Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE
Corso di Laurea in Matematica

ON THE IWASAWA DECOMPOSITION

Tesi di Laurea in Algebra

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Anno Accademico 2021-2022

Ai miei genitori,
che da sempre credono in me.

Introduzione

La decomposizione di Iwasawa è una decomposizione di un gruppo di Lie semisemplice G in cui i fattori sono sottogruppi chiusi di G e il cui prototipo è il procedimento di ortonormalizzazione di Gram-Schmidt. Per esempio, siano $G = \mathrm{SL}_n(\mathbb{C})$, $U(n)$ il gruppo unitario, $K = G \cap U(n)$, A il sottogruppo di G delle matrici diagonali con elementi diagonali positivi ed U il sottogruppo delle matrici unipotenti, triangolari superiori. La decomposizione di Iwasawa è $G = KAU$ nel senso che la moltiplicazione $K \times A \times U \to G$ è un omeomorfismo (e, in effetti, un diffeomorfismo). Questa decomposizione si estende a tutti i gruppi di Lie semisemplici. In tal caso si ottiene una decomposizione dell'algebra di Lie e si solleva ai gruppi. I dettagli di questa costruzione generale possono essere trovati, ad esempio, in [K].

Obbiettivo principale di questo lavoro è studiare la decomposizione di Iwasawa del gruppo $\mathrm{SL}_2(\mathbb{R})$ approfondendone alcune rilevanti applicazioni topologiche ed algebriche.

La tesi si sviluppa intorno ad un capitolo principale, il Capitolo 2, che si apre con la dimostrazione della decomposizione di Iwasawa per $\mathrm{SL}_2(\mathbb{R})$. Il Teorema 2.2.1 mostra poi che $\mathrm{SL}_2(\mathbb{R})$ può essere pensato come la parte interna di un toro solido, dunque è possibile calcolarne il gruppo fondamentale e il rivestimento universale. Il paragrafo 2.4 è dedicato allo studio delle classi di coniugio degli elementi di $\mathrm{SL}_2(\mathbb{R})$, descritte in termini dei sottogruppi che compaiono nella decomposizione di Iwasawa (Teorema 2.4.1). Infine nel paragrafo 2.5 la decomposizione di Iwasawa viene ottenuta utilizzando l'azione di $\mathrm{SL}_2(\mathbb{R})$ sul semipiano superiore. La referenza principale per questo capitolo è una nota di Keith Conrad sull'argomento [C].

Il Capitolo 1 raccoglie il materiale introduttivo utile ad una lettura autocontenuta della tesi. In particolare vengono introdotti i gruppi topologici $\mathrm{GL}_n(\mathbb{F})$ e $\mathrm{SL}_n(\mathbb{F})$, l'azione di un gruppo su un insieme e l'esponenziale di matrici.

Infine, nel Capitolo 3 viene estesa la dimostrazione della decomposizione di Iwasawa ai gruppi $\mathrm{SL}_n(\mathbb{R})$ e $\mathrm{SL}_n(\mathbb{C})$.

Introduction

The Iwasawa decomposition is a decomposition of a semisimple Lie group G in which the factors are closed subgroups of G and whose prototype is the Gram-Schmidt orthonormalization process. For instance, let $G = \mathrm{SL}_n(\mathbb{C})$, $U(n)$ the unitary group, $K = G \cap U(n)$, A the subgroup of G of diagonal matrices with positive diagonal entries and U the subgroup of unipotent upper triangular matrices. The Iwasawa decomposition is $G = KAU$, meaning that the product $K \times A \times U \to G$ is a homomorphism (and, in fact, a diffeomorphism). This decomposition extends to all semisimple Lie groups. In that case, a decomposition of the Lie algebra is obtained and lifted to the group. The details of this general construction can be found, for example, in [K].

The main goal of this thesis is to study the Iwasawa decomposition of $\mathrm{SL}_2(\mathbb{R})$, examining some relevant topological and algebraic applications.

The thesis is developed around a main chapter, Chapter 2, which opens with a proof of the Iwasawa decomposition for $\mathrm{SL}_2(\mathbb{R})$. Theorem 2.2.1 shows that $\mathrm{SL}_2(\mathbb{R})$ can be thought as the inside of a solid torus, so it is possible to calculate its fundamental group and universal covering. Section 2.4 concerns the study of the conjugacy classes of elements of $\mathrm{SL}_2(\mathbb{R})$, described in terms of the subgroups that appear in the Iwasawa decomposition (Theorem 2.4.1). Lastly, in Section 2.5, the Iwasawa decomposition is obtained using the action of $\mathrm{SL}_2(\mathbb{R})$ on the upper half-plane. The main reference for this chapter is a note by Keith Conrad on the topic [C].

Chapter 1 presents introductory material useful for reading the thesis.

In particular, the following topics are introduced: the topological groups $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{SL}_n(\mathbb{F})$, the action of a group on a set, and matrix exponential.

Lastly, in Chapter 3 the Iwasawa decomposition is generalized to the groups $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{C})$.

Contents

Chapter 1

Preliminaries

1.1 The groups $\mathbf{GL}_n(\mathbb{F})$ and $\mathbf{SL}_n(\mathbb{F})$

Let $\mathbb{F}$ be a field and let n be a positive integer.
We denote by $\mathrm{M}_n(\mathbb{F})$ the set of $n \times n$ matrices with coefficients in $\mathbb{F}$.
The *general linear group* of order n is the subset of $\mathrm{M}_n(\mathbb{F})$ of invertible matrices:

$$\mathrm{GL}_n(\mathbb{F}) = \{g \in \mathrm{M}_n(\mathbb{F}) \mid \det g \neq 0\}.$$

Endowing it with the matrix product operation it gains a group structure.
We define $\mathrm{SL}_n(\mathbb{F})$ as the set of matrices in $\mathrm{M}_n(\mathbb{F})$ with determinant 1:

$$\mathrm{SL}_n(\mathbb{F}) = \{g \in \mathrm{M}_n(\mathbb{F}) \mid \det g = 1\}.$$

It is a subgroup of $\mathrm{GL}_n(\mathbb{F})$ due to Binet Theorem.
In this thesis we are interested in the cases $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

Topological structure

A *topological group* G is a topological space with a group structure such that the group operations

$$\begin{aligned} G \times G &\to G & \qquad G &\to G \\ (x, y) &\mapsto x \cdot y & \qquad x &\mapsto x^{-1} \end{aligned}$$

are continuous.
In the special cases $\mathbb{F} = \mathbb{R}, \mathbb{C}$ we can see $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{SL}_n(\mathbb{F})$ as topological groups endowing them with the euclidean induced topology, identifying $\mathrm{M}_n(\mathbb{F})$ with $\mathbb{F}^{n\times n}$.
The interesting maps between two topological groups are the continuous homomorphisms.

Remark 1.1.1. Requiring both continuity and homomorphism conditions is not redundant at all: neither condition implies the other. For instance we can examine $G := (\mathbb{Q}[e], +)$ with the euclidean induced topology from $\mathbb{R}$ and the functions

$$\begin{aligned} f\colon G &\to G \\ x &\mapsto 3 \end{aligned} \qquad\qquad \begin{aligned} g\colon G &\to G \\ q &\mapsto q \quad \text{for } q \in \mathbb{Q} \\ e &\mapsto 0; \end{aligned}$$

the map f is continuous because it is constant, but it is not a homomorphism since $f(0) \neq 0$; conversely g is a homomorphism by definition but it is not continuous because there exist open neighbourhoods of 0 whose inverse image is not open.

1.2 Group action

Let X be a set and let G be a group.
A map $\sigma\colon X \to X$ is called a *permutation* of X if it is bijective. Let us denote by $\mathrm{Perm}(X)$ the set of all the permutations of X. It's easy to prove that it gains a group structure when endowed with function composition operation.
We define an *action of the group G on X* a map

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto gx \end{aligned}$$

satisfying the two following properties:

- denoted by e the identity of the group G, for every $x \in X$, $ex = x$;
- for every $g_1, g_2 \in G$ and for every $x \in X$, $g_1(g_2x) = (g_1g_2)x$.

It follows from these properties that the map $\rho_g \colon x \mapsto gx$ is a permutation of X (with $\rho_{g^{-1}} \colon x \mapsto g^{-1}x$ as inverse), hence the map

$$\begin{aligned} \rho \colon G &\to \mathrm{Perm}(X) \\ g &\mapsto \rho_g \end{aligned}$$

is a group homomorphism.
Conversely such a homomorphism gives rise to an action of G on X.
Given $x \in X$ we denote by O_x^G the subset of elements $y \in X$ such that $y = gx$ for some $g \in G$, called the *orbit of x in G*, and by $\mathrm{Stab}(x)$ the subset of elements g in G such that $gx = x$ (relative to a fixed action), called the *stabilizer of x* (in G).
It's easy to prove that the latter is a subgroup of G.
An action of G on X is called *transitive* if for any two elements $x, y \in X$ there exists an element $g \in G$ such that $y = gx$; in other words an action is transitive on X if there is a unique orbit. Equivalently we will say that G *acts transitively* on X.

1.3 Exponential and logarithm of matrices

Let us introduce a norm on $\mathrm{M}_n(\mathbb{R})$. We define

$$\begin{aligned} \|\cdot\| : \mathrm{M}_n(\mathbb{R}) &\to \mathbb{R} \\ A &\mapsto \sup_{x \in \mathbb{R}^n, \|x\|_{\mathbb{R}^n} = 1} \|Ax\|_{\mathbb{R}^n} . \end{aligned} \tag{1.1}$$

It is well defined because the sphere $\{x \in \mathbb{R}^n, \|x\|_{\mathbb{R}^n} = 1\}$ is compact and the map $x \mapsto \|Ax\|_{\mathbb{R}^n}$ is continuous, and it is actually a norm (cf. [A]).
We can define, as in the scalar case, the exponential of a matrix $X \in \mathrm{M}_n(\mathbb{R})$ as follows

$$\exp X := \sum_{j=0}^{\infty} \frac{X^j}{j!}. \tag{1.2}$$

Proposition 1.3.1. *For every $X \in \mathrm{M}_n(\mathbb{R})$, the series* (1.2) *is convergent for the norm* (1.1).

Proof. For the proof see [A]. $\square$

We will also use e^X to denote $\exp X$.

Cases of diagonal and nilpotent matrices

Let us examine the cases of diagonal and nilpotent matrices. Narrowing down to these two cases greatly simplifies the calculation of the sum of the series. In the first case because the series of matrices becomes a matrix whose components are scalar series; in the second one because the series becomes a polynomial: let $A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in \mathrm{M}_n(\mathbb{R})$ be a diagonal matrix and $N \in \mathrm{M}_n(\mathbb{R})$ be a nilpotent matrix (notice that its order of nilpotence must be $\leq n$).
We have

$$e^A = \sum_{i=0}^{\infty} \frac{1}{i!} A^i = I_n + \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} + \frac{1}{2} \begin{pmatrix} a_1^2 & & \\ & \ddots & \\ & & a_n^2 \end{pmatrix} + \cdots =$$

$$= \begin{pmatrix} e^{a_1} & & \\ & \ddots & \\ & & e^{a_n} \end{pmatrix};$$

$$e^N = \sum_{i=0}^{\infty} \frac{1}{i!} N^i = \sum_{i=0}^{n-1} \frac{1}{i!} N^i + \frac{1}{n!} 0 + \frac{1}{(n+1)!} 0 + \cdots = \sum_{i=0}^{n-1} \frac{1}{i!} N^i,$$

where we have denoted by 0 the null matrix.
Let us fix the following notation:

- $\mathfrak{n}_n$ is the set of strictly upper triangular matrices (which are nilpotent of degree $\leq n$);

- $U_n := \{I_n + X \mid X \in \mathfrak{n}_n\}$ is the set of *unipotent upper triangular matrices*, i.e. the set of the upper triangular matrices with 1's as diagonal entries;
- D_n is the set of diagonal matrices;
- A_n is the set of diagonal matrices with positive diagonal entries.

We define, for every $Y = I_n + X \in U_n$,

$$\log(I_n + X) := \sum_{j=1}^{n-1} (-1)^{j+1} \frac{X^j}{j}$$

and for every $A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in A_n$,

$$\log A := \begin{pmatrix} \log a_1 & & \\ & \ddots & \\ & & \log a_n \end{pmatrix}.$$

It is proven that

$$\begin{aligned} \exp\colon \mathfrak{n}_n &\to U_n, \\ X &\mapsto e^X \end{aligned} \qquad\qquad \begin{aligned} \log\colon U_n &\to \mathfrak{n}_n \\ Y &\mapsto \log Y \end{aligned}$$

are inverse functions over any field of characteristic 0 (cf. [L1]).
Notice that for $n = 2$ the bijection is just $X \longleftrightarrow I_2 + X$.
Currently it's easy to prove that

$$\begin{aligned} \exp\colon D_n &\to A_n, \\ D &\mapsto e^D \end{aligned} \qquad\qquad \begin{aligned} \log\colon A_n &\to D_n \\ A &\mapsto \log A \end{aligned}$$

are inverse functions over any field of characteristic 0.
We can also define for $Y \in U_n, A_n$ and for $t \in \mathbb{R}$

$$Y^t := \exp(t \cdot \log Y).$$

In the specific case of $n = 2$, if we set $U = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, and $A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$, we have

$$U^t = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^t = \exp\left(t \cdot \log \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) = \exp \begin{pmatrix} 0 & tx \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & tx \\ 0 & 1 \end{pmatrix};$$

$$A^t = \exp(t \cdot \log \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} = \begin{pmatrix} e^{t \cdot \log a_1} & 0 \\ 0 & e^{t \cdot \log a_2} \end{pmatrix} = \begin{pmatrix} a_1^t & 0 \\ 0 & a_2^t \end{pmatrix}.$$

We will use these formulas in Section 2.2.

Chapter 2

$\mathbf{SL}_2(\mathbb{R})$ Iwasawa decomposition

In this chapter we will pay attention on $\mathrm{SL}_2(\mathbb{R})$.
We will derive a product decomposition for it, called the *Iwasawa Decomposition*, and provide some applications.

2.1 Iwasawa decomposition

Consider in $\mathrm{SL}_2(\mathbb{R})$ the following three subgroups:

$$K = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \, \theta \in \mathbb{R} \right\}, \quad A = \left\{ \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \middle| \, r > 0 \right\},$$
$$U = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}.$$

Theorem 2.1.1. *For every $g \in \mathrm{SL}_2(\mathbb{R})$ there exist three unique matrices $k \in K$, $a \in A$ and $u \in U$ such that $g = kau$.*

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in $\mathrm{SL}_2(\mathbb{R})$ and apply it to the standard basis $\{e_1, e_2\}$ of $\mathbb{R}^2$. Since $det(g) = 1$, the vectors

$$ge_1 = \begin{pmatrix} a \\ c \end{pmatrix}, \qquad ge_2 = \begin{pmatrix} b \\ d \end{pmatrix}$$

form a basis of $\mathbb{R}^2$.
We want to apply a series of transformations in $\mathrm{SL}_2(\mathbb{R})$ to bring back ge_1 and ge_2 respectively to e_1, e_2.
Let θ be the counterclockwise angle from the positive x-axis to ge_1 and let ρ_θ be the counterclockwise rotation around the origin by θ; $\rho_{-\theta} = (\rho_\theta)^{-1}$ and $\rho_{-\theta} g e_1$ lies on the positive x-axis, i.e., it is a positive scalar multiple of e_1
Since $\rho_{-\theta}$ is an orthogonal transformation, then

$$r := \|\rho_{-\theta} g e_1\| = \|ge_1\| = \sqrt{a^2 + c^2}.$$

Therefore we can rescale $\rho_{-\theta} g e_1$ applying $\begin{pmatrix} 1/r & 0 \\ 0 & 1/r \end{pmatrix}$ to obtain e_1, but this matrix doesn't have determinant 1, whereas $\begin{pmatrix} 1/r & 0 \\ 0 & r \end{pmatrix}$ acts in the same way on $\rho_{-\theta} g e_1$ and has determinant 1.
Call $B := \begin{pmatrix} 1/r & 0 \\ 0 & r \end{pmatrix} \rho_{-\theta} g$; how does it transform e_2? B acts as the identity on e_1, so it has the form $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$ and it has determinant 1 (since it is product of matrices in $\mathrm{SL}_2(\mathbb{R})$), thus the bottom right entry must be 1, i.e. B sends e_2 to $\begin{pmatrix} x \\ 1 \end{pmatrix}$, for some $x \in \mathbb{R}$.
Then consider the linear transformation $\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$, which has determinant 1 and fixes the x-axes; it sends e_1 to itself and $\begin{pmatrix} x \\ 1 \end{pmatrix}$ to e_2.
We have finally returned to the standard basis using a sequence of transformations in $\mathrm{SL}_2(\mathbb{R})$, namely the map

$$\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/r & 0 \\ 0 & r \end{pmatrix} \rho_{-\theta} g$$

is the identity over e_1 and e_2, so it is the identity on $\mathbb{R}^2$. Solving for g, we obtain

$$g = \rho_\theta \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in KAU.$$

Moreover we can write the parameters θ, r and x in terms of the entries of g: if

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \\ &= \begin{pmatrix} r\cos\theta & rx\cos\theta - 1/r\sin\theta \\ r\sin\theta & rx\sin\theta + 1/r\cos\theta \end{pmatrix} \end{aligned} \tag{2.1}$$

we have

$$a^2 + c^2 = r^2,\ r > 0 \implies r = \sqrt{a^2 + c^2}, \tag{2.2}$$

$$\begin{cases} \cos\theta = a/r \\ \sin\theta = c/r \end{cases} \implies \theta \text{ is uniquely determined,} \tag{2.3}$$

$$\begin{pmatrix} x \\ 1 \end{pmatrix} = Be_2 = \begin{pmatrix} \frac{1}{r}(b\cos\theta + d\sin\theta) \\ r(-b\sin\theta + d\cos\theta) \end{pmatrix} \implies x = \frac{ab + cd}{a^2 + c^2}. \tag{2.4}$$

In this way we have also shown the uniqueness of the decomposition, since r and θ do not depend on the construction, but on formula (2.1), and from the same formula we can solve for x, obtaining (2.4). $\square$

Remark 2.1.1. Notice that saying that there is a unique $G = UAK$ decomposition or saying that there is a unique decomposition $G = KAU$ is the same thing.
Indeed, if $G = UAK$, then for $g \in G$ we can consider $g = uak$ for some $u \in U$, $a \in A$, $k \in K$, so that $g{-}1 = k^{-1}a^{-1}u^{-1}$. In this way we obtain a decomposition $G = KAU$ (and vice versa).

2.2 Topological Applications

In Section 1.1 we discussed the topological structure of the groups $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{R})$, endowed with the topology induced by the euclidean one from

$\mathbb{R}^{n\times n}$.

We might wonder how to visualize $\mathrm{SL}_2(\mathbb{R})$, assuming it is possible. In this section we will use the Iwasawa decomposition of $\mathrm{SL}_2(\mathbb{R})$ to get a concrete image of it.

Theorem 2.2.1. *As a topological space,* $\mathrm{SL}_2(\mathbb{R})$ *is homeomorphic to the inside of a solid torus.*

Proof. Let $\varphi\colon K\times A\times U\to \mathrm{SL}_2(\mathbb{R})$ be the application that sends (k,a,u) to kau. It is surjective by the existence of the Iwasawa decomposition and it is injective because of the uniqueness. Its inverse is $\psi\colon\ \mathrm{SL}_2(\mathbb{R})\to K\times A\times U$, $g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\mapsto (k(g),a(g),u(g))$, where

$$k(g):=\begin{pmatrix} a/r(g) & -c/r(g)\\ c/r(g) & a/r(g)\end{pmatrix},\ a(g):=\begin{pmatrix} r(g) & 0\\ 0 & 1/r(g)\end{pmatrix},\ u(g):=\begin{pmatrix} 1 & x(g)\\ 0 & 1\end{pmatrix},$$

$$r(g):=\sqrt{a^2+c^2},\qquad x(g):=\frac{ab+cd}{a^2+c^2}.$$

Notice that the maps φ and ψ are both continuous.

Topologically $K\cong S^1$, $A\cong\mathbb{R}_{\geq 0}\cong\mathbb{R}$, $U\cong\mathbb{R}$. Since $\mathbb{R}^2\cong \mathrm{D}^2$ (the open disc with radius 1) through the map $h\mapsto\frac{h}{1+\|h\|}$, then $\mathrm{SL}_2(\mathbb{R})\cong K\times A\times U\cong S^1\times\mathbb{R}^2\cong S^1\times D^2$, i.e. $\mathrm{SL}_2(\mathbb{R})$ is homeomorphic to the inside of a solid torus. $\square$

Remark 2.2.1. Notice that the map $\varphi\colon K\times A\times N\to\mathrm{SL}_2(\mathbb{R})$ is not a group homomorphism. Indeed, we have, for example, with $\theta=\pi/2$, $r=2$, $x=0$,

$$\varphi\left(\left(\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix},\begin{pmatrix} 2 & 0\\ 0 & \frac{1}{2}\end{pmatrix},\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\right)^2\right)=$$

$$=\varphi\left(\begin{pmatrix} -1 & 0\\ 0 & -1\end{pmatrix},\begin{pmatrix} 4 & 0\\ 0 & \frac{1}{4}\end{pmatrix},\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\right)=\begin{pmatrix} -4 & 0\\ 0 & -\frac{1}{4}\end{pmatrix}$$

but

$$\left(\varphi\left(\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix},\begin{pmatrix} 2 & 0\\ 0 & \frac{1}{2}\end{pmatrix},\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\right)\right)^2=\begin{pmatrix} 0 & -\frac{1}{2}\\ 2 & 0\end{pmatrix}^2=\begin{pmatrix} -1 & 0\\ 0 & -1\end{pmatrix}.$$

Definition 2.2.1. Given a topological space X and its subspace A, we say that A is a *deformation retract of* X if there exists a continuous map

$$r\colon X \times [0,1] \to X$$

such that

$$\begin{aligned} &r(x,0) \in A \text{ for every } x \in X;\\ &r(x,1) = x \text{ for every } x \in X,\\ &r(a,0) = a \text{ for every } a \in A. \end{aligned}$$

If we strengthen the third condition with $r(a,t) = a$ for every $a \in A$ and for every $t \in [0,1]$, then we say that A is a *strong* deformation retract of X.

Remark 2.2.2. The inside of the solid torus $S^1 \times D^2$ has $S^1 \times \{(0,0)\}$ as a strong deformation retract, by the homotopy

$$\begin{aligned} r\colon (S^1 \times D^2) \times [0,1] &\to S^1 \times D^2\\ ((e^{i\theta},(a,b)),t) &\mapsto (e^{i\theta},(ta,tb)), \end{aligned}$$

in fact

$$\begin{aligned} &r((e^{i\theta},(a,b)),0) = (e^{i\theta},(0,0)),\\ &r((e^{i\theta},(a,b)),1) = (e^{i\theta},(a,b)),\\ &r((e^{i\theta},(0,0)),t) = (e^{i\theta},(0,0)) \quad \forall t \in [0,1]. \end{aligned}$$

Therefore $\pi_1(\mathrm{SL}_2(\mathbb{R})) \cong \pi_1(S^1 \times D^2) \cong \pi_1(S^1) \cong \mathbb{Z}$.
We can see that the counterpart of S^1 in $\mathrm{SL}_2(\mathbb{R})$ is precisely K, writing down the explicit homotopy that retracts $\mathrm{SL}_2(\mathbb{R})$ to K:

$$\begin{aligned} \tilde{r}\colon \mathrm{SL}_2(\mathbb{R}) \times [0,1] &\to \mathrm{SL}_2(\mathbb{R})\\ (kau,t) &\mapsto ka^t u^t = k \begin{pmatrix} r^t & 0\\ 0 & 1/r^t \end{pmatrix} \begin{pmatrix} 1 & tx\\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\tilde{r}(kau,0) = k, \qquad \tilde{r}(kau,1) = kau, \qquad \tilde{r}(k,t) = k \quad \forall t \in [0,1],$$

i.e. $\pi_1(\mathrm{SL}_2(\mathbb{R})) \cong \pi_1(K) \cong \mathbb{Z}$.
We also point out that the universal covering space of $\mathrm{SL}_2(\mathbb{R})$ is homeomorphic to the inside of a solid infinite cylinder $\mathbb{R} \times D^2 \cong \mathbb{R}^3$; further from the Galois correspondence between subgroups of the fundamental group and connected covering spaces of a topological space, $\mathrm{SL}_2(\mathbb{R})$ admit a unique covering space of degree n for every integer n.
It has relevant importance the degree-2 covering space, called the *Metaplectic Group* (cf. [HT]).

2.3 Algebraic Applications

From now on we will set:

$$k_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad a_r := \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}, \quad u_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Given two subgroups H, K of a group G, let us denote by HK the set of the elements $g \in G$ such that $g = hk$ for some $h \in H$, $k \in K$.

Lemma 2.3.1. *Let H, K be two subgroups of a group G. If $HK = KH$ then HK (or equivalently KH) is a subgroup of G.*

Proof. First of all notice that the identity of G belongs to HK because it belongs to H and K.
Let $h_1k_1, h_2k_2 \in HK$, with $h_i \in H, k_i \in K$, for $i = 1, 2$. Since $k_1h_2 \in KH$ and by hypothesis $KH = HK$, there exist two elements $h_3 \in H, k_3 \in K$ such that $k_1h_2 = h_3k_3$. Therefore

$$h_1k_1h_2k_2 = h_1h_3k_3k_2 \in HK;$$

moreover

$$(h_1k_1)^{-1} = k_1^{-1}h_1^{-1} \in KH = HK.$$

We have proven that HK is closed under group operations, i.e. it is a subgroup. □

Lemma 2.3.2. *Let A, U be the subgroups introduced in Section 2.1. Then $AU = UA$. It follows from Lemma 2.3.1 that AU (or equivalently UA) is a subgroup of* $\mathrm{SL}_2(\mathbb{R})$.

Proof. Let $a_r \in A$, $u_x \in U$, then

$$a_r u_x = \begin{pmatrix} r & rx \\ 0 & 1/r \end{pmatrix} = u_{r^2 x} a_r \in UA$$

Conversely

$$u_x a_r = \begin{pmatrix} r & x/r \\ 0 & 1/r \end{pmatrix} = a_r u_{\frac{x}{r^2}} \in AU.$$

□

Theorem 2.3.1. *The trivial homomorphism is the unique continuous homomorphism* $\mathrm{SL}_2(\mathbb{R}) \to \mathbb{R}$.

Proof. Let $f\colon (\mathrm{SL}_2(\mathbb{R}), \cdot) \to (\mathbb{R}, +)$ be a continuous homomorphism. We will show that $f \equiv 0$.
For every $kau \in \mathrm{SL}_2(\mathbb{R})$, $f(kau) = f(k) + f(a) + f(u)$. Let's study how f acts on K, A, U.
If an element $b \in \mathrm{SL}_2(\mathbb{R})$ has finite order, i.e. there exists a positive integer n such that $b^n = 1_{\mathrm{SL}_2(\mathbb{R})}$, then $0 = f(1) = f(b^n) = nf(b)$, hence $f(b) = 0$.
Consider $D := \{k \in K \mid \exists\, n \in \mathbb{N}, n > 0 \text{ such that } k^n = 1_{\mathrm{SL}_2(\mathbb{R})}\} = \{k_{q\pi} | q \in \mathbb{Q}\}$. D is dense in K and $f \equiv 0$ on D, thus, by the continuity of f, $f \equiv 0$ over K.
We recall that all continuous homomorphisms $h\colon \mathbb{R} \to \mathbb{R}$ are of the form $h_t\colon x \mapsto tx$ for some $t \in \mathbb{R}$ ($t = f(1)$).
Since $(A, \cdot) \cong (\mathbb{R}, +)$ through $l_A\colon a_r \mapsto \log r$ and $(U, \cdot) \cong (\mathbb{R}, +)$ through $l_U\colon u_x \mapsto x$, then there exist t, $s \in \mathbb{R}$ such that for every $r > 0$, $x \in \mathbb{R}$

$$f(a_r) = f(l_A^{-1}(\log r)) = h_t(\log r) = t \log r,$$
$$f(u_x) = f(l_U^{-1}(x)) = h_s(x) = sx.$$

By Lemma 2.3.2, for every $r > 0, x \in \mathbb{R}$, $f(a_r u_x) = f(u_{r^2x} a_r)$, but

$$\begin{aligned} f(a_r u_x) &= f(a_r) + f(u_x) = t \log r + sx \\ f(u_{r^2x} a_r) &= f(u_{r^2x}) + f(a_r) = sr^2 x + t \log r. \end{aligned}$$

It follows (for example placing $r = 2, x \neq 0$) that $s = 0$, i.e., $f \equiv 0$ over U.
In order to show that f is trivial over A we will use the fact that a homomorphism with image in $(\mathbb{R}, +)$ is invariant on conjugate classes:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1/r & 0 \\ 0 & r \end{pmatrix} = a_r^{-1},$$

therefore $f(a_r) = f(a_r^{-1}) = -f(a_r)$, which implies $f(a_r) = 0$ for every $r > 0$. □

Corollary 2.3.1. *Every continuous homomorphism* $(\mathrm{SL}_2(\mathbb{R}), \cdot) \to (\mathrm{GL}_n(\mathbb{R}), \cdot)$ *has image in* $\mathrm{SL}_n(\mathbb{R})$.

Proof. Let f be a continuous homomorphism as in the statement and consider $det\colon (\mathrm{GL}_n(\mathbb{R}), \cdot) \to (\mathbb{R}^*, \cdot)$.
The topological group $\mathrm{SL}_2(\mathbb{R})$ is connected (Theorem 2.2.1) and $(det \circ f)(I_2) = 1$, so $Im(det \circ f)$ lies in $\mathbb{R}_{>0}$. Hence we can compose $det \circ f$ with $h\colon x \mapsto \log x$.
Notice that $h \circ det \circ f$ is a continuous homomorphism from $\mathrm{SL}_2(\mathbb{R})$ to $(\mathbb{R}, +)$, so it's trivial by Theorem 2.3.1. Since h is bijective, $det \circ f \equiv 1$, namely for every $g \in \mathrm{SL}_2(\mathbb{R})$, $f(g)$ has determinant 1. □

Example 2.3.1. Let $V := \mathbb{R}[x, y]_2$ be the $\mathbb{R}$-vector space of homogeneous polynomials in x and y of degree 2.
Let's consider the action of G:=$\mathrm{GL}_2(\mathbb{R})$ on V

$$\begin{aligned} \rho\colon \mathrm{G} &\to \mathrm{Hom}(V) \\ g &\mapsto \rho_g, \quad \rho_g(p) = gp, \end{aligned}$$

where g acts on p as a linear change of variables: if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $gp := p(ax + cy, bx + dy)$.

The map ρ is well defined (the action on V is linear) and it is in fact an action, indeed, if $g_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $g_2 = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, then

$$\begin{aligned}
g_1(g_2(p)) &= g_1(p(ex+gy, fx+hy)) = \\
&= p(e(ax+cy)+g(bx+dy), f(ax+cy)+h(bx+dy)), \\
g_1g_2 &= \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}, \\
(g_1g_2)p &= p((ae+bg)x+(ce+dg)y, (af+bh)x+(cf+dh)y), \\
\implies g_1(g_2(p)) &= (g_1g_2)p.
\end{aligned}$$

If we fix the basis $B = \{x^2, xy, y^2\}$ of V we can identify $\mathrm{Hom}(V)$ with $\mathrm{M}_3(\mathbb{R})$. Notice that $\operatorname{Im}\rho \subseteq \mathrm{GL}_3(\mathbb{R})$ because ρ_g is invertible with inverse $\rho_{g^{-1}}$.
Let us calculate ρ_g in terms of matrices:

$$\begin{aligned}
\rho_g(x^2) &= (ax+cy)^2 = a^2x^2 + 2acxy + c^2y^2, \\
\rho_g(xy) &= (ax+cy)(bx+dy) = abx^2 + (ad+bc)xy + cdy^2, \\
\rho_g(y^2) &= (bx+dy)^2 = b^2x^2 + 2bdxy + d^2y^2.
\end{aligned}$$

Hence we obtain

$$f(g) := M_B^B(\rho_g) = \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}.$$

Therefore f is a continuous homomorphism from $\mathrm{GL}_2(\mathbb{R})$ to $\mathrm{GL}_3(\mathbb{R})$. Notice that

$$\begin{aligned}
\det f(g) &= a^2(ad^3 - bcd^2) - ab(2acd^2 - 2bdc^2) + b^2(ac^2d - bc^3) = \\
&= a^2d^2(ad-bc) - 2adbc(ad-bc) + b^2c^2(ad-bc) = \\
&= (ad-bc)^3 = (\det g)^3,
\end{aligned}$$

which shows f(g) is in fact invertible and that if $g \in \mathrm{SL}_2(\mathbb{R})$ then $\det f(g) = (\det g)^3 = 1^3 = 1$, i.e. $f(g) \in \mathrm{SL}_3(\mathbb{R})$.

2.4 Conjugacy classes in $\mathbf{SL}_2(\mathbb{R})$

Given a group G and a subgroup H of G we can define the following equivalence relations, called conjugation relations:

$$a \sim_G b \iff \exists\, g \in G : a = g^{-1}bg,$$
$$a \sim_H b \iff \exists\, h \in H : a = h^{-1}bh,$$

where a, b are any two elements of G.
Given $l \in G$ we can consider the orbits of l with respect to these relations:

$$O_l^G = \{g^{-1}lg \mid g \in G\},$$
$$O_l^H = \{h^{-1}lh \mid h \in H\}.$$

Of course $O_l^H \subseteq O_l^G$. This inclusion could be strict or an equality, depending on l.
In this section we will focus on understanding the nature of this inclusion in the specific case of $G := \mathrm{GL}_2(\mathbb{R})$ and its subgroup $H := \mathrm{SL}_2(\mathbb{R})$.

Remark 2.4.1. We recall that two real matrices are conjugate in $\mathrm{GL}_2(\mathbb{C})$ if and only if they are conjugate in $\mathrm{GL}_2(\mathbb{R})$.

Remark 2.4.2. Notice that trace and determinant are invariant under conjugation in G, but they are not characterizing. In other words, if $(a, b \in G)$ $a \sim_G b$, then $\operatorname{Tr} a = \operatorname{Tr} b$ and $\det a = \det b$, but in general, if $\operatorname{Tr} a = \operatorname{Tr} b$ or $\det a = \det b$, not necessarily $a \sim_G b$.
On the contrary, for every $h \in H$ other than $\pm I_2$, the orbit of h depends only on the trace t of h, i.e., excluding $\pm I_2$, the trace is characterizing for $\sim_G$ in H. Let's see why.
It is well known that the matrices in the same G-orbit as h are all and only those which seen in $\mathrm{GL}_2(\mathbb{C})$ have the same Jordan normal form. The latter depends uniquely on the trace of h, as in this case the characteristic polynomial is $p_h(x) = x^2 - tx + 1$ and its eigenvalues in $\mathbb{C}$ are $\lambda_{1,2} = \frac{t \pm \sqrt{t^2-4}}{2}$, which are

- distinct if $t \neq \pm 2$. In this case t uniquely determines (unless the order of the diagonal elements) the diagonal form $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ of h;
- equal if $t = \pm 2$. In this case, since we have excluded the cases $h = \pm I_2$, h will be conjugate (in $\mathrm{GL}_2(\mathbb{C})$) to the Jordan matrix $\begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}$.

By Remark 2.4.1, we can summarize the above discussion as follows:

- if $h \in \mathrm{SL}_2(\mathbb{R})$ with $\operatorname{Tr} h \neq \pm 2$, then $O_h^G = \{l \in \mathrm{SL}_2(\mathbb{R}) \mid \operatorname{Tr} l = \operatorname{Tr} h\}$;
- if $h \in \mathrm{SL}_2(\mathbb{R})$ with $\operatorname{Tr} h = \pm 2$, $h \neq \pm I_2$ then $O_h^G = \{l \in \mathrm{SL}_2(\mathbb{R}) \mid \operatorname{Tr} l = \operatorname{Tr} h, l \neq \pm I_2\}$;
- $O_{I_2}^G = \{I_2\}, \quad O_{-I_2}^G = \{-I_2\}$.

We now focus on the orbit O_h^H, for $h \in H$.

In the following theorem, where not specified, when we talk about conjugacy in $\mathrm{SL}_2(\mathbb{R})$ we will refer to the equivalence relation $\sim_H$.

Let $v_1, v_2 \in \mathbb{R}^2$ be two vectors. We will denote by $([v_1][v_2])$ the 2×2 matrix with v_1 on the first column and v_2 on the second one.

Theorem 2.4.1. *Let $h \in H$ and let $t = \operatorname{Tr} h$.*
If $t^2 > 4$ then h is conjugate to a unique matrix of the form $\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$ with $|\lambda| > 1$.
If $t^2 = 4$ then h is conjugate to exactly one matrix among $\pm I_2$, $\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\pm \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$.
If $t^2 < 4$ then h is conjugate to a unique matrix of the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ other than $\pm I_2$.

Proof. Let $h \in \mathrm{SL}_2(\mathbb{R})$ and let $p_h(x) = x^2 - tx + 1$ be its charateristic polynomial. The nature of the eigenvalues of the matrix depends on the discriminant $d_h = t^2 - 4$ of this polynomial. We thus have the following three cases:

- $d_h > 0 \iff t^2 > 4$
  The matrix h has two real distinct eigenvalues $\lambda > \mu$ with respective eigenvectors v, w; since $\det h = 1$, $\mu = 1/\lambda$. Notice that $|\lambda| > 1$.
  The matrix h is conjugate in $\mathrm{GL}_2(\mathbb{R})$ to $\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$ by $c := ([v][w])$.
  Since scaling a vector doesn't change its nature of eigenvector, we can rescale w so that c has determinant 1. We observe that $\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$ and $\begin{pmatrix} 1/\lambda & 0 \\ 0 & \lambda \end{pmatrix}$ are conjugate to each other by an element of $\mathrm{SL}_2(\mathbb{R})$ (specifically $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$). Therefore we can choose as representative the orbit of h the first one, which has as top left entry an element λ of absolute value greater than 1.
  Moreover h cannot be conjugated to any other matrix of the form $\begin{pmatrix} \eta & 0 \\ 0 & 1/\eta \end{pmatrix}$ with $|\eta| > 1, \eta \neq \lambda$, because of the eigenvalues.

- $d_h = 0 \iff t^2 = 4 \iff t = \pm 2$
  The eigenvalues of h are ± 1, with algebraic multiplicity $m_a = 2$. If the geometric multiplicity m_g is 2 too then h is necessarily $\pm I_2$, which is conjugated only to itself.
  From now on, to fix the ideas, let's assume $t = 2$; the case $t = -2$ is analogous.
  In the case when the geometric multiplicity is 1, consider an eigenvector v and extend it to a basis $\{v, \tilde{w}\}$ of $\mathbb{R}^2$. Rescaling as in the previous case $\tilde{w}$ to w we obtain $\det c = 1$, where $c := ([v][w])$.
  So $h = c \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} c^{-1}$ for some $x \in \mathbb{R}$, $x \neq 0$.

Let's set $r := \sqrt{|x|}$. Then

$$\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/r & 0 \\ 0 & r \end{pmatrix} = \begin{pmatrix} r & \pm r \\ 0 & 1/r \end{pmatrix} \begin{pmatrix} 1/r & 0 \\ 0 & r \end{pmatrix} = \\ = \begin{pmatrix} 1 & \pm r^2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \pm |x| \\ 0 & 1 \end{pmatrix},$$

namely h is conjugate to either $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ according to the sign of x.
Observe that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \nsim_{\mathrm{SL}_2(\mathbb{R})} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, indeed, if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, then

$$g \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} g^{-1} = \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1-ac & a^2 \\ -c^2 & 1+ac \end{pmatrix},$$

i.e. the top right entry of a matrix conjugated to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ by an element of $\mathrm{SL}_2(\mathbb{R})$ must be non negative.

- $d_h < 0 \iff t^2 < 4$
  In this case h has two non-real complex conjugate eigenvalues of module 1, so we can write them as $e^{i\theta}, e^{-i\theta}$. Let v be a vector such that $hv = e^{i\theta}v$. Then $h\overline{v} = \overline{hv} = \overline{e^{i\theta}v} = e^{-i\theta}\overline{v}$. This shows that $\{v, \overline{v}\}$ is a basis of complex eigenvectors for h.
  Let $u := v + \overline{v}$ and $w := i(v - \overline{v})$, which belong to $\mathbb{R}^2$ and are linearly independent over $\mathbb{R}$ (because of the linear independence of v and $\overline{v}$ over $\mathbb{C}$). We have:

$$\begin{aligned} h(u) &= e^{i\theta}v + e^{-i\theta}\overline{v} = \cos\theta v + i\sin\theta v + \cos\theta\overline{v} - i\sin\theta\overline{v} = \\ &= \cos\theta u + \sin\theta w \\ h(w) &= e^{i\theta}iv - e^{-i\theta}i\overline{v} = \cos\theta iv - \sin\theta v - \cos\theta i\overline{v} - \sin\theta\overline{v} = \\ &= -\sin\theta u + \cos\theta w \end{aligned}$$

Thus, setting $c := ([u][w])$, we have $h = c \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} c^{-1}$.
Setting now $r := \sqrt{|\det c|}$, $\tilde{u} := u/r$, $\tilde{w} := w/r$, $\tilde{c} := ([\tilde{u}][\tilde{w}])$ we obtain

$$h = \tilde{c} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tilde{c}^{-1}, \qquad \text{with } \det\tilde{c} = \pm 1.$$

If $\det\tilde{c} = 1$ then h is conjugate to $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ in $\mathrm{Sl}_2(\mathbb{R})$, otherwise we can reverse the order of the columns of $\tilde{c}$ to give it determinant 1. In this way h is conjugate to $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{R})$.
It remains to check that $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ are conjugate to each other in $\mathrm{SL}_2(\mathbb{R})$ if and only if they are equal. This is true because in this case we should have $\{e^{i\theta}, e^{-i\theta}\} = \{e^{i\alpha}, e^{-i\alpha}\}$, namely $\alpha = \pm\theta + 2k\pi$ for some $k \in \mathbb{Z}$; lastly α cannot be $-\theta + 2k\pi$ because (setting $x := \cos\theta = \cos\alpha, y := \sin\theta = -\sin\alpha \neq 0$) if there exists $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ such that $g \begin{pmatrix} x & y \\ -y & x \end{pmatrix} g^{-1} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies$$

$$\implies \begin{pmatrix} ax - by & ay + bx \\ cx - dy & cy + dx \end{pmatrix} = \begin{pmatrix} ax - cy & bx - dy \\ ay + cx & by + dx \end{pmatrix} \implies$$

$$\implies \begin{cases} by = cy \\ ay = -dy \end{cases} \overset{y\neq 0}{\implies} \begin{cases} b = c \\ a = -d \end{cases}$$

and this is absurd because we would have $1 = \det g = ad - bc = -d^2 - c^2 < 0$.

$\square$

Remark 2.4.3. Theorem 2.4.1 tells us that, for $h \in \mathrm{SL}_2(\mathbb{R})$, if we set $t_h = \operatorname{Tr} h$, the following hold:

- if $t_h < 2$ then $O_h^H = O_h^G$;
- if $t_h = 2$, $h \neq \pm I_2$ then $O_h^H \subsetneq O_h^G$;
- if $t_h > 2$ then $O_h^H \subsetneq O_h^G$, indeed $\operatorname{Tr}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \operatorname{Tr}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ but these matrices aren't conjugate to each other in $\mathrm{SL}_2(\mathbb{R})$.

Remark 2.4.4. Let K, A, U be the subgroups introduced in Section 2.1 and let h, t_h as in Remark 2.4.3. Theorem 2.4.1 shows that

- if $t_h < 2$ then O_h^H has a representative in K;
- if $t_h = 2$ then O_h^H has a representative in U;
- if $t_h > 2$ then O_h^H has a representative in A.

2.5 Action on the upper half-plane $\mathcal{H}$

In this section G will always denote $\mathrm{SL}_2(\mathbb{R})$ and K, A, U its subgroups introduced in Section 2.1. We will also use the notation introduced at the beginning of Section 2.3.

Consider the upper half-plane

$$\mathcal{H} := \{x + iy \mid x, y \in \mathbb{R},\ y > 0\}.$$

We can define on $\mathcal{H}$ a metric other than the Euclidean one, the Poincaré metric. This provides a hyperbolic geometric model, where the length of a curve $\gamma\colon [a, b] \to \mathcal{H}$, $\gamma(t) = x(t) + iy(t)$, is measured by integrating from a to b the line element $\frac{|\gamma'(t)|}{y(t)^2}$, where $|\cdot|$ is the euclidean distance of the complex plane.

Therefore we can define the distance $d_{\mathcal{H}}$ between two points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ of $\mathcal{H}$ as the infimum of the length of the curves having z_1

and z_2 as extremes.
It is proven (cf. [P]) that it is

$$d_{\mathcal{H}}(z_1, z_2) = 2\operatorname{arctanh}\frac{|z_2 - z_1|}{|z_2 - \overline{z_1}|} = 2\ln\frac{|z_2 - z_1| + |z_2 - \overline{z_1}|}{\sqrt{4y_1y_2}}.$$

We can define an action of G on $\mathcal{H}$ as follows: for every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and for every $z \in \mathcal{H}$, set

$$g(z) := \frac{az+b}{cz+d}.$$

Proposition 2.5.1. *The map $\rho\colon \mathrm{SL}_2(\mathbb{R}) \times \mathcal{H} \to \mathcal{H}$, $(g, z) \mapsto g(z)$ is a group action.*

Proof. First of all let us show that for every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in G and for every z in $\mathcal{H}$, $g(z)$ belongs to $\mathcal{H}$:

$$g(z) = \frac{az+b}{cz+d} \cdot \frac{\overline{cz+d}}{\overline{cz+d}} = \frac{ac|z|^2 + bd + adz + bc\overline{z}}{|cz+d|^2}$$
$$\implies \operatorname{Im} g(z) = \frac{(ad-bc)\operatorname{Im} z}{|cz+d|^2} = \frac{\operatorname{Im} z}{|cz+d|^2} > 0,$$

hence ρ is well-defined.
Let us now verify the two properties of an action:

- $1_G(z) = z$ for every $z \in \mathcal{H}$:

$$I_2(z) = \frac{1 \cdot z + 0}{0 \cdot z + 1} = z;$$

- $g_1(g_2(z)) = (g_1g_2)(z)$ for every $g_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $g_2 = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in G$, $z \in \mathcal{H}$:

$$g_1(g_2(z)) = \frac{a(\frac{ez+f}{gz+h}) + b}{c(\frac{ez+f}{gz+h}) + d} = \frac{aez + bgz + af + bh}{cez + dgz + cf + dh},$$
$$(g_1g_2)(z) = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}(z) = \frac{(ae+bg)z + af + bh}{(ce+dg)z + cf + dh}.$$

$\square$

Proposition 2.5.2. *The stabilizer of i in G is K.*

Proof. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$,

$$g(i) = i \iff ai + b = (ci + d)i \iff ai + b = di - c \iff$$
$$\iff \begin{cases} a = d \\ b = -c \end{cases} \iff g = \begin{pmatrix} a & -c \\ c & a \end{pmatrix} \underset{a^2+c^2=1}{\iff} g \in K.$$

$\square$

Remark 2.5.1. Every $a_r \in A$ acts on $\mathcal{H}$ as a dilation by a factor r^2 and every $u_x \in U$ acts on $\mathcal{H}$ as a translation parallel to the real axis by x. Indeed

$$a_r(z) = \frac{rz + 0}{0 \cdot z + \frac{1}{r}} = r^2 z;$$
$$u_X(z) = \frac{z + x}{0 \cdot z + 1} = z + x.$$

Proposition 2.5.3. *The action of* $\mathrm{SL}_2(\mathbb{R})$ *on* $\mathcal{H}$ *is transitive.*

Proof. In order to prove the statement we will show that for every element $z = x + iy \in \mathcal{H}$ there exists an element $g \in G$ such that $g(i) = z$; in this way every two elements of $\mathcal{H}$ are in the same orbit of i, i.e., they are in the same orbit. Let $r := \sqrt{y}$, $a_r := \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}$ and $u_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. Thus, by Remark 2.5.1,

$$u_x a_r(i) = u_x(r^2 i) = u_x(yi) = x + iy.$$

$\square$

Remark 2.5.2. In the proof above we got a matrix g in terms of the image of i we were aiming for. Conversely given $g \in G$ the Iwasawa decomposition $g = u_x a_r k_\theta$ immediately tells us how g acts on i:

$$g(i) = u_x a_r k_\theta(i) = u_x a_r(i) = x + ir^2.$$

We can see $\mathcal{H}$ as the cosets space $\mathrm{SL}_2(\mathbb{R})/K = \{[g]_\sim \mid g \in \mathrm{SL}_2(\mathbb{R})\}$, where $g_1 \sim g_2$ if and only if there exists $k \in K$ such that $g_1 = g_2 k$.
Notice that G/K is not a quotient group, because K is not normal in G.

Proposition 2.5.4. *The map*

$$\begin{aligned} \varphi\colon \mathrm{SL}_2(\mathbb{R})/K &\to \mathcal{H} \\ [g] &\mapsto g(i) \end{aligned}$$

is a bijection.

Proof. We observe that φ is well defined because if $g_1 = g_2 k$ for some $k \in K$ then $g_1(i) = g_2 k(i) = g_2(i)$ (see Proposition 2.5.2).
For the same reason if $g_1(i) = g_2(i)$ then $g_2^{-1} g_1(i) = i$, namely $g_2^{-1} g_1 \in K$, so $[g_1] = [g_2]$. This proves the injectivity of φ.
Lastly, φ is surjective by Proposition 2.5.3. □

We can derive an alternative proof of the Iwasawa decomposition using the action of G on $\mathcal{H}$:

Theorem 2.5.1. *For every $g \in G$ there exist three unique matrices $u \in U$, $a \in A$, $k \in K$ such that $g = uak$.*

Proof. Let $g \in G$ and let $x + iy := g(i)$ $(x, y \in \mathbb{R})$. By Remark 2.5.1, if we set $r := \sqrt{y}$ then $g(i) = u_x a_r(i)$. Hence by Proposition 2.5.4 there exists $k_\theta \in K$ such that $g = u_x a_r k_\theta$. This proves the existence.
If there exist $u_{x_1} \in U$, $a_{r_1} \in A$, $k_{\theta_1} \in K$ such that $u_x a_r k_\theta = u_{x_1} a_{r_1} k_{\theta_1}$ then

$$\begin{aligned} u_x a_r k_\theta(i) &= u_{x_1} a_{r_1} k_{\theta_1}(i) \\ u_x a_r(i) &= u_{x_1} a_{r_1}(i) \\ x + iy &= x_1 + i r_1^2. \end{aligned}$$

It follows that $u_x = u_{x_1}$ and $a_r = a_{r_1}$, thus $k_\theta = k_{\theta_1}$, namely we have proven the uniqueness of the decomposition. □

Remark 2.5.3. Let $g \in G$ and let $t_g = \operatorname{Tr} g$. In Section 2.4 the difference $t_g^2 - 4$ gave us information about the conjugacy class of g; the sign of this

quantity is also relevant to the research of fixed points under the action of g on $\mathcal{H}$.
Indeed, if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the fixed-point condition $g(z) = z$ is equivalent to $az + b = z(cz + d)$, so z is a fixed point for g if and only if z is a zero of $q_g(x) = cx^2 + (d-a)x - b$.
Excluding the cases $g = \pm I_2$ (in which every $z \in \mathcal{H}$ is a fixed point) and the cases where $c = 0$, $g \neq \pm I_2$ (in which we would obtain $(d-a)x = b$, that has no non-real roots), the nature of the zeros of $q_g(x)$ are determined by its discriminant

$$(d-a)^2+4bc = d^2+a^2-2ad-2ad+2ad+4bc = (a+d)^2-4(ad-bc) = t_g^2-4.$$

If $t_g^2 - 4 > 0$ then q_g has two real distinct roots, so g has no fixed points on $\mathcal{H}$ (but it has two independent real eigenvectors when seen as an action on $\mathbb{R}^2$); if $t_g^2 - 4 = 0$ then q_g has one unique real root, i.e. g has no fixed point on $\mathcal{H}$ (and it is not diagonalizable); finally if $t_g^2 - 4 < 0$ then g has two complex conjugate roots, so g has one and only one fixed point on $\mathcal{H}$ (but it has no real eigenvalues when seen as an action on $\mathbb{R}^2$).

Remark 2.5.4. It is interesting to compare the role of the subgroup K in the action on $\mathbb{R}^2$ and on $\mathcal{H}$.
As a transformation of $\mathbb{R}^2$, an element $k \in K$ acts like a rotation around the origin $(0, 0)$ and the K-orbit of a nonzero vector v is the subset of $\mathbb{R}^2$ of the elements which have the same distance of v from $(0, 0)$; in other words the K-orbit of v is the circle of radius $\|v\|$ centered at the origin.
As a transformation of $\mathcal{H}$, an element of K fixes i and it can be shown that it acts like a rotation around i relative to the hyperbolic metric on $\mathcal{H}$, namely the K-orbit of an element z is the subset of the points $y \in \mathcal{H}$ such that $d_{\mathcal{H}}(y, i) = d_{\mathcal{H}}(z, i)$.

Chapter 3

$\mathbf{SL}_n(\mathbb{R})$ and $\mathbf{SL}_n(\mathbb{C})$ Iwasawa decompositions

In this chapter we will generalize the Iwasawa decomposition to the case of $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{C})$ for a generic n.

3.1 $\mathbf{SL}_n(\mathbb{R})$ Iwasawa decomposition

Let us define the following subgroups of $G := \mathrm{SL}_n(\mathbb{R})$:

$$K_n := \{k \in \mathrm{SL}_n(\mathbb{R}) \mid k^T = k^{-1}\},$$

the group of orthogonal matrices with determinant 1, usually denoted by $\mathrm{SO}_n(\mathbb{R})$;

$$A_n := \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in \mathrm{SL}_n(\mathbb{R}) \;\middle|\; a_i > 0 \;\forall i = 1, \dots, n \right\},$$

the group of diagonal matrices with positive entries;

$$U_n := \left\{ \begin{pmatrix} 1 & x_{12} & \cdots & x_{1n} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & x_{n-1\,n} \\ 0 & 0 & \cdots & 1 \end{pmatrix} \middle| x_{ij} \in \mathbb{R} \right\},$$

the group of upper triangular matrices with 1's on the diagonal.
The letters we use to denote these three subgroups are chosen on purpose, namely, K stands for *compact* (K_n is closed and bounded in $\mathbb{R}^{n\times n}$), A stands for *abelian* and U stands for *unipotent*, because every $u \in U_n$ can be written as $u = I_n + x$ with x nilpotent.

From now on, to lighten the notation, n will be fixed and we will denote U_n, A_n, K_n by U, A, K, respectively.
Recalling Remark 2.1.1, we will state the theorem in terms of the $G = UAK$ Iwasawa decomposition.

Theorem 3.1.1. *For every $g \in G = \mathrm{SL}_n(\mathbb{R})$ there exist unique $u \in U$, $a \in A$, $k \in K$ such that $g = uak$. In other words the map*

$$\begin{aligned} \Phi\colon U \times A \times K &\to G \\ (u, a, k) &\mapsto uak \end{aligned}$$

is a bijection.

Proof. Let us first prove the existence. Consider $(g_{ij})_{i,j=1,\ldots,n} \in G$ and let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^n$.

$$ge_j = \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1_j \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} g_{1j} \\ \vdots \\ g_{nj} \end{pmatrix} =: g_j.$$

We can orthonormalize $\{g_j\}_{j=1,\ldots,n}$ to $\{k_j\}_{j=1,\ldots,n}$ using the Gram-Schmidt process: it consists of subtracting a linear combination of $g_1, \ldots, g_{j-1}$ to g_j

and then normalizing the result in order to get mutually perpendicular unit vectors. We observe that this is a triangular process:

$$\begin{aligned}
&b_{11}g_1 && = k_1\\
&b_{12}g_1 + b_{22}g_2 && = k_2\\
&\vdots && \vdots\\
&b_{1j}g_1 + b_{2j}g_2 + \dots + b_{jj}g_j && = k_j\\
&\vdots && \vdots\\
&b_{1n}g_1 + b_{2n}g_2 + \quad \dots \quad + b_{nn}g_n && = k_n
\end{aligned}$$

Notice that $b_{jj} > 0$ for every $j = 1, \dots, n$ and that the matrix $k := ([k_1] \cdots [k_n])$ is orthogonal. Setting $b_{ij} = 0$ for $i > j$, let $B := (b_{ij})_{i,j=1,\dots,n}$. We have

$$\begin{aligned}
B^T g^T = &\begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ b_{12} & b_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ b_{1n} & \dots & \dots & b_{nn} \end{pmatrix} \begin{pmatrix} g_1^T \\ g_2^T \\ \vdots \\ g_n^T \end{pmatrix}\\
\implies (B^T g^T)_j &= b_{1j}g_1^T + b_{2j}g_2^T + \dots + b_{jj}g_j^T = k_j^T\\
&\implies B^T g^T = k^T \implies gB = k.
\end{aligned}$$

We observe that B is upper triangular with positive diagonal entries and, since $\pm 1 = \det k = \det g \cdot \det B$, B must have determinant 1.

Consider, for $i = 1, \dots, n$, $a_i := b_{ii}$ and $a := \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$, that is invertible, so we can multiply $a^{-1}B =: u$, which is strictly upper triangular and has 1's as diagonal elements.

In this way we have obtained $k = gau$, i.e. $g = u^{-1}a^{-1}k$, with $u^{-1} \in U$, $a^-1 \in A$, $k \in K$ (we have proven the surjectivity of Φ).

It remains to prove the uniqueness.

Let's suppose that $g = u_1 a_1 k_1 = u_2 a_2 k_2$, with $u_1, u_2 \in U$, $a_1, a_2 \in A$,

$k_1, k_2 \in K$. Consider gg^T. It results

$$\begin{aligned} u_1 a_1 k_1 k_1^T a_1^T u_1^T &= u_2 a_2 k_2 k_2^T a_2^T u_2^T \\ u_1 a_1^2 u_1^T &= u_2 a_2^2 u_2^T \\ u_2^{-1} u_1 a_1^T &= a_2^2 u_2^T u_1^{-T}. \end{aligned}$$

On the left we have an upper triangular matrix, whereas on the right we have a lower triangular matrix, so it must be diagonal. Since a is invertible and $u_2^{-1}u_1$ has 1's as diagonal entries, $u_2^{-1}u_1 = I_n$, i.e. $u_2 = u_1$. Thus $a_1^2 = a_2^2$; both a_1 and a_2 are diagonal matrices with positive entries, so $a_1 = a_2$. It follows that $k_1 = k_2$ too, proving the uniqueness (or the injectivity of Φ).

□

3.2 $\mathrm{SL}_n(\mathbb{C})$ Iwasawa decomposition

In order to generalize the Iwasawa decomposition to the complex case we have to introduce some notions which are parallel to the real case ones.

We define the standard hermitian scalar product on $\mathbb{C}^n$ as follows: if $z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}, w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{C}^n$, $\langle z, w \rangle := \sum_{i=1}^n z_i \overline{w_i}$.
Notice that $\langle\ ,\ \rangle$ induces a norm on $\mathbb{C}^n$:

$$\|z\| := \sqrt{\langle z, z \rangle}. \tag{3.1}$$

For $g \in \mathrm{M}_n(\mathbb{C})$, we set $g^* = \overline{g}^T$.
A basis $\{v_i\}_{i=1,\dots,n}$ of $\mathbb{C}^n$ (or in general a set of linear independent vectors $\{v_i\}_{i=1,\dots,k}$ over $\mathbb{C}$) is called *orthonormal* (with respect to the hermitian product) if $\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

Consider the following three subgroups of $\mathrm{SL}_n(\mathbb{C})$:

$$K := \{X \in \mathrm{SL}_n(\mathbb{C}) \mid X^* X = I_n\};$$

$$A := \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \middle| \, a_i \in \mathbb{R}, a_i > 0 \ \forall i = 1, \dots, n, \ \prod_{i=1}^{n} a_i = 1 \right\};$$

$$U := \left\{ \begin{pmatrix} 1 & x_{12} & \cdots & x_{1n} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & x_{n-1\,n} \\ 0 & 0 & \cdots & 1 \end{pmatrix} \middle| x_{ij} \in \mathbb{C} \right\}.$$

The group K is called the *complex unitary subgroup* and its elements are called *unitary* matrices; notice that a matrix is unitary if and only if its columns form an orthonormal basis of $\mathbb{C}^n$. The elements of A are diagonal matrices with positive diagonal elements. The elements of U are called *(complex) unipotent upper triangular* matrices.
We can now state the Iwasawa decomposition for $\mathrm{SL}_n(\mathbb{C})$.

Theorem 3.2.1. *For every $g \in \mathrm{SL}_n(\mathbb{C})$ there exist unique $u \in U$, $a \in A$, $k \in K$ such that $g = uak$.*

Proof. The proof is the same as in Theorem 3.1.1, using the Gram-Schmidt process with respect to the standard hermitian scalar product, orthonormalizing with respect to the norm (3.1). □

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Ringraziamenti

Vorrei dedicare questo spazio a coloro che, da sempre o in quest'ultimo periodo, con il loro supporto mi hanno permesso di credere in me stessa e arrivare dove sono oggi.

Voglio ringraziare prima di chiunque altro i miei genitori, senza i quali nulla di tutto questo sarebbe stato possibile.

Ringrazio la mia relatrice Nicoletta Cantarini, senza la quale non sarebbe stata possibile la stesura di questa tesi, per avermi seguito e introdotto al magnifico mondo dell'algebra.

Ringrazio i miei fratelli Lorenzo, Riccardo e Tommaso, che sanno sempre come farmi sorridere.

Ringrazio le mie amiche Arianna, Elena, Valerie e Megan per essere da sempre un punto fermo della mia vita, per essermi sempre state vicine, anche da lontano.

Ringrazio le mie coinquiline Anna, Agata ed Eleonora, perché so che in ogni momento posso contare su di loro.

Ringrazio immensamente tutti i compagni di questa fantastica avventura, che ogni giorno spero non finisca mai.
In particolare voglio ringraziare i miei capi Davide e Federico, per avermi insegnato cosa significhi essere una squadra.
Ringrazio Elena, che in questi tre anni è stata la mia partner in ogni esaurimento, e Chiara, con cui ho imparato che le discussioni non incrinano i rapporti, bensì li fortificano.

Ringrazio Leonardo, per essere fin dal principio la persona su cui posso contare, e Filippo, che mi ha ricordato quanto leggero possa essere parlare di cose profonde.
Infine ringrazio Matteo, per essere la persona che da sempre condivide tutti i miei dubbi e crede nelle mie capacità.