Alma Mater Studiorum · University of Bologna

School of Science
Department of Physics and Astronomy
Master Degree in Physics

HIDDEN SCHRÖDINGER SYMMETRY IN FLRW AND BIANCHI I MINISUPERSPACE MODELS OF COSMOLOGY AND BLACK HOLES SCHWARZSCHILD DYNAMICS

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Academic Year 2021/2022

Abstract

A broad sector of literature focuses on the relationship between fluid dynamics and gravitational systems. This thesis presents results that suggest the existence of a new kind of fluid/gravity duality not based on the holographic principle.

The goal is to provide tools that allow us to systematically unearth hidden symmetries for reduced models of cosmology. The focus is on the field space of these models, i.e. the superspace. In fact, conformal isometries of the supermetric leave geodesics in the field space unaltered; this leads to symmetries of the models. An innovative aspect is the use of the Eisenhart-Duval's lift. Using this method, systems constrained by a potential can be treated as free ones. Moreover, charges explicitly dependent on time, i.e. dynamical, can be found.

A detailed analysis is carried out on three basic models of homogenous cosmology: i) flat Friedmann-Lemaître-Robertson-Walker's isotropic universe filled with a massless scalar field; ii) Schwarzschild's black hole mechanics and its extension to vacuum (A)dS gravity; iii) Bianchi's anisotropic type I universe with a massless scalar field. The results show the presence of a hidden Schrödinger's symmetry which, being intrinsic to both Navier-Stokes' and Schrödinger's equations, indicates a correspondence between cosmology and hydrodynamics.

Furthermore, the central extension of this algebra explicitly relates two concepts. The first is the number of particles coming from the fluid picture; while the second is the ratio between the *IR* and *UV* cutoffs that weighs how much a theory has of "classical" over "quantum". This suggests a spacetime that emerges from an underlying world which is described by quantum building blocks. These quanta statistically conspire to appear as gravitational phenomena from a macroscopic point of view.

Sommario

Un vasto settore della letteratura si concentra sul rapporto tra fluidodinamica e sistemi gravitazionali. In questa tesi sono presentati risultati che suggeriscono l'esistenza di una nuova dualità non basata sul principio olografico.

L'obiettivo è quello di fornire degli strumenti che permettano di scoprire in modo sistematico simmetrie nascoste nei modelli ridotti della cosmologia. La fisica qui presa in considerazione ha luogo nello spazio dei campi di questi modelli, chiamato superspazio. Infatti le isometrie conformi della supermetrica lasciano inalterate le geodetiche nello spazio dei campi; questo conduce a simmetrie dei modelli. Un aspetto innovativo è l'impiego del metodo di innalzamento ad uno spazio con maggiori dimensioni introdotto da Eisenhart-Duval. Questo strumento permette sia di trattare i sistemi che sono limitati da un potenziale al pari di quelli liberi, che di trovare cariche che sono esplicitamente dipendenti dal tempo, ovvero cariche dinamiche.

Un'analisi dettagliata è svolta su tre semplici modelli di cosmologia omogenea: i) l'universo piatto e isotropo descritto dal modello di Friedmann-Lemaître-Robertson-Walker in presenza di un campo scalare senza massa; ii) la meccanica dei buchi neri modellata da Schwarzschild e l'estensione al vuoto nello spazio (A)dS; iii) l'universo anisotropo in presenza di un campo non massivo descritto dal modello chiamato secondo la classificazione Bianchi di tipo I. I risultati mostrano la presenza di una simmetria di Schrödinger che, essendo intrinseca alle equazioni sia di Navier-Stokes che di Schrödinger, fa ipotizzare una corrispondenza tra la cosmologia e l'idrodinamica.

Inoltre, l'estensione centrale dell'algebra di Schrödinger da un lato assume il valore di numero di particelle mentre dall'altro quello di rapporto tra i cutoff infrarosso e ultravioletto della teoria. Il primo valore lo deve all'interpretazione che si ha in fluidodinamica mentre il secondo soppesa quanto una teoria abbia di "classico" o di "quantistico". L'identificazione tra questi due valori risuona con gli altri risultati e suggerisce uno spaziotempo emergente da un mondo sottostante descritto per mezzo di mattoni quantistici che statisticamente si comportano in modo tale da generare fenomeni gravitazionali a livello macroscopico.

Contents

Introduction

Uncovering symmetries of general relativity and gravitational systems beyond the gauge symmetry under spacetime diffeomorphisms is a crucial endeavour to understand the theory, both at the classical and quantum levels, thus it presents a clear intrinsic interest. Beyond that, it also opens the road to build fruitful dictionaries with other physical systems sharing the same symmetries. In this work the focus will be hold on to this kind of interconnection with fluid dynamics.

This thesis accounts for results already published in [16] hence will follow quite thoroughly the steps taken therein while adding more literature theoretical context.

The first chapter gives a general idea on the mirroring between hydrodynamical systems and gravitational ones. In particular in order to convey context to this duality the reader will be introduced with a brief summary of the very well know result from literature that is analogue gravity. Here sound waves both in classical fluids and Bose-Einstein's condensates are shown to behave as discrete particles coupling to an arising effective curved geometry that only depends on internal parameters of the fluids and not at all on the laboratory frame. What takes shape is a kinetical description at perturbation level carrying a serious intrinsic limitation in a possible search for the dynamical description of geometries built therein. Indeed, the fluid taken as the starting point to realize the analogue model lives on the fixed background of the laboratory frame and only the excitations that move inside it do not couple with this background. It is clear that, what is probably the main characteristic of a genereal relativistic theory, i.e. the diffeomorphism invariance surely does not come easily nor for free in this picture. Nevertheless, analogue gravity provides a nice starting point, making the pair with other results, into indicate some deeper hidden structure that may link hydrodynamics with gravity. These further results will be shown coming from previous discoveries made using the moments method to map the hydrodynamics of a general nonlinear Schrödinger's fluid onto the superspace of cosmological reduces models. Lastly, the reader is going to meet a description of the Schrödinger's group symmetry since this is the associated structure describing transformations that leave invariant the Navier-Stokes' equations. Showing Schrödinger's is the group of non relativistic conformal symmetries of the compressible fluids as well as, of course, of the Schrödinger's equation describing quantum condensates, in its non linear variants.

The interplay of the central extension of the Schrödinger's algebra is a quite important notion that will be encountered across the whole document. On one hand it behaves as the number of particles in the hydrodynamical picture while on the other it shows up as a ratio – between the IR and UV cut-offs of the gravitational models at study – that gauges for the level of "classical" as opposed as "quantum" nature of a system. This, clearly, is going to be discussed in more depth in the dissertation but can already be understood that entails a quite important new kind of duality between fluid and gravity.

Previous line of work started with [91], continued with the analysis of isotropic Friedmann-Lemaître-Robertson-Walker's models filled with a massless scalar field in [8, 11, 12], extended to take into account also the effects in precence of the cosmological constant and an arbitrary homogeneous spatial curvature as well as a detailed discussion on the role of the gauge fixing in [6, 13] and finally to cover the case of the anisotropic Kantowski-Sachs' model describing the Schwarzschild's black hole interior with [51, 52] then extended also to Schwarzschild-(A)dS in [14] to show that the same results characterize both the exterior and interior of the black hole geometry, as well as the region beyond the cosmological horizon. The central core result is the existence of a $\mathfrak{sl}(2,\mathbb{R})$ symmetry in all those. In the following chapters the search for a systematic derivation of this symmetric structure will be undertaken as well as the unearthing of a broader $\mathfrak{sch}(2)$ one.

In particular in chapter 2 is described why it is a correct procedure to use minisuperspace approach as well as its extension called Eisenhart-Duval's lift to find conserved charges for the system one wants to study. In particular, detailed prescriptions are going to be provided step by step on how to construct these reduced models and how to get correct observables. This offers us a systematic approach to tackle both the three examples presented in this thesis and several others. Moreover it serves the purpose to open a geometric interpretation and origin for the conformal symmetry that are going to be found with it. Note that the two methods are not equivalent and actually provide very different sets of charges. Indeed dynamical ones can be unearthed only through the use of the lift.

Chapter 3 is devoted to the study of the flat Friedmann-Lemaître-Robertson-Walker's cosmological model filled with a massless scalar field; while in chapter 4 the Schwarzschild's black hole mechanics and its (A)dS extension are undertaken. In both cases an introduction to the models as well as the construction of the superspace and lifted line elements are given; charges and algebras they satisfy are computed in detail and the transformations that generate them are shown. Through the use of the founded dynamical observables is also going to be possible to find solutions for the dynamics in the field space providing a complete description of the system. For the Friedmann-Lemaître-Robertson-Walker's reduced model an extra section is dedicated to show how Schrödinger's group can be embedded in a broader one: the conformal symmetry group $SO(4,2)$.

Viceversa is also true that a $\mathfrak{so}(4,2)$ algebra contains a $\mathfrak{sch}(2)$. This is shown in the description of the simplest anisotropic cosmological model, the Bianchi's I type. Indeed in this case the conformal group algebra is naturally given already by the use of the superspace approach while the Schrödinger's one can be only recovered as a subalgebra of it. The extrapolated subgroup is though very much differently from the two presented for the other models since it does not present any dynamical observables. These once again can be found implementing the Eisenhart-Duval's lift but only the protected $\mathfrak{sl}(2,\mathbb{R})$ algebra is recovered this way.

In each example the Casimir operators for the Schrödinger's algebra are reckoned and explicitly computed. The 2d version of this algebra is equipped with a total of three communting operators. Two of which do not really give us newer information than the ones already provided by literature. The third one instead is the central extension that, as already mentioned above, has the duplicitous interpretation of number of particles in the hydrodynamical picture and here reveils itself as the gauging parameter for classical vs. quantum factor of the studied theory.

The new duality revealed between homogeneous gravity and non relativistic fluid systems, even if brings it to mind, is very much different from the fluid/gravity duality built in the AdS/CFT correspondence framework. The latter, deeply studied in literature, gravitates around the central focal point of holography. While the new kind that this thesis is going to deal with, is an actual correspondence with equations of the dynamics of a fluid living on superspaces. This new duality leaves one curious about the very nature of spacetime itself. Would it be possible, as suggest by [83], for it to just be a coarse grained approximation of an underlying fluid made of pregeometrical quantum building blocks and for gravity, as well, to emerge as a macroscopical effect due to statistical conspiring of microscopical quantum components?

Chapter 1

Hydrodynamical mirroring of gravity

The interplay between relativistic fluid dynamics and gravitational physics is quite a lively branch of ongoing theoretical research. Since the early works on the membrane paradigm [93] several attempts have been made in order to thoroughly and formally introduce a theory describing the interconnections and possibly a deeper duality between the two. Probably the most known correspondence framework developed up until now is the AdS/CFT duality, formulated firstly in the anti de Sitter framework and more recently in the asymptotically flat context [17, 18, 22, 61, 68, 90].

An interesting question is whether a similar correspondence can also be uncovered between the dynamics of the gravitational field and that of non relativistic fluids. Another question, perhaps even more important, is if it can exist in a non holographic context, i.e. where the fluid system is not understood as defined in the boundary of the gravitational one whereas instead lives on its superspace. This conundrum was tried to be answered through the use of the analysis method of the wavefunction moments in the picture of a possible duality between nonlinear Schrödinger's fluids – as a Bose-Einstein's condensate could be – and homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker's cosmology [64, 71, 72]. This approach was successful to some degree since indeed a mapping between the two was uncovered as we will see in section 1.2 on page 7.

In the following sections a first intuition about fluid/gravity analogy is going to be given using possibly the simplest example among all, that is analogue gravity. Both sound waves moving inside a classical fluid and in a Bose-Einstein's condensate will be shown coupling to an effective curved geometry arising naturally at the hydrodynamical level of these condensed matter systems, very much as particle of sounds, namely *phonons*, moving in a gravitationally bent spacetime.

It will also be introduced in more detail the Schrödinger's group that is going to be heavily employed through the whole document to establish a new sort of fluid/gravity duality resonating with some investigations on the non relativistic regime of the AdS/CFT and fluid/gravity correspondences [41, 43, 59, 94], the construction of the aging/gravity duality [65, 70, 76] and parallelling with ongoing efforts to extend the mapping shown just few lines above to other systems.

1.1 Analogue gravity: a first intuition

The foundative work of Barceló, Liberati and Visser [3], pillar in the field of analogue gravity, start its second chapter with the words

> Acoustics in a moving fluid is the simplest and cleanest example of an analogue model. The basic physics is simple, the conceptual framework is simple, and specific computations are often simple (whenever, that is, they are not impossibly hard).

In order to give a first intuition of what fluid/gravity duality means, in this section the main steps they used to shed light on the subject through the use of – by their own admission – a simple example will be reviewed.

1.1.1 Classical fluids

Let us start by considering an inviscid, irrotational and barotropic fluid. This means, respectively, that the viscosity is null $\eta = 0$; that the rotor of fluid velocity[1] vanish $\nabla \times \mathbf{v} = 0$ and that fluid density can be seen as a function only of the preassure $\rho = \rho(p)$. The foundamental equations governing the dynamics of this fluid, it is well know, are the continuity one

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{1.1.1}$$

and, from standard manipulations of the Cauchy's equation first and further into the Navier-Stokes' ones, the Euler's equation

$$\rho \left[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\, \mathbf{v}\right] = -\nabla p - \nabla V_{\text{ext}} \tag{1.1.2}$$

here we assume the only external forces applied to the fluid are due to pressure, hence $V_{\text{ext}} = 0$. Then by considering the assumption of irrotationality it is possible to write the fluid velocity in terms of a potential $\mathbf{v} = -\nabla\theta$ and using the assumption of barotropicity that allows to define the enthalpy as

$$h(p) = \int \frac{\mathrm{d}p}{\rho(p)} \qquad \Longrightarrow \qquad \nabla h = \frac{1}{\rho}\nabla p \tag{1.1.3}$$

[1] Note that bold notation is as usual for 3d vectors.

with standard manipulations[2] the Euler's equation becomes the following version of the Bernoulli's

$$-\partial_t\theta + \frac{1}{2}(\nabla\theta)^2 + h(p) = 0 \qquad (1.1.5)$$

The purpose of this reasoning is to show that fluid motion directly affects the dynamics of perturbations over a backgroud. These perturbations are nothing else than sound wave and we can imagine them as phonons in the quantum analogy. Of course to reach the goal a background and fluctuations around it must be introduced, namely we need to linearise eq. (1.1.5) around the fixed backgroud (ρ_0, p_0, θ_0) assuming

$$\begin{aligned} \rho &= \rho_0 + \epsilon\rho_1 + \mathcal{O}(\epsilon^2) \\ p &= p_0 + \epsilon p_1 + \mathcal{O}(\epsilon^2) \\ \theta &= \theta_0 + \epsilon\theta_1 + \mathcal{O}(\epsilon^2) \end{aligned} \qquad (1.1.6)$$

using the linearized version of p in the definition of the density and linearizing the enthalpy as well, the following relations are straightforwardly recovered

$$\rho_1 = \frac{\partial\rho}{\partial p}p_1\,, \qquad h(p) = h(p_0) + \epsilon\frac{p_1}{\rho_0} \qquad (1.1.7)$$

thanks to this expression for the enthalpy is easy to write eq. (1.1.2) at first order and from it to find a profile for the pressure exitation p_1. This promptly gives ρ_1 that can be plugged in to the first order continuity equation eq. (1.1.1) to get to a dynamical equation for θ_1

$$\partial_t\left[\frac{\partial\rho}{\partial p}\rho_0\left(\partial_t\theta_1 + \mathbf{v}_0\cdot\nabla\theta_1\right)\right] - \nabla\cdot\left[\rho_0\nabla\theta_1 - \frac{\partial\rho}{\partial p}\rho_0\mathbf{v}_0\left(\partial_t\theta_1 + \mathbf{v}_0\cdot\nabla\theta_1\right)\right] = 0 \qquad (1.1.8)$$

Being people of habit, a more familiar form is far better. So recast this into

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\theta_1\right) = 0 \qquad (1.1.9)$$

to easily recognize that phonons satisfy the d'Alembert's equation $\Box\theta_1 = 0$.

The most important result here though is not merely that θ_1 satisfy an equation of motion but that the fluid present a curved geometry described by the acoustic

[2] Indeed the follwing relation holds true

$$(\mathbf{v}\cdot\nabla)\,\mathbf{v} = \nabla\left(\frac{\mathbf{v}^2}{2}\right) - \mathbf{v}\times(\nabla\times\mathbf{v}) \qquad (1.1.4)$$

that alogn with the vorticity free condition produces the presented result.

metric

$$g_{\mu\nu} = \frac{\rho_0}{c_s} \begin{pmatrix} -\left(c_s^2 - v_0^2\right) & v_0^j \\ v_0^i & \delta_{ij} \end{pmatrix} \tag{1.1.10}$$

that depends only on macroscopic properties of the fluid itself, like its background velocity $\mathbf{v}_0$, its density ρ_0 and the velocity at which sound propagates inside it

$$c_s \equiv \sqrt{\frac{\partial p}{\partial \rho}} \tag{1.1.11}$$

regardless of the laboratory frame. This has been revolutionary both for the gates it opens as for possible direct experiments to study gravitational effects – otherwise secluded to indirect astrophysical analysis– and for the analogy it depicts between hydrodynamics and gravity. Indeed the effective curved geometry that arises naturally from the hydrodynamical level allows us to think about spacetime, and the gravitational effects its bending causes, as a fluid in which phonons are affected by this structure.

Figure 1.1: River analogy: salmon swim upstream to try escaping the ominous fate that awaits them in the fall at the end of the river. They can outrun the danger if they move faster than the current, just like phonons moving in a supersonic fluid are doomed to fall and be trapped into dumb holes. Picture from R. Aguero-Santacruz, D. Bermudez. *Hawking radiation in optics and beyond.* 2020.

This is just like the river analogy, pictured in Fig. 1.1 on the preceding page in which salmon are dragged by the stream of the river and are completely unaware of the bigger picture in which the river is immersed. If the stream has favourable flow their swimming is help, viceversa if the current oppose them their efforts are not rewarder. The analogy can be even brought to the extrem by considering a fall, where the velocity of the water is so high that the ill-fated salmon cannot outrun it and are perpetually dragged into a, so called, *dumb hole*.

1.1.2 Bose-Einstein's condensate

Results from previous section can be directly traslated to the case of a Bose-Einstein's condensate, in what follows is shown a way to retrieve this generalization.

From literature it is well known that Bose-Einstein's condensates are governed by the nonlinear Schrödinger's equation called Gross-Pitaevskii's equation

$$i\hbar\partial_t\psi = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{x}) + \kappa(a)\psi^\dagger\psi\right]\psi \tag{1.1.12}$$

in which a potential term due to external forces is present as well as a term due to self interaction of the bosons components of the quantum condensate. The latter is parametrised by $\kappa(a)$ that depend on the scattering amplitute via $\kappa(a) = 4\pi a\hbar^2 m^{-1}$.

Now the wavefunction can be written following the Madelung's prescription

$$\psi(t, \mathbf{x}) = \sqrt{\rho(t, \mathbf{x})}\, e^{-i\theta(t,\mathbf{x})/\hbar} \tag{1.1.13}$$

and plugged into the Gross-Pitaevskii's equation, eq. (1.1.12) to get two relations, the first for the real and the second for the imaginary part

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{1.1.14}$$

$$m\left[\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}\right] = -\nabla\left[\kappa\rho + V_{\text{ext}} + V_{\text{quant}}\right] \tag{1.1.15}$$

We made two crucial definitions in order to retrieve them in this form. Respectively the velocity field and the quantum potential

$$\mathbf{v} = \frac{\nabla\theta}{m}\,, \qquad V_{\text{quant}} = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} \tag{1.1.16}$$

It is immediate to see that eq. (1.1.14) is the equivalent of the continuity equation encountered in eq. (1.1.1) and that the matter density therein is here the wavefunction amplitute; in what is a very straightforward correspondance through the usual probability interpretation of the quantum framework that wants the number of particles being computed as the integral of $|\psi|^2$

$$n = \int_V \mathrm{d}^3x\;\psi^\dagger\psi = \int_V \mathrm{d}^3x\;\rho(t, \mathbf{x}) \tag{1.1.17}$$

As for eq. (1.1.15), the quantum potential is induced by interaction between bosons inside the fluid and can be seen as producing a sort of external effective quantum force while the term $\kappa\rho$ can be identified with the pressure of the classical fluid described in the previous section. The immaginary part equation then matches with Euler's equation in eq. (1.1.2).

Now that the two equations at the foundation of the entire reasoning done above have been found, the same prescriptions can be used to introduce infinitesimal exitations of the Bose-Einstein's condensate as

$$\begin{aligned} \rho &= \rho_0 + \epsilon\rho_1 + \mathcal{O}(\epsilon^2) \\ \theta &= \theta_0 + \epsilon\theta_1 + \mathcal{O}(\epsilon^2) \end{aligned} \tag{1.1.18}$$

and after quite some computations a second order wave equation for the velocity field perturbation θ_1, remembrance of eq. (1.1.8), is found. It can be recasted in the very well known – and much less cumbersome – form that is looked for

$$\Box\theta_1 = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\theta_1\right) = 0 \tag{1.1.19}$$

once the analogue metric tensor and the velocity of sound are, respectively, defined as

$$g_{\mu\nu} = \frac{\rho_0}{mc_s}\begin{pmatrix} -\left(c_s^2 - v_0^2\right) & v_0^j \\ v_0^i & \delta_{ij} \end{pmatrix}, \qquad c_s = \sqrt{\frac{\rho_0\kappa}{m}} \tag{1.1.20}$$

Generic nonlinear Schrödinger's fluid

Once again a further generalization can be developed by disentangling from the Gross-Pitaevskii's approximation. Quite straightforwardly one can substitute the self interaction term in eq. (1.1.12) with a generic one that depends only on the amplitude of the wavefunction and time

$$\kappa(a)\psi^\dagger\psi \qquad \longrightarrow \qquad \mathsf{g}\left(|\psi|^2, t\right) \tag{1.1.21}$$

The only real difference in the results found above, if this change were to be carried on, would be that the velocity of sound would take the general form

$$c_s = \sqrt{\frac{\rho_0\mathsf{g}'}{m}} \tag{1.1.22}$$

in which the derivative indicated with a primed notation is w.r.t. the modulo squared of the wavefunction $\mathsf{g}' = \partial_\rho\mathsf{g}$.

1.2 NLS' hydrodynamics mapping onto cosmology

1.2.1 Method of the moments

Additional insights into the dynamics of the nonlinear Schrödinger's equation and a bridge between it and cosmology can be gained through the use of the, so called, method of the moments described in detailed by [72, 88]. This kind of mapping, that further suggest a fluid/gravity duality, is reviewed here.

To keep the exposition of computations – that are completely generalisable to higher dimensions – at the most basic level possible[3] let us consider a 1d generic nonlinear Schrödinger's model trapped by an external harmonic potential; eq. (1.1.12) reads

$$i\partial_t\psi = \left[-\frac{1}{2m}\partial_r^2 + \frac{\lambda(t)}{2}r^2 + \mathtt{g}\left(|\psi|^2, t\right)\right]\psi \tag{1.2.1}$$

Now following the prescriptions from the method of moments one can find the following relevant ones

$$\begin{aligned}
\mathcal{I}_2(t) &= \frac{m}{4}\int \mathrm{d}r\ r^2\psi^\dagger\psi \\
\mathcal{I}_3(t) &= \frac{i}{4}\int \mathrm{d}r\ r\left(\psi\partial_r\psi^\dagger - \psi^\dagger\partial_r\psi\right) \\
K(t) &= \frac{1}{2m}\int \mathrm{d}r\ \partial_r\psi^\dagger\partial_r\psi \\
J(t) &= \int \mathrm{d}r\ \mathtt{G}\left(|\psi|^2, t\right)
\end{aligned} \tag{1.2.2}$$

the use of $\partial_\rho \mathtt{G} = \mathtt{g}$ was made. If they are plugged into the nonlinear Schrödinger's equation four equations of motion can be recovered. Though they are not a closed set, meaning an infinite tower of differential equations would need to be dealt with in order to get a solution for the dynamics. A more realistic way to get passed this issue is to implement the combined moment

$$\mathcal{I}_4(t) = K(t) + J(t) \tag{1.2.3}$$

so to get the following linear set of moment equations describing the dynamics

$$\dot{\mathcal{I}}_2 = \mathcal{I}_3\ , \qquad \dot{\mathcal{I}}_3 = -\frac{2\lambda}{m}\mathcal{I}_2 + \mathcal{I}_4\ , \qquad \dot{\mathcal{I}}_4 = -\frac{2\lambda}{m}\mathcal{I}_3 \tag{1.2.4}$$

[3]To ease up notation we also use $\hbar = 1$.

This system can be a closed set but, in order for it to be so, a further assumption is necessary; that being $\mathsf{g} = \kappa\rho^2$ with $\kappa \in \mathbb{C}$. This is also the condition to get a quintic nonlinear Schrödinger's equation.

A first intuitive interpretation from the physical side of these moments sees $\mathcal{I}_2$ being associated with the square of the wavefunction width, $\mathcal{I}_3$ with the momentum of it and finally $\mathcal{I}_4$ with the total energy given by the sum of its kinetic and potential contributions.

The following step an interested person can take is to rescast a sector of the just found dynamics into the well known Ermakov-Pinney's equation

$$m\ddot{X} + \lambda X - \frac{Q}{X^3} = 0 \tag{1.2.5}$$

by making use of the definition of the notation $X = \sqrt{\mathcal{I}_2}$ and of the constant of motion for the dynamics eq. (1.2.4)

$$\frac{4Q}{m} = 2\mathcal{I}_2\mathcal{I}_4 - \mathcal{I}_3^2 \tag{1.2.6}$$

This kind of equations, and all the implications that follow, have been deeply and formally studied since 1880 starting from Ermakov. A detailed analysis of the concerned facets is carried out both in the work of [88] and in the references therein. The most important aspect in this contingency is that one just needs to find a solution for the Ermakov-Pinney's equation so that the entire dynamics described in eq. (1.2.4) be solved as well.

A note that needs to be made is for sure that in absence of a trapping potential, namely when $\lambda(t) = 0$, that is the case of a free quintic Bose-Einstein's condensate, eq. (1.2.5) describes the motion of the 1d conformal mechanics, i.e. the description of the conformal particle subjected to conformal symmetries.

1.2.2 Mapping onto cosmology

It is then straightforward to map the nonlinear Schrödinger's hydrodynamics onto cosmology by making some definitions and identifications. In particular we define the new functions $\phi(\tau)$, $\rho(\tau)$ and $a(\tau)$ as well as the real constant G as per the following relations

$$\frac{8\pi G}{a^2}\dot{\phi}^2(\tau) = \frac{2\lambda(\tau)}{m}\,, \qquad \rho(\tau) = \frac{3}{4\pi G}\mathcal{I}_4\,, \qquad a(\tau) = 2\sqrt{\mathcal{I}_2} \tag{1.2.7}$$

where the dotted notation – henceforth – stand as derivatives in the new time parameter defined as $\mathrm{d}\tau = a^{-1}\mathrm{d}t$. Then the fourth defining relation

$$k = 16\frac{Q}{m} \tag{1.2.8}$$

is automatically recovered.

The associations are already made clear by the use of standard cosmological notation, but for the sake of completeness: function ϕ is a scalar matter field with energy density ρ, while a stands for the scale factor and k for the curvature constant and finally G is the Newton's constant. By appling these variable redefinitions to some of the equations found earlier, these can be directly rewritten as

$$\begin{aligned} \frac{4Q}{m} &= 2\mathcal{I}_2\mathcal{I}_4 - \mathcal{I}_3^2 &\longmapsto\quad& \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \\ m\ddot{X} + \lambda X - \frac{Q}{X^3} &= 0 &\longmapsto\quad& \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} = -4\pi G\dot{\phi}^2 \\ \dot{\mathcal{I}}_4 &= -\frac{2\lambda}{m}\mathcal{I}_3 &\longmapsto\quad& \dot{\rho} = -3H\dot{\phi}^2 \end{aligned} \tag{1.2.9}$$

the defining relation for the constant of motion eq. (1.2.6) becomes the constrain Friedmann's equation; the Ermakov-Pinney's equation for the wavefunction width eq. (1.2.5) maps to the dynamical Friedmann's equation – hence does not add any new information basically just being the derivative of the first – and finally the equation of motion for the moment $\mathcal{I}_4$, i.e. the energy, is brought to the energy-momentum conservation equation for the scalar field. This manifestly shows the mapping painting the duality between hydrodynamics and cosmological models. Note that this bridge is natural only if thought with one end in the minisuperspace of the gravitational theories. Indeed the variables identifications are made between the moments and variables living on reduced phase space like ϕ or a. More detail about minisuperspaces and Friedmann-Lemaître-Robertson-Walker's reduced model can be found in chapters 2 and 3.

A final remark is that the same kind of mapping procedure can be undertaken also for other models like Schwarzschild's black hole dynamics and Bianchi's I anisotropic universe to find dualities with different kinds of Bose-Einstein's condensate's hydrodynamic.

1.3 Schrödinger's group: Navier-Stokes' symmetry

Schrödinger's group is the non relativistic conformal symmetry of the compressible Navier-Stokes' equations [67]

$$\rho\left[\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}\right] = -\nabla p + \nu\nabla^2\mathbf{v} + \left(\zeta + \frac{1}{3}\nu\right)\nabla\left(\nabla\cdot\mathbf{v}\right) \tag{1.3.1}$$

in which we recognise $\mathbf{v}$ as the vector velocity of the fluid, p as the pressure, ν and ζ as, respectively, the shear and bulk viscosity and finally ρ as the fluid density. Of course it is also the symmetry of the Schrödinger's equation – which clearly due its name to – and some nonlinear extensions of it [54, 66, 80]. Meaning its associated transformations leave untouched also the dynamics of superfluids and Bose-Einstein's condensates that can be described with it. The mapping encountered in section 1.2 on page 7 can be also seen throught the eyes of groups and algebras of symmetry formalisms [64, 71, 72] by looking for matching sets of these kind of structures between the two systems and, more in general, frameworks.

In this section a brief review of the main points that are used in the following chapters is given. But for further insight on the subject the interested reader is referred to [43, 62, 63] in addition to the other references already mentioned. In the end a very interesting bridge between the gravitational dynamics of some minisuperspace models and non relativistic hydrodynamics of fluids described by the Navier-Stokes' equations and hosted on an auxiliary background will be given. As well as an hint of the possibility of a deeper understanding and describing methodology for spacetime itself as a quantum many bodies system where gravitational effects would emerge from the collective behaviour of its constituents.

1.3.1 Charges and algebra in arbitrary dimensions

Schrödinger's group in arbitrary dimensions d is defined as the semidirect product of the Heisenberg's one with the 2d real special linear group and a rotations one. In the algerba formalism its Levi's decomposition reads

$$\mathfrak{sch}(d) = \mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{so}(d) \ltimes \mathfrak{h}(d) \tag{1.3.2}$$

Generators associated with the special linear algebra can be called $Q_\pm$ and Q_0; the ones related to rotations J_{jk} which indices goes as $1 \leq j < k \leq d$ and the ones that generate the Heisenberg's group are named P_i for the translations and B_i for the Galilean boosts where in these cases the index can assume values $1 \leq i \leq d$. Finally only the central extension n of the algebra is left to be mentioned.

A possible representation for these can be realized in the differential operator form – as shown in [2] and references therein – respectively, for the Q's as

$$\begin{aligned} Q_+ &= 2\partial_t \,, \qquad Q_0 = 1 + \frac{x_k}{2}\partial_k + t\partial_t \\ Q_- &= \frac{1}{4}\left(t - \frac{n x_k}{2} + 2t x_k + 2t^2 \partial_t\right) \end{aligned} \tag{1.3.3}$$

and for the others as

$$P_i = -2\partial_i \,, \qquad B_i = \frac{n x_i}{2} - t\partial_i \,, \qquad J_{jk} = x_k \partial_j - x_j \partial_k \tag{1.3.4}$$

It is possible to show that the Schrödinger's generator satisfy the following non trivial commutation relations

$$
\begin{aligned}
&[Q_+, Q_-] = 2Q_0 \ , \qquad [Q_0, Q_\pm] = \mp Q_\pm \ , \qquad [B_i, P_j] = \delta_{ij} n \\
&[Q_-, P_i] = -B_i \ , \qquad [Q_0, P_i] = -\frac{1}{2} P_i \ , \qquad [J_{jk}, P_i] = \delta_{ji} P_k - \delta_{ki} P_j \\
&[Q_+, B_i] = +P_i \ , \qquad [Q_0, B_i] = \frac{1}{2} B_i \ , \qquad [J_{jk}, B_i] = \delta_{ji} B_k - \delta_{ki} B_j \\
&\qquad [J_{ij}, J_{kl}] = \delta_{ik} J_{jl} + \delta_{jl} J_{ik} - \delta_{il} J_{jk} - \delta_{jk} J_{il}
\end{aligned}
\tag{1.3.5}
$$

that is the Schrödinger's algebra $\mathfrak{sch}(d)$.

1.3.2 The 2d case and its Casimirs

In the following, in particular in chapter 3 and 4, we are going to encounter the 2d realization of the Schrödinger's algebra. Let us show here the explicit forms the relations from the previous section take in this specific dimension.

The Levi's decomposition directly becomes

$$\mathfrak{sch}\,(2) = (\mathfrak{sl}\,(2, \mathbb{R}) \times \mathfrak{so}\,(2)) \ltimes \left(\mathbb{R}^2 \times \mathbb{R}^2\right) \tag{1.3.6}$$

then the generators associated with each subalgebra can be recognised respectively as the Q's, a single rotation $J_{jk} = J_{12} = J$ since $1 \le j < k \le d = 2$ and for the two $\mathbb{R}^2$ sectors the two sets of translations $P_\pm$ and boosts $B_\pm$, identified for easier readability with the $\pm$ notation. And the commutation relations simplify to

$$
\begin{aligned}
&[Q_+, Q_-] = 2Q_0 \ , \qquad [Q_0, Q_\pm] = \mp Q_\pm \ , \qquad [B_\pm, P_\pm] = n \\
&[Q_-, P_\pm] = -B_\pm \ , \qquad [Q_0, P_\pm] = -\frac{1}{2} P_\pm \ , \qquad [J, P_\pm] = \pm P_\mp \\
&[Q_+, B_\pm] = +P_\pm \ , \qquad [Q_0, B_\pm] = \frac{1}{2} B_\pm \ , \qquad [J, B_\pm] = \pm B_\mp
\end{aligned}
\tag{1.3.7}
$$

In the general case the Beltrametti-Blasi formula [2] predicts a total of

$$\text{rank}\,[\mathfrak{sl}(2, \mathbb{R})] + \text{rank}\,[\mathfrak{so}(d)] + 1 = \frac{d}{2} + 2 \tag{1.3.8}$$

functionally independent Casimir operators that in this particular instance will add up to the amount of three.

The first one is quadratic in the generators and takes the form

$$\mathcal{C}_1 = P_+B_- - P_-B_+ + nJ \tag{1.3.9}$$

while the second is cubic and reads

$$\begin{aligned}\mathcal{C}_2 = 2nQ_+Q_- + \left(B_+^2 + B_-^2\right)Q_+ + \left(P_+^2 + P_-^2\right)Q_- - 2nQ_0\left(Q_0 + 2\right)\\ - 2\left(P_+B_+ + P_-B_-\right)Q_0 - \left(P_+B_- - P_-B_+ + \frac{1}{2}nJ\right)J\end{aligned} \tag{1.3.10}$$

Finally the third one is the central extension itself

$$n = [B_\pm, P_\pm] \tag{1.3.11}$$

1.3.3 The $n = IR/UV$ ratio: an hint for emergent QG?

The central extension of the Schrödinger's group here called n is associated and interpreted as the number of particles – or indifferently non relativistic mass – of the fluid the group is symmetry of. Indeed the invariance of the Schrödinger's equation that generates this charge is w.r.t. phase space transformations. These of course leave untouched the probability and its interpretation [62, 63, 66, 67]. Namely if one considers the Madelung's fluid representation already introduced in eq. (1.1.13) the transformations related with n take the form

$$\begin{aligned} t &\longmapsto \tilde{t} = t\\ x^i &\longmapsto \tilde{x}^i = x^i\\ \rho &\longmapsto \tilde{\rho}(\tilde{t},\tilde{x}^i) = \rho(t,x^i)\\ \theta &\longmapsto \tilde{\theta}(\tilde{t},\tilde{x}^i) = \theta(t,x^i) + \varphi \end{aligned} \tag{1.3.12}$$

and the central extension can be found by the use of

$$n = \int \mathrm{d}x^i\ \rho(t,x^i) \tag{1.3.13}$$

that coincides with the definition of number of particles via the standard interpretation of the probability amplitude of wavefunction of a quantum system, eq. (1.1.17).

In the present work the reader will be presented with some examples of restricted models for gravitational systems. And in particular, in eq. (3.1.4) for what concerns the Friedmann-Lemaître-Robertson-Walker's model and in eq. (4.1.3) for the Schwarzschild's black hole dynamics description, it is going to be shown that the central extension for the Schrödinger's algebra the two models satisfy is closely related to strictly geometrical parameters. In general terms can be said that n encodes both the IR and UV cut-offs of the theories as the ratio

$$n = \frac{IR}{UV} \tag{1.3.14}$$

The former comes from the fiducial volume introduced as a regulator in the symmetry reduction process to prevent the theories from diverging while the latter derives from the gauging action the Planck length has over the arising of gravitational effects in the microscopical realm.

This is probably the most important result due to the possible repercussions which might have. Indeed if one consider a scenario of emergent gravity from quantum building blocks of spacetime itself it would appear that the central extension actually plays the role of average number of these Planckian cells which compone the fabric of the universe. Spacetime would be described as a quantum fluid and gravity would emerge as a macroscopic effect determined by the statistical behaviour of the microscopical elements that compone it.

While this remains a suggestion at this stage, it nevertheless matches the situation in real fluids where n plays indeed the role of the conserved number of particles of the system. The identification of hidden Schrödinger's algebras of observables naturally introduces a new notion of *number of particles* for cosmology and black hole mechanics as well. This fits remarkably well with the standard interpretation of these systems as thermodynamical – or hydrodynamical – objects.

Emergent gravity scenario

The notion of gravity emergent from an underlining microscopical world, completely pregeometrical and quantistic that would allow to describe gravitational effects as a mascoscopical manifestation determined by the collective behaviour of quantum building blocks living in this concealed world, is a very interesting one.

Emergence itself is almost a more philosophical rather than physical puzzle [83, 86] but if it is grounded in an elemental quantum theory can depict a scenario worth researching to get a better understanding of fundamental physics. This could be the framework of the Tensorial Group Field Theory [57, 58, 74, 81, 82, 85, 92].

Indeed the objective of this theory is to describe the nature of spacetime as emerging from microscopical building blocks conspiring to behave macroscopically as a condensate. The core idea at the foundation of the theory are these pregeometrical quanta described by tetrahedra with discrete geometry encoded into group structures where group elements are associated with triangles boundaries of the tetrahedron as it is shown in Fig. 1.2 on the next page. Therein is also pictured, in an intuitive form, the way in which one can build a Fock space of the theory

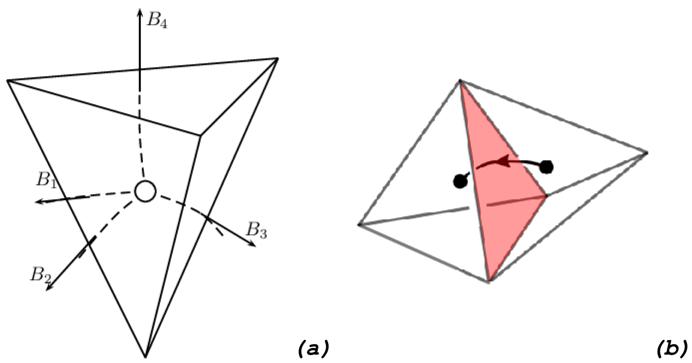

Figure 1.2: In (a) tetrahedron description of pregeometrical quanta in which each triangles boundaries is associated with a group element B_i. Picture from S. Gielen, L. Sindoni. *Quantum Cosmology from Group Field Theory Condensates: a Review.* 2016. In (b) tetrahedra gluing by mean of group elements identification to construct a Fock space. Picture from M. Dupuis, J. P. Ryan, S. Speziale. *Discrete Gravity Models and Loop Quantum Gravity: a Short Review.* 2012.

by means of identification of group elements that give names to faces of different tetrahedra. This gluing procedure basically produce a continuum that can be thought off as a quantum fluid giving rise to spacetime.

Group field theory can be mathematically manipulated, starting from a partition function, to obtain several results that can be confronted with known aspects of the current accepted physical descriptions. Two of the main achievable results are the volume profile for the universe and phantom like dark energy effects. The former is provable to be non vanishing for each pregeometrical configuration state hence implying two possibilities to describe the birth of the univese. The first one is a quantum bounce; the second, quite more realistic, a *geometrogenesis*[4] in place of the Big Bang. Since, indeed, quantum behaviours have to kick in at some point and spacetime has to leave the place for the pregeometrical fluid of tetrahedra. The latter computable evidence we mentioned comes from quantum corrections to the theory that naturally bring to life an accelerated dynamics at late times. These effects could explain and give a less arbitrariness to effects at the moment explained with dark energy [84].

In this work we do not aim to give a full nor exact introductory description to this theory but nevertheless humbly refer the interested reader to it.

[4]With the word geometrogenesis is meant the emergent creation of spacetime from the underlying quantum structure described up until here.

Chapter 2

Geometrization of system dynamics

The general idea of this chapter is to show how to geometrize the dynamics of systems that one wants to take under consideration in terms of geodesic motion on auxiliary backgrounds. Identifying then the conformal isometries of these backgrounds provides a geometrization for symmetries of the systems themselves. This is a very natural approach motivated by the fact that the general relativistic dynamics of the gravitational field can be recasted in form of geodesic motion of a point-like particle on the superspace [60].

Two paths can be followed to pursuit this endeavour. The first one has been developed and employed quite much by and in the works [30–33, 40, 98] and more recently in [53] and is based on the analysis of field space, to wit the superspace. The second bases its roots on an extension of the latter and permits to perform a sort of a second geometrization of gravity. Indeed this formalism exibits a metric equipped with a covariantly constant null Killing Vector; meaning it can be viewed as a pp-wave. This leads to a correspondence between conformal isometries of pp-waves and dynamical symmetries of mechanical systems [42,43]. But applied to minisuperspace models of gravity – already describing homogeneous gravitational fields – it configures itself as a second geometrization of gravity. This is the Eisenhart-Duval's lift, introduced firstly in [46] and explained thoroughly in [24, 28]. This has already been used fruitfully in several other sets of systems like [25–27, 38, 42, 49, 50, 55, 56, 96].

2.1 Minisuperspace

Minisuperspaces are reduced models of gravity where some assumpions are made to simplify the systems at study. Notably homogeneity or some sort of geometrical symmetry, among other assumptions, restrict the dynamics to a finite set of degrees of freedom.

As introduced above, in this picture the dynamics can be seen as motion on a different space: the field space, because of its geometrization. Conformal Killing Vectors found for this space directly provide Noetherian symmetries of the system.

Some examples of minisuperspaces, the ones implemented in the following, are Friedmann-Lemaître-Robertson-Walker's homegeneous and isotropic model of cosmology, Bianchi's I model used to describe homogenious but anisotropic universes and also Schwarzschild's dynamics for black holes. Details for these will be found in respective chapters, below.

2.1.1 Line element and action

Let us start by considering a restriction of the gravitational theory in the ADM decomposition framework where no mixing terms of the $\mathrm{d}t\mathrm{d}x^i$ kind are present, the line element then reads

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -N^2(t)\mathrm{d}t^2 + h_{ij}[\chi^a(t), x^i]\mathrm{d}x^i \mathrm{d}x^j \tag{2.1.1}$$

where N is the lapse function and the dependence of the projected metric h_{ij} on the fields χ^a and spatial coordinates is explicitated. Note that in the fields both gravitational and matter ones are considered.

Then the Einstein-Hilbert's action of matter in a gravitational setting can be recasted in the one of a pointlike particle on the superspace

$$S = c\ell_P \int \mathrm{d}t \left[\frac{1}{2N} g_{ab}\dot{\chi}^a\dot{\chi}^b - NV(\chi)\right] \tag{2.1.2}$$

where a potential V, dependent on the fields and reflecting both the intrinsic curvature of the spacelike hypersurfaces and self interaction of the matter fields, has been introduced along with the notation ℓ_P for the Planck length and the very important constant c. The $c\ell_P$ factor comes from the standard $\ell_P^{-2} \propto (\hbar G)^{-2}$ coefficient in front of the Einstein-Hilbert action combined with the 3d fiducial volume V_0 descending from the integration over the spacial hypersurface appropriately cut-off. Since the Planck length gauges the UV scale while the fiducial volume the IR one, the factor c indeed encodes the powerful notion of the ratio

$$c \propto \frac{IR}{UV} \tag{2.1.3}$$

this is going to came quite useful in interpreting results both from arguments on the Schrödinger's group in section 1.3 on page 9 and computations for the minisuperspaces in analysis in chapters 3 and 4. The kinetic terms of the action include both the extrinsic curvature of constant t hypersurfaces and the kinetic term of the Lagrangian for the matter sector. It is expressed in language of the field space metric, known as supermetric[1].

As it is well know from literature, the equations of motion from this action are derived by requiring its stationarity w.r.t. lapse variations δN and field variations $\delta\chi^a$. These should be nothing but equivalent to imposing the Einstein's equations on the spacetime metric and the field equations for matter evolution over this same metric. The former gives the Hamiltonian constraint of the ADM formalism stating the time evolution of the spatial metric, while the latter leads to the momentum constraints, namely the projection of the Einstein's equations on the spacelike slices. In particular once we move to the Hamiltonian formalism

$$\pi_a \equiv \frac{\partial \mathcal{L}}{\partial \dot{\chi}^a} = \frac{1}{N} g_{ab}\dot{\chi}^b \,, \qquad H \equiv \pi_a \dot{\chi}^a - L = \frac{N}{2} g^{ab}\pi_a\pi_b + NV = Nh \tag{2.1.4}$$

from the invariance of the action w.r.t. the lapse function, the constraint reads

$$h \mathrel{\hat{=}} 0 \tag{2.1.5}$$

where we factored out of the Hamiltonian the laps function.

2.1.2 Symmetries from Conformal Killing Vectors

For free systems with vanishing potential the action in eq. (2.1.2) is simply a geodesic Lagrangian; the equations of motion of which – in particular the one descending from lapse variations invariance – impose that the spacetime metric field components χ^a follow null geodesics in superspace, with supermetric g_{ab}. Since conformal isometries of the supermetric map the set of null geodesics onto itself, and thus define automorphisms of the set of solutions of the equations of motion, Conformal Killing Vectors of the supermetric naturally define symmetries of the minisuperspace.

More precisely, consider a Conformal Killing Vector $\xi^a\partial_a$ on the field space, hence satisfying

$$\pounds_\chi g_{ab} = \varphi g_{ab} \tag{2.1.6}$$

where $\varphi(\chi)$ is a conformal rescaling factor. It is then a standard result that the scalar product between the velocity vector and Killing vector is conserved along null geodesics, being thus a constant of motion. So, introducing $\mathcal{O}_\xi = \xi^a\pi_a$, a straightforward calculation allows to check that

$$\{\mathcal{O}_\xi, H\} = -\frac{1}{2}\pi^a\pi^b \pounds_\xi g_{ab} = -\varphi H \underset{(H=0)}{\hat{=}} 0 \tag{2.1.7}$$

[1]Note that here indices in greek letters $\mu, \nu, \ldots$ are used for the spacetime, middle roman letters $i, j, k, \ldots$ for constant t hypersufaces and early roman letters $a, b, c, \ldots$ for superspace.

where use of the Hamiltonian constraint from eq. (2.1.5) was made to shows $\mathcal{O}_\xi$ is a weak Dirac observable of the system. This means that it is conserved along classical trajectories and subsequently generates a symmetry by Noether's theorem. Listing all the Conformal Killing Vectors of the field space metric therefore provides a simple way to identify conserved charges and symmetries of the model.

Note however that this procedure misses some tipe of symmetries. Firstly, the observables constructed in such way, by definition, are always linear in the momenta hence do not cover conserved charges of higher order in π_a −this issue could probably be solved by looking at Conformal Killing Tensors−. But further missed symmetries are the ones generated by charges dependent on time, i.e. evolving constants of motion

$$\frac{\mathrm{d}\mathcal{O}_\xi}{\mathrm{d}\tau} = \partial_\tau \mathcal{O}_\xi + \{\mathcal{O}_\xi, H\} = 0 \tag{2.1.8}$$

One elegant way to obtain these is through the Eisenhart-Duval's lift that is going to be presented in section 2.2 on the next page.

2.1.3 Note on generic non vanishing potential systems

We have seen how Conformal Killing Vectors of the superspace are related to Noetherian symmetries of systems with vanishing potential. In the opposite case in which instead it is not vanishing a sort of trick can be applied to recover the situation already taken in consideration. But more careful is needed and some extra terms must be taken into account in the end.

The original action in eq. (2.1.2) can be recasted to hide the potential factor as

$$S = c\ell_P \int \mathrm{d}\eta \left[\frac{1}{2} G_{ab}\dot{\chi}^a\dot{\chi}^b - 1\right] , \qquad G_{ab} = V(\chi) g_{ab} \tag{2.1.9}$$

in which the proper time $\mathrm{d}\eta = NV\mathrm{d}t$ was introduced along with the conformal rescale of the supermetric. Notice that in here the dot means derivations w.r.t. η.

The constant shift in the Lagrangian plays the non-trivial role of shifting the physical value of the Hamiltonian but would not contribute to the bulk equations of motion. Hence it can be dropped recovering once again a geodesic Lagrangian: solutions $\chi^a(\eta)$ now follow geodesics of the conformally rescaled supermetric G_{ab}.

Care is needed though in what concerns the lapse that seems vanished from the action but, on the contrary, is very much present hidden inside the definition of the new proper time. The original equation of motion w.r.t. the lapse variation then implies now the constraint

$$\frac{1}{2} G_{ab}\dot{\chi}^a\dot{\chi}^b - 1 = 0 \tag{2.1.10}$$

i.e. it restricts to geodesics that now have a fixed mass. If this is the case, the need to introduce correction terms proportional to –what could be called– *supermass* is clear; these will have to be added to the conserved charges for null geodesics. An example will be given in chapter 4.

2.2 Eisenhart-Duval lift

The Eisenhart-Duval's lift is a general method to geometrize any mechanical system, being it free or bounded by a potential, by mapping it onto null geodesic Lagrangians. It has been introduced in [46] and later on presented thoroughly in [24, 28]. The idea it bases on is to consider two (1+1) extra dimensions to embed the superspace into following some standardised prescription in the construction of the lifted line element and action. This result in a peculiar structure which provides a natural host for the geometrical realization of non-relativistic conformal symmetry as show by [44, 45].

In the present section the same path followed to analyse the superspace procedure is undertaken, starting from the bulding of the line element towards the proof that Conformal Killing Vectors of the lift are connected to symmetries of the system.

2.2.1 Line element and action

As mentioned above, the Eisenhart-Duval's lift prescribes a precise procedure to go from the superspace action to the lifted one. In particular starting from eq. (2.1.2) one finds the line element

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}w + g_{ab}\mathrm{d}\chi^a\mathrm{d}\chi^b - 2V(\chi)\mathrm{d}u^2 \tag{2.2.1}$$

and the action

$$S = \int \mathrm{d}\lambda \, \frac{1}{\mathcal{N}} \left[\dot{u}\dot{w} + \frac{1}{2N} g_{ab}\dot{\chi}^a\dot{\chi}^b - V(\chi)\dot{u}^2 \right] \tag{2.2.2}$$

where of course the new pair of variables is (u, w), the dot denotes derivative in the new affine parameter λ and, finally, the dinamical factor $\dot{u}^2$ is added to the potential term, yielding once again to a geodesic Lagrangian in the lifted space.

Then one can move to the Hamiltonian formalism by computing the conjugate momenta

$$p_u \equiv \frac{\partial\mathcal{L}}{\partial\dot{u}} = \frac{\dot{w} - 2V\dot{u}}{\mathcal{N}} \,, \qquad p_w \equiv \frac{\partial\mathcal{L}}{\partial\dot{w}} = \frac{\dot{u}}{\mathcal{N}} \,, \qquad p_a \equiv \frac{\partial\mathcal{L}}{\partial\dot{\chi}^a} = \frac{g_{ab}\dot{\chi}^b}{\mathcal{N}} \tag{2.2.3}$$

and Legendre transforming the Lagrangian

$$\mathcal{H} = \mathcal{L} = \mathcal{N} \left[p_u p_w + \frac{1}{2} g^{ab} p_a p_b + V p_w^2 \right] \tag{2.2.4}$$

that incidentally reads exactly like the Lagrangian itself. Once again, the invariance of the action w.r.t. the infinitesimal variation of the laps function – here $\delta\mathcal{N}$ – gives the constraint

$$\mathcal{H} \hat{=} 0 \tag{2.2.5}$$

2.2.2 Equivalence with the original system

By use of the Hamiltonian just found it is straightforward to see the remaining equations of motion to be

$$\dot{u} = \frac{\mathrm{d}u}{\mathrm{d}\lambda} = \{u, \mathcal{H}\} = \mathcal{N} p_w \,, \qquad \dot{w} = \frac{\mathrm{d}w}{\mathrm{d}\lambda} = \{w, \mathcal{H}\} = \mathcal{N}\left(p_u + 2V\right) \tag{2.2.6}$$

for the two newly considered coordinates,

$$\dot{\chi}^a = \frac{\mathrm{d}\chi^a}{\mathrm{d}\lambda} = \{\chi^a, \mathcal{H}\} = \mathcal{N} g^{ab} p_b \tag{2.2.7}$$

for the superspace fields and, finally, for their momenta

$$\dot{p}_a = \frac{\mathrm{d}p_a}{\mathrm{d}\lambda} = \{p_a, H\} = -\mathcal{N}\left(\frac{1}{2} p_b p_c \partial_a g^{bc} + p_w^2 \partial_a V\right) \tag{2.2.8}$$

Of course, since the variables u and w are cyclic – i.e. do not enter explicitely in the Lagrangian – their conjugate momenta are constant of motion; it is useful to set

$$p_w = 1 \tag{2.2.9}$$

and to see that following this assumption the constraint in eq. (2.2.5) is rewritable into the quite interesting relation

$$p_u + h \underset{p_w=1}{\hat{=}} 0 \tag{2.2.10}$$

where the Hamiltonian h – defined in eq. (2.1.4) – is the one from the original action. It is clear the association between variable u with the proper time coordinate in the superspace $\mathrm{d}\tau = N\mathrm{d}t$. In particular, it is possible to deparametrize the equations of motion of the lifted system in terms of $u \leftrightarrow \tau$ so to recover the equations of motion of the original system, eq. (2.1.2) showing that the geodesic equation for the massless test particle on the lift indeed reproduces the equations of the initial mechanical system. The fields part, eq. (2.2.7), becomes

$$\left.\frac{\mathrm{d}\chi^a}{\mathrm{d}\tau}\right|_{p_w=1} = \left.\frac{\mathrm{d}\chi^a}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}u}\right|_{p_w=1} = g^{ab} p_b = \{\chi^a, h\} \tag{2.2.11}$$

while the momenta part, eq. (2.2.8), reads

$$\left.\frac{\mathrm{d}p_a}{\mathrm{d}\tau}\right|_{p_w=1} = \left.\frac{\mathrm{d}p_a}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}u}\right|_{p_w=1} = \frac{1}{2}p_b p_c \partial_a g^{bc} + p_w^2 \partial_a V = \{p_a, h\} \tag{2.2.12}$$

The only thing left to show is that indeed Conformal Killing Vectors of the lift give symmetries of the system.

2.2.3 Symmetries from Conformal Killing Vectors

We have seen in section 2.1.2 on page 17 how Conformal Killing Vectors of the superspace generate symmetries of the system at study when this is described by the free motion – i.e. not restricted by any potential – of a pointlike particle in field space. With the Eisenhart-Duval's lift this condition is automatically satisfied since the potential term is made kinetical by construction, hence Conformal Killing Vectors of the lifted supermetric are generators of symmetries of the system.

In particular if we call $\mathcal{G}_{\hat{a}\hat{b}}$ the lifted supermetric[2] whose components read

$$\mathcal{G}_{uw} = 1 \ , \qquad \mathcal{G}_{uu} = -2V \ , \qquad \mathcal{G}_{ab} = g_{ab} \tag{2.2.13}$$

and starting from the Conformal Killing Vector of the superspace ξ costruct the vector X as

$$X^{\hat{a}}\partial_{\hat{a}} = X^u\partial_u + X^w\partial_w + \xi^a\partial_a \tag{2.2.14}$$

we have that this is a Conformal Killing Vector of the lifted space only if, in this same space, satisfy – as in eq. (2.1.6) – the defining relation

$$\pounds_X \mathcal{G}_{\hat{a}\hat{b}} = \varphi \mathcal{G}_{\hat{a}\hat{b}} \tag{2.2.15}$$

These indeed contains – in their (a, b) component – also the relation $\pounds_\xi g = \varphi g$; but in this case $\xi(\chi^a, u)$ can depend both on the fields and, most remarkably, time $u \leftrightarrow \tau$. Now to finally prove that these Conformal Killing Vectors of the Eisenhart-Duval's lift are indeed generators for symmetries of the system remains to build conserved charges and show that these are constants of motion w.r.t. time.

Following the same path seen in section 2.1.2 on page 17, it is possible to use the projection rule $\mathcal{O}_X = X^{\hat{a}}p_{\hat{a}}$ to find the charges that are looked for. These satisfy the relation $\left\{X^{\hat{a}}p_{\hat{a}}, \mathcal{H}\right\} = -\varphi\mathcal{H} \hat{=} 0$ and using the assumption of eq. (2.2.9) it is possible to translate it back to the original system.

$$\left.\mathcal{O}_X\right|_{p_w=1} = \left. -hX^u + X^w + \xi^a p_a \right|_{u=\tau} \tag{2.2.16}$$

[2] Here is introduced the hatted notation of the early roman letters in use to denote indices of the lifted superspace, in particular $\hat{a} = (u, w, a)$.

Upon further assuming that the vector X does not depend on the coordinate w, a straightforward calculation allows to check that this is indeed a constant of motion for the original system

$$\left| \begin{array}{l} \partial_w X = 0 \\ \pounds_X \mathcal{G} = \varphi \mathcal{G} \end{array} \right. \qquad \Longrightarrow \qquad \frac{\mathrm{d}\mathcal{O}_X}{\mathrm{d}\tau} = \partial_\tau \mathcal{O}_X + \{\mathcal{O}_X, h\} = 0 \tag{2.2.17}$$

The key remarks coming out of this treatment are mainly two. First is that this method produces evolving constants of motion, i.e. conserved charges which explicitly depend on time, which was not possible before but is indeed very much needed to find Schrödinger's charges. This follows directly by the dependence on u of the Conformal Killing Vector fields X. Second, despite working also on free systems and straightforwardly producing conserved charges, the Eisenhart-Duval's lift is complementary to the superspace method. Indeed these charges are non-trivial and explicitly time-dependent and thus come on top of the ones descending from the Conformal Killing Vectors of the supermetric.

2.3 Dimensionality and maximal symmetry

A final note is required on Conformal Killing Vectors and their relation with the dimensionality of the space they live on. In this section the main two focus will be on the number of Killing Vectors and the way to understand whether it is the maximal possible or not.

All results presented here are just a very brief summary of subjects thoroughly studied in the general relativity literature or specific ones, respectively, presented and recovered in much more detail in [4].

Number of Killing Vectors The maximal number of Killing Vectors for a d-dimensional geometry is given by

$$n_{\mathrm{KV}} = \frac{d\,(d+1)}{2} \tag{2.3.1}$$

Number of Conformal Killing Vectors The maximal number of Conformal Killing Vectors that a $(d \geq 3)$-dimensional curved manifold can admit is

$$n_{\mathrm{CKV}} = \frac{(d+1)\,(d+2)}{2} \tag{2.3.2}$$

while it can admit an infinity of linearly independent Conformal Killing Vectors if it is 2-dimensional.

Maximally symmetric spaces A d-dimensional curved background is maximally symmetric – i.e. it has the maximal number of Conformal Killing Vectors – if and only if its Weyl's and Cotton's tensors both vanish identically. The Weyl's tensor is defined as

$$W_{\kappa\lambda\mu\nu} = R_{\kappa\lambda\mu\nu} + \frac{2}{d-2}\left(g_{\kappa[\nu}R_{\mu]\lambda} + g_{\lambda[\mu}R_{\nu]\kappa}\right) + \frac{2Rg_{\kappa[\mu}g_{\nu]\lambda}}{(d-2)(d-1)} \tag{2.3.3}$$

and basically is a traceless Riemann's tensor since behaves as an indicator of the tidal forces an object feels moving along geodesics, encoding changing in shape but not in volume. While the Cotton's is a tensor concomitant on the metric defined as follows

$$C_{\lambda\mu\nu} = 2\nabla_{[\lambda}R_{\mu]\nu} - \frac{1}{(d-1)}\nabla_{[\lambda}Rg_{\mu]\nu} \tag{2.3.4}$$

Quite important to note is that the former always vanishes for $d \leq 3$ while the latter is not defined for $d \leq 2$. Therfore only Weyl's tensor is to be checked in 3-d spaces while there is no need to check neither of them for the patological 2d case.

The difference could not be clearer: for a system with d degree of freedom, the superspace is d-dimensional while the Eisenhart-Duval's lift is an higher $(d+2)$-dimensional manifold; it follows that their curvature properties and thus the maximal number of Conformal Killing Vectors they can have can be drastically different. By construction, this reflects on the underlying symmetries of the mechanical system which stands as their shadow. The Friedmann-Lemaître-Robertson-Walker's model of cosmology – that is going to be studied in chapter 3 – along with all the other systems with two degrees of freedom, is a very striking example: the associated superspace is a 2d background which is thus conformally flat. Therefore, it possesses an infinite set of linearly independent Conformal Killing Vectors. On the contrary, the Eisenhart-Duval's lift of these systems is a 4d background which possesses at most fifteen linearly independent Conformal Killing Vectors. Hence, some symmetries of the superspace can not have their counterpart in the Eisenhart-Duval's lift. Thus identifying the underlying symmetries of a given mechanical system via the Conformal Killing Vectors of its superspace or its Eisenhart-Duval's lift should be considered as two distinct approaches which can reveal different sets of symmetries.

Chapter 3

FLRW's cosmology

In this chapter, as well as in the following ones, explicit computations of what has been presented in chapter 2 are going to be carried out. In particular here the case of Friedmann-Lemaître-Robertson-Walker's (FLRW) minisuperspace model for cosmology is undertaken as example. Line elements and actions both in the superspace and Eisenhart-Duval's lift methodologies are found; from these, charges from the projection rule are going to be shown to satisfy $\mathfrak{sch}(2)$ algebra; explicit form of transformations but also the solutions to the dynamics in terms of conserved charges are going to be presented; as well as the Casimir operators; finally we will see that this Schrödinger's algebra is embeded in a broader conformal symmetry kind of structure.

3.1 Superspace approach

3.1.1 Line element and action

Friedmann-Lemaître-Robertson-Walker's is the simplest cosmological model consisting in a flat geometry filled with a massless scalar field and describes an homogeneous and isotropic universe.

Let us start from its well known line element

$$\mathrm{d}s^2 = -N^2(t)\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{3.1.1}$$

in which clearly $N(t)$ stands for the lapse function and $a(t)$ for the scale factor. From standard manipulations the Einstein-Hilbert's action in the presence of a matter scalar field ϕ is

$$S = \int_{\mathcal{M}} \mathrm{d}^4x\, \sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right] = V_0\int \mathrm{d}t\, \left[\frac{a^3}{2N}\dot{\phi}^2 - \frac{3}{8\pi G}\frac{a\dot{a}^2}{N}\right] \tag{3.1.2}$$

where the total derivative term has been dropped and V_0 is the fiducial volume of spacial integration. Introducing then the volume $z = a^{3/2}$ and using the Planck's length $\ell_P = \sqrt{12\pi G}$ one gets

$$S = c\ell_P \int \mathrm{d}t \left[\frac{\ell_P^2 z^2 \dot{\phi}^2 - 4\dot{z}^2}{2N} \right] \tag{3.1.3}$$

In remebrance of what said about eq. (2.1.3) the very important constant c has been factored out

$$c = \frac{V_0}{\ell_P^3} \tag{3.1.4}$$

which encodes the ratio between the IR and UV cutoffs of the symmetry reduced system.

Now it is direct to see that this corresponds to the action of the kind we saw in eq. (2.1.2) in the absence of a potential, meaning of a free system, with variables $\chi^a = \{\phi, z\}$ and with supermetric given by $g_{\phi\phi} = \ell_P^2 z^2$, $g_{zz} = -4$. Following step by step what done in chapter 2, the momenta and Hamiltonian are computed

$$\pi = \frac{c\ell_P^3}{N} z^2 \dot{\phi} \,, \qquad p = -\frac{4c\ell_P}{N}\dot{z} \,, \qquad H = Nh = \frac{N}{2c\ell_P^3}\left(\frac{\pi^2}{z^2} - \frac{\ell_P^2 p^2}{4}\right) \tag{3.1.5}$$

finally then, the equations of motion read

$$\frac{\dot{\phi}}{N} = \frac{\pi}{c\ell_P^3 z^2} \,, \qquad \frac{\dot{\pi}}{N} = 0 \,, \qquad \frac{\dot{z}}{N} = -\frac{p}{4c\ell_P} \,, \qquad \frac{\dot{p}}{N} = \frac{\pi^2}{c\ell_P^3 z^3} \tag{3.1.6}$$

along, of course, with the Hamiltonian constraint given by the variation w.r.t. the lapse function, $h \mathrel{\hat{=}} 0$.

3.1.2 Charges and $\mathfrak{W}$itt's algebra

Following the approach painted in section 2.1 on page 15 to find conserved charges from the Conformal Killing Vectors of the superspace one may note that from eq. (3.1.3) the supermetric clearly produces the line element

$$\mathrm{d}s^2 = z^2 \left(\ell_P^2 \mathrm{d}\phi^2 - \frac{4\mathrm{d}z^2}{z^2}\right) \tag{3.1.7}$$

This is a 2d manifold which is therefore conformally flat and, as remembered in section 2.3 on page 22 like any other 2d space, admits an infinite set of Conformal Killing Vectors which provides an infinite amount of observables for this cosmological system. In particular, conformally rescaling the supermetric and defining two new fields

$$\mathrm{d}s^2 = \ell_P^2 \mathrm{d}\phi^2 - \frac{4\mathrm{d}z^2}{z^2} = \mathrm{d}\chi^+ \mathrm{d}\chi^- \,, \qquad \chi^\pm = \ell_P \phi \pm 2 \log z \tag{3.1.8}$$

the exact situation described by eq. (B.0.2) is recovered. So if one applies the results found later on in Appendix B, finds the two families labelled by $\pm$ of Conformal Killing Vectors of eq. (B.0.4) and by implementing the conjugate momenta to $\chi^\pm$

$$p_\pm = \frac{1}{2}\left(\frac{\pi}{\ell_P} \pm \frac{zp}{2}\right) \tag{3.1.9}$$

can project them with the usual rule $\mathcal{O}_\xi = \xi^a p_a$ to find the charges of interest, thus explicitly writable as

$$W_F^\pm = F^\pm p_\pm = \frac{F^\pm(\ell_P \phi \pm 2 \log z\,)}{2}\left(\frac{\pi}{\ell_P} \pm \frac{zp}{2}\right) \tag{3.1.10}$$

The dependance of the arbitrary function on the new variables is highlighted.

From eq. (B.0.5) –but it is straightforward to check for the specific case at hand, too– the charges found in this way satisfy the $\mathfrak{Witt}$'s algebra

$$\left\{W_F^\pm, W_G^\mp\right\} = 0 \,, \qquad \left\{W_F^\pm, W_G^\pm\right\} = 2W_{[F,G]}^\pm \tag{3.1.11}$$

where square brackets of the arbitrary functions mean

$$[F, G] = F'G - G'F \tag{3.1.12}$$

Finally can be shown that they indeed are weak Dirac observables of the original system as per

$$\left\{W_F^\pm, h\right\} = \left(F'_\pm \pm \frac{F_\pm}{2}\right) h \underset{(H=0)}{\hat{=}} 0 \tag{3.1.13}$$

As a final note, let us point that demanding for strong Dirac observables by this construction imposes that $F'_\pm = \mp F/2$ resulting in the charges

$$W^\pm = \frac{e^{\mp \ell_P \phi/2}}{2}\left(\frac{\pi}{\ell_P} \pm \frac{zp}{2}\right) \tag{3.1.14}$$

that coincide, up to a meaningless constant factor, with the generators of translation in the superspace that are going to be found with the Eisenhart-Duval's lift in eq. (3.2.3) and are there called $P_\pm$.

3.2 Eisenhart-Duval's lift approach

As expected from the discussions in section 2.3 on page 22 on the number of Conformal Killing Vectors a generic d-dimensional space posses, it has been shown that Friedmann-Lemaître-Robertson-Walker's minisuperspace model entails an infinite number of them: a set that satisfy a couple of copies of the $\mathfrak{Witt}$'s algebra. But the aim of the present work is to unearth the hidden Schrödinger's algebra present in this model. We encountered the way to do so in chapter 2 when Eisenhart-Duval's lift was presented. Let us then apply it to find what has been looked for.

Following the directions given by the lift procedure in eq. (2.2.1) onto the action recovered in eq. (3.1.3), the following lifted line element is directly found

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}w + c\ell_P\left(\ell_P^2 z^2\mathrm{d}\phi^2 - 4\mathrm{d}z^2\right) \tag{3.2.1}$$

It is possible to show that the Weyl's and Cotton's tensors associated to this 4d geometry both vanish. Therefore, it is conformally flat and thus maximally symmetric. It follows that it possesses the maximal number of Conformal Killing Vectors, i.e. fifteen. This set can be organized w.r.t. their commutativity with the covariant constant null Killing vector defined by $N^{\hat{a}}\partial_{\hat{a}} = \partial_w$. For the moment the focus is on the ones that commute; an in-deep analisys of the remaining ones is postponed to section 3.4 on page 33 but for the complete set the reader is refered to Appendix C.

3.2.1 Charges and $\mathfrak{sch}(2)$ algebra

Having found the Conformal Killing Vectors it is possible to follow the usual projection rule to find the charges but the rules set in eq. (2.2.9) must also be taken into account in this case. Indeed we must remember that in the Eisenhart-Duval's lift the variable u can be identified with the time of the system, once the constant conjugate momenta to w is set to one. This has deep repercussions, for example in the case $\partial_u \to p_u = -h$. The charges, communting with the null vector, found this way are

$$\left|\begin{aligned} &Q_+ = \ell_P h \\ &Q_0 = \frac{1}{2}zp + \tau h \\ &Q_- = cz^2 - \frac{1}{2\ell_P}\left(\tau z p + \tau^2 h\right) \end{aligned}\right. \tag{3.2.2}$$

that correspond to the charges already discussed at length in [6, 8, 11, 13] and that there are shown to form a $\mathfrak{sl}(2,\mathbb{R})$ algebra; and

$$\begin{aligned} P_\pm &= e^{\mp \ell_P \phi/2}\left(\frac{p}{2} \pm \frac{\pi}{\ell_P z}\right) , \qquad J = \frac{\pi}{\ell_P} \\ B_\pm &= e^{\mp \ell_P \phi/2}\left[2cz - \frac{\tau}{\ell_P}\left(\frac{p}{2} \pm \frac{\pi}{\ell_P z}\right)\right] \end{aligned} \tag{3.2.3}$$

It is straightforward to check that these charges are indeed conserved. In particular, while the charges $P_\pm$ and J are strong Dirac observables, the charges Q_0, Q_- and $B_\pm$ depend explicitly on time and play the role of evolving constants of motion which generate dynamical symmetries of the system.

These charges satisfy the algebra

$$\begin{aligned} &\{Q_+, Q_-\} = Q_0 , \qquad \{Q_0, Q_\pm\} = \pm Q_\pm \\ &\{Q_-, P_\pm\} = \frac{1}{2}B_\pm , \qquad \{Q_0, P_\pm\} = +\frac{1}{2}P_\pm , \qquad \{J, P_\pm\} = \pm\frac{1}{2}P_\pm \\ &\{Q_+, B_\pm\} = P_\pm , \qquad \{Q_0, B_\pm\} = -\frac{1}{2}B_\pm , \qquad \{J, B_\pm\} = \pm\frac{1}{2}B_\pm \end{aligned} \tag{3.2.4}$$

that is recognisable as a two dimensional centrally extended Schrödinger's algebra as the one presented in eq. (1.3.6), with central extension given by the bracket

$$n = \{B_\mp, P_\pm\} = 2c \tag{3.2.5}$$

The generators $P_\pm$ and $B_\pm$ constitute the two $\mathbb{R}^2$ sectors of the algebra, while J is the $\mathfrak{so}(2)$ one and, as already stated, the Q's, form the $\mathfrak{sl}(2,\mathbb{R})$ remaining part. They respectively generate translations, Galilean boosts, rotations in the P's and B's planes and Möbius transformations. Detailed forms of these transformations are going to be seen in section 3.3 on the following page. Note, already, though that all of these are not at all transformations in space or spacetime but in field space.

3.2.2 Casimirs

The three Casimir operators associated with the 2d Schrödinger's algebra have been introduced generically in section 1.3 on page 9, here the explicit computations for the specific case are undertaken. In particular eq. (1.3.9) results into

$$\mathcal{C}_1 = \frac{2c\pi}{\ell_P} \tag{3.2.6}$$

which corresponds to the conserved kinetic energy of the scalar matter field; while the second Casimir, the cubic one, can be seen as proportional to

$$\mathcal{C}_2 \propto \pi^2 \tag{3.2.7}$$

Finally the one given by the central extension from eq. (1.3.11) here can be recovered by comparing eq. (3.2.5) with eq. (3.1.4)

$$n = \frac{2V_0}{\ell_P^3} \tag{3.2.8}$$

The fiducial volume V_0, introduced as a regulator in the symmetry reduction process, embodies the *IR* cut-off of the Friedmann-Lemaître-Robertson-Walker's model while the Planck length ℓ_P gives us the UV scale at which gravitational effects arise. Their ratio stands as the effective number of fundamental Planckian cells contained in the region of fiducial volume.

The ideas that resonate with this result are presented in section 1.3.3 on page 12 and grow their roots in the hint of an underlying discrete structure – as in lattice quantum gravity approaches or loop quantum gravity [75] –. Moreover, if the foundamental spatiotemporal cells are associated to pregeometrical quantum constituents – as in tensorial group field theories [81, 82] – this central extension can be understood as directly measuring the average population of the quantum many bodies states whose dynamics is collectively described by the gravitational model at study.

3.3 Transformations and solutions of the dynamics

3.3.1 Symmetry transformations

In this breif section symmetry transformations of the action eq. (3.1.3), generated by the conserved charges identified above but also the explicit invariance of action itself w.r.t. them are given.

Translations and Galilean boosts generated respectively by the conserved charges $P_\pm$ and $B_\pm$ can be compactly written as

$$\begin{aligned} \tau &\longmapsto \tilde{\tau} = \tau \\ z &\longmapsto \tilde{z}(\tilde{\tau}) = z(\tau) + \frac{e^{\mp \ell_P \phi/2}}{2}\zeta(\tau) \\ \phi &\longmapsto \tilde{\phi}(\tilde{\tau}) = \phi(\tau) \pm \frac{e^{\mp \ell_P \phi/2}}{\ell_P z}\zeta(\tau) \end{aligned} \tag{3.3.1}$$

where $\zeta(\tau)$ is an arbitrary function. Considering the infinitesimal version of these transformations, the action transforms as

$$\delta_\zeta S = \frac{c\ell_P}{2}\int \mathrm{d}\tau\, \left\{\ddot{\zeta} z e^{\mp\ell_P\phi/2} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\dot{\zeta} z e^{\mp\ell_P\phi/2}\right) + \mathcal{O}(\zeta^2)\right\} \tag{3.3.2}$$

which shows that the above transformations are indeed Noether symmetries provided $\ddot{\zeta} = 0$, thus implying

$$\zeta(\tau) = \beta + \vartheta\tau \tag{3.3.3}$$

the two constants β and ϑ are introduced to parametrize the symmetry.

The remaining symmetries generated by the charges Q's correspond to the following conformal transformations

$$\begin{aligned} \tau &\longmapsto \tilde{\tau} = f(\tau) \\ z &\longmapsto \tilde{z}(\tilde{\tau}) = \dot{f}^{1/2}(\tau) z(\tau) \\ \phi &\longmapsto \tilde{\phi}(\tilde{\tau}) = \phi(\tau) \end{aligned} \tag{3.3.4}$$

at the infinitesimal level is possible to rewrite the generic function $f(\tau) \simeq \tau + \varsigma(\tau)$ to get the variation of the action

$$\delta_\varsigma S = c\ell_P \int \mathrm{d}\tau\, \left\{\dddot{\varsigma} z^2 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[2\ddot{\varsigma} z^2 + \varsigma\left(\ell_P^2 z^2 \dot{\phi}^2 - 4\dot{z}^2\right)\right] + \mathcal{O}(\varsigma^2)\right\} \tag{3.3.5}$$

this is the variation related to a Noether symmetry provided $\dddot{\varsigma} = 0$ which consequently imposes

$$\varsigma(\tau) = \sigma_- + \sigma_0\tau + \sigma_+\tau^2 \tag{3.3.6}$$

once again the real constants σ's are introduced to parametrize the symmetry. The transformations presented initially in [8] for the flat FLRW cosmological model have been retrieved.

Of course applying the Noether theorem, we recover the conserved Noether charges from the total derivative terms of the action variations computed above. This leads back to the Schrödinger charges derived in eq.s (3.2.2 − 3.2.3).

3.3.2 Solutions of the dynamics

Now, let us show that the knowledge of this charge algebra allows to algebraically solve the Friedmann-Lemaître-Robertson-Walker's dynamics. It is possible to distinguish between the information encoded on one hand in the strong Dirac observables, such as $P_\pm$ and J, and on the other hand in the evolving constants of motion, i.e. the charges which depend explicitly on the time coordinate.

The first charges allow to solve for the deparametrized dynamics. For instance, the translation charges $P_\pm$ permit to directly solve the deparametrized trajectories $v(\phi)$ that reads

$$v(\phi) \underset{(H=0)}{\hat{=}} e^{\mp \ell_P \phi} \left(\frac{Q_0 \pm J}{P_\pm} \right)^2 \tag{3.3.7}$$

These conserved charges are to be understood as relational observables, in the sense that they encode time independent relations between the dynamical variables, telling how these vary in terms of each others. Having a complete set of such observables it is feasible to deparametrize the system and tell the evolution of all dynamical variables in terms of any single variable of the system. Identifying those relational observables is thus exactly equivalent to solve for the timeless deparametrized trajectories of the system itself.

On the contrary, the evolving constants of motion allow to algebraically determine the physical trajectories w.r.t. the time coordinate τ. Remember that on-shell, the Hamiltonian of the system has to vanish which reflects its relativistic nature. We have therefore $Q_+ \hat{=} 0$. Combining the expressions of Q_- and Q_0 allows to solve for the physical trajectory $v(\tau)$ which reads

$$v(\tau) \underset{(H=0)}{\hat{=}} \frac{2}{N} \left[Q_0 \frac{\tau}{\ell_P} + Q_- \right] \tag{3.3.8}$$

and we recover the linear growth of the volume in presence of a scalar field driving the cosmic expansion. Notice that the sign of Q_0 determines the expanding or contracting behavior of the universe. This linear evolution of the spatial volume clearly leads to a finite time crash, either in the past if Q_0 is positive or in the future if it is negative. This singularity is a standard feature of flat Friedmann-Lemaître-Robertson-Walker's cosmology.

In order to solve the profile of the scalar field, notice that

$$Q_+ = -\frac{1}{2c} P_+ P_- \underset{(H=0)}{\hat{=}} 0 \tag{3.3.9}$$

so that physical trajectories correspond to either $P_- = 0$ and $P_+ \neq 0$ or $P_- \neq 0$ and $P_+ = 0$. Choosing the first case, we can use the profile of $z(\tau)$ and the charge P_+ to find

$$\phi(\tau) \underset{(H=0)}{\hat{=}} -\frac{1}{\ell_P} \log \left[\frac{2}{N} \left(Q_0 \frac{\tau}{\ell_P} + Q_- \right) \right] + \frac{2}{\ell_P} \log \left[\frac{Q_0 + J}{P_+} \right] \tag{3.3.10}$$

Thus, as expected, the conserved charges provide a complete set of observables which fully encode the Friedmann-Lemaître-Robertson-Walker's dynamics. As a result, the cosmological dynamics can be algebraically solved solely based on the knowledge of the Schrödinger's cosmological observables.

3.4 $SO(4,2)$ embedding

Has already been anticipated that a total of fifteen Conformal Killing Vectors can be computed for the Friedmann-Lemaître-Robertson-Walker's model starting from the Eisenhart-Duval's lift. Up until now only the ones commuting with the null Killing vector $N^{\hat{a}}\partial_{\hat{a}} = \partial_w$ have been taken into consideration to show the presence of the Schrödinger's algebra within the system field space. But, if also the others are considered, they can be recasted to show they actually form the broader $\mathfrak{so}(4,2)$ algebra introduced in detail in the present work in appendix A.

This section does not aim to provide for the explicit form of these not commuting extra Conformal Killing Vectors – that are listed in appendix C – nontheless it is important to note that the lifted line element found in eq. (3.2.1) is invariant under the swap $u \leftrightarrow w$. Hence it is possible to define the concept of *duality* w.r.t. this swap and find both the null Killing vector itself $N^{\hat{a}}\partial_{\hat{a}} = \partial_w = \tilde{Q}_+^{\hat{a}}\partial_{\hat{a}}$ and four of the six charges left: $\tilde{Q}_0$, $\tilde{Q}_-$ and $\tilde{B}_\pm$. Intresting to note is the fact that the latters are cubic in the momenta, as opposed to the charges found previously that are quadratic.

Finally, the last two charges, also cubic in the momenta, are found to be the $Y_\pm$ that are clearly conserved since are polynomial combinations of the already derived constants of motion with extra Hamiltonian factors in h.

Let us get to the main objective of the present section, then. Considering the flat metric and the coordinates[1]

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \qquad x^\alpha = \begin{pmatrix} -u+w \\ u+w \\ ze^{+\phi} \\ ze^{-\phi} \end{pmatrix} \tag{3.4.1}$$

a given vector field can be decomposed as

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha} = \left(-\partial_u + \partial_w \ , \ \ \partial_u + \partial_w \ , \ \ e^{-\phi}\left[\partial_z + \frac{\partial_\phi}{z}\right] \ , \ \ e^{+\phi}\left[\partial_z - \frac{\partial_\phi}{z}\right] \right) \tag{3.4.2}$$

The generators of the $\mathfrak{so}(4,2)$ Lie algebra given by eq. (A.0.2) which satisfy the standard commutation relations therein can then be rewritten in terms of the

[1] Where the indeces in early greek letters assume values in the set $\{0, 1, +, -\}$.

coordinates of the lift; starting from the rotational generators $M_{\alpha\beta}$ as

$$\left|\begin{aligned} & M_{01} = 2\left(u\partial_u - w\partial_w\right) \\ & M_{0\pm} = e^{\mp\phi}\left[(u-w)\left(\partial_z \pm \frac{\partial_\phi}{z}\right) - z\left(-\partial_u + \partial_w\right)\right] \\ & M_{1\pm} = e^{\mp\phi}\left[(u+w)\left(\partial_z \pm \frac{\partial_\phi}{z}\right) - z\left(\partial_u + \partial_w\right)\right] \\ & M_{+-} = -2\partial_\phi \end{aligned}\right. \tag{3.4.3}$$

while the special conformal transformation generators are given by

$$\left|\begin{aligned} & K_0 = 4\left(u^2 + \frac{z^2}{2}\right)\partial_u - 4\left(w^2 + \frac{z^2}{2}\right)\partial_w + 4z\left(u-w\right)\partial_z \\ & K_1 = 4\left(u^2 - \frac{z^2}{2}\right)\partial_u + 4\left(w^2 - \frac{z^2}{2}\right)\partial_w + 4z\left(u+w\right)\partial_z \\ & K_\pm = e^{\mp\phi}\left[4z\left(u\partial_u + w\partial_w\right) + \left(2z^2 - 4uw\right)\partial_z \mp \frac{(2z^2+4uw)}{z}\partial_\phi\right] \end{aligned}\right. \tag{3.4.4}$$

Finally, the dilation takes the form

$$D = 2\left(u\partial_u + w\partial_w + z\partial_z\right) \tag{3.4.5}$$

At this point, by reorganizing the $\mathfrak{so}(4,2)$ generators, the Schrödinger's ones can be extracted. In particular they are given by the three $\mathfrak{sl}(2,\mathbb{R})$ charges

$$Q_+ = \frac{1}{8}\left(K_1 + K_0\right) \qquad Q_0 = -\frac{1}{4}\left(D + M_{01}\right) \qquad Q_- = \frac{1}{2}\left(T_1 - T_0\right) \tag{3.4.6}$$

together with

$$P_\pm = P_\pm \qquad B_\pm = -\frac{1}{2}\left(M_{1\pm} + M_{0\pm}\right) \qquad J = -\frac{1}{2}M_{+-} \tag{3.4.7}$$

The dual generators coming from the $u \leftrightarrow w$ swap instead read

$$\tilde{Q}_+ = \frac{1}{8}\left(K_1 - K_0\right) \qquad \tilde{Q}_0 = \frac{1}{4}\left(D - M_{01}\right) \qquad \tilde{Q}_- = \frac{1}{2}\left(T_1 + T_0\right) \tag{3.4.8}$$

and

$$\tilde{B}_\pm = -\frac{1}{2}\left(M_{1\pm} - M_{0\pm}\right) \tag{3.4.9}$$

Finally, the algebra is completed by the two remaining generators

$$Y_{\pm} = \frac{1}{2} K_{\pm} \tag{3.4.10}$$

As a final remark, this $\mathfrak{so}(4,2)$ algebra plays a very similar role to the conformal symmetries of the quantum two body problems like the spectral analysis of the hydrogen atom [69]. Thus the finite transformations do not seem especially relevant at the classical level but they will definitely be relevant to the quantization and spectrum analysis of the quantum theory.

Chapter 4

Schwarzschild's black hole dynamics

As mentioned at the beginning of chapter 3, the present one will show the explicit computations introduced in chapter 2 about the geometrization of a space applied specifically to the case of Schwarzschild's black hole mechanics.

Once again, following the same steps taken before, the line element and action both in the superspace and Eisenhart-Duval's lift approaches will be computed; from these Conformal Killing Vectors and, through the usual projection rule, associated charges are going to be shown to satisfy the $\mathfrak{sch}(2)$ algebra; transformations and solutions to the dynamics will be given explicitly; as well as the Casimir operators for the algebra. Moreover, since this case possesses two degree of freedom, as the Friedmann-Lemaître-Robertson-Walker's one, and satisfies the conditions to be maximally symmetric in its lift, it will be possible to find a wider embedding algebra. Finally an extension to the (anti) de Sitter case will be taken into consideration, since only an simple corrective term is to be added for this purpose.

Previous investigations on the dynamical symmetries of the Schwarzschild's mechanics on which this chapter laying its foundations are presented in [14, 51, 52].

4.1 Superspace approach

4.1.1 Line element and action

Consider the spherically symmetric geometry with line element

$$\mathrm{d}s^2 = \epsilon\left(-N^2(x)\mathrm{d}x^2 + A^2(x)\mathrm{d}y^2\right) + \ell_s^2 B^2(x)\mathrm{d}\Omega^2 \tag{4.1.1}$$

where $N(x)$ is the lapse. ℓ_s is a fiducial length scale encoding the curvature on the fiducial 2d sphere and $\epsilon = \pm 1$ dictates the region of spacetime covered by the metric allowing us to treat both the interior and the exterior regions of the black hole at once. To map the line element onto the Schwarzschild's metric, the coordinate x is the radial coordinate, while the y coordinate is the time coordinate. The sector $\epsilon = -1$ describes the usual exterior of the Schwarzschild's metric, for which x is spacelike and y is timelike and for which we use a timelike foliation of the bulk. While the sector $\epsilon = +1$ describes the interior of the Schwarzschild's black hole, for which x becomes timelike and y spacelike and where we use a spacelike foliation of the geometry. In this regard, Fig. 4.1 is reported from [14]; there the interested reader may find a more detailed treatment and presentation of the subject.

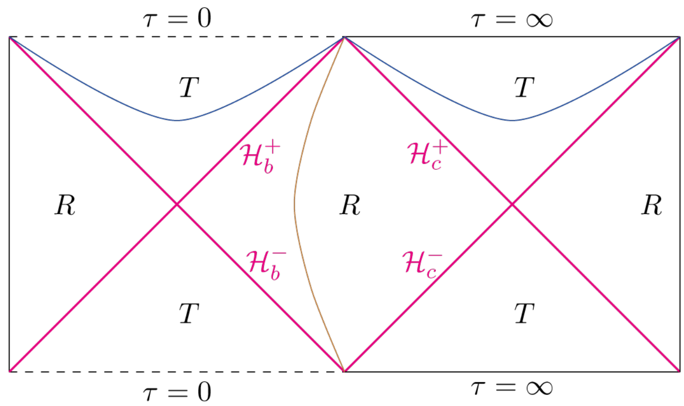

Figure 4.1: Penrose diagram of the Schwarzschild-dS black hole. The T regions correspond to the region inside the black and white hole horizons $\mathcal{H}_b^{\pm}$ and the regions beyond the cosmological horizon $\mathcal{H}_c^{\pm}$. They are foliated by spacelike hypersurface of constant time (blue). The R region corresponds to the region between the black hole horizon and the cosmological horizon and is foliated by timelike hypersurface (brown).

The associated Einstein-Hilbert's action governing the dynamics then reads

$$S_\epsilon = \int_{\mathcal{M}} \mathrm{d}^4x \, \sqrt{-g}\frac{R}{2\ell_P^2} = \epsilon c \ell_P \int \mathrm{d}\tau \, \left[\frac{\epsilon}{\ell_s^2} - A^2\dot{B}^2 - 2AB\dot{B}\dot{A}\right] \tag{4.1.2}$$

where a new gauge fixed time coordinate $\mathrm{d}\tau = NA\mathrm{d}x$ is introduced so that the potential term in the action is simplified to a constant; the total derivative term coming from direct computation of the Ricci's curvature is dropped as not influencing the dynamics and, clearly, the dotted variables are derived w.r.t. the new variable τ.

The form of eq. (2.1.2) is recovered and the important constant c is factored out, explicitly in this case it entails

$$c = \frac{\ell_0 \ell_s^2}{\ell_P^3} \tag{4.1.3}$$

in which the length ℓ_0 sets the size of the system in the y direction.

To show that these reasonings indeed correspond to Schwarzschild's black hole dynamics we solve the equations of motion that throughout a straightforward computation give the following profiles for the A and B fields

$$A^2(\tau) = -\frac{\epsilon}{\mathcal{C}^2\ell_s^2}\frac{\tau-\tau_+}{\tau-\tau_-}\,, \qquad B(\tau) = \mathcal{C}\,(\tau-\tau_-) \tag{4.1.4}$$

with $\mathcal{C}$ and $\tau_\pm$ constant of integration. Then upon performing both the translation $\tau \to \tau + \tau_-$ and the coordinates rescalings $\tau \to \mathcal{C}\tau/\ell_s$ and $y \to \mathcal{C}y/\ell_s$, the line element in eq. (4.1.1) trasforms to the usual Schwarzschild's form

$$\mathrm{d}s^2 = -\left(1-\frac{\ell_M}{\tau}\right)\mathrm{d}\tau^2 - \left(1-\frac{\ell_M}{\tau}\right)^{-1}\mathrm{d}y^2 + \tau^2\mathrm{d}\Omega^2 \tag{4.1.5}$$

with the Komar's mass of the Schwarzschild's black hole reading $\ell_M = \tau_+ - \tau_-$.

As far as the aim of the present work is concerned – i.e. looking for the hidden Schrödinger's symmetry – it is easier to perform computations with redefined fields. In particular it is more comfortable to write

$$V_1 = B^2\,, \qquad V_2 = \frac{A^2B^2}{2} \tag{4.1.6}$$

so that the action in eq. (4.1.2) transforms to

$$S_\epsilon = \epsilon c\ell_P\int \mathrm{d}\tau\,\left[\frac{\epsilon}{\ell_s^2} + \frac{V_2\dot{V}_1^2 - 2V_1\dot{V}_1\dot{V}_2}{2V_1^2}\right] \tag{4.1.7}$$

from which the conjugate momenta and the Hamiltonian are get

$$P_1 = \epsilon c\ell_P\left(\frac{V_2\dot{V}_1 - V_1\dot{V}_2}{V_1^2}\right)\,, \qquad P_2 = -\epsilon c\ell_P\frac{\dot{V}_1}{V_1} \tag{4.1.8}$$

$$h = -\frac{1}{\epsilon c\ell_P}\left(V_1P_1P_2 + \frac{1}{2}V_2P_2^2\right) - \frac{c\ell_P}{\ell_s^2} \tag{4.1.9}$$

Now it is clear that, being the potential term of the Hamiltonian a mere constant, it is convenient to work with a shifted Hamiltonian defined as $\tilde{h} = h + c\ell_P/\ell_s^2$ so that on-shell the vanishing Hamiltonian conditions reads[1] $\tilde{h}\,\hat{=}\,c\ell_P/\ell_s^2 \equiv \varpi$.

In this framework the equations of motion can be retrieved

$$\dot{V}_1 = -\frac{V_1P_2}{\epsilon c\ell_P}\,, \qquad \dot{P}_1 = \frac{P_1P_2}{\epsilon c\ell_P}\,, \qquad \dot{V}_2 = -\frac{C}{c\ell_P}\,, \qquad \dot{P}_2 = \frac{P_2^2}{2\epsilon c\ell_P} \tag{4.1.10}$$

the following phase space function[2] was introduced

$$C = -\epsilon\,(V_1P_1 + V_2P_2) \tag{4.1.11}$$

---

[1]It will reveal useful to define the constant Hamiltonian shift ϖ.

[2]Note that it is a weak Dirac observable $\{C, h\} = -\epsilon h \,\hat{=}\, 0$.

4.1.2 Charges and $\mathfrak{Witt}$'s algebra

We are using the superspace approach and we ended up with the set of variables $\chi^a = \{V_1, V_2\}$; from eq. (4.1.7) the supermetric clearly produces the line element

$$\mathrm{d}s^2 = \frac{V_2}{V_1^2}\mathrm{d}V_1^2 - \frac{2}{V_1}\mathrm{d}V_1\mathrm{d}V_2 \tag{4.1.12}$$

It is imperative to already notice that this case is quite different from the one treated in chapter 3 since, there, the action eq. (3.1.3) does not contain any potential term, such that the cosmological dynamics can be mapped to null geodesics in superspace. On the contrary, the symmetry reduced action for the black hole mechanics eq. (4.1.7) contains a constant potential. What remains unchanged is the fact that this is a 2d manifold which is therefore conformally flat and, as any 2d space – like we remembered in section 2.3 on page 22 – admits an infinite set of Conformal Killing Vectors. Nevertheless, as introduced in section 2.1.3 on page 18, this does not provide anymore conserved charges of the system since the potential term of the action at the level of the superspace plays the role of a non vanishing mass[3]. Corrective terms proportional to this supermass needs then to be added in order to recover well defined weak observables. In general, though, this has the consequence that the two associated copies of the $\mathfrak{Witt}$'s charges do not commute anymore [39].

Let us get to the explicit computation of the charges and the $\mathfrak{Witt}$'s algebra then. Following the same procedure of chapter 3, the line element can be conformally rescaled into the form eq. (B.0.2)

$$\mathrm{d}s^2 = \mathrm{d}\chi^+\mathrm{d}\chi^- \ , \qquad \chi^+ = \sqrt{V_1} \ , \qquad \chi^- = -\frac{V_2}{\sqrt{V_1}} \tag{4.1.13}$$

and from the considerations done in appendix B it is possible to use the conjugate momenta defined as

$$p_+ = 2\sqrt{V_1}P_1 + \frac{V_2 P_2}{\sqrt{V_1}} \ , \qquad p_- = 2\sqrt{V_1}P_2 \tag{4.1.14}$$

through the usual projection rule $\mathcal{O}_\xi = \xi^a p_a$ applied on the Conformal Killing Vectors, to recover the charges $W_F^\pm = F^\pm p_\pm$. As previously anticipated, this charges are not conserved on-shell indeed

$$\left\{W_F^\pm, h\right\} = F'_\pm \tilde{h} \underset{(H=0)}{\hat{=}} F'_\pm \varpi \tag{4.1.15}$$

[3]Depending on the sign of this mass contribution, the dynamics is either mapped to timelike or spacelike geodesics of the supermetric

and in order to correct this issue one can take into account the extra terms due to the supermass contribution to get to the corrected charges

$$\overline{W}_F^+ = F^+ \left(p_+ - \frac{2\epsilon c\ell_P \varpi}{p_-}\right) , \qquad \overline{W}_F^- = F^- \left(p_- + \frac{\varpi p_-}{\tilde{h}}\right) \tag{4.1.16}$$

so that on-shell $\left\{\overline{W}_F^\pm, h\right\} = F'_\pm \left(\tilde{h} - \varpi\right) \hat{=} 0$ meaning these charges are conserved. Once again, as a final note, is worth mentionig that requiring for strong Dirac obsevables means in this case to ask $F'_\pm = 0$ hence, up to a meaningless constant factor, the charges coincide with the generators of translation in the superspace that are going to be found in eq. (4.2.4) with the Eisenhart-Duval's lift method.

Finally, it is straightforward to show that each set of charges form a $\mathfrak{Witt}$'s algebra, eq. (B.0.5); but do not commute between themselves

$$\left\{W_F^+, W_G^-\right\} = 2\epsilon c\ell_P\varpi \left[\frac{2F'_+G^-}{p_+}\left(1 + \frac{\varpi}{2\tilde{h}}\right) + \frac{F^+G'_-}{2p_-}\left(1 + \frac{2\varpi}{\tilde{h}}\right)\right] \neq 0 \tag{4.1.17}$$

this is due to the supermass corrective terms.

4.2 Eisenhart-Duval's lift approach

We have seen the infinite dimensional $\mathfrak{Witt}$'s algebra prouced by the two sets of generators found with the superspace analysis. But in order to find the hidden Schrödinger's algebra that is looked for, the Eisenhart-Duval's lift has to be implemented.

Indeed like for the Friedmann-Lemaître-Robertson-Walker's case in chapter 3, the lifted space is 4d and it is possible to compute both the Weyl's and Cotton's tensors associated to it and to show that they are vanishing. The line element found applying eq. (2.2.1) on the action[4] in eq. (4.1.7)

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}w + \epsilon c\ell_P \left(\frac{V_2}{V_1^2}\mathrm{d}V_1^2 - \frac{2}{V_1}\mathrm{d}V_1\mathrm{d}V_2\right) \tag{4.2.2}$$

hence is of a conformally flat, and so maximally symmetric, manifold. That, we have already seen, is associated to a total of fifteen Conformal Killing Vectors.

[4]Note that in this case the constant potential does not really enter meaningfully in the computations for the algebra. Indeed the line element, considered the potential, would have been

$$\mathrm{d}s^2 = \frac{2c\ell_P}{\ell_s^2}\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}w + \epsilon c\ell_P \left(\frac{V_2}{V_1^2}\mathrm{d}V_1^2 - \frac{2}{V_1}\mathrm{d}V_1\mathrm{d}V_2\right) \tag{4.2.1}$$

but both the conformally flatness and the charges are left unspoilt even if the explicit form of the associated Conformal Killing Vectors is not exactly the same. A very similar thing happens when the (A)dS extension is developed, see section 4.4 on page 45.

4.2.1 Charges and $\mathfrak{sch}(2)$ algebra

The complete set of Conformal Killing Vectors found for the lift is reported in appendix C, whereas here it will be possible to read the explicit forms of the charges whose vectors are commuting with the covariant constant null Killing vector $N^{\hat{a}}\partial_{\hat{a}} = \partial_w$. This is present by definition of the lift itself and once again gives important information about which charges to consider to retrieve the Schrödinger's algebra. As it has became customary throughout this thesis, the projection rule $\mathcal{O}_\xi = \xi^a \partial_a$ is applied to find what follows.

$$\left| \begin{aligned} & Q_+ = \tilde{h}\,, \\ & Q_0^\epsilon = \epsilon C + \tau \tilde{h}\,, \\ & Q_-^\epsilon = -2\epsilon c \ell_P V_2 + 2\epsilon\tau C + \tau^2 \tilde{h} \end{aligned} \right. \tag{4.2.3}$$

these are the charges for the $\mathfrak{sl}(2,\mathbb{R})$ algebra, already shown in the previous works of [14, 51, 52]. Also the other five

$$\begin{aligned} P_+ &= 2\sqrt{V_1}P_1 + \frac{V_2 P_2}{\sqrt{V_1}}\,, \qquad B_+^\epsilon = \frac{2\epsilon c \ell_P V_2}{\sqrt{V_1}} + \tau\left(2\sqrt{V_1}P_1 + \frac{V_2P_2}{\sqrt{V_1}}\right) \\ P_- &= 2\sqrt{V_1}P_2\,, \qquad J = V_1 P_1\,, \qquad B_-^\epsilon = 4\epsilon c \ell_P \sqrt{V_1} + 2\tau\sqrt{V_1}P_2 \end{aligned} \tag{4.2.4}$$

Notice that also here the charges Q_0^ϵ, Q_-^ϵ and $B_\pm^\epsilon$ are evolving constants of motion and depend explicitly on the time coordinate. Therefore, they do not commute with the hamiltonian and change the energy. On the other hand, the other charges are strong Dirac observables of the system.

These eight charges alongside with the N one form, indeed, a $\mathfrak{sch}(2)$ algebra. The first three, as mentioned few lines above, are a $\mathfrak{sl}(2,\mathbb{R})$ subalgebra

$$\left\{Q_+, Q_-^\epsilon\right\} = 2Q_0^\epsilon\,, \qquad \left\{Q_0^\epsilon, Q_+\right\} = -Q_+\,, \qquad \left\{Q_0^\epsilon, Q_-^\epsilon\right\} = +Q_-^\epsilon \tag{4.2.5}$$

while the others satisfy the brackets

$$\begin{aligned} &\left\{Q_-^\epsilon, P_\pm\right\} = -B_\pm^\epsilon\,, \qquad \left\{Q_0^\epsilon, P_\pm\right\} = -\frac{1}{2}P_\pm\,, \qquad \left\{J, P_\pm\right\} = \pm\frac{1}{2}P_\pm \\ &\left\{Q_+, B_\pm^\epsilon\right\} = P_\pm\,, \qquad \left\{Q_0^\epsilon, B_\pm^\epsilon\right\} = \frac{1}{2}B_\pm^\epsilon\,, \qquad \left\{J, B_\pm^\epsilon\right\} = \pm\frac{1}{2}B_\pm^\epsilon \end{aligned} \tag{4.2.6}$$

and finally the central extension is given by the relation

$$n = \left\{B_\mp^\epsilon, P_\pm\right\} = 4\epsilon c \tag{4.2.7}$$

The generators $P_\pm$ and $B_\pm$ constitute the two $\mathbb{R}^2$ sectors of the algebra, while J is the $\mathfrak{so}(2)$ one and, as already stated, the Q's, form the $\mathfrak{sl}(2,\mathbb{R})$ remaining part. They respectively generate translations, Galilean boosts, rotations in the P's and B's planes and the Möbius transformations.

4.2.2 Casimirs

From section 1.3 on page 9 we know that the 2d Schrödinger's algebra has three Casimirs. Following explicit computation for the Schwarzschild's black hole dynamics case eq. (1.3.9) takes the value

$$\mathcal{C}_1 = 4\epsilon c V_1 P_1 \tag{4.2.8}$$

and it is possible to show that this is proportional to the Schwarzschild's mass ℓ_M introduced in eq. (4.1.5). While the second Casimir, the cubic one, is proportional to the squared black hole mass

$$\mathcal{C}_2 \propto \ell_M^2 \tag{4.2.9}$$

Thus, the invariant Casimir operators labelling the state of the black hole correspond to its mass as expected but also encode the information on the effective size of the system since the last one is the central extension from eq. (1.3.11).

For the model at study, comparing eq. (4.2.7) with eq. (4.1.3), can be recovered

$$n = 4\epsilon \frac{\ell_0 \ell_s^2}{\ell_P^3} \tag{4.2.10}$$

In this last Casimir the fiducial volume, regulator of the symmetry reduction process, can be recognised in the product of the two constants ℓ_0 that sets the size along the y direction and the ℓ_s^2 that encodes the properties of the 2d sphere. What has been said about the corresponding Casimir in the case of the Friedmann-Lemaître-Robertson-Walker's model in section 3.2.2 on page 29 holds true also in the present case. Indeed this fiducial volume supplies the ratio with the *IR* cut-off of the theory while the Planck length ℓ_P, once again, behaves like a gauge for the *UV* scale. Thus, this central extension allows for the association between the new notion of number of particles introduced in section 1.3.3 on page 12 to the Schwarzschild's black hole dynamics theory. Notion which is intimately related to the symmetry reduction procedure followed in this chapter since all information on the boundary of the fiducial region turn out to be encapsulated in this parameter.

This sort of interpretation parallels the standard one for this minisuperspace as a many bodied system emerging from an underlying microscopic theory, providing a new symmetry based argument for an hydrodynamic like description of it.

4.3 Transformations and solutions of the dynamics

4.3.1 Symmetry transformations

In this section will be shown that the Schrödinger's charges presented above generate well defined Noetherian symmetries of the gauge fixed Schwarzschild's mechanical action from eq. (4.1.7). Charges $P_\pm$ and $B_\pm$ are related to transformations that can compactly be written as

$$\begin{aligned} \tau &\longmapsto \tilde{\tau} = \tau \\ V_1 &\longmapsto \tilde{V_1}(\tilde{\tau}) = V_1(\tau) + 2\sqrt{V_1}(\tau)\zeta_2(\tau) \\ V_2 &\longmapsto \tilde{V_2}(\tilde{\tau}) = V_2(\tau) + \sqrt{V_1}(\tau)\zeta_1(\tau) + \frac{V_2(\tau)}{\sqrt{V_1}(\tau)}\zeta_2(\tau) \end{aligned} \tag{4.3.1}$$

with two arbitrary functions of the time coordinate ζ_1 and ζ_2. The infinitesimal transformation of the action is then given by

$$\delta_\zeta S = 2\epsilon c\ell_P \int \mathrm{d}\tau \, \left\{ \ddot{\zeta}_1 \sqrt{V_1} + \ddot{\zeta}_2 \frac{V_2}{\sqrt{V_1}} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\dot{\zeta}_1\sqrt{V_1} + \dot{\zeta}_2\frac{V_2}{\sqrt{V_1}} \right) + \mathcal{O}\left(\zeta_1^2\,, \zeta_2^2\right) \right\} \tag{4.3.2}$$

These are symmetries provided $\ddot{\zeta}_1 = \ddot{\zeta}_2 = 0$, meaning

$$\zeta_1(\tau) = \beta_1 + \vartheta_1\tau \,, \qquad \zeta_2(\tau) = \beta_2 + \vartheta_2\tau \tag{4.3.3}$$

The four constants β's and ϑ's thus parametrize the symmetry associated with translations and boost in field space.

Symmetries generated by the Q's instead have been introduced and deeply investigated in previous works [14, 51, 52]; here we just list the transformations that produce them and for detailed expression of the action variation the reader is referred to the original sources

$$\begin{aligned} \tau &\longmapsto \tilde{\tau} = f(\tau) \\ V_1 &\longmapsto \tilde{V_1}(\tilde{\tau}) = \dot{f}(\tau)V_1(\tau) \\ V_2 &\longmapsto \tilde{V_2}(\tilde{\tau}) = \dot{f}(\tau)V_2(\tau) \end{aligned} \tag{4.3.4}$$

in which we have to assume that

$$f(\tau) = \frac{a\tau + b}{c\tau + d}\,, \qquad ad - bc \neq 0 \tag{4.3.5}$$

4.3.2 Solutions of the dynamics

The Schwarzschild's black hole dynamics is governed by the relations in eq. (4.1.4) where profiles for the A and B fields are provided in terms of the integration constants $\mathcal{C}$ and $\tau_\pm$. In order to prove that the Schrödinger's charges fully encode the Schwarzschild's geometry, i.e. that their knowledge is enough to reconstruct the whole geometry, these constants of integration have to be expressed in terms of the conserved charges.

New variables defined for easier computations in eq. (4.1.6) can be retrieved in terms of Schrödinger's charges as

$$4\epsilon c\ell_P\sqrt{V_1} = B^\epsilon_- - \tau P_- \,, \qquad 2\epsilon c\ell_P\frac{V_2}{\sqrt{V_1}} = B^\epsilon_+ - \tau P_+ \tag{4.3.6}$$

hence, for the fields we are looking for, these values can be substitute in to find

$$A^2(\tau) = 4\frac{B^\epsilon_+ - \tau P_+}{B^\epsilon_- - \tau P_-} \,, \qquad B(\tau) = \frac{B^\epsilon_- - \tau P_-}{4\epsilon c\ell_P} \tag{4.3.7}$$

So it is then straightforward to find a dictionary between the observables of the Schrödinger's algebra and the constants of integration entering the expressions for the classical trajectories

$$\mathcal{C} = \frac{P_-}{4\epsilon c\ell_P} \,, \qquad \tau_\pm = \frac{B^\epsilon_\pm}{P_\pm} \,, \qquad \varpi = Q_+ \tag{4.3.8}$$

This underlines the role of constants of integration as constants of motion, and viceversa the interpretation of conserved charges as initial conditions for trajectories.

4.4 (A)dS extension

With small modifications to what done above it is possible to describe the dynamics of the vacuum (A)dS gravity. The present work does not aspire to give a full treatment of that model but only to briefly show that the same considerations done for the Schwarzschild's black hole dynamics hold true there. For this reason what will be presented in this section is going to be in large part taken for granted from previous works [14] and only the modifications needed to recover it from the Schwarzschild's case will be highlighted.

Let us start by noticing that the action for the (A)dS case is

$$S = \int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-g}\left[\frac{R-2\Lambda}{2\ell_P^2}\right] \tag{4.4.1}$$

and that after the same computations and considerations done in section 4.1.1 on page 37 the following reduced gauged fixed action can be retrieved

$$S_\epsilon = \epsilon c\ell_P\int \mathrm{d}\tau\,\left[\frac{\epsilon}{\ell_s^2} - \frac{\epsilon V_1}{\ell_\Lambda^2} + \frac{V_2\dot{V}_1^2 - 2V_1\dot{V}_1\dot{V}_2}{2V_1^2}\right] \tag{4.4.2}$$

It differs from the action already presented in eq. (4.1.7) only by the term $-\epsilon V_1/\ell_\Lambda^2$.

Then it is straightforward to find its Eisenhart-Duval's lift to be

$$\mathrm{d}s^2 = -\frac{c\ell_P V_1}{\ell_\Lambda^2}\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}w + \epsilon c\ell_P\left(\frac{V_2}{V_1^2}\mathrm{d}V_1^2 - \frac{2}{V_1}\mathrm{d}V_1\mathrm{d}V_2\right) \tag{4.4.3}$$

then the associated charges read[5]

$$\left|\begin{aligned} Q_+^\epsilon &= \tilde{h} - \frac{V_1P_2}{\ell_\Lambda^2}\left[\epsilon\tau + \frac{\tau^2}{4G}\right] \\ Q_0^\epsilon &= \epsilon C + \tau\tilde{h} - \frac{V_1P_2}{\ell_\Lambda^2}\left[\tau G + \epsilon\tau^2 + \frac{\tau^3}{6G}\right] \\ Q_-^\epsilon &= -2\epsilon c\ell_P V_2 + 2\epsilon\tau C + \tau^2\tilde{h} - \frac{V_1P_2}{\ell_\Lambda^2}\left[\tau^2 G + \frac{2\epsilon\tau^3}{3} + \frac{\tau^4}{12G}\right]\end{aligned}\right. \tag{4.4.4}$$

these are the modifications to the charges introduced in eq. (4.2.3) and the convention $G = c\ell_P/P_2$ was used to simplify the notation. The other five, changing the ones introduced in eq. (4.2.4), read

$$\begin{gathered} P_+ = 2\sqrt{V_1}P_1 + \frac{V_2P_2}{\sqrt{V_1}} + \frac{\sqrt{V_1}P_2}{\ell_\Lambda^2}\left[2\tau G + \frac{\epsilon\tau^2}{2}\right] \\ P_- = 2\sqrt{V_1}P_2\ , \qquad J = V_1P_1 \\ B_+^\epsilon = \frac{2\epsilon c\ell_P V_2}{\sqrt{V_1}} + \tau\left(2\sqrt{V_1}P_1 + \frac{V_2P_2}{\sqrt{V_1}}\right) + \frac{\sqrt{V_1}P_2}{\ell_\Lambda^2}\left[\tau^2 G + \frac{\epsilon\tau^3}{6}\right] \\ B_-^\epsilon = 4\epsilon c\ell_P\sqrt{V_1} + 2\tau\sqrt{V_1}P_2 \end{gathered} \tag{4.4.5}$$

Finally, the algebra they satisfy, that results being a $\mathfrak{sch}(2)$, is

$$\begin{aligned} &\{Q_+^\epsilon, Q_-^\epsilon\} = 2Q_0^\epsilon\ , && \{Q_0^\epsilon, Q_\pm^\epsilon\} = \mp Q_\pm^\epsilon\ , && \{B_\mp^\epsilon, P_\pm\} = 4\epsilon c \\ &\{Q_-^\epsilon, P_\pm\} = -B_\pm^\epsilon\ , && \{Q_0^\epsilon, P_\pm\} = -\frac{1}{2}P_\pm\ , && \{J, P_\pm\} = \pm\frac{1}{2}P_\pm \\ &\{Q_+^\epsilon, B_\pm^\epsilon\} = P_\pm\ , && \{Q_0^\epsilon, B_\pm^\epsilon\} = \frac{1}{2}B_\pm^\epsilon\ , && \{J, B_\pm^\epsilon\} = \pm\frac{1}{2}B_\pm^\epsilon \end{aligned} \tag{4.4.6}$$

[5] All the extra terms added in order to take into consideration the new piece of the lift are segrageted to show them explicitly and depend on the patameter $1/\ell_\Lambda^2$.

Chapter 5

Bianchi's I anisotropic universe

This chapter is focused on presenting a different situation from the ones depicted in chapters 3 and 4. Bianchi's cosmologies [47, 78] describe homogeneous but anisotropic universes classifying every possible 3d real Lie algebras and the symmetries therein [23, 36, 37, 53, 87]. The specific case used here as example is the Bianchi's type I universe that is the simplest possible, being it a generalization of the Friedmann-Lemaître-Robertson-Walker's model with indipendent scale factors for each space direction. It assumes a particularly relevant purpose in this work because it allows us to show how different approaches to the geometrization described in chapter 2 behave with a higher dimensional space and different curvature properties.

The reader will be shown how already through the use of the superspace approach is possible to reach a maximally symmetric description of the dynamics and that the associated charges form a $\mathfrak{so}(4,2)$ algebra. From this the Schrödinger's observables are going to be retrieved as a subalgebra. The use of the Eisenhart-Duval's lift will still be essential to get dynamical charges that can be implemented to solve for the equations of motion.

5.1 Superspace approach

5.1.1 Line element and action

Line element from eq. (3.1.1) can be immediately generalized to consider each spacelike direction indipendently from the others as far as the scale factor is concerned

$$\mathrm{d}s^2 = -N^2(t)\mathrm{d}t^2 + a_+^2(t)(\mathrm{d}x^+)^2 + a_0^2(t)(\mathrm{d}x^0)^2 + a_-^2(t)(\mathrm{d}x^-)^2 \tag{5.1.1}$$

then, in order to ease up the computations that need to be perfomed in the following, it is possible to introduce the Misner's variables [77]

$$\log a_\pm^2 = \frac{2}{3}\log z \pm \beta_2 - \frac{\beta_1}{\sqrt{3}} \,, \qquad \log a_0^2 = \frac{2}{3}\log z + \frac{2\beta_1}{\sqrt{3}} \tag{5.1.2}$$

so that the symmetry reduced gauged fixed action associated with the Bianchi's I universe filled with a massless scalar field ϕ can be rewritten, performing step by step what done in order to achive eq. (3.1.3), as

$$S = \frac{c\ell_P}{2}\int \mathrm{d}\tau \, \left[z^2\left(\ell_P^2\dot{\phi}^2 + \dot{\beta}_1^2 + \dot{\beta}_2^2\right) - \dot{z}^2\right] \tag{5.1.3}$$

in which the z variable takes the role, similarly as in the Friedmann-Lemaître-Robertson-Walker's case, of the square root of the volume; that case can be retrieved by asking that the time derivatives of the β's vanish. Another interesting fact to highlight is that in this framework the new variables β's, that account for the anisotropy of the described universe, basically behave as two additional fields in an isotropic universe.

Let us move on to the Hamiltonian formalism description of this model. The conjugate momenta to the $\chi^a = \{\phi, \beta_1, \beta_2, z\}$ fields read respectively

$$\pi = c\ell_P^3 z^2\dot{\phi} \,, \qquad p_i = c\ell_P z^2\dot{\beta}_i \,, \qquad p_z = -c\ell_P\dot{z} \tag{5.1.4}$$

while the Hamiltonian, found Legendre transforming of the Lagrangian, results in

$$h = \frac{1}{2c\ell_P}\left(\frac{\ell_P^{-2}\pi^2 + p_1^2 + p_2^2}{z^2} - p_z^2\right) \tag{5.1.5}$$

5.1.2 Charges and $\mathfrak{so}(4,2)$ algebra

Following the approach described in section 2.1 on page 15 to find conserved charges from the Conformal Killing Vectors of the superspace we need to start reading the line element[1] from eq. (5.1.3)

$$\mathrm{d}s^2 = \frac{c\ell_P}{2}\left[z^2\left(\mathrm{d}\varphi^2 + \mathrm{d}\beta_1^2 + \mathrm{d}\beta_2^2\right) - \mathrm{d}z^2\right] \tag{5.1.6}$$

This it is associated with a 4d manifold whose Weyl's and Cotton't tensors vanish, hence is conformally flat as per the theorem in section 2.3 on page 22. Moreover, the maximal number of Conformal Killing Vectors described therein tells that for the specificity of this case fifteen are the conserved charges that can be found

[1]Here the field redefinition $\mathrm{d}\varphi = \ell_P^2\mathrm{d}\phi$ was introduced to get lighter equations in what follows.

through the usual projection rule. The complete set of Conformal Killing Vectors is given in appendix C; the projected charges instead are

$$T_1 = p_1 \ , \qquad T_2 = p_2 \ , \qquad T_\varphi = \pi \ , \qquad T_0 = z p_z \tag{5.1.7}$$

that behave like translations

$$\left| \begin{array}{l} M_{1\,2} = \beta_1 p_2 - \beta_2 p_1 \\ M_{1\,\varphi} = \beta_1 \pi - \varphi p_1 \\ M_{2\,\varphi} = \beta_2 \pi - \varphi p_2 \end{array} \right| \begin{array}{l} M_{1\,0} = \beta_1 z p_z + \log z \, p_1 \\ M_{2\,0} = \beta_2 z p_z + \log z \, p_2 \\ M_{\varphi\,0} = \varphi z p_z + \log z \, \pi \end{array} \tag{5.1.8}$$

that follow the rotations and boosts, while the dilation is given by

$$D = \log z \, z p_z + \beta_1 p_1 + \beta_2 p_2 + \varphi \pi \tag{5.1.9}$$

and finally the special conformal transformations read

$$\left| \begin{array}{l} K_0 = -2 \log z \left[\left(\log z + \dfrac{F}{2 \log z} \right) z p_z + \beta_1 p_1 + \beta_2 p_2 + \varphi \pi \right] \\ K_1 = 2\beta_1 \left[\log z \, z p_z + \left(\beta_1 - \dfrac{F}{2\beta_1} \right) p_1 + \beta_2 p_2 + \varphi \pi \right] \\ K_2 = 2\beta_2 \left[\log z \, z p_z + \beta_1 p_1 + \left(\beta_2 - \dfrac{F}{2\beta_2} \right) p_2 + \varphi \pi \right] \\ K_\varphi = 2\varphi \left[\log z \, z p_z + \beta_1 p_1 + \beta_2 p_2 + \left(\varphi - \dfrac{F}{2\varphi} \right) \pi \right] \end{array} \right. \tag{5.1.10}$$

where the function F is defined as

$$F(z, \beta_1, \beta_2, \varphi) = \beta_1^2 + \beta_2^2 + \varphi^2 - \log^2 z \tag{5.1.11}$$

This not only looks and behaves like the set of charges of a $\mathfrak{so}(4,2)$ algebra in their representation from eq. (A.0.2) but can be shown to actually match them exactly if the Minkowski metric and coordinates[2] x^α are taken

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \qquad x^\alpha = \begin{pmatrix} \log z \\ \beta_1 \\ \beta_2 \\ \varphi \end{pmatrix} \tag{5.1.12}$$

[2]Where the indeces in early greek letters assume values in the set $\{0, 1, 2, \varphi\}$.

so that

$$x_\alpha = \Big(-\log z\ ,\ \ \beta_1\ ,\ \ \beta_2\ ,\ \ \varphi\Big)\ , \qquad \partial_\alpha = \frac{\partial}{\partial x^\alpha} = \Big(z\partial_z\ ,\ \ \partial_1\ ,\ \ \partial_2\ ,\ \ \partial_\varphi\Big) \tag{5.1.13}$$

This implies they actually form the $\mathfrak{so}(4,2)$ algebra described in appendix A and the 4d conformal group $SO(4,2)$ stands as a physical symmetry of the Bianchi's I universe filled with a massless scalar field. Something to note is that they are not independent but actually related to each other as per

$$\begin{gathered} T_0 M_{12} + T_1 M_{20} + T_2 M_{01} = 0\ , \qquad T_1 K_2 - T_2 K_1 + 2M_{12} D = 0 \\ T_\varphi M_{12} + T_1 M_{2\varphi} + T_2 M_{\varphi 1} = 0\ , \qquad T_0 K_\varphi - T_\varphi K_0 + 2M_{0\varphi} D = 0 \\ M_{01} M_{2\varphi} + M_{02} M_{\varphi 1} + M_{0\varphi} M_{12} = 0 \end{gathered} \tag{5.1.14}$$

A final important remark is that all of these observables are actually strong Dirac ones, meaning time independent constants of motion, giving algebraic relations between the dynamical variables. They fully characterize how the variables evolve w.r.t. each other and give the deparametrized trajectories, i.e. the evolution of all degrees of freedom in terms of an arbitrary singled out variable. In this sense, they play the role of relational observables. In section 5.3.1 on page 52 will be shown how applying the Eisenhart-Duval's lift also time dependent charges can be found.

5.2 Schrödinger extrapolation and Casimirs

5.2.1 Schrödinger extrapolation

Since the aim of the present work is to show hidden Schrödinger's algebra in several models of cosmology, it is imperative to find the relations that allow to explicitely see this hidden subalgebra in the $\mathfrak{so}(4,2)$ one just found.

It is always possible to embedd a Schrödinger's algebra $\mathfrak{sch}(d)$ into a wider conformal $\mathfrak{so}(d+2)$ one but also viceversa is then possible. Based on [95] one can construct Schrödinger's charges in the form seen above by combining the observables found in the previous section. In particular the three that form the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra are given by

$$q_+ = \frac{1}{2}\left(K_\varphi - K_0\right)\ , \qquad q_0 = \frac{1}{2}\left(D + M_{\varphi 0}\right)\ , \qquad q_- = \frac{1}{2}\left(T_0 + T_\varphi\right) \tag{5.2.1}$$

while translations, rotations and boosts generators can be found as

$$P_+ = T_1\ , \qquad P_- = T_2\ , \qquad J = -M_{12} \tag{5.2.2}$$

$$B_+ = M_{10} + M_{\varphi 1}\ , \qquad B_- = M_{20} + M_{\varphi 2} \tag{5.2.3}$$

finally the central extension – given by the $u \leftrightarrow w$ swap of Q_+ in the Eisenhart-Duval's lift of the previous models analysed – reads

$$n = \frac{1}{2}\left(T_0 - T_\varphi\right) \tag{5.2.4}$$

These charges satisfy indeed the $\mathfrak{sch}(2)$ algebra

$$\begin{aligned}
&\{q_+, q_-\} = q_0\ , && \{q_0, q_\pm\} = \pm q_\pm\ , && n = \{B_\mp, P_\pm\} \\
&\{q_-, P_\pm\} = \frac{1}{2}B_\pm\ , && \{q_0, P_\pm\} = +\frac{1}{2}P_\pm\ , && \{J, P_\pm\} = \pm\frac{1}{2}P_\pm \\
&\{q_+, B_\pm\} = P_\pm\ , && \{q_0, B_\pm\} = -\frac{1}{2}B_\pm\ , && \{J, B_\pm\} = \pm\frac{1}{2}B_\pm
\end{aligned} \tag{5.2.5}$$

5.2.2 Casimirs

The new Schrödinger's charge algebra is remarkably different from the ones identified for the two previous models. Not only it is constructed from strong Dirac observables and it does not include any evolving constants of motion – such that the symmetry transformations do not involve the time coordinate – but also the Casimirs labelling the classical state of this system have a different interpretations.

Both the Casimir from eq. (1.3.9) and the cubic one vanishes identically

$$\mathcal{C}_1 = 0\ , \qquad C_2 = 0 \tag{5.2.6}$$

While the central extension corresponds to the center of mass of the isotropic motion. This algebra provides thus a different algebraic structure to organize the phase space of this anisotropic model.

5.3 Eisenhart-Duval's lift approach

As anticipated the Eisenhart-Duval's lift is necessary to recover dynamical charges to use and solve the dynamics profiles of the system since with the superspace approach only time indipendent ones are found.

Following the prescriptions introduced in eq. (2.2.1) with regard to the construction of the lift from the action, here eq. (5.1.3), it is straightforward to find

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}w + c\ell_P\left[z^2\left(\mathrm{d}\varphi^2 + \mathrm{d}\beta_1^2 + \mathrm{d}\beta_2^2\right) - \mathrm{d}z^2\right] \tag{5.3.1}$$

this is 6d and its Cotton's tensor does not vanish, meaning it is not conformally flat and maximally symmetric. But it is not prevented to still compute the associated Conformal Killing Vectors; the only point of care is to know that not all the vectors found with the superspace approach will be comprehended in the set found here. In particular only T_1, T_2, T_φ, M_{12}, $M_{1\varphi}$ and $M_{2\varphi}$ are recovered.

5.3.1 Dynamical charges from the lift

The dynamical charges found from the lift match perfectly the ones of the Friedmann-Lemaître-Robertson-Walker's case

$$\left| \begin{aligned} & Q_+ = h \\ & Q_0 = \frac{1}{2} z p_z + \tau h \\ & Q_- = \frac{c\ell_P z^2}{2} + \tau z p_z + \tau^2 h \end{aligned} \right. \tag{5.3.2}$$

along with their respective duals under the $u \leftrightarrow w$ swap.

These charges, with the six from the superspace approach we stated few lines above, form a $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{iso}(3)$ algebra[3]

$$\begin{aligned} &\{Q_+, Q_-\} = Q_0 \\ &\{Q_0, Q_\pm\} = \mp Q_\pm \end{aligned} \quad , \quad \begin{aligned} &\{M_{\alpha\beta} M_{\gamma\delta}\} = \eta_{\alpha\delta} M_{\beta\gamma} + \eta_{\beta\gamma} M_{\alpha\delta} + \eta_{\alpha\gamma} M_{\delta\beta} + \eta_{\beta\delta} M_{\gamma\alpha} \\ &\{T_\alpha, M_{\beta\gamma}\} = \eta_{\beta\alpha} T_\gamma - \eta_{\gamma\alpha} T_\beta \end{aligned} \tag{5.3.3}$$

5.3.2 Solutions of the dynamics

To complete the discussion on this model, it is left to show how the dynamics of the four degrees of freedom can be integrated using the conserved charges. Thanks to the dynamical ones Q_0 and Q_- is possible to find the time dependance; in particular the profile for $z(\tau)$ reads

$$z^2(\tau) = \frac{2Q_+ - 4\tau Q_0}{c\ell_P} \tag{5.3.4}$$

that subsequentially allows to write the other three

$$\beta_i(\tau) = \frac{2M_{i0} - T_i \log z^2(\tau)}{2T_0} \quad , \qquad \varphi(\tau) = \frac{2M_{\varphi 0} - T_\varphi \log z^2(\tau)}{2T_0} \tag{5.3.5}$$

[3]where indices in early greek letters take values from the set $\{1, 2, \varphi\}$

Chapter 6

Conclusions and open directions

Three symmetry reduced gravitational models have been presented above.

- The Friedmann-Lemaître-Robertson-Walker's model filled with a massless scalar field, that is the simplest cosmological model for a flat geometry describing an homogeneous and isotropic universe.
- The Schwarzschild's description for black hole mechanics, that for what concerns dimensionality and curvature properties resembles closely the first case studied.
- Finally the Bianchi's I model which takes into account the effects of anisotropies of an homogeneous universe.

We have seen in chapter 2 what a symmetry reduction process entails and, above all, why the superspace and its lifted extension analysis methods can be used to look for conserved charges. In particular the geometrization procedure of a given reduced model onto an auxiliary background has been presented along with detailed prescriptions on how to projects Conformal Killing Vectors of its field space onto the observables looked for. Indeed, reduced systems can be seen as pointlike particles bounded to move along geodesics in their field space and conformal isometries of the supermetric map these null geodesics onto themselves. Then Conformal Killing Vectors of the superspace, defining automorphisms of the solutions to the equations of motion, provide symmetries for the systems. The superspace extension called Eisenhart-Duval's lift has proven itself crucial in the analysis of systems that present a non vanishing potential – and for which extra terms proportional to the supermass have to be considered – as well as to recover dynamical observables. Indeed using the minisuperspace approach one can only found weak Dirac observables that hence do not depend explicitly on what is identified as the time parameter. But implementing the lifted method those can be found and the dynamics of the system can be completely parametrized and solved.

This kind of study techniques deeply pose their roots into the dimensionality and curvature properties of the models taken under scrutiny. Indeed it is a result from literature the dependence of the number of Conformal Killing Vectors, and whether a space is maximally symmetric or not, on the conformal flatness of the space itself. What has been shown is that both the first and second models taken as examples share the same properties, as far as this is concerned, causing equivalent results. In chapter 3 and chapter 4, respectively the Friedmann-Lemaître-Robertson-Walker's and the Schwarzschild's chapters, through the use of the superspace methodology are recovered two sets each of infinite dimentional $\mathfrak{Witt}$'s algebra. Characterisation by use of those is not very useful because is straightforward to understand, as a basic example, that with the addition of extra fields the dimensionality of the space changes and this algebra drastically fails to live on.

While deploying the Eisenhart-Duval's lift we have been able to show that both models hide a Schrödinger's 2d algebra

$$\mathfrak{sch}\,(2) = (\mathfrak{sl}\,(2,\mathbb{R}) \times \mathfrak{so}\,(2)) \ltimes \left(\mathbb{R}^2 \times \mathbb{R}^2\right) \tag{6.0.1}$$

in themselves. This generates the non relativistic conformal group which is precisely the dynamical symmetry of the compressible Navier-Stokes' equations for classical fluids as well as of the nonlinear Schrödinger's equation describing quantum superfluids like the Bose-Einstein's condensate. In the first chapter a brief introduction on fluid/gravity duality was given and this new findings suggest that the dynamics of gravitational minisuperspace models could be recasted in terms of Schrödinger's systems as non relativistic fluid motion hosted on an auxiliary background. Using the Madelung fluid representation to do the mapping, and on the base of shared symmetries among the two systems, a new form of non holographic duality is evident. This analogy and even just the newly found algebraic structure gives an innovative way to arrange phase space from a group theoretical point of view especially relevant for quantization. A final remarkable note to make is that such a non relativistic conformal symmetry can be realized in such relativistic gravitational system because it lives in field space and not in spacetime per se. Indeed the hydrodynamical equations, describing the physics subject to the symmetry, govern quantities that belong to the superspace and are relational between them. Leaving completely uninfluential the manifold beneath them.

As far as the Bianchi's I model goes all the considerations made are not reciprocated. Indeed having completely different properties of dimensionality and curvature, when the superspace approach is applied a full set of weak Dirac observables is found. These can be arranged to satisfy a wider $\mathfrak{so}(4,2)$ algebra and the $\mathfrak{sch}(2)$ one, that can be extrapolated as a subalgebra of it, has a totally different flavour w.r.t. the ones found in the other two cases. Indeed being formed by non dynamical observables it cannot be used to fully algebraically solve the dynamics, i.e. in a non relational way. Nontheless some further charges can be obtained with the lifted approach; they of course are dynamical – hence usable to find solutions – but they can be shown to form only the protected $\mathfrak{sl}(2,\mathbb{R})$ algebra. This reveals that the anisotropies break the Schrödinger's symmetry identified in the isotropic case, while leaving the $SL(2,\mathbb{R})$ subgroup of transformations encoding the isotropic motion untouched, thus parralelling the results found in [10, 53].

Finally, Casimir operators for the unearthed Schrödinger's algebra are taken into account for each model in their own chapter. Literature teaches that the 2d version of this algebra presents itself with three associated communting operators. The first two are quadratic and cubic in the momenta and do not really provide for new information. Indeed in the Friedmann-Lemaître-Robertson-Walker's case is related to the conservation of the kinetic energy of the scalar matter field, while in the Schwarzschild's case, as expected, is related to the mass of the black hole described by the model. For the Bianchi's I model everything is different since the Schrödinger's algebraic structure has not the same value as for the other examples, being it recovered from a set of only weak Dirac observables. In this case two Casimirs are null; the third Casimir is the central extension and in this case is related to the center of mass of the isotropic motion. In the first two cases instead assumes a way more interesting value being the ratio of the IR and UV cut-offs of the theories [19, 21, 75]. The former one derives from the symmetry reduction process that brings along the fiducial volume of integration as a regulator in order not to stumble upon divergences. The latter instead is brought in by the Planck length that, it is well know, encodes the quantum nature of gravity. The ratio hence is a gauging parameter for the classical vs. quantum nature of the theory at study and, matching this with the notion of number of particles in the hydrodynamical side of the mirror, brings to mind a new dictionary for the fluid/gravity duality.

The question that rises into the mind of the curious is whether all of this is a symptom of a deeper kind of interpretation needed to understand the nature of spacetime as a quantum fluid made of pregeometrical building blocks conspiring in a statistical fashion to manifest to us as gravitational effects [83].

∗ ∗ ∗

A list of possible future open directions is given with detail in [16], here is just a summary of the most remarkable ones.

A first further step forward could be the generalization of the analysis described in this thesis to more complex models. For example, considering the presence of additional fields or by undertaking the study of the whole set of Bianchi's models.

To get to a whole other level of difficulties, inhomogeneous systems could be tackled. But, in order to do that, one would need a field theoretical formulation of the Eisenhart-Duval's lift which, at the moment, is sadly just at an intial stage of development [29, 48]. Otherwise perturbative inhomogeneities and investigations on symmetries of the background could be considered to reach the same goal.

The interplay between the fluid picture and the gravitational side suggests a possible recasting of quantum gravitational dynamics as a fluid propagating on the (lifted) superspace, hence opening a new type of dictionary for reduced classical and quantum models of gravity. Following this perspective, it could also be interesting to consider whether this can offer a new way to construct analogue gravity models beyond the kinematical level. And specifically overcome the intrinsic limitations therein by step away from the standard treatment of the fluid in this theory while getting closer to the new interpretation of a hydrodynamics living on the field space, i.e. the superspace.

Finally, using the newly found group theoretical formulation of these gravitational minisuperspaces could be profitable to address their quantization and narrow the ambiguities coming from standard quantization procedure of the quantum field theory framework.

Appendix A

$\mathfrak{so}(4,2)$ algebra

The $SO(4,2)$ group is a 15d extension of the Poincaré group that describes conformal symmetries. The extension, beside the usual ten degree of freedom from the Poincaré group, includes four for special conformal transformations and one for dilation.

As it is customary to do for the Lorentz and Poincaré cases, also here it is possible to cluster the generators of the group in a matrix form

$$C_{\alpha\beta} = \begin{pmatrix} 0 & T_1 & T_2 & T_3 & T_4 & D \\ -T_1 & 0 & M_{12} & M_{13} & M_{14} & K_1 \\ -T_2 & -M_{12} & 0 & M_{23} & M_{24} & K_2 \\ -T_3 & -M_{13} & -M_{23} & 0 & M_{34} & K_3 \\ -T_4 & -M_{14} & -M_{24} & -M_{34} & 0 & K_4 \\ -D & -K_1 & -K_2 & -K_3 & -K_4 & 0 \end{pmatrix} \tag{A.0.1}$$

the matrix is antisymmetric, i.e. $C_{\alpha\beta} = -C_{\beta\alpha}$, hence the elements with same indices, namely $\alpha = \beta$, are null and the generators have the following representation

$$\begin{aligned} &\text{4 translations} && T_\alpha = \partial_\alpha \\ &\text{6 Lorentz} && M_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha \\ &\text{1 dilation} && D = x^\alpha \partial_\alpha \\ &\text{4 conformals} && K_\alpha = 2x_\alpha x^\beta \partial_\beta - x^\beta x_\beta \partial_\alpha \end{aligned} \tag{A.0.2}$$

where, of course, Poincaré generators include Lorentz plus translations and Lorentz generators are given by the three boosts and three rotations.

Across all sections of the present work, associated charges are found implementing the projection rule $\mathcal{O} = \xi^a \pi_a$ where ξ is the Conformal Killing Vector hidden in eq. (A.0.2); for example, in the case of translation, $T_\alpha = \xi^\alpha_T \partial_\alpha \Longrightarrow \mathcal{O}^T_\alpha = \pi_\alpha$. The generators and, in particular, the charges found satisfy the algebra

$$
\begin{aligned}
&\{M_{\alpha\beta} M_{\gamma\delta}\} = \eta_{\alpha\delta} M_{\beta\gamma} + \eta_{\beta\gamma} M_{\alpha\delta} + \eta_{\alpha\gamma} M_{\delta\beta} + \eta_{\beta\delta} M_{\gamma\alpha} \\
&\{T_\alpha, T_\beta\} = 0 \,, \qquad \{T_\alpha, M_{\beta\gamma}\} = \eta_{\beta\alpha} T_\gamma - \eta_{\gamma\alpha} T_\beta \\
&\{K_\alpha, K_\beta\} = 0 \,, \qquad \{K_\alpha, M_{\beta\gamma}\} = \eta_{\beta\alpha} K_\gamma - \eta_{\gamma\alpha} K_\beta \\
&\{D, K_\alpha\} = K_\alpha \,, \qquad \{D, M_{\alpha\beta}\} = 0 \,, \qquad \{T_\alpha, D\} = T_\alpha \\
&\{T_\alpha, K_\beta\} = 2\left(\eta_{\alpha\beta} D - M_{\alpha\beta}\right)
\end{aligned}
$$

in which $\eta_{\alpha\beta}$ is the Minkowski metric tensor.

Finally, transformations related to this group are given by

$$
x^\alpha \longmapsto x'^\alpha = \frac{x^\alpha - \epsilon^\alpha x^2}{1 - 2\epsilon_\beta x^\beta + \epsilon^2 x^2} \tag{A.0.3}
$$

here ϵ is the vector parameter of the transformation and with the square is meant $x^2 = x_\beta x^\beta$.

Appendix B

$\mathfrak{Witt}$' algebra

It is a known result of literature that the algebra of conformal transformations on 2d spaces is given by two copies of the $\mathfrak{Witt}$'s algebra. For a detailed analysis of these and also on the connection between this algebra, the Schrödinger's one and the conformal symmetries, [95] and all the references therein are suggested to the interested reader.

In this appendix the focus is just on how to find and show that Conformal Killing Vectors of a 2d superspace satisfy the $\mathfrak{Witt}$'s algebra. Let us start from a generic action and notice that is always possible to redefine fields in the following suitable form

$$S = \int \mathrm{d}\tau \; g_{ab}\dot{\chi}^a\dot{\chi}^b = \int \mathrm{d}\tau \; \dot{\chi}^+\dot{\chi}^- \tag{B.0.1}$$

hence the superspace line element reads

$$\mathrm{d}s^2 = 2\mathrm{d}\chi^+\mathrm{d}\chi^- \tag{B.0.2}$$

Now, from the usual Conformal Killing Vectors equation $\pounds_\xi g_{ab} = \varphi g_{ab}$ are recovered the conditions

$$\partial_-\xi^- + \partial_+\xi^+ = \varphi \; , \qquad \partial_\pm\xi^\mp = 0 \tag{B.0.3}$$

that imply the existence of two distinct families of Conformal Killing Vectors respectively given by

$$\left(W_F^+\right)^a \partial_a = F^+(\chi^+)\partial_+ \; , \qquad \left(W_F^-\right)^a \partial_a = F^-(\chi^-)\partial_- \tag{B.0.4}$$

Once again the projection rule $\mathcal{O}_\xi = \xi^a p_a$ can be applied and both Conformal Killing Vectors and charges found are shown to satisfy the algebra

$$\left\{W_F^\pm, W_G^\pm\right\} = 2W_{[F,G]}^\pm \; , \qquad [F, G] = F'G - G'F \tag{B.0.5}$$

If the hamiltonian of the sistem $h = p_+ p_-$ is considered and it satisfies the on-shell condition $h \hat{=} 0$, than it is straightforward to prove that the set of 𝔚𝔦𝔱𝔱's charges is indeed a set of weak Dirac observables of the system, i.e.

$$\left\{ W_F^{\pm}, h \right\} = F'_{\pm} h \underset{(H=0)}{\hat{=}} 0 \tag{B.0.6}$$

One final note to make is that in the case of free systems the two sets of charges commute between themselves $\left\{ W_F^{\pm}, W_G^{\mp} \right\} = 0$ while this relation gets spoiled once the corrective terms to consider the effects of the supermass are added as introduced in section 2.1.3 on page 18 and better exemplified in chapter 4.

Appendix C

Conformal Killing Vectors

C.1 FLRW's Eisenhart-Duval's lift

$$Q_+^{\hat{a}}\partial_{\hat{a}} = -c\partial_u \ , \qquad Q_0^{\hat{a}}\partial_{\hat{a}} = \frac{1}{2}z\partial_z + u\partial_u$$
$$Q_-^{\hat{a}}\partial_{\hat{a}} = c\ell_P z^2\partial_w + \frac{1}{2}\left(uz\partial_z + u^2\partial_u\right)$$

$$P_\pm^{\hat{a}}\partial_{\hat{a}} = e^{\mp\ell_P\phi/2}\left[\frac{\ell_P}{2}\partial_z \pm \frac{1}{z}\partial_\phi\right] \ , \qquad J^{\hat{a}}\partial_{\hat{a}} = \partial_\phi$$
$$B_\pm^{\hat{a}}\partial_{\hat{a}} = e^{\mp\ell_P\phi/2}\left[2c\ell_P^2 z\partial_w + u\left(\frac{\ell_P}{2}\partial_z \pm \frac{1}{z}\partial_\phi\right)\right]$$

$$-cN^{\hat{a}}\partial_{\hat{a}} = \tilde{Q}_+^{\hat{a}}\partial_{\hat{a}} = -c\partial_w \ , \qquad \tilde{Q}_0^{\hat{a}}\partial_{\hat{a}} = \frac{1}{2}z\partial_z + w\partial_w$$
$$\tilde{Q}_-^{\hat{a}}\partial_{\hat{a}} = c\ell_P z^2\partial_u + \frac{1}{2}\left(wz\partial_z + w^2\partial_w\right)$$
$$\tilde{B}_\pm^{\hat{a}}\partial_{\hat{a}} = e^{\mp\ell_P\phi/2}\left[2c\ell_P^2 z\partial_u + w\left(\frac{\ell_P}{2}\partial_z \pm \frac{1}{z}\partial_\phi\right)\right]$$

$$Y_\pm^{\hat{a}}\partial_{\hat{a}} = e^{\mp\ell_P\phi/2}\left[2c\ell_P^2 z\left(u\partial_u + w\partial_w\right) + \left(2c\ell_P z^2 + uw\right)\left(\frac{\ell_P}{2}\partial_z \pm \frac{1}{z}\partial_\phi\right)\right]$$

C.2 Schwarzschild's Eisenhart-Duval's lift

$$Q_+^{\hat{a}}\partial_{\hat{a}} = \partial_u \ , \qquad Q_0^{\hat{a}}\partial_{\hat{a}} = V_2\partial_{V_2} + u\partial_u$$
$$Q_-^{\hat{a}}\partial_{\hat{a}} = 2V_2\partial_w + \frac{2u}{\epsilon c}\left(V_1\partial_{V_1} + V_2\partial_{V_2}\right) + \frac{u^2}{\epsilon c}\partial_u$$

$$P_+^{\hat{a}}\partial_{\hat{a}} = 2\sqrt{V_1}\partial_{V_1} + \frac{V_2}{\sqrt{V_1}}\partial_{V_2}$$
$$P_-^{\hat{a}}\partial_{\hat{a}} = \sqrt{V_1}\partial_{V_2} \ , \qquad J^{\hat{a}}\partial_{\hat{a}} = V_1\partial_{V_1}$$
$$B_+^{\hat{a}}\partial_{\hat{a}} = \frac{2V_2}{\sqrt{V_1}}\partial_w + \frac{u}{\epsilon c}\left(2\sqrt{V_1}\partial_{V_1} + \frac{V_2}{\sqrt{V_1}}\partial V_2\right)$$
$$B_-^{\hat{a}}\partial_{\hat{a}} = 2\sqrt{V_1}\partial_w + \frac{u}{\epsilon c}\sqrt{V_1}\partial_{V_2}$$

$$N^{\hat{a}}\partial_{\hat{a}} = \tilde{Q}_+^{\hat{a}}\partial_{\hat{a}} = \partial_w \ , \qquad \tilde{Q}_0^{\hat{a}}\partial_{\hat{a}} = V_2\partial_{V_2} + w\partial_w$$
$$\tilde{Q}_-^{\hat{a}}\partial_{\hat{a}} = 2V_2\partial_u + \frac{2w}{\epsilon c}\left(V_1\partial_{V_1} + V_2\partial_{V_2}\right) + \frac{w^2}{\epsilon c}\partial_w$$
$$\tilde{B}_+^{\hat{a}}\partial_{\hat{a}} = \frac{2V_2}{\sqrt{V_1}}\partial_u + \frac{w}{\epsilon c}\left(2\sqrt{V_1}\partial_{V_1} + \frac{V_2}{\sqrt{V_1}}\partial V_2\right)$$
$$\tilde{B}_-^{\hat{a}}\partial_{\hat{a}} = 2\sqrt{V_1}\partial_u + \frac{w}{\epsilon c}\sqrt{V_1}\partial_{V_2}$$

$$Y_+^{\hat{a}}\partial_{\hat{a}} = \frac{2V_2}{\sqrt{V_1}}\left(u\partial_u + w\partial_w\right) + 2uw\sqrt{V_1}\partial_{V_1} + \left(uw + 2V_2\right)\frac{V_2}{\sqrt{V_1}}\partial_{V_2}$$
$$Y_-^{\hat{a}}\partial_{\hat{a}} = 2\sqrt{V_1}\left(u\partial_u + w\partial_w\right) + 4V_1\sqrt{V_1}\partial_{V_1} + \left(uw + 2V_2\right)\sqrt{V_1}\partial_{V_2}$$

C.3 Bianchi's I superspace

$$T_1^{\hat{a}}\partial_{\hat{a}} = \partial_1\ , \qquad T_2^{\hat{a}}\partial_{\hat{a}} = \partial_2\ , \qquad T_\varphi^{\hat{a}}\partial_{\hat{a}} = \partial_\varphi\ , \qquad T_0^{\hat{a}}\partial_{\hat{a}} = z\partial_z$$

$$\begin{aligned} M_{1,2}^{\hat{a}}\partial_{\hat{a}} &= \beta_1\partial_2 - \beta_2\partial_1\ , \qquad & M_{1,0}^{\hat{a}}\partial_{\hat{a}} &= \beta_1 z\partial_z + \log z\,\partial_1 \\ M_{1,\varphi}^{\hat{a}}\partial_{\hat{a}} &= \beta_1\partial_\varphi - \varphi\partial_1\ , \qquad & M_{2,0}^{\hat{a}}\partial_{\hat{a}} &= \beta_2 z\partial_z + \log z\,\partial_2 \\ M_{2,\varphi}^{\hat{a}}\partial_{\hat{a}} &= \beta_2\partial_\varphi - \varphi\partial_2\ , \qquad & M_{\varphi,0}^{\hat{a}}\partial_{\hat{a}} &= \varphi z\partial_z + \log z\,\partial\varphi \end{aligned}$$

$$D^{\hat{a}}\partial_{\hat{a}} = \log z\, z\partial_z + \beta_1\partial_1 + \beta_2\partial_2 + \varphi\partial_\varphi$$

$$\begin{aligned} K_0^{\hat{a}}\partial_{\hat{a}} &= -2\log z\ \left[\left(\log z\ + \frac{F}{2\log z}\right) z\partial_z + \beta_1\partial_1 + \beta_2\partial_2 + \varphi\partial_\varphi\right] \\ K_1^{\hat{a}}\partial_{\hat{a}} &= 2\beta_1\left[\log z\, z\partial_z + \left(\beta_1 - \frac{F}{2\beta_1}\right)\partial_1 + \beta_2\partial_2 + \varphi\partial_\varphi\right] \\ K_2^{\hat{a}}\partial_{\hat{a}} &= 2\beta_2\left[\log z\, z\partial_z + \beta_1\partial_1 + \left(\beta_2 - \frac{F}{2\beta_2}\right)\partial_2 + \varphi\partial_\varphi\right] \\ K_\varphi^{\hat{a}}\partial_{\hat{a}} &= 2\varphi\left[\log z\, z\partial_z + \beta_1\partial_1 + \beta_2\partial_2 + \left(\varphi - \frac{F}{2\varphi}\right)\partial_\varphi\right] \end{aligned}$$

C.4 Bianchi's I Eisenhart-Duval's lift

$$Q_+^{\hat{a}}\partial_{\hat{a}} = \partial_u\ , \qquad Q_0^{\hat{a}}\partial_{\hat{a}} = \frac{1}{2}z\partial_z + u\partial_u$$

$$Q_-^{\hat{a}}\partial_{\hat{a}} = \frac{c\ell_P}{2}z^2\partial_w + uz\partial_z + u^2\partial_u$$

$$\tilde{Q}_+^{\hat{a}}\partial_{\hat{a}} = \partial_w\ , \qquad \tilde{Q}_0^{\hat{a}}\partial_{\hat{a}} = \frac{1}{2}z\partial_z + w\partial_w$$

$$\tilde{Q}_-^{\hat{a}}\partial_{\hat{a}} = \frac{c\ell_P}{2}z^2\partial_u + wz\partial_z + w^2\partial_w$$

Bibliography

[1] A. Alonso-Serrano, M. Bojowald, and D. Brizuela. *Quantum approach to a Bianchi I singularity*. 2020. DOI: 10.48550/arXiv.2001.11488.

[2] F. Alshammari, P. S. Isaac, and I. Marquette. *A differential operator realisation approach for constructing Casimir operators of non-semisimple Lie algebras*. 2018. DOI: 10.48550/arXiv.1706.09970.

[3] C. Barcelo, S. Liberati, and M. Visser. *Analogue Gravity*. 2011. DOI: 10.48550/arXiv.gr-qc/0505065.

[4] C. Batista. *On Spaces with the Maximal Number of Conformal Killing Vectors*. 2017. DOI: 10.48550/arXiv.1711.01337.

[5] X. Bekaert and X. Morand. *Embedding nonrelativistic physics inside a gravitational wave*. 2013. DOI: 10.48550/arXiv.1307.6263.

[6] J. Ben Achour. *Proper time reparametrization in cosmology: Möbius symmetry and Kodama charges*. 2021. DOI: 10.48550/arXiv.2103.10700.

[7] J. Ben Achour and E. R. Livine. *Thiemann complexifier in classical and quantum FLRW cosmology*. 2017. DOI: 10.48550/arXiv.1705.03772.

[8] J. Ben Achour and E. R. Livine. *Cosmology as a CFT_1*. 2019. DOI: 10.48550/arXiv.1909.13390.

[9] J. Ben Achour and E. R. Livine. *Polymer Quantum Cosmology: Lifting quantization ambiguities using a $SL(2,\mathbb{R})$ conformal symmetry*. 2019. DOI: 10.48550/arXiv.1806.09290.

[10] J. Ben Achour and E. R. Livine. *Protected $SL(2,\mathbb{R})$ Symmetry in Quantum Cosmology*. 2019. DOI: 10.48550/arXiv.1904.06149.

[11] J. Ben Achour and E. R. Livine. *Conformal structure of FLRW cosmology: spinorial repre- sentation and the $\mathfrak{so}(2,3)$ algebra of observables*. 2020. DOI: 10.48550/arXiv.2001.11807.

[12] J. Ben Achour and E. R. Livine. *Cosmological spinor*. 2020. DOI: 10.48550/arXiv.2004.06387.

[13] J. Ben Achour and E. R. Livine. *The cosmological constant from conformal transformations: Möbius invariance and Schwarzian action*. 2020. DOI: 10.48550/arXiv.2004.05841.

[14] J. Ben Achour and E. R. Livine. *Symmetries and conformal bridge in Schwarzschild-(A)dS black hole mechanics*. 2021. DOI: 10.48550/arXiv.2110.01455.

[15] J. Ben Achour, E. R. Livine, S. Mukohyama, and J. P. Uzan. *Hidden symmetry of the static response of black holes: Applications to Love numbers*. 2022. DOI: 10.48550/arXiv.2202.12828.

[16] J. Ben Achour, E. R. Livine, D. Oriti, and G. Piani. *Schrödinger symmetry in cosmology and black hole mechanics*. 2022. DOI: 10.48550/arXiv.2207.07312.

[17] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani. *Nonlinear Fluid Dynamics from Gravity*. 2008. DOI: 10.48550/arXiv.0712.2456.

[18] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla, et al. *Conformal Nonlinear Fluid Dynamics from Gravity in Arbitrary Dimensions*. 2008. DOI: 10.48550/arXiv.0809.4272.

[19] N. Bodendorfer and F. Haneder. *A note on coarse graining and group representations*. 2021. DOI: 10.48550/arXiv.2011.12749.

[20] M. Bojowald. *Canonical Gravity and Applications. Cosmology, Black Holes, and Quantum Gravity*. 2010. DOI: 10.1017/CBO9780511921759.

[21] M. Bojowald and S. Brahma. *Minisuperspace models as infrared contributions*. 2016. DOI: 10.48550/arXiv.1509.00640.

[22] I. Bredberg, C. Keeler, V. Lysov, and A. Strominger. *From Navier-Stokes To Einstein*. 2012. DOI: 10.48550/arXiv.1101.2451.

[23] S. Capozziello, G. Marmo, C. Rubano, and P. Scudellaro. *Noether symmetries in Bianchi universes*. 1997. DOI: 10.48550/arXiv.gr-qc/9606050.

[24] M. Cariglia and F. K. Alves. *The Eisenhart lift: a didactical introduction of modern geometrical concepts from Hamiltonian dynamics*. 2015. DOI: 10.48550/arXiv.1503.07802.

[25] M. Cariglia, C. Duval, G. W. Gibbons, and P. A. Horvathy. *Eisenhart lifts and symmetries of time-dependent systems*. 2016. DOI: 10.48550/arXiv.1605.01932.

[26] M. Cariglia, A. Galajinsky, G. W. Gibbons, and P. A. Horvathy. *Cosmological aspects of the Eisenhart–Duval lift*. 2018. DOI: 10.48550/arXiv.1802.03370.

[27] M. Cariglia and G. Gibbons. *Generalised Eisenhart lift of the Toda chain.* 2014. DOI: 10.48550/arXiv.1312.2019.

[28] M. Cariglia, G. W. Gibbons, P. A. van Holten J. W. adn Horvathy, and P. M. Zhang. *Conformal Killing Tensors and covariant Hamiltonian Dynamics.* 2014. DOI: 10.48550/arXiv.1404.3422.

[29] S. Chanda and P. Guha. *Eisenhart lift and Randers–Finsler formulation for scalar field theory.* 2021. DOI: 10.48550/arXiv.1905.06188.

[30] T. Christodoulakis, N. Dimakis, and P. A. Terzis. *Lie point and variational symmetries in minisuperspace Einstein gravity.* 2014. DOI: 10.48550/arXiv.1304.4359.

[31] T. Christodoulakis, N. Dimakis, P. A. Terzis, G. Doulis, et al. *Conditional Symmetries and the Canonical Quantization of Constrained Minisuperspace Actions: the Schwarzschild case.* 2013. DOI: 10.48550/arXiv.1208.0462.

[32] T. Christodoulakis, N. Dimakis, P. A. Terzis, B. Vakili, et al. *Minisuperspace canonical quantization of the Reissner-Nordström black hole via conditional symmetries.* 2014. DOI: 10.48550/arXiv.1309.6106.

[33] T. Christodoulakis, A. Karagiorgos, and A. Zampeli. *Symmetries in Classical and Quantum Treatment of Einstein's Cosmological Equations and Mini-Superspace Actions.* 2018. DOI: 10.3390/sym10030070.

[34] L. Ciambelli, C. Marteau, A. C. Petkou, P. M. Petropoulos, et al. *Covariant Galilean versus Carrollian hydrodynamics from relativistic fluids.* 2018. DOI: 10.48550/arXiv.1802.05286.

[35] L. Ciambelli, C. Marteau, A. C. Petkou, P. M. Petropoulos, et al. *Flat holography and Carrollian fluids.* 2018. DOI: 10.48550/arXiv.1802.06809.

[36] S. Cotsakis, P. Leach, and H. Pantazi. *Symmetries of homogeneous cosmologies.* 1998. DOI: 10.48550/arXiv.gr-qc/0011017.

[37] O. Coussaert and M. Henneaux. *Bianchi cosmological models and gauge symmetries.* 1993. DOI: 10.48550/arXiv.gr-qc/9301001.

[38] S. Dhasmana, A. Sen, and Z. K. Silagadze. *Equivalence of a harmonic oscillator to a free particle and Eisenhart lift.* 2021. DOI: 10.48550/arXiv.2106.09523.

[39] N. Dimakis. *Hidden symmetries from distortions of the conformal structure.* 2022. DOI: 10.48550/arXiv.2205.14973.

[40] N. Dimakis, P. A. Terzis, and T. Christodoulakis. *Contact symmetries of constrained systems and the associated integrals of motion.* 2015. DOI: 10.1088/1742-6596/633/1/012040.

[41] V. K. Dobrev. *Non-Relativistic Holography - A Group-Theoretical Perspective*. 2014. DOI: 10.48550/arXiv.1312.0219.

[42] C. Duval, G. W. Gibbons, and P. Horvathy. *Celestial mechanics, conformal structures and gravitational waves*. 1991. DOI: 10.48550/arXiv.hep-th/0512188.

[43] C. Duval, M. Hassaine, and P. A. Horvathy. *The Geometry of Schrodinger symmetry in gravity background/non-relativistic CFT*. 2009. DOI: 10.48550/arXiv.0809.3128.

[44] C. Duval and P. A. Horvathy. *Non-relativistic conformal symmetries and Newton-Cartan structures*. 2009. DOI: 10.48550/arXiv.0904.0531.

[45] C. Duval and S. Lazzarini. *Schrödinger Manifolds*. 2012. DOI: 10.48550/arXiv.1201.0683.

[46] L. P. Eisenhart. *Dynamical trajectories and geodesics*. 1928. DOI: 10.2307/1968307.

[47] G. F. R. Ellis and H. van Elst. *Cosmological models: Cargese lectures 1998*. 1999. DOI: 10.48550/arXiv.gr-qc/9812046.

[48] K. Finn, S. Karamitsos, and A. Pilaftsis. *Eisenhart lift for field theories*. 2018. DOI: 10.48550/arXiv.1806.02431.

[49] K. Finn, S. Karamitsos, and A. Pilaftsis. *Quantizing the Eisenhart Lift*. 2021. DOI: 10.48550/arXiv.2012.15288.

[50] A. P. Fordy and A. Galajinsky. *Eisenhart Lift of 2-Dimensional Mechanics*. 2019. DOI: 10.48550/arXiv.1901.03699.

[51] M. Geiller, E. R. Livine, and F. Sartini. *Symmetries of the black hole interior and singularity regularization*. 2021. DOI: 10.48550/arXiv.2010.07059.

[52] M. Geiller, E. R. Livine, and F. Sartini. *BMS_3 mechanics and the black hole interior*. 2022. DOI: 10.48550/arXiv.2107.03878.

[53] M. Geiller, E. R. Livine, and F. Sartini. *Dynamical symmetries of homogeneous minisuperspace models*. 2022. DOI: 10.48550/arXiv.2205.02615.

[54] P. K. Ghosh. *Conformal symmetry and the nonlinear Schrodinger equation*. 2002. DOI: 10.48550/arXiv.cond-mat/0102488.

[55] G. W. Gibbons. *Lifting the Eisenhart-Duval Lift to a Minimal Brane*. 2020. DOI: 10.48550/arXiv.2003.06179.

[56] G. W. Gibbons and C. N. Pope. *Kohn's Theorem, Larmor's Equivalence Principle and the Newton-Hooke Group*. 2011. DOI: 10.48550/arXiv.1010.2455.

[57] S. Gielen, D. Oriti, and L. Sindoni. *Homogeneous cosmologies as group field theory condensates*. 2014. DOI: 10.48550/arXiv.1311.1238.

[58] S. Gielen and L. Sindoni. *Quantum Cosmology from Group Field Theory Condensates: a Review*. 2016. DOI: 10.48550/arXiv.1602.08104.

[59] W. D. Goldberger. *AdS/CFT duality for non-relativistic field theory*. 2009. DOI: 10.48550/arXiv.0806.2867.

[60] J. Greensite. *Field theory as free fall*. 1996. DOI: 10.48550/arXiv.gr-qc/9508033.

[61] M. Haack and A. Yarom. *Nonlinear viscous hydrodynamics in various dimensions using AdS/CFT*. 2008. DOI: 10.48550/arXiv.0806.4602.

[62] M. Hassaine and P. A. Horvathy. *Field dependent symmetries of a nonrelativistic fluid model*. 2000. DOI: 10.48550/arXiv.math-ph/9904022.

[63] M. Hassaine and P. A. Horvathy. *Symmetries of fluid dynamics with polytropic exponent*. 2001. DOI: 10.48550/arXiv.hep-th/0009092.

[64] R. M. Hawkins and J. E. Lidsey. *The Ermakov-Pinney equation in scalar field cosmologies*. 2002. DOI: 10.48550/arXiv.astro-ph/0112139.

[65] M. Henkel, M. Pleimling, C. Godreche, and J. M. Luck. *Aging, Phase Ordering, and Conformal Invariance*. 2001. DOI: 10.48550/arXiv.hep-th/0107122.

[66] P. A. Horvathy and J. C. Yera. *The non-linear Schrodinger equation and the conformal properties of non-relativistic space-time*. 2009. DOI: 10.48550/arXiv.math-ph/9903001.

[67] P. A. Horvathy and P. M. Zhang. *Non-relativistic conformal symmetries in fluid mechanics*. 2010. DOI: 10.48550/arXiv.0906.3594.

[68] V. E. Hubeny, S. Minwalla, and M. Rangamani. *The fluid/gravity correspondence*. 2011. DOI: 10.48550/arXiv.1107.5780.

[69] G. Jordan Maclay. *Dynamical symmetries of the H Atom, One of the most Important Tools of Modern Physics: $SO(4)$ to $SO(4,2)$, Background, Theory and Use in Calculating Radiative Shifts*. 2020. DOI: 10.3390/sym12081323.

[70] J. I. Jottar, R. G. Leigh, D. Minic, and L. A. Pando Zayas. *Aging and Holography*. 2010. DOI: 10.48550/arXiv.1004.3752.

[71] J. E. Lidsey. *Scalar Field Cosmologies Hidden Within the Nonlinear Schrodinger Equation*. 2013. DOI: 10.48550/arXiv.1309.7181.

[72] J. E. Lidsey. *Inflationary Cosmology, Diffeomorphism Group of the Line and Virasoro Coadjoint Orbits*. 2018. DOI: 10.48550/arXiv.1802.09186.

[73] P. Malkiewicz, P. Peter, and S. D. P. Vitenti. *Quantum empty Bianchi I spacetime with internal time*. 2020. DOI: 10.48550/arXiv.1911.09892.

[74] L. Marchetti and D. Oriti. *Effective relational cosmological dynamics from Quantum Gravity*. 2021. DOI: 10.48550/arXiv.2008.02774.

[75] F. M. Mele and J. Münch. *The Physical Relevance of the Fiducial Cell in Loop Quantum Cosmology*. 2021. DOI: 10.48550/arXiv.2109.10663.

[76] D. Minic and M. Pleimling. *Non-relativistic AdS/CFT and Aging/Gravity Duality*. 2008. DOI: 10.48550/arXiv.0807.3665.

[77] C. W. Misner. *Quantum Cosmology. I.* 1969. DOI: 10.1103/PhysRev.186.1319.

[78] G. Montani, M. V. Battisti, R. Benini, and G. Imponente. *Classical and Quantum Features of the Mixmaster Singularity*. 2008. DOI: 10.48550/arXiv.0712.3008.

[79] K. Morand. *Embedding Galilean and Carrollian geometries I. Gravitational waves*. 2020. DOI: 10.48550/arXiv.1811.12681.

[80] U. Niederer. *The maximal kinematical symmetry group of the free Schrödinger equation*. 1972. DOI: 10.5169/seals-114417.

[81] D. Oriti. *The microscopic dynamics of quantum space as a group field theory*. 2011. DOI: 10.48550/arXiv.1110.5606.

[82] D. Oriti. *The universe as a quantum gravity condensate*. 2017. DOI: 10.48550/arXiv.1612.09521.

[83] D. Oriti. *Tensorial Group Field Theory condensate cosmology as an example of spacetime emergence in quantum gravity*. 2022. DOI: 10.48550/arXiv.2112.02585.

[84] D. Oriti and X. Pang. *Phantom-like dark energy from quantum gravity*. 2021. DOI: 10.48550/arXiv.2105.03751.

[85] D. Oriti, L. Sindoni, and E. Wilson-Ewing. *Emergent Friedmann dynamics with a quantum bounce from quantum gravity condensates*. 2016. DOI: 10.48550/arXiv.1602.05881.

[86] T. Padmanabhan. *Emergent Gravity Paradigm: Recent Progress*. 2015. DOI: 10.48550/arXiv.1410.6285.

[87] A. Paliathanasis, L. Karpathopoulos, A. Wojnar, and S. Capozziello. *Wheeler–DeWitt equation and Lie symmetries in Bianchi scalar-field cosmology*. 2016. DOI: 10.48550/arXiv.1601.06528.

[88] V.M. Perez-Garcia, P. Torres, and G.D. Montesinos. *The method of moments for Nonlinear Schrodinger Equations: Theory and Applications*. 2005. DOI: 10.48550/arXiv.nlin/0510044.

[89] A. C. Petkou, P. M. Petropoulos, D. R. Betancour, and K. Siampos. *Relativistic Fluids, Hydrodynamic Frames and their Galilean versus Carrollian Avatars*. 2022. DOI: 10.48550/arXiv.2205.09142.

[90] P. M. Petropoulos and K. Siampos. *Integrability, Einstein spaces and holographic fluids*. 2015. DOI: 10.48550/arXiv.1510.06456.

[91] B. Pioline and A. Waldron. *Quantum cosmology and conformal invariance*. 2003. DOI: 10.48550/arXiv.hep-th/0209044.

[92] A. G. A. Pithis and M. Sakellariadou. *Group field theory condensate cosmology: An appetizer*. 2019. DOI: 10.48550/arXiv.1904.00598.

[93] R. H. Price and K. S. Thorne. *Membrane Viewpoint on Black Holes: Properties and Evolution of the Stretched Horizon*. 1986. DOI: 10.1103/PhysRevD.33.915.

[94] M. Rangamani, S. F. Ross, D. T. Son, and E. G. Thompson. *Conformal non-relativistic hydrodynamics from gravity*. 2009. DOI: 10.48550/arXiv.0811.2049.

[95] C. Roger and J. Unterberger. *The Schrodinger-Virasoro Lie group and algebra: From geometry to representation theory*. 2006. DOI: 10.48550/arXiv.math-ph/0601050.

[96] A. Sen, B. K. Parida, S. Dhasmana, and Z. K. Silagadze. *Eisenhart lift of Koopman-von Neumann mechanics*. 2022. DOI: 10.48550/arXiv.2207.05073.

[97] M. Taylor. *Non-relativistic holography*. 2008. DOI: 10.48550/arXiv.0812.0530.

[98] P. A. Terzis, N. Dimakis, T. Christodoulakis, A. Paliathanasis, et al. *Variational contact symmetries of constrained Lagrangians*. 2016. DOI: 10.48550/arXiv.1503.00932.

[99] L. Vinet and A. Zhedanov. *Representations of the Schrödinger group and Matrix orthogonal polynomials*. 2011. DOI: 10.48550/arXiv.1105.0701.

[100] P. M. Zhang, M. Cariglia, M. Elbistan, and P. A. Horvathy. *Scaling and conformal symmetries for plane gravitational waves*. 2020. DOI: 10.48550/arXiv.1905.08661.