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Rational algorithms for the decomposition of Feynman Integrals via Intersection Theory

Supervisor:
Dr. Tiziano Peraro

Submitted by:
Gaia Fontana

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Abstract

The decomposition of Feynman integrals into a basis of independent master integrals is an essential ingredient of high-precision theoretical predictions, that often represents a major bottleneck when processes with a high number of loops and legs are involved. In this thesis we present a new algorithm for the decomposition of Feynman integrals into master integrals with the formalism of intersection theory. Intersection theory is a novel approach that allows to decompose Feynman integrals into master integrals via projections, based on a scalar product between Feynman integrals called intersection number. We propose a new purely rational algorithm for the calculation of intersection numbers of differential $n-$forms that avoids the presence of algebraic extensions. We show how expansions around non-rational poles, which are a bottleneck of existing algorithms for intersection numbers, can be avoided by performing an expansion in series around a rational polynomial irreducible over $\mathbb{Q}$, that we refer to as $p(z)-$adic expansion. The algorithm we developed has been implemented and tested on several diagrams, both at one and two loops.

Contents

1 | Introduction

Particle physics studies the tiniest building blocks of the universe: particles and their interactions. From a theoretical point of view, particles are described as excited quantum states of physical fields. By now, their interacting nature can be described only perturbatively, within the framework of Quantum Field Theory (QFT). Within QFT, a special role is played by gauge theories, that describe fields and their interactions following a symmetry group. So far, the most accurate description of particles and their interactions is the Standard Model (SM) of particle physics. In a nutshell, it is a gauge theory based on the symmetry group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$, where $SU(3)_C$ contains the description of strong interactions with the theory of *Quantum Chromodynamics (QCD)* and $SU(2)_L \otimes U(1)_Y$ describes *electroweak interactions*, that combine *Quantum Electro-Dynamics (QED)* and weak interactions. While $SU(3)_C$ is an exact symmetry of nature, $SU(2)_L \otimes U(1)_Y$ is broken into $U(1)_{EM}$ via the *Higgs mechanism*, that is responsible for giving mass to all the particles in nature.

The discovery of the Higgs boson at the Large Hadron Collider (LHC) [1, 9] is only the last of the innumerable verifications of the Standard Model. Behind this enormous success, however, there are also many aspects that are not accounted for and therefore we cannot regard at it as our final answer to the mysteries of the subatomic world. The most obvious example is that it fails in incorporating a description of gravity into the framework of QFT, but it also does not address the problem of the observed neutrino oscillations, the asymmetry between matter and antimatter and the existence of dark matter and dark energy. It is therefore of extreme importance to obtain precise theoretical predictions to be compared with experimental data. This necessity is twofold, in order to both investigate with high accuracy the properties of known particles, such as the *top quark* and the Higgs boson, and to look for deviations from experimental data, in order to see clearly where the theory deviates from reality and seek signals of new physics.

The main tool to investigate fundamental interactions, both at the theoretical and at the experimental level, are scattering processes. The proba-

bility for a certain process to happen is encoded in the so-called *S-matrix*, or scattering matrix, that contains as its main ingredient the scattering amplitude. Therefore, scattering amplitudes are a topic of central importance. From the theoretical point of view, they help in enclosing hidden properties of the theory that are not manifest in the Lagrangian formulation. From a phenomenological point of view, they constitute the bridge between theory and experiment, allowing to make theoretical predictions that can be falsified against the experimental data. For a given order in perturbation theory, scattering amplitudes are calculated as the sum of the Feynman diagrams contributing to the process at study, with each diagram associated to a corresponding Feynman integral. Depending on the process, one can count hundreds or thousands of Feynman integrals that need to be calculated. This seemingly insurmountable task is actually simplified by noticing that Feynman integrals obey linear relations, and therefore not all of them are independent. The task, in the end, reduces to the calculation of the minimal set of independent integrals, that are known as *master integrals* (MI). It is therefore of extreme importance to have efficient methods that perform the decomposition of Feynman integrals into master integrals and for the calculation of the latter. The current state of the art for Feynman integrals decomposition is given by the Laporta algorithm [25], that performs the reduction of a set of Feynman integrals via the solution of a large system of identities, known as *integration-by-parts (IBP) system*. This method is computationally challenging when applied to integrals corresponding to diagrams with a large number of loops and legs, that is precisely what happens when one considers high-precision predictions. Because of the upcoming High Luminosity phase at LHC, the accuracy that needs to be reached in order to be comparable with the experiments corresponds to percent level [7, 50, 15] which usually corresponds to at least Next-to-Next-to-Leading-Order (NNLO).

It is extremely interesting to look for alternative methods relying on direct decompositions and bypassing the resolution of large linear systems. The use of intersection theory is a recent proposal in this direction. Its approach is purely algebraic and stems from a recent branch of algebraic geometry and topology, namely intersection theory of differential forms applied to the study of Aomoto-Gel'fand hypergeometric functions [12, 34, 35, 2]. This class of multivalued functions has the property that their integrand vanishes on the boundary of the integration contour, exactly as Feynman integrals in dimensional regularization. Via a change of variables (in this thesis we will use the so-called *Baikov representation*) it is possible to cast the Feynman integral in consideration in a form that reflects the one of a generic hypergeometric integral, allowing us to exploit the tools of intersection theory for its study.

The most important property of hypergeometric integrals that can be used to obtain a new strategy of decomposition of Feynman integrals is the fact that they obey a vector space structure. Identifying the basis vectors as the master integrals, we can reduce the problem of decomposition into master integrals into the problem of projecting an element belonging to a vector space into the space basis. To do this, we introduce a scalar product between Feynman integrals called *intersection number*. In this way we find a strategy that can be used on a single Feynman integral at time and that bypasses completely the creation and the solution of a large system of identities. In this approach it suffices to build the metric matrix of scalar products between the basis vectors, that is, between the master integrals, and decompose the Feynman integral in consideration by finding the coefficients of the projection. Our problem therefore reduces to the computation of the intersection numbers between the Feynman integral and each one of the master integrals.

The standard procedure to carry out intersection numbers is based on a recursive algorithm [17, 16] that considers at each step univariate integrations, even if recent progresses are paving a way for new strategies that directly carry out a multivariate integration [10]. At the time of the writing, these algorithms present as a downside the appearance of non-rational poles in intermediate steps. There have been attempts in proposing algorithms that avoid non-rational poles [56] but at the price of introducing complicated integral transformations in order to satisfy certain conditions on the nature of poles of the functions appearing in the algorithm.

In this thesis we propose a new approach that solves the problem of finding a purely rational algorithm without posing conditions on the order of the poles of the function and without the need of changes of bases or integral transformations. The main novelty in our proposal is constituted by the concept of series expansion of a rational function around a polynomial irreducible over $\mathbb{Q}$, that we refer to as $p(z)-$*adic expansion*. It allows us to consider the contribution to the intersection number of all the non-rational poles satisfying a certain irreducible rational polynomial equation at once, completely avoiding the explicit appearance of non-rational poles which are the roots of such polynomial. The results presented here show that the algorithm is in agreement with the results obtained with the traditional Laporta method and is a promising candidate to be implemented on frameworks based on finite fields. The latter is a technology that can be used to efficiently implement rational algorithms and that in the last years has seen many applications in the field of high-energy physics (see [42] and reference therein).

The algorithm that we are going to present is the first example of a purely rational algorithm for the computation of intersection numbers for Feynman

integrals decomposition that does not rely on complex integral transformations or changes of bases. Its full-automation can open the possibility of using these novel techniques for integrals decomposition for future precision calculations.

The thesis is organized as follows. In Ch. 2 we give a brief overview of Feynman integrals and their properties, introducing some terminology that will be used throughout this work. We also summarize the main steps of the Laporta algorithm. In Ch. 3 we introduce the formalism of intersection theory for Feynman integrals. We define a scalar product between Feynman integrals and the necessary tools to obtain a decomposition into master integrals. We describe how to obtain a representation of Feynman integrals that mirrors the one of hypergeometric integrals. In particular, we focus on the strategies used to find the number of elements in the vector space and we propose a new procedure that we refer to as *master monomial analysis*. In Ch. 4 we present the main novelty of this thesis. We describe our rational algorithm, the procedure to obtain the $p(z)-$adic expansion of a rational function and its usage in computing intersection numbers. In Ch. 5 we apply our new algorithm to several examples at one and two loops and we check them against the decomposition obtained via the Laporta algorithm. The appendices complement the work by giving insights on various aspects of algebraic geometry (App. A), on Baikov representation (App. B), on the decomposition into basis vectors via intersection theory (App. C), on the main formula used to compute intersection numbers (App. E) and on finite fields technology (App. F).

2 | Feynman integrals

> The very advanced counting system used by elementary particle theorists for counting the loops is: 'One, two, many,'
>
> E. Remiddi [57]

We introduce the main object of study of this thesis, namely scalar Feynman integrals, starting from their definition and discussing their properties. We expand on the procedure of decomposing them into a linearly independent set of master integrals and on the state of the art to conduct this procedure, the Laporta algorithm. We set the main notation used throughout the work. In the following, we assume the reader to be familiar with Quantum Field Theory (QFT) concepts and we do not attempt in making an exhaustive review of the subject, which can be found in many textbooks [46, 44].

2.1 | Feynman integrals

Multiloop Feynman integrals are an essential ingredient to make theoretical predictions about scattering processes via perturbative QFT. Every scattering process is in turn characterized by a cross section, an important constituent of which is the scattering amplitude, that encodes the probability for a certain asymptotic initial state to evolve into another in the distant future. Scattering amplitudes are computed at a given order in perturbation theory as the sum of all the Feynman diagrams contributing to the process at study. The number of diagrams and their complexity increases with the order in perturbation theory and the number of external particles involved in the process. Therefore Feynman diagrams, and the associated Feynman integrals, are the building blocks of scattering amplitudes. In order to reach a certain level of precision in theoretical calculations, that can be compared to

the experimental one, we need to consider diagrams presenting an increasing number of loops and legs.

As opposed to *tree level* diagrams, when dealing with diagrams presenting *loops*, we are faced with the problem of them being divergent. Such divergencies need to be accounted for by introducing a regularization prescription. For the purposes of our work we will consider Feynman integrals in dimensional regularization, analytically continuing the number of spacetime dimensions from 4 to d. This allows us to avoid the divergencies that manifest when integrating over loop momenta in 4 spacetime dimensions and to exploit the properties obeyed by Feynman integrals in dimensional regularization.

As we have seen, the calculation of scattering amplitudes leads to multiloop Feynman integrals. As an example, consider an L loop Feynman diagram with $E+1$ external legs. We indicate with $p_1, \dots, p_E$ the independent external momenta, their number being E as one is fixed by momentum conservation. The associated Feynman integral can be written as

$$\int \prod_{i=1}^{L} \mathrm{d}^d k_i \frac{N(k_i^\mu)}{z_1^{\alpha_1} \dots z_{t'}^{\alpha_{t'}}}, \tag{2.1}$$

where the numerator $N(k_i^\mu)$ presents an arbitrary complicated tensor structure that we can write as

$$N(k_i^\mu) = k_1^{\mu_1} \dots k_1^{\mu_n} \dots k_L^{\nu_1} \dots k_L^{\nu_m}. \tag{2.2}$$

The denominators z_i, or *inverse propagators*, have a quadratic structure

$$z_i = P_i^2 - M_i^2, \tag{2.3}$$

where P_i and M_i are respectively linear combinations of momenta $k_1, \dots, k_L$, $p_1, \dots, p_E$ and masses imposed by the structure of the diagram in consideration. We indicate with t' the number of propagators appearing the integral, out of which t are independent.

It is desirable to write such a complicated integral in a simpler form in order to manipulate it and ultimately solving it. The standard procedure for dealing with the tensorial structure of multiloop Feynman integrals goes under the name of *tensor reduction* and consists in factoring out the tensorial part, leaving the full dependence on loop momenta in scalar factors, called scalar Feynman integrals. The amplitude therefore takes the form

$$A = \sum_i a_i I_i, \tag{2.4}$$

where the coefficients a_i are rational functions of the kinematic invariants and of the dimensional regulator. This can be done in several ways: with

Passarino-Veltman [40] reduction, decomposition into tensors and form factors or integrand reduction [37, 19, 32, 4, 58, 33]. After this procedure, we obtain the factorization of the tensorial multiloop integral in the product of two terms, one containing the whole tensorial structure, whose form is only determined by Lorentz invariance and symmetry considerations, and the other consisting in scalar integrals with the entire dependence on loop variables. The scalar Feynman integral takes the form

$$\int \prod_{i=1}^{L} \mathrm{d}^d k_i \frac{1}{z_1^{\alpha_1} \dots z_n^{\alpha_n}}, \tag{2.5}$$

where n is the number of independent scalar products between loop momenta and external momenta and between the loop momenta themselves. Scalar Feynman integrals presents also $n - t$ auxiliary propagators, or irreducible scalar products, that we will define in Sect. 2.1.1. Scalar Feynman integrals in dimensional regularization obey a set of linear relations classified in integration-by-parts identities (IBP), Lorentz invariance identities (LI) and symmetry relations (SR) that allow to write a generic scalar Feynman integral in terms of a smaller set of integrals, called *master integrals*. It is of great importance to solve these relations in order to perform the decomposition of a generic Feynman integral into master integrals for a variety of reasons, most notably:

- only after the reduction to master integrals certain properties of the theory, such as gauge invariance, are manifest and can be effectively checked;
- typically, when calculating an amplitude, one is faced with the computation of a large number of scalar Feynman integrals. It is desirable to reduce the ones to be effectively calculated to a smaller set of independent integrals that usually present a less complicated and more manageable structure;
- these relations allow to write linear first-order differential equations for the master integrals with respect to the kinematic invariants. Solving the differential equations is the most effective strategy to calculate the master integrals as opposed to a direct integration.

We therefore concentrate on how to obtain these relations and how to apply them in a suitable way in order to decompose a generic Feynman integral into master integrals. The endpoint of this Chapter is the description of the Laporta algorithm, the by-now standard procedure used for the decomposition into master integrals. Before starting, we introduce a bit of terminology, to set a common ground for the rest of the work.

2.1.1 | Families and sectors

We begin by introducing a classification of integrals that is useful to organize a calculation. For clarity, we rewrite our generic scalar Feynman integral with L loops, $E+1$ external legs in d dimensions [3]

$$I[\alpha_1,\dots,\alpha_n] = \int \prod_{i=1}^{L} \left(\frac{\mathrm{d}^d k_i}{i\pi^{d/2}}\right) \frac{1}{z_1^{\alpha_1}\dots z_n^{\alpha_n}}. \tag{2.6}$$

We recall that only E external momenta are independent because of momentum conservation and we indicate them with $p_1,\dots,p_E$. We suppose that the integral presents t independent propagators, their number determined by the structure of the diagram under study. We recall that propagators are objects defined as

$$\frac{1}{z_i}, \tag{2.7}$$

the z_i are therefore inverse propagators, with the quadratic structure

$$z_i = P_i^2 - M_i^2, \tag{2.8}$$

with P_i a combination of loop and external momenta and M_i internal masses. Within scalar Feynman integrals, the propagators alone are often not enough to express every scalar product between loop momenta or loop momenta and external momenta. In fact, we can have n independent scalar products between any independent external momenta and a loop momenta, or between two internal loop momenta, obtained as:

$$\begin{aligned} &k_i \cdot p_j \qquad 1 \le i \le L, \quad 1 \le j \le E, \\ &k_i \cdot k_j \qquad 1 \le i \le j \le L. \end{aligned} \tag{2.9}$$

n can be therefore calculated as

$$n = L\,E + \frac{L(L+1)}{2}, \tag{2.10}$$

where the first term takes into account the total number of possible scalar product between independent external momenta and a loop momenta, whereas the second term calculates the combinatorics for the scalar products between two loop momenta.

Therefore, t of the n the scalar products can be written in terms of the inverse propagators, the remaining $q = n - t$ are called *irreducible scalar products* or *ISPs* and will be used as auxiliary variables. We can divide the denominators appearing in eq. (2.6) as

$$I[\alpha_1,\dots,\alpha_n] = \int \prod_{i=1}^{L} \left(\frac{\mathrm{d}^d k_i}{i\pi^{d/2}}\right) \frac{1}{z_1^{\alpha_1}\dots z_t^{\alpha_t} z_{t+1}^{\alpha_{t+1}}\dots z_n^{\alpha_n}}, \tag{2.11}$$

with

$$\alpha_i \begin{cases} \geq 0 & i = 1, \ldots, t \\ \leq 0 & i = t+1, \ldots, n \end{cases} . \tag{2.12}$$

The only difference between irreducible scalar products and denominators ("true" propagators) is therefore the sign of the exponent. Here and in the following we will refer to both ISPs and denominators as propagators, without distinction.

Example 2.1.1. For example, considering the massless double box integral (fig. 2.1, all figures in this thesis were drawn with Jaxodraw [5] based on Axodraw [51]) we have

- $L = 2$ loops, $E + 1 = 4$ external momenta and $N = 2 \cdot 3 + (2 \cdot 3)/2 = 9$ independent scalar products;
- it counts 7 propagators, therefore we need to add 2 irreducible scalar products.

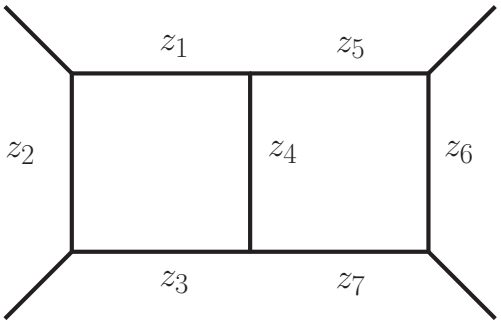

FIGURE 2.1: Massless double box diagram with indicated the seven propagators $z_1, \ldots, z_7$.

The scalar integral in consideration is identified by a vector of integer indices $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)$.

We introduce the concept of family of Feynman integrals:

Family [29]: A *family* of Feynman integrals is identified with integrals having the same complete set of propagators $\{z_1, \ldots, z_n\}$.

As an example, the box integral lies in the same family of the four-points bubble integral, that is in fact obtained from the box by "pinching" (removing) two propagators (fig. 2.2). We will make use of the following:

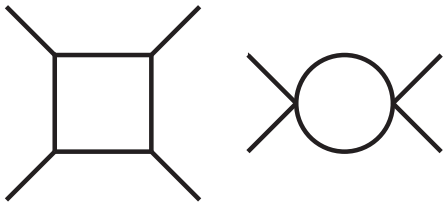

FIGURE 2.2: Box diagram and four-points bubble.

Dot notation [3]: the presence of n *dots* on a certain propagator indicates that it is elevated to the power $n + 1$.

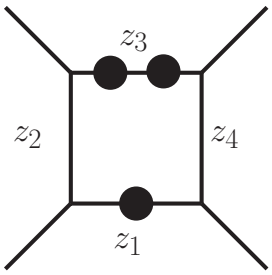

FIGURE 2.3: Box diagram where the propagator z_1 is elevated to the power 2 and the propagator z_3 to the power 3.

Finally, we define what we mean by "cutting" a propagator:

Cut of a propagator: To *cut* a propagator z_i means imposing on it the on-shell condition, that corresponds to the substitution:

$$\frac{1}{z_i} \to \delta(z_i) \tag{2.13}$$

The configuration where all the denominators are on the cut corresponds to the so-called *maximal cut*.

We now define the concept of sectors:

Sector [29]: Two integrals belong to the same *sector* if and only if their subset of propagators with positive powers is the same, i.e. if they have the same denominators, possibly raised to different powers and/or with different irreducible scalar products.

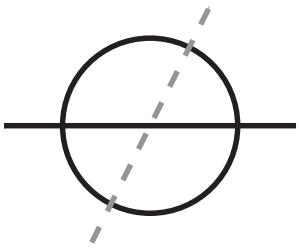

FIGURE 2.4: Sunrise diagram on the max cut.

For each $\boldsymbol{\alpha}$ we define a vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ where the θ_i are set to one if and only if $\alpha_i > 0$ and zero otherwise [29]:

$$\theta_i = \boldsymbol{\Theta}\left(\alpha_i - \frac{1}{2}\right), \tag{2.14}$$

where $\boldsymbol{\Theta}$ is Heaviside step function. All scalar integrals within one sector lead to the same $\boldsymbol{\theta}$.

Corner integral [29] The scalar integral for which $\boldsymbol{\alpha}_C = \boldsymbol{\theta}(\boldsymbol{\alpha}_C)$ is called the *corner integral* of the sector.

The corner integral of a sector is then the integral presenting no numerators and all denominators elevated to 1. If n is the total number of generalized propagators, the total number of sectors is 2^n.

We are now equipped with a "vocabulary" that is useful in the organization of calculations and algorithms involving many Feynman integrals. The next step is the introduction of some numbers giving us an idea of the complexity of each Feynman integral, in order to identify the more complex ones and express them as a function of the simpler ones using the linear relations. To this end, we define the vector of positive propagators powers $(r_1, \ldots, r_t)$ which is obtained from $\boldsymbol{\alpha}$ by removing all non-positive indices while preserving the order and the vector of negative propagators powers, $(s_1, \ldots, s_{n-t})$, again taken from $\boldsymbol{\alpha}$ while preserving the order. We formally define t, the number of independent propagators, as

$$t = \sum_{i=1}^{n} \theta_i. \tag{2.15}$$

Within a given sector, we define the sum of all positive powers and the negative sum of all non-positive powers to be a measure of the complexity of an

integral [29]. To this end, we define:

$$r = \sum_{i=1}^{t} r_i, \qquad s = -\sum_{i=1}^{n-t} s_i. \tag{2.16}$$

We further notice that the number of dots of an integral is given by

$$n_{\text{dots}} = r - t. \tag{2.17}$$

It is generally understood that the higher r, s and t are, the more complicated is the integral. Before reducing to master integrals, we must introduce a weight associated to each integral, which indicates how "complicated" we perceive that integral. The basis integrals present the lowest weight, as we prefer to express every other integral as a function of them. Whenever we solve any linear system satisfied by Feynman integrals we therefore express integrals with higher weight in terms of those with lower weight. Associating a weight to an integral means creating an *ordering* between them. The ordering is arbitrary, and therefore the list of master integrals is not unique, but every ordering yields the same number of master integrals. It is equivalent to the choice of the basis of a vector space. Some bases may be more effective than others, depending on the particular problem we are considering, but the dimension of the vector space, therefore the number of elements in the basis, stays the same. Examples of orderings can be found in [42, 29, 30].

We finally highlight that there exists also other criteria of ordering, such as finiteness [38] or analytic properties which make integration easier [22].

2.2 | Reduction to master integrals

After introducing the necessary terminology to organize Feynman integrals and some numbers that gauge their complexity, we turn to the description of the linear relations obeyed by Feynman integrals in dimensional regularization and to the algorithm that exploits them to obtain their decomposition into master integrals.

Each integral in the set constituting our amplitude obeys a set of linear relations, making the integrals in the set not all linearly independent. We want to write all of them as linear combinations of a smaller set of independent integrals in order to reduce the number of them to be computed. This procedure is called *reduction to master integrals*, which is the name we give to the integrals in the minimal, linearly independent set. Linear relations between Feynman integrals are classified in

- integration-by-parts identities (IBP)
- Lorentz invariance identities (LI)
- symmetry relations (SR)

We will describe how to obtain each of them in the following sections.

2.2.1 | Integration-by-parts identities

Integration-by-parts identities were first proposed around 1980 by Chetyrkin and Tkachov [11, 49] and are a remarkable property of dimensionally regulated Feynman integrals. The core idea is the extension to arbitrary d dimensions of Gauss' theorem for the vanishing of the integral of a divergence. Integration-by-parts identities derive from the vanishing of total derivatives in dimensional regularization, which can be derived following [59, 55].

Exploiting the Poincarè invariance of a generic loop integral $\int \mathrm{d}^d k f(k)$ in dimensional regularization under loop momentum shift by an arbitrary momentum $k \to k + p$

$$\int \mathrm{d}^d k\, f(k+p) = \int \mathrm{d}^d k\, f(k), \tag{2.18}$$

and expanding around small values of the shift p,

$$\int \mathrm{d}^d k\, f(k+p) = \int \mathrm{d}^d k\, f(k) + p^\mu \int \mathrm{d}^d k \frac{\partial f(k)}{\partial k^\mu} + \mathcal{O}\big(p^2\big). \tag{2.19}$$

That implies the vanishing of total derivatives

$$\int \mathrm{d}^d k \frac{\partial}{\partial k^\mu} f(k) = 0. \tag{2.20}$$

Therefore we can find linear relations between Feynman integrals by applying integration-by-parts identities. We consider a complete set of identities which take the following form

$$\begin{aligned} &\int \mathrm{d}^d k_1 \dots \mathrm{d}^d k_L \, \frac{\partial}{\partial k_i^\mu} \cdot \left(\frac{v_j^\mu}{z_1^{a_1} \dots z_N^{a_N}} \right) = 0, \\ &v_j^\mu = \begin{cases} k_1^\mu \dots k_L^\mu \\ p_1^\mu \dots p_E^\mu \end{cases}, \end{aligned} \tag{2.21}$$

where we omitted the normalization of the integrals as it is not relevant to our discussion.

After the differentiation, one has to express the scalar products in terms of the denominators and of the chosen irreducible scalar products to obtain a set of relations of the form [47]

$$\sum_i c_i I[\alpha_1 + \beta_{i,1}, \ldots, \alpha_n + \beta_{i,n}] = 0, \tag{2.22}$$

with c_i being polynomials depending on the dimensional regulator d, kinematic invariants, masses and indices, while $I[\alpha_1, \ldots, \alpha_n]$ are Feynman integrals of the same sector or of *subsectors*, if some of the indices are zero or negative. By solving these relations, one can express a generic Feynman integral in terms of a linear combination of *linearly independent* master integrals. It has been proved [47] that the number of master integrals is finite. In general, the reduction to master integrals is not a unique process, since, as we said, there is arbitrariness in choosing an ordering and one usually tries to reduce the integral and calls master integrals the ones that cannot be reduced further.

2.2.2 | Lorentz invariance identities

We review Lorentz invariance identities following [3, 48]. Lorentz invariance identities arise by exploiting the nature of Lorentz scalars of the integrals in consideration. Considering an infinitesimal Lorentz transformation on the external momenta $p_i^\mu \to p_i^\mu + \delta p_i^\mu$ with $\delta p_i^\mu = \omega^{\mu\nu} p_{i\nu}$ and $\omega^{\mu\nu}$ a totally antisymmetric tensor, we have

$$\begin{aligned} I[\alpha_1, \ldots, \alpha_n] &= I[\alpha_1, \ldots, \alpha_n] + \sum_n \frac{\partial I[\alpha_1, \ldots, \alpha_n]}{\partial p_n^\mu} \\ &= I[\alpha_1, \ldots, \alpha_n] + \omega^{\mu\nu} \sum_n p_{n\nu} \frac{\partial I[\alpha_1, \ldots, \alpha_n]}{\partial p_n^\mu}, \end{aligned} \tag{2.23}$$

therefore

$$\sum_n \left(p_{n\mu} \frac{\partial}{\partial p_n^\nu} - p_{n\nu} \frac{\partial}{\partial p_n^\mu} \right) I[\alpha_1, \ldots, \alpha_n] = 0, \tag{2.24}$$

to be contracted with all possible antisymmetric combinations of the form

$$p_i^\mu \, p_j^\nu - p_i^\nu \, p_j^\mu, \tag{2.25}$$

that leads to $E(E-1)/2$ equations between integrals with shifted indices. For example, following [3, 48], consider the three-points and four-points functions (respectively known as "vertex" and "box" topologies, fig. 2.5). In the former

case, we have two independent momenta, p_1 and p_2. We can build only one LI identity for I_{vertex},

$$(p_1^\mu p_2^\nu - p_1^\nu p_2^\mu) \sum_{i=1}^{2} \left(p_{i\mu} \frac{\partial}{\partial p_i^\nu} - p_{i\nu} \frac{\partial}{\partial p_i^\mu} \right) I_{\text{vertex}} = 0. \tag{2.26}$$

While for the four-points function I_{box} we have three independent momenta, p_1, p_2 and p_3. We can therefore build 3 LIs,

$$\begin{aligned}
(p_1^\mu p_2^\nu - p_1^\nu p_2^\mu) \sum_{i=1}^{3} \left(p_{i\mu} \frac{\partial}{\partial p_i^\nu} - p_{i\nu} \frac{\partial}{\partial p_i^\mu} \right) I_{\text{box}} = 0, \\
(p_1^\mu p_3^\nu - p_1^\nu p_3^\mu) \sum_{i=1}^{3} \left(p_{i\mu} \frac{\partial}{\partial p_i^\nu} - p_{i\nu} \frac{\partial}{\partial p_i^\mu} \right) I_{\text{box}} = 0, \\
(p_2^\mu p_3^\nu - p_2^\nu p_3^\mu) \sum_{i=1}^{3} \left(p_{i\mu} \frac{\partial}{\partial p_i^\nu} - p_{i\nu} \frac{\partial}{\partial p_i^\mu} \right) I_{\text{box}} = 0.
\end{aligned} \tag{2.27}$$

Since it has been proved that LIs [26] are not linearly independent from IBPs, in principle we could avoid using Lorentz invariance identities in the Laporta algorithm. LIs can be reproduced with only IBPs by generating and solving a larger system of IBPs identities. This, however, would make the procedure computationally much more expensive, therefore it is not reasonable to avoid using LIs.

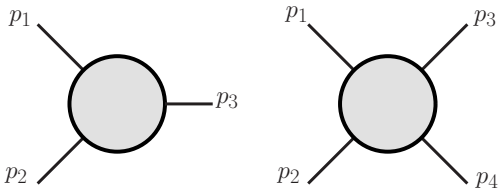

FIGURE 2.5: Vertex and box topologies.

2.2.3 | Symmetry relations

In general, other identities between Feynman integrals emerge when it is possible to shift the loop momenta such that the integral value does not change, but its integrand is expressed as a combination of different integrands. Then, considering the original integral and imposing its equivalence with the combination of integrals obtained via the shift, it is possible to write identities between integrals belonging to the same sector or even to different ones.

As an example, considering the one loop bubble

$$I[\alpha_1, \alpha_2] = \int \mathrm{d}^d k \frac{1}{(k^2 - m^2)^{\alpha_1} ((k+p)^2 - m^2)^{\alpha_2}}, \tag{2.28}$$

we have the symmetry relation

$$I[\alpha_1, \alpha_2] = I[\alpha_2, \alpha_1], \tag{2.29}$$

corresponding to the momentum shift

$$k \to k^\star = -k - p. \tag{2.30}$$

In general, symmetry relations can be derived from loop momentum shifts

$$k_i^\star = \sum_{j=1}^{L} M_{ij} k_j + \sum_{j=1}^{E} c_j^{(i)} p_j, \quad i = (1, \ldots, L),\ M_{ij}, c_j^{(i)} \in \{0, \pm 1\}. \tag{2.31}$$

Applying it to an integral containing also negative powers (ISPs elevated to nonzero exponents) results in a linear combination of integrals. Symmetries that relate integrals within the same sector are referred to as *sector symmetries*. Those relating different sectors of the same or of different families are called *sector mappings*. We also highlight an observation about symmetry relations obtained for diagrams presenting symmetries with respect to permutations of external legs, i.e. symmetries involving permutations of external momenta preserving the kinematic invariants [42]. For these diagrams we call *normal mappings* the shift of loop momenta mapping a sector into another one. We call *generalized mappings* a permutation of external momenta that preserves the kinematic invariants that can be or cannot be followed by a shift of loop momenta. It has been verified [42] that for a given set of seed integrals (see the definition of seed integrals in Sect. 2.3) the identities generated for a generalized mapped sector are not all covered by the identities generated, for the same set of seed integrals, by combining the sector mappings with identities generated for the unmapped sectors. The missing identities can be recovered by adding more seed integrals in the unique and mapped sectors and therefore complicating the system of equations. It is therefore reasonable to include IBPs,LIs and SRs obtained for sectors that are "generalized" mapped in our system of identities.

2.2.4 | Identification of trivial sectors

Another important step when dealing with a huge system of identities is the possibility of a-priori identification of integrals yielding zero. In fact,

within dimensional regularization, scaleless integrals are set to zero. We can prove it by exploiting the property of scaling of the measure under a transformation of the type $k \to sk$

$$\int \mathrm{d}^d\, kk^{2a} = \int \mathrm{d}^d\, (sk)(sk)^{2a} = s^{d+2a} \int \mathrm{d}^d\, kk^{2a}, \tag{2.32}$$

therefore

$$\int \mathrm{d}^d\, kk^{2a} = 0. \tag{2.33}$$

Which implies that for generic a and s, $\int k^{2a} = 0$. This implies that whenever we find a scaleless integral, we can safely ignore it in all our identities.

An important clue to look for is the vanishing of the corner integral of a given sector. To test it, a possible strategy consists in performing a reduction for it and see whether it is zero. If it is, then all other integrals belonging to the sector are zero as well [27]. A sector whose corner integral is zero is called *trivial sector* or *zero sector*.

2.3 | Laporta algorithm

By exploiting the properties of dimensionally regulated Feynman integrals, we end up with large sets of linear identities among integrals, with rational coefficients depending on the dimension d and on the kinematic invariants. Not all integrals are independent in the sense of these identities and therefore can be expressed as linear combinations of a minimal set of integrals, called master integrals.

At first glance, the solution of such a system of identities may look easy, but in reality the number of equations can grow up to hundreds, or thousands, or even millions, depending on the process under study. High-performance computer algebra is needed to handle the complexity of the expressions obtained and, even with that, the size of the intermediate expressions makes difficult the inversion of such enormous system.

Recent progresses in this direction include approaches based on finite fields to sidestep the appearance of large intermediate expressions [43, 31].

The Laporta algorithm constitutes the standard procedure used for performing Feynman integrals decomposition. It exploits a huge system of identities, called *IBP system*, in order to express complex Feynman integrals to master integrals. The procedure works as follows. It first builds the *IBP system* obeyed by the set of integrals we want to decompose. To create these identities, it exploits integration-by-parts identities, Lorentz invariance identities and symmetry relations. The identities are written for integrals of the

form (2.6) with *symbolic* exponents α_i and the equations obtained are called *template equations*. Then, it evaluates the α_is on explicit numerical values, called *seeds*. The key point is that the system eventually "closes", presenting more equations than unknowns and can therefore be solved for the set of integrals we want to reduce. It is also interesting to point out that, when writing down the equations for the seed integrals, other integrals not in the set to be reduced may appear. They are known as auxiliary integrals. We can summarize the entire procedure in three points:

- creation of the IBP identities made of template equations;
- for a set of seeds, evaluation of the template equations for explicit numerical values for the α_is and generation of the system of seed equations;
- solution of the system of seed equations with the chosen ordering.

At the end of the procedure the integrals $\{I_i\}$ in the set to be decomposed are expressed as linear combinations of a few master integrals $\{G_i\}$

$$I_i = \sum_j c_{ij} G_j. \tag{2.34}$$

2.3.1 | Example: one loop Bubble I_{α_1,α_2}

We propose a simple example of reduction to master integrals. We consider the so-called "bubble" diagram with a massive loop in d-dimensions, also known as the *two-points one loop topology* [3]:

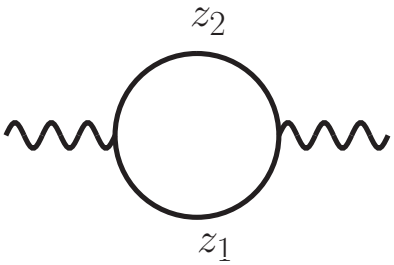

FIGURE 2.6: "Bubble" diagram.

We write a generic bubble integral as

$$\begin{aligned} I[\alpha_1, \alpha_2] &= \int \mathrm{d}^d k \frac{1}{(k^2 - m^2)^{\alpha_1} ((k-p)^2 - m^2)^{\alpha_2}} \\ &= \int \mathrm{d}^d k \frac{1}{z_1^{\alpha_1} z_2^{\alpha_2}}. \end{aligned} \tag{2.35}$$

It is parametrized by the powers α_1 and α_2 of the two propagators. For convenience we adopt the notation $I[\alpha_1, \alpha_2] \equiv I_{\alpha_1,\alpha_2}$. The vanishing of the total derivative is given by the following two integration-by-parts identities, obtained by considering the total derivative with respect to the internal momentum k_μ contracted either with the loop momentum (first IBP) or with the external momentum (second IBP):

$$\int \mathrm{d}^d k \frac{\partial}{\partial k_\mu}\left(\frac{v^\mu}{z_1^{\alpha_1} z_2^{\alpha_2}}\right) = 0 \begin{cases} \int \mathrm{d}^d k \frac{\partial}{\partial k_\mu}\left(\frac{k^\mu}{z_1^{\alpha_1} z_2^{\alpha_2}}\right) = 0 \\ \int \mathrm{d}^d k \frac{\partial}{\partial k_\mu}\left(\frac{p^\mu}{z_1^{\alpha_1} z_2^{\alpha_2}}\right) = 0 \end{cases}, \tag{2.36}$$

$v_\mu = k_\mu, p_\mu$.

Thanks to the IBPs we can show that the generic I_{α_1,α_2} can be written as a linear combination of two master integrals: the *bubble* $I_{1,1}$ and *tadpole* $I_{1,0}$. The other tadpole, identified by $I_{0,1}$ is *mapped* to $I_{1,0}$ with a symmetry relation (see Sect. 2.2.3), that amounts to shifting the loop momentum $k_\mu \to k_\mu - p_\mu$.

We build the system of *template equations* for generic exponents α_1, α_2 by calculating the IBPs in eq. (2.36). The first template equation, that we indicate with [bubble$_1$], is obtained from the first IBP:

$$\int \frac{\partial}{\partial k_\mu}\left(\frac{k^\mu}{z_1^{\alpha_1} z_2^{\alpha_2}}\right) = 0, \tag{2.37}$$

and gives

$$\int \frac{d - 2\alpha_1 - \alpha_2}{z_1^{\alpha_1} z_2^{\alpha_2}} - \int \frac{2\alpha_1 m^2}{z_1^{\alpha_1+1} z_2^{\alpha_2}} - \int \frac{\alpha_2(z_1 - p^2 + 2m^2)}{z_1^{\alpha_1} z_2^{\alpha_2+1}} = 0, \tag{2.38}$$

while the second template equation, [bubble$_2$], is obtained from the second IBP:

$$\int \frac{\partial}{\partial k_\mu}\left(\frac{p^\mu}{z_1^{\alpha_1} z_2^{\alpha_2}}\right) = 0, \tag{2.39}$$

we obtain:

$$\int \frac{\alpha_2 - \alpha_1}{z_1^{\alpha_1} z_2^{\alpha_2}} - \int \frac{\alpha_1(p^2 - z_2)}{z_1^{\alpha_1+1} z_2^{\alpha_2}} - \int \frac{\alpha_2(z_1 - p^2)}{z_1^{\alpha_1} z_2^{\alpha_2+1}} = 0. \tag{2.40}$$

We can now give explicit values to α_1 and α_2 in order to build explicit relations between integrals of the bubble and tadpole families. We consider the seeds:

$$\begin{aligned} &\alpha_1 = 0,\ \alpha_2 = 1, \\ &\alpha_1 = 1,\ \alpha_2 = 0, \\ &\alpha_1 = 1,\ \alpha_2 = 1. \end{aligned} \tag{2.41}$$

For $\alpha_1 = 0, \alpha_2 = 1$ we obtain

$$\begin{aligned} &\int \frac{d-1}{z_2} - \int \frac{z_1 + 2m^2 - p^2}{z_2^2} = 0 \qquad [\text{bubble}_1]\,, \\ &-\int \frac{1}{z_2} + \int \frac{z_1 - p^2}{z_2^2} = 0 \qquad [\text{bubble}_2]\,. \end{aligned} \tag{2.42}$$

That allows us to we find the following relation (fig. 2.7)

$$2m^2 \int \frac{1}{z_2^2} = (d-2) \int \frac{1}{z_2}. \tag{2.43}$$

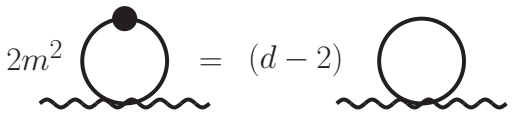

FIGURE 2.7: Diagrammatic identity for eq. (2.43).

For $\alpha_1 = 1, \alpha_2 = 0$ we obtain

$$\begin{aligned} &\int \frac{d-2}{z_1} - \int \frac{2m^2}{z_1^2} = 0 \qquad [\text{bubble}_1]\,, \\ &\int \frac{z_1 - z_2 + p^2}{z_1^2} = 0 \qquad [\text{bubble}_2]\,. \end{aligned} \tag{2.44}$$

That allows us to find the same identity of eq. (2.43) for the propagator z_1. This is a consequence of the fact that, as we said, $I_{1,0}$ and $I_{0,1}$ are mapped and therefore satisfy the same identities. For $\alpha_1 = 1, \alpha_2 = 1$ we obtain

$$\begin{aligned} &\int \frac{d-3}{z_1 z_2} - \int \frac{4m^2}{z_1^2 z_2} + \int \frac{p^2}{z_1 z_2^2} - \int \frac{1}{z_2^2} = 0 \qquad [\text{bubble}_1]\,, \\ &\int \frac{1}{z_1^2 z_2} - \int \frac{1}{z_1 z_2^2} = 0 \qquad [\text{bubble}_2]\,. \end{aligned} \tag{2.45}$$

That allows us to find the identities (fig. 2.8)

$$\int \frac{1}{z_1 z_2^2} = \int \frac{1}{z_1^2 z_2}, \tag{2.46}$$

and (fig. 2.9)

$$\left(4m^2 - p^2\right) \int \frac{1}{z_1^2 z_2} = (d-3) \int \frac{1}{z_1 z_2} - \int \frac{1}{z_2^2}. \tag{2.47}$$

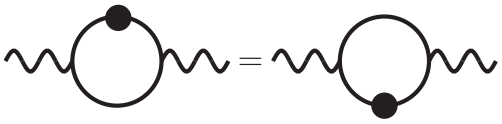

FIGURE 2.8: Diagrammatic identity for eq. (2.46).

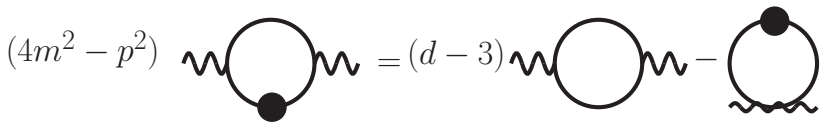

FIGURE 2.9: Diagrammatic identity for eq. (2.47).

We generated enough identities and relations that can be used to decompose, for example, $I[1,2]$ as a function of the two master integrals $I[1,1]$ and $I[1,0]$

$$I_{1,2} = \frac{d-3}{4m^2-p^2} I_{1,1} - \frac{1}{4m^2-p^2}\frac{d-2}{2m^2} I_{1,0}. \tag{2.48}$$

Giving more values to α_1 and α_2 and generating more identities allows to decompose integrals presenting more dots in the propagators.

2.4 | Differential equations

Feynman integrals decomposition into master integrals is also used to obtain differential equations for the master integrals, in order to compute them without recurring to a direct integration. We follow [48, 3] for a brief review of this approach.

The differential-equations method consists in using IBPs to derive *linear first-order* differential equations in the external kinematic invariants satisfied by MIs. It is one of the most successful and effective methods devised to tackle the problem of evaluation of Feynman integrals without employing direct integration. Historically, it was first proposed by Kotikov [24], who considered only differential equations with respect to the internal masses and was then extended and generalized to differential equations with respect to all external invariants by Remiddi and Gehrmann [45, 18], who made this method feasible to be applied to diagrams with massless internal propagators. For clarity, let us refer to a process with $E+1$ external legs with independent external momenta $p_1, \ldots, p_E$. At any loop order the process depends up to

$(E+1)E/2$ kinematic invariants, i.e. $s_{ij} \equiv p_i \cdot p_j$. Following [48] we adopt the notation

$$\mathbf{x} = \left(x_1, \ldots, x_{E(E+1)/2}\right) = (s_{11}, \ldots, s_{1E}, \ldots, s_{EE}). \tag{2.49}$$

We can write E relations using the chain rule:

$$\frac{\partial}{\partial p_i^\mu} = \sum_j \frac{\partial x_j}{\partial p_i^\mu} \frac{\partial}{\partial x_j}, \quad (i = 1, \ldots, E). \tag{2.50}$$

That can be contracted with any of external momenta producing E^2 scalar equations:

$$p_k^\mu \frac{\partial}{\partial p_i^\mu} = p_k^\mu \sum_j \frac{\partial x_j}{\partial p_i^\mu} \frac{\partial}{\partial x_j}, \quad (i, k = 1, \ldots, E). \tag{2.51}$$

The relations in eq. (2.51) can be inverted to re-express $E(E+1)/2$ derivatives in the external invariants as linear combinations of derivatives in the external momenta. The system is often overcostrained since

$$E^2 - \frac{(E+1)E}{2} = \frac{E(E-1)}{2} \geq 0 \quad \forall (E+1) \geq 2. \tag{2.52}$$

Lorentz invariance —see Sect. 2.2.2— provide $E(E-1)/2$ relations among the derivatives with respect to the external momenta. Using LIs one can prove that these different representations are all equivalent to one another. All these considerations are independent of the loop order we are considering, but only depend on the Lorentz invariance properties of the process. Once a choice is made, we end up with

$$\frac{\partial}{\partial x_j} = \sum_{ik} A_{ik}^{(j)} p_l^\mu \frac{\partial}{\partial p_i^\mu}, \tag{2.53}$$

with $A_{ik}^{(j)}$ rational functions in the dimensional regulator ϵ and of the $E(E+1)/2$ external invariants $\mathbf{x}$. Considering any scalar integral I we can then write its derivative with respect to invariants as

$$\frac{\partial I}{\partial x_j} = \sum_{ik} A_{ik}^{(j)}\, p_l^\mu\, \frac{\partial I}{\partial p_i^\mu}. \tag{2.54}$$

We can apply this procedure to write differential equations for a set of master integrals $\{G_i\}$

$$\frac{\partial G_n}{\partial x_j} = \sum_{ik} A_{ik}^{(j)}\, p_l^\mu\, \frac{\partial G_n}{\partial p_i^\mu}, \qquad (n = 1, \ldots, n_{MI}), \tag{2.55}$$

where the right hand side (r.h.s.), having undergone derivatives, it is not expressed as a linear combination of master integrals. This does not pose a problem since we can now decompose the integrals appearing on the r.h.s. into master integrals and substitute them again in eq. (2.55). In this way we obtain a differential equation for the MIs

$$\frac{\partial G_n}{\partial x_j} = \sum_k M_{nk}^{(j)} G_k \qquad (n = 1, \ldots, n_{MI}). \tag{2.56}$$

To summarize, the method of differential equations consists of:

- writing down derivatives of the MIs with respect to external invariants;
- reducing to MIs the right hand side;
- obtaining a system of differential equations for the MIs.

3 | Intersection Theory

> One of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane
>
> R.J. Eden, P.V. Landshoff, D.I. Olive, J.C. Polkinghorne [14]

In this Chapter we introduce the formalism of intersection theory and its applications to the decomposition of multiloop Feynman integrals in master integrals. We include the presentation of a novel procedure to analyze the number of master integrals for a given diagram. Throughout the whole chapter we closely follow the presentation developed in [17, 16].

3.1 | Overview

Intersection theory of differential forms is a branch of algebraic geometry and topology, originally developed to study Aomoto-Gel'fand hypergeometric functions [12, 34, 36] and only recently its techniques have been borrowed and applied to the study of Feynman integrals [17, 16, 35]. By drawing a connection between Feynman integrals and hypergeometric functions, it allows to describe in a pure algebraic way the problem of decomposition of a Feynman integral into master integrals. This rather novel procedure is an alternative to the regularly used Laporta method. It presents as its main asset the direct decomposition of multiloop integrals and therefore the advantage of sidestepping the solution of large and sparse systems of IBP identities. This is done by noticing that Feynman integrals obey a vector space structure and therefore, after having identified the vector space basis with the master integrals, their decomposition turns into the projection of a vector into the vector space basis. In order to show that Feynman integrals

obey a vector space structure, it is necessary to change their representation from the commonly used momentum parametrization to one that mirrors the structure of an hypergeometric function. In our work we choose the Baikov representation, where integration variables are changed from loop momenta to generalized propagators. Other alternatives are possibles, such as representations that integrate over Lorentz invariant scalar products [8]. The vector space is called *twisted cohomology* and it consists of equivalence classes of Feynman integrals. The equivalence relation is conveniently expressed as an integration-by-parts identity. That is, in our case, we consider Feynman integrals whose integrands are equivalent up to an IBP in Baikov space.

To summarize, first we write Feynman integrals in a suitable representation to be used with the formalism of intersection theory. Then, we decompose them into a basis of master integrals via the calculation of several scalar products, called intersection numbers. The main point in the calculation of intersection numbers consists in summing over the residues, calculated around a set of poles, of the product of two functions, one of them being the solution of a differential equation around each pole in consideration. The calculation of intersection numbers of differential $n-$forms is done via a recursive procedure.

3.2 | Hypergeometric functions vector space

As we anticipated, intersection theory allows for a direct decomposition of Feynman integrals into master integrals in terms of projections into a basis, bypassing the resolution of large systems of identities. In this section we expand on the Feynman integrals parametrization we use in order to perform intersection theory calculations and on the decomposition strategies. There are various ways in which we can represent a generic Feynman integral. As discussed in Ch. 2, the usual representation we use for an L loop Feynman integral in d-dimensions is the momentum parametrization, that we rewrite here for convenience

$$I[\alpha_1, \ldots, \alpha_n] = \int \left(\prod_{i=1}^{L} \frac{\mathrm{d}^d k_i}{i\pi^{d/2}} \right) \frac{1}{z_1^{\alpha_1} \ldots z_n^{\alpha_n}}, \tag{3.1}$$

where z_i are generalized inverse propagators of the form $z_i = P_i^2 - M_i^2$ as described in Ch. 2. We first look at the so called Aomoto-Gel'fand generalized hypergeometric integrals, in order to find a representation for Feynman integrals mimicking their structure. Our goal is to take advantage of the properties obeyed by generalized hypergeometric integrals to decompose our

Feynman integrals written in a convenient representation. We closely follow [16, 17]. *Aomoto-Gel'fand* hypergeometric functions are integrals of the type

$$I = \int_{\mathcal{C}} u(\mathbf{z})\varphi(\mathbf{z}), \tag{3.2}$$

where u is a multivalued function, called *twist*. In the context of Feynman integrals decomposition it takes the form

$$u = \prod_i \mathcal{B}_i^{\gamma_i}, \tag{3.3}$$

with γ_i "generic" exponents. φ is a rational single-valued and holomorphic differential n-form. Both u and φ depend on n variables $\mathbf{z} = (z_1, \ldots, z_n)$. We indicate φ as

$$\varphi(\mathbf{z}) = \frac{f(\mathbf{z})}{z_1^{\alpha_1} \ldots z_n^{\alpha_n}} \, \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n = \frac{f(\mathbf{z})}{z_1^{\alpha_1} \ldots z_n^{\alpha_n}} \, \mathrm{d}^n z, \tag{3.4}$$

with $f(\mathbf{z})$ a rational function and $\alpha_i \in \mathbb{Z}$.

A key assumption is that u regulates all the poles present in φ.

$$u(\mathbf{z})\,\varphi(\mathbf{z}) \xrightarrow{\mathbf{z} \to \text{ pole of } \varphi} \text{ finite}. \tag{3.5}$$

Within Feynman integrals, this assumption is often violated and, in order to accommodate for it, we need to introduce some regulators, as we explain in Sect. 3.4

Another fundamental assumption is for the function $u(\mathbf{z})$ to vanish on the boundary of $\mathcal{C}$:

$$u(\partial\mathcal{C}) = 0. \tag{3.6}$$

This property is the key to give rise to the vector space structure, that is in turn the key object of the decomposition via intersection theory. We consider the integral of the total derivative of the product of two functions, u, which is our multivalued function, and ξ which is a differential $(m-1)-$form.

By Stokes' theorem and eq. 3.6, the integral is zero:

$$\int_{\mathcal{C}} \mathrm{d}[u\xi] = \int_{\partial\mathcal{C}} u\xi = 0. \tag{3.7}$$

Expanding the differential we obtain

$$\begin{aligned} \int_{\mathcal{C}} \mathrm{d}[u\xi] &= \int_{\mathcal{C}} \mathrm{d}u \wedge \xi + u\,\mathrm{d}\xi \\ &= \int_{\mathcal{C}} u\left[\frac{\mathrm{d}u}{u} \wedge + \mathrm{d}\right]\xi \\ &\equiv \int u\,\nabla_\omega \xi. \end{aligned} \tag{3.8}$$

For ease of notation we introduce the function ω and we rewrite all inside the brackets as a covariant derivative ∇_ω

$$\frac{\mathrm{d}u}{u} = \mathrm{d}\log u \equiv \omega, \qquad \nabla_\omega \equiv \mathrm{d} + \omega \wedge . \tag{3.9}$$

Therefore we find that the integral of the product of the multivalued function u and the *covariant derivative* of a differential $(m-1)$-form, $\nabla_\omega \xi$ (which is an $m-$form), is zero:

$$\int u \nabla_\omega \xi = 0. \tag{3.10}$$

So we can see that if we have an integral of a certain u and an $m-$form φ we can add to the integrand the product of u times the covariant derivative of any $(m-1)-$form and the integral remains the same. This is equivalent to say that two integrands are the same up to integration-by-parts identities.

This leads us to define the concept of *twisted cohomology* class, namely the class of functions ${}_\omega\langle\varphi|$ integrating to the same result and that differ by a total covariant derivative:

$$\varphi \sim \varphi + \nabla_\omega \xi. \tag{3.11}$$

The functions φ modulo the equivalence class obey a vector space structure called *twisted cohomology*, where the word "twisted" comes from the replacing of the usual derivative by the twisted version represented by the covariant derivative ∇_ω. This is the same generalization that happens when considering the gauge transformation in an abelian gauge theory, like QED, and generalizing it to a non abelian one, like QCD [54]. We indicate the equivalence class of φ with the bra notation:

$${}_\omega\langle\varphi| : \{\, \psi \,|\, \exists \xi \,,\, \psi + \nabla_\omega \xi = \varphi \,\}, \tag{3.12}$$

and the twisted cohomology group with H^n_ω.

In the same fashion we can define a *twisted homology group* consisting of the equivalence classes of contours of integration that give rise to the same result. It can be seen that it also obeys a vector space structure, this time indicated with H^ω_n, with its members identified by the square ket notation $|\mathcal{C}]$.

Here and in the following we will not be interested in the homology group, as in order to have integration-by-parts identities it is not necessary to specify the contour of integration, it suffices to just choose integrands obeying certain properties, such as the vanishing of u on the boundaries. Therefore we focus our discussion only on the twisted cohomology group and its properties.

It is now manifest that hypergeometric integrals can be seen as bilinear pairings between an element of the twisted cohomology and one of the twisted homology, the result being independent of the representative chosen:

$$I = \int_{\mathcal{C}} u(\mathbf{z})\varphi(\mathbf{z}) = \langle\varphi|\,\mathcal{C}]. \tag{3.13}$$

The pairing means that in practice we have to perform an integration over the contour $\mathcal{C}$ of the product between φ and u.

Fixing the contour of integration $\mathcal{C}$, we can see that the integral $I = \langle\varphi|\,\mathcal{C}]$ inherits the vector space structure of the $\langle\varphi|$. As we will present in the following sections, connections are made between integrals I and the multiloop Feynman integrals we are interested in decomposing. The fact that I obeys a vector space structure opens up the possibility to perform the decomposition into master integrals via a direct projection into the vector space basis.

In order to perform projections into the vector space basis we need to define a scalar product and therefore firstly introduce a dual vector space. This is done by introducing a dual integral:

$$I^\star = \int_{\mathcal{C}^\star} u^{-1}\,\varphi^\star = [\mathcal{C}^\star\,|\varphi^\star\rangle\,, \tag{3.14}$$

where we consider the following covariant derivative:

$$\nabla_{-\omega} = \mathrm{d} - \omega\wedge, \qquad \omega = \mathrm{d}\log u\,, \tag{3.15}$$

which just switches $\omega \to -\omega$ since the dual integral presents u^{-1} instead of u. The equivalence classes of $\varphi^\star$ are therefore given by

$$\varphi^\star \sim \varphi^\star + \nabla_{-\omega}\xi^\star. \tag{3.16}$$

We can define the *dual twisted cohomology* vector space as the vector space of the equivalence classes:

$$|\varphi^\star\rangle_\omega : \{\,\psi^\star \,|\, \exists\xi^\star\,,\, \psi^\star + \nabla_{-\omega}\xi^\star = \varphi^\star\,\}. \tag{3.17}$$

We will indicate the dual twisted cohomology vector space with $(H^n_\omega)^\star = H^n_{-\omega}$ and its members with $|\varphi^\star\rangle$.

As above, the equivalence relation can also be found between duals of integration contours, obtaining the dual twisted homology vector space $(H^\omega_n)^\star = H_n^{-\omega}$ with its members indicated with $[\mathcal{C}^\star|$.

This ultimately leads to the interpretation of the dual integral as a pairing between two equivalence classes, belonging respectively to the dual twisted homology and cohomology vector spaces.

Having introduced a vector space and its dual we can now proceed to define a scalar product between the two spaces. As mentioned before, we will focus on the twisted cohomology group, leaving the contour of integration constant.

Given two integrals $I = \langle \varphi_L | \mathcal{C}]$ and $I^\star = [\mathcal{C}^\star | \varphi_R \rangle$, the scalar product between φ_L, belonging to the twisted cohomology vector space and φ_R belonging to its dual is called *intersection number* and is defined by Cho and Matsumoto as [12]

$$\langle \varphi_L | \varphi_R \rangle_\omega = \frac{1}{(2\pi i)^n} \int_X \iota_\omega(\varphi_L) \wedge \varphi_R, \tag{3.18}$$

where we used the L and R notation instead of the $\star$ one in order to recognize the φs from their position in the bra-ket product. the contour of integration X represent the whole complex space $\mathbb{C}^n$ stripped of the poles of ω. $\iota_\omega(\varphi_L)$ [6] performs a regularization on φ_L by mapping it to an equivalent form (so the value of the integral is unchanged) with compact support. Since both φ_L and φ_R are holomorphic on the domain of integration X, omitting the mapping ι_ω would cause the intersection number to vanish. Following [6] we report a brief justification of why this is true and so this prescription actually yields an integral that is nonzero.

We focus on the $1-$dimensional case in the variable $z \in \mathbb{C}$. X is therefore $\mathbb{C}$ without the points corresponding to the poles of ω. We can always decompose any function $f(x, y)$ over $\mathbb{C}$ in its real and imaginary part $f(x, y) = u+iv$. By introducing the following change of variables

$$\begin{cases} z = x + iy \\ \bar{z} = x - iy \end{cases}, \tag{3.19}$$

we end up with a function $\tilde{f}(z, \bar{z})$. Because of the Cauchy-Riemann conditions, if a function $\tilde{f}$ is holomorphic, then its dependence on $z, \bar{z}$ reduces to $\tilde{f}(z)$. In general, any $1-$form φ can be decomposed as

$$\varphi = \varphi_z \, \mathrm{d}z + \varphi_{\bar{z}} \, \mathrm{d}\bar{z}. \tag{3.20}$$

When doing the wedge product between two forms, because of the antisymmetric nature of the wedge product, only the term proportional to $\mathrm{d}z \wedge \mathrm{d}\bar{z}$ survives. If although the forms are holomorphic, then they are decomposed as $\varphi = \varphi_z \, \mathrm{d}z$ giving rise to only terms proportional to $\mathrm{d}z \wedge \mathrm{d}z$ that inevitably vanish.

In the multivariate case, the integration over X is done by iteration, splitting it into one-dimensional fibers and proceeding one variable at time, considering the others as constant parameters.

Intersection numbers obey interesting properties such as

- independence from the representative chosen for each equivalence class of the twisted and dual twisted cohomology groups

$$\langle\varphi_L|\varphi_R\rangle = \langle\tilde{\varphi_L}|\varphi_R\rangle = \langle\varphi_L|\tilde{\varphi_R}\rangle = \langle\tilde{\varphi_L}|\tilde{\varphi_R}\rangle\,, \tag{3.21}$$

  with

$$\begin{aligned}\tilde{\varphi}_L &= \varphi_L + \nabla_\omega \xi_L,\\ \tilde{\varphi}_R &= \varphi_R + \nabla_{-\omega} \xi_R,\end{aligned} \tag{3.22}$$

  where the $(n-1)$-differential forms ξ_L and ξ_R are chosen so that they have poles regulated by u;

- obey the symmetry relation

$$\langle\varphi_L|\varphi_R\rangle_\omega = (-1)^n\ \langle\varphi_R|\varphi_L\rangle_{-\omega}\,. \tag{3.23}$$

To summarize, in this section we introduced hypergeometric integrals as bilinear pairings between cohomology and homology classes of equivalence, we gave a brief account of their properties, the most important one being the fact that hypergeometric integrals with the same contour of integration obey a vector space structure. We introduced a dual vector space in order to perform scalar products and ultimately projections. Now we turn to the description of the dimension of the vector space, ultimately corresponding to the number of master integrals.

3.3 | Dimension of twisted cohomology group

An important characteristic in the description of the twisted cohomology vector space is its dimension, that we indicate with ν and can be calculated in different ways. The first way is presented in [16], and we report it for completeness: using the complex Morse (Picard-Lefschetz) theory, we can determine ν as the number of *critical points* of the function $\log u(\mathbf{z})$ [28]. Critical points are defined as the solutions of $\partial_{z_i} \log u(\mathbf{z}) = 0$, which can be expressed more conveniently using ω. Since $\omega = \mathrm{d}\log(u)$, then

$$\omega = \sum_{i=1}^{n} \hat{\omega}_i \mathrm{d}z_i, \qquad \hat{\omega}_i = \frac{\partial \log(u)}{\partial z_i} = \frac{\partial_{z_i} u}{u}. \tag{3.24}$$

Therefore counting the number of critical points means counting the number of solutions of

$$\hat{\omega}_i = \partial_{z_i} \log u(\mathbf{z}) = 0, \qquad (i = 1, \ldots, n). \tag{3.25}$$

A strategy based on the Shape Lemma (see App. A) allows to count the number of solutions without explicitly solving the system. Since in our applications to the decomposition of Feynman integrals the function u takes the form

$$u = \prod_j \mathcal{B}_j^{\gamma_j}, \tag{3.26}$$

eq. (3.25) turns into

$$\omega = \mathrm{d}\log(u) = \sum_j \gamma_j \log(\mathcal{B}_j), \tag{3.27}$$

where each $\hat{\omega}_i$ is given by

$$\hat{\omega}_i = \sum_j \gamma_j \frac{\partial_{z_i}\mathcal{B}_j}{\mathcal{B}_j}, \qquad (i = 1, \ldots, n). \tag{3.28}$$

Then, assuming we do not have critical points at infinity, the number of solutions equals the dimension of the quotient space of the ideal $\mathcal{I}$:

$$\mathcal{I} = \left\langle \beta_1, \ldots, \beta_n, z_0 \prod_j \mathcal{B}_j - 1 \right\rangle, \quad \text{with } \beta_k \equiv \sum_i \gamma_i (\partial_{z_k}\mathcal{B}_i) \prod_{i\neq j} \mathcal{B}_j, \tag{3.29}$$

where the β_i are the numerators of each term in the sum constituting $\hat{\omega}_i$. The last term with the additional variable z_0

$$z_0 \prod_j \mathcal{B}_j - 1, \tag{3.30}$$

is introduced in order to exclude from the solutions of the system the case for any of the $\mathcal{B}_j$ to be zero (as they appear in the denominator of the $\hat{\omega}_i$). The Shape Lemma ensures that the number of zeroes of $\mathcal{I}$, and therefore the number of solutions of eq. (3.29), is the dimension of the quotient ring (App. A)

$$\nu = \dim(\mathcal{V}(\mathcal{I})/\mathcal{I}), \tag{3.31}$$

where we need to consider the ideal $\mathcal{I}$ as generated by a Gröbner basis $\mathcal{G}_\mathcal{I}$ and we indicate with $\mathcal{V}(\mathcal{I})$ the algebraic variety of polynomials vanishing on the zeroes of $\mathcal{I}$.

We now present our strategy to find the dimension of the twisted cohomology group.

3.3.1 | Master monomial analysis

In our work we use a method inspired by the Laporta algorithm to calculate the dimension of the twisted cohomology group, that we name *master monomials analysis*.

As we anticipated and we will explain thoroughly in Sect. 3.4.2, in our applications the function u has the form

$$u = \prod_j \mathcal{B}_j^{\gamma_j}. \tag{3.32}$$

In particular

$$u = \prod_j z_j^{\rho_j} \mathcal{B}^{\gamma}, \tag{3.33}$$

where $\mathcal{B}$ is the Baikov polynomial and $z_j^{\rho_j}$ are denominators elevated to parameters ρ_j acting as regulators of the poles of ω. As the Shape Lemma ensures, the basis of the quotient ring $\mathcal{V}(\mathcal{I})/\mathcal{I}$ is given by the number of points in $\mathcal{V}(\mathcal{I})$, therefore we are looking for the solution of the system generated by the generators of the ideal $\mathcal{I}$. The generators are the ones appearing in eq. (3.29)

$$\{\beta_i\}_{i=1}^n \cup \{z_0 \prod_j \mathcal{B}_j - 1\}, \quad \text{with } \beta_k \equiv \sum_i \gamma_i (\partial_{z_k} \mathcal{B}_i) \prod_{i \neq j} \mathcal{B}_j. \tag{3.34}$$

Suppose our Feynman diagram in Baikov representation counts n propagators, n_{isp} of which are irreducible scalar products. As we will show later, irreducible scalar products do not need to be regulated by any ρ_j. Therefore we have three types of equations to put to zero in our system.

The first one is the case in which the variable z_i is a propagator that needs to be regulated. In that case $\hat{\omega}_i$ gives:

$$\begin{aligned}
\hat{\omega}_i &= \frac{\partial_{z_i}\left(z_1^{\rho_1} \dots z_i^{\rho_i} \dots z_{n-n_{\text{isp}}}^{\rho_{n-n_{\text{isp}}}} \mathcal{B}^{\gamma}\right)}{z_1^{\rho_1} \dots z_i^{\rho_i} \dots z_{n-n_{\text{isp}}}^{\rho_{n-n_{\text{isp}}}} \mathcal{B}^{\gamma}} \\
&= \frac{\rho_i z_1^{\rho_1} \dots z_i^{\rho_i - 1} \dots z_{n-n_{\text{isp}}}^{\rho_{n-n_{\text{isp}}}} \mathcal{B}^{\gamma}}{z_1^{\rho_1} \dots z_i^{\rho_i} \dots z_{n-n_{\text{isp}}}^{\rho_{n-n_{\text{isp}}}} \mathcal{B}^{\gamma}} + \frac{z_1^{\rho_1} \dots z_i^{\rho_i} \dots z_{n-n_{\text{isp}}}^{\rho_{n-n_{\text{isp}}}} \partial_{z_i} \mathcal{B}^{\gamma}}{z_1^{\rho_1} \dots z_i^{\rho_i} \dots z_{n-n_{\text{isp}}}^{\rho_{n-n_{\text{isp}}}} \mathcal{B}^{\gamma}} \\
&= \frac{\rho_i}{z_i} + \frac{\gamma \mathcal{B}^{\gamma-1} \partial_{z_i} \mathcal{B}}{\mathcal{B}^{\gamma}} \\
&= \frac{\rho_i \mathcal{B} + \gamma z_i \partial_{z_i} \mathcal{B}}{z_i \mathcal{B}^{\gamma}}.
\end{aligned} \tag{3.35}$$

The second case is when z_k is an irreducible scalar product, in that case $\hat{\omega}_k$ simplifies with respect to the previous equation as:

$$\begin{aligned}\hat{\omega}_k &= \frac{\partial_{z_k}\left(z_1^{\rho_1}\dots z_{n-n_{\text{isp}}}^{\rho_{n-n_{\text{isp}}}}\mathcal{B}^\gamma\right)}{z_1^{\rho_1}\dots z_{n-n_{\text{isp}}}^{\rho_{n-n_{\text{isp}}}}\mathcal{B}^\gamma}\\ &= \frac{z_1^{\rho_1}\dots z_{n-n_{\text{isp}}}^{\rho_{n-n_{\text{isp}}}}\partial_{z_k}\mathcal{B}^\gamma}{z_1^{\rho_1}\dots z_{n-n_{\text{isp}}}^{\rho_{n-n_{\text{isp}}}}\mathcal{B}^\gamma}\\ &= \frac{\gamma\mathcal{B}^{\gamma-1}\partial_{z_k}\mathcal{B}}{\mathcal{B}^\gamma}\\ &= \frac{\gamma\partial_{z_k}\mathcal{B}}{\mathcal{B}}.\end{aligned} \tag{3.36}$$

While the third case is an equation, indicated with $\hat{\omega}_{extra}$, that we add in order to prevent the appearance of solutions as $\mathcal{B}_j = 0$ for any j:

$$\hat{\omega}_{extra} = z_0 \prod_{j=1}^{n-n_{\text{isp}}} z_j^{\rho_j}\,\mathcal{B}^\gamma - 1. \tag{3.37}$$

Our goal is then to determine the number of solutions of the system

$$\begin{cases}\hat{\omega}_i = 0 & (i = 1,\dots,n-n_{\text{isp}})\\ \hat{\omega}_k = 0 & (k = n-n_{\text{isp}},\dots,n)\\ \hat{\omega}_{extra} = 0\end{cases}, \tag{3.38}$$

that reads

$$\begin{cases}\rho_i\mathcal{B} + \gamma z_i\partial_{z_i}\mathcal{B} = 0 & (i = 1,\dots,n-n_{\text{isp}})\\ \gamma\partial_{z_k}\mathcal{B} = 0 & (k = n-n_{\text{isp}},\dots,n)\\ z_0\prod_{j=1}^{n-n_{\text{isp}}} z_j^{\rho_j}\,\mathcal{B}^\gamma - 1 = 0\end{cases}. \tag{3.39}$$

That is a multivariate polynomial system in the variables $z_1,\dots,z_n$. To solve this system of equations we adopt the following strategy. We see a polynomial equation as a linear relation satisfied by monomials, which relates monomials elevated to different powers. Following this reasoning, the monomials appearing in the equations are not all linearly independent (in the sense of the relations drawn by the polynomial equations). We look for the number of independent monomials, which corresponds to the number of solutions of the system and, therefore, to the dimension of the quotient space. Consider the following example

Example 3.3.1. The polynomial equation

$$z^2 + bz + c = 0, \tag{3.40}$$

can be seen as a linear relation between the monomials z^2, z and 1. We can write a template equation for symbolic n for eq. (3.40) by multiplying it for an arbitrary monomial z^n

$$z^{n+2} + bz^{n+1} + cz^n = 0. \tag{3.41}$$

By giving several seeds (explicit values) to n, we build a system of equations. Its solution yields that any $z^n (n \geq 2)$ can be expressed as linear combination of a factor linear in z and a term of order 0 in z. Therefore, the linearly independent quantities with respect to the relation (3.41) are z and 1. A possible monomial basis contains two "master monomials" and can be identified with the set $\{z, 1\}$. In fact, eq. (3.40) has usually two solutions.

Therefore, in order to find the number of solution of eq. (3.39), we generate more polynomial equations, called *template equations*, by multiplying the equations in the system (3.39) by monomials elevated to arbitrary powers, that is

$$\begin{cases} z^\alpha(\rho_i \mathcal{B} + \gamma z_i \partial_{z_i} \mathcal{B}) = 0 \\ z^\alpha(\gamma \partial_{z_k} \mathcal{B}) = 0 \\ z^\alpha\Big(z_0 \prod_{j=1}^{n-n_{isp}} z_j^{\rho_j}\, \mathcal{B}^\gamma - 1\Big) = 0 \end{cases} . \tag{3.42}$$

For clarity of notation, we indicate a monomial elevated to arbitrary powers $\alpha_1, \ldots, \alpha_n$ in the variables $z_1, \ldots, z_n$ with z^α where both z and α are vectors with n components,$z = (z_1, \ldots, z_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$, and the expanded notation reads:

$$z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}. \tag{3.43}$$

We then set the symbolic powers α in the seed equations (3.42) to integer numbers to obtain a large number of equations called *seed equations*. As in the Laporta procedure, we generate a large number of identities and then we solve them in order to find the independent variables. In this case we solve the system of seed equations we generated in order to find the master monomials. Their number corresponds to the dimension of the twisted cohomology vector space.

Before solving it we need to specify an order for the monomials, to express which are the criteria for the monomials we prefer to have in our basis. For this procedure we implemented a MATHEMATICA routine which relies on the framework FINITEFLOW [42, 43] in order to generate the seed equations and solve them.

3.3.2 | Basis projection

Considering the twisted cohomology group H^n_ω equipped with a basis of *master forms* $\{\langle e_1|, \ldots, \langle e_\nu|\}$ and the dual twisted cohomology group $H^n_{-\omega}$ with the dual master forms' basis $\{|h_1\rangle, \ldots, |h_\nu\rangle\}$, we can decompose any vector $\langle\varphi|$:

$$\langle\varphi| = \sum_{i=1}^{\nu} c_i \langle e_i|, \qquad c_i = \sum_{j=1}^{\nu} \langle\varphi|h_j\rangle \left(\mathbf{C}^{-1}\right)_{ji}, \tag{3.44}$$

and any dual vector $|\varphi^\star\rangle$:

$$|\varphi^\star\rangle = \sum_{i=1}^{\nu} c_i^\star |h_i\rangle, \qquad c_i^\star = \sum_{j=1}^{\nu} \left(\mathbf{C}^{-1}\right)_{ij} \langle e_j|\varphi^\star\rangle. \tag{3.45}$$

Therefore, pairing the $\langle\varphi|$ ($|\varphi^\star\rangle$) with a constant contour $|\mathcal{C}]$ ($[\mathcal{C}^\star|$), we can write the decomposition of any hypergeometric integral I ($I^\star$) in terms of the basis integrals J_i ($J_i^\star$).

$$\begin{aligned} I &= \sum_{i=1}^{\nu} c_i J_i \qquad I = \langle\varphi|\,\mathcal{C}] \qquad J_i = \langle e_i|\,\mathcal{C}], \\ I^\star &= \sum_{i=1}^{\nu} c_i^\star J_i^\star \qquad I^\star = [\mathcal{C}^\star\,|\varphi^\star\rangle \qquad J_i^\star = [\mathcal{C}^\star\,|h_i\rangle. \end{aligned} \tag{3.46}$$

The basis integrals will correspond to the master integrals when dealing with Feynman integrals decompositon. In eqq. (3.44),(3.45) we introduced the metric $\mathbf{C}_{ij}$ as the $\nu \times \nu$ matrix of intersections of the master forms:

$$\mathbf{C}_{ij} = \langle e_i|h_j\rangle, \tag{3.47}$$

which in general differs from the identity, as we will not be using orthonormal bases. We also used the resolution of the identity in the cohomology space

$$\mathbb{I}_{\mathrm{c}} = \sum_{i,j=1}^{\nu} |h_i\rangle \left(\mathbf{C}^{-1}\right)_{ij} \langle e_j|. \tag{3.48}$$

We give an account for the formulae used for the decomposition of a vector of the (dual) twisted cohomology space into its (dual) basis in App. C. The same reasoning can be applied to the twisted homology vector space, defining a scalar product between representatives of the twisted homology and its dual.

As we have seen, linear relations between elements of the cohomology vector space translate into vector decomposition and ultimately in integrals decomposition by keeping constant the integration contour.

3.4 | Feynman Integrals decomposition

The intersection-theory approach allows us to identify Feynman integrals with elements of the twisted cohomology vector space, $I = \langle \varphi | \mathcal{C}]$, the master integrals being identified with integration over the same contour of the basis forms $J_i = \langle e_i | \mathcal{C}]$. Within this framework, we see the problem of decomposition into master integrals as finding the coefficients of the projection of a vector into its basis elements via the scalar product we defined in Sect. 3.2.

Since we want to decompose a Feynman integral I into a basis of MIs $\{J_i\}_{i=1}^{\nu}$, we first need to put the Feynman Integrals into a suitable representation to be used in combination with intersection theory. In the following we describe two possibilities, which do not exhaust all the possible strategies but are the ones used in our work.

Throughout this work we performed calculations starting from Feynman integrals in Baikov representation.

3.4.1 | Baikov representation

One way to transform our Feynman integrals so they look like an hypergeometric function is by means of the Baikov representation, which was invented recently, in the 1990s.

We follow closely the treatment presented in [21, 55, 39]. We consider an L loop Feynman integral with $E+1$ external legs in $d = 4 - 2\epsilon$ dimensions. We label loop momenta as $k_1, \ldots, k_L$. Because of momentum conservation, only E of the external momenta will be linearly independent and we indicate them with $p_1, \ldots, p_E$. We collectively indicate loop and external momenta as q with $q_i = k_i$, $i = 1, \ldots, L$ and $q_{L+i} = p_i$, $i = 1, \ldots, E$. As we said in Ch. 2 we have $n = L(L+1)/2 + LE$ independent scalar products that can be obtained as:

$$\begin{aligned} &k_i \cdot p_j \qquad 1 \leq i \leq L, \quad 1 \leq j \leq E, \\ &k_i \cdot k_j \qquad 1 \leq i \leq j \leq L. \end{aligned} \tag{3.49}$$

and we will collectively indicate them as $s_{i,j} = q_i \cdot q_j$ with $s_{i,j} = s_{j,i}$. For clarity we report the form of the Feynman integral in consideration in momentum parametrization, that is written as

$$I[\alpha_1, \ldots, \alpha_n] = \int \left(\prod_{i=1}^{L} \frac{\mathrm{d}^d k_i}{i\pi^{d/2}} \right) \frac{1}{z_1^{\alpha_1} \ldots z_n^{\alpha_n}}, \tag{3.50}$$

with $z_i = P_i^2 - M_i^2$, P_i and M_i respectively indicating a combination of momenta and masses imposed by the kinematics of the diagram in consideration.

The main idea behind Baikov representation is to switch integration variable from the loop momenta to the denominators:

$$\{k_i^\mu\}_{i=1}^L \to \{z_i\}_{i=1}^n. \tag{3.51}$$

After the change of variables the integral takes the form

$$I[\alpha_1, \ldots, \alpha_n] = K \int_{\mathcal{C}} \frac{\mathrm{d}z_1 \ldots \mathrm{d}z_n}{z_1^{\alpha_1} \ldots z_n^{\alpha_n}} B^{\frac{D-L-E-1}{2}}, \tag{3.52}$$

where B is the Baikov polynomial calculated as a determinant of the Gram matrix of scalar products $q_i \cdot q_j$

$$B = G(q_1, \ldots, q_n) = \mathrm{Det}(q_i \cdot q_j), \tag{3.53}$$

K is a constant prefactor depending on the kinematic invariants and on the regulator d and $\mathcal{C}$ is an integration contour such that the Baikov polynomial vanishes on its boundaries. From this representation we can recognize the structure of an hypergeometric integral by writing

$$I[\alpha_1, \ldots, \alpha_n] = K \int_{\mathcal{C}} u\, \varphi, \tag{3.54}$$

with

$$u = B^{\frac{D-L-E-1}{2}}, \quad \varphi = \frac{\mathrm{d}z_1 \ldots \mathrm{d}z_n}{z_1^{\alpha_1} \ldots z_n^{\alpha_n}}. \tag{3.55}$$

More details on the Baikov representation and how to obtain it can be found in App. B. Cuts of propagators are implemented in Baikov representation simply setting the cut propagators z_is to zero in the Baikov polynomial. For example, cutting the set of propagators $\tau = \{z_1 \ldots z_m\}$ we get

$$u_\tau = u(\mathbf{z})\Big|_{z_i \in \tau \to 0}. \tag{3.56}$$

The cut prescription in Baikov representation coincides with the replacement

$$\frac{1}{z_i} \to \delta(z_i), \tag{3.57}$$

inside the φ, when the diagram in consideration has no dots on the cut propagator. The integration contour, on the other hand, is deformed into

$$\mathcal{C}_{\tau\,\mathrm{cut}} = \circlearrowleft_1 \wedge \cdots \circlearrowleft_m \wedge \mathcal{C}', \tag{3.58}$$

where $\circlearrowleft_i$ indicates a small circle in the complex plane around the pole at $z_i = 0$ and $\mathcal{C}^{'}$ is given by the intersection of the original domain with the on-shell conditions $z_i = 0, (i = 1, \ldots, m)$

$$\mathcal{C}^{'} = \bigcap_{z_i \in \tau} \{z_i = 0\} \cap \mathcal{C}. \tag{3.59}$$

Integrating out the cut variables, the hypergeometric integrals takes the form

$$I[\alpha_1, \ldots, \alpha_n]\Big|_{\tau\text{cut}} \sim \int_{\mathcal{C}'} u^{'} \varphi^{'}. \tag{3.60}$$

with

$$u^{'} = u\Big|_{z_i \in \tau,\, z_i = 0}, \qquad \varphi^{'} = \hat{\varphi}^{'}\, \mathrm{d}^{n-m} z, \quad \mathrm{d}^{n-m} z = \mathrm{d}^{m+1} z \wedge \cdots \wedge \mathrm{d}^{n} z. \tag{3.61}$$

where $\hat{\varphi}^{'}$ has an expression accommodating for the presence of dotted cut propagators

$$\hat{\varphi}^{'} = \frac{f(z_{m+1}, \ldots, z_n)}{z_{m+1}^{\alpha_{m+1}} \ldots z_n^{\alpha_n}} \frac{1}{u} \left(\prod_{i=1}^{m} \frac{\partial_{z_i}^{(\alpha_i - 1)} u}{(\alpha_i - 1)!} \right). \tag{3.62}$$

We expand a bit on eq. (3.62). The complicated expression actually just indicates the treatment that the function φ needs to undergo in order to apply the cut prescription in the form of eq. (3.57) whenever it presents one or more dots on the cut propagator. For clarity, we propose an example to explicitly see what happens when the cut propagator has one dot.

Example 3.4.1. We consider the box diagram with one dot on the propagator z_2, indicated with $I[1, 2, 1, 1]$.

$$I[1, 2, 1, 1] = \int_{\mathcal{C}} \mathrm{d}^4 z\, u \frac{1}{z_1 z_2^2 z_3 z_4}. \tag{3.63}$$

Suppose we want to decompose this diagram but on the cut $\{2, 4\}$. We can not do it directly because the prescription of eq. (3.57) is limited to un-dotted propagators. Therefore, we need to recast the integral in a suitable form such that it keeps the hypergeometric structure of the integrand while removing

the dot. By integrations by parts identities we get

$$\begin{aligned} I[1,2,1,1] &= \int_{\mathcal{C}} \mathrm{d}^4 z\, u \frac{1}{z_1 z_2^2 z_3 z_4} \\ &= -\int_{\mathcal{C}} \mathrm{d}^4 z\, u \partial_{z_2} \left(\frac{1}{z_1 z_2 z_3 z_4} \right) \\ &= -\frac{u}{z_1 z_2 z_3 z_4}\bigg|_{\partial \mathcal{C}} + \int_{\mathcal{C}} \mathrm{d}^4 z\, \partial_{z_2}(u) \frac{1}{z_1 z_2 z_3 z_4} \\ &= -0 + \int_{\mathcal{C}} \mathrm{d}^4 z\, \partial_{z_2}(u) \frac{1}{z_1 z_2 z_3 z_4}, \end{aligned} \tag{3.64}$$

where the last passage is justified by the vanishing of boundary terms in dimensional regularization. We write the remaining term in the form $u \cdot \varphi'$

$$\begin{aligned} \int_{\mathcal{C}} \mathrm{d}^4 z\, \partial_{z_2}(u) \frac{1}{z_1 z_2 z_3 z_4} &= \int_{\mathcal{C}} \mathrm{d}^4 z\, u \frac{\partial_{z_2}(u)}{u} \frac{1}{z_1 z_2 z_3 z_4} \\ &= \int_{\mathcal{C}} \mathrm{d}^4 z\, u \frac{\partial_{z_2} \log(u)}{z_1 z_2 z_3 z_4}. \end{aligned} \tag{3.65}$$

So that we can apply the cut prescription. Therefore we have that the differential form associated to $I[1,2,1,1]$ on the cut $\{2,4\}$ reads

$$\varphi' = \frac{\partial_{z_2} \log(u)}{z_2 z_4}\bigg|_{z_2 = z_4 = 0} \mathrm{d}z_2 \wedge \mathrm{d}z_4. \tag{3.66}$$

The procedure for the presence of a higher number of dots is the same, just repeated, and leads to the expression for φ' given in eq. (3.62)

3.4.2 | Decomposition strategies

Our goal is finding the coefficients c_i of the decomposition of a generic Feynman integral I in a basis of master integrals $\{J_i\}$

$$I = \sum_i c_i J_i. \tag{3.67}$$

In [16] three strategies are presented: we report here the two we use throughout the work. We start by describing the first step, common to all the strategies, that consists in finding the master integrals.

Finding the MIs

We label the propagators of the Feynman integral in consideration with integer numbers and indicate the set with Σ.

Any *sector* (following the definition given in Ch. 2) of the integral in consideration is identified by its denominators, which form a subset σ of Σ. Considering a sector of a given integral means cutting all $z_i \in \sigma$. This means that we have to operate on u by sending to zero all the $z_i \in \sigma$:

$$u_\sigma = u(\mathbf{z})\Big|_{z_i \in \sigma \to 0}. \tag{3.68}$$

where u is the Baikov polynomial to the power γ. In order to find the number of master integrals we analyze all sectors of the integral family in consideration following one of the procedure outlined in Sect. 3.3. In particular, we follow the master monomial analysis presented in Sect. 3.3.1. The total number of MIs is given by the sum of the MIs in each sector

$$\nu = \sum_\sigma \nu_\sigma. \tag{3.69}$$

where symmetry relations are not taken into account, therefore the number is usually higher than the one found via Laporta algorithm. The master integrals are then arbitrarily chosen and identified with the pairing

$$J_i = \langle e_i |\, \mathcal{C}]. \tag{3.70}$$

Straight decomposition

We directly decompose the integral I following eq. (3.67), that expanded looks like

$$I = \int_{\mathcal{C}} u\,\varphi = \langle \varphi |\, \mathcal{C}] = \sum_{i=1}^{\nu} c_i\ \langle e_i |\, \mathcal{C}] = \sum_{i=1}^{\nu} c_i \int_{\mathcal{C}} u\, e_i = \sum_{i=1}^{\nu} c_i\, J_i, \tag{3.71}$$

where c_i are given by the decomposition as seen in eq. (3.44). This approach takes the Feynman integral and directly decomposes it into the basis of chosen master integrals. It implies that one needs to calculate intersection numbers of $(n_{\text{den}} + n_{\text{isp}})-$differential forms, since all Baikov variables are taken into account, making it the approach that needs to calculate the intersection numbers with higher n and is therefore the one computationally more demanding.

In the definition of hypergeometric functions in Sect. 3.2, we said that u must regulate all poles of φ. This assumption is often violated in Feynman

integrals, and is taken care of via the introduction of a regulated u, denoted by u_ρ that contains all the monomials $z_i^{\rho_i}$ for each non regulated pole present in the differential form φ

$$u_\rho = \left(\prod_{i\in\Sigma} z_i^{\rho_i}\right) u(\mathbf{z}). \tag{3.72}$$

ω takes the regulated form, ω_ρ

$$\omega_\rho = \mathrm{d}\log(u_\rho) = \mathrm{d}\log(u) + \sum_{i\in\Sigma} \rho_i \frac{\mathrm{d}z_i}{z_i} = \omega + \sum_{i\in\Sigma} \rho_i \frac{\mathrm{d}z_i}{z_i}. \tag{3.73}$$

Intersection numbers calculated through the regulated u_ρ and ω_ρ lead to a set of coefficients that are a function of the regulators, $c_i(\rho_k)$. The coefficients c_i appearing in eq. (3.67) are obtained taking the limit

$$c_i = \lim_{\rho_k\to 0} c_i(\rho_k) = \lim_{\rho_k\to 0} \sum_{j=1}^{\nu} \langle\varphi|h_j\rangle_\rho \left(\mathbf{C}_\rho^{-1}\right)_{ji}, \qquad (\mathbf{C}_\rho)_{ij} = \langle e_i|h_j\rangle_\rho . \tag{3.74}$$

Bottom up decomposition

For this strategy, called *bottom up approach*, we first need to make some definitions, following [16]. We define a *spanning set of cuts* as the minimal set of cuts in which each master integral appears at least once and we denote an element of the set, called a spanning cut, with τ.

The bottom up approach consist in performing the decomposition on the set of spanning cuts of the integral in consideration. This implies calculating intersection number of n forms with n being the number of propagators and irreducible scalar products on the set of cuts in consideration. Clearly, $n < n_{\text{den}} + n_{\text{isp}}$, therefore making this approach less computationally demanding than the straight decomposition.

Considering a spanning cut τ we indicate with S_τ the set of sectors surviving on the cut τ, namely the sectors that contain all the denominators that are on the cut on τ

$$S_\tau = \{\sigma \,|\, \sigma \subseteq \tau\}. \tag{3.75}$$

On the spanning cut τ, u becomes

$$u_\tau = u(\mathbf{z})\Big|_{z_i\in\tau\to 0}, \tag{3.76}$$

and the decomposition is performed considering u_τ instead of u. The coefficient $c_{i,\tau}$ obtained are identical to those appearing in the straight decomposition as coefficients are invariants under cuts in unitarity-based approaches.

The coefficients obtained from the straight decomposition can be obtained combining all the coefficients obtained from all τ in the set of spanning cuts. Integral decomposition on the spanning cut τ takes the form

$$\begin{aligned} I_\tau &= \int_{\mathcal{C}_\tau} u_\tau \, \varphi_\tau = \langle \varphi_\tau | \, \mathcal{C}_\tau] \\ &= \sum_{i=1}^{\nu_{S_\tau}} c_i \, \langle e_{i,\tau} | \, \mathcal{C}_\tau] = \sum_{i=1}^{\nu_{S_\tau}} c_i \int_{\mathcal{C}_\tau} u_\tau \, e_{i,\tau} \\ &= \sum_{i=1}^{\nu_{S_\tau}} c_i \, J_{i,\tau}, \end{aligned} \tag{3.77}$$

where ν_{S_τ} is the number of MIs surviving on the spanning cut. The coefficients are given by

$$c_i = \sum_{j=1}^{\nu_{S_\tau}} \langle \varphi_\tau | h_{j,\tau} \rangle \left(\mathbf{C}_\tau^{-1} \right)_{ji}, \qquad (\mathbf{C}_\tau)_{ij} = \langle e_{i,\tau} | h_{j,\tau} \rangle \, . \tag{3.78}$$

Also in this case we need to introduce a regularized u_τ, indicated with $u_{\tau,\rho}$ whenever φ_τ presents poles non regulated by u_τ. The decomposition takes the form analogous to the one obtained in the case of straight decomposition, but everything is considered on the cut τ. We report for clarity the form of the functions $u_{\tau,\rho}$

$$u_{\tau,\rho} = \left(\prod_{i \in \Sigma \setminus \tau} z_i^{\rho_i} \right) u_\tau, \tag{3.79}$$

and $\omega_{\tau,\rho}$

$$\omega_{\tau,\rho} = \mathrm{d} \log(u_{\tau,\rho}) = \mathrm{d} \log(u_\tau) + \sum_{i \in \Sigma \setminus \tau} \rho_i \frac{\mathrm{d}z_i}{z_i} = \omega_\tau + \sum_{i \in \Sigma \setminus \tau} \rho_i \frac{\mathrm{d}z_i}{z_i}. \tag{3.80}$$

The coefficients obtained after taking the limit

$$c_i = \lim_{\rho_k \to 0} c_i(\rho_k) = \lim_{\rho_k \to 0} \sum_{j=1}^{\nu_{S_\tau}} \langle \varphi_\tau | h_{j,\tau} \rangle_\rho \left(\mathbf{C}_{\tau,\rho}^{-1} \right)_{ji}, \quad (\mathbf{C}_{\tau,\rho})_{ij} = \langle e_{i,\tau} | h_{j,\tau} \rangle_\rho \, . \tag{3.81}$$

3.5 | Calculation of Intersection numbers

We finally turn to the description of the actual calculation of intersection numbers and their properties. We follow [16, 17], highlighting that the

method hereby described leads to the appearance of non-rational poles in intermediate steps of the calculation. In Ch. 4 we propose a novel approach that allows to avoid the presence of non-rational poles. We start by describing how to calculate intersection numbers in the univariate case as the sum of residues around a set of poles of the product of two functions. For each residue, we express one of the functions as the local solution of a differential equation around the pole in consideration. For the multivariate case we introduce a recursive approach.

3.5.1 | Univariate case

We start by describing how to calculate intersection numbers of differential one-forms, or one-forms intersection numbers.

Given two differential one-forms $\langle\varphi_L| = \varphi_L \,\mathrm{d}z$ and $|\varphi_R\rangle = \varphi_R \,\mathrm{d}z$, respectively associated to the integral $I = \int_{\mathcal{C}} \mathrm{d}z\, u\, \varphi_L$ and the dual integral $I^\star = \int_{\mathcal{C}^\star} \mathrm{d}z\, u^{-1}\, \varphi_R$, we calculate their intersection number as

$$\langle\varphi_L|\varphi_R\rangle = \sum_{p\in\mathcal{P}_\omega} \mathrm{Res}_{z=p}(\psi\varphi_R), \tag{3.82}$$

where the sum is extended over $p \in \mathcal{P}_\omega$:

$$\mathcal{P}_\omega = \{z \,|\, z \text{ is a pole of } \omega\} \cup \{\infty\}, \qquad \omega = \mathrm{d}\log u. \tag{3.83}$$

A more detailed derivation of eq. (3.82) can be found in App. E. ψ is the local solution around the point p of the differential equation

$$\nabla_\omega \psi = \varphi_L, \tag{3.84}$$

with the covariant derivative defined as

$$\nabla_\omega \psi \equiv \frac{\mathrm{d}\psi}{\mathrm{d}z} + \omega\psi. \tag{3.85}$$

As it is not necessary (and most of the times it is not even possible) to find a global solution, it is preferable to look for ψ as a Laurent expansion around $z = p$:

$$\psi = \sum_{i=\min}^{\max} c_i (z-p)^i + O\big((z-p)^{\max+1}\big), \tag{3.86}$$

by plugging the ansatz in eq. (3.84) and solving for c_i.

Endpoints of the expansion: univariate case

We make a brief digression over the maximum and the minimum orders in the expansion of ψ in eq. (3.86). We use the following notation: for a function $f(z)$ we indicate with

$$\mathrm{ord}_p f(z), \tag{3.87}$$

the order of the first term of the expansion of $f(z)$ around a point p. We indicate with

$$\mathrm{end}_p f(z), \tag{3.88}$$

the maximum order we are interested in when considering the expansion of $f(z)$ around the point p. Considering the expansion of ψ, for the minimum, *min*, we need to expand ψ so that the left hand side of eq. (3.84) has the same order as φ_L around p, otherwise the system is impossible, so we want that

$$\min[\mathrm{ord}_p(\psi) - 1, \mathrm{ord}_p(\omega) + \mathrm{ord}_p(\psi)] = \mathrm{ord}_p(\varphi_L). \tag{3.89}$$

Since ω in the case of $1-$forms can only have simple poles, because it is a logarithmic form, we have that

$$\mathrm{ord}_p(\omega) + \mathrm{ord}_p(\psi) = \mathrm{ord}_p(\psi) - 1, \tag{3.90}$$

that is the same result obtained for the term under the derivative. ω can only have simple poles because it is defined as $\omega = u'(z)/u(z)$. In fact, suppose $u(z) \sim O(\tau^n)$, then $u'(z) \sim O(\tau^{n-1})$ and therefore

$$\omega \sim O\big(\tau^{-1}\big). \tag{3.91}$$

So we have that

$$\mathrm{ord}_p(\psi) - 1 = \mathrm{ord}_p(\varphi_L), \tag{3.92}$$

therefore

$$\mathrm{ord}_p(\psi) = \mathrm{ord}_p(\varphi_L) + 1. \tag{3.93}$$

For the maximum, *max*, we need to expand ψ so that the product $\psi\varphi_R$ presents a pole in order to have a residue different from zero, that is:

$$\begin{aligned} &\mathrm{end}_p(\psi) + \mathrm{ord}_p(\varphi_R) = -1, \\ &\mathrm{end}_p(\psi) = -1 - \mathrm{ord}_p(\varphi_R). \end{aligned} \tag{3.94}$$

These consideration bring up the following formulae for the orders of the expansion of the ansatz of ψ

$$\begin{aligned} &\min = \mathrm{ord}_p(\varphi_L) + 1, \\ &\max = -1 - \mathrm{ord}_p(\varphi_R). \end{aligned} \tag{3.95}$$

The calculation of any residue around p therefore presents as a preliminary step the calculation of the orders of the functions φ_L, φ_R and ω and proceeds through if and only if it is satisfied the requirement max $\geq$ min.

3.5.2 | Multivariate case

We then proceed to the generalization of the procedure outlined in the previous section to the calculation of intersection numbers of two differential $n-$forms. As before, we start from two differential n-forms, $\left\langle \varphi_L^{(n)} \right|$ associated to I:

$$\begin{aligned} \left\langle \varphi_L^{(n)} \right| &= \varphi_L(z_1, \ldots, z_n)\, \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n = \varphi_L(z_1, \ldots, z_n)\, \mathrm{d}^n z, \\ I &= \int_{\mathcal{C}^{(n)}} \mathrm{d}^n z\ u(z_1, \ldots, z_n)\, \varphi_L(z_1, \ldots, z_n), \end{aligned} \tag{3.96}$$

and $\left| \varphi_R^{(n)} \right\rangle$ associated to $I^\star$

$$\begin{aligned} \left| \varphi_R^{(n)} \right\rangle &= \varphi_R(z_1, \ldots, z_n)\, \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n = \varphi_R(z_1, \ldots, z_n)\, \mathrm{d}^n z, \\ I^\star &= \int_{\mathcal{C}^{(n)\star}} \mathrm{d}^n z\ u^{-1}(z_1, \ldots, z_n)\, \varphi_R(z_1, \ldots, z_n). \end{aligned} \tag{3.97}$$

To extend the univariate procedure, we first need to take into account the fact that, being n-forms, they depend on n variables and not just one. The recursive procedure we are going to outline was first presented in [35]. We follow the description presented in [16]. The recursive procedure proceeds through one variable at time and, during the calculation, considers the other variables as constant parameters. It divides the problem in the calculation of intersection numbers in $z_1, \ldots, z_{n-1}$ and then the intersection number depending only on z_n. Each passage turns out to be a sort of generalization of the $1-$forms algorithm presented in Sect. 3.5.1 with an additional step.

Variables' ordering

First of all, since we will proceed one variable at time, we need to order the n variables from the innermost (the first to be integrated) to the outermost (the last to be integrated) and proceed to calculate the intersection number following this ordering. For clarity, let's assume that our n variables are ordered from the innermost to the outermost as

$$\begin{array}{ccccc} z_1 & \to & z_2 & \to \cdots \to & z_n. \\ \downarrow & & & & \downarrow \\ \text{inner} & & & & \text{outer} \end{array} \tag{3.98}$$

Imagine we are integrating the variable z_i, the dimension of the vector space is given by considering the variable we are integrating (z_i) and the ones in the inner layers with respect to it $\{z_j\}_{j<i}$ and calculating the number of solutions of the system

$$\{\hat{\omega}_k = \partial_{z_k}\omega = 0, \qquad \text{with } k \,|\, z_k \leq z_i, \tag{3.99}$$

or by calculating the number of master monomials in the variables $\{z_j\}_{j\leq i}$ following the procedure outlined in Sect. 3.3.1, which is the one we follow.

Algorithm

After having ordered the variables and calculated the number of MIs for each step, in order to calculate the intersection numbers of two n-forms we turn to the *recursive* procedure that relies on the intersection numbers of $(n-1)$-forms, that are assumed to be known, and consists in a generalization of the intersection numbers of 1−forms for the variable we are considering where in the final part it is added a resolution of a differential equation that generalize the one present in the 1−forms algorithm.

We first have to make a distinction between the *inner* and the *outer* space, the inner space consisting of the variables from the 1-st one to the $(n-1)$-th one, with dimension $\nu_{(n-1)}$ where we introduce the basis of master forms $\left\{\left\langle e_1^{(n-1)}\right|, \ldots, \left\langle e_{\nu_{(n-1)}}^{(n-1)}\right|\right\}$ and the dual basis $\left\{\left|h_1^{(n-1)}\right\rangle, \ldots, \left|h_{\nu_{(n-1)}}^{(n-1)}\right\rangle\right\}$. The outer space instead consists of the n-th variable we are considering, say z_n, and has dimension ν_n.

$$\underbrace{z_1 \to z_2 \to \ldots z_{n-1}}_{\text{inner space}} \to \underbrace{z_n}_{\text{outer space}} \tag{3.100}$$

The differential n-forms $\left\langle \varphi_L^{(n)}\right|$ and $\left|\varphi_R^{(n)}\right\rangle$ can be projected onto the $(n-1)$-forms space as

$$\begin{aligned}
\left\langle \varphi_L^{(n)}\right| &= \sum_{i=1}^{\nu_{(n-1)}} \varphi_{L,i} \left\langle e_i^{(n-1)}\right| \wedge \mathrm{d}z_n, \\
\left|\varphi_R^{(n)}\right\rangle &= \sum_{i=1}^{\nu_{(n-1)}} \varphi_{R,i} \left|h_i^{(n-1)}\right\rangle \wedge \mathrm{d}z_n,
\end{aligned} \tag{3.101}$$

with

$$\begin{aligned}
\varphi_{L,i} &= \varphi_{L,i}(z_n), \\
\varphi_{R,i} &= \varphi_{R,i}(z_n),
\end{aligned} \tag{3.102}$$

and the sums

$$\begin{aligned}\sum_{i=1}^{\nu_{(n-1)}} \varphi_{L,i} \left\langle e_i^{(n-1)} \right| &= \left\langle \varphi_L^{(n-1)} \right|, \\ \sum_{i=1}^{\nu_{(n-1)}} \varphi_{R,i} \left| h_i^{(n-1)} \right\rangle &= \left| \varphi_R^{(n-1)} \right\rangle,\end{aligned} \tag{3.103}$$

are the decomposition on the inner space of $\left\langle \varphi_L^{(n-1)} \right|, \left| \varphi_R^{(n-1)} \right\rangle$. The coefficients are obtained via a projection similar to the one used for the intersections of 1-forms: we first introduce a metric on the inner space formed by the intersection numbers of the inner space basis and dual basis vectors:

$$\left(\mathbf{C}_{(n-1)}\right)_{ij} = \left\langle e_i^{(n-1)} \middle| h_j^{(n-1)} \right\rangle, \tag{3.104}$$

and we calculate the coefficients as

$$\begin{aligned}\varphi_{L,i}\, \mathrm{d}z_n &= \left\langle \varphi_L^{(n)} \middle| h_j^{(n-1)} \right\rangle \left(\mathbf{C}^{-1}_{(n-1)}\right)_{ji}, \\ \varphi_{R,i}\, \mathrm{d}z_n &= \left(\mathbf{C}^{-1}_{(n-1)}\right)_{ij} \left\langle e_j^{(n-1)} \middle| \varphi_R^{(n)} \right\rangle.\end{aligned} \tag{3.105}$$

The intersection number of two $n-$forms is given by

$$\left\langle \varphi_L^{(n)} \middle| \varphi_R^{(n)} \right\rangle = \sum_{p \in \mathcal{P}_{\mathbf{\Omega},n}} \mathrm{Res}_{z_n=p}\Big(\psi_i^{(n)} \left(\mathbf{C}_{(n-1)}\right)_{ij} \varphi_{R,j} \Big). \tag{3.106}$$

$\psi_i^{(n)}$, analogously to the 1-forms case, is the local solution around the pole p of the equation

$$\partial_{z_n} \psi_i^{(n)} + \psi_j^{(n)}\, \mathbf{\Omega}_{ji}^{(n)} = \varphi_{L,i}, \qquad (i = 1, \ldots, n). \tag{3.107}$$

This time is an $\nu_{(n-1)}-$dimensional vector. In order to apply eq. (3.106) it is not necessary to find a global solution but it is sufficient to have an ansatz of it as a holomorphic Laurent series expansion around the pole p in consideration:

$$\psi_k^{(n)} = \sum_{i=\text{global-min}}^{\text{global-max}} c_{ki} (z_n - p)^i + \mathcal{O}\big((z_n - p)^{\text{global-max}+1}\big). \tag{3.108}$$

$\mathbf{\Omega}_{ji}^{(n)}$ is an $\nu_{(n-1)} \times \nu_{(n-1)}$ matrix defined as

$$\mathbf{\Omega}_{ji}^{(n)} = \left\langle (\partial_{z_n} + \omega_n) e_j^{(n-1)} \middle| h_k^{(n-1)} \right\rangle \left(\mathbf{C}^{-1}_{(n-1)}\right)_{ki}. \tag{3.109}$$

And the sum in eq. (3.106) runs over the set $\mathcal{P}_n$:

$$\mathcal{P}_{\mathbf{\Omega},n} = \{z \,|\, z \text{ is a pole of } \mathbf{\Omega}^{(n)}\} \cup \{\infty\}, \tag{3.110}$$

Combining eq. (3.105) and eq. (3.106) we notice that we can rewrite the formula for the intersection of two n-forms as

$$\left\langle \varphi_L^{(n)} \middle| \varphi_R^{(n)} \right\rangle = \sum_{p \in \mathcal{P}_{\mathbf{\Omega},n}} \text{Res}_{z_n=p}\left(\psi_i^{(n)} \left\langle e_i^{(n-1)} \middle| \varphi_R^{(n)} \right\rangle \right). \tag{3.111}$$

Endpoints of the expansion: multivariate case

In order to find a local solution for $\psi^{(n)}$ we need to expand it as we did in the case of 1-forms. There is an additional complication due to the fact that eq. (3.107) is a vectorial equation. Each term $\psi_i^{(n)}$ enters all the equations because of the product $\psi^{(n)} \cdot \mathbf{\Omega}^{(n)}$. A conservative solution can be found by expanding each $\psi_i^{(n)}$ with the same maximum and minimum, that are found as global maximum, *global-max*, and global minimum, *global-min*, of the equations.

The global-min is given by the same procedure outlined in Sect. 3.5.1, that we can summarize as

$$\min[\text{ord}_p(\psi_i) - 1, \text{ord}_p(\mathbf{\Omega}_{ji}) + \text{ord}_p(\psi_j)] = \text{ord}_p(\varphi_{L,i}), \tag{3.112}$$

this time $\mathbf{\Omega}_{ji}$ can have higher poles than simple ones and since we look for a common global minimum for each component of ψ, we can calculate it as

$$\text{ord}_p(\psi_i) = \min_k[\text{ord}_p(\varphi_{L,k})] - \max_{kj}[\text{ord}_p(\mathbf{\Omega}_{kj})], \quad (i,j,k = 1,\ldots,n). \tag{3.113}$$

while for the global-max we need to expand each $\psi_i^{(n)}$ so that its product with $\left(\mathbf{C}_{(n-1)}\right)_{ij} \varphi_{R,j}$ has a pole. So that:

$$\text{end}_p(\psi_i) = \max_k\left[\left(\mathbf{C}_{(n-1)}\right)_{kj} \varphi_{R,j}\right], \qquad (i,j,k = 1,\ldots,n). \tag{3.114}$$

and the sum over j is implicit.

3.5.3 | Summary

We summarize the necessary steps in order to calculate the intersection numbers of two differential forms, starting from the univariate case.

Univariate case

Suppose we are integrating in the variable z:

- find the poles of the function ω;
- consider the order of φ_L, φ_R and ω around each one of the poles and calculate the orders for the ψ expansion;
- if max<min, then the contribution of the considered pole to the intersection number is zero. Otherwise, build the differential eq. (3.84) and solve it close to the pole using the ansatz in eq. (3.86);
- calculate the residue of the product $\psi\varphi_R$ around the pole in consideration.

Multivariate case

We need a preliminary step:

- order the variables from the innermost to the outermost;
- for each "layer" calculate the number of master forms.

Then, one can calculate intersection numbers of two $n-$forms by relying on the already-calculated intersection numbers of $(n-1)-$forms

- project $\varphi_L^{(n)}$ and $\varphi_R^{(n)}$ on the basis of $(n-1)-$forms;
- find the poles of the function $\mathbf{\Omega}^{(n)}$;
- for each pole calculate the order of each term in $\varphi_{L,i}, \varphi_{R,i}$ and $\mathbf{\Omega}_{ij}^{(n)}$. Calculate global-max and global-min. If global-max<global-min, the the contribution of the considered pole to the intersection number is zero;
- if not, build the differential eq. (3.106) and solve it close to the pole using the ansatz in eq. (3.108);
- the contribution to the intersection number is given by the residue of the product $\psi \cdot \tilde{\varphi}_R$ around the pole in consideration.

4 | A Rational algorithm for intersection numbers

In this chapter we introduce the main novelty presented in this thesis: a purely rational algorithm for the computation of intersection numbers. We give an account of the key ideas behind the algorithm and discuss the implementation details. The algorithm is implemented in a MATHEMATICA routine and has been successfully tested on a variety of diagrams, as summarized in Ch. 5.

4.1 | Eluding non-rational poles

The algorithm for the calculation of intersection numbers presented in Ch. 2 allows for the presence of non-rational contributions, such as square roots, appearing when looking at the poles of the functions ω and Ω and entering the calculations because at a certain stage we write the solution of a differential equation as an expansion in series around them. The intersection number resulting from the calculation, instead, is a rational function of the kinematic invariants and of the dimensional regulator. Therefore, a cancellation of the non-rational terms must happen in intermediate stages of the calculation. In computer algebra systems, if possible, it is preferable to work with algorithms avoiding the presence of square roots, to have better computational efficiency and performance. Such algorithms are called purely rational, as they do not present algebraic extensions in any step of the calculation. Another reason is the possibility to use purely rational algorithms in combination with finite-fields based technologies (see App. F), a widely used tool to achieve analytical results in a fast and exact way, that recently has found applications in the context of theoretical predictions in high energy physics [42, 43, 31, 23, 29]. Before presenting our rational algorithm we first need to introduce some tools that are necessary to avoid the explicit appearance of non-rational contributions. In the calculation of intersection numbers

we are faced with the necessity to calculate the sum of the residues of the product of two functions, one of them, ψ, being the solution of a differential equation, calculated as the series expansion around the poles of the function ω (or Ω in the multivariate case). The alarming point is that these poles can be non-rational. To avoid dealing with them explicitly we instead consider the fact that they combine in order to satisfy rational polynomial equations in the form $p(z) = 0$. We therefore try to calculate the contribution to the residues of the non-rational poles "all at once" by considering expansions around the rational polynomial they satisfy. In order to do this, we exploit techniques such as the calculation of the polynomial reminder modulo $p(z)$, the univariate global residue theorem and we draw an analogy from $p-$adic numbers to obtain an expansion of a generic rational function with respect to polynomials prime over the field of rational numbers.

4.1.1 | Polynomial reminder modulo $p(z)$

Since it is a concept that we will make large use of, we begin by introducing the concept of polynomial reminder modulo a univariate polynomial $p(z)$ in the variable z. We consider, as an example, a polynomial $p(z)$ of degree deg p = 2, so that it can be explicitly written as $p(z) = az^2 + bz + c$, with $a, b, c \in \mathbb{Q}$. Moreover, we assume it has two distinct real solutions, z_1 and z_2, that can be non-rational. The polynomial reminder modulo $p(z)$ of a function $f(z)$ consists in the recursive substitution of

$$z^2 \to \frac{-bz - c}{a}, \tag{4.1}$$

which is valid on the solutions of $p(z)$, namely when evaluating $f(z_1)$ and $f(z_2)$. This means that whenever we find terms with degree ≥ 2 in z we substitute eq. (4.1) until we end up with terms that are at most linear in z. For example, if we have a term proportional to z^3 we need to substitute two times:

$$\begin{aligned} z^3 &= z \cdot z^2 && \text{eq. (4.1)} \\ &= z \cdot \left(\frac{-bz - c}{a}\right) = -\frac{bz^2}{a} - \frac{cz}{a} && \text{eq. (4.1)} \\ &= -\frac{b}{a}\left(\frac{-bz - c}{a}\right) - \frac{cz}{a} \\ &= \frac{b^2 z + bc}{a^2} - \frac{cz}{a}. \end{aligned} \tag{4.2}$$

We introduce the following notation for the polynomial reminder modulo $p(z)$ of a generic function $f(z)$

$$\lfloor f(z) \rfloor_{p(z)} \equiv f(z) \mod p(z). \tag{4.3}$$

The polynomial reminder modulo $p(z)$ can be generalized to the case when $f(z)$ is a rational function, $f(z) = n(z)/d(z)$. It consists in rewriting $1/d(z)$ as its so called *multiplicative inverse*, namely a function $\tilde{d}(z)$ that multiplied by $d(z)$ gives 1 modulo $p(z)$

$$\tilde{d}(z) = \frac{1}{d(z)} \mod p(z) \iff \tilde{d}(z)d(z) = 1 \mod p(z). \tag{4.4}$$

For a more in-depth explanation of polynomial division and the multiplicative inverse see App. A.

4.1.2 | A case study

We consider an explicit example with the intent of showing that, in order to calculate the contribution of a given set of irrational poles to the intersection number, it is not necessary to know the exact position of the irrational poles, but it suffices to know the rational polynomial they satisfy. A more general and systematic approach for achieving this will be presented in Sect. 4.3. We highlight that all the calculations are performed by keeping the irrational poles symbolic, without giving them an explicit numerical value, and only at the end we will see how they combine giving rise to rational quantities that are effectively evaluated.

For simplicity, we consider the case of univariate intersection numbers in the variable z. Suppose we have at most a single pole with respect to the solutions z_1, z_2 of $p(z)$ in all the functions of our interest $\omega, \varphi_L, \varphi_R$. In that case, we can write $\omega, \varphi_L, \varphi_R$ as

$$\begin{aligned}
\omega &= \frac{\bar{\omega}}{p(z)} = \frac{\bar{\omega}}{a(z-z_1)(z-z_2)}, \\
\varphi_L &= \frac{\bar{\varphi}_L}{p(z)} = \frac{\bar{\varphi}_L}{a(z-z_1)(z-z_2)}, \\
\varphi_R &= \frac{\bar{\varphi}_R}{p(z)} = \frac{\bar{\varphi}_R}{a(z-z_1)(z-z_2)}.
\end{aligned} \tag{4.5}$$

The expansion highlights the order of the poles of the functions around the solutions of $p(z)$ in the denominator. Considering the expansion around one of the solutions of $p(z)$, for example $z = z_1$ (but the exact same reasoning can be applied to $z = z_2$), we make the substitution $\tau = z - z_1$. ψ is expanded

around τ with endpoints max = min = 0, and we can write $\psi = \psi_0 + \mathcal{O}(\tau)$. We get the following differential equation:

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau} + \frac{\omega_\tau}{\tau}\right)\psi_0 = \frac{\varphi_{L,\tau}}{\tau}, \qquad \tau = z - z_1, \tag{4.6}$$

where we wrote $\omega_\tau = \bar{\omega}/(z - z_2)$ and $\varphi_{L,\tau} = \bar{\varphi}_L/(z - z_2)$ because of the τ expansion. Notice that in these expansions we are not considering the explicit value of the pole z_1 (z_2) but we are treating them as symbolic quantities.

Then ψ is obtained as

$$\psi_0 = \frac{\varphi_{L,\tau}}{\omega_\tau} = \frac{\bar{\varphi}_L(z_1)}{\bar{\omega}(z_1)}, \tag{4.7}$$

where the last equivalence comes from the fact that in this case we had a simple pole in both functions and this lead to the cancellation of the factors $(z - z_2)$. Then the residue is expressed as

$$\begin{aligned} \mathrm{Res}_{z=z_1}(\psi(z)\varphi_R(z)) &= \mathrm{Res}_{z=z_1}\left(\psi(z)\frac{\bar{\varphi}_R(z)}{a(z - z_1)(z - z_2)}\right) \\ &= \psi_0\frac{\bar{\varphi}_R(z_1)}{a(z_1 - z_2)} \\ &= \frac{\bar{\varphi}_L(z_1)\bar{\varphi}_R(z_1)}{a\bar{\omega}(z_1)(z_1 - z_2)}. \end{aligned} \tag{4.8}$$

Then, we perform the polynomial reminder modulo $p(z_1)$ on the function $(\bar{\varphi}_L\,\bar{\varphi}_R)/\bar{\omega}$ to obtain $\lfloor\bar{\varphi}_L\bar{\varphi}_R\bar{\omega}'\rfloor_{p(z)}$, where $\bar{\omega}'$ is the multiplicative inverse of $1/\bar{\omega}$. This step is necessary since, when quantities that have already undergo polynomial division with respect to $p(z)$ are multiplied, we can have the appearance of terms proportional to z^2, that still need to undergo polynomial division. After this passage, we obtain an expression linear in z_1:

$$\mathrm{Res}_{z=z_1}(\dots) = \frac{\lfloor\bar{\varphi}_L\bar{\varphi}_R\bar{\omega}'\rfloor_{p(z)}\big|_{z=z_1}}{a(z_1 - z_2)} = \frac{c_0 + c_1 z_1}{a(z_1 - z_2)}, \qquad c_0, c_1 \in \mathbb{Q}. \tag{4.9}$$

Doing the same reasoning again around the pole $z = z_2$, we get an analogous expression, this time linear in $z = z_2$. In the end, as prescribed by the algorithm, we need to sum over the contributions from the solutions of $p(z)$

and therefore we sum the two terms

$$\begin{aligned}\frac{\bar{\varphi}_L(z_1)\bar{\varphi}_R(z_1)}{a\bar{\omega}(z_1)(z_1-z_2)}+\frac{\bar{\varphi}_L(z_2)\bar{\varphi}_R(z_2)}{a\bar{\omega}(z_2)(z_2-z_1)} &= \frac{\lfloor\bar{\varphi}_L\bar{\varphi}_R\bar{\omega}'\rfloor_{p(z)}\big|_{z=z_1}}{a(z_1-z_2)}+\frac{\lfloor\bar{\varphi}_L\bar{\varphi}_R\bar{\omega}'\rfloor_{p(z)}\big|_{z=z_2}}{a(z_2-z_1)}\\ &= \frac{c_0+c_1z_1}{a(z_1-z_2)}+\frac{c_0+c_1z_2}{a(z_2-z_1)}\\ &= \frac{c_1}{a}.\end{aligned} \tag{4.10}$$

In the end we see that the only terms containing non-rational poles are proportional to $z_1 - z_2$ and they simplify against the denominator. This allows us to have our intersection number written as a combination of only rational factors, without introducing explicitly the non-rational solutions. This can be done in computer algebra systems by the calculation of polynomial reminders modulo $p(z)$ and symbolically introducing the solutions z_1 and z_2 without giving their explicit expression that contains square roots. After successive polynomial division the expression simplifies in a way that does not contain the single solutions but only rational combinations of them.

This example explicitly shows how non-rational poles combine to give rise to rational contributions to the intersection number and that the exact position of the non-rational poles it is not a necessary information in order to calculate their contribution to the intersection number. We now generalize this example to ultimately obtain a rational algorithm calculating intersection numbers. The generalization conceptually is the same as the example hereby presented, and has as its core step the definition of expansion of a rational function with respect to a polynomial irreducible over $\mathbb{Q}$, that we refer to as $p(z)-$adic expansion. We define this operation in analogy to the $p-$adic expansion of a rational number, of which we now give a brief account.

4.2 | $p-$adic numbers

We review a few concepts about $p-$adic numbers, well-known mathematical objects that have recently found some applications in loop calculations [13] and that we use as foundations for building an analogy justifying the procedure of expansion of a rational function in series of polynomials. We indicate the field of $p-$adic numbers with $\mathbb{Q}_p$, where p indicates a prime number. The $p-$adic expansion of any integer number can be written as a finite expansion that is equal to its representation in base p. For example, the $7-$adic expansion of 18 can be written as

$$18 = 4 + 2 \cdot 7, \tag{4.11}$$

so $18 = 24$ in $7-$adics. Instead, considering rational numbers $r = n/d$, their $p-$adic representation can be written as a formal series in p with integer coefficients

$$r = \sum_{i=-k}^{\infty} a_i p^i = a_{-k}p^{-k} + \cdots + a_{-1}p^{-1} + a_0 + a_1 p + \ldots, \tag{4.12}$$

where the integers a_is lie in the range $[0, p-1]$. The expansion is obtained by first checking whether r presents factors of p the denominator or if $\gcd(p, d) = 1$. In the former case, we factor out the power p^k in the denominator to get an expression of the form $r = p^{-k}\frac{n}{d}$. Then, we use repeated division with reminder on n/d to obtain the coefficient of the expansion. The division step is given by first calculating the multiplicative inverse of $1/d$ with the extended Euclidean algorithm. It exploits the Bézout identity, which, starting from $r = \frac{n}{d}$ with p, d coprime, implies that exists u, t that satisfy $td + up = 1$. Therefore we can write r as

$$r = \frac{n}{d} = \frac{n}{d}(td + up) = tn + p\frac{un}{d}. \tag{4.13}$$

Then, we do the Euclidean division of nt by p obtaining

$$nt = qp + a, \tag{4.14}$$

with reminder $0 \leq a < p$. So the first term in the expansion is given by a

$$\begin{aligned} r &= (qp + a) + p\frac{un}{d} \\ &= a + p\frac{qd + un}{d} \\ &= a + pr'. \end{aligned} \tag{4.15}$$

The iteration proceeds considering the quotient r' as the new r and repeating the procedure. We present an example to clarify the procedure.

Example 4.2.1. We calculate the first terms of the $5-$adic expansion of $\frac{1}{2}$, following [20]. Since 2 and 5 are coprime we can use the Bézout identity to find u, t such that

$$2t + 5u = 1, \tag{4.16}$$

that is satisfied for $t = 3$ and $u = -1$ since

$$2 \cdot 3 - 5 \cdot 1 = 6 - 5 = 1. \tag{4.17}$$

Therefore we can easily divide $\frac{1}{2}$ by 5 as

$$\frac{1}{2} = \frac{1}{2}(3 \cdot 2 - 5) = \frac{-1}{2} \cdot 5 + 3, \tag{4.18}$$

and we take 3 as the first term in the expansion. We continue treating the quotient $-1/2$ as the new rational number to expand. We find the Bézout identity for $-1/2$

$$-2t + 5u = 1, \tag{4.19}$$

satisfied for $t = 2, u = 1$

$$-2 \cdot 2 + 5 = -4 + 5 = 1, \tag{4.20}$$

then we can divide $-1/2$ by 5 to obtain

$$-\frac{1}{2} = -\frac{1}{2}(-2 \cdot 2 + 5) = \frac{-1}{2} \cdot 5 + 2. \tag{4.21}$$

Therefore, the second term in the expansion is 2. The second quotient is again $-1/2$ so successive division will always lead to reminder 2. We can therefore write the 5−adic expansion of $1/2$ as

$$\frac{1}{2} = 3 + 2 \cdot 5 + 2 \cdot 5^2 + \cdots = 3 + \sum_k 2 \cdot 5^k. \tag{4.22}$$

This may seem strange but notice that it works: if we multiply by 2 we get 1 on the left hand side while on the right hand side we obtain $2 \cdot 3 = 6$ but the coefficient of the expansion must lie in the range $[0, 4]$ so we write 1 and carry a 1 to be added to the successive terms. In the second term we get $2 \cdot 2 + 1 = 5$ so again we write 0 and we carry 1. The same for the third term and for all the successive.

4.3 | *p(z)*−adic rational functions

In close analogy to $p-$adic numbers, we build a procedure for the expansion of rational functions with respect to a polynomial $p(z)$ prime over $\mathbb{Q}$, that is, irreducible over $\mathbb{Q}$. In this thesis we refer to it as $p(z)-$adic expansion of rational functions or simply as polynomial expansion.

Consider a function $f(z)$ and a polynomial $p(z)$ irreducible over $\mathbb{Q}$ with degree deg p. We want to express $f(z)$ as a series expansion with respect to $p(z)$, in the form

$$f(z) = \sum_{i=\min}^{\infty} c_i(z) p^i(z), \tag{4.23}$$

where the coefficients $c_i(z)$ are given by

$$c_i(z) = \sum_{j=0}^{\deg \mathrm{p}-1} c_{ij} z^i. \tag{4.24}$$

To compute this, we start with the case where $f(z)$ is a polynomial. We use repeated polynomial division with reminder to obtain the coefficients of the expansion. To obtain the first term in the expansion, $c_0(z)$, it is sufficient to write $f(z)$ as

$$f(z) = c_0(z) + q_1(z)p(z), \tag{4.25}$$

where $q(z)$ is the quotient of the division of $f(z)$ modulo $p(z)$ and $c_0(z)$ is the remainder. Then, we consider $q_1(z)$ as our "new" $f(z)$ and we repeat the polynomial division to get the successive terms

$$q_1(z) = c_1(z) + q_2(z)p(z), \tag{4.26}$$

where $c_1(z)$ is the term of order $\mathcal{O}(p(z))$ in the expansion. We continue with the same procedure on $q_1(z)$ and so on

$$\begin{aligned} q_2(z) &= c_2(z) + q_3(z)p(z), \\ q_3(z) &= c_3(z) + q_4(z)p(z), \\ &\dots \quad \dots \quad \dots \quad \dots . \end{aligned} \tag{4.27}$$

This procedure allows us to compute the coefficients $c_i(z)$ up to the order we need in our expansion.

If, instead, the function $f(z)$ is a rational function, we can write it as

$$f(z) = \frac{n(z)}{d(z)}, \quad \text{assuming} \quad \gcd(d(z), p(z)) = 1. \tag{4.28}$$

If the assumption $\gcd(d(z), p(z)) = 1$ is not satisfied, and therefore $f(z)$ has a singularity in $p(z)$, we first need to write it in the form

$$f(z) = \frac{n(z)}{d(z)} \frac{1}{p(z)^k}, \tag{4.29}$$

obtained by factoring out the singularity in the denominator. Then, we can calculate its polynomial expansion by exploiting the concept of multiplicative inverse modulo a polynomial $p(z)$ and successive divisions with respect to $p(z)$. The multiplicative inverse can be calculated via the Euclidean algorithm or through an ansatz and allows us to write $1/d(z)$ as

$$\tilde{d}(z) = \frac{1}{d(z)} \mod p(z) \tag{4.30}$$

with $\tilde{d}(z)$ a polynomial of degree deg p $-$ 1 (for more details see App. A). Then we can write $f(z)$ as

$$f(z) \mod p(z) = n(z)\tilde{d}(z) \mod p(z), \tag{4.31}$$

and calling

$$\tilde{f}(z) \equiv f(z) \mod p(z), \tag{4.32}$$

we can write

$$\tilde{f}(z)d(z) \mod p(z) = n(z) \mod p(z). \tag{4.33}$$

Therefore, we can write the numerator $n(z)$ decomposed as

$$n(z) = \tilde{f}(z)d(z) + q(z)p(z), \tag{4.34}$$

where $q(z)$ is the quotient of the division of $n(z) - \tilde{f}(z)d(z)$ with respect to the polynomial $p(z)$. In this way we obtained the first term of the expansion of $f(z)$, namely $\tilde{f}(z)$,

$$f(z) = \frac{n(z)}{d(z)} = \frac{\tilde{f}(z)d(z) + q(z)p(z)}{d(z)} = \tilde{f}(z) + \frac{q(z)p(z)}{d(z)}, \tag{4.35}$$

in the case of $f(z)$ being a rational function.

We now look at the expansion we want to obtain as showed in eq. (4.23). Since we factored out the terms proportional to $p(z)$ in $d(z)$, we have that $d(z)$ and $p(z)$ satisfy $\gcd(d(z), p(z)) = 1$, so the first term in the expansion is of order $\mathcal{O}(p(z)^0)$

$$f(z) = \sum_{i=0}^{\infty} c_i(z)p^i(z). \tag{4.36}$$

The first term can be obtained as

$$c_0(z) = \frac{n(z)}{d(z)} \mod p(z), \tag{4.37}$$

so we can write $n(z)$ following eq. (4.34)

$$n(z) = c_0(z)d(z) + q_1(z)p(z), \tag{4.38}$$

and we can calculate the difference between $f(z)$ and $c_0(z)$, that amounts to

$$f(z) - c_0(z) = \frac{n(z)}{d(z)} - c_0(z) = \frac{n(z) - c_0(z)d(z)}{d(z)} = \frac{q_1(z)p(z)}{d(z)}. \tag{4.39}$$

We continue, calculating the first term as

$$c_1(z) = \frac{q_1(z)}{d(z)} \mod p(z), \tag{4.40}$$

that amounts to considering $q_1(z)/d(z)$ as the new $f(z)$. We can calculate the difference between $q_1(z)/d(z)$ and $c_1(z)$ as

$$\frac{q_1(z)}{d(z)} - c_1(z) = \frac{q_1(z) - c_1(z)d(z)}{d(z)} = \frac{q_2(z)p(z)}{d(z)}, \tag{4.41}$$

that allows us to obtain $c_2(z)$

$$c_2(z) = \frac{q_2(z)}{d(z)} \mod p(z). \tag{4.42}$$

We can iterate this procedure to obtain all the terms we need in our expansion. Then, for the purpose of our calculations, we consider the limit of our expansion for $p(z) \to 0$. This is because in the calculation of intersection numbers we often need to expand around the poles of either ω or Ω. The irrational poles, that we ultimately want to avoid, satisfy polynomials $p(z)$ irreducible over $\mathbb{Q}$. Therefore our expansion around $p(z)$ for $p(z) \to 0$ ultimately means to consider the expansion of a function near all the poles satisfying $p(z) = 0$ at once.

4.3.1 | Symbolic shortcut

The repeated divisions can be avoided by means of a shortcut to compute the $p(z)-$adic expansion of a rational function. Assume the degree of the polynomial $p(z)$ is n. The whole procedure explained above can be easily done in two steps. We first perform the polynomial reminder with respect to $p(z) - \delta$ where δ represent a symbolic small quantity that will later be expanded in series around 0. In this way what we really are doing to calculate the polynomial reminder is to recursively substitute each z^n with

$$z^n = -\frac{q_{n-1}}{q_n} z^{n-1} - \cdots - \frac{q_1}{q_n} z - \frac{q_0}{q_n} + \delta, \tag{4.43}$$

so we will obtain an expression that has a dependence on δ and whose degree with respect to z is less or equal to $n-1$. We highlight that this procedure actually amounts to a simple redefinition of $p(z) \equiv \delta$[1]. In this way we can expand in series for arbitrary small $p(z)$ by expanding in series around $\delta \to 0$. The notation for the polynomial reminder introduced in Sect. 4.1 in this case becomes

$$\lfloor f(z) \rfloor_{p(z)-\delta} \equiv f(z) \mod p(z) - \delta, \tag{4.44}$$

[1]This redefinition eliminates the explicit dependence of $p(z)$ from z, therefore needs to be done with care. For example, when we need to calculate the $p(z)-$adic expansion of the derivative in z of a function $f(z)$ we first need to perform the derivative and then take the polynomial reminder with respect to $p(z) - \delta$.

and effectively amounts to the substitution $p(z) = \delta$. We then expand the obtained expression in series for small δ,

$$\lfloor f(z) \rfloor_{p(z)-\delta} = \sum_{i}^{\max} \sum_{j=0}^{\deg \mathrm{p}-1} c_{ij} z^j \delta^i + \mathcal{O}\big(\delta^{\max+1}\big). \tag{4.45}$$

In this way we obtained the $p(z)-$adic expansion of a generic rational function around $p(z)$ up to the order we need.

We summarize the two steps necessary to implement the shortcut for the $p(z)-$adic expansion of a rational function $f(z)$:

- from $f(z)$ obtain $\lfloor f(z) \rfloor_{p(z)-\delta}$ by taking the polynomial reminder w.r.t. $p(z) - \delta$, i.e. substituting $p(z)$ with δ;
- expansion of $\lfloor f(z) \rfloor_{p(z)-\delta}$ for small δ up to $\mathcal{O}(\delta^{\max+1})$.

This procedure is in close analogy with the $p-$adic expansion of a rational number. For $p-$adic numbers we expand a rational number with respect to a prime p, our expansion allows to expand a rational function $f(z)$ with respect to a polynomial $p(z)$ that is prime over $\mathbb{Q}$. We can thus formally take the limit $p(z) \to 0$, considering at once the expansion around all the z_i satisfying $p(z) = 0$. The point is that such expansion can be used when dealing with a polynomial presenting non-rational roots in its solutions. We consider the expansion around the whole polynomial, and, for the limit of vanishing polynomial, we effectively account for all the solutions at once, instead of multiple expansions around its non-rational roots.

4.4 | Application to intersection numbers

The generic outline of the algorithm for calculating n-forms intersection numbers remains similar but with some modifications when looking for the poles of the functions ω and Ω and in the operations carried out for calculating the residue of the product of $\psi\varphi_R$ (or $\psi \cdot \tilde{\varphi}_R$), that varies depending on the nature of the pole in consideration. Let us first consider the part concerning the search of the poles of the functions ω and Ω. Suppose we are integrating the variable z. If we are in the univariate case, since ω is a function, we simply factor its denominator over $\mathbb{Q}$. If, instead, we are in the multivariate case, since Ω is a matrix, we run through all of its entries and we factor over $\mathbb{Q}$ the denominator of each one of them. We thus group all distinct factors. In both cases we end up with a list of factors that are irreducible over $\mathbb{Q}$. For

convenience we call each factor $p_i(z)$ and the whole list, both in the univariate and in the multivariate case, takes the form

$$\text{factors} = \{p_1(z), \ldots, p_n(z)\}. \tag{4.46}$$

We calculate the intersection number $\langle\varphi_L|\varphi_R\rangle$ as the sum of all the contributions from all the factors $p_i(z)$ in the list. We indicate the contribution given from the factor $p_i(z)$ with $\langle\varphi_L|\varphi_R\rangle_{p_i(z)}$, where we highlight that looking at the contribution from $p_i(z)$ means that we are, at once, considering the contribution of all its roots,

$$\langle\varphi_L|\varphi_R\rangle = \sum_{p_i(z)\in\text{factor}} \langle\varphi_L|\varphi_R\rangle_{p_i(z)}\,. \tag{4.47}$$

We then check each $p_i(z)$ to see what is its degree with respect to z. We consider all other variables as constant parameters just like the dimensional regulator, the regulators ρ_i and the kinematic invariants. We have two possibilities, namely whether the $p_i(z)$ we are considering is linear or not in the integration variable z. For convenience, in the following we call a generic factor $p_i(z)$ simply $p(z)$.

Factor linear in z

If $p(z)$ is linear with respect to z then it has the form

$$p(z) = z - q_0, \tag{4.48}$$

where q_0 is a rational number. This case does not need any further inquiry and we can safely proceed with the algorithm described in Sect. 3.5, calculating the contribution that this rational pole gives to the sum of the residues for the intersection number in consideration.

Factor non linear in z

The other possibility consist in $p(z)$ having a degree higher than 1 in the integration variable z. Heuristically, we have noticed that the degree is usually 2, but there are no reasons we should restrict to this case. Supposing the factor $p(z)$ has degree n, we can write it in the form

$$p(z) = q_n z^n + q_{n-1} z^{n-1} + \cdots + q_0, \tag{4.49}$$

where each q_i is a rational number. This is the case where the factorization over $\mathbb{R}$ or $\mathbb{C}$ would lead to the presence of algebraic extensions, as the

factorization over $\mathbb{Q}$ could not reduce the polynomial to the product of linear factors. This is the case where we need to make some changes in the algorithm in order to prevent the appearance of non-rational contributions.

As explained above, our general line of reasoning will be to consider separately each $p(z)$, which is a polynomial irreducible over $\mathbb{Q}$, and expand our functions of interest with respect to it, using the concept of $p(z)-$adic expansion explained in Sect. 4.3. We will then exploit the univariate global residue theorem [56] (for more details see App. A) in order to obtain the residue of the product $\psi\varphi_R$ (or $\psi_i\tilde{\varphi}_{R,i}$ in the multivariate case) around the non-rational solutions of the polynomial in consideration. The univariate global residue theorem [56] allows us to do it without ever explicitly write the solutions.

We begin by describing what happens in the univariate case. We first check whether the irreducible factor $p(z)$ presents a *leading coefficient*, l_c, different than 1 in front of its higher degree variable, z^n. If so, we save it for later use,

$$l_c = q_n. \tag{4.50}$$

We then find the order of the $p(z)-$adic expansion of the functions φ_L, φ_R and ω with respect to z, obtained by first taking the polynomial reminder with respect to the polynomial $p(z) - \delta$, where δ is an auxiliary parameter, as we explained in Sect. 4.3.1 and then expanding in series for small δ. We find the endpoints max and min in the univariate case given by

$$\begin{aligned} &\text{min} = \mathcal{O}(\varphi_L) - \mathcal{O}(\omega), \\ &\text{max} = -\mathcal{O}(\varphi_R) - 1. \end{aligned} \tag{4.51}$$

So that in the univariate case the expansion for ψ in terms of $p(z)$ looks like

$$\psi = \sum_{i=\text{min}}^{\text{max}} \sum_{j=0}^{\text{deg p}-1} c_{ij} z^j p^i(z) + \mathcal{O}\big(p(z)^{\text{max}+1}\big). \tag{4.52}$$

We build the univariate differential equation, eq. (3.84), that for convenience we rewrite here

$$\frac{\partial}{\partial z}\psi + \omega\psi = \varphi_L, \tag{4.53}$$

and we perform a $p(z)-$adic expansion around $p(z) - \delta$ (as explained in Sect. 4.3.1) in order to obtain

$$\Big\lfloor \frac{\partial}{\partial z}\psi \Big\rfloor_{p(z)-\delta} + \lfloor \omega\psi \rfloor_{p(z)-\delta} = \lfloor \varphi_L \rfloor_{p(z)-\delta}. \tag{4.54}$$

Then, we expand in series for $\delta \to 0$ and we solve it for the coefficients c_{ji}. We substitute the obtained solutions for c_{ji} in $\lfloor \psi \rfloor_{p(z)-\delta}$ and we multiply it by

φ_R. We $p(z)-$adic expand the product $\psi\varphi_R$, namely we take the polynomial reminder of the product $\lfloor\psi\rfloor_{p(z)-\delta}\,\varphi_R = \lfloor\psi\varphi_R\rfloor_{p(z)-\delta}$ and we expand in series for small δ up to order $\mathcal{O}(\delta^0)$

$$\big\lfloor\psi\varphi_R\big\rfloor_{p(z)-\delta} = \sum_{i}^{-1}\sum_{j=1}^{\deg\,\mathrm{p}-1}\tilde{c}_{ij}z^j\delta^i + \mathcal{O}\big(\delta^0\big), \tag{4.55}$$

where the sum over i starts from the order of $\lfloor\psi\varphi_R\rfloor_{p(z)-\delta}$ around $\delta\to 0$. Following the univariate global residue theorem [56], we calculate the residue of the product $\psi\varphi_R$ around the solutions of $p(z)$ by taking in the term of order $\mathcal{O}(\delta^{-1})$ (that corresponds to the term of order $\mathcal{O}(p(z)^{-1})$) the coefficient of term proportional to $z^{\deg\,\mathrm{p}-1}$ and dividing it by the leading coefficient. The contribution to the intersection number around the irreducible polynomial $p(z)$ is therefore

$$\langle\varphi_L|\varphi_R\rangle_{p(z)} = \frac{\tilde{c}_{-1,\deg\,\mathrm{p}-1}}{l_c}. \tag{4.56}$$

In the multivariate case the situation is analogous, the main difference being the vectorial nature of $\varphi_L,\tilde{\varphi}_R$ and ψ and the fact that $\mathbf{\Omega}^{(n)}$ is a matrix. We begin by considering the order of each one of the $p(z)-$adic expansion of all each entry of $\varphi_{L,i}$,$\tilde{\varphi}_{R,i}$ and $\mathbf{\Omega}^{(n)}_{ij}$ and calculate global-max and global-min as follows

$$\begin{aligned}&\text{global-min} = \min_i(\mathcal{O}(\varphi_{L,i})) - \max_{ij}\Big(\mathcal{O}(\mathbf{\Omega}^{(n)}_{ij})\Big),\\ &\text{global-max} = \max_i(-\mathcal{O}(\tilde{\varphi}_{R,i}) - 1),\end{aligned} \tag{4.57}$$

where i and j run over the number of MIs in the inner space. Remember that in the multivariate case we have $\tilde{\varphi}_{R,i}\equiv C_{ij}\varphi_{R,j}$. Then, the expansion of each ψ_k in terms of $p(z)$ looks like

$$\psi^{(n)}_k = \sum_{i=\text{global-min}}^{\text{global-max}}\sum_{j=1}^{\deg\,\mathrm{p}-1} c_{kij}z^j p^i(z). \tag{4.58}$$

We build the multivariate differential equation, eq. (3.107), that we rewrite here for convenience

$$\frac{\partial}{\partial z}\psi^{(n)}_i + \psi^{(n)}_j\mathbf{\Omega}^{(n)}_{ji} = \varphi_{L,i}, \tag{4.59}$$

and we apply to it the $p(z)-$adic expansion around $p(z)-\delta$ (as explained in Sect. 4.3.1) in order to obtain

$$\Big\lfloor\frac{\partial}{\partial z}\psi^{(n)}_i\Big\rfloor_{p(z)-\delta} + \lfloor\psi^{(n)}_j\cdot\mathbf{\Omega}^{(n)}_{ji}\rfloor_{p(z)-\delta} = \lfloor\varphi_{L,i}\rfloor_{p(z)-\delta}. \tag{4.60}$$

Then, we expand around $\delta \to 0$ in all the terms constituting our differential equation and we solve it for the coefficients c_{kij}. We then substitute the obtained expressions for c_{kij} in $\lfloor \psi_i^{(n)} \rfloor_{p(z)-\delta}$ and we multiply it by $\tilde{\varphi}_{R,i}$. We then take the polynomial reminder of the product $\lfloor \psi_i^{(n)} \rfloor_{p(z)-\delta} \tilde{\varphi}_{R,i} = \lfloor \psi_i^{(n)} \tilde{\varphi}_{R,i} \rfloor_{p(z)-\delta}$ and we expand it for $\delta \to 0$,

$$\lfloor \psi_i^{(n)} \tilde{\varphi}_{R,i} \rfloor_{p(z)-\delta} = \sum_{i}^{-1} \sum_{j=1}^{\deg \mathrm{p}-1} \tilde{c}_{ij} z^j \delta^i + \mathcal{O}\big(\delta^0\big). \tag{4.61}$$

Again, following the univariate global residue theorem [56], we calculate the residue of the product $\psi_i \, \tilde{\varphi}_{R,i}$ around the solutions of $p(z)$ by taking, in the term of order $\mathcal{O}(\delta^{-1})$, the coefficient of the term proportional to $z^{\deg \mathrm{p}-1}$ and dividing it for the leading coefficient. The contribution to the intersection number around the irreducible polynomial $p(z)$ is therefore

$$\left\langle \varphi_L^{(n)} \middle| \varphi_R^{(n)} \right\rangle_{p(z)} = \frac{\tilde{c}_{-1,\deg \mathrm{p}-1}}{l_c}. \tag{4.62}$$

After calculating for each factor in the list of eq. (4.46) its contribution to the intersection number, we sum all the contributions to obtain the value of the intersection number.

5 | Applications to Feynman integrals

In the following we use the new method presented in Ch. 4 to compute the decomposition of several Feynman integrals belonging to different families, both at one and two loops. All these results have been checked against the decomposition obtained with the traditional Laporta approach implemented on the framework FiniteFlow. For each family we report the relevant master forms considered at each layer of the recursive algorithm.

5.1 | One Loop cases

We begin with some examples of Feynman integrals at one loop.

5.1.1 | Triangle with a massive loop

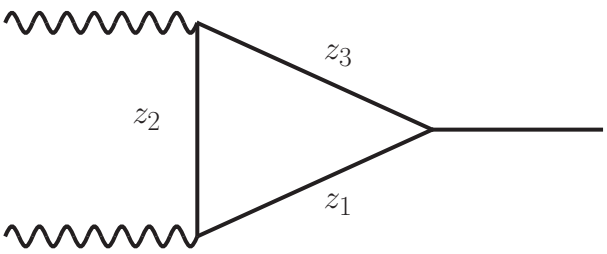

Figure 5.1: Massive triangle at one loop.

We compute the decomposition of several integrals belonging to the family of massive one loop triangle. We begin with an overview of the kinematics. The family has one Mandelstam invariant,

$$s = 2p_1 \cdot p_2. \tag{5.1}$$

For the external momenta we have

$$\begin{aligned} &p_1^2 = 0,\, p_2^2 = 0, \\ &p_3 = -p_1 - p_2. \end{aligned} \tag{5.2}$$

We identify the propagators with

$$z_1 = k^2 - m^2, \quad z_2 = (k+p_1)^2 - m^2, \quad z_3 = (k+p_1+p_2)^2 - m^2. \tag{5.3}$$

therefore we are computing intersection numbers of differential 3−forms. The Baikov polynomial is given by

$$B = -\frac{1}{4} s\big(m^2 s + s z_2 - (z_1 - z_2)(z_2 - z_3)\big). \tag{5.4}$$

Therefore the regularized twist takes the form

$$u_\rho = B^{\frac{d-4}{2}} z_1^\rho z_2^\rho z_3^\rho. \tag{5.5}$$

We consider the following ordering for the integration variables, from the innermost to the outermost:

$$z_2 \to z_3 \to z_1. \tag{5.6}$$

The dimension and the choice for the master forms in the inner spaces found as illustrated in Sect. 3.3.1 are given by

$$\begin{aligned} &\text{integration in } z_2\text{:} \quad \Big\{1, \frac{1}{z_2}\Big\}, \\ &\text{integration in } z_3\text{:} \quad \Big\{1, \frac{1}{z_3}, \frac{1}{z_2 z_3}\Big\}. \end{aligned} \tag{5.7}$$

For the outermost integration we have that master forms correspond to master integrals and we make the following choice (fig. 5.2)

$$I_{\text{tri}} = I_{1,1,1}, \quad I_{\text{bub}} = I_{1,0,1}, \quad I_{\text{tad}} = I_{0,0,1}. \tag{5.8}$$

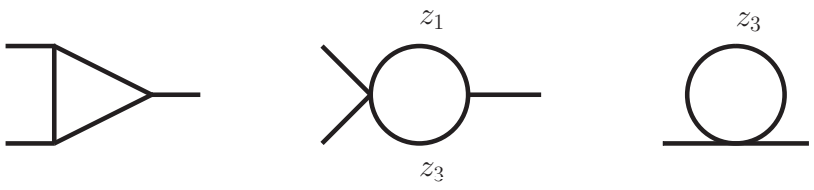

FIGURE 5.2: Master integrals for the massive triangle.

Following the procedure outlined in Sect. 3.5.2, we calculate the metric $\mathbf{C}_\rho$ given by the intersection numbers between the master integrals

$$\mathbf{C}_\rho = \begin{pmatrix} \langle I_{\rm tri}|I_{\rm tri}\rangle & \langle I_{\rm tri}|I_{\rm bub}\rangle & \langle I_{\rm tri}|I_{\rm tad}\rangle \\ \langle I_{\rm bub}|I_{\rm tri}\rangle & \langle I_{\rm bub}|I_{\rm bub}\rangle & \langle I_{\rm bub}|I_{\rm tad}\rangle \\ \langle I_{\rm tad}|I_{\rm tri}\rangle & \langle I_{\rm tad}|I_{\rm bub}\rangle & \langle I_{\rm tad}|I_{\rm tad}\rangle \end{pmatrix}, \tag{5.9}$$

where each intersection number is given by

$$\langle I_{\rm tri}|I_{\rm tri}\rangle = \frac{6\epsilon}{\rho^3(2\epsilon-3\rho)}, \tag{5.10}$$

$$\langle I_{\rm tri}|I_{\rm bub}\rangle = \frac{\epsilon s}{\rho^2(2\epsilon-3\rho)(2\epsilon-3\rho+1)}, \tag{5.11}$$

$$\langle I_{\rm tri}|I_{\rm tad}\rangle = -\frac{\epsilon s(4m^2(2\epsilon^2+\epsilon(3-9\rho)+(1-3\rho)^2)+\rho s(\epsilon-2\rho+1))}{\rho^2(2\epsilon-3\rho+1)(2\epsilon-3\rho+2)(2\epsilon^2+\epsilon(2-9\rho)+3\rho(3\rho-1))}, \tag{5.12}$$

$$\langle I_{\rm bub}|I_{\rm tri}\rangle = \frac{\epsilon s}{\rho^2(2\epsilon-3\rho-1)(2\epsilon-3\rho)}, \tag{5.13}$$

$$\langle I_{\rm bub}|I_{\rm bub}\rangle = -\frac{\epsilon s(\epsilon^2(s-4m^2)+2\epsilon\rho(9m^2-2s)+2\rho^2(2s-9m^2))}{\rho^2(2\epsilon-3\rho)(2\epsilon-3\rho+1)(2\epsilon^2-\epsilon(9\rho+1)+3\rho(3\rho+1))}, \tag{5.14}$$

$$\begin{aligned}\langle I_{\rm bub}|I_{\rm tad}\rangle = {} & \frac{1}{\rho(2\epsilon-3\rho+1)(2\epsilon-3\rho+2)(2\epsilon^2+\epsilon(2-9\rho)+3\rho(3\rho-1))} \\ & \frac{1}{(2\epsilon^2-\epsilon(9\rho+1)+3\rho(3\rho+1))}\Big(\epsilon s^2(\epsilon^3(s-8m^2) \\ & +\epsilon^2(2m^2(24\rho-5)-6\rho s+s)-2\epsilon(m^2(48\rho^2-20\rho+1) \\ & +2\rho(1-3\rho)s)+\rho(m^2(63\rho^2-39\rho+4)+4\rho(1-2\rho)s))\Big),\end{aligned} \tag{5.15}$$

$$\langle I_{\rm tad}|I_{\rm tri}\rangle = \frac{\epsilon s(\rho s(-\epsilon+2\rho+1)-4m^2(2\epsilon^2-3\epsilon(3\rho+1)+(3\rho+1)^2))}{\rho^2(\epsilon-3\rho-1)(2\epsilon-3\rho-2)(2\epsilon-3\rho-1)(2\epsilon-3\rho)}, \tag{5.16}$$

$$\langle I_{\text{tad}}|I_{\text{bub}}\rangle = \frac{1}{\rho(\epsilon-3\rho-1)(2\epsilon-3\rho-2)(2\epsilon-3\rho)(2\epsilon-3\rho+1)} \frac{1}{(2\epsilon^2-\epsilon(9\rho+1)+3\rho(3\rho+1))}\Big(\epsilon s^2(\epsilon^3(s-8m^2) + \epsilon^2(2m^2(24\rho+5)-(6\rho+1)s)-2\epsilon m^2(48\rho^2+20\rho+1) + 4\epsilon\rho(3\rho+1)s+\rho(m^2(63\rho^2+39\rho+4)-4\rho(2\rho+1)s))\Big), \quad (5.17)$$

$$\langle I_{\text{tad}}|I_{\text{tad}}\rangle = \frac{1}{(\epsilon-3\rho-1)(2\epsilon-3\rho-2)(2\epsilon-3\rho+1)(2\epsilon-3\rho+2)} \frac{1}{(2\epsilon^2+\epsilon(2-9\rho)+3r(3\rho-1))(2\epsilon^2-\epsilon(9\rho+1)+3\rho(3\rho+1))} \Big(\epsilon s^2(-s^2(\epsilon-2\rho)^2(\epsilon^2-4\epsilon\rho+4\rho^2-1)+m^4(-48\epsilon^4 + 360\epsilon^3\rho-28\epsilon^2(36\rho^2-1)+6\epsilon\rho(207\rho^2-17)-567\rho^4 + 99\rho^2-4)+2m^2s(6\epsilon^4-47\epsilon^3\rho+\epsilon^2(139\rho^2-6) + \epsilon(23\rho-183\rho^3)+90\rho^4-22\rho^2))\Big), \quad (5.18)$$

and we use it to perform the decomposition of the following integrals following the *straight decomposition* procedure outlined in Sect. 3.4.2

$$I_{1,2,1},\ I_{1,1,2},\ I_{2,1,1}, \quad (5.19)$$

where the coefficients are obtained following eq. (3.74) after taking the limit $\rho \to 0$. We successfully computed the decomposition of

$$I_{1,2,1} = \frac{(-2\epsilon^2+3\epsilon-1)}{m^4 s}\, I_{\text{tad}} + \frac{(1-2\epsilon)}{m^2 s}\, I_{\text{bub}} - \frac{\epsilon}{m^2}\, I_{\text{tri}}. \quad (5.20)$$

$$I_{1,1,2} = \frac{(\epsilon^2 m^2 - \frac{\epsilon^2 s}{4} - \frac{3\epsilon m^2}{2} + \frac{\epsilon s}{4} + \frac{m^2}{2})}{m^6 s - \frac{m^4 s^2}{4}}\, I_{\text{tad}} + \frac{(\epsilon-\frac{1}{2})}{m^2 s - \frac{s^2}{4}}\, I_{\text{bub}} = I_{2,1,1}. \quad (5.21)$$

5.1.2 | Massless box diagram

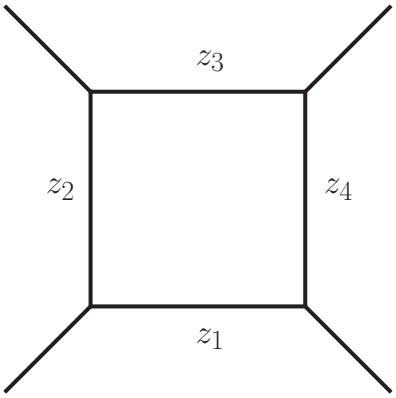

FIGURE 5.3: Massless box at one loop.

We compute the decomposition of several integrals belonging to the family of massless one loop box. We begin with an overview of the kinematics. The family has two Mandelstam invariants

$$s_{12} = 2p_1 \cdot p_2, \qquad s_{13} = 2p_1 \cdot p_3. \tag{5.22}$$

For the external momenta we have

$$\begin{aligned} &p_1^2 = 0,\, p_2^2 = 0,\, p_3^2 = 0,\, p_4^2 = 0, \\ &p_4 = -p_1 - p_2 - p_3. \end{aligned} \tag{5.23}$$

We identify propagators with

$$\begin{aligned} &z_1 = k^2, \qquad z_2 = (k+p_1)^2, \qquad z_3 = (k+p_1+p_2)^2, \\ &z_4 = (k+p_1+p_2+p_3)^2. \end{aligned} \tag{5.24}$$

The Baikov polynomial is given by

$$\begin{aligned} B = \frac{1}{16}\Big(&s_{12}^4 + 2s_{12}^3(s_{13} - z_1 + z_2 - z_3 + z_4) + s_{12}^2(s_{13}^2 + 2s_{13}(-2z_1 + z_2 - 2z_3 + z_4) \\ &+ (z_1 - z_2 + z_3 - z_4)^2) - 2s_{12}s_{13}(s_{13}(z_1 + z_3) - z_1^2 + z_1(z_2 + z_4) \\ &+ z_2z_3 - 2z_2z_4 - z_3^2 + z_3z_4) + s_{13}^2(z_1 - z_3)^2\Big). \end{aligned} \tag{5.25}$$

Therefore the regularized twist u takes the form

$$u_\rho = B^{\frac{d-5}{2}} z_1^\rho z_2^\rho z_3^\rho z_4^\rho. \tag{5.26}$$

From the *master monomials analysis*, performed out of the cut, we find three master integrals, that we identify as

$$\begin{aligned} I_{\text{box}} &= I_{1,1,1,1}, \\ I_{\text{bub1}} &= I_{1,0,1,0}, \\ I_{\text{bub2}} &= I_{0,1,0,1}. \end{aligned} \tag{5.27}$$

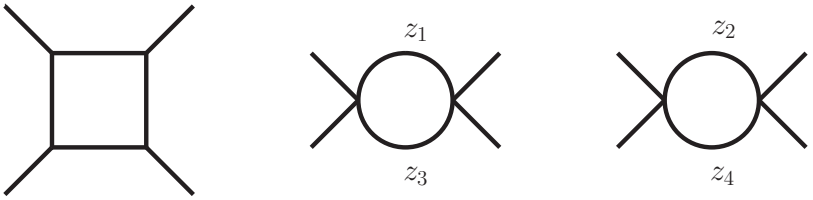

FIGURE 5.4: Master integrals for the massless box.

We consider the decomposition of the following integrals

$$\begin{aligned} &I_{2,1,1,1}, \qquad I_{1,2,1,1}, \qquad I_{1,1,2,1}, \\ &I_{1,1,1,2}, \qquad I_{1,2,2,1}. \end{aligned} \tag{5.28}$$

We follow the *bottom-up decomposition* as presented in Sect. 3.4.2, based on spanning-cuts. Since the decomposition of a generic integral, performed without cuts, can be written as

$$I = c_0 I_{\text{box}} + c_1 I_{\text{bub1}} + c_2 I_{\text{bub2}}. \tag{5.29}$$

we need two spanning cuts to obtain all the coefficients of the decomposition: the cut $\{1,3\}$ corresponding to the cut of propagators z_1 and z_3 and the cut $\{2,4\}$ where we cut the propagators z_2, z_4. These two set of cuts allow us to find the coefficients

$$\begin{aligned} &\text{cut } \{1,3\}: \qquad I = c_0 I_{\text{box}} + c_1 I_{\text{bub1}}, \\ &\text{cut } \{2,4\}: \qquad I = c_0 I_{\text{box}} + c_2 I_{\text{bub2}}. \end{aligned} \tag{5.30}$$

By combining the results obtained in these two sets of cuts, we are able to recover all the coefficients of the decomposition.

Spanning cut $\{1,3\}$

In the spanning cut $\{1,3\}$ the regularized twist takes the form

$$u_{\rho,1,3} = B_{1,3}^{\frac{d-5}{2}} z_2^\rho z_4^\rho. \tag{5.31}$$

where $B_{1,3}$ is the Baikov polynomial on the cut $\{1,3\}$

$$\begin{aligned} B_{1,3} = \frac{1}{16}\Big(& s_{12}^4 + 2s_{12}^3(s_{13}+z_2+z_4) + s_{12}^2(s_{13}^2 \\ & + 2s_{13}(z_2+z_4) + (-z_2-z_4)^2) + 4s_{12}s_{13}z_2z_4\Big). \end{aligned} \tag{5.32}$$

we calculate 2−forms intersection numbers in the variables z_2, z_4. We consider the following ordering from the innermost to the outermost

$$z_2 \to z_4. \tag{5.33}$$

with the following internal master forms in z_2

$$\text{integration in } z_2\text{:} \quad \Big\{1, \frac{1}{z_2}\Big\}. \tag{5.34}$$

We write the integrals we want to decompose as (following the cut prescription for dotted propagators explained in Sect. 3.4.1)

$$\begin{aligned} I_{1,2,1,1} &= \int u_{\rho,1,3}\frac{1}{z_2^2 z_4}, \qquad I_{1,1,1,2} = \int u_{\rho,1,3}\frac{1}{z_2 z_4^2}, \\ I_{2,1,1,1} &= \int u_{\rho,1,3}\frac{\partial_{z_1}\log(u)}{z_2 z_4}\Big|_{z_1=z_3=0}, \qquad I_{1,1,2,1} = \int u_{\rho,1,3}\frac{\partial_{z_3}\log(u)}{z_2 z_4}\Big|_{z_1=z_3=0}, \\ I_{1,2,2,1} &= \int u_{\rho,1,3}\frac{\partial_{z_3}\log(u)}{z_2^2 z_4}\Big|_{z_1=z_3=0}. \end{aligned} \tag{5.35}$$

As explained above, this gives us c_0 and c_1.

Spanning cut $\{2,4\}$

In the spanning cut $\{2,4\}$ the regularized twist takes the form

$$u_{\rho,2,4} = B_{2,4}^{\frac{d-5}{2}} z_1^\rho z_3^\rho. \tag{5.36}$$

where $B_{2,4}$ is the Baikov polynomial on the cut $\{2,4\}$

$$\begin{aligned} B_{2,4} = \frac{1}{16}\Big(& s_{12}^4 + 2s_{12}^3(s_{13}-z_1-z_3) + s_{12}^2(s_{13}^2 \\ & + 2s_{13}(-2z_1-2z_3) + (z_1+z_3)^2) - 2s_{12}s_{13}(s_{13}(z_1+z_3) \\ & - z_1^2 - z_3^2) + s_{13}^2(z_1-z_3)^2\Big). \end{aligned} \tag{5.37}$$

we have 2−forms intersection numbers in the variables z_1, z_3. We consider the following ordering for the variables from the innermost to the outermost

$$z_1 \to z_3. \tag{5.38}$$

with the following internal master forms in z_1

$$\text{integration in } z_1\text{:} \quad \Big\{1, \frac{1}{z_1}\Big\}. \tag{5.39}$$

Similarly to the previous cut, we write the integrals that we decompose as

$$\begin{aligned}
I_{1,2,1,1} &= \int u_{\rho,2,4} \frac{\partial_{z_2} \log(u)}{z_1 z_3}\bigg|_{z_2=z_4=0}, \qquad I_{1,1,1,2} = \int u_{\rho,2,4} \frac{\partial_{z_4} \log(u)}{z_1 z_3}\bigg|_{z_2=z_4=0}, \\
I_{2,1,1,1} &= \int u_{\rho,2,4} \frac{1}{z_1 z_3}, \qquad I_{1,1,2,1} = \int u_{\rho,2,4} \frac{1}{z_1 z_3}, \\
I_{1,2,2,1} &= \int u_{\rho,2,4} \frac{\partial_{z_2} \log(u)}{z_1 z_3^2}\bigg|_{z_2=z_4=0}.
\end{aligned} \tag{5.40}$$

This cut yields the coefficients c_0 and c_2. Combined with the previous one we obtain

$$\begin{aligned}
I_{1,2,1,1} = &\frac{(8\epsilon^2 - 2)}{\epsilon s_{12}^3 + \epsilon s_{12}^2 s_{13} + s_{12}^3 + s_{12}^2 s_{13}} I_{\text{bub1}} \\
&+ \frac{(-2\epsilon - 1)}{s_{12} + s_{13}} I_{\text{box}} = I_{1,1,1,2}.
\end{aligned} \tag{5.41}$$

$$\begin{aligned}
I_{2,1,1,1} = &\frac{(2 - 8\epsilon^2)}{\epsilon s_{12}^3 + 2\epsilon s_{12}^2 s_{13} + \epsilon s_{12} s_{13}^2 + s_{12}^3 + 2 s_{12}^2 s_{13} + s_{12} s_{13}^2} I_{\text{bub2}} \\
&+ \frac{(2\epsilon + 1)}{s_{12}} I_{\text{box}} = I_{1,1,2,1}.
\end{aligned} \tag{5.42}$$

$$\begin{aligned}
I_{1,2,2,1} = &\frac{(-4\epsilon^2 - 6\epsilon - 2)}{s_{12}^2 + s_{12} s_{13}} I_{\text{box}} + \frac{(16\epsilon^2 - 4)}{s_{12}^4 + s_{12}^3 s_{13}} I_{\text{bub1}} \\
&+ \frac{(16\epsilon^2 - 4)}{s_{12}^4 + 3 s_{12}^3 s_{13} + 3 s_{12}^2 s_{13}^2 + s_{12} s_{13}^3} I_{\text{bub2}}.
\end{aligned} \tag{5.43}$$

5.2 | Two Loops cases

We present the results obtained in the decomposition of several Feynman integrals at two loops.

5.2.1 | Massless boxtriangle diagram

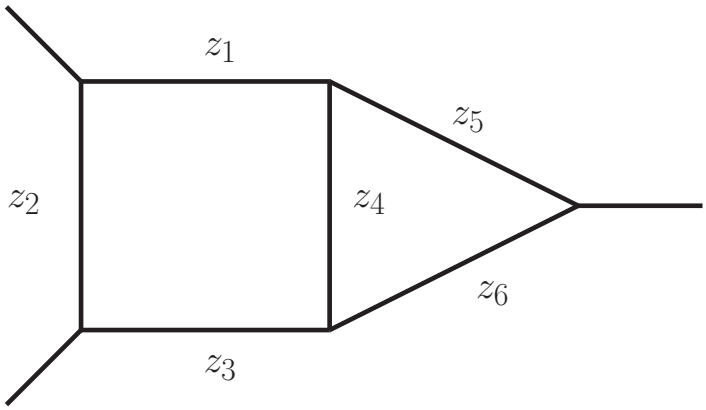

FIGURE 5.5: Massless boxtriangle.

We compute the decomposition of several integrals belonging to the family of massless boxtriangle. We begin with an overview of the kinematics. The family has one Mandelstam invariant,

$$s_{12} = 2p_1 \cdot p_2. \tag{5.44}$$

For the external momenta we have

$$\begin{aligned} &p_1^2 = 0,\, p_2^2 = 0, \\ &p_3 = -p_1 - p_2. \end{aligned} \tag{5.45}$$

We identify the propagators with

$$\begin{aligned} &z_1 = k_1^2, \quad z_2 = (k_1 - p_1)^2, \quad z_3 = (k_1 - p_1 - p_2)^2, \\ &z_4 = (k_1 - k_2)^2, \quad z_5 = k_2^2, \quad z_6 = (k_2 - p_1 - p_2)^2, \\ &z_7 = (k_2 - p_1)^2. \end{aligned} \tag{5.46}$$

The master integrals are

$$\begin{aligned} &I_{\mathrm{m1}} = I_{0,1,0,1,1,1,0}, \qquad I_{\mathrm{m2}} = I_{1,0,1,0,1,1,0}, \\ &I_{\mathrm{m3}} = I_{1,0,0,1,0,1,0}, \qquad I_{\mathrm{m4}} = I_{0,0,1,1,1,0,0}. \end{aligned} \tag{5.47}$$

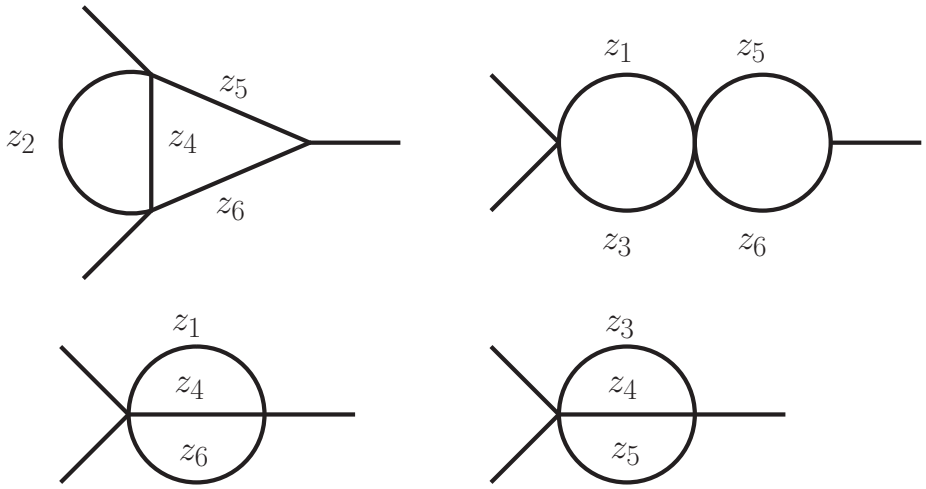

FIGURE 5.6: Master integrals for the massless boxtriangle.

We compute the decomposition of the following integrals, using the spanning cuts approach,

$$\begin{aligned} &I_{1,1,1,1,1,1,-1}, \quad I_{1,1,1,1,1,1,-2}, \quad I_{2,1,1,1,1,1,-1}, \\ &I_{1,1,1,2,1,1,-2}, \quad I_{1,1,0,1,1,1,-1}, \quad I_{1,2,1,1,1,1,0}, \\ &I_{1,2,1,2,1,1,0}, \quad I_{2,1,2,1,1,1,-1}. \end{aligned} \tag{5.48}$$

Since the decomposition of a generic integral can be written as

$$I = c_1 I_{\mathrm{m1}} + c_2 I_{\mathrm{m2}} + c_3 I_{\mathrm{m3}} + c_4 I_{\mathrm{m4}}. \tag{5.49}$$

It is sufficient to consider the following spanning cuts: $\{2,4,5,6\}$, $\{1,3,5,6\}$, $\{1,4,6\}$ and $\{3,4,5\}$ to get all the coefficients of the decomposition of a generic Feynman integral belonging to the boxtriangle family. In particular we see the coefficient corresponding to each spanning cuts are

$$\begin{aligned} &\text{cut } \{2,4,5,6\}: \quad I = c_1 I_{\mathrm{m1}}, \\ &\text{cut } \{1,3,5,6\}: \quad I = c_2 I_{\mathrm{m2}}, \\ &\text{cut } \{1,4,6\}: \quad I = c_3 I_{\mathrm{m3}}, \\ &\text{cut } \{3,4,5\}: \quad I = c_4 I_{\mathrm{m4}}. \end{aligned} \tag{5.50}$$

At the end of the section we report the integrals for which we calculated all the coefficients of the decomposition.

Spanning cut $\{2,4,5,6\}$

We compute 3−forms intersection numbers in the variables z_7, z_1, z_3. We chose the ordering, from the innermost to the outermost, as

$$z_7 \to z_1 \to z_3. \tag{5.51}$$

with the following choice for the internal master forms

$$\begin{aligned} &\text{integration in } z_7\text{:} \quad \Big\{1\Big\}, \\ &\text{integration in } z_1\text{:} \quad \Big\{1, \frac{1}{z_1}\Big\}. \end{aligned} \tag{5.52}$$

Spanning cut $\{1,3,5,6\}$

We compute 3−forms intersection numbers in the variables z_7, z_2, z_4. We chose the ordering, from the innermost to the outermost, as

$$z_7 \to z_2 \to z_4, \tag{5.53}$$

with the following choice for the internal master forms

$$\begin{aligned} &\text{integration in } z_7\text{:} \quad \Big\{z_7\Big\}, \\ &\text{integration in } z_2\text{:} \quad \Big\{\frac{1}{z_2}\Big\}. \end{aligned} \tag{5.54}$$

Spanning cut $\{1,4,6\}$

We compute 4−forms intersection numbers in the variables z_7, z_2, z_5, z_3. We chose the ordering, from the innermost to the outermost, as

$$z_7 \to z_2 \to z_5 \to z_3, \tag{5.55}$$

with the following choice for the internal master forms

$$\begin{aligned} &\text{integration in } z_7\text{:} \quad \Big\{1\Big\}, \\ &\text{integration in } z_2\text{:} \quad \Big\{\frac{1}{z_2}\Big\}, \\ &\text{integration in } z_5\text{:} \quad \Big\{\frac{1}{z_2 z_5}\Big\}. \end{aligned} \tag{5.56}$$

Spanning cut $\{3,4,5\}$

We compute 4−forms intersection numbers in the variables z_1, z_2, z_6, z_7. We chose the ordering, from the innermost to the outermost, as

$$z_7 \to z_2 \to z_6 \to z_1, \tag{5.57}$$

with the following choice for the internal master forms

$$\begin{aligned} &\text{integration in } z_7\text{:} \quad \left\{z_7\right\}, \\ &\text{integration in } z_2\text{:} \quad \left\{\frac{1}{z_2}\right\}, \\ &\text{integration in } z_6\text{:} \quad \left\{\frac{1}{z_6}\right\}. \end{aligned} \tag{5.58}$$

We report the complete decompositions

$$\begin{aligned} I_{1,1,1,1,1,1,-1} = & \\ & - \frac{2(6\epsilon^2-5\epsilon+1)}{\epsilon(2\epsilon-1)s_{12}}\, I_{\text{m1}} \\ & - \frac{2(2\epsilon^3 s_{12}-3\epsilon^2 s_{12}+\epsilon s_{12})}{\epsilon^2(2\epsilon-1)s_{12}^2}\, I_{\text{m2}} \\ & - \frac{2(-9\epsilon^3+18\epsilon^2-11\epsilon+2)}{\epsilon^2(2\epsilon-1)s_{12}^2}\, I_{\text{m3}} \\ & - \frac{2(-9\epsilon^3+18\epsilon^2-11\epsilon+2)}{\epsilon^2(2\epsilon-1)s_{12}^2}\, I_{\text{m4}}. \end{aligned} \tag{5.59}$$

$$\begin{aligned} I_{1,1,1,1,1,1,-2} = & \\ & - \frac{(1-2\epsilon)(2\epsilon-1)}{(\epsilon-1)\epsilon}\, I_{\text{m1}} \\ & - \frac{(2\epsilon-1)(2\epsilon s_{12}-\epsilon^2 s_{12})}{2(\epsilon-1)\epsilon^2 s_{12}}\, I_{\text{m2}} \\ & - \frac{(2\epsilon-1)(3\epsilon^2-8\epsilon+4)}{2(\epsilon-1)\epsilon^2 s_{12}}\, I_{\text{m3}} \\ & - \frac{(2\epsilon-1)(3\epsilon^2-8\epsilon+4)}{2(\epsilon-1)\epsilon^2 s_{12}}\, I_{\text{m4}}. \end{aligned} \tag{5.60}$$

$$I_{2,1,1,1,1,1,-1} =$$

$$
\begin{aligned}
&- \frac{3(2\epsilon+1)(6\epsilon^3+\epsilon^2-4\epsilon+1)}{\epsilon(\epsilon+1)^2(2\epsilon-1)s_{12}^2}\, I_{\mathrm{m1}} \\
&- \frac{(2\epsilon+1)(4\epsilon^5 s_{12}+2\epsilon^4 s_{12}-6\epsilon^3 s_{12}-2\epsilon^2 s_{12}+2\epsilon s_{12})}{\epsilon^2(\epsilon+1)^2(2\epsilon-1)s_{12}^3}\, I_{\mathrm{m2}} \\
&- \frac{(2\epsilon+1)(-54\epsilon^5+108\epsilon^4-93\epsilon^3+66\epsilon^2-33\epsilon+6)}{\epsilon^2(\epsilon+1)^2(2\epsilon-1)s_{12}^3}\, I_{\mathrm{m3}} \\
&+ \frac{3(2\epsilon+1)(9\epsilon^4-9\epsilon^3-7\epsilon^2+9\epsilon-2)}{\epsilon^2(\epsilon+1)^2(2\epsilon-1)s_{12}^3}\, I_{\mathrm{m4}}. \qquad (5.61)
\end{aligned}
$$

$$
\begin{aligned}
I_{1,1,1,2,1,1,-2} = & \\
&- \frac{2(2\epsilon+1)(6\epsilon^2-5\epsilon+1)}{\epsilon(2\epsilon-1)s_{12}}\, I_{\mathrm{m1}} \\
&- \frac{(2\epsilon+1)(2\epsilon^4 s_{12}-3\epsilon^3 s_{12}-3\epsilon^2 s_{12}+2\epsilon s_{12})}{\epsilon^2(\epsilon+1)^2(2\epsilon-1)s_{12}^2}\, I_{\mathrm{m2}} \\
&- \frac{(2\epsilon+1)(18\epsilon^5-72\epsilon^4+85\epsilon^3-21\epsilon^2-12\epsilon+4)}{\epsilon^2(\epsilon+1)^2(2\epsilon-1)s_{12}^2}\, I_{\mathrm{m3}} \\
&- \frac{(2\epsilon+1)(18\epsilon^5-72\epsilon^4+85\epsilon^3-21\epsilon^2-12\epsilon+4)}{\epsilon^2(\epsilon+1)^2(2\epsilon-1)s_{12}^2}\, I_{\mathrm{m4}}. \qquad (5.62)
\end{aligned}
$$

5.2.2 | Massless doublebox diagram

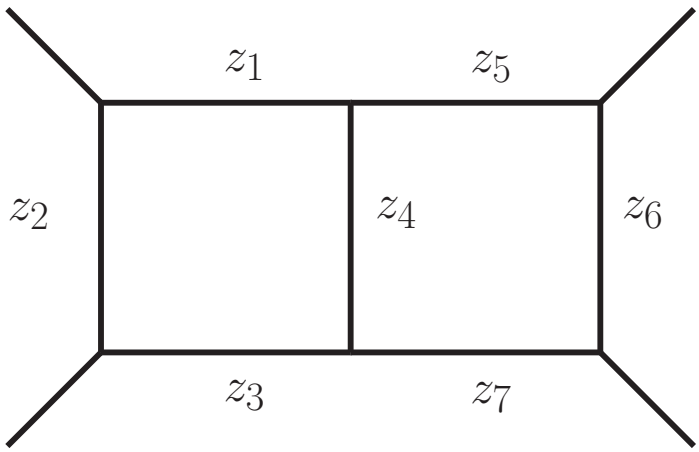

FIGURE 5.7: Massless doublebox.

We consider the decomposition of several integrals of the doublebox family with massless propagators. We first give a brief review of the kinematics. We

have two Mandelstam invariants,

$$s_{12} = p_1 \cdot p_2, \quad s_{13} = p_1 \cdot p_3, \tag{5.63}$$

while for the external momenta we have

$$\begin{aligned} &p_i^2 = 0, (i = 1, \ldots, 4), \\ &p_4 = -\sum_{i=1}^{3} p_i. \end{aligned} \tag{5.64}$$

We indicate the loop momenta with k_1 and k_2. The denominators $z_1, \ldots, z_7$ and the ISPs z_8, z_9 are

$$\begin{aligned} &z_1 = k_1^2, \quad z_2 = (k_1 - p_1)^2, \quad z_3 = (k_1 - p_1 - p_2)^2, \\ &z_4 = (k_1 - k_2)^2, \quad z_5 = k_2^2, \\ &z_6 = (k_2 - p_1 - p_2 - p_3)^2, \\ &z_7 = (k_2 - p_1 - p_2)^2, \\ &z_8 = (k_1 - p_1 - p_2 - p_3)^2, \quad z_9 = (k_2 - p_1)^2. \end{aligned} \tag{5.65}$$

We consider the decomposition of a set of integrals on three different cuts, as illustrated below.

Maximal cut

We analyze the integrals first on the maximal cut, which corresponds to the calculations of 2−forms intersection numbers in z_8, z_9. The Baikov polynomial on the maximal cut is given by

$$B_{\text{max cut}} = \frac{1}{16} s_{12} z_8 z_9 (s_{12}^2 + s_{12}(s_{13} + z_8 + z_9) + z_8 z_9), \tag{5.66}$$

and the regularized twist takes the form

$$u_{\rho,\text{max cut}} = B_{\text{max cut}}^{\frac{d-6}{2}} z_8^\rho z_9^\rho. \tag{5.67}$$

The master monomials analysis on the max cut yields two master integrals

$$I_{1,1,1,1,1,1,1,0,0} \equiv I_0, \qquad I_{1,1,1,1,1,1,1,0,-1} \equiv I_{-1}. \tag{5.68}$$

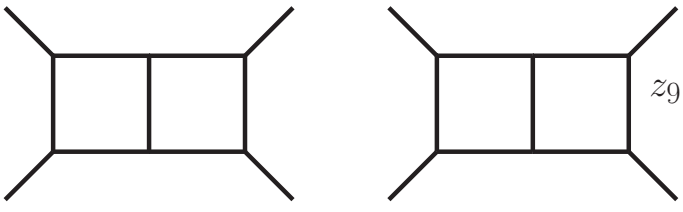

FIGURE 5.8: Master integrals for the massless double box on the max cut.

We calculate intersection numbers of differential 2−forms with the ordering

$$z_9 \to z_8, \tag{5.69}$$

with the following internal master forms

$$\text{integration in } z_9\text{:} \quad \Big\{1, z_9\Big\} \tag{5.70}$$

We consider the decomposition of the following integrals

$$\begin{aligned} &I_{1,1,1,1,1,1,1,-1,-1}, \qquad I_{1,1,1,1,1,1,1,-2,-1}, \qquad I_{1,1,1,1,2,1,1,-1,-2}, \\ &I_{1,1,1,1,1,1,1,-2,-2}, \qquad I_{1,2,1,1,1,1,1,0,0}, \qquad I_{1,2,1,1,1,1,1,-1,-1}, \\ &I_{1,1,1,1,1,1,2,-2,-1}. \end{aligned} \tag{5.71}$$

We obtain the following decompositions,

$$I_{1,1,1,1,1,1,1,-1,-1} = -\frac{s_{12}(s_{12}+s_{13})}{2}\, I_0 - \frac{(3s_{12})}{2}\, I_{-1}. \tag{5.72}$$

$$\begin{aligned} I_{1,1,1,1,1,1,1,-2,-1} = &\frac{(3\epsilon-1)s_{12}^2(s_{12}+s_{13})}{2(2\epsilon-1)}\, I_0 \\ &+ \frac{s_{12}((7\epsilon-3)s_{12}-2\epsilon s_{13})}{2(2\epsilon-1)}\, I_{-1}. \end{aligned} \tag{5.73}$$

$$\begin{aligned} I_{1,1,1,1,1,1,1,-1,-2} = &\frac{(3\epsilon-1)s_{12}^2(s_{12}+s_{13})}{2(2\epsilon-1)}\, I_0 \\ &+ \frac{s_{12}((7\epsilon-3)s_{12}-2\epsilon s_{13})}{2(2\epsilon-1)}\, I_{-1}. \end{aligned} \tag{5.74}$$

$$\begin{aligned} I_{1,1,1,1,1,1,1,-2,-2} = &\\ &- \frac{s_{12}^2(s_{12}+s_{13})((7\epsilon^2-6\epsilon+1)s_{12}+(-2\epsilon^2+3\epsilon-1)s_{13})}{2(1-2\epsilon)^2}\, I_0 \\ &- \frac{(3\epsilon-1)s_{12}^2((5\epsilon-3)s_{12}+(3-4\epsilon)s_{13})}{2(1-2\epsilon)^2}\, I_{-1}. \end{aligned} \tag{5.75}$$

$$I_{1,2,1,1,1,1,1,0,0} = -\frac{6\epsilon^2+5\epsilon+1}{(\epsilon+1)(s_{12}+s_{13})}\, I_0 - \frac{2\epsilon(2\epsilon+1)}{(\epsilon+1)s_{12}(s_{12}+s_{13})}\, I_{-1}. \tag{5.76}$$

$$I_{1,2,1,1,1,1,1,-1,-1} = 2\epsilon\, I_{-1}. \tag{5.77}$$

$$\begin{aligned} I_{1,1,1,1,1,1,2,-2,-1} = &\frac{\epsilon(3\epsilon-1)s_{12}(s_{12}+s_{13})}{2\epsilon-1}\, I_0 \\ &+ \frac{\epsilon((7\epsilon-3)s_{12}-2\epsilon s_{13})}{2\epsilon-1}\, I_{-1}. \end{aligned} \tag{5.78}$$

Maximal cut $-z_5$

We then consider the family of massles doublebox where all the denominators are on the cut except z_5. The master integrals on this cut are the same that we found in the maximal cut, namely I_0 and I_{-1}. We calculate intersection numbers of differential 3−forms with the ordering (from the innermost to the outermost)

$$z_9 \to z_8 \to z_5. \tag{5.79}$$

We choose the following internal master forms

$$\begin{aligned} &\text{integration in } z_9\text{:} \quad \Big\{1\Big\}, \\ &\text{integration in } z_8\text{:} \quad \Big\{1, z_9\Big\}. \end{aligned} \tag{5.80}$$

We consider the decomposition of the following integrals

$$\begin{aligned} &I_{1,1,1,1,1,1,1,-1,-1}, \quad I_{1,1,1,1,0,1,1,-1,-1}, \quad I_{1,1,1,1,2,1,1,-1,-1}, \\ &I_{1,1,1,1,2,1,1,-2,-1}, \quad I_{1,1,1,1,2,1,1,-2,-2}, \quad I_{2,1,1,1,1,1,1,-1,-1}, \\ &I_{2,1,1,1,1,1,1,-1,-2}. \end{aligned} \tag{5.81}$$

The decomposition obtained for $I_{1,1,1,1,1,1,1,-1,-1}$ is in agreement with the one obtained on the maximal cut. The other decompositions yield

$$I_{1,1,1,1,0,1,1,-1,-1} = 0. \tag{5.82}$$

$$I_{1,1,1,1,2,1,1,-1,-1} = -\epsilon(s_{12}+s_{13})\, I_0 - 3\epsilon\, I_{-1} \tag{5.83}$$

$$I_{2,1,1,1,1,1,1,-1,-1} = -\epsilon(s_{12}+s_{13})\, I_0 - 3\epsilon\, I_{-1}. \tag{5.84}$$

$$I_{1,1,1,1,2,1,1,-2,-1} = \frac{\epsilon(3\epsilon-1)s_{12}(s_{12}+s_{13})}{2\epsilon-1}\, I_0 + \frac{\epsilon((7\epsilon-3)s_{12}-2\epsilon s_{13})}{2\epsilon-1}\, I_{-1}. \tag{5.85}$$

$$\begin{aligned} I_{1,1,1,1,2,1,1,-2,-2} = &\frac{s_{12}(s_{12}+s_{13})((7\epsilon^2-6\epsilon+1)s_{12}+(-2\epsilon^2+3\epsilon-1)s_{13})}{2-4\epsilon}\, I_0 \\ &+ \frac{(3\epsilon-1)s_{12}((5\epsilon-3)s_{12}+(3-4\epsilon)s_{13})}{2-4\epsilon}\, I_{-1}. \end{aligned} \tag{5.86}$$

$$I_{2,1,1,1,1,1,1,-1,-2} = \frac{\epsilon(3\epsilon-1)s_{12}(s_{12}+s_{13})}{2\epsilon-1}\, I_0 + \frac{\epsilon((7\epsilon-3)s_{12}-2\epsilon s_{13})}{2\epsilon-1}\, I_{-1}. \tag{5.87}$$

Maximal cut $-z_5 - z_3$

We then consider the massles doublebox where all the denominators are on the cut except z_5 and z_3. The master integrals on this cut are

$$\begin{aligned} I_{-1}, \qquad & I_0, \\ I_{1,1,0,1,0,1,1,0,0} \equiv & I_{\text{diag}}. \end{aligned} \tag{5.88}$$

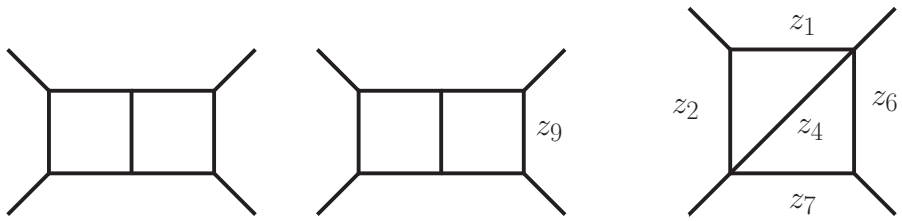

FIGURE 5.9: Master integrals for the massless double box where the denominators z_5, z_3 are not on the cut.

We calculate intersection numbers of differential 4−forms with the variables' ordering (from the innermost to the outermost)

$$z_8 \to z_3 \to z_5 \to z_9. \tag{5.89}$$

We choose the following internal master forms

$$\begin{aligned} &\text{integration in } z_8\text{:} \quad \Big\{1\Big\}, \\ &\text{integration in } z_3\text{:} \quad \Big\{\frac{1}{z_3}\Big\}, \\ &\text{integration in } z_5\text{:} \quad \Big\{1, \frac{1}{z_5}, \frac{1}{z_3}\Big\}. \end{aligned} \tag{5.90}$$

We calculate the decomposition of the following integrals

$$\begin{aligned} &I_{1,1,1,1,1,1,1,-1,-1}, \qquad I_{1,1,1,1,0,1,1,-1,-1}, \qquad I_{1,1,1,1,2,1,1,-1,-1}, \\ &I_{1,1,1,1,2,1,1,-2,-2}, \qquad I_{2,1,1,1,1,1,1,-1,-2}. \end{aligned} \tag{5.91}$$

Obtaining:

$$I_{1,1,1,1,1,1,1,-1,-1} = -\frac{1}{2}s_{12}(s_{12}+s_{13})\, I_0 - \frac{1}{2}(3s_{12})\, I_{-1} + \frac{2s_{12}+9s_{13}}{2s_{12}}\, I_{\text{diag}}. \tag{5.92}$$

$$I_{1,1,1,1,0,1,1,-1,-1} = \frac{(s_{12}+s_{13})(2\epsilon^2 s_{12} - 3\epsilon s_{12} - 3\epsilon s_{13} + s_{12} + s_{13})}{(6\epsilon^2 - 5\epsilon + 1)s_{13}}\, I_{\text{diag}}. \tag{5.93}$$

$$I_{1,1,1,1,2,1,1,-1,-1} = -\epsilon(s_{12}+s_{13})\, I_0 - 3\epsilon\, I_{-1} + \frac{3(3\epsilon s_{13} + s_{13})}{s_{12}^2}\, I_{\text{diag}}. \tag{5.94}$$

$$\begin{aligned} I_{1,1,1,1,2,1,1,-2,-2} = {} & \frac{s_{12}(s_{12}+s_{13})((7\epsilon^2 - 6\epsilon + 1)s_{12} + (-2\epsilon^2 + 3\epsilon - 1)s_{13})}{2-4\epsilon}\, I_0 \\ & + \frac{(3\epsilon - 1)s_{12}((5\epsilon - 3)s_{12} + (3-4\epsilon)s_{13})}{2-4\epsilon}\, I_{-1} \\ & + \frac{1}{2(6\epsilon^2 - 5\epsilon + 1)s_{12}s_{13}}\Big((4\epsilon^2 - 6\epsilon + 2)s_{12}^3 \\ & + 2(2\epsilon^2 - 9\epsilon + 3)s_{12}^2 s_{13} + (135\epsilon^3 - 171\epsilon^2 \\ & + 45\epsilon - 1)s_{12}s_{13}^2 + 3(-18\epsilon^3 + 27\epsilon^2 - 16\epsilon + 3)s_{13}^3\Big)\, I_{\text{diag}}. \end{aligned} \tag{5.95}$$

$$\begin{aligned} I_{2,1,1,1,1,1,1,-1,-2} = {} & \frac{\epsilon(3\epsilon - 1)s_{12}(s_{12}+s_{13})}{2\epsilon - 1}\, I_0 + \frac{\epsilon((7\epsilon - 3)s_{12} - 2\epsilon s_{13})}{2\epsilon - 1}\, I_{-1} \\ & + \frac{1}{(2\epsilon^2 + \epsilon - 1)s_{12}s_{13}}\Big(\epsilon((4\epsilon^2 - 6\epsilon + 2)s_{12}^2 \\ & + 4(\epsilon^2 - 4\epsilon + 1)s_{12}s_{13} - 3(7\epsilon^2 + 8\epsilon - 3)s_{13}^2)\Big)\, I_{\text{diag}}. \end{aligned} \tag{5.96}$$

5.2.3 | Massless pentabox diagram

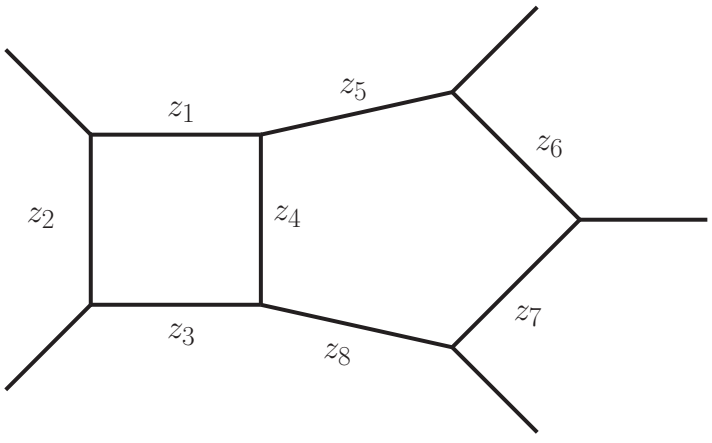

FIGURE 5.10: Massless pentabox.

We consider the decomposition of several diagrams of the pentabox topology with massless propagators. The kinematics presents five external momenta, $p_1, \ldots, p_5$ of which only p_1, p_2, p_3, p_4 are independent due to momentum conservation

$$\begin{aligned} p_i^2 &= 0, (i = 1, \ldots, 5), \\ p_5 &= -\sum_{i=1}^{4} p_i. \end{aligned} \tag{5.97}$$

There are 5 Mandelstam invariants $s_{ij} = p_i \cdot p_j$, that we chose cyclically as

$$\begin{aligned} &s_{12} = p_1 \cdot p_2, \quad s_{23} = p_2 \cdot p_3, \quad s_{34} = p_3 \cdot p_4, \\ &s_{45} = p_4 \cdot p_5, \quad s_{51} = p_5 \cdot p_1. \end{aligned} \tag{5.98}$$

We indicate the loop momenta with k_1 and k_2. The propagators $z_1, \ldots, z_8$ and the ISPs z_9, z_{10}, z_{11} are identified as

$$\begin{aligned} &z_1 = k_1^2, \quad z_2 = (k_1 - p_1)^2, \quad z_3 = (k_1 - p_1 - p_2)^2, \\ &z_4 = (k_1 - k_2)^2, \quad z_5 = k_2^2, \\ &z_6 = (k_2 - p_1 - p_2 - p_3 - p_4)^2, \\ &z_7 = (k_2 - p_1 - p_2 - p_3)^2, \quad z_8 = (k_2 - p_1 - p_2)^2, \\ &z_9 = (k_1 - p_1 - p_2 - p_3 - p_4)^2, \\ &z_{10} = (k_1 - p_1 - p_2 - p_3)^2, \quad z_{11} = (k_2 - p_1)^2. \end{aligned} \tag{5.99}$$

Working on the maximal cut where all the "real" propagators are set to zero, we are left with only the ISPs,z_9, z_{10}, z_{11}, and, therefore, with the calculation of 3-forms intersection numbers. Since we work on the maximal cut, for simplicity we adopt the notation:

$$I_{1,1,1,1,1,1,1,1,a_1,a_2,a_3} \equiv I_{a_1,a_2,a_3}. \tag{5.100}$$

We find as master integrals

$$\begin{aligned} I_{0,0,0} &= \frac{1}{z_1 z_2 z_3 z_4 z_5 z_6 z_7 z_8}, \\ I_{0,-1,0} &= \frac{z_{10}}{z_1 z_2 z_3 z_4 z_5 z_6 z_7 z_8}, \\ I_{0,0,-1} &= \frac{z_{11}}{z_1 z_2 z_3 z_4 z_5 z_6 z_7 z_8}. \end{aligned} \tag{5.101}$$

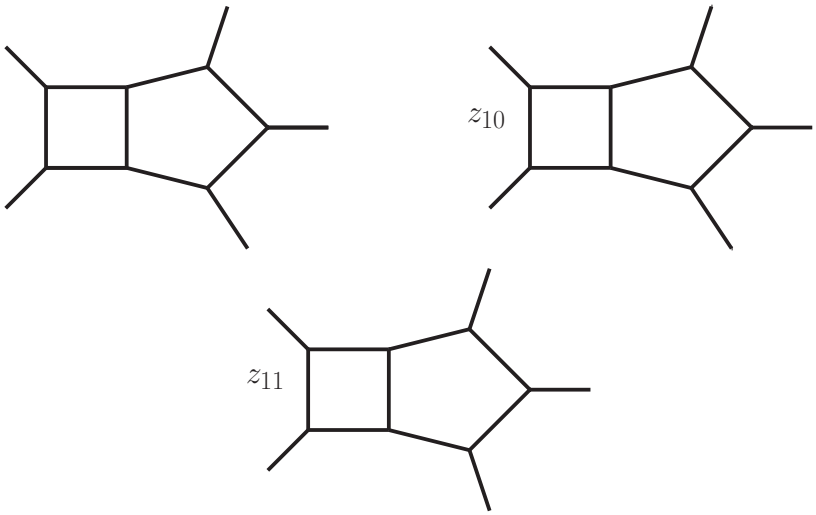

FIGURE 5.11: Master integrals for the massless penta box on the max cut.

And we calculate the decomposition into master integrals of the following integrals

$$\begin{aligned} &I_{-1,0,0}, \quad I_{-1,-1,0}, \quad I_{0,-2,0}, \\ &I_{-1,-1,-1}, \quad I_{-1,-1,-2}, \quad I_{-1,-2,-1}, \\ &I_{-2,-1,-1}. \end{aligned} \tag{5.102}$$

We choose the following ordering of variables, from the innermost to the outermost

$$z_{10} \to z_9 \to z_{11}. \tag{5.103}$$

And the internal master forms as

$$\begin{aligned} &\text{integration in } z_{10}\text{:} \quad \Big\{1\Big\}, \\ &\text{integration in } z_9\text{:} \quad \Big\{1\Big\}. \end{aligned} \tag{5.104}$$

We find the following decompositions. In our calculation with the intersection theory approach we set the Mandelstam invariants to random numerical values, keeping the ϵ dependence analytic. We report the analytic form of decompositions as obtained with the reconstruction performed with FINITEFLOW by solving the Laporta system, which is in agreement with our numerical results.

$$\begin{aligned} I_{-1,-1,-1} = \\ \frac{1}{2(2\epsilon-1)(s_{12}+s_{23}-s_{45})}\Big(&\epsilon s_{12}s_{23}(-3s_{12}^2 - s_{12}(2s_{23} - 3s_{34} - 3s_{45} \end{aligned}$$

$$
\begin{aligned}
&+ s_{51}) + 2s_{23}s_{34} + s_{45}(s_{51} - 3s_{34}))\Big) I_{0,0,0} \\
&+ \frac{1}{4\epsilon - 2}\Big(\epsilon(3s_{12}^2 - 3s_{12}(s_{34} + s_{45}) + s_{34}s_{45})\Big) I_{0,0,-1} \\
&+ \frac{1}{(2\epsilon - 1)(s_{12} + s_{23} - s_{45})}\Big(\epsilon s_{12}(3s_{12}^2 + s_{12}(2s_{23} - 3s_{34} \\
&- 3s_{45} + 2s_{51}) - 2s_{23}s_{34} + s_{23}s_{51} + 3s_{34}s_{45} - 2s_{45}s_{51})\Big) I_{0,-1,0}.
\end{aligned} \tag{5.105}
$$

$$
\begin{aligned}
I_{-1,-1,0} = &\\
&\frac{1}{2(2\epsilon - 1)(s_{12} + s_{23} - s_{45})}\Big(s_{23}((2\epsilon + 1)s_{12}^2 + s_{12}(\epsilon s_{23} \\
&- 2\epsilon s_{34} - 2\epsilon s_{45} + \epsilon s_{51} + s_{23} - s_{34} - s_{45}) - (\epsilon + 1)s_{23}s_{34} \\
&+ s_{45}(2\epsilon s_{34} - \epsilon s_{51} + s_{34}))\Big) I_{0,0,0} \\
&+ \frac{1}{2 - 4\epsilon}\Big((2\epsilon + 1)(s_{12} - s_{34} - s_{45})\Big) I_{0,0,-1} \\
&+ \frac{1}{(2\epsilon - 1)(s_{12} + s_{23} - s_{45})}\Big(- ((2\epsilon + 1)s_{12}^2) - s_{12}(\epsilon s_{23} - 2\epsilon s_{34} \\
&- 2\epsilon s_{45} + \epsilon s_{51} + s_{23} - s_{34} - s_{45} + s_{51}) + s_{23}(\epsilon s_{34} + s_{34} - s_{51}) \\
&+ s_{45}(-2\epsilon s_{34} + \epsilon s_{51} - s_{34} + s_{51})\Big) I_{0,-1,0}.
\end{aligned} \tag{5.106}
$$

$$
\begin{aligned}
I_{0,-2,0} = &\\
&\frac{1}{2(2\epsilon - 1)(s_{12} - s_{34} + s_{51})}\Big(s_{23}((2\epsilon + 1)s_{12}^2 + s_{12}(\epsilon s_{23} - 2\epsilon s_{34} - 2\epsilon s_{45} \\
&+ \epsilon s_{51} + s_{23} - s_{34} - s_{45}) - (\epsilon + 1)s_{23}s_{34} + \epsilon s_{45}(2s_{34} - s_{51}))\Big) I_{0,0,0} \\
&- \frac{1}{2(2\epsilon - 1)(s_{12} - s_{34} + s_{51})}\Big((2\epsilon + 1)(s_{12} + s_{23} - s_{45})(s_{12} - s_{34} - s_{45})\Big) I_{0,0,-1} \\
&+ \frac{1}{(2\epsilon - 1)(s_{12} - s_{34} + s_{51})}\Big(- ((2\epsilon + 1)s_{12}^2) + s_{12}(2\epsilon s_{34} + 2\epsilon s_{45} - 2\epsilon s_{51} \\
&- 2s_{23} + s_{34} + s_{45}) + 2\epsilon s_{45}(s_{51} - s_{34}) + s_{23}(2s_{34} - s_{51})\Big) I_{0,-1,0}.
\end{aligned} \tag{5.107}
$$

$$
\begin{aligned}
I_{-1,0,0} = &\\
&\frac{1}{2(s_{12} + s_{23} - s_{45})}\Big(s_{12}(s_{51} - s_{23}) + s_{23}s_{34} - s_{45}s_{51}\Big) I_{0,0,0}
\end{aligned}
$$

$$+\frac{s_{12}-s_{34}+s_{51}}{s_{12}+s_{23}-s_{45}}\,I_{0,-1,0}. \tag{5.108}$$

$I_{-1,-2,-1} =$

$$\begin{aligned}
&\frac{1}{4(2\epsilon^2-3\epsilon+1)(s_{12}+s_{23}-s_{45})(s_{12}-s_{34}+s_{51})}(\epsilon s_{12}s_{23}((9\epsilon-3)s_{12}^4\\
&+s_{12}^3(2\epsilon(5s_{23}-9s_{34}-9s_{45}+5s_{51})+6s_{34}+6s_{45}-5s_{51})+s_{12}^2(2(\epsilon+1)s_{23}^2\\
&-s_{23}(\epsilon(20s_{34}+10s_{45}-9s_{51})+2s_{51})+(9\epsilon-3)s_{34}^2+s_{34}(34\epsilon s_{45}-10\epsilon s_{51}\\
&-12s_{45}+5s_{51})+9\epsilon s_{45}^2-20\epsilon s_{45}s_{51}+2\epsilon s_{51}^2-3s_{45}^2+10s_{45}s_{51}-s_{51}^2)\\
&+s_{12}(-((\epsilon+1)s_{23}^2(4s_{34}-s_{51}))+\epsilon s_{23}(10s_{34}^2\\
&+18s_{34}s_{45}-9s_{34}s_{51}-9s_{45}s_{51}+s_{51}^2)\\
&+(6-16\epsilon)s_{34}^2s_{45}+s_{34}s_{45}((6-16\epsilon)s_{45}\\
&+9(2\epsilon-1)s_{51})+(2\epsilon-1)s_{45}s_{51}(5s_{45}-2s_{51})\\
&+2s_{23}s_{51}(s_{34}+s_{45}))+(\epsilon+1)s_{23}^2s_{34}(2s_{34}-s_{51})-s_{23}s_{45}(\epsilon(8s_{34}^2-7s_{34}s_{51}+s_{51}^2)\\
&+s_{34}s_{51})+s_{45}^2((7\epsilon-3)s_{34}^2+(4-8\epsilon)s_{34}s_{51}+(2\epsilon-1)s_{51}^2)))\,I_{0,0,0}+\\
&-\frac{1}{4(2\epsilon^2-3\epsilon+1)(s_{12}-s_{34}+s_{51})}(\epsilon((9\epsilon-3)s_{12}^4\\
&+s_{12}^3((4\epsilon+2)s_{23}+(6-18\epsilon)s_{34}-18\epsilon s_{45}+7\epsilon s_{51}+6s_{45}-4s_{51})\\
&+s_{12}^2(-(2\epsilon+1)s_{23}(4s_{34}+2s_{45}-s_{51})+(9\epsilon-3)s_{34}^2+s_{34}(28\epsilon s_{45}-7\epsilon s_{51}\\
&-10s_{45}+4s_{51})+s_{45}(9\epsilon s_{45}-14\epsilon s_{51}-3s_{45}+8s_{51}))+s_{12}((2\epsilon+1)s_{23}(s_{34}\\
&+s_{45})(2s_{34}-s_{51})+s_{45}((4-10\epsilon)s_{34}^2+(4-10\epsilon)s_{34}s_{45}\\
&+(8\epsilon-5)s_{34}s_{51}+(7\epsilon-4)s_{45}s_{51}))+(\epsilon-1)s_{34}s_{45}^2(s_{34}-s_{51})))\,I_{0,0,-1}+\\
&-\frac{1}{2(2\epsilon^2-3\epsilon+1)(s_{12}+s_{23}-s_{45})(s_{12}-s_{34}+s_{51})}(\epsilon s_{12}((9\epsilon-3)s_{12}^4\\
&+s_{12}^3(\epsilon(10s_{23}-18s_{34}-18s_{45}+13s_{51})+6(s_{34}\\
&+s_{45}-s_{51}))+s_{12}^2(2(\epsilon+1)s_{23}^2-s_{23}(\epsilon(20s_{34}+10s_{45}-13s_{51})+2s_{51})\\
&+(9\epsilon-3)s_{34}^2+s_{34}(34\epsilon s_{45}-13\epsilon s_{51}-12s_{45}+6s_{51})\\
&+9\epsilon s_{45}^2-26\epsilon s_{45}s_{51}+4\epsilon s_{51}^2-3s_{45}^2+12s_{45}s_{51}-2s_{51}^2)+s_{12}(-2(\epsilon+1)s_{23}^2(2s_{34}\\
&-s_{51})+\epsilon s_{23}(10s_{34}^2+18s_{34}s_{45}-13s_{34}s_{51}-13s_{45}s_{51}+3s_{51}^2)+(6-16\epsilon)s_{34}^2s_{45}\\
&+s_{34}s_{45}((6-16\epsilon)s_{45}+(24\epsilon-11)s_{51})+s_{45}s_{51}((13\epsilon-6)s_{45}\\
&+(4-8\epsilon)s_{51})+2s_{23}s_{51}(s_{34}+s_{45}))+s_{23}^2(2(\epsilon+1)s_{34}^2\\
&-2(\epsilon+1)s_{34}s_{51}+s_{51}^2)-s_{23}s_{45}(\epsilon(8s_{34}^2-11s_{34}s_{51}\\
&+3s_{51}^2)+s_{34}s_{51})+s_{45}^2(s_{34}-s_{51})((7\epsilon-3)s_{34}+(2-4\epsilon)s_{51})))\,I_{0,-1,0}.
\end{aligned} \tag{5.109}$$

$I_{-2,-1,-1} =$

$$
\begin{aligned}
&\frac{1}{4(2\epsilon^2-3\epsilon+1)(s_{12}+s_{23}-s_{45})^2}\Big(\epsilon s_{12}s_{23}((9\epsilon-3)s_{12}^4\\
&+s_{12}^3((13\epsilon-6)s_{23}+(6-18\epsilon)s_{34}-18\epsilon s_{45}+7\epsilon s_{51}+6s_{45}+s_{51})\\
&+s_{12}^2((4\epsilon-2)s_{23}^2+s_{23}((12-26\epsilon)s_{34}-13\epsilon s_{45}+8\epsilon s_{51}+6s_{45})+(9\epsilon-3)s_{34}^2\\
&+s_{34}(34\epsilon s_{45}-7\epsilon s_{51}-12s_{45}-s_{51})+9\epsilon s_{45}^2-14\epsilon s_{45}s_{51}+\epsilon s_{51}^2\\
&-3s_{45}^2-2s_{45}s_{51}+s_{51}^2)+s_{12}(s_{23}^2(-8\epsilon s_{34}+\epsilon s_{51}+4s_{34}+s_{51})+s_{23}((13\epsilon-6)s_{34}^2\\
&+(24\epsilon-11)s_{34}s_{45}-8\epsilon s_{34}s_{51}+\epsilon s_{51}(s_{51}-8s_{45}))+s_{45}((6-16\epsilon)s_{34}^2\\
&+2s_{34}((3-8\epsilon)s_{45}+6\epsilon s_{51}+s_{51})+s_{51}(7\epsilon s_{45}-2(\epsilon+1)s_{51}+s_{45})))\\
&+s_{23}^2s_{34}((4\epsilon-2)s_{34}-(\epsilon+1)s_{51})+s_{23}s_{45}((5-11\epsilon)s_{34}^2+s_{34}(6\epsilon s_{51}+s_{51})\\
&-\epsilon s_{51}^2)+s_{45}^2((7\epsilon-3)s_{34}^2-s_{34}(5\epsilon s_{51}+s_{51})+(\epsilon+1)s_{51}^2))\Big)I_{0,0,0}\\
&+\frac{1}{4(2\epsilon^2-3\epsilon+1)(s_{12}+s_{23}-s_{45})}\Big(\epsilon((3-9\epsilon)s_{12}^4\\
&-s_{12}^3((7\epsilon-4)s_{23}+2((3-9\epsilon)s_{34}-9\epsilon s_{45}+2\epsilon s_{51}+3s_{45}+s_{51}))\\
&+s_{12}^2(s_{23}(2(7\epsilon-4)s_{34}+7\epsilon s_{45}-2\epsilon s_{51}-4s_{45}-s_{51})\\
&+(3-9\epsilon)s_{34}^2+2s_{34}(-14\epsilon s_{45}+2\epsilon s_{51}+5s_{45}+s_{51})+s_{45}(-9\epsilon s_{45}\\
&+8\epsilon s_{51}+3s_{45}+4s_{51}))+s_{12}(s_{23}((4-7\epsilon)s_{34}^2+s_{34}(-8\epsilon s_{45}\\
&+2\epsilon s_{51}+5s_{45}+s_{51})+(2\epsilon+1)s_{45}s_{51})+2s_{45}(s_{34}+s_{45})((5\epsilon-2)s_{34}\\
&-(2\epsilon+1)s_{51}))+(\epsilon-1)s_{34}^2s_{45}(s_{23}-s_{45}))\Big)I_{0,0,-1}\\
&-\frac{1}{2(2\epsilon^2-3\epsilon+1)(s_{12}+s_{23}-s_{45})^2}\Big(\epsilon s_{12}((9\epsilon-3)s_{12}^4\\
&+s_{12}^3((13\epsilon-6)s_{23}+2((3-9\epsilon)s_{34}-9\epsilon s_{45}+5\epsilon s_{51}+3s_{45}))\\
&+s_{12}^2((4\epsilon-2)s_{23}^2+s_{23}((12-26\epsilon)s_{34}-13\epsilon s_{45}+13\epsilon s_{51}\\
&+6s_{45}-2s_{51})+(9\epsilon-3)s_{34}^2+2s_{34}(17\epsilon s_{45}-5\epsilon s_{51}-6s_{45})+9\epsilon s_{45}^2\\
&-20\epsilon s_{45}s_{51}+2\epsilon s_{51}^2-3s_{45}^2+2s_{51}^2)+s_{12}(s_{23}^2((4-8\epsilon)s_{34}\\
&+3\epsilon s_{51})+s_{23}((13\epsilon-6)s_{34}^2+(24\epsilon-11)s_{34}s_{45}+(2-13\epsilon)s_{34}s_{51}\\
&+(2-13\epsilon)s_{45}s_{51}+2(\epsilon+1)s_{51}^2)-2s_{45}((8\epsilon-3)s_{34}^2\\
&+(8\epsilon-3)s_{34}s_{45}-9\epsilon s_{34}s_{51}+s_{51}(-5\epsilon s_{45}+2\epsilon s_{51}+2s_{51})))\\
&+s_{23}^2((4\epsilon-2)s_{34}^2-3\epsilon s_{34}s_{51}+s_{51}^2)-s_{23}s_{45}((11\epsilon-5)s_{34}^2\\
&+s_{34}(s_{51}-11\epsilon s_{51})+2(\epsilon+1)s_{51}^2)+s_{45}^2((7\epsilon-3)s_{34}^2\\
&-8\epsilon s_{34}s_{51}+2(\epsilon+1)s_{51}^2))\Big)I_{0,-1,0}. \qquad (5.110)
\end{aligned}
$$

$$
\begin{aligned}
&I_{-1,-1,-2}=\\
&\frac{1}{4(2\epsilon-1)(s_{12}+s_{23}-s_{45})(s_{12}-s_{34}-s_{45})}\Big(s_{12}s_{23}((9\epsilon-3)s_{12}^4
\end{aligned}
$$

$$
\begin{aligned}
&+ s_{12}^3((7\epsilon-3)s_{23} - 2((9\epsilon-3)s_{34} + 9\epsilon s_{45} - 2\epsilon s_{51} - 3s_{45} + s_{51})) \\
&+ s_{12}^2((6-14\epsilon)s_{23}s_{34} + (3-7\epsilon)s_{23}s_{45} + 2(\epsilon-1)s_{23}s_{51} + (9\epsilon-3)s_{34}^2 \\
&+ (28\epsilon-9)s_{34}s_{45} + (2-4\epsilon)s_{34}s_{51} + (9\epsilon-3)s_{45}^2 + (4-8\epsilon)s_{45}s_{51}) \\
&+ s_{12}(s_{23}((7\epsilon-3)s_{34}^2 + (8\epsilon-3)s_{34}s_{45} - 2(\epsilon-1)s_{34}s_{51} \\
&- 2(\epsilon-1)s_{45}s_{51}) - s_{45}(s_{34}+s_{45})((10\epsilon-3)s_{34} \\
&+ (2-4\epsilon)s_{51})) + \epsilon s_{34}^2 s_{45}(s_{45}-s_{23}))\Big) I_{0,0,0} \\
&+ \frac{1}{4(2\epsilon-1)(s_{12}-s_{34}-s_{45})}\Big((3-9\epsilon)s_{12}^4 + s_{12}^3(-\epsilon s_{23} \\
&+ 6(3\epsilon-1)s_{34} + 18\epsilon s_{45} - \epsilon s_{51} + s_{23} - 6s_{45} + s_{51}) \\
&+ s_{12}^2((\epsilon-1)s_{23}(2s_{34}+s_{45}) + (3-9\epsilon)s_{34}^2 + (7-22\epsilon)s_{34}s_{45} \\
&+ (\epsilon-1)s_{34}s_{51} + (3-9\epsilon)s_{45}^2 + 2(\epsilon-1)s_{45}s_{51}) \\
&+ s_{12}(s_{23}s_{34}(-\epsilon s_{34} + s_{34} + s_{45}) + s_{45}((4\epsilon-1)s_{34}^2 + s_{34}(4\epsilon s_{45} \\
&- s_{45} + s_{51}) - (\epsilon-1)s_{45}s_{51})) + \epsilon s_{34}s_{45}(s_{45}(s_{34}-s_{51}) - s_{23}s_{34})\Big) I_{0,0,-1} \\
&+ \frac{1}{2(2\epsilon-1)(s_{12}+s_{23}-s_{45})(s_{12}-s_{34}-s_{45})}\Big(s_{12}((3-9\epsilon)s_{12}^4 \\
&+ s_{12}^3((3-7\epsilon)s_{23} + 6(3\epsilon-1)s_{34} + 18\epsilon s_{45} - 7\epsilon s_{51} - 6s_{45} \\
&+ 3s_{51}) + s_{12}^2(s_{23}(2(7\epsilon-3)s_{34} + 7\epsilon s_{45} - 5\epsilon s_{51} - 3s_{45} \\
&+ 3s_{51}) + (3-9\epsilon)s_{34}^2 + s_{34}(-28\epsilon s_{45} + 7\epsilon s_{51} + 9s_{45} - 3s_{51}) \\
&+ s_{45}(-9\epsilon s_{45} + 14\epsilon s_{51} + 3s_{45} - 6s_{51})) + s_{12}(s_{23}((3-7\epsilon)s_{34}^2 \\
&+ (3-8\epsilon)s_{34}s_{45} + (5\epsilon-3)s_{34}s_{51} + (5\epsilon-3)s_{45}s_{51}) \\
&+ s_{45}((10\epsilon-3)s_{34}^2 + (10\epsilon-3)s_{34}s_{45} + (3-8\epsilon)s_{34}s_{51} \\
&+ (3-7\epsilon)s_{45}s_{51})) + \epsilon s_{34}s_{45}(s_{23}-s_{45})(s_{34}-s_{51}))\Big) I_{0,-1,0}. \quad (5.111)
\end{aligned}
$$

6 | Conclusions

Scattering amplitudes are a fundamental object in the study of fundamental interactions. They enter in the calculation of the S-matrix, whose modulus squared represents the probability of a certain process to occur and therefore plays a fundamental role in the connection between theory and experiment. Moreover, their theoretical study allows to unveil properties of the theory that are not manifest in the Lagrangian formulation. Their building blocks are the Feynman diagrams participating in the process, thus making the study of the associated Feynman integrals of primary importance. Among the several aspects to be considered when dealing with the so-called *Feynman integrals calculus*, in this thesis we focused on the techniques to perform the decomposition of Feynman integrals into a minimal, linearly independent set, the so-called *master integrals*.

The importance of decomposing Feynman integrals into master integrals lies in the possibility of reducing the number of integrals to be effectively calculated from hundreds or thousands into a smaller set. Up to now, the main decomposition strategy has been the one based on the Laporta algorithm, which exploits linear relations obeyed by Feynman integrals in order to build a large system of identities, whose resolution yields the decomposition of the set of *needed* Feynman integrals into master integrals. The solution of such a system is a major bottleneck in terms of computational resources when diagrams with an high number of loops and legs are involved. It is therefore very interesting to look for decomposition strategies that avoid large systems of identities, preferring direct approaches. *Intersection theory* is a purely algebraic approach to the decomposition of Feynman integrals, that exploits their nature of elements of a vector space. It allows to reduce any Feynman integral into a basis of master integrals in the same way a vector is decomposed into its basis elements via projections. This is made possible by the introduction of a scalar product in the space of Feynman integrals, called *intersection number*, that is a rational function of the kinematic invariants and of the dimensional regulator. The state of the art for decomposition via intersection theory is given by a recursive algorithm in which non-rational

poles explicitly appear in intermediate stages. In order to compute intersection numbers several steps need to be carried out for each pole of a function: it is necessary to solve a differential equation around it, followed by taking the residue of the product of two functions, one of which being the solution of the differential equation.

In this thesis we focused on the implementation of a purely rational algorithm that avoids the explicit use of non-rational poles. In order to do so, we introduced the concept of $p(z)-$adic expansion of a rational function, in analogy to the already known concept of $p-$adic expansion of a rational number, that allowed us to expand any rational function in terms of the polynomial irreducible over $\mathbb{Q}$, whose zeroes are the non-rational poles. The final result consists in an algorithm that treats non-rational solutions satisfying the same rational polynomial equation "all at once" and calculates their contribution to the intersection number by expanding around the irreducible polynomial in consideration. The algorithm has been implemented on a MATHEMATICA routine and has been tested on the decomposition of several Feynman integrals, both at one and two loops. The results have been checked against the decomposition obtained using the traditional Laporta algorithm in the framework FINITEFLOW.

It is the first time that a purely rational algorithm for the computation of intersection numbers that does not require complex transformations of the integrals or any change of basis is presented. We see this as a promising result, that can lead, in the future, to an effective implementation of the algorithm in frameworks based on finite fields, such as FINITEFLOW. This and similar tools allow very efficient implementations of rational algorithms, and have already been used for cutting-edge calculations in high energy physics. The results presented in this thesis thus opened up the possibility of using this technology for computing intersection numbers between Feynman integrals in the context of multi-loop predictions. In conclusion, the effective full-automation of the algorithm and its distribution could lead to improvements in making cutting-edge predictions for theoretical high energy physics.

A | A short tour of Algebraic Geometry

In this Appendix we review some basic concepts about algebraic geometry. We don't seek mathematical rigorousness, preferring to focus on ideas and concepts. We closely follow the presentation in Appendix B of [41].

Algebraic geometry is the branch of mathematics studying polynomial equations and their solutions. We indicate with $P[\mathbf{z}] = P[z_1, \ldots, z_n]$ the ring of polynomials in the variables $\mathbf{z} = \{z_i\}_{i=1}^n$ over the field $\mathbb{F}$, which is usually $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Z}/p$.

We recall some basic definitions about polynomials and rational functions, following [43]. We indicate a *monomial* in the variables $\mathbf{z} = z_1, \ldots, z_n$ with

$$\mathbf{z}^\alpha = \prod_{i=1}^n z_i^{\alpha_i}, \tag{A.1}$$

with $\alpha = (\alpha_1, \ldots, \alpha_n)$ an n-dimensional vector with $\alpha_i \geq 0$. Its *total degree* is defined as

$$|\alpha| = \sum_{i=1}^n \alpha_i. \tag{A.2}$$

A *polynomial* in $\mathbf{z}$ is defined as a finite linear combination of monomials with coefficients in a field $\mathbb{F}$. The set of polynomials in the variables $\mathbf{z}$ with coefficients in $\mathbb{F}$ forms a ring that we indicate with $\mathbb{F}[\mathbf{z}]$. Any polynomial $f \in \mathbb{F}[\mathbf{z}]$ is uniquely identified by a set of exponents $\{\alpha\}$ and coefficients $c_\alpha \in \mathbb{F}$

$$f(\mathbf{z}) = \sum_\alpha c_\alpha \mathbf{z}^\alpha. \tag{A.3}$$

We indicate with $\mathbb{F}(\mathbf{z})$ the field of rational functions in the variables $\mathbf{z}$ with coefficients in $\mathbb{F}$. Any rational function $f \in \mathbb{F}(\mathbf{z})$ can be expressed as the ratio of two polynomials p,$q \in \mathbb{F}[\mathbf{z}]$

$$f(\mathbf{z}) = \frac{p(\mathbf{z})}{q(\mathbf{z})} = \frac{\sum_\alpha n_\alpha \mathbf{z}^\alpha}{\sum_\beta d_\beta \mathbf{z}^\beta}, \qquad n_\alpha, d_\beta \in \mathbb{F}. \tag{A.4}$$

A.1 | Polynomial ideals

An important concept in algebraic geometry is the *ideal*. Suppose we have a set of polynomials $\mathbf{p} = \{p_1, \ldots, p_m\}$ in the variables $\mathbf{z} = (z_1, \ldots, z_n)$. We define the *ideal* $\mathcal{J}$ generated by $\mathbf{p}$ as the set of linear combination of generic polynomials $h_k(\mathbf{z}) \in P[\mathbf{z}]$ multiplied by elements of $\mathbf{p}$:

$$\mathcal{J} \equiv < p_1, \ldots, p_m > = \left\{ \sum_{k=1}^{m} h_k(\mathbf{z}) p_k(\mathbf{z}),\ h_k(\mathbf{z}) \in P[\mathbf{z}] \right\}. \tag{A.5}$$

Notice that if $\mathbf{z^*}$ is a solution of the set of polynomials $\mathbf{p}$, that is,

$$\mathbf{z^*} \mid \quad p_1(\mathbf{z^*}) = 0, \ldots, p_m(\mathbf{z^*}) = 0, \tag{A.6}$$

then all elements of $\mathcal{J}$ are zero on $\mathbf{z^*}$

$$\forall q \in \mathcal{J}, \quad q(\mathbf{z^*}) = 0. \tag{A.7}$$

The inverse is not always true: if a polynomial vanishes on all the solutions of the system $\mathbf{p} = 0$ it doesn't necessarily belong to the ideal $\mathcal{J}$.

Given an ideal $\mathcal{J}$, we call *algebraic variety* the set of points on which all the polynomials of the ideal vanish:

$$\mathcal{V}(\mathcal{J}) = \{\mathbf{z} \mid \quad q(\mathbf{z}) = 0, \quad \forall q \in \mathcal{J}\}. \tag{A.8}$$

The algebraic variety is associated to the generators of the ideal $\mathcal{J}$. The set of polynomials vanishing on the elements of an algebraic variety $\mathcal{V}$ is itself an ideal and we call it $\mathcal{I}(\mathcal{V})$:

$$\mathcal{I}(\mathcal{V}) = \{h \in P[\mathbf{z}] \mid \quad h(\mathbf{z}) = 0,\ \forall \mathbf{z} \in \mathcal{V}\}. \tag{A.9}$$

For any ideal $\mathcal{J}$ we can see that

$$\mathcal{J} \subseteq \mathcal{I}(\mathcal{V}(\mathcal{J})). \tag{A.10}$$

That is, the ideal is contained in the set of polynomials that vanish on the zeroes of the ideal. But, as we said, these two sets are not always equal. The equality is reached when the ideal $\mathcal{J}$ has a certain property and is called *radical*. The *radical of an ideal* is indicated as $\sqrt{\mathcal{J}}$ and is defined as

$$\sqrt{\mathcal{J}} = \{p \in P[\mathbf{z}] \mid p^k \in \mathcal{J} \text{ for some integer } k > 0\}. \tag{A.11}$$

Therefore an ideal is called radical if it is true that $\sqrt{\mathcal{J}} = \mathcal{J}$.

An important theorem is Hilbert's Nullstellensatz: it proves the equality between an ideal and the set of polynomials vanishing on the elements of the algebraic variety associated to that ideal. The theorem states

$$\mathcal{I}(\mathcal{V}(\mathcal{J})) = \sqrt{\mathcal{J}}. \tag{A.12}$$

It is obvious that, if $\mathcal{J}$ is radical we have

$$\mathcal{I}(\mathcal{V}(\mathcal{J})) = \mathcal{J}. \tag{A.13}$$

In the case when the variety $\mathcal{V}(\mathcal{J}) = \varnothing$, that is, the system of equations $p_1(\mathbf{z}) = \cdots = p_m(\mathbf{z}) = 0$ generated by the ideal $\mathcal{J}$ has zero solutions, it holds the weak version of Hilbert's Nullstellensatz theorem:

$$\mathcal{V}(\mathcal{J}) = \varnothing \iff \exists\, h_1, \ldots, h_m \in P[\mathbf{z}] \mid \sum_{i=1}^{m} h_i(\mathbf{z})\, p_i(\mathbf{z}) = 1. \tag{A.14}$$

If the conditions of Hilbert's weak Nullstellensatz are satisfied, then $\mathcal{J} = P[\mathbf{z}]$ and any polynomial p in $P[\mathbf{z}]$ can be written as a linear combination of the generators of $\mathcal{J}$.

A.2 | Monomial ordering

Given a polynomial ring $P[\mathbf{z}]$, it is important to fix an *ordering* between the monomials. In one variable it is easy as there is only one possible ordering, but when it comes to more than one variable there are several possibilities. A *monomial order*, or *monomial ordering*, in the ring of polynomials $P[\mathbf{z}]$ is a total order relation in the set of monomials of the ring that satisfies

- if $m_1 < m_2$ then $m_1\, m_3 < m_2\, m_3$,
- $\forall m_1 \neq 1$ then $m_1 > 1$.

with m_1, m_2, m_3 monomials in $P[\mathbf{z}]$. In the case of univariate polynomials in the variable z the only monomial ordering satisfying these conditions is

$$1 < z < z^2 < \ldots .$$

In the case of multivariate polynomials in the variables $z_1, \ldots, z_n$ several orderings can be defined, such as the *lexicographical order* and the *degree reverse lexicographical order* (see [41] for more details).

A.3 | Polynomial division

Given a set of polynomials $\{p_1, \ldots, p_m\}$ in the variables $\mathbf{z}$ and a monomial ordering, the polynomial division algorithm allows to write any polynomial $f \in P[\mathbf{z}]$ as

$$f(\mathbf{z}) = \mathcal{Q}(\mathbf{z}) + \mathcal{R}(\mathbf{z}), \qquad \text{with } \mathcal{Q}(\mathbf{z}) = \sum_{i=1}^{m} q_i(\mathbf{z})\, p_i(\mathbf{z}). \tag{A.15}$$

$\mathcal{Q}(\mathbf{z})$ and $\mathcal{R}(\mathbf{z})$ respectively identified as the *quotient* and the *reminder* of the polynomial division. Notice that, following what we have defined, $\mathcal{Q}(\mathbf{z}) \in \mathcal{J}$, $\mathcal{J} =< p_1, \ldots, p_m >$. It is also said that f is *reduced* to $\mathcal{R}$ modulo the polynomials $\{p_1, \ldots, p_m\}$ or:

$$f(\mathbf{z}) = \mathcal{R}(\mathbf{z}) \mod \{p_i\}_{i=1}^{m}. \tag{A.16}$$

Unfortunately, this decomposition is in general not unique and the quotient and the reminder can depend on the order in which the polynomials p_k are taken. Moreover, the reminder could be different from zero even if f belongs to the ideal $\mathcal{J}$, a situation that we necessarily want to avoid. These downsides can be avoided by introducing the concept of *Gröbner basis*, defined as the set $\{g_1, \ldots, g_\ell\}$ satisfying the following:

- generates $\mathcal{J}$
$$\mathcal{J} =< p_1, \ldots, p_m >=< g_1, \ldots, g_\ell >; \tag{A.17}$$
- the reminder of the polynomial division modulo $\{g_i\}_{i=1}^{\ell}$ is unique once the monomial ordering is fixed. In particular:
  - the reminder of the division modulo the Gröbner basis $\mathcal{R}_{\mathcal{G}_\mathcal{J}}$ depends only the ideal $\mathcal{J}$ and on the dividend f;
  - $\mathcal{R}_{\mathcal{G}_\mathcal{J}}(\mathbf{z}) = 0$ if and only if the dividend is a member of the ideal, $f \in \mathcal{J}$.

It can be shown that every ideal has a Gröbner basis. In general the number of its elements can be different from the number of the generators used to define the ideal, i.e. $\ell \neq m$. In the univariate case the polynomial division has already the "good properties" that we obtain using a Gröbner basis in the multivariate case. Having introduced the Gröbner basis in the multivariate case, we now can talk, both in the univariate and in the multivariate case, of *reduction modulo a polynomial ideal*. Consider a polynomial ring $P[\mathbf{z}]$, an ideal $\mathcal{J}$ and a Gröbner basis $\mathcal{G}_\mathcal{J}$. A polynomial $f(\mathbf{z})$ is said to be reduced

to $\mathcal{R}(\mathbf{z})$ modulo $\mathcal{J}$ if $\mathcal{R}(\mathbf{z})$ is the reminder of the polynomial division of $f(\mathbf{z})$ modulo the Gröbner basis $\mathcal{G}_\mathcal{J}$. We indicate the reduced f with $\lfloor f \rfloor_\mathcal{J}$ and the definition takes the form

$$\lfloor f \rfloor_\mathcal{J} = f \qquad \mod \mathcal{G}_\mathcal{J}. \tag{A.18}$$

In $P[\mathbf{z}]$, for any ideal $\mathcal{J}$, we can define an equivalence relation $\sim_\mathcal{J}$ between polynomials: two polynomials $p(\mathbf{z}), q(\mathbf{z})$ are in the same class of equivalence if their difference belongs to the ideal $\mathcal{J}$

$$p \sim_\mathcal{J} q \iff p - q \in \mathcal{J}. \tag{A.19}$$

Another way to say this is that polynomials are equivalent if the reduction of their difference modulo a Gröbner basis of $\mathcal{J}$ is zero.

It can be shown that the set of equivalence classes defined by $\sim_\mathcal{J}$ is a polynomial ring, called *quotient ring* and denoted by

$$P[\mathbf{z}]/\mathcal{J} \equiv \{[p]_\mathcal{J} \,|\, p \in P[\mathbf{z}]\}. \tag{A.20}$$

We can perform algebraic operation between classes by doing that operation on the representative (as the result does not depend on the representative). The most natural way to identify an equivalence class with a representative is by doing

$$[\, p \,]_\mathcal{J} \leftrightarrow \lfloor\, p \,\rfloor_\mathcal{J}. \tag{A.21}$$

That is, each equivalence class is identified with the reminder of a representative modulo $\mathcal{G}_\mathcal{J}$. This reasoning allows us to perform operation between polynomials in the quotient ring using the same concepts of modular arithmetic. For each nonzero polynomial $q(\mathbf{z}) \in P[\mathbf{z}]$, assuming $q(\mathbf{z})$ and each generator of the ideal have no common zeroes, we can define the *multiplicative inverse* modulo the ideal $\mathcal{J}$ generated by the Gröbner basis $\{g_i(\mathbf{z})\}_{i=1}^{\ell}$. By Hilbert's Nullstellensatz, in fact, there exist polynomials $\tilde{q}(\mathbf{z}), \{\tilde{g}_i(\mathbf{z})\}_{i=1}^{\ell}$ such that

$$\tilde{q}(\mathbf{z})q(\mathbf{z}) + \sum_{i=1}^{\ell} \tilde{g}_i(\mathbf{z})g_i(\mathbf{z}) = 1 \tag{A.22}$$

that is,

$$\tilde{q}(\mathbf{z}) = \frac{1}{q(z)} \mod \mathcal{G}_\mathcal{J} \tag{A.23}$$

or, in other words

$$\tilde{q}(\mathbf{z})q(\mathbf{z}) = 1 \mod \mathcal{G}_\mathcal{J} \tag{A.24}$$

The multiplicative inverse can be calculated via the Euclidean algorithm or through an ansatz.

A.4 | Zero-dimensional ideals

We collect some result that we use as hidden hypotheses when performing the master monomials analysis in Sect. 3.3.1. Statements of theorems are reported from [41]. We begin with the *Finiteness theorem*.

Theorem A.4.1. *Finiteness. Let $\mathcal{J}$ be an ideal in $P[\mathbf{z}]$. The following two conditions are equivalent:*

- *$P[\mathbf{z}]/\mathcal{J}$ is a finite-dimensional space;*
- *$\mathcal{V}(\mathcal{J})$ is a finite set.*

If the conditions in the *Finiteness Theorem* are satisfied, the ideal $\mathcal{J}$ is said to be *zero dimensional*. The following proposition holds:

Lemma A.4.2. *Let $\mathcal{J}$ be a zero-dimensional ideal.*

$$\left(\mathcal{J} = \sqrt{\mathcal{J}}\right) \iff \textit{every solution in } \mathcal{V}(\mathcal{J}) \textit{ has multiplicity } 1. \tag{A.25}$$

We now turn to the *Shape Lemma* which is the theorem used to prove the dimension of the twisted cohomology vector space.

Theorem A.4.3. *Shape Lemma. Let $\mathcal{J}$ be a zero dimensional radical ideal in $P[\mathbf{z}]$. Then, the following condition is satisfied: if $\mathcal{V}(\mathcal{J})$ has n_s points, then the quotient ring $P[\mathbf{z}]/\mathcal{J}$ is n_s-dimensional, i.e. admits a basis of n_s monomials.*

To summarize, if a system $\{p_i(\mathbf{z}) = 0\}$ has n_s solutions with multiplicity 1, then the dimension of the quotient ring $P[\mathbf{z}]/\mathcal{J}$ ($\mathcal{J} =< p_i >$) is n_s. This justifies our procedure of counting the number of master integrals presented in Sect. 3.3.1.

A.5 | Univariate global residue

We give an account of the univariate global residue theorem as reported in [56], that is the main result we use to calculate intersection numbers through the rational algorithm proposed in Ch. 4.

The result is obtained for one variable, indicated with z, and it suits our needs as intersection numbers are calculated proceeding with one variable at time, leaving the other variables as constant parameters at each passage.

Given a polynomial $p(z) \in \mathbb{K}[z]$ and $f(z) \in \mathbb{K}(z)$ we denote the ideal generated by $p(z)$ with $< p >$. Let $f(z)$ be rational function (in our algorithm, $\mathbb{K} = \mathbb{Q}$) that we write as

$$f(z) = \frac{n(z)}{d(z)}, \qquad n(z), d(z) \in \mathbb{K}[z], \quad \text{with}\, \gcd(n(z), d(z)) = 1. \tag{A.26}$$

We want to compute the *global residue*

$$\text{Res}_{<p>}(f(z)), \tag{A.27}$$

where the global residue of a function $f(z)$ is defined as

$$\text{Res}_{<p>}(f) \equiv \sum_{z_i \in \mathcal{V}(<p>)} \text{Res}_{z=z_i}\left(\frac{f(z)}{p(z)}\right). \tag{A.28}$$

That is, the residue computed on all the zeroes of the ideal generated by $p(z)$ of the function $f(z)/p(z)$. Since $f(z)$ is rational, assuming $\gcd(d(z), p(z)) = 1$, we can calculate the inverse of $d(z)$ modulo $p(z)$ exploiting the concept of multiplicative inverse. We denote it with $\tilde{d}(z)$. This procedure allows us to write the global residue as

$$\text{Res}_{<p>}(f(z)) = \text{Res}_{<p>}\Big(n(z)\tilde{d}(z)\Big). \tag{A.29}$$

That allows us to replace a calculation with rational functions with a calculation with polynomials. This can be proved as follows: because of Hilbert's Nullstellensaz there exist two polynomials in z, denoted with $\tilde{p}(z)$ and $\tilde{d}(z)$ such that

$$\tilde{p}(z)p(z) + \tilde{d}(z)d(z) = 1, \tag{A.30}$$

so we can multiply $f(z)$ by 1 and substituting the l.h.s. of eq. (A.30) in the definition of global residue

$$\begin{aligned}
\text{Res}_{<p>}(f(z)) &= \sum_{z_i \in \mathcal{V}(<p>)} \text{Res}_{z=z_i}\left(\frac{f(z)}{p(z)}\right) = \sum_{z_i \in \mathcal{V}(<p>)} \text{Res}_{z=z_i}\left(\frac{n(z)}{d(z)}\frac{1}{p(z)}\right) \\
&= \sum_{z_i \in \mathcal{V}(<p>)} \text{Res}_{z=z_i}\left(\frac{n(z)}{d(z)}\frac{1}{p(z)}\Big(\tilde{p}(z)p(z) + \tilde{d}(z)d(z)\Big)\right) \\
&= \sum_{z_i \in \mathcal{V}(<p>)} \text{Res}_{z=z_i}\left(\frac{n(z)}{d(z)}\tilde{p}(z) + n(z)\tilde{d}(z)\frac{1}{p(z)}\right) \\
&= \sum_{z_i \in \mathcal{V}(<p>)} \text{Res}_{z=z_i}\left(n(z)\tilde{d}(z)\frac{1}{p(z)}\right) = \text{Res}_{<p>}\Big(n(z)\tilde{d}(z)\Big).
\end{aligned} \tag{A.31}$$

where in the last expression the first term has been removed because it gives zero contribution to the residue as it is an holomorphic function for all z_i and, as we assumed, $\gcd(d(z), p(z)) = 1$. We have the following proposition [56], that we refer to as *univariate global residue theorem*

Theorem A.5.1. *Univariate global residue. Let $p(z), n(z), d(z) \in P[z]$ with $gcd(p(z), d(z)) = gcd(n(z), d(z)) = 1$. Let deg $p = n$ and let $\tilde{d}(z)$ be the multiplicative inverse of $d(z)$ with respect to the ideal $< p >$. Then*

$$Res_{<p>}\left(\frac{n(z)}{d(z)}\right) = \frac{a_{n-1}}{c_n}. \tag{A.32}$$

where a_{n-1} is the coefficient of z^{n-1} in the reduction of $n(z)\tilde{d}(z)$ modulo $p(z)$ and c_n is the coefficient of z^n of $p(z)$ (namely, its leading coefficient).

B | Baikov representation

We write more in detail the derivation of Baikov representation, following closely [21, 55]. Starting from an L loop Feynman diagram in d dimensions with $E+1$ external legs and $p_1, \ldots, p_E$ independent external momenta, we can write its momentum parametrization as

$$\int \left(\prod_{i=1}^{L} \frac{\mathrm{d}^d k_i}{i\pi^{d/2}} \right) \frac{1}{z_1^{\alpha_1} \ldots z_N^{\alpha_N}}, \tag{B.1}$$

with $z_i = P_i^2 - M_i^2$, with P_i and M_i respectively indicating a combination of momenta (both internal and external) and masses, dictated by the structure of the diagram in consideration. Let $q_i = k_1, \ldots, k_L, p_1, \ldots, p_E$. We introduce the following notation for scalar products

$$s_{i,j} = q_i \cdot q_j, \qquad (i, j = 1, \ldots, L+E), \quad j \geq i. \tag{B.2}$$

where the last condition is added to take into account that $s_{i,j} = s_{j,i}$ and we consider only the independent scalar products. Since we have L loop momenta and E independent external momenta, the number of independent $s_{i,j}$ corresponds to the number of independent Lorentz invariant scalar products involving loop momenta and is $n = L(L+1)/2 + LE$.

The basic idea behind Baikov representation consists in splitting each loop momentum into two sets, where the integrand of (B.1) depends trivially on one set and not trivially on the other. In the former case the strategy is to integrate over the trivial-dependence set, in the latter one needs to perform a change of variables to the Baikov variables. We split each loop momentum k_i in a parallel and orthogonal space

$$k_i = k_{i\parallel} + k_{i\perp}. \tag{B.3}$$

So that

$$k_i^2 = k_{i\parallel}^2 + k_{i\perp}^2, \tag{B.4}$$

where the parallel component of k_i lies in the subspace spanned by

$$k_{i\|} \in \text{span}\langle k_{i+1}, \dots, k_L, p_1, \dots, p_E\rangle. \tag{B.5}$$

The integration measure becomes

$$\begin{aligned} &\mathrm{d}^d k_1\, \mathrm{d}^d k_2\, \dots\, \mathrm{d}^d k_L \\ &= \mathrm{d}^{E+L-1} k_1\, \mathrm{d}^{d-E-L+1} k_1\, \mathrm{d}^{E+L-2} k_2\, \mathrm{d}^{d-E-L+2} k_2 \dots\, \mathrm{d}^E k_L \mathrm{d}^{d-E} k_L \\ &= \prod_{i=1}^{L} \mathrm{d}^{E+L-i} k_{i\|}\, \mathrm{d}^{d-E-L+i} k_{i\perp}. \end{aligned} \tag{B.6}$$

We now have to treat separately the integration over the parallel and the orthogonal spaces, starting from the parallel. We can write each volume element $\mathrm{d}^{E+L-i} k_{i\|}$ as

$$\mathrm{d}^{E+L-i} k_{i\|} = \frac{\mathrm{d}s_{i,i+1} \dots \mathrm{d}s_{i,L+E}}{G(k_{i+1}, \dots k_L, p_1, \dots p_E)^{1/2}}, \tag{B.7}$$

where

$$G(v_1, \dots, v_n) = \text{Det}(v_i \cdot v_j). \tag{B.8}$$

On the other hand, we can write the orthogonal space measure for each k_i by going in spherical coordinates

$$\begin{aligned} \mathrm{d}^{d-E-L+i} k_{i\perp} &= k_{i\perp}^{d-E-L+i-1} \mathrm{d}k_{i\perp} \Omega_{d-E-L+i} \\ &= \frac{1}{2} (k_{i\perp})^{\frac{d-E-L+i-2}{2}} \mathrm{d}k_{i\perp}^2 \Omega_{d-E-L+i}, \end{aligned} \tag{B.9}$$

where Ω_n is the $n-$dimensional sphere volume. Then, noticing that

$$k_\perp^2 = k^2 - k_\|^2, \tag{B.10}$$

we see that $s_{i,i} = k_i^2$ hence $\mathrm{d}k_{i\perp}^2 = \mathrm{d}s_{i,i}$.

Computing $k_{i\perp}^2$ as a function of $k_i, \dots, k_L, p_1, \dots, p_E$ one can show

$$k_{i\perp}^2 = \frac{G(k_i, \dots, k_L, p_1, \dots, p_E)}{G(k_{i+1}, \dots, k_L, p_1, \dots, p_E)}, \tag{B.11}$$

therefore

$$\mathrm{d}^{d-E-L+i} k_{i\perp} = \frac{1}{2} \left(\frac{G(k_i, \dots, k_L, p_1, \dots, p_E)}{G(k_{i+1}, \dots, k_L, p_1, \dots, p_E)} \right)^{\frac{d-E-L+i-2}{2}} \mathrm{d}s_{i,i}\, \Omega_{d-E-L+i}. \tag{B.12}$$

Therefore we can rewrite eq. (B.1) as

$$\begin{aligned}
&\int \left(\prod_{i=1}^{L} \frac{\mathrm{d}^d k_i}{i\pi^{d/2}}\right) \frac{1}{z_1^{\alpha_1} \dots z_n^{\alpha_n}} \\
&= \frac{1}{(i\pi^{d/2})^L} \int \frac{\mathrm{d}s_{1,2}\mathrm{d}s_{1,3}\dots\mathrm{d}s_{1,L+E}}{G(k_2,\dots,k_L,p_1,\dots,p_E)^{1/2}} \dots \frac{\mathrm{d}s_{L,L+1}\dots\mathrm{d}s_{L,L+E}}{G(p_1,\dots,p_E)^{1/2}} \\
&\quad \times \frac{1}{2}\Omega_{d-E-L+1}\left(\frac{G(k_1,\dots,k_L,p_1,\dots,p_E)}{G(k_2,\dots,k_L,p_1,\dots,p_E)}\right)^{\frac{d-E-L-1}{2}} \mathrm{d}s_{1,1}\dots \\
&\quad \times \frac{1}{2}\Omega_{d-E}\left(\frac{G(k_L,p_1,\dots,p_E)}{G(p_1,\dots,p_E)}\right)^{\frac{d-E-2}{2}} \mathrm{d}s_{L,L} \frac{1}{z_1^{\alpha_1} \dots z_n^{\alpha_n}} \\
&= \frac{1}{(2i\pi^{d/2})^L}\left(\prod_{i=1}^{L} \Omega_{d-E-L+i}\right) G(p_1,\dots,p_E)^{(-d+E+1)/2} \\
&\quad \int \left(\prod_{i=1}^{L}\prod_{j=1}^{L+E} \mathrm{d}s_{i,j}\right) G(k_1,\dots,k_L,p_1,\dots,p_E)^{(d-E-L-1)/2} \frac{1}{z_1^{\alpha_1} \dots z_n^{\alpha_n}}.
\end{aligned} \tag{B.13}$$

Then we can exchange variables from the $s_{i,j}$ to the denominators. Since we said that the independent scalar products are n, we can indicate the n independent $s_{i,j}$ as σ_k, $k = (1,\dots,n)$, in order to write the generalized inverse propagators as

$$z_i = \sum_{j=1}^{n} C_{ij}\sigma_j + f_i, \tag{B.14}$$

where we introduced the $n \times n$ matrix C and the term f_i is independent of the loop momenta. We can invert the relation in order to go into the Baikov variables

$$\sigma_i = \sum_j (C^{-1})_{ij}(z_j - f_j), \tag{B.15}$$

and we therefore have an additional jacobian that takes into account this change of variables

$$\prod_{i=1}^{L}\prod_{j=1}^{L+E} \mathrm{d}s_{i,j} \equiv \mathrm{d}\sigma_1 \dots \mathrm{d}\sigma_n = \mathrm{Det}(C^{-1})\mathrm{d}z_1 \dots \mathrm{d}z_n. \tag{B.16}$$

Rewriting everything we have the Baikov representation written as

$$\begin{aligned}
&\frac{1}{(2i\pi^{d/2})^L}\left(\prod_{i=1}^{L} \Omega_{d-E-L+i}\right) \frac{G(p_1,\dots,p_E)^{(-d+E+1)/2}}{\mathrm{Det}C} \\
&\quad \times \int \frac{\mathrm{d}z_1 \dots \mathrm{d}z_n}{z_1^{\alpha_1} \dots z_n^{\alpha_n}} G(k_1,\dots,k_L,p_1,\dots,p_E)^{(d-E-L-1)/2}
\end{aligned} \tag{B.17}$$

We now give a brief justification for the expression of the integration measures in eqq. (B.11),(B.7) starting from the expression of $\mathrm{d}k_{i\|}$ in eq. (B.7). Since for each k_i we treat the "outer" loop momenta $k_{i+1}, \ldots, k_L$ as if they were additional external momenta, it is sufficient to prove that the formula holds at one loop. The generalization to more loops comes naturally by simply considering a larger set of external momenta that includes the additional $k_{i+1}, \ldots, k_L$. For simplicity we indicate $k_i^\mu \equiv k^\mu$.

The vector $k_\|^\mu$ can be written as linear combinaton of $p_1, \ldots, p_E$ as

$$k_\|^\mu = \sum_{i=1}^{E} c_i p_i^\mu. \tag{B.18}$$

We first change the variables to c_i and later to $s_i \equiv k_\| \cdot p_i$.

$$\mathrm{d}^E k_\| = \mathrm{d}c_1 \ldots \mathrm{d}c_E |J|. \tag{B.19}$$

we need to compute the Jacobian J

$$J = \mathrm{Det}\left(\frac{\partial k_\|^\mu}{\partial c_i}\right), \tag{B.20}$$

with $\partial k_\|^\mu / \partial c_i$ seen as a matrix with indices μ, i. It is easier to compute

$$J^2 = \mathrm{Det}\left(\frac{\partial k_\|^\mu}{\partial c_i}\frac{\partial k_{\|\mu}}{\partial c_j}\right), \tag{B.21}$$

with matrix indices i, j. Since

$$\frac{\partial k^\mu}{\partial c_i} = p_i^\mu, \tag{B.22}$$

then

$$J^2 = \mathrm{Det}(p_i \cdot p_j) = G(p_1, \ldots, p_E), \tag{B.23}$$

and therefore

$$J = \pm G(p_1, \ldots, p_E)^{1/2}. \tag{B.24}$$

Now we change the variables to the s_i

$$\mathrm{d}^E k_\| = \mathrm{d}c_1 \ldots \mathrm{d}c_E |J| = \mathrm{d}s_1 \ldots \mathrm{d}s_E |J J'|, \tag{B.25}$$

with the jacobian J' given by

$$J' = \mathrm{Det}\left(\frac{\partial c_i}{\partial s_j}\right) = \frac{1}{\mathrm{Det}\left(\frac{\partial s_i}{\partial c_j}\right)}, \tag{B.26}$$

where $\partial s_i/\partial c_j$ is a matrix with indices i, j

$$\frac{\partial s_i}{\partial c_j} = \frac{\partial}{\partial c_j}\big(k_\parallel \cdot p_i\big) = p_i \cdot p_j, \tag{B.27}$$

and so

$$\frac{1}{J'} = \mathrm{Det}(p_i \cdot p_j) = G(p_1, \ldots, p_E). \tag{B.28}$$

From eq. (B.25), replacing J and J'

$$\mathrm{d}^n k = \mathrm{d}s_1 \ldots \mathrm{d}s_n \frac{1}{G(p_1, \ldots, p_E)^{1/2}}. \tag{B.29}$$

Now for the orthogonal measure we prove eq. (B.11). Also in this case it is sufficient to prove it at one loop and we use the notation $k_i^\mu \equiv k^\mu$. We expand k^μ as

$$\begin{cases} k^\mu = k_\parallel^\mu + k_\perp^\mu \\ k_\parallel^\mu = \sum_{i=1}^E c_i p_i^\mu \end{cases}, \tag{B.30}$$

where we observe the vanishing of the scalar products

$$k_\parallel \cdot k_\perp = p_i \cdot k_\perp = 0, \tag{B.31}$$

and therefore

$$k \cdot k_\perp = k_\perp^2. \tag{B.32}$$

Because of eq. (B.30) $k, k_\perp, p_1, \ldots, p_E$ are linearly dependent and therefore

$$G(k_\perp, k, p_1, \ldots, p_E) = 0. \tag{B.33}$$

Expanding it, we obtain

$$
\begin{aligned}
G(k_\perp, k, p_1, \ldots, p_E) &= \mathrm{Det}\begin{pmatrix} k_\perp^2 & k_\perp\cdot k & k_\perp\cdot p_1 & \ldots & k_\perp\cdot p_E \\ k_\perp\cdot k & k^2 & k\cdot p_1 & \ldots & k\cdot p_E \\ k_\perp\cdot p_1 & k\cdot p_1 & p_1^2 & \ldots & p_1\cdot p_E \\ \vdots & & & \ldots & \vdots \\ k_\perp\cdot p_E & k\cdot p_E & p_1\cdot p_E & \ldots & p_E^2 \end{pmatrix} \\
&= \mathrm{Det}\begin{pmatrix} k_\perp^2 & k_\perp^2 & 0 & \ldots & 0 \\ k_\perp^2 & k^2 & k\cdot p_1 & \ldots & k\cdot p_E \\ 0 & k\cdot p_1 & p_1^2 & \ldots & p_1\cdot p_E \\ \vdots & & & \ldots & \vdots \\ 0 & k\cdot p_E & p_1\cdot p_E & \ldots & p_E^2 \end{pmatrix} \\
&= k_\perp^2 G(k, p_1, \ldots p_E) \\
&\quad - k_\perp^2 \mathrm{Det}\begin{pmatrix} k_\perp^2 & k\cdot p_1 & \ldots & k\cdot p_E \\ 0 & p_1^2 & \ldots & p_1\cdot p_E \\ \vdots & & \ldots & \vdots \\ 0 & p_1\cdot p_E & \ldots & p_E^2 \end{pmatrix} \\
&= k_\perp^2 G(k, p_1, \ldots, p_E) - k_\perp^4 G(p_1, \ldots, p_E) = 0.
\end{aligned}
\tag{B.34}
$$

Therefore

$$
k_\perp^2 = \frac{G(k, p_1, \ldots, p_E)}{G(p_1, \ldots, p_E)}, \tag{B.35}
$$

which proves eq. (B.11).

C | Master decomposition formula

After introducing a basis for the space of twisted cocycles, $\{\langle e_1|, \ldots, \langle e_\nu|\}$, a basis for the dual space of twisted cocycles $\{|h_1\rangle, \ldots, |h_\nu\rangle\}$ and a metric $\mathbf{C}_{ij} = \langle e_i|h_j\rangle$ we now look at finding the decomposition formula for arbitrary $\langle\varphi|$ and $|\varphi^\star\rangle$ in terms of the basis vectors. We follow closely the treatment presented in [16].

We build the $(\nu+1) \times (\nu+1)$ matrix $\mathbf{M}$ of intersection numbers, for arbitrary $\langle\varphi|$ and $|\varphi^\star\rangle$, defined as

$$\mathbf{M} \equiv \begin{pmatrix} \langle\varphi|\varphi^\star\rangle & \langle\varphi|h_1\rangle & \ldots & \langle\varphi|h_\nu\rangle \\ \langle e_1|\varphi^\star\rangle & \langle e_1|h_1\rangle & \ldots & \langle e_1|h_\nu\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle e_\nu|\varphi^\star\rangle & \langle e_\nu|h_1\rangle & \ldots & \langle e_\nu|h_\nu\rangle \end{pmatrix} \equiv \begin{pmatrix} \langle\varphi|\varphi^\star\rangle & \mathbf{A}^\intercal \\ \mathbf{B} & \mathbf{C} \end{pmatrix}. \tag{C.1}$$

The second equivalence re-express the matrix $\mathbf{M}$ using the column vectors $\mathbf{A}_i = \langle\varphi|h_i\rangle$, $\mathbf{B}_i = \langle e_i|\varphi^\star\rangle$ with $i = (1, \ldots, \nu)$ and the metric $\mathbf{C}_{ij} = \langle e_i|h_j\rangle$.

Since we assume the dimension of the twisted cocycles space to be ν and that each entry of $\mathbf{M}$ is bilinear, then the determinant of $\mathbf{M}$ vanishes. Using the identity for the determinant of a block matrix we get

$$\operatorname{Det}\mathbf{M} = \operatorname{Det}\mathbf{C}\big(\langle\varphi|\varphi^\star\rangle - \mathbf{A}^\intercal\mathbf{C}^{-1}\mathbf{B}\big) = 0, \tag{C.2}$$

because the metric matrix is nonzero. Therefore,

$$\langle\varphi|\varphi^\star\rangle - \mathbf{A}^\intercal\mathbf{C}^{-1}\mathbf{B} = 0, \tag{C.3}$$

that can be rewritten as

$$\langle\varphi|\varphi^\star\rangle = \mathbf{A}^\intercal\mathbf{C}^{-1}\mathbf{B}. \tag{C.4}$$

This last equivalence can be expanded to get the decomposition of a generic intersection number in terms of intersection numbers between basis vectors

$$\langle\varphi|\varphi^\star\rangle = \sum_{i,j=1}^{\nu} \langle\varphi|h_j\rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle e_i|\varphi^\star\rangle. \tag{C.5}$$

Given the arbitrariness of $\langle\varphi|$ and $|\varphi^\star\rangle$ we can extract from eq. (C.5) some really important relations:

- because of the equivalence between the r.h.s and the l.h.s. we have the expression of identity operator in the cohomology space

$$\langle\varphi|\,\mathbb{I}_{\mathrm{c}}\,|\varphi^\star\rangle = \sum_{i,j=1}^{\nu} \langle\varphi|h_j\rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle e_i|\varphi^\star\rangle\,. \tag{C.6}$$

  obtained by stripping the r.h.s. and the l.h.s. of $\langle\varphi|$ and $|\varphi^\star\rangle$ because of their arbitrariness and getting:

$$\mathbb{I}_{\mathrm{c}} \equiv \sum_{i,j=1}^{\nu} |h_j\rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle e_i|\,; \tag{C.7}$$

- because of the arbitrariness of $|\varphi^\star\rangle$ we can strip it out of eq. (C.5) and obtain the decomposition of $\langle\varphi|$ in terms of the basis vectors $\{\langle e_i|\}_{i=1}^{\nu}$

$$\langle\varphi| = \sum_{i=1}^{\nu} \underbrace{\sum_{j=1}^{\nu} \langle\varphi|h_j\rangle \left(\mathbf{C}^{-1}\right)_{ji}}_{c_i} \langle e_i|\,, \tag{C.8}$$

  where we isolated the coefficients c_i;

- the same reasoning can be applied to strip out $\langle\varphi|$ from eq. (C.5) and obtain the decomposition of $|\varphi^\star\rangle$ in terms of the dual basis vectors $\{|h_i\rangle\}_{i=1}^{\nu}$

$$|\varphi^\star\rangle = \sum_{i=1}^{\nu} \underbrace{\sum_{j=1}^{\nu} \left(\mathbf{C}^{-1}\right)_{ij} \langle e_j|\varphi^\star\rangle}_{c_i^\star} |h_i\rangle\,, \tag{C.9}$$

  where we isolated the coefficients $c_i^\star$.

D | Derivation of $\boldsymbol{\Omega}$

We closely follow [16] to give a brief account on how the connection $\boldsymbol{\Omega}$ appears in the algorithm for multivariate intersection numbers.

We start from the univariate case in the variable z_1, recalling how we obtain the univariate connection $\omega = \boldsymbol{\Omega}^{(1)} = \mathrm{d}_{z_1} \log u$. It comes from the total derivative of the product of $u = u(z_1)$ and an $(n-1)$-differential form $\xi = \xi(z_1)$

$$\begin{aligned} 0 &= \int_{\mathcal{C}} \mathrm{d}_{z_1}(\xi u) = \int_{\mathcal{C}} (\mathrm{d}_{z_1}\xi + \mathrm{d}_{z_1} \log u \wedge \xi) u \\ &= \int_{\mathcal{C}} \nabla_{\boldsymbol{\Omega}^{(1)}} \xi u = \langle \nabla_{\boldsymbol{\Omega}^{(1)}} \xi | \, \mathcal{C}], \end{aligned} \tag{D.1}$$

where we used a new notation $\nabla_{\boldsymbol{\Omega}^{(1)}}$ that corresponds to ∇_ω: $\nabla_{\boldsymbol{\Omega}^{(1)}} = \mathrm{d}_{z_1} + \boldsymbol{\Omega}^{(1)} = \mathrm{d}_{z_1} + \omega = \nabla_\omega$. Now we extend the reasoning to n-forms in the variables $\mathbf{z} = (z_1, \ldots, z_n)$, with $u = u(\mathbf{z})$, considering the integral I

$$\begin{aligned} I = \left\langle \varphi^{(n)} \middle| \, \mathcal{C}^{(n)} \right] &= \int_{\mathcal{C}^{(n)}} \varphi(\mathbf{z}) u \\ &= \sum_{i=1}^{\nu_{(n-1)}} \int_{\mathcal{C}^{(n)}} \varphi_i(z_n) \int_{\mathcal{C}^{(n-1)}} e_i^{(n-1)}(z_1, \ldots, z_n) u \\ &= \sum_{i=1}^{\nu_{(n-1)}} \int_{\mathcal{C}^{(n)}} \varphi_i(z_n) \left\langle e_i^{(n-1)} \middle| \, \mathcal{C}^{(n-1)} \right], \end{aligned} \tag{D.2}$$

where the role of u in the univariate case is now played by $\left\langle e_i^{(n-1)} \middle| \, \mathcal{C}^{(n-1)} \right]$. In principle there exist many $1-$forms $\varphi_i(z_n)$ integrating to the same result. In particular, we consider the vanishing of the total derivative in z_n of $\left\langle e_i^{(n-1)} \middle| \, \mathcal{C}^{(n-1)} \right]$ times the function (0-form) $\xi_i(z_n)$

$$0 = \int_{\mathcal{C}^{(n)}} \mathrm{d}_{z_n} \left(\xi_i(z_n) \left\langle e_i^{(n-1)} \middle| \, \mathcal{C}^{(n-1)} \right] \right), \tag{D.3}$$

and $\left\langle e_i^{(n-1)} \middle| \mathcal{C}^{(n-1)} \right]$ satisfies the differential equation in z_n

$$\mathrm{d}_{z_n} \left\langle e_i^{(n-1)} \middle| \mathcal{C}^{(n-1)} \right] = \mathbf{\Omega}_{ij}^{(n)} \left\langle e_j^{(n-1)} \middle| \mathcal{C}^{(n-1)} \right], \tag{D.4}$$

with $\mathbf{\Omega}^{(n)}$ a $\nu_{(n-1)} \times \nu_{(n-1)}$ matrix. Inserting (D.4) in (D.3) we get

$$\begin{aligned} 0 &= \int_{\mathcal{C}^{(n)}} \left(\left(\delta_{ij} \mathrm{d}_{z_n} + \mathbf{\Omega}_{ij}^{(n)} \right) \xi_i(z_n) \right) \left\langle e_j^{(n-1)} \middle| \mathcal{C}^{(n-1)} \right] \\ &= \int_{\mathcal{C}^{(n)}} \left(\nabla_{\mathbf{\Omega}^{(n)}} \right)_{ij} \xi_i(z_n) \left\langle e_j^{(n-1)} \middle| \mathcal{C}^{(n-1)} \right], \end{aligned} \tag{D.5}$$

where we define the connection at the n-th integration step in z_n

$$\nabla_{\mathbf{\Omega}^{(n)}} \equiv \mathbb{I} \mathrm{d}_{z_n} + \mathbf{\Omega}^{(n)}. \tag{D.6}$$

And the matrix $\mathbf{\Omega}^{(n)}$ is obtained as

$$\begin{aligned} \mathrm{d}_{z_n} \left\langle e_i^{(n-1)} \middle| \mathcal{C}^{(n-1)} \right] &= \mathrm{d}_{z_n} \int_{\mathcal{C}^{(n-1)}} e_i(z_1, \ldots, z_n) u \\ &= \int_{\mathcal{C}^{(n-1)}} (\mathrm{d}_{z_n} e_i(z_1, \ldots, z_n) + \mathrm{d}_{z_n} \log u \wedge e_i(z_1, \ldots, z_n)) u \\ &= \int_{\mathcal{C}^{(n-1)}} (\mathrm{d}_{z_n} + \omega_n \wedge) e_i(z_1, \ldots, z_n) u \\ &= \left\langle (\mathrm{d}_{z_n} + \omega_n \wedge) e_i^{(n-1)} \middle| \mathcal{C}^{(n-1)} \right], \end{aligned} \tag{D.7}$$

with $\omega_n = \mathrm{d}_{z_n} \log u$. We can employ the decomposition formula into basis vectors to further simplify the last line and obtain

$$\left\langle (\mathrm{d}_{z_n} + \omega_n \wedge) e_i^{(n-1)} \right| = \left\langle (\mathrm{d}_{z_n} + \omega_n \wedge) e_i^{(n-1)} \middle| h_k \right\rangle \left(\mathbf{C}_{(n-1)}^{-1} \right)_{kj} \left\langle e_j^{(n-1)} \right|, \tag{D.8}$$

and we can identify $\mathbf{\Omega}^{(n)}$

$$\mathbf{\Omega}_{ij}^{(n)} = \left\langle (\mathrm{d}_{z_n} + \omega_n \wedge) e_i^{(n-1)} \middle| h_k \right\rangle \left(\mathbf{C}_{(n-1)}^{-1} \right)_{kj}. \tag{D.9}$$

E | Intersection numbers of differential $1-$forms

We give a brief account on the justification for the formula presented in eq. (3.82) that is used for the calculation of intersection numbers of differential $1-$forms. We closely follow the treatment presented in [6]. We consider a $1-$form φ_L and we indicate its poles with $z_i \in \mathbb{C}$. We will show how, from the definition of intersection number given in eq. (3.18) we end up in eq. (3.82). In order to do so, we need to build the compactification $\iota_\omega(\varphi_L)$: the key point will be the definition of circular regions around each pole z_i.

We consider V_i and U_i to be two discs centered in z_i such that $V_i \subset U_i$. Each disc is defined such that $U_i \cap U_j = \varnothing$ if $i \neq j$. Then, for each i we consider

- ψ_i, a holomorphic function satisfying

$$\nabla_\omega \psi_i = \varphi_L, \tag{E.1}$$

  on $U_i \setminus \{z_i\}$.

- ξ_i, a function such that

$$\xi_i = \begin{cases} 1 \text{ on } V_i; \\ 0 \leq \xi_i \leq 1 \text{ smooth interpolation on } U_i \setminus V_i; \\ 0 \text{ out of } U_i. \end{cases} \quad . \tag{E.2}$$

Then we can write the mapped function $\iota_\omega(\varphi_L)$, a function lying in the same cohomology class as φ_L, as the difference between φ_L and the covariant derivative of a $0-$form (that is, a function)

$$\iota_\omega(\varphi_L) = \varphi_L - \sum_i \nabla_\omega(\xi_i \psi_i) = \varphi_L \sum_i (\mathrm{d}\xi_i \psi_i + h_i \nabla_\omega \psi_i). \tag{E.3}$$

Notice a few things:

- since $\xi_i = 0$ out of U_i, we are not modifying φ_L in that region, (i.e. $\iota(\varphi_L) = \varphi_L$ outside U_i);
- since, inside V_i, $\xi_i = 1$, the whole φ_L is subtracted around the singular poles. Therefore $\iota_\omega(\varphi_L) = 0$ in the innermost region V_i;
- in the intermediate ring $U_i \setminus V_i$ we subtract φ_L smoothly and moreover it's the only region where the term $\mathrm{d}\xi_i\psi_i \neq 0$.

It can be proved that a function ψ_i satisfying eq. (E.1) exists and is unique [6]. We can rewrite the intersection number $\langle\varphi_L|\varphi_R\rangle_\omega$ as

$$\begin{aligned}\langle\varphi_L|\varphi_R\rangle_\omega = \\ &= \frac{1}{2\pi i}\int_X \left[\varphi_L - \sum_i (\mathrm{d}\xi_i)\psi_i - \sum_i \xi_i \nabla_\omega \psi_i\right] \wedge \varphi_R \\ &= -\frac{1}{2\pi i}\sum_i \int_{U_i\setminus V_i} \mathrm{d}\xi_i\psi_i \wedge \varphi_R.\end{aligned} \tag{E.4}$$

The second equality is obtained by noticing that

- $\varphi_L \wedge \varphi_R = 0$ leads the first and the third term to vanish;
- the second term survives only where $\mathrm{d}\xi_i \neq 0$, that is on the ring $U_i \setminus V_i$;
- it holds
$$\mathrm{d}\xi_i\psi_i \wedge \varphi_R = \mathrm{d}(\xi_i\psi_i\varphi_R), \tag{E.5}$$
because both $\xi_i\mathrm{d}\psi_i \wedge \varphi_R$ and $\xi_i\psi_i\mathrm{d}\varphi_R$ vanish as $\mathrm{d}\psi_i$ and φ_R are both holomorphic and φ_R is a closed form.

We can rewrite eq. (E.4) using Stokes' theorem

$$\begin{aligned}\langle\varphi_L|\varphi_R\rangle_\omega &= -\frac{1}{2\pi i}\sum_i \int_{U_i\setminus V_i} \mathrm{d}(\xi_i\psi_i\varphi_R) = -\frac{1}{2\pi i}\sum_i \int_{\partial(U_i\setminus V_i)} \xi_i\psi_i\varphi_R \\ &= \frac{1}{2\pi i}\sum_i \int_{\partial(V_i)} \psi_i\varphi_R,\end{aligned} \tag{E.6}$$

and, since V_i is a closed path, we can rewrite the last term as a sum of residues

$$\langle\varphi_L|\varphi_R\rangle_\omega = \sum_i \mathrm{Res}_{z=z_i}(\psi_i\varphi_R), \tag{E.7}$$

obtaining eq. (3.82).

F | Finite fields technology

-We give a brief review of finite fields technology, nowadays widely employed [42, 43, 23, 31, 29] for performing cutting-edge calculations relevant for high-precision theoretical predictions. In this thesis we use the framework FiniteFlow for two reasons:

- to solve the linear system of equations necessary for the "master monomials" analysis;
- to obtain independently the decomposition of Feynman integrals with the Laporta algorithm, in order to check the results obtained via intersection theory.

Moreover, the rational algorithm described in Ch. 4 is suitable for a future implementation over finite fields. We start by introducing what are finite fields and we then give a brief review of the framework FiniteFlow referring for further details to [42, 43].

Finite fields are fields with a finite number of elements. We indicate with $\mathbb{Z}_n$ the set of non-negative integers smaller than n

$$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}. \tag{F.1}$$

In $\mathbb{Z}_n$ one can define arithmetic operations such as addition,multiplication and subtractions using modular arithmetic: they are done by performing the operation over the field of integer numbers $\mathbb{Z}$ and taking the reminder of the integer division of the result modulo n. In particular, FiniteFlow employs finite fields of integer numbers modulo a prime number p, indicated with $\mathbb{Z}_p$.

We can also define the inverse a^{-1} of an element $a \in \mathbb{Z}_n, a \neq 0$, with respect to multiplication if and only if a and n are coprime. The multiplicative inverse is an element $b \in \mathbb{Z}_n$ such that

$$a^{-1} \bmod n \equiv b \iff (ab) \bmod n = 1. \tag{F.2}$$

Since FiniteFlow uses finite fields modulo a prime number p, $\mathbb{Z}_p$, the existence of an inverse is always guaranteed for every non-vanishing element

of $\mathbb{Z}_p$. The multiplicative inverse is calculated using the extended Euclidean algorithm (for more details refer to [43], Appendix *A*.1). Therefore, $\mathbb{Z}_p$ is a field where all rational operations are well-defined.

The existence of multiplicative inverse allows us to define a non-invertible map between rational numbers $\mathbb{Q}$ and elements of the finite field $\mathbb{Z}_p$

$$\begin{array}{ccc} \mathbb{Q} & \longrightarrow & \mathbb{Z}_p \\ q = \dfrac{a}{b} & \longrightarrow & q \bmod p, \end{array} \tag{F.3}$$

where the map is defined as

$$q \bmod p \equiv \left(a \times \left(b^{-1} \bmod p\right)\right) \bmod p. \tag{F.4}$$

This implies that polynomials and rational functions are well defined objects. In particular, we refer to the definition of polynomials and rational functions presented in App. A. Therefore, any numerical algorithm consisting in a sequence of rational operations, can be implemented over finite fields $\mathbb{Z}_p$. The map between rational numbers and a finite field is not invertible, as $\mathbb{Q}$ is infinite and $\mathbb{Z}_p$ is not. However, one can exploit an important result in modular arithmetic, Wang's *rational reconstruction* algorithm [52, 53]. It allows, under certain assumptions, to successfully invert the map between $\mathbb{Q}$ and $\mathbb{Z}_n$, therefore allowing to reconstruct q from its images over $\mathbb{Z}_n$ with n sufficiently large. More in detail, given a rational number $q = a/b$, Wang's algorithm is successful in reconstructing q from its image in $\mathbb{Z}_n$ if and only if $|a|, |b| < \sqrt{n/2}$. However, since the main reason for the use of finite fields is the possibility of performing calculations efficiently using machine size integers, the prime numbers p considered are bounded from above, namely $p < 2^{64}$, since most modern machines have integers with a size of 64 bits. This can seem a problem for the usability of the rational reconstruction algorithm, but it is a limitation that can be easily sidestepped by means of the Chinese reminder theorem. In fact, the latter allows to deduce a number $a \in \mathbb{Z}_n$, where $n = \prod_i n_i$, from its images $a_i \in \mathbb{Z}_{n_i}$, where it is required that the integers n_i have no common factors. Within FINITEFLOW, one can deduce the image of a number over $\mathbb{Z}_{p_1 p_2 \ldots}$ from its images over several prime fields $\mathbb{Z}_{p_1}$,$\mathbb{Z}_{p_2}$,.... Therefore, once the product of the selected sequence of primes $\{p_1, \ldots, p_n\}$ is large enough, Wang's reconstruction algorithm will be successful.

More in detail, the algorithm works as follows: the coefficients of any reconstructed rational function, n_α and d_α, are mapped to the rational field using Wang's algorithm. The result is checked numerically against evaluations of the reconstructed function over finite fields not used for the reconstruction, if it's successful, the algorithm reaches the terminating condition.

If the check is unsuccessful, the reconstruction is done again over more finite fields $\mathbb{Z}_{p_i}$ and the results are combined using the Chinese reminder theorem, in order to obtain a new result over $\mathbb{Z}_{p_1 p_2 \ldots}$ that will be mapped to $\mathbb{Q}$.

Before performing the rational reconstruction of the coefficients of the rational function, it is necessary to reconstruct the rational function itself over $\mathbb{Z}_{p_i}(\mathbf{z})$. This is done via a multivariate functional reconstruction algorithm described in detail in [42]. This algorithm solves the so-called *black box interpolation problem* (F.5), that is, the problem of inferring with very high probability the analytic expression of a function from its numerical evaluations. In this case, these evaluations are performed over finite fields. Assuming we have a procedure to evaluate an n-variate rational function f of which we do not know the analytic form, the procedure takes as input numerical values for $\mathbf{z}$ and a prime p and returns the function evaluated over the finite field $\mathbb{Z}_p$ at the point $\mathbf{z}$. The evaluation may fail in some points called *bad points*, or *singular points*, that do not necessarily correspond to singularities in the analytic expression of f, but may also due to spurious singularities appearing in intermediate steps of the evaluation. When a *bad point* is encountered, it is replaced with a different one. In realistic cases, however, the occurrence of these points is extremely rare. The functional reconstruction algorithm allows to identify the monomials appearing in the analytic expression of f and the value of their coefficients n_α and d_α,

$$(\mathbf{z}, p) \longrightarrow \boxed{\ f\ } \longrightarrow f(\mathbf{z}) \bmod p. \tag{F.5}$$

FiniteFlow allows to reconstruct the analytic expression of a function by exploiting first a multivariate functional reconstruction algorithm, that allows to obtain the monomials and the coefficients evaluated over the finite field in consideration, followed by Wang's rational reconstruction algorithm, that maps the coefficients over the finite field into their rational expression. This procedure reduces the problem of computing any multivariate and multivalued rational function to the one of providing a numerical implementation of it over finite fields. FiniteFlow is characterized by good flexibility and performance. It has more the structure of a framework rather than of a single-use specialized program. It relies on a special kind of computational graphs, known as *dataflow graphs*, in order to allow the user to build any needed numerical algorithm by concatenating basic operations whose algorithms are encoded in low-level languages. To be more clear, dataflow graphs are directed acyclic graphs used to represent a numerical calculation. Any graph is made of nodes (see fig. F.1) and arrows. The latter represents data (in the case of FiniteFlow, numerical values) and the former represents algorithms

operating on the data received as inputs (identified by incoming arrows) and producing data as output (identified by outgoing arrows). In FINITEFLOW an arrow represents a list of values and a node a basic numerical algorithm. A node can have zero or more incoming arrows but only one outgoing arrow as output.

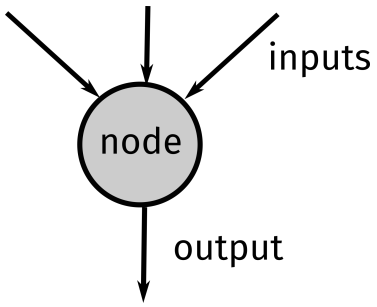

FIGURE F.1: Node presenting three input arrows and only one output arrow. Image taken from [42].

Nodes represent basic algorithms that are implemented once and for all in a low level language such as C++. Such algorithms include the ones for the evaluation of rational functions, dense and sparse linear solvers, matrix multiplication (useful for linear substitutions), and the ones to perform linear fits and Laurent expansions. The user can define more complex algorithms by combining the nodes into a computational graph representing a complete calculation, chaining the nodes so that the output of each building block is used as input for the others. This allows the user to define algorithms using the built-in algorithms as building blocks without concerning about the low-level details of the numerical implementation. In this sense, FINITEFLOW can be seen as a toolkit that provides the user with the necessary instruments to build any rational algorithm.

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