ALMA MATER STUDIORUM
UNIVERSITY OF BOLOGNA

SCHOOL OF SCIENCE
Department of Mathematics
Master Degree in Mathematics

Highlighting Interdisciplinarity between Physics and Mathematics in Historical Papers on Special Relativity:

Development of an Analytical Tool for Characterising Boundary Crossing Mechanisms

Thesis in Mathematics Education

Supervisor:
Prof. Olivia Levrini

Submitted by:
Lucia Modica

Co-Supervisor:
Prof. Alessia Cattabriga

Co-Supervisor:
Dott. Lorenzo Miani

Academic year 2021/2022

L’ambizione e il puro senso del dovere
non danno frutti veramente importanti,
che invece derivano dall’amore e
dalla dedizione verso gli uomini e le cose.
A. Einstein

ABSTRACT

This thesis is part of the Erasmus + IDENTITIES project, the aim of which is to develop interdisciplinary teaching materials for pre-service teacher education. Specifically, it follows up on research conducted by Lorenzo Miani, aimed at highlighting how the Special Theory of Relativity (STR) historically arose from a special interaction between mathematics and physics. This co-evolution was sought, and highlighted, through the analysis of the four foundational articles of STR written by Lorentz (1904), Poincaré (1906), Einstein (1905) and Minkowski (1908). For the analysis of these articles, we used the metaphor of the 'boundary', set out in Akkerman and Bakker's (2011) meta-theory, referring to the boundary between Mathematics and Physics.

An operational tool for analysing original articles was developed to extract the relationship between the two disciplines. Such an analysis can make a considerable contribution to the Justification Problem, intercepting the possibility of investigating the identity of Mathematics as a discipline. This type of analysis allowed us to understand the 'boundary styles' of each author and the nature of Lorentz Transformations as boundary objects.

The design of an activity for pre-service teacher education is also illustrated. This takes the form of a tutorial for group work and was tested in the Physics Education course at the University of Bologna, held by Professor Olivia Levrini. Thanks to the activity, it was possible to reflect on disciplinary identities and the importance of boundary experiences to overcome stereotypes.

The tool elaborated in the thesis is open to future developments since it can be used for the analysis of a wide variety of texts and for the construction of "boundary zones", which are increasingly desired and encouraged in European reports.

Italian abstract

Questa tesi di laurea si colloca all'interno del progetto Erasmus + IDENTITIES, il cui obiettivo è sviluppare materiali didattici interdisciplinari per la formazione iniziale degli insegnanti. Nello specifico, si dà seguito ad una ricerca condotta da Lorenzo Miani, finalizzata a mettere in evidenza come la Teoria della Relatività Speciale (STR) sia storicamente nata da una speciale interazione tra matematica e fisica. Tale co-evoluzione è stata cercata, e messa in evidenza, attraverso l'analisi dei quattro articoli fondativi della STR scritti da Lorentz (1904), Poincaré (1906), Einstein (1905) e Minkowski (1908). Per l'analisi di questi articoli abbiamo utilizzato la metafora del "confine", esposta nella metateoria di Akkerman e Bakker (2011), riferendosi al confine tra Matematica e Fisica.

È stato sviluppato uno strumento operativo di analisi di articoli originali per estrarne il rapporto tra le due discipline. Un'analisi di questo tipo può portare un contributo considerevole al Justification Problem, intercettando la possibilità di indagare sull'identità della Matematica, intesa come disciplina. Questo tipo di analisi ha permesso di comprendere gli "stili al confine" di ogni autore, e la natura delle Trasformazioni di Lorentz in quanto oggetto di confine.

È inoltre illustrata la progettazione di un'attività per la formazione iniziale degli insegnanti. Questa si configura come un tutorial per lavori di gruppo, ed è stata sperimentata nel corso di Didattica della Fisica dell'Università di Bologna, tenuto dalla Professoressa Olivia Levrini. Grazie all'attività, è stato possibile riflettere sulle identità disciplinari e sull'importanza di fare "esperienze di confine" per superare stereotipi.

Lo strumento elaborato nella tesi si apre a sviluppi futuri, dal momento che si presta ad essere utilizzato per l'analisi di una grande varietà di testi e per la costruzione di "boundary zone", sempre più auspicate e incentivate nei report europei.

TABLE OF CONTENTS

CHAPTER 1 – INTRODUCTION

This thesis is part of the Erasmus + IDENTITIES project, the aim of which is to develop interdisciplinary teaching materials for pre-service teacher education. Specifically, it follows up on research conducted by Lorenzo Miani, in his Master's Thesis in Physics, focused on the design of a blended module on the Special Theory of Relativity (STR), aimed at highlighting how the theory was historically born from a special interaction, a "co-evolution" between mathematics and physics (Tzanakis, 2016). This co-evolution was sought, and highlighted, through the analysis of the four founding articles of STR: *Electromagnetic phenomena in a system moving with any velocity smaller than that of light*, Lorentz (1904); *Sur la dynamique de l'électron* [*On the Dynamics of the Electron*], Poincaré (1906); *On the electrodynamics of moving bodies*, Einstein (1905) and *Space and Time*, Minkowski (1908). For the analysis of these articles, Lorenzo Miani used the metaphor of the 'boundary', set out in Akkerman and Bakker's (2011) meta-theory, referring to the boundary between Mathematics and Physics.

My thesis arose from the need to make the analysis more detailed, trying to construct an operational tool aimed to recognise, in physics texts, the type of interdisciplinarity they, eventually, implement. This opens the door to the design of useful classroom activities, with students involved in pre-service teacher training, but also with high school students, in order to develop interdisciplinary sensitivity and skills. Moreover, an analysis of this kind can make a considerable contribution to the *Justification Problem*, intercepting the possibility of investigating the identity of Mathematics as a discipline.

Therefore, in the second chapter an exploration is made of the following themes in the Mathematics Education: the *Justification Problem*; the role of the history of mathematics in the training of future teachers; the theme of the nature of mathematics; the theme of disciplinary identity; the theme of interdisciplinarity; the importance of original sources; the concept of metadiscourse. The study was conducted in order to answer this question: why, despite the fact that research in the Mathematics Education has led to a vast number of useful strategies and materials for teaching the discipline, students continue to ask "what is mathematics for?".

The idea is that, in addition to responding to the increasing demands of modernity, interdisciplinarity provides an opportunity to compare attitudes towards different disciplines. In the chapter, therefore, the definition of interdisciplinarity chosen to conduct the study is made explicit. This implies that the disciplinary dimension is not renounced, but on the contrary, disciplinary identities are valorised in order to compare, integrate and complement different knowledge areas.

Chapter three describes Akkerman and Bakker's framework (*Boundary Crossing and Boundary Objects*, 2011), in which the boundary metaphor is elaborated to describe interdisciplinarity in terms of the exchange of 'boundary objects' and in terms of 'boundary crossing mechanisms' between different disciplinary fields. After presenting it, the chapter describes how an operational tool of 'fine-grained' analysis of original articles was elaborated to extract, from the texts, the relationship between mathematics and physics. Akkerman and Bakker identify four boundary crossing mechanisms: *Identification*, *Coordination*, *Reflection* and *Transformation*. Within each of these they define processes through which the mechanisms are implemented. The analysis grid constructed in the thesis includes:

(i) operational definitions of the processes;
(ii) sample sentences illustrating each process;
(iii) words serving as markers in the text.

The fourth chapter presents the application of the grid in the analysis of articles by Lorentz, Poincaré, Einstein and Minkowski. Through the application, it was possible to shed light on the boundary crossing mechanisms that the four authors adopt throughout their articles and, in a second, finer analysis, the way in which the Lorentz Transformations are derived and treated by each of them. This type of analysis allowed us to understand the 'boundary styles' of each author, and the nature of Lorentz Transformations as boundary objects.

The analysis also led to epistemological reflections on the identities of mathematics and physics as disciplines. The results and their discussion are set out in chapter five.

In chapter six, the design of an activity for pre-service teacher education is illustrated. The activity takes the form of a tutorial for group work, constructed to guide future teachers to analyse excerpts from the original texts and recognise in them the different approaches to interdisciplinarity, as well as the different meanings attributed to the Lorentz Transformations. The activity was tested in the Physics Education course at the University of Bologna, held by Professor Olivia Levrini and attended by both physics and mathematics master students. Thanks to the activity, it was possible to reflect on disciplinary identities and the importance of boundary experiences to overcome stereotypes.

Chapter seven is the chapter of conclusions. The tool elaborated in the thesis is open to future developments since it can be used for the analysis of a wide variety of texts and for the construction of "boundary zones", which are increasingly desired and encouraged in European reports.

Italian introduction

Questa tesi fa parte del progetto Erasmus + IDENTITIES, il cui obiettivo è sviluppare materiali didattici interdisciplinari per la formazione degli insegnanti pre-servizio. Nello specifico, dà seguito alla ricerca condotta da Lorenzo Miani, nella sua tesi di laurea magistrale in Fisica, incentrata sulla progettazione di un modulo blended sulla Teoria della Relatività Speciale (STR), volto a evidenziare come la teoria sia storicamente nata da una particolare interazione, una "co-evoluzione" tra matematica e fisica (Tzanakis, 2016). Questa co-evoluzione è stata ricercata, ed evidenziata, attraverso l'analisi dei quattro articoli fondanti della STR: *Electromagnetic phenomena in a system moving with any velocity smaller than that of light*, Lorentz (1904); *Sur la dynamique de l'électron* [*On the Dynamics of the Electron*], Poincaré (1906); *On the electrodynamics of moving bodies*, Einstein (1905) e *Space and Time*, Minkowski (1908). Per l'analisi di questi articoli, Lorenzo Miani ha utilizzato la metafora del "confine", esposta nella meta-teoria di Akkerman e Bakker (2011), riferendosi al confine tra matematica e fisica.

La mia tesi è nata dall'esigenza di rendere più dettagliata l'analisi, cercando di costruire uno strumento operativo volto a riconoscere, nei testi di fisica, il tipo di interdisciplinarità che essi, eventualmente, mettono in atto. Questo apre le porte alla progettazione di attività utili in classe, con gli studenti impegnati nella formazione degli insegnanti pre-servizio, ma anche con gli studenti delle scuole superiori, al fine di sviluppare sensibilità e competenze interdisciplinari. Inoltre, un'analisi di questo tipo può dare un notevole contributo al *Justification Problem*, intercettando la possibilità di indagare sull'identità della Matematica, intesa come disciplina.

Dunque, nel secondo capitolo è riportato un approfondimento sui seguenti temi della Didattica della Matematica: il *Justification Problem*; il ruolo della storia della matematica nella formazione dei futuri docenti; il tema della natura della matematica; il tema dell'identità disciplinare; il tema dell'interdisciplinarità; l'importanza delle fonti originali; il concetto di metadiscorso. L'analisi è stata condotta per rispondere a questa domanda: come mai, nonostante la ricerca in Didattica della Matematica abbia portato ad una grande quantità di strategie e materiali utili all'insegnamento della disciplina, gli studenti continuano a chiedersi "a cosa serve la matematica?". L'idea è che l'interdisciplinarità, oltre a rispondere alle richieste sempre più frequenti della modernità, dia l'occasione per fare un confronto di atteggiamenti verso diverse discipline. Nel capitolo, quindi, viene esplicitata la definizione di interdisciplinarità scelta per condurre lo studio. Questa implica che non si rinunci alla dimensione disciplinare, ma al contrario si valorizzino le identità disciplinari per confrontare, integrare e rendere complementari diverse aree del sapere.

Nel capitolo tre è descritto il framework di Akkerman e Bakker (Boundary Crossing and Boundary Objects, 2011), nel quale si elabora la metafora del confine per descrivere l'interdisciplinarità in termini di scambio, tra i diversi ambiti disciplinari, di "boundary objects" e in termini di "boundary crossing mechanisms". Dopo averlo presentato, nel capitolo si descrive come sia stato elaborato uno strumento operativo di analisi "fine-grained" di articoli originali per estrarne il rapporto tra matematica e fisica. Akkerman e Bakker identificano infatti quattro meccanismi di attraversamento dei confini: *Identification, Coordination, Reflection* e *Transformation*. All'interno di ognuno di questi definiscono i processi attraverso i quali i meccanismi vengono implementati. La griglia di analisi costruita nella tesi include:

(i) definizioni operative dei processi;
(ii) esempi di frasi che illustrano ogni processo;
(iii) parole che fungono da marker nel testo.

Nel quarto capitolo si presenta l'applicazione della griglia nell'analisi degli articoli di Lorentz, Poincaré, Einstein e Minkowski. Attraverso l'applicazione è stato possibile mettere in luce i meccanismi di attraversamento del confine che i quattro autori adottano lungo i loro articoli e, ed una seconda analisi più fine, il modo in cui le Trasformazioni di Lorentz vengono ricavate e trattate da ognuno di loro. Questo tipo di analisi ha permesso di comprendere gli "stili al confine" di ogni autore, e la natura delle Trasformazioni di Lorentz in quanto oggetto di confine.

L'analisi ha altresì condotto a riflessioni epistemologiche sulle identità di matematica e fisica in quanto discipline. I risultati e la loro discussione sono esposti nel capitolo cinque.

Nel sesto capitolo è illustrata la progettazione di un'attività per la formazione degli insegnanti pre-servizio. L'attività si configura come un tutorial per lavori di gruppo, costruito per guidare i futuri insegnanti ad analizzare estratti dai testi originali e riconoscere in essi i diversi approcci alla interdisciplinarità, nonché i diversi significati attribuiti alle Trasformazioni di Lorentz. L'attività è stata sperimentata nel corso di Didattica della Fisica dell'Università di Bologna, tenuto dalla Professoressa Olivia Levrini, frequentato sia da studenti magistrali sia di fisica sia di matematica. Grazie all'attività, è stato possibile riflettere sulle identità disciplinari e sull'importanza di fare esperienze di confine per superare stereotipi.

Il capitolo sette è quello delle conclusioni. Lo strumento elaborato nella tesi si apre a sviluppi futuri, dal momento che si presta ad essere utilizzato per l'analisi di una grande varietà di testi e per la costruzione di "boundary zone", sempre più auspicate e incentivate nei report europei.

CHAPTER 2 – OVERVIEW

The purpose of this chapter is to show what our main references are, what interest mathematics education has in the study of interdisciplinarity and why, what do we mean by interdisciplinarity, through which concepts we must necessarily go to have clear ideas, why we decided to work on the original papers on the Special Theory of Relativity. Finally, we summarise the work done so far in the context of the IDENTITIES project.

2.1 Mathematics education towards Interdisciplinarity

Justification problem

In Denmark in the year 2000, the Ministry of Education and other official bodies appointed a committee to conduct a project to address various problems and challenges concerning the teaching and learning of mathematics. This is how the ‘KOM project’ (Kompetencer Og Matematiklæring – in Danish – stands for “Competencies and the Learning of Mathematics”) was born. The terms of reference for the project were formulated by means of a series of questions:

- To what extent is there a need for innovation of the prevalent forms of mathematics education?
- Which mathematical competencies need to be developed with students at different stages of the education system?
- How do we ensure progression and coherence in mathematics teaching and learning throughout the education system?
- How do we measure mathematical competence?
- What should be the content of up-to-date mathematics curricula?
- How do we ensure the ongoing development of mathematics as an education subject as well as of its teaching?
- What does society demand and expect of mathematics teaching and learning?
- What will mathematical teaching materials look like in the future?
- How can we, in Denmark, make use of international experiences with mathematics teaching?
- How should mathematics teaching be organised in the future?

The chair of the committee was Mogens Niss, he made it clear that the Danish KOM project was not conceived as a research project in the traditional sense (Niss, 2003). The idea was not to answer all the complex questions posed for reference. Rather, its task was to produce a thoughtful analysis of the issue, to make recommendations for the reform of mathematics education in Denmark and to provide ideas and insights for the further development of mathematics teaching and learning in Denmark. During the project, a new concept emerged that also turned out to be of interest outside Denmark: *the justification problem*. This concept manifests itself on two levels: a social one and an individual one. The important social aspect is that society seems to increasingly need people with good mathematical skills (e.g., technological development, the need to use information technology, the increasingly central role of mathematical models for finance). Here, however, lies the so-called "relevance paradox": at the individual level, difficulties in mathematics are common and the love for it is felt by very few. Most people ask: "what is the point of mathematics?". To address this question, the committee members identified several topics for discussion during the KOM project: first, should mathematics really be taught to everyone? There are different currents of thought regarding the answer to this question; what kind of preparation should mathematics teachers have? How to solve the problem of students' transitions from one school grade to the next? Last, how to give students a coherent idea of the identity of mathematics?

So, to provide some sort of solution, even if only partial, the KOM project went on to say that to teach mathematics we need mathematical competence. In particular, they identified and defined eight different competences, divided into two groups. The first group of competencies are to do with the ability to *ask and answer questions in and with mathematics*:

1. Thinking mathematically (mastering mathematical modes of thought)
2. Posing and solving mathematical problems
3. Modelling mathematically (i.e., analysing and building models)
4. Reasoning mathematically

The other group of competencies are to do with the ability to deal with and *manage mathematical language and tools*:

5. Representing mathematical entities (objects and situations)
6. Handling mathematical symbols and formalisms
7. Communicating in, with, and about mathematics
8. Making use of aids and tools (IT included)

All these eight competencies concern mental or physical processes, activities, and behaviours. In other words, the focus is on what individuals can do. The feeling, however, is that somehow, we are skirting around the issue. Students keep asking "why do I have to study mathematics?". Giorgio Bolondi and Bruno D'Amore (2021) in "La matematica non serve a nulla [Mathematics is useless]", briefly discuss the answers that have been given to this question over time:

"There are those who have said that mathematics is a science which is cultivated *only for the honour of the human spirit*, and therefore we care little that it "serves" anything." (Bolondi & D'Amore, 2021)

In fact, it must be said that, among the many answers from illustrious figures that the authors quote, many try to explain *what* mathematics is, not why it is useful. It is therefore natural to think that perhaps this question should be answered first. If we search online for "what is mathematics?" we are very likely to run into Courant and Robbins' classic: "What is Mathematics? An Elementary Approach to Ideas and Methods" (1996). To describe this volume, we use the words Reuben Hersh (1997) uses in the preface of his book "What is Mathematics, Really?":

"They never answered their question; or rather, they answered it by *showing* what mathematics is, not by telling what it is. After devouring the book with wonder and delight, I was still left asking, 'But what is mathematics, really?'" (Hersh, 1997).

It is clearly not a simple matter to respond to. It is easy that in attempting to do this, one ends up in a discussion with a very philosophical character. However, the philosophy of mathematics is not without its currents of thought and complications. We may often come across books entitled "Philosophy of Mathematics", or similar;

the problem is that they frequently deal with mathematical logic. These kinds of volumes do not meet our needs. It is not our intention to get into the conflict between 'positivists' and 'empiricists', or between Platonism and formalism. We do not want to ask the question of what notion of existence to adopt, or whether mathematical objects really exist or not. Our focus remains on the context of mathematics education.

In his book, Hersh recognises among the causes of failure in mathematics education the misconception of the nature of mathematics. Its work doesn't propose teaching strategies; however, he argues that a credible philosophy of mathematics must accord with the experiences of teaching and learning mathematics. The author strongly defends the idea that mathematics must be understood as a human activity, a social phenomenon, part of human culture, evolved historically, and only intelligible in a social context. He discusses at length what he calls criteria for the philosophy of mathematics, identifying three essential ones:

1. To recognize the scope and variety of mathematics;
2. To fit into general epistemology and philosophy of science;
3. To be compatible with mathematical practice-research, application, teaching, history, calculation, and mathematical intuition.

It is clear that the question "what is mathematics?" requires a long, complex and multi-layered answer. We cannot give such comprehensive treatment. For something close to that, we recommend Hersh's book. However, a contribution can be made.

Let us return to Niss and the KOM project, because in his presentation article we find just before the conclusions an interesting reflection. In addition to identifying and defining the eight mathematical competences, the committee found that it was essential to focus on mathematics as a discipline. More specifically, they have identified three kinds of overview and judgement regarding mathematics as a discipline that students should develop throughout their study of mathematics. These overviews and judgements concern:

- the *actual application* of mathematics in other subjects and fields of practice, of scientific or societal significance;
- the *historical development* of mathematics, internally as well as externally;
- the special *nature of mathematics* as a discipline.

In this work we will try to go throughout these aims, but to do so we have to explain first what we intend for discipline.

Disciplinary identity

The focus of this work is interdisciplinarity. Interdisciplinarity has become increasingly central to scientific debates in recent years. Regarding this there are different opinions, some find that interdisciplinarity is the future, others that there is a risk of losing disciplinary identities. To explore it and address it, it is necessary to focus on what is a discipline, and in particular, what defines a discipline. To do so, we analyse the work from Krishnan (2009) "What are Academic Disciplines?", in which the author focuses on the problem of defining disciplines. The author starts from the etymological origin of the term 'discipline':

"The term 'discipline' originates from the Latin words *discipulus*, which means pupil, and *disciplina*, which means teaching (noun). […] A dictionary definition will give a whole range of quite different meanings of the term from training to submission to an authority to the control and self-control of behaviour." (Krishnan, 2009, p. 8).

The author restricts the field to academic disciplines and the criteria they should meet to be considered as such: research object, a body of accumulated specialist knowledge, theories and concepts, specific language, methodologies, institutionalism. The main interest in this work, however, is to find a way to distinguish disciplines from each other, especially "because of" interdisciplinarity. Thus, the concept of disciplinarity is approached from different paradigmatic angles: from the philosophical perspective, from the anthropological, sociological, historical perspective, from the perspective of market and organisation, and finally from the perspective of teaching and learning. These are all dimensions through which a discipline can find a definition, recognise itself, but also find new stimuli and seek interdisciplinarity. In summary, Krishnan argues that academic disciplines are under attack from many sides, and that some of them are in serious danger of being wiped out. He suggests three strategies of resistance: retreat and reinforce identity and boundaries; move closer to a stronger discipline and form a strategic alliance; or finally, reconstitute within a newer and broader field of study with the aim of dominating the new discourse. There is the possibility of adopting more than one at different times, but each has its advantages and disadvantages. Of course, he goes on to point out that with the latter two, one risks losing identity, but interestingly, he also recognises some risk in adopting the former:

"The great danger of this strategy is that the discipline loses touch with its societal and science environment and thus just speeds up its own obsolescence and irrelevance […]" (Krishnan, 2009, p. 48).

Therefore, a discipline may become increasingly irrelevant in the course of time, perhaps even to the point of disappearing in the future. However, it is understood that with regard to the reinforcement of identity, just considering the philosophical/historical perspective, the problem could also be the past. It is recognised that long ago there was only one knowledge:

"In Ancient times education and philosophy was interdisciplinary (or rather predisciplinary) in the sense that philosophers did not accept any boundaries or limitations to the validity of the truths they uncovered by the way of thinking." (Krishnan, 2009, p. 13).

Thus, it may have happened that some discipline lost some piece of identity. Therefore, is the answer we are looking for in the deep study of history and philosophy? Or would it be better to give up the idea of understanding the nature of mathematics?

Sibel Erduran and Zoubeida R. Dagher (2014) proposed a way to define the identity of a discipline without giving up the teaching of nature of science, aware that this is a goal fraught with controversy. Indeed, their aim has been to extend the definition of 'science' to include the epistemic, cognitive and social aspects of science in a coordinated manner, so that a wider range of students can potentially be involved in science. The authors started from the Family Resemblance Approach (FRA) proposed by the philosophers of science Gurol Irzik and Robert Nola during the plenary lecture at the *International History and Philosophy in Science Teaching [IHPST]* conference held in Thessaloniki, Greece, in 2011. They expanded the framework and added more categories to it. Irzik and Nola's (2014) version of FRA is illustrated in figure 1. Erduran & Dagher propose a visual tool to show, at a glance, how all components of the cognitive-epistemic and socio-institutional systems interact with each other, enhancing or influencing scientific activity.

Figure 1 - FRA wheel (Erduran & Dagher, 2014, p. 28)

"The family resemblance model of nature of science conceptualizes science in terms of a cognitive-epistemic and a social-institutional system. […] Science as a cognitive-epistemic system encompasses processes of inquiry, aims and values, methods and methodological rules, and scientific knowledge, while science as a social-institutional system encompasses professional activities, scientific ethos, social certification and dissemination of scientific knowledge, and social values." (Erduran & Dagher, 2014, p. 20).

Irzik and Nola begin the articulation of the FRA with reference to processes of inquiry followed by aims and values, and so on. Erduran & Dagher deemed it more appropriate to start the articulation and extension of the framework by focusing on the aims and values of science. They also have extended FRA including three additional categories: “social organizations and interactions”, “political power structures” and “financial systems”. Since 2014, the framework has been developed, analysed and expanded by many. It has been used in many different ways: analysis of curriculum documents, analysis of textbooks and teacher training. Through these empirical studies, gaps in the design of some curriculum documents have been identified, and its potential as a pedagogical strategy in shaping the content of teacher training has emerged.

This tool therefore seems to work well in many ways, perhaps even to redefine the boundaries of a particular discipline. It must be said, however, that underlying the framework is the idea of a family resemblance to science, based on the understanding that all scientific disciplines share certain characteristics.
We can say that this fact is particularly present within the domain of STEM disciplines (Science, Technology, Engineering and Mathematics).

Interdisciplinarity

The growing interest in STEM disciplines arose from around the 1990s onwards. This growing interest it is mainly due to the growing demand for people with special skills and competences from the world of industry (Martín-Páez, Aguilera, Perales-Palacios, Vílchez-González, 2018). Over time, these societal needs have increasingly led to the development of a true STEM identity. The academic world has moved accordingly, meeting this growing interest. Therefore, the field of education has also had to take an active involvement. And within this very sphere, attention to interdisciplinarity is particularly alive. The belief is that students must be prepared to face the challenges of contemporary society (e.g., climate change, artificial intelligence, nanotechnology) that are significantly influencing their perception of the future and their role in current and future society. These challenges require deep interdisciplinary preparation and a profound redefinition of traditional disciplinary teaching (Branchetti & Levrini, 2019). Indeed, as was also argued in Krishnan’s work, especially in Europe and USA, interdisciplinarity is seen as a desirable concept to develop, both in terms of research, and in terms of education. So, interdisciplinarity is a central concept, connected to the needs of society, the world of education and the concept of disciplinary identity. However, it must be made clear what we mean by interdisciplinarity.

The taxonomy about interdisciplinarity is rich and variegated. Julie Thompson Klein proposes a review in her 2010 paper: "A taxonomy of interdisciplinarity". Around the 1970s, the Organisation for Economic Co-operation and Development (OECD) published the first interdisciplinary typology, and from then on, there have been many different definitions, classifications and characterisations on the subject. The most general classification on the subject is widely accepted:

- When the approach is to juxtapose disciplines, we talk about *multidisciplinarity*. This is most frequently the case in school curricula. The relationship between disciplines seems loose and restricted.
- *Interdisciplinarity* is described as a more interactive and proactive process than multidisciplinarity. It concerns integrated projects, restructuring existing approaches through explicit *focus* and *blending*, *linking* issues and questions that are not specific to individual disciplines.
- *Transdisciplinarity* was defined as a common system of axioms that transcends the narrow scope of disciplinary worldviews through an overarching synthesis.

These three approaches differ from each other according to the level of interaction and integration between the disciplines involved. For example, regarding the first two, we talk of 'complementing', while for the third, we talk of 'hybridizing'. There are then internal classifications within each of these three categories depending on the scope of the approach, or on what is shared, whether methodologies, problems, epistemologies, or history. The author stresses the point that care was taken to distinguish those that may be significant collaborations from those that are not. For example, an "intrinsic" interdisciplinarity is said to exist when analytical tools such as mathematical computer simulation models are shared. Instead:

"In Integrated ID, which Boden pronounces 'the only true interdisciplinarity', the concepts and insights of one discipline contribute to the problems and theories of another [...]. Individuals may find their original disciplinary methods and theoretical concepts modified as a result of cooperation, fostering new conceptual categories and methodological unification." (Thompson Klein, 2010, p. 20).

Clearly, the boundaries between multidisciplinarity, interdisciplinarity and transdisciplinarity are not clear-cut. However, for our purposes we adopt the concept of interdisciplinarity, because:

"[…] an interdisciplinary learning approach integrates the disciplines and diffuses their limits, passing through different levels of cognitive ability in pursuit of developing a holistic thought process. In this manner, students can make meaningful connections that allow them to process knowledge to produce an interdisciplinary understanding that is applicable to reality" (Martín-Páez et al., 2018, p. 802).

Now the question naturally is: how can STEM issues, and thus interdisciplinarity, help to develop and clarify the identity of mathematics as a discipline?

At the ESERA2019 conference Laura Branchetti and Olivia Levrini had a presentation entitled "Disciplines and interdisciplinarity in STEM education to foster scientific authenticity and develop epistemic skills". They argued that epistemic skills can be formed more effectively in a comparative and interdisciplinary perspective: that is, if different disciplines are compared and if both specific and transversal skills are emphasised. In the presentation, it is explained that two examples of genuine STEM (interdisciplinary) issues were proposed to university students. The first concerns the need to cross the boundaries between physics and mathematics in order to understand the nature and significance of the quantum breakthrough induced by the black-body radiation problem that baffled scientists at the end of the 19th century. The second issue concerns the teaching, at secondary school level, of STEM topics such as climate change and artificial intelligence. The interdisciplinary approach allowed students to grasp the disciplinary turn and reflect on what kinds of knowledge and epistemic practices (ways of reasoning, representing, modelling, arguing, communicating) characterise a mathematical and a physical approach.

Thus, in this thesis we argue that the question "what is the nature of mathematics as a discipline?" can be answered, at least partially, by conducting a good analysis of the boundaries between mathematics and other disciplines. We think the boundary between mathematics and physics is particularly interesting. Generally, these two disciplines are taught as if they live on two different planets, leading to different learning problems for students with respect to both.

Problems also arise in terms of image and idea that students (even university students) have of the two disciplines. Usually, when one thinks about the connection between mathematics and physics, one mostly remembers the practice of modelling. Since there are different types of models in science, and in science education, it is worth explaining further what we mean by a model. In the most general sense, a model is a representation of a phenomenon, system, object, or idea through something familiar and known (or not). Funda Ornek, in his 2008 paper "Models in Science Education: Applications of Models in Learning and Teaching Science", proposes a literature-based categorisation of the different types of models. She distinguishes mental models from conceptual models. The formers are defined as psychological representations of real or imaginary situations. Whereas conceptual models are external representations that are shared by a given community and have their coherence with the scientific knowledge of that community. These external representations can be mathematical formulations, analogies, graphs, or material objects (Ornek, 2008). Within this type of models, she places mathematical models, computer models, physical models, and physics models. The definitions of mathematical model and physics model, and how they are realised, are particularly interesting:

"A mathematical model is the use of mathematical language to describe the behaviour of a system. That is, it is a description or summarization of important features of a real-world system or phenomenon in terms of symbols, equations, and numbers." (Ornek, 2008, p. 37).

Moreover, the modelling process consists of a series of steps in and out of the mathematical world:

1. Formulate real problem.

2. Assumptions for model.
3. Formulate mathematical problem.
4. Solve mathematical problem.
5. Interpret solution.
6. Validate model.
7. Use model to explain, predict, decide, design.

By physics model, on the other hand, we mean:

"A physics model in the physics-education community is considered as a simplified and idealized physical system, phenomenon, or idealization. Also, a mathematical model can be a component of a physics model." (Ornek, 2008, p. 42).

In addition, in modelling, the following processes are followed:

- Start from fundamental principles
- Estimate quantities
- Make assumptions and approximations
- Decide how to model the system
- Explain / predict a real physical phenomenon in the system
- Evaluate the explanation or prediction

Therefore, these two models are defined in two different ways, suggesting disciplinary affiliation, although they are two very similar ways of approaching problems, one may include the other. All this certainly has its value and validity, but it is only one part of the relationship between mathematics and physics.

The history and development of these two sciences are closely interconnected. The emergence of a real dividing line started no more than 100 years ago. Tzanakis (2016) argues that:

"[…] putting emphasis on integrating historical and epistemological issues in the teaching and learning of mathematics and physics constitutes a possible natural way for exposing them in the making that may lead to a better understanding of their specific parts, and to a deeper awareness of what they are as disciplines." (Tzanakis, 2016, p. 3).

These two disciplines have many commonalities in terms of epistemology. The author illustrates how this is not just a comfortable collaboration, but a strong interconnection. For physics, mathematics is not merely the most convenient language to adopt; for mathematics, physics is not merely a source of contexts in which to find application. There are several typical examples in history of how the development of mathematics and physics has been parallel, and how one has provided precious stimuli to the other: Infinitesimal calculus and classical mechanics in the 17th century; Riemannian geometry and tensor calculus as the indispensable framework for Einstein's formulation of General Relativity; Dirac's introduction and use of his δ-function in quantum mechanics as a key initial step for the development of distribution theory; and many others. It is a historically-evidenced mutual process of dialectical interplay that seems to be based on a deeper epistemological affinity of the two disciplines (Tzanakis, 2016). Because of this bidirectional relationship, the interest in interdisciplinarity between the two disciplines seems legitimate. The belief is that from a good analysis of the boundaries between mathematics and physics, something meaningful can emerge about the individual identities of the two. Tzanakis illustrates three examples to support his theses on the importance of the integration of history, epistemological similarities and the interconnection between mathematics and physics:

1. Measuring the distance of inaccessible objects.
2. Rotations, Space-Time and the Special Theory of Relativity.
3. Differential Equations, (Functional) Analysis and Quantum Mechanics.

The first reveals the strong cultural significance of mathematical thinking and its applications to important real-world problems throughout history; the third deals with subjects that are usually treated separately, but which are historically strongly interconnected and have motivated, stimulated, and guided the course of development to their present form.

Regarding the second example, Tzanakis argues that it can be an excellent argument to show a concrete application of linear algebra topics (matrices, vectors, groups, etc.), which usually seem to live in total abstraction. He also points out that it is an argument that helps to answer the question “how did we get here?”. Indeed, we go on to show how fruitful can be the choice of treating, analysing and/or presenting the Special Theory of Relativity.

2.2 Why the original papers on Special Theory of Relativity?

<u>The importance of original sources</u>

First of all, it must be said that the study of original documents has value in its own right for science education. It has been made very evident over the years that the integration of history and epistemology in the teaching of mathematics and physics is a fruitful element for learning. One very direct way of dealing with epistemology and history of science is to engage with historical documents. In “The use of original sources in the mathematics classroom” from 2002 by Hans Niels Jahnke et al. we read:

“In principle, the aims and effects which might be pursued by way of an original source will not be different from those attained by other types of historical activities. However, there are three general ideas which might best be suited for describing the special effects of studying a source. These are the notions of *replacement*, *reorientation* and *cultural understanding*” (Jahnke, 2002, p. 291).

What the author means to say is that there is an opportunity to develop in students an idea of mathematics that respects its nature as an intellectual activity, historically evolved and that goes beyond what is generally taught. Moreover, primary sources offer a non (or at least few) mediated contact with the way ideas were defined at a certain time, by their authors directly.

“The source then becomes an interlocutor to be interpreted, to be questioned, to be answered and to be argued with.” (Jahnke, 2002, p. 296).

Moreover, the analysis of original texts turns out to be a fundamental activity for teacher education. It contributes to disciplinary, historical and epistemological preparation, and also brings up interesting debates at the cognitive level.

In addition, the feeling is that there is no acknowledgement that science, too, has its own narrative strategies and linguistic peculiarities. They tend to be neglected at school and university until research education, this fact contributes to stereotypes about what science is and how it operates. In this regard, the concept of metadiscourse, born in the field of applied linguistics, is helpful:

“Metadiscourse is self-reflective linguistic material referring to the evolving text and to the writer and imagined reader of that text. It is based on a view of writing as social engagement and in academic contexts reveals the ways that writers project themselves into their discourse to signal their attitude towards both the propositional content and the audience of the text. [...]” (Hyland, Tse, 2004, p.156).

So, the point is that a text is not only its content, but it also transmits the implicit beliefs, ideas, and goals of the writers who, in turn, take the readers into account. And in this sense, we are interested in metadiscourse, because it is interpersonal, it considers the reader’s knowledge, textual experiences and processing needs and it provides writers with an armoury of rhetorical appeals.

The concept of metadiscourse has interested many researchers in the humanities, of both social constructionist and functional orientation to discourse, but also researchers interested by the possibility of finding patterns of interaction across texts. British linguist Ken Hyland has long been involved in the study of discourse, with a particular focus on academic content in which a relationship between author identity and disciplinarity is discernible. And speaking of disciplines, in the article co-written with Tse in 2004 “Metadiscourse in Academic Writing: A Reappraisal”, we read:

“[...] disciplines are not only distinguished by their objects of study. The fact that academics actively engage in knowledge construction as members of professional groups means that their decisions concerning how propositional information should be presented are crucial. It is these decisions which socially ground their discourses, connecting them to the broad inquiry patterns and knowledge structures of their disciplines and revealing something of the way academic communities understand the things they investigate and conceptualize appropriate writer-reader interactions. [...]" (Hyland, Tse, 2004, p.174).

Hyland draws on several sources to explore how authors convey aspects of their identities within the constraints placed upon them by their disciplines' rhetorical conventions. In his 2012 book "Disciplinary Identities: Individuality and Community in Academic Discourse", he demonstrates the effectiveness of keyword and collocation analysis in highlighting both the norms of a particular genre and the author's idiosyncratic choices. Thus, it is possible to identify in a text, by a reader with a specific background knowledge, both traces of the identities of the disciplines involved and the underlying intentions of the writer, in this sense Hyland refers to the concept of "voice" (Hyland, 2008).

The concept of metadiscourse, and the possibility of detecting and analysing it in academic texts, has attracted many researchers, although it is an under-theorised and empirically vague concept. There are no simple linguistic criteria for identifying it because, not only is it an open category to which new items can be added to fit the writer's needs, but the items themselves can act as metadiscourse detectors in some parts of the text and in others not (Hyland, Tse, 2004, p. 158). However:

"A classification schema nevertheless performs a valuable role. Not only does it help reveal the functions that writers perform, but it also provides a means of comparing generic practices and exploring the rhetorical preferences of different discourse communities. [...]" (Hyland, Tse, 2004, p.175).

Adriano Demattè, in "Documenti storici per la didattica della matematica/1" of 2008, also argues that, as modern readers, we can derive from the original documents both a conscious intention of the author regarding the communication of mathematical knowledge and a range of extra-mathematical directions that contribute to a wider perspective. Indeed, the analysis of a historical document generally requires us to broaden our thinking by integrating points of view from other disciplines. It is therefore a more operational, and thus comprehensive, way of deepening themes of the history and epistemology of mathematics, which ensures that it cannot be confined to pure narrative. Essentially the same applies in the field of physics education.

The historical/epistemological and interdisciplinary value of the Special Theory of Relativity

The Special Theory of Relativity (STR) has had a great deal of attention as a scientific theory, both at the academic research level and at the societal level. It was a big step in scientific thinking and the first real step towards modern physics, for example Giannetto (2009) talks about revolution in XX century physics induced by relativity theories. It is therefore a topic whose analysis and discussion can always lead to interesting reflections.

STR is treated in university physics courses and also in mathematics courses, although in less depth and with different approaches. Moreover, it is present in the curricula of some secondary school courses, so it happens to be addressed by high school students too. However, the difficulties that students of all levels address when approaching the Special Theory of Relativity are more than well documented (see for example Scherr et al., 2002). It is a subject that has been used to develop, compare and discuss approaches to conceptual change, like the classical approach of Posner, Strike, Henson and Gerzog and the coordination class theory.

Lorenzo Miani, in his master’s degree theses, has done a remarkable structured review of studies on the teaching and learning of the Special Theory of Relativity in high schools and universities, within the framework of Physical Education Research (PER) and Mathematics Education Research (MER). The aim was to identify what are the main research topics, research methodologies and techniques related to the STR in PER and MER, what are the main difficulties encountered by students and teachers in dealing with the topic, and what are the least considered aspects of this topic (Miani, 2021).

The investigation led Miani to select several articles within the framework of Physics Education Research, in the end he identified 49 papers that met his criteria (e.g., the fact that they addressed the Special Theory of Relativity from an educational perspective). In this case, he was able to categorise them according to their focus:

- 14 articles are on the topic of Students' Difficulties.
- 13 articles are on the topic of Digital Tools Development.
- 8 articles are on the topic of History and Epistemology.
- 8 articles are on the topic of Curricula Development.
- 7 articles are on the topic of Conceptual Change.
- 6 articles are on the topic of In/Pre-Service Teachers Formation.
- 2 articles are on the topic of Interdisciplinarity.

Instead, the small number (6) of articles found in Mathematics Education Research did not allow such a detailed categorisation. Five can be described as focusing on the role of the Group of Transformations in the Special Theory of Relativity, while one seems to propose a narrative approach to the subject. Even from this first glance we can deduce that this is an under-explored topic in terms of Interdisciplinarity and In/Pre-Service Teachers Formation, especially from the perspective of Mathematics Education Research.

Moreover, STR represents one of the most illuminating examples of the intimate relationship between mathematics and physics (Galili, 2018). During the nineteenth century, optics and electricity underwent a slow mathematisation. "A survey of physics literature in the years 1895-1905 shows that the electrodynamics of moving bodies then was a widely discussed topic" (Darrigol, 2006, p. 2). Many scientists became involved, both mathematicians and physicists. One of these was James Clerk Maxwell, responsible for the admirable theory of electromagnetism (a field theory). However, controversies were not absent. The substantial problem was the survival of the concept of the ether, and the impossibility of revealing its effects. Evidently Albert Einstein was interested by these difficulties. But people often make the mistake of believing that the Special Theory of Relativity was born "in a single stroke of genius" (Darrigol, 2006, p.1) by him alone. Even the books that are adopted in schools have the fault of favouring this myth that Einstein, strongly influenced by the results of the Michelson & Morley experiment, is the sole father of the theory. While historians and philosophers debate the genesis, dividing themselves between those who believe in this version and those who argue that the theory was already discovered by Hendrik Lorentz and Henri Poincaré (Levrini, 2014).

History therefore shows us how STR emerged from the need of many to solve problems in the theory of electromagnetism. For this reason, many mathematical formulae of Special Relativity (e.g., Lorenz transformations) were already known before Einstein wrote his 1905 paper. However, it must be said that he brought a new physical structure, based on a particular set of basic principles and concepts, to the theory. This indeed represented an incredible conceptual revolution of space and time (Galili, 2018). Moreover, the development of the Special Theory of Relativity did not stop with Einstein's 1905 article, especially regarding the ontological aspects of the theory. Hermann Minkowski, a mathematician by profession, proposed his vision in 1908, which had profound implications for the concept of space-time (Levrini, 1999).

One thing is sure: the question has been addressed by at least two physicists (Hendrik Lorentz and Albert Einstein) and at least two mathematicians (Henri Poincaré and Hermann Minkowski).

To summarise: the study of interdisciplinarity between mathematics and physics in the context of the original articles on the Special Theory of Relativity, begun by Lorenzo Miani, finds reason in Tzanakis's work, in the fact that STR is a subject that is treated at school and university, and that is a source of difficulties in learning, and conceptual change. What emerges in Miani's thesis is above all the fact that STR is an understudied topic from an interdisciplinary point of view. This thesis aims to refine the analysis of the texts in order to highlight more clearly the role that interdisciplinarity has played in the development of the Special Theory of Relativity.

2.3 The IDENTITIES project

This thesis has been developed within the project IDENTITIES (*Integrate Disciplines to Elaborate Novel Teaching approaches to InTerdisciplinarity and Innovate pre-service teacher Education for STEM challenges*). IDENTITIES is an ERASMUS + project coordinated by the research group in Physics Education of Bologna. In this project the partners assume that the search for the meaning of interdisciplinarity cannot ignore the meaning of disciplines and their epistemological identities. The project IDENTITIES started in September 2019 (https://identitiesproject.eu). Together with the group of Bologna there are other 4 different partners coming from Italy (University of Parma), France (University of Montpellier), Spain (University of Barcelona) and Greece (University of Crete). The main goal of the project is to build innovative and transferable teaching modules and courses to be used in contexts of pre-service teacher education (e.g., curricula in Physics Education, Mathematics Education or Computer Science Education within master's degree courses). The central theme of the modules is interdisciplinarity in STEM education, with a focus on the links and interweaving between physics, mathematics, and computer science. A core idea of the project IDENTITIES is that, in order to develop interdisciplinary pathways that cross boundaries between disciplines, it is essential to maintain and value the identities and peculiarities of each of them.

The 2019 article "Interplay between mathematics and physics to grasp the nature of a scientific breakthrough: the case of the black body" by Laura Branchetti, Alessia Cattabriga and Olivia Levrini, well represents the project's main theoretical references (Branchetti, Cattabriga, Levrini, 2019). This paper uses the model that Udhen, Karam, Pietrocola and Pospiech developed to highlight the interaction between physics and mathematics in the context of teaching and learning practices (Uhden et al., 2012) to analyse Planck's original articles. The analysis made it possible to reconstruct the author's reasoning, and revealed the structural role played by mathematics, which opened the door to the quantum scientific breakthrough. The results of the analysis also led to the design of a teaching tutorial that was implemented with undergraduate mathematics and physics students.

As a pilot activity of the IDENTITES project, between November-December 2019, a training course was organised for in-service secondary school teachers on the interdisciplinarity between mathematics and physics, on the topic of the parabola and parabolic motion. The course was held as part of the PLS (Piano Lauree Scientifiche) activities of the Physics and Mathematics Research Group of the University of Bologna, in collaboration with the POT (Piano Orientamento Tutorato) activities of the Humanities and other university departments. A presentation on this was also made by Levrini, Branchetti, Cattabriga, Moruzzi and Viale at the GEO-CRUI conference held from 15 to 17 June 2020.

As a pilot study, the course was the subject of various analyses and reflections in many dissertations within the IDENTITIES project (Gombi 2020, Quadrelli 2020, Sicignano 2020, Pollani 2021, Polverini 2022). In "The foundational case of the parabolic motion: design of an interdisciplinary activity for the IDENTITIES project" of 2020, by Gombi, a didactic activity is created, aimed at future teachers, on the topic of the parabola and parabolic motion. The activity was designed with the aim of guiding through the main passages that characterised, from an epistemological point of view, the evolution of physical thought from Tartaglia's theory of the motion of the projectile to the demonstration of the parabolic trajectory of Galileo's projectile. In other dissertations, the activities are analysed to investigate how the interdisciplinary dimension between mathematics and physics is perceived in two spheres: the historical-epistemological sphere and the school sphere. Or the reactions of high school students to interdisciplinary modules on the topic of the parabola are analysed. The thesis "Exploring interdisciplinarity between physics and mathematics: the design of a linguistic and an epistemological tool for analysing texts about the parabolic motion" of 2022 by Polverini, aims to contribute to the project by developing linguistic and epistemological tools that future teachers can use to explore and analyse both the disciplinary identity of physics and mathematics and their intertwining that emerges from high school textbooks. In particular, again, the focus was on the topic of parabolic motion.

However, not all dissertations have dealt with this issue. In "La mappa logistica come caso di studio per riflettere sull'interdisciplinarità nei sistemi complessi" of 2019, by Chiusoli, an analysis is conducted from a didactic perspective of the logistic map as a model of a complex system and an interdisciplinary reflection is developed from this, identifying the contributions that mathematics, physics and computer science make to understanding the model. And indeed, other studies within the IDENTITIES project also addressed the issue of complexity. Eleonora Barelli gave a presentation entitled “Le simulazioni computazionali come strumenti interdisciplinari di decisione: risultati di un’indagine con studenti universitari. [Computational simulations as interdisciplinary decision-making tools: results of a study with university students]” at the SIF conference which was held from 14 to 18 September 2020. It is an overview of an activity, which involved 50 Physics and Mathematics students enrolled in the Physics Education course, that took place between March and April 2020. The core phases consisted of: a lecture on the model of the spread of a virus; the analysis of a simulation on the dynamics of opinion (Axelrod model); a role-play activity for the analysis of a simulation on the formation of terrorist groups. The research question was: “which forms of reasoning do students put into play when discussing a complex societal problem through a simulation? In particular, which forms of reasoning are triggered by different elements of the simulation (data, graph, actions and scenarios, model)?” (Barelli & Levrini, 2021). Among other things, it emerged that the potential of simulations also lies in their interdisciplinary character involving a variety of forms of knowledge from different subject areas.

Another topic on which a module for secondary school students was developed is the topic of quantum technologies. Satanassi, Ercolessi and Levrini presented it in “Un approccio interdisciplinare alle tecnologie quantistiche: un modulo per studenti di scuola secondaria. [An interdisciplinary approach to quantum technologies: A module for secondary school students]” at the SIF conference held from 14 to 18 September 2020. The research questions were: “What can a multi/interdisciplinary approach to classical and quantum random walk highlight? How can it highlight the differences between classical and quantum logic? What picture of probability emerges?” (Satanassi, Ercolessi, Levrini, 2020).

The IDENTITIES partnership also participated in the discussion on interdisciplinarity within the ESERA community by presenting a symposium at the ESERA2021 Conference, held virtually from August 30th to September 3rd, entitled "Disciplinary Identities In Interdisciplinary Topics: Challenges and Opportunities for Teacher Education". In the paper, four research contributions are presented that refer to modules developed within the IDENTITIES project and already implemented in local and international contexts. Two studies focus on advanced, intrinsically interdisciplinary, STEM topics that are societally relevant but difficult to include in official curricula: coronavirus evolution and nanotechnologies. Other two focus on curricular themes that curricula and teaching tend to separate in different fields: cryptography and parabola and parabolic motion. The four studies also differ in their approach to didactics research: in the first two, instruments developed in the context of mathematics education are used, while in the other two, the instruments used were developed in the context of science education. The four perspectives presented in the paper lead to the delineation of the common features of the interdisciplinary approach that the IDENTITIES project pursues. First of all, the importance of founding interdisciplinary reflection on the identities of the disciplines involved. The belief is that disciplines are still fundamental to interdisciplinary education and to achieving productive interdisciplinarity. The focus on the preparation of future teachers is crucial in this regard. If the aim is to achieve a change in the way students learn, then it is necessary to start with teaching in pre-service teacher education programmes. It should also be mentioned that in all four modules presented, there is a submodule in which the concepts developed in the Akkerman and Bakker metatheory of 2011 are used. Indeed, in the IDENTITIES' conceptualization of interdisciplinarity, it is crucial the metaphor of *boundary*, borrowed from this metatheory. For instance, in the modules about cryptography, and parabola and parabolic motion, interdisciplinarity is used as a *lever* to uncover the epistemological cores of the disciplines by confronting them on a common *boundary theme*. Pre-service student teachers were stimulated to conduct several reflections on the concepts of *boundary object* and *boundary crossing mechanism*, which belong to the Akkerman and Bakker framework. This allowed them to be guided in the passage from separate disciplines to interdisciplinarity and so they recognised the types of *interaction between disciplines* that occur in interdisciplinary contexts.

The Akkerman and Bakker framework also plays a key role in Lorenzo Miani's previously mentioned master's degree thesis from 2021: "Highlighting Interdisciplinarity between Physics and Mathematics in Historical Papers on Special Relativity: Design of Blended Activities for Pre-Service Teacher Education", part of the IDENTITIES project too. As already mentioned, this dissertation arose from the need to develop his work. Therefore, in the next chapter, the Akkerman and Bakker metatheory will be presented in depth, and we will discuss the methodology we adopted to extrapolate a finer analysis tool from it.

CHAPTER 3 – BOUNDARY FRAMEWORK AND METHODOLOGY

This chapter first presents in detail the metatheory exposed by Akkerman and Bakker in their 2011 article: “Boundary Crossing and Boundary Object”. Afterwards, we describe the methodology adopted to apply the framework and analyse the foundational papers of the Special Theory of Relativity.

3.1 Akkerman and Bakker Boundary concepts

Akkerman and Bakker’s article arose from a desire to determine current insights into the learning (employed in a very broad sense) potential of *boundaries*. They explain that, particularly in the context of educational theory, the concepts of *boundary crossing* and *boundary object* have become more and more central. This is because diversity and mobility increasingly characterise the world of instruction and work. However, it is unclear how the consideration of *boundaries* leads to enhanced learning. Therefore, they conduct a review of the literature produced in the contexts of the social sciences and educational sciences concerning the concepts of *boundary crossing* and *boundary object*.

<u>Brief description of the genesis of the concepts</u>

The concept of *boundaries* is broad, and it applies to different contexts. The authors remind us that educational research mostly studies learning within the boundaries of practices, focusing on particular groups of people or areas of expertise. However, since around the 1980s or 1990s, many studies have recognised the heterogeneity of new forms of work, as they involve different professional cultures. Akkerman and Bakker (2011) explain that the term *boundary crossing* was introduced to indicate how professionals at work may need to enter unfamiliar territory, where they are not fully qualified, and face the challenge of negotiating and combining ingredients from different contexts to achieve hybrid situations. While the concept of *boundary objects* is used to indicate how artefacts can play a specific function in linking intersecting practices. Boundary objects are those objects that both inhabit different intersecting worlds and fulfil the information requirements of each of them. These are plastic enough to adapt to the local needs and constraints of the several parties using them, but robust enough to maintain a common identity across sites. They are weakly structured in common use and become strongly structured in individual site use.

But the concept of boundary has also been of interest to fields of research such as social psychology, where the human mind is no longer studied solely in terms of a unified subject, but as a multiple and discontinuous self intrinsically related to other individuals and generalised. Indeed, the authors suggest that "a boundary can be seen as a sociocultural difference that leads to a discontinuity in action or interaction. Boundaries simultaneously suggest an identity and a continuity, in the sense that within the discontinuity two or more sites are relevant to each other in a particular way." (Akkerman & Bakker, 2011, p. 133).

In the context of the educational sciences, the interest in boundaries has grown through different phases. They mostly speak of sociocultural boundaries (e.g., the transition from theoretical learning to practical application). In the beginning, differences were seen as an obstacle, something to be overcome. Over time, the concepts of boundaries and boundary crossing assumed an increasingly positive meaning. There are now explicit views in which they are seen as vital forces for change and development, without which communities of practice would become stale and lose dynamism.

Basically, the belief is that confrontation, dialogue, diversity are always good opportunities for learning. Therefore, boundaries should be valued over barriers. The Akkerman and Bakker review allowed them to clarify what the nature of boundaries is, and what mechanisms of dialogic learning take place at the boundary.

Boundary nature

The analysis revealed how boundaries meet within and between the domains of work (science, technology, but also healthcare and teaching), school and everyday life. In the first domain, these are mostly boundaries between different professions, due to the need for interdisciplinarity. In the second, these are boundaries between teachers and students of various kinds. In the third, the boundaries are within social groups: between adolescents and adults, or between different ethnic groups for example. Equally interesting are the boundaries between these domains: the boundary between school and work with regard to the social, economic or psychological status of the individual, or with regard to the difference between theoretical and practical knowledge. In short, the authors tried to consider all possible types of boundaries between and within different possible domains. This is because they decided to look for similarities between different conceptualisations in order to arrive at a definition of the nature of the boundary.

Summarising the basic points, boundaries are:

- Always between at least two sites.
- Ambiguous in nature, as a middle ground, the boundary belongs to both one world and another, but also to neither.
- Crossed or created by people and objects.

In fact, the authors highlight, among others, the concept of *boundary people*. These are persons who encounter discontinuities in their actions and interactions. They may have the task of building bridges between worlds, but at the same time, these people are held responsible in each world. Here again, the ambiguous nature of the boundary manifests itself. It is not easy to be a boundary people, because it can happen that one is seen as an outsider by both worlds involved.

Boundary Objects

As already mentioned, boundary objects are defined as "artifacts that articulate meaning and address multiple perspectives. […] [They] have different meanings in different social worlds but at the same time have a structure that is common enough to make them recognizable across these worlds." (Akkerman & Bakker, 2011, p. 140-141). Moreover, they allow different worlds to collaborate by facilitating the crossing of boundaries. Often their main function is to facilitate communication, however, this is not all, their significance emerges strongly when different perspectives meet. The ambiguous nature of the boundary also manifests in these objects, making them usable in different ways by different people at different times. For these reasons too, their function as boundary crossing facilitators is not taken for granted. To get an even clearer idea, the metaphor of the black box may be inspiring: "As black boxes, boundary objects tend to be invisible or taken-for-granted mediations that translate across sites but, when carefully considered or opened up, may provide learning opportunities" (Akkerman & Bakker, 2011, p. 141).

Boundary Crossing

Akkerman and Bakker focused on how the consideration of boundaries, and the eventual crossing of them, can lead to learning. They have discerned four potential mechanisms of learning at the boundary, which they summarize as identification, coordination, reflection, and transformation. Underlying each of these, they also identified processes. These are more practical in nature, consisting of actions that can be performed at the boundary which then define the mechanism adopted for crossing.

Identification

Boundary learning can be described in terms of identification, in the sense that the dividing lines between the practices involved are uncertain or destabilised. This can be caused by feelings of threat or by increasing similarities or overlaps between practices. So, it is a questioning of the basic identity of each of the intersecting sites. This questioning leads to a new understanding of what the different practices are about. It can be implemented through two processes:

- *Othering:* It is the act of negotiating different identities, which may not coexist harmoniously. This occurs through the distinctions that people use, and make others use, to define themselves at the boundary between one system and another. Thus, it is not a process of crossing fixed boundaries, but a redefinition of these, emphasising what differentiates one world from another.

- *Legitimating coexistence:* It is often highly political and sensitive for the people involved. It is about the need to consider the interferences between multiple memberships, and the acceptance of the existence of other groups.

"What is typical in identification processes is that the boundaries between practices are encountered and reconstructed, without necessarily overcoming discontinuities. The learning potential resides in a renewed sense making of different practices and related identities." (Akkerman & Bakker, 2011, p. 143).

Coordination

The learning from the adoption of a *coordination mechanism* lies in the search for effective means and procedures that allow different practices to cooperate efficiently, even in the absence of consensus. In these cases, dialogue between different partners is established only to the extent necessary to maintain the workflow. At a practical level, four processes are recognised as enabling coordination:

- *Communicative connection:* It can be established by instrumentalities (boundary objects) that are shared by multiple parties; such instrumentalities are read differently by different actors.

- *Efforts of translation:* By this they do not trivially mean just the translation of different languages between different cultures (for example). This translation work strongly relates to finding a balance in the aforementioned ambiguity of boundaries. Translations involve both an intersubjective terrain and a diversity of possible understandings, e.g., it occurs with the translation of research results into concrete commercial applications.

- *Enhancing boundary permeability:* These are those actions and interactions that are performed at the boundary for their characteristic of not incurring problems, costs, or excessive discontinuity. Like going through convenient tunnels in a mountain rather than climbing it. Permeability can also be enhanced by the repetition of ritual practices.

- *Routinization:* Linked to the previous one, it consists of the search for procedures through which coordination becomes part of an automated or operational practice.

"The potential in the coordinative mechanism resides not in reconstructing but in overcoming the boundary, in the sense that continuity is established, facilitating future and effortless movement between different sites." (Akkerman & Bakker, 2011, p. 144).

Reflection

This kind of boundary crossing emphasises not only understanding, but also the formulation of distinctive perspectives. It enables the realisation and explication of differences between practices and thus learning something new about one's own and others' practices. The authors distinguish two processes through which this can take place:

- *Perspective making:* It is the making explicit of one's understanding and knowledge of a particular issue. This process can take place through boundary objects, which facilitate communication between different systems of activity. In particular, they allow one to formulate and represent one's perspective, which can reflexively give access to implicit and unstated assumptions, making explicit the knowledge and assumptions mobilised in the interpretation of the object.
- *Perspective taking:* It is the act of looking at oneself through the eyes of other worlds. This sometimes has a reflexive impact, i.e., one is led to see things in another way, and thus to look at his own practice with a new perspective. In short, it is a matter of taking the other into account; by taking another perspective, one can begin to see things in a different light.

"A consequence of *perspective making* and *perspective taking* is that people's ways of looking into the world are enriched so that one enriches one's identity beyond its current status" (Akkerman & Bakker, 2011, p. 146).

Transformation

We speak of transformation when the learning involved engages the boundary worlds in profound changes in practices, potentially even in the creation of a new intermediate practice, sometimes called boundary practice. Often this happens in response to a disruption in the workflow, to problems, but which connect the intersecting practices. The focus in the crossing mechanism is precisely the dialogue, which is not relegated to the minimum, but it is the object of interest. Six are the processes that achieve transformation:

- *Confrontation:* It is a necessary condition for transformation. It is the act of serious reconsideration of practices and interrelationships between intersecting worlds. Groups are encouraged to consider each other's arguments, even if they are strange or new. This often occurs because of the presence of some problem or lack, or a third perspective.

- *Recognizing a shared problem space:* It is closely related to the process of *confrontation*. It consists in recognising that there is a problem, an issue, which affects both worlds at the boundary and is therefore shared and mutually agreeable.

- *Hybridization:* It occurs when worlds on the boundary engage in a creative process in which something hybrid, i.e., a new cultural form, emerges. Ingredients from different contexts are combined into something new and unknown. This can take the form of new tools, such as the formation of a new concept. The hybrid result can also take the form of a completely new practice that is placed among established practices.

- *Crystallization:* It can only happen if there was *hybridization* first. Therefore, it applies to what has been created. It is the act of incorporating the thing into practice so that it has real consequences. A boundary object can be something hybrid to be crystallised. However, a pragmatic commitment to new activities is required, which does not happen through the object itself, it facilitates it. For instance, through the development of new routines or procedures that embody what has been created or learned. The authors point out that this is rarely realised.

- *Maintaining uniqueness of the intersecting practices:* It is about maintaining the integrity of the family field. Transformation into new or modified practices does not occur without a certain level of reinforcement of established practices, as happens in identification processes. It is precisely difference that sustains the relevance and value of intersecting practices.

- *Continuous joint work at the boundary:* It is the effort or awareness to find a way to cross practices. People from different activity systems meet to discuss and work together, reflecting with shared problems at the boundary. It is often described as a process of negotiation of meaning.

“The various processes of transformation indicate how difficult it is to achieve but, if successful, also imply sustainable impact.” (Akkerman & Bakker, 2011, p. 150).

Conclusion

Finally, Akkerman and Bakker conclude by stating that these four mechanisms allow us to think more precisely about boundary crossing and boundary objects.

Some significant differences and links between the four are also highlighted:

- On a general level, it seems that identification does not involve boundary crossing, something that the other three mechanisms do.
- It seems that the mechanisms of identification and reflection are mostly about perspectives and identity, whereas coordination and transformation are based more on practice and activity.
- With regard to boundary crossing, the *coordination mechanism* seems to be the most ‘convenient’ and easy to adopt, while transformation seems to be the most ‘laborious’, employing discontinuity, confrontation, dialogue and changes.
- Identification and reflection seem to condition transformation because in the latter, boundaries must be encountered and contested before being used to develop practices.

Furthermore: “There is one conclusion that holds for all four of the mechanisms: Dialogical engagement at the boundary does not mean a fusion of the intersecting social worlds or a dissolving of the boundary. Hence, boundary crossing should not be seen as a process of moving from initial diversity and multiplicity to homogeneity and unity but rather as a process of establishing continuity in a situation of sociocultural difference. This holds also for the *transformation mechanism*, in which something new is generated in the interchange of the existing practices, precisely by virtue of their differences.” (Akkerman & Bakker, 2011, p. 152).

Interdisciplinary correlation

Clearly, the "worlds", "sites" or "groups" of which Akkerman and Bakker speak could be of any kind. It was their precise intention to remain as general as possible, this also makes their theory so interesting. The authors themselves admit that, in this way, it becomes possible for researchers to adopt not only a systemic or macro perspective, but also a situated or micro perspective. In order to gain a better understanding of how they defined boundary objects and boundary crossing mechanisms, they often gave different types of examples, naturally taken from the literature they reviewed. In the context of the description of the *transformation mechanism*, Palmer's "Structures and strategies of interdisciplinary science" from 1999 is quoted: "Interdisciplinary research requires a balance between established core knowledge and the infusion of new knowledge. As researchers explore new problem areas, they do not necessarily abandon their disciplinary concentrations" (Palmer, 1999, p. 250).

We believe that Akkerman and Bakker's framework can also be useful in describing the boundaries (and attitudes towards boundaries) between different disciplines. The disciplines, in our view, should be understood according to the FRA wheel, i.e., as cognitive-epistemic and socio-institutional systems. Therefore, consisting in processes of inquiry, aims and values, methods and methodological rules, scientific knowledge, professional activities, scientific ethos, social certification and dissemination of scientific knowledge, and social values. This makes the application of the framework possible and appropriate.

In this thesis, this is done on the foundational articles of the Special Theory of Relativity, in order to detect and examine the possible interdisciplinarity between mathematics and physics in this context. Thus, it will be a situated analysis. We also decide to follow the useful research direction indicated by Akkerman and Bakker: that of identifying a series of indicators or methodological markers with which diversity and the resulting discontinuities can be empirically detected.

The way in which a useful tool was created for this purpose is described below.

3.2 Methodology

Our objectives for the analysis of the boundary between mathematics and physics, in the context of the original founding articles of the Special Theory of Relativity, were primarily:

- To create a tool that at the same time best reflects Akkerman and Bakker's boundary concepts, is specific for the boundary between mathematics and physics, and is practical to use.
- Detect the boundary attitudes of the four authors (Lorentz, Poincaré, Einstein and Minkowski), and understand what kind of role interdisciplinarity played in the development of the Special Theory of Relativity.
- Verify if the Lorentz Transformations can be considered a boundary object.

The most time-consuming work was done to achieve the first objective, since we were convinced that the other two could be achieved as a consequence.

To obtain the analysis tool, the work was carried out in four macro phases:

- The in-depth study of the Akkerman and Bakker (2011) metatheory.
- The search for feedback of the ideas developed by reading the original texts on the Special Theory of Relativity.
- The development of a draft analysis tool.
- The checking and refining of the analysis tool.

While, for the detection of the boundary attitudes of the four authors, and of the boundary object nature of the Lorentz Transformations, we also wanted to graphically elaborate the data detected through the analysis tool.

In this section, we will explain how we developed an analysis grid and the graphical representations resulting from the analyses carried out with it on the articles.

Akkerman and Bakker framework transposed into an analysis grid

The in-depth study

The first step was to read in depth the Akkerman and Bakker (2011) article. The aim was to gain a greater understanding of the external and internal functioning of each learning mechanism. Two artefacts were created: Lorenzo Miani provided a graphic visualisation representing the boundary between two worlds before and after each type of crossing (Fig.2).

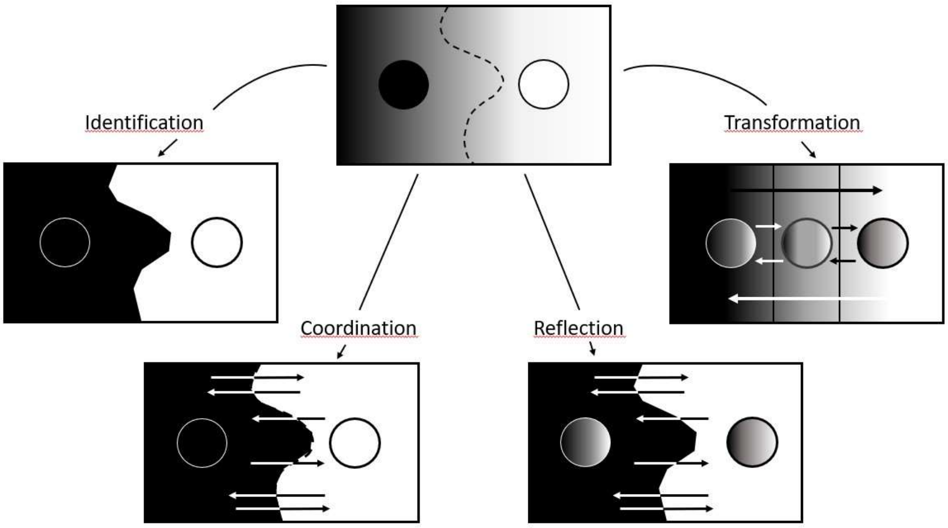

Figure 2 - Graphical representations of the boundary between two worlds before and after each type of mechanism

The focus is therefore on what we think happens at an overall level, regardless of what processes have been used, on the sites involved. I have provided a conceptual map taking into account the processes underlying the mechanisms.

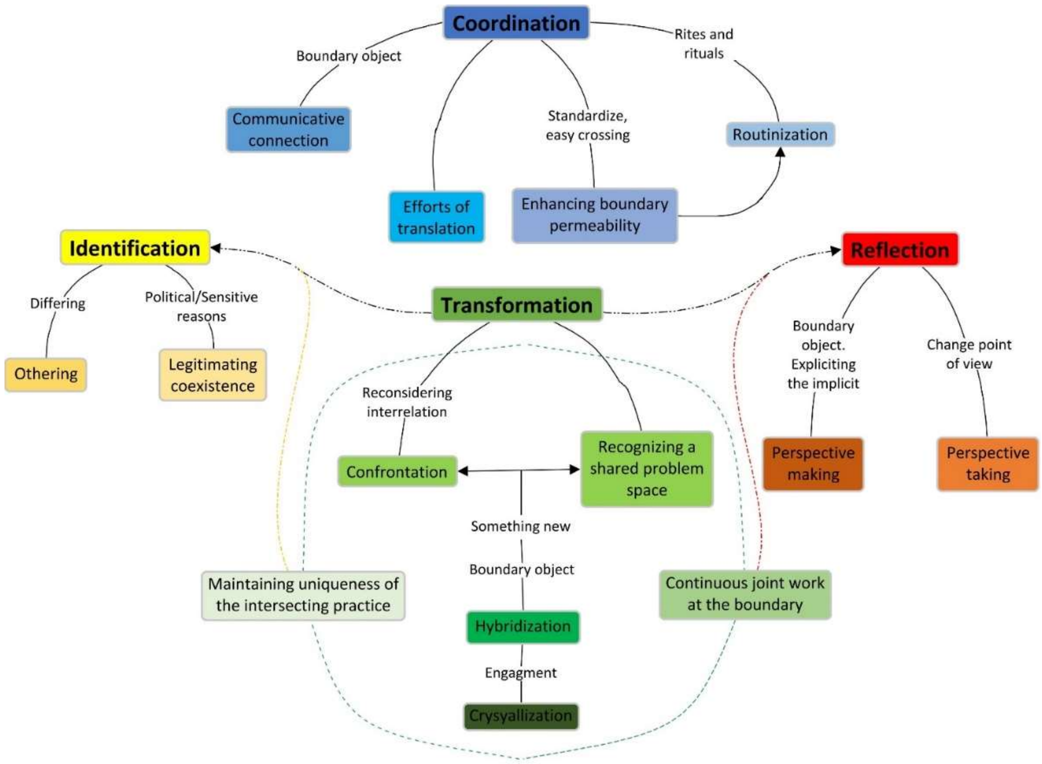

Figure 3 - Mechanisms and processes

I felt it was important not to leave out any processes. The intention was to show the internal hierarchy of each mechanism. The images were realised trying to respect the descriptions provided by Akkerman and Bakker as best as possible.

In our opinion, the power of these images lies in their ability to highlight operational aspects of the framework. First of all, to clearly distinguish the learning mechanisms, it is necessary not to lose focus on the end result of the boundary crossing (if any). In all four cases, the boundaries are redrawn, but in four different ways: the *Identification mechanism* clearly marks the differences, and there is not boundary crossing; the *Coordination mechanism* allows fruitful collaboration over clear distinctions; the *Reflection mechanism* enriches the two boundary fronts with new knowledge, in addition to the fact that it makes the different points of view clear; the *Transformation mechanism* brings out something new and shareable, which lies in the middle.

Furthermore, it has been realised that learning mechanisms possess their own dynamism that is embodied in the way processes are adopted. It was realised that the processes that implement the mechanisms of Identification and Reflection are disjointed from each other and can therefore be detected independently. In contrast, the hierarchy within the *Coordination* and *Transformation mechanism*s is more complicated.

Within the *Coordination mechanism*, a link is established between *Enhancing boundary permeability* and *Routinization*: the former implies the latter.

Within the *Transformation mechanism*, on the other hand, the hierarchy is even more complex:

- Between *Confrontation* and *Recognizing a shared problem space* there is a double implication.
- *Confrontation* and *Recognizing a shared problem space* are a necessary condition for there to be *Hybridization*.
- There is no *Crystallization* if there is no *Hybridization*.
- *Maintaining uniqueness of the intersecting practices* and *Continuous joint work at the boundary* operate like a milieu, in which the other processes of the mechanism are immersed.

In addition, the influence that the *Transformation mechanism* experiences from the Identification and *Reflection mechanism*s seems to be embodied in the processes of *Maintaining uniqueness of the intersecting practices* and *Continuous joint work at the boundary* respectively.

Finally, another aspect to consider was the fact that the mechanisms resulting from boundary crossing (Coordination, Reflection and Transformation) possess one particularly appropriate process for the use of boundary objects. Akkerman and Bakker do not exclude the use of these through different processes, but we perceived a preference for:

- The *Communicative connection* process within the *Coordination mechanism*.
- The *Perspective making* process within the *Reflection mechanism*.
- The *Hybridization* process within the *Transformation mechanism*.

All these characteristics allowed us to formulate an initial general vision of an operational framework. We therefore decided to proceed by testing this model on the original articles.

The search for feedback

During the second phase, we read several times the founding articles of the Special Theory of Relativity: “Electromagnetic phenomena in a system moving with any velocity smaller than that of light” (Lorentz, 1904), “Sur la dynamique de l'électron [On the dynamic of the electron]” (Poincaré, 1906), “On the Electrodynamics of Moving Bodies” (Einstein, 1905) and “Space and time” (Minkowski, 1908).
The idea was to try to identify mechanisms directly on the texts, depending on what is done by the author, how he does it, and for what reason.

So, each of us decided to highlight parts of the texts with different colours, depending on what we felt might be the mechanism adopted by the authors in those particular passages. We chose the colours:

- Yellow for the *Identification mechanism*.
- Blue for the *Coordination mechanism*.
- Red for the *Reflection mechanism*.
- Green for the *Transformation mechanism*.

We carried out this work keeping in mind the boundary general model that we obtained during the first phase. In particular, Lorenzo Miani and I conducted this first analysis with two slightly different approaches: I tried to identify the processes, and consequently determine the mechanisms; instead, he tried to identify the mechanisms directly, taking into account the processes meanwhile.

So, we exchanged our observations with the supervision of Professor Olivia Levrini. From this comparison, new evidence emerged to guide us towards a better analysis, and towards specialisation and refinement of the model. We both felt the need to highlight certain parts of the text as belonging strictly to the world of physics or the world of mathematics; we agreed that it would be rude to detect the adoption of a single mechanism for each of the four articles. Finally, we agreed on the possibility of making process detection the main action of the analysis tool.

Therefore, in order to pursue the development of a real analysis tool that is practical to use, we decided to provide definitions of processes that were specific to the boundary between mathematics and physics. To make them easier to detect, we have also associated some examples to them. These are small phrases extracted from the articles, or explanations of typical attitudes of practices, familiar to mathematicians and physicists. A table description follows (Tab. 1, 2).

Observations must be made on how each boundary crossing mechanism has been redefined through the decomposition into processes, hence this specialisation of the framework. Also in this case, we have tried to preserve and respect the definitions provided by Akkerman and Bakker as much as possible.

Mechanism	Process	Definition	Examples
Identification	Othering	It occurs by defining one discipline in light of another, delineating how it differs from the other one.	
	Legitimating coexistence	It is often a process highly political and sensitive to those involved.	
Coordination	Efforts of translation	The "natural" translation of physics into mathematics and vice versa; translations entail both an intersubjective ground as well as a diversity of possible understandings.	- "Let its length be l". - Inferring the nature of a physical quantity from dimensional analysis.
	Communicative connection	It's a process established using instrumentalities (boundary objects) that are shared by both disciplines.	- "With the help of this result we easily determine the quantities ξ, η, ζ". - The use of certain types of formulas and equations (boundary object) for the description of a motion.
	Enhancing boundary permeability	Try to make boundaries permeable through the repetition of different ritual practices. Make "comfortable" hypotheses that facilitate the crossing, that standardize the work.	- Assume that a motion complies with a known law. - Modelling in order to find known theories.
	Routinization	Finding procedures through which coordination becomes part of the automated or operational practice.	Perform calculations, simple steps, almost obvious.
Reflection	Perspective making	Making explicit one's understanding and knowledge (the implicit) using boundary objects as a kind of headlight.	- Looking at an object, like an equation, a result, seeing what hadn't been seen before. - Looking at mathematical results and see a new way of describing a physical phenomenon.
	Perspective taking	Look at the problem, or theme, from the perspective of the other discipline, changing point of view.	Don't limit yourself to physical interpretations, but find new evidence, new mathematical structures, which can enrich the discourse.

Table 1 - Processes definition and examples part 1

Mechanism	Process	Definition	Examples
Transformation	Confrontation, Recognizing a shared problem space	Face up to some lack, problem or inconsistency that forces the intersecting disciplines to seriously reconsider their current practices and the interrelations.	
	Hybridization	A creative process in which something hybrid emerges, something new, through boundary objects, or which consists in the reconceptualization of boundary objects.	
	Crystallization	The reasoning is that it is one thing to create something hybrid at the boundary but quite another has coherence and to embed it in practice so that it has real consequences. Crystallization can occur by means of something called reification, that is, to "congeal this experience into 'thingness'". Boundary object can be an example of reification.	- Do some numerical checks. - Apply the hybrid object to a situation to find consistency.
	Continuous joint work at the Boundary	It is often described as a process of negotiation of meaning. It is like a reflection mechanism but extremely dynamic, you have the feeling of being able to move from one point of view to another in a fluid and fast way, like a ping pong ball.	Physical and mathematical arguments alternate continuously.
	Maintaining uniqueness of intersecting practice	It is like the identification mechanism but incorporated in that of transformation.	

Table 2 - Processes definition and examples part 2

In our reinterpretation emerged that the *Coordination mechanism* plays the role of the typical modelling practice in the sense of Ornek (2008). We identified this connection because we had the feeling that concepts related to modelling resonate with the definitions of *Coordination mechanism* processes. That is, we are implementing the *Efforts of translation* process when we describe a phenomenon, or a system, in terms of symbols, equations, and numbers, when we decide how to formulate the problem, when we interpret the solution. We implement the

Communicative connection process when we want to use a model, principle, or object to model. We succeed in *Enhancing boundary permeability* when we make good assumptions, approximations, predictions, and interpretations. And finally, the calculations required to solve the problem constitute the process of *Routinization*. In addition, one feature is common to all modelling and the *Coordination mechanism*: the desire to make things simple, standard, familiar and easy.

Regarding the *Reflection mechanism*, what has emerged is that it is characterised by *discovery*, or *surprise*, which are part of learning. To give an example: many times, we use models to confirm predictions, to find what we already expected, but other times we don't really know where it can take us. For example, if we look at a problem, a phenomenon, or a system from the point of view of the other discipline, we may learn something more, which may be a revelation of meaning: a new application of a mathematical theory, significant mathematical properties for an equation related to a physical phenomenon, or similar situations. We made use also of an analogy to define the process of *Perspective making*: the boundary object involved acts as a headlight, which sheds light on new knowledge.

The *Transformation mechanism* is the most difficult to characterise, however we have tried to decline it as best we could. We point out that we have combined the processes of *Confrontation and Recognising a shared problem space*, for two main reasons: first, a co-dependency, a double implication, already emerged in the original framework; secondly, this coincidence also appeared in the text detection. An interesting role was recognised to the process of *Crystallization*: the demonstrative and verification role. It is one of the most frequent processes within the practices of physics and mathematics, however, let us remember that we can speak of *Crystallization* only if it is applied to a new, unknown and hybrid object or concept. Another analogy emerged in the definition of the process of *Continuous joint work at the Boundary*: that of the ping pong ball, which passes from one side of the net (boundary) to the other very quickly. This is an image evocative of a feeling that a reader may experience in the presence of this process. Finally, the process of Maintaining uniqueness of intersecting practices was defined as an *Identification mechanism*, but internal to the *Transformation mechanism*; the idea is that these are passages in which there is a perceptible desire to emphasise characteristics that differentiate mathematics from physics, but in a transformative context. Considering the hierarchy between the processes, internal to the *Transformation mechanism*, is crucial in detection.

Our work continued in other readings of the original articles, with the aim of returning to exchange ideas and refine the analysis.

The development of a draft analysis tool

We have already remarked how we were often guided by our subjective impressions during the reading of the four articles. This may seem a risky choice, however, proceeding in the search for clear signals in the text, the question became increasingly relevant. We decided to create an operational analysis grid, in the sense that, to definitions and examples of the processes, we also added marker words to be searched in the text. These words have been chosen for the processes considering their meaning, the position they occupy in certain contexts and the frequency with which they appear in these contexts; we have therefore drawn up a preliminary list. What we tried to detect and quantify in the texts is a form of metadiscourse in the sense of Hyland and Tse (2004).

But we wish to clarify that we did not draw up a list of words based on rules of linguistics, we are not capable of it. The words were *found* in the text. We chose some of them because they are familiar from a socio-institutional point of view. We had the idea that the analysis of the text, through these words, could be predominantly cognitive-epistemic in nature. Because we felt justified in conducting such work, we proceeded to read the original texts trying to analyse them, trusting the draft analysis grid that had been created. We chose different colours to highlight the marker words of the processes on the text: shades of yellow for the *Identification mechanism*, shades of blue for the *Coordination mechanism*, shades of red for the *Reflection mechanism* and shades of green for the *Transformation mechanism*. Thus, the final phase consisted of a definitive refinement to achieve an analysis tool.

The checking and refining of the analysis tool

The draft analysis grid was predictably incomplete in some points, and redundant in others. The last revision led us to decide which words should definitely be taken into account and which, on the other hand, did not work. The result is given below (Tab. 3, 4, 5, 6). We decided to include verbs, conjunctions, connectives, and small expressions. Regarding verbs, we specify that they must be highlighted in all their conjugations and tenses during the analysis.

We performed the last analysis by meticulously underlining marker words each time we encountered them in the text. In doing so, we realised several things:

- The words associated to the same mechanism must be in consistent quantity to deduce a coherent conclusion.
- When there is an isolated word, i.e., belonging to a different mechanism from the predominant one in the paragraph, we must be careful: maybe the author has made an alternative semantic use of it to the one we recognised, in that case we can avoid underlining it, it is not relevant.

We emphasise, however, that in our objective search for marker words, we have never lost sight of our interpretive capacity. This work must be done without ever losing focus on the context in which the words are detected. So, underlining words by the mere fact that they appear in the text is an operation that sometimes leads to seeing something that was previously unseen, or to confirming that there is a process that was already expected to be detected, but other times it leads to nothing. It is essential to always keep in mind the meaning of what we are reading. This last point is particularly important also for practical reasons: not every word marks a unique process, and there are processes for which it was impossible to find markers, in this last case we rely on definitions and examples.

The grid clearly shows that there are processes that are more detectable than others because there are few marker words, while others, being the most common, have a large set of markers, so identifying them might seem a more dispersive exercise. In any case, we have bolded the most recurrent words for each process.

We are convinced that this long work has led to the creation of a practical, effective, and complete tool. We arrived to define in detail the individual parts, and the connections between them, of the mechanisms, which in turn are interconnected, through an internal hierarchy, and an external one that we explained explicitly. All these aspects make the analysis through the grid quite effective to uncover fine-grained details of the text.

	Description	Examples	Markers words
Identification			
• Othering	It occurs by defining one discipline in light of another, delineating how it differs from the other one.		
• Legitimating coexistence	It is often a process highly political and sensitive to those involved.		
Coordination			
• Efforts of translation	The "natural" translation of physics into mathematics and vice versa; translations entail both an intersubjective ground as well as a diversity of possible understandings	▪ "Let its length be *l*" ▪ Inferring the nature of a physical quantity from dimensional analysis	**Represent**, **let**, **denote**, define, **call**, correspond, **where**, express, relate, be, indicate, write, identify, associate, describe, mean, consist, constitute, interpret, designate, notation.
○ Communicative connection	It's a process established using instrumentalities (boundary objects) that are shared by both disciplines	▪ "With the help of this result we easily determine the quantities ξ, η, ς" ▪ The use of certain types of formulas and equations (boundary object) for the description of a motion	**Take**, **apply**, **use**, introduce, with the help of, with the aid of.

Table 3 - Analysis grid part 1

	Description	Examples	Markers words
Coordination			
○ Enhancing boundary permeability	Try to make boundaries permeable through the repetition of different ritual practices. Make "comfortable" hypotheses that facilitate the crossing, that standardize the work.	▪ Assume that a motion complies with a known law. ▪ Modelling in order to find known theories.	**May**, must, shall, **consider**, put, satisfy, **determine**, **if**, **suppose**, **choose**, hypothesis, condition, impose, want, place, insert, seek, in accordance, on account of, agreement, decide, envisage, introduce, in connection, assign, assume, set, adopt, take.
○ Routinization	Finding procedures through which coordination becomes part of the automated or operational practice	▪ Perform calculations, simple steps, almost obvious.	Determine, give, substitute, get, have, write, by the formulae, reduce, so that, **become**, multiply, **calculate**, replace, result, according to, derived by, **follow**, **hence**, thus, then, solve, **obtain**, integrate, decompose, equal, consequently.

Table 4 - Analysis grid part 2

	Description	Examples	Markers words
Reflection			
• Perspective making	Making explicit one's understanding and knowledge (the implicit) using boundary objects as a kind of headlight.	▪ Looking at an object, like an equation, a result, seeing what hadn't been seen before. ▪ Looking at mathematical results and see a new way of describing a physical phenomenon.	**Find**, lead, led, **show**, explain, make, learn, allow, rise, ensue, mean, emerges, raise.
• Perspective taking	Look at the problem, or theme, from the perspective of the other discipline, changing point of view.	▪ Don't limit yourself to physical interpretations, but find new evidence, new mathematical structures, which can enrich the discourse.	**Appear**, regard, **see**, **note**, present, observe, arrive, remark, visualize, manifest, view, look.
Transformation			
• Confrontation, Recognizing a shared problem space	Face up to some lack, problem or inconsistency that forces the intersecting disciplines to seriously reconsider their current practices and the interrelations.		Problem, divergence, **question**, contrast, contradiction, asymmetries, dogma, discuss, disharmony, disdain, impossibility, incompatible.

Table 5 - Analysis grid part 3

	Description	Examples	Markers words
Transformation			
○ Hybridization	A creative process in which something hybrid emerges, something new, through boundary objects, or which consists in the reconceptualization of boundary objects.		Deduce, **modify**, state, establish, **assume**, develop, remodel, grasp, introduce, give, take, construct, amend.
○ Crystallization	The reasoning is that it is one thing to create something hybrid at the boundary but quite another has coherence and to embed it in practice so that it has real consequences. Crystallization can occur by means of something called reification, that is, to "congeal this experience into 'thingness'". Boundary object can be an example of reification.	▪ Do some numerical checks. ▪ Apply the hybrid object to a situation to find consistency.	Force, compatible, **prove**, **accept**, confirm, reject, connection, **understand**, necessarily, demonstrate, undertake, admit, **valid**, allow.
• Continuous joint work at the Boundary	It is often described as a process of negotiation of meaning. It is like a reflection mechanism but extremely dynamic, you have the feeling of being able to move from one point of view to another in a fluid and fast way, like a ping pong ball.	▪ Physical and mathematical arguments alternate continuously.	
• Maintaining uniqueness of intersecting practice	It is like the identification mechanism but incorporated in that of transformation		

Table 6 - Analysis grid part 4

Method of processing and return of analysis

Let us note that the original articles analysed are quite extensive, some of them very long. Therefore, we decided to find a way to quantify, visualise and return the results at a glance with graphical representations. At first, we wanted to create a graph for each article, each of them able to summarise the analyses made through the analysis grid developed.

Lorenzo Miani and I tried to obtain these items separately, to compare later. We did not have any precise guidelines, however, we both used Microsoft Excel's "stacked bar chart" tool, in horizontal bars; moreover, we both considered where the authors of the original articles remain on one side of the boundary, that of mathematics or that of physics. Two colours were chosen for the bars in which this occurs: violet for mathematics, grey for physics. We have maintained the colours yellow, blue, red and green for the mechanisms of Identification, Coordination, Reflection and Transformation respectively. However, we took two slightly different approaches.

Beyond aesthetics, the relevant methodological difference in the construction of the graphs consisted in the quantification of the interval of "time" in which a mechanism, or a strictly disciplinary argument, is adopted by the authors: personally, I counted the individual lines, with the exception of the lines where calculations are performed, in that case I counted only one line, even if more appear; instead Lorenzo Miani counted groups of five lines, without differentiating the method for the lines with calculations. Therefore, in his plot, it was possible to see overlaps between mechanisms, but this was not a problem; on the contrary, it showed us even more clearly the complexity of boundary crossing. In fact, mechanisms are not to be understood as closed compartments, the authors' "journey" is dynamic, and this is a feature of their thinking that should not be overlooked. There is also a practical advantage in choosing to divide by groups of five lines, or to count them individually: the papers have different lengths, and the paragraphs can also be either very long or very short, so if we chose them as units, it would be difficult to understand the distribution and proportion of the mechanisms along the texts.

In our confrontation we found a good level of agreement, however, in the end we chose to show the graphs obtained by counting individual lines and only one line in the case of several calculations. The horizontal bar graph serves as a spectrum of learning mechanisms. On the horizontal axis we have placed the numbering of lines and chapters of each article, while on the vertical axis the mechanisms and disciplines of mathematics and physics. Looking at the spectra from left to right, it is possible to follow the progression at the boundary between mathematics and physics of each author along the articles.

Zoom on Lorentz Transformations

Following the advice of supervisor Olivia Levrini, to be able to fully describe the narratives of the authors, we decided that we had to try to go back to the content of the discourses, with a focus on the Lorentz Transformations. Therefore, we set a new roadmap for each article:

- To describe the author's goals, specifying whether they were proposed explicitly or implicitly;
- To place the sections on Lorentz Transformations within the narrative, i.e., say with what premises the authors arrive to them and for what purposes they introduce them;
- To highlight and return, through other graphs of the same type, the adoption of processes within the sections on Lorentz Transformations.

Regarding this last point: the methodology used for the creation of the ulterior spectra is almost the same methodology we used for the mechanisms. One difference lies in the fact that on the vertical axis, instead of mechanisms, there are the processes, in addition to the disciplines of mathematics and physics. The colours used for the bars are the same colours we used for highlighting the words on the texts: shades of blue for the processes underlying the *Coordination mechanism*, red for those underlying the *Reflection mechanism*, and green for those underlying the *Transformation mechanism*. Another difference resides in the way certain lines of the text were associated with more than one process, seeking to respect the objectiveness of the grid's detection. This led to overlaps in the graphical result which, however, as already mentioned, are permissible considering the complexity of the rhetoric of the authors' discourses.

Once this was done, we tried to comment on the mathematics-physics relationship of each author with respect to their epistemological visions, and the ideas of space-time they had in mind.

In the following chapters, the analyses performed on the texts using the grid will be shown and the results discussed.

CHAPTER 4 – PAPERS' ANALYSIS

In the first section of this chapter we present the content of the articles we have analysed:

- Electromagnetic phenomena in a system moving with any velocity smaller than that of light, H.A. Lorentz, 1904 (English Translation by Perrett, W. & Jeffery, G. B. 1952);
- Sur la dynamique de l'électron (On the Dynamics of the Electron), H. Poincaré, 1906 (English Translation by Popp, B. D. 2020);
- On the Electrodynamics of Moving Bodies, A. Einstein, 1905 (English Translation by Perrett, W. & Jeffery, G. B. 1952);
- Space and Time, H. Minkowski, 1908 (English Translation by Lewertoff, F., Petkov, V. 2012).

They have been selected since they have deeply contributed to the development of the Special Theory of Relativity.

After this presentation, there are two sections in which a macro analysis conducted on the articles in their entirety is exposed, and an analysis that focuses only on the Lorentz transformations within the articles.

4.1 Papers' presentation

For our purposes, it was important to consider the structure of the articles and the type of publications they were. A brief presentation of these follows here.

<u>H.A. Lorentz, Electromagnetic phenomena in a system moving with any velocity smaller than that of light</u>

Around the second half of the nineteenth century many experiments were performed that had the objective to confirm or refute various hypotheses on optics and moving bodies, also Maxwell's Electrodynamics of moving bodies was improved. Lorentz actively participated in the discussion by publishing several works on the subject.

Following some criticisms made in particular by Poincaré, Lorentz wanted to offer a new version of his theory, so this article was published in 1904, in the proceedings of the Royal Dutch Academy of Art and Science (KNAW). We analysed the version with notes by A. Sommerfeld and translated by W. Perrett and G. B. Jeffery in "The

Principle of Relativity: a Collection of Original Memoirs on the Special and General Theory of Relativity London", published in 1952 by Methuen and Co., Ltd. It is composed of 14 short sections:

1. In the first section Lorentz illustrates the issue he aims to address: "The problem of determining the influence exerted on electric and optical phenomena by a translation, such as all systems have in virtue of the Earth's annual motion […]" (Lorentz, 1904, p. 809). To this end he continues with the description of some of the most relevant experiments of that time: the interference-experiment of Michelson, the new experiments of Rayleigh and Brace and finally those of Trouton and Noble, which are treated more in detail.
2. In this section Lorentz states that he has taken into account the notes made by Poincaré and that he is ready for a new treatment of the problem.
3. From here the author addresses the question directly. He considers the "fundamental equations of electron theory" (Lorentz, 1904, p. 811), i.e., Maxwell's equations, and he adapts them to the case of an electron moving with velocity v in a system moving wholly in the direction of x with a constant velocity w with respect to the ether.
4. In this section it is his intention to transform the formulas obtained with a change of variables. Putting $\frac{c^2}{c^2 - w^2} = k^2$, he obtains $x'=klx$, $y'=ly$, $z'=lz$, $t' = \frac{l}{k}t - kl\frac{w}{c^2}x$. These will be the Lorentz Transformations. He also says: "The variable t' may be called the "local time"; indeed, for $k = 1$, $l = 1$ it becomes identical with what I have formerly understood by this name." (Lorentz, 1904, p. 813).
5. Here the author aims to represent the vectors defined through the transformations by a scalar potential and a vector potential, then he performs a series of calculations.
6. Lorentz considers two special cases: "The first is that of an electrostatic system, i.e. a system having no other motion but the translation with the velocity w." (Lorentz, 1904, p. 815) and he adapts the found equations to this case. Then he says: "The result may be put in a simple form if we

compare the moving system *Σ* with which we are concerned, to another electrostatic system *Σ'* which remains at rest […]" (Lorentz, 1904, p. 815). In simple terms he continues to make calculations by using his "change of variables".

7. Still here he writes: "Our second special case is that of a particle having an electric moment […]" (Lorentz, 1904, p. 816). In this way, in the last 3 sections, the author has obtained several formulas that characterize his electrodynamic systems, in fact this section ends with a formula that wholly determines the field produced by a polarized particle.
8. Only at this point, by the author's own admission, he suggests two hypotheses:
   - "I shall now suppose *that the electrons, which I take to be spheres of radius R in the state of rest, have their dimensions changed by the effect of a translation, the dimensions in the direction of motion becoming kl times and those in perpendicular direction l times smaller.*" (Lorentz, 1904, p. 818).
   - "In the second place I shall suppose *that the forces between uncharged particles, as well as those between such particles and electrons, are influenced by a translation in quite the same way as the electric forces in an electrostatic system.*" (Lorentz, 1904, p. 819).

   And he argues the significance of what we've seen so far along with these assumptions.
9. In this section Lorentz calculates the electromagnetic momentum of a single electron. Since he considers really complicated the theory of rapidly varying motions of an electron, he seeks a good approximation by setting in the case of a quasi-stationary motion, so he finds transverse mass and longitudinal mass and then arrives to the electromagnetic moment.
10. In this section he wants to proceed by examining the influence of Earth's motion on optical phenomena in a system of transparent bodies. He says: "I shall show that, if we start from any given state of motion in a system without translation, we may deduce from it a corresponding state that can exist in the same system after a translation has been imparted to it […]"

(Lorentz, 1904, p. 821). In what follows the author uses several results obtained previously and the assumptions made in section 8. Finally, he concludes: “We are therefore led to suppose *that the influence of a translation on the dimension* [...] *is confined to those that have the direction of the motion, these becoming k times smaller than they are in the state of rest*. If this hypothesis is added to those we have already made, we may be sure that two states, the one in the moving system, the other in the same system while at rest, corresponding as stated above, may both be possible. [...]” (Lorentz, 1904, p. 824).

11. This is the section where the author argues that what we have seen so far justifies the results of the experiments mentioned in the first section, although he admits: “Our assumption about the contraction of the electrons cannot in itself be pronounced to be either plausible or inadmissible.” (Lorentz, 1904, p. 824).
12. In this section, the author quickly discusses cases where molecular motion is also considered: “We may conceive that bodies in which this has a sensible influence or even predominates, undergo the same deformation as the systems of particles of constant relative position of which alone we have spoken till now [...]” (Lorentz, 1904, p. 826).
13. In this section Lorentz discusses two sets of measurements published by Kaufmann in 1902, to show that these agree with the values obtained in his theory as much as those obtained by Abraham.
14. In the last section, the author takes the opportunity to mention an additional experiment performed by Trouton at the suggestion of Fitz Gerald.

<u>H. Poincaré, Sur la dynamique de l’électron (On the Dynamics of the Electron)</u>

“In France, the mathematician Henri Poincaré had been teaching electrodynamics at the Sorbonne for several years. After reviewing the theories of Maxwell, Helmholtz, Hertz, Larmor, and Lorentz, he judged that the latter was the one that best accounted for the whole range of optic and electromagnetic phenomena. Yet he was not entirely satisfied with Lorentz’s theory, because he believed it contradicted fundamental principles of physics” (Darrigol, 2006, p. 12).

As we have already mentioned, Lorentz wrote in 1904 in response to some of his criticisms. Poincaré in turn wanted to respond by publishing the results of his reflections on the subject. We have dealt with the long version published in 1906 in the "Rendiconti of the Circolo matematico di Palermo". We analysed the version translated by B. D. Popp in Henri Poincaré: Electrons to Special Relativity, published in 2020 by Springer Nature Switzerland AG. This consists of an introduction, which is a kind of summary of what is set forth in detail along the entire article, and 9 sections:

1. Lorentz Transformation: in which, effectively, Poincaré introduces the transformations in a personal form, i.e., he chooses the units of length and time so that the speed of light is equal to one. He starts from the fundamental formulas, and from a law that describes the mechanical force experienced by an element of matter of volume: $X = \rho f + \rho(\eta\gamma + \zeta\beta)$. He says: "These equations are subject to a remarkable transformation discovered by Lorentz and which is of interest because it explains why no experiment is able to let us know the absolute motion of the universe. [...]" (Poincaré, 1906, p. 48). He then goes on to verify that his transformations are equivalent to those written by Lorentz, that they respect a set of conditions (e.g., the continuity condition), and that they leave a set of laws unchanged.
2. Principle of Least Action: in this section instead, he describes the state of the system using the principle of least action. After several calculations, wanting to determine the force acting on electrons, passing from calculus of variations to ordinary differential calculus, making several assumptions on electrons, finally he can solve the functional *J* finding exactly the law of the first section: $X = \rho f + \rho(\eta\gamma + \zeta\beta)$.
3. Lorentz Transformation and the Principle of Least Action: Poincaré starts saying: "We are going to see if the principle of least action gives us the reason for the success of the Lorentz transformation [...]" (Poincaré, 1906, p. 61). In fact, the author proceeds by applying Lorentz Transformations to the *J* functional, finding that $J = J'$. To justify this equation, he considers an electron whose position at time *t* is $x = x_0 + U$, $y = y_0 + V$, $z = z_0 + W$, and a corresponding electron considered after Lorentz transformations. As a

result, from his calculations, he obtains that “The principle of least action therefore leads us to the same result as the analysis from §1.” (Poincaré, 1906, p. 64).

4. The Lorentz Group: in this section Poincaré shows that Lorentz Transformations form a group and uses this to prove that the coefficient *l=1.*
5. Langevin waves: this is a kind of parenthesis in which the author shows how Lorentz Transformations can also facilitate some demonstrations.
6. Contraction of Electrons: in this section the author wishes to test which of the hypotheses of Abraham, Lorentz, and Langevin is the correct one with respect to the significance of what is obtained when the Lorentz Transformations are applied to the case in which a single electron is traveling in a motion of straight and uniform translation. To do this he determines the total energy given by the motion of the electron, the corresponding action, and the electromagnetic momentum to calculate the electromagnetic masses of the electron. Poincaré finds some contradictions that he explains using what he obtained in section 3. “The conclusion is that if the electron is subject to a binding between its three axes, *and if no other force is involved apart from the binding forces*, the shape that this electron will take, when driven at a uniform speed, can only be that of the ideal electron corresponding to a sphere [...]” (Poincaré, 1906, p. 80). Thus, to incorporate Lorentz’s law the author argues that it is necessary to assume that there is an additional force resulting from a special potential derived from the three axes of the ellipsoid.
7. Quasi-Stationary Motion: “It remains to be seen whether this hypothesis about the contraction of electrons reflects the impossibility of showing absolute motion; I will start by studying quasi-stationary motion of an electron which is isolated or only subject to the action of other distant electrons.” (Poincaré, 1906, p. 82). So, the author applies the Lorentz Transformations to the equations of quasi-stationary motion. He finds that they remain unaltered but argues that this is not enough to prove that the Lorentz hypothesis is the only one that leads to this result. So, he extends hypotheses that will lead him to calculations, and then says: “Thus Lorentz’s

hypothesis is the only one which is compatible with the impossibility of showing absolute motion; if this impossibility is accepted, it must be accepted that moving electrons contract so as to become ellipsoids of revolution two axes of which remain constant; the existence of an additional potential proportional to the volume of the electron also has to be accepted, as we showed in the previous section. Lorentz's analysis is therefore found to be fully confirmed [...]" (Poincaré, 1906, p. 88).

8. Arbitrary Motion: In this section Poincaré extends the result obtained for quasi-stationary motions to the general case using what he had obtained in the third section. So, he concludes saying that if the inertia of electron, or the forces to which it is subject are only of electromagnetic origin, or give rise to the potential assumed in the previous section, then no experiment will be able to show the absolute motion. He asks what are these forces that give origin to this potential. He finds that they can be compared to a pressure that governs the inside of the electron. He concludes saying: "[...] one is tempted to conclude that there is some relation between the cause which gives rise to gravitation and that which gives rise to this additional potential." (Poincaré, 1906, p. 90).
9. Hypotheses on Gravitation: "Thus Lorentz's theory would fully explain the impossibility of showing absolute motion, if all the forces were of electromagnetic origin. But there are other forces to which an electromagnetic origin cannot be attributed, such as gravitation for example. [...] It is now appropriate to go into the details and examine more closely this hypothesis." (Poincaré, 1906, p. 91). The author proceeds by advancing several hypotheses and checking their compatibility with fundamental laws, then extending the hypotheses even further. He does not close the discussion completely, the article concludes by saying that a deeper discussion would be necessary.

<u>A. Einstein, On the Electrodynamics of Moving Bodies</u>

Albert Einstein had an early interest in electrodynamics: at the age of just 26, he had already published many works. We have analysed the edition of Einstein's On the Electrodynamics of Moving Bodies based on the English translation of his original 1905 German-language paper (published as Zur Elektrodynamik bewegter Korper, in Annalen der Physik. 17:891, 1905) which appeared in the book The Principle of Relativity, published in 1923 by Methuen and Company, Ltd. of London.

The article presents an introduction and 10 sections divided into two parts: the Kinematical part and the Electrodynamical part:

- Introduction:

  Einstein says that he wants to address the problem of asymmetries, which seem to be not inherent in phenomena, that are encountered when Maxwell's electrodynamics is applied to moving bodies. He gives some examples of this and says that these together with the unsuccessful attempts to discover any motion of the earth with respect to the "light medium" suggest that the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics are valid. So already at this point the author raises to postulates the Principle of Relativity and the fact that light propagates in empty space always with a defined velocity c that does not depend on the state of motion of the emitting body.
- Kinematical part:

1. Einstein discusses the problem of defining "time", then describes how one can move to the more convenient concept of simultaneity. An operational definition of simultaneity follows, that is, he suggests that the time of point A, for example, is measured with a clock in the neighbourhood of A, the same for point B. Einstein calls t_A the time of a starting ray of light from A in the direction of B, t_B the time in which it reflects from B in the direction of A, and t'_A the time in which it arrive again at A, the two clocks are synchronous if: $t_B - t_A = t'_A - t_B$. And he assumes that this definition is free from contradictions. Also, he assumes: $\frac{2AB}{t'_A - t_A} = c$.
2. In this section the author conducts a reflection that follows from his two postulates and the definition of simultaneity just shown. He describes how one can measure the length of a rod in a reference system solid to it, and how to measure it if one is in uniform parallel translational motion with

velocity *v*, using the operational definition of simultaneity. The author reflects on the fact that we get two different lengths by virtue of the two postulates. He proceeds by adding two clocks A and B at the extremities of the rod, and with similar reasoning he shows how for an observer in solidarity with the rod the clocks are synchronized, while for an observer in uniform motion of parallel translation with velocity *v* they are not.

3. He derives the transformations required to go from a stationary reference system to one that is in uniform translational motion with respect to the former, always using the operational definition of simultaneity and its postulates. Next, thanks to the same transformations, he shows that his two postulates are compatible; he concludes by finding that $\phi(v) = 1$.
4. Here Einstein shows that transformations turn a sphere of radius R into an ellipsoid of revolution with axes $R\sqrt{1-\frac{v^2}{c^2}}$, R, R. This leads to the claim that the speed of light plays the role of an infinitely large velocity. Finally, he shows that time is not absolute.
5. Here the author derives the law of composition of velocities, at the end of the section he also mentions the fact that the transformations form a group.

- Electrodynamical part:

6. In this section Einstein shows that the transformations leave the Maxwell-Hertz equations for empty space unchanged. He concludes by saying that electric and magnetic forces do not exist independently of the state of motion of the system of coordinates, and that the asymmetries mentioned in the introduction in this way disappear.
7. Using the results obtained in the previous section and the transformations, Einstein calculates the aberration law in its most general form, the amplitude of the wave as it appears from the moving system, and finally he concludes: "It follows from these results that to an observer approaching a source of light with the velocity *c*, this source of light must appear of infinite intensity." (Einstein, 1905, p. 57).
8. "Einstein derives the transformation law for the energy of a light pulse, and uses this law to derive the work done by radiation pressure on a moving mirror." (Darrigol, 2006, p. 25).

9. In this section Einstein starts again from the Maxwell-Hertz equations, this time also considering the convention-currents. So, he says: “If we imagine that electric charges are invariably coupled to small rigid bodies (ions, electrons), these equations are the electromagnetic basis of electrodynamics and Lorentzian optics of moving bodies.” (Einstein, 1905, p. 60). Then he uses again what he obtained in sections 5, 6 and the transformations, so he concludes: “[...] we have the proof that, on the basis of our kinematical principles, the electrodynamic foundation of Lorentz’s theory of the electrodynamics of moving bodies is in agreement with the principle of relativity.” (Einstein, 1905, p. 60).
10. In this last section Einstein considers an electron moving in an electromagnetic field, then he writes the laws of its motion and applies to them the transformations and equations obtained in section 6. He adopts precise definitions of force and acceleration, consequently he obtains an expression for the longitudinal mass and another for the transverse mass. Thanks to these, he calculates the kinetic energy of the electron, also deriving the fact that there can be no speed greater than that of light. The author goes on to emphasize three points:
    - the first is actually a suggestion for possible future experiments;
    - the second one suggests that there must be a relationship between potential difference and acquired speed of electron;
    - in the third he calculates the radius of curvature of the path of the electron when there is a magnetic force acting perpendicular to the speed of the electron.

    He concludes by saying: “These three relationships are a complete expression for the laws according to which, by the theory here advanced, the electron must move.” (Einstein, 1905, p. 64-65).

H. Minkowski, Space and Time

Minkowski’s involvement with the electrodynamics of moving bodies was documented as early as the summer of 1905, when Einstein’s and Poincaré’s papers had not yet been published. After Minkowski’s lecture “Space and Time” given on September 21, 1908, he claimed that it came as a shock to him to discover that

Einstein had arrived at the same conclusions independently of him, who was waiting to conclude the mathematical structure of the theory (Petkov, 2012).
We have analysed the lecture given at the 80th Meeting of Scientists in Cologne on September 21, 1908, translated by Lewertoff F., Petkov V. in Space and Time: Minkowski's Papers on Relativity, published in 2012 by Minkowski Institute Press. This begins with what has become the author's most famous statement: "Gentlemen! The views of space and time which I want to present to you arose from the domain of experimental physics, and therein lies their strength. Their tendency is radical. From now onwards space by itself and time by itself will recede completely to become mere shadows and only a type of union of the two will still stand independently on its own." (Minkowski, 1908, p. 39). After that the article is divided into 5 sections:

1. In this first section Minkowski states the problem he wants to focus on: "The equations of Newtonian mechanics show a twofold invariance." (Minkowski, 1908, p. 39) and expresses his displeasure with the fact that this has never really been emphasized. The author decides to visualize the situation graphically, so he builds a four-dimensional space, explains that one of the two known groups of homogeneous linear transformations leaves unchanged the expression $x^2 + y^2 + z^2$; while the other allows us to replace x, y, z, t by $x - \alpha t$, $y - \beta t$, $z - \gamma t$, t, without altering the expressions of the laws of mechanics. Minkowski decides to explain completely the theme considering the structure $c^2t^2 - x^2 - y^2 - z^2 = 1$, where c is a positive parameter, it is a two-sheeted hyperboloid. At this point he considers only the sheet in the region $t > 0$ and he geometrically finds transformations that leave it unchanged. He obtains transformations that form a group that depends on the parameter c, which he will call group G_c. Thanks to this he shows us that for $c = \infty$ we get G_∞, which is the complete group associated with Newtonian mechanics. Only at this point Minkowski reveals that c is the velocity of propagation of light in empty space. Finally, he emphasizes how the invariance of the laws of nature with respect to the G_c group allows us to arbitrarily choose an x, y, z, t, space-time reference system.

2. In this section Minkowski introduces his fundamental axiom: "*With appropriate setting of space and time the substance existing at any worldpoint can always be regarded as being at rest.*" (Minkowski, 1908, p. 43). The author then tells us that this means that the expression $c^2dt^2 - dx^2 - dy^2 - dz^2$ is always positive, i.e., that any velocity v is always less than c. At this point he wants to bring arguments in favour of accepting the group G_c, in this regard he gives a couple of examples, but above all he shows geometrically that the Lorentzian hypothesis is equivalent to the new concept of space and time. He concludes by saying that he prefers to call the *relativity postulate* by the name of the *postulate of the absolute world*, or shortly *world postulate*.
3. In this section the author wants: "[…] to explain now how, as a result of this, we gain more understanding of the forms under which the laws of physics present themselves. Especially the concept of acceleration acquires a sharply prominent character." (Minkowski, 1908, p. 45). Therefore, he goes on to define the past lightcone, future lightcone, timelike vectors, spacelike vectors and so on. This is in effect an explanatory section of the geometric space created.
4. Here he goes on to explain the new "rules" of the new geometric space, at least how to interpret force when it is transformed into a force in a new reference system, so what laws of motion will be respected from a substantial point P. He defines the motive force vector, the momentum vector, the force vector of motion at P, so he says: "According to these definitions, the law of motion for a point mass with a given force vector is: *The force vector of the motion is equal to the motive force vector*." (Minkowski, 1908, p. 50). Then he derives the kinetic energy of the point explaining also what role the law of conservation of energy has for the description of the motion.
5. Minkowski in this last section determines the field induced by an electron using the past lightcone and a whole series of geometric properties of the space, explaining to us the physical meaning of the objects described and found. Then he says: "Then it emerges in the description itself of the field

caused by the electron that the division of the field into electric and magnetic forces is a relative one with respect to the specified time axis […]" (Minkowski, 1908, p. 52). He goes on to describe the ponderomotive action of an arbitrarily moving point charge exerted on another arbitrarily moving point charge and revises the Newtonian law of attraction with respect to the world postulate. He restores Kepler's laws and underlines, therefore, how his theory returns harmony between Newtonian mechanics and modern electrodynamics, as well as between mathematics and physics.

4.2 Papers' macro analysis

In this section we expose a first general analysis of the four articles of Lorentz, Poincaré, Einstein and Minkowski. This was achieved by applying to the papers in their entirety the analysis grid, presented in the previous chapter, detecting therefore the boundary crossing mechanisms that the authors adopt along their work. Four main colours were used to identify and highlight on the papers the marker words associated with the four mechanisms: yellow for the *identification mechanism*, blue for the *coordination mechanism*, red for the *reflection mechanism* and green for the *transformation mechanism*. Other two colours were used to identify the parts belonging strictly to the mathematical side (violet), or to the physical side (grey); we present below the legend (Fig 4.).

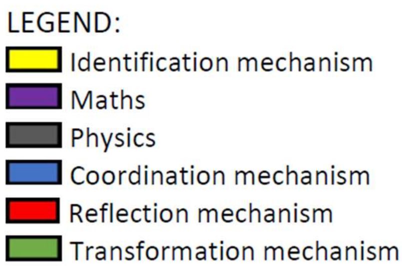

Figure 4 – Colours legend for the mechanisms analysis

We expose the analysis done by showing just a few excerpts from the articles, which we believe exemplify the long work done, also showing how the lens we developed performs. Finally, we also show a graphical representation obtained from the results of the analysis; this was created, as was explained in the previous chapter, by counting the lines of the articles that are attributable to a certain boundary crossing mechanism. Moreover, it has the property of returning at a glance the whole analysis made through the analysis grid: the visualization allows us to recognize what kind of journey the authors have made, what is their style, in the development of their theory and in crossing the relative boundary between mathematics and physics.

Lorentz, 1904

To understand Lorentz's style and path in his article, it is sufficient to show in detail the analysis made on excerpts taken from the first section, to understand what his starting point is, from section 6, to understand how he proceeds after introducing the transformations, and from section 11, which is almost conclusive, in order to understand what his point of arrival is, since section 12 discusses very quickly molecular motion and last two sections of the article are dedicated to the discussion of some experiments.

In the first section, the author starts by discussing some experiments that had the objective of detecting the effects on optical and electrical phenomena of motion through the ether. These are passages that we have largely identified as belonging to the world of physics, since they are practical descriptions of how these experiments were implemented, and what conclusions they led to. In fact, mathematics only comes into play marginally towards the end of the section, and just to translate hypotheses that had been advanced by Trouton and Noble in the context of their experiments.

Let's look the excerpt from this section:

"[…] In the second place Trouton and Noble have endeavoured to detect a turning couple acting on a charged condenser, whose plates make a certain angle with the direction of translation. The theory of electrons, unless it be **modified** by some new **hypothesis**, would undoubtedly require the existence of such a couple. In order to **see** this, it will suffice to **consider** a condenser with aether as dielectricum. It **may** be **show** that in every electrostatic system, moving with a velocity *w*, there is a certain amount of "electromagnetic momentum", **if** we **represent** this, in direction and magnitude, by a vector $\mathfrak{G}$, the couple in question will be **determined** by the vector product

$$[\mathfrak{G} \cdot w] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

[…]” (Lorentz, 1904, p. 809-810).

We clarify that:

- The term *modified* is highlighted in green because we have associated it with the *transformation mechanism*, but in this case, it does not lead to detect such a mechanism since it is a modification that the author wants to make by advancing a hypothesis that descends from the world of physics alone.
- The terms *hypothesis*, *if*, *consider*, *may* are typical of the *coordination mechanism*, in particular of the process of enhancing the boundary permeability, and in this particular case they precisely signal the intention to set the condition in which a chosen physical situation can be described in mathematical language.
- The term *represent* signals the process of *efforts of translation* which generally precedes the transcription into formulas of what the author wants to express;
- The terms *see* and *show*, typical of the mechanism of reflection, are less than those of the mechanism of coordination, moreover we do not perceive any desire to be on the other side of the boundary by Lorentz.

After having introduced the transformations, Lorentz uses mathematics mostly to calculate, formulate, and describe the characteristics of the electrostatic and electrodynamic systems he considers. In section 6 he says:

“[…] Then we **shall obtain** the forces acting on the electrons of the moving system Σ, **if** we first **determine** the corresponding forces in Σ', and next **multiply** their components in the direction of the axis of x by l^2, and their components perpendicular to that axis by $\frac{l^2}{k}$. This is conveniently **expressed** by the formula

$$\mathfrak{F}(\Sigma) = (l^2, \frac{l^2}{k}, \frac{l^2}{k})\mathfrak{F}(\Sigma') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

[…]” (Lorentz, 1904, p. 815).

In this excerpt we find also one of the moments in which the author uses the *routinization* process too: he says “we *shall obtain* the forces acting on the electrons in the moving system”, i.e., he uses the comfort of the other system, so it is enhancing the permeability of the boundary. Then, he chooses to “*multiply* (the components, ndr) by l^2 [...] and $\frac{l^2}{k}$", which is an operation, so it is routine. Finally, he admits that it is convenient to *express* everything with a formula.

The section 11 begins with the sentence:

"It is easily **seen** that the proposed theory can account for a large number of facts. […]" (Lorentz, 1904, p. 824).

The following is a summary of what he has shown up to this point, saying also that:

"[…] They [the above conclusions, ndr] also contain an **explanation** of MICHELSON'S negative result, more general and of somewhat different form than the one previously given, and they **show** why RAYLEIGH and BRACE could **find** no signs of double refraction produced by the motion of the Earth. […]" (Lorentz, 1904, p. 825).

He uses:

- The words *explanation* and *show*, which we have associated with the process of *perspective making*, that is, he makes explicit the signifying content of the calculations, which was implicit. Also the word *find* is associated with this process, but in this case it has a contextual use.
- The word *seen,* that means he places himself on the other side of the boundary, looking at the problem from the mathematical point of view.

The excerpt from the first section, in our opinion, well shows how Lorentz lives in the world of (experimental) physics and coordinates with the world of mathematics, using the latter mostly as a language through which to translate his physical hypotheses. The author maintains this attitude predominantly throughout his work, as shown in the excerpt from section 6. So, for Lorentz it seems often a matter of comfort, convenience, language. Thus, we associated the central body of the article with a *Coordination mechanism*.

However, it would be incorrect to say that Lorentz is confined to this. As mentioned in the previous section, the author does not advance any new hypotheses until section 8, but from there on we recognize an attempt to give an explanation for what he has found. This is evident enough in section 11: he comes to confirm the *hypothesis of contraction*, and in his view, it is this that justifies the results of the experiments. In our opinion this piece reveals a different attitude of the author, in that he recognizes that in the calculations he has made, and the formulas he has found, there may be a solution to his problem, these are not meaningless. It seems that in this part of the paper he adopts a mechanism of reflection.

We conclude underlining that Lorentz defines what he has obtained also *more general*: this is an aspect not to be underestimated, in that it tells us that among the objectives of the author there was the desire to find something, a fact, a truth, that has general value, that does not have to be adapted case by case. This property of generality seems to be guaranteed by mathematics, since it emerges in the course of a *reflection mechanism*.

The latter excerpt is one of those parts in which the analysis grid allowed us to detect a boundary attitude, and thus a narrative style, that we had not been able to see before the application of the lens.

Finally, we show the graphical representation of the analysis (Fig. 5) performed on the article:

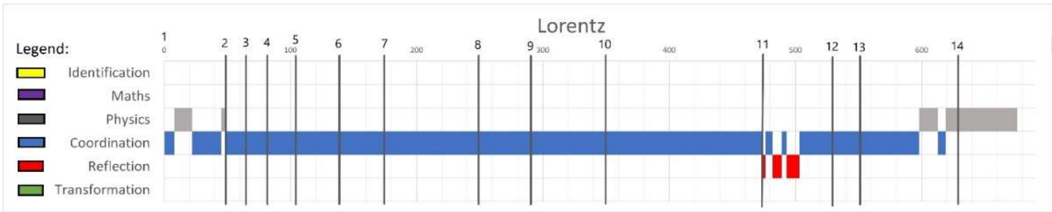

Figure 5 - Lorentz's pattern

This pattern shows succinctly what boundary crossing mechanisms we found along the paper, or the parts of the paper where the author is strictly on one side. One can see what we have found, that is, the fact that Lorentz starts his work from the physical world, then he makes extensive use of the *coordination mechanism*, which always happens by staying mostly on the side of physics. After the 10th section he adopts a mechanism of reflection, which is almost conclusive, since the next sections are devoted mostly to the discussion of other experiments. The latter have therefore been identified as parts that belong to the world of physics.

The identification of the *reflection mechanism* in section 11 allowed us to highlight one of Lorentz goals, that is to find a more *general* explanation with respect to the ones given previously (p. 825). After this, he concludes that his theory has this quality.

Poincaré, 1906

For the analysis of Poincaré's paper it was important not to overlook that there is a long introduction which serves as a summary of the whole article. Of course, we did not limit ourselves to the analysis of this one, but we applied the analysis grid to the entire article. In doing so, we had to consider the fact that the author first explains everything he wants to do, then he does it focusing mostly on the calculations. We noticed also that the introduction is not free from a style that we could define rhetorical. This way of writing makes the analysis a little more complicated, but this does not mean that it has not led to interesting results. So, we will also show a small excerpt from the introduction, because it is illustrative of the attitude the author adopts towards the topic.

We will proceed by discussing what his style is in the main body of the article, showing an exemplifying excerpt from section 5. We conclude with excerpts from sections 8 and 9, not only to understand how Poincaré intends to close his work, but also to emphasize traits that distinguish him. These traits come out in a particular way in these sections and in the fourth, which, however, we will discuss in depth in the next section.

Let's look at the excerpt from the introduction:

"The importance of the **question** led me to take it up again; the **results** that I **obtained** are in **agreement** with those of Lorentz on all important points; I was only led to **amend** and supplement them in some points of detail. The **differences**, which are of secondary importance, will be **seen** later." (Poincaré, 1906, p. 46).

You can see that:

- The terms *question* and *differences* signal a desire to confront within a shared problem space, as by the author's own admission, he feels he must take up again the topic. Considering that the topic is a physical one and that the author is a mathematician, this attitude puts him on the boundary.

- The terms *results*, *obtained,* and *agreement* indicate a *coordination mechanism*, that the author effectively adopts in most of his work. In fact, the article is quite long also because Poincaré does not leave anything to chance, it is his precise intention to justify every step formally.
- We have highlighted in green the term *amend*, and in particular we have identified it as a marker of the *hybridization* process, also for the reasons mentioned above, i.e. it seems to signal the will to take Lorentz theory, a physical theory, and give it some characteristics dear to mathematicians: coherence, consistency, rigour. This puts the author's work again on the boundary.
- The term *seen* signals a mechanism of reflection. In this case we think it signals the intention to show the theory developed through the eyes and the typical practices of a mathematician.

There follow four sections almost entirely devoted to Lorentz Transformations. Now we can consider section 5, in which the author performs several calculations with the purpose of showing the results already obtained by Langevin, but in a different way: using Lorentz Transformations. Poincaré uses words like *means*, *find* and *show* which are associated with the process of *perspective making*, and indeed they highlight this process, since the author uses them referring to the formulas he obtains. He concludes by saying:

"[…] Therefore, at a very distant point, the acceleration wave dominates and can consequently be **regarded** as being the same as the total wave. Additionally, the law of homogeneity **shows** us that the acceleration wave is self-similar at a distant point and at an arbitrary point. […]" (Poincaré, 1906, p. 73).

By using the word *regarded* the author signals to us that it is his mathematical point of view that leads to this new awareness; using the word *shows* he signals that it is the law of homogeneity that brings out physical evidence.

Poincaré justifies Lorentz's hypothesis on the contraction of the electron assuming that there is a special force comparable to an external pressure which is exerted on the deformable and compressible electron. In section 8, on arbitrary motion, he wants to determine this force. Towards the end we find this sentence:

“The theorem is therefore general; at the same time, it **gives** us a solution to the **question** that we asked at the end of §1: to **find** additional forces unchanged by the Lorentz transformation. The additional potential (F) **satisfies** that **condition**.” (Poincaré, 1906, p. 89).

We find:

- Another time the word *question*, which reminds us that Poincaré wanted to solve a problem that concerned both the mathematical apparatus of the theory and the physical concepts that compose it.
- The term *gives,* which tells us that his answer lies in having introduced and justified a special force, something new.
- The term *find* indicates to us that this (finding additional forces unchanged by the Lorentz Transformation) was possible thanks to his mathematical point of view: in fact, he arrives at these conclusions also thanks to his precise calculations and the hypothesis he introduces to make the theory complete. We can consider this approach abundantly mathematical.
- The terms *satisfies* and *conditions*, which remind us of the way he has coordinated in detail (with calculations and hypotheses) to arrive at these conclusions.

In the last section, as mentioned in the previous section, the author wants to examine in detail assumptions about gravitation that follow from Lorentz theory (and transformations). In particular, he wants to see the significance of the affection of the Newtonian force by Lorentz Transformations. Let’s look at this excerpt:

“[…] It can be **seen** that **if** the two bodies are simply driven in a shared translation, the force which acts are the attracted body is normal to an ellipsoid that has the attracting body at its center.

To go further, we need to **seek** the *invariants of the Lorentz group*.

We know that the substitutions from this group (with the assumption $l = 1$) are linear substitutions which do not change the quadratic form:

$$x^2 + y^2 + z^2 - t^2.$$

[…]” (Poincaré, 1906, p. 93).

We can see that the word *if* is used, which also by its meaning tends to precede a hypothetical attitude. The word *seek* announces to us that calculations will be made, while the word *seen* indicates that the author has arrived at certain conclusions by reviewing what he has obtained through a mathematical perspective, in the sense that he feels deep confidence in this and is guided by this feeling. The last sentence of the excerpt is just the beginning of a discrete segment in which, as the author himself says, he searches for invariants of the Lorentz group: this is a work that could be extrapolated from the article and would still make sense for the mathematical world, and for this reason we decided to identify it as belonging strictly to mathematics.

We discuss now the results of the analysis. In the introduction the author declares one of his main goals: to introduce the Principle of Relativity in the theory of optical and electrical phenomena. So, he recognizes a shared problem space, he compares himself with already existing theories and he wants to verify and/or modify them in order to make everything rigorous and coherent. Therefore, what has prevailed through our lens is a mechanism of transformation. However, there is more: we also detected both a *coordination mechanism* and a *reflection mechanism*. For these reasons, the first excerpt anticipates what will be the general attitude of the author. There follow four sections devoted almost entirely to Lorentz Transformations. The fourth section is fully devoted to showing that Lorentz Transformations do form a group, but in the first three, he proceeds by explaining briefly at the beginning what he wants to do, and continues by performing many calculations, concluding by explaining what is the meaning of the formulas he obtained. Indeed, it is interesting to note that throughout the article we have detected a good number of passages in which the author inevitably adopts a mechanism of reflection, sometimes placing himself on the side of physics, sometimes on the side of mathematics, and in particular it is often used in the incipit and/or conclusion of the sections. This can be seen well from the excerpt we took from section 5, in which we can observe how the author uses the Lorentz Transformations as a sort of headlight, looking at mathematical results as a new way to describe a physical phenomenon.
The intentions stated in the introduction are definitely realized in the eighth section. The analysis we have made on this one, and the excerpt we have chosen, show how

the author’s transformative approach leads to the assumption of something new: a special force. The mechanism of transformation we have detected is justified by the fact that we are dealing with a physical object in a physical theory, but it is obtained through the rigour and approach of a mathematician.

Finally, the analysis also highlighted the fact that the author actually remains a mathematician throughout the article. This may seem like a predictable result, but it is worth pointing out. Much of Poincaré’s accomplishments are due to the fact that he still remains anchored to the world of mathematics. This is well seen in the section where he proves that Lorentz transformations form a group, and in the last section. In the latter we found other passages in which the author adopts a mechanism of reflection, placing himself on the side of the mathematical world to study a physical phenomenon. At some point, however, he evidently feels the need to fully immerse himself in the world of mathematics, as if the *reflection mechanism* acted as a trampoline.

At this point it may help to visualize the diagram that is derived from our analysis of the article (Fig. 6).

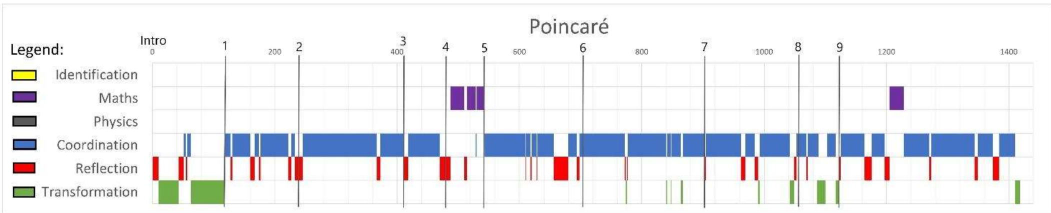

Figure 6 - Poincaré's pattern

Thanks to this image, it is easier to realize the complexity of the path that Poincaré undertakes. However, we can summarize what we have said so far: the author begins with a great desire to confront by recognizing a shared problem space, he proceeds with coordination passages that are often preceded and/or concluded with reflection passages, which sometimes led him to reside completely in the world of mathematics. There are also moments in which the mechanism of transformation is adopted, in the sense that full *rigour*, coherence and consistency (mathematical world) are restored to the Lorentz theory (physical world).

Einstein, 1905

We will expose the analysis done on Einstein's paper by discussing the introduction and showing two excerpts from it, in order to observe the way it poses and begins its journey. We will proceed with two exemplary excerpts from the first section, which are equally important to detect the basis of the author's work. Finally, we will show an excerpt from section 8, which will be sufficient to give concrete evidence of the results obtained from the analysis done on the main body of the article.

Starting with the introduction, the author immediately states what is the problem he wants to address:

"It is known that Maxwell's electrodynamics—as usually **understood** at the present time—when **applied** to moving bodies, **leads** to **asymmetries** which do not **appear** to be inherent in the phenomena. […]" (Einstein, 1905, p. 37).

We clarify that:

- The word *asymmetries* reveals, in this context, a problem related to Maxwell's electrodynamics, as it is *understood*, the latter is a word we have associated with the process of *crystallization*, which lies behind the *transformation mechanism*, and in fact signals the fact that the theory Einstein wants to focus on is more than accepted, although it has problems.
- These problems arise when it is *applied* to moving bodies: it is not a coincidence, in our opinion, that here the author refers to a coordination practice between mathematics and physics.
- In the moment in which he uses the word *leads*, he is referring to the fact that it is mathematics that points out something that, from the point of view of the physical world, does not *appear* inherent in phenomena.

So, it seems that Einstein is moving quickly from one side to the other, discussing a shared problem, leading us to think he is moving right on the boundary between mathematics and physics.

The author proceeds by using in his favour the example of the reciprocal electrodynamic action between a magnet and a conductor, describing in detail how the phenomenon depends only on their relative motion. This long argument has an extremely physical character, so we have identified it as such. For Einstein it is this example that justifies raising the Principle of Relativity to a postulate. Now let's see another excerpt:

"These two postulates suffice for the attainment of a simple and consistent theory of the electrodynamics of moving bodies based on Maxwell's theory for stationary bodies. The introduction of a "luminiferous ether" will **prove** to be superfluous inasmuch as the **view** here to be **developed** will not require an "absolutely stationary space" provided with special properties [...]" (Einstein, 1905, p. 37).

Einstein's words are clear: what he develops is a *view*. In particular, it owes its consistency to the type of reasoning he adopts, namely axiomatic reasoning, typical of the mathematical world. We have associated the words *developed* and *prove* respectively to the processes of *hybridization* and *crystallization* typical of the *transformation mechanism*: even though the former refers to a view, as mentioned above, and the latter seems to be mostly used in a contextual way, it is always the view that proves that the ether is superfluous.

In the first section Einstein proceeds with a brief passage aimed mostly at describing the objects he wants to deal with, such as the stationary system. In talking about these objects, he focuses on the concepts of time and simultaneity. Here we found a remarkable sentence:

"Now we must bear carefully in mind that a mathematical description of this kind has no physical meaning unless we are quite clear as to what we understand by "time."" (Einstein, 1905, p. 39).

Applying the analysis grid, we found processes that were discordant with each other and not inherent to the context. For example, we find the word *must*, which in other cases has signalled processes of enhancing the boundary permeability, but we felt that in this context the author is not looking for a way to coordinate with the world of mathematics. The sentence has a rhetorical character, so we decided to try to interpret it in its entirety, without using the marker words. At first, we thought we were detecting an *identification mechanism*, because it may seem that Einstein is making a distinction between what mathematics does (a description) and what physics does (giving meaning to the concept of time). However, we were not entirely convinced of this decision, because it can be interpreted in the opposite way: physical meaning and mathematical description need each other. We were able to make a final decision after we finished our analysis of the whole section. Continuing there follows a considerable section in which the concept of time and the concept of simultaneity are argued by talking about trains, clocks, positions, ways of determining time. We have identified this last piece as belonging to the world of physics. However, the one that follows, the operational definition of simultaneity, was particularly interesting through our lenses, and it also seems to make sense of the sentence above. In fact, said that Einstein calls t_A the time of a starting ray of light from A in the direction of B, t_B the time in which it reflects from B in the direction of A, and t'_A the time in which it arrives again at A, he goes on to say:

"In **accordance** with **definition** the two clocks synchronize **if**

$$t_B - t_A = t'_A - t_B.$$

We **assume** that this **definition** of synchronism is free from **contradictions**, and possible for any number of points; and that the following relations are universally **valid**: -

1. **If** the clock at B synchronizes with the clock at A, the clock at A synchronizes with the clock at B.
2. **If** the clock at A synchronizes with the clock at B and also with the clock at C, the clocks at B and C also synchronize with each other." (Einstein, 1904, p. 40).

Although the markers of the *transformation mechanism* may seem few, they were significant because they stimulated us to realize that the operational definition given by Einstein makes the simultaneity relation an equivalence relation:

- The words *accordance* and *if* indicates that he's moving from the physical world of clocks, time, etc., to the mathematics world enhancing the boundary permeability.
- The word *definition* concludes his first move by translating in mathematical language the physical concept of synchronism.

- The word *contradiction* signals to us that Einstein continues to move toward the boundary, he confronts in a shared problem space.
- With the words *assume* and *valid* he decides to hybridize and crystallize the concept of simultaneity by associating to it the symmetric property (1) and the reflexive and transitive properties (2).

For the rest of the article the author adopts a precise and recursive style. Take for example section 8, which like most other sections, begins almost immediately with a series of calculations. In this case they were aimed at finding the laws describing the energy of a light ray in the moving system and the pressure of light exerted on a reflecting surface. However, the section ends with the following sentence:

"What is essential is, that the electric and magnetic force of the light which is influenced by a moving body, be transformed into a system of co-ordinates at rest relatively to the body. By this **means** all **problems** in the optics of moving bodies will be **reduced** to a series of **problems** in the optics of stationary bodies." (Einstein, 1905, p. 59).

The word *means* signals that the mathematical apparatus of the theory has brought out a new way of approaching *problems* in the optics of moving bodies. It is the mathematics that allows the author to *reduce* these problems to problems in the optics of stationary bodies.

The analysis we conducted on Einstein's article revealed, among other things, the fact that the author had a very clean style. His line of thought is clear, and his style is recognizable throughout the entire paper.
We see from the first excerpt that he starts with the processes of *confrontation* and joint work at the boundary, identified through the detection of a dynamic *reflection mechanism*, i.e., he goes from a mathematical to a physical perspective very quickly. For these reasons, we identified this short sentence as a step that belongs to the *transformation mechanism*. This is followed by a segment that we have identified as belonging to the physics side which let us understand that the author's home is the physical world. Again, this may seem like a natural outcome, but it gives us a way to better understand his boundary behaviour after this last step. In

fact, in this way we can say even more strongly that in the second excerpt from the introduction there is a *reflection mechanism*: he uses axiomatic reasoning.
In the first section, as mentioned above, Einstein adopts a mostly descriptive style: it is a translation of his physical thoughts into mathematical objects, which is why we have identified this part with a *coordination mechanism*. Really relevant is the excerpt on the operational definition of simultaneity: this tells us how much Einstein placed himself on the boundary between mathematics and physics and found strength in the interdisciplinarity between them; maybe that's what he was trying to tell us with that sentence that seemed so ambiguous, i.e., in this case, for full understanding, we need to stand on the boundary between mathematics and physics. This is a key step in his paper as the operational definition of simultaneity plays a key role, along with its two postulates, in the development of his theory.
Regarding the body of the article: for each section he remains mostly on the side of the world of physics, coordinates with the world of mathematics, making mostly calculations, and finally concludes with a brief reflection piece. We have shown this by discussing section 8 and the analysis performed on the extracted sentence that concludes it.
We now show the graphical representation of our analysis (Fig. 7):

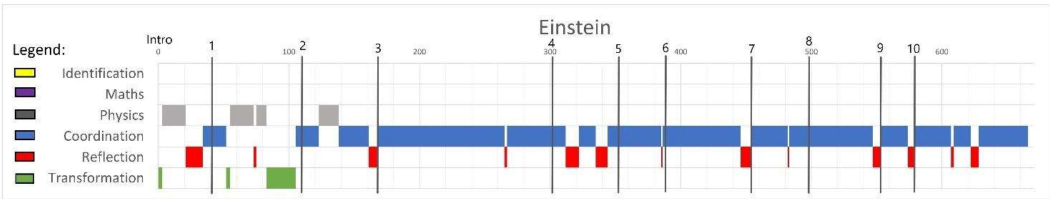

Figure 7 - Einstein's pattern

The image shows well what we have said so far: Einstein's home is the world of physics, i.e., a world of observations, experience, measurements, etc., and he starts from this for the development of his theory. Afterwards the author moves to the boundary, uses axiomatic reasoning, in order to create a consistent theory. Once the foundation is laid, he proceeds by describing and characterizing the theory by making the appropriate calculations, but never leaving anything unexplained.

Minkowski, 1908

When we analysed Minkowski's paper, we tried not to forget that it is a transcript of a speech, so it inevitably has a rhetorical style. We will not discuss in depth the first two sections of the paper here: this will be done in the next section. In any case, there are also other passages in the author's paper that can help us understand his style. We will see the analysis on the famous introductory sentence of the paper, since it has an undoubted summarizing function of the whole work. We will discuss two short sentences extracted from the first section, to understand with which approach Minkowski deals with the subject. In order to understand how he performs after introducing and developing Lorentz Transformations, we will expose the analysis made on a sentence from the third section, and another short excerpt which is the beginning sentence of the fourth section. Finally, to understand what the author has led us to, we will analyse the last proposition of the last section.

Let's start seeing how the famous sentence that opens the article is detected by our lenses:

"Gentlemen! The **views** of space and time which I want to **present** to you **arose** from the domain of experimental physics, and therein lies their strength. Their tendency is radical. From now onwards space by itself and time by itself will recede completely to **become** mere shadows and only a type of union of the two will still stand independently on its own." (Minkowski, 1906, p. 39).

Minkowski openly states his intentions:

- The term *view* says per se that the author has adopted a certain perspective regarding the concepts of space and time.
- The term *present* signals, in our opinion, the will to adopt a certain (mathematical) style in exposing his theory.
- The term *arose* signals the fact that the author also considers the point of view of the world of physics.
- The term *become* belongs usually to the mechanism of coordination, but in this case, we have not attributed particular meanings to it, in that in this case it seems to have mostly a contextual use.

Minkowski is a mathematician who claims to present a new view of space and time, which moreover arose from the domain of experimental physics.

Regarding the first section, we believe that Minkowski wanted to emphasize the importance of *continuous joint work at the boundary* between mathematics and physics, in fact, the first section begins by saying:

"I want to show first how to move from the currently adopted mechanics through purely mathematical reasoning to modified ideas about space and time." (Minkowski, 1908, p. 39).

Afterwards he talks about the fact that only one of the two groups that leave the equations of Newtonian mechanics unchanged was considered. This was considered a piece where the author recognizes a shared problem space and wants to confront this physical topic like a true mathematician. In fact, our analysis has revealed that the author initially approaches the subject in a very mathematical and technical manner. The first steps he takes are aimed at constructing a four-dimensional space, and he states in this regard: "[…] the somewhat greater abstraction associated with the number 4 does not hurt the mathematician." (Minkowski, 1908, p. 40), as if he wanted to maintain the uniqueness of the intersecting practices by signalling that, although the topic is physical in nature, he is also speaking about mathematics and as a mathematician. The author continues by devoting the first and second sections to Lorentz Transformations, the world postulate and the physical meanings these can have.

To see the mathematics approaches of Minkowski, we can also consider parts of the article where the author moves starting from the mathematical world. In the third section, where his style is mostly descriptive, each mathematical object in the geometric space he created obtains a physical meaning. We can look at an excerpt taken nearly from the beginning of the section:

"[…] The first, the *past lightcone* of O, **consists**, we can say, of all worldpoints which "send light to O", the second, the *future lightcone* of O, **consists** of all worldpoints which "receive light from O". The area bounded solely by the past lightcone may be **called** *before* O, whereas the area bounded solely by the future lightcone - *after* O. […]" (Minkowski, 1908, p. 46).

We have associated the words *consists* and *called* to the process of *efforts of translation*. Moreover, in this section, as already mentioned, he gives many definitions and performs some calculations, to show how the space he has created can be associated with physical concepts and can lead to laws on phenomena.
In any case, our analysis led us to detect a boundary attitude more complex than this.

Considering section four, here the author focuses a lot on the meaning of force in the new space. Let's extract the first sentence:

"To **demonstrate** that the adoption of the group G_c for the laws of physics never leads to a **contradiction**, it is inevitable to **undertake** a revision of all physics based on the **assumption** of this group. […]" (Minkowski, 1908, p. 49).

At this point Minkowski wants to show that even the laws of mechanics are consistent with the new apparatus and vice versa. In fact, we find:

- The word *assumption,* that signals the creation of something new, in this case the G_c group that also acts as a generator of a four-dimensional space, then a space-time; also for this reason we can associate the word to the process of *hybridization* that is part of the mechanism of transformation.
- The word *contradiction* signals the author's desire to confront, to show that there are no problems with this new view.
- He also uses the words *demonstrate* and *undertake* to refer to the G_c group: these have been associated with the *crystallization* process because they signal that there is a commitment in wanting to confirm and validate the object he has hybridized.

We noticed that this style is repeated, i.e., all sections begin with such an attitude. In the rest of the sections, he often continues to describe objects and make some calculations, as seen for section three. It happens that he alternates these kinds of moments with moments in which he explains some physical meanings, for example in section 5 they are frequent.

A piece we can't overlook is the end of this section, i.e., the last one:

"[…] With the **development** of the mathematical consequences of this postulate, sufficient **findings** of its experimental **validity** will be arrived at so that even those to whom it seems unsympathetic or painful to abandon the prevailing **views** become reconciled through the thought of a pre-stabilized harmony between mathematics and physics." (Minkowski, 1906, p. 53).

Here there is a part where the author adopts a physical perspective: this is well signalled by the word *findings*, but we believe it was his intention to conclude at the boundary. In particular we have entirely highlighted the proposition as a process of *continuous joint work at the boundary*, since there are markers words of the processes of *hybridization* and *crystallization* (*development* and *validity*), so belonging to the *transformation mechanism*, but especially because Minkowski explicitly speaks of harmony restored between mathematics and physics.

Analysing Minkowski's paper with an objective eye was not easy, due to the rhetorical style. However, this did not make us give up on the idea that an interdisciplinary objective was consciously present among the author's intentions. We want to think that it is consistent to identify the introductory sentence as a passage of reflection, also given all that we have said, namely that the author always remains very much connected to his mathematical world, but that he strongly wants to address a physical theme.

In the first section Minkowski immediately places himself in a shared problem space, making a great confrontation, so he moves to the boundary. And not only that, but we also became convinced that for the author the new view of space and time, the four-dimensional space, the G_c group and the new mechanics are actually one object, one entity, which has both physical and mathematical nature, thus a transformed nature. The article is focused mostly on the development of a theoretical apparatus concerning these four things and their connection, so it turns out to be associated for the most part with a mechanism of transformation.

For the rest of the article, in fact, he chooses to begin each section always with a brief transformation sentence, because his principal objective is crystallizing this new hybrid object.

Despite this, let us not forget that in the central body of the sections, as well as in the first one, a *coordination mechanism* that starts from the world of mathematics prevails. The moments when the author brings out physical meanings from his work have been identified as reflection passages, that is, Minkowski takes the physical perspective.
In any case, the last statements of the article remind us of what was the great force that led the author to obtain his revolutionary view of space-time: interdisciplinarity. In particular, the use of *abstraction*, a typical mathematical practice, in a world that usually does not use it, physics.

Now we can summarize what we found by looking at the analysis representation (Fig. 8):

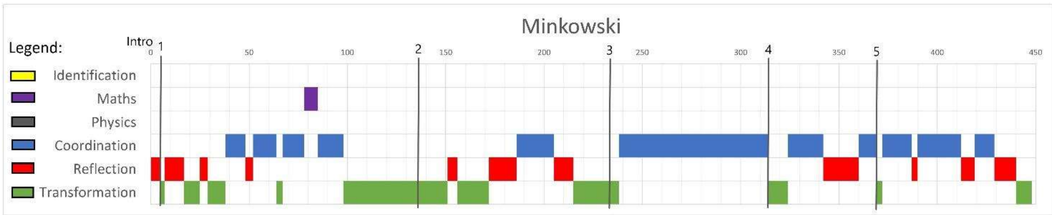

Figure 8 – Minkowski's pattern

We can see that Minkowski starts by self-reflecting on the physics side of the boundary. It lays the groundwork for a mechanism of transformation: *confrontation and recognizing a shared problem space*. So, he constructs the four-dimensional space and finds the G_c group beginning from the mathematical world and coordinating with the world of physics. After that he moves to the boundary showing the connections between four-dimensional space, G_c group, space-time, and new mechanics, emphasizing the importance that abstraction has for this journey. He continues and concludes by crystallizing the object he has hybridized.

4.3 Sections on the Lorentz Transformations analysis

In this section we present a finer analysis, conducted only on those sections of the four articles that feature Lorentz Transformations. In fact, we are convinced that it makes sense to focus on the latter for several reasons:

- they are a fundamental aspect of the theory;
- all four authors derive and interpret them in different ways;
- they form the basis of what is generally taught in schools.

Each of the four authors has a different starting point and different goals, but they all go through Lorentz Transformations. Therefore, we were led to think, using an allegorical image, that the transformations acted as a bridge between the starting point and the arrival for each of them. All four crossed the same bridge, but in four different ways. Therefore, for the type of research that concerns us, it seemed particularly appropriate to focus on the sections devoted to these. We decided to highlight the individual processes underlying the boundary crossing mechanisms too. We assigned to each of these a colour that belongs to the range of shades of the colour of the mechanism to which they belong. Shades of yellow for processes belonging to the *identification mechanism*, shades of blue for processes belonging to the *coordination mechanism*, shades of red for processes belonging to the *reflection mechanism*, shades of green for processes belonging to the *transformation mechanism*. We have maintained the colour violet for mathematics, and the colour grey for physics. Legend follows (Fig. 9). So, as in the previous section, we will show the analysis conducted on some exemplary excerpts and graphical representations similar to the previous ones. These were also created by counting the number of lines of articles attributable to the different processes. However, overlaps are visible in these, a natural consequence of a more detailed, and thus more complex, analysis. In this section, we also show the actual lines count, which will help us formulate further hypotheses on the styles of the four authors and the nature they attributed to the Lorentz Transformations.

Figure 9 - Colours legend for the processes analysis

Lorentz's Transformations

Lorentz derives and treats his transformations in sections 3 and 4. We will show an excerpt from each of these and the corresponding analyses.

Of section 3 we will show the incipit, to get an idea of how the author sets up his work:

"I **shall** start from the fundamental equations of the theory of electrons. **Let** $\mathfrak{d}$ be the dielectric displacement in the aether, […]. Then, **if** we **use** a fixed system of coordinates,

$$\left.\begin{aligned} div\mathfrak{d} = \varrho,\ \ div\mathfrak{h} = 0, \\ rot\mathfrak{h} = \frac{1}{c}\left(\dot{\mathfrak{d}} + \varrho\mathfrak{v}\right), \\ rot\mathfrak{d} = -\frac{1}{c}\dot{\mathfrak{h}}, \\ \mathfrak{f} = \mathfrak{d} + \frac{1}{c}[\mathfrak{v}\cdot\mathfrak{h}]. \end{aligned}\right\} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (2)$$

I **shall** now **suppose** that the system as a whole moves in the direction of *x*, with a constant velocity *w,* […]" (Lorentz, 1904, p. 811).

We find clear markers:

- The word *let* is one of the most significant markers of the *efforts of translation* process, and indeed in this passage it indicates to us precisely the willingness to assign names and letters to physical quantities, thus an actual translation. The first step towards modelling the situation.

- The words *shall*, *if* and *suppose* are, by their very nature, signal a hypothetical process, which can be associated with the process of *enhancing boundary permeability*. Recall that we have assigned to this process precisely the hypothetical role of modelling.

- The word *use* betrays the utilitarian nature of the author's action. Thus, he refers to an object (a fixed coordinate system) that may be convenient for modelling purposes, which is also why we have associated it with the process of *communicative connection*.

Lorentz goes on to perform mostly calculations, hence routine processes. It is in section 4 that he first shows the transformations, right from the start:

"We **shall** further transform these formulae by a change of variables. **Putting**

$$\frac{c^2}{c^2 - w^2} = k^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and understanding by *l* another numerical quantity, to be **determined** further on, I **take** as new independent variables

$$x' = klx\,, \;\; y' = ly\,, \;\; z' = lz\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$t' = \frac{l}{k}t - kl\frac{w}{c^2}x\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

[…]" (Lorentz, 1904, p. 812).

In addition to the word *shall*, already mentioned, we find the words *putting* and *determined* that are part of the process of *enhancing boundary permeability*. In fact, we are led to think that the author is coordinating with the world of mathematics, manipulating the objects he has defined, with the only aim, as by his own admission, of obtaining a convenient change of variables. The word *take*, like the word *use* above, signals to us the utilitarian meaning of the object in question, but this time it is precisely the Lorentz Transformations.

The author proceeds with more or less the same approach until the end of the section. It may be useful at this point to show the detailed graphical representation of the path (Fig. 10):

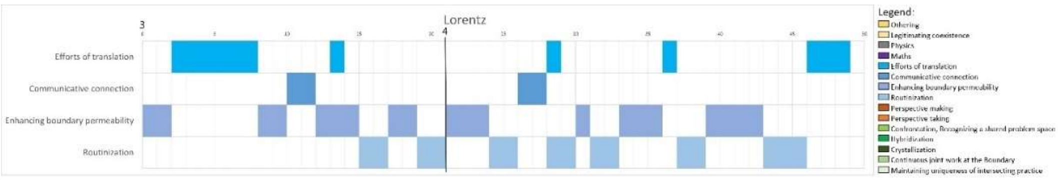

Figure 10 - Lorentz's processes

Interestingly, Lorentz does not spend much time on transformations, neither on their meaning nor on how he derives them. Our analysis led us to think that he saw them as a mathematical object with no physical meaning, but useful to better develop his theory. One of the factors that led us to this conclusion was the emergence of the predominance of the process of *enhancing boundary permeability*. This can be perceived from the graph above, but to confirm this we report the line count (Tab. 7):

Processes	%	Absolutes
Efforts of translation	23.5	12
Communicative connection	7.8	4
Enhancing boundary permeability	**39,2**	**20**
Routinization	29,4	15

Table 7 – Lorentz's count

So, we can call Lorentz's style: hypothetical-deductive style. The author goes through transformations, but he has a very strong physical apparatus of knowledge and beliefs, so he does not let them play a leading role. Their placement within the article also suggests this. They are preceded by arguments about various experiments, thus belonging essentially to the world of physics. Their purpose also seems to be only to facilitate calculations. We are therefore inclined to think that for Lorentz the transformations played the simple role of a useful object.

Lorentz Group

Unlike Lorentz, Poincaré discusses the Lorentz Transformations at length. He dedicates no less than four sections to them. Let us remember, however, that Poincaré's article is the longest of the four analysed, and that his style on the boundary is dynamic, sometimes complex. We will show the analysis performed on one excerpt from section 1, two excerpts from section 2, two excerpts from section 3 and three excerpts from section 4.

Poincaré begins by showing a set of formulae that suits the physical situation under study. The excerpt from section 1 can be found almost at the beginning of the section:

"[…] An element of matter of volume *dxdydz* experiences a mechanical force whose components *Xdxdydz*, *Zdxdydz*, *Ydxdydz* are **determined** from the formula:

$$X = \rho f + \rho(\eta\gamma - \zeta\beta). \quad (2)$$

These equations are subject to a **remarkable** transformation discovered by Lorentz and which is of interest because it **explains** why no experiment is able to let us know the absolute motion of the universe. **Let** us **set**:

$$x' = kl(x + \varepsilon t)\ ,\ t' = kl(t + \varepsilon x)\ ,\ y' = ly\ ,\ z' = lz \qquad (3)$$

[…]" (Poincaré, 1906, p.48).

We find markers of the *coordination mechanism* already discussed: *determined* and *let*, to which we add the word *set*, which we associated to the process of *enhancing boundary permeability*. We think it is normal that when a physical situation is modelled, there is always a trace of *efforts of translation* and *enhancing boundary permeability*. What is interesting in this excerpt is the position the author takes when he uses the words *remarkable* and *explains*. The former has been associated with the process of *perspective taking*: in fact, it announces to us that Poincaré has taken the mathematician's point of view, so he cannot overlook the nature of the Lorentz Transformations. It is his intention to give due credit to this object. Consistently, the second word has been associated with the process of *perspective making*. The author states that, in his opinion, it is the Lorentz Transformations that make explicit the knowledge that no experiment will ever have the desired outcome. Poincaré looks at transformations from another point of view and wants to give them new importance.
He proceeds by meticulously verifying that they leave all the formulas of the initial set unchanged.

In the second section he discusses the problem in its most general way. We first show the analysis of a quasi-initial sentence:

"[…] I will however go back over the **question** because I prefer to **present** it in a slightly different form which will be useful for my purpose. […]" (Poincaré, 1906, p. 54).

The word *question* gives us a reminder that we have already noted that the author has a strong desire for confrontation in a shared problem space. On the other hand, the word *present* again indicates to us that Poincaré adopts a personal way of approaching the problem, which derives from his mathematical knowledge and practice. This is also suggested by the way the author proceeds: rigorously, making nearly all the calculations explicit. In fact, he concludes the section as follows:

"[…] it **follows**:

$$X = \rho f + \rho(\eta\gamma - \zeta\beta)$$

This is equation (2) from the previous section." (Poincaré, 1906, p. 61).

The word *follows* occurs very frequently throughout the calculations of all four authors, so it is typical of the *routinization* process.

In section 3, Poincaré works to clarify the physical meaning of the Lorentz Transformations. The first excerpt we will show is the opening sentence of the section and is a clear statement of intent:

"We are going to **see** **if** the principle of least action gives us the reason for the success of the Lorentz transformation. […]" (Poincaré, 1906, p. 61).

Again, the word *if* foretells a hypothesis. While the word *see*, also by its nature, has been associated with the process of *perspective taking*. In fact, we are convinced that it indicates the author's positioning on the side of the world of physics: it is a physical meaning he is seeking for the Lorentz Transformations.

Following a long calculative development there is a clear concluding sentence of the section:

"[…] The principle of least action therefore **leads** us to the same result as the analysis from §1. […]" (Poincaré, 1906, p. 64).

In this case, the word *leads* indicates the extraction of new knowledge, typical of the process of *perspective making*.

Section 4 gives us the opportunity to observe in detail an attitude at the boundary that is peculiar in Poincaré: the use of the *reflection mechanism* as a kind of trampoline to return to the world of mathematics. The section begins simply with the sentence:

“It is important to **note** that the Lorentz transformation do form a group. […]” (Poincaré, 1906, p. 64).

It seems quite reasonable to think that such an awareness comes more naturally to a mathematician than to a physicist. This is why the word *note* has been attributed to the process of *perspective taking*: this time Poincaré looks at transformations from a mathematical point of view. However, in this section the author does not simply adopt a perspective. The section could be excerpted almost completely from the article, and would still make sense for the mathematical world:

“[…] We are therefore led to consider a continuous group that we will call the Lorentz group in which will allow as infinitesimal transformations:

1) the transformation T_0 which will be permutable with all the others;

2) the three transformations T_1, T_2, T_3; and

3) the three rotations $[T_1, T_2]$, $[T_2, T_3]$, $[T_3, T_1]$. […]” (Poincaré, 1906, p. 66).

However, the author never loses focus, in fact in this section he will also say:

“[…] But for our purposes, we **should** only **consider** a part of the transformations from this group […]” (Poincaré, 1906, p. 66).

The words *should* and *consider* indicate a hypothetical process of the author. In our opinion, this signals that he still wanted to coordinate with the world of physics.

In conclusion, in our opinion, for Poincaré the Lorentz Transformations are an object that possesses various properties from both a physical and mathematical point of view. However, he want to show them as rigorously as possible, not omitting any steps. For this reason, reflexive passages are frequent, but there is no shortage of coordinative ones. To get a synthetic idea of this, it is useful to show the graphical representation obtained (Fig. 11):

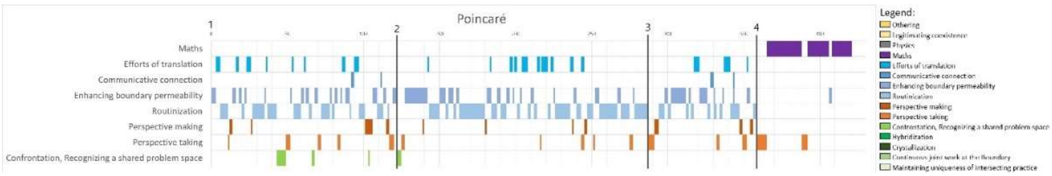

Figure 11 - Poincaré's processes

This analysis shows another characteristic of the author's style: the long adoption of the *routinization* process. We have not discussed this kind of passages in detail, because these are mostly lines in which formulae, calculations and operations of various kinds appear. Furthermore, let us recall that, even in the case of several formulae one after the other, it was decided to count only one line. It may therefore be useful for the discussion to add the actual count now (Tab. 8):

Processes	%	Absolutes
Maths	11,5	50
Efforts of translation	10,8	47
Communicative connection	1,4	6
Enhancing boundary permeability	23	100
Routinization	**36,4**	**158**
Perspective making	5	22
Perspective taking	9	39
Confrontation, Recognizing a shared problem space	2,7	12

Table 8 – Poincaré's count

Finally, let us also emphasise the way in which the author places the Lorentz Transformations within the article: he discusses them in the beginning, he justifies and characterises them, with the intention of later being able to use them in full confidence to simplify calculations, to verify the correctness of various hypotheses of the theory of electron dynamics and to attempt to investigate gravitation. Thus, we can say that they act as a true headlight.

Transformation of co-ordinate and their meaning

We have identified two sections in which the Lorentz Transformations play a leading role in Einstein's work: section 3 entitled "Theory of the Transformation of

Co-ordinates and Times from a Stationary System to another System in Uniform Motion of Translation Relatively to the Former”, and section 4 entitled “Physical Meaning of the Equations Obtained in Respect to Moving Rigid Bodies and Moving Clocks”. We show the analysis made on three excerpts from the former, and two excerpts from the latter.

The first excerpt we present is located almost at the beginning of the section. It follows a large segment in which the author describes the situation and assigns variables, which is why we have identified it as an adoption of the *efforts of translation* process.

Let us therefore see how he proceeds:

“[…] To any system of values *x, y, z, t*, which completely **defines** the place and time of an event in the stationary system, there belongs a system of values ξ, η, ζ, τ, **determining** that event relatively to the system *k*, and our task is now to **find** the system of equations connecting these quantities.
In the first place it is clear that the equations **must** be linear **on account of** the properties of homogeneity which we attribute to space and time. […]” (Einstein, 1905, p. 43-44).

We therefore see the following markers:

- The word *defines*, which belongs to the *efforts of translation* process since it involves a transposition of physical concepts (place and time of an event) into a mathematical object (a system of values).

- The word *determining*, which in this case is associated with the *routinization* process since the author is referring to a frequent practice: switching to other variables.

- The word *must* and the expression *on account of* which reveal a process of *enhancing boundary permeability*. Clearly Einstein starts from his physical assumptions, and he expects that these will correspond to a certain type of mathematical property, so it is a deductive hypothetical process.

At this point he goes on to do various calculations, arriving at a first version of the Lorentz Transformations:

"[…] **Substituting** for *x'* its value, we **obtain**

$$\tau = \phi(v)\beta(t - vx/c^2),$$
$$\xi = \phi(v)\beta(x - vt),$$
$$\eta = \phi(v)y,$$
$$\zeta = \phi(v)z,$$

where

$$\beta = \frac{1}{\sqrt{1 - v^2/c^2}},$$

and ϕ is an as yet unknown function of *v.* […]" (Einstein, 1905, p. 45-46).

In fact, we find typical markers of the *routinization* process: *substituting* and *obtain*, since they signal mathematical operations and calculations. Instead, the word *where* signals to us that a name has been given to a quantity, hence a process of *efforts of translation*.

However, there is also a moment that must be noted: Einstein considers a spherical wave emitted with velocity *c* at the moment $t = \tau = 0$, therefore he finds that it has the same equation in both reference systems, and states:

"[…] The wave under **consideration** is therefore no less a spherical wave with velocity of propagation *c* when viewed in the moving system. This **shows** that our two fundamental principles are compatible. […]" (Einstein, 1905, p. 46).

We find the word *consideration*, which indicates that the author has modelled a spherical wave. Especially interesting is the use of the word *shows*, which is a marker of the process of *perspective making*. In effect the author is telling us that from the calculations we deduce that the physical apparatus adopted is consistent. We can therefore think that Einstein adopted a mathematical point of view by making explicit and confirming something that was previously assumed and implicit.

In section 4, the author maintains this attitude. He begins by considering a rigid sphere in motion with respect to the system K and at rest with respect to the system k, then says:

“[…] The equation of the surface of this sphere moving relatively to the system *K* with velocity *v* is

$$\xi^2 + \eta^2 + \zeta^2 = R^2.$$

The equation of this surface **expressed** in *x, y, z* at the time *t = 0* is

$$\frac{x^2}{(\sqrt{1 - v^2/c^2})^2} + y^2 + z^2 = R^2.$$

[…] For velocities greater than that of light our deliberations become meaningless; we **shall**, however, **find** in what follows, that the velocity of light in our theory plays the part, physically, of an infinitely great velocity. […]” (Einstein, 1905, p. 48).

Also in this excerpt we find a natural process of *efforts of translation* signalled by the word *expressed*. The word *shall* should indicate a process of *enhancing boundary permeability*, but it appears subordinate to the word *find*. As above, we have detected a process of *perspective making* because it seems that Einstein is guided by mathematics, trusts what he obtains through calculations and expresses physical concepts accordingly. Therefore, we believe he is adopting a *reflection mechanism*.

Even the relativity of time is made explicit and justified through this new perspective provided by mathematics:

“[…] Between the quantities *x, t*, and *τ*, which refer to the position of the clock, we **have**, evidently, *x* = *vt* and

$$\tau = \frac{1}{\sqrt{1 - v^2/c^2}}(t - vx/c^2).$$

Therefore,

$$\tau = t\sqrt{1 - v^2/c^2} = t - (1 - \sqrt{1 - v^2/c^2})t$$

whence it **follows** that the time marked by the clock (viewed in the stationary system) is slow by $1 - \sqrt{1 - v^2/c^2}$ seconds per second […].
From this there **ensues** the following peculiar consequence. […]” (Einstein, 1905, p. 49).

In addition to other markers of the *routinization* process such as *have* and *follows*, used because calculations are still taking place, we find the word *ensues*. Even the meaning of the word itself suggests that something has emerged. We feel we can say that Einstein has again adopted the process of *perspective making*.

Finally, we can say that Einstein derives and approaches the Lorentz Transformations in two ways. Firstly, he trusts his postulates and derives the transformations as a consequence of these. Secondly, he shows that his theory is consistent precisely because the mathematics he derives from them says so. So, the author starts from the physical world, coordinates with the world of mathematics, and finds an object that will incarnate his theory so much that it will be able to make it explicit and valid.
To get a more visual idea of the analysis, let us now show the graph we obtained (Fig. 12):

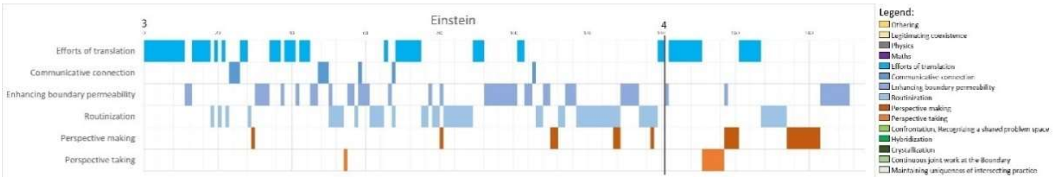

Figure 12 – Einstein’s processes

It must also be emphasised that the Lorentz Transformations play a fundamental role in the rest of Einstein’s work. They are derived and treated following two long sections that are mostly conceptual, but it is also true that from section 4 onwards the author makes massive use of them. As already mentioned, he uses them to make explicit the consistency of his axiomatic reasoning. They are then used to deal with the Maxwell - Hertz equations in empty space without incurring asymmetries, and to study the dynamics of the electron. However, it cannot be said that Einstein’s style is like Poincaré’s for this reason. For the latter, the Lorentz Transformations were a headlight for the development of his theory, on the other hand, for Einstein they are not just that. They follow from a specific choice: the elevation of the Principle of Relativity and the constancy of the velocity of light to postulates. In our opinion, axiomatic reasoning may also be the reason why the *efforts of translation* process is predominant. We show the line count (Tab. 9):

Processes	%	Absolutes
Efforts of translation	**29,7**	**59**
Communicative connection	4,5	9
Enhancing boundary permeability	25,1	50
Routinization	27,1	54
Perspective making	10	20
Perspective taking	3,5	7

Table 9 - Einstein's count

This good quantity is present in the rest of the article too. This seems consistent with his choice to begin his discussion with two postulates, once set out it is enough to proceed accordingly. Another reason why this predominance seems to make sense is the peculiar use of mental experiments.

G_c group and space-time

Minkowski derives and interprets the Lorentz Transformations in the first two sections of his article. We will show the analysis performed on four excerpts from section 1 and three from section 2.

We have already briefly discussed in the previous section how the author sets his starting point. Let us now show in detail the analysis made:

"[…] The equations of Newtonian mechanics **show** a twofold invariance. […] One always tends to treat the second group with **disdain** […]. But it is the composed complete group as a whole that **gives** us to think." (Minkowski, 1908, p. 39).

This is the part where Minkowski reveals what is the problem:

- The word *show* indicates the adoption of the mathematical point of view. In particular, it is a process of *perspective making* because the author is emphasising a mathematical property of an object (the equations of Newtonian mechanics) that is closely related to the physical world. So, it is a change of perspective, a mechanism of reflection.

- The word *disdain* indicates to us the fact that Minkowski wants to move to the boundary. In our opinion he uses this to denounce a problem that needs confrontation.

- To confirm this, he anticipates his intentions by using the word *gives*. We have identified it as a marker of the *hybridization* process because he is considering something new, unexpected: the whole group.

The author goes on to construct a four-dimensional space:

"[…] **Let** *x, y, z* be orthogonal coordinates for space and let *t* **denote** time. […] **let** us imagine that everywhere and at any time something perceivable exists. […]" (Minkowski, 1908, p. 40).

The words *let* and *denote* are clear markers of the effort of translation process. However, it is a coordinative act that starts from the world of mathematics, this is shown well in the next excerpt:

"[…] What has now the requirement of orthogonality in space to do with this complete freedom of choice of the direction of the time axis upwards?
To **establish** the connection we **take** a positive parameter *c* and **look** at the structure

$$c^2t^2 - x^2 - y^2 - z^2 = 1.$$

[…] A full understanding of the rest of those transformations can be **obtained** by **considering** such among them for which *y* and *z* remain unchanged. We draw (Fig.1) […]" (Minkowski, 1908, p. 41).

The words we find are:

- The word *establish*, which signals to us the strong desire to create something hybrid, which has its own mathematical dignity (a geometric space), but also a strong physical meaning (space-time).

- The word *take*, which indicates a process of *communicative connection* through the object paramenter *c*.

- The word *look* signalling a process of *perspective taking*, in fact it is an action that is performed through a point of view. In this case, it is the mathematical point of view since it involves a mathematical object that apparently does not yet act a physical role.

- The words *obtained* and *considering* are part of the *coordination mechanism*. We have already discussed these quite a bit before, they are because the author decided to model the situation leaving the y- and z-axis unchanged.

From there on, there is a long, purely geometric disquisition, which we have therefore identified as mathematical. However, once the group of transformations that leave the considered hyperboloid unchanged has been obtained, Minkowski changes his approach and places himself totally on the boundary.

For example, we have identified the following excerpt in its entirety as an adoption of the process of *continuous joint work at the boundary*:

"[…] at the end natural phenomena do not actually possess an invariance with the group G_∞, but rather with a group G_c with a certain finite c, which is extremely great only in the ordinary units of measurement. […]" (Minkowski, 1908, p. 42).

This is how the author tells us how fruitful the connection between mathematical and physical knowledge can be.

We can proceed to the analysis of the excerpts from section 2:

"[…] We now want to **introduce** this fundamental axiom:
With appropriate setting of space and time the substance existing at any worldpoint can always be regarded as being at rest.
This axiom **means** that at every worldpoint the **expression**

$$c^2dt^2 - dx^2 - dy^2 - dz^2$$

is always positive, which is equivalent to saying that any velocity v is always smaller than c. […]" (Minkowski, 1908, p. 43).

The word *introduce* has been associated with the process of *hybridization*, as it is an axiom full of physical concepts closely related to the nature of the G_c group. Therefore, it is a hybrid awareness. Instead, the word *means* indicates to us that for the author there is a certain interpretation from the mathematical point of view. It is a process of *perspective making* because it makes explicit knowledge on that side.

We think it is natural, for the same reason, the presence of a marker of the process of *efforts of translation* as *expression*.

Immediately afterwards Minkowski states:

"[…] The *impulse* and true motivation for ***accepting*** *the group* G_c came from **noticing** that the differential equation for the propagation of light waves in the empty space possesses that group G_c. […]" (Minkowski, 1908, p. 43).

The word *accepting* reveals a commitment and involvement with the G_c group, which Minkowski treated as a hybrid object. So, we associated it with the process of *crystallization*. Instead, the word *noticing* serves to make us realise that the mathematical object (the group) also makes sense from a physical point of view. So, it is *perspective taking*.

Let us take the last excerpt from section 2. This shows us that throughout the author's work, there are also moments when he takes care to specify certain details:

"[…] To step over the concept of space in such a way is an instance of what can be achieved only due to the audacity of mathematical culture. […]" (Minkowski, 1908, p. 45).

We decided to identify the whole sentence as an adoption of the process of maintaining uniqueness of intersecting practice. This is because he seems to want to emphasise the peculiarities of mathematics in relation to physics, but in the context of a collaborative achievement.

In conclusion, we are convinced that, despite the complexity of his argument, Minkowski wanted to show and crystallize the hybrid nature of the Lorentz Transformations.
We agree that it seems that Minkowski often recurs to mechanisms of reflection or coordination. However, the work he does on the transformations is transformational. In fact, we have noted:

- Processes of *confrontation/recognising a shared problem space* at the beginning of the article.
- *Hybridization* processes in section 1.

- Increasingly frequent processes of *crystallization* of the hybrid object from section 2 onwards.

The graph we obtained is as complex as Minkowski's style (Fig. 13):

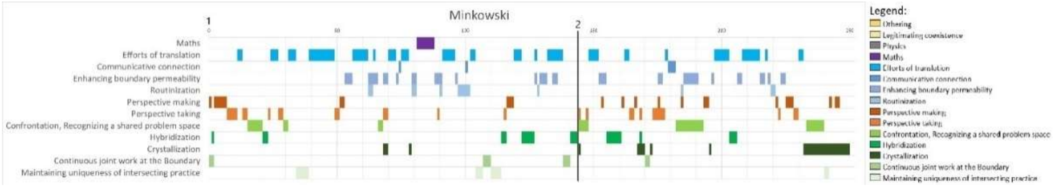

Figure 13 – Minkowski's processes

We recognise the purely mathematical work he does in section 1, and the lengthy *crystallization* process at the end of section 2.

Instead, the line count led to a result that can be interpreted in several ways. There is a good presence of the effort of translation process. We show the count (Tab. 10):

Processes	%	Absolutes
Maths	2,3	7
Efforts of translation	**24**	**71**
Communicative connection	1,7	5
Enhancing boundary permeability	13,8	41
Routinization	4,7	14
Perspective making	8,4	25
Perspective taking	9,4	28
Confrontation, Recognizing a shared problem space	10,8	32
Hybridization	7,7	23
Crystallization	9,1	27
Continuous joint work at the Boundary	3,4	10
Maintaining uniqueness of intersecting practice	4,7	14

Table 10 – Minkowski's count

This can be the result of Minkowski's geometric approach: he has to assign many variables, define many objects, and construct a space. In any case, we are convinced that the absolute protagonist of the author's work is the G_c group/space-time. The first two sections are devoted to deriving the transformations, to asserting their hybrid nature as full of physical meaning. The following sections simply describe and characterise the new concept. Also in conclusion, Minkowski emphasises the importance of the discovery made. It is his wish that we may grasp the consequences of the reshaping of the view of nature.

CHAPTER 5 - DISCUSSION OF THE RESULTS OF THE ANALYSIS

In Chapter 4, we showed two types of analysis: a macro-analysis and a micro-analysis. In this chapter, we will discuss the results obtained from both, showing how these results made us reflect on the style of each author, the characteristics that differentiate them, the way in which the relationship between mathematics and physics influences their work, and the role and meaning that each author attributes to the Lorentz Transformations. Finally, we make some general observations.

5.1 Boundary styles, differences between authors and their ideas on the mathematics-physics relationship

We summarise here the considerations we made for each author.

Lorentz begins his work by discussing the results of experiments that were carried out at the time to detect the effects of the earth's motion through the ether. His starting point is therefore strictly physical. He continues by adopting the *Coordination mechanism* at length to construct a physical theory that explains why the experiments did not return the expected results. In fact, his work essentially consists of modelling, quantifying, and translating into formulae the physical phenomena, systems, and hypotheses he makes to proceed with his treatment. Every step that the author takes is based on a very strong apparatus of knowledge and beliefs about the physical concepts he is interested in, something that he never questions. In chapter 11, in summing up what he achieved so far, we noted a brief adoption of the mechanism of Reflection, this is because it is in this chapter that Lorentz attributes physical significance to the numerical results he has obtained. In our opinion, the change of perspective resides in the fact that the author trusts what mathematics led him to, he does not trash the work he has done because it is unexpected, so a new knowledge emerges: the contraction hypothesis. Once Lorentz has concluded his discussion of the theory, he returns to discuss other experiments, thus returning to a strictly physical treatment.

For Poincaré too, the *Coordination mechanism* is prevalent, but in his case, it is a boundary crossing that starts more from the mathematical side. In principle, his approach is transformative because he states that he wants to deal with the lacks and problems of a theory that does not only concern the physicists, the physical ideas that constitute it and the consequences for experiments, but also the mathematical structure that is its basis, and therefore it is something that needs to be treated jointly. The fact that the author takes up the entire work done by Lorentz and others and rearranges it makes his *Coordination mechanism*s very long and dense of processes aimed to routinise the procedure and of calculations. Moreover, his mathematical point of view helps him to attribute more than one meaning to the Lorentz Transformations, so he uses the *Reflection mechanism* several times, also in the rest of his work when he tries to bring out new consequences of the theory using the transformations. In his attempt to make everything work, Poincaré creates something new: a special force. It comes both from the physical ideas that were already at the basis of the theory and from his rearrangement of the mathematical structure. In these brief passages, we have noted the mechanism of Transformation.

Einstein's style seems to follow a kind of predefined agenda, he starts predominantly with arguments of strong physical character, but he proceeds by taking two fundamental steps, namely the operational definition of simultaneity and the elevation to postulates of the Principle of Relativity and the constancy of the velocity *c* of light propagation for inertial reference systems, adopting the *Transformation mechanism* and the *Reflection mechanism* respectively. The first mechanism makes simultaneity an object with its own physical conceptual force, but also with its own strong mathematical coherence, that is an equivalence relation; the second mechanism allows him to move on to axiomatic reasoning, which makes his subsequent work consistent. From there on, until the end, he adopts long *Coordination mechanism*s, which are concluded by significant *Reflection mechanism*s to extrapolate the meaning of what he did with the calculations. Actually, the main part of the boundary work is done in the beginning, with these two basic moves, after which the theory comes naturally.

Minkowski's article is the one in which we detected a greater amount of *Transformation mechanism*, although the *Coordination mechanism* remains prevalent. At the beginning of his article, the author openly states that he is a mathematician interested in a physical theory, so he consciously changes perspective. He starts the core part of the paper with an almost purely mathematical approach, Minkowski merely coordinates with the world of physics to keep his 4D space, and the G_c group, in touch with the targets of physical interest. He proceeds, then, with the specific intention of finding an ontological meaning to what he develops; it is not enough for him that physical concepts correspond to good mathematical structures, he wants a real fusion. Therefore, the *Transformation mechanism* is predominant, because this approach is present right to the end of his article. Once the hybrid space-time object has been created, the Coordination moments become more frequent because they are aimed to the characterisation of the space and the attribution of physical meanings to geometric objects.

So, the boundary approaches of the four authors played a fundamental role in the development of the STR. Although Lorentz never questioned the existence of the ether, or the nature of space and time, he trusted the mathematical path he took in constructing the structure of the theory, which led him to the contraction hypothesis. The Coordination with the world of mathematics was important in *creating a theory* to explain the results of the experiments, a theory that he wanted to be as general as possible. Poincaré, and his rigorous mathematical approach, helped to *solve many problems and errors*, especially the omission of the Principle of Relativity. Moreover, he set the basis for the Lorentz Transformations to become central to the theory. Einstein was certainly a visionary, and he was the first of the four authors to realise that *ether would not be needed*, and that *time need not be absolute*. But he succeeded not only through physical visions, but also through the great power of mathematics. The Special Theory of Relativity was also made great by the 'simplicity' with which it is explained mathematically. Minkowski went even further, his transformative approach to the boundary between mathematics and physics *changed the idea of space and time* forever, thanks in part to his search for a strong link between realities belonging to the two disciplines.

We can summarise the authors' ideas of the mathematics-physics relationship as follows:

- for Lorentz, mathematics is a useful instrument for physics;
- for Poincaré, mathematics reveals meanings for physics too, and is a solid source of coherence;

- for Einstein, mathematics constitutes the foundation on which a good and consistent physical theory needs to stand;
- for Minkowski, mathematics and physics together describe the nature of the world.

5.2 Role and meaning given by authors to Lorentz Transformations

Remember the importance of the Lorentz Transformations for STR, and for these four articles in particular. The micro-analysis we conducted on the sections devoted to the subject allowed us, among other things, to better understand the role and meaning each author attributed to them.

For Lorentz, transformations are a way of establishing a *communicative connection* within a *Coordination mechanism*. The analysis we have conducted highlights the adoption of a utilitarian approach; the author has no intention of finding deep meaning in what for him is the result of a simple change of variables. In fact, even when he gives an interpretation to the variable *t'*, and its link to the variable *t*, he merely says that "The variable *t'* may be called the "local time"" (Lorentz, 1904, p. 813), suggesting that there is no conceptual implication behind these new independent variables. Lorentz uses transformations throughout his paper to facilitate calculations, which allows him to develop the theory comfortably, but not to change, or question, the apparatus of knowledge on which he relies for all his work.

For Poincaré, the Lorentz Transformations establish perspective within different mechanisms of Reflection. First of all, with an initial adoption of the *Reflection mechanism*, he gives them the central role in the theory created by Lorentz, he explicitly says that the transformations are the ones that give an explanation to the results of experiments, he also meticulously verifies that they leave a large set of fundamental formulae unchanged. In a second reflective moment, the author adopts the physical perspective, and explains how to obtain these transformations through the Principle of Least Action, giving them a new conceptual aspect. In a third reflective moment, Poincaré jumps into the world of mathematics, and brings out a new fundamental knowledge: the Lorentz Transformations form a group, so they also have a high mathematical value. In the end, he makes them a medium for investigation, research, and verification of the entire Lorentz theory.

For Einstein, they are both a way of creating a *communicative connection* within a *Coordination mechanism* and a way of creating perspective within a *Reflection mechanism*. The author's line of thought is clear and direct: Lorentz Transformations are those that allow one to move from one inertial reference system to another, thus a useful tool within his theory. However, in the next section, their meaning is sought, and it emerges that they are also a way of verifying the compatibility of the two postulates, thus a way of guaranteeing the validity of the entire theory, and also a way of bringing out new consequences, namely the relativity of lengths and time. Also in the rest of Einstein's work transformations play a fundamental role, in fact they are also able to prove that ether it is not necessary to exists.

For Minkowski, they are a hybrid and crystallized object within a mechanism of Transformation. The author's entire work focuses on Lorentz Transformations, which acquire a new way of being derived, a new form, a rediscovered completeness, new meanings, new responsibilities. He fuses four concepts, two physical (mechanics and space-time) and two mathematical (4D space and G_c group). Minkowski is the only one who extrapolated everything possible from the transformations, this led him to the realisation that they really suffice to develop the entire theory and to be able to say even more than anyone who came before him. Not just from a mathematical, or physical, point of view, but also ontological.

So, we can say that the type of approach to the boundary that the authors adopted has been crucial for the Lorentz Transformations: the pure *Coordination mechanism* was not enough for Lorentz to understand their true meaning, he needed at least a change of perspective, a *Reflection mechanism*, to learn the implications for time and ether. The learning was then complete when Minkowski adopted the *Transformation mechanism*, creating a theory that revolutionised the concept of space-time. In any case, we believe that this analysis clearly revealed the nature of the Lorentz Transformations as a boundary object. Indeed, these have their own meaning in both worlds, but they also facilitated and enabled the collaboration and crossing of the boundary between the two disciplines, they assumed different meanings depending on the perspectives, and they expressed their true strength when these perspectives met.

5.3 General observations

The macro analysis allowed us to see how each author adopted several mechanisms in the article. In particular, the boundary crossing mechanism most adopted by all four is the Coordination. In our opinion, this result underlines the intrinsic relationship between mathematics and physics. In short, physics cannot avoid mathematics, especially because of the need to model physical phenomena, systems, and situations. And also mathematics is not indifferent to the role of physics, especially when it is engaged in developing a theory as dense in conceptual meaning as STR. Moreover, physics in general has often been a source of interest and stimulation for mathematicians throughout history. At least, for a theory such as Special Relativity, it seems that the two disciplines cannot escape seeking collaboration.

Another interesting fact that emerges from the analysis is the total absence of the *Identification mechanism*. This is used in situations in which there is a need to redefine the boundaries of a world, perhaps because it came in contact with another world, and the connection is such that it creates identity confusion. The four articles we have analysed have very specific purposes, none of which have anything to do with this kind of desire to redefine boundaries. We think that this absence of the *Identification mechanism* is explained by the fact that the four authors had no intention of placing a barrier between the disciplines of mathematics and physics, quite the contrary.

So, it seems clear that the STR has been a source of encounter, collaboration, reflection, and joint work between the world of mathematics and the world of physics. Moreover, we may say that the Special Theory of Relativity emerges from the relationship between mathematics and physics.

The micro analysis on the sections concerning the Lorentz Transformations allowed us to highlight their nature as boundary objects. Not only this, however, going into detail regarding the internal functioning of the boundary crossing mechanisms allowed us to highlight how the relationship between mathematics and physics respects a form of pattern when considered in its simplest form, and how this pattern is broken if a greater depth of interdisciplinarity between the two disciplines is achieved. A good example of this pattern is the process spectrum of Lorentz's work (Fig. 14).

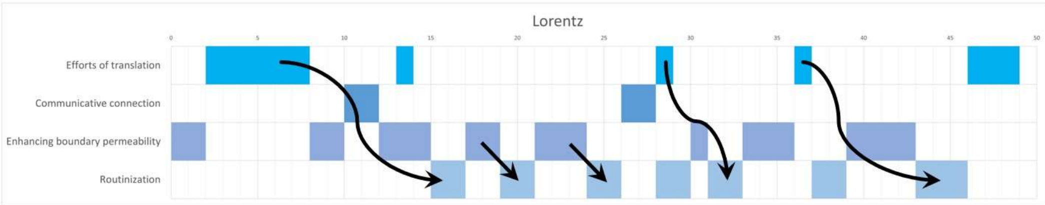

Figure 14 - Coordination pattern

What clearly emerges is a kind of iterative procedure: first one efforts to translate, then one enhances the boundary permeability, eventually passing through the use of boundary object, then creating a *communicative connection*, after which one goes into a routine, finally translating everything and starting again. In the graph, this iterative procedure looks like a kind of succession of steps. It is possible to see this pattern also in the spectra of Einstein and Poincaré, in the parts in which they adopt the *Coordination mechanism*. But this kind of pattern is completely broken when a different mechanism is adopted, as happens in Einstein's work (Fig. 15).

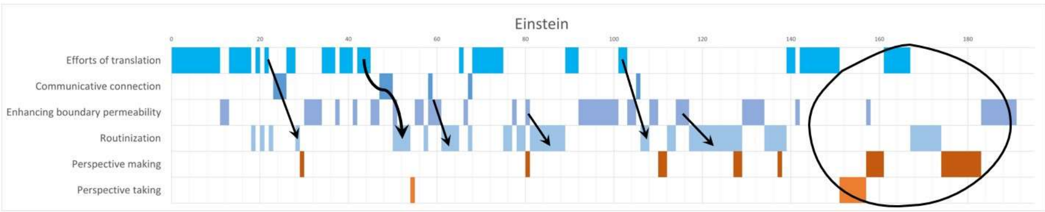

Figure 15 – Reflection's influence on the pattern

Something similar happens in Poincaré's fourth section, because the author shifts completely to the mathematical side, but where it is most evident is in Minkowski's article, i.e., the spectrum of processes does not seem to follow any kind of pattern, it is even difficult to read (Fig. 13). We think that this is a manifestation of ambiguity and that this ambiguity is essentially part of the action of crossing the boundary, of *hybridization*, of creating something new, because fundamentally it is a complex approach.

In our study, we also focused on the general relationship between mathematics and physics. We, therefore, tried to analyse in what ways these two disciplines talk to each other, what aspects they have in common and what aspects are instead identity-based. The macro-analysis and micro-analysis of the sections concerning the Lorentz transformations led us to make observations on the ways in which the authors treat the theory. With regard to these 'ways', we decided that the Italian word that best describes what we mean is *procedimento*. Let us therefore clarify what we mean by *procedimento*:

"Way, method by which a mental, manual, technical operation is conducted: *demonstrative, explanatory p.; calculation p.; following an analytical, inductive, deductive p*. In particular, in mathematics and its applications, logical reasoning to arrive at the solution to a given problem: *solving a question by algebraic, geometric p.; the p. is right, but the result is wrong because there was an error in calculation; by heuristic p. and iterative p.* […]" (Vocabolario on line, Treccani, https://www.treccani.it/vocabolario/procedimento/, my translation).

The English translation chosen for this word is *proceeding*, although the English meaning is slightly different, especially since it is used in legal settings mostly, but the substance is similar:

"[…] Outside of the law, the noun is also used to show the steps taken, that there is involvement with a process or venture, or that progress is being made: "He took the necessary proceedings to make sure everything was okay."" (Vocabulary.com, https://www.vocabulary.com/dictionary/proceeding).

We wondered whether these proceedings could be a way to distinguish or identify the two disciplines.

Regarding Lorentz, the only time he does not adopt a physical approach or the *Coordination mechanism*, is a small section in Chapter 11, where he adopts the *Reflection mechanism*. This fact allowed us to pay more attention to the words he uses, and in particular, it made us notice this sentence:

"[…] They [the above conclusions, ndr] also contain an explanation of MICHELSON'S negative result, more **general** […]" (Lorentz, 1904, p. 825).

So, we became convinced that the proceeding that accompanies Lorentz throughout his treatment may be *generalisation*. Instead, the proceeding that, in our opinion, characterises Poincaré's work is *rigour*. This conviction comes from the evident abundance of calculations and precision in performing them within his article, a fact also confirmed by the line count from the micro-analysis. For Einstein, it seems natural to argue that the proceeding he adopts is *axiomatic reasoning*, a fact that emerges well from both the macro and micro analysis. Finally, with regard to Minkowski, we believe that the proceeding that belongs most to him in his work is *abstraction*; we quote a sentence in which this is mentioned:

"[…] the somewhat greater **abstraction** associated with the number 4 does not hurt the mathematician." (Minkowski, 1908, p. 40).

Therefore, having identified these four proceedings, which are not without overlaps between them, we asked ourselves whether they belonged more to the world of mathematics or that of physics, or whether they were even links between the two disciplines. So, we can say that the study we conducted on the four original articles on the Special Theory of Relativity, in terms of boundary between mathematics and physics, triggered in us deep reflections on the identity of the two disciplines. We can also say that these are reflections of epistemological nature. We refer to the classical philosophical notion of epistemology as a theory of knowledge, increasingly identified with theories of scientific knowledge, but also to the so-called personal epistemology, which refers instead to individual beliefs about knowledge and learning. In short, we refer to a notion of epistemology that implies the identification of transversal themes, activities and ideas that can structure the knowledge of pre-service teacher education students at a meta-level and foster their reflection on the nature of disciplines and knowledge in general. Ravaioli in his 2020 Ph.D. thesis, developed within the Physics Education Research (PER) group of the University of Bologna, define these kinds of themes/activities/ideas as 'epistemological activators', in the sense that they can foster the activation of epistemological reflections on the nature of knowledge and science itself, more specifically:

“An epistemological activator is an idea, a theme, or an activity, that has the potential either (a) to organize knowledge on a higher abstractive level, or (b) to set a new context where specific ideas can become key concepts. Because and by means of this potential, an epistemological activator also owns the power (c) to raise questions about the nature and the role of science itself.” (Ravaioli, 2020, p. 44).

It is not obvious to state with certainty that the four proceedings we identified are epistemological activators, or that it is the analysis made of the boundaries between mathematics and physics.

To verify their potential, we designed an activity to be carried out with students in pre-service teacher training courses, and we tested it within the Physics Education course held by Professor Olivia Levrini at the University of Bologna. In the next chapter, we explain how we designed and implemented this activity.

CHAPTER 6 - ACTIVITY DESIGN AND PILOT TESTING

In this chapter, we will present the design of an activity, based on the analysis made, that can be proposed to students in pre-service teacher training courses. This activity was also tested with students on the Physics Education course held by Professor Olivia Levrini of the University of Bologna, and we will present and discuss the results obtained.

6.1 Activity design

The activity is designed to be carried out during 2 or 3 lessons of 2 or 3 hours and consists of 5 main phases.

The first phase provides an introduction to the concepts of:

- Boundary, in its broadest meaning;
- Community of practice, in the sense of the learning theory developed by Jean Lave and Etienne Wenger in the 1990s;
- Discipline, according to Krishnan (2009);
- Proceeding, in our definition given above;
- Interdisciplinarity, according to Thompson Klein (2010), and the IDENTITIES project approach.

During this first phase, we also ask students to participate by answering the following question on www.Wooclap.com:

In your opinion, which proceedings do you recognise as being proper to or characterising disciplines such as mathematics or physics?

The resulting word cloud is projected and displayed for all students so that they can view it.

Before entering the second phase, it is explained that we sought to analyse the role of interdisciplinarity in the original texts that founded the Special Theory of Relativity. Therefore, the second phase consists of showing, in a concise manner, how the four authors (Lorentz, Poincaré, Einstein and Minkowski) derive, treat, and interpret the Lorentz Transformations. The objective is to provide a guide, and to make the third phase quicker and more immediate so that students do not have to spend too much time understanding the calculations the authors make. It is also possible to conclude with an initial small discussion on the proceedings the students inserted to create the word cloud through Wooclap.

The third phase consists of giving students about an hour to work in groups with the aim of analysing excerpts from the original articles. In detail, we distribute:

- Chapters 3, 4 and 11 from Lorentz's article;
- Chapters 1, 2 (a small part, almost only the beginning and the ending), 3 and 4 from Poincaré's article;
- Chapter 3 from Einstein's article;
- Introductory sentence and Chapter 1 from Minkowski's article.

The pages with these excerpts have sentences highlighted in bold to emphasise the key passages of the authors' treatment. The choice of these excerpts is mainly determined by the need to distribute, more or less, the same amount of material for each author, without leaving out the most important concepts and passages. The trace that guides the students during this phase is the following:

a) *What is the proceeding(s) that characterises each author?*
b) *In your opinion, this proceeding belongs to mathematics, physics, or to both? Why?*

In addition, during the entire hour, the word cloud that resulted from the Wooclap done during the first phase remains projected.

The fourth phase consists of feedback from the students' reflections. In addition, a Google form is proposed with the aim of giving everyone the opportunity to answer in detail, for each author, the questions posed during the group work.

The fifth phase provides students with a summary of the analysis work carried out in this thesis. It explains:

- that our purpose is to study the interdisciplinarity between mathematics and physics;
- the fundamental steps of our methodology;
- the adaptation of Akkerman and Bakker's framework;
- the way in which we applied the analysis grid;

- the results obtained;
- the proceedings we have identified.

Following this, we take time for questions, perplexities, and possible debates.

6.2 Pilot testing

Sample on which the activity was tested

The Physics Education course at the University of Bologna, taught by Professor Olivia Levrini, is attended by students on the Bachelor Degree Course in Physics, students on the Master Degree Course in Physics, and students on the Master Degree Course in Mathematics. Specifically, the class that Lorenzo Miani and I addressed consisted of approximately 38 students, of which approximately 9 had a mathematical background and the remaining 29 had a physics background. In previous lessons, the class addressed the topic of the Special Theory of Relativity from several points of view (e.g., its historical evolution, the way it is proposed at school, and the different didactic approaches with which it can be taught).

Data collection method

As designed, we first proposed the first question to the class:

In your opinion, which proceedings do you recognise as being proper to or characterising disciplines such as mathematics or physics?

through www.Wooclap.com. We also allowed them to answer more than once, and we stored both the word cloud image and the individual answers through a Microsoft Excel spreadsheet.

For the group work hour, the students were divided into 9 groups, in order to have one student with a mathematical background per group. Each group consisted of 5 or 4 students. We then took notes, during the restitution phase four, regarding the trace:

a) *What is the proceeding(s) that characterises each author?*
b) *In your opinion, this proceeding belongs to mathematics, physics, or to both? Why?*

Finally, we kept the answers obtained through the Google form.

We show the results below.

<u>Results of the pilot test</u>

In total, we received 89 answers to the first question posed on www.Wooclap.com. Below we show the image of the resulting word cloud (Fig. 16).

Figure 16 - Wooclap word cloud

We tried to group some of the answers we received into larger categories when an affinity was evident. Thus, we also show the table with the answers (Tab. 11).

This first result convinced us that we rendered at least partially the idea of proceeding that we wanted the students to grasp.

Answers	Categories	N°
Demonstration, Demonstrations, Demonstrate, Logical demonstrations, Rigorous demonstration, Verification, Verification of theory, The need to justify everything	Demonstration	19
Deductive, Deduction, Deductive logic, Deductive reasoning, Inductive, Induction	Deductive – Inductive	8
Reasoning, Formal reasoning, Formalism, Proceeding characterised by logical and rational reasoning described by specific language with properties of universality, The fact that it uses languages that make it a universal and necessary knowledge, Mathematical language, The high use of mathematical language, Universality	Reasoning – Language – Formalism – Universality	8
Experimentation, Experiments, Experimental method, The Scientific Method, Empirical control, Comparison of theoretical predictions and experimental data	Experimentation	8
Modelling, Construction of new models, Model construction	Modelling	6
Observational, Observation phase, Systematic observations, Objective observation	Observational	6
Exact sciences, Science, Applications to other sciences, Applications to technology	Science and Application	4
Analytical, Analytics, Data analysis	Analytical	3
Conjecture, Creating conjecture, Plausible conjecture	Conjecture	3
Rigour		3
Simulation, Numerical simulations	Simulation	2
Mental visualisation, Vision	Vision	2
Graphs, Tables	Graphs and Tables	2
Interpretation, Interpretation of results	Interpretation	2
Generalisation		2
Abstraction		2
Classification		1
Axiom system and inference from axioms and theorem		1
Constructing theories		1
Seeking regularity and law		1
Prevision		1
Approximations		1
Solving an equation		1
Search for symmetries		1
Measure		1

Table 11 - Wooclap answers

Below is a summary of the reflections that each working group shared with us regarding each author during phase four. We specify, anyway, that all groups felt the need to freely explain what they thought, and each of them found an original way to approach the activity.

Group 1 - (5 persons) This group decided to rank the authors according to who they considered "most mathematical". Thus, we report their considerations on the proceedings used by the authors, following their order. Minkowski was considered the "most mathematical" because of the proceedings he adopts and the way he presents his work. In fact, the students associated the author with the *geometric* proceeding and especially the *abstraction* proceeding, and they also considered his way of presenting theory to be "constraining", to the point of being considered "sneaky" and "suffocating", because he "leaves no room" for future developments or different interpretations. Poincaré is ranked second on the "most mathematical" list because the students felt that he had more physical references than Minkowski, although the author gave a more mathematical location to the Lorentz Transformations. The proceedings they associated with him are *abstraction*, *classification*, and above all *generalisation*. About Lorentz's work, the students recognised a path that earned him third place in the list of the "most mathematical", namely the fact that the author begins and ends with physical arguments, passing through mathematics, with the aim of returning to talk about experiments. The proceeding they associated with him is *modelling*. Einstein is the most physical of the four authors for the students in this group because he elicits strong physical meanings, he has a logical and operationist approach: therefore, they called his proceeding: *to demonstrate operationally*.

Group 2 - (4 persons) This group also recognized in Lorentz's work a physics-mathematics-physics path. They associated him with the *formalization* proceeding, as well as with Poincaré, which, however, is additionally characterized by what the students called the *theoretical control* proceeding, in that he gives the feeling of wanting to do things in many different ways to make sure things "return". An approach that students recognized to be more mathematical. In contrast, Einstein was associated with what they called the *operational control* proceeding since it is more practical. Finally, with Minkowski they also associated the term *constraining*, this time understood precisely as a proceeding, mostly mathematical in nature. They also recognized him as having a more philosophical position, in the sense that they identified among his goals the intention to give an “idea of the world”.

Group 3 - (4 persons) This group recognised in Lorentz’s work a *deductive* proceeding, through mathematical algorithms, and the proceedings of *physical interpretation* and *physical verification*, so they considered the author to be somewhere between mathematics and physics. The proceedings they identified in Poincaré’s work, on the other hand, are characterised by the adoption of different points of view, hence what the students called the *perspective* proceeding, they also attributed to it f*ormalisation*, *rigour*, and *generalisation* more characteristic of mathematics. Einstein’s work was perceived as “purely physical”, characterised by the adoption of the proceeding of *visualisation*. Students also recognised the importance of *postulates* and a proceeding that moves the author from hypotheses to results through *abstraction*. In contrast, Minkowski was perceived as purely (or “atrociously”) mathematical, as his work leaves no doubt. The proceeding they decided to associate with him is that of *generalisation*.

Group 4 - (4 persons) This group could not say what proceeding there might be in Lorentz’s work. However, for these students, the author has a physical approach, and mathematics is used to “drive the theory forward”, in fact in their opinion he starts from physical hypotheses (derived from Maxwell’s theory) and uses mathematics improperly because he does not provide sufficient physical explanations for the substitutions he makes for the calculations. In Poincaré, on the other hand, they recognised the proceeding of *generalisation*, which in their opinion is more mathematical, since they perceived that the author’s objective is to search for a mathematical structure that can fill the holes in the theory. Einstein was associated with the proceedings of *experience* and *intuition*, which they considered to be purely physical since the desire is to begin the discussion with the minimum number of hypotheses. Minkowski was perceived as specular compared to Einstein, in the sense that they perceived the centrality of mathematical structure and the importance of 4D space, almost as if physics was only a tool. However, they could not say whether his proceedings were more mathematical or physical, they called him a "theoretical physicist".

Group 5 - (4 persons) These students associated Lorentz's work with a proceeding they called *mathematical experimentation*, since the author starts with a physical problem, goes through formulas, and arrives at a solution. They associated the *mathematical demonstration/justification* proceeding with Poincaré because he demonstrates several things about the Lorentz Transformations in more ways than one. With Einstein, they associated the proceedings of *physical interpretation* and *representation*, which they considered very physical. With Minkowski, they associated the proceeding of *mathematical visualisation/representation*, and the *geometric* proceeding considered very mathematical.

Group 6 - (5 persons) This group generally missed the objective of indicating proceedings. Lorentz was considered neither a mathematician nor a physicist because, in the opinion of these students, nothing new emerges in his work on either side. What characterises him is that he does a lot of calculations. They did not say much about Poincaré because they admitted that they did not understand him, however they had the feeling that the author "pretended to be interested in physics". In these students' opinion, Einstein is the most physical because he does few calculations. Finally, Minkowski's approach was defined by the expression "complex discourse", because the author mixes mathematics, physics and philosophy in a way that they called transdisciplinary, because he "uses mathematical words, but physical arguments".

Group 7 - (4 persons) In this group's view, Lorentz lies somewhere between mathematics and physics because his starting point and end point are physical in nature, but he goes through mathematics. In particular, they recognised the use of what they called the *mathematical inference* proceeding, however, in the opinion of these students, the author makes instrumental use of it, having assumed the existence of the ether at the beginning. Poincaré was also considered to be somewhere between mathematics and physics in his use of algebra to start from and arrive at physical concepts. Einstein has been associated with the proceedings of *physical inference*, *mental experiments*, *the use of postulates* and *coherence*. All of these were perceived as belonging to the world of physics. Finally, Minkowski was associated with the proceeding they called *geometrical assumptions*, and they recognised it as purely mathematical.

Group 8 - (4 persons) This group produced a kind of form to determine the authors' positioning at the boundary between mathematics and physics, however, they did not provide any proceeding. In any case, these students considered Lorentz to have an interdisciplinary approach, so he belongs to both the mathematical and the physical world in an "equal" way. Poincaré, on the other hand, was perceived as purely mathematical, hence with a disciplinary approach. For Einstein, they recognised a multidisciplinary approach because they perceived the author to be strongly anchored in the world of physics but using mathematical models. On the other hand, they saw Minkowski's work as something that brings out new knowledge, that makes mathematics different from before, able to be a "world-builder", so they defined his approach as transdisciplinary.

Group 9 - (4 persons) This group merely told us which of the authors, in their opinion, is more mathematical or more physical. They did not perceive a sufficient amount of physics in Poincaré's work, so he was ranked as the most mathematical of the four. The "second mathematician" is Lorentz because he also provides physical passages. Einstein was perceived as the most physical. Whereas, since they recognised a particular use of geometry to represent physics, they decided that Minkowski is "outstanding".

About the Google form, we received responses from only 17 students; the answers, however, do not completely overlap with the previous summary, perhaps because the questionnaire guided the way students responded more strongly. In addition, new proceedings emerged; again, we tried to group similar proceedings into broader categories.

We begin by looking at the answers concerning Lorentz's work, the proceedings most attributed to him are the *hypothetical-deductive*, the *demonstrative* and the so-called "*He does calculus*", which often goes together with the *change of variables* (Fig. 17). Many have recognised the physics-mathematics-physics path. Thus, the opinions found through the Google form did not differ too much from those collected during phase four of the activity.

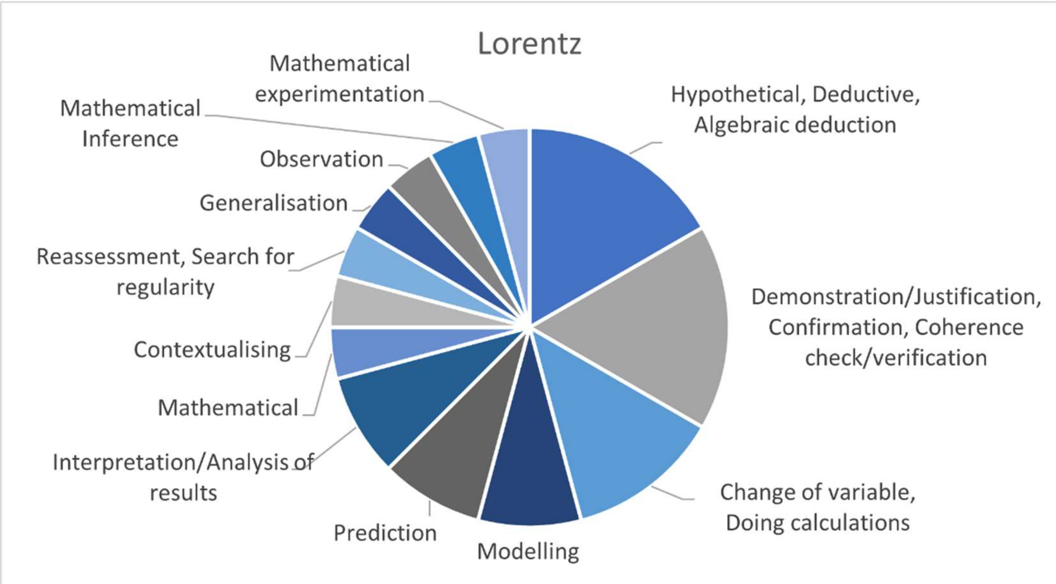

Figure 17 – Lorentz' proceedings in students' opinion

The proceeding of *generalisation*, which we had identified, was mentioned, but only minimally.

Instead, it appears among the proceedings most attributed to Poincaré's work, which are the *demonstrative* and indeed the *generalisation*. Among those moderately attributed are the so-called "*reassessment*", or *search for regularities*, and *rigour*, which we had identified (Fig. 18). In any case, it remains the author that the students say they understood least.

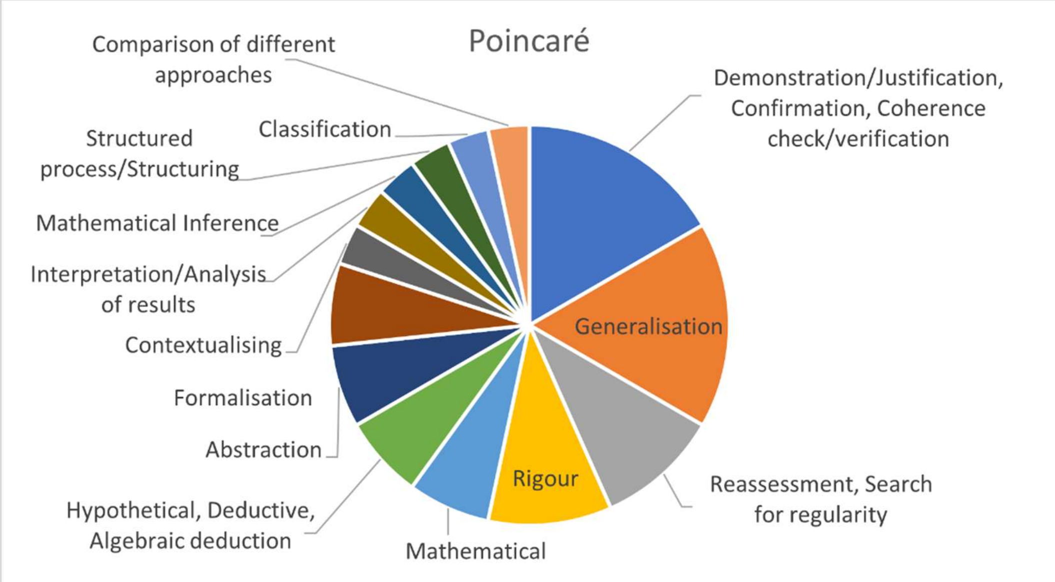

Figure 18 – Poincaré's proceedings in students' opinion

Whereas they seem to have a very clear understanding of Einstein's work. The proceedings they most attributed to him are *axiomatic reasoning*, *operationism* and *hypothetical-deductive*. Those moderately attributed to him are *visualisation* and *imagination* (Fig. 19).

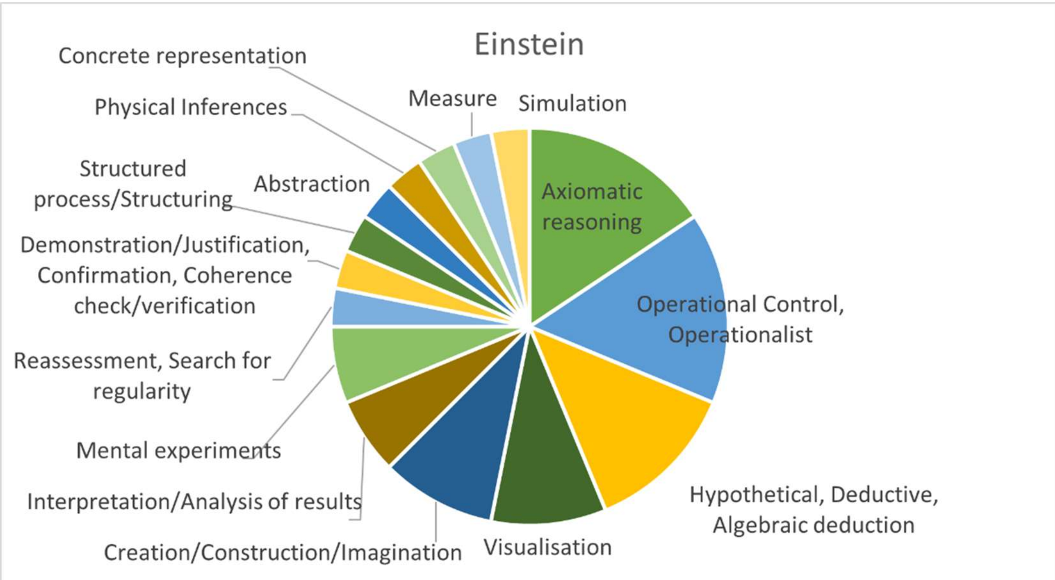

Figure 19 - Einstein's proceedings in students' opinion

An interesting fact is the emergence of axiomatic reasoning, which seemed almost absent in classroom reflections. However, we must specify, it is almost never called in this way, but described with expressions such as: "deduction from physical principles", "[...] physical assumptions. What emerges, however, from the structure of the reasoning is that of a mathematical theory (postulates and theorems from them)", "primary principles". As also emerged from the reflections of some groups who had noticed "the use of postulates".

The answers on Minkowski are less heterogeneous compared to those on the other authors, this was also true during the restitution phase in class. However, they are not the same responses, in fact the "suffocating" or "constraining" character that had characterised the groups' reflections has disappeared. *Geometric* and *abstraction* proceedings definitely dominate, followed by *generalisation* and *visualisation* (Fig. 20).

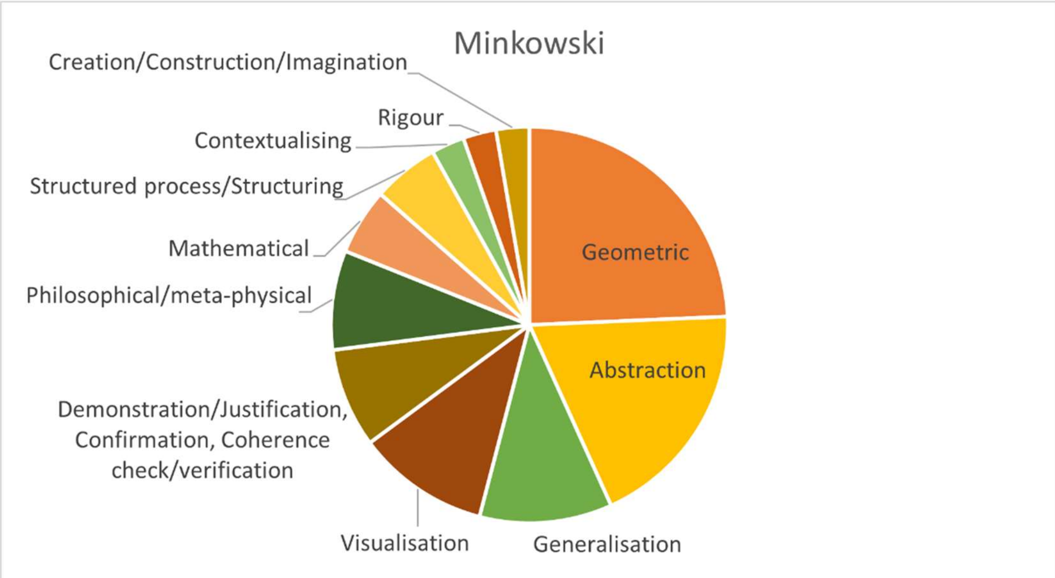

Figure 20 - Minkowski's proceedings in students' opinion

It is interesting to see how the students placed, with respect to the boundary between mathematics and physics, the proceedings they identified within the excerpts from the original articles. We report a graph with the results (Fig. 21).

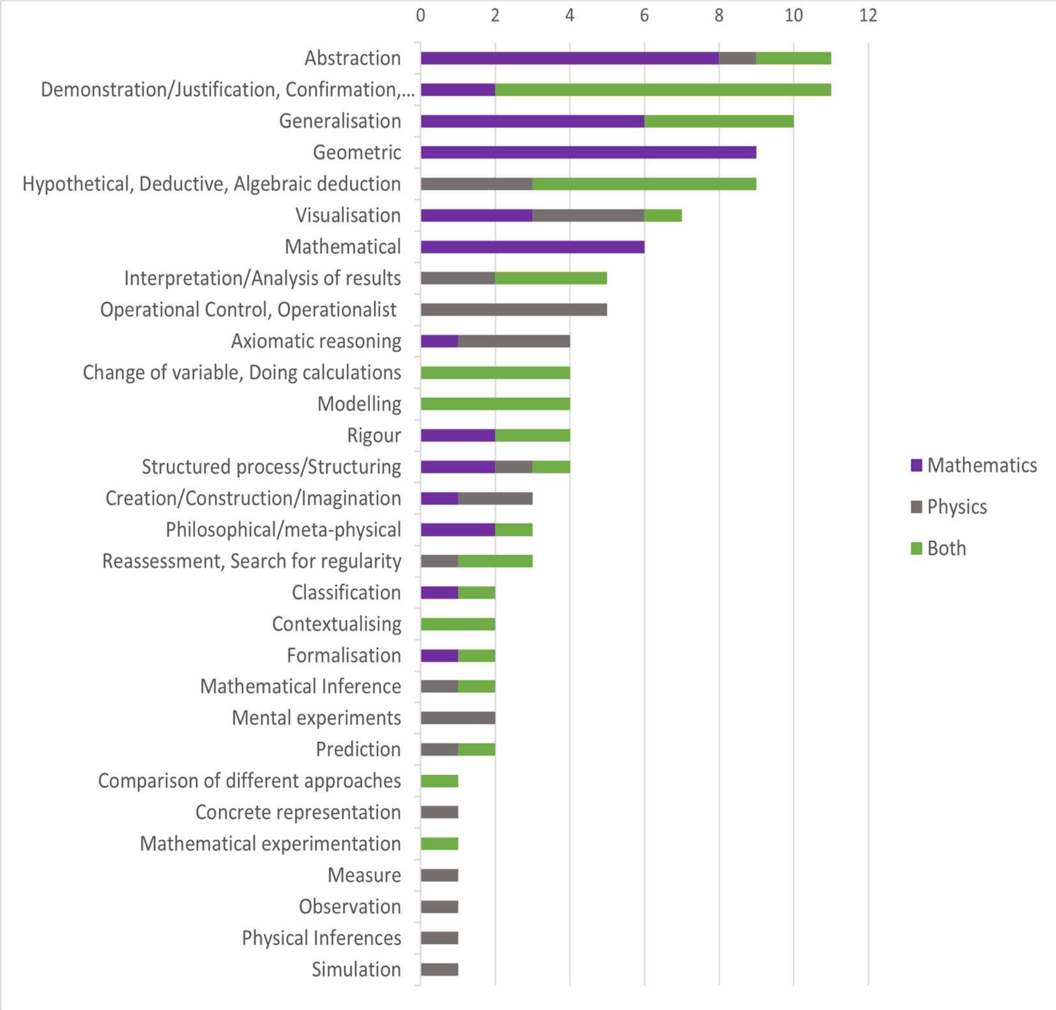

Figure 21 - Disciplinary attribution of proceedings

We discuss the main ones:

- The proceeding of *abstraction* is mostly considered mathematical, although some students have rightly argued that it is a proceeding that is also adopted in physics, to be precise, especially in theoretical physics. It is interesting to note that at the time of the Wooclap, the proceeding of abstraction was mentioned only twice, whereas after the group work, it became the most indicated.
- The *demonstrative* proceeding was the one that was mentioned most when the question was asked through www.Wooclap.com. Interestingly, the students refused to dismiss it as such, and they attributed it a little to all four authors. It could be said that this is an obvious fact, indeed the students too

placed it as something that belongs to both disciplines, claiming precisely that neither of them can dispense with it.

- In contrast, *generalisation* found limited attention in the first phase of the activity, but evidently emerged more strongly during the analysis of the articles. This proceeding is perceived as mostly mathematical, someone even claims that “it is a mathematician’s typical worry”.
- The *geometric* proceeding did not appear at all in the first phase, and it does a little during the restitution. However, in the answers to the form, it is the proceeding most attributed to Minkowski and is considered strictly mathematical.
- The *hypothetical-deductive* proceeding is mostly indicated as belonging to both disciplines, although someone perceives it as more typical of physics.
- A surprising result is the association of *axiomatic reasoning* mostly with the discipline of physics. Recall that the students never referred to it by this name, but by alternative expressions. What has emerged, both during the restitution phase and in the responses to the Google form, is that students perceive the act of starting from a minimum number of hypotheses as typically physical, while they attribute to mathematicians the strange and improbable attitude of abounding with hypotheses, inserting them in their arguments almost at random.
- The proceeding of *rigour* was mentioned little in general but gained some mention with group work. It was associated with both disciplines by two students, while two others associated it with mathematics. So, it seems that it is perceived as tending to be mathematical.

Finally, we note that these results seem to be slightly vitiated by the fact that many students were influenced by their previous ideas about authors. For example, Einstein was perceived as “the most physicist of all” by almost everyone, perhaps this is what led to the association of axiomatic reasoning with the world of physics. In fact, it seems that it was not clear that we wanted the proceedings, not the authors, to be placed, instead they focused very much on the authors’ intentions.

Furthermore, we point out that certain stereotypes are not easy to be questioned, but if a rich and nuanced language is provided, they can be discussed, and much awareness can be gained about aspects of the disciplines to which not enough consideration has been given.

CHAPTER 7 - CONCLUSIONS

The aim of this thesis was to make Lorenzo Miani's work of analysing the interdisciplinarity between mathematics and physics, in the context of the founding articles of the Special Theory of Relativity, more refined and operational. Moreover, we were motivated by the conviction that this approach brings out more clearly the identity of the disciplines involved.

The theme of the disciplinary identity of mathematics is fundamental to addressing the Justification Problem. We have therefore made a review with the objective of deepening the themes dear to the Mathematics Education, and explaining how creating 'boundary zones', particularly between mathematics and physics, can be a valuable resource from many points of view.

We therefore extrapolated from Akkerman and Bakker's (2011) metatheory a grid capable of highlighting, analysing and "quantifying" the boundary between the two disciplines on the original articles by Lorentz (1904), Poincaré (1906), Einstein (1905) and Minkowski (1908).

This gave us a clearer understanding of the 'boundary narrative styles' of the four authors. In fact, we highlighted that Lorentz created a general theory through the use of the *Coordination mechanism*; that Poincaré ordered, adjusted and completed the theory by applying different *mechanisms* of *Reflection* and *Coordination*; that Einstein went beyond the concept of ether and the concept of absolute time through two fundamental adoptions of the *Transformation* and *Reflection mechanism*s; finally, that Minkowski changed the concept of space-time using the *Transformation mechanism* predominantly.

Furthermore, the analysis highlighted that Lorentz Transformations are particularly suitable to be analysed as a boundary object and that this interdisciplinarity perspective is powerful to compare the different roles they played in the development of the STR. For Lorentz they are a useful tool to facilitate calculations and to be able to develop the theory; for Poincaré they are an object with different meanings, both physical, due to the Principle of Least Action, and mathematical, since the transformations form a group; for Einstein they are a mathematical tool that embodies his theory and demonstrates the compatibility of his two postulates; for Minkowski they are the very concept of space-time.

This kind of work stimulated in us deep epistemological reflections on the identity of mathematics and physics. The different ways in which authors have developed theory have emerged. We decided to call these ways proceeding. Wondering whether they are peculiar to only one of the two disciplines, or belong to both, inspired the design of the activity that we experimented with in the Physics Education course held by Professor Olivia Levrini at the University of Bologna. The aim of the activity is to create interdisciplinary sensitivity and skills in students on teacher training courses.

The results of the experimentation have convinced us that we have, at least in part, achieved the objective. In particular, regarding disciplinary identities, reflections emerged that helped the students to recognise what kind of idea they had of mathematics and physics, in such a way as to question previous stereotypes.

Finally, the grid developed, and used to analyse the four original articles on the Special Theory of Relativity, seems open to future development. We do not exclude that it could be applied to other academic articles involving the disciplines of mathematics and physics, to test its validity in other contexts, and perhaps be able to bring out the authors' use of boundary-crossing mechanisms in the development of other theories too.

BIBLIOGRAPHY/SITOGRAGHY

[1] Akkerman, S. F., Bakker, A. (2011). Boundary Crossing and Boundary Objects. *Review of Educational Research*, 81(2), 132-169. https://doi.org/10.3102/0034654311404435

[2] Baccaglini – Frank, A., Di Martino, P., Natalini, R., Rosolini, G. (2018), *Didattica della matematica*. Mondadori università.

[3] Barelli, E. (2020). Le simulazioni computazionali come strumenti interdisciplinari di decisione: risultati di un'indagine con studenti universitari. [Computational simulations as interdisciplinary decision-making tools: results of a study with university students]. Presented at the SIF conference – 2020, September 14th-18th (online).

[4] Barelli, E., Barquero, B., Romero, O., Aguada, M. R., Giménez, J., Pipitone, C., Sala-Sebastià, G., Nipyrakis, A., Kokolaki, A., Metaxas, I., Michailidi, E., Stavrou, D., Lodi, M., Sbaraglia, M., Bartzia, EI., Modeste, S., Martini, S., Durand-Guerrier, V., Satanassi, S., Fantini, P., Bagaglini, V., Kapon, S., Branchetti, L., Levrini, O. (2021). Disciplinary Identities In Interdisciplinary Topics: Challenges And Opportunities For Teacher Education. Presented at the ESERA2021 conference – 2021, August 30th - September 3rd (online).

[5] Barelli, E., Levrini, O. (2021). Computational Simulations as Citizenship Tools: A Study With University Physics and Mathematics Students. Presented at AERA – 2021, April (online).

[6] Boden, M.A. (1999). What is interdisciplinarity? In: R. Cunningham (ed.) *Interdisciplinarity and the organization of knowledge in Europe*, pp. 13–24. Luxembourg: Office for Official Publications of the European Communities.

[7] Bolondi, G., & D'Amore, B. (2021). La matematica non serve a nulla. In D'Amore, B. (Ed.) (2021). *La didattica della matematica: riflessioni teoriche e proposte concrete*. Atti del Convegno *Incontri con la matematica* n. 35, Castel San Pietro Terme (Bo), 5-6-7 novembre 2021. Pp. 7-10.

[8] Branchetti, L., Cattabriga, A., Levrini, O. (2019). Interplay between mathematics and physics to grasp the nature of a scientific breakthrough: the case of the black body. Physical Review – *Physics Education Research*, 15(2), 020130 (2019). 10.1103/PhysRevPhysEducRes.15.020130

[9] Branchetti, L., Levrini, O. (2019). Disciplines and interdisciplinarity in STEM education to foster scientific authenticity and develop epistemic skills. Presented at ESERA2019 conference within the invited symposium "Crossing boundaries – problematizing interdisciplinarity in science education" – 2019, August 26th-30th (Bologna-Italy).

[10] Chiusoli, S. (2019). *La mappa logistica come caso di studio per riflettere sull'interdisciplinarità nei sistemi complessi [The logistic map as a case study to reflect on interdisciplinarity in complex systems]*. Bachelor thesis in Physics, Alma Mater Studiorum – University of Bologna. Advisor: Prof. O. Levrini. Co-advisor: E. Barelli

[11] Courant R. – Robbins H. (1996), *What is Mathematics?: An Elementary Approach to Ideas and Methods*, 2nd edition, with additional material by Ian Stewart, Oxford University Press, New York.

[12] Darrigol, O. (2006). The Genesis of the Theory of Relativity. In Damour, T., Darrigol, O., Rivasseau, V. (2006). *Einstein, 1905-2005. Poincaré Seminar 2005*, 1-31. Birkhäuser Basel. https://doi.org/10.1007/3-7643-7436-5

[13] Demattè, A. (January 2008). Documenti storici per la didattica della matematica/1. *Didattica delle Scienze.* 253, 9 – 17. https://www.researchgate.net/publication/340340290_Documenti_storici_per_la_didattica_della_matematica1

[14] Einstein, A. (1905). On the electrodynamics of moving bodies. Original paper "Zur Elektrodynamik bewegter Körper", Annalen der Physik. 17. In Lorentz, H. A., Einstein, A., Minkowski, H., Weyl, H. (1952). *The Principle of Relativity: a Collection of Original Memoirs on the Special and General*

Theory of Relativity. With Notes by A. Sommerfeld. Translated by W. Perrett and G. B. Jeffery. London: Methuen and Co., Ltd., 35-65.

[15] Engeström, Y., Engeström, R., & Kärkkäinen, M. (1995). Polycontextuality and boundary crossing in expert cognition: Learning and problem solving in complex work activities. *Learning and Instruction*, 5, 319–336.

[16] Engeström, Y., Engeström, R., & Kärkkäinen, M. (1997). The emerging horizontal dimension of practical intelligence: Polycontextuality and boundary crossing in complex work activities. In R. Sternberg & E. Grigorenko (Eds.), Intelligence, heredity, and environment (pp. 440–462). New York, NY: Cambridge University Press.

[17] Erduran S., Dagher Z., Mcdonald C. V. (2019). *Contributions of the Family Resemblance Approach to Nature of Science in Science Education: A Review of Emergent Research and Development*, Science & Education, pp 311−332.

[18] Erduran, S. & Dagher, Z. (2014a). *Reconceptualizing the nature of science for science education: scientific knowledge, practices and other family categories*. Dordrecht: Springer

[19] Galili, I. (2018). Physics and Mathematics as Interwoven Disciplines in Science Education. *Science & Education*, 27, 7-37. https://doi.org/10.1007/s11191-018-9958-y

[20] Giannetto, E. (2009). The electromagnetic conception of nature at the root of the special and general relativity theories and its revolutionary meaning. *Science & Education*, 18, 765–781.

[21] Gombi, A. (2020). *The foundational case of the parabolic motion: design of an interdisciplinary activity for the IDENTITIES project*. Master thesis in Physics, Alma Mater Studiorum – University of Bologna. Advisor: Prof. O. Levrini. Co-advisor: S. Satanassi.

[22] Hersh R. (1997), *What is mathematics really?*, Oxford University Press, Oxford.

http://www.math.chalmers.se/Math/Grundutb/CTH/mve375/1112/docs/KOMkompetenser.pdf

[23] Hyland, K. (2008). Disciplinary voices. English Text Construction, 1(1), 5–22. https://doi.org/10.1075/ETC.1.1.03HYL

[24] Hyland, K., Tse, P. (2004, June 01). Metadiscourse in Academic Writing: A Reappraisal. Applied Linguistics, 25(2), 156–177. https://doi.org/10.1093/applin/25.2.156

[25] Jahnke, H.N. et al. (2002). The use of original sources in the mathematics classroom. In: Fauvel, J., Van Maanen, J. (eds) History in Mathematics Education. New ICMI Study Series, vol 6. Springer, Dordrecht. https://doi.org/10.1007/0-306-47220-1_9

[26] Krishnan, A. (2009). *What Are Academic Disciplines? Some Observations on the Disciplinarity vs. Interdisciplinarity Debate*. NCRM Working Paper. National Centre for Research Methods. http://eprints.ncrm.ac.uk/783/1/what_are_academic_disciplines.pdf

[27] Levrini, O. (1999). Relatività ristretta e concezioni di spazio. *Giornale di Fisica*, 11(4), 205-220.

[28] Levrini, O. (2014). The Role of History and Philosophy in Research on Teaching and Learning of Relativity. In M.R. Matthews (ed.), *International Handbook of Research in History, Philosophy and Science Teaching*, 157-181. Springer Science+Business Media Dordrecht.

[29] Levrini, O., Branchetti, L., Fantini, P. (2019). Le discipline (scientifiche) nell'epoca della frammentazione della conoscenza. [The scientific disciplines at the time of knowledge fragmentation]. Presented at the 3rd National Seminar on Licei Matematici – 2019, September 18th (Fisciano).

[30] Lorentz, H. A. (1904). Electromagnetic phenomena in a system moving with any velocity smaller than that of light. Huygens Institute - Royal Netherlands Academy of Arts and Sciences (KNAW), Proceedings, 6, 1903-1904, Amsterdam, 1904, 809-831. In Lorentz, H. A., Einstein, A., Minkowski, H., Weyl, H. (1952). *The Principle of Relativity: a Collection of Original Memoirs on the Special and General Theory of Relativity*. With Notes by A. Sommerfeld. Translated by W. Perrett and G. B. Jeffery. London: Methuen and Co., Ltd., 9-34.

[31] Martín-Páez, T., Aguilera, D., Perales-Palacios, F.J., Vílchez-González, J.M. (2018). What are we talking about when we talk about STEM education? A review of literature. *Science Education*, 103, 799-822. https://doi.org/10.1002/sce.21522

[32] Miani, L. (2021). *Highlighting interdisciplinarity between physics and mathematics in historical papers on special relativity: design of blended activities for pre-service teacher education.* Master Degree theses. Alma Mater Studiorum - Università di Bologna. Advisor: Prof. O. Levrini. Co-advisor: Prof. Dimitris Stavrou

[33] Minkowski, H. (1908). Space and Time. Original paper: "Raum und Zeit", *Physikalische Zeitschrift,* 10: 75–88. Translated by Lewertoff, F., Petkov, V. (2012). Space and Time: Minkowski's Papers on Relativity. Minkowski Institute Press, 39-55.

[34] Minkowski, H., Petkov, V. (2012). *Space and Time: Minkowski's Papers on Relativity*. Minkowski Institute Press.

[35] Niss, M. A. (2003). Mathematical competencies and the learning of mathematics: the Danish KOM project. In A. Gagatsis, & S. Papastavridis (Eds.), *3rd Mediterranean Conference on Mathematical Education - Athens, Hellas 3-4-5 January 2003* (pp. 116-124). Hellenic Mathematical Society.

[36] Ornek, F. (2008). Models in Science Education Applications of Models in Learning and Teaching Science. *International Journal of Environmental & Science Education*, 2008, 3 (2), 35 – 45.

[37] Palmer, C. L. (1999). Structures and strategies of interdisciplinary science. *Journal of the American Society for Information Science*, 50, 242–253.

[38] Poincaré, H. (1906). Sur la dynamique de l'électron. *Rendiconti del circolo matematico di Palermo*, 21, 129–176. Session of July 23, 1905. Translated by B. D. Popp in Popp, B. D. (2020). Henri Poincaré: Electrons to Special Relativity, 45-102. Springer Nature Switzerland AG 2020. https://doi.org/10.1007/978-3-030-48039-4_5

[39] Pollani, L. (2021). *Interdisciplinarità tra Matematica e Fisica: analisi di testi in termini di razionalità e spiegazione [Interplay between Mathematics and Physics: an analysis of texts using rationality and explanation]*. Master thesis in Mathematics, University of Genova. Advisor: Prof. F. Morselli. Co-advisor: Prof. L. Branchetti.

[40] Polverini, G. (2022). *Exploring interdisciplinarity between physics and mathematics: the design of a linguistic and an epistemological tool for analysing texts about the parabolic motion*. Master thesis in Physics, Alma Mater Studiorum – University of Bologna. Advisor: Prof. O. Levrini. Co-advisors: Dr. V. Bagaglini, Dr. S. Satanassi

[41] Procedimento. *Vocabolario on line*. Istituto della Enciclopedia Italiana fondata da Giovanni Treccani S.p.A., treccani.it. Accessed 02 Aug 2022, https://www.treccani.it/vocabolario/procedimento/

[42] Proceeding. *Vocabulary.com*. A division of IXL Learning, Inc. *Vocabulary.com*. Accessed 02 Aug 2022, https://www.vocabulary.com/dictionary/proceeding

[43] Quadrelli, N. (2020). *Un approccio interdisciplinare alla parabola per riflettere sulla nascita della fisica moderna: risultati di uno studio con studenti di classe terza di Liceo Scientifico. [An interdisciplinary approach to parabola to reflect on the origin of modern physics: Results of a study with students of the 3rd year of Scientific Lyceum]*. Bachelor thesis in Physics, Alma Mater Studiorum – University of Bologna. Advisor: Prof. O. Levrini. Co-advisor: S. Satanassi.

[44] Ravaioli, G. (2020). Epistemological activators and students' epistemologies in learning modern STEM topics. Ph.D. theses. Alma Mater Studiorum – University of Bologna. PhD coordinator: Prof. Silvia Arcelli. Advisor: Prof. O. Levrini.

[45] Satanassi, S., Ercolessi, E., Levrini, O. (2020). Un approccio interdisciplinare alle tecnologie quantistiche: un modulo per studenti di scuola secondaria. [An interdisciplinary approach to quantum technologies: A module for secondary school students]. Presented at the SIF conference – 2020, September 14th-18th (online).

[46] Scherr, R. E., Shaffer, P. S. and Vokos, S. (2002). The challenge of changing deeply held student beliefs about the relativity of simultaneity. *American Journal of Physics*, 70(12), 1238-1248. https://doi.org/10.1119/1.1509420

[47] Sicignano, M. (2020). *Indagare l'interdisciplinarità tra fisica e matematica tramite la teoria delle reti: analisi di un corso di formazione insegnanti sul tema della parabola. [Investigating interdisciplinarity between physics and mathematics through network theory: analysis of a professional development course for teachers on the issue of parabola]*. Master thesis in Physics, Alma Mater Studiorum – University of Bologna. Advisor: Dr. G. Tasquier. Co-advisor: Dr. E. Giampieri

[48] Star, S. L., & Griesemer, J. R. (1989). Institutional ecology, "translations" and boundary objects: Amateurs and professionals in Berkeley's Museum of Vertebrate Zoology, 1907–39. *Social Studies of Science*, 19, 387–420.

[49] Suchman, L. (1994). Working relations of technology production and use. *Computer Supported Cooperative Work*, 2, 21–39.

[50] Thompson Klein, J. (2010). A taxonomy of interdisciplinarity. In R. Frodeman, J. T. Klein, & C. Mitcham (Eds.), *The Oxford handbook of interdisciplinarity*, 15–30. Oxford and New York: Oxford University Press. https://doi.org/10.1093/oxfordhb/9780198733522.001.0001

[51] Tzanakis, C. (2016). Mathematics & physics: an innermost relationship. Didactical implications for their teaching & learning. *History and Pedagogy of Mathematics*, Montpellier, France. https://hal.archives-ouvertes.fr/hal-01349231

[52] Uhden, O., Avelar Sotomaior Karam, R., Pospiech, G., & Pietrocola, M. (2012). Modelling Mathematical Reasoning in Physics Education. *Science & Education*, 20(4). https://doi.org/10.1007/s11191-011-9396-6

[53] Wenger, E. (1998). Communities of practice, learning, meaning and identity. Cambridge, UK: Cambridge University Press.

RINGRAZIAMENTI

Quando si vuole insegnare/studiare la storia ci si imbatte sempre in un dilemma: procedere cronologicamente o per temi? Lo stesso dilemma sorge nel momento in cui si ha una lista di persone da ringraziare. Io ho deciso di adottare l'ordine cronologico, così che non ci possano essere interpretazioni errate sul grado di importanza di ognuno degli individui che nominerò.

Naturalmente, comincerò dai miei genitori: se non fosse per loro oggi non sarei qui. Non solo per motivi scontati, ma soprattutto perché mi hanno capita quando ancora non riuscivo io stessa. Hanno creduto in delle mie potenzialità che ancora oggi stento a riconoscere. Mi hanno indicato una strada che si è rivelata la più giusta in assoluto. Non potrò mai ripagarvi per tutto quello che avete sopportato e fatto per me.

Il cammino è stato arduo, e tra le persone che lo hanno reso percorribile c'è senza dubbio il professore Pace. Non la ringrazio solo d'avermi dato le necessarie conoscenze disciplinari, ma anche d'aver illuminato il sentiero con una perla di saggezza: "Modica a te piace il dramma, ma sappi che per un insegnante un'aula non è altro che un palcoscenico."

Tra gli insegnanti che ho incontrato nel corso della mia laurea triennale, ce n'è una in particolare, che mi ha donato la più importante delle frecce al mio arco: la stima di me. Ringrazio la professoressa Gallo dell'università di Roma Tre che, oltre ad avermi fatto amare la fisica, ha speso del tempo per me, per farmi capire che era giunto il momento di non accontentarmi più della mediocrità. L'esame con lei è tra quelli che ricordo con più emozione. Ho decisamente cambiato passo dal nostro incontro.

Poi si sa che nella vita si hanno dei successi accademici e lavorativi anche grazie agli affetti ed alla loro stabilità. Ringrazio Chiara e Matteo che non sono dei semplici amici, ma una vera e propria colonna portante della mia felicità. Siete la coppia di persone più onesta, sincera e amorevole che abbia mai incontrato. Ogni giorno il mio affetto per voi mi dà forza e gioia.

Ringrazio il mio compagno di vita: Luigi. Sei l’unica persona al mondo che ha la sfortuna di vivere tutti i miei difetti più tremendi, e di amarmi lo stesso. Senza di te forse sarei comunque arrivata a certi traguardi, ma la gioia di festeggiare insieme, nel corso della nostra dolce quotidianità, è qualcosa a cui non vorrò rinunciare mai. Grazie del supporto e dell’impegno che dimostri in tutto ciò che fai.

Grazie alla professoressa Cattabriga d’avermi instillato pensieri positivi e propositivi nel corso dei suoi insegnamenti. Con lei ho imparato che non basta criticare gli errori del passato, è fondamentale guardare al futuro. Per i consigli, il sostegno, la fiducia, l'appoggio, grazie infinite.

Grazie alla professoressa Levrini. Grazie d’aver riaperto porte della conoscenza e del pensiero che pensavo chiuse per sempre. Mi ha restituito parte di me, di ciò che ero, e mi ha donato nuova energia per cercare di essere sempre di più. Grazie d’aver creduto in me, d’avermi dato delle opportunità incredibili, e d’essere un modello da seguire ed ammirare.

Grazie a Lorenzo, che non si è limitato ad essere un ottimo correlatore, ma anche una persona con cui avere sempre conversazioni stimolanti. Grazie d’aver ceduto nell’essermi anche un po’ amico, hai avuto tantissima pazienza. Grazie delle critiche costruttive di cui ho fatto tesoro, ed anche d’avermi donato la giusta quantità di punti da usare per l’ortografia di questa tesi.

Grazie ai miei compagni di viaggio del corso di laurea magistrale. Grazie dei consigli, delle uscite, delle risate e d’aver fatto fronte comune contro l’isolamento da pandemia.

Infine, grazie a tutte le persone che hanno fatto parte di questi miei primi 29 anni di vita, nel bene e nel male, a chi c’è ancora, e a chi non c’è più. Si può sempre imparare tanto da tutto.

Grazie,
Lucia