]\ \text{Sin}[\phi] \\ \text{Sin}[\theta]\ \text{Sin}[\phi] & \text{Cos}[\theta]\ r\ \text{Sin}[\phi] & \text{Cos}[\phi]\ r\ \text{Sin}[\theta] \\ \text{Cos}[\theta] & -r\ \text{Sin}[\theta] & 0 \end{pmatrix}^{a}{}_{b} \right\}$$

This sets a default change of basis. The command `SetBasisChange[direct,inverse]` provides a pair of CTensor objects, which are tensors written in the coordinates determined by the above basis, with the direct and inverse basis changes. The change of basis in both charts reads

```
Simplify[ToCTensor[Basis, {sph, -cart}]][a, -b]
Simplify[ToCTensor[Basis, {sph, -cart}] /. cart2sphRules][a, -b]
```

$$\begin{pmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \\ \bullet & \bullet & -\frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix}^{a}{}_{b}$$

$$\begin{pmatrix} \text{Cos}[\phi]\ \text{Sin}[\theta] & \text{Sin}[\theta]\ \text{Sin}[\phi] & \text{Cos}[\theta] \\ \frac{\text{Cos}[\theta]\ \text{Cos}[\phi]}{r} & \frac{\text{Cos}[\theta]\ \text{Sin}[\phi]}{r} & -\frac{\text{Sin}[\theta]}{r} \\ -\frac{\text{Csc}[\theta]\ \text{Sin}[\phi]}{r} & \frac{\text{Cos}[\phi]\ \text{Csc}[\theta]}{r} & 0 \end{pmatrix}^{a}{}_{b}$$

The columns of the above matrix are the spherical basis components of the Cartesian basis vectors.

```
Simplify[ToCTensor[Basis, {cart, -sph}]][a, -b]
Simplify[ToCTensor[Basis, {cart, -sph}] /. cart2sphRules][a, -b]
```

$$\begin{pmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{x\ z}{\sqrt{x^2+y^2}} & -y \\ \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{y\ z}{\sqrt{x^2+y^2}} & x \\ \frac{z}{\sqrt{x^2+y^2+z^2}} & -\sqrt{x^2+y^2} & 0 \end{pmatrix}^{a}{}_{b}$$

$$\begin{pmatrix} \text{Cos}[\phi]\ \text{Sin}[\theta] & \text{Cos}[\theta]\ \text{Cos}[\phi]\ r & -r\ \text{Sin}[\theta]\ \text{Sin}[\phi] \\ \text{Sin}[\theta]\ \text{Sin}[\phi] & \text{Cos}[\theta]\ r\ \text{Sin}[\phi] & \text{Cos}[\phi]\ r\ \text{Sin}[\theta] \\ \text{Cos}[\theta] & -r\ \text{Sin}[\theta] & 0 \end{pmatrix}^{a}{}_{b}$$

The columns of the above matrix are the Cartesian components of the spherical basis vectors.
We can also compute the determinant of the Jacobian of the transformation

```
Jacobian[cart, sph][] // ToValues
```

$$\frac{1}{\sqrt{(x^2+y^2)\ (x^2+y^2+z^2)}}$$

and its inverse

```
Jacobian[sph, cart][] // ToValues
```

$$\sqrt{(x^2+y^2)\ (x^2+y^2+z^2)}$$

where the command ToValues replaces all known tensor-values for the tensors in a given expression.
A rather illuminating example lies in computing the Gradient, the Divergence and the Laplacian of scalars in the new spherical coordinates. In order to do so, we define a scalar field S[] and an orthonormal basis Osph on the manifold R3, related to spherical coordinates by

```
SetBasisChange[CTensor[{{1, 0, 0}, {0, r[], 0}, {0, 0, r[] Sin[θ[]]}},
  {-sph, Osph}], sph]
```

2.1.1 Gradient

The expression in Cartesian coordinates and in spherical coordinates are, respectively

```
CDcart[-i][S[]]
CD[-i][S[]]
```

$\mathcal{D}_i\,S$

$$\left|\begin{matrix}\mathcal{D}_1\,S\\ \mathcal{D}_2\,S\\ \mathcal{D}_3\,S\end{matrix}\right|_i$$

The components of the gradient in the orthonormal basis are

```
% // ReplaceBases[Osph]
```

$$\left|\begin{matrix}\mathcal{D}_1\,S\\ \dfrac{\mathcal{D}_2\,S}{r}\\ \dfrac{\text{Csc}[\theta]\ (\mathcal{D}_3\,S)}{r}\end{matrix}\right|_i$$

Another way to compute the gradient is the default Mathematica command Grad[f, $x_1, ..., x_n$, *chart*], which gives the gradient in the coordinates chart

```
In[]:= Grad[S[r, θ, ϕ], {r, θ, ϕ}, "Spherical"] // MatrixForm
```

Out[]//MatrixForm=

$$\begin{pmatrix} S^{(1,0,0)}[r,\theta,\phi]\\ \dfrac{S^{(0,1,0)}[r,\theta,\phi]}{r}\\ \dfrac{\text{Csc}[\theta]\,S^{(0,0,1)}[r,\theta,\phi]}{r}\end{pmatrix}$$

The agreement between the standard **Mathematica** and the **xAct** results confirms the goodness of the tensor algebra addon implementation.

2.1.2 Divergence

Let's define a vector field `V[i]` on the manifold `R3` and its associated CTensor vector in different bases

```
Vsph = ToCTensor[V, {sph}, 0]
Vcart = ToCTensor[V, {cart}, 0]
VOsph1 = ToCTensor[V, {Osph}, 0]
```

CTensor[{V^1, V^2, V^3}, {sph}, 0]

CTensor[{V^1, V^2, V^3}, {cart}, 0]

CTensor[{V^1, V^2, V^3}, {Osph}, 0]

With Cartesian coordinates

```
CDcart[-i]@Vcart[i]
```

$$\mathcal{D}_1 V^1 + \mathcal{D}_2 V^2 + \mathcal{D}_3 V^3$$

whereas, with spherical coordinates

```
CD[-i]@Vsph[i]
```

$$\mathcal{D}_1 V^1 + \frac{V^1 + r\,(\mathcal{D}_2 V^2)}{r} + \frac{V^1 + \mathrm{Cot}[\theta]\, r\, V^2 + r\,(\mathcal{D}_3 V^3)}{r}$$

```
CD[-i]@VOsph[i] // Expand
```

$$\frac{2 V^1}{r} + \frac{\mathrm{Cot}[\theta]\; V^2}{r} + \mathcal{D}_1 V^1 + \frac{\mathcal{D}_2 V^2}{r} + \frac{\mathrm{Csc}[\theta]\,(\mathcal{D}_3 V^3)}{r}$$

The same expression could be obtained using the standard command `Div[`$f_1, \ldots, f_n$`,` $x_1, \ldots, x_n$`, chart]`, which gives the divergence in the coordinates chart

```
Div[{v1[r, θ, ϕ], v2[r, θ, ϕ], v3[r, θ, ϕ]}, {r, θ, ϕ},
  "Spherical"] // Expand
```

$$\frac{2\, v1[r, \theta, \phi]}{r} + \frac{Cot[\theta]\, v2[r, \theta, \phi]}{r} + \frac{Csc[\theta]\, v3^{(0,0,1)}[r, \theta, \phi]}{r} + \frac{v2^{(0,1,0)}[r, \theta, \phi]}{r} + v1^{(1,0,0)}[r, \theta, \phi]$$

Again, we see the agreement between the two ways of computing the divergence.

2.1.3 Laplacian of scalars

With Cartesian coordinates

```
CDcart[i][CDcart[-i][S[]]] /. sph2cartRules // Simplify
```

$$\mathcal{D}_1 \mathcal{D}_1 S + \mathcal{D}_2 \mathcal{D}_2 S + \mathcal{D}_3 \mathcal{D}_3 S$$

instead, using spherical coordinates, we get

```
CD[i][CD[-i][S[]]] // Expand
```

$$\frac{2\, (\mathcal{D}_1 S)}{r} + \mathcal{D}_1 \mathcal{D}_1 S + \frac{Cot[\theta]\, (\mathcal{D}_2 S)}{r^2} + \frac{\mathcal{D}_2 \mathcal{D}_2 S}{r^2} + \frac{Csc[\theta]^2\, (\mathcal{D}_3 \mathcal{D}_3 S)}{r^2}$$

The above expression can be compared to what we get using the standard command `Laplacian`

```
CD[i][CD[-i][S[]]] // Expand
```

$$\frac{2\, (\mathcal{D}_1 S)}{r} + \mathcal{D}_1 \mathcal{D}_1 S + \frac{Cot[\theta]\, (\mathcal{D}_2 S)}{r^2} + \frac{\mathcal{D}_2 \mathcal{D}_2 S}{r^2} + \frac{Csc[\theta]^2\, (\mathcal{D}_3 \mathcal{D}_3 S)}{r^2}$$

2.2 Change to cylindrical coordinates

Similarly, we can apply the same procedure to change from cylindrical to Cartesian coordinates. This time we won't show all the computational steps (which are very similar to the previous change), but only the most important results.
The change between cylindrical and Cartesian coordinates can be parametrized with the three coordinates ρ, ϕ and h, as in the following figure.

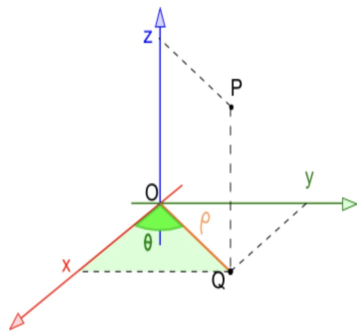

Figure 2.2: *Cylindrical coordinates.*

$$\phi : \begin{cases} x = \rho\cos(\phi) \\ y = \rho\sin(\phi) \\ z = h \end{cases} , \tag{2.2.1}$$

with $(\rho, \phi, h) \in [0, +\infty[\times[0, \pi[\times R$.

The determinant of the Jacobian matrix of φ equals

$$\det J_\varphi(\rho, \phi, h) = \det \begin{pmatrix} \cos(\phi) & -\rho\sin(\phi) & 0 \\ \sin(\phi) & \rho\cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \rho\,. \tag{2.2.2}$$

Using the same commands as in the spherical coordinates case above, the change of basis reads

```
cart2cil = {Sqrt[x[]^2 + y[]^2], ArcTan[x[], y[]], z[]}
```

$\left\{\sqrt{x^2 + y^2}\text{, ArcTan[x, y], z}\right\}$

```
cil2cart = {r[] Cos[θ[]], r[] Sin[θ[]], h[]}
```

```
{Cos[θ] r, r Sin[θ], h}
```

```
cart2cilRules = Thread[{x[], y[], z[]} → cil2cart]
```

```
{x → Cos[θ] r, y → r Sin[θ], z → h}
```

```
cil2cartRules = Thread[{r[], θ[], h[]} → cart2cil]
```

$\left\{r \to \sqrt{x^2 + y^2}\text{, }\theta \to \text{ArcTan[x, y], h} \to \text{z}\right\}$

The Jacobian matrices are given by

```
cart2cilJacobian = CTensor[jcart2cil, {cil, -cart}]
cil2cartJacobian = CTensor[jcil2cart, {cart, -cil}]
```

$$\text{CTensor}\left[\left\{\left\{\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 0\right\}, \left\{-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right\}, \{0, 0, 1\}\right\}, \{\text{cil}, -\text{cart}\}, 0\right]$$

```
CTensor[{{Cos[θ], -r Sin[θ], 0},
  {Sin[θ], Cos[θ] r, 0}, {0, 0, 1}}, {cart, -cil}, 0]
```

```
{cart2cilJacobian[a, -b], cil2cartJacobian[a, -b]}
```

$$\left\{ \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{a}{}_{b}, \begin{pmatrix} \text{Cos}[\theta] & -r\,\text{Sin}[\theta] & 0 \\ \text{Sin}[\theta] & \text{Cos}[\theta]\, r & 0 \\ 0 & 0 & 1 \end{pmatrix}^{a}{}_{b} \right\}$$

We can define once again an orthonormal basis `Ocil`, related to cylindrical coordinates by

```
SetBasisChange[CTensor[{{1, 0, 0}, {0, r[], 0}, {0, 0, 1}},
  {-cil, Ocil}], cil]
```

The form of the gradient in cylindrical coordinates is

```
In[]:= Grad[S[r, θ, h], {r, θ, h}, "Cylindrical"] // MatrixForm
```

Out[]//MatrixForm=

$$\begin{pmatrix} S^{(1,0,0)}[r, \theta, h] \\ \frac{S^{(0,1,0)}[r,\theta,h]}{r} \\ S^{(0,0,1)}[r, \theta, h] \end{pmatrix}$$

while the divergence becomes

```
Div[{v1[r, θ, h], v2[r, θ, h], v3[r, θ, h]}, {r, θ, h},
  "Cylindrical"] // Expand
CDcart[i][CDcart[-i][S[]]] /. cil2cartRules // Simplify
```

$$\frac{v1[r,\theta,h]}{r} + v3^{(0,0,1)}[r,\theta,h] + \frac{v2^{(0,1,0)}[r,\theta,h]}{r} + v1^{(1,0,0)}[r,\theta,h]$$

Finally, the Laplacian of a scalar is given by

```
Laplacian[S[r, θ, h], {r, θ, h}, "Cylindrical"] // Expand
CDcart[j][CDcart[-j][Vcart[k]]] /. cil2cartRules //
 Simplify
```

$$S^{(0,0,2)}[r,\theta,h] + \frac{S^{(0,2,0)}[r,\theta,h]}{r^2} + \frac{S^{(1,0,0)}[r,\theta,h]}{r} + S^{(2,0,0)}[r,\theta,h]$$

Chapter 3

The ΛCDM model

The goal of our research is to employ the tools of `Mathematica` and `xAct` to create a pipeline that allows us, starting from a given action, to find the most general results for that model of the Universe (i.e. the equations of motion, density parameters...). The analytic procedure is the following: we choose the action, vary a field and expand at first order, integrate by part and set the variation to be null. The following step consists in extracting Friedmann and Klein-Gordon equations and solving them in order to obtain the behaviour of $a(t)$, $H(t)$ and of the components Ω_i. The true strength of this algorithm relies in its capacity to describe any type of universe, in particular a universe governed by any arbitrary action or scalar field instead of the cosmological constant Λ.

We firstly study the well-known ΛCDM Model as a test for our algorithm. The results we will get in this case must correspond to the experimental evidence collected so far: if they do, the validity of our workflow is confirmed, and we can proceed extending our study to more models with additional degrees of freedom.

3.1 Friedmann equations

The action is simply given by

$$S = \int \sqrt{-g} \left[\frac{m_p^2 R[\nabla]}{2} - \Lambda \right] d^4x \,. \tag{3.1.1}$$

As we know from the theory, the FLRW metric reduces Einstein equations to just two Friedmann equations

$$3 \left[(\frac{\dot{a}}{a})^2 + \frac{k}{a^2} \right] = 8\pi G_N \rho(t) \,, \tag{3.1.2}$$

$$3 \frac{\ddot{a}}{a} = -4\pi G(\rho(t) + 3P(t)) \,. \tag{3.1.3}$$

If we assume a flat Universe as confirmed by recent experiments, i.e., $k = 0$, the first one becomes

$$H(t)^2 = \frac{8\pi G\rho(t)}{3}\,. \tag{3.1.4}$$

3.2 Evolution of parameters

In this section we solve Friedmann equations numerically using `Mathematica`. Since these equations are differential equations, we have to set the initial parameters of our model to certain values (note that, for both a_0 and t_0, we can only assign non-zero values, since the computation would naturally break down dealing with infinities). Moreover, we consider the cosmological parameters $\Omega_r^0 = 9 \cdot 10^{-5}$, $\Omega_m^0 = \Omega_b^0 + \Omega_{cdm}^0 = 0.0486 + 0.2589$ and, consequently, $\Omega_\Lambda^0 = 1 - \Omega_r^0 - \Omega_m^0$. We set the initial parameters to

$$t_0 = 10^{-10}\,, \tag{3.2.1}$$

$$a_0 = 10^{-6}\,, \tag{3.2.2}$$

and write ρ(t) and $P(t)$ as

$$\rho(t) = \frac{8\pi G}{3} H_0^2 \left[\frac{\Omega_m^0}{a(t)^3} + \frac{\Omega_r^0}{a(t)^4} + \Omega_\Lambda^0\right]\,, \tag{3.2.3}$$

$$P(t) = \frac{8\pi G}{3} H_0^2 \left[\frac{1}{3}\frac{\Omega_m}{a(t)^3} - \Omega_\Lambda\right]\,, \tag{3.2.4}$$

where H_0 is defined as $\sqrt{\frac{8\pi G}{3}}$. We can now solve the second order differential Friedmann equations with the commands `Solve` and `NDSolveValue`, and refining the integration process. We firstly compute the value of the time derivative of the scale factor `a'[t]` from the first Friedmann equation, and insert it as initial condition for the derivative when solving the second Friedmann equation. Therefore the commands used are

```
s0 = Solve[(a'[t] / a[t])^2 - (8 G π ρ[t])/3 == 0, a'[t]][[2]];
```

```
s = NDSolveValue[{a[t] * 4/3 * G * Pi (3 P[t] + ρ[t]) + a''[t] == 0, a[t0] == at0,
    a'[t0] == a'[t] /. s0 /. a[t] → at0, WhenEvent[a[t] == 1, "StopIntegration"]},
   a, {t, t0, Infinity}, Method → {"TimeIntegration" → "ExplicitRungeKutta"},
   PrecisionGoal → Infinity]
tf = s["Domain"][[1, 2]]
```

Remembering that the densities of the cosmic fluids can be written in the following way

$$\Omega_r(t) = \frac{\Omega_m^0}{a(t)^3} \frac{H_0^2}{H(t)^2}\,, \tag{3.2.5}$$

$$\Omega_r(t) = \frac{\Omega_r^0}{a(t)^4} \frac{H_0^2}{H(t)^2}\,, \tag{3.2.6}$$

$$\Omega_\Lambda(t) = \Omega_\Lambda^0 \frac{H_0^2}{H(t)^2}\,, \tag{3.2.7}$$

we finally obtain the desired plots for $a(t)$, $H(t)$ and Ω_i(t)

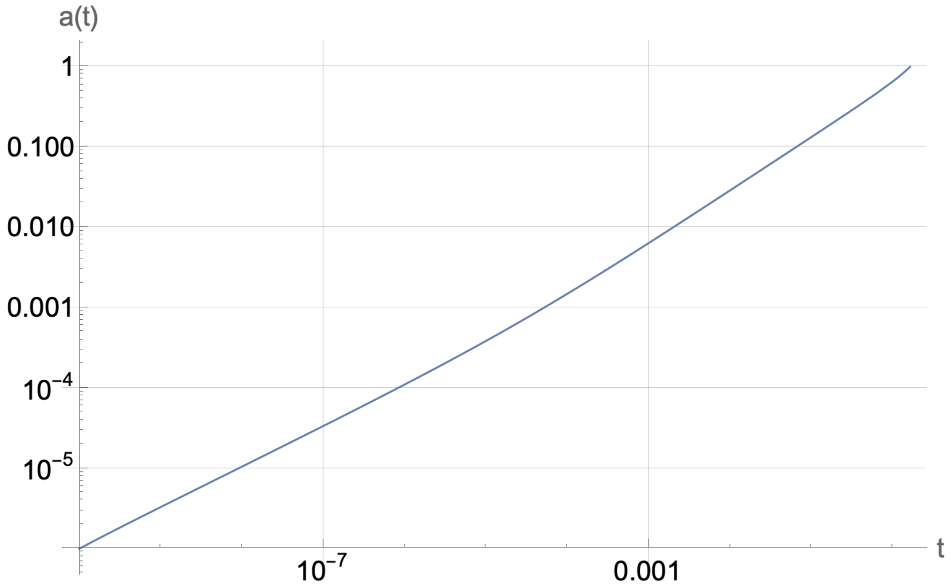

Figure 3.1: *Evolution of a for ΛCDM as a function of the cosmological time t. The scale factor is $a = 1$ at final value of time in the plot $t_f = 1.65$ (in the units introduced above).*

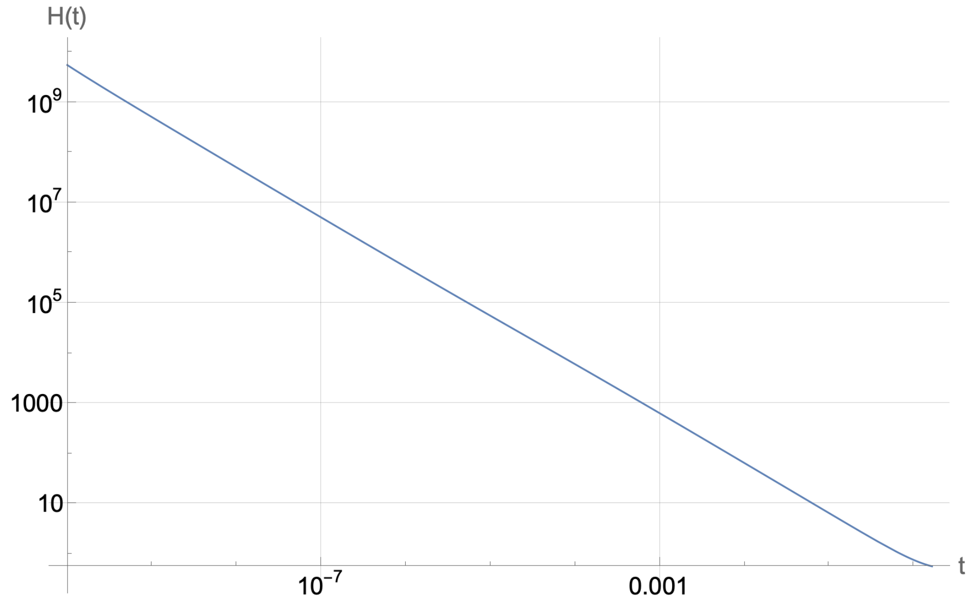

Figure 3.2: *Evolution of $H(t)$ for ΛCDM as a function of the cosmological time t. The scale factor is $a = 1$ at final value of time in the plot.*

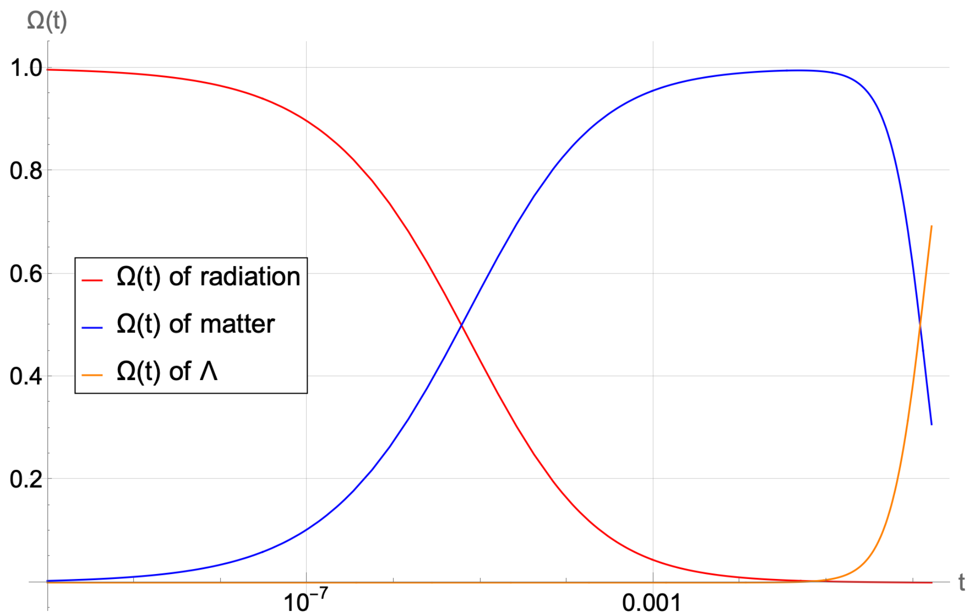

Figure 3.3: *Evolution of Ω_r, Ω_m and Ω_Λ for ΛCDM as functions of the cosmological time t. The scale factor is $a = 1$ at final value of time in the plot.. The dark energy density is increasing at late times, while radiation density becomes negligible.*

The previous graphics perfectly reproduce the experimental evidence we have, in fact

- $a(t)$ started from a very low value (approximately zero) and has since then increased up to today's accepted value of 1;
- the radiation density has undergone a constant diminution since the beginning of the universe, whereas matter density has been increasing ever since, up to redshift 0.7, after which it has drastically dropped, leaving space to dark energy. At every moment, the relation $\Omega_r + \Omega_m + \Omega_\Lambda = 1$ holds.

We are now ready to substitute the cosmological constant Λ with a scalar potential, and see if we recover the same results of the ΛCDM model.

Chapter 4

Modified Gravity

In this chapter, we introduce the Horndeski action, which is a generalization of the Einstein-Hilbert action of General Relativity, and study the equations of motion and the evolution of parameters for some cases corresponding to different choices of this action. In particular, we will apply our workflow first to the *Extended Quintessence* model and then to a new model, in order to test its validity.

4.1 Horndeski theory

4.1.1 An extension of General Relativity

In the previous section we have shown how to derive the equations of motion, Friedmann equations and the evolution of all parameters from the standard ΛCDM action, where the source of dark energy is encompassed in a non-zero cosmological constant Λ. However, there are some theoretical caveats that come along with this description, such as the fine-tuning problem and the coincidence problem.
Let's briefly take a look at the latter. We know from countless observational data that $\Omega_m^0 \simeq 0.3$ and $\Omega_\Lambda^0 \simeq 0.7$. Moreover, the current value of the vacuum energy is of the same order of magnitude as the matter density; this is a huge dilemma, since the vacuum energy and the matter density evolve rapidly with respect to each other, as can be seen from their ratio

$$\frac{\Omega_\Lambda}{\Omega_m} \propto a^3\,, \tag{4.1.1}$$

which means that, being $\Omega_m^0 = 1 - \Omega_\Lambda^0$, the matter-dark energy equality scale factor becomes

$$a_{eq} = \sqrt[3]{\frac{1-\Omega_\Lambda^0}{\Omega_\Lambda^0}}\,. \tag{4.1.2}$$

Now, if we consider $\Omega_\Lambda^0 \simeq 0.7$, we obtain $a_{eq} \cong 0.75$, very close to the value of the scale factor today $a = 1$. If the two densities are comparable today, then, in the past the vacuum energy must have been extremely small, while in the future the matter density will be insignificant. Thus, since this seems an effect tailored for late Universe observations, this poses a coincidence problem.

Therefore, for the caveats mentioned above, General Relativity might not to be the complete theory of gravity at large scales. Or, at least, it is important to explore other possibilities. For instance, an approach is to consider some extension theories, which add new degrees of freedom to the standard theory. These theories are called Modified Gravity theories. We can generalize Einstein-Hilbert action as

$$S = \int d^4x\sqrt{-g}f(R, R_{\mu\nu}, R_{\mu\nu\rho\sigma}, \phi, V^\mu, T^{\mu\nu}, ...) \tag{4.1.3}$$

where ϕ, V^μ and $T^{\mu\nu}$ are generic scalar, vector and tensor fields that we add as new degrees of freedom. Theoretically, we have an infinite arbitrariness in choosing the type and the number of additional degrees of freedom, but in practice there are many constraints that should be considered. The main mathematical condition our modified theory of gravity has to satisfy is that the the equations of motion should be second order differential equations: higher differential orders would introduce the so-called Ostrogradsky instabilities [11, 12].

4.1.2 Action

The Hordenski gravity is a Modified Gravity model theory with one additional (scalar) degree of freedom with respect to General Relativity. Its action is [7]

$$S[g_{\mu\nu}] = \frac{1}{8\pi G}\int d^4x\sqrt{-g}\sum_{i=2}^{5}\mathcal{L}_i[g_{\mu,\nu}, \phi] + S_M[g_{\mu\nu}]\,, \tag{4.1.4}$$

where the four Lagrangian densities $\mathcal{L}_i$ are defined as

$$\mathcal{L}_2 = G_2[\phi, X]\,, \tag{4.1.5}$$
$$\mathcal{L}_3 = -G_3[\phi, X]\Box\phi\,, \tag{4.1.6}$$
$$\mathcal{L}_4 = G_4[\phi, X]R + G_{4X}[\phi, X]\left[(\Box\phi)^2 - \phi_{;\mu\nu}\phi^{;\mu\nu}\right]\,, \tag{4.1.7}$$
$$\mathcal{L}_5 = G_5[\phi, X]G_{\mu\nu}\phi^{;\mu\nu} - \frac{1}{6}G_{5X}[\phi, X]\left[(\Box\phi)^3 + 2\phi_{;\mu}{}^{\nu}\phi_{;\nu}{}^{\alpha}\phi_{;\alpha}{}^{\mu} - 3\phi_{;\mu\nu}\phi^{;\mu\nu}\Box\phi\right]\,. \tag{4.1.8}$$

The $G_i[\phi, X]$ are four generic functions of a scalar field ϕ and $X = \partial_\mu\phi\partial^\mu\phi$, where repeated indices are summed over following Einstein's convention. Moreover, $\Box = \nabla_\mu\nabla^\mu$, while the subscript ϕ or X on the G_i functions represents respectively the derivative with respect to ϕ and X, i.e. $G_{iX} \equiv \frac{\partial G_i}{\partial X}$ and $G_{i\phi} \equiv \frac{\partial G_i}{\partial \phi}$.

The structure of this Lagrangian leads to equations of motion which are second order differential equations. In fact, it can be proven that Horndeski's theory is the most general theory of gravity in four dimensions whose Lagrangian is constructed out of the metric tensor and a scalar field and leads to second order equations of motion. The well known action of General Relativity (see Ch.3) is obtained considering

$$G_2[\phi, X] = -\Lambda, G_4[\phi, X] = \frac{m_p^2}{2} \; and \; G_3[\phi, X] = G_5[\phi, X] = 0\,. \tag{4.1.9}$$

If we vary the action (4.1.4) and impose the FLRW metric, we obtain the equations of motion which are the correspondent of the ΛCDM Friedmann equations for the Horndeski theory. Those are

$$H^2 = \frac{8\pi G}{3m_p^2}(\rho + \rho_\phi)\,, \tag{4.1.10}$$

$$\dot{H} + H^2 = -\frac{4\pi G}{3m_p^2}(\rho + \rho_\phi + 3P + 3P_\phi)\,, \tag{4.1.11}$$

where ρ and P are the energy density and pressure of standard matter, whereas ρ_ϕ and P_ϕ are the energy density and pressure of the scalar field, whose expressions depend on the choice of the Horndeski G_i functions. In the next sections we analyse some choices, using `xAct` to compute their equations of motion.

4.2 Extended Quintessence

Since we know from the experiments that the dark energy is distributed rather uniformly through space and is evolving slowly with time, we can try substituting the cosmological constant Λ with a slow-roll scalar field with a kinetic term in the action, and using a particular choice of a potential $V(\phi)$. Moreover, we non-minimally couple the scalar field with gravity introducing a coupling function $F(\phi)$ as below. The action for this model is

$$S = \sqrt{-g}\left\{\frac{m_p^2}{2}[1 + F(\phi)]R - \frac{1}{2}(\nabla_\mu\phi)(\nabla^\mu\phi) - V[\phi]\right\}\,, \tag{4.2.1}$$

meaning the G_i of the Lagrangian densities in Horndeski's action become

$$G_2[\phi, X] = X - V(\phi),\; G_3[\phi, X] = X,\; G_4[\phi, X] = (1 + F(\phi))\frac{m_p^2}{2},\; G_5[\phi, X] = 0\,. \tag{4.2.2}$$

4.2.1 `xPert`: covariant equations

Using the `xPert` package in `xAct`, we start as usual by defining a manifold M and the metric tensor `g`. For the variation of the action, it is also necessary to define the perturbed metric `h` and the perturbation parameter ϵ. The general equation is $g_{ab} = g^0_{ab} + h_{ab}$, where h_{ab} is the first order correction (h^1_{ab} in the code)

```
DefMetricPerturbation[metric, metpert, ϵ];
PrintAs[metpert] ^= "h";
** DefParameter: Defining parameter ϵ.
** DefTensor: Defining tensor metpert[LI[order], -a, -b].
Perturbation[metric[-a, -b], 1]
```

$h^1{}_{ab}$

It is possible to compute the metric perturbation at any order. For instance, we can find the metric perturbation at third order with the command `Perturbation[tensor,order]` or with the command `Perturbed`, which gives the whole equation for the perturbed metric up to the order considered

```
Perturbed[metric[a, b], 3]
Perturbation[metric[a, b], 3]
```

$$g^{ab} + \epsilon \Delta[g^{ab}] + \frac{1}{2}\epsilon^2 \Delta^2[g^{ab}] + \frac{1}{6}\epsilon^3 \Delta^3[g^{ab}]$$

$$\Delta^3[g^{ab}]$$

We can also evaluate the perturbation directly in the latter case, using the following command

```
ExpandPerturbation@%
```

$$-6\, h^{1ac}\, h^1{}_c{}^d\, h^1{}_d{}^b + 3\, h^1{}_c{}^b\, h^{2ac} + 3\, h^{1ac}\, h^2{}_c{}^b - h^{3ab}$$

We find the equations of motion with the following steps: we write the action, vary a field and expand at first order. Firstly we define the Horndeski scalar field ϕ (`Phi`) and its perturbation at first order $\delta\phi$ (`PertPhi`).
Then the action is defined as

```
L = Sqrt[-Detmetric[]]
   (MPlanck ^ 2 / 2 RicciScalarCD[] * (1 + F[Phi[]]) -
     1 / 2 * CD[-b][Phi[]] * CD[b][Phi[]] - V[Phi[]])
```

$$\sqrt{-\tilde{\tilde{g}}}\left(\frac{1}{2} m_p{}^2\, (1 + F[\varphi])\, R[\nabla] - V[\varphi] - \frac{1}{2}(\nabla_b \varphi)\left(\nabla^b \varphi\right)\right)$$

and the perturbed action becomes

```
Lpert = ToCanonical@ContractMetric@ExpandPerturbation@Perturbation@L
```

$$-\frac{1}{2} m_p{}^2 \sqrt{-\tilde{\tilde{g}}}\, h^{1ba}\, R[\nabla]_{ba} - \frac{1}{2} m_p{}^2 \sqrt{-\tilde{\tilde{g}}}\, F[\varphi]\, h^{1ba}\, R[\nabla]_{ba} + \frac{1}{4} m_p{}^2 \sqrt{-\tilde{\tilde{g}}}\, h^{1b}{}_b\, R[\nabla] + \frac{1}{4} m_p{}^2 \sqrt{-\tilde{\tilde{g}}}\, F[\varphi]\, h^{1b}{}_b\, R[\nabla] - \frac{1}{2} \sqrt{-\tilde{\tilde{g}}}\, h^{1b}{}_b\, V[\varphi] - \frac{1}{2} m_p{}^2 \sqrt{-\tilde{\tilde{g}}}\, h^{1b}{}_b{}^{;a}{}_{;a} - \frac{1}{2} m_p{}^2 \sqrt{-\tilde{\tilde{g}}}\, F[\varphi]\, h^{1b}{}_b{}^{;a}{}_{;a} + \frac{1}{2} m_p{}^2 \sqrt{-\tilde{\tilde{g}}}\, h^{1ba}{}_{;b;a} + \frac{1}{2} m_p{}^2 \sqrt{-\tilde{\tilde{g}}}\, F[\varphi]\, h^{1ba}{}_{;b;a} - \sqrt{-\tilde{\tilde{g}}}\, (\nabla_b \varphi)\, \delta\varphi^{1;b} + \frac{1}{2} \sqrt{-\tilde{\tilde{g}}}\, h^{1}{}_{ba}\, (\nabla^a \varphi)\, (\nabla^b \varphi) - \frac{1}{4} \sqrt{-\tilde{\tilde{g}}}\, h^{1a}{}_a\, (\nabla_b \varphi)\, (\nabla^b \varphi) + \frac{1}{2} m_p{}^2 \sqrt{-\tilde{\tilde{g}}}\, \delta\varphi^1\, R[\nabla]\, F'[\varphi] - \sqrt{-\tilde{\tilde{g}}}\, \delta\varphi^1\, V'[\varphi]$$

We can find different sets of equations of motion; in order to get the equations of motion for the scalar field V, we can use the functional derivative on the varied function with respect to the field. The command `VarD` computes the functional derivative of `Lpert` with respect to `PertPhi`, using the covariant derivative `CD` to make the integration by parts

```
0 == VarD[PertPhi[LI[1]], CD][Lpert]
```

$$0 == \delta_1{}^1 \sqrt{-\tilde{\tilde{g}}}\, (\nabla_a \nabla^a \varphi) + \frac{1}{2} m_p{}^2\, \delta_1{}^1 \sqrt{-\tilde{\tilde{g}}}\, R[\nabla]\, F'[\varphi] - \delta_1{}^1 \sqrt{-\tilde{\tilde{g}}}\, V'[\varphi]$$

Dividing by a constant factor (the determinant) and considering the delta equal to 1 (since it is a *non-physical* delta on the perturbation order indices), we get the field evolution equation called the Klein–Gordon equation

$$\nabla_a \nabla^a \phi + \frac{m_p^2 R[\nabla]}{2} F'(\phi) - V'(\phi) = 0\,. \tag{4.2.3}$$

Furthermore, we can find the correct Einstein equations, using a functional derivative with respect to the metric. The expression we obtain is the following

$$\begin{aligned} m_p^2 \{ & G[\nabla]_{ab} + G[\nabla]_{ab} F(\phi) - (\nabla_b \phi \nabla_a \phi) F'(\phi) + g_{ab} (\nabla_c \phi)(\nabla^c \phi) F'(\phi) \\ & - (\nabla_a \phi)(\nabla_b \phi) F''(\phi) + g_{ab} (\nabla_c \phi)(\nabla^c \phi) F''(\phi) \} \\ & - (\nabla_a \phi)(\nabla_b \phi) + g_{ab} [V(\phi) + \frac{1}{2} (\nabla_c \phi)(\nabla^c \phi)] = 0\,, \end{aligned} \tag{4.2.4}$$

4.2.2 `xCoba`: equations on FLRW

We now want to write the equations using the FLRW coordinates. We define a useful set of coordinates to write Eq. (4.2.3) and Eq. (4.2.4), in order to extract the Einstein equations. Using the package `xCoba`, we start as usual by defining our manifold `M` and the metric tensor `g`. Upon defining a tensor ϕ and a scalar field `V`, we define the same action as in Eq. (4.2.1). Using the basis defined by the comoving coordinates t, r, θ, ϕ, we get the FLRW metric

```
In[∘]:= MatrixForm[metricarray = {
        {-1, 0, 0, 0},
        {0, F[t[]]^2/(1-Curv*r[]^2), 0, 0},
        {0, 0, F[t[]]^2*r[]^2, 0},
           {0, 0, 0, F[t[]]^2*r[]^2 Sin[θ[]]^2}
        }]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a[t]^2}{1-\mathrm{Curv}\, r^2} & 0 & 0 \\ 0 & 0 & a[t]^2\, r^2 & 0 \\ 0 & 0 & 0 & a[t]^2\, r^2\, \mathrm{Sin}[\theta]^2 \end{pmatrix}$$

Note that, within this file, the curvature constant k and the evolution parameter a(t) are denoted, respectively, as `Curv` and `F[t]`, in order to distinguish them from the generic set of indices (`a,...,l`) of the manifold. With `xCoba` we can then compute all the quantities in FLRW coordinates. For instance, the Chrystoffel symbols for the metric are as follows.

```
ChristoffelCDPDB[a, -b, -c] // ToBasis[B] // ComponentArray //
 ToValues
```

$$\left\{\left\{\{0,0,0,0\},\left\{0,\frac{a[t]\,a'[t]}{1-\text{Curv}\,r^2},0,0\right\},\{0,0,a[t]\,r^2\,a'[t],0\},\{0,0,0,a[t]\,r^2\,\text{Sin}[\theta]^2\,a'[t]\}\right\},\left\{\left\{0,\frac{a'[t]}{a[t]},0,0\right\},\left\{\frac{a'[t]}{a[t]},\frac{\text{Curv}\,r}{1-\text{Curv}\,r^2},0,0\right\},\{0,0,r\,(-1+\text{Curv}\,r^2),0\},\{0,0,0,r\,(-1+\text{Curv}\,r^2)\,\text{Sin}[\theta]^2\}\right\},\left\{\left\{0,0,\frac{a'[t]}{a[t]},0\right\},\left\{0,0,\frac{1}{r},0\right\},\left\{\frac{a'[t]}{a[t]},\frac{1}{r},0,0\right\},\{0,0,0,-\text{Cos}[\theta]\,\text{Sin}[\theta]\}\right\},\left\{\left\{0,0,0,\frac{a'[t]}{a[t]}\right\},\left\{0,0,0,\frac{1}{r}\right\},\{0,0,0,\text{Cot}[\theta]\},\left\{\frac{a'[t]}{a[t]},\frac{1}{r},\text{Cot}[\theta],0\right\}\right\}\right\}$$

We now focus on the Einstein equations. The Einstein tensor is

```
EinsteinCD[-a, -b] // ToBasis[B] // ComponentArray // ToValues
```

$$\left\{\left\{\frac{3\,(\text{Curv}+a'[t]^2)}{a[t]^2},0,0,0\right\},\left\{0,\frac{\text{Curv}+a'[t]^2+2\,a[t]\,a''[t]}{-1+\text{Curv}\,r^2},0,0\right\},\{0,0,-r^2\,(\text{Curv}+a'[t]^2+2\,a[t]\,a''[t]),0\},\{0,0,0,-r^2\,\text{Sin}[\theta]^2\,(\text{Curv}+a'[t]^2+2\,a[t]\,a''[t])\}\right\}$$

We also need to define the stress-energy tensor $T_{\mu\nu}$, using the velocity field u^μ and two scalar functions ρ and P

```
T[b_, c_] := (ρ[] + P[]) * u[b] * u[c] + P[] * metric[b, c]
```

```
T[b, -b] // ToBasis[B] // ToValues
```

$3\,P-\rho$

Thus the Einstein equations become

```
EinsteinEquation =
  (EinsteinCD[-a, -b] + EinsteinCD[-a, -b] × F[Phi[]] + (8 * Pi * G) *
                    metric[-a, -b] × V[Phi[]] - (8 * Pi * G) * (CD[-a][Phi[]])
                    (CD[-b][Phi[]]) + (8 * Pi * G) * 1 / 2 metric[-a, -b]
                    (CD[-c][Phi[]]) (CD[c][Phi[]]) - (CD[-b][CD[-a][Phi[]]])
                    Derivative[1][F][Phi[]] + metric[-a, -b] (CD[-c][CD[c][Phi[]]])
                    Derivative[1][F][Phi[]] - (CD[-a][Phi[]]) (CD[-b][Phi[]])
                    Derivative[2][F][Phi[]] + metric[-a, -b] (CD[-c][Phi[]])
                    (CD[c][Phi[]]) Derivative[2][F][Phi[]] - (8 * Pi * G) * T[-a, -b] /.
                 Curv → 0 // SeparateMetric[metric] // ToBasis[B] // ComponentArray //
             TraceBasisDummy // ToValues // ToValues // ToBasis[B] // ComponentArray //
        TraceBasisDummy // ToValues // ToValues)
```

$$\Big\{\Big\{\frac{3\,\mathrm{Curv}}{a[t]^2}+\frac{3\,\mathrm{Curv}\,F[\phi[t]]}{a[t]^2}-8G\pi V[\phi[t]]-8G\pi\rho+\frac{3\,a'[t]^2}{a[t]^2}+\frac{3F[\phi[t]]\,a'[t]^2}{a[t]^2}+\frac{3F'[\phi[t]]\,a'[t]\,\phi'[t]}{a[t]}-4G\pi\phi'[t]^2,\ 0,\ 0,\ 0\Big\},$$

$$\Big\{0,\ -\frac{8G\pi a[t]^2 P}{1-\mathrm{Curv}\,r^2}+\frac{\mathrm{Curv}}{-1+\mathrm{Curv}\,r^2}+\frac{\mathrm{Curv}\,F[\phi[t]]}{-1+\mathrm{Curv}\,r^2}+\frac{8G\pi a[t]^2 V[\phi[t]]}{1-\mathrm{Curv}\,r^2}+\frac{a'[t]^2}{-1+\mathrm{Curv}\,r^2}+\frac{F[\phi[t]]\,a'[t]^2}{-1+\mathrm{Curv}\,r^2}-\frac{2a[t]F'[\phi[t]]\,a'[t]\,\phi'[t]}{1-\mathrm{Curv}\,r^2}-\frac{4G\pi a[t]^2\phi'[t]^2}{1-\mathrm{Curv}\,r^2}-\frac{a[t]^2\phi'[t]^2F''[\phi[t]]}{1-\mathrm{Curv}\,r^2}+\frac{2a[t]\,a''[t]}{-1+\mathrm{Curv}\,r^2}+\frac{2F[\phi[t]]\,a[t]\,a''[t]}{-1+\mathrm{Curv}\,r^2}-\frac{a[t]^2F'[\phi[t]]\,\phi''[t]}{1-\mathrm{Curv}\,r^2},\ 0,\ 0\Big\},$$

$$\Big\{0,\ 0,\ -\mathrm{Curv}\,r^2-\mathrm{Curv}\,F[\phi[t]]\,r^2-8G\pi a[t]^2Pr^2+8G\pi a[t]^2r^2V[\phi[t]]-r^2a'[t]^2-F[\phi[t]]\,r^2a'[t]^2-2a[t]\,r^2F'[\phi[t]]\,a'[t]\,\phi'[t]-4G\pi a[t]^2r^2\phi'[t]^2-a[t]^2r^2\phi'[t]^2F''[\phi[t]]-2a[t]\,r^2a''[t]-2F[\phi[t]]\,a[t]\,r^2a''[t]-a[t]^2r^2F'[\phi[t]]\,\phi''[t],\ 0\Big\},$$

$$\Big\{0,\ 0,\ 0,\ -\mathrm{Curv}\,r^2\mathrm{Sin}[\theta]^2-\mathrm{Curv}\,F[\phi[t]]\,r^2\mathrm{Sin}[\theta]^2-8G\pi a[t]^2Pr^2\mathrm{Sin}[\theta]^2+8G\pi a[t]^2r^2\mathrm{Sin}[\theta]^2V[\phi[t]]-r^2\mathrm{Sin}[\theta]^2a'[t]^2-F[\phi[t]]\,r^2\mathrm{Sin}[\theta]^2a'[t]^2-2a[t]\,r^2\mathrm{Sin}[\theta]^2F'[\phi[t]]\,a'[t]\,\phi'[t]-4G\pi a[t]^2r^2\mathrm{Sin}[\theta]^2\phi'[t]^2-a[t]^2r^2\mathrm{Sin}[\theta]^2\phi'[t]^2F''[\phi[t]]-2a[t]\,r^2\mathrm{Sin}[\theta]^2a''[t]-2F[\phi[t]]\,a[t]\,r^2\mathrm{Sin}[\theta]^2a''[t]-a[t]^2r^2\mathrm{Sin}[\theta]^2F'[\phi[t]]\,\phi''[t]\Big\}\Big\}$$

from which we can extract the single Einstein equations as follows

```
(Extract[EinsteinEquation, {1, 1}] / 3 /. K'[t[]] → H[t[]] * K[t[]] /.
    K[t[]] → a /. Curv → 0 // Simplify) == 0
```

$$(1 + F[\phi[t]])\, H[t]^2 + H[t]\, F'[\phi[t]]\, \phi'[t] - \frac{4}{3} G \pi \left(2 V[\phi[t]] + 2 \rho + \phi'[t]^2\right) == 0$$

```
((-Extract[EinsteinEquation, {2, 2}] * (1 - Curv * r[] ^ 2) / K[t[]] ^ 2 /.
        K'[t[]] → H[t[]] * K[t[]] /.
       K''[t[]] → K[t[]] * (H[t[]]' + H[t[]] ^ 2) /. K[t[]] → a) / 2 /.
    Curv → 0 // Simplify // Expand) == 0
```

$$\frac{3 H[t]^2}{2} + \frac{3}{2} F[\phi[t]]\, H[t]^2 + 4 G \pi P - 4 G \pi V[\phi[t]] + H[t]' + F[\phi[t]]\, H[t]' + H[t]\, F'[\phi[t]]\, \phi'[t] + 2 G \pi \phi'[t]^2 + \frac{1}{2} \phi'[t]^2 F''[\phi[t]] + \frac{1}{2} F'[\phi[t]]\, \phi''[t] == 0$$

where we neglect the other two spatial equations since they are equivalent to the second. Rearranging the terms we then get

$$(1 + F[\phi(t)])H(t)^2 + H(t)F'[\phi(t)]\phi'(t) - \frac{4\pi G}{3}(2V[\phi(t)] + 2\rho + \phi'(t)^2) = 0\,, \tag{4.2.5}$$

$$\begin{aligned} &\frac{3}{2}(1 + F[\phi(t)])H(t)^2 + 4\pi G(P - V[\phi(t)]) + (1 + F[\phi(t)])H'(t) + \\ &H(t)F'[\phi(t)]\phi'(t) + 2\pi G\phi'(t) + \frac{1}{2}(\phi'(t)^2 F''[\phi(t)] + F'[\phi(t)]\phi''(t)) = 0\,. \end{aligned} \tag{4.2.6}$$

The last step required to obtain all the necessary equations to study the evolution of $a(t)$ and $H(t)$ is to obtain Klein-Gordon equation. With a procedure similar to the one used for the Einstein equation, we obtain

$$3H(t)\phi'(t) + V'[\phi(t)] + 3F'[\phi(t)]\frac{(a'(t)^2 + a(t)a''(t))}{8\pi G a(t)^2} + \phi''(t) = 0\,. \tag{4.2.7}$$

4.2.3 Time evolution

We are ready to estimate some relevant quantities of our system. Using the previous definition for the densities Ω_i, we set the following initial conditions to

$$\begin{aligned} \alpha &= 0.1\,, \\ t_0 &= 10^{-10}\,, \\ a_0 &= 10^{-6}\,, \\ \phi_0 &= 10^{-10}\,, \\ \phi_0' &= 10^{-5}\,, \end{aligned} \tag{4.2.8}$$

where α is a parameter which modulates the quadratic dependence of the function F with respect to ϕ, and ϕ_0 and ϕ_0' are, respectively, the initial values of the field ϕ and its velocity ϕ'. The units and the functional form of V ad F considered during the numerical evaluation are defined by

$$\begin{aligned} H_0 &= \sqrt{\frac{8\pi G}{3}}, \\ \Lambda &= 3 \cdot H_0^2 \cdot \Omega_\Lambda, \\ F(\phi) &= \alpha \cdot \phi^2, \\ V(\phi, X) &\equiv \frac{\Lambda}{\sqrt{\phi}}\,. \end{aligned} \tag{4.2.9}$$

We can now define $\rho_{\Lambda CDM}(t)$, $P_{\Lambda CDM}(t)$, $\rho(t)$ and $P(t)$ (respectively the energy density in the ΛCDM model, the pressure in the ΛCDM model, the energy density for Extended Quintessence and the pressure for Extended Quintessence) in the following way

$$\rho_{\Lambda CDM} = \frac{3}{8\pi G} H_0^2 \left[\frac{\Omega_m}{a(t)^3} + \frac{\Omega_r}{a(t)^4} + \Omega_\Lambda\right]\,, \tag{4.2.10}$$

$$P_{\Lambda CDM} = \frac{3}{8\pi G} H_0^2 \left[\frac{1}{3}\frac{\Omega_r}{a(t)^4} - \Omega_\Lambda\right]\,, \tag{4.2.11}$$

$$\rho(t) = \frac{3}{8\pi G} H_0^2 \left[\frac{\Omega_m}{a(t)^3} + \frac{\Omega_r}{a(t)^4}\right]\,, \tag{4.2.12}$$

$$P(t) = \frac{3}{8\pi G} H_0^2 \left[\frac{1}{3}\frac{\Omega_r}{a(t)^4}\right]\,, \tag{4.2.13}$$

and use again `Solve` and `NDSolveValue` to solve the differential equations (4.2.5), (4.2.6) and (4.2.7). We still have to define some important quantities: $\rho_\phi(t)$, $P_\phi(t)$ and $\omega(t) \equiv \frac{P_\phi(t)}{\rho_\phi(t)}$, where the first two terms are the energy density and the pressure of the scalar field and ω is the state parameter (see Ch.1),

$$\rho_\phi(t) = \frac{3}{8\pi G}\left\{-H(t)F'[\phi(t)]\phi'(t) + \frac{8\pi G}{3}V[\phi(t)] + \frac{4\pi G}{3}\phi'(t)^2 - H(t)^2F[\phi(t)]\right\}, \tag{4.2.14}$$

$$\begin{aligned} P_\phi(t) = \frac{1}{4\pi G}(&\frac{3}{2}F[\phi(t)]H(t)^2 - 4\pi G V[\phi(t)] + F[\phi(t)]H'(t) \\ &+ H(t)F'[\phi(t)]\phi'(t) + 2\pi G\phi'(t)^2 + \frac{1}{2}\phi'(t)^2F''[\phi(t)] + \frac{1}{2}\phi''(t)^2F'[\phi(t)]) . \end{aligned} \tag{4.2.15}$$

At this point, we are finally able to plot the scale factor $a(t)$, the Hubble parameter $H(t)$, the fractional densities $\Omega_i(t)$ and the equation of state parameter $\omega(t)$. In particular, we can study the time evolution of the densities Ω_i for Extended Quintessence with respect to the ΛCDM model.We find that the behaviour of the fractional densities for the ΛCDM model is very well reproduced by the choice of the potential and the coupling function within the Extended Quintessence theory.

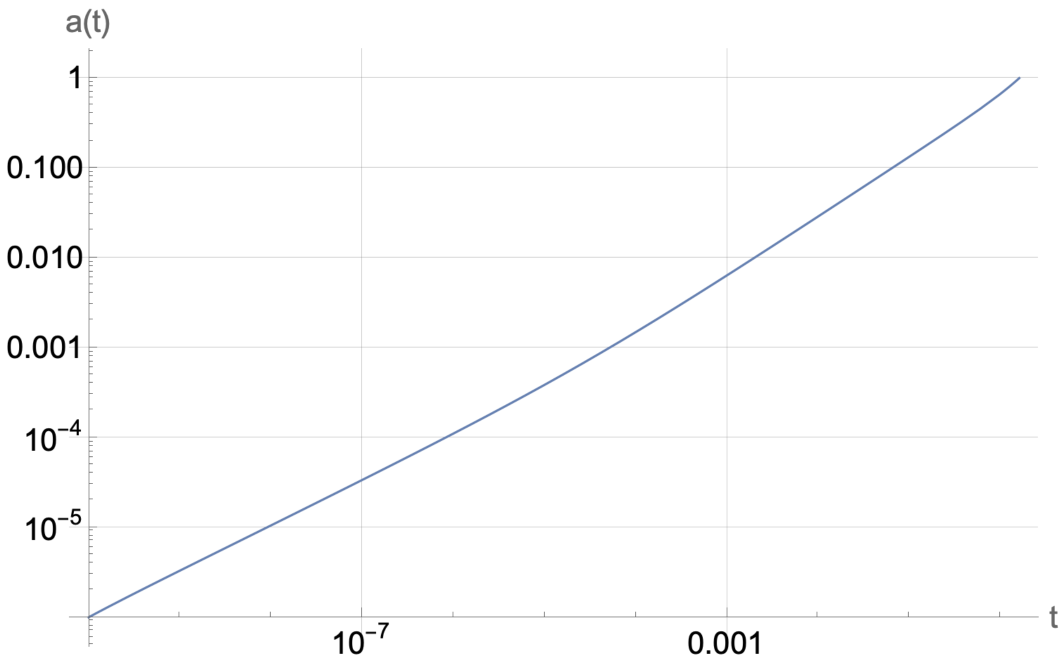

Figure 4.1: *Evolution of a for ΛCDM as a function of the cosmological time t. The scale factor is $a = 1$ at final value of time in the plot $t_f = 1.62$ (in the units introduced above).*

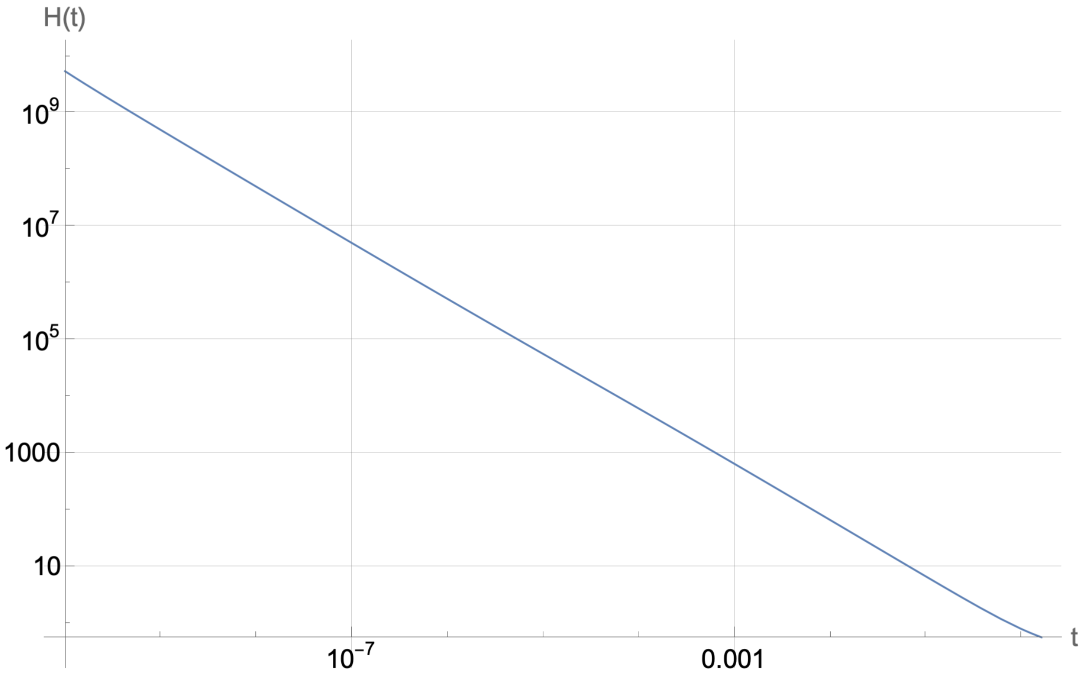

Figure 4.2: *Evolution of H for Extended Quintessence as a function of the cosmological time t. The scale factor is $a = 1$ at final value of time in the plot.*

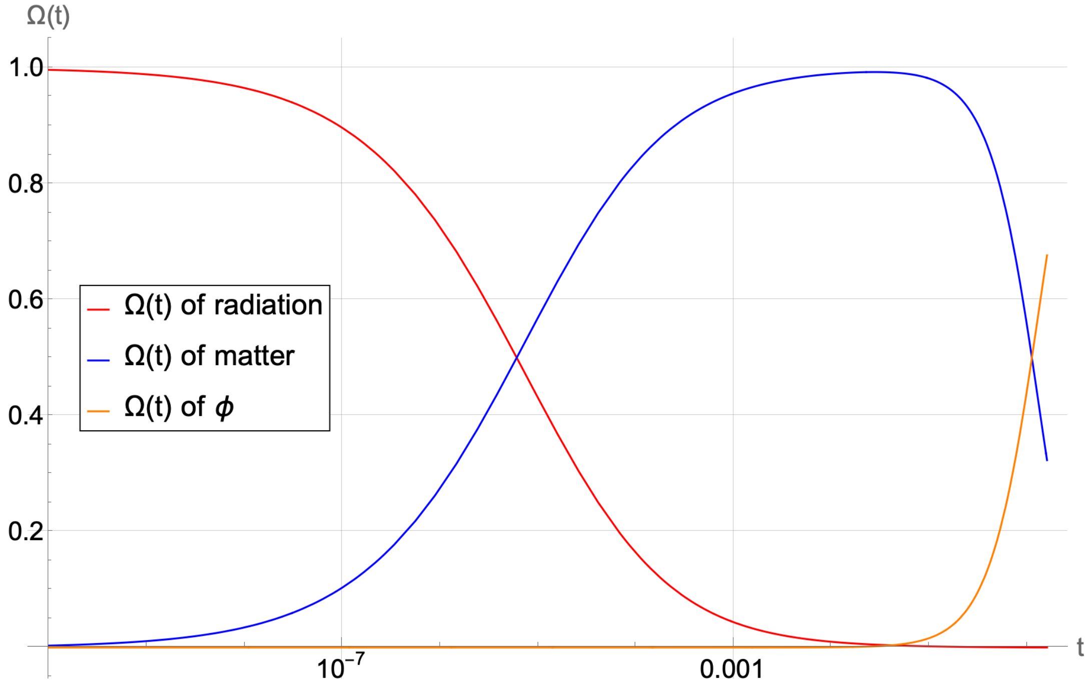

Figure 4.3: *Evolution of Ω_r, Ω_m and Ω_Λ for Extended Quintessence as a function of the cosmological time t. The scale factor is $a = 1$ at final value of time in the plot.*

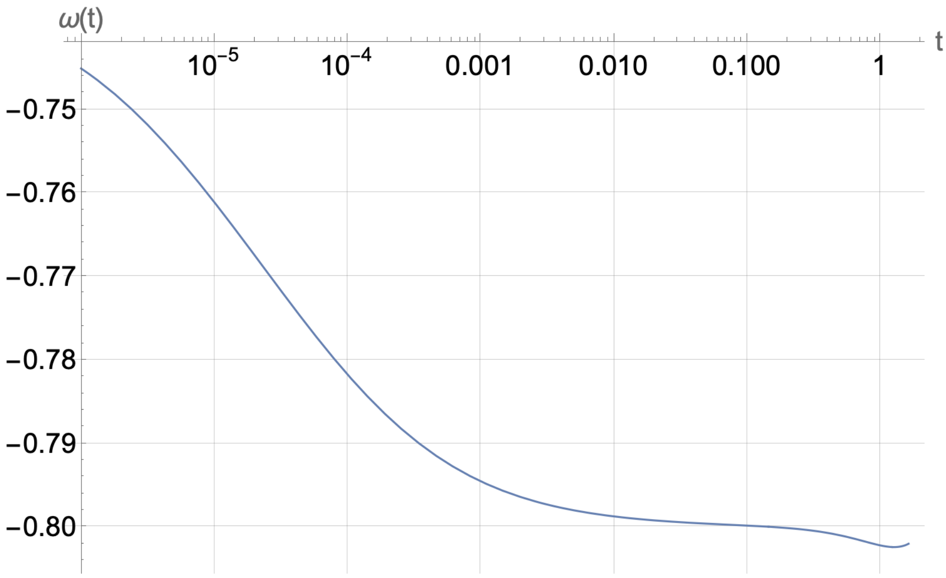

Figure 4.4: *Evolution of the state parameter ω as a function of the cosmological time t. The scale factor is $a = 1$ at final value of time in the plot. Note that ω starts from a non-zero value and gradually settles to $\omega \simeq -0.8$, very close to the theoretical value of the cosmological constant in the ΛCDM model, $\omega = -1$.*

It is also important to plot the evolution of the Hubble parameter and the fractional densities with respect to the cosmic scale factor. In fact, the time where $a(t) = 1$ might change with respect to ΛCDM depending on the model considered (see for instance the value of the time when $a = 1$ in ΛCDM from Fig. 3.1 and the one in Extended Quintessence from Fig. 4.1), and therefore the best comparison can be made when we follow the scale factor evolution rather than the time.

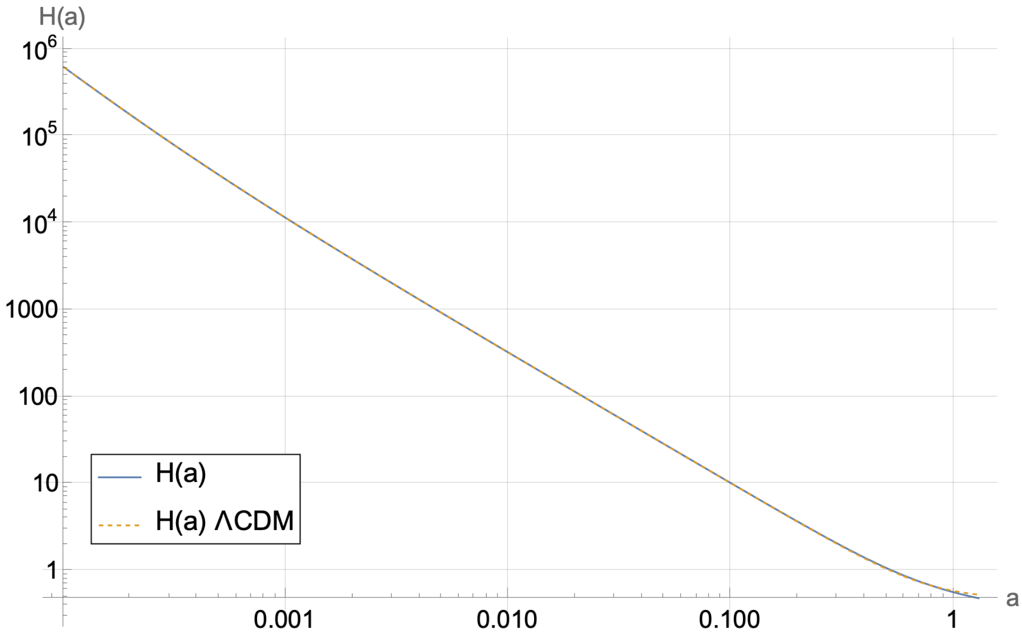

Figure 4.5: *Evolution of H as a function of a for Extended Quintessence, compared to the ΛCDM case. We note the accordance between the two models up to slight differences at late times.*

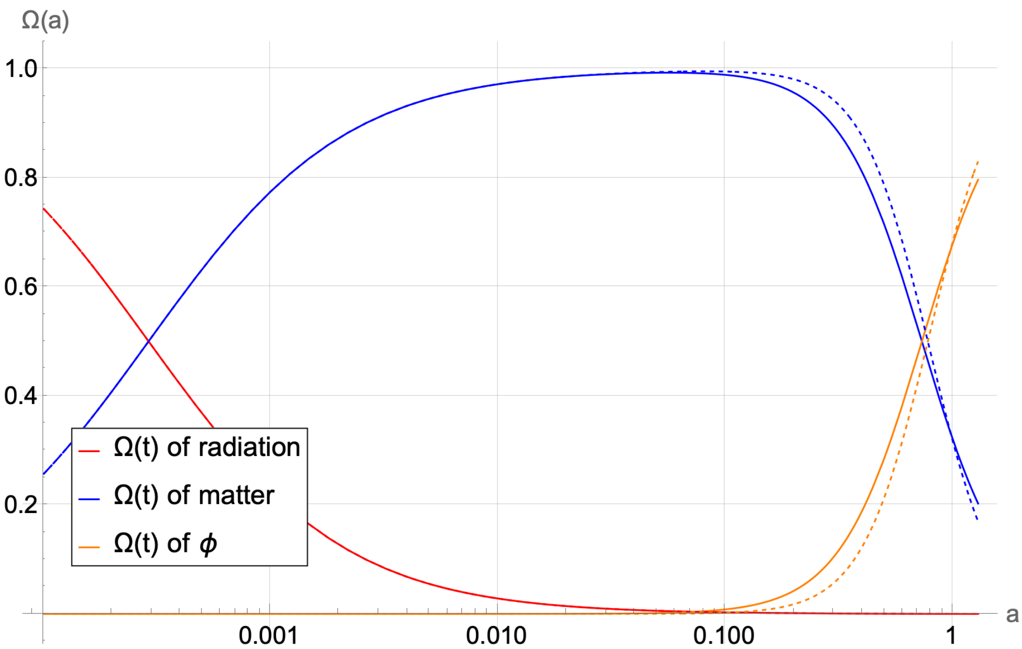

Figure 4.6: *Evolution of Ω_r, Ω_m and Ω_Λ as function of a for Extended Quintessence, compared to the ΛCDM case.*

4.3 Testing a new model

With the pipeline we implemented in *xAct*, it is possible to describe the dark energy source varying freely the Lagrangian densities of the Horndeski action and obtain the correspondent equations of motion.
In this last section we are going to describe an extension of the previous model. The new action is

$$S = \sqrt{-g}\left\{\frac{m_p^2}{2}(1+F(\phi))R[\nabla] - V[\phi,(\nabla_\mu\phi)(\nabla^\mu\phi)] - \frac{1}{2}(\nabla_\mu\phi)(\nabla^\mu\phi)\right\}, \tag{4.3.1}$$

where V now depends both on ϕ and $X \equiv (\nabla_b\phi)(\nabla^b\phi)$, which can be viewed as a kinetic term (up to some constant factor). The functions G_i in Hordenski's action are then given by

$$G_2[\phi,X] = X - V(\phi,X), G_4[\phi,X] = (1+F(\phi))\frac{m_p^2}{2}, G_3[\phi,X] = X, G_5[\phi,X] = 0\,. \tag{4.3.2}$$

We also choose a scalar potential V that depends both on ϕ and X, using the following function

$$V(\phi,X) \equiv (1+2\beta X)e^{-\phi}\Lambda\,. \tag{4.3.3}$$

With this potential we are coupling the kinetic part X with the scalar field itself via an exponential function which depends on the parameter β (for this reason, we will refer to it from now on as β-Quintessence). In fact, the overall kinetic part contribution in the action with this potential is now $(1+2\beta\Lambda e^{-\phi})X$ (also considering the standard kinetic part $-\nabla\phi\nabla\phi/2$ part in the action).

Following the same steps shown in the previous cases, we obtain the Einstein equations

$$(1+F[\phi(t)])H(t)^2 + H(t)F'[\phi(t)]\phi'(t) - \frac{4\pi G}{3}(2V[\phi(t),X]+ \\ 2\rho + \phi'(t)^2) + \frac{2}{3}\phi'(t)^2\frac{\partial V[\phi(t),X]}{\partial X} = 0\,, \tag{4.3.4}$$

$$\frac{3}{2}(1+F[\phi(t)])H(t)^2 + 4\pi G(P - V[\phi(t),X]) + (1+F[\phi(t)])H'(t)+ \\ H(t)F'[\phi(t)]\phi'(t) + 2\pi G\phi'(t) + \frac{1}{2}(\phi'(t)^2F''[\phi(t)] + F'[\phi(t)]\phi''(t)) = 0\,. \tag{4.3.5}$$

The Klein-Gordon equation reads

$$\begin{aligned} &\frac{3F'[\phi(t)][a'(t)^2 + a(t)a''(t)]}{8\pi G a(t)^2} - \\ &\frac{3a'(t)\phi'(t)[1 + 2\frac{\partial V[\phi,X]}{\partial X}]}{a(t)} - \\ \phi''(t)\Bigg\{1 + 2&\frac{\partial V[\phi, X]}{\partial X} - 4\phi'(t)^2\frac{\partial^2 V[\phi, X]}{\partial X^2} - \\ &\frac{\partial V[\phi, X]}{\partial \phi} - 2\phi'(t)^2\frac{\partial^2 V[\phi, X]}{\partial\phi\partial X}\Bigg\} = 0\,. \end{aligned} \tag{4.3.6}$$

Using the same values of Extended Quintessence for Ω_i, we set the following initial conditions to

$$\begin{aligned} \alpha &= 0.1\,, \\ t_0 &= 10^{-10}\,, \\ a_0 &= 10^{-6}\,, \\ \phi_0 &= 10^{-10}\,, \\ \phi'_0 &= 10^{-5}\,, \end{aligned} \tag{4.3.7}$$

and define the new parameters

$$\begin{aligned} \Lambda &= 3H_0^2\Omega_\Lambda\,, \\ H_0 &= \sqrt{\frac{8\pi G}{3}}\,, \\ F(\phi) &= \alpha\phi^2\,, \\ V(\phi, X) &\equiv (1 + 2\beta X)e^{\phi}\Lambda\,. \end{aligned} \tag{4.3.8}$$

Using the same method as before, we can easily solve Eqs. (4.3.4), (4.3.5) and (4.3.6) for different values of β, attaining our purpose of finding the cosmic evolution of $a(t)$,$H(t)$,$\Omega_i(t)$ and $\omega(t)$. The bigger the value of β, the better our model will reproduce the ΛCDM behaviour. We choose $\beta = 0.05$, in order to appreciate the difference between the two models. In particular, the plots obtained are the following

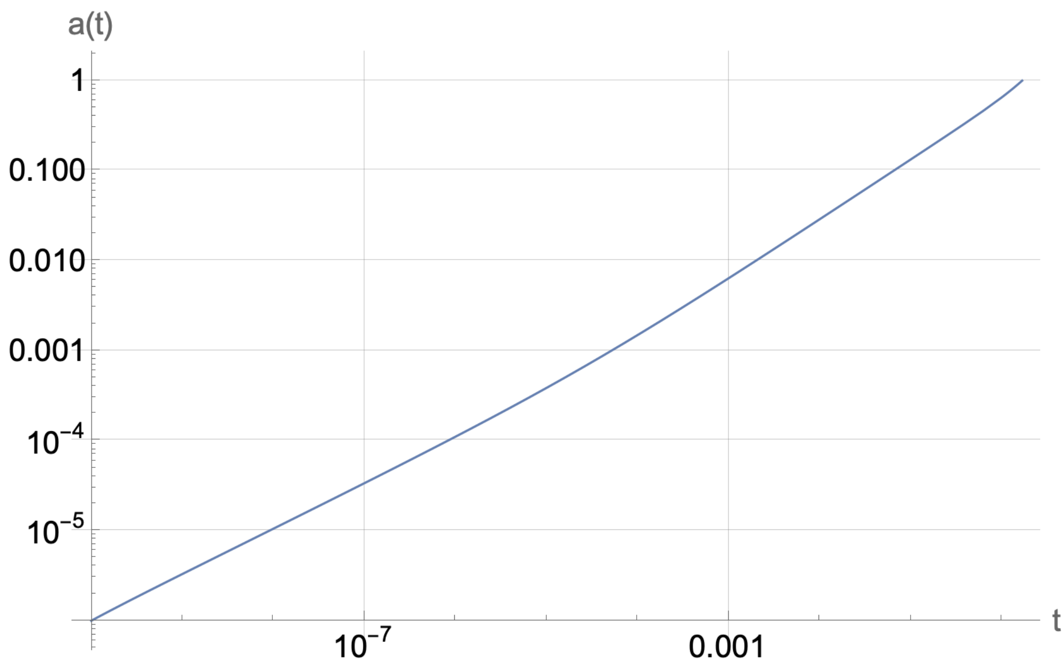

Figure 4.7: *Evolution of a for β-Quintessence as a function of the cosmological time t. The scale factor is $a = 1$ at final value of time in the plot $t_f = 1.69$ (in the units introduced above).*

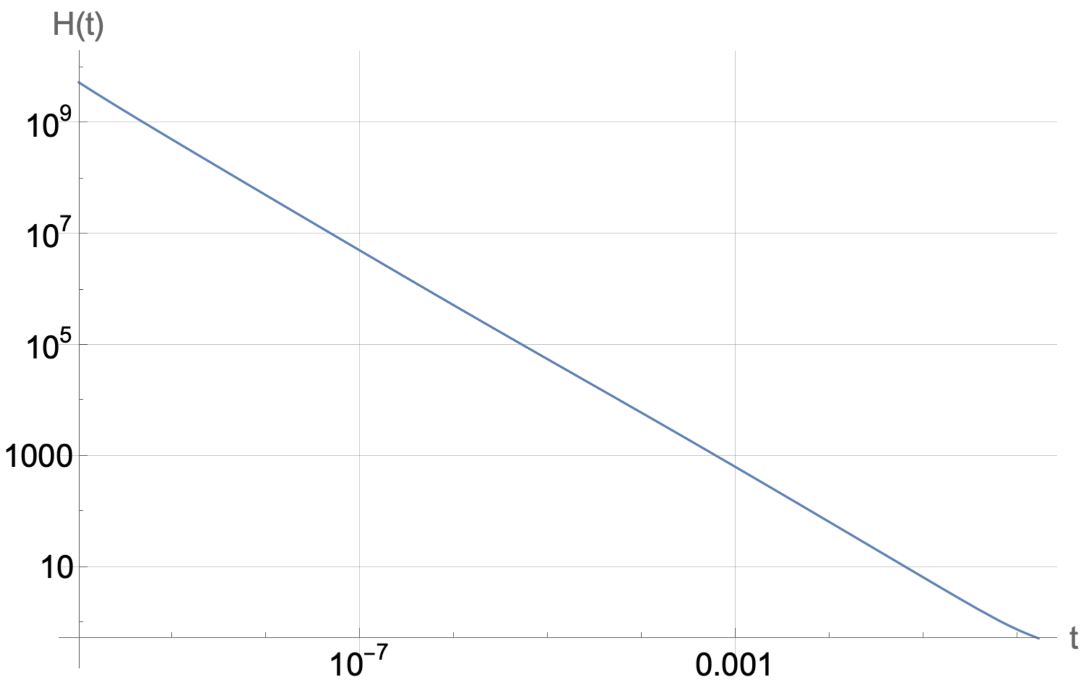

Figure 4.8: *Evolution of H for β-Quintessence as a function of the cosmological time t. The scale factor is $a = 1$ at final value of time in the plot.*

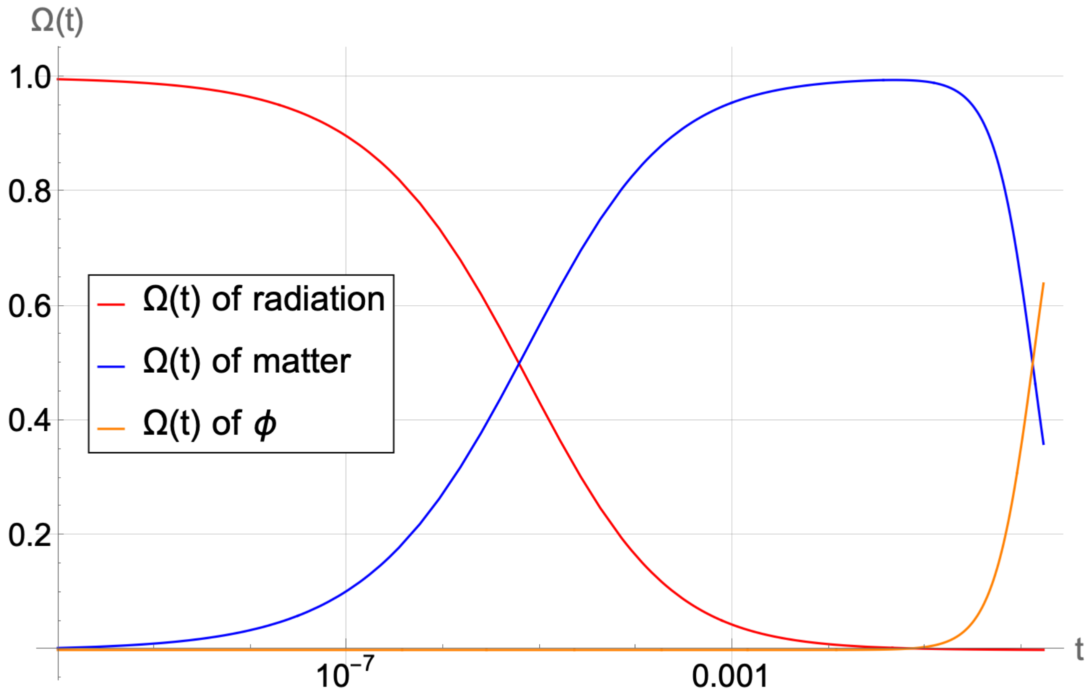

Figure 4.9: *Evolution of* Ω_r, Ω_m *and* Ω_Λ *for* β*-Quintessence, with* $\beta = 0.05$ *as a function of the cosmological time* t*. The scale factor is* $a = 1$ *at final value of time in the plot.*

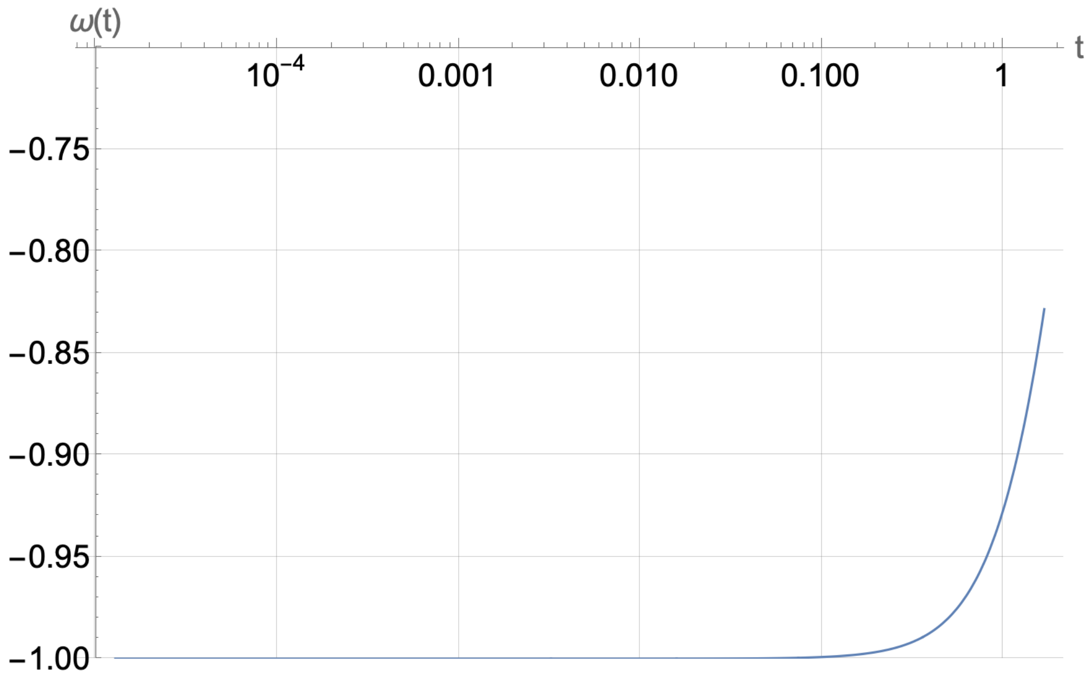

Figure 4.10: *Evolution of the state parameter of the scalar field* ω *for the* β*-Quintessence model as a function of the cosmological time* t*. The scale factor is* $a = 1$ *at final value of time in the plot.*

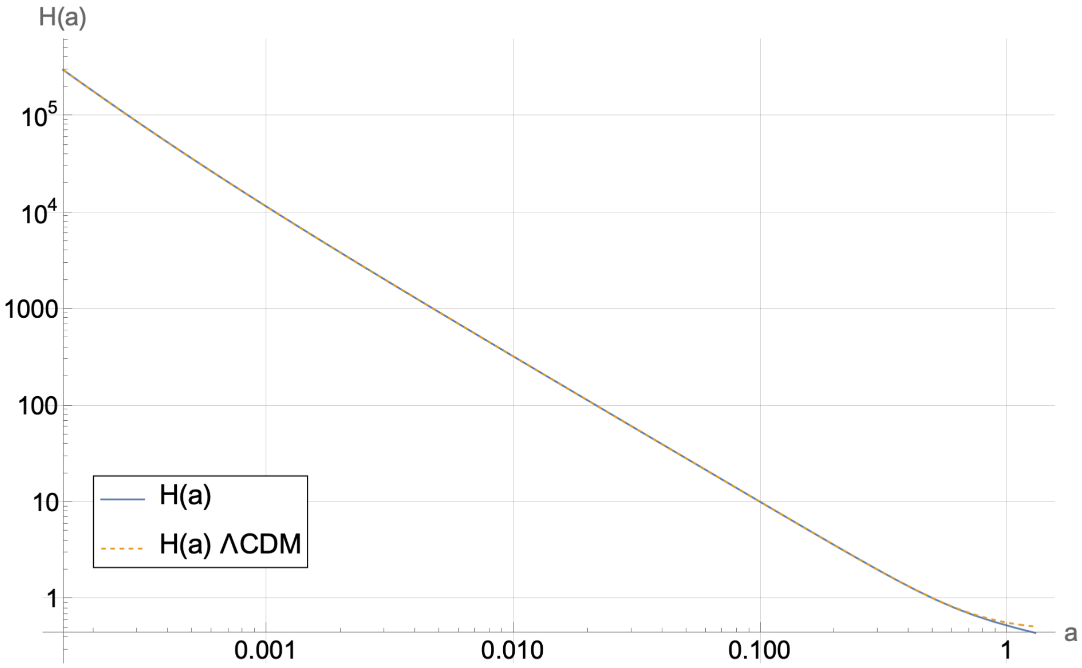

Figure 4.11: *Evolution of H for β-Quintessence as a function of a, compared to the ΛCDM case.*

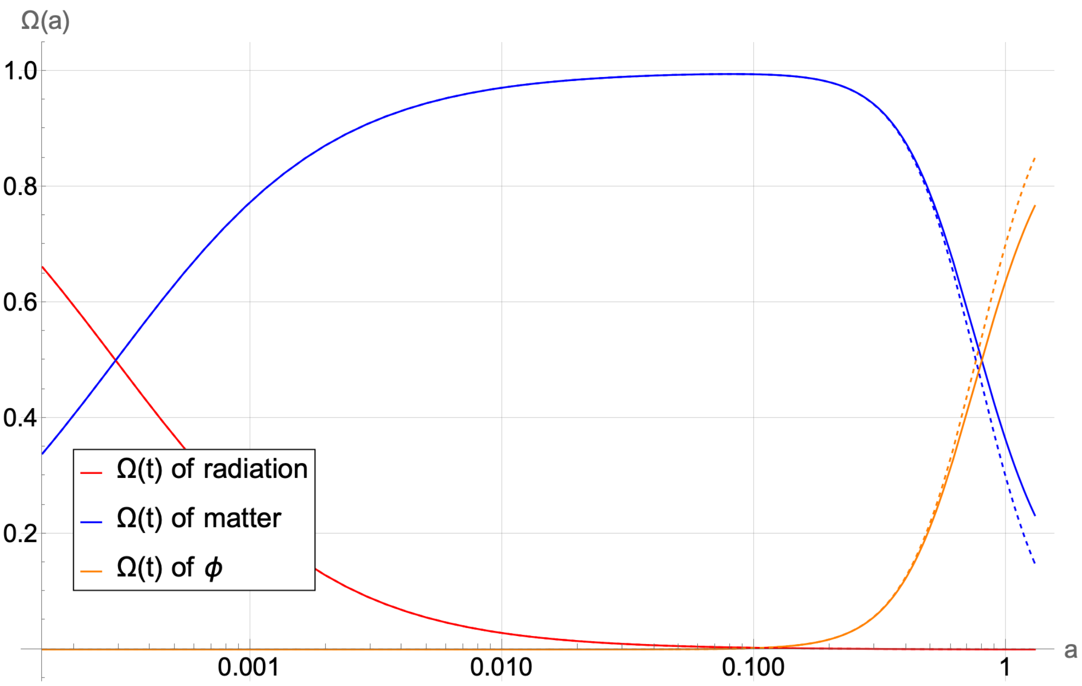

Figure 4.12: *Evolution of Ω_r, Ω_m and Ω_Λ for β-Quintessence as a function of a, compared to the ΛCDM case.*

We have shown that β-Quintessence is a theoretical model which well reproduces the evolution of the ΛCDM model. With this value of β, the state parameter ω perfectly corresponds to the theoretical value of the cosmological constant Λ, $\omega = -1$, up to a certain time, while evolves differently in the late time.

We have thus tested the validity of our workflow by applying it to an arbitrary model to study its equations and behaviour.

Conclusion

In the thesis, we studied the standard model of cosmology (ΛCDM) in the context of General Relativity and alternative models within the Modified Gravity models, using the numerical methods of the tensor algebra tool `xAct`. In the first chapter we provided an outline of the most relevant equations of General Relativity and cosmology, focusing in particular on the ΛCDM model. The following chapter contains a technical explanation of the crucial commands in `xAct`, useful for the understanding of the following chapters. In the third chapter, we derived the equations of motion and the time evolution plots for the ΛCDM model, which accurately reproduce the experimental data. In the last chapter, we applied the same workflow implemented on `xAct` to study two models: *Extended Quintessence* and *β-Quintessence*, the latter being an original implementation. In particular, both models display an accurate description of the time evolution as in the ΛCDM case.

The crucial aspect about this thesis was indeed the implementation of a universal pipeline which can be applied to the most general cases, both to extract the relevant quantities and prove or reject the validity of the model.

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