Alma Mater Studiorum · Università di Bologna

Scuola di Scienze
Dipartimento di Fisica e Astronomia
Corso di Laurea in Fisica

THE QUANTUM TIME-DEPENDENT HARMONIC OSCILLATOR

Relatore:
Prof. Alexandr Kamenchtchik

Presentata da:
Gabriele Farnè

Anno Accademico 2021/2022

Sommario

La presente tesi si propone di mostrare che l'oscillatore armonico quantistico dipendente dal tempo è un sistema risolvibile in maniera esatta.

La trattazione è articolata in tre capitoli: nel primo viene richiamata la teoria dell'oscillatore armonico quantistico indipendente dal tempo, al fine di recuperare i concetti e le metodologie che sono comuni anche alla sua controparte dipendente dal tempo. Nel secondo capitolo viene fornita una breve introduzione alla teoria degli operatori invarianti dipendenti dal tempo, di cui ci interessa la loro relazione con le soluzioni dell'equazione di Schrödinger. Infine, nel terzo capitolo viene presentato il problema dell'oscillatore armonico quantistico dipendente dal tempo e discussa la sua soluzione esatta. In aggiunta se ne individuano gli stati coerenti.

Abstract

The purpose of this thesis is to show that there exists an exact solution to the problem of the quantum time-dependent harmonic oscillator.

The treatment is organized in three chapters: in the first one the theory of the quantum time-independent harmonic oscillator is recalled, in order to recover the concepts and methodologies which are common also to its time-dependent counterpart. The second chapter consists in a brief introduction to the theory of time-dependent invariant operators, whose relation with the solutions of the Schrödinger equation is of interest. Finally, in the third chapter the problem of the quantum time-dependent harmonic oscillator is presented and its exact solution is discussed. Furthermore, its coherent states are specified.

Summary

Chapter 1

The quantum simple harmonic oscillator

1.1 Introduction to the problem

The quantum time-independent harmonic oscillator is nothing else but the well-known quantum simple harmonic oscillator. A simple harmonic oscillator of mass m and frequency ω has the following Hamiltonian operator:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2, \tag{1.1}$$

where $\hat{q}$ and $\hat{p}$ are the position and momentum operators of the harmonic oscillator, respectively. These obey the canonical commutation relation:

$$[\hat{q}, \hat{p}] = i\hbar\hat{1}. \tag{1.2}$$

The Hamiltonian (1.1) and the quantum condition (1.2) define the system completely. The Heisenberg equations for the position and momentum operators are given by

$$\frac{d\hat{q}_H}{dt} = \frac{1}{i\hbar}[\hat{q}_H, \hat{H}_H] = \frac{\hat{p}_H}{m}, \tag{1.3a}$$

$$\frac{d\hat{p}_H}{dt} = \frac{1}{i\hbar}[\hat{p}_H, \hat{H}_H] = -m\omega^2\hat{q}_H, \tag{1.3b}$$

where we use the subscript "H" to distinguish the Heisenberg picture from the standard Schrödinger picture. Equations (1.3) combined together give the equation of motion

$$\frac{d^2\hat{q}_H}{dt^2} + \omega^2\hat{q}_H = 0. \tag{1.4}$$

This is a typical problem of *quantum statics*, in which the potential energy function is independent of time. Therefore, in order to solve the time-dependent Schrödinger equation, it suffices to work out the associated time-independent one. There exist two distinct approaches to this problem: the first one consists in solving indirectly the eigenvalue problem of the Hamiltonian by means of an algebraic operator technique; the second one aims to a straightforward solution to the eigenvalue equation of the Hamiltonian.

1.2 Algebraic solution

In this section we follow the treatment given by Cohen-Tannoudji and Co [1]. Consider the following operators:

$$\hat{a} = \frac{1}{(2m\hbar\omega)^{1/2}}(\hat{p} - im\omega\hat{q}), \tag{1.5}$$

$$\hat{a}^\dagger = \frac{1}{(2m\hbar\omega)^{1/2}}(\hat{p} + im\omega\hat{q}). \tag{1.6}$$

The operator $\hat{a}$ and its adjoint $\hat{a}^\dagger$ satisfy the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = \hat{1}. \tag{1.7}$$

The operators $\hat{a}$ and $\hat{a}^\dagger$ constitute a *destruction and creation operator pair*. With every such pair of operators there is associated a *number operator* $\hat{N}$, defined as

$$\hat{N} = \hat{a}^\dagger\hat{a}. \tag{1.8}$$

The operator $\hat{N}$ is Hermitian and, together with $\hat{a}$ and $\hat{a}^\dagger$, obeys the commutation relations

$$[\hat{N}, \hat{a}] = -\hat{a}, \tag{1.9a}$$

$$[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger. \tag{1.9b}$$

By direct calculation one can show that

$$\hat{N} = \frac{1}{\hbar\omega}\left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2\right) - \frac{\hat{1}}{2}. \tag{1.10}$$

Thus one deduces

$$\hat{H} = \hbar\omega\left(\hat{N} + \frac{\hat{1}}{2}\right). \tag{1.11}$$

Relation (1.11) implies that the eigenvalue problem of the Hamiltonian operator is automatically solved once we find the solution to the eigenvalue problem of the number operator. From this point forward we shall assume that $\hat{N}$ and $\hat{H}$ are selfadjoint operators, that is there exists an orthonormal basis for the space of the dynamical states of the system formed by their eigenvectors.

Then, consider the eigenvalue equation

$$\hat{N} |n\rangle = n |n\rangle , \tag{1.12}$$

$|n\rangle$ being an eigenket of $\hat{N}$ belonging to the eigenvalue n. The solution of (1.12) is based on the commutation relations (1.7) and (1.9). The following lemmas can be proven.

Lemma 1. *The eigenvalues n of the operator $\hat{N}$ are non-negative numbers.*

Lemma 2. *Let $|n\rangle$ be a non-zero eigenket of $\hat{N}$ with eigenvalue n. Then:*

(i) if $n = 0$, the ket $\hat{a} |0\rangle$ is zero;

(ii) if $n > 0$, the ket $\hat{a} |n\rangle$ is a non-zero eigenket of $\hat{N}$ belonging to the eigenvalue $n-1$.

Lemma 3. *Let $|n\rangle$ be a non-zero eigenket of $\hat{N}$ with eigenvalue n. Then:*

(i) the ket $\hat{a}^\dagger |n\rangle$ is always non-zero;

(ii) the ket $\hat{a}^\dagger |n\rangle$ is a non-zero eigenket of $\hat{N}$ belonging to the eigenvalue $n+1$.

From Lemma 1 and Lemma 2 one can derive that an arbitrary eigenvalue n of $\hat{N}$ must be a non-negative integer. Then Lemma 3 can be used to show that the spectrum of $\hat{N}$ indeed coincides with the set of non-negative integers.

Therefore, an eigenket $|n\rangle$ of the number operator $\hat{N}$ belonging to the eigenvalue n will be also an eigenket of $\hat{H}$, by solving the eigenvalue equation

$$\hat{H} |n\rangle = E_n |n\rangle , \tag{1.13}$$

with the energy eigenvalue E_n given by

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right) \tag{1.14}$$

where $n = 0, 1, 2, \ldots$ So the energy of a quantum simple harmonic oscillator is quantized, resulting in a discrete spectrum. In particular its energy eigenvalues are equispaced and the smallest value, corresponding to the ground state, is not zero, but $\hbar\omega/2$.

In addition to the above properties, the energy spectrum is completely non-degenerate. It can be proven by induction. Firstly, according to Lemma 2, if $|0\rangle$ is an eigenket of the number operator belonging to the eigenvalue 0, it must satisfy the condition

$$\hat{a} |0\rangle = 0. \tag{1.15}$$

Substituting the definition (1.5) of $\hat{a}$, (1.15) reads as

$$\frac{1}{(2m\hbar\omega)^{1/2}}(\hat{p} - im\omega\hat{q})\,|0\rangle = 0. \tag{1.16}$$

In the configuration space representation $\{|x\rangle\}$ equation (1.16) becomes the first-order differential equation

$$\left(l\frac{d}{dx} + \frac{x}{l}\right)\langle x|0\rangle = 0, \tag{1.17}$$

where l is the characteristic length scale of the quantum simple harmonic oscillator considered, defined as

$$l = \left(\frac{\hbar}{m\omega}\right)^{1/2}. \tag{1.18}$$

The general solution of (1.17) is

$$\langle x|0\rangle = c\exp\left(-\frac{x^2}{2l^2}\right) \tag{1.19}$$

where c is an integration constant. Since there is only one linearly independent solution to (1.17), there is only one linearly independent ket which satisfy (1.15). Therefore the eigenvalue 0 is non-degenerate. Secondly, suppose that the eigenvalue n is non-degenerate. Then consider an eigenket $|n+1\rangle$ belonging to the eigenvalue $n+1$. Because of Lemma 2, the ket $\hat{a}\,|n+1\rangle$ is a non-vanishing eigenket of $\hat{N}$ with eigenvalue n. Since n is non-degenerate by assumption, there exist a multiplicative constant factor c such that

$$\hat{a}\,|n+1\rangle = c\,|n\rangle \tag{1.20}$$

holds. By inverting this equation we get

$$\begin{aligned}
\hat{a}^\dagger\hat{a}\,|n+1\rangle &= c\hat{a}^\dagger\,|n\rangle \\
\hat{N}\,|n+1\rangle &= c\hat{a}^\dagger\,|n\rangle \\
(n+1)\,|n+1\rangle &= c\hat{a}^\dagger\,|n\rangle \\
|n+1\rangle &= \frac{c}{n+1}\hat{a}^\dagger\,|n\rangle\,.
\end{aligned} \tag{1.21}$$

Because of Lemma 3, $\hat{a}^\dagger\,|n\rangle$ is an eigenket of $\hat{N}$ belonging to the eigenvalue $n+1$. As $|n+1\rangle$ is an arbitrary eigenket with eigenvalue $n+1$, from (1.21) follows that any eigenket with eigenvalue $n+1$ is proportional to $\hat{a}^\dagger\,|n\rangle$. Therefore, $n+1$ is non-degenerate and this concludes our proof by induction.

Since the spectrum of $\hat{N}$ is completely non-degenerate, $\hat{N}$ alone is a complete set of commuting selfadjoint operators, that is there exists a unique orthonormal basis of its eigenvectors up to a phase choice. Clearly, the same applies to $\hat{H}$. This basis unique up to normalization can be easily constructed in terms of $|0\rangle$. To begin with, we shall assume $|0\rangle$ to be normalizable with $\langle 0|0\rangle = 1$. Then, because of Lemma 3, the kets $\hat{a}^\dagger |n\rangle$ and $|n+1\rangle$ are the same up to a multiplicative constant. Requiring this constant to be real and positive by convention, one can show that

$$\hat{a}^\dagger |n\rangle = (n+1)^{1/2} |n+1\rangle . \tag{1.22}$$

By recurrence one thus obtains

$$|n\rangle = (n!)^{-1/2} \hat{a}^{\dagger n} |0\rangle . \tag{1.23}$$

The orthonormality of the set of kets obtained through the formula (1.23) is guaranteed by the hermiticity of $\hat{H}$, since each ket of such set belongs to a different eigenvalue, and by the fact that $|0\rangle$ is normalized to 1. Furthermore, the set exhausts all the linearly independent eigenkets of $\hat{H}$, given the uniqueness of $|0\rangle$. Thus, being $\hat{H}$ selfadjoint, the above set constitute an orthonormal basis.

It is worth observing that one can prove also the following relation:

$$\hat{a} |n\rangle = n^{1/2} |n-1\rangle , \quad n > 0. \tag{1.24}$$

So, $\hat{a}$ and its Hermitian conjugate $\hat{a}^\dagger$ act on the eigenkets of $\hat{N}$ as *ladder operators*. We can always guarantee the validity of relations (1.22) and (1.24) by specifying the relative phases of the kets $|n\rangle$ in an appropriate way.

For completeness, we show how to compute the energy eigenfunctions $\phi_n(x)$ in this framework. We have already obtained the eigenfunction corresponding to the ground state in (1.17). The conventionally normalized ground state eigenfunction has the form

$$\phi_0(x) = \frac{1}{l^{1/2}\pi^{1/4}} \exp\left(-\frac{x^2}{2l^2}\right). \tag{1.25}$$

Now, the normalized energy eigenfunction $\phi_n(x)$ of the harmonic oscillator can be computed using (1.23) as follows:

$$\phi_n(x) = \langle x|n\rangle = (n!)^{-1/2} \langle x| \hat{a}^{\dagger n} |0\rangle . \tag{1.26}$$

Using (1.6) and the action of the position and momentum operators on the $\langle x|$ one gets

$$\phi_n(x) = \frac{(-1)^n i^n}{(2^n n!)^{1/2}} \left(l\frac{d}{dx} - \frac{x}{l}\right)^n \langle x|0\rangle = \frac{(-1)^n i^n}{l^{1/2}\pi^{1/4}(2^n n!)^{1/2}} \left(l\frac{d}{dx} - \frac{x}{l}\right)^n \exp\left(-\frac{x^2}{2l^2}\right) \tag{1.27}$$

Finally, we can rewrite (1.27) as

$$\phi_n(x) = \frac{(-1)^n i^n}{l^{1/2}\pi^{1/4}(2^n n!)^{1/2}} H_n\left(\frac{x}{l}\right) \exp\left(-\frac{x^2}{2l^2}\right), \tag{1.28}$$

where we introduced the Hermite polynomial of degree n

$$H_n\left(\frac{x}{l}\right) = \exp\left(\frac{x^2}{2l^2}\right)\left(l\frac{d}{dx} - \frac{x}{l}\right)^n \exp\left(-\frac{x^2}{2l^2}\right). \tag{1.29}$$

1.3 Analytical solution

We have seen that the algebraic technique of the destruction and creation operators is extremely powerful and succeeds in finding basically all the fundamental properties of the quantum simple harmonic oscillator. However, it is worth illustrating also the analytical approach since it constitutes an emblematic application of wave mechanics. In what follows, we will refer ourselves to the treatment given by Griffiths [2].

Consider the time-indipendent Schrödinger equation for the harmonic oscillator:

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} + \frac{1}{2}m\omega^2 x^2\phi = E\phi. \tag{1.30}$$

We want to find the values of E for which a non-vanishing solution $\phi(x)$ of (1.30) exists. Given the form of the potential, we require the solution to vanish at infinity.

Let us first introduce the dimensionless variable

$$\xi = \frac{x}{l}, \tag{1.31}$$

where l is given by (1.18), and rewrite (1.30) in terms of it:

$$\frac{d^2\phi}{d\xi^2} = (\xi^2 - K)\phi. \tag{1.32}$$

We have introduced K as the energy in units of $\hbar\omega/2$, that is

$$K = \frac{2E}{\hbar\omega}. \tag{1.33}$$

Now, note that at very large ξ (or equivalently at very large x) (1.32) takes the asymptotic form

$$\frac{d^2\phi}{d\xi^2} \approx \xi^2\phi, \tag{1.34}$$

which admits an approximate solution that is bounded, namely

$$\phi(\xi) \approx c\exp\left(-\frac{\xi^2}{2}\right), \tag{1.35}$$

c being an integration constant. The asymptotic form of the physically acceptable solutions of (1.32) suggests the ansatz

$$\phi(\xi) = H(\xi)\exp\left(-\frac{\xi^2}{2}\right). \tag{1.36}$$

Since $\phi(\xi)$ must vanish at infinity, $H(\xi)$ must have at most a polynomial growth. Plugging (1.36) into (1.32) we obtain the following differential equation for the function $H(\xi)$:

$$\frac{d^2H}{d\xi^2} - 2\xi\frac{dH}{d\xi} + (K-1)H = 0. \tag{1.37}$$

Equation (1.37) is a standard differential equation of mathematical physics known as *Hermite equation*. It can be solved with the power series method. In particular, one can show that for physically acceptable solutions we must have $K = 2n + 1$, for some non-negative integer n. This implies that

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots \tag{1.38}$$

recovering the fundamental quantization condition we found algebraically. Then, the physically acceptable solutions which are admitted by (1.32) are polynomials of degree n in ξ, precisely the Hermite polynomials, $H_n(\xi)$.

Thus, by (1.31) and (1.36) we can state that the energy eigenfunction corresponding to the energy eigenvalue E_n is given by

$$\phi_n(x) = \frac{1}{l^{1/2}\pi^{1/4}(2^n n!)^{1/2}} H_n\left(\frac{x}{l}\right) \exp\left(-\frac{x^2}{2l^2}\right), \tag{1.39}$$

where the normalization constant is determined by imposing the normalization condition

$$\int_{-\infty}^{+\infty} dx\, \phi^*_{n'}(x)\phi_n(x) = \delta_{n',n}. \tag{1.40}$$

Notice that (1.39) coincides with (1.28) up to a constant phase factor.

1.4 Coherent states

Coherent states constitute a class of special quantum states in which the quantum simple harmonic oscillator exhibits classical motion.

A coherent state is codified by a ket $|\alpha\rangle$ of the form

$$|\alpha\rangle = \exp\left(-|\alpha|^2/2\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{1/2}} |n\rangle, \tag{1.41}$$

where α is a complex number that parametrize the coherent states and the $|n\rangle$ are the eigenkets of the number operator $\hat{N}$. By construction, the kets $|\alpha\rangle$ are normalized to 1.

The coherent state $|\alpha\rangle$ is an eigenket of the destruction operator $\hat{a}$ with eigenvalue given by α:

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle\,. \tag{1.42}$$

Relation (1.42) can be considered as an alternative definition for the coherent states. In fact one can derive (1.41) searching an explicit normalized expression in terms of the kets $|n\rangle$ for the eigenkets of $\hat{a}$ (see [3] for the details).

To show what we mentioned at the beginning of this section we shall consider the dynamics of these states. Suppose that at an initial time t_0 the quantum simple harmonic oscillator is prepared in the state $|\psi(t_0)\rangle = |\alpha\rangle$. The Hamiltonian operator of the quantum simple harmonic oscillator does not depend on time, thus the evolution operator $\hat{U}(t, t_0)$ is simply given by

$$\hat{U}(t, t_0) = \exp\left(-i\frac{(t-t_0)}{\hbar}\hat{H}\right). \tag{1.43}$$

Then, take $t_0 = 0$ for simplicity. The state of the system at any time $t > 0$ is given by

$$|\psi(t)\rangle = \hat{U}(t, 0)\,|\psi(0)\rangle = \exp\left(-i\frac{(t-t_0)}{\hbar}\hat{H}\right)|\alpha\rangle\,, \tag{1.44}$$

which substituting (1.41) turns into

$$|\psi(t)\rangle = \exp\big(-|\alpha|^2/2\big)\sum_{n=0}^{\infty}\frac{(\alpha\exp(-i\omega t))^n}{(n!)^{1/2}}\,|n\rangle\exp(-i\omega t/2) = |\alpha\exp(-i\omega t)\rangle\exp(-i\omega t/2). \tag{1.45}$$

Now, setting $\alpha = 1$, (1.45) becomes

$$|\psi(t)\rangle = |\exp(-i\omega t)\rangle\exp(-i\omega t/2). \tag{1.46}$$

Using (1.42) and expressing the position and momentum operators in terms of the destruction and creation operators, we can compute the expectation values of position and momentum in a generic coherent state $|\alpha\rangle$, obtaining the following results:

$$< q > = \langle\alpha|\,\hat{q}\,|\alpha\rangle = -2^{1/2} l\,\mathrm{Im}\,\alpha, \tag{1.47a}$$

$$< p > = \langle\alpha|\,\hat{p}\,|\alpha\rangle = 2^{1/2}\hbar l^{-1}\,\mathrm{Re}\,\alpha. \tag{1.47b}$$

Relations (1.47) can then be used to compute the time-dependent mean values of position and momentum of a harmonic oscillator whose state evolves according to (1.46). We get

$$< q >_t = 2^{1/2} l\sin(\omega t), \tag{1.48a}$$

$$< p >_t = 2^{1/2}\hbar l^{-1}\cos(\omega t). \tag{1.48b}$$

These two expectation values evolve in time as the position and momentum of a classical simple harmonic oscillator.

Another important feature of the coherent states is that they minimize the uncertainty product $\Delta q \Delta p$. Given a generic coherent state $|\alpha\rangle$, the uncertainties associated to the position and the momentum of the harmonic oscillator in such state can be computed following the same reasoning used for the corresponding expectation values. One thus obtains

$$\Delta q = \langle\alpha| \left(\hat{q} - \langle\alpha| \hat{q} |\alpha\rangle\right)^2 |\alpha\rangle^{1/2} = 2^{-1/2} l, \tag{1.49a}$$

$$\Delta p = \langle\alpha| \left(\hat{p} - \langle\alpha| \hat{p} |\alpha\rangle\right)^2 |\alpha\rangle^{1/2} = 2^{-1/2} \hbar l^{-1}, \tag{1.49b}$$

and we can clearly see that

$$\Delta q \Delta p = \frac{\hbar}{2}. \tag{1.50}$$

Chapter 2

Explicitly time-dependent invariants

2.1 Generalities

Consider a system whose Hamiltonian operator $\hat{H}(t)$ depends explicitly on time. Of course, such a system is not closed, in the sense that some external influence, which do not need to be specified, may change the parameters of the system and therefore alter its total energy. Assume the existence of another explicitly time-dependent non-trivial operator $\hat{I}(t)$, which is Hermitian and *invariant*. That is to say $\hat{I}(t)$ satisfies the two following conditions:

$$\hat{I}^\dagger = \hat{I}, \tag{2.1}$$

$$\frac{d\hat{I}_H}{dt} = \left(\frac{d\hat{I}}{dt}\right)_H + \frac{1}{i\hbar}[\hat{I}_H, \hat{H}_H] = 0. \tag{2.2}$$

Let $|\psi(t)\rangle$ be a solution of the time-dependent Schrödinger equation

$$i\hbar\frac{d\,|\psi(t)\rangle}{dt} = \hat{H}(t)\,|\psi(t)\rangle\,. \tag{2.3}$$

Consider the same ket in the Heisenberg picture:

$$|\psi(t)\rangle_H = \hat{U}(t,0)^{-1}\,|\psi(t)\rangle = |\psi(0)\rangle\,. \tag{2.4}$$

By applying the left-hand side of (2.2) to $|\psi(t)\rangle_H$ we obtain

$$\begin{aligned}
\frac{d\hat{I}_H}{dt}\,|\psi(t)\rangle_H &= \left[\left(\frac{d\hat{I}}{dt}\right)_H + \frac{1}{i\hbar}[\hat{I}_H, \hat{H}_H]\right]|\psi(t)\rangle_H = \\
&= \left(\frac{d\hat{I}}{dt}\right)_H |\psi(t)\rangle_H + \frac{1}{i\hbar}[\hat{I}_H, \hat{H}_H]\,|\psi(t)\rangle_H = \\
&= \left(\frac{d\hat{I}}{dt}\right)_H |\psi(t)\rangle_H + \frac{1}{i\hbar}(\hat{I}_H\hat{H}_H - \hat{H}_H\hat{I}_H)\,|\psi(t)\rangle_H =
\end{aligned}$$

$$= \Big(\frac{d\hat{I}}{dt}\Big)_H |\psi(t)\rangle_H + \frac{1}{i\hbar}\hat{I}_H\hat{H}_H |\psi(t)\rangle_H - \frac{1}{i\hbar}\hat{H}_H\hat{I}_H |\psi(t)\rangle_H . \tag{2.5}$$

Relation (2.2) so implies

$$i\hbar\Big(\frac{d\hat{I}}{dt}\Big)_H |\psi(t)\rangle_H + \hat{I}_H\hat{H}_H |\psi(t)\rangle_H - \hat{H}_H\hat{I}_H |\psi(t)\rangle_H = 0. \tag{2.6}$$

Returning to the Schrödinger picture we have

$$i\hbar\frac{d\hat{I}}{dt} |\psi(t)\rangle + \hat{I}\hat{H} |\psi(t)\rangle - \hat{H}\hat{I} |\psi(t)\rangle = 0. \tag{2.7}$$

Observing that

$$\frac{d\hat{I}}{dt} |\psi(t)\rangle = \frac{d}{dt}(\hat{I} |\psi(t)\rangle) - \hat{I}\frac{d |\psi(t)\rangle}{dt}, \tag{2.8}$$

we can rewrite the left-hand side of equation (2.7) as

$$\begin{aligned} &i\hbar\frac{d}{dt}(\hat{I} |\psi(t)\rangle) - i\hbar\hat{I}\frac{d |\psi(t)\rangle}{dt} + \hat{I}\hat{H} |\psi(t)\rangle - \hat{H}\hat{I} |\psi(t)\rangle = \\ &\quad = i\hbar\frac{d}{dt}(\hat{I} |\psi(t)\rangle) - \hat{I}\Big(i\hbar\frac{d |\psi(t)\rangle}{dt} - \hat{H} |\psi(t)\rangle\Big) - \hat{H}\hat{I} |\psi(t)\rangle = \\ &\quad = i\hbar\frac{d}{dt}(\hat{I} |\psi(t)\rangle) - \hat{H}\hat{I} |\psi(t)\rangle \end{aligned} \tag{2.9}$$

where in the last step we used the fact that $|\psi(t)\rangle$ is a solution of the time-dependent Schrödinger equation. Then equation (2.7) becomes

$$i\hbar\frac{d}{dt}(\hat{I} |\psi(t)\rangle) = \hat{H}(\hat{I} |\psi(t)\rangle), \tag{2.10}$$

that means that the action of the invariant operator $\hat{I}$ on a solution of the Schrödinger equation produces another solution of the Schrödinger equation. It is worth noting that this result holds for any invariant operator, even if it involves the operation of time differentiation. However, as we will see in section 2.2, if the invariant operator does not involve time differentiation, we can derive an explicit rule to choose the phases of the eigenkets of $\hat{I}$ in such a way that these kets themselves satisfy the Schrödinger equation. Henceforth, we shall assume that $\hat{I}$ does not involve time-differentiation.

We assume that the invariant operator belongs to a complete set of commuting self-adjoint operators, so that there is an orthonormal basis of eigenkets of $\hat{I}$ which is unique up to a phase choice. Let us denote the eigenkets of such basis by $|\lambda, \kappa\rangle$, where λ represents the eigenvalue of $\hat{I}$ associated to the eigenket considered and the label κ stands for the collection of quantum numbers other than λ necessary for identifying the same eigenket. Then, these kets satisfy the eigenvalue equation

$$\hat{I}(t) |\lambda, \kappa\rangle = \lambda |\lambda, \kappa\rangle \tag{2.11}$$

and the orthonormality relation

$$\langle \lambda', \kappa' | \lambda, \kappa \rangle = \delta_{\lambda',\lambda} \delta_{\kappa',\kappa}. \tag{2.12}$$

The eigenvalues λ are real because of the condition (2.1). We can also give a proof of their time-independence. We first apply the left-hand side of (2.2) to the ket $|\lambda, \kappa\rangle_H$ and obtain

$$\begin{aligned}
\frac{d\hat{I}_H}{dt} |\lambda, \kappa\rangle_H &= \left[\left(\frac{d\hat{I}}{dt} \right)_H + \frac{1}{i\hbar} [\hat{I}_H, \hat{H}_H] \right] |\lambda, \kappa\rangle_H = \\
&= \left(\frac{d\hat{I}}{dt} \right)_H |\lambda, \kappa\rangle_H + \frac{1}{i\hbar} [\hat{I}_H, \hat{H}_H] \, |\lambda, \kappa\rangle_H = \\
&= \left(\frac{d\hat{I}}{dt} \right)_H |\lambda, \kappa\rangle_H + \frac{1}{i\hbar} (\hat{I}_H \hat{H}_H - \hat{H}_H \hat{I}_H) \, |\lambda, \kappa\rangle_H = \\
&= \left(\frac{d\hat{I}}{dt} \right)_H |\lambda, \kappa\rangle_H + \frac{1}{i\hbar} \hat{I}_H \hat{H}_H \, |\lambda, \kappa\rangle_H - \frac{1}{i\hbar} \hat{H}_H \hat{I}_H \, |\lambda, \kappa\rangle_H = \\
&= \left(\frac{d\hat{I}}{dt} \right)_H |\lambda, \kappa\rangle_H + \frac{1}{i\hbar} \hat{I}_H \hat{H}_H \, |\lambda, \kappa\rangle_H - \frac{\lambda}{i\hbar} \hat{H}_H \, |\lambda, \kappa\rangle_H \, ,
\end{aligned} \tag{2.13}$$

where in the last step we used (2.11). Relation (2.2) so implies

$$i\hbar \left(\frac{d\hat{I}}{dt} \right)_H |\lambda, \kappa\rangle_H + \hat{I}_H \hat{H}_H \, |\lambda, \kappa\rangle_H - \lambda \hat{H}_H \, |\lambda, \kappa\rangle_H = 0. \tag{2.14}$$

Then we take the scalar product of equation (2.14) with a ket $|\lambda', \kappa'\rangle_H$:

$$\begin{aligned}
&i\hbar \, {}_H\langle \lambda', \kappa' | \left(\frac{d\hat{I}}{dt} \right)_H |\lambda, \kappa\rangle_H + {}_H\langle \lambda', \kappa' | \hat{I}_H \hat{H}_H \, |\lambda, \kappa\rangle_H - \lambda \, {}_H\langle \lambda', \kappa' | \hat{H}_H \, |\lambda, \kappa\rangle_H = 0 \\
&i\hbar \, {}_H\langle \lambda', \kappa' | \left(\frac{d\hat{I}}{dt} \right)_H |\lambda, \kappa\rangle_H + \lambda' \, {}_H\langle \lambda', \kappa' | \hat{H}_H \, |\lambda, \kappa\rangle_H - \lambda \, {}_H\langle \lambda', \kappa' | \hat{H}_H \, |\lambda, \kappa\rangle_H = 0 \\
&i\hbar \, {}_H\langle \lambda', \kappa' | \left(\frac{d\hat{I}}{dt} \right)_H |\lambda, \kappa\rangle_H + (\lambda' - \lambda) \, {}_H\langle \lambda', \kappa' | \hat{H}_H \, |\lambda, \kappa\rangle_H = 0.
\end{aligned} \tag{2.15}$$

Therefore, if $\lambda' = \lambda$, relation

$${}_H\langle \lambda, \kappa' | \left(\frac{d\hat{I}}{dt} \right)_H |\lambda, \kappa\rangle_H = 0 \tag{2.16}$$

holds regardless of the relation between κ' and κ. Clearly, an equation of the same form holds in the Schrödinger picture:

$$\langle \lambda, \kappa' | \, \frac{d\hat{I}}{dt} \, |\lambda, \kappa\rangle = 0. \tag{2.17}$$

Now, by differentiating (2.11) with respect to time, we obtain

$$\frac{d\hat{I}}{dt}\left|\lambda,\kappa\right\rangle + \hat{I}\frac{d}{dt}\left|\lambda,\kappa\right\rangle = \frac{d\lambda}{dt}\left|\lambda,\kappa\right\rangle + \lambda\frac{d}{dt}\left|\lambda,\kappa\right\rangle . \tag{2.18}$$

Taking the scalar product of equation (2.18) with the ket $|\lambda,\kappa\rangle$, we get

$$\begin{aligned}
&\left\langle\lambda,\kappa\right|\frac{d\hat{I}}{dt}\left|\lambda,\kappa\right\rangle + \left\langle\lambda,\kappa\right|\hat{I}\frac{d}{dt}\left|\lambda,\kappa\right\rangle = \frac{d\lambda}{dt}\left\langle\lambda,\kappa|\lambda,\kappa\right\rangle + \lambda\left\langle\lambda,\kappa\right|\frac{d}{dt}\left|\lambda,\kappa\right\rangle \\
&\left\langle\lambda,\kappa\right|\frac{d\hat{I}}{dt}\left|\lambda,\kappa\right\rangle + \lambda\left\langle\lambda,\kappa\right|\frac{d}{dt}\left|\lambda,\kappa\right\rangle = \frac{d\lambda}{dt} + \lambda\left\langle\lambda,\kappa\right|\frac{d}{dt}\left|\lambda,\kappa\right\rangle \\
&\left\langle\lambda,\kappa\right|\frac{d\hat{I}}{dt}\left|\lambda,\kappa\right\rangle = \frac{d\lambda}{dt},
\end{aligned} \tag{2.19}$$

where in the second step we used the orthonormality of the kets $|\lambda,\kappa\rangle$. Finally, using (2.17), equation (2.19) implies

$$\frac{d\lambda}{dt} = 0. \tag{2.20}$$

Of course, since the eigenvalues are time-independent, the eigenkets must be time-dependent.

2.2 Relation to solutions of the time-dependent Schrödinger equation

In order to investigate the connection between eigenkets of $\hat{I}$ and solutions of the Schrödinger equation, we first rewrite equation (2.18) using (2.20) as follows:

$$(\lambda\hat{1} - \hat{I})\frac{d}{dt}\left|\lambda,\kappa\right\rangle = \frac{d\hat{I}}{dt}\left|\lambda,\kappa\right\rangle . \tag{2.21}$$

Then, by taking the scalar product of equation (2.21) with the ket $|\lambda',\kappa'\rangle$, we obtain

$$\begin{aligned}
&\left\langle\lambda',\kappa'\right|(\lambda\hat{1} - \hat{I})\frac{d}{dt}\left|\lambda,\kappa\right\rangle = \left\langle\lambda',\kappa'\right|\frac{d\hat{I}}{dt}\left|\lambda,\kappa\right\rangle \\
&i\hbar\lambda\left\langle\lambda',\kappa'\right|\frac{d}{dt}\left|\lambda,\kappa\right\rangle - i\hbar\lambda'\left\langle\lambda',\kappa'\right|\frac{d}{dt}\left|\lambda,\kappa\right\rangle = i\hbar\left\langle\lambda',\kappa'\right|\frac{d\hat{I}}{dt}\left|\lambda,\kappa\right\rangle \\
&i\hbar(\lambda-\lambda')\left\langle\lambda',\kappa'\right|\frac{d}{dt}\left|\lambda,\kappa\right\rangle = (\lambda-\lambda')\left\langle\lambda',\kappa'\right|\hat{H}\left|\lambda,\kappa\right\rangle ,
\end{aligned} \tag{2.22}$$

where in the last step we used equation (2.15) read in the Schrödinger picture to rewrite the right-hand side. For $\lambda' \neq \lambda$, equation (2.22) implies

$$i\hbar\left\langle\lambda',\kappa'\right|\frac{d}{dt}\left|\lambda,\kappa\right\rangle = \left\langle\lambda',\kappa'\right|\hat{H}\left|\lambda,\kappa\right\rangle . \tag{2.23}$$

Nonetheless, we cannot infer

$$i\hbar \langle \lambda, \kappa' | \frac{d}{dt} | \lambda, \kappa \rangle = \langle \lambda, \kappa' | \hat{H} | \lambda, \kappa \rangle . \tag{2.24}$$

If equation (2.23) held for $\lambda' = \lambda$ as well as for $\lambda' \neq \lambda$, we would then immediately deduce that the kets $|\lambda, \kappa\rangle$ are solutions of the Schrödinger equation. We shall specify the phases of the kets $|\lambda, \kappa\rangle$, which have not been fixed by our definitions, in order to achieve the validity of equation (2.23) for $\lambda' = \lambda$ as well.

We assume that some definite phase has been chosen, but we are still free to multiply $|\lambda, \kappa\rangle$ by an arbitrarily time-dependent phase factor. So, we can define a new set of eigenkets of $\hat{I}(t)$ related to our initial set by a time-dependent gauge transformation:

$$|\lambda, \kappa\rangle_\alpha = \exp(i\alpha_{\lambda\kappa}(t)) \, |\lambda, \kappa\rangle \tag{2.25}$$

where the $\alpha_{\lambda\kappa}(t)$ are arbitrary real functions of time. Since by assumption $\hat{I}(t)$ does not contain time-derivative operators, the kets $|\lambda, \kappa\rangle_\alpha$ are still orthonormal eigenkets of $\hat{I}$. For $\lambda' = \lambda$, equation (2.23) also holds for matrix elements with respect to the new eigenkets. Now, suppose that the new eigenkets satisfy (2.23) for $\lambda' = \lambda$ as well. Then, we have

$$i\hbar \, {}_\alpha\langle \lambda, \kappa' | \frac{d}{dt} | \lambda, \kappa \rangle_\alpha = {}_\alpha\langle \lambda, \kappa' | \hat{H} | \lambda, \kappa \rangle_\alpha \tag{2.26}$$

Substituting (2.25) into equation (2.26), we obtain

$$\begin{aligned} i\hbar \Big[\exp(-i\alpha_{\lambda\kappa'}(t)) \, \langle \lambda, \kappa' | \Big] \frac{d}{dt} \Big[\, |\lambda, \kappa\rangle \exp(i\alpha_{\lambda\kappa}(t)) \Big] = \\ = \Big[\exp(-i\alpha_{\lambda\kappa}(t)) \, \langle \lambda, \kappa' | \Big] \hat{H} \Big[\, |\lambda, \kappa\rangle \exp(i\alpha_{\lambda\kappa}(t)) \Big] \end{aligned} \tag{2.27}$$

and a further development of the equation brings us to

$$i\hbar \langle \lambda, \kappa' | \frac{d}{dt} | \lambda, \kappa \rangle - \hbar \frac{d\alpha_{\lambda\kappa}}{dt} \langle \lambda, \kappa' | \lambda, \kappa \rangle = \langle \lambda, \kappa' | \hat{H} | \lambda, \kappa \rangle . \tag{2.28}$$

At last, using the orthonormality of the kets $|\lambda, \kappa\rangle$, we get

$$\hbar \delta_{\kappa', \kappa} \frac{d\alpha_{\lambda\kappa}}{dt} = \langle \lambda, \kappa' | \left(i\hbar \frac{d}{dt} - \hat{H} \right) | \lambda, \kappa \rangle . \tag{2.29}$$

For $\kappa' \neq \kappa$, the left-hand side of (2.29) vanishes and the equation reduces to

$$\langle \lambda, \kappa' | \left(i\hbar \frac{d}{dt} - \hat{H} \right) | \lambda, \kappa \rangle = 0. \tag{2.30}$$

This diagonalization condition is always possible since the operator $i\hbar \, d/dt - \hat{H}$ is Hermitian. For $\kappa' = \kappa$, instead, we obtain the following equation for the phases of the gauge transformation:

$$\hbar \frac{d\alpha_{\lambda\kappa}}{dt} = \langle \lambda, \kappa | \left(i\hbar \frac{d}{dt} - \hat{H} \right) | \lambda, \kappa \rangle . \tag{2.31}$$

Therefore, the new eigenkets defined in (2.25) are solutions of the Schrödinger equation if and only if the initial eigenkets $|\lambda, \kappa\rangle$ are chosen to satisfy equation (2.30) and the phases $\alpha_{\lambda\kappa}(t)$ are chosen to be solutions of the equation (2.31).

Since each ket of the new set of eigenkets of $\hat{I}(t)$, $|\lambda, \kappa\rangle_{\alpha}$, satisfies the Schrödinger equation, the general solution of this latter can be written as

$$|t\rangle = \sum_{\lambda,\kappa} c_{\lambda\kappa} \, |\lambda, \kappa; t\rangle_{\alpha} = \sum_{\lambda,\kappa} c_{\lambda\kappa} \exp(i\alpha_{\lambda\kappa}(t)) \, |\lambda, \kappa; t\rangle \,, \tag{2.32}$$

where the $c_{\lambda\kappa}$ are time-independent coefficients given by

$$c_{\lambda\kappa} = {}_{\alpha}\langle\lambda, \kappa; t|t\rangle = \exp(-i\alpha_{\lambda\kappa}(t)) \, \langle\lambda, \kappa; t|t\rangle \,. \tag{2.33}$$

We revised the notation in equations (2.32) and (2.33) by indicating the time dependences explicitly: here the Schrödinger state vector is denoted by $|t\rangle$ and the eigenkets of the invariant by $|\lambda, \kappa; t\rangle$. However, in order to simplify the notation, we shall continue to suppress the time dependence in the following.

Chapter 3

The quantum time-dependent harmonic oscillator

3.1 Introduction to the problem

A quantum time-dependent harmonic oscillator of mass $m(t)$ and frequency $\omega(t)$ is characterized by the following Hamiltonian operator:

$$\hat{H}(t) = \frac{\hat{p}^2}{2m(t)} + \frac{1}{2}m(t)\omega^2(t)\hat{q}^2, \tag{3.1}$$

where $\hat{q}$ and $\hat{p}$ are the position and momentum operators, respectively, which obey the canonical commutation relation $[\hat{q}, \hat{p}] = \hat{1}$. The Heisenberg equations for $\hat{q}$ and $\hat{p}$ are given by

$$\frac{d\hat{q}_H}{dt} = \frac{1}{i\hbar}[\hat{q}_H, \hat{H}_H] = \frac{\hat{p}_H}{m(t)}, \tag{3.2a}$$

$$\frac{d\hat{p}_H}{dt} = \frac{1}{i\hbar}[\hat{p}_H, \hat{H}_H] = -m(t)\omega^2(t)\hat{q}_H. \tag{3.2b}$$

Equations (3.2) combined together give the equation of motion

$$\frac{d^2\hat{q}_H}{dt^2} + \mu(t)\frac{d\hat{q}_H}{dt} + \omega^2(t)\hat{q}_H = 0, \tag{3.3}$$

where we have introduced

$$\mu(t) = \frac{d}{dt}[\ln(m(t))]. \tag{3.4}$$

We assume $m(t)$ and $\omega(t)$ to be arbitrary, piecewise-continuous functions of time. In particular, we assume the former to be real and positive. We do not require $\omega(t)$ to be real because our treatment is valid even when $\omega(t)$ is imaginary, until $\omega^2(t)$ is a real function. Therefore, henceforth we shall assume $\omega^2(t)$ to be a either positive or negative real function of time. Of course, if one wants to recover the quantum simple harmonic oscillator in the limit of time-independence, $\omega(t)$ needs to be real.

The quantum time-dependent harmonic oscillator is a problem of *quantum dynamics*, since the potential which characterises it is time-dependent. We are interested in showing the existence of an exact solution to the associated time-dependent Schrödinger equation. The key to do this is the derivation of a class of exact Hermitian invariants.

3.2 A family of invariants for a time-dependent harmonic oscillator

We assume the existence of a Hermitian invariant operator of the homogeneous, quadratic form

$$\hat{I}(t) = \frac{1}{2}[\alpha(t)\hat{q}^2 + \beta(t)\hat{p}^2 + \gamma(t)\{\hat{q},\hat{p}\}_+], \tag{3.5}$$

where $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are real functions of time and the numerical factor is inserted for convenience. Here we use the same notation used by Lewis and Riesenfeld in [4] for the anticommutator of two operators, that is $\{\hat{q},\hat{p}\}_+ = \hat{q}\hat{p} + \hat{p}\hat{q}$.

Being an invariant, $\hat{I}(t)$ must satisfy equation (2.2). So, let us first take the derivative of (3.5) with respect to time,

$$\frac{d\hat{I}}{dt} = \frac{1}{2}\Big(\frac{d\alpha}{dt}\hat{q}^2 + \frac{d\beta}{dt}\hat{p}^2 + \frac{d\gamma}{dt}\{\hat{q},\hat{p}\}_+\Big), \tag{3.6}$$

and then, for the sake of clarity, rewrite this and the previous equation in the Heisenberg picture:

$$\hat{I}_H = \frac{1}{2}(\alpha\hat{q}_H^2 + \beta\hat{p}_H^2 + \gamma\{\hat{q}_H,\hat{p}_H\}_+), \tag{3.7}$$

$$\Big(\frac{d\hat{I}}{dt}\Big)_H = \frac{1}{2}\Big(\frac{d\alpha}{dt}\hat{q}_H^2 + \frac{d\beta}{dt}\hat{p}_H^2 + \frac{d\gamma}{dt}\{\hat{q}_H,\hat{p}_H\}_+\Big). \tag{3.8}$$

Now we compute the commutator $[\hat{I}_H, \hat{H}_H]$:

$$\begin{aligned}
[\hat{I}_H, \hat{H}_H] &= \left[\frac{1}{2}(\alpha\hat{q}_H^2 + \beta\hat{p}_H^2 + \gamma\{\hat{q}_H,\hat{p}_H\}_+), \frac{\hat{p}_H^2}{2m} + \frac{1}{2}m\omega^2\hat{q}_H^2\right] = \\
&= \frac{1}{2m}\left[\frac{1}{2}(\alpha\hat{q}_H^2 + \beta\hat{p}_H^2 + \gamma\{\hat{q}_H,\hat{p}_H\}_+), \hat{p}_H^2\right] \\
&\quad + \frac{1}{2}m\omega^2\left[\frac{1}{2}(\alpha\hat{q}_H^2 + \beta\hat{p}_H^2 + \gamma\{\hat{q}_H,\hat{p}_H\}_+), \hat{q}_H^2\right] = \\
&= \frac{\alpha}{4m}[\hat{q}_H^2, \hat{p}_H^2] + \frac{\beta}{4m}[\hat{p}_H^2, \hat{p}_H^2] + \frac{\gamma}{4m}[\{\hat{q}_H,\hat{p}_H\}_+, \hat{p}_H^2]
\end{aligned}$$

$$+\frac{m\omega^2\alpha}{4}[\hat{q}_H^2,\hat{q}_H^2]+\frac{m\omega^2\beta}{4}[\hat{p}_H^2,\hat{q}_H^2]+\frac{m\omega^2\gamma}{4}[\{\hat{q}_H,\hat{p}_H\}_+,\hat{q}_H^2]. \tag{3.9}$$

Substituting every commutator appearing in (3.9) with its explicit result (see Appendices for more details) and rearranging the expression thus obtained, we get

$$[\hat{I}_H,\hat{H}_H]=-i\hbar m\omega^2\gamma\,\hat{q}_H^2+i\hbar\frac{\gamma}{m}\hat{p}_H^2+\Big(\frac{i\hbar}{2}\frac{\alpha}{m}-\frac{i\hbar}{2}m\omega^2\beta\Big)\{\hat{q}_H,\hat{p}_H\}_+. \tag{3.10}$$

Finally, we can obtain an explicit expression for the left-hand side of (2.2) using (3.8) and (3.10), namely

$$\begin{aligned}\frac{d\hat{I}_H}{dt}&=\Big(\frac{d\hat{I}}{dt}\Big)_H+\frac{1}{i\hbar}[\hat{I}_H,\hat{H}_H]=\\&=\frac{1}{2}\Big[\Big(\frac{d\alpha}{dt}-2m\omega^2\gamma\Big)\hat{q}_H^2+\Big(\frac{d\beta}{dt}+\frac{2\gamma}{m}\Big)\hat{p}_H^2+\Big(\frac{d\gamma}{dt}+\frac{\alpha}{m}-m\omega^2\beta\Big)\{\hat{q}_H,\hat{p}_H\}_+\Big].\end{aligned} \tag{3.11}$$

Relation (2.2) so implies

$$\Big(\frac{d\alpha}{dt}-2m\omega^2\gamma\Big)\hat{q}_H^2+\Big(\frac{d\beta}{dt}+\frac{2\gamma}{m}\Big)\hat{p}_H^2+\Big(\frac{d\gamma}{dt}+\frac{\alpha}{m}-m\omega^2\beta\Big)\{\hat{q}_H,\hat{p}_H\}_+=0. \tag{3.12}$$

In order to satisfy equation (3.12), we demand

$$\dot{\alpha}=2m\omega^2\gamma, \tag{3.13a}$$

$$\dot{\beta}=-\frac{2\gamma}{m}, \tag{3.13b}$$

$$\dot{\gamma}=-\frac{\alpha}{m}+m\omega^2\beta, \tag{3.13c}$$

where we shifted to the notation $\dot{\alpha}=d\alpha/dt$ to lighten the expressions with which we will deal in the following. It is convenient to introduce another function $\sigma(t)$ defined by

$$\beta(t)=\sigma^2(t). \tag{3.14}$$

Because $\beta(t)$ is a real function of time, $\sigma^2(t)$ is a real function of time as well. Equation (3.13b) then becomes

$$\gamma=-m\sigma\dot{\sigma}. \tag{3.15}$$

Taking the time derivative of equation (3.15), we obtain

$$\dot{\gamma}=-\dot{m}\sigma\dot{\sigma}-m\dot{\sigma}^2-m\sigma\ddot{\sigma}, \tag{3.16}$$

and comparing the right-hand side of this equation with that of equation (3.13c) we get

$$\alpha=m\dot{m}\sigma\dot{\sigma}+m^2\dot{\sigma}^2+m^2\sigma\ddot{\sigma}+m^2\omega^2\sigma^2. \tag{3.17}$$

Similarly, taking the time derivative of equation (3.17), we obtain

$$\begin{aligned}\dot{\alpha} &= \dot{m}^2\sigma\dot{\sigma} + m\ddot{m}\sigma\dot{\sigma} + 3m\dot{m}\dot{\sigma}^2 + 3m\dot{m}\sigma\ddot{\sigma} + 3m^2\dot{\sigma}\ddot{\sigma} \\ &\quad + m^2\sigma\,\dddot{\sigma} + 2m\dot{m}\omega^2\sigma^2 + 2m^2\omega\dot{\omega}\sigma^2 + 2m^2\omega^2\sigma\dot{\sigma},\end{aligned} \tag{3.18}$$

and comparing the right-hand side of this equation with that of equation (3.13a) we get

$$\begin{aligned}&\dot{m}^2\sigma\dot{\sigma} + m\ddot{m}\sigma\dot{\sigma} + 3m\dot{m}\dot{\sigma}^2 + 3m\dot{m}\sigma\ddot{\sigma} + 3m^2\dot{\sigma}\ddot{\sigma} \\ &\quad + m^2\sigma\,\dddot{\sigma} + 2m\dot{m}\omega^2\sigma^2 + 2m^2\omega\dot{\omega}\sigma^2 + 4m^2\omega^2\sigma\dot{\sigma} = 0.\end{aligned} \tag{3.19}$$

The non-linear differential equation so obtained imposes a constraint on $\sigma(t)$. We can render it more manageable as follows:

$$\begin{aligned}&3\dot{\sigma}(m^2\ddot{\sigma} + m^2\omega^2\sigma + m\dot{m}\dot{\sigma}) + \sigma[m^2\omega^2\dot{\sigma} + 2(m\dot{m}\omega^2 + m^2\omega\dot{\omega})\sigma \\ &\quad + \dot{m}^2\dot{\sigma} + m\ddot{m}\dot{\sigma} + m\dot{m}\ddot{\sigma}] + \sigma(2m\dot{m}\ddot{\sigma} + m^2\,\dddot{\sigma}) = 0 \\ &3\dot{\sigma}(m^2\ddot{\sigma} + m^2\omega^2\sigma + m\dot{m}\dot{\sigma}) + \sigma\frac{d}{dt}(m^2\omega^2\sigma + m\dot{m}\dot{\sigma}) + \sigma\frac{d}{dt}(m^2\ddot{\sigma}) = 0 \\ &\sigma\frac{d}{dt}(m^2\ddot{\sigma} + m^2\omega^2\sigma + m\dot{m}\dot{\sigma}) + 3\dot{\sigma}(m^2\ddot{\sigma} + m^2\omega^2\sigma + m\dot{m}\dot{\sigma}) = 0.\end{aligned} \tag{3.20}$$

In order to find a first integral, we rewrite (3.20) as

$$\frac{1}{m^2\ddot{\sigma} + m^2\omega^2\sigma + m\dot{m}\dot{\sigma}}\frac{d}{dt}(m^2\ddot{\sigma} + m^2\omega^2\sigma + m\dot{m}\dot{\sigma}) = -3\frac{\dot{\sigma}}{\sigma} \tag{3.21}$$

and then integrate it to get

$$\ln\left(m^2\ddot{\sigma} + m^2\omega^2\sigma + m\dot{m}\dot{\sigma}\right) = -3\ln\sigma + c, \tag{3.22}$$

where c is an integration constant. From equation (3.22) thus follows

$$m^2\ddot{\sigma} + m^2\omega^2\sigma + m\dot{m}\dot{\sigma} = \frac{c}{\sigma^3}. \tag{3.23}$$

Exploiting (3.23), we can rewrite (3.17) as

$$\alpha = m^2\dot{\sigma}^2 + \frac{c}{\sigma^2}. \tag{3.24}$$

Now, returning to the Schrödinger picture and using (3.14), (3.15) and (3.24), we can express the invariant in the form

$$\hat{I}(t) = \frac{1}{2}\left[\frac{c}{\sigma^2}\hat{q}^2 + (\sigma\hat{p} - m\dot{\sigma}\hat{q})^2\right], \tag{3.25}$$

with (3.23) as an auxiliary equation. After making the scale transformation

$$\sigma(t) = c^{1/4}\rho(t), \tag{3.26}$$

$\rho(t)$ being a new auxiliary function of time, equation (3.25) turns into

$$\hat{I}(t) = \frac{c^{1/2}}{2}\left[\frac{1}{\rho^2}\hat{q}^2 + (\rho\hat{p} - m\dot{\rho}\hat{q})^2\right], \tag{3.27}$$

while the subsidiary condition (3.23) becomes

$$m^2\ddot{\rho} + m^2\omega^2\rho + m\dot{m}\dot{\rho} = \frac{1}{\rho^3}. \tag{3.28}$$

So the constant multiplicative factor $c^{1/2}$ in (3.27) can be discarded, thus showing the illusory arbitrariness implied by the presence of the constant c in equation (3.25). Finally, we can write the invariant in the form

$$\hat{I} = \frac{1}{2}\left[\frac{1}{\rho^2}\hat{q}^2 + (\rho\hat{p} - m\dot{\rho}\hat{q})^2\right], \tag{3.29}$$

where $\rho(t)$ satisfies the auxiliary equation

$$\ddot{\rho} + \mu\dot{\rho} + \omega^2\rho = \frac{1}{m^2\rho^3}. \tag{3.30}$$

Note that only the real solutions of this equation make $\hat{I}(t)$ Hermitian.

With any particular solution of equation (3.30) there is associated an invariant operator of the form given by equation (3.29). In this way we can construct a family of invariants which is in one-to-one correspondence with the set of solutions of the non-linear differential equation (3.30). According to the theory of chapter 2, chosen an invariant of the form (3.29), we can compute the appropriate time-dependent phase factors that make the eigenstates of the invariant solutions of the Schrödinger equation. Of course, before pursuing such issue, we have to determine the eigenvalues and eigenstates of the invariant. This problem may be approached with an algebraic method or an analytical one, just as the energy eigenvalue problem for the quantum simple harmonic oscillator.

3.3 Algebraic solution

The algebraic approach is completely analogous to the operator technique presented in section 1.2. Consider the time-dependent operator

$$\hat{a} = \frac{1}{(2\hbar)^{1/2}}\left[\frac{1}{\rho}\hat{q} + i(\rho\hat{p} - m\dot{\rho}\hat{q})\right] \tag{3.31}$$

and its adjoint

$$\hat{a}^\dagger = \frac{1}{(2\hbar)^{1/2}}\Big[\frac{1}{\rho}\hat{q} - i(\rho\hat{p} - m\dot{\rho}\hat{q})\Big]. \tag{3.32}$$

The operators $\hat{a}$ and $\hat{a}^\dagger$ constitute a pair of destruction and creation operators. In fact, they obey the commutation relation $[\hat{a}, \hat{a}^\dagger] = \hat{1}$. This can be easily shown by direct calculation:

$$\begin{aligned}
[\hat{a}, \hat{a}^\dagger] = &\Big[\frac{1}{(2\hbar)^{1/2}}\Big[\frac{1}{\rho}\hat{q} + i(\rho\hat{p} - m\dot{\rho}\hat{q})\Big], \frac{1}{(2\hbar)^{1/2}}\Big[\frac{1}{\rho}\hat{q} - i(\rho\hat{p} - m\dot{\rho}\hat{q})\Big]\Big] = \\
&= \frac{1}{(2\hbar)^{1/2}\rho}\Big[\frac{1}{(2\hbar)^{1/2}}\Big[\frac{1}{\rho}\hat{q} + i(\rho\hat{p} - m\dot{\rho}\hat{q})\Big], \hat{q}\Big] \\
&\quad - \frac{i\rho}{(2\hbar)^{1/2}}\Big[\frac{1}{(2\hbar)^{1/2}}\Big[\frac{1}{\rho}\hat{q} + i(\rho\hat{p} - m\dot{\rho}\hat{q})\Big], \hat{p}\Big] \\
&\quad + \frac{im\dot{\rho}}{(2\hbar)^{1/2}}\Big[\frac{1}{(2\hbar)^{1/2}}\Big[\frac{1}{\rho}\hat{q} + i(\rho\hat{p} - m\dot{\rho}\hat{q})\Big], \hat{q}\Big] = \\
&= \frac{1}{2\hbar\rho^2}[\hat{q}, \hat{q}] + \frac{i}{2\hbar}[\hat{p}, \hat{q}] - \frac{i\dot{\rho}m}{2\hbar\rho}[\hat{q}, \hat{q}] - \frac{i}{2\hbar}[\hat{q}, \hat{p}] + \frac{\rho^2}{2\hbar}[\hat{p}, \hat{p}] \\
&\quad - \frac{m\rho\dot{\rho}}{2\hbar}[\hat{q}, \hat{p}] + \frac{im\dot{\rho}}{2\hbar\rho}[\hat{q}, \hat{q}] - \frac{m\rho\dot{\rho}}{2\hbar}[\hat{p}, \hat{q}] + \frac{m^2\dot{\rho}^2}{2\hbar}[\hat{q}, \hat{q}] = \\
&= -\frac{i}{2\hbar}[\hat{q}, \hat{p}] - \frac{i}{2\hbar}[\hat{q}, \hat{p}] - \frac{m\rho\dot{\rho}}{2\hbar}[\hat{q}, \hat{p}] + \frac{m\rho\dot{\rho}}{2\hbar}[\hat{q}, \hat{p}] = \\
&= -\frac{i}{\hbar}[\hat{q}, \hat{p}] = \\
&= \hat{1}.
\end{aligned} \tag{3.33}$$

So we can define a number operator $\hat{N} = \hat{a}^\dagger\hat{a}$. We can derive an explicit expression of $\hat{N}$ again by direct calculation:

$$\begin{aligned}
\hat{N} = &\hat{a}^\dagger\hat{a} = \\
&= \frac{1}{2\hbar}\Big[\frac{1}{\rho}\hat{q} - i(\rho\hat{p} - m\dot{\rho}\hat{q})\Big]\Big[\frac{1}{\rho}\hat{q} + i(\rho\hat{p} - m\dot{\rho}\hat{q})\Big] = \\
&= \frac{1}{2\hbar}\Big[\frac{1}{\rho^2}\hat{q}^2 + (\rho\hat{p} - m\dot{\rho}\hat{q})^2 + \frac{i}{\rho}\hat{q}(\rho\hat{p} - m\dot{\rho}\hat{q}) - \frac{i}{\rho}(\rho\hat{p} - m\dot{\rho}\hat{q})\hat{q}\Big] = \\
&= \frac{1}{2\hbar}\Big[\frac{1}{\rho^2}\hat{q}^2 + (\rho\hat{p} - m\dot{\rho}\hat{q})^2 + i\hat{q}\hat{p} - \frac{im\dot{\rho}}{\rho}\hat{q}^2 - i\hat{p}\hat{q} + \frac{im\dot{\rho}}{\rho}\hat{q}^2\Big] = \\
&= \frac{1}{2\hbar}\Big[\frac{1}{\rho^2}\hat{q}^2 + (\rho\hat{p} - m\dot{\rho}\hat{q})^2 + i[\hat{q}, \hat{p}]\Big] = \\
&= \frac{1}{2\hbar}\Big[\frac{1}{\rho^2}\hat{q}^2 + (\rho\hat{p} - m\dot{\rho}\hat{q})^2\Big] - \frac{\hat{1}}{2} =
\end{aligned}$$

$$= \frac{1}{\hbar}\hat{I} - \frac{\hat{1}}{2}, \tag{3.34}$$

where in the last step we recognized (3.29). Thus one deduces

$$\hat{I} = \hbar\Big(\hat{N} + \frac{\hat{1}}{2}\Big). \tag{3.35}$$

Before proceeding with our discussion, it is worth noting that all the results presented in section 1.2 concerning the eigenvalue problem of the number operator introduced for the quantum simple harmonic oscillator (except for the non-degeneracy of the spectrum, which is a property characteristic of the problem considered) are equally valid for any operator of the form (1.8), with $\hat{a}$ and its adjoint $\hat{a}^\dagger$ satisfying the commutation relation (1.7), since they are all direct consequences of this very same commutation relation.

So, the spectrum of $\hat{N}$ is the set of non-negative integer numbers. Furthermore, relation (3.35) implies that the normalized eigenkets $|\lambda\rangle$ of $\hat{I}$ coincide with the normalized eigenkets $|n\rangle$ of $\hat{N}$:

$$\hat{N}\,|n\rangle = n\,|n\rangle\,, \tag{3.36}$$

$$\hat{I}\,|n\rangle = \lambda_n\,|n\rangle\,, \tag{3.37}$$

where $n = 0, 1, 2, \dots$ and the eigenvalues λ_n of $\hat{I}$ are given by

$$\lambda_n = \hbar\Big(n + \frac{1}{2}\Big). \tag{3.38}$$

We shall assume $\hat{N}$ and $\hat{I}$ to be selfadjoint operators, for the purpose of having an orthonormal basis for the space of the dynamical states of the system formed by their eigenkets. It can be proven that this basis is unique up to normalization, since the spectrum of $\hat{N}$ is completely non-degenerate. To begin with, for the reasons exposed in the previous paragraph, if $|0\rangle$ is an eigenket of $\hat{N}$ belonging to the eigenvalue 0, it must obey the following relation:

$$\hat{a}\,|0\rangle = 0. \tag{3.39}$$

Then, consider the action of the destruction operator $\hat{a}$ on a bra $\langle x|$, conjugate of the ket $|x\rangle$ of the configuration space representation:

$$\begin{aligned}
\langle x|\,\hat{a} &= \frac{1}{(2\hbar)^{1/2}}\,\langle x|\,\Big[\frac{1}{\rho}\hat{q} + i(\rho\hat{p} - m\dot{\rho}\hat{q})\Big] = \\
&= \frac{1}{(2\hbar)^{1/2}}\Big[\frac{1}{\rho}\,\langle x|\,\hat{q} + i\rho\,\langle x|\,\hat{p} - im\dot{\rho}\,\langle x|\,\hat{q}\Big] = \\
&= \frac{1}{(2\hbar)^{1/2}}\Big[\frac{1}{\rho}x\,\langle x| + \hbar\rho\frac{d}{dx}\,\langle x| - im\dot{\rho}x\,\langle x|\,\Big] = \\
&= \frac{\hbar\rho}{(2\hbar)^{1/2}}\Big[\Big(\frac{1}{\hbar\rho^2} - \frac{im\dot{\rho}}{\hbar\rho}\Big)x\,\langle x| + \frac{d}{dx}\,\langle x|\,\Big] =
\end{aligned}$$

$$= \frac{\hbar\rho}{(2\hbar)^{1/2}}\Big[\Big(\frac{1}{\hbar\rho^2} - \frac{im\dot\rho}{\hbar\rho}\Big)x + \frac{d}{dx}\Big]\langle x| \,. \tag{3.40}$$

From (3.39) $\langle x|\,\hat a\,|0\rangle = 0$ follows. Using (3.40), this condition reads as

$$\Big[\Big(\frac{1}{\hbar\rho^2} - \frac{im\dot\rho}{\hbar\rho}\Big)x + \frac{d}{dx}\Big]\langle x|0\rangle = 0. \tag{3.41}$$

This is a first-order differential equation, whose general solution is given by

$$\langle x|0\rangle = c\exp\Big[-\frac{1}{2}\Big(\frac{1}{\hbar\rho^2} - \frac{im\dot\rho}{\hbar\rho}\Big)x^2\Big], \tag{3.42}$$

c being an integration constant that may depend on time, but is independent of x. Since there is only one linearly independent solution to (3.41), there is only one linearly independent ket which satisfies (3.39). Therefore the eigenvalue 0 is non-degenerate. To complete the proof of the total non-degeneracy of the spectrum of $\hat N$, one can follow an argument similar to that presented in section 1.2, showing by recurrence the non-degeneracy of each other eigenvalue of the spectrum.

Hence the orthonormal basis $\{|n\rangle\}$ constitutes a representation. We choose the relative phases of the normalized kets $|n\rangle$ so that the standard lowering and raising relations for $\hat a$ and $\hat a^\dagger$ are satisfied:

$$\hat a\,|n\rangle = n^{1/2}\,|n-1\rangle\,, \tag{3.43}$$

$$\hat a^\dagger\,|n\rangle = (n+1)^{1/2}\,|n+1\rangle\,. \tag{3.44}$$

Now our aim is to make the eigenkets $|n\rangle$ solutions of the Schrödinger equation, by effecting the transformation of equations (2.25) and (2.31). So we need to find the diagonal matrix elements of the operators $\hat H$ and d/dt. Let us start from the former. In order to express $\hat H$ in terms of $\hat a$ and $\hat a^\dagger$, we first use equations (3.31) and (3.32) to express $\hat q$ and $\hat p$ in terms of them. We get

$$\hat q = \Big(\frac{\hbar}{2}\Big)^{1/2}\rho\,(\hat a + \hat a^\dagger) \tag{3.45}$$

and

$$\hat p = \Big(\frac{\hbar}{2}\Big)^{1/2}\frac{1}{i\rho}[(1+im\rho\dot\rho)\hat a - (1-im\rho\dot\rho)\hat a^\dagger]. \tag{3.46}$$

Then we compute the square of these operators:

$$\hat q^2 = \frac{\hbar}{2}\rho^2(\hat a+\hat a^\dagger)^2 = \frac{\hbar}{2}\rho^2(\hat a^2+\hat a^{\dagger 2}+\hat a\hat a^\dagger+\hat a^\dagger\hat a) = \frac{\hbar}{2}\rho^2\hat a^2 + \frac{\hbar}{2}\rho^2\hat a^{\dagger 2} + \frac{\hbar}{2}\rho^2\{\hat a,\hat a^\dagger\}_+, \tag{3.47}$$

$$\hat p^2 = -\frac{\hbar}{2\rho^2}[(1+im\rho\dot\rho)\hat a - (1-im\rho\dot\rho)\hat a^\dagger]^2 =$$

$$
\begin{aligned}
&= -\frac{\hbar}{2\rho^2}[(1+im\rho\dot{\rho})^2\hat{a}^2 + (1-im\rho\dot{\rho})^2\hat{a}^{\dagger 2} - (1+m^2\rho^2\dot{\rho}^2)\hat{a}\hat{a}^\dagger - (1+m^2\rho^2\dot{\rho}^2)\hat{a}\hat{a}^\dagger] = \\
&= -\frac{\hbar}{2\rho^2}(1+2im\rho\dot{\rho} - m^2\rho^2\dot{\rho}^2)\hat{a}^2 - \frac{\hbar}{2\rho^2}(1-2im\rho\dot{\rho} - m^2\rho^2\dot{\rho}^2)\hat{a}^{\dagger 2} \\
&\quad + \frac{\hbar}{2\rho^2}(1+m^2\rho^2\dot{\rho}^2)(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) = \\
&= \frac{\hbar}{2}\Big(-\frac{1}{\rho^2} - 2im\frac{\dot{\rho}}{\rho} + m^2\dot{\rho}^2\Big)\hat{a}^2 + \frac{\hbar}{2}\Big(-\frac{1}{\rho^2} + 2im\frac{\dot{\rho}}{\rho} + m^2\dot{\rho}^2\Big)\hat{a}^{\dagger 2} \\
&\quad + \frac{\hbar}{2}\Big(\frac{1}{\rho^2} + m^2\dot{\rho}^2\Big)\{\hat{a}, \hat{a}^\dagger\}_+ . \qquad (3.48)
\end{aligned}
$$

Substituting (3.47) and (3.48) into equation (3.1), we obtain

$$
\begin{aligned}
\hat{H} = \frac{\hbar}{4m}\Big[&\Big(-\frac{1}{\rho^2} - 2im\frac{\dot{\rho}}{\rho} + m^2\dot{\rho}^2 + m^2\omega^2\rho^2\Big)\hat{a}^2 \\
&+ \Big(-\frac{1}{\rho^2} + 2im\frac{\dot{\rho}}{\rho} + m^2\dot{\rho}^2 + m^2\omega^2\rho^2\Big)\hat{a}^{\dagger 2} \\
&+ \Big(\frac{1}{\rho^2} + m^2\dot{\rho}^2 + m^2\omega^2\rho^2\Big)\{\hat{a}, \hat{a}^\dagger\}_+\Big]. \qquad (3.49)
\end{aligned}
$$

Now, the diagonal matrix elements of the operators $\hat{a}^2$ and $\hat{a}^{\dagger 2}$ are zero, as one can see using relations (3.43) and (3.44) and the orthonormality of the kets $|n\rangle$:

$$
\langle n|\,\hat{a}^2\,|n\rangle = \langle n|\,\hat{a}\,|n-1\rangle\, n^{1/2} = \langle n|n-2\rangle\,[n(n-1)]^{1/2} = 0, \qquad (3.50)
$$

$$
\langle n|\,\hat{a}^{\dagger 2}\,|n\rangle = \langle n|\,\hat{a}^\dagger\,|n+1\rangle\,(n+1)^{1/2} = \langle n|n+2\rangle\,[(n+1)(n+2)]^{1/2} = 0. \qquad (3.51)
$$

Conversely, the diagonal matrix elements of the operator $\{\hat{a}, \hat{a}^\dagger\}_+$ are not zero. Precisely, they are given by

$$
\langle n|\,\{\hat{a}, \hat{a}^\dagger\}_+\,|n\rangle = \langle n|\,(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})\,|n\rangle = \langle n|\,(2\hat{N} + \hat{1})\,|n\rangle = 2\Big(n + \frac{1}{2}\Big), \qquad (3.52)
$$

where in the second step we added and subtracted $\hat{a}^\dagger\hat{a}$ inside the parenthesis and used (3.33), while in the last step we used the orthonormality of the kets $|n\rangle$. Therefore, only the term proportional to $\{\hat{a}, \hat{a}^\dagger\}_+$ contributes non-trivially to the diagonal matrix elements of $\hat{H}$, which are given by

$$
\langle n|\,\hat{H}\,|n\rangle = \frac{\hbar}{2m}\Big(\frac{1}{\rho^2} + m^2\dot{\rho}^2 + m^2\omega^2\rho^2\Big)\Big(n + \frac{1}{2}\Big). \qquad (3.53)
$$

As Lewis suggests in [5], “It is interesting to note that the expectation values of $\hat{H}$ are equally spaced at every instant and that the lowest value is always obtained with

$n = 0$", just as with the quantum simple harmonic oscillator. Clearly, since the $\{|n\rangle\}$ representation does not diagonalize the Hamiltonian operator, $\hat{H}$ has also non-diagonal matrix elements. The computation of these is not necessary for our discussion, but it is reported in the Appendices for completeness.

In order to evaluate the diagonal matrix elements of d/dt, we begin by considering the raising relation (3.44), which we rewrite for convenience as

$$\hat{a}^\dagger |n-1\rangle = n^{1/2} |n\rangle . \tag{3.54}$$

By taking the time derivative of this equation, and then the scalar product with the ket $|n\rangle$, we obtain

$$\begin{aligned}
\langle n| \frac{d\hat{a}^\dagger}{dt} |n-1\rangle + \langle n| \hat{a}^\dagger \frac{d}{dt} |n-1\rangle &= n^{1/2} \langle n| \frac{d}{dt} |n\rangle \\
\langle n| \frac{d\hat{a}^\dagger}{dt} |n-1\rangle + n^{1/2} \langle n-1| \frac{d}{dt} |n-1\rangle &= n^{1/2} \langle n| \frac{d}{dt} |n\rangle \\
\langle n| \frac{d}{dt} |n\rangle = \langle n-1| \frac{d}{dt} |n-1\rangle &+ n^{-1/2} \langle n| \frac{d\hat{a}^\dagger}{dt} |n-1\rangle ,
\end{aligned} \tag{3.55}$$

where in the second step we used the conjugate of the equation

$$\hat{a} |n\rangle = n^{1/2} |n-1\rangle . \tag{3.56}$$

So next we compute the expression for $d\hat{a}^\dagger/dt$ in terms of $\hat{a}$ and $\hat{a}^\dagger$:

$$\begin{aligned}
\frac{d\hat{a}^\dagger}{dt} &= \frac{1}{(2\hbar)^{1/2}} \Big[-\frac{\dot{\rho}}{\rho^2}\hat{q} - i(\dot{\rho}\hat{p} - \dot{m}\dot{\rho}\hat{q} - m\ddot{\rho}\hat{q}) \Big] = \\
&= \frac{1}{(2\hbar)^{1/2}} \Big[-\frac{\dot{\rho}}{\rho^2}\hat{q} - i\dot{\rho}\hat{p} + i\dot{m}\dot{\rho}\hat{q} + im\ddot{\rho}\hat{q} \Big] = \\
&= \frac{1}{(2\hbar)^{1/2}} \Big(-\frac{\dot{\rho}}{\rho^2} + i\dot{m}\dot{\rho} + im\ddot{\rho} \Big)\hat{q} - \frac{i\dot{\rho}}{(2\hbar)^{1/2}}\hat{p} = \\
&= \frac{1}{2}\Big(-\frac{\dot{\rho}}{\rho} + i\dot{m}\rho\dot{\rho} + im\rho\ddot{\rho} \Big)(\hat{a} + \hat{a}^\dagger) - \frac{1}{2}\frac{\dot{\rho}}{\rho}(1 + im\rho\dot{\rho})\hat{a} + \frac{1}{2}\frac{\dot{\rho}}{\rho}(1 - im\rho\dot{\rho})\hat{a}^\dagger = \\
&= \frac{1}{2}\Big[-\frac{2\dot{\rho}}{\rho} + im(\rho\ddot{\rho} - \dot{\rho}^2) + i\dot{m}\rho\dot{\rho} \Big]\hat{a} + \frac{1}{2}[im(\rho\ddot{\rho} - \dot{\rho}^2) + i\dot{m}\rho\dot{\rho}]\hat{a}^\dagger,
\end{aligned} \tag{3.57}$$

where in the fourth step we substituted $\hat{q}$ and $\hat{p}$ with their expressions in terms of $\hat{a}$ and $\hat{a}^\dagger$. Using (3.57), the raising and lowering relations for $\hat{a}$ and $\hat{a}^\dagger$ and the orthonormality of the kets $|n\rangle$, we can rewrite the second term at the right-hand side of equation (3.55) as follows:

$$n^{-1/2} \langle n| \frac{d\hat{a}^\dagger}{dt} |n-1\rangle = \frac{n^{-1/2}}{2} \Big[-\frac{2\dot{\rho}}{\rho} + im(\rho\ddot{\rho} - \dot{\rho}^2) + i\dot{m}\rho\dot{\rho} \Big] \langle n| \hat{a} |n-1\rangle$$

$$
\begin{aligned}
&+\frac{n^{-1/2}}{2}[im(\rho\ddot{\rho}-\dot{\rho}^2)+i\dot{m}\rho\dot{\rho}]\,\langle n|\,\hat{a}^\dagger\,|n-1\rangle =\\
&=\frac{n^{-1/2}}{2}\Big[-\frac{2\dot{\rho}}{\rho}+im(\rho\ddot{\rho}-\dot{\rho}^2)+i\dot{m}\rho\dot{\rho}\Big](n-1)^{1/2}\,\langle n|n-2\rangle\\
&+\frac{n^{-1/2}}{2}[im(\rho\ddot{\rho}-\dot{\rho}^2)+i\dot{m}\rho\dot{\rho}]n^{1/2}\,\langle n|n\rangle =\\
&=i\frac{m}{2}(\rho\ddot{\rho}-\dot{\rho}^2+\mu\rho\dot{\rho}). \qquad (3.58)
\end{aligned}
$$

Equation (3.55) thus becomes

$$\langle n|\,\frac{d}{dt}\,|n\rangle = \langle n-1|\,\frac{d}{dt}\,|n-1\rangle + i\frac{m}{2}(\rho\ddot{\rho}-\dot{\rho}^2+\mu\rho\dot{\rho}), \qquad (3.59)$$

and by recurrence we get

$$\langle n|\,\frac{d}{dt}\,|n\rangle = \langle 0|\,\frac{d}{dt}\,|0\rangle + i\frac{n}{2}m(\rho\ddot{\rho}-\dot{\rho}^2+\mu\rho\dot{\rho}). \qquad (3.60)$$

It can be easily shown that the anti-hermiticity of d/dt requires all diagonal matrix elements of d/dt to be purely imaginary. Nevertheless, no further information about $\langle 0|\,d/dt\,|0\rangle$ can be determined from equation (3.60). In fact, the choice of relative phases that guarantees the validity of relations (3.43) and (3.44) leaves the phase of the ket $|0\rangle$ undetermined. In general this time-dependent ket may have a time-dependent phase factor, which can be specified arbitrarily. For convenience we thus demand

$$\langle 0|\,\frac{d}{dt}\,|0\rangle = i\frac{m}{4}(\rho\ddot{\rho}-\dot{\rho}^2+\mu\rho\dot{\rho}). \qquad (3.61)$$

In this way $\langle 0|\,d/dt\,|0\rangle$ vanishes in the limit that $\rho(t)$ becomes a constant. With this convention the general diagonal matrix element of d/dt is given by

$$\langle n|\,\frac{d}{dt}\,|n\rangle = i\frac{m}{2}(\rho\ddot{\rho}-\dot{\rho}^2+\mu\rho\dot{\rho})\Big(n+\frac{1}{2}\Big). \qquad (3.62)$$

At this point we have all the ingredients to calculate the phases required to perform the transformation of equation (2.25). Using equations (3.53) and (3.62), we can rewrite the right-hand side of equation (2.31) as follows:

$$
\begin{aligned}
\langle n|\Big(i\hbar\frac{d}{dt}-\hat{H}\Big)|n\rangle &= i\hbar\,\langle n|\,\frac{d}{dt}\,|n\rangle - \langle n|\,\hat{H}\,|n\rangle =\\
&=-\frac{\hbar m}{2}(\rho\ddot{\rho}-\dot{\rho}^2+\mu\rho\dot{\rho})\Big(n+\frac{1}{2}\Big)\\
&\quad-\frac{\hbar}{2m}\Big(\frac{1}{\rho^2}+m^2\dot{\rho}^2+m^2\omega^2\rho^2\Big)\Big(n+\frac{1}{2}\Big) =
\end{aligned}
$$

$$= -\frac{\hbar}{2m}\Big(m^2\rho\ddot{\rho} - m^2\dot{\rho}^2 + m^2\mu\rho\dot{\rho} + \frac{1}{\rho^2} + m^2\dot{\rho}^2 + m^2\omega^2\rho^2\Big)\Big(n + \frac{1}{2}\Big) =$$
$$= -\frac{\hbar}{2m}\Big(m^2\rho\ddot{\rho} + m^2\mu\rho\dot{\rho} + \frac{1}{\rho^2} + m^2\omega^2\rho^2\Big)\Big(n + \frac{1}{2}\Big) =$$
$$= -\frac{\hbar}{m\rho^2}\Big(n + \frac{1}{2}\Big), \tag{3.63}$$

where we exploited the subsidiary condition (3.30) to simplify the result in the last step. Therefore, equation (2.31) becomes

$$\frac{d\alpha_n}{dt} = -\frac{1}{m\rho^2}\Big(n + \frac{1}{2}\Big). \tag{3.64}$$

Finally, the phase factors may be written in the form

$$\alpha_n(t) = -\Big(n + \frac{1}{2}\Big)\int_0^t dt'\, \frac{1}{m(t')\rho^2(t')}. \tag{3.65}$$

3.4 Analytical solution

The analytical approach is conducted in the framework of wave mechanics. We follow the treatment given by Pedrosa in [6]. Consider the eigenvalue equation

$$\mathrm{I}(t)\phi(x,t) = \lambda\phi(x,t), \tag{3.66}$$

where $\mathrm{I}(t)$ is the invariant operator (3.29) in the configuration space representation, that is given by

$$\mathrm{I} = \frac{1}{2}\Big[\frac{x^2}{\rho^2} + \Big(-i\hbar\rho\frac{d}{dx} - m\dot{\rho}x\Big)^2\Big], \tag{3.67}$$

$\phi(x,t)$ is an eigenfunction of $\mathrm{I}(t)$ and λ the time-independent eigenvalue to which $\phi(x,t)$ belongs. We assume that the eigenfunctions of the invariant form a complete orthonormal set. Thus

$$\int_{-\infty}^{\infty} dx\, \phi'^*(x,t)\phi(x,t) = 0 \tag{3.68}$$

and

$$\int_{-\infty}^{\infty} dx\, \phi^*(x,t)\phi(x,t) = \int_{-\infty}^{\infty} dx\, |\phi(x,t)|^2 = 1. \tag{3.69}$$

In order to solve (3.66), we effect the unitary transformation

$$\tilde{\phi}(x,t) = \mathrm{U}\phi(x,t), \tag{3.70}$$

with the unitary operator U given by

$$\mathrm{U} = \exp\Big(-\frac{im\dot{\rho}}{2\hbar\rho}x^2\Big). \tag{3.71}$$

Under this transformation the eigenvalue equation (3.66) turns into

$$\tilde{\mathrm{I}}\tilde{\phi}(x,t) = \lambda\tilde{\phi}(x,t), \tag{3.72}$$

where

$$\tilde{\mathrm{I}} = \mathrm{U}\mathrm{I}\mathrm{U}^\dagger. \tag{3.73}$$

To obtain the explicit form of the transformed invariant, we start by plugging equation (3.67) into (3.73):

$$\begin{aligned}
\tilde{\mathrm{I}} &= \frac{1}{2}\mathrm{U}\Big[\frac{x^2}{\rho^2} + \Big(-i\hbar\rho\frac{d}{dx} - m\dot{\rho}x\Big)^2\Big]\mathrm{U}^\dagger = \\
&= \frac{1}{2}\mathrm{U}\Big[\frac{x^2}{\rho^2} - \hbar^2\rho^2\frac{d^2}{dx^2} + m^2\dot{\rho}^2x^2 + i\hbar m\rho\dot{\rho} + 2i\hbar m\rho\dot{\rho}x\frac{d}{dx}\Big]\mathrm{U}^\dagger = \\
&= \frac{x^2}{2\rho^2} - \frac{\hbar^2\rho^2}{2}\mathrm{U}\frac{d}{dx}\Big(\frac{d\mathrm{U}^\dagger}{dx} + \mathrm{U}^\dagger\frac{d}{dx}\Big) + \frac{m^2\dot{\rho}^2x^2}{2} + \frac{i\hbar m\rho\dot{\rho}}{2} \\
&\quad + i\hbar m\rho\dot{\rho}x\mathrm{U}\Big(\frac{d\mathrm{U}^\dagger}{dx} + \mathrm{U}^\dagger\frac{d}{dx}\Big) = \\
&= \frac{x^2}{2\rho^2} - \frac{\hbar^2\rho^2}{2}\mathrm{U}\Big(\frac{d^2\mathrm{U}^\dagger}{dx^2} + 2\frac{d\mathrm{U}^\dagger}{dx}\frac{d}{dx} + \mathrm{U}^\dagger\frac{d^2}{dx^2}\Big) + \frac{m^2\dot{\rho}^2x^2}{2} + \frac{i\hbar m\rho\dot{\rho}}{2} \\
&\quad + i\hbar m\rho\dot{\rho}x\mathrm{U}\frac{d\mathrm{U}^\dagger}{dx} + i\hbar m\rho\dot{\rho}x\frac{d}{dx} = \\
&= \frac{x^2}{2\rho^2} - \frac{\hbar^2\rho^2}{2}\mathrm{U}\frac{d^2\mathrm{U}^\dagger}{dx^2} - \hbar^2\rho^2\mathrm{U}\frac{d\mathrm{U}^\dagger}{dx}\frac{d}{dx} - \frac{\hbar^2\rho^2}{2}\frac{d^2}{dx^2} + \frac{m^2\dot{\rho}^2x^2}{2} + \frac{i\hbar m\rho\dot{\rho}}{2} \\
&\quad + i\hbar m\rho\dot{\rho}x\mathrm{U}\frac{d\mathrm{U}^\dagger}{dx} + i\hbar m\rho\dot{\rho}x\frac{d}{dx}.
\end{aligned} \tag{3.74}$$

In the calculations above we used the condition $\mathrm{U}\mathrm{U}^\dagger = 1$. Next we compute the first and second derivative of $\mathrm{U}^\dagger$ with respect to x:

$$\frac{d\mathrm{U}^\dagger}{dx} = \frac{d}{dx}\exp\Big(\frac{im\dot{\rho}}{2\hbar\rho}x^2\Big) = \frac{im\dot{\rho}}{\hbar\rho}x\exp\Big(\frac{im\dot{\rho}}{2\hbar\rho}x^2\Big) = \frac{im\dot{\rho}}{\hbar\rho}x\mathrm{U}^\dagger, \tag{3.75}$$

$$\frac{d^2\mathrm{U}^\dagger}{dx^2} = \frac{d}{dx}\Big(\frac{im\dot{\rho}}{\hbar\rho}x\mathrm{U}^\dagger\Big) = \frac{im\dot{\rho}}{\hbar\rho}\mathrm{U}^\dagger + \frac{im\dot{\rho}}{\hbar\rho}x\frac{d\mathrm{U}^\dagger}{dx} = \frac{im\dot{\rho}}{\hbar\rho}\mathrm{U}^\dagger - \frac{m^2\dot{\rho}^2}{\hbar^2\rho^2}x^2\mathrm{U}^\dagger =$$

$$= \Big(\frac{im\dot{\rho}}{\hbar\rho} - \frac{m^2\dot{\rho}^2}{\hbar^2\rho^2}x^2\Big)\mathrm{U}^\dagger. \tag{3.76}$$

Then, substituting equations (3.75) and (3.76) into (3.74) and using again the condition $\mathrm{UU}^\dagger = 1$, we get

$$\begin{aligned}\tilde{\mathrm{I}} = &\frac{x^2}{2\rho^2} - \frac{\hbar^2\rho^2}{2}\Big(\frac{im\dot{\rho}}{\hbar\rho} - \frac{m^2\dot{\rho}^2}{\hbar^2\rho^2}x^2\Big) - i\hbar m\rho\dot{\rho}x\frac{d}{dx} - \frac{\hbar^2\rho^2}{2}\frac{d^2}{dx^2} + \frac{m^2\dot{\rho}^2x^2}{2} + \frac{i\hbar m\rho\dot{\rho}}{2}\\ &- m^2\dot{\rho}^2x^2 + i\hbar m\rho\dot{\rho}x\frac{d}{dx} =\\ = &\frac{x^2}{2\rho^2} - \frac{i\hbar m\rho\dot{\rho}}{2} + \frac{m^2\dot{\rho}^2x^2}{2} - i\hbar m\rho\dot{\rho}x\frac{d}{dx} - \frac{\hbar^2\rho^2}{2}\frac{d^2}{dx^2} + \frac{m^2\dot{\rho}^2x^2}{2} + \frac{i\hbar m\rho\dot{\rho}}{2}\\ &- m^2\dot{\rho}^2x^2 + i\hbar m\rho\dot{\rho}x\frac{d}{dx}.\end{aligned} \tag{3.77}$$

At last, by simplifying, we obtain

$$\tilde{\mathrm{I}} = -\frac{\hbar^2\rho^2}{2}\frac{d^2}{dx^2} + \frac{1}{2}\frac{x^2}{\rho^2}. \tag{3.78}$$

Therefore, specifying the action of $\tilde{\mathrm{I}}$, equation (3.72) can be written as

$$-\frac{\hbar^2\rho^2}{2}\frac{\partial^2\tilde{\phi}}{\partial x^2} + \frac{1}{2}\frac{x^2}{\rho^2}\tilde{\phi} = \lambda\tilde{\phi}, \tag{3.79}$$

where the ordinary derivative becomes a partial one since $\tilde{\phi}$ depends also on t.

Now, it is convenient to introduce a new variable σ, defined by

$$\sigma = \frac{x}{\rho(t)}. \tag{3.80}$$

Taking into account this change of variable, equation (3.79) can be written in the form

$$-\frac{\hbar^2}{2}\frac{\partial^2\tilde{\phi}}{\partial\sigma^2} + \frac{\sigma^2}{2}\tilde{\phi} = \lambda\tilde{\phi}. \tag{3.81}$$

The normalization condition (3.69) can also be rewritten as

$$\int_{-\infty}^{\infty} d\sigma\rho(t)\,\tilde{\phi}^*(\sigma,t)\tilde{\phi}(\sigma,t) = \int_{-\infty}^{\infty} d\sigma\rho(t)\,|\tilde{\phi}(\sigma,t)|^2 = 1. \tag{3.82}$$

Equation (3.81) is a partial differential equation which does not involve mixed derivatives. This suggests to solve it by separation of variables, by adopting the ansatz:

$$\tilde{\phi}(\sigma,t) = \Gamma(t)\varphi(\sigma). \tag{3.83}$$

Plugging (3.83) into (3.81), the temporal part $\Gamma(t)$ gets cancelled and we obtain an ordinary differential equation for $\varphi(\sigma)$, that is

$$-\frac{\hbar^2}{2}\frac{d^2\varphi}{d\sigma^2}+\frac{\sigma^2}{2}\varphi=\lambda\varphi. \tag{3.84}$$

The function $\Gamma(t)$ is hence left undetermined. We notice that, substituting the ansatz in (3.82) as well, we get

$$|\Gamma(t)|^2\rho(t)\int_{-\infty}^{\infty}d\sigma\,|\varphi(\sigma)|^2=1. \tag{3.85}$$

So a convenient choice for $\Gamma(t)$ is one that makes the normalization condition

$$\int_{-\infty}^{\infty}d\sigma\,\varphi^*(\sigma)\varphi(\sigma)=\int_{-\infty}^{\infty}d\sigma\,|\varphi(\sigma)|^2=1 \tag{3.86}$$

hold. With this convention, we have

$$|\Gamma(t)|^2=\frac{1}{\rho(t)}. \tag{3.87}$$

Since the introduction of a phase factor would be irrelevant, we choose $\Gamma(t)$ such that

$$\Gamma(t)=\frac{1}{\rho^{1/2}(t)}. \tag{3.88}$$

We proceed to solve the ordinary differential equation (3.84). Let us first make the change of variable

$$\xi=\frac{\sigma}{\hbar^{1/2}} \tag{3.89}$$

and rewrite the differential equation as

$$\frac{d^2\varphi}{d\xi^2}=(\xi^2-K)\varphi, \tag{3.90}$$

where we have introduced $K=\frac{2\lambda}{\hbar}$. Now notice that equation (3.90) is formally identical to equation (1.32). Therefore, using a similar reasoning to that of section 1.3, we can get the values of λ for which there exists a physical acceptable solution of (3.90). These are given by

$$\lambda_n=\hbar\Big(n+\frac{1}{2}\Big), \tag{3.91}$$

where n is a non-negative integer number, and the corresponding solutions of (3.90) are

$$\varphi_n(\sigma)=\frac{1}{\pi^{1/4}\hbar^{1/4}(2^n n!)^{1/2}}H_n\Big(\frac{\sigma}{\hbar^{1/2}}\Big)\exp\Big(-\frac{\sigma^2}{2\hbar}\Big), \tag{3.92}$$

where H_n is the standard Hermite polynomial of degree n. The normalization constant was determined imposing (3.86). Thus, by using (3.70), (3.71), (3.80), (3.83), (3.88) and (3.92), we find that the eigenfunctions of I(t) are given by

$$\phi_n(x,t) = \frac{1}{\pi^{1/4}\hbar^{1/4}(2^n n!)^{1/2}\rho^{1/2}(t)} H_n\Big(\frac{x}{\hbar^{1/2}\rho(t)}\Big) \exp\left[\frac{im(t)}{2\hbar}\Big(\frac{\dot\rho(t)}{\rho(t)} + \frac{i}{m(t)\rho^2(t)}\Big)x^2\right], \tag{3.93}$$

with the corresponding time-independent eigenvalues given by (3.91).

We now propose an alternative method to that presented in section 3.3 for the computation of the phase factors of the transformation (2.25). Let us denote by $|n\rangle$ the eigenket of $\hat I$ corresponding to the eigenfunction $\phi_n(x,t)$, for continuity of notation, and let $\hat U$ be the unitary operator acting on the space of dynamical states whose form in the configuration space representation is given by U:

$$\langle x|\, \hat U = \exp\Big(-\frac{im\dot\rho}{2\hbar\rho}x^2\Big)\langle x|\,. \tag{3.94}$$

Then, consider the second member of equation (2.31):

$$\begin{aligned}\langle n|\Big(i\hbar\frac{d}{dt} - \hat H\Big)|n\rangle &= \langle n|\,\hat U^\dagger \hat U\Big(i\hbar\frac{d}{dt} - \hat H\Big)\hat U^\dagger \hat U\,|n\rangle = \\ &= \langle \tilde n|\,\hat U\Big(i\hbar\frac{d}{dt} - \hat H\Big)\hat U^\dagger\,|\tilde n\rangle = \\ &= \langle \tilde n|\Big(i\hbar \hat U\frac{d\hat U^\dagger}{dt} + i\hbar\frac{d}{dt} - \hat U\hat H\hat U^\dagger\Big)|\tilde n\rangle\,,\end{aligned} \tag{3.95}$$

where $|\tilde n\rangle = \hat U\,|n\rangle$ corresponds to the transformed eigenfunction $\tilde\phi_n(x,t)$. In order to calculate the expectation value (3.95), we look for the form of every operator in the parenthesis in configuration space representation. From (3.94)

$$\hat U^\dagger\,|x\rangle = \exp\Big(\frac{im\dot\rho}{2\hbar\rho}x^2\Big)|x\rangle \tag{3.96}$$

follows. Since this relation is true for any $|x\rangle$, $\hat U^\dagger$ is a function of the operator $\hat q$, precisely given by

$$\hat U^\dagger = \exp\Big(\frac{im\dot\rho}{2\hbar\rho}\hat q^2\Big). \tag{3.97}$$

Hence, we have

$$\begin{aligned}i\hbar\hat U\frac{d\hat U^\dagger}{dt} &= i\hbar\exp\Big(-\frac{im\dot\rho}{2\hbar\rho}\hat q^2\Big)\frac{d}{dt}\exp\Big(\frac{im\dot\rho}{2\hbar\rho}\hat q^2\Big) = \\ &= i\hbar\exp\Big(-\frac{im\dot\rho}{2\hbar\rho}\hat q^2\Big)\frac{i}{2\hbar}\Big(\frac{\dot m\dot\rho}{\rho} + \frac{m\ddot\rho}{\rho} - \frac{m\dot\rho^2}{\rho^2}\Big)\hat q^2\exp\Big(\frac{im\dot\rho}{2\hbar\rho}\hat q^2\Big) =\end{aligned}$$

$$= \Big(-\frac{\dot{m}\dot{\rho}}{2\rho} - \frac{m\ddot{\rho}}{2\rho} + \frac{m\dot{\rho}^2}{2\rho^2}\Big)\hat{q}^2, \tag{3.98}$$

and

$$\langle x|\, i\hbar\hat{U}\frac{d\hat{U}^\dagger}{dt} = \Big(-\frac{\dot{m}\dot{\rho}}{2\rho} - \frac{m\ddot{\rho}}{2\rho} + \frac{m\dot{\rho}^2}{2\rho^2}\Big)\hat{x}^2 \langle x|\,. \tag{3.99}$$

In the calculations above we used the fact that $\hat{q}$ and $\hat{U}$ commute. For this very same fact, we can write

$$\hat{U}\hat{H}\hat{U}^\dagger = \frac{1}{2m}\hat{U}\hat{p}^2\hat{U}^\dagger + \frac{1}{2}m\omega^2\hat{q}^2. \tag{3.100}$$

Therefore, the action of the operator $\hat{U}\hat{H}\hat{U}^\dagger$ on $\langle x|$ is given by

$$\begin{aligned}
\langle x|\,\hat{U}\hat{H}\hat{U}^\dagger &= \langle x|\left(\frac{1}{2m}\hat{U}\hat{p}^2\hat{U}^\dagger + \frac{1}{2}m\omega^2\hat{q}^2\right) = \\
&= \frac{1}{2m}\langle x|\,\hat{U}\hat{p}^2\hat{U}^\dagger + \frac{1}{2}m\omega^2 \langle x|\,\hat{q}^2 = \\
&= \frac{1}{2m}\exp\Big(-\frac{im\dot{\rho}}{2\hbar\rho}x^2\Big)\Big(-i\hbar\frac{d}{dx}\Big)^2\Big[\exp\Big(\frac{im\dot{\rho}}{2\hbar\rho}x^2\Big)\langle x|\Big] + \frac{1}{2}m\omega^2x^2\langle x| = \\
&= -\frac{\hbar^2}{2m}\exp\Big(-\frac{im\dot{\rho}}{2\hbar\rho}x^2\Big)\frac{d}{dx}\Big[\frac{im\dot{\rho}}{\hbar\rho}x\exp\Big(\frac{im\dot{\rho}}{2\hbar\rho}x^2\Big)\langle x| + \exp\Big(\frac{im\dot{\rho}}{2\hbar\rho}x^2\Big)\frac{d}{dx}\langle x|\Big] \\
&\quad + \frac{1}{2}m\omega^2x^2\langle x| = \\
&= -\frac{\hbar^2}{2m}\exp\Big(-\frac{im\dot{\rho}}{2\hbar\rho}x^2\Big)\Big[\frac{im\dot{\rho}}{\hbar\rho}\exp\Big(\frac{im\dot{\rho}}{2\hbar\rho}x^2\Big)\langle x| - \frac{m^2\dot{\rho}^2}{\hbar^2\rho^2}x^2\exp\Big(\frac{im\dot{\rho}}{2\hbar\rho}x^2\Big)\langle x| \\
&\quad + 2\frac{im\dot{\rho}}{\hbar\rho}x\exp\Big(\frac{im\dot{\rho}}{2\hbar\rho}x^2\Big)\frac{d}{dx}\langle x| + \exp\Big(\frac{im\dot{\rho}}{2\hbar\rho}x^2\Big)\frac{d^2}{dx^2}\langle x|\Big] + \frac{1}{2}m\omega^2x^2\langle x| = \\
&= -\frac{i\hbar\dot{\rho}}{2\rho}\langle x| + \frac{m\dot{\rho}^2}{2\rho^2}x^2\langle x| - \frac{i\hbar\dot{\rho}}{\rho}x\frac{d}{dx}\langle x| - \frac{\hbar^2}{2m}\frac{d^2}{dx^2}\langle x| + \frac{1}{2}m\omega^2x^2\langle x| = \\
&= \Big(-\frac{i\hbar\dot{\rho}}{2\rho} + \frac{m\dot{\rho}^2}{2\rho^2}x^2 - \frac{i\hbar\dot{\rho}}{\rho}x\frac{d}{dx} - \frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2m\rho^4}x^2 - \frac{m\ddot{\rho}}{2\rho}x^2 - \frac{\dot{m}\dot{\rho}}{2\rho}x^2\Big)\langle x|\,,
\end{aligned} \tag{3.101}$$

where in the last step we used the auxiliary equation (3.30) to substitute ω. Using (3.99), (3.101) and subsequently (3.78) and (3.72), we obtain

$$\begin{aligned}
&\langle \tilde{n}|\Big(i\hbar\hat{U}\frac{d\hat{U}^\dagger}{dt} + i\hbar\frac{d}{dt} - \hat{U}\hat{H}\hat{U}^\dagger\Big)|\tilde{n}\rangle = \\
&\quad = \int_{-\infty}^{\infty} dx\,\tilde{\phi}_n^*(x,t)\Big(i\hbar\frac{d}{dt} + \frac{i\hbar\dot{\rho}}{2\rho} + \frac{i\hbar\dot{\rho}}{\rho}x\frac{d}{dx} + \frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \frac{1}{2m\rho^4}x^2\Big)\tilde{\phi}_n(x,t) = \\
&\quad = \int_{-\infty}^{\infty} dx\tilde{\phi}_n^*(x,t)\Big(i\hbar\frac{d}{dt} + \frac{i\hbar\dot{\rho}}{2\rho} + \frac{i\hbar\dot{\rho}}{\rho}x\frac{d}{dx} - \frac{\tilde{I}}{m\rho^2}\Big)\tilde{\phi}_n(x,t) =
\end{aligned}$$

$$= i\hbar \int_{-\infty}^{\infty} dx\, \tilde{\phi}_n^*(x,t) \frac{d\tilde{\phi}_n(x,t)}{dt} + \frac{i\hbar\dot{\rho}}{2\rho} \int_{-\infty}^{\infty} dx\, \tilde{\phi}_n^*(x,t)\tilde{\phi}_n(x,t)$$
$$+ \frac{i\hbar\dot{\rho}}{\rho} \int_{-\infty}^{\infty} dx\, \tilde{\phi}_n^*(x,t) x \frac{\partial\tilde{\phi}_n(x,t)}{\partial x} - \frac{\lambda_n}{m\rho^2}. \tag{3.102}$$

By means of the change of variable (3.80), (3.102) becomes

$$\langle\tilde{n}|\Big(i\hbar\hat{U}\frac{d\hat{U}^\dagger}{dt} + i\hbar\frac{d}{dt} - \hat{U}\hat{H}\hat{U}^\dagger\Big)|\tilde{n}\rangle =$$
$$= i\hbar \int_{-\infty}^{\infty} d\sigma\rho\, \tilde{\phi}_n^*(\sigma,t) \frac{d\tilde{\phi}_n(\sigma,t)}{dt} + \frac{i\hbar\dot{\rho}}{2\rho} \int_{-\infty}^{\infty} d\sigma\rho\, \tilde{\phi}_n^*(\sigma,t)\tilde{\phi}_n(\sigma,t)$$
$$+ \frac{i\hbar\dot{\rho}}{\rho} \int_{-\infty}^{\infty} d\sigma\rho\, \tilde{\phi}_n^*(\sigma,t)\sigma \frac{\partial\tilde{\phi}_n(\sigma,t)}{\partial\sigma} - \frac{\lambda_n}{m\rho^2}. \tag{3.103}$$

Now, substituting

$$\tilde{\phi}_n(\sigma,t) = \frac{1}{\rho^{1/2}(t)}\varphi_n(\sigma) \tag{3.104}$$

into (3.103), we get

$$\langle\tilde{n}|\Big(i\hbar\hat{U}\frac{d\hat{U}^\dagger}{dt} + i\hbar\frac{d}{dt} - \hat{U}\hat{H}\hat{U}^\dagger\Big)|\tilde{n}\rangle =$$
$$= i\hbar \int_{-\infty}^{\infty} d\sigma\rho^{1/2}\, \varphi_n^*(\sigma)\Big(-\frac{\dot{\rho}}{2\rho^{3/2}}\varphi_n(\sigma) - \frac{\dot{\rho}}{\rho^{3/2}}\sigma\frac{d\varphi_n(\sigma)}{d\sigma}\Big) + \frac{i\hbar\dot{\rho}}{2\rho} \int_{-\infty}^{\infty} d\sigma\, \varphi_n^*(\sigma)\varphi_n(\sigma)$$
$$+ \frac{i\hbar\dot{\rho}}{\rho} \int_{-\infty}^{\infty} d\sigma\, \varphi_n^*(\sigma)\sigma\frac{d\varphi_n(\sigma)}{d\sigma} - \frac{\lambda_n}{m\rho^2} =$$
$$= -\frac{i\hbar\dot{\rho}}{2\rho} \int_{-\infty}^{\infty} d\sigma\, \varphi_n^*(\sigma)\varphi_n(\sigma) - \frac{i\hbar\dot{\rho}}{\rho} \int_{-\infty}^{\infty} d\sigma\, \varphi_n^*(\sigma)\sigma\frac{d\varphi_n(\sigma)}{d\sigma} + \frac{i\hbar\dot{\rho}}{2\rho} \int_{-\infty}^{\infty} d\sigma\, \varphi_n^*(\sigma)\varphi_n(\sigma)$$
$$+ \frac{i\hbar\dot{\rho}}{\rho} \int_{-\infty}^{\infty} d\sigma\, \varphi_n^*(\sigma)\sigma\frac{d\varphi_n(\sigma)}{d\sigma} - \frac{\lambda_n}{m\rho^2} =$$
$$= -\frac{\hbar}{m\rho^2}\Big(n + \frac{1}{2}\Big), \tag{3.105}$$

where in the last step we inserted the explicit expression of the eigenvalue λ_n. Therefore, using (3.95) and (3.105), equation (2.31) becomes

$$\frac{d\alpha_n}{dt} = -\frac{1}{m\rho^2}\Big(n + \frac{1}{2}\Big). \tag{3.106}$$

By integrating this differential equation, we reach the same result obtained in section 3.3, that is

$$\alpha_n(t) = -\Big(n + \frac{1}{2}\Big) \int_0^t dt'\, \frac{1}{m(t')\rho^2(t')}. \tag{3.107}$$

Hence the general Schrödinger wave function may be written as

$$\psi(x,t) = \sum_n c_n \exp[i\alpha_n(t)]\phi_n(x,t), \tag{3.108}$$

with c_n time-independent constants and $\alpha_n(t)$ and $\phi_n(x,t)$ given by equations (3.107) and (3.93), respectively.

3.5 Coherent states

We can construct coherent states also for the time-dependent harmonic oscillator. They are codified by the wave functions

$$\phi_\alpha(x,t) = \frac{1}{\rho^{1/2}(t)} \exp\left[\frac{im(t)\dot{\rho}(t)}{2\hbar\rho(t)}x^2\right]\varphi_\alpha(\sigma,t), \tag{3.109}$$

where

$$\varphi_\alpha(\sigma,t) = \exp\bigl(-|\alpha|^2/2\bigr)\sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{1/2}} \exp[i\alpha_n(t)]\varphi_n(\sigma), \tag{3.110}$$

with α an arbitrary complex parameter. In fact these normalized wave functions are eigenvectors of the destruction operator, that is to say they satisfy the eigenvalue equation

$$a\phi_\alpha(x,t) = \alpha(t)\phi_\alpha(x,t), \tag{3.111}$$

where a is the form taken by the destruction operator (3.31) in the configuration space representation and $\alpha(t)$ is given by

$$\alpha(t) = \alpha\exp[2i\alpha_0(t)], \tag{3.112}$$

with

$$\alpha_0(t) = -\frac{1}{2}\int_0^t dt' \, \frac{1}{m(t')\rho^2(t')}. \tag{3.113}$$

We now show a proof of equation (3.111). First, using (3.71), we write (3.109) more compactly as

$$\phi_\alpha(x,t) = \frac{1}{\rho^{1/2}}\mathrm{U}^\dagger\varphi_\alpha(\sigma,t). \tag{3.114}$$

Then, we apply a to the expression above. Using the unitarity of U, we obtain

$$a\phi_\alpha(x,t) = \frac{1}{\rho^{1/2}}a\mathrm{U}^\dagger\varphi_\alpha(\sigma,t) = \frac{1}{\rho^{1/2}}\mathrm{U}^\dagger\mathrm{U}a\mathrm{U}^\dagger\varphi_\alpha(\sigma,t) = \frac{1}{\rho^{1/2}}\mathrm{U}^\dagger\tilde{a}\varphi_\alpha(\sigma,t), \tag{3.115}$$

where $\tilde{a} = \mathrm{U}a\mathrm{U}^\dagger$.

Before proceeding to determine the action of $\tilde{a}$ on $\varphi_\alpha(\sigma, t)$, we need the following argument. Suppose we have chosen the relative phases of the eigenkets $|n\rangle$ of $\hat{I}$ to make relations (3.43) and (3.44) hold. Then, by projecting equation (3.43) on the $\{|x\rangle\}$ representation, by applying the unitary transformation (3.70), by making the change of variable (3.80) and by substituting (3.104), we get

$$\begin{aligned}
&\langle x|\, \hat{a}\, |n\rangle = n^{1/2} \langle x|n-1\rangle \\
&a\phi_n(x,t) = n^{1/2}\phi_{n-1}(x,t) \\
&\mathrm{U}a\mathrm{U}^\dagger \mathrm{U}\phi_n(x,t) = n^{1/2}\mathrm{U}\phi_{n-1}(x,t) \\
&\tilde{a}\tilde{\phi}_n(x,t) = n^{1/2}\tilde{\phi}_{n-1}(x,t) \\
&\tilde{a}\tilde{\phi}_n(\sigma,t) = n^{1/2}\tilde{\phi}_{n-1}(\sigma,t) \\
&\tilde{a}\varphi_n(\sigma) = n^{1/2}\varphi_{n-1}(\sigma),
\end{aligned} \tag{3.116}$$

where in the last step we used the fact that the destruction operator does not involve time derivatives.

Using (3.116) we can determine the action of $\tilde{a}$ on $\varphi_\alpha(\sigma, t)$:

$$\begin{aligned}
\tilde{a}\tilde{\varphi}_\alpha(\sigma,t) &= \tilde{a}\Big\{ \exp\big(-|\alpha|^2/2\big) \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{1/2}} \exp[i\alpha_n(t)]\varphi_n(\sigma)\Big\} = \\
&= \exp\big(-|\alpha|^2/2\big) \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{1/2}} \exp[i\alpha_n(t)]\tilde{a}\varphi_n(\sigma) = \\
&= \exp\big(-|\alpha|^2/2\big) \sum_{n=1}^{\infty} \frac{\alpha^n}{(n!)^{1/2}} \exp[i\alpha_n(t)]n^{1/2}\varphi_{n-1}(\sigma) = \\
&= \exp\big(-|\alpha|^2/2\big) \sum_{n=0}^{\infty} \frac{\alpha^{n+1}}{[(n+1)!]^{1/2}} \exp[i\alpha_{n+1}(t)](n+1)^{1/2}\varphi_n(\sigma) = \\
&= \exp\big(-|\alpha|^2/2\big) \sum_{n=0}^{\infty} \frac{\alpha\alpha^n}{(n+1)^{1/2}(n!)^{1/2}} \exp[2i\alpha_0(t)] \exp[i\alpha_n(t)](n+1)^{1/2}\varphi_n(\sigma) = \\
&= \alpha \exp[2i\alpha_0(t)] \exp\big(-|\alpha|^2/2\big) \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{1/2}} \exp[i\alpha_n(t)]\varphi_n(\sigma) = \\
&= \alpha \exp[2i\alpha_0(t)]\varphi_\alpha(\sigma,t).
\end{aligned} \tag{3.117}$$

Substituting (3.117) into (3.115), we obtain

$$a\phi_\alpha(x,t) = \alpha \exp[2i\alpha_0(t)] \frac{1}{\rho^{1/2}} \mathrm{U}^\dagger \varphi_\alpha(\sigma,t). \tag{3.118}$$

Finally, using (3.112) and (3.114), we achieve equation (3.111).

We can show the position-momentum uncertainty relation satisfied by these states. Let us denote by $|\alpha\rangle$ the normalized time-dependent ket corresponding to the wave function (3.109). Using (3.45) and (3.111) written in Dirac's formalism, we can compute the expectation value for the position operator $\hat{q}$:

$$\begin{aligned} < q > = \langle\alpha|\,\hat{q}\,|\alpha\rangle = (2\hbar)^{1/2}\frac{\rho}{2}\,\langle\alpha|\,(\hat{a}+\hat{a}^\dagger)\,|\alpha\rangle = (2\hbar)^{1/2}\frac{\rho}{2}(\langle\alpha|\,\hat{a}\,|\alpha\rangle + \langle\alpha|\,\hat{a}^\dagger\,|\alpha\rangle) = \\ = (2\hbar)^{1/2}\frac{\rho}{2}(\alpha(t)\,\langle\alpha|\alpha\rangle + \alpha^*(t)\,\langle\alpha|\alpha\rangle) = (2\hbar)^{1/2}\rho\frac{\alpha(t)+\alpha^*(t)}{2} = \\ = (2\hbar)^{1/2}\rho\,\mathrm{Re}\{\alpha(t)\}. \end{aligned} \tag{3.119}$$

In order to compute the uncertainty in q we exploit the following formula:

$$(\Delta q)^2 = < (q - < q >)^2 > = < q^2 > - < q >^2 . \tag{3.120}$$

So we need $< q^2 >$. From (3.47) we derive

$$\begin{aligned} < q^2 > &= \langle\alpha|\,\hat{q}^2\,|\alpha\rangle = \frac{\hbar}{2}\rho^2\,\langle\alpha|\,(\hat{a}^2+\hat{a}^{\dagger 2}+2\hat{a}^\dagger\hat{a}+\hat{1})\,|\alpha\rangle = \\ &= \frac{\hbar}{2}\rho^2(\langle\alpha|\,\hat{a}^2\,|\alpha\rangle + \langle\alpha|\,\hat{a}^{\dagger 2}\,|\alpha\rangle + 2\,\langle\alpha|\,\hat{a}^\dagger\hat{a}\,|\alpha\rangle + \langle\alpha|\,\hat{1}\,|\alpha\rangle) = \\ &= \frac{\hbar}{2}\rho^2(\alpha^2(t)\,\langle\alpha|\alpha\rangle + \alpha^{*2}(t)\,\langle\alpha|\alpha\rangle + 2|\alpha(t)|^2\,\langle\alpha|\alpha\rangle + \langle\alpha|\alpha\rangle) = \\ &= \frac{\hbar}{2}\rho^2(\alpha^2(t)+\alpha^{*2}(t)+2|\alpha(t)|^2+1) = \\ &= \frac{\hbar}{2}\rho^2\alpha^2(t) + \frac{\hbar}{2}\rho^2\alpha^{*2}(t) + \hbar\rho^2|\alpha(t)|^2 + \frac{\hbar}{2}\rho^2. \end{aligned} \tag{3.121}$$

Then, since from (3.119)

$$\begin{aligned} < q >^2 &= \frac{\hbar}{2}\rho^2(\alpha(t)+\alpha^*(t))^2 = \frac{\hbar}{2}\rho^2(\alpha^2(t)+\alpha^{*2}(t)+2|\alpha(t)|^2) = \\ &= \frac{\hbar}{2}\rho^2\alpha^2(t) + \frac{\hbar}{2}\rho^2\alpha^{*2}(t) + \hbar\rho^2|\alpha(t)|^2 \end{aligned} \tag{3.122}$$

follows, using equations (3.120), (3.121) and (3.122), we have immediately

$$(\Delta q)^2 = \frac{\hbar}{2}\rho^2. \tag{3.123}$$

Using (3.46) and (3.111) written in Dirac's formalism, we can evaluate the expectation value for the momentum operator $\hat{p}$:

$$< p > = \langle\alpha|\,\hat{p}\,|\alpha\rangle = \frac{(2\hbar)^{1/2}}{2i\rho}\,\langle\alpha|\,[(1+im\rho\dot{\rho})\hat{a} - (1-im\rho\dot{\rho})\hat{a}^\dagger]\,|\alpha\rangle =$$

$$
\begin{aligned}
&= \frac{(2\hbar)^{1/2}}{2i\rho}[(1+im\rho\dot{\rho})\langle\alpha|\,\hat{a}\,|\alpha\rangle - (1-im\rho\dot{\rho})\langle\alpha|\,\hat{a}^\dagger\,|\alpha\rangle] = \\
&= \frac{(2\hbar)^{1/2}}{2i\rho}[(1+im\rho\dot{\rho})\alpha(t)\langle\alpha|\alpha\rangle - (1-im\rho\dot{\rho})\alpha^*(t)\langle\alpha|\alpha\rangle] = \\
&= \frac{(2\hbar)^{1/2}}{2i\rho}[(1+im\rho\dot{\rho})\alpha(t) - (1-im\rho\dot{\rho})\alpha^*(t)] = \\
&= \frac{(2\hbar)^{1/2}}{2i\rho}\alpha(t) + \frac{(2\hbar)^{1/2}m\dot{\rho}}{2}\alpha(t) - \frac{(2\hbar)^{1/2}}{2i\rho}\alpha^*(t) + \frac{(2\hbar)^{1/2}m\dot{\rho}}{2}\alpha^*(t) = \\
&= \frac{(2\hbar)^{1/2}}{\rho}\frac{\alpha(t)-\alpha^*(t)}{2i} + (2\hbar)^{1/2}m\dot{\rho}\frac{\alpha(t)+\alpha^*(t)}{2} = \\
&= \frac{(2\hbar)^{1/2}}{\rho}\,\mathrm{Im}\{\alpha(t)\} + (2\hbar)^{1/2}m\dot{\rho}\,\mathrm{Re}\{\alpha(t)\}.
\end{aligned}
\tag{3.124}
$$

In order to compute the uncertainty in p we use the following formula:

$$
(\Delta p)^2 = < (p - < p >)^2 > = < p^2 > - < p >^2 \tag{3.125}
$$

Thus we need $< p^2 >$. From (3.48) we derive

$$
\begin{aligned}
< p^2 > &= \langle\alpha|\,\hat{p}^2\,|\alpha\rangle = \\
&= -\frac{\hbar}{2\rho^2}\langle\alpha|\,[(1+im\rho\dot{\rho})^2\hat{a}^2 + (1-im\rho\dot{\rho})^2\hat{a}^{\dagger 2} - 2(1+m^2\rho^2\dot{\rho}^2)\hat{a}^\dagger\hat{a} \\
&\quad - (1+m^2\rho^2\dot{\rho}^2)\hat{1}]\,|\alpha\rangle = \\
&= -\frac{\hbar}{2\rho^2}[(1+im\rho\dot{\rho})^2\langle\alpha|\,\hat{a}^2\,|\alpha\rangle + (1-im\rho\dot{\rho})^2\langle\alpha|\,\hat{a}^{\dagger 2}\,|\alpha\rangle] \\
&\quad - 2(1+m^2\rho^2\dot{\rho}^2)\langle\alpha|\,\hat{a}^\dagger\hat{a}\,|\alpha\rangle - (1+m^2\rho^2\dot{\rho}^2)\langle\alpha|\,\hat{1}\,|\alpha\rangle] = \\
&= -\frac{\hbar}{2\rho^2}[(1+im\rho\dot{\rho})^2\alpha^2(t)\langle\alpha|\alpha\rangle + (1-im\rho\dot{\rho})^2\alpha^{*2}(t)\langle\alpha|\alpha\rangle \\
&\quad - 2(1+m^2\rho^2\dot{\rho}^2)|\alpha(t)|^2\langle\alpha|\alpha\rangle - (1+m^2\rho^2\dot{\rho}^2)\langle\alpha|\alpha\rangle] = \\
&= -\frac{\hbar}{2\rho^2}[(1+im\rho\dot{\rho})^2\alpha^2(t) + (1-im\rho\dot{\rho})^2\alpha^{*2}(t) - 2(1+m^2\rho^2\dot{\rho}^2)|\alpha(t)|^2 \\
&\quad - (1+m^2\rho^2\dot{\rho}^2)] = \\
&= -\frac{\hbar}{2\rho^2}(1-m^2\rho^2\dot{\rho}^2+2im\rho\dot{\rho})\alpha^2(t) - \frac{\hbar}{2\rho^2}(1-m^2\rho^2\dot{\rho}^2-2im\rho\dot{\rho})\alpha^{*2}(t) \\
&\quad + \frac{\hbar}{\rho^2}(1+m^2\rho^2\dot{\rho}^2)|\alpha(t)|^2 + \frac{\hbar}{2\rho^2}(1+m^2\rho^2\dot{\rho}^2) = \\
&= \Big(-\frac{\hbar}{2\rho^2} + \frac{\hbar m^2\dot{\rho}^2}{2} - \frac{i\hbar m\dot{\rho}}{\rho}\Big)\alpha^2(t) + \Big(-\frac{\hbar}{2\rho^2} + \frac{\hbar m^2\dot{\rho}^2}{2} + \frac{i\hbar m\dot{\rho}}{\rho}\Big)\alpha^{*2}(t)
\end{aligned}
$$

$$+\Big(\frac{\hbar}{\rho^2}+\hbar m^2\dot\rho^2\Big)|\alpha(t)|^2+\frac{\hbar}{2\rho^2}+\frac{\hbar m^2\dot\rho^2}{2}. \tag{3.126}$$

Then, since from (3.124)

$$\begin{aligned}
<p>^2 &= \Big[\frac{(2\hbar)^{1/2}}{\rho}\frac{\alpha(t)-\alpha^*(t)}{2i}+(2\hbar)^{1/2}m\dot\rho\frac{\alpha(t)+\alpha^*(t)}{2}\Big]^2=\\
&=\frac{2\hbar}{\rho^2}\frac{[\alpha(t)-\alpha^*(t)]^2}{4i^2}+2\hbar m^2\dot\rho^2\frac{[\alpha(t)+\alpha^*(t)]^2}{4}+\frac{\hbar m\dot\rho}{i\rho}[\alpha^2(t)-\alpha^{*2}(t)]=\\
&=-\frac{\hbar}{2\rho^2}[\alpha^2(t)+\alpha^{*2}(t)-2|\alpha(t)|^2]+\frac{\hbar m^2\dot\rho^2}{2}[\alpha^2(t)+\alpha^{*2}(t)+2|\alpha(t)|^2]\\
&\quad-\frac{i\hbar m\dot\rho}{\rho}[\alpha^2(t)-\alpha^{*2}(t)]=\\
&=\Big(-\frac{\hbar}{2\rho^2}+\frac{\hbar m^2\dot\rho^2}{2}-\frac{i\hbar m\dot\rho}{\rho}\Big)\alpha^2(t)+\Big(-\frac{\hbar}{2\rho^2}+\frac{\hbar m^2\dot\rho^2}{2}+\frac{i\hbar m\dot\rho}{\rho}\Big)\alpha^{*2}(t)\\
&\quad+\Big(\frac{\hbar}{\rho^2}+\hbar m^2\dot\rho^2\Big)|\alpha(t)|^2
\end{aligned} \tag{3.127}$$

follows, using equations (3.125), (3.126) and (3.127), we immediately deduce

$$(\Delta p)^2=\frac{\hbar}{2}\Big(\frac{1}{\rho^2}+m^2\dot\rho^2\Big). \tag{3.128}$$

Therefore, using (3.123) and (3.128), the product of the uncertainties in q and p is given by

$$\Delta q\Delta p=\frac{\hbar}{2}(1+m^2\rho^2\dot\rho^2)^{1/2}. \tag{3.129}$$

Notice the difference with the uncertainty relation (1.50): the product of the uncertainties, in general, does not take its minimum value.

Appendices

A Commutation relations involving $\hat{q}$ and $\hat{p}$

The expression obtained in (3.9) for the commutator $[\hat{I}_H, \hat{H}_H]$ is given by a sum of several other commutators, which involve the square and/or the product of the operators $\hat{q}$ and $\hat{p}$. Here are reported the explicit calculations for those commutators.

The commutator $[\hat{q}^2, \hat{p}]$ is given by

$$[\hat{q}^2, \hat{p}] = [\hat{q}\hat{q}, \hat{p}] = [\hat{q}, \hat{p}]\hat{q} + \hat{q}[\hat{q}, \hat{p}] = i\hbar\hat{1}\hat{q} + i\hbar\hat{q}\hat{1} = 2i\hbar\hat{q}. \tag{A.1}$$

Similarly, we can compute the commutator $[\hat{q}, \hat{p}^2]$ as follows:

$$[\hat{q}, \hat{p}^2] = [\hat{q}, \hat{p}\hat{p}] = [\hat{q}, \hat{p}]\hat{p} + \hat{p}[\hat{q}, \hat{p}] = i\hbar\hat{1}\hat{p} + i\hbar\hat{p}\hat{1} = 2i\hbar\hat{p}. \tag{A.2}$$

Either (A.1) or (A.2) can be used to evaluate $[\hat{q}^2, \hat{p}^2]$:

$$[\hat{q}^2, \hat{p}^2] = [\hat{q}^2, \hat{p}]\hat{p} + \hat{p}[\hat{q}^2, \hat{p}] = 2i\hbar\hat{q}\hat{p} + 2i\hbar\hat{p}\hat{q} = 2i\hbar(\hat{q}\hat{p} + \hat{p}\hat{q}) = 2i\hbar\{\hat{q}, \hat{p}\}_+. \tag{A.3}$$

To determine the commutator $[\{\hat{q}, \hat{p}\}_+, \hat{q}^2]$ we use (A.1):

$$\begin{aligned}[\{\hat{q}, \hat{p}\}_+, \hat{q}^2] &= [\hat{q}\hat{p} + \hat{p}\hat{q}, \hat{q}^2] = [\hat{q}\hat{p}, \hat{q}^2] + [\hat{p}\hat{q}, \hat{q}^2] = [\hat{q}, \hat{q}^2]\hat{p} + \hat{q}[\hat{p}, \hat{q}^2] + [\hat{p}, \hat{q}^2]\hat{q} + \hat{p}[\hat{q}, \hat{q}^2] = \\ &= -2i\hbar\hat{q}^2 - 2i\hbar\hat{q}^2 = -4i\hbar\hat{q}^2. \end{aligned} \tag{A.4}$$

To determine the commutator $[\{\hat{q}, \hat{p}\}_+, \hat{p}^2]$ we use (A.2) instead:

$$\begin{aligned}[\{\hat{q}, \hat{p}\}_+, \hat{p}^2] &= [\hat{q}\hat{p} + \hat{p}\hat{q}, \hat{p}^2] = [\hat{q}\hat{p}, \hat{p}^2] + [\hat{p}\hat{q}, \hat{p}^2] = [\hat{q}, \hat{p}^2]\hat{p} + \hat{q}[\hat{p}, \hat{p}^2] + [\hat{p}, \hat{p}^2]\hat{q} + \hat{p}[\hat{q}, \hat{p}^2] = \\ &= 2i\hbar\hat{p}^2 + 2i\hbar\hat{p}^2 = 4i\hbar\hat{p}^2. \end{aligned} \tag{A.5}$$

B Off-diagonal matrix elements of $\hat{H}$ and d/dt

In order to compute the off-diagonal matrix elements of $\hat{H}$ and d/dt with respect to the kets $|n\rangle$, we first determine the off-diagonal matrix elements of $\hat{a}^2$, $\hat{a}^{\dagger 2}$ and $\{\hat{a}, \hat{a}^\dagger\}_+$:

$$\langle n'| \hat{a}^2 |n\rangle = \langle n'| \hat{a} |n-1\rangle\, n^{1/2} = \langle n'|n-2\rangle\, [n(n-1)]^{1/2} = [n(n-1)]^{1/2}\delta_{n',n-2}, \tag{B.1}$$

$$\langle n'| \hat{a}^{\dagger 2} |n\rangle = \langle n'| \hat{a}^\dagger |n+1\rangle\, (n+1)^{1/2} = \langle n'|n+2\rangle\, [(n+1)(n+2)]^{1/2} = [(n+1)(n+2)]^{1/2}\delta_{n',n+2}, \tag{B.2}$$

$$\langle n'| \{\hat{a}, \hat{a}^\dagger\}_+ |n\rangle = \langle n'| (\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) |n\rangle = \langle n'| (2\hat{N} + \hat{1}) |n\rangle = \langle n'|n\rangle\, (2n+1) = (2n+1)\delta_{n',n}. \tag{B.3}$$

Secondly, we rewrite the auxiliary equation (3.30) as

$$-\frac{1}{\rho^2} + m^2\omega^2\rho^2 = -m^2\rho\ddot{\rho} - m\dot{m}\rho\dot{\rho}, \tag{B.4}$$

and then we substitute (B.4) into (3.49):

$$\hat{H} = \frac{\hbar}{4m}\Big\{\Big[m^2(\dot{\rho}^2 - \rho\ddot{\rho}) - m\dot{\rho}\Big(\dot{m}\rho + \frac{2i}{\rho}\Big)\Big]\hat{a}^2 + \Big[m^2(\dot{\rho}^2 - \rho\ddot{\rho}) - m\dot{\rho}\Big(\dot{m}\rho - \frac{2i}{\rho}\Big)\Big]\hat{a}^{\dagger 2} + \Big(\frac{1}{\rho^2} + m^2\dot{\rho}^2 + m^2\omega^2\rho^2\Big)\{\hat{a}, \hat{a}^\dagger\}_+\Big\}. \tag{B.5}$$

Then, for $n' \neq n$, expression (B.5) immediately yields

$$\langle n'|\,\hat{H}\,|n\rangle = \frac{\hbar}{4}\Big\{\Big[m(\dot{\rho}^2 - \rho\ddot{\rho}) - \dot{\rho}\Big(\dot{m}\rho + \frac{2i}{\rho}\Big)\Big][n(n-1)]^{1/2}\delta_{n',n-2} + \Big[m(\dot{\rho}^2 - \rho\ddot{\rho}) - \dot{\rho}\Big(\dot{m}\rho - \frac{2i}{\rho}\Big)\Big][(n+1)(n+2)]^{1/2}\delta_{n',n+2}\Big\}. \tag{B.6}$$

Furthermore, from equation (2.23) we derive

$$\langle n'|\,\frac{d}{dt}\,|n\rangle = \frac{1}{i\hbar}\,\langle n'|\,\hat{H}\,|n\rangle\,, \quad n' \neq n. \tag{B.7}$$

References

[1] C. Cohen-Tannoudji, P. Davies, D. Betts, B. Diu, F. Laloë, B. Dui, S. Hemley, N. Ostrowsky, and D. Ostrowsky. Quantum Mechanics. A Wiley - Interscience publication v. 1, (Wiley, 1977).

[2] D. Griffiths. Introduction to Quantum Mechanics. (Cambridge University Press, 2017).

[3] G. Auletta, M. Fortunato, and G. Parisi. Quantum Mechanics. (Cambridge University Press, 2009). 467, 468.

[4] J. Lewis H. R. and W. B. Riesenfeld, An Exact Quantum Theory of the Time-Dependent Harmonic Oscillator and of a Charged Particle in a Time-Dependent Electromagnetic Field, http://dx.doi.org/10.1063/1.1664991 *Journal of Mathematical Physics* **10** 1458 (1969).

[5] J. Lewis H. R., Class of Exact Invariants for Classical and Quantum Time-Dependent Harmonic Oscillators, http://dx.doi.org/10.1063/1.1664532 *Journal of Mathematical Physics* **9** 1976 (1968).

[6] I. A. Pedrosa, Exact wave functions of a harmonic oscillator with time-dependent mass and frequency, http://dx.doi.org/10.1103/PhysRevA.55.3219 **55** 3219 (1997).