Alma Mater Studiorum · University of Bologna

School of Science
Department of Physics and Astronomy
Master Degree in Physics

Detectors in black hole quantum coherent states

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Academic Year 2021/2022

Abstract

The corpuscular model describes black holes as leaky bound states of gravitons. To account for the role of matter, a coherent state is built and a semiclassical description is given to the gravitational field by connecting the classical source with the quantum state for gravitons. The properties of this state can be analysed with the help of an Unruh-DeWitt detector, coupled to the quantum state of the system. The presence of a detector in general regularises the usual diverging behaviour of the field in the deep ultraviolet region, and will allow us to probe the coherent state structure and the graviton emission. In particular, a Newtonian analogue of the Unruh effect will be discussed and the coherent state will be modified to properly account for the spherical symmetry of the potential at the level of the quantum state. This correction will ensure that vacuum contributions responsible for the Unruh thermal spectrum are present in a coherent state emission process.

Contents

Notation and conventions

In this work, greek indices run over the values $\mu = 0, 1, 2, 3$, and latin indices over $i = 1, 2, 3$. The Einstein convention is adopted for sums over repeated indices

$$g_{\mu\nu}u^{\mu} = \sum_{\mu=0}^{3} g_{\mu\nu}u^{\mu}.$$

The convention for the metric tensor is the "mostly plus" one, meaning that the signature of the Minkowski metric is

$$\eta_{\mu\nu} = \mathrm{diag}(-, +, +, +).$$

Indices are raised and lowered with the metric tensor $g_{\mu\nu}$ or its inverse $g^{\mu\nu}$, defined by $g_{\mu\nu}g^{\nu\lambda} = \delta_{\mu}^{\lambda}$:

$$u_{\mu} = g_{\mu\nu}u^{\nu}$$
$$u^{\mu} = g^{\mu\nu}u_{\nu}.$$

Where otherwise indicated, in this work the speed of light will be set equal to one, $c = 1$, while the Planck constant $\hbar$ and the Newton constant G will be left explicit.
The scalar product between two solutions of the Klein-Gordon equation, f_1 and f_2, is written as

$$(f_1, f_2) = i \int d^3x (f_1 \partial_t f_2^* - f_2^* \partial_t f_1).$$

Furthermore, various notations for recurring quantities are summarized here:
R_h: event horizon,
R_S: size of the source,
R_∞: length associated to the time of collapse,
m_p: Planck mass,
l_p: Planck length,
UV, IR: ultraviolet and infrared,
$\langle \hat{A} \rangle$: mean value for the operator $\hat{A}$,
$\mathcal{R}(\alpha), \mathcal{I}(\alpha)$: real and imaginary parts of a complex number α,
(A): superscript relative to the areal coordinate,
(H): superscript relative to the harmonic coordinate.

Preface

The most predictive framework describing gravitational systems is the Einstein theory of General Relativity. Gravity is seen as the curvature of a manifold, the spacetime, whose effects can be measured through test body motions. The curvature is determined by the energy-momentum content of the spacetime itself and this feature is encoded by the Einstein field equations. Among all the possible gravitational systems, described by solutions of the field equations, the most intriguing ones are black holes. Those systems are formed when the collapse of a star occurs, and the gravitational pull within a region called event horizon is so strong that even light cannot escape from it.
Black holes are one of the most outstanding predictions of General Relativity, but they also flag the loss of predictability of the theory. The so called singularities arise in the deep UV region, where physical quantities become infinite. It is believed that a quantum theory for the gravitational interaction will fix this inconsistency, as happened in the past for the electromagnetic interaction. To this end, many models for black holes have been built in the past years, trying to reproduce the known classical features and, at the same time, attempting to fix the singularity problem. Some of these models rely on the most advanced framework for quantum mechanical processes at dispose, that is, Quantum Field Theory. Even though gravity is not treated like the other interactions in the Einstein theory, it can also be regarded from a field theoretic point of view. Constructing an effective field theory for gravity should give the geometrical description as an emerging and classical feature of the theory itself.
This approach to Quantum Gravity follows the so called corpuscular pictures, that see black holes as bound states of gravitons. In this context, the number of gravitons N_g is the only parameter of the theory, and can be regarded as a measure of the "classicality" of the system. Such parameter can be related to the mass of the black hole through the fundamental corupuscular relation $M^2 \sim N_g m_p^2$.
By following this perspective, a corpuscular approach is taken to reconstruct the Newtonian potential from a coherent state for a scalar field of gravitons in flat spacetime. The quantum version of this function gets corrections from the coherent nature of gravity, specifically by the requirement that the coherent state should be properly normalized. The Newton potential, once treated quantum mechanically, could then be used to reconstruct the full Schwarzschild metric function through a mean field approach. Test

particles will then follow geodesics given by the quantum corrected metric tensor given by the coherent state. In the context of measuring coherent state effects, the role of a detector as a probe for the quantum field should not be neglected. This conception is inspired by a series of works that inserted a test body profile inside the definition of a quantum field, in order to avoid the well known UV divergences coming from vacuum fluctuations. However, in this work the introduction of a detector is made at the quantum level, in the very state of the system. This is consistent with the view that nature does not have a net division between what is classical and what is quantum, and even a macroscopic device deserves its own quantum state. The price to pay to introduce a detector is the loss of coherence of the quantum state of the system. Anyway, the very act of taking a measurement with a device that is coupled to the quantum system has, on one hand, the aim of describing a more physically meaningful process, while on the other hand it should fix technical problems in the mathematical setup of Quantum Field Theory. To this end, a detector approach can heal the well known divergences coming from the calculation of the variance of a quantum field. The quantum nature of the measuring device will be explicit from the "competition" of the parameters of the detector with the parameters of the gravitational source defining the coherent state.
A detector approach also comes to help when the very concept of particles is questioned: in Quantum Field Theory, a particle is regarded as an irreducible representation of the Poincaré group. But when such symmetry group breaks down, as for the case of curved spacetimes or, in general, for non inertial coordinate systems, what is the meaning of a particle? The issue of defining a particle becomes observer-dependent and the point of view where a particle is seen as a "clic" in a measuring device whose motion should be specified seems to be more pragmatic. Therefore, testing the coherent state model with a detector model (the Unruh-DeWitt one) was interesting from at least two points of view: first of all, it allowed to understand how coherent state transitions look like when described by non inertial observers. Some features of the results were model dependent, i.e. they are peculiar of the Newtonian configuration, but other ones can be seen as general properties of a coherent state. Secondarily, it was useful to have a deeper understanding of certain aspects of the model that were somewhat hidden. In fact, in facing polynomials of the field, it was found how all the symmetries of the field configuration should be imposed at the level of the coherent state. This allows to recover the full spectrum for the transition probability of a coherent state measured by a detector in uniformly accelerated motion. Furthermore, when discussing the possibility of recovering the Hawking radiation from a coherent state perspective, the non perturbative role played by coordinates in General Relativity and in the mean field approach, used to reconstruct the quantum corrected Schwarzschild metric, was better understood.
This thesis dissertation is organized as follows. In chapter 1 the corpuscular model of a black hole as a bound state of gravitons [1–3] is presented, and the emerging nature of the Einstein geometrical view of gravity is stressed. In chapter 2 coherent states for a quantum field are briefly introduced, mainly following [4, 5]. chapter 3 presents the

coherent state approach to the corpuscular model; here, a coherent state for the scalar graviton field is built with the aim of reproducing the Newtonian configuration coming from the bound state. The Newton potential gets quantum corrections [6] and can be regarded as the Schwarzschild metric function [7]. This means that also the metric tensor is modified [8]. chapter 4 discusses the role of a detector in a quantum field measurement process, as exposed in [9], but with the perspective that the detector should be described in a quantum language. This point of view is supported by recent studies [10, 11], and is used to fix the divergence of the uncertainty of a quantum field in a very simple setup. In chapter 5 the Unruh-DeWitt detector model [12, 13] is presented, and the Unruh effect is recovered in this context. The analogies between Schwarzschild coordinates and Rindler coordinates, as well as between Kruskal and Minkowski ones, are shown in order to stress the role of gravity seen as spacetime curvature. The analogies are also presented to understand how curvature effects can be simulated in flat spacetime through the Equivalence Principle. chapter 6 is where the Unruh-DeWitt detector is employed in the presence of the coherent state describing gravitons, and where the coefficients of the coherent state are modified to enforce the spherical symmetry after quantizing the field. This procedure ensures that there is an Unruh thermal contribution to the final transition probability even for a coherent state. Finally, the role of coordinates is briefly discussed in the context of recovering the Hawking radiation.

Chapter 1

Corpuscular Picture

To study the gravitational interaction from a quantum point of view, it is necessary to consider regimes where gravity is the dominant force with respect to the other fundamental ones. To this end, the most interesting and intriguing objects that can be modelled with Quantum Mechanics are black holes: first of all, even at the classical level, black holes are very simple objects, since they are completely characterized by their mass, their charge and their angular momentum as stated by the No Hair Theorem. Furthermore, the gravitational field in the region approaching the event horizon of a black hole is extremely strong, and much more stronger then the field of other gravitational systems. The physics of this region is not challenged by direct experiments yet, and it is therefore subject to theoretical speculation involving the quantum nature of gravity, or the effects that gravity could trigger in quantized systems. Starting from the assumption that a full quantum theory of gravity remains nowadays out of reach, it is worth to ask if the available mathematical tools and physical frameworks, can effectively describe the classical and semiclassical effects involving black holes. These effects are already described by General Relativity, but a more refined quantum description can give some extra information or intuition about the physics behind these systems.

To speak about classical or quantum gravitational effects, it is crucial to ask at which scale do these effects become strong, and thus measurable. General Relativity couples the gravitational field to any form of energy and matter through the Newton constant G, which has dimensions of

$$[G] = \frac{[length]}{[mass]}. \tag{1.1}$$

Then, it is possible to build a characteristic length for classical gravity by multiplying the Newton constant for something which has the dimensions of a mass, such as the mass of a gravitational source. This length scale, known as Schwarzschild radius, points out the distance from the gravitational source at which the classical gravitational effects become dominant

$$R_h = 2GM. \tag{1.2}$$

Let us stress that the above quantity is purely classical, and does not tell anything about the length at which the quantum gravitational interaction becomes relevant. For this purpose, the Planck constant $\hbar$ should be included, its dimensions given by

$$[\hbar] = [mass][length]. \tag{1.3}$$

From this constant and the Newtonian one, the length at which quantum gravitational fluctuations become strong can be built as

$$l_p = \sqrt{\hbar G}. \tag{1.4}$$

At this distance, called Planck length, any Quantum Field Theory perturbative approach will break down and will not be able to produce physical results anymore.
Therefore, a first definition of classicality for the gravitational field produced by a source can be written as

$$l_p \ll R_h. \tag{1.5}$$

In this region, an effective theory of gravity can, in principle, be safely built. Moreover, from the Planck length it is possible to find something with the dimensions of a mass, that is, the Planck mass:

$$m_p = \frac{l_p}{G}. \tag{1.6}$$

To summarize, a typical gravitational source will show three different regions over which the interaction can be analysed: the first one covers distances $r \in [R_h, \infty)$ and here gravity can be considered weak with respect to the other forces, as it can be seen by comparing the coupling constants already existing in classical laws. The second one covers the range $r \in [l_p, R_h]$ and here classical gravitational effects become strong. With a certain amount of approximation at distances where Quantum Mechanics becomes relevant, this region is the domain of General Relativity: the classical Newtonian theory needs corrections to take into account the true non linear nature of gravity. The last region is probed by distances on the order of the Planck scale l_p and here the full quantum theory of gravity is needed.
As previously stated, since the full theory is not at hand, the third region is ruled out. The other two regions could be modelled within a quantum field framework, and it is crucial to understand how to recover classical physics from a quantum description of the system taken into exam. As it will be discussed later, a system described by a state will show classical features when it is highly occupied, that is, called N the eigenvalue of the number operator, telling the number of quanta of a field in a given state, the condition of classicality can be generically indicated as

$$N \gg 1. \tag{1.7}$$

It is natural then to associate the occupation number of gravitons (quanta of the gravitational field) with the region involving the event horizon scale. This fundamental idea is the cornerstone of the so called corpuscular picture of black holes. The corpuscular theory presented here [1, 2] describes black holes as quantum condensates of gravitons, and recovers classical and semiclassical effects coming from black hole mechanics from the physics of condensates, while completely neglecting the role of matter and other interactions.

As a starting point, let us build an effective fine structure constant by taking into account the dimensions of the Newton constant

$$\alpha_g = \frac{\hbar G}{\lambda^2}, \tag{1.8}$$

where λ identifies the typical wavelength of the gravitational interacting particles. It is clear that for large distances/lengths (and thus low energies) the coupling constant will be rather small, and the interaction between the gravitational quanta making up the source will be very weak. Intuitively, this condition on the weakness of the interaction should be satisfied if (1.5) is true, that is, for every astrophysical black hole; therefore, black holes themselves are just weakly interacting ensembles of (spin two) particles, with the interaction depending on the energy of the system.

Imagine now to have a source of radius $r \gg R_h$: at this level, General Relativity is undoubtedly well approximated by the Newtonian theory and thus the gravitational energy of the probe will be

$$E_g \sim \frac{M R_h}{r}. \tag{1.9}$$

On the other hand, the total energy of the system, approximated by the weakness of the interaction with the sum of the energies of the single gravitons, is

$$E_g \sim \sum_\lambda \hbar \frac{N_\lambda}{\lambda}. \tag{1.10}$$

The above expression can be approximated with the dominant term of the sum, which is just the energetic level determined by the most occupied wavelength, which can be thought as the distance characterizing the gravitational source $\lambda \sim r$, i.e.

$$E_g \sim \frac{N\hbar}{r}. \tag{1.11}$$

Matching the two expressions given for the energy results in

$$N = \frac{M R_s}{\hbar}, \tag{1.12}$$

which is valid until quantum gravitational effects will not become dominant.

However, having neglected from the beginning any energetic contribution coming from matter, the only available energy is the gravitational one, and it has a negative sign since

it is a binding energy. A weakly interacting bosonic condensate can be self-sustained, and thus exist, only if its compressibility is strictly positive [14]. The compressibility of such systems turns out to be proportional to the interacting potential, which in the present case is strictly negative. Therefore, a gravitational source, in which only the Newtonian potential is present, cannot be self-sustained and must necessary collapse. This is an effect coming from condensate mechanics, but it allows to understand, at an intuitive and approximated level, the gravitational collapse of a star whose internal radiation pressure is no more able to match the self gravity given by the heavy elements forming the stellar structure.
By keeping ignoring matter contributions, the collapse of the condensate can be stopped only when another term coming from gravity balances the Newtonian one. If the source has a size of $r \sim R_h$, the number of gravitons can still be estimated with (1.12), because the size is still $r \gg l_p$: in practice, at distances on the order of R_h, a collective interaction can be set up and it plays a central role, while the graviton-graviton interaction is still negligible. Moreover, the (1.12) can be written in a more useful way by means of (1.6):

$$N \sim \frac{M^2}{m_p^2}. \tag{1.13}$$

From this expression, it is clear how the number of gravitons is proportional to the mass of the source and that classical physics is valid in a regime where [15]

$$M \gg m_p. \tag{1.14}$$

This statement is important to define what is (or what is not) a gravitational source: an electron will never satisfy the above condition, since it contains $N \ll 1$ gravitons (and thus no gravitons at all). Obviously, this does not forbid any kind of gravitational interaction for the electron, which can still exchange gravitons in scattering processes, but this means that the electron cannot be considered a source of gravity in the physical sense, as specified above. By keeping $r \sim R_h$, the gravitational energy of a black hole can be estimated as

$$E_g \sim M, \tag{1.15}$$

and again from (1.12), it brings to

$$M \sim N\frac{\hbar}{\lambda}. \tag{1.16}$$

Furthermore, by using (1.13), the number of gravitons can be written as

$$N = \frac{\lambda^2}{l_p^2}, \tag{1.17}$$

which again confirms that $N \gg 1$ characterizes sources composed by gravitons with a typical wavelength of $\lambda \gg l_p$. Moreover, even for the wavelength, a scaling law can be found by simply inverting the above relation:

$$\lambda = \sqrt{N} l_p. \tag{1.18}$$

With equation (1.8), it can be then noticed how

$$\alpha_g = \frac{1}{N}, \tag{1.19}$$

which means that the graviton-graviton interaction is way weaker if there are more gravitons making up the source.

Now, it can be understood how the sustainability of the condensate is possible thanks to the non linearity of the gravitational field. This can be seen by looking at the energy of a graviton: a graviton feeling the collective binding interaction of the other $N - 1$ gravitons is subjected to a potential like

$$U_G \sim -\alpha_g N \frac{\hbar}{r}, \tag{1.20}$$

and its kinetic energy will be

$$K \sim \frac{\hbar}{\lambda}. \tag{1.21}$$

But for $r \sim \lambda$ and (1.19) the marginally bound condition stemming the equilibrium between the kinetic energy and the self energy is reached, i.e.

$$K + U_G \sim 0. \tag{1.22}$$

Therefore, a balance between the two behaviours can be reached, and the graviton condensate can become self-sustained near the event horizon, which means that the gravitational source becomes sustainable when gravity becomes strong and nonlinear, and a black hole is formed [3].

At this point, it is worth to notice two facts: the first one is that in a black hole at $N \gg 1$ the wavelength expressed from (1.18), combined with the expression of N as a function of the mass M, can be used to estimate the characteristic magnitude of the event horizon of the system

$$\lambda \sim R_h. \tag{1.23}$$

This means that the corpuscular picture reproduces the geometric aspect of General Relativity by stating how the event horizon is nothing but the dominant wavelength of gravitons making up black holes. Moreover, N is fixed because if the mass of the source rises, also the event horizon grows and this implies a bigger characteristic wavelength of gravitons that, in turn, make them softer. This tuning happens because the gravitational coupling depends on the energy of the system and it is also the key aspect which makes gravity simple at the semiclassical level, where with simple it is meant that few parameters are necessary to completely describe the theory. In this case, only one parameter is needed, N; such simplicity at the quantum level reflects the simplicity at the classical level dictated by the No Hair Theorem.

To conclude, in the corpuscolar picture a black hole is characterized by only one parameter, N, the occupation number of gravitons. This is because gravity interacts through energy. Gravitons interact with a constant that goes like $\alpha_g \sim 1/N$, they have a wavelength going like $\lambda \sim \sqrt{N} l_p$ and together they form a source of mass $M \sim \sqrt{N} m_p$. Specifically, they form a condensate of N weakly interacting particles, and the black hole can be seen as a bound state; this is possible because of the self interacting nature of gravity, which makes the condensate self-sustained at distances close to the event horizon. In this context, it is remarkable how the condition (1.8) makes the collective potential energy felt by each graviton equal to the energy necessary to escape the bound state, i.e. $\lambda \sim \lambda_{escape}$. This implies that a graviton can detach from the bound state and escape to infinity, giving rise to an emission process that could be interpreted as the Hawking radiation. The very novelty brought by the corpuscular picture is the absence of any geometry at the fundamental level. The geometric picture can be recovered from the underlying quantum description of the gravitational field, and this crucial aspect is the starting point of the coherent state description.

Chapter 2

Coherent states in QFT

Coherent states were introduced for the first time by Schroedinger in a seminal paper [16] to study the dynamics of the harmonic oscillator, with the aim of recovering Classical Mechanics from Quantum Mechanics. The adjective "coherent" deserves a deep understanding; in the original paper, this word never appears, but what was intended there for coherent is probably "semiclassical". Coherent states for the harmonic oscillator show mean values for the dynamical variables of position and momentum that closely mimics their counterparts in classical mechanics. But this feature is actually shared by any mean value, as proved by the Ehrenfest Theorem. Therefore, coherent states are more than that: they are able to minimize the uncertainty of given observables, where for "minimize" it is meant that the Uncertainty Principle involving these observables is saturated with the equality.
At a first sight, this feature seems to involve only the kinematics of the system, i.e. considered two observables for any kind of system, coherent states could be built in order to minimize the variances. This conclusion may be supported by the fact that Gaussian wave-packets can be built for any system and these functions are special from the point of view of the position-momentum duality in Fourier space. However, a state is really said to be coherent not just for the minimized uncertainties that it gives, but also because such quantities are constant in time. This fact can be easily proved [4] for the harmonic oscillator, thanks to the special quadratic form of its Hamiltonian. It does not hold for arbitrary systems: a Gaussian wave-packet built for the free particle will spread out in its time evolution, and with it, its uncertainty will spread as well. A coherent wavefunction for the harmonic oscillator will show temporal stable variances for the associated observables. Thus, the definition of coherent states is not only kinematical, but also dynamical, and the harmonic oscillator was the first system for which these states have been built. It is quite natural to ask if coherent states could be constructed for field operators, since the latter can be written as a superposition of infinitely many harmonic oscillators in Fourier space. The answer is positive, and even if the price to pay is having to deal with some technicalities [17], the reward gained by defining coherent states for

quantum fields is a clearer understanding of what is meant for a quantum field to behave classically. This mathematical setup will be useful later in the construction of a quantum theory which can effectively reproduce the gravitational field.

2.1 Coherent states for the simple harmonic oscillator

Let us consider a Hamiltonian $H(x,p)$ and the two dynamical variables x and p, which in Quantum Mechanics are promoted to operators $\hat{x}$ and $\hat{p}$ acting on a Hilbert space $\mathcal{H}$ with commutation relations

$$[\hat{x},\hat{p}] = i\hbar. \tag{2.1}$$

Since the two operators do not commute, a complete set of simultaneous eigenstates cannot be found. This means that two simultaneous measurements of the above quantities cannot be taken, and the Heisenberg Principle is represented by the following inequality

$$\Delta\hat{x}\Delta\hat{p} \geq \frac{|\langle[\hat{x},\hat{p}]\rangle|}{2}, \tag{2.2}$$

where for a generic operator $\hat{A}$ the variance, or the uncertainty, is defined as

$$\Delta\hat{A} = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2. \tag{2.3}$$

In this case, the best thing to do is to find a state that can saturate this inequality. Such a task can be done by means of the following

Theorem. *Let $\hat{A}$ and $\hat{B}$ be two non commuting operators and $|\psi\rangle$ a normalized vector in the domains of $\hat{A}\hat{B}$ and $\hat{B}\hat{A}$.*
Then (2.2) is saturated if, and only if, $|\psi\rangle$ is an eigenstate for one of the following operators:
i) $\hat{A}$
ii) $\hat{B}$
iii) $\hat{A} - i\gamma\hat{B}$, for γ real.

It can be shown [18] that for $\gamma = \frac{\Delta\hat{A}}{\Delta\hat{B}}$, the factor acquires the meaning of a weight for the two uncertainties. For the position and momentum observables, the operator $\hat{A} - i\gamma\hat{B}$ can be defined in such a way that its eigenvalue equation shows the desired states

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle. \tag{2.4}$$

This definition is purely kinematical, but these states gain special properties when the time evolution is dictated by the Hamiltonian of the harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{1}{2}m^2\omega^2x^2. \tag{2.5}$$

For this system, the operator $\hat{a}$ can be written as

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p} \right). \tag{2.6}$$

It takes the role of an annihilation operator acting on a Fock space spanned by the eigenstates of the Hamiltonian, that is

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle , \tag{2.7}$$

with $\hat{a}|0\rangle = 0$. The state $|\alpha\rangle$ can be expanded as

$$|\alpha\rangle = \sum_n c_n |n\rangle , \tag{2.8}$$

and by applying iteratively (2.7) the coefficients c_n are found to be

$$c_n = c_0 \frac{\alpha^n}{\sqrt{n!}}. \tag{2.9}$$

The number c_0 can be used to normalize the state $|\alpha\rangle$ so that at the end

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle . \tag{2.10}$$

By means of the creation operator

$$\hat{a}^\dagger |n-1\rangle = \sqrt{n} |n\rangle \tag{2.11}$$

the $|\alpha\rangle$ state can also be rewritten in another useful way, which is

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha \hat{a}^\dagger} |0\rangle . \tag{2.12}$$

With this state written with respect to the occupation number basis, it can be seen how the uncertainties for $\hat{x}$ and $\hat{p}$ do not only saturate the Heisenberg relation, but they also remain constant in time, with

$$\Delta\hat{p} = \Delta\hat{x} = \sqrt{\frac{\hbar}{2}}. \tag{2.13}$$

This can be shown with the help of the Ehrenfest Theorem [4], stemming how the expectation value of a quantum mechanical operator $\hat{A}$ follows the classical equations of motion

$$\frac{d}{dt} \langle \hat{A} \rangle = \langle \partial_t \hat{A} \rangle + \frac{1}{i\hbar} \langle [\hat{A}, \hat{H}] \rangle . \tag{2.14}$$

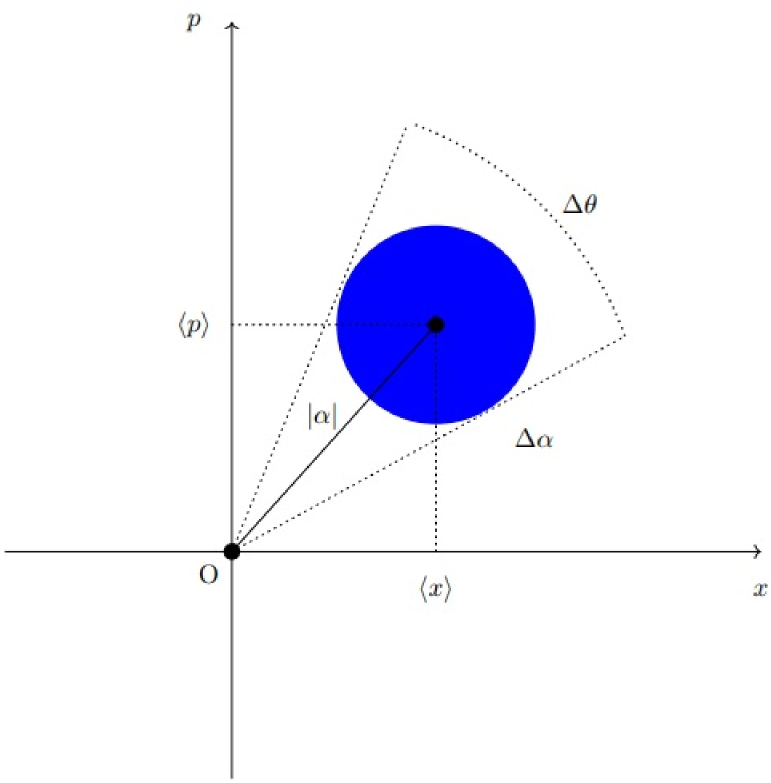

Figure 2.1: Phase space plot of a coherent state.

Therefore, when the system is in the state $|\alpha\rangle$, called coherent state, it will be in a quantum state extremely close to its classical counterpart. From the point of view of the uncertainties, such state can be represented by a disk of $\frac{\hbar}{2}$ diameter centered around $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$ in an $x-p$ plane, rather then a point in the phase space (see figure 2.1). Given the fact that $\alpha = |\alpha|e^{i\theta}$, the disk can be moved radially by changing $|\alpha|$ and rotated by changing θ. Moreover, it can be shown [19] that the set of states $|\alpha\rangle$, for α varying, is an overcomplete set of vectors, which means that these states can be used in place of the usual occupation number basis for the Fock space. However, coherent states are not strictly speaking a basis in the usual sense, since

$$\langle\beta|\alpha\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)+\alpha^*\beta} \tag{2.15}$$

for $\beta \neq \alpha$, as can be readily seen by using (2.12). Thus, coherent states belonging to the same family overlap to each other. This fact does not imply that coherent states cannot be used in the same way of occupation number states, since $\forall\,|\alpha\rangle$ coherent state in $\mathcal{H}$, and for $|\Phi\rangle \in \mathcal{H}$, it turns out that $\langle\alpha|\Phi\rangle = 0 \iff |\Phi\rangle = |0\rangle$, leading to a basis-like behaviour. Furthermore, coherent states resolve the identity like complete basis, but with the relation given by

$$1 = \frac{1}{\pi}\int d\mathcal{R}(\alpha)d\mathcal{I}(\alpha)\,|\alpha\rangle\,\langle\alpha|\,, \tag{2.16}$$

with $d\mathcal{R}(\alpha)$, $d\mathcal{I}(\alpha)$ denoting the integration measures over the real and imaginary parts of the complex number α.
To summarize, for the harmonic oscillator a special family of states with the following features can be defined:

- Minimum uncertainties for $\hat{x}$ and $\hat{p}$ operators.
- Temporal stable uncertainties for $\hat{x}$ and $\hat{p}$ operators.
- Overcompleteness.

2.2 Coherent states for a field

Let us now lift the previous considerations from a simple one dimensional harmonic oscillator to a free scalar field. Taking initially a single mode operator, the field can be expanded as

$$\hat{\Phi}_k(x) = \hat{a}_k e^{ikx} + \hat{a}_k^\dagger e^{-ikx}, \tag{2.17}$$

and its conjugate momentum as

$$\hat{\Pi}_k(x) = -i\omega_k(\hat{a}_k e^{ikx} - \hat{a}_k^\dagger e^{-ikx}). \tag{2.18}$$

The only non vanishing canonical commutation relation is given by

$$[\hat{a}_k, \hat{a}_k^\dagger] = 1, \tag{2.19}$$

thus leading to the same algebra of the harmonic oscillator and the same definition of the coherent state with $\gamma = \frac{1}{\omega_k}$

$$\hat{a}_k |\alpha_k\rangle = \alpha_k |\alpha_k\rangle . \tag{2.20}$$

The classical counterpart of the field operator $\hat{\Phi}_k(x)$ would be a monochromatic wave, and it can be seen that, by writing again $\alpha_k = |\alpha_k| e^{i\theta_k}$,

$$\langle\alpha_k| \hat{\Phi}_k(x) |\alpha_k\rangle = 2|\alpha_k| \cos(\vec{k} \cdot \vec{x} - \omega_k t + \theta_k) \tag{2.21}$$

so that the expectation value of the quantum field over a coherent state reproduces the classical wave configuration.
When the field is in a general superposition of modes,

$$\hat{\Phi}(x) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{h}{2\omega_k}} (\hat{a}_k e^{ikx} + \hat{a}_k^\dagger e^{-ikx}), \tag{2.22}$$

it splits into a positive and a negative frequency part

$$\hat{\Phi}(x) = \hat{\Phi}^{(+)}(x) + \hat{\Phi}^{(-)}(x) \tag{2.23}$$

with

$$\hat{\Phi}^{(+)}(x) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{\hbar}{2\omega_k}} \hat{a}_k e^{ikx}, \tag{2.24}$$

and a coherent state can still be defined as the eigenstate of the positive frequency part of the field

$$\hat{\Phi}^{(+)} |\Phi\rangle = \Phi |\Phi\rangle , \tag{2.25}$$

which again reproduces the classical behaviour for

$$\langle\Phi| \hat{\Phi}(x) |\Phi\rangle = \Phi(x). \tag{2.26}$$

The expression for this state, with respect to the occupation number basis, can be found by generalizing (2.12) and by taking care of counting all the modes

$$|\Phi\rangle = e^{-\frac{1}{2}\int d^3k |\alpha_k|^2} e^{\int d^3k \alpha_k \hat{a}_k^\dagger} |0\rangle . \tag{2.27}$$

The above expression can also be rewritten in terms of the single mode coherent states $|\alpha_k\rangle$ by tensoring them together

$$|\Phi\rangle = \bigotimes |\alpha_k\rangle , \tag{2.28}$$

with $\bigotimes |\alpha_k\rangle = |\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \ldots$, and the family of $|\alpha_k\rangle$ can be used in place of the usual basis $|n_k\rangle$. This shows trivially how $|\alpha_k\rangle$ is also eigenstate of $\hat{\Phi}^{(+)}(x)$, and how the vector $|\Phi\rangle$ also reproduces the classical expression of the conjugate field $\hat{\Pi}(x)$. The resolution of the identity can be generalized as

$$1 = \frac{1}{\pi} \prod_k \int d\mathcal{R}(\alpha_k) d\mathcal{I}(\alpha_k) \, |\alpha_k\rangle \langle\alpha_k| \,, \tag{2.29}$$

and $\mathcal{R}(\alpha_k)$ and $\mathcal{I}(\alpha_k)$ are again the real and imaginary parts of the kth complex number α_k.

This framework can be transposed from the free theory to any theory that shows a linear coupling between the field and an external source $j(x)$ [5]. Suppose the source for the field to be switched on and off, that is, it will be a non zero function only in a given time interval. The Hamiltonian of the system is still diagonalizable, because a general solution can be written as the sum of a free solution and a particular solution, which can be taken to be the convolution of the retarded propagator with the source, i.e.

$$\Phi(x) = \Phi^{(0)}(x) + i \int d^4y D_r(x-y) j(y), \tag{2.30}$$

with $(\Box - m^2)\Phi^{(0)}(x) = 0$ and

$$D_r(x-y) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{\hbar}{2\omega_p} \theta(x^0 - y^0)(e^{ip(x-y)} - e^{-ip(x-y)}). \tag{2.31}$$

After the source is switched off $\theta(x^0 - y^0) = 1$, and the general solution can be expanded in terms of the Fourier transform of the source $\tilde{j}(p)$

$$\Phi(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{\hbar}{2\omega_p}} \left[\left(a_p + \frac{i}{\sqrt{2\omega_p}} \tilde{j}(p) \right) e^{ipx} + \left(a_p^* - \frac{i}{\sqrt{2\omega_p}} \tilde{j}(p) \right) e^{-ipx} \right]. \tag{2.32}$$

From this expression, new creation and annihilation operators can be identified by

$$\hat{a}_p \to \hat{a}_p + \frac{i}{\sqrt{2\omega_p}} \tilde{j}(p) \tag{2.33}$$

and

$$\hat{a}_p^\dagger \to \hat{a}_p^\dagger - \frac{i}{\sqrt{2\omega_p}} \tilde{j}(p). \tag{2.34}$$

Now, it is well known that in the interacting picture fields evolve with the free Hamiltonian, while states evolve with the interacting Hamiltonian, which in this case is

$$H_{int} = \int d^4x j(x) \Phi(x). \tag{2.35}$$

Thus, starting from the vacuum $|0\rangle$, when the interaction is switched off the system will be in a state of the kind

$$|\Phi\rangle = e^{\frac{i}{\hbar}\int d^4x j(x)\Phi(x)} |0\rangle \,. \tag{2.36}$$

Expanding the field in the new creation and annihilation operators leads to

$$|\Phi\rangle = \bigotimes |j_k\rangle \,, \tag{2.37}$$

where $|j_k\rangle$ has the expression (2.12) with

$$j_k = \frac{i}{\sqrt{(2\pi)^3 2\omega_k \hbar}} \int d^4x j(x) e^{ikx}, \tag{2.38}$$

and the exponential with the annihilation operators naturally vanishes on the vacuum. The above result shows that for an interaction like (2.35) the coherent state family provides a natural basis.

2.3 The classical limit of a quantum field

Classical fields usually assume configurations involving waves with given amplitudes and phases. Since in the canonical quantization procedure fields are reinterpreted as operators, it could be asked if there are operators associated to the amplitude and the phase of a field. Together with a positive answer to this question, the effort to find such operators will make the meaning of taking the classical limit of a quantum field clearer. For the sake of simplicity, such operators will be found only for single modes quantum fields.

A first naive attempt would be to write down a generic Fourier coefficient for the field, $a = Ae^{i\Theta}$, as the product of a positive definite modulus and an exponential. Subsequently, interpret such terms as the amplitude and the phase of the field and then promote them to the role of operators, i.e. $\hat{a} = \hat{A}e^{i\hat{\Theta}}$, with $\hat{A}^2 = \hat{N}$. However, as it is well known, the definition of an operator related to a periodic observable is tricky (for a review of this problem, see [20]), since the very definition of a phase is only up to a 2π shift. Such an interpretation would lead to nonsensical results such as the following. Using the above decomposition inside the well known commutation relation for ladder operators brings to

$$[\hat{A}^2, e^{i\hat{\Theta}}] = -e^{i\hat{\Theta}}. \tag{2.39}$$

This relation directly follows if a complementarity between $\hat{A}^2$ and $\hat{\Theta}$ is assumed, i.e.

$$[\hat{A}^2, \hat{\Theta}] = i. \tag{2.40}$$

However, taking the vacuum expectation value of this commutator leads to

$$\langle[\hat{A}^2, \hat{\Theta}]\rangle_0 = 0 = i. \tag{2.41}$$

The above problem can be circumvented by means of the Polar Decomposition Theorem, which allows to write an operator as a product of a unitary operator and a semipositive definite operator, provided the space of states to be finite dimensional. Doing that for the creation operator of a field brings to

$$\hat{a} = \hat{E}\widehat{\sqrt{N}}, \tag{2.42}$$

where for clarity the operator $\widehat{\sqrt{N}}$ shows a square root just to point out how $(\widehat{\sqrt{N}})^2 = \hat{N}$. The definition of the operator $\hat{E}$ will be more careful now, and a consistent operator corresponding to a quantum phase will be found without directly providing an operator for the angular variable itself. Of course, the Fock space related to a field is not finite dimensional, and to make use of the Polar Decomposition Theorem it should be truncated in some way. So let us redefine the creation operator as follows

$$\begin{cases} \hat{a}^\dagger \left|n\right\rangle = \sqrt{n+1} \left|n+1\right\rangle & n < N \\ \hat{a}^\dagger \left|N\right\rangle = 0 & n > N, \end{cases} \tag{2.43}$$

where N is the maximum occupation number, i.e. the cut-off imposed to the Fock space. The algebra now is finite dimensional and the fundamental commutation relation reads

$$[\hat{a}, \hat{a}^\dagger] = 1 - (N+1) \left|N\right\rangle \left\langle N\right| . \tag{2.44}$$

It is immediate to notice, from the explicit expression of coherent states in occupation number basis (2.10), how the eigenstates of the number operator $\left|n\right\rangle$ have as probability density on the coherent state a Poissonian distribution. This distribution can be further approximated by a Gaussian one in the limit $\bar{N} = |\alpha|^2 \gg 1$:

$$|\left\langle n|\alpha\right\rangle|^2 \sim \frac{1}{\sqrt{2\pi\bar{N}}} e^{-\frac{1}{2}\frac{(n-\bar{N})^2}{\bar{N}}} . \tag{2.45}$$

As can be seen by the Polar Decomposition Theorem, the operator $\hat{N}$ is related to the amplitude of the field, and the above limit hints that a classical behaviour is shown once we have a lot of quanta in the state.
The operator $\hat{E}$ can instead be identified with

$$\hat{E} = \sum_{j=0}^{N-1} \left|j\right\rangle \left\langle j+1\right| + \left|N\right\rangle \left\langle 0\right| , \tag{2.46}$$

which has the same action of $\hat{a}$ on the Fock states $\left|n\right\rangle$, but without the factorial eigenvalue. It is indeed unitary, and it is then a good candidate to represent some kind of phase operator. To see this, impose the eigenvalue equation

$$\hat{E} \left|\Theta\right\rangle = e^{i\Theta} \left|\Theta\right\rangle \tag{2.47}$$

and write the eigenstates with respect to the coherent states; this will allow to capture the uncertainty of the phase operator. In fact, the eigenstates of this operator can initially be rewritten in terms of the Fock basis as

$$|\Theta\rangle = \sum_{k=0}^{N} d_k \, |k\rangle \,. \tag{2.48}$$

By inserting the above equation and (2.46) inside (2.47), the following recursive relation is found

$$d_{k+1} = d_k e^{-i\Theta}, \tag{2.49}$$

with

$$d_0 = d_N e^{-i\Theta}. \tag{2.50}$$

These equations can be combined to give

$$d_k = d_0 e^{ik\Theta}, \tag{2.51}$$

with a condition for the phase resulting in

$$\Theta = \frac{2\pi m}{N+1}, \tag{2.52}$$

and m an integer number. The coefficient d_0 is used to normalize $|\Theta\rangle$, leading to

$$|\Theta\rangle = \frac{1}{\sqrt{N+1}} \sum_{k=0}^{N} e^{i\frac{2\pi m}{N+1}} \, |k\rangle \,. \tag{2.53}$$

Now, taking again the limit $\bar{N} \gg 1$, the probability of the phase state in the coherent state basis can be calculated as

$$| \langle\Theta|\alpha\rangle |^2 \sim e^{-2\bar{N}(\theta-\Phi)^2}, \tag{2.54}$$

with θ the usual phase of the eigenvalue α corresponding to the classical phase of the field. Also, this is the squared modulus of the eigenfunction of the phase operator and presents a Gaussian-like profile. As all the Gaussian wavefunctions, the coefficient inside the exponent can be interpreted as the uncertainty of the packet. This uncertainty is given by $\Delta\Theta = \frac{1}{2\Delta N}$, pointing out a "minimum uncertainty" behaviour for the amplitude and the phase of the field

$$\Delta N \Delta\Theta = \frac{1}{2}. \tag{2.55}$$

In contrast, having considered a Fock state in place of a coherent one would have led $\Delta N = 0$, as the number operator is diagonal in this basis. For the phase operator instead, considering only the real part of $\hat{E}$, namely $\hat{C} = \frac{1}{2}(\hat{E} + \hat{E}^\dagger)$, it can be found that [17]

$$\begin{cases} \langle\hat{C}\rangle_n = 0 \\ \langle\hat{C}^2\rangle_n = \frac{1}{2}. \end{cases} \tag{2.56}$$

Therefore

$$\langle \Delta \hat{C} \rangle_n = \frac{1}{\sqrt{2}}, \tag{2.57}$$

thus showing that the uncertainty in θ, or rather $cos\theta$, is maximal and the above state can be seen as an infinitely thin disk in a $x - p$ space, as the phase is spread uniformly over $[0, 2\pi]$ (see figure 2.2).

It can also be seen how these distributions imply an occupancy amplitude of the states $|0\rangle$ and $|N\rangle$ which becomes negligible for the condition $N \gg \bar{N} \gg 1$, that is:

$$|\langle N|\alpha\rangle| \sim \frac{e^{\frac{(N-\bar{N})}{2}}}{(2\pi N)^{\frac{1}{4}}} \left(\frac{\bar{N}}{N}\right)^{\frac{N}{2}} \ll 1 \tag{2.58}$$

and

$$|\langle 0|\alpha\rangle| \sim e^{-\frac{\bar{N}}{2}} \ll 1. \tag{2.59}$$

Thus truncating the Fock space much above the high occupation number of the field does not exclude significant amplitudes.

Now, it is possible to ask whether a field behaves in a classical manner. From the probability distributions found, the field state needs to have a high occupation number to get simultaneously an almost sharp amplitude, with $\bar{N} \gg \Delta N = \sqrt{N}$ (but still $\Delta N \gg 1$), as well as an almost sharp phase with $\Delta\Theta \ll 1$. Notice that to achieve this result, the field should be bosonic, since the Pauli Exclusion Principle forbids the occupation of the same mode for two fermionic quanta.

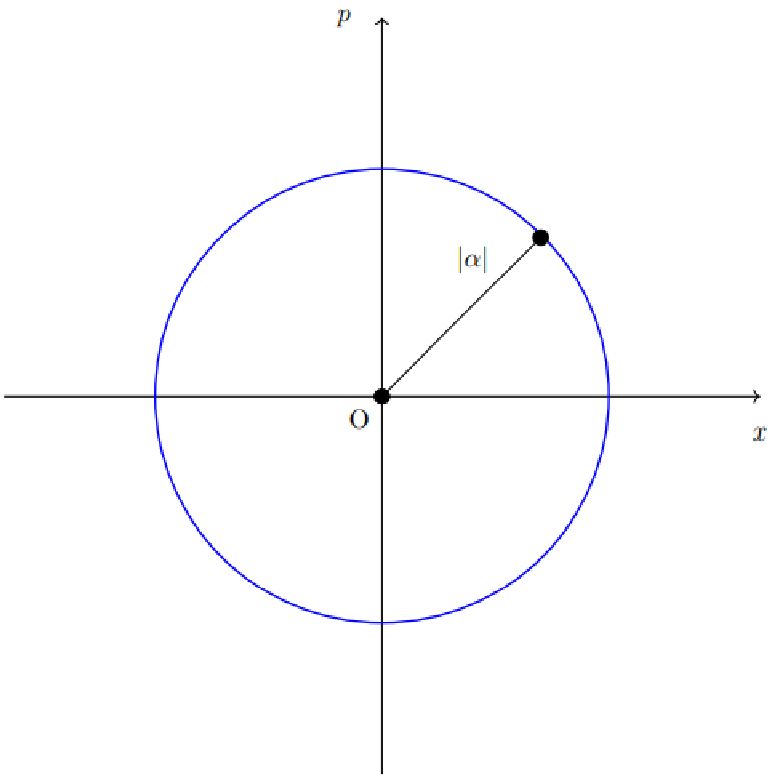

Figure 2.2: Phase space plot of a Fock state. The uncertainty on the phase is maximal as the possible value for the angle is spread over the interval $[0, 2\pi]$. The amplitude is instead sharply specified by the value $|\alpha|$, as can be expected by an eigenstate of the number operator.

Chapter 3

Scalar coherent state description of the gravitational field

The corpuscular picture of a black hole presents various qualities: it is a very simple effective theory from the point of view of the parameters which are needed to specify it; moreover, it takes into account classical and semiclassical aspects of black hole physics under a new perspective. The very important message which should be taken from the corpuscular picture is that black holes are bound states of gravitons, and that such bound state reproduces the geometric aspect of gravity as an emerging feature [1]. However, aside from the fact that the corpuscular approach presented above gives rather qualitative explanations and does not go into the details and subtleties of some important issues, like how to precisely recover the geometric picture, it completely neglects any role of matter from which the black hole had formed. In particular, it is generically stated that any classical notion could, in principle, be recovered from graviton mechanics. Stating this in a more refined way, the geometrical picture should hide an effective quantum description which is appropriate for gravity, that will be now described as a quantum field among the others. A careful semiclassical treatment of the underlying quantum theory that takes into account the most interesting aspects of the corpuscular picture shows that the role of matter cannot be neglected for the sake of consistency of the theory itself. This is what will be shown below, together with the approach to gravity coming from a theory that quantizes a graviton field and effectively tries to reproduce its classical physics [6, 8, 21, 22].

In the weak field limit, where the source of the field is small if compared to other scales, let us say the Planck scale ($M_{source} \ll m_p$), the classical theory can be linearized as

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}, \tag{3.1}$$

with $\epsilon \ll 1$.

The perturbation $h_{\mu\nu}$ can be quantized over the Minkowski space where the vacuum state $|0\rangle$ is defined as the state where no matter nor metric are excited, and the Fock space is built with the usual occupation number basis. In this regime, $h_{\mu\nu}$ excitations can be seen as spin 2 massless particles, the gravitons, and the gravitational field is linearly coupled at tree level to the energy momentum tensor of matter fields. This approach is useful to overcome the difficulties arising from the non linearity of the self interactions characterizing strong gravitational regimes and the gauge fixing problem of the full General Relativistic Theory.

In a strong regime ($M_{source} \gg m_p$), which is the one involving black holes, the gravitational field becomes strong and the quantization procedure over the flat background becomes obscure, since in (3.1) $h_{\mu\nu}$ will be no more a small quantity; one may think to start with a non flat background solution of the field equations $g^{(0)}_{\mu\nu}$ and write the graviton field as a perturbation of the above metric,

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \epsilon h_{\mu\nu}, \tag{3.2}$$

with $g^{(0)}_{\mu\nu}$ that can be any known solution such as the Schwarzschild one. This expansion, which works in some contexts such as gravitational waves physics, looks conceptually suspicious from at least two points of view. First of all, the background is still rather arbitrary, and another choice of $g^{(0)}_{\mu\nu}$ would lead to a different definition for excitations. Secondarily, when dealing with the corpuscular picture of gravity, the (geometrical) Einstein equations themselves, and their solutions, are seen as emerging from the pure quantum theory in a suitable limit, and therefore a classical solution cannot be taken a priori to quantize the theory itself.

This knot is not easy to untie. Therefore, instead of looking at the full non interacting quantum theory, let us look for a quantum state built out from the Fock space of gravitons for the linearized theory that can reproduce classical field configurations. On top of the conceptual issues, there are also technical difficulties arising from treating metric tensors. A possible solution to that problem, which makes calculations simpler, is to replace the metric tensor $h_{\mu\nu}$ with a scalar field Φ, since it is well known how difficulties in computations rise with the spin of the fields. The choice of a scalar may seem strange at first sight, since General Relativity as a field theory is regarded as a spin 2 theory. But it is also true that in the non relativistic limit for the amplitude of a graviton exchange, the scalar Newtonian potential is recovered as the non propagating temporal component of the metric tensor [7]. Thus, after having recognized how the Newtonian potential is embedded in the metric tensor for the Schwarzschild solution at the classical level, a scalar mean field approach [8] will be employed with the aim of grasping as many information as possible by barely looking at what are the features that a metric function shows when it comes from a quantization procedure.

Thus, the desired quantum state $|g\rangle$ can be defined as

$$V_q = \langle g| \hat{\Phi} |g\rangle \,, \tag{3.3}$$

that is, it can be required that a scalar quantum field representing gravity gives back, through the expectation value over this state, the Newtonian potential (and thus the

Schwarzschild metric function) up to quantum corrections. Then, the geometrical description will be recovered with a corrected Schwarzschild metric of the type

$$ds^2 = -(1 + 2V_q)\, dt^2 + (1 + 2V_q)^{-1}\, dr^2 + r^2 d\Omega. \tag{3.4}$$

Let us stress that, in this perspective, the Newtonian configuration comes from a totally quantized theory in Minkowski spacetime. The state $|g\rangle$, if existing, should correctly reproduce the gravitational potential V_q, with the hope of finding new features on the nature of gravity coming from Quantum Mechanics. These corpuscular corrections to the Newtonian potential will then enter the metric function through a non perturbative mean field approach. It is not surprising that, after having discussed how coherent states can make what does it mean to recover a notion of "classicality" very clear, $|g\rangle$ will be a coherent state itself.

3.1 Newton potential from the metric tensor

Let us begin by looking at how Newtonian gravity emerges from General Relativity. In the linearized regime, representing the weak field limit of the metric, the gauge fixed Einstein-Hilbert action in the presence of an external source (or current) reads

$$S = \int d^4x \left[-\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + \frac{1}{4}\partial^\mu h \partial_\mu h + \sqrt{8\pi G} h_{\mu\nu} T^{\mu\nu} \right]. \tag{3.5}$$

The equations of motion for $h_{\mu\nu}$ can be written as

$$\Box h_{\mu\nu} = -\sqrt{8\pi G}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T \right). \tag{3.6}$$

They can be transformed in momentum space,

$$\tilde{h}_{\mu\nu} = \frac{\sqrt{8\pi G}}{k^2}\left(\tilde{T}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\tilde{T} \right), \tag{3.7}$$

where $\tilde{T}$ is the Fourier transform of T, so that the interaction term in the Lagrangian can be seen as the interaction between two external sources

$$\frac{8\pi G}{k^2}\left(\tilde{T}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\tilde{T} \right)\tilde{T}'^{\mu\nu}. \tag{3.8}$$

The calculation of amplitudes for given processes is made by means of the propagator connecting sources. Such amplitudes depend on the form of the interaction term between the current and the field or, in term of the sources, between two sources.
Using the energy-momentum conservation

$$\partial_\mu T^{\mu\nu} = 0, \tag{3.9}$$

and choosing the reference frame in such a way that the momentum vector is given by $k^\mu = (\omega, 0, 0, k)$, the components of the energy-momentum tensor satisfy the relation

$$\tilde{T}^{3\nu} = \frac{\omega}{k}\tilde{T}^{0\nu}. \tag{3.10}$$

Expanding (3.8) in all its indices and using (3.10), the following expression for the amplitude of the process is given by

$$\begin{aligned} &-\frac{4\pi G}{k^2}\tilde{T}^{00}\tilde{T}'^{00} + \frac{4\pi G\omega^2}{k^4}\tilde{T}^{00}\tilde{T}'^{00} - \frac{4\pi G}{k^2}\tilde{T}^{00}(\tilde{T}'^{11} + \tilde{T}'^{22}) \\ &-\frac{4\pi G}{k^2}\tilde{T}'^{00}(\tilde{T}^{11} + \tilde{T}^{22}) + \frac{16\pi G}{k^2}(\tilde{T}^{01}\tilde{T}'^{01} + \tilde{T}^{02}\tilde{T}'^{02}) \\ &-\frac{4\pi G}{k^2-\omega^2}(\tilde{T}^{11} - \tilde{T}^{22})(\tilde{T}'^{11} - \tilde{T}'^{22}) - \frac{16\pi G}{k^2-\omega^2}\tilde{T}^{12}\tilde{T}'^{12}. \end{aligned} \tag{3.11}$$

It is immediate to notice how the amplitude naturally splits into a $\frac{1}{k^2}$ part coupled to the temporal components of the source and a $\frac{1}{k^2-\omega^2}$ part coupled to the spatial components. The former will give the Newtonian instantaneous interaction in the non relativistic limit, while the latter is the causal amplitude propagating in the full field theory, with a pole giving the dispersion relation for the wave.
By taking the non relativistic limit, and remembering that the spatial components of the energy momentum tensor scale with the velocity of the source v, it is expected that only the $\frac{1}{k^2}\tilde{T}^{00}\tilde{T}'^{00}$ survives, since the term $\frac{\omega^2}{k^4}\tilde{T}^{00}\tilde{T}'^{00}$ can be rewritten as a $\tilde{T}^{03}\tilde{T}'^{03}$ term using again (3.10). The only consistent way to recover Newtonian gravity from a relativistic theory is to consider stationary sources $\tilde{T} = \tilde{T}(\vec{k})$. This is because Newtonian physics propagates instantaneously any kind of signal, lacking the notion of causality, and therefore no time propagation should be considered in order to match the two regimes. Going back in position space, it is found that

$$-\int \frac{d^4k}{(2\pi)^4}e^{ik(x-x')}\frac{4\pi G}{k^2}\tilde{T}^{00}\tilde{T}'^{00} = -4\pi G\int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\frac{\tilde{T}^{00}\tilde{T}'^{00}}{k^2}\delta(t-t'), \tag{3.12}$$

which is indeed instantaneous. To see that the above term really represents the Newton potential, suppose the two sources to be point-like, i.e.

$$\begin{cases} \tilde{T}^{00} = M \\ \tilde{T}'^{00} = M'. \end{cases} \tag{3.13}$$

Then (3.12) reads

$$-\frac{GMM'}{\pi}\int_0^\infty dk\int_{-1}^1 d(\cos\theta)e^{i|\vec{k}||\vec{x}-\vec{x}'|\cos\theta}\delta(t-t') = -\frac{GMM'}{\pi|\vec{x}-\vec{x}'|}\int_{-\infty}^\infty dz\frac{\sin z}{z}\delta(t-t'), \tag{3.14}$$

that is, the non relativistic and weak field amplitude for two stationary particles reduces to the usual two body gravitational interaction energy

$$U(|\vec{x}-\vec{x}'|) = -\frac{GMM'}{|\vec{x}-\vec{x}'|}, \tag{3.15}$$

which can be derived from the potential

$$V\left(|\vec{x}-\vec{x}'|\right) = -\frac{GM}{|\vec{x}-\vec{x}'|}. \tag{3.16}$$

Now, let us look at the Schwarzschild solution generated by a source of mass M

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2. \tag{3.17}$$

By identifying $r = |\vec{x}-\vec{x}'|$[1], the Newtonian potential can be written as a function of the areal radius $r = \sqrt{\frac{A}{4\pi}}$, with $A = \int d\Omega\sqrt{|g|}|_{t,r=const.}$ the area of the two dimensional hypersurface defined by $t, r = const.$

It is now clear how to embed (3.16) inside the g_{00} and g_{11} components. For example, considering a test particle radially falling from infinity in the gravitational field, its four-velocity components are given by

$$u^0 = -\frac{1}{1-\frac{2GM}{r}} \tag{3.18}$$

and

$$u^r = \sqrt{\frac{2GM}{r}}, \tag{3.19}$$

where the energy conservation along the geodesic, as well as the normalization of the velocity vector, were used. Thus, the geodesic equation reads

$$\frac{d^2}{d\tau^2}r = -\frac{GM}{r^2} = -\frac{d}{dr}V(r), \tag{3.20}$$

like the Newtonian equation of motion in the presence of a gravitational source.

[1]This point will be further discussed in section 6.4 and it will be crucial to understand how non perturbative aspects of gravity enter into the scalar coherent state description.

3.2 Coherent state description

As extensively discussed, the states which better reproduce a semiclassical behaviour, with the world "semiclassical" understood as minimum and constant uncertainty of the field configuration, are coherent states. Thus a coherent state for a free massless scalar field of gravitons can be built in order to reproduce the Newtonian potential, and thus the Schwarzschild metric function, which comes out as a configuration in the mean field approach [8].
First of all, let us quantize a canonically normalized field $\sqrt{G}\Phi$ that satisfies the Klein-Gordon equation in flat spacetime

$$(-\partial_t^2 + \Delta)\Phi(x) = 0. \tag{3.21}$$

The best choice for the basis modes is given by the Bessel functions, since the Newtonian potential has a radial dependence and polar coordinates are best suited to write it. Specifically, imposing the expected spherical symmetry of the system gives the modes

$$u_k(x) = e^{-ikt} j_0(kr), \tag{3.22}$$

that lead to

$$\hat{\Phi}(x) = \int_0^\infty \frac{dk}{2\pi^2} k^2 \sqrt{\frac{\hbar}{2k}} (\hat{a}_k u_k + \hat{a}_k^\dagger u_k^*) \tag{3.23}$$

and

$$\hat{\Pi}(x) = i \int_0^\infty \frac{dk}{2\pi^2} k^2 \sqrt{\frac{\hbar k}{2}} (\hat{a}_k u_k - \hat{a}_k^\dagger u_k^*). \tag{3.24}$$

These fields satisfy

$$[\hat{\Phi}(t,r), \hat{\Pi}(t,r')] = \frac{i\hbar}{4\pi r^2} \delta(r - r'), \tag{3.25}$$

with the factor $1/4\pi r^2$ coming from the scalar product of two Bessel functions. The creation and annihilation operators satisfy then

$$[\hat{a}_k, \hat{a}_p] = \frac{2\pi^2}{k^2} \delta(k - p) \tag{3.26}$$

and, as already said, the Fock space is built out from the Minkowski vacuum, where no excitations of matter or gravitons are present.
Then the coherent state for the graviton field is given by (2.27):

$$|g\rangle = e^{-\frac{N_g}{2}} \exp\left\{ \int_0^\infty \frac{dk}{2\pi^2} k^2 g_k \hat{a}_k^\dagger \right\} |0\rangle\,, \tag{3.27}$$

and satisfies the equation

$$a_k |g\rangle = g_k e^{i\gamma_k(t)} |g\rangle\,. \tag{3.28}$$

The quantity

$$N_g = \int_0^\infty \frac{dk}{2\pi^2} k^2 g_k^2 \tag{3.29}$$

is identified with the graviton number because it is the result of $\hat{N} = \hat{a}_k^\dagger \hat{a}_k$ on the coherent state, and thus measures the distance of the state from the vacuum, where $N_g = 0$. Such a quantity also ensures the proper normalization for the coherent state itself. It is clear how $N_g \gg 1$, since the quantum field $\hat{\Phi}$ should reproduce a classical field configuration (and thus the classical metric function), making the whole approach non perturbative.
Let us remark that the $\hat{a}_k^\dagger$ operators superposed in the coherent state definition are the creation operators for the Fock space of the free scalar theory. It is worth to highlight that the coherent state for the field configuration is built from this linear theory by means of (3.30), and not from the fully solved interacting theory (which is well known to be not solvable at all, as any interacting Relativistic Quantum Field Theory). This is possible because the Newtonian configuration is already known, despite being static. Of course, nothing is really static at the very fundamental level in any relativistic theory, but in a first, and most importantly simple, treatment of gravity such property is necessary to recover the classical and corrected behaviour for the field. The free theory is also better than a theory where the field is linearly coupled to an external classical current. In fact, even if a coherent state built for a linearly interacting theory were considered in place of $|g\rangle$, the linearity, and thus the very definition of the coherent state, would be broken by loop corrections that inevitably arise when a theory with an interaction is quantized.
The Schwarzchild metric function is therefore recovered as

$$\sqrt{G} \langle \Phi \rangle_g = V, \tag{3.30}$$

with the function V which satisfies the classical Poisson equation,

$$\Delta V = 4\pi G \rho, \tag{3.31}$$

with ρ a classical matter density function. By Fourier transforming the Poisson equation, the relation in momentum space between the potential and the source is given by

$$\tilde{V} = -\frac{4\pi G \tilde{\rho}}{k^2}. \tag{3.32}$$

Considering now the most simple solution for the spherically symmetric case, which comes from a point-like source described by a density function

$$\rho(r) = \frac{M}{4\pi r^2} \delta(r), \tag{3.33}$$

with M the mass of the source, the above relation becomes

$$\tilde{V} = -\frac{4\pi G M}{k^2}. \tag{3.34}$$

By expanding in momentum space both sides of (3.30),

$$\sqrt{G}\,\langle\hat{\Phi}\rangle_g = \sqrt{G}\int_0^\infty \frac{dkk^2}{2\pi^2}\sqrt{\frac{\hbar}{2k}}\frac{\sin(kr)}{kr}\left(g_k e^{i\gamma_k(t)}e^{-ikt} + g_k^* e^{-i\gamma_k(t)}e^{ikt}\right) \tag{3.35}$$

$$V(r) = -\int_0^\infty \frac{dkk^2}{2\pi^2}\frac{4\pi GM}{k^2}\frac{\sin(kr)}{kr}, \tag{3.36}$$

the equation is satisfied by

$$g_k = -\frac{4\pi M}{\sqrt{2k^3}m_p} \tag{3.37}$$

only provided that

$$\gamma_k(t) = kt. \tag{3.38}$$

This gives an explicit expression for the graviton number (3.29)

$$N_g = \frac{4M^2}{m_p^2}\int_0^\infty \frac{dk}{k} \tag{3.39}$$

that diverges both in the IR and UV.
A quantum state exists only if provided with a well defined normalization, which is not the case for $|g\rangle$. The problematic expression for N_g arises from the sharpness of the point-like source, which can be regularized. However, the Poisson equation, unlike Einstein field equations, accepts point-like sources on its right hand side, and therefore it is simpler to put two cut-offs in the theory rather than smearing the source to regularize the normalization factor. Such cut-offs will be just a mathematically simple way to describe the fact that the very existence of a proper coherent state requires the g_k to be modified with respect to their classical values in the deep IR and UV. The cut-offs can be physically justified by thinking on a finite size R_s and on a finite time of collapse R_∞ for the source. These two length scales naturally introduce momentum scales that can be regarded as cut-offs,

$$k_{UV} = \frac{1}{R_s} \tag{3.40}$$

$$k_{IR} = \frac{1}{R_\infty}, \tag{3.41}$$

leading to

$$N_g = 4\frac{M^2}{m_p^2}\log\left(\frac{R_\infty}{R_s}\right). \tag{3.42}$$

The above cut-offs hint that the quantum coherent state for a black hole needs not to contain all the possible (high and low) frequency modes. This conclusion is further strengthened by the fact that the scalar field expectation value should reproduce the classical metric function only outside the horizon R_h, since the role played by the matter

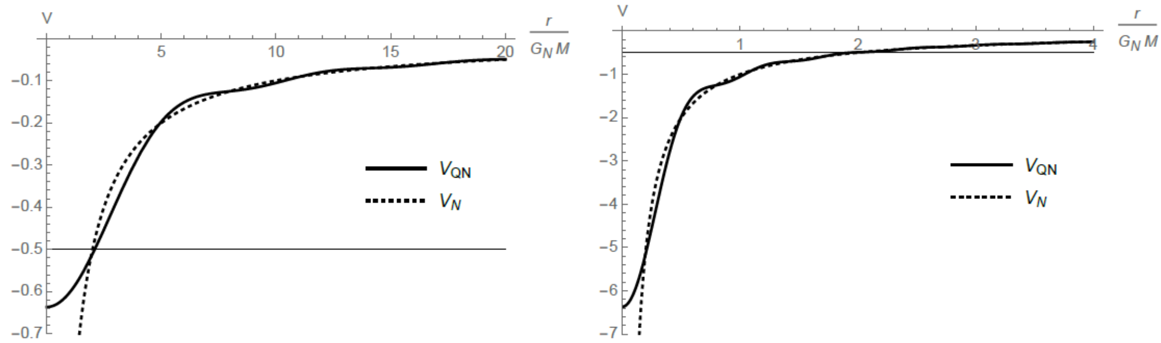

Figure 3.1: The quantum metric function and its horizon plotted for $R_s = \frac{R_h}{2}$ and for $R_s = \frac{R_h}{20}$. Figure taken from [8].

content inside the horizon is unknown and experimental bounds can be placed only in the outer region of communication,

$$\sqrt{G}\,\langle g|\hat{\Phi}|g\rangle \simeq V_q(r) \qquad r \gtrsim R_h. \tag{3.43}$$

Notice that the expression (3.42) differs from the first result (1.13) by the presence of the logarithmic term involving the features of the source, bringing no hair violations at the level of the occupation number. The healing of the divergences explicitly depends on the choice of the cut-offs, which was arbitrary, but nevertheless the cut-offs can give a taste of what can occur when Quantum Mechanics describes a function that enters in a gravity description. Furthermore, as can be seen from figures 3.1, 3.2, the oscillations of the quantum solution around the classical one can be made arbitrarily small by changing the size of the source R_s. In particular, plugging the cut-offs in the Fourier expansion for the solution of the Poisson equation modifies the metric function as

$$V_q = -\frac{2GM}{\pi r}\int_0^{\frac{r}{R_s}} \frac{\sin(z)}{z}dz. \tag{3.44}$$

Therefore, inserting the cut-offs brings quantum corrections to the classical metric function through V_q, and the reconstructed Schwarzschild solution is[2]

$$ds^2 = -(1+2V_q)dt^2 + (1+2V_q)^{-1}dr^2 + r^2 d\Omega. \tag{3.45}$$

Besides the presence of an event horizon even in this mean field metric tensor, it is interesting to look at the classical spacetime singularity of the Schwarzschild solution.

[2]This reconstruction of the metric tensor works thanks to the classical analogy between the Newtonian potential and the Schwarzschild metric function. The identification of the "flat" distance r with the areal Schwarzschild coordinate r hides all the ambiguities given by non perturbative aspects of gravity. See again section 6.4

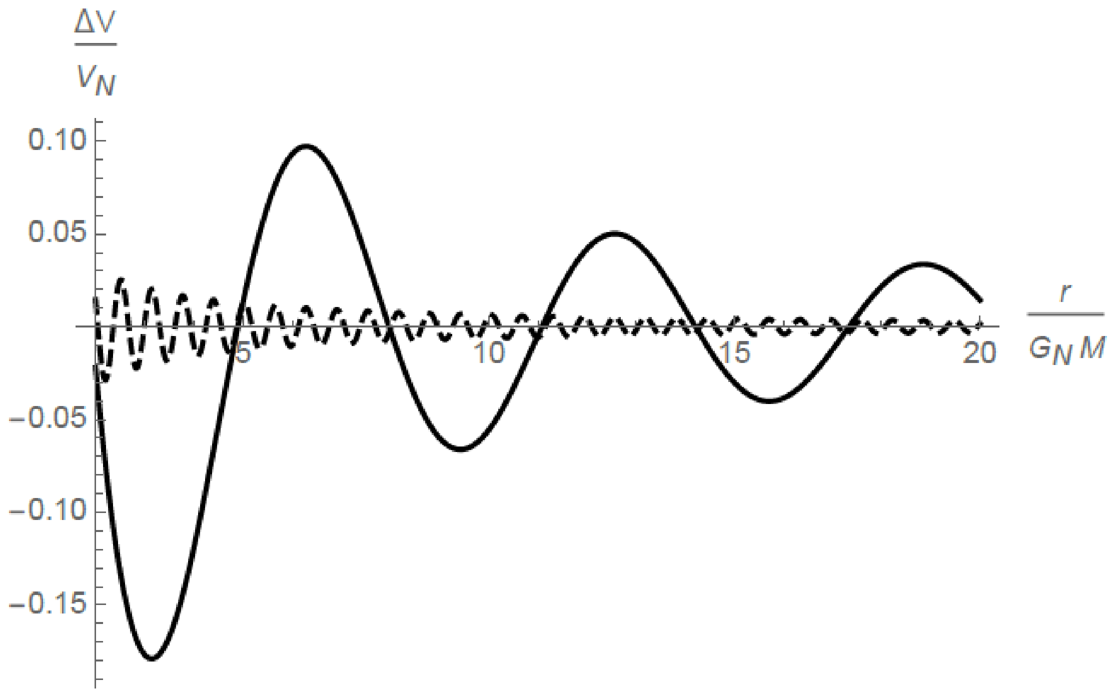

Figure 3.2: Oscillations of the quantum metric function around the classical solution. The solid line has $R_s = \frac{R_h}{2}$, while the dashed one has $R_s = \frac{R_h}{20}$. Figure taken from [8].

This quantum picture cannot try to solve exactly the singularity at $r = 0$; anyway, the quantum corrected function V_q is regular when approaching the origin:

$$V_q \sim -\frac{2GM}{\pi r}\left(\frac{r}{R_s} - \frac{1}{3\cdot 3!}\frac{r^3}{R_s^3}\right) = \frac{2GM}{\pi R_s}\left(1 - \frac{r^2}{18R_s^2}\right), \tag{3.46}$$

suggesting how the gravitational tidal forces will not show any singular behaviour. This conclusion is further supported by the evaluation of the Kretschmann invariant, which now goes like r^{-4}. This can be seen by computing the relative acceleration between radial geodesics, i.e.

$$\frac{\ddot{\delta r}}{\delta r} \simeq \frac{8G^2M^2}{9\pi^2 R_s^2}\left(1 - \frac{\pi R_s}{4GM}\right). \tag{3.47}$$

So, it could be said that $r = 0$ is now an integrable singularity.
Apart from N_g, the source cut-offs regularize also other interesting quantities which would otherwise diverge, such as the mean value for the gravitons momentum

$$\langle k\rangle_g = \frac{4M^2}{m_p^2}\int_0^\infty dk \tag{3.48}$$

that becomes

$$\langle k\rangle_g = \frac{4M^2}{m_p^2}\left(\frac{1}{R_s} - \frac{1}{R_\infty}\right). \tag{3.49}$$

From this expression, the wavelength of gravitons can be calculated by

$$\lambda = \frac{N_g}{\langle k\rangle_g}, \tag{3.50}$$

that gives R_h, the result (1.23) hinted by the corpuscular picture, only if the cut-offs satisfy

$$\frac{\log\frac{R_\infty}{R_s}}{\frac{1}{R_s} - \frac{1}{R_\infty}} = R_h, \tag{3.51}$$

so that the radius of the source and the radius of the outer region containing the gravitational field are actually connected at the quantum level.

Chapter 4

The uncertainty of a quantum field

It was shown how the semiclassical configuration for the gravitational field can be obtained by a coherent state; such state can be seen as a superposition of gravitons with a mean characteristic wavelength $\lambda \sim 2GM$. Coherent states ensure the most classical behaviour for the field, where, as previously said, this is understood as a minimum and constant uncertainty $\Delta\hat{\Phi}(x)$, in complete analogy with the harmonic oscillator case. Actually, what really matters when speaking of classicality is the ratio $\frac{\Delta\hat{\Phi}(x)}{\langle\Phi(x)\rangle}$, which should bring to an expression depending from the number of gravitons that goes to zero, as the latter is sent to infinity in the classical regime. This feature is peculiar for any theory relying on a corpuscular picture [15].
In the calculation of the variance, infrared and ultraviolet divergences arise, and the problem is fixed with the introduction of a detector. This solution, even if presented in a very simple manner, has its grounds on the questions regarding the validity of the standard Quantum Field Theory approach. For example, technical difficulties and conceptual issues, such as superluminar signalling, arise when Quantum Information Theory is brought in such a framework [11], so the introduction of a Detector Based Quantum Field Theory has been proposed several times [10] in order to avoid inconsistencies. These ideas have their foundations on a very seminal paper by Bohr and Rosenfeld [9], but also on thought experiments such as the Heisenberg's microscope.
It is therefore worth to ask what does it mean to introduce a quantum device for the measurement process even in basic Quantum Field Theory calculations, such as the ones involving the uncertainty of fields. In this context, the detector approach solves a problem that was already treated from another perspective. The divergence of the expectation value $\langle\hat{\Phi}^2\rangle$ is understood as a consequence of the locality of $\hat{\Phi}$ itself [17]: the field operator is not an honest observable, as much as the operator $\hat{x}$ in ordinary Quantum Mechanics. It brings finite norm states, such as $|0\rangle$ or $|g\rangle$, to infinite norm states. The physical counterpart of this mathematical issue is that it is required an infinite amount of energy to measure a particle precisely in the position x. This behaviour is thus healed by changing the observable associated with the field, which can be done

by using a smeared field operator. From the detector point of view, such divergence is instead healed with the introduction of a detector wavefunction that is able to regulate the high momentum infinities.

4.1 Bohr-Rosenfeld approach

At this level one could, in principle, verify that coherent states (2.25) saturate the uncertainty relation between $\hat{\Phi}(x)$ and its conjugate field $\hat{\Pi}(x)$, making their variances constant in time. However, evaluating the mean value of $\hat{\Phi}^2(x)$ shows that

$$\langle \hat{\Phi}^2 \rangle_g = \infty. \tag{4.1}$$

In fact, the bracket can be written as

$$\begin{aligned} \langle g|\hat{\Phi}^2(x)|g\rangle = &\int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{\hbar}{2\sqrt{\omega_p\omega_q}} \\ &\cdot [\langle g|\hat{a}_p\hat{a}_q|g\rangle\, e^{-i\omega_p t+i\vec{p}\cdot\vec{x}}e^{-i\omega_q t+i\vec{q}\cdot\vec{x}} + \langle g|\hat{a}_p^\dagger\hat{a}_q^\dagger|g\rangle\, e^{+i\omega_p t-i\vec{p}\cdot\vec{x}}e^{i\omega_q t-i\vec{q}\cdot\vec{x}} \\ &+ \langle g|\hat{a}_p^\dagger\hat{a}_q|g\rangle\, e^{i\omega_p t-i\vec{p}\cdot\vec{x}}e^{-i\omega_q t+i\vec{q}\cdot\vec{x}} + \langle g|\hat{a}_p\hat{a}_q^\dagger|g\rangle\, e^{-i\omega_p t+i\vec{p}\cdot\vec{x}}e^{i\omega_q t-i\vec{q}\cdot\vec{x}}]. \end{aligned} \tag{4.2}$$

In the last term, the commutation relation between the ladder operators can be used, leading to

$$\begin{aligned} \langle g|\hat{\Phi}^2(x)|g\rangle = &\int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{\hbar}{2\sqrt{\omega_p\omega_q}} \\ &\cdot [\langle g|\hat{a}_p\hat{a}_q|g\rangle\, e^{-i\omega_p t+i\vec{p}\cdot\vec{x}}e^{-i\omega_q t+i\vec{q}\cdot\vec{x}} + \langle g|\hat{a}_p^\dagger\hat{a}_q^\dagger|g\rangle\, e^{+i\omega_p t-i\vec{p}\cdot\vec{x}}e^{i\omega_q t-i\vec{q}\cdot\vec{x}} \\ &+ \langle g|\hat{a}_p^\dagger\hat{a}_q|g\rangle\, e^{i\omega_p t-i\vec{p}\cdot\vec{x}}e^{-i\omega_q t+i\vec{q}\cdot\vec{x}} + \langle g|\hat{a}_q^\dagger\hat{a}_p|g\rangle\, e^{-i\omega_p t+i\vec{p}\cdot\vec{x}}e^{i\omega_q t-i\vec{q}\cdot\vec{x}}] \\ &+ \int \frac{d^3p}{(2\pi)^3}\frac{\hbar}{2\omega_p}. \end{aligned} \tag{4.3}$$

The first double integral can be recognized as $\langle g|\hat{\Phi}(x)|g\rangle^2$, since the eigenvalues of the coherent state kill the temporal part of the modes, and the spatial part is used to reconstruct the mean value. The second integral is a purely vacuum contribution, which diverges in the ultraviolet region of momentum, meaning that

$$(\Delta\hat{\Phi})_g^2 = \int \frac{d^3p}{(2\pi)^3}\frac{\hbar}{2\omega_p} = \frac{\hbar}{4\pi^2}\int_0^\infty dpp. \tag{4.4}$$

In 1933, Bohr and Rosenfeld [9] treated the problem in a pioneeristic work, while they were looking for an alternative way to find the equal time commutation relations between fields at different points in space. In this paper, the Heisenberg relations were found by means of a “smeared operator”

$$\hat{\bar{\Phi}} = \int d^3x f(x)\hat{\Phi}(x), \tag{4.5}$$

where $f(x)$ is a function associated to the device which takes the measurement of the field $\hat{\Phi}$, e.g. a charged test body for an electric field. In the original work, the smearing test function was taken just as a normalization factor for the average of the field in a given volume, and such a volume was interpreted to be the one covered by the test body detector. The Bohr-Rosenfeld smearing approach was the first one which made use of a detector in a Quantum Field Theory computations: the detector is described as a device capable of feeling properties and effects of the field, and enters in the theory to fix the measurement process.
Bohr's detector has the feature of being a macroscopic and classical object, i.e. the probe of the electric field should be a test charge with a charge much bigger then the electron charge. This allows for measurements of macroscopic field properties over a given spacetime region, by minimizing the expected response of the body to the field and the influence of the body to the source of the field itself. This point of view, enforced by (4.5), gives the same definition of smeared operators that are present in Haag and Wightman's Axiomatic Quantum Field Theory [23]. But it should be stressed that $f(x)$ plays a different role, since it is not the wavefunction associated to the particle created by the field operator; this point of view looks suspicious when dealing with states such as $|g\rangle$ that should reproduce a configuration with a huge (when not infinite!) amount of particles, as could be a Coulombian or Newtonian static configuration.
Then, the source of the problem could be thought not as coming from the ill-defined nature of the operator $\hat{\Phi}(x)$, but instead as coming from the introduction of a device which reads all the possible frequencies of a field, which are infinite.
So, let us insert the detector profile

$$\hat{\bar{\Phi}} = \int \frac{d^3x}{(2\pi\sigma^2)^{\frac{3}{2}}} e^{-\frac{r^2}{2\sigma^2}} \hat{\Phi}(x), \tag{4.6}$$

with σ the variance of the weighting function. The smeared uncertainty now reads

$$(\Delta\hat{\bar{\Phi}})^2_g = \int \frac{d^3k d^3p}{(2\pi)^6} \int \frac{d^3x d^3y}{(2\pi\sigma^2)^3} \frac{\hbar}{2\sqrt{\omega_k\omega_p}} e^{-\frac{r^2}{2\sigma^2}} e^{-\frac{r'^2}{2\sigma^2}} e^{-i\omega_k t + i\vec{k}\cdot\vec{x} + i\omega_p t - i\vec{p}\cdot\vec{y}} \langle g|[\hat{a}_k, \hat{a}_p^\dagger]|g\rangle . \tag{4.7}$$

By employing the commutation relation of the ladder operators, $[\hat{a}_k, \hat{a}_p^\dagger] = (2\pi)^3\delta_{kp}$, the above expression becomes

$$(\Delta\hat{\bar{\Phi}})^2_g = \frac{\hbar}{(2\pi)^3} \int \frac{d^3k}{2\omega_k} \left(\int \frac{d^3x}{(2\pi\sigma^2)^{\frac{3}{2}}} e^{-\frac{r^2}{2\sigma^2} + i\vec{p}\cdot\vec{x}} \right) \left(\int \frac{d^3y}{(2\pi\sigma^2)^{\frac{3}{2}}} e^{-\frac{r'^2}{2\sigma^2} - i\vec{p}\cdot\vec{y}} \right). \tag{4.8}$$

Now, with the help of the Gaussian integral

$$\int d^3x e^{-\frac{r^2}{2\sigma^2}\pm i\vec{p}\cdot\vec{x}} = (2\pi)^{\frac{3}{2}}\sigma^3 e^{-\frac{p^2\sigma^2}{2}}, \tag{4.9}$$

the variance is

$$(\Delta\hat{\hat{\Phi}})^2_g = \frac{\hbar}{2(2\pi)^2}\int_0^\infty \frac{dp2p^2}{\omega_p}e^{-\sigma^2p^2}, \tag{4.10}$$

and by making use of

$$2\int_0^\infty dzze^{-z^2} = 1, \tag{4.11}$$

it becomes

$$(\Delta\hat{\hat{\Phi}})^2_g = \frac{\hbar}{8\pi^2\sigma^2}. \tag{4.12}$$

The final result can be approximately written as

$$(\Delta\hat{\hat{\Phi}})_g \sim \frac{\sqrt{\hbar}}{\sigma}. \tag{4.13}$$

In the same way, the computation for the conjugate field variance $\Delta\hat{\hat{\Pi}}$ can be done straightforwardly and gives

$$(\Delta\hat{\hat{\Pi}})_g \sim \frac{\sqrt{\hbar}}{\sigma^2}, \tag{4.14}$$

while the smeared commutator of $\hat{\hat{\Phi}}$ and $\hat{\hat{\Pi}}$ is

$$\langle[\hat{\hat{\Phi}},\hat{\hat{\Pi}}]\rangle_g \sim \frac{\hbar}{\sigma^3}. \tag{4.15}$$

It could be verified that the above uncertainties saturate the Heisenberg Principle

$$(\Delta\hat{\hat{\Phi}})_g(\Delta\hat{\hat{\Pi}})_g = \frac{|\langle[\hat{\hat{\Phi}},\hat{\hat{\Pi}}]\rangle_g|}{2}. \tag{4.16}$$

This saturation comes from having chosen the smearing profile as a Gaussian function, which notoriously has this property. However, it could be checked by means of the Ehrenfest Theorem that the uncertainties are also constant in time.

4.2 The detector quantum state

The Bohr-Rosenfeld argument lies on considerations which could be modified in order to get a conceptually consistent picture for the measurement of a field. First of all, it is not clear how and why the detector profile should enter (4.5) in this way. If $f(x)$ is associated

to the measuring device, it is reasonable to think that it should modify the results from a measuring process of the observable $\hat{\Phi}(x)$ itself, instead of entering in the definition of a different operator $\hat{\Phi}(f)$. Secondarily, since the integration with $f(x)$ smears the point x over which the field is measured, the interpretation of the function $f(x)$ is of a profile function for the detector; that is, $f(x)$ describes the spatial extension of the detector, according to the original idea of having a macroscopic object probing the field. Thinking about the detector this way clashes with the idea that the world is basically quantum, as the detector would be a purely classical object coupled to a purely quantum one.
In order to fix these key aspects, it is more reasonable to think the detector as a quantum object itself, and reinterpret the function $f(x)$ as a wavefunction describing it. Such quantum state will enter in the measurement process of the state of the field, and its effects on the field properties should be explicit in the results of the uncertainty calculation by means of the parameters defining the detector itself. This point of view is supported by studies on the introduction of a detector in the very framework of Quantum Field Theory, in order to fix inconsistencies coming from mixing Relativistic Quantum Mechanics and Information Theory [10, 24]. This conception can also be found at the beginnings of Quantum Mechanics, for example in the so called "Heisenberg's microscope". In this thought experiment, a cone of light coming from a microscope strikes an electron and registers the position of the electron up to an uncertainty given by its optical resolution. This is reflected in the uncertainty over the recoiling momentum of the particle, and the product of the two uncertainties gives back the Heisenberg relation. So, the quantum uncertainty was supposed to come from the very act of measuring, and thus from a device, the microscope, which is a detector; the uncertainty is not intrinsic of the particle, and it is necessary to add an extra ingredient to get it, a detector. In this new perspective, what will be found should be matched with the Bohr-Rosenfeld results, that are kept as guidelines. In what follows, such effects of the detectors will enter in an heuristic and simple way, because the goal is to observe the qualitative dependence of the uncertainty and the ratio $\frac{\Delta\hat{\Phi}(x)}{\langle\hat{\Phi}(x)\rangle}$ from the parameters of the detector.
In a first attempt, imagine the detector to be described by a properly normalized Gaussian wave-packet in position space

$$f(x) = \frac{1}{(2\pi\sigma^2)^{\frac{3}{4}}} e^{-\frac{r^2}{4\sigma^2}}, \tag{4.17}$$

where σ^2, the width of the packet, can be seen as the counterpart of the classical spatial extension of the detector. With a little notation abuse, the detector wavefunction can be inserted in the quantum state as $|\text{State}\rangle = |d\rangle \otimes |g\rangle$. The abuse lies in the fact that the detector and the field are not really decoupled, and the two states are not independent, meaning that $|\text{State}\rangle$ cannot be factorized in a direct product. This is because to register the presence of the field, the Hamiltonian of the detector should contain an interacting term. This is what happens, for example, in the Unruh-DeWitt detector model that will be extensively treated later. However, since this approach is rather heuristic, keep the notation abuse and assume that the action of the field over the detector quantum state in position space is

$$\hat{\Phi}(x)f(x_d) = f(x_d)\hat{\Phi}(x_d), \tag{4.18}$$

that is, imagine that the field is measured at the point where the detector packet is centered. Using the completeness relation for the eigenstates of the position operator,

$$1 = \int d^3x_d \, |x_d\rangle \, \langle x_d| \, , \tag{4.19}$$

the new integral should be given by

$$\begin{aligned}
(\Delta\hat{\Phi})^2_{\text{State}} &= \langle \text{State}\,|\hat{\Phi}^2(x)|\,\text{State}\rangle - \langle \text{State}\,|\hat{\Phi}(x)|\,\text{State}\rangle^2 \\
&= \langle g| \otimes \langle d| \, \hat{\Phi}(x)\hat{\Phi}(x) \, |d\rangle \otimes |g\rangle - \Big(\langle g| \otimes \langle d|\hat{\Phi}(x)|d\rangle \otimes |g\rangle\Big)^2 \\
&= \int d^3x_d| \, \langle d|x_d\rangle \, |^2 \, \langle g|\hat{\Phi}^2(x_d)|g\rangle - \left(\int d^3x_d| \, \langle d|x_d\rangle \, |^2 \, \langle g|\hat{\Phi}(x_d)|g\rangle\right)^2 \\
&= \int d^3x_d|f(x_d)|^2 \, \langle g|\hat{\Phi}^2(x_d)|g\rangle - \left(\int d^3x_d|f(x_d)|^2 \, \langle g|\hat{\Phi}(x_d)|g\rangle\right)^2 .
\end{aligned} \tag{4.20}$$

Dismissing the subscript $-d$ for the position variable, the first term can be written again as

$$\begin{aligned}
\int d^3x|f(x)|^2 \, \langle g|\hat{\Phi}^2(x)|g\rangle = &\left(\int d^3x|f(x)|^2 \, \langle g|\hat{\Phi}(x)|g\rangle\right)^2 \\
&+ \int d^3x|f(x)|^2 \int \frac{d^3p}{(2\pi)^3}\frac{\hbar}{2\omega_p},
\end{aligned} \tag{4.21}$$

and thus

$$(\Delta\hat{\Phi})^2_{\text{State}} = \int d^3x|f(x)|^2 \int \frac{d^3p}{(2\pi)^3}\frac{\hbar}{2\omega_p}. \tag{4.22}$$

The first thing to notice is that, again, the uncertainty goes to infinity in the region of large momenta. This is because the Gaussian wavefunction did not enter in a convolution with the field, but only as a detector state. In the smeared field method, there was a Gaussian profile for each field operator because the operator itself was modified and regulated in the UV region. The detector state method instead fails at the level of (4.18), because the effect of the detector quantum state on the field operator locks the field to the detector position, but does not tell anything about the possible detectable momenta. In other words, (4.18) implicitly assumes that the detector is capable of measuring every possible frequency, which is not physically reasonable given its finite spatial extension.

Therefore, the new heuristic action should take into account the finite window of detectable momenta by regulating the field operator itself, once it meets the detector state. A new action could be assumed as

$$\hat{\Phi}(x)f(x_d) = f(x_d)\hat{\Phi}_R(x_d), \tag{4.23}$$

where

$$\hat{\Phi}_R(x) = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{\hbar}{2\omega_p}}R(p, p_{min}, p_{max})(\hat{a}_p e^{-i\omega_p t + i\vec{p}\cdot\vec{x}} + \hat{a}_p^\dagger e^{i\omega_p t - i\vec{p}\cdot\vec{x}}). \tag{4.24}$$

$R(p, p_{min}, p_{max})$ is the regulating function that cuts the high momentum region and can heal the UV divergence. The choice of such function is detector dependent, and thus rather arbitrary in this computation. To make things easier, for the moment just choose a sharp window function of the kind

$$R(p, p_{min}, p_{max}) = \theta(p_{max} - p)\theta(p - p_{min}). \tag{4.25}$$

The p_{min} cut-off can be safely sent to zero since the field uncertainty goes to infinity for high momenta. This means that the variance now reads

$$\begin{aligned}(\Delta\hat{\Phi})^2_{\text{State}} &= \int d^3x|f(x)|^2 \int \frac{dpp^2}{2\pi^2}\frac{\hbar}{2\omega_p}\theta(p_{max} - p) \\ &= \int d^3x|f(x)|^2 \int_0^{p_{max}} \frac{dp}{4\pi^2}\hbar p \\ &= \frac{\hbar}{8\pi^2}p^2_{max},\end{aligned} \tag{4.26}$$

where in the last line the normalization of the Gaussian packet was used. It could be asked then what is p_{max}; the limited range of measurable momenta is related to the limited extension of the detecting device. This means that, given the duality between space and momentum variables, the cut-off can be written as

$$p_{max} \sim \frac{1}{\sigma}, \tag{4.27}$$

and the variance finally reads

$$(\Delta\hat{\Phi})^2_{\text{State}} = \frac{\hbar}{8\pi^2\sigma^2}, \tag{4.28}$$

and

$$(\Delta\hat{\Phi})_{\text{State}} \sim \frac{\sqrt{\hbar}}{\sigma}, \tag{4.29}$$

which is the same result of (4.13). The same calculation can be performed for the conjugate field $\hat{\Pi}(x)$, but now the vacuum contribution reads

$$\begin{aligned}(\Delta\hat{\Pi})^2_{\text{State}} &= \int d^3x|f(x)|^2 \int \frac{dpp^2}{2\pi^2}\frac{\hbar\omega_p}{2}\theta(p_{max}-p) \\ &= \int d^3x|f(x)|^2 \int_0^{p_{max}} \frac{dp}{4\pi^2}\hbar p^3 \\ &= \frac{\hbar}{16\pi^2}p^4_{max},\end{aligned} \tag{4.30}$$

leading to

$$(\Delta\hat{\Pi})_{\text{State}} \sim \frac{\sqrt{\hbar}}{\sigma^2}, \tag{4.31}$$

finding again (4.14). It is straightforward to check that now the Uncertainty relation is not saturated, unlike in (4.16). This is because the system is neither in a coherent state (since the state $|d\rangle$, and thus $|\text{State}\rangle$, does not satisfy the dynamical and kinematical requirements discussed in chapter 2) nor in a purely Gaussian state anymore. To put it in another way, the cut momenta for the field expansion makes the coherent state $|g\rangle$ not coherent for the operator $\hat{\Phi}_R(x)$.

4.2.1 Different choice of the regulating function

The precise form of the function $R(p, p_{max}, p_{min})$ depends from the detector, but different choices could be tested in order to look at the changes imposed by another regulating function. To see this, suppose that

$$R(p, p_{max}) = e^{-\frac{p^2}{p^2_{max}}}, \tag{4.32}$$

that now gives

$$\begin{aligned}(\Delta\hat{\Phi})^2_{\text{State}} &= \int_0^\infty \frac{dpp^2}{2\pi^2}\frac{\hbar}{2\omega_p}e^{-\frac{2p^2}{p^2_{max}}} \\ &= \frac{\hbar}{16\pi^2\sigma^2}\end{aligned} \tag{4.33}$$

and

$$\begin{aligned}(\Delta\hat{\Pi})^2_{\text{State}} &= \int_0^\infty \frac{dpp^2}{2\pi^2}\frac{\hbar\omega_p}{2}e^{-\frac{2p^2}{p^2_{max}}} \\ &= \frac{\hbar}{32\pi^2\sigma^4}.\end{aligned} \tag{4.34}$$

This brings to

$$(\Delta\hat{\Phi})_{\text{State}} \sim \frac{\sqrt{\hbar}}{\sigma} \tag{4.35}$$

for the field $\hat{\Phi}$, while for the conjugate field $\hat{\Pi}$

$$(\Delta\hat{\Pi})_{\text{State}} \sim \frac{\sqrt{\hbar}}{\sigma^2}. \tag{4.36}$$

This shows how the choice of the function $R(p, p_{max})$ does not affect the σ scaling of the variances.

4.3 Mean value of the field

In the previous setup, it is clear that the Gaussian wavefunction of the detector plays no significant role in the regulation of the vacuum divergences, that are healed by the regulator $R(p, p_{max}, p_{min})$. In fact, in (4.26) the position space wavefunction of the detector is completely factorized. However, the uncertainties found mean nothing if they are not compared with the quantity which mostly mimics classical physics, that is, the expectation value of the fields over a given state. If the configuration space wavefunction is not inserted, the ratio $\frac{\Delta\hat{\Phi}(x)}{\langle\hat{\Phi}(x)\rangle}$ would not be correctly adimensional, with extra scaling of σ. Thus, it is necessary to keep $f(x_d)$ not because it is necessary to heal the vacuum divergence, but because an incorrect dimensional analysis suggests that both the position and momentum sides of the detector quantum state should enter the computation. In what follows, the system is again composed by the detector and the gravitational field, with the former in a Gaussian and momentum regulating state and the latter in its coherent state.
The following bracket can be evaluated

$$\langle\hat{\Phi}\rangle = \int d^3x |f(x)|^2 \langle g|\hat{\Phi}_R(x)|g\rangle\,. \tag{4.37}$$

This expression has a direct physical interpretation: it is the quantum counterpart of the mean gravitational potential energy of the detector, since $|f(x)|^2$ has the role of a probability density in the quantum theory, and takes the place of the classical matter density. The above integral will now depend from the source of the gravitational field, which features are encoded inside the coherent state $|g\rangle$. The latter is required to reproduce, up to quantum corrections, the classical potential solution of the Poisson equation. Then the mean value reads in general

$$\sqrt{G}\,\langle\hat{\Phi}\rangle = \frac{1}{(2\pi\sigma^2)^{\frac{3}{2}}} \int dx^3 e^{-\frac{r^2}{2\sigma^2}} \int_0^\infty dk \frac{k^2}{2\pi^2} \sqrt{\frac{\hbar G}{2\omega_k}} R(k, k_{min}, k_{max}) g_k \frac{\sin(kr)}{kr}, \tag{4.38}$$

with $r^2 = |\vec{x}|^2$. For the moment, ignore the momentum function inside the g_k definition coming from the quantum nature of the state $|g\rangle$, and suppose the eigenvalues of the coherent state to be their classical counterpart. This is possible because the oscillations around the classical expression of $V_q(r)$, coming from the quantum part of g_k, can be made arbitrarily small [8]. Thus, the mean value reads

$$\begin{aligned}
\sqrt{G}\,\langle \mathrm{State}\,|\hat{\Phi}(x)|\,\mathrm{State}\rangle &= \int d^3x|f(x)|^2\,\langle g|\sqrt{G}\hat{\Phi}_R(x)|g\rangle \\
&= -4\pi\int drr^2|f(r)|^2\frac{GM}{r} \\
&= -4\pi\int_0^\infty drr^2\frac{e^{-\frac{r^2}{2\sigma^2}}}{(2\pi\sigma^2)^{\frac{3}{2}}}\frac{GM}{r} \\
&= -\frac{4\pi GM}{(2\pi\sigma^2)^{\frac{3}{2}}}\int_0^\infty drre^{-\frac{r^2}{2\sigma^2}} \\
&= -\frac{4\pi GM\sigma^2}{(2\pi\sigma^2)^{\frac{3}{2}}} = \frac{2GM}{\sqrt{2\pi}\sigma}.
\end{aligned} \tag{4.39}$$

In the second line, the regulating function $R(p, p_{max}, p_{min})$ was neglected. Recalling that $R = \theta(p_{max} - p)$, it could be said that the detector is placed in a region where $p_{UV} < p_{max}$ and thus, neglecting p_{UV} allows also for p_{max} to be ignored. Squaring this expression gives

$$\langle \mathrm{State}\,|\hat{\Phi}(x)|\,\mathrm{State}\rangle^2 = \frac{2G^2M^2}{\pi\sigma^2}. \tag{4.40}$$

To take the ratio between the uncertainty and the mean value, it should be recalled that for the latter the field was taken to be canonically normalized. Thus, $\Delta\hat{\Phi}^2_{\mathrm{State}} \to \Delta\hat{\Phi}^2_{\mathrm{State}}G$ and

$$\frac{(\Delta\hat{\Phi})^2_{\mathrm{State}}}{\langle\hat{\Phi}\rangle^2_{\mathrm{State}}} \sim \frac{\hbar}{GM^2} = \frac{\hbar}{Gm_p^2N_g}, \tag{4.41}$$

where the relation $N_g = M^2/m_p^2$ was used.
The above result correctly goes to zero in the so called "classical limits", i.e. for $\hbar \to 0$ and for $N \to \infty$. The former limit stems for the absence of any vacuum (and thus quantum) fluctuation, while the latter one points out the highly occupancy of the state. Both the limits make the uncertainty ratio a vanishing quantity. The dependence on the detector width σ disappears, and this could not be achieved if the regulation was not carried out both in space and momentum. In other words, both the regulating functions $R(p, p_{max}, p_{min})$ and $f(x_d)$ were necessary in order to get the right adimensional expression for $\frac{\Delta\hat{\Phi}(x)}{\langle\hat{\Phi}(x)\rangle}$. The fact that the above ratio is σ independent happens specifically for point-like sources of the gravitational field, as it will be clear in the next section.

4.4 Discussing the sources

The above calculations were performed in the absence of cut-offs, i.e. it was neglected the quantum sign coming from the coherent state. This fact imposed also to relax the cut-off condition coming from the finite momentum window of the detector. Therefore, it is now worth to test the form of the source giving rise to the coefficients g_k. The sources tested for the mean value are a point-like source and a Gaussian source, as presented in [6]. The homogeneous ball source is not treated because the coherent state reproduces the classical solution only outside the source, and the gravitational field outside a ball of matter is the same of the field outside a point-like source, as stated by Gauss theorem.

4.4.1 Pointlike source

The coefficients of the coherent state are already known

$$g_k = -\frac{4\pi M}{m_p\sqrt{2k^3}}\theta(k - k_{IR})\theta(k_{UV} - k), \tag{4.42}$$

and $k_{IR} \to 0$ could be safely taken. The quantum nature of g_k enters in the mean value of the regulated field, so let us evaluate

$$\sqrt{G}\,\langle \text{State}\,|\hat{\Phi}|\,\text{State}\rangle = -\frac{2GM}{\pi}\int d^3x|f(x)|^2\frac{1}{r}\int_0^{r/R_s} dz\frac{\sin z}{z}\theta(z/r - p_{max}). \tag{4.43}$$

Now, the second θ-function can at most change the cut-off in the z integral. If $p_{max} > p_{UV}$, then the quantum nature of the coherent state can be measured. On the other hand, if $p_{max} < p_{UV}$ this is not the case. To see how a cut-off modifies the result previously obtained, just call the momentum cut-off (whether it is p_{max} or p_{UV}) $\bar{p}$:

$$\sqrt{G}\,\langle \text{State}\,|\hat{\Phi}|\,\text{State}\rangle = -\frac{2GM}{\pi}\int d^3x|f(x)|^2\frac{1}{r}\int_0^{\bar{p}r} dz\frac{\sin z}{z}. \tag{4.44}$$

By Taylor expanding the sine, the bracket gives

$$
\begin{aligned}
\sqrt{G}\,\langle \mathrm{State}\,|\hat{\Phi}|\,\mathrm{State}\rangle &= -\frac{8GM}{(2\pi\sigma^2)^{\frac{3}{2}}}\int_0^\infty drre^{-\frac{r^2}{2\sigma^2}}\int_0^{\bar{p}r} dz\left(1-\frac{z^2}{3!}+\dots\right)\\
&= -\frac{8GM}{(2\pi\sigma^2)^{\frac{3}{2}}}\int_0^\infty drre^{-\frac{r^2}{2\sigma^2}}\left[\bar{p}r-\frac{(\bar{p}r)^3}{3\cdot 3!}+\dots\right]\\
&= -\frac{8GM}{(\pi\sigma^2)^{\frac{3}{2}}}\left[\bar{p}\sigma^3\int_0^\infty dyy^2e^{-y^2}-2\frac{\bar{p}^3\sigma^5}{3\cdot 3!}\int_0^\infty dyy^4e^{-y^2}+\dots\right]\\
&= -\frac{8GM}{(\pi\sigma^2)^{\frac{3}{2}}}\left[\bar{p}\sigma^3\frac{\sqrt{\pi}}{4}-2\frac{\bar{p}^3\sigma^5}{3\cdot 3!}\frac{3\sqrt{\pi}}{8}+\dots\right]\\
&= -\frac{2GM}{\pi\sigma}\left[\bar{p}\sigma-\frac{\bar{p}^3\sigma^3}{3!}+\dots\right]\\
&= -\frac{2GM}{\pi}\frac{\sin(\bar{p}\sigma)}{\sigma}.
\end{aligned}
\tag{4.45}
$$

Therefore, the addition of a cut-off just brings an extra oscillating factor $\sin(\bar{p}\sigma)$ with an adimensional argument. The squared ratio between the uncertainty and the mean value now reads, with the help of (3.42),

$$
\frac{(\Delta\hat{\Phi})^2_{\mathrm{State}}}{\langle\hat{\Phi}\rangle^2_{\mathrm{State}}} \sim \frac{\hbar}{Gm_p^2N_g}\frac{\log\left(\frac{p_{UV}}{p_{IR}}\right)}{\sin^2(\bar{p}\sigma)}. \tag{4.46}
$$

It could be noticed that there is a competition between the coherent state parameter and the detector one in the term $\sin^2(\bar{p}\sigma)$. This term depends both on the source and the detector functions, and gives small deviations from the main term $\hbar/Gm_p^2N_g$. It could be checked that (4.46) correctly reproduces the results already obtained by Bohr and Rosenfeld up to the deviations coming from the cut-off of the detector, which are the true novelty of the detector state approach.

4.4.2 Gaussian source

Let us look at the case of a Gaussian source

$$
\rho(r) = \frac{M_0}{(2\pi\delta^2)^{\frac{3}{2}}}e^{-\frac{r^2}{2\delta^2}}, \tag{4.47}
$$

with δ defining the width of the source, and

$$
M_0 = 4\pi\int_0^\infty drr^2\rho(r) \tag{4.48}
$$

the total mass of the source. It is immediate to find the Fourier transform for the matter distribution $\rho(r)$:

$$\tilde{\rho}(p) = M_0 e^{-\frac{\delta^2 p^2}{4}}, \tag{4.49}$$

from which the coefficients building the state $|g\rangle$ are given by the general formula

$$g_p = -\frac{4\pi\tilde{\rho}(k)}{\sqrt{2k^3}m_p}, \tag{4.50}$$

that for the point-like source brings to (3.37), while for the Gaussian one gives

$$g_p = -\frac{4\pi M_0}{\sqrt{2p^3}m_p} e^{-\frac{\delta^2 p^2}{4}}. \tag{4.51}$$

This brings to [6]

$$\begin{aligned}\sqrt{G}\,\langle g|\hat{\Phi}(r)|g\rangle &= -\frac{2GM_0}{\pi}\int_0^\infty dp j_0(pr) e^{-\frac{\delta^2 p^2}{4}} \\ &= -\frac{GM_0}{r}\,\mathrm{Erf}\left(\frac{r}{\delta}\right),\end{aligned} \tag{4.52}$$

with

$$\mathrm{Erf}\left(\frac{r}{\sigma}\right) = \frac{2}{\sqrt{\pi}}\int_0^{\frac{r}{\sigma}} dt e^{-\frac{t^2}{2}}. \tag{4.53}$$

Now, for an easy evaluation of the mean value over $|\mathrm{State}\rangle$, employ the Gaussian regulator

$$R(p, p_{max}) = e^{-\frac{p^2}{4p_{max}^2}} = e^{-\frac{\sigma^2 p^2}{4}}, \tag{4.54}$$

where the numerical factor at the exponent was conveniently chosen to match the one in the Gaussian source function. Then

$$\begin{aligned}
\sqrt{G}\,\langle \text{State}\,|\hat{\Phi}|\,\text{State}\rangle &= -GM_0 \int d^3x |f(x)|^2 \frac{1}{r}\,\text{Erf}\left(r/\sqrt{\delta^2+\sigma^2}\right) \\
&= -\frac{4\pi G M_0}{(2\pi\sigma^2)^{\frac{3}{2}}} \int_0^\infty dr r e^{-\frac{r^2}{2\sigma^2}}\,\text{Erf}\left(r/\sqrt{\delta^2+\sigma^2}\right) \\
&= -\frac{4\pi G M_0}{(2\pi\sigma^2)^{\frac{3}{2}}} \int_0^\infty dr r e^{-\frac{r^2}{2\sigma^2}} \frac{2}{\sqrt{\pi}} \int_0^{\frac{r}{\sqrt{\delta^2+\sigma^2}}} dz \left[1 - \frac{z^2}{2} + \dots\right] \\
&= -\frac{8\sqrt{\pi} G M_0}{(2\pi\sigma^2)^{\frac{3}{2}}} \int_0^\infty dr r e^{-\frac{r^2}{2\sigma^2}} \left[\frac{r}{\sqrt{\delta^2+\sigma^2}} - \frac{r^3}{6\left(\sqrt{\delta^2+\sigma^2}\right)^3} + \dots\right] \\
&= -\frac{8\sqrt{\pi} G M_0}{(2\pi\sigma^2)^{\frac{3}{2}}\sqrt{\delta^2+\sigma^2}} \left[\int_0^\infty dr \left(r^2 e^{-\frac{r^2}{2\sigma^2}} - \frac{r^4}{6(\delta^2+\sigma^2)} e^{-\frac{r^2}{2\sigma^2}}\right) + \dots\right] \\
&= -\frac{8\sqrt{\pi} G M_0}{(2\pi\sigma^2)^{\frac{3}{2}}\sqrt{\delta^2+\sigma^2}} \left[(\sqrt{2}\sigma)^3 \frac{\sqrt{\pi}}{4} - \frac{3(\sqrt{2}\sigma)^5}{6\cdot 8(\delta^2+\sigma^2)}\right] \\
&= -\frac{2GM_0}{\sqrt{\pi}\sqrt{\delta^2+\sigma^2}} \left[1 - \frac{\sigma^2}{2(\delta^2+\sigma^2)} + \dots\right] \\
&= -\frac{2GM_0}{\sqrt{\pi}\sigma\sqrt{1+\frac{\delta^2}{\sigma^2}}} e^{-\frac{\sigma^2}{2(\delta^2+\sigma^2)}}.
\end{aligned} \tag{4.55}$$

To express the square of the ratio between the uncertainty and the mean value as a function of the occupation number of gravitons, it is necessary to express N_g as a function of the profile of the source.
To this purpose, the Gaussian source alone is not able to regularize the IR divergence of the occupation number. In complete analogy with the point-like source, let us introduce also an IR cutoff that can be safely sent to zero in the evaluation of $\langle\hat{\Phi}\rangle_{\text{State}}$:

$$\begin{aligned}
N_g &= 4\frac{M_0^2}{m_p^2} \int_{p_{IR}}^\infty dk \frac{e^{-\frac{\delta^2 k^2}{2}}}{k} \\
&= 2\frac{M_0^2}{m_p^2} \int_{\frac{\delta p_{IR}}{2}}^\infty dy \frac{e^{-y}}{y} \\
&= 2\frac{M_0^2}{m_p^2} \Gamma\left(0, \frac{\delta p_{IR}}{2}\right),
\end{aligned} \tag{4.56}$$

where the lower incomplete Gamma function function is defined as

$$\Gamma(0,x) = \int_x^\infty dy \frac{e^{-y}}{y}. \tag{4.57}$$

This leads to

$$\frac{(\Delta\hat{\Phi})^2_{\text{State}}}{\langle\hat{\Phi}\rangle^2_{\text{State}}} \sim \frac{\hbar\Gamma\left(0, \frac{\delta p_{IR}}{2}\right)}{G m_p^2 N_g}\left(1+\frac{\delta^2}{\sigma^2}\right) e^{\frac{\sigma^2}{\sigma^2+\delta^2}}. \tag{4.58}$$

It could be checked that even this result agrees with the Bohr-Rosenfeld approach, but now $\delta \to \sqrt{\delta^2+\sigma^2}$. The difference with respect to the point-like source lies in the presence of M_0 (instead of M). The competition between the source size and the detector sensitivity is made explicit by the ratio of the two quantities δ and σ, whereas in the point-like case it was shown by the cut-off $\bar{p}$. Furthermore, the ratio $\frac{\Delta\hat{\Phi}(x)}{\langle\hat{\Phi}(x)\rangle}$ is again correctly adimensional, but now for $\sigma \to 0$ it blows up. This is because the ratio between the length scales is $\frac{\delta^2+\sigma^2}{\sigma^2}$, meaning that the point-like case was rather special.
The above analysis shows that treating the detector from a quantum perspective is consistent with the results discussed by Bohr and Rosenfeld only if the detector state regulates the momentum behaviour of the field and enters in the scalar product for the mean value. This is explicit thanks to the presence of $R(p, p_{max}, p_{min})$ in the field expansion that cuts the vacuum modes, and also thanks to $f(x)$ in the mean value that evaluates the mean potential energy of the detector. Moreover, the ratio of the uncertainty over the mean value leads to a scaling with N_g consistent with the corpuscular picture described by a coherent state. Quantum deviations from the Bohr-Rosenfeld results are explicit with this heuristic method, and appear in terms where the detector and the source parameters are both present, specifically $\sin^{-2}(\bar{p}\sigma)$ in (4.46), and $(\delta^2+\sigma^2)\exp\left(\frac{\sigma^2}{\delta^2+\sigma^2}\right)$ in (4.58). Pursuing this point of view brings the coherent state model to be tested with a "detector formalism"; specifically, coherent state emission will be treated with the so called Unruh-DeWitt detector model.

Chapter 5

Hawking and Unruh effect

The following chapter is dedicated to the classical treatment of the Hawking radiation and the Unruh effect, that is, the emission of particles in a Planckian spectrum due to the acceleration of the observer. Given the previous discussion about the introduction of detectors in Quantum Field Theory computations, this phenomenon will be described through the so called Unruh-DeWitt detector model.

In the Hawking radiation, a field which is quantized on a curved spacetime has its modes excited by the curvature, and produces an emission of particles. Such spontaneous process is triggered by the gravitational collapse and is sustained by the presence of an event horizon. Furthermore, particles are emitted following a Planckian spectrum, which naturally associates a temperature to black holes that is proportional to the inverse of its mass; in this sense, black holes are not totally black, but (almost) black bodies.

However, the Equivalence Principle links geodesic motion in curved spacetimes to the concept of inertiality already given in Special Relativity. In particular, an accelerating (and thus non inertial) observer on a rocket ship far from any source of gravity is expected to measure the same radiation spectrum detected by an observer stationing near a black hole. This is precisely what is meant by Unruh effect, and the analogy will be made more refined in the next sections. For the Unruh effect found in [12], the trigger lies in the acceleration of the observer which enters in place of the mass of the black hole in the definition of the temperature inside the Planckian spectrum. Even if the Unruh effect can be extracted with the same geometrical machineries of the Hawking effect, nearly after its first appearance, Unruh and DeWitt [12, 13] found an alternative way to extract the Planckian spectrum for uniformly accelerated observers in flat spacetime. The Unruh-DeWitt detector model couples a quantum field and an operator associated to a particle detector. The device follows a trajectory, and its motion determines if the quantum field appears as excited or not. The excitations are recorded by a jump of the energy eigenstates describing the internal structure of the detector. In particular, given the weakness of the linear interaction between the field and the detector, it is possible to perturbatively evaluate the transition amplitude for the emission of particles

from the initial vacuum state. The probability coming from this computation shows the characteristic thermal spectrum of the Unruh effect. The process is again triggered by the non inertial trajectory of the detector, but this time the motion is directly enforced inside the quantum field through the contact interaction term. The detector gets excited because there is a non trivial correlation between field excitations along its trajectory. This approach soon became crucial in understanding the relativistic quantum physics of non inertial observers, and the particle content of the vacuum they can measure [25].
Therefore, with the Unruh-DeWitt detector, the Unruh effect will be regarded as the result of vacuum state measurements for a field during the accelerated motion of the probe. The results for the vacuum will then be compared, in the next chapter, with the results for the coherent state, in order to understand how particle emission from the corpuscular model differs from the vacuum one, and how this setup can reproduce the Hawking radiation.

5.1 Particle creation in curved spacetime: Hawking effect

General Relativity treats any reference frame on equal foot. Inertial frames are linked to spacetime geodesics through the Equivalence Principle, and they are no more privileged frames. Lorentz invariance keeps holding only in local frames, built from an orthonormal tetrad that can be, at most, extended only through entire curves which are geodesics. Lorentz invariance is a building block of Quantum Field Theory [17], and the possibility of having this symmetry only at a local level questions concepts that can globally hold in General Relativity (such as the particle one). For example, Lorentz invariance constraints Lagrangians and thus the equations of motion for any theory. Taking for instance a free massless scalar field theory

$$\mathcal{L}_{free} = -\frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi, \tag{5.1}$$

a general solution

$$\Phi = \sum_k (a_k u_k + a_k^* u_k^*) \tag{5.2}$$

can be built by a Fourier superposition of plane waves

$$u_k \sim e^{-ikx} \qquad u_k^* \sim e^{ikx}, \tag{5.3}$$

which are an orthonormal and complete set of solutions of the Klein Gordon equation. The modes are properly separated into positive (u_k) and negative (u_k^*) frequency modes by the presence of the trivial time translation symmetry of Minkowski spacetime, that naturally introduces a Killing vector $\xi^\mu = (1,0,0,0)$ which in turn gives an operator $\xi^\mu \partial_\mu$ for which the positive (negative) frequency modes are eigenvectors with positive

(negative) eigenvalues ω_k.
Through a properly defined scalar product with the field, these modes give the Fourier coefficients that will become creation and annihilation operators

$$a_k = (\Phi, u_k), \tag{5.4}$$

and thus, once the theory is quantized, a vacuum state

$$\hat{a}_k |0\rangle = 0 \qquad \forall k \in \mathbb{N} \tag{5.5}$$

which has all the symmetries of the theory, together with the Fock space of states.
Now, the vacuum state of a field theory in flat spacetime is not unique, but any vacuum defined starting by a Lorentz frame can be linked to each other by a unitary transformation (it is said that two vacua are unitary equivalent), and correspondingly each mode and each creation and annihilation operator in two different frames can be linked with a given unitary transformation. For non inertial frames, such as the ones usually present in General Relativity, there are no more unitary links between different vacua, modes or operators and therefore they can be non equivalent in the sense explained above. In details, the transformation between states, modes and operators of two different frames is given by a set of coefficients defining the so called Bogoliubov transformations. Denoting with a prime the quantities in the second frame, the transformation rules can be summarized as follows [26]

$$\begin{cases} u'_k = \sum_i (\alpha_{ki} u_i + \beta_{ki} u_i^*) \\ \hat{a}'_k = \sum_i (\alpha_{ki}^* \hat{a}_i - \beta_{ki}^* \hat{a}_i^\dagger) \\ |0\rangle = \langle 0'|0\rangle \exp\left\{-\frac{1}{2}\sum_{i,j,k} \beta_{jk}^* \alpha_{ik}^{-1} \hat{a}_i'^\dagger \hat{a}_j'^\dagger\right\} |0'\rangle \,. \end{cases} \tag{5.6}$$

This implies that a choice for the vacuum should be made, but in general two vacua, each one defined by a different observer, will not have the same particle content: depending on the observer which sets up the reference frame and chooses the vacuum, the other vacuum will not be in general a zero particle state.
The questions are then how to choose a vacuum state, and how is this vacuum seen by other observers when a gravitational field is present. A simple answer to the first question can be naturally found for spacetimes which show Minkowskian regions, such as asymptotically flat spacetimes, where the distances between events in regions spatially far at early and late times are described by the Minkowski metric

$$g_{\mu\nu} \to \eta_{\mu\nu} \qquad t \to \pm\infty \cup r \to \infty. \tag{5.7}$$

For example, the solution to any field equation can be expanded in terms of Minkowski plane waves, solution to the Klein-Gordon equation in region $t \to -\infty \cup r \to \infty$, and the usual vacuum state can be introduced. Asking for the particle content of this state means taking the mean value of the number operator defined by the creation and annihilation operators

$$\langle 0|\hat{N}'_k|0\rangle = \sum_i |\beta_{ik}|^2, \tag{5.8}$$

implying that in any other spacetime region this vacuum will in general no more be an empty state, due to the non-unitary relation between the ladder operators.

If, to make it easier, the new region where the initial vacuum is questioned is again an asymptotic flat region, that is, the region $t \to +\infty \cup r \to \infty$, finding what kind of particle content the new inertial observer perceives is equivalent to finding the relation among the old and the new ladder operators, as it is made explicit by (5.8). This is ultimately given by the relation among the two inertial coordinates in the two flat regions, commonly called "in" and "out" regions, which is made non trivial by the presence of the gravitational field itself. Therefore, even at coordinate level, it could be said that the gravitational field seen as the curvature of spacetime modifies the vacuum content of the theory, like an interaction in Quantum Field Theory which takes into account particle creation.

However, not all spacetimes can induce spontaneous emission effects: as in Quantum Mechanics, where any transition between particle states is driven by the time dependence of an interacting Hamiltonian, the gravitational field, that acts like a potential, should likewise be time dependent. The very presence of a time dependent metric makes the particle interpretation of the Fock states ambiguous: since, in general, a curved spacetime is not stationary, and thus there is not any time translation Killing field that could be used in order to introduce a global notion of time, the very splitting between positive and negative frequencies, necessary to introduce the whole Fock state apparatus, becomes problematic. Asymptotic flat regions naturally allows for different Minkowskian Killing fields, but the time dependence in the intermediate region where there are gravitational effects changes the modes and the ladder operators, as would be expected for a wave with given amplitude and frequency that suffers effects such as redshift when it passes by a gravitational source.

Another key role is played by horizons, and specifically for the case of black hole emission by the event horizon; this surface is crucial for the existence of particle creation outside a black hole, since such particles are paired with negative Killing energy particles, called partners, that ensure energy conservation. Moreover, the strength of the gravitational field at the horizon imposes a high redshift to the particles trying to escape it, and this allows only for the most energetic particles to be detected at infinity, giving a certain degree of model independence in the spectrum of particles evaluated with (5.8). The resulting law, neglecting the backscattering of the modes, the backreaction of the gravitational field and any other interaction, is a universal law showing a Planckian spectrum with a temperature depending on the mass of the black hole M, that is, the

Hawking radiation [1]:

$$\langle 0|\hat{N}'_k|0\rangle = \frac{1}{\exp\left[\hbar\omega_k/k_B T_H\right] - 1}, \tag{5.9}$$

with $T_H = \frac{\hbar}{8\pi G M k_B}$.

5.2 Rindler coordinates and the uniformly accelerated particle

Let us choose a uniformly accelerated trajectory. The equations of motion for an accelerated particle with a constant acceleration a can be solved in a more practical way [28] by considering the system of equations

$$\begin{cases} u_\alpha u^\alpha = -1 \\ a_\alpha a^\alpha = a^2 \\ a_\alpha u^\alpha = 0, \end{cases} \tag{5.10}$$

where u^α and a^α are respectively the four velocity and the four acceleration of the particle. Combining the equations of the system leads to the following laws

$$\begin{cases} \frac{d^2u^0}{d\tau^2} = a^2 u^0 \\ \frac{d^2u^x}{d\tau^2} = a^2 u^x, \end{cases} \tag{5.11}$$

which solutions are known once the conditions of no velocity and no acceleration at $\tau = 0$ are provided. This gives

$$\begin{cases} u^0 = \cosh(a\tau) \\ u^x = \sinh(a\tau), \end{cases} \tag{5.12}$$

that can be further integrated in the proper time to get the worldline

$$\begin{cases} t = \frac{1}{a}\sinh(a\tau) \\ x = \frac{1}{a}\cosh(a\tau). \end{cases} \tag{5.13}$$

The above equations can also be regarded as a coordinate transformation between an inertial and a non inertial observer if another coordinate ζ is introduced such that

$$\begin{cases} t = \frac{1}{a}e^{a\zeta}\sinh(a\tau) \\ x = \frac{1}{a}e^{a\zeta}\cosh(a\tau). \end{cases} \tag{5.14}$$

[1] For a derivation of the Planckian spectrum in a simple model, see [27]

The coordinates ζ and τ, called Rindler coordinates, are the one associated to accelerated observers: for a fixed ζ the uniformly accelerated case is recovered. Picking $\zeta = 0$ as an example, the worldline in a Minkowski plane for such observers is given by a hyperbola

$$x^2 - t^2 = \frac{1}{a^2}. \tag{5.15}$$

On the other hand, each time slice $\tau = const.$ is a straight line in a Minkowski plane

$$\tau = \frac{1}{a} \tanh^{-1}\left(\frac{t}{x}\right). \tag{5.16}$$

The grid obtained by intersecting the $\tau = const.$ and $\zeta = const.$ lines is the Rindler coordinate system in a Minkowski plane.
The Rindler metric is given by

$$ds^2 = e^{2a\zeta}(-d\tau^2 + d\zeta^2), \tag{5.17}$$

and it is clear that the metric tensor has a time translation symmetry $\tau \to \tau + \epsilon$. This symmetry will be crucial for the analysis of particle creation out from the vacuum with an Unruh-DeWitt detector model. A Killing vector

$$K_r^\alpha = (1, 0) \tag{5.18}$$

is associated to the time translation symmetry. The subscript r points out that these components are written with respect to the Rindler coordinates. It is interesting to look at the expression of this Killing vector in a Minkowskian coordinate system. Knowing the transformation rule (5.14), it is straightforward to find (subscript m standing for "Minkowski")

$$K_m^\alpha = \frac{\partial x_m^\alpha}{\partial x_r^\beta} K_r^\beta = a(x, t). \tag{5.19}$$

This expression may be obscure, but it is well known that the number of Killing vectors is fixed by the metric tensor, and should not be affected by any coordinate transformation. Therefore, K_m^α is clearly not the Killing vector associated with time translations in Minkowski coordinates, since the latter would be $A^\alpha = (1, 0)$, but it will be another Killing vector. Let us consider then a Lorentz transformation of the vector $x^\mu = (t, x)$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{5.20}$$

and in particular its infinitesimal version

$$\Lambda^\mu{}_\nu \sim \delta^\mu{}_\nu + \epsilon^\mu{}_\nu, \tag{5.21}$$

with $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$. Calling $\epsilon_{01} = \epsilon$, the change in the vector x^ν is given by

$$\delta x^\mu = \epsilon^\mu{}_\nu x^\nu = \epsilon(x, t). \tag{5.22}$$

Up to a constant, which is irrelevant, it is clear that a time translation for an accelerated observer corresponds to a Lorentz transformation for an inertial observer.

5.2.1 Near Horizon approximation and correspondence with Schwarzschild spacetime

Consider the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{5.23}$$

and suppose to place a probe at a distance near the event horizon. Parametrize this distance with x such that

$$r = 2GM + \frac{x^2}{8GM}, \tag{5.24}$$

and $0 < x \ll 2GM$.
It is straightforward now to expand the metric tensor with x in place of r, and in particular

$$1 - \frac{2GM}{r} \sim \frac{x^2}{16G^2M^2} = (Kx)^2, \tag{5.25}$$

where the surface gravity $K = 1/4GM$ was introduced. Ignoring the angular part by fixing θ, ϕ thanks to spherical symmetry, the metric is now

$$ds^2 = (Kx)^2(-dt^2 + dx^2), \tag{5.26}$$

which is the Rindler metric. This makes sense since accelerated observers in flat spacetime perceive the same effects of static observers in curved spacetime, as stemmed by the Equivalence Principle. To see this correspondence, consider the transformation law which brings the metric of a spherically symmetric solution for the field equations from its Schwarzschild form to its Kruskal form:

$$\begin{cases} U = \mp K e^{-Ku} \\ V = \pm K e^{Kv}, \end{cases} \tag{5.27}$$

with

$$\begin{cases} u = t - r^* \\ v = t + r^* \end{cases} \tag{5.28}$$

the Eddington-Finkelstein advanced and retarded coordinates, and r^* the tortoise coordinate for the Schwarzschild solution. The plus and minus signs depend on the region considered in the manifold.
The null coordinates U, V can be transformed into a timelike and a spacelike coordinate by the laws

$$\begin{cases} T = \frac{U+V}{2} = e^{Kr^*} \sinh{(Kt)} \\ X = \frac{V-U}{2} = e^{Kr^*} \cosh{(Kt)}, \end{cases} \tag{5.29}$$

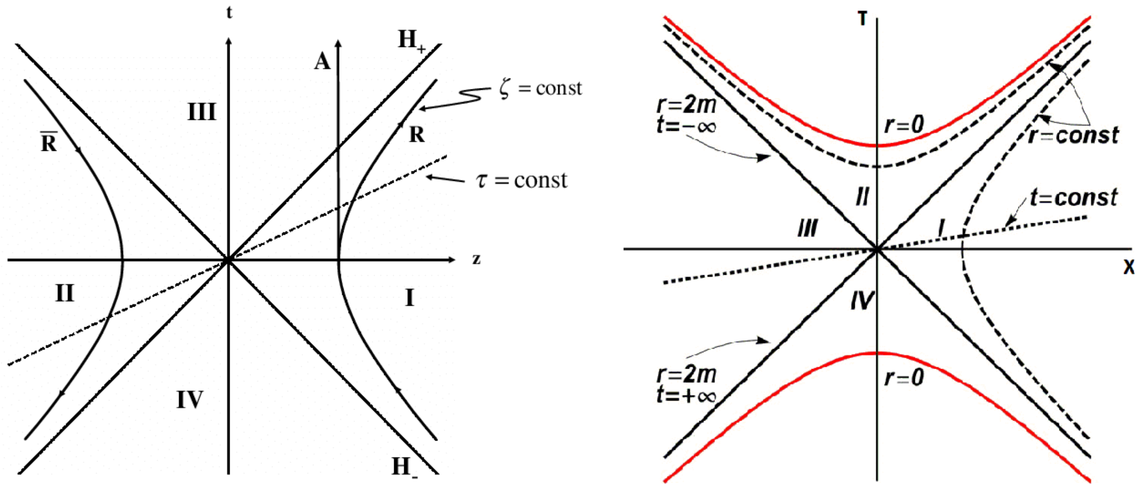

Figure 5.1: Minkowski and Kruskal manifolds respectively splitted by Rindler and Schwarzschild coordinates. Images taken from [29] and [30].

and the Kruskal metric reads

$$ds^2 = -\frac{32M^3G^3}{r}e^{-\frac{r}{2GM}}(-dT^2 + dX^2) + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{5.30}$$

with r that is now an implicit function of X, T. The transformation laws found above have exactly the same form of the ones that bring the Minkowski metric to the Rindler metric

$$\begin{cases} T = \frac{1}{a}e^{a\zeta}\sinh{(a\tau)} \\ X = \frac{1}{a}e^{a\zeta}\cosh{(a\tau)}, \end{cases} \tag{5.31}$$

which leads to

$$ds^2 = e^{2a\zeta}(-d\tau^2 + d\zeta^2), \tag{5.32}$$

and (5.26) can be recovered with $Kx = \frac{1}{a}e^{a\zeta}$. Rindler coordinates are known to have a horizon given by the special relativistic bounds imposed on the velocity and the acceleration. In particular, they can map only one quarter of the Minkowski spacetime, the one delimited by the asymptotes $T = \pm X$ on the positive X semiplane. This is exactly the same situation happening in the Schwarzschild manifold, where the Schwarzschild metric covers only one portion of spacetime, while Kruskal coordinates cover the whole spacetime. Therefore, the correspondence is between Schwarzschild $\rightarrow$ Rindler and Kruskal $\rightarrow$ Minkowski, as shown by figure 5.1.

The above idea, based on the Equivalence Principle, should not be mistaken with a true global similarity between the two metric tensors nor between the two different spacetimes: the Schwarzschild metric is related to an observer placed very far from the spherically symmetric source of the gravitational field, while the Rindler metric describes the point of view of an observer accelerating in a given direction in flat spacetime. While the Schwarzschild spacetime is curved, with a non vanishing Riemann tensor, the Rindler metric essentially describes Minkowski spacetime from another perspective. However, this analogy hints that an effect of vacuum particle production could occur even in flat spacetime, when it is described by a non inertial observer. As anticipated, this is what is meant by Unruh effect.

5.3 Particle creation in flat spacetime: Unruh effect

As previously seen, in Minkowski spacetime particles are defined with the help of the Poincaré group. Here, the modes of the field equations associated to particles are positive/negative frequency modes with respect to the Minkowskian coordinate t. With a Poincaré transformation, positive frequency modes always go into positive frequency modes, and all inertial observers agree on the particle concept and the number of particles in a given state; the vacuum state, defined by the annihilation operators associated to the modes, contains zero particles for all these observers.
The problem arises in arbitrary curved spacetimes, since General relativity tells that all coordinate systems are equivalent, and thus there is in general no preferred time-like coordinate that defines something with the property of being "positive frequency". Moreover, the Poincaré group breaks down in an arbitrary spacetime, while the Lorentz group becomes local. This problem affects also the physics described by non inertial observers that are not linked through a Poincaré transformation to inertial ones. Even for them, the particle concept becomes ambiguous due to the mixing of the modes.
A pragmatic solution to the problem, suggested by Unruh and deWitt, was to think of particles as something which is detected by a "clic" in a particle detector. The detector originally considered by Unruh [12] represents a particle in a box coupled to a quantum field in a curvilinear background. Here, a particle has been detected when the detector makes a transition between energy levels under the influence of the quantum field. This in turn corresponds to absorption or emission of particles of the field. De Witt [13] introduced a simpler version of Unruh's original idea by considering as a detector a point-like quantum particle with two or more energy levels which is coupled to the quantum field through a monopole operator. The equivalence between the two approaches has been proven in [31].
Let us then consider the most simple detector model known, the point-like one. Its internal energy levels are labelled by E and it is linearly coupled through its monopole operator $\hat{m}(\tau)$ to a scalar field $\hat{\Psi}(x)$, while it is moving along a worldline $x^{\mu}(\tau)$ in flat spacetime. The coupling is made by a contact interaction which evaluates the field on

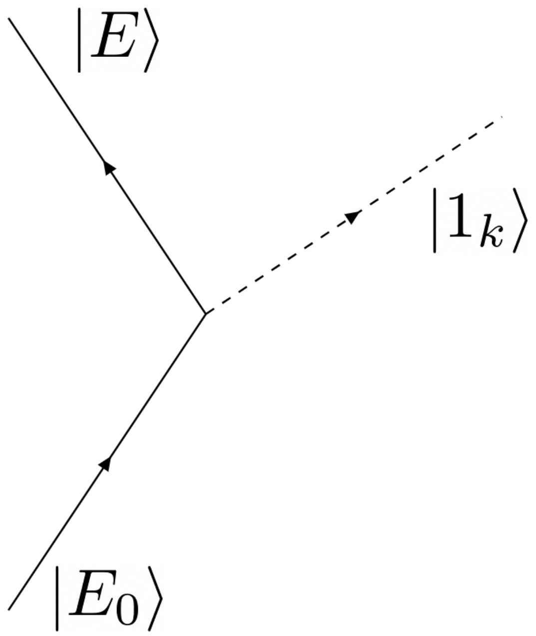

Figure 5.2: One particle emission process.

the points belonging to the worldline of the detector. The interacting Hamiltonian is thus given by

$$\hat{H}_{int}(\tau) = c\hat{m}(\tau)\hat{\Psi}(x(\tau)), \tag{5.33}$$

where c is a coupling constant, which smallness will be useful to apply time-dependent perturbation theory, and which constancy points out that the interaction is eternal. In this section, $\hbar = 1$, and it will be suitably restored at the end of the calculations performed.

Suppose that the detector starts in its ground state E_0: the energy spectrum is supposed to be bounded from below, in such a way that any E will be $E > E_0$, and discrete. The field $\hat{\Psi}$ is in its vacuum state $|0\rangle$. It will be shown that the transition to possible final states, different from the vacuum, is allowed if the trajectory of the detector is non inertial. This reflects the vacuum particle content ambiguity for non inertial observers. The detector will make a transition from E_0 to E, while the field can make a jump from $|0\rangle$ to $|1_p\rangle$, its one particle state (figure 5.2). This happens because at first order in perturbation theory the only field final state which gives a non-vanishing amplitude is the one containing a particle with a given momentum p. Later on, the particle content of the final state will be relaxed to look at all the transition probability to any state.

In the transition process, a key role is played by the detector trajectory, which kills the whole amplitude in the case of an inertial motion[2]. Actually, a smooth switching function of the proper time could be added [33], but since the focus will be given to the possible processes happening firstly for the vacuum and then for the coherent state, the details regarding the nature of the interaction coupling, and the internal structure of the detector, will be minimally discussed. Thus the interaction will last for an infinite amount of proper time. For a non inertial motion, such as a uniformly accelerated trajectory, both the field and the detector gain energy from the process.

It is worth to stress that such definition of a particle is not directly linked to the particle ambiguity illustrated in section 5.1, since it was observed how for generic trajectories a vanishing Bogoliubov coefficient $\beta_k = 0$ (and thus of $|\beta_k|^2$) does not imply a vanishing probability for detector transitions $P(\Omega) \neq 0$, with Ω the transition frequency. Conversely, a vanishing probability $P(\Omega) = 0$ does not imply zero Bogoliubov coefficients, i.e. it is $\beta_k \neq 0$. However, it was shown how for inertial and uniformly accelerating trajectories there is a correspondence that gives the same results. The above assertions were discussed in [34], as well as the nature of the excitations of the detector. At a first sight, the fact that both the detector and the field are gaining energy seems to violate energy conservation; actually, the missing energy is supplied by the agent force, and thus the external potential, that accelerates the measuring device. The detector climbs its internal energy and at the same time there is an emission of a scalar quantum through the detector coupling; in this sense, the field acquires the role of a friction force which resists to the change in the motion of the detector.

Furthermore, the Unruh-DeWitt approach is based on an interesting fact: the detector has quantum mechanical properties, since it has associated an Hilbert space over which its energy states are defined, and its monopole operator acts, but at the same time it follows a trajectory, which is a purely classical concept. This means that its center of mass is uncorrelated to the field and is looked in a classical way, being its motion prescribed. The detector motion could also be related to the coupled field, but only when the latter is treated as an external potential, independent from the field operator entering the interaction.

Historically, many trajectories have been explored [35–37], and the subject of particle emission for non inertial detectors gained an interest on its own. For example, vacuum emission given by uniform circular motion can be calculated, even if the computation cannot be performed in a closed form, and numerical techniques should be instead used [35]. Actually, there are special classes of interesting trajectories, which will be discussed later.

It is also worth noticing that the presence of an horizon is not a necessary ingredient for emission processes due to curvilinear backgrounds. In fact, as already said, a radiative

[2] However, it is worth mentioning that this is not totally true: in fact, if the detector is switched on and off in a finite time interval, the amplitude will be finite and non vanishing. This is a consequence of the Heisenberg Principle, since the switching actions related to given instants of time carry an uncertainty in energy. Such uncertainty can make "tics" appear in the particle counting of the detector, see [32].

process of this type happens for circular motions, which associated coordinates do not show any horizon; notice also that a circular orbit around a black hole is a spacetime geodesic, but there is still particle emission [35], meaning that also the non-geodesic motion does not play a fundamental role. On the other hand, a fixed test body above a gravitational source without an horizon does not detect anything. Therefore, precise statements on "Unruh type" effects with a detector model can be made only for uniformly accelerating trajectories. For the reasons explained above, and since it mimics gravitational effects in curved spacetime through the Equivalence Principle, the non inertial motion of a uniformly accelerated trajectory will be treated with particular attention.
To conclude this brief detour on the Unruh-DeWitt detector model, it is worth mentioning that, besides its application as a physical probe of vacuum effects, this setup has nowadays entered in the discussions about detector based Quantum Field Theories [10]; the introduction of a detector as a necessary measurement device explicitly present in the quantum state, and as a tool to define field-related concepts, enters the game consistently with what discussed in chapter 4.

5.3.1 Transition amplitude

Let us look at what previously told in details: using time dependent perturbation theory to first order the following amplitude should be evaluated

$$A(i \to f) = -ic \langle E, \psi| \int_{-\infty}^{+\infty} d\tau \hat{m}(\tau) \hat{\Psi}(x(\tau)) |E_0, 0\rangle \,. \tag{5.34}$$

In the interacting picture, the time evolution of the monopole operator is dictated by the free Hamiltonian of the detector, i.e.

$$\hat{m}(\tau) = e^{iH_0\tau} \hat{m}(0) e^{-iH_0\tau}, \tag{5.35}$$

and since $\hat{H}_0 |E\rangle = E |E\rangle$, that is, the detector is in an eigenstate of the free (and not of the full) Hamiltonian of the detector, the amplitude can be written as

$$A(i \to f) = -ic \langle E|\hat{m}(0)|E_0\rangle \int_{-\infty}^{+\infty} d\tau e^{i\Omega\tau} \langle \psi|\hat{\Psi}(x(\tau))|0\rangle \,, \tag{5.36}$$

with $\Omega = E - E_0$. For the field $\hat{\Psi}$, considering the usual mode expansion, the only surviving operators are the $a_k^\dagger$, for every k, since they give non-vanishing matrix elements due to the presence of $|0\rangle$. However, since the momentum in the final state is fixed, the amplitude does not vanish only if

$$|\psi\rangle = |1_p\rangle \,. \tag{5.37}$$

Of course, this choice is forced by the first order expansion in perturbation theory, as a higher order calculation would lead to polynomials in $\hat{a}_k$ and $\hat{a}_k^\dagger$ with more non trivial contributions. Let us assume that the motion happens only in the x, t plane and omit the other two spatial coordinates by setting them constant and equal to zero. Evaluating the bracket for the field gives

$$\langle 1_p|\hat{\Psi}(x)|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\langle 1_p|a_k^\dagger|0\rangle\, e^{-ikx+i\omega_k t} = \frac{4\pi}{\sqrt{2\omega_p}}e^{-ipx+i\omega_p t}. \tag{5.38}$$

As mentioned above, the position variables for the field are locked to the ones of the detector. It is straightforward to see that for an inertial trajectory the transition amplitude vanishes: the equations of motion for an uniform velocity motion along the x-direction give as a result

$$x = x_0 + v(1-v^2)^{-\frac{1}{2}}\tau, \tag{5.39}$$

with $v = const. < 1$. From the above equation, suppressing the y and z coordinates, the amplitude becomes

$$A(i\to f) = -ic\frac{4\pi}{\sqrt{2\omega_p}}e^{-ipx_0}\langle E|\,\hat{m}(0)\,|E_0\rangle\,\delta\left(\Omega + \frac{\omega_p - pv}{\sqrt{1-v^2}}\right). \tag{5.40}$$

Reminding that $\Omega > 0$, and that $pv < \omega_p = |p|$, the delta function has no support, and the transition amplitude vanishes. Another choice from (5.39) would have led to a different integral, and thus no delta function arises at all, giving a non vanishing matrix element. Therefore, let us plug the trajectory for an accelerated observer in the amplitude, in place of (5.39):

$$A(i\to f) = -ic\frac{4\pi}{\sqrt{2\omega_p}}\langle E|\,\hat{m}(0)\,|E_0\rangle\int_{-\infty}^{+\infty} d\tau e^{i\Omega\tau}e^{i\omega_p a^{-1}\sinh a\tau - ipa^{-1}\cosh a\tau}. \tag{5.41}$$

The proper time integral

$$\int_{-\infty}^{+\infty} d\tau e^{i\Omega\tau + i\frac{\omega_p}{a}\sinh a\tau - i\frac{p}{a}\cosh a\tau} \tag{5.42}$$

can be rewritten as

$$\int_{-\infty}^{+\infty} d\tau e^{i\Omega\tau + i\frac{\omega_p}{2a}(e^{a\tau}-e^{-a\tau}) - i\frac{p}{2a}(e^{a\tau}-e^{-a\tau})}, \tag{5.43}$$

and with the following position

$$e^{-a\tau} = x, \tag{5.44}$$

as

$$\frac{1}{a}\int_0^\infty dx x^{-1-i\frac{\Omega}{a}}e^{i\frac{\omega_p - p}{2a}\frac{1}{x} - i\frac{\omega+p}{2a}x}. \tag{5.45}$$

Considering that $\omega_p = p$, it further simplifies

$$\frac{1}{a}\int_0^\infty dx x^{-1-i\frac{\Omega}{a}}e^{-i\frac{\omega_p}{a}x}, \tag{5.46}$$

and with another position

$$i\frac{\omega_p}{a}x = z, \tag{5.47}$$

it becomes

$$\frac{1}{a}\left(\frac{a}{i\omega_p}\right)^{-i\frac{\Omega}{a}}\int_0^\infty dz z^{-1-i\frac{\Omega}{a}}e^{-z}. \tag{5.48}$$

The z-integral can be recognized as a Γ function,

$$\Gamma(z) = \int_0^\infty dx x^{z-1}e^{-x}, \tag{5.49}$$

and after having rewritten

$$(-i)^{-i\frac{\Omega}{a}} = e^{(-i\frac{\pi}{2})(-i\frac{\Omega}{a})} = e^{-\frac{\pi\Omega}{2a}}, \tag{5.50}$$

the result is then

$$\frac{1}{a}e^{-\frac{\pi\Omega}{2a}}\left(\frac{\omega_p}{a}\right)^{i\frac{\Omega}{a}}\Gamma\left(-i\frac{\Omega}{a}\right), \tag{5.51}$$

which is of course non zero. The probability is then

$$P(i\to f) = \frac{c^2}{4\pi\omega_p}|\langle E|\,\hat{m}(0)\,|E_0\rangle|^2\frac{1}{a^2}e^{-\frac{\pi\Omega}{a}}\left|\Gamma\left(-i\frac{\Omega}{a}\right)\right|^2. \tag{5.52}$$

By means of the Euler reflection formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin\pi z}, \tag{5.53}$$

it holds that

$$\left|\Gamma\left(-i\frac{\Omega}{a}\right)\right|^2 = \frac{\pi}{\frac{\Omega}{a}\sinh\left(\pi\frac{\Omega}{a}\right)}. \tag{5.54}$$

Using the definition $\sinh\left(\pi\frac{\Omega}{a}\right) = \frac{1}{2}(e^{\pi\frac{\Omega}{a}} - e^{-\pi\frac{\Omega}{a}})$, and combining it with the factor $e^{-\frac{\pi\Omega}{a}}$, the probability becomes

$$P(i\to f) = \frac{16\pi^3c^2\hbar}{a\omega_p(E-E_0)}|\langle E|\,\hat{m}(0)\,|E_0\rangle|^2\frac{1}{\exp\left[\frac{2\pi(E-E_0)}{\hbar a}\right]-1}, \tag{5.55}$$

where it was restored $\hbar$ by dimensional analysis, since $\Omega = (E-E_0)/\hbar$. The last factor is a Planckian factor if the following identity is provided

$$T_U \equiv \frac{\hbar a}{2\pi k_B}. \tag{5.56}$$

Therefore, the more the detector accelerates, the more probable will be the emission of a particle.

5.3.2 Transition to all final states

Generically, it is interesting to work out the transition probability for all the available energy levels E given by all the possible final states $|\psi\rangle$. Thus, it is worth evaluating the following amplitude, with the sum over all final states

$$A(i \to f) = -ic \langle E| \hat{m}(0) |E_0\rangle \int_{-\infty}^{+\infty} d\tau e^{i\Omega\tau} \langle \psi| \hat{\Psi}(x(\tau)) |0\rangle . \tag{5.57}$$

To take the probability, sum over all final states such that

$$\begin{aligned} P(i \to f) = c^2 \sum_E \sum_\psi | \langle E|\hat{m}(0)|E_0\rangle |^2 \\ \cdot \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau' e^{-i\Omega(\tau-\tau')} \langle 0|\hat{\Psi}(x(\tau))|\psi\rangle \langle \psi|\hat{\Psi}(x(\tau'))|0\rangle \end{aligned} \tag{5.58}$$

or, using the completeness relation,

$$\sum_\psi |\psi\rangle \langle\psi| = 1, \tag{5.59}$$

it is

$$\begin{aligned} P(i \to f) = c^2 \sum_E | \langle E|\hat{m}(0)|E_0\rangle |^2 \\ \cdot \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau' e^{-i\Omega(\tau-\tau')} \langle 0|\hat{\Psi}(x(\tau))\hat{\Psi}(x(\tau'))|0\rangle . \end{aligned} \tag{5.60}$$

The term involving the matrix elements of the monopole operator $| \langle E| \hat{m}(0) |E_0\rangle |^2$ embeds the detector details (that is, its internal structure) and it is called Selectivity. The second term is independent from these details and it is called the Response function

$$F(E - E_0) = \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau' e^{-i\Omega(\tau-\tau')} D^+(x, x'). \tag{5.61}$$

The Response function embeds the interplay between the kinematic of the detector and the dynamics of the field through the presence of the trajectory of the detector inside the field itself. The above bracket involving the vacuum states is called the (positive frequency) Wightman Green function

$$D^+(x, x') = \langle 0| \hat{\Phi}(x)\hat{\Phi}(x') |0\rangle , \tag{5.62}$$

with the shorthands $x = x(\tau)$ and $x' = x(\tau')$.
The Wightman function is the crucial term which points out the particle content of

the vacuum and it is an universal feature of the field. This peculiarity will lead to the universal Planckian factor characterizing the Unruh effect, which is thus dependent on the detector motion. This view enforces the fact that radiative processes of this type are kinematical effects, as they depend from the motion rather than from the specific field involved in the process. Then, let us perform the calculation to express the positive frequency Wightman Green function in position space explicitly. Expanding the field operator in Minkowski plane waves gives the following expression for the Wightman function

$$D^+(x,x') = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{-i\omega_p(t-t')+i\vec{p}\cdot(\vec{x}-\vec{x}')}. \tag{5.63}$$

The above three dimensional integral can be rewritten in a four dimensional equivalent expression with the introduction of poles [38]

$$iD^+(x,x') = \int \frac{d^4p}{(2\pi)^4}\frac{e^{-i\omega_p(t-t')+i\vec{p}\cdot(\vec{x}-\vec{x}')}}{\omega_p^2-|\vec{p}|^2}. \tag{5.64}$$

The prescription to get the positive frequency Wightman function from the above integral, when it is mapped into a complex one, tells that the positive pole $\omega_p = |\vec{p}|$ should be encapsulated with a circular contour γ^+, while the negative one should be avoided, see figure (5.3). Therefore, consider

$$iD^+(x,x') = \int \frac{d^3p}{(2\pi)^4}e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}\oint_{\gamma^+} dz\frac{e^{-iz(t-t')}}{z^2-|\vec{p}|^2}, \tag{5.65}$$

with $\omega_p \to z, \quad z\in\mathbb{C}$. The complex integral can be evaluated with the Cauchy theorem and gives

$$\begin{aligned}\oint_{\gamma^+} dz\frac{e^{-iz(t-t')}}{z^2-|\vec{p}|^2} &= 2\pi i\,\mathrm{Res}\left[\frac{e^{-iz(t-t')}}{z^2-|\vec{p}|^2},|\vec{p}|\right] = \\ &= 2\pi i\frac{e^{-i|\vec{p}|(t-t')}}{2|\vec{p}|}.\end{aligned} \tag{5.66}$$

With this result, the Wightman function becomes

$$iD^+(x,x') = \frac{i}{(2\pi)^3}\int d^3p\frac{e^{-i|\vec{p}|(t-t')+i\vec{p}\cdot(\vec{x}-\vec{x}')}}{2|\vec{p}|}. \tag{5.67}$$

This integral should be regulated. To make it convergent, apply the prescription $t-t' \to t-t'-i\epsilon$. The regulator ϵ can be interpreted by means of the Sokhotski–Plemelj formula, which in general reads

$$\int_{-\infty}^{+\infty} dx\frac{f(x)}{x-x_0-i\epsilon} = \mathrm{Pr}\int_{-\infty}^{+\infty} dx\frac{f(x)}{x-x_0} + i\pi f(x_0), \tag{5.68}$$

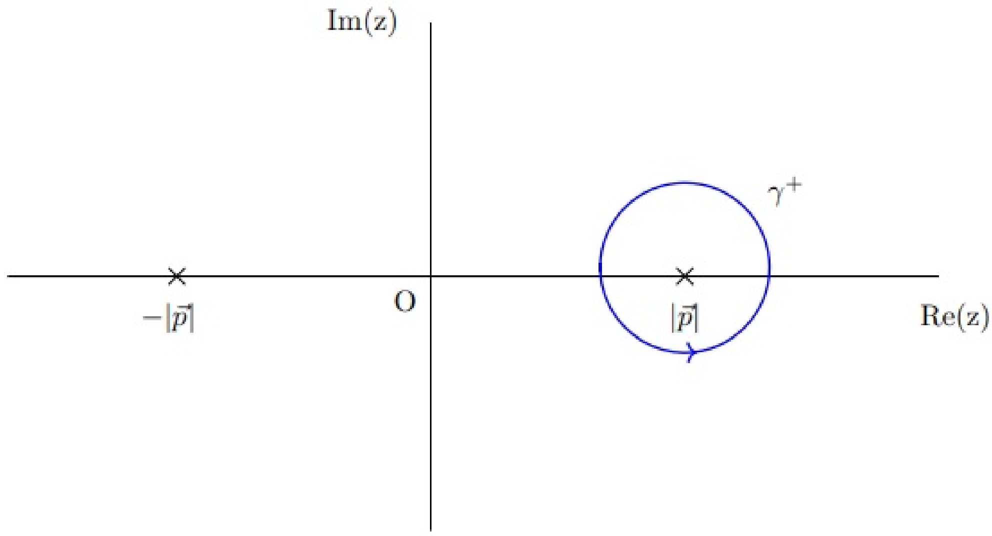

Figure 5.3: Integration path for the Positive Wightman function.

with $f(z)$ an analytic function which uniformly goes to zero at infinity and Pr standing for the principal value of the integral. The three dimensional integral in momentum space can straightforwardly be evaluated

$$
\begin{aligned}
iD^+(x,x') &= \frac{i}{(2\pi)^3}\int d^3p \frac{e^{-i|\vec{p}|(t-t'-i\epsilon)+i\vec{p}\cdot(\vec{x}-\vec{x}')}}{2|\vec{p}|} \\
&= \frac{i}{(2\pi)^2}\int_0^\infty dp\frac{p}{2}e^{-p[i(t-t')+\epsilon]}\int_{-1}^{+1} d(\cos\theta)e^{ip|\vec{x}-\vec{x}'|\cos\theta} \\
&= \frac{1}{8\pi^2|\vec{x}-\vec{x}'|}\left(\int_0^\infty dpe^{-p[\epsilon+i(t-t')]+ip|\vec{x}-\vec{x}'|} - \int_0^\infty dpe^{-p[\epsilon+i(t-t')]-ip|\vec{x}-\vec{x}'|}\right) \\
&= \frac{1}{8\pi^2|\vec{x}-\vec{x}'|}\left(\frac{1}{\epsilon+i(t-t')-i|\vec{x}-\vec{x}'|} - \frac{1}{\epsilon+i(t-t')+i|\vec{x}-\vec{x}'|}\right) \\
&= \frac{1}{8\pi^2|\vec{x}-\vec{x}'|}\frac{2i|\vec{x}-\vec{x}'|}{[\epsilon+i(t-t')]^2+|\vec{x}-\vec{x}'|^2} \\
&= -\frac{i}{4\pi^2}\frac{1}{(t-t'-i\epsilon)^2-|\vec{x}-\vec{x}'|^2},
\end{aligned}
\tag{5.69}
$$

and therefore

$$D^+(x,x') = -\frac{1}{4\pi^2[(t-t'-i\epsilon)^2-|\vec{x}-\vec{x}'|^2]}. \tag{5.70}$$

It is worth to notice that in the special cases that will be treated, the Wightman function is proper time translational invariant. The reason behind this fact will be discussed when the result for the vacuum transition will be matched with the one coming from the coherent state transition, but for now it will be explicit from the considered trajectories. For these cases, the double integration in proper time for the probability can be simplified by means of a substitution like

$$\delta\tau = \tau - \tau', \tag{5.71}$$

which brings to

$$P(i\to f) = c^2\sum_E |\langle E|\,\hat{m}(0)\,|E_0\rangle|^2\int_{-\infty}^{+\infty} d\tau'\int_{-\infty}^{+\infty} d(\delta\tau)e^{-i\Omega\delta\tau}D^+(\delta\tau). \tag{5.72}$$

Therefore, the integral in $\delta\tau$ is performed to get the Fourier transform of the Wightman function, while the other one is free to diverge. This means that the transition probability diverges. As a consequence, the transition rate per unit time is constant. Then, the above issue can be healed in two ways: the first option is to consider an adiabatic switching of the interaction. This puts an upper and lower bound to the τ integrals given by the particular form of the coupling. This will also introduce a finite time interval for the interaction, which shows the aspects related to the Uncertainty Principle previously noted. The second option makes things simpler, since the probability per unit time is directly evaluated

$$\frac{dP(i \to f)}{d\tau} = c^2 \sum_E |\langle E|\,\hat{m}(0)\,|E_0\rangle|^2 \int_{-\infty}^{\infty} d(\delta\tau) D^+(\delta\tau) e^{-i\Omega\delta\tau}. \tag{5.73}$$

The probability rate is usually a more reliable quantity then the probability itself due to this healed divergence.
Now evaluate the probability rate: by plugging (5.39) inside (5.70), the Wightman function becomes

$$D^+(\delta\tau) = -\frac{1}{4\pi(\delta\tau - i\epsilon)^2}, \tag{5.74}$$

where the Lorentz factor γ in the trajectory was absorbed inside ϵ. The above expression allows to consider the integral in $\delta\tau$ as an integral in the complex plane by means of the following mapping $\delta\tau \to z, \quad z \in \mathbb{C}$, and by considering

$$\oint_\Gamma dz f(z), \tag{5.75}$$

with $f(z) = e^{-i\Omega z} \frac{-1}{4\pi(z-i\epsilon)^2}$.
In order to solve the integral, the contour should be taken as in figure (5.4):

$$\Gamma = l^{(+)} \cup \gamma^{(-)}, \tag{5.76}$$

with $l^{(+)}$ a straight line on the real axis, and $\gamma^{(-)}$ a semicircle on the lower half plane. In such a way, when the radius of the semicircle is sent to infinity, the integral over $\gamma^{(-)}$ is zero, as stated by Jordan Lemma. At the same time, the double pole in the Wightman function is shifted in the upper-half plane, and thus does not contribute at all to the complex integral, which gives a net zero result, as stated by the Cauchy Theorem. This implies that also the integral in $l^{(+)}$, which is just the integral in $\delta\tau$, is zero. Therefore, there is no energy transition due to the presence of the field $\hat{\Phi}$ for the detector when it moves on an inertial trajectory.
Next, plugging (5.13) inside (5.70) gives

$$D^+(\delta\tau) = -\frac{1}{16\pi^2 a^{-2} \sinh^2\left(\frac{a\delta\tau}{2} - ia\epsilon\right)}. \tag{5.77}$$

The Wightman function can be usefully rewritten considering the identity $\sinh x = -i\sin(ix)$ and using the Mittag-Leffler expansion of the cosecant squared function (see section B.1)

$$\csc^2 x = \frac{1}{\pi^2} \sum_{k=-\infty}^{+\infty} (x-k)^{-2}, \tag{5.78}$$

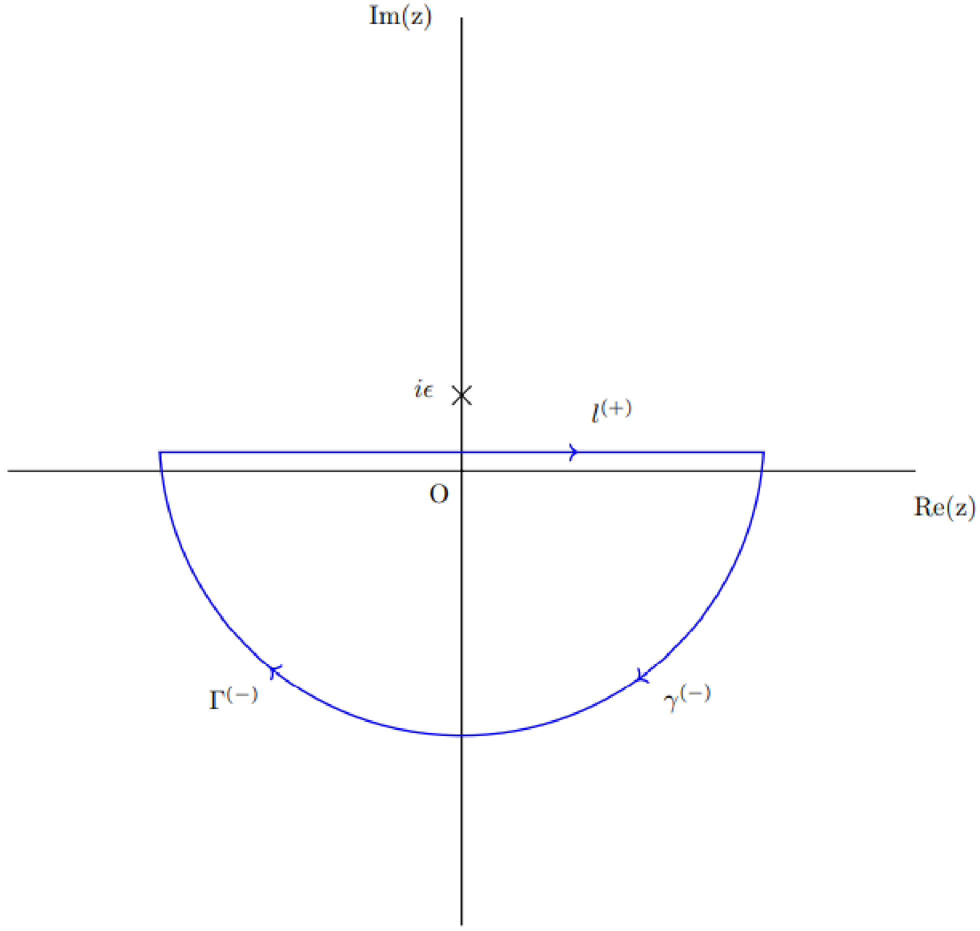

Figure 5.4: Integration path for the function $f(z)$.

that is,

$$D^{+}(\delta\tau) = -\frac{1}{4\pi^2}\sum_{k=-\infty}^{+\infty}(\delta\tau - 2i\epsilon + 2\pi i k a^{-1})^{-2}. \tag{5.79}$$

This gives an integral that can be mapped in the complex plane as

$$\oint_{\Gamma} dz e^{-i\Omega z}\frac{1}{4\pi^2}\sum_{k=-\infty}^{+\infty}\frac{-1}{(z + 2\pi i k a^{-1} - 2i\epsilon)^2}, \tag{5.80}$$

with again Γ given by (5.76).

For the function $f(z) = \frac{\exp(-i\Omega z)}{(z+2\pi i k a^{-1}-2i\epsilon)^2}$, there are now double poles in the lower half plane (see figure 5.5), $z_k = -2\pi i a^{-1}k + 2i\epsilon$, and each of them gives a residue

$$\begin{aligned} &\mathrm{Res}\,[f(z), z_k] \\ &= 2\pi i \lim_{z\to z_k}\frac{d}{dz}e^{-i\Omega z} \\ &= 2\pi\Omega e^{-\frac{2\pi\Omega}{a}}e^{2\Omega\epsilon}. \end{aligned} \tag{5.81}$$

Sending safely $\epsilon \to 0$, the integral can be evaluated through the Cauchy Theorem and it becomes

$$\frac{\Omega}{2\pi}\sum_{k=0}^{+\infty}\left(e^{-\frac{2\pi\Omega}{a}}\right)^k, \tag{5.82}$$

and having $\frac{2\pi\Omega}{a} > 0$, the series converges to the usual sum for a geometric series. The probability per unit time then becomes

$$\frac{dP(i\to f)}{d\tau} = \frac{c^2}{2\pi}\sum_E |\langle E|\,\hat{m}(0)\,|E_0\rangle|^2\frac{\Omega}{\exp\left[\frac{2\pi(E-E_0)}{a\hbar}\right]-1}, \tag{5.83}$$

thus showing that the thermal spectrum is recovered, again with a temperature given by (5.56). It is worth to remark that the Unruh spectrum is given only by the factor $1/(\exp[(E-E_0)/T_U]-1)$ and therefore only this part of the probability should be taken as representative of the thermal behaviour of the quantum state when the motion is not inertial.

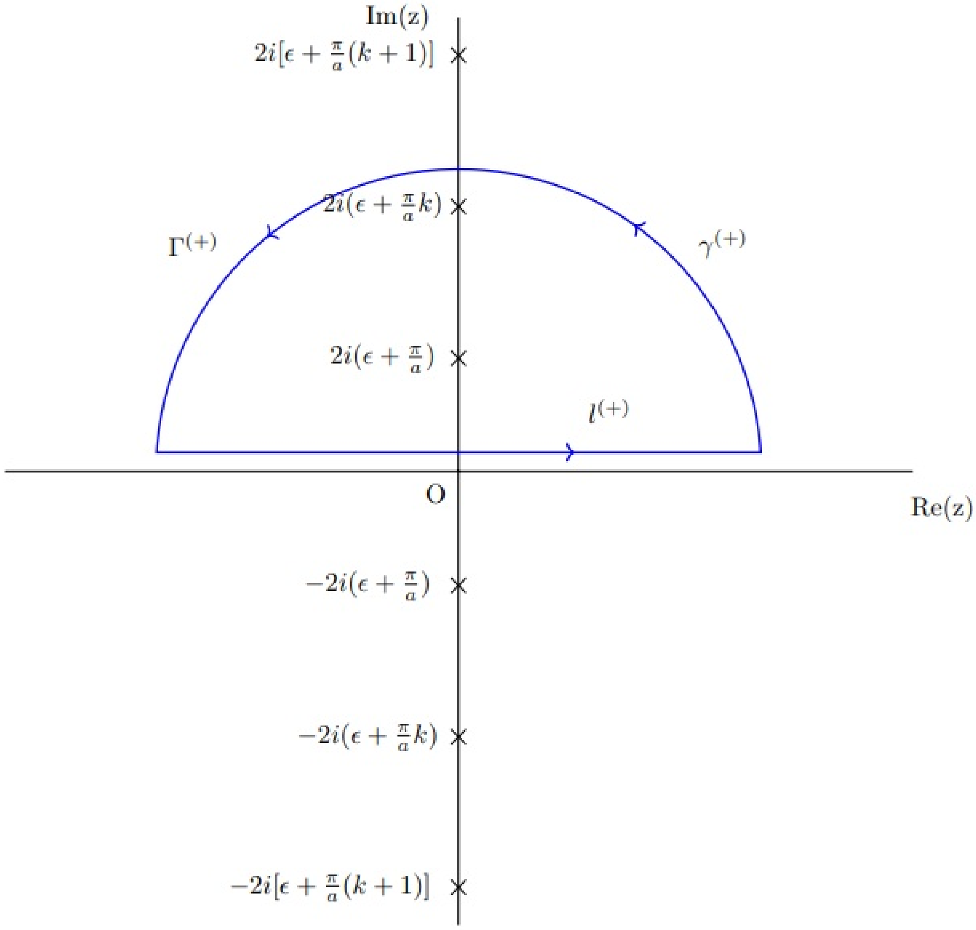

Figure 5.5: Integration path for the function $f(z)$.

Chapter 6

Particle emission in coherent state spacetime

In the previous chapters, the Hawking effect was discussed by many perspectives: in the original treatment, the radiation was interpreted as given by the excitation of the modes of a quantized field triggered by the collapse of an astrophysical object. The key ingredients involving gravity were the non stationarity of the metric tensor, in a very similar fashion to other interacting quantum field theories, and the presence of an event horizon. The strength of the field was such that the details of the collapse could also be neglected, and it was possible to find a universal Planckian spectrum with a characteristic temperature involving the black hole surface gravity (and thus the mass). At the grounds of the Hawking radiation lies the ambiguity of the particle concept, or rather, of the particle content in a given state. This is because a vacuum state defined for inertial observers is not perceived as the true "zero particle" state by non inertial ones, for which the true vacuum is defined starting from the modes of their own reference frames, the latter not linked by a Poincaré transformation to any inertial frame.
Moreover, even for non inertial observers in flat spacetime a horizon could exist, albeit it is not linked to the presence of a gravitational field, but to the causal structure of Special Relativity. The phenomenon of "particle creation" in flat spacetime, the Unruh effect, was successfully explained by Unruh and DeWitt, and modelled by coupling an accelerated detector with a quantum field, finding that the detector could get excited as a consequence of the non vanishing transition from the vacuum to any particle state of the given field. The choice of a uniformly accelerated detector was dictated by the Equivalence Principle. However, it is fundamental to keep in mind that the only requirement needed here was the non inertiality of the motion.
From the corpuscular approach, instead, the main lesson learnt was that the event horizon (and thus the surface gravity) is not a fundamental ingredient, since the geometrical picture described with General Relativity is regarded as an emerging feature of the particular (effective) field theory describing a gravitational source. Gravity is described as

a field among the others.
As already explained, the approach to gravity carried out in this work tries to take inspiration from the corpuscular picture with a more refined theory. The present model uses a coherent state to describe the Newtonian potential, and thus the Schwarzschild metric function, outside a black hole. It could be asked how a coherent state looks like in an accelerated frame, if something looking like the Unruh effect can be recovered and if the Hawking radiation for gravitons can be extracted from this formalism. In this sense, the approach followed by Unruh and DeWitt is useful when the gravitational field is described by a quantum state whose properties are fixed by classical arguments, and having such quantum state in place of the vacuum allows to make comparisons with the known results explored in the previous chapter. In fact, it could be expected that a coherent state will give deviations from vacuum state results, but nevertheless these differences could be measured and carry a sign of the coherent nature of gravity. On the other hand, it could be asked if the results found actually reproduce modifications in the Hawking spectrum or are associated to other physical situations.
Therefore, the aim is to set up an Unruh-DeWitt-like calculation involving the gravitational field, in order to see if the Unruh effect could be used to detect the quantum properties of the black hole state, and if such are related to the Hawking effect. The detector-gravity interaction will follow the point-like monopole coupling already given by DeWitt. This simple linear interaction can be regarded as a quantum version of the usual matter-gravity Newtonian interaction. This simplification can be justified by the treatment of the gravitational field based on the scalar field toy model. The goal is to grasp coherent state effects and specifically what kind of emission a coherent state produces in a Unruh-DeWitt detector model.
As a result, it is found that an effect, that is analogous to the Unruh one, occurs for accelerated trajectories. However, to get the expected result, the coefficients of the coherent state should be modified to account for the spherical symmetry at the level of the quantum state. However, this emission of gravitons cannot be related to the Hawking radiation. The reasons behind that lie on the perturbative approach carried out in the detector-gravity interaction, and on the dynamical treatment of the gravitational field. This means that the Hawking radiation could be reproduced by the coherent state picture in an Unruh-DeWitt setup only if the latter is able to reproduce the relativistic and non perturbative aspects of gravity, that is, with a background approach. Nevertheless, the transition probability encountered could be interpreted as a particle emission from a coherent state describing a static field configuration due to the non intertial motion of a detector. This means that such effect could happen for accelerated detectors in regions of space where gravity is well approximated by its non relativistic Newtonian form.

6.1 Transition probability

Let us describe the gravity-detector coupling as

$$H_{int} = c\hat{m}(\tau)\sqrt{G}\hat{\Phi}(x(\tau)), \tag{6.1}$$

and rewrite the calculations performed in the previous chapter by mapping

- $c \to c\sqrt{G}$: the coupling now should include the right power of G in order to automatically ensure the canonical normalization for the scalar field, that gives back the Newton constant for the coherent state computation. The condition $c << 1$ is imposed and perturbation theory can still be applied.
- $\hat{\Psi} \to \hat{\Phi}$: now the scalar field is the gravitational field itself and the particles emitted are gravitons. $|0\rangle$ is the vacuum state of gravity where no excitation of the reference metric over which the theory is quantized is present.

Therefore, (5.55) is interpreted as the probability for the emission of a graviton starting by the vacuum state

$$P(i \to f) = \frac{16\pi^3 c^2 G\hbar}{a\omega_p(E - E_0)} |\langle E|\hat{m}(0)|E_0\rangle|^2 \frac{1}{\exp\left[\frac{E-E_0}{T_U}\right] - 1}, \tag{6.2}$$

and (5.83) is viewed as the transition probability rate to all final states of the vacuum

$$\frac{dP(i \to f)}{d\tau} = \frac{c^2 G}{2\pi} \sum_E |\langle E|\hat{m}(0)|E_0\rangle|^2 \frac{\Omega}{\exp\left[\frac{E-E_0}{T_U}\right] - 1}. \tag{6.3}$$

Let us stress again that the previous results are reinterpreted from the corpuscular picture point of view. Originally, the above probability and probability rate had been interpreted as coming from a system (the field plus the detector) described in curvilinear coordinates that mimicked the effects of curved spacetime. Now, they are seen as vacuum gravitational emission processes given by non inertial motion: gravity enters in the quantum state and not as a background spectator.

With the choice of the interacting Hamiltonian (6.1), the only difference with respect to the vacuum case lies in the state of the gravitational field. The initial state is the coherent state reproducing the static Newton potential of a point-like source $|g\rangle$, that was already constructed and it is conveniently reported here:

$$\begin{cases} \hat{a}_k |g\rangle = g_k e^{i\omega_k t} |g\rangle \\ |g\rangle = e^{-\frac{N_g}{2}} e^{\int d^3k g_k \hat{a}_k^\dagger} |0\rangle \\ g_k = -\frac{4\pi M}{\sqrt{2k^3} m_p}. \end{cases} \tag{6.4}$$

The final state will be again a coherent state $|g'\rangle$, with a prime denoting the different particle content of the state: since in the corpuscular picture the Hawking radiation is seen as a loss of gravitons from the coherent state, the state $|g'\rangle$ will be less occupied then $|g\rangle$, having $N'_g = N_g - n$ with, for instance, $n << N_g$[1]. As will be seen later, this difference in the occupation number is reflected in a mass loss for the source from M to, say, $M - dm$. Moreover, it is intriguing that a process of this type cannot happen if the initial state is the vacuum state, since the latter is also the ground state of the system, and thus the least occupied. The coherent state, instead, can undergo transitions to lower occupied states and this possibility will appear as a factor in the amplitude depending from n/N_g.

6.1.1 Inertial trajectories

As a first try, it can be checked that a spontaneous emission cannot happen if the detector follows an inertial trajectory in Minkowski spacetime, even if now the gravitational field is in its coherent state and not in the vacuum anymore. Supposing for simplicity that the detector is standing still at $r = const.$, by means of first order time dependent perturbation theory, the amplitude reads

$$A(i \to f) = -ic \int_{-\infty}^{+\infty} d\tau \, \langle E, g'| \, \hat{m}(\tau)\sqrt{G}\hat{\Phi}(x(\tau)) \, |g, E_0\rangle \, . \tag{6.5}$$

Using the time evolution for the operator $\hat{m}(\tau)$ leads to

$$A(i \to f) = -ic \, \langle E| \, \hat{m}(0) \, |E_0\rangle \int_{-\infty}^{+\infty} d\tau e^{i\Omega\tau} \, \langle g'| \, \sqrt{G}\hat{\Phi}(x(\tau)) \, |g\rangle \, . \tag{6.6}$$

It is now convenient to expand the scalar field taking advantage of the spherical symmetry. Once the creation and annihilation operators acted on the coherent state, what is left is

$$A(i \to f) = -ic \, \langle E| \, \hat{m}(0) \, |E_0\rangle \int_{-\infty}^{+\infty} d\tau e^{i\Omega\tau} \int \frac{dk k^2}{2\pi^2} \sqrt{\frac{\hbar G}{2k}} \frac{\sin{(kr)}}{kr} (g_k + g'_k) \, \langle g'|g\rangle \, . \tag{6.7}$$

It is useless now to proceed in the calculation, since there is not any proper time dependence in the above integrals furnishing a support to the δ-function or modifying the

[1] The transition amplitude to highly occupied states, that is with $N'_g = N_g + n$, is the same of the one calculated below. In this sense, emission and absorption are rather symmetrical effects, having the same dependence from the occupation number of the coherent state, the energy and the temperature of the system.

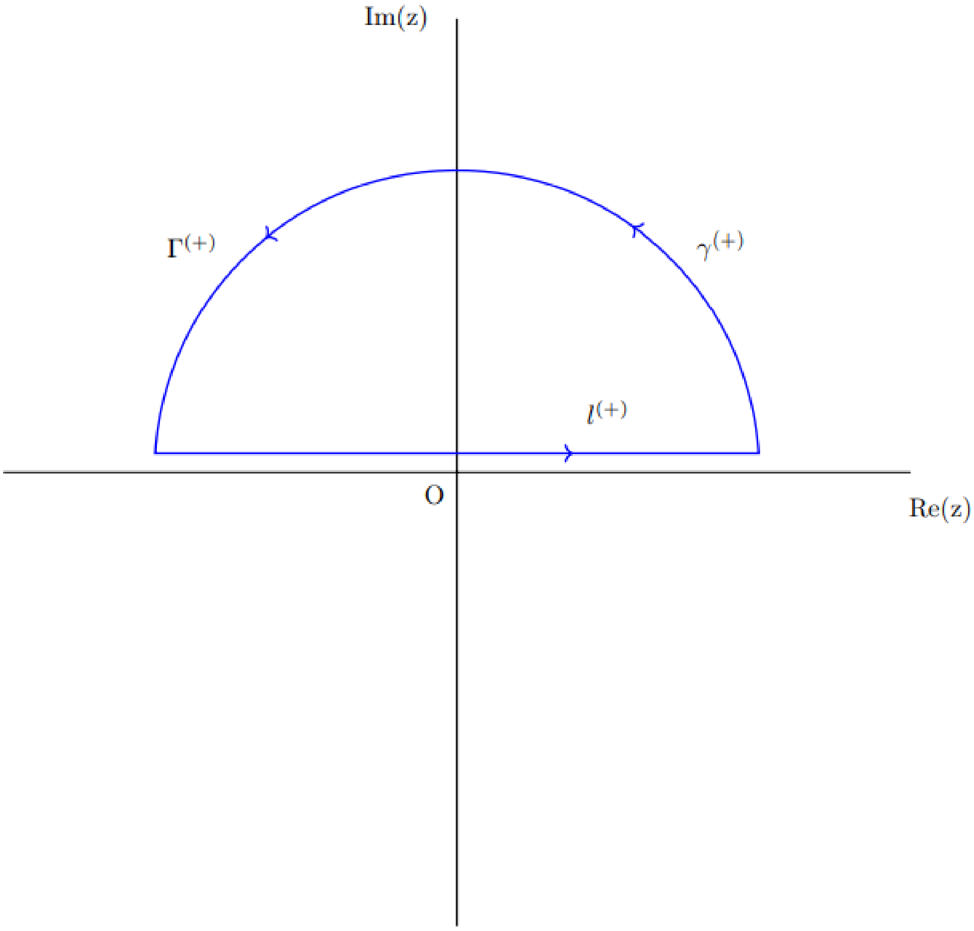

Figure 6.1: Integration path for the function $f(z)$.

integrand. Thus, there is no emission from the coherent state. Also for radial inertial motion

$$\begin{cases} t(\tau) = \gamma\tau \\ r(\tau) = r_0 + v\gamma\tau \end{cases} \tag{6.8}$$

there is no emission, as can be checked by

$$A(i \to f) = -ic \langle E| \hat{m}(0) |E_0\rangle \int_{-\infty}^{+\infty} d\tau e^{i\Omega\tau} \int \frac{dkk^2}{2\pi^2} \sqrt{\frac{\hbar G}{2k}} \frac{\sin[kr(\tau)]}{kr(\tau)} (g_k + g'_k) \langle g'|g\rangle \,, \tag{6.9}$$

from which the proper time integral gives

$$\int_{-\infty}^{+\infty} d\tau f(\tau). \tag{6.10}$$

with $f(\tau) = e^{i\Omega\tau} \frac{\sin[k(r_0+v\gamma\tau)]}{k(r_0+v\gamma\tau)}$.
Mapping $\tau \to z, \quad z \in \mathbb{C}$, it is straightforward to see that the integral evaluated on a semicircle in the upper half complex plane vanishes, by means of the Cauchy Theorem. This is because the integrand has no poles and the Jordan Lemma can be applied by closing the contour at infinity, see figure 6.1.
Therefore, two key conclusions can be deduced: first of all, even here inertial Minkowskian trajectories kill any emission amplitude. Secondarily, if the gravitational field is described by a coherent state, then a precise recovering of the Hawking radiation would require that $r = const.$ trajectories give the same results for accelerated ones in a vacuum state, as dictated by the Equivalence Principle. This is because in the Unruh effect the accelerated motion is responsible for emission in place of the gravitational field itself.

6.1.2 Uniformly accelerated radial trajectory

The trajectory of the detector will be now a radially accelerated trajectory in flat spacetime, and the equations of motion are formally identical to the ones already calculated in the $t - x$ Minkowski spacetime:

$$\begin{cases} t(\tau) = \frac{1}{a}\sinh(a\tau) \\ r(\tau) = \frac{1}{a}\cosh(a\tau), \end{cases} \tag{6.11}$$

with the distance which is again the (τ- dependent) radial distance from the center of the gravitational source, with $a = const.$ the modulus of the acceleration. Considering again the amplitude of the process (6.9), it is now worth exploring what the dependence from the proper time does to the coherent state transition.

Let us first evaluate $\langle g'|g\rangle$: expanding the coherent states through their definition gives ($g'_p = g_p(M - dm)$)

$$\langle g'|g\rangle = e^{-\frac{N_g - n}{2}} e^{-\frac{N_g}{2}} \langle 0| e^{\int \frac{dpp^2}{2\pi^2} g_p(M-dm)\hat{a}_p} e^{\int \frac{dkk^2}{2\pi^2} g_k(M)\hat{a}_k^\dagger} |0\rangle \,, \tag{6.12}$$

where dm is the mass loss due to the escape of gravitons.
Next, joining the two exponentials by means of the BCH formula

$$e^{\hat{X}} e^{\hat{Y}} = e^{\hat{X}+\hat{Y}+\frac{1}{2}[\hat{X},\hat{Y}]+\dots}, \tag{6.13}$$

where $\hat{X}$ and $\hat{Y}$ are two operators, leads to

$$\langle g'|g\rangle = e^{-N_g + \frac{n}{2}} \langle 0| e^{\int \frac{dpp^2}{2\pi^2} \frac{dkk^2}{2\pi^2} g_p(M-dm) g_k(M)[\hat{a}_p,\hat{a}_k^\dagger]} |0\rangle \,, \tag{6.14}$$

and by making use of $[\hat{a}_k, \hat{a}_p^\dagger] = \frac{2\pi^2}{p^2}\delta(p-k)$, the bracket becomes

$$\langle g'|g\rangle = e^{-N_g + \frac{n}{2}} e^{\int \frac{dpp^2}{2\pi^2} g_p(M-dm) g_p(M)}. \tag{6.15}$$

Now, inserting the expressions for the coefficients g_k reproducing the field of a point-like source, the bracket is

$$\langle g'|g\rangle = e^{-N_g + \frac{n}{2}} e^{4\int_{k_{IR}}^{k_{UV}} dp \frac{M^2}{m_p^2} \frac{1}{p} \frac{M-dm}{M}}. \tag{6.16}$$

Recalling that $N_g = 4\int_{k_{IR}}^{k_{UV}} dk \frac{M^2}{m_p^2}\frac{1}{k}$, the two N_g parameters cancel each other, and what's left is

$$\langle g'|g\rangle = e^{\frac{n}{2} - \frac{dm}{M} N_g}. \tag{6.17}$$

The quantity dm should be related to the universal parameter N_g and to the mass of the source. This can be done by means of the relations coming from the corpuscular scaling law (3.42),

$$M = \frac{m_p}{\log\left(\frac{k_{UV}}{k_{IR}}\right)} \sqrt{N_g} \tag{6.18}$$

and

$$M - dm = \frac{m_p}{\log\left(\frac{k_{UV}}{k_{IR}}\right)} \sqrt{N_g - n}. \tag{6.19}$$

Taking their ratio leads to

$$\frac{M - dm}{M} = \sqrt{1 - \frac{n}{N_g}}, \tag{6.20}$$

and so

$$-\frac{dm}{M} = \sqrt{1 - \frac{n}{N_g}} - 1. \tag{6.21}$$

For $N >> 1$, the square root can be expanded

$$\sqrt{1-\frac{n}{N_g}} \sim 1-\frac{n}{2N_g}-\frac{n^2}{8N_g^2} \tag{6.22}$$

to get

$$-\frac{dm}{M}N_g = -\frac{n}{2}-\frac{n^2}{8N_g}, \tag{6.23}$$

which finally gives

$$\langle g'|g\rangle = e^{-\frac{n^2}{8N_g}}. \tag{6.24}$$

Now, the amplitude is

$$A(i\to f) = ic\sqrt{G\hbar}\,\langle E|\,\hat{m}(0)\,|E_0\rangle\, e^{-\frac{n^2}{8N_g}} \int_{-\infty}^{+\infty} d\tau e^{i\Omega\tau}\frac{2M-dm}{2m_p}\frac{2}{\pi}\int_0^{k_{UV}} dk \frac{\sin\left[kr(\tau)\right]}{kr(\tau)}, \tag{6.25}$$

where the coefficients g_k and g'_k were explicited. To simplify, let us send the cut-off $k_{UV} \to \infty$. This is possible since k_{UV} is needed to heal the diverging occupation number, but its action on the Newtonian field brings oscillating corrections to the classical expression, that can be made arbitrarily small (as discussed in chapter 3).
Calling

$$\bar{M} = \frac{M+M-dm}{2}, \tag{6.26}$$

the average of the mass before and after the transition, the amplitude can be compactly rewritten as

$$A(i\to f) = icG\bar{M}\,\langle E|\,\hat{m}(0)\,|E_0\rangle\, e^{-\frac{n^2}{8N_g}} \int_{-\infty}^{+\infty} d\tau e^{i\Omega\tau}\frac{1}{r(\tau)}. \tag{6.27}$$

Making explicit the radial trajectory $r(\tau) = \frac{1}{a}\cosh{(a\tau)}$, the integral reads

$$\int_{-\infty}^{+\infty} d\tau e^{i\Omega\tau}\frac{1}{\cosh{(a\tau)}}. \tag{6.28}$$

To compute this, first of all notice that the following identities hold:

$$\operatorname{sech} x = (\cosh x)^{-1} = [\cos{(ix)}]^{-1} = \sec{(ix)}. \tag{6.29}$$

Then, the Mittag-Leffler expansion calculated in section B.2

$$\sec z = \sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)\pi}{(k+\frac{1}{2})^2\pi^2-z^2} \tag{6.30}$$

with the argument $z = ia\tau$ can be used to obtain

$$\operatorname{sech}(a\tau) = \frac{\pi}{a^2}\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)}{\tau^2+\frac{\pi^2}{4a^2}(2k+1)^2}. \tag{6.31}$$

Plugging this in the integral and swapping the sum and the integral signs results in

$$\int_{-\infty}^{+\infty} d\tau e^{i\Omega\tau}\frac{1}{\cosh(a\tau)} = \frac{\pi}{a^2}\sum_{k=0}^{\infty}(-1)^k(2k+1)\int_{-\infty}^{+\infty} d\tau f(\tau). \tag{6.32}$$

with $f(\tau) = \frac{\exp(i\Omega\tau)}{\left[\tau - i\frac{\pi}{2a}(2k+1)\right]\left[\tau + i\frac{\pi}{2a}(2k+1)\right]}$.

The above integral can be mapped into the complex plane by $\tau \to z, \quad z \in \mathbb{C}$, and becomes a contour integral. Since $\Omega > 0$, the contour Γ should be closed in the upper-half plane, as in figure 6.2. The function $f(z)$ goes asymptotically to zero, and thus the contribution of the contour at infinity vanishes thanks to the Jordan Lemma. The contribution on the real line is exactly the starting integral in τ. The chosen contour encapsules only half of the poles of the function, specifically $z_k = i\frac{\pi}{2a}(2k+1)$. By means of the Cauchy Theorem, only the residues of these poles contribute to the evaluation of the complex integral

$$\oint_{\Gamma} f(z) = 2\pi i \operatorname{Res}[f(z), z_k]. \tag{6.33}$$

These poles are simple poles, so the residues are evaluated as

$$\operatorname{Res}[f(z), z_k] = \lim_{z\to z_k}\frac{e^{i\Omega z}}{(z-z_k)(z+z_k)}(z-z_k) = \frac{2ae^{-\frac{\Omega\pi(2k+1)}{2a}}}{2\pi i(2k+1)}, \tag{6.34}$$

and the integral finally becomes

$$\int_{-\infty}^{+\infty} d\tau \frac{e^{i\Omega\tau}}{\left[\tau - i\frac{\pi}{2a}(2k+1)\right]\left[\tau + i\frac{\pi}{2a}(2k+1)\right]} = \frac{2ae^{-\frac{\Omega\pi(2k+1)}{2a}}}{2k+1}. \tag{6.35}$$

The amplitude then becomes

$$A(i\to f) = \frac{2\pi i c\sqrt{G}\bar{M}}{m_p}\langle E|\,\hat{m}(0)\,|E_0\rangle\, e^{-\frac{n^2}{8N_g}}\sum_{k=0}^{\infty}(-1)^k(e^{-\frac{\Omega\pi}{2a}})^{2k+1}. \tag{6.36}$$

Given the fact that $\frac{\Omega\pi}{2a} > 0$ everywhere, the exponential function, denoted as

$$x \equiv e^{-\frac{\Omega\pi}{2a}}, \tag{6.37}$$

always satisfies $x < 1$. The series can then be manipulated as follows

$$\sum_{k=0}^{\infty}(-1)^k x^{2k+1} = x\sum_{k=0}^{\infty}(-1)^k(x^2)^k = x\sum_{k=0}^{\infty}(-x^2)^k = \frac{x}{1+x^2}, \tag{6.38}$$

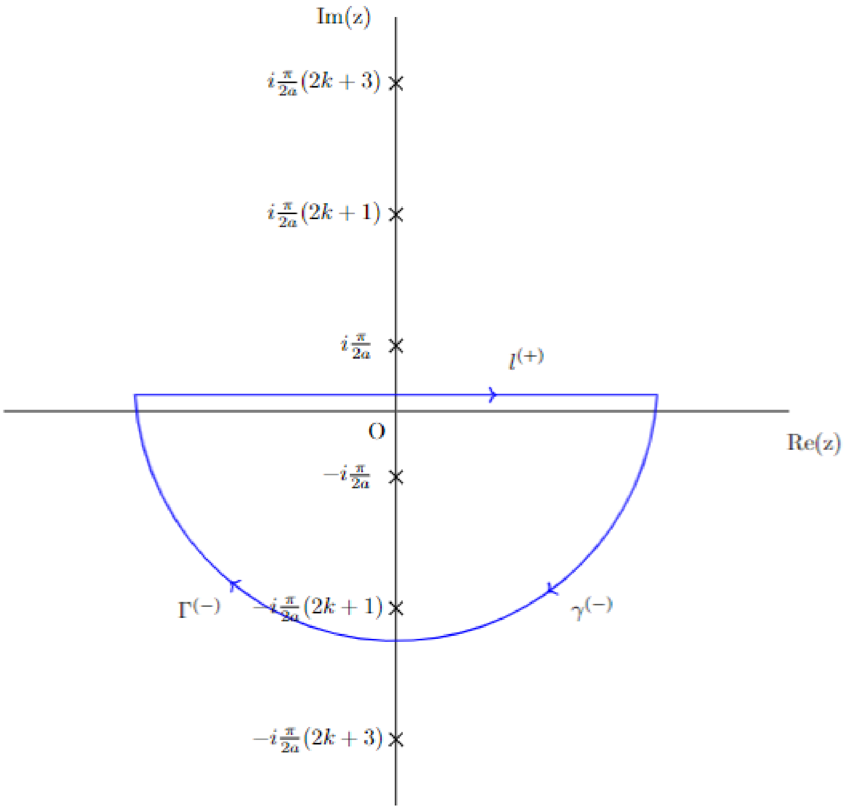

Figure 6.2: Integration path for the function $f(z)$.

leading to

$$A(i \to f) = 2\pi i c G \bar{M} \langle E| \hat{m}(0) |E_0\rangle \, e^{-\frac{n^2}{8N_g}} \frac{e^{-\frac{\Omega\pi}{2a}}}{1+e^{-\frac{\Omega\pi}{a}}}. \tag{6.39}$$

Making use of the definition for the $\cosh x$ function, the amplitude finally reads

$$A(i \to f) = \pi i c G \bar{M} e^{-\frac{n^2}{8N_g}} \langle E| \hat{m}(0) |E_0\rangle \frac{1}{\cosh\left(\frac{\Omega\pi}{2a}\right)}. \tag{6.40}$$

Dimensional analysis and the analogy with the Unruh-DeWitt case naturally introduce a temperature, defined as

$$T_N \equiv \frac{2\hbar a}{\pi k_B}, \tag{6.41}$$

which is four times bigger then the Unruh temperature defined for an accelerating observer perceiving the vacuum as a bath of particles of temperature (5.56). The probability density

$$P(i \to f) = \pi^2 G^2 \bar{M}^2 |\langle E| \hat{m}(0) |E_0\rangle|^2 e^{-\frac{n^2}{4N_g}} \operatorname{sech}^2\left(\frac{E-E_0}{k_B T_N}\right) \tag{6.42}$$

describes the distribution of particles contained in the coherent state as perceived by an accelerated detector.

6.2 Transition to all final states

Let us work out the probability for the system to make a transition to all final states. From the amplitude

$$A(i \to f) = -ic \int_{-\infty}^{+\infty} d\tau \, \langle E| \hat{m}(\tau) |E_0\rangle \langle \Psi| \sqrt{G}\hat{\Phi}(x(\tau)) |g\rangle\,, \tag{6.43}$$

the total probability is obtained by summing over all the possible states

$$P(i \to f) = c^2 \sum_E |\langle E| \hat{m}(0) |E_0\rangle|^2 F(\Omega), \tag{6.44}$$

where the Selectivity is again given by $\sum_E |\langle E| \hat{m}(0) |E_0\rangle|^2$, but now the Response function (5.61) is

$$F(\Omega) = G \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau' e^{-i\Omega(\tau-\tau')} \langle g| \hat{\Phi}(x)\hat{\Phi}(x') |g\rangle\,. \tag{6.45}$$

Notice that, considering the transition to any possible coherent state imposes to use the appropriate completeness relation which is now (2.29), bringing an extra π factor. Again,

the focus will be on the Response function, since the latter completely characterizes the particle content of the state, and it is independent from the specifics of the detector which is probing the gravitational field.
First of all, handling the bracket for the two point function by means of the usual spherically symmetric expansion of the field gives

$$\begin{aligned} G \langle g| \hat{\Phi}(x)\hat{\Phi}(x') |g\rangle = &\int \frac{dk}{2\pi^2} k^2 \sqrt{\frac{\hbar G}{2k}} 2 g_k \frac{\sin[kr(\tau)]}{kr(\tau)} \int \frac{dp}{2\pi^2} p^2 \sqrt{\frac{\hbar G}{2p}} 2 g_p \frac{\sin[pr(\tau')]}{pr(\tau')} \\ &+ G \int \frac{dp}{2\pi^2} \frac{\hbar}{2p} \frac{\sin[pr(\tau)]}{pr(\tau)} \frac{\sin[pr(\tau')]}{pr(\tau')} e^{-i\omega_p t(\tau) + i\omega_p t(\tau')}. \end{aligned} \tag{6.46}$$

The first term is related to the square of the Newton potential and represents the classical part of the above bracket, while the second term comes from $[\hat{a}_k, \hat{a}_p^\dagger]$, and is thus related to the vacuum of the field.
Plugging this result inside the Response function and switching the momentum integrals with the proper time integrals results in

$$\begin{aligned} F(\Omega) = &\int \frac{dk}{2\pi^2} k^2 \sqrt{\frac{\hbar G}{2k}} 2 g_k \int_{-\infty}^{+\infty} d\tau e^{-i\Omega\tau} \frac{\sin[kr(\tau)]}{kr(\tau)} \\ &\cdot \int \frac{dp}{2\pi^2} p^2 \sqrt{\frac{\hbar G}{2p}} 2 g_p \int_{-\infty}^{+\infty} d\tau' e^{+i\Omega\tau'} \frac{\sin[pr(\tau')]}{pr(\tau')} \\ &+ G \int \frac{dp}{2\pi^2} \frac{\hbar}{2p} \int_{-\infty}^{+\infty} d\tau e^{-i\Omega\tau} e^{-i\omega_p t(\tau)} \frac{\sin[pr(\tau)]}{pr(\tau)} \int_{-\infty}^{+\infty} d\tau' e^{i\Omega\tau'} e^{i\omega_p t(\tau')} \frac{\sin[pr(\tau')]}{pr(\tau')}. \end{aligned} \tag{6.47}$$

The first term can be rewritten in such a way that each momentum and proper time integral gives the same result performed for the $|g\rangle \to |g'\rangle$ transition, and thus will lead to the same function of $\frac{\Omega\pi}{2a}$ found in (6.42).
Focus on the last term: it contains a double copy of the same integral, up to minus signs in the exponents. Consider just one of them, for example

$$I^+ \equiv \int_{-\infty}^{+\infty} d\tau e^{-i\Omega\tau} e^{-i\omega_p t(\tau)} \frac{\sin[pr(\tau)]}{pr(\tau)} = \int_{-\infty}^{+\infty} d\tau e^{-i\Omega\tau} e^{-i\frac{p}{a}\sinh(a\tau)} \frac{\sin[\frac{p}{a}\cosh(a\tau)]}{\frac{p}{a}\cosh(a\tau)}. \tag{6.48}$$

To tame this integral, map $\tau \to z, \quad z \in \mathbb{C}$, and consider the complex integral on the integration contour closed by an infinite radius semicircle in the lower half of the complex plane $\Gamma = l^{(+)} \cup \gamma^{(-)}$. The contribution on $\gamma^{(-)}$ is zero because the function

$$f(z) = e^{-i\frac{p}{a}\sinh(az)} \frac{\sin[\frac{p}{a}\cosh(az)]}{\frac{p}{a}\cosh(az)} \tag{6.49}$$

absolutely converges when the radius of the semicircle goes to infinity. On the other hand, using the Cauchy Theorem, the integral on Γ is given by the sum of the residues inside the contour itself. However, as can be readily seen, the function has no pole inside Γ. The only possible candidates to be poles are the zeroes of $\cosh{(az)}$, which are displaced along the imaginary axis at $z = z_k = \frac{i\pi}{2a}(2k+1), \quad k \in \mathbb{Z}$. But it is straightforward to check that

$$\begin{aligned}
&\lim_{z\to z_k} e^{-i\Omega z - i\frac{p}{a}\sinh{(az)}} \frac{\sin{[\frac{p}{a}\cosh{(az)}]}}{\frac{p}{a}\cosh{(az)}} \\
&= e^{-i\Omega z_k - i\frac{p}{a}\sinh{(az_k)}} \lim_{z\to z_k} \frac{\sin{[\frac{p}{a}\cosh{(az)}]}}{\frac{p}{a}\cosh{(az)}} \\
&= e^{-i\Omega z_k - i\frac{p}{a}\sinh{(az_k)}} \cos(\cosh z_k) \\
&= e^{-i\Omega z_k - i\frac{p}{a}\sinh{(az_k)}},
\end{aligned} \tag{6.50}$$

where in the third line the De L'Hopital Theorem was used to make clear how every zero of $\cosh z$ is neutralized by the cosine function. The same result could be obtained by noticing that when approaching one of its zeroes, the sine function goes like its argument. Thus no pole means no residue, and no residue means a net zero result for the complex integral on Γ. As a consequence, also $I^{(+)}$ which is just the integral over $l^{(+)}$ is zero. The same result is obtained for the integral with the positive exponents. Therefore, the vacuum term brings no contribution to the Response function. It can be anticipated that this result is wrong. This problem will be analyzed in the next section. Focusing on the first term, the Response function is

$$F(\Omega) = \frac{\pi^2 G^2 M^2}{\cosh^2{(\frac{\Omega\pi}{2a})}}, \tag{6.51}$$

and the probability is

$$P(i\to f) = \pi^3 G^2 M^2 \sum_E |\langle E|\,\hat{m}(0)\,|E_0\rangle|^2 \operatorname{sech}^2\left(\frac{E-E_0}{k_B T_N}\right) \tag{6.52}$$

6.3 Analysis of the results

There are two main aspects which should be commented by looking at (6.52) before analysing the results. First of all, (6.52) is a transition probability in the coherent state case, while (6.2) is a probability rate in the vacuum case. The probability for the coherent state transition is constant and this means that evaluating the rate of the process gives a vanishing result. To understand this difference, let us go back to the vacuum probability calculation: in that case, a central role was played by the Wightman function. This function satisfies the covariant Klein-Gordon equation

$$\Box_x D^+(x,x') = 0, \tag{6.53}$$

with solution given by (5.70). It is useful now to understand why the Wightman function is special. The key aspect is given by its τ translation invariance. To understand from where this invariance comes from, the Killing vectors for the Minkowski metric should be discussed in more details.
Among all the possible trajectories in Minkowski spacetime, the most interesting ones for the discussion of vacuum transitions have constant acceleration and angular velocity. It turns out that these features can be identified with geometric invariants of the curves, such as the curvature and the torsion [35, 36]. Curves with τ independent invariants are called stationary worldlines, and an observer following them can naturally introduce a coordinate system in which the observers are at rest, and the transformed metric tensor is time independent. Therefore, the interval between two events will depend at most from the proper time interval between them.
The crucial point is that these worldlines are uniquely associated to the timelike Killing vector fields of the Minkowski metric. The Minkowski metric is, by definition, left invariant by transformations belonging to the Poincaré group (three boosts, three rotations, three space translations and time translation). To each symmetry of the metric tensor, a Killing vector is associated. Killing vectors can be timelike and spacelike, but not all of the timelike vectors are directly useful for being associated to a stationary worldline. However, since a combination of Killing fields is again a Killing field, it is possible to find six timelike Killing vectors to which families of stationary worldlines can be associated. Actually, each stationary worldline is an integral curve of a timelike Killing vector field, and each timelike Killing vector field is tangent to a stationary worldline. The proof of such statements can be found in [35].
To put this in concrete, think about the example extensively treated before, that is the uniformly accelerated particle: the only non-vanishing geometric invariant of its trajectory is the acceleration a, which is constant. Therefore the worldline is stationary. The equations of motion were found to be

$$\begin{cases} t(\tau) = \frac{1}{a}\sinh(a\tau) \\ x(\tau) = \frac{1}{a}\cosh(a\tau) \end{cases} . \tag{6.54}$$

The interval between two events $x^\mu(\tau)$ and $x^\mu(\tau')$ is

$$\begin{aligned} (x-x')_\mu(x-x')^\mu &= -(t)^2 - (t')^2 + 2tt' + x^2 + x'^2 - 2xx' = \\ &= \frac{2}{a^2}[1-\cosh(a\delta\tau)], \end{aligned} \tag{6.55}$$

and thus trivially proper time translation invariant, while the vector tangent to such a curve, the four velocity, is

$$u^\mu = (\cosh(a\tau), \sinh(a\tau), 0, 0). \tag{6.56}$$

The four velocity was already shown to be the Killing vector associated to boosts in the direction of the acceleration, as can be seen by rewriting it as

$$u^\mu = a(x, t, 0, 0). \tag{6.57}$$

Another, rather trivial, example is the one of a static particle: the equations of motion are just

$$\begin{cases} t(\tau) = \tau \\ x = 0 \end{cases}, \tag{6.58}$$

with the position fixed at the origin. The interval between two events depends only on $\Delta\tau$

$$\begin{aligned} (x - x')_\mu (x - x')^\mu &= -(t - t')^2 \\ &= -(\delta\tau)^2, \end{aligned} \tag{6.59}$$

and the tangent vector to its curve is the Killing vector associated to time translations

$$u^\mu = (1, 0, 0, 0). \tag{6.60}$$

The other four classes of stationary worldlines associated to Killing vectors involve rotations and combinations of radial motion and rotations.
Now, it is immediate to see that, up to an $i\epsilon$ factor with $\epsilon << 1$, the Wightman function (5.70) is the inverse of the invariant distance between two events. Therefore, if the Wightman function is evaluated on a stationary worldline, it will only depend on the proper time interval, that is, it will be proper time translation invariant. The above invariance is responsible for the factorization of the proper time integrals and thus for the constance of the transition probability rate. In this sense, it is explicit how the Wightman function is completely characterized by the detector motion.
Now, consider the term in (6.52) coming from $\langle g| \hat{\Phi}(x)\hat{\Phi}(x') |g\rangle$. All the calculations were performed using

$$\hat{a}_k |g\rangle = g_k e^{ikt} |g\rangle, \tag{6.61}$$

with the g_k defined by the equation

$$\sqrt{G} \langle g| \hat{\Phi}(x) |g\rangle \simeq V(r) \qquad r \gtrsim R_h. \tag{6.62}$$

The coherence of the state reproduces the classical configuration. This configuration, in a quadratic term such as the correlation function (6.46), gives the square of the Fourier transform of the field (plus a vacuum contribution), and this is peculiar of coherent states. The Fourier transform uses the proper time integral to bring the field in the dual space of energy and thus will always lead to τ independent terms.
The second aspect that should be noticed is that a temperature was introduced to compare the expression for the coherent state probability with the vacuum probability. This

temperature is not the Unruh temperature, and instead satisfies the relation $T_N = 4T_U$. Therefore, it seems that the emission of particles happens at a higher temperature if the initial state is a Newtonian configuration. To understand why, look again at the expression (5.77) for the Wightman function for an accelerated trajectory: thanks to the proper time translation symmetry, it was possible to arrange its expression in such a way that the Wightman function depends only on $\delta\tau/2$. The extra $1/2$ factor is crucial here, because it changes the position of poles inside the proper time integral and leads, after the evaluation of the residues, to an exponential factor of the kind $\exp(2\Omega\pi/a)$. As explained before, it is impossible to arrange the expression for the term in (6.52) coming from $\langle g|\,\hat{\Phi}(x)\hat{\Phi}(x')\,|g\rangle$ in such a way that it becomes only $\delta\tau$-dependent, and there is no such $1/2$ prefactor able to change the position of poles in the calculation. Therefore, the difference in temperature emissions arises again from the coherent configuration for the field represented by the g_k coefficients of $|g\rangle$.

6.3.1 Fixing the coherent state coefficients

In the evaluation of the probability rate from a coherent state to all final states, it was found that such quantity split into two pieces. From the Response function

$$F(\Omega) = \int d\tau d\tau' e^{-i\Omega(\tau-\tau')} G\,\langle g|\hat{\Phi}(x)\hat{\Phi}(x')|g\rangle\,, \tag{6.63}$$

and by expanding the field as in [8]

$$\hat{\Phi}(t,r) = \int \frac{dp}{2\pi^2} p^2 \sqrt{\frac{\hbar}{2p}} \left[\hat{a}_p e^{-i\omega_p t} \frac{\sin(pr)}{pr} + \hat{a}_p^\dagger e^{i\omega_p t} \frac{\sin(pr)}{pr} \right], \tag{6.64}$$

it was found that

$$\begin{aligned} F(\Omega) = \int d\tau d\tau' e^{-i\Omega(\tau-\tau')} \{ &\int \frac{dk}{2\pi^2} k^2 \sqrt{\frac{\hbar G}{2k}} 2g_k \frac{\sin[kr(\tau)]}{kr(\tau)} \int \frac{dp}{2\pi^2} p^2 \sqrt{\frac{\hbar}{2p}} 2g_p \frac{\sin[pr(\tau')]}{pr(\tau')} \\ &+ \int \frac{dp}{2\pi^2} \frac{\hbar G}{2p} \frac{\sin[pr(\tau)]}{pr(\tau)} \frac{\sin[pr(\tau')]}{pr(\tau')} e^{-i\omega_p t(\tau) + i\omega_p t(\tau')} \}. \end{aligned} \tag{6.65}$$

As previously commented, the first piece almost gives back the squared transition probability for the process $|g\rangle \to |g'\rangle$ with $|g'\rangle$ a less occupied coherent state. The second term was instead found to be zero.
As already said, the first term is the square of the Fourier transform for the Newtonian potential and does not depend on $\hbar$, hinting that it is some kind of classical contribution to the probability amplitude. The second term is instead a vacuum contribution, as

it arises from the commutation relations between ladder operators, and it is therefore completely quantum. Thus, the question is why such quantum term vanishes. Let us remember that the mode expansion followed the requirement of spherical symmetry, i.e. the field configuration under analysis, even at the classical level, was supposed to be invariant under rotations. This means that, in order to recover a radial configuration, the mode expansion was written using Bessel and harmonic functions, with the requirement that the only possible value for the angular momentum was $l = 0$. Thus, the modes were

$$u_p(t, r) = e^{-i\omega_p t} j_0(pr). \tag{6.66}$$

Now, imagine not to impose any kind of symmetry, and write the modes as plane waves

$$u_p(t, \vec{r}) = e^{-i\omega_p t + i\vec{p}\cdot\vec{r}}. \tag{6.67}$$

The vacuum contribution to the Response function would be

$$G \int \frac{d^3p}{(2\pi)^3} \frac{\hbar}{2p} \exp\left[-i\omega_p(t - t') + i\vec{p}\cdot(\vec{r} - \vec{r}')\right]. \tag{6.68}$$

This is the three dimensional integral expression of the positive frequency Wightman function (5.63), which is the crucial term giving rise to the Unruh effect in flat spacetime. Of course, this effect cannot be basis dependent, since (6.66) gives zero while (6.67) gives the Planckian spectrum. The mistake made was to impose the spherical symmetry at the classical level by keeping only the $j_0(kr)$ term. This procedure sounds innocent on a first sight, since the Newtonian configuration was correctly reproduced by the expectation value over the coherent state. But when polynomials of the field evaluated at different spacetime points are considered, like in correlation functions, all of the modes must be included in order to propagate field excitations, even those with $l \neq 0$. These polynomials, when acting on coherent states, usually give rise to vacuum contributions, which should not neglect basis elements that are solutions with arbitrarily large angular momentum. The spherical symmetry happens at the level of the coherent state $|g\rangle$, and it is on this state that one should impose the symmetry through the eigenvalues g_k.
The mathematical counterpart of this problem is that a plane wave, solution of the Klein-Gordon equation and entering in the mode expansion, cannot be reconstructed from the spherical wave $j_0(kr)$ alone. This is because plane waves are eigenfunctions of the three dimensional momentum vector and naturally point out a direction in momentum space. Spherical waves, instead, are eigenfunctions for the modulus squared of the momentum. Plane waves do not have a definite angular momentum, while spherical waves do not have a definite direction. Therefore, reconstructing a plane wave from a spherical one requires infinite angular momentum terms; in fact, given

$$e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} = e^{ipr\cos\theta}, \tag{6.69}$$

a linear combination of spherical waves can be written as

$$e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} = \sum_l c_l j_l(kr) P_l(\cos\theta). \tag{6.70}$$

The $P_l(\cos\theta)$ are the Legendre polynomials coming from the spherical harmonics which compose the angular part of the expansion (since a plane wave does not depend on the angle ϕ, the terms in m are absent). The coefficients c_l can be fixed through a normalization procedure. It is found that

$$e^{i\vec{k}\cdot\vec{r}} = \sum_l (2l+1) i^l j_l(kr) P_l(\cos\theta). \tag{6.71}$$

The crucial term in this expansion is the i^l, that makes the linear combination complex both in the spatial and in the temporal part, like the plane wave mode. Therefore, the vacuum contribution can be obtained only with a complete set of Klein-Gordon solutions, and this set cannot be complete without $l \neq 0$ terms. The spherical symmetry can be imposed at the coherent state level by a proper redefinition of the coefficients g_k.
The new expression could be found again by employing polar coordinates for plane waves

$$\hat{\Phi}(\vec{r},t) = \sum_l \int \frac{dp d(\cos\theta)}{(2\pi)^2} p^2 \sqrt{\frac{\hbar}{2p}} j_l(pr) P_l(\cos\theta)(2l+1)\left[\hat{a}_p e^{-i\omega_p t} i^l + \hat{a}_p^\dagger e^{i\omega_p t}(i^l)^*\right], \tag{6.72}$$

and the bracket over the coherent state, for the canonically normalized scalar field, should be set equal to the Fourier transform of the classical configuration of the field

$$\begin{aligned}&\sqrt{G}\sum_l \int \frac{dp d(\cos\theta) p^2}{(2\pi)^2}(2l+1)\sqrt{\frac{\hbar}{2p}}\left[g_{p,(l)} e^{-i\omega_p t} i^l + g^*_{p,(l)} e^{i\omega_p t}(i^l)^*\right] j_l(pr) P_l(\cos\theta)\\ &= \sum_l \int \frac{dp d(\cos\theta) p^2}{(2\pi)^2} i^l \tilde{V}_l(k) j_l(pr) P_l(\cos\theta)(2l+1).\end{aligned} \tag{6.73}$$

This brings to

$$\sum_l \sqrt{\frac{G\hbar}{2p}}[g_{p,(l)} e^{-i\omega_p t} i^l + g^*_{p,(l)} e^{i\omega_p t}(i^l)^*] = \sum_l i^l \tilde{V}_l(p). \tag{6.74}$$

The right hand side expansion of (6.73) is completely general. Let us impose the spherical symmetry at this level: the left hand side of the above equation can be rewritten with the help of

$$e^{-i\omega_p t} i^l = e^{-i(\omega_p t - l\frac{\pi}{2})}, \tag{6.75}$$

and by writing

$$g_{p,(l)} = g_p e^{i\gamma_{t,l}(p)}, \tag{6.76}$$

it becomes

$$\sum_l \sqrt{\frac{G\hbar}{2p}} 2g_p \cos\left[\gamma_{t,l}(p) - \omega_p t + l\frac{\pi}{2}\right] = \sum_l i^l \tilde{V}_l(p). \tag{6.77}$$

Imposing the spherical symmetry condition

$$\tilde{V}_l(p) = \tilde{V}(p), \tag{6.78}$$

that is, the only non vanishing term in the decomposition is the one with $l = 0$, brings to a left hand side that for consistency should be zero for $l \neq 0$. This can be achieved by

$$\gamma_{t,l}(p) = \begin{cases} \omega_p t & l = 0 \\ \omega_p t - (l+1)\frac{\pi}{2} & l \neq 0 \end{cases} . \tag{6.79}$$

The arbitrary choice of putting the angular dependence of the coefficients inside the phase is the same performed for the time dependence. In this sense, imposing a symmetry after the quantization makes the coherent state more complicated, but now all the solutions in the mode expansion are recovered. Furthermore, the spherical symmetry is now dictated by the quantum state $|g\rangle$, and this is explicit in the γ factor. By means of the Fourier transform of the classical solution

$$\tilde{V}(p) = -\frac{4\pi l_p \tilde{\rho}(k)}{m_p k^2}, \tag{6.80}$$

which is the same expression already employed for the previous g_k (alternatively, the choice of spherical symmetry could be shifted to the matter configuration $\tilde{\rho}(k)$), it is found that

$$g_{p,(l)} = -\frac{4\pi M}{\sqrt{2k^3} m_p} e^{i\gamma_{l,t}(k)}. \tag{6.81}$$

This choice assures that $g_p \neq 0 \quad \forall l \in \mathbb{N}$, but also that the cosine assumes the value of 1 for $l = 0$ and 0 for the other values of l. This can be checked using the mode expansion in spherical coordinates

$$
\begin{aligned}
\sqrt{G}\,\langle g|\hat{\Phi}(t,\vec{r})|g\rangle &= \sqrt{G}\sum_{l}\int\frac{dkd\cos\theta}{(2\pi)^2}k^2\sqrt{\frac{\hbar}{2k}}j_l(kr)P_l(\cos\theta)\\
&\cdot\left[\langle g|\hat{a}_k|g\rangle\, c_l e^{-i\omega_k t}+\langle g|\hat{a}_k^\dagger|g\rangle\, c_l^* e^{i\omega_k t}\right]\\
&= \sqrt{G}\int\frac{dkd\cos\theta}{(2\pi)^2}k^2\sqrt{\frac{\hbar}{2k}}2g_k j_0(kr)P_0(\cos\theta)\cos\left[\gamma_{t,0}-\omega_k t\right]\\
&+\sqrt{G}\sum_{l\geq 1}\int\frac{dkd\cos\theta}{(2\pi)^2}k^2\sqrt{\frac{\hbar}{2k}}2g_k j_l(kr)P_l(\cos\theta)\cos\left[\gamma_{t,l}-\omega_k t+l\frac{\pi}{2}\right]\\
&= \sqrt{G}\int\frac{dk}{2\pi^2}\sqrt{\frac{\hbar}{2k}}g_k j_0(kr)\simeq V_q(r).
\end{aligned}
\tag{6.82}
$$

Now, starting from (6.67), the correlation function for the coherent state reads

$$
\begin{aligned}
G\,\langle g|\hat{\Phi}(x)\hat{\Phi}(x')|g\rangle &= \sum_{l,l'}\int\frac{dpd(\cos\theta)}{(2\pi)^2}p^2\sqrt{\frac{\hbar G}{2p}}j_l(pr)P_l(\cos\theta)(2l+1)\\
&\cdot\int\frac{dkd(\cos\theta')}{(2\pi)^2}k^2\sqrt{\frac{\hbar G}{2k}}j_{l'}(kr')P_{l'}(\cos\theta')(2l'+1)\\
&\cdot\left[\langle g|\hat{a}_p\hat{a}_k|g\rangle\, e^{-i\omega_p t}i^{l+l'}+\langle g|\hat{a}_p^\dagger\hat{a}_k^\dagger|g\rangle\, e^{i\omega_p t}(i^{l+l'})^*\right.\\
&\left.+\langle g|\hat{a}_p^\dagger\hat{a}_k|g\rangle\, i^l(i^{l'})^*+\langle g|\hat{a}_k^\dagger\hat{a}_p|g\rangle\, i^{l'}(i^l)^*\right]\\
&+G\int\frac{d^3p}{(2\pi)^3}\frac{\hbar}{2p}\exp\left[-i\omega_p(t-t')+i\vec{p}\cdot(\vec{r}-\vec{r}')\right].
\end{aligned}
\tag{6.83}
$$

For $l=0$ it is immediate to notice that the first term gives back the one previously analysed, that is, the squared Fourier transform of the potential. This is because $P_0(\cos\theta)=1$, and the only Bessel function is $j_0(pr)$. For $l\geq 1$, the $\gamma_{t,l}(p)$ term makes the brackets of $\langle g|\hat{a}_p\hat{a}_k|g\rangle$ and $\langle g|\hat{a}_p^\dagger\hat{a}_k^\dagger|g\rangle$ gain an extra -1 factor, while the terms $\langle g|\hat{a}_p^\dagger\hat{a}_k|g\rangle$ and $\langle g|\hat{a}_k^\dagger\hat{a}_p|g\rangle$ do not have any extra phase. Therefore the coherent part of the two point function for $l\neq 0$ is identically zero.
It is immediate to check that, even now, for $r=const.$ the transition probability to all final states is τ-independent

$$
\begin{aligned}
G\,\langle g|\,\hat{\Phi}(x)\hat{\Phi}(x')\,|g\rangle &= \int\frac{dk}{2\pi^2}k^2\sqrt{\frac{\hbar G}{2k}}2g_k\frac{\sin(kr)}{kr}\int\frac{dp}{2\pi^2}p^2\sqrt{\frac{\hbar G}{2p}}2g_p\frac{\sin(pr)}{pr}\\
&+G\int\frac{d^3p}{(2\pi)^3}\frac{\hbar}{2p}\exp\left[-i\omega_p(t-t')+i\vec{p}\cdot(\vec{r}-\vec{r}')\right].
\end{aligned}
\tag{6.84}
$$

The second term vanishes as vacuum terms with inertial trajectories give a net zero contribution to the probability amplitude. Notice that, had the g_k coefficients been time dependent, the first term, and therefore the transition, would have been non zero. This shows how the static nature of the Newtonian potential is reflected into the proper time independence for the correlation function for inertial trajectories, as expected from a non relativistic eternal field configuration. Therefore, the vanishing of the term $P_{coherent}$ for inertial motions is peculiar of configurations with no explicit time dependence.
For radially accelerated trajectory, it is then found

$$\begin{aligned} P(i \to f) &= P_{coherent} + P_{vacuum} \\ &= \pi c^2 \sum_E | \langle E | \hat{m}(0) | E_0 \rangle |^2 [\pi^2 G^2 M^2 \operatorname{sech}^2 \left(\frac{E - E_0}{k_B T_N} \right) \\ &+ \frac{G}{2\pi} \int_{-\infty}^{+\infty} d\tau \frac{\Omega}{\exp\left[\frac{E-E_0}{k_B T_U}\right] - 1}]. \end{aligned} \tag{6.85}$$

6.3.2 Further comments

Let us evaluate the so called classical limit $\hbar \to 0$. The Planckian term comes from the vacuum contribution, and is expected to go to zero in the classical limit, as there is no classical counterpart for the vacuum state. This limit can be performed by making explicit any $\hbar$ dependence in the probability. Keeping $E - E_0$ fixed, leads to

$$\lim_{\hbar \to 0} \frac{E - E_0}{\hbar \left(\exp\left[\frac{2\pi(E-E_0)}{\hbar a}\right] - 1 \right)} = 0. \tag{6.86}$$

The $P_{coherent}$ instead gives

$$\lim_{\hbar \to 0} \operatorname{sech}^2 \left[\frac{\pi(E - E_0)}{2\hbar a} \right] = \infty, \tag{6.87}$$

which blows up. This result could point out how this term can be seen as "super" classical. Of course, the divergence is not physical and if such contribution to the probability is regarded as classical, the relation $\Omega = (E - E_0)/\hbar$ is not consistent, as it is a purely quantum relation and the frequency of a classical system is not related to the energy through $\hbar$. Supposing that, the first term can be regarded as $\hbar$ independent and gives a finite, classical, result. This conclusion is further strengthened by the proportionality factor in front of the hyperbolic secant function. Since

$$P_{coherent} \propto | \langle E | \hat{m}(0) | E_0 \rangle |^2 G^2 M^2, \tag{6.88}$$

such proportionality resembles a field strength squared contribution to energy transitions, like the ones happening in classical electrodynamics for accelerated charges. In what

follows, Ω is still regarded as a function of $\hbar$ in order to compare the two probabilities. Another interesting limit to test is $M \to 0$; in this regime, the coherent contribution vanishes while the vacuum one remains unaffected. The usual Unruh effect found in chapter 5 is recovered. It is worth to remind, however, how both the $\hbar$ and M limits are formal, since those dimensionful quantities are fixed constants. The two regions described above could be thought as respectively coming from the limits of $N_g \to \infty$ (as explained in chapter 4, where quantum fluctuations represented by $\Delta\hat{\Phi}$ vanished for $\hbar \to 0$ and $N_g \to \infty$) and $N_g \to 0$ (as explicited by (3.42)).
The two terms $P_{coherent}$ and P_{vacuum} allow for a non vanishing probability both in the classical and quantum regimes. Furthermore, the two terms can be compared: the vacuum contribution has the usual diverging integral in proper time. To avoid considering the transition probability rate, let us switch on the detector at a time $\tau = -T$ and switch it off at $\tau = +T$. This leads to

$$P_{vacuum} = c^2 GT \frac{\Omega}{\exp\left[\frac{E-E_0}{k_B T_U}\right] - 1}. \tag{6.89}$$

It is worth to remark that the switching procedure is not totally innocent, as discussed in [32]. Let us call the particle distributions appearing in the probabilities $f(x)$, with $x = (E - E_0)/k_B T$. These two functions can be plotted like in figure 6.3. It is worth to see what is their behaviour in two different limits, i.e. for small and large x.
For $x \to \infty$, the two profiles go to zero. They actually become indistinguishable even at finite, albeit big, values of the ratio. Such ratio can become large as $E - E_0$ grows or as T shrinks, that is for a large quantity of energy emitted or for small accelerations.The drop of the probability is almost the same because both the transitions are triggered by the acceleration of the detector. Instead, the suppression for highly energetic particles in the coherent state case confirms how the field prefers to emit low energy quanta. For $x \to 0$, the Planckian profile blows up. In the usual derivation of the Hawking spectrum, this infrared divergence is a consequence of having neglected the backscattering of the gravitational potential, and it is customary to deal with it by including grey body factors. The coherent state profile instead does not suffer of this behaviour and shows a smooth and normalized profile near the origin. There exists also a point $x_{int} = 0.70$ at which one spectrum overcomes the other; such a value can be found to be $f(x_{int}) \sim 0.97$.

Finally, it is interesting to understand how the cut-offs in the coherent state could modify the emission spectrum. The cut-off k_{UV} was neglected in (6.25) to simplify computations and the procedure was justified because k_{UV} is not necessary to regularize the expectation value which reproduces the Newtonian potential for finite sources, and most importantly because the classical configuration can be recovered with arbitrary accuracy. However, even if such cut-off adds a multiplicative factor in the proper time integral which depends on the acceleration of the detector, the positions of the poles in the integrand do not change by this modification. This is because the factor can be approximately seen as a function of $k_{UV} r(\tau)$ which, since both k_{UV} and $r(\tau)$ are strictly

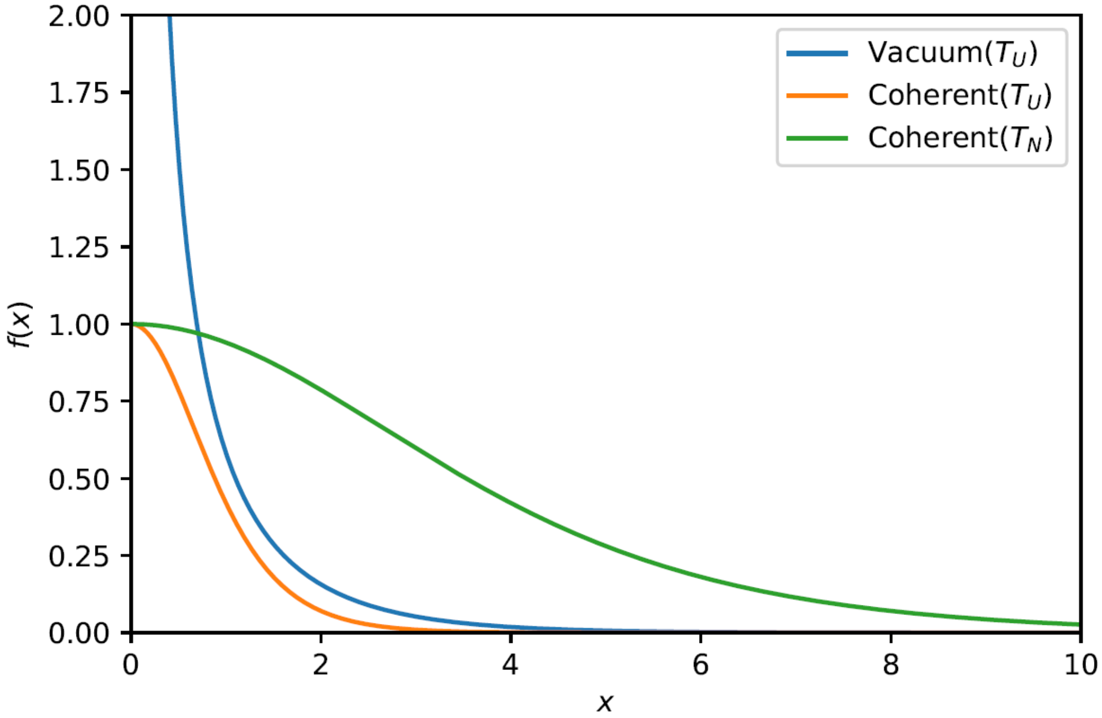

Figure 6.3: The blue line represents the vacuum distribution (Unruh thermal spectrum). The green line is the coherent distribution (Newton thermal spectrum). In orange, the Newtonian spectrum if the temperature of emission were the Unruh one.

positive, oscillates around the value of $\pi/2$. Such a function will make the integrand oscillate around the value $e^{i\Omega\tau}/\cosh(a\tau)$, but will not add any further pole or modify the position of the poles of $1/\cosh(a\tau)$. This means that the residues remain unchanged, and the integrand will be evaluated on the same values of $z_k = \frac{i\pi}{2a}(2k+1)$, leading to the same exponential factor which depends on $\Omega\pi/2a$.
Now, even if the evaluation of the series in (6.36) will be a difficult task due to the presence of an extra function of a, k and k_{UV} (and this will be reflected in a different profile for the emission spectrum), the emission temperature will not be affected, as it is extracted only by the exponential factor $e^{i\Omega\tau}$, which is the only one involving a quantity with the right dimensions of a temperature. Therefore, the quantity T_N is independent of the UV behaviour of the model, i.e. it is independent of the inclusion of a cut-off on the high frequency modes, as could be expected from a classical quantity. This suggests that the temperature T_N depends on the coherence of the state (and, of course, from the chosen accelerated trajectory), rather than from the specific configuration reproduced. This conclusion is further supporter by the fact that T_U comes from the τ translation invariance of the Wightman function that, in general, is not shared by other correlation functions. At the same time, it is always true that a coherent state correlation function splits in two terms, one coming from a vacuum contribution and the other one coming from the squared Fourier transform of the reproduced configuration, and the latter is the responsible for particle emission at temperature T_N.

6.4 Is it possible to recover the Hawking spectrum?

Having analyzed the properties of (6.85), it could be asked what kind of phenomenon is described by the former setup, and if it is related with the Hawking radiation. The extra term coming from the coherent nature of the quantum state forbids a direct interpretation of the emission spectrum as a thermal one. As already said, the latter is still present as a vacuum contribution, but it only appears dominant when quantum effects are stronger, i.e. for N_g small. It is tempting to look at this regime of highly accelerated detectors and plug by hand $a = (4GM)^{-1}$ to recover the Hawking spectrum, but the point is that such procedure is not consistent with the discussions made over the coherent state: the mass M comes from the quantum state and cannot be enforced in the kinematical features of the detector. The acceleration and the field are still not connected at the level of the perturbative coupling. The gravitational field, once treated perturbatively and dynamically as a quantum field, does not modify the trajectories in an Unruh-DeWitt detector model. Therefore, what is described in this computation is the "Newtonian version" of the Unruh effect, where the coherent state term points out that there is a source of gravity, and the particle emission spectrum is modified by it. This calculation could also be extended to the evaluation of the Unruh effect in presence of other fields such as the Coulomb one, or generalized to arbitrary coherent states. A physical situation

where this effect could be measured is the one of a detector in a rocketship accelerating away from a gravitational source, in a region of spacetime where General Relativity is well approximated by the Newtonian theory. In this sense, the acceleration impressed by the engines of the rocketship is somewhat combined with the one impressed by the Newtonian potential, and this brings to an "hyperacceleration" that modifies the overall emission temperature. Notice, however, that a measuring procedure becomes difficult not only because the acceleration needed to emit particles is very high, but also because the gravitational interaction is weaker then the other interactions. In this sense, an analogous experiment involving electric fields could be easier to perform.
In light of the above analysis it could be asked if a quantum perturbative approach to the gravitational field could reproduce the Hawking radiation. This would require that $r = const.$ should be recognized as a non inertial trajectory leading to the same kind of effect of a uniformly accelerating observer in flat spacetime, as suggested by the Equivalence Principle. In this sense, an operator depending from the gravitational field should act on the coherent state and enforce the right trajectory to a "test field". This test field could subsequently be interpreted as a field for the emitted gravitons. This interpretation neglects the mechanism of emission, but it agrees with the idea given by the corpuscular picture that the Hawking radiation comes from gravitons escaping the black hole leaky bound state. The former idea would be implemented by the following interaction

$$H_{int} = c\sqrt{G}\hat{O}[\Phi]\hat{m}(\tau)\Psi(x) \tag{6.90}$$

with $\hat{\Psi}$ the test field, $\hat{m}(\tau)$ again a detector associated operator, and $\hat{O}[\Phi]$ the operator which, acting on $|g\rangle$, should realize the change of trajectory from Minkowski to Schwarzschild spacetime.
This approach is hard to apply, as the operator $\hat{O}[\Phi]$ should change the kinematical features of a given spacetime, i.e. it should map Minkowski geodesics to Schwarzschild geodesics. But Schwarzschild geodesics are not "small" perturbations of Minkowski geodesics. This difficulty is explicit when, for example, one notes that the operator $\hat{O}[\Phi]$ should map the Minkowskian coordinate r to the tortoise coordinate r^*. Despite the difficulty to define in a strict manner this operator, the meaning of r^* would be of a function of r and M, and not of a radial distance. Furthermore, the normal modes of a field in a Schwarzschild spacetime are not small perturbations of the normal modes in Minkowski spacetime. In an Unruh-DeWitt model, the impossibility of changing from a Minkowski to a Schwarzschild spacetime in a perturbative way is reflected in the impossibility of linking perturbatively the coordinates of two different spacetimes.
In this sense, an Unruh-DeWitt detector model always distinguishes three main aspects: the detector, the field and the spacetime, that is encoded in the trajectory fixed by hand. What is perturbative is the coupling between the detector (but not its motion) and the emitting field. The motion is pre-determined by the equations governing its dynamics that, in the case where the detector center of mass is not affected by the quantum field,

should be known from the beginning. Even if the field acts as a non perturbative background, like it would be for an electron in an electric field, the trajectory would not give back the known result. This is because only in a curved spacetime a fixed position can be seen as a non geodesic motion.
The non perturbative regime of gravity can be expressed by the coordinates themselves, in the sense that coordinates adapted to strong gravitational regimes cannot be consistently introduced in a flat spacetime formalism. Therefore, if a curved spacetime trajectory should be simulated in a Minkowskian context, the only possible way is to work with accelerated trajectories as already seen in chapter 5.
Thus, it seems that the only possibility for the Unruh-DeWitt detector to enforce the $r = const.$ trajectory in a coherent state spacetime is to fix the background geometry through the coherent state itself, and then to evaluate the transition amplitude, as in Appendix C. This implementation follows what was presented in chapter 3, where the metric was reconstructed through a mean field approach with the coherent state. But the reconstructed metric function suffers of the ambiguity of which kind of coordinates are used: this fact was hidden by the mean field method where the Schwarzschild metric was recovered by

$$\sqrt{G}\,\langle g|\hat{\Phi}(t,r)|g\rangle \simeq V_q(r) \qquad r \gtrsim R_h. \tag{6.91}$$

This definition was given in a flat spacetime context, with t the Minkowski time and r the harmonic Minkowski radius. However, in reconstructing the metric tensor

$$ds^2 = -\left(1+2V_q\right)dt^2 + \left(1+2V_q\right)^{-1}dr^2 + r^2 d\Omega^2, \tag{6.92}$$

and r is the areal radius in Schwarzschild spacetime. The areal radius in the Schwarzschild manifold is defined by

$$r_s^{(A)} = \sqrt{\frac{A}{4\pi}}, \tag{6.93}$$

with the subscript s standing for "Schwarzschild", and A the area of the two dimensional surface defined by $t, r = const.$ as in chapter 3. The areal radius is the spacelike coordinate that makes the spherical symmetry explicit in the metric tensor. This means that it has the same form also in Minkowski spacetime. However, in the latter manifold, the link between the harmonic and the areal radius is

$$r_m^{(H)} = r_m^{(A)}, \tag{6.94}$$

with m standing for "Minkowski", while in the former one it is [39]

$$r_s^{(H)} = r_s^{(A)} - GM. \tag{6.95}$$

This shows that linking $r_m^{(H)}$ and $r_s^{(H)}$ requires to switch off the mass, and this operation is clearly non perturbative, or, put it in another way, a source cannot be built perturbatively by smoothly adding small pieces of mass to the vacuum. Nevertheless, a mean field method brings quantum corrections to the classical metric function, and such corrections enter the event horizon expression (and thus in the surface gravity). This means that coherent states reproducing a quantum corrected metric function could modify the Hawking emission temperature.

Conclusions and outlooks

The aim of this work was to test the corpuscular theory described by the coherent state model with the help of a detector. The corpuscular picture [1] briefly reviewed in chapter 1 describes black holes as leaky bound states in an effective field theory that gives back the Einstein geometrical theory as emerging. Since this picture is rather qualitative, and neglects the role of matter in the condensate formation, coherent states were employed in chapter 3, in order to look at the quantum corpuscular corrections to the Newtonian potential [6] (and thus to the Schwarzschild metric function [8]). The Newtonian potential comes from a semiclassical approach where gravity is described by a coherent state, and the source is treated in a classical manner.
Furthermore, it was discussed how the properties of a quantum field can only be measured with a detector [9], and the presence of such a device should be explicit in the quantum description of the system [10]. As a warm up, the detector formalism was employed in chapter 4 in order to fix the UV behaviour of the uncertainties of a quantum field. The UV illness for the variances was addressed with the non existence of a detector that can measure every possible frequency. This resulted in the scaling law for the ratio $(\Delta\hat{\Phi})^2_{\text{State}}/\langle\hat{\Phi}\rangle^2_{\text{State}} \sim \hbar/N_g$, which is consistent with the corpuscular picture. The detector parameters entered the variance to regularize vacuum contributions, but also entered the mean value expression thanks to the source profile. The main difference with respect to the Bohr-Rosenfeld method [9] is the presence of a competition between the parameters of the detector and the source.
Then, the quantum corpuscular (and coherent) nature of the Newtonian potential was tested by means of an Unruh-DeWitt detector; this model was presented in chapter 5. This approach to the radiation process was consistent with the concept that a measurement can be carried out only if the field is coupled to a detector, and a particle seen as a "clic" in a measuring device. In chapter 6, a detector was considered with an inertial and a non inertial trajectory in the presence of a coherent state: first of all, because deviations were expected with respect to the usual Unruh effect; secondarily, because we wanted to see if there was a way to recover the Hawking radiation seen as an emission of gravitons. In order to deal with polynomials of the field that give rise to vacuum contributions, the description of the coherent state had to be refined, since the spherical symmetry should be enforced at the quantum level, i.e. by the quantum state itself that

now depends on the quantum number for the angular momentum.
The presence of a quantum state different from the vacuum splits the spontaneous emission probability in two pieces, $P_{coherent}$ and P_{vacuum}. The vacuum term is the same of the Unruh effect, and it is dominant in a quantum regime where $\hbar$ is no more negligible (or, equivalently, where N_g is small), while the first term depends on the initial state. For a coherent state, $P_{coherent}$ dominates the classical limit and a temperature $T_N = 4T_U$ appears. Such a temperature does not change if the cut-off, i.e. the corpuscular correction to the Newton potential necessary to normalize the coherent state, is added. It is therefore believed that this temperature depends only on the coherent nature of the quantum state. Since $P_{coherent}$ is the square of the Fourier transform of the classical field configuration, it is naturally a constant quantity in the proper time. This conclusion seems to be independent of the configuration reproduced by the coherent state. The static nature of the reproduced configuration is instead reflected in the absence of particle emission for inertial trajectories, as $P_{coherent} = 0$ thanks to the independence from the t coordinate of the squared Fourier transform of the potential.
Finally, it was debated how a quantum approach to the gravitational field, as intended in chapter 6, could not reproduce the Hawking spectrum. The non perturbative nature of gravity that modifies the detector motion forbids to map perturbatively Minkowski trajectories to Schwarzschild trajectories. This behaviour was reflected in the non perturbative link given by the source parameter M between coordinates belonging to different spacetimes, and also by the independence of the hyperbolic trajectory of the detector from the source. The only consistent way to recover the Hawking radiation from a coherent state with an Unruh-DeWitt detector model is to rely on a non perturbative mean field approach for the metric tensor, as in chapter 3, and then evaluate the perturbative amplitude for an emission process. This could bring significant deviations from the classical emission temperature.

Appendix A

Proof of the Theorem 2.1

Let us consider two observables $\hat{A}$ and $\hat{B}$ and a normalized vector $|\psi\rangle$. From these, define two new operators $\hat{A}'$ and $\hat{B}'$ such that

$$\begin{cases} \hat{A}' = \hat{A} - \langle \hat{A} \rangle_\psi \\ \hat{B}' = \hat{B} - \langle \hat{B} \rangle_\psi \end{cases} \tag{A.1}$$

Then, using the Schwarz inequality, it holds that

$$\langle \hat{A}'^2 \rangle_\psi \langle \hat{B}'^2 \rangle_\psi \geq | \langle \hat{A}' \hat{B}' \rangle_\psi |^2, \tag{A.2}$$

and given that

$$| \langle \hat{A}' \hat{B}' \rangle_\psi | \geq |\mathcal{I}(\langle \hat{A}' \hat{B}' \rangle_\psi)|, \tag{A.3}$$

by using the definition for the imaginary part of a complex number, together with the hermitianity of the operators, the Uncertainty inequality is found to be

$$\langle \hat{A}' \rangle_\psi^2 \langle \hat{B}' \rangle_\psi^2 \geq \frac{1}{4} | \langle [\hat{A}', \hat{B}'] \rangle_\psi |^2. \tag{A.4}$$

Since

$$\langle \hat{A}'^2 \rangle_\psi = (\psi \hat{A}', \hat{A}' \psi) = (\Delta \hat{A}_\psi)^2, \tag{A.5}$$

and

$$\langle [\hat{A}', \hat{B}'] \rangle_\psi = \langle [\hat{A}, \hat{B}] \rangle_\psi \,, \tag{A.6}$$

the Uncertainty Principle then becomes

$$(\Delta \hat{A}_\psi)^2 (\Delta \hat{B}_\psi)^2 \geq \frac{1}{4} | \langle [\hat{A}, \hat{B}] \rangle_\psi |^2. \tag{A.7}$$

To get the equality for (A.7), the Schwarz inequality (A.2) should be saturated, as much as (A.3). This can trivially happen if

$$\hat{A}' |\psi\rangle = 0, \tag{A.8}$$

or if

$$\hat{B}' |\psi\rangle = 0, \tag{A.9}$$

that is, if ψ is an eigenvector of the operator $\hat{A}$ or $\hat{B}$.
However, a non trivial saturation can happen if the following relation is supposed

$$\hat{A}' |\psi\rangle = c\hat{B}' |\psi\rangle , \tag{A.10}$$

with $c \in \mathbb{C}$. Then, (A.2) is saturated since the action of $\hat{A}'$ is the same of $\hat{B}'$ up to the c constant, while (A.3) is saturated if the action of $\hat{A}'\hat{B}'$ on $|\psi\rangle$ leaves only a purely imaginary quantity, that is, if $c = i\gamma$ with γ in $\mathbb{R}$. Therefore the following equality holds

$$\hat{A}' |\psi\rangle = i\gamma\hat{B}' |\psi\rangle , \tag{A.11}$$

or, by means of the definition for the primed operators,

$$(\hat{A} - i\gamma\hat{B}) |\psi\rangle = (\langle\hat{A}\rangle_\psi - i\gamma \langle\hat{B}\rangle_\psi) |\psi\rangle \equiv \lambda |\psi\rangle . \tag{A.12}$$

Thus the Uncertainty Principle is saturated if the vector $|\psi\rangle$ is an eigenstate of the operator $\hat{A} - i\gamma\hat{B}$ with γ real.
Conversely, suppose that $|\psi\rangle$ is an eigenstate of the operator $\hat{A} - i\gamma\hat{B}$ with eigenvalue $\lambda = a + ib$ with $a, b \in \mathbb{R}$. Then the following equalities hold

$$\begin{aligned} \lambda|\psi|^2 &= \langle\psi| \lambda |\psi\rangle \\ &= \langle\psi| (\hat{A} - i\gamma\hat{B}) |\psi\rangle. \\ &= \langle\hat{A}\rangle_\psi - i\gamma \langle\hat{B}\rangle_\psi \end{aligned} \tag{A.13}$$

Equating the real and imaginary parts of this expression leads to

$$\begin{cases} \langle\hat{A}\rangle_\psi = a \\ \langle\hat{B}\rangle_\psi = b, \end{cases} \tag{A.14}$$

and thus to

$$(\hat{A} - \langle\hat{A}\rangle_\psi) |\psi\rangle = i\gamma(\hat{B} - \langle\hat{B}\rangle_\psi) |\psi\rangle . \tag{A.15}$$

which is again the necessary and sufficient condition (A.11) that saturates the Uncertainty relation.

Appendix B

Mittag-Leffler calculations

B.1 Cosecant squared function

Consider the cosecant squared function defined by

$$\csc^2 \pi z = \frac{1}{\sin^2(\pi z)}. \tag{B.1}$$

This function has an infinite and countable number of poles defined by

$$z_k = k \qquad k \in \mathbb{Z}, \tag{B.2}$$

and thus it can be expanded in a Mittag-Leffler way. Each pole is a pole of order two, as can be seen by the computation of the principal part of the Laurent series of the function around the $k-$th pole:

$$\begin{aligned}
\frac{1}{\sin^2(\pi z)} &= \frac{1}{1-\cos^2(\pi z)} \\
&= \frac{1}{1-\left(1-\frac{(\pi z)^2}{2}+\frac{(\pi z)^4}{4!}+\dots\right)^2} \\
&= \frac{1}{1-\left(1-(\pi z)^2+\frac{1}{3}(\pi z)^4+\dots\right)} \\
&= \frac{1}{(\pi z)^2}\frac{1}{1-\frac{1}{3}(\pi z)^2+\dots} \\
&= \frac{z^{-2}}{\pi^2}\sum_{k=0}^{\infty}\left(\frac{(\pi z)^2}{3}\right)^k \\
&= \frac{z^{-2}}{\pi^2}+1+\frac{\pi^2}{3}z^2+\dots
\end{aligned} \tag{B.3}$$

By this expression, it is immediate that the Laurent coefficients are

$$a_k = (\pi)^{-2}. \tag{B.4}$$

Thus the cosecant squared function can be written as

$$\csc^2(\pi z) = h(z) + \frac{1}{\pi^2}\sum_{k\in\mathbb{Z}} \frac{1}{(z-k)^2}. \tag{B.5}$$

To evaluate the entire function $h(z)$, let us look at the asymptotic behaviour of $\csc^2 z$. Since this function has an infinite amount of poles, its limit to infinity should be taken by dodging them. This can be done if the cosecant squared is evaluated on the sequence

$$a_j = \left\{ j + \frac{1}{2} \right\}_{j=0}^{\infty}, \tag{B.6}$$

which obviously goes to infinity as $j \to \infty$. Then evaluating

$$\lim_{j\to\infty} \frac{1}{\sin^2(\pi a_j)} = 1 \tag{B.7}$$

tells that the cosecant squared does not diverge at infinity and therefore also the entire function $h(z)$. But an entire function which is everywhere bounded is a constant, as stated by the First Liouville Theorem. Given this fact, the cosecant squared can be rewritten as

$$\csc^2(\pi z) = h_0 + \frac{1}{\pi^2}\sum_{k\in\mathbb{Z}} \frac{1}{(z-k)^2}. \tag{B.8}$$

To find the value of h_0, evaluate the above functions on a convenient point, such as $z = \frac{1}{2}$:

$$h_0 = 1 - \frac{4}{\pi^2}\sum_{k\in\mathbb{Z}} \frac{1}{(1-2k)^2}. \tag{B.9}$$

The next step is to evaluate the sum of the above series. This can be done with the help of another function

$$F(w) = \frac{\pi\cos(\pi w)}{\sin(\pi w)}\frac{1}{(1-2w)^2}. \tag{B.10}$$

$F(w)$ shows a double pole in $w = \frac{1}{2}$ and an infinite amount of simple poles at $w_k = k, \quad k \in \mathbb{Z}$. By means of the Cauchy Theorem, the task is then to evaluate

$$\frac{1}{2\pi i}\oint_{|w|=r_n} F(w)dw = \text{Res}\left[F(w), \frac{1}{2}\right] + \sum_{n=-\infty}^{+\infty} \text{Res}[F(w), k], \tag{B.11}$$

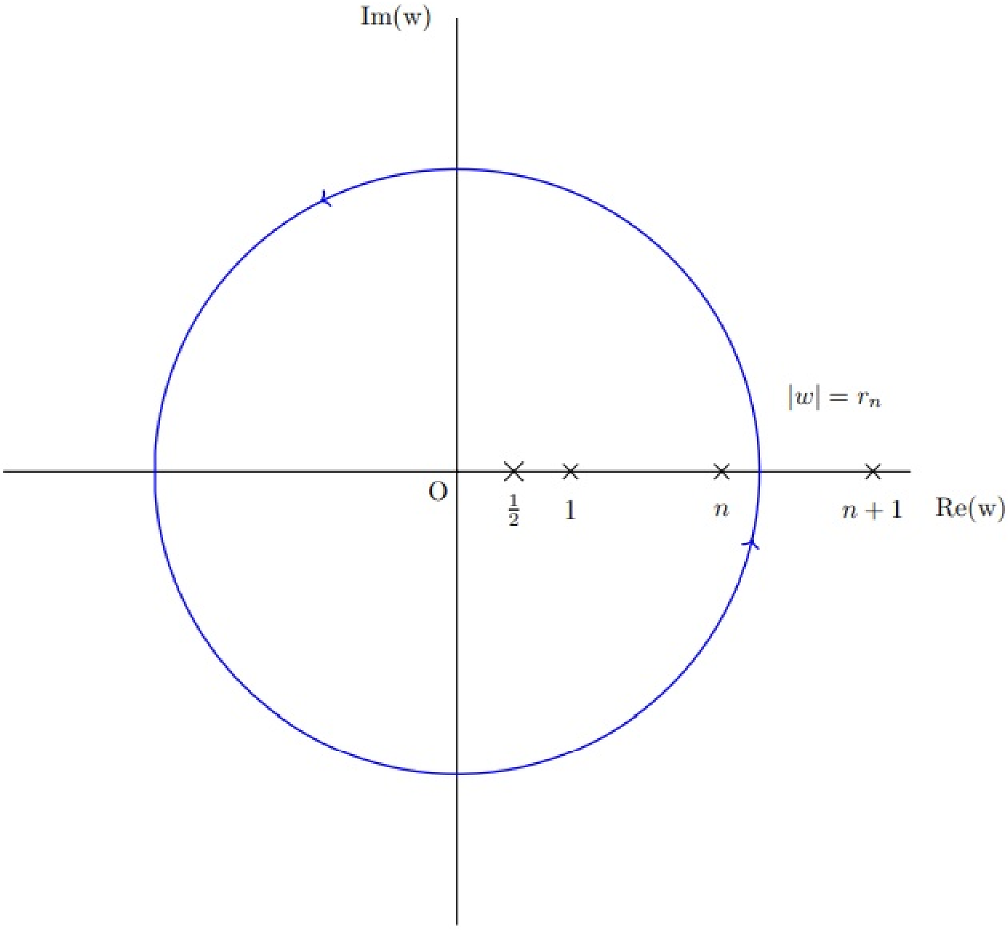

Figure B.1: Integration path for the function $F(w)$.

with $|w| = r_n$ standing for the path described by a circumference of radius $r_n = \frac{3}{2} + n$, see figure B.1. This choice of the radius dodges all the poles of $F(w)$ and goes to infinity as $n \to \infty$. Taking this limit is precisely what is needed to recover the unknown series from this integral. With the help of a well known lemma over infinite arcs, the integral on r_n can be computed. The function $wF(w)$, evaluated on r_n, with $n \to \infty$, gives

$$\lim_{n\to\infty} |r_n F(r_n)| = \lim_{n\to\infty} \left| \frac{r_n \pi \cos(\pi r_n)}{\sin(\pi r_n)} \frac{1}{(1-2r_n)^2} \right| = 0. \tag{B.12}$$

Since $\cos(\pi r_n) = 0 \quad \forall n \in \mathbb{N}$, the result is

$$\frac{1}{2\pi i} \oint_{|w|=r_n} F(w) = 0. \tag{B.13}$$

The residue in $w = \frac{1}{2}$ can readily be evaluated by

$$\lim_{w\to\frac{1}{2}} \frac{d}{dw} \left[(w - \frac{1}{2})^2 \frac{\pi}{(1-2w)^2} \frac{\cos(\pi w)}{\sin(\pi w)} \right] = \lim_{w\to\frac{1}{2}} \frac{d}{dw} \left[\frac{\pi}{4} \cot(\pi w) \right] = -\frac{\pi^2}{4}. \tag{B.14}$$

The sum of the residues for the poles of the inverse sine is the series (B.9), and therefore

$$\sum_{k\in\mathbb{Z}} \frac{1}{(1-2k)^2} = \frac{\pi^2}{4}. \tag{B.15}$$

Plugging this inside (B.9) gives

$$h_0 = 0, \tag{B.16}$$

and finally

$$\csc^2 z = \pi^{-2} \sum_{k\in\mathbb{Z}} (z-k)^{-2}. \tag{B.17}$$

B.2 Secant function

Consider the secant function defined by

$$\sec z = \frac{1}{\cos z}. \tag{B.18}$$

This function has an infinite and countable number of poles defined as

$$z_k = \pi(k + \frac{1}{2}) \qquad k \in \mathbb{Z}, \tag{B.19}$$

and thus it can be expanded in a Mittag-Leffler way. Each pole is a simple pole and the sum of the Laurent principal parts can readily be evaluated by

$$a_k = \frac{\text{Res}[\sec z, z_k]}{2\pi i} = \lim_{z \to z_k} \frac{z - z_k}{\cos z} = (-1)^{k-1}. \tag{B.20}$$

Thus the secant function can be written as

$$\sec z = h(z) + \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1}}{z - (k + \frac{1}{2})\pi}. \tag{B.21}$$

To evaluate the entire function $h(z)$, let us look at the asymptotic behaviour of $\sec z$. Since this function has an infinite amount of poles, its limit to infinity should be taken by dodging the poles. This can be done if the secant is evaluated on the sequence

$$a_j = \{2\pi j\}_{j=0}^{\infty}, \tag{B.22}$$

which obviously goes to infinity as $j \to \infty$. Then evaluating

$$\lim_{j \to \infty} \frac{1}{\cos a_j} = 1 \tag{B.23}$$

tells that the secant does not diverge at infinity, and therefore also the entire function $h(z)$. But an entire function which is everywhere bounded is a constant, as stated by the First Liouville Theorem. Given this fact, the secant can be rewritten as

$$\sec z = h_0 + \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1}}{z - (k + \frac{1}{2})\pi}. \tag{B.24}$$

To find the value of h_0, evaluate the above functions on a convenient point, such as $z = 0$:

$$h_0 = 1 - \frac{2}{\pi} \sum_{k \in \mathbb{Z}} \frac{(-1)^k}{2k + 1}. \tag{B.25}$$

The next step is to evaluate the sum of the above series. This can be done with the help of another function

$$F(w) = \frac{\pi \cos(2\pi w)}{\sin(\pi w)} \frac{1}{2w + 1}. \tag{B.26}$$

$F(w)$ shows a simple pole in $w = -\frac{1}{2}$, and an infinite amount of simple poles at $w_k = k, \quad k \in \mathbb{Z}$. By means of the Cauchy Theorem, the task is then to evaluate

$$\frac{1}{2\pi i} \oint_{|w|=r_n} F(w) dw = \text{Res}\left[F(w), -\frac{1}{2}\right] + \sum_{k=-n}^{k=+n} \text{Res}[F(w), k], \tag{B.27}$$

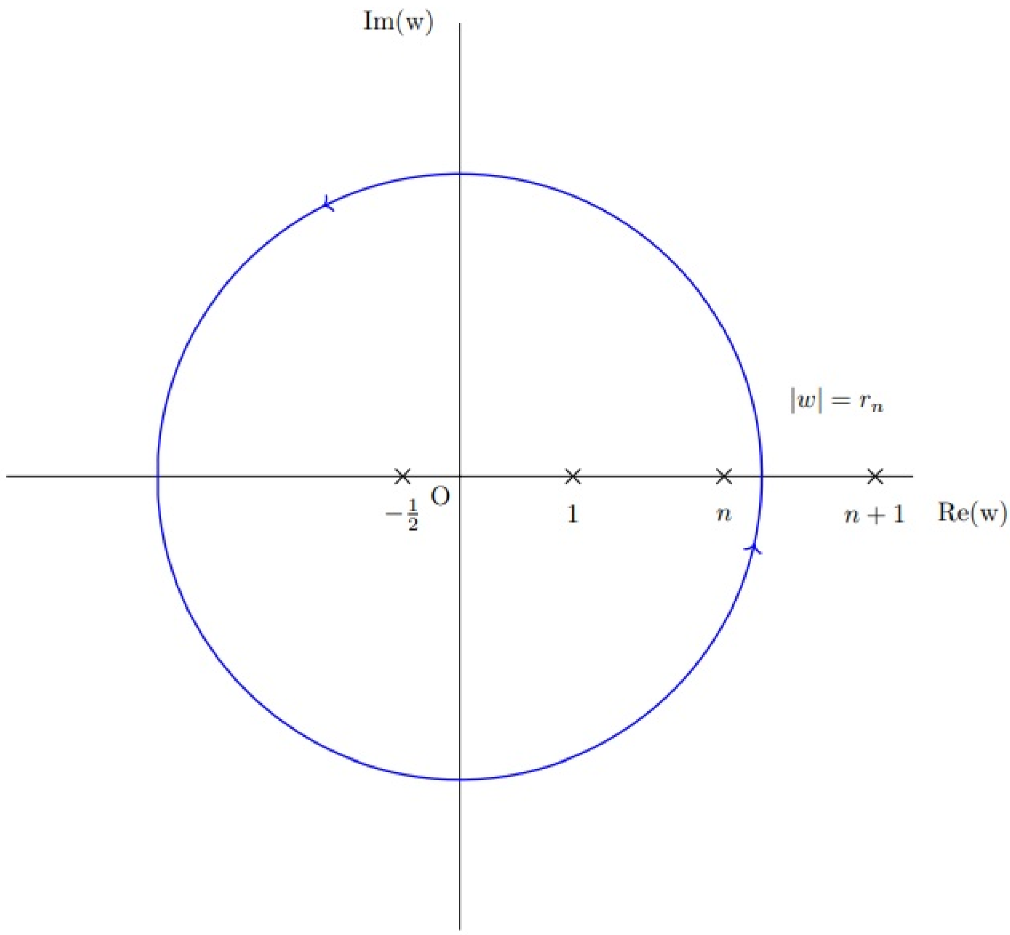

Figure B.2: Integration path for the function $F(w)$.

with $|w| = r_n$ standing for the path described by a circumference of radius $r_n = \frac{3}{2} + n$, see figure B.2. This choice of the radius dodges all the poles of $F(w)$ and goes to infinity as $n \to \infty$. Taking this limit is precisely what is needed to recover the unknown series from this integral. With the help of a lemma over infinite arcs, the integral on r_n can be computed: the function $wF(w)$, evaluated on r_n, with $n \to \infty$ gives

$$\lim_{n\to\infty} |r_n F(r_n)| = \lim_{n\to\infty} \left| \frac{r_n \pi \cos(2\pi r_n)}{\sin(\pi r_n)} \frac{1}{2r_n + 1} \right| = \frac{\pi}{2}, \tag{B.28}$$

and therefore the result is

$$\lim_{n\to\infty} \frac{1}{2\pi i} \oint_{|w|=r_n} F(w) = \frac{1}{2\pi i} 2\pi i \frac{\pi}{2} = \frac{\pi}{2}. \tag{B.29}$$

The residue in $w = -\frac{1}{2}$ gives a vanishing result, as can be readily seen by

$$\lim_{w\to -\frac{1}{2}} \left(w + \frac{1}{2} \right) \frac{\pi}{2w+1} \frac{\cos(2\pi w)}{\sin(\pi w)} = 0, \tag{B.30}$$

since the cosine vanishes at $\frac{\pi}{2}$.
The sum of residues for the poles of the sine is the series in (B.25), and therefore

$$\sum_{k\in\mathbb{Z}} \frac{(-1)^k}{2k+1} = \frac{\pi}{2}. \tag{B.31}$$

Plugging this inside (B.25) gives

$$h_0 = 0, \tag{B.32}$$

and finally

$$\sec z = \sum_{k\in\mathbb{Z}} \frac{(-1)^{k-1}}{z - \left(k + \frac{1}{2}\right)\pi}. \tag{B.33}$$

This sum can be rewritten in a more useful way by multiplying and dividing for $z + (k + \frac{1}{2})\pi$:

$$\sum_{k\in\mathbb{Z}} \frac{(-1)^{k-1}\left[z + \left(k + \frac{1}{2}\right)\pi\right]}{z^2 - \left(k + \frac{1}{2}\right)^2 \pi^2}. \tag{B.34}$$

Splitting the numerator, the first term is an odd term with respect z and should vanish since $\sec z$ is an even function:

$$\sum_{k\in\mathbb{Z}} \frac{(-1)^{k-1} z}{z^2 - \left(k + \frac{1}{2}\right)^2 \pi^2} = 0. \tag{B.35}$$

The other term is even in k and thus the sum over $k \in \mathbb{Z}$ can be reduced to a sum over $k \in \mathbb{N}$ multiplied by a factor of two. From this, the secant function can be ultimately written as

$$\sec z = \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)\pi}{\left(k + \frac{1}{2}\right)^2 \pi^2 - z^2}. \tag{B.36}$$

Appendix C

Hawking radiation from an Unruh-DeWitt detector model

Let us consider a free massless scalar field over the Schwarzschild solution. The general covariant Lagrangian that couples the scalar field with gravity is given by

$$\mathcal{L} = \int d^4x \sqrt{|g|} \frac{1}{2} g^{\mu\nu} \nabla_\mu \Psi \nabla_\nu \Psi. \tag{C.1}$$

The metric tensor can be imagined as reproduced by the coherent state $|g\rangle$, but the cut-offs will be ignored; this can be seen as a situation where the scalar field modes that will be emitted (and thus detected) are inside the range given by k_{IR} and k_{UV}. To be consistent with the corpuscular picture, the scalar field $\hat{\Psi}$ could be interpreted as the one describing the propagation of gravitons that leaks out from the bound state described by the state $|g\rangle$; in this sense, the gravitational field appears as an external potential (through the metric tensor defined by the coherent state) and as a quantized field (through the operator $\hat{\Psi}$), like in atomic transition processes, where the vector potential is quantized and the Coulomb field is an external function.
The metric tensor is in the Schwarzschild form, thus meaning that $V_q(r) = -GM/r$ and the cut-offs are suppressed. Provided the following transformation laws,

$$\begin{cases} T = e^{\frac{r^*}{4GM}} \sinh\left(\frac{t}{4GM}\right) \\ X = e^{\frac{r^*}{4GM}} \cosh\left(\frac{t}{4GM}\right), \end{cases} \tag{C.2}$$

the metric can be written into its Kruskal extension

$$ds^2 = \frac{32 G^3 M^3}{r} e^{-\frac{r}{2GM}} (-dT^2 + dX^2) + r^2 d\Omega^2, \tag{C.3}$$

with r^* again the Regge-Wheeler coordinate and $r = r(T, X)$ a function defined by

$$-T^2 + X^2 = \left(\frac{r}{2GM} - 1\right) e^{\frac{r}{2GM}} = e^{\frac{r^*}{2GM}}. \tag{C.4}$$

Asking again for $r^* = const.$ implies that the detector will follow a hyperbolic trajectory in Kruskal coordinates

$$-T^2 + X^2 = const. \tag{C.5}$$

The advantage of the Kruskal form of the metric is that it is conformal to the Minkowski one, provided the angular dimensions to be suppressed. The conformal transformation reads

$$\eta_{\mu\nu} = \omega^2 g_{\mu\nu}, \tag{C.6}$$

and the conformal factor is given by

$$\omega^2 = \frac{r}{32G^3M^3} e^{\frac{r}{2GM}}. \tag{C.7}$$

It is now remarkable to notice that even (C.1) is conformally invariant in two dimensions [40], and thus, once the Lagrangian is transformed, the field can be expanded like in flat spacetime with plane wave modes in the T, X coordinates. With these modes, a vacuum state $|0_\omega\rangle$ can be introduced. The operator associated to the detector $\hat{m}$ is assumed to evolve with its free Hamiltonian with respect to Schwarzschild time t. The initial state is called the conformal vacuum $|0_\omega\rangle$ and can be interpreted as the one with no free gravitons, and the only available final state at first order in perturbation theory is the one particle state. The transition amplitude reads

$$A(i \to f) = -iG \langle E|\hat{m}(0)|E_0\rangle \int_{-\infty}^{+\infty} dt \sqrt{\frac{\hbar}{4\pi\omega_p}} e^{i\Omega t} e^{i\omega_p T(\tau) - ipX(\tau)}. \tag{C.8}$$

It is immediate to notice how (C.2) for constant r^* are formally equivalent to the equations defining an hyperbolic and uniformly accelerated motion in flat spacetime, with $a \to (4GM)^{-1}$. This is not surprising, once the Minkowski-Kruscal relation was already exploited in chapter 5. Therefore, the calculation for the amplitude follows the very same steps of the original Unruh-DeWitt one, and leads to

$$P(i \to f) \propto 1/[\exp(8\pi GM\Omega) - 1] \tag{C.9}$$

and to the definiton of the Hawking temperature $T_H = \hbar/8\pi GMk_B$.

The transition amplitude can be evaluated by relaxing the requirement on the final states and summing over them, like in the flat spacetime case. The only difference here lies on the form of the Wightman function, which is now evaluated in two (actually 1+1) dimensions:

$$D^{(+)}(x, x') = -\frac{1}{4\pi} \log(|T - T' - i\epsilon|^2 - |X - X'|^2). \tag{C.10}$$

The interval inside the logarithm can be rewritten in terms of the Schwarzschild time like

$$D^{(+)}(x, x') = -\frac{1}{2\pi} \log\left|2\sinh\left(\frac{\delta t}{8GM} - i\epsilon\right)\right|. \tag{C.11}$$

The δt dependence brings the usual diverging t' factor in the evaluation of the probability transition. It is therefore necessary to take the transition rate also for the Kruskal calculation. It reads

$$\frac{dP(i \to f)}{dt} = -\sum_E |\langle E|\hat{m}(0)|E_0\rangle|^2 \int_{-\infty}^{+\infty} \delta t e^{-i\Omega\delta t} \frac{1}{2\pi} \log \left| 2 \sinh \left(\frac{\delta t}{8GM} - i\epsilon \right) \right|. \quad \text{(C.12)}$$

Integrating by part twice yields to the familiar expression

$$\frac{dP(i \to f)}{dt} = -\sum_E |\langle E|\hat{m}(0)|E_0\rangle|^2 \int_{-\infty}^{+\infty} \delta t e^{-i\Omega\delta t} \left[\frac{4GM\Omega}{\pi} \sinh\left(\delta t/8GM - i\epsilon\right) \right]^{-2}, \quad \text{(C.13)}$$

that gives the same integral performed in flat spacetime and thus the same result with $a \to (4GM)^{-1}$:

$$\frac{dP(i \to f)}{dt} \propto \sum_E |\langle E|\hat{m}(0)|E_0\rangle|^2 / [\exp(8\pi GM\Omega) - 1]. \quad \text{(C.14)}$$

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