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Knot Theory and its applications

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Introduzione

Questa tesi si occuperá dare una introduzione alla teoria dei nodi. Nei tre capitoli verranno presi in considerazione tre aspetti. Si descriverá come la teoria dei nodi si sia sviluppata nel corso degli anni in relazione alle diverse scoperte scientifiche avvenute. Si potr quindi subito avere una idea di come questa teoria sia estremamente connessa a diverse altre. Nel secondo capitolo ci si occuper degli aspetti pi formali di questa teoria. Si introdurrá il concetto di nodi equivalenti e di invariante dei nodi. Si definiranno diversi invarianti, dai piú elementari, le mosse di Reidemeister, il numero di incroci e la tricolorabilitá, fino ai polinomi invarianti, tra cui il polinomio di Alexander, il polinomio di Jones e quello di Kauffmann. Infine si spiegheranno alcune applicazioni della teoria dei nodi in chimica, fisica e biologia. Sulla chimica, si definirá la chiralitá molecolare e si mostrerá come la chiralitá dei nodi possa essere utile nel determinare quella molecolare. In campo fisico, si mostrerá la relazione che esiste tra l'equazione di Yang-Baxter e i nodi. E in conclusione si mostrerá come modellare un importante processo biologico, la recombinazione del DNA, grazie alla teoria dei nodi.

Introduction

"When Alexander reached Gordium, he was seized with a longing to ascend to the acropolis, where the palace of Gordius and his son Midas was situated, and to see Gordius wagon and the knot of the wagons yoke:. Over and above this there was a legend about the wagon, that anyone who untied the knot of the yoke would rule Asia. The knot was of cornel bark, and you could not see where it began or ended. Alexander was unable to find how to untie the knot but unwilling to leave it tied, in case this caused a disturbance among the masses; some say that he struck it with his sword, cut the knot, and said it was now untied."

- Lucius Flavius Arrianus, "Anabasi Alexandri", Book II-

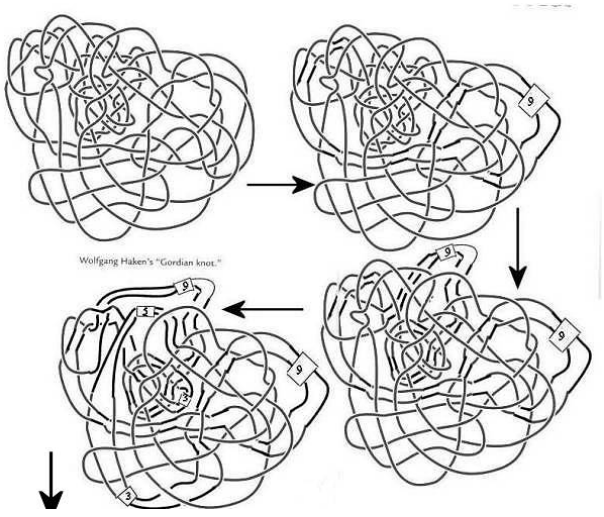


Figure 1: Wolfagang Haken's gordian knot.

Alexander unties the gordian knot, according to the legend, cutting it. The history wants him to became the ruler of Asia, but mathematically speaking, Alexander have never solved that enigma. In fact a knot can be untied if it can be deformed smoothly into a circle, that is without any cut!

Given a knot, is it possible to have different deformation of the same.
This give raise to a mathematical problem: to state if, among all the different deformations of a given knot, is it possible to find an unknotted loop.
That is the central idea of the Knots Theory. The main problem of this subject is in fact, given two different knot, to state if they are (topologically) equivalent, *isotopic*, that is if they can be smoothly deformed one into the other. In this sense, a loop is unknotted if is equivalent to a circle in the plane. Different way to the approach that problem were used during the past years. The first idea was to consider the knot as 1-surface and studying it through the property of their complement in a 3-dimensional space.
This method is the one that give birth to the *topology*. The term "topology" was, in fact, coined by Johann Listing in the 1848, in order to describe a subject that was interested in the qualitative property of an object, instead than in the quantitative property.
Let's take a step back. Leibniz wrote in 1679:

> *I consider that we need yet another kind of analysis,... which deals directly with position*

and he called it "geometry of position"(geometria situs). That is, the history of the knot theory is also the history of the development of this new geometry theorized by Leibniz.

In this dissertation, we are going to give a brief introduction of knot theory, looking at different aspects.

In the first chapter, we will see how the research on this subject changed during the time. Starting from the prehistorical discovers of knots, we will see how the interest in this field increase whit the idea that the matter is made by vortex. This idea brought several scientists to be interested in knots: Tait, Maxwell, Thompson and Kirkman among the others.
The last section is about the modern knot theory. In particular about the first main breakdown due to Alexander and Reidemeister's discoveries of new invariants and the second one due to Jones and its new polynomial.

In the second chapter we will consider knot as curves in $\mathbb{R}^{\not 3}$ and we will be interested in their mathematical properties. In this chapter we investigate a way for determining whether or not two knot are equivalent. Different knots invariants are going to be introduced.
The first section will give us some basic concept about knot theory.
The second section is dedicated to the classical knots invariant: the Reidemeister moves, the bridge number, the linking number and the tricolorability.
In the third section the Seifert matrix is introduce, in order to compute the Alexander polynomial and Alexander-Conway polynomial.
The fourth section is about the Jones polynomial. And for computing it an introduction on braids groups is required. The last invariant to be introduced is the Kauffman polynomial.

In the last chapter we will show how knot theory influenced others discipline: chemistry, physics and biology.

Chapter 1

History of Knot Theory

> My soul is an amphicheiral knot Upon a liquid vortex wrought By Intellect in the Unseen residing, While thou dost like a convict sit With marlinspike untwisting it Only to find my knottiness abiding
>
> James Clerk Maxwell, *A paradoxical ode*

1 The Firsts Discoveries

Knots have been center of interest for humans beings since the prehistory. The first example we have is a seal found in Anatolia, dated 1700 B.C., represented braids and knots. However the earliest discovery of knots is attributed to a Greek physician named Heraklas, who lived during the first century A.D.

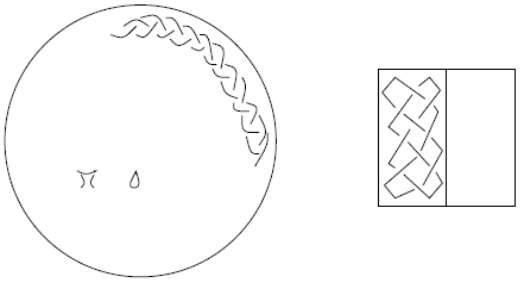


Figure 1.1: Stamp seal, about 1700 BC (the British Museum).

In fact he wrote an essay on surgeons slings in which he explains eighteen

ways to tie orthopedic slings. Even if it was not properly knot theory, it should be taken as the first example of knot theory's application we have in the scientific literature. Although we could think of knot theory as a ancient discipline, in the mathematical scene is quite a new arrival.
In a letter to Christian Huygens (1629-1695), written in 1679 [9], Gottfried Willhelm Leibniz, German philosopher and mathematician wrote:

> *I am not content with algebra, in that it yields neither the shortest proofs nor the most beautiful construction of geometry. Consequently, in view of this, I consider that we need yet another kind of analysis, geometric or linear, which deal directly with position, as algebra deals with magnitude.*

With those word Leibniz laid the basis for a new science that he called *geometria situs*, geometry of position, and that is now known as *topology*.
The first example of Leibniz's new conception of geometry was given by Leonard Euler (1707-1783). Solving the bridges of Königsberg problem, Euler did not worry about the exact position of the bridges, instead he recognized that the key information was to understand which properties derive from their reciprocal position.
He wrote in his paper *Solutio problematis ad geometriam situs pertinentis*[4], 1736 :

> *The branch of geometry that deals with magnitudes has been zealously studied throughout the past, but there is another branch that has been almost unknown up to now; Leibniz spoke of it first, calling it the "geometry of position". This branch of geometry deals with relations dependent on position: it does not take magnitudes into considerations, nor does it involve calculation with quantities. But as yet no satisfactory definition has been given of the problems that belongs to this geometry of position or of the method to be used in solving them.*

But only in the 1771, for the first time, knot theory was mentioned as a field of studies by Alexander-Theophile Vandermonde (1735-1796). He studied braids and knots as examples of this new science of position envisioned by Leibniz.

He wrote in his paper *Remarques sur les problmes de situation* (Remarks on problems of position) [20]:

> *Whatever the twist and turns of a system of threads in space, one can always obtain an expression for the calculation of its dimensions, but this expression will be of little use in practice. The craftsman who fashions a braid, a net, or some knots will be concerned, not with questions of measurement, but with those of position: what he sees there is the manner in which the threads are interlaced.*
>
> *One of the oldest notes by Gauss to be found among his papers is a sheet of paper with the date* 1794*. It bears the heading "A collection of knots" and contains thirteen neatly sketched views of knots with English names written beside them . . . With it are two additional pieces of paper with sketches of knots. One is dated* 1819*: the other is much later, . . .* [16]

Carl Friedrich Gauss' studies were essential for the creation of the knot theory.
Between 1825 and 1844, he worked on the classification of closed plane curves with a finite number of self-intersections, which he called *Tractfiguren* (we may think of them as knot projections). His method consisted in giving an orientation to these curves and then in labeling the crossing with letters. Therefore he created a sequence, starting from a chosen starting point all over the tractfiguren. Thus a curve with n crossing would have a sequence of length 2^n. For example, the trefoil knot would be recorded as $ABCABC$.

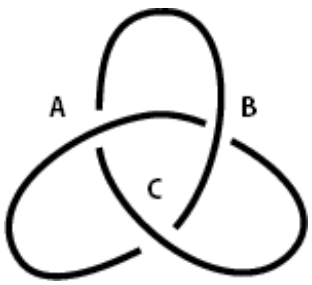


Figure 1.2: The Gauss' labeled scheme of the trefoil $ABCABC$.

Moreover, working in electrodynamics, Gauss discovered the first relevant result in Knot Theory. His aim was to find a method that could estimate how much work is done on a magnetic pole moving along a closed curve in presence of a loop current. During his investigation he discovered what is called *Gauss linking number*.
The Gauss linking number is the first invariant discovered[1]

> *A main task (that lies) on the border between geometria situs and geometria magnitudinis is to count the windings of two closed or infinite lines . . . m number of the windings. This value is shared, i.e., it remains the same if the lines are interchanged.*[13]

The Gauss linking number was reconsidered by many.
James Clerk Maxwell (1831-1879) showed his interest in this topic and found two loops that cannot be separated even if their Gauss linking number equals to 0.

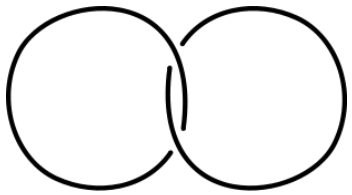


Figure 1.3: In the Hopflink the linking number is 0, but the two loop cannot be separated.

In 1876 O. Boeddicker, observed that the linking number is nothing but the number of times a curve winds around another. Later, in 1892, Hermann Karl Brunn gave a easy method to determinate the linking number of a two-component link. If the link has components K_1 and K_2, we consider any diagram of the link and count each point at which K_1 crosses under K_2 as in the figure 1.4. The sum of these $+1$ or -1 is the Gauss linking number.
Another relevant contribute to the develop of knots theory is due to Johann Benedict Listing (1808-1882). Listing, who was Gauss' student, published in 1847 his monograph *Vorstudien zur Topologie*[10]. Most of the paper is dedicated to the studies of mathematical knots and their classification. He was the first to use the term *topology*. He wrote:

[1] In Gauss' formulation: $m = -\frac{1}{4\pi}\int_C\int_{C'}\frac{(r-r')\times dl dl'}{|r-r'|}$

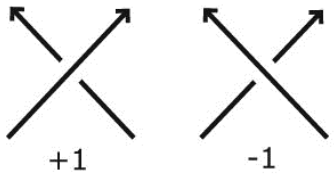


Figure 1.4: The linking number.

> *I hope you let me use the name "Topology for this kind of studies of spatial images, rather than suggested by Leibniz name "geometria situs, reminding of notion of "measure" and "measurement", playing here absolutely subordinate role and confiding with "géométrie de position" which is established for a different kind of geometrical studies. Therefore, by Topology we will mean the study of modal relation and spacial image, or of laws of connectedness, mutual disposition and traces of points, lines surface, bodies and their parts or their unions in space, independently of relations of measures and quantities.*

Listing was interested in developing an algebraic calculus of knot diagrams so that it could easily be determined when two diagrams represented the same knot. In particulary, he showed interest in the chirality of a knots, i.e. the isotopy between knot and its mirror image. He was the first to state that the right handed trefoil and the left handed trefoil are not isotopic. On the contrary the figure eight knot also known as the Listing knot is achiral, that is it's equivalent to his mirror image. (see 1.5)

2 Physic's interest in knot theory

During the 1860's the scientific community was divided in two: the ones that believed that the matter is composed by atoms ("corpuscular theory"), and the others, that believe that the matter was behaving as waves. In 1867, William Thomson, later to be Lord Kelvin (1824-1907), recognized that the particular shape of a vortex was not as important as the underlying topological structure, and felt that an understanding of such vortices would lead to a complete understanding of matter. Inspired by Helmholtz's considerations[5],

he wrote all his ideas in the essay, "On vortex atoms"[17], in which he assume that the matter consisted of *vortex atoms*.

> *After noticing Helmholtz's admirable discovery of the law of vortex motion in a perfect liquid, that is, in a fluid perfectly destitute of viscosity (or fluid friction), the author said that this discovery inevitably suggest that the Helmholtz's ring are the only true atoms.*

According to Thomson's idea, atoms have the same shape as a vortex that is flowing in the ether, a perfect fluid, that can possibly be knotted, but still maintain its identity. Then atoms are simply knots and a molecule is a composition of knots.
This seems so absurd now that we have knowledge of atoms and constituents of matter, but one should think about string theory, our new model in the infinite small, which does not look more reasonable or realistic from vortex theory.
With this new theory, the interest for knots theory increased in the scientific society. If the atoms were modeled by knots, then the next problem was trying to classify them. A first attempt in this direction is due to the physicist Peter Guthrie Tait (1831-1901). Despite his important role in the development of Thomsons idea, Tait initially felt that Thomson was wrong. He felt that vortex motions principal application would be in the theory of electromagnetism. Anyway, despite Taits initial objection, Thomson continued thinking about atoms as vortices, sparking the interest of James Clerk Maxwell. Maxwell had been doing work in electromagnetism for some time. However, he was also open to the idea that knotted vortices could be the fundamental building blocks of matter. So he started a discussions which went on in letters exchanged with Tait and Thomson about some of his ideas and discoveries. They were very interested in his ideas. In fact he rediscovered an integral formula counting the linking number of two closed curves which Gauss had discovered, but had not published and also gave equations for knotted curves in a three dimensional space. Moreover, in his letter, he noted that the trefoil was the simplest knot that was truly knotted that consisted of a single strand. He went on to recognize a parameter in his equations that

could determine if the trefoil thus produced was right-handed or left-handed and claimed (without proof) that there was no way to change a right-handed trefoil into a left-handed one or vice versa.

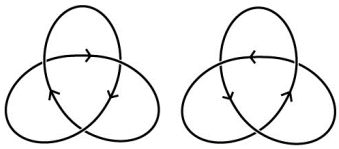


Figure 1.5: The left-handed and right-hand trefoil.

Thus in a couple of days, Maxwell had anticipated much of what would happen in knot theory over the next 80 years. Maxwell's research was directed to determinate whether two projections of a knot represented the same knot. In order to answer his question, he created a labeling scheme for the crossing points of a knot projection and then showed that every knot diagram must contain a region bounded by fewer than four arcs, where he defined an arc to be a segment of the projection from one crossing point to the next. So, he began to determine all of the possibilities for such regions using this scheme. In the case of a region bounded by one arc, this was simply a twist as shown in 1.6, which could easily be undone without changing the knot. For regions bounded by two arcs, he found two possibilities. Namely, a region created as a strand passed over another strand at two consecutive points or a region created as a strand passed over and then under another. (See 1.6.) In the

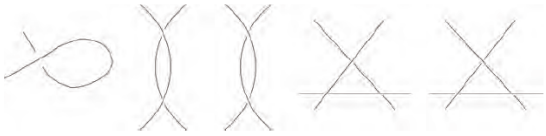


Figure 1.6: The regions bounded by less than 4 arcs.

first case, the top strand can be moved so that it no longer crosses over the bottom strand without changing the link type; the second however, could not be undone. Surprisingly, the situation gets no more complicated with regions bounded by three arcs, where there are two possible cases as shown in Figure 1.6. Regarding this situation he wrote in [11]

> *In the first case, any one curve can be moved past the intersection of the other two without disturbing them. In the second case this*

> *cannot be done and the intersection of two curves is a bar to the motion of the third in that direction.*

Although his approach contained no mathematical rigor, it is remarkable to note that Maxwell had defined the "Reidemeister moves" which would be shown to be the fundamental moves in modifying knots only in the 1920s.
In the meantime, Tait gradually changed his mind, and started to tabulate knots, hoping to create a table of elements to go with Thomson's atomic theory. By 1876 he had set out to make a complete table of knots up to seven crossings. However he understood that the complexity of the knots he was producing would prevent them from all being stable enough as vortices to represent atoms, meaning that a tabulation of knots with higher crossing numbers would be required[14]. Such a tabulation would require more efficient methods for determining if two knot diagrams were the same, an idea that would not be realized for over 100 years.

> *The development of this subject promises absolutely endless work, but work of a very interesting and useful kind, because it is intimately connected with the theory of knots, which (especially as applied in Sir W. Thomsonś Theory of Vortex Atoms) is likely soon to become an important branch of mathematics.*

Taits serious investigation of knots began, as for Gauss and Maxwell, with the development of a way to symbolically encode the crossings of a knot projection. Tait developed his own encoding scheme. During his research he introduced three principles, known as *Tait conjectures*. In order to simplify his work he decided to work only with *alternating knots*, namely those which have an upper crossing followed by an under crossing. Guided by the evidence from these alternating diagram, he stated his first conjecture.

Conjecture 1. *A reduced alternating knot has minimal crossing number.*

Tait's second conjecture was cryptically stated as *If the simplest is* $+\ -\ +-+-$ *then irreducible* [2]. The interpretation we have today is the following:

Conjecture 2. *An alternating knot diagram without nugatory crossings, those that separate two nontrivial distinct portions of the knot, cannot be manipulated to have fewer crossings.*

For example, the trefoil and figure-8 knot cannot be drawn with fewer crossings, as they are both depicted as alternating knots without nugatory crossings. This conjecture was first rigorously proved by Murasugi in [12], only after the discovery of the Jones polynomial. Tait was not so sure of his third conjecture, today best known as Tait's flyping conjecture, which is usually stated as

Conjecture 3. *Any two reduced alternating diagrams of a given knot are related via a sequence of flypes*[2]*, diagrammatic moves such as the one depicted in Figure1.7.*

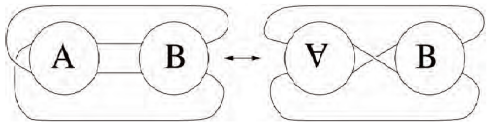


Figure 1.7: The flype move.

Tait's flyping conjecture remained an important open problem in knot theory until Menasco and Thistlethwaite proved it in 1993.
In May 1884, the Reverend Thomas Kirkman, who had spent a significant portion of the previous 30 years considering combinatorial problems involving graphs and hypergraphs, sent Tait his table of knots, up to 9 crossing. Kirkman viewed the knot enumeration problem as a problem of enumerating particular 4-regular planar graphs that could be projections of alternating knots or links and started to work on the problem, but he went to argue with Tait about whether or not knots projections are sufficient to determinate a knots isotopy. Kirkman, being a true combinatorialist and not a topologist, felt that twisting moves were not of interest and equivalences via such operations should not be considered. So he didn't solve the problem of determinate which of the projections represent same knots, but Tait completed his work, without having any proof of the inequivalence of each of the knots.

> *...the disadvantage of being to a greater or less extent tentative. Not that the rules laid down ... leave any room for mere guessing,*

[2]We should note that Tait applied the name twist to what modern knot theorists call a flype

> *but they are too complex to be always completely kept in view. Thus we cannot be absolutely certain that by means of such processes we have obtained all the essentially different forms which the definition we employ comprehends.*

In 1899 Charles Newton Little classified non alternating knots up to 10 crossing. He took 6 years to complete his work.
Many more scientists continued to tabulate knots for the next years. These tables were partially extended in M.G. Hasemans doctoral dissertation of 1916. Knots up to 11 crossings were enumerated by John H. Conway before 1969. Knots up to 13 crossings were enumerated by C .H .Dowker and M. B. Thistlethwaite in 1983. Nowadays, computerization of the problem has made knot enumeration considerably easier, but the rapid growth in the number of knots is still astonishing. For example in a July 2003 tabulation of all prime, alternating knots through 22 crossings performed by S. Rankin, J. Schermann, and O. Smith have been found $6,217,553,258$ knots. The counting of knots is one part of the history of knot theory, however other are the disciplines that make knot theory develop.

3 The modern knot theory

Even though Thomson's vortex atoms were of course abandoned, pure Mathematicians were still interesting in knot theory. Poincaré introduced the Fundamental group about 1900. The fundamental group was a significant advance in the study of topology, since it created a way for using the tools of abstract algebra for studying the field of topology. Shortly after its discovery, it was applied to knot theory. In 1908, Heinrich Tietze (1880-1964) used the fundamental group of the complement of a knot in $\mathbb{R}^3$, called the *knot group*, in order to distinguish the unknot from the trefoil knot [18]. The Austrian mathematician Wilhelm Wirtinger (1865-1945), whose work was actually motivated by the study of algebraic functions of a single complex variable, showed in his lecture delivered at a meeting of the German Mathematical Society in 1905 a method for finding a knot group presentation (it is called now the *Wirtinger presentation of* a knot group). Later, Max Dehn, a German mathematician who studied under Hilbert, became interested in

the theory of knots as he worked to prove the Poincaré conjecture. Of course, Dehn did not prove the Poincaré conjecture, but he did develop another algorithm (distinct from Wirtingers) for constructing the fundamental group of the complement of a link. Using this, Dehn showed that a knot is nontrivial if and only if its fundamental group is non-abelian. He went on to show that a trefoil knot and its mirror image are different. That is, Dehn showed that the trefoil knot is not achiral. The work of Dehn and his colleagues came to a halt with the outbreak of World War I, and it would not be until after the war that any significant work in knot theory resumed. The breakthrough in knot theory is due to James W. Alexander at Princeton and Kurt Reidemeister in Vienna. In the 1920s, independently, both of them arrived at the same knot invariant. Alexander used the homology groups while Reidemeister used the fundamental groups.

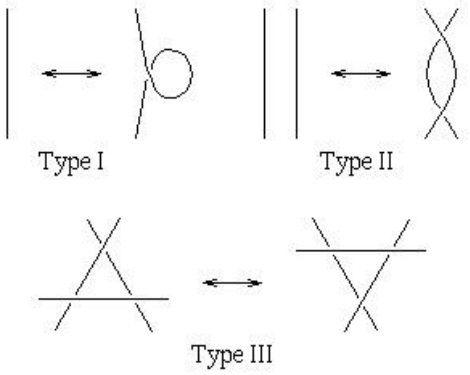


Figure 1.8: The Reidemeister moves.

Alexander later went on to develop a polynomial invariant of knots (*the Alexander polynomial*), while Reidemeister showed that all projections of a knot were related by a sequence of the three moves shown in Figure 1.8. In 1933 Reidemeister lost his professorship in Könisberg, for being "politically unreliable" [3]. Alexander and others not in German-controlled areas went to work on war-related problems and left knot theory behind, at least temporarily. Again the knot theory was interrupted by the war. After the war, Princeton again became a center for knot-theoretical research in the United States. The leader of post-war knot theory in the United States was Ralph H. Fox. Fox felt that the classical definitions of knots, as polygonal curves,

caused knot theory to be too disconnected from the rest of topology. He redefined a knot with topologically-defined set of curves and instead of $\mathbb{R}^3$ he propose to insert a knot in other compact 3-manifolds. Fox's work led to a number of new geometric knot invariants. In the 1970s, J.H. Conway was doing more than simply tabulating knots as discussed in the previous section. He devised a new way to calculate the Alexander polynomial using an algorithm on knot diagrams. In fact, his work actually led to a refinement of the Alexander polynomial that is often called the Conway polynomial. The main breakthrough in knot theory occurred in 1984 when Vaughan Jones developed a new polynomial invariant of knots (the Jones polynomial) as he conducted research on von Neumann algebras. The Jones polynomial was a significant improvement over the earlier polynomial invariants as it was able to distinguish many knots from their mirror images. Four months after the discovery of the Jones polynomial, an invariant called the *HOMFLY* polynomial was discovered. Its name derived from the name of the scientists who discovered it, Jim Hoste, Adrian Ocneanu, Kenneth Millett, Peter J. Freyd, W. B. R. Lickorish, and David Nelson Yetter. This polynomial is a generalization of the Jones polynomial, and in most of the cases it detects the chirality of knots. In August 1985 mathematician Louis H. Kauffman (1945,-) employing techniques used in the study of statistical physics, discovered another approach to the Jones polynomial, the *Kauffman bracket*. This new polynomial produced an alternative method for computing the Jones polynomial.

So far, we have seen how knot theory as been developed during the history, and how new invariants have been discovered according to the different mathematical theory, used at that precise time. In the next chapter we will introduce knot theory with a formal approach. We will study different invariants, starting from the simplest ones and introduce their mathematical proprieties.

Chapter 2

Knot invariants

O time! thou must untangle this, not I; It is too hard a knot for me to untie!

William Shakespeare, *Twelfth Night, Act II, Scene 2*

1 Basic concepts

The intuitive notion of a knot is that of a knotted loop of rope. This notion leads naturally to the definition of a knot as a continuous simple closed curve in R. Such a curve consists of a continuous function $f : [0, 1] \to \mathbb{R}^3$ with $f(0) = f(1)$ and with $f(x) = f(y)$ implying one of three possibilities:$x = y$; $x = 0$ and $y = 1$; $x = 1$ and $y = 0$. Unfortunately this definition admits pathological or so called *wild knots* into our studies. The solution is either to introduce the concept of differentiability or to use polygonal curves instead of differentiable ones in the definition. The simplest definitions in knot theory are based on the following approach.

Definition 2.1. A *knot* is a closed curve in $\mathbb{R}^{\not\perp}$ that does not intersect itself.

Projections of a knot on some plane allow the representation of a knot as a knot diagram. Certain knot projections are better than others as in some projections too much information is lost.

Definition 2.2. A knot projection is called a *regular projection* if no three points on the knot project to the same point, and no two edges of the knot

are mapped onto one another.

Suppose to have two projections of the same knot. If we made a knot out of a string that modeled the first of the two projections, then we should be able to rearrange the string to resemble the second projection. This rearranging of the string, that is, the movement of the string through three-dimensional space without letting it pass through itself, an *ambient isotopy*.

Definition 2.3. An *planar isotopy* is a deformation of a knot projection within the plane that keeps every crossing intact.

A knot diagram is the regular projection of a knot to the plane with broken lines indicating where one part of the knot undercrosses the other part. Informally, an orientation of a knot can be thought of as a direction of travel around the knot.

Definition 2.4. The *connected sum of two knots*, K_1 and K_2, is formed by removing a small arc from each knot and then connecting the four endpoints by two new arcs in such a way that no new crossings are introduced, the result being a single knot, denoted by $K_1 \# K_2$

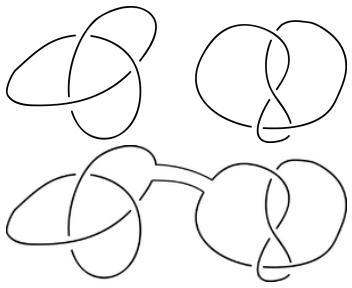


Figure 2.1: Connected sum of the trefoil knot and eight-figure knots.

2 Classical knot invariants

In this section we will see some of the *invariant of knots* (i.e. a "quantity" that is the same for equivalent knots). Two knots are said to be equivalent if there exists an ambient isotopy between them. The problem of equivalence between knots is essentially a problem of equivalence between their regular

projections. In fact the first invariant we will show consist of three moves that Reidemeister discovered, that can deform a knot projection into another one, keeping the knots represented by those two projections, equivalent. We will also see other invariants that are quite elementary, that follow from the observation of the knot projections, such as the number of crossing points, the bridge number, the linking number and the tricolorability.

2.1 The Reidemeister moves

In 1926, the German mathematician Kurt Reidemeister (1893-1971) proved that if we have two diagrams of the same knot, we can get from the one diagram to the other by a series of Reidemeister moves and planar isotopies.

Definition 2.5. A *Reidemeister move* is one of three following ways to change a projection of the knot.

- *First Reidemeister move*: allows us to put in or take out a twist in the knot, as in fig.2.2;
- *Second Reidemeister move*: allows us to either add two crossings or remove two crossings, as in fig.2.2;
- *Third Reidemeister move*: allows us to slide a strand of the knot from one side of a crossing to the other side of the crossing, as in fig.2.2 .

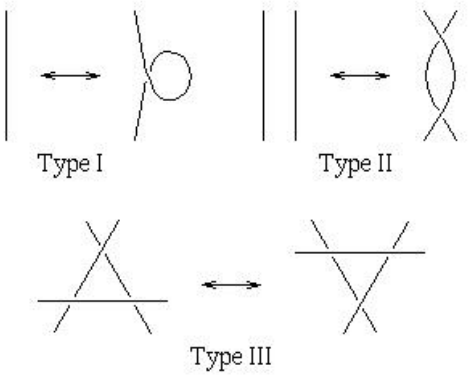


Figure 2.2: The Reidemeister moves.

Reidemeister proved the following theorem [8]:

Theorem 2.1. *Two knots K and K' with diagram D and D' are equivalent if and only $D = D_0, D_1, \ldots, D_n =$ "D' of intermediate diagrams such that each differs from its predecessor by one of the three Reidemeisteir moves.*

In the light of Reidemeister theorem it may seems that the problem of determining whether two projections represent the same knot is easy: it is enough to check whether or not there is a sequence of Reidemeister moves taking from one projection to the other. But taking from one projection to the other could require an infinite number or Reidemeister moves. For instance, the trefoil knot is not achiral, but there is no proof in term of Reidemeister moves. Even if we could prove that we cannot get from the standard projection of the trefoil knot to its mirror image in 10000006 Reidemeister moves, maybe we could do it with 10000007 moves.

2.2 The minimum number of crossing points

A diagram D of a knot K has at most a finite number of crossing points. However, this number $c(D)$ is not a knot invariant. For example, the trivial knot has two regular projections D and D', which have a different number of crossing points.

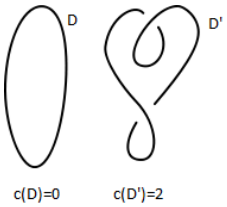


Figure 2.3: Two diagrams of the unknotted with different crossing number.

Consider all the diagrams of K and let $\bar{c}(K)$ be the minimum number of crossing points of all the diagrams. This $c(K)$ is a knot invariant.

Theorem 2.2.

$$\bar{c}(K) = \min_{\mathcal{D}} c(D)$$

is a knot invariant, where $\mathcal{D}$ is the set of all the diagrams, D, of K.

Proof. Suppose that D_0 is the minimum diagram of K. Let K' be a knot that is equivalent to K, and suppose that D_0' is its minimum diagram. Since

D_0' is a diagram also for K (K and K' are equivalent), from the definition we have that $c(D_0) \leq c(D_0')$. However, since D_0 is a diagram of K', it again follows from the definition that $c(D_0') \leq c(D_0)$. Hence, combining the two inequalities, we obtain $c(D_0) = c(D_0')$, i.e., $c(D_0)$ is the minimum number of crossing points for all knots equivalent to K. Consequently, it is a knot invariant. □

2.3 The bridge number

At each crossing point of a diagram, D, of a knot K, let us remove a small segment AB that passes over the crossing point. The result of removing these small segment is a collection of disconnected polygonal curves (see *figure* 2.40). We may think of the original diagram as the resulting projection that occurs when we attach the segments $AB, \ldots,$ that pass over to the endpoints of these disconnected polygonal curves on the plane.

Figure 2.4: The bridges of the trefoil knot.

These segments are called *bridges*, since they pass above the segment on the plane. For a given projection D, the number of bridges is called *bridge number*.

Definition 2.6. Let D be a diagram of a knot, K. If we can divide up D into $2n$ polygonal curves $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$, i.e.,

$$D = \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n \cup \beta_1 \cup \beta_2 \ldots \beta_n$$

that satisfy the following condition:

- α_1, α_2, …, α_n are mutually disjoint, simple polynomial curves.
- β_1, β_2, …, β_n are mutually disjoint, simple curves.
- At the crossing point of D, α_1, α_2, …, α_n are segments that pass over the crossing points. While at the crossing points of D, β_1, β_2, …, β_n are segments that pass under the crossing points.

then the bridge number of D, $br(D)$, is said to be at most n.

If $br(D) \leq n$ but $br(D) \not\leq n-1$, we define $br(D) = n$.
The bridge number of a diagram D is not a knot invariant. There exist knots that have diagrams with different bridge numbers.

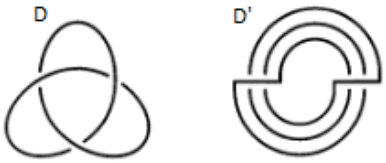


Figure 2.5: These are two diagrams of the trefoil knot with different bridge number. In fact $br(D) = 3$ and $br(D') = 2$.

As for the crossing number, if we consider all the diagrams of a given knot K, then the minimum bridge number of all these diagrams is an knot invariant.

Theorem 2.3.

$$br(K) = \min_{\mathcal{D}} br(D)$$

is an invariant for K, where $\mathcal{D}$ is the set of all diagrams of K. This quantity is called the bridge number of K.

Proof. Same as the proof of theorem 2.2.2. Suppose that D_0 is the minimum diagram of K. Let K' be a knot that is equivalent to K, and suppose that D_0' is its minimum diagram. Since D_0' is a diagram also for K (K and K' are equivalent), from the definition we have that $br(D_0) \leq br(D_0')$. However, since D_0 is a diagram of K', it again follows from the definition that $br(D_0') \leq br(D_0)$. Hence, combining the two inequalities, we obtain $br(D_0) = br(D_0')$, i.e., $br(D_0)$ is the minimum number of crossing points for all knots equivalent to K. Consequently, it is a knot invariant. □

Remark 1. In the specific case of $br(K) = 2$, there are many knots with this bridge number, including the trefoil knot and the figure eight knot. These knots, called the 2-bridge knots, have been studied to the point that they have been completely classified. However, in general, no method has been found yet to allow us to determinate $br(K)$ for an arbitrary knot.

2.4 The linking number

A *link* is a set of knotted loops all tangled up together. Here are two projections of two famous links, the Whitehead link and the Borromean rings.

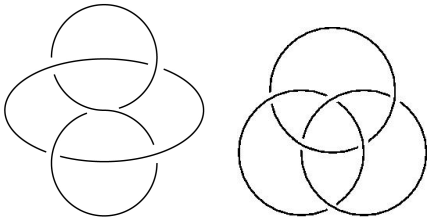


Figure 2.6: The Whitehead link and the Borromean rings.

Since the Whitehead link is made up of two loops knotted with each other, we say that it is a *link of two components*. Instead, the Borromean rings had three components.2.6 A knot is considered as a link with one component. An easy way to check whether or not two links are equivalent is two count the components in the link. If the numbers are different, the two links have to be different. So, obviously, the trefoil knot, the Whitehead link and the Borromean rings are all different.
Now consider the *unlink* and the *Hopf link*2.7.

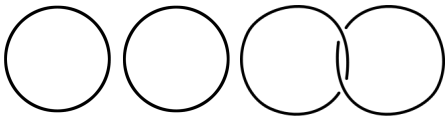


Figure 2.7: The unlink of two components and the Hopf link.

Both of these are two-components link, but they cannot be equivalent. In fact the unlink is splittable, that is, can be deformed so that the two components lie on different sides of a plane in three-space. Instead in the Hopf link the two components do link each other once. If we have the projection of two links, each with the same number of components, to check if the two link are equivalent we have to define the *linking number*. At a crossing point, c of an oriented diagram we have two possible configurations, as in fig.2.8. In the first picture we assign $sign(c) = +1$, while in the second we assign $sign(c) = -1$. The first crossing point is said to be positive, while the second is said to be negative.

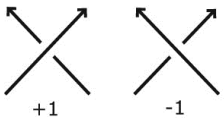


Figure 2.8: Computing linking number.

Definition 2.7. Suppose that D is an oriented diagram of a two component link, $K = \{K_1, K_2\}$. Now, suppose that the crossing points of D at which the projection of K_1 and K_2 intersect are c_1, c_2, ..., c_m. (We ignore the crossing points of the projections of K_1 and K_2, which are self intersections of the knot component). Then

$$\frac{1}{2}\{sign(c_1) + sign(c_2) + \cdots + sign(c_m)\}$$

is called the *linking number* of K_1 and K_2 which will be denoted by $lk(K_1, K_2)$.

Theorem 2.4. *The linking number $lk(K_1, K_2)$ is a knot invariant.*

Proof. We show this by proving that the Reidemeister moves do not change the linking number. Let's first look at the effect of the first Reidemeister move on the linking number. It can create or eliminate a self-crossing in one of the two components, but it will not affect the crossings that involve both of the components, so it leaves the linking number unchanged. Let's now have at the

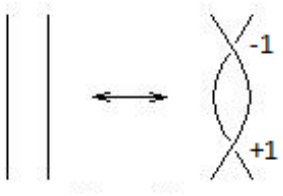


Figure 2.9: The second Reidemeister move does not effect linking number.

effect of the second Reidemeister move. We are assuming that the two strands correspond to the two different components, because otherwise the move has no effect on linking number. One of the new crossings contributes a $+1$ to the sum, and the other crossing contributes a -1, so the net contribution to the linking number is 0. Even if we change the orientation on one of the strands, we will still have one $+1$ and one -1 contribution, so the second Reidemeister move leaves the linking number unchanged. Finally, the third Reidemeister

move doesn't effect linking number either. Once orientation is chosen for each of the three strands and $+1$ and -1 are assigned to each of the crossing, it is clear that sliding the strand over during the third Reidemeister move doesn't change the number of $+1$ and -1, and so the linking number is preserved.

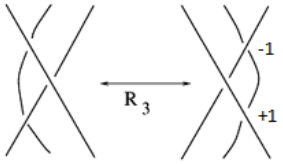


Figure 2.10: The third Reidemeister move does not effect linking number.

□

Unfortunately the linking number is not such a perfect invariant. If we compute the linking number for the Whitehead link in figure 2.6, it has linking number 0, just like the unlink of two components. So, we cannot even show that the Whitehead link is different from the unlink of two components, just by looking at its linking number.

2.5 The tricolorability

In this final subsection, we introduce a new invariant that will allow us to prove that there are other knots besides the unknot. In fact in principle every projection of a knot could be a messy projection of the unknot and could be turned into it through a series of Reidemeister moves. So we will prove that there is at least one other knot besides the unknot. In order to prove this we need to introduce the concept of *tricolorability*.

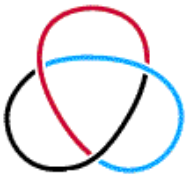


Figure 2.11: The Trefoil is tricolorable.

Definition 2.8. A projection of a knot or a link is said to be *tricolorable* if each of the strands in the projection can be colored one of three different

chosen colors, so that at each crossing, either three different colors come together or all the same color comes together. In order that a projection be tricolorable, is also required that at least two colors are used.

Example 1. *The unknot is not tricolorable. We certainly cannot use at least two colors on it.*

Theorem 2.5. *The tricolorability of a knot is a knot invariant.*

Proof. We have to show that the Reidemeister moves will preserve the tricolorability. If we do a first Reidemeister move and introduce a crossing, we can just leave all the strands involved the same color, and the new crossing will satisfy the requirements for tricolorability. Similarly, removing a crossing with the first Reidemeister move preserves tricolorability.

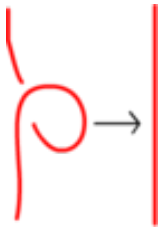


Figure 2.12: The first Reidemeister move preserves tricolorability.

If we do a second Reidemeister move and introduce two crossings and the two original strands are different colors, we can change the color of the new strand to the third color, and the resulting knot projection is tricolorable. If the two original strands are the same colors we can leave the new strand and the new crossings all the same color.

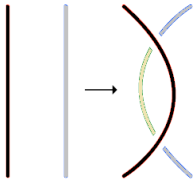


Figure 2.13: The second Reidemeister move preserves tricolorability.

Similarly, using a second Reidemeister move to reduce the number of crossings by two will also preserve tricolorability. Either all of the strands that appear in the projection for the Reidemeister move are the same color, in which case

we can color the strands that result from the Reidemeister move that same color, or three distinct colors come together at each of the two crossings, in which case we can color the two resulting strands in two different colors. In both cases, since the original projection was colored with at least two distinct colors, the resulting projection will also be colored with at least two colors. Finally, to check that the tricolorability is invariant under the third Reidemeister move we have to check five different cases. The fig.2.14 shows all of them and from the picture is easy to understand that the in all the cases that the third Reidemeister move preserves tricolorability.

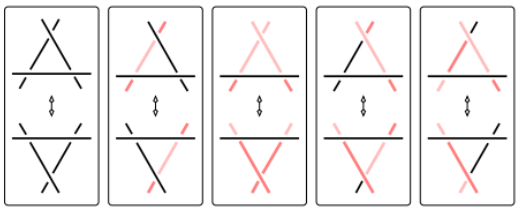


Figure 2.14: The third Reidemeister move preserves tricolorability.

□

Therefore Reidemeister moves leave tricolorability unaffected and whether or not a projection is tricolorable depends only on the knot given by the projection. Thus, we can state the following:

Proposition 2.6. *Either every projection of a knot is tricolorable or no projection of that knot is tricolorable.*

Thus, many knots can be shown to be non trivial using tricolorability furthermore we can conclude that the any tricolorable knot must be distinct from any knot that is not tricolorable. However, tricolorability is a weak knot invariant. We can prove that the figure-eight is not tricolorable(see 2.15), hence it is not the same as the trefoil knot(see 2.11). But we cannot use tricolorability to show that the figure-eight is nontrivial.
Notice that the unlink of two component is tricolorable. This is the reverse of what happened for tricolorability for knots. Now if we have a link of two components that is not tricolorable, we know it cannot be the unlink.

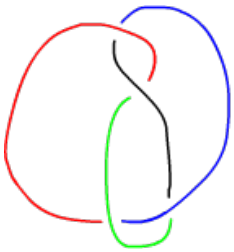


Figure 2.15: The figure-eight knot is not tricolorable.

3 Seifert matrix and its invariants

In this section we will introduce the concept of seifert surface, and show how this concept is related to the knots. Given any knot, in fact, there is a collection of orientable surfaces whose boundary is the knot. We introduce this new concepts, in order to obtain an important knot invariant, the Alexander polynomial.

3.1 Seifert matrix

In order to define a Seifert surface, we will start with the following theorem:

Theorem 3.1. *Given an arbitrary knot K, then there exists in $\mathbb{R}^3$ an orientable, connected surface, F, that has as its boundary K.*

The *Seifert algorithm* will give as a proof of the theorem. We briefly sketch it.
Suppose that K is an oriented knot and D is a regular diagram for K. Now, we have to decompose D into several simple closed curves. First we need to draw a small circle with one of the crossing points of D as its center. This circle intersects D at four points, say, a, b, c, and d. Let us splice this crossing points and connect a and d, and b and c as in fig.2.16.

Figure 2.16: Splicing of a knot.

This operation is called the *splicing* of a knot K at a crossing point of D. Performing the splicing operation at every crossing of D we shall remove all the crossing points from D. The result is that D becomes decomposed into several simple closed curves. These curves are called *Seifert curves*. D itself has been transformed into a diagram of a link on the plane without crossing, i.e. the trivial link. Each of these simple curves marks off a disk. Since we do not want the disks to intersect one another, we will choose them to be at different heights rather than having them all in the same plane.

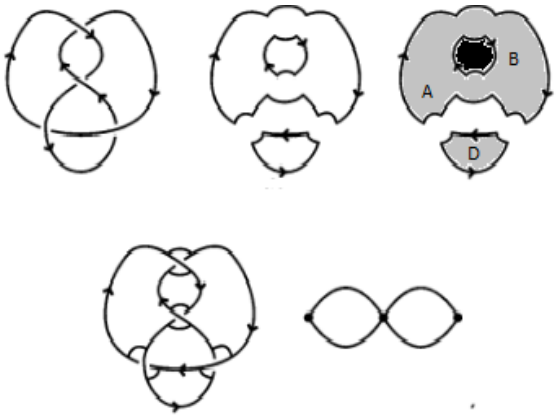


Figure 2.17: Seifert surface of the figure-eight knot.

Now we would like to connect each of such disk to another one of them at the crossings of the knot using twisted bands. To do this, first, we take a square *abcd* and give it a simple positive or negative twist. These twisted square are the required bands.

Figure 2.18: Positive and negative twist, respectively.

If we attach positive (or negative) bands at the places of D that corresponded to positive (or negative) crossing points before they were spliced, then we obtain a connected, orientable surface F. The boundary of this surface is plainly the original knot K. This suggests the following definition:

Definition 2.9. An orientable, connected surface that has as its boundary an oriented knot K is called a *Seifert surface* of K.

Now, if we contract each disk to a point, and at the same time the width of the bands is shrunk, ideally, into segments, then from these points and segments a graph in space is formed. Such a graph is called the *Seifert graph* of K.
The Seifert surface of a link is not unique in any way, and even Seiferts algorithm applied to different projections of the same knot will result different. Since the Seifert surface itself cannot be an invariant of the knot, we need to look for other information. The first of these is the genus. A fundamental theorem in topology states that any orientable closed surface (i.e. with no boundary), F, is topologically equivalent to a sphere with p handles attached to it. The number of these handles is called the *genus* of F, and is denoted by $g(F)$. When F is a Seifert surface, it has boundary. Then we redefine the genus of F as the genus of the corresponding surface without boundary obtained by attaching to each boundary component a disc.

Definition 2.10. The genus of a knot is the minimal genus of all the Seifert surfaces of the knot.

The genus is then a knot invariant. But to calculate the genus of an arbitrary knot is a difficult task. However, it has some useful properties, the main one being that it is additive, i.e. $g(K1\#K2) = g(K1) + g(K2)$. Also, for alternating knots (i.e. a knot whose oriented diagram alternates between over and under crossings as you follow it around), Seifert's algorithm does indeed provide a minimal genus surface. Although the determination of the genus of an arbitrary knot is a difficult problem, to determinate the genus of constructed orientable surface is quite easy.

Theorem 3.2. *[8] Let F be a Seifert surface of a link K. If s is the number of Seifert circles and c is the number of crossings of the projection of a link, this produces a surface of genus:*

$$g(F) = \frac{1 - s + c - n}{2}$$

where n is the number of components of the link.

Proposition 3.3. *[8] If the genus of the Seifert surface, F, of a knot (or link) is $g(F)$, then on F there are $2g(F)+n-1(=m)$ closed curves α_1, α_2, ..., α_m, where n is the number of components of the link.*

These $2g(F)+n-1(=m)$ closed curves will indicate certain characteristics of the knot K that is the boundary of the surface F. Individually, however, these closed curves are of little interest, but as a collection they will provide us with a knot invariant.
Since Seifert surfaces of a given knot, K, are orientable by construction, inheriting the orientation from the knot, we may think of Seifert surface as having a positive and negative side. In order to make these side more distinct, we could make the surface thicker by some ϵ, where $\epsilon > 0$ and sufficient small to not disturb the topology of the surface. Then $F \times 0$ is the negative side of F, while $F \times \epsilon$ is the positive side. Now, consider any closed curve, α on F. Let α^* be the curve α which lies on the positive side of F, i.e. α^* is the closed curve lying on $F \times \epsilon$.
Then we can define the Seifert matrix.

Definition 2.11. Let F be a Seifert surface for a knot with m closed curves α_1,α_2, ..., α_m where m is defined above. Then the *Seifert matrix* is an $m \times m$ matrix,

$$M = [lk(\alpha_i, \alpha_j^*)]_{i,j=1,2,\ldots,m}$$

In general, the linking number $lk(\alpha_i, \alpha_j^*)$ and $lk(\alpha_j, \alpha_i^*)$ are not equal, so the matrix is not a symmetric matrix. In the case when $g(F) = 0$, the Seifert matrix of K is defined to be the empty matrix (K is the trivial knot).
To better understand these concepts, let us consider the figure-eight knot. In the figure 2.17 we can see the figure-eight knot, its Seifert surface and the subsequent Seifert graph. Removing the edges from the Seifert graph we found the *spanning tree*, that is just three vertices in a line. Adding in the removed edges one at a time gives us two cycles, one between the top two vertices, and one between the bottom two vertices.
These correspond to a closed curve, a, between the top and main levels of the Seifert surface, and a closed curve, b between the base levels of the Seifert surface respectively. The closed curves are raised slightly above the surface to better illustrate their path through the crossings. In order to compute the

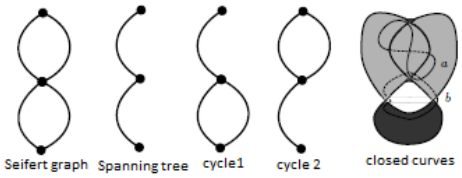


Figure 2.19: How two find two closed curves in the Seifert surface of the figure-eight knot.

Seifert matrix for the figure-eight knot we need to find how these two closed curves link with themselves and each other.

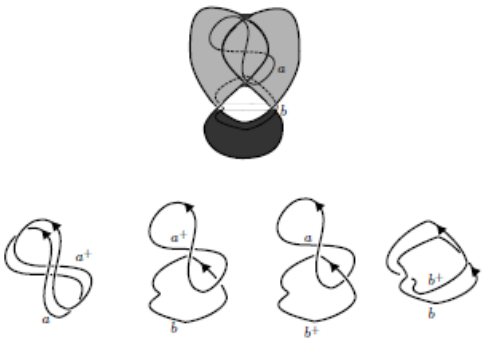


Figure 2.20: The Seifert surface with two closed curves and how these link with themselves and each other.

From the above four projections, it follows that

$$lk(a, a^*) = -1, \qquad lk(a, b^*) = 1, \qquad lk(b, a^*) = 0, \qquad lk(b, b^*) = 1$$

Therefore, the Seifert matrix for the figure-eight knot is

$$M = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

In order to obtain an invariant of a knot from a Seifert matrix, we need to examine the relationship between the Seifert matrices of the same knot. We need to introduce the concept of the S-equivalence of two square matrices.

Theorem 3.4. *Two Seifert matrices, M_1 and M_2, obtained from two equivalent knots can be changed from one to the other by applying, a finite number of times, the following two operation, Λ_1 and Λ_2, and their inverses:*

$$\Lambda_1 : M_1 \to PM_1P^T,$$

where P is an invertible integer matrix, with $detP = \pm 1$ ($detP$ is just the usual determinant of P), and P^T denotes the transpose matrix of P.

$$\Lambda_2 : M_1 \to M_2 = \begin{bmatrix} & & & * & 0 \\ & M_1 & & \dots & \dots \\ & & & * & 0 \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} & & & 0 & 0 \\ & M_1 & & \dots & \dots \\ & & & 0 & 0 \\ * & \dots & * & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}$$

*where * denotes an arbitrary integer.*

The operation Λ_1 either interchanges two rows, say i^{th} and j^{th} rows, and then interchanges the i^{th} and j^{th} columns; or it adds k times the i^{th} row to the j^{th} row, and then adds k times the i^{th} column to the j^{th} column. This operation is called an *elementary symmetric matrix operation.* The operation Λ_2, on the other hand, is a matrix operation that is particular to knot theory. This operation has been defined so that it corresponds to the change in the genus of the Seifert surface due to a Reidemeister move, i.e. it makes the Seifert matrix either smaller or larger.

Definition 2.12. Two square matrices M, M' obtained one from the other by applying the operations Λ_1, Λ_2 and the inverse Λ_2^{-1} a finite number of times, are said to be *S-equivalent*, and are denoted by $M \overset{S}{\sim} M'$

It follows, from the above theorem, that two Seifert matrices obtained from the two equivalent knots are S-equivalent.
We shall conclude by proving two properties of Seifert matrices. We denote by M_K the Seifert matrix of a knot of K.

Proposition 3.5. *Suppose that K is an oriented knot and $-K$ is the knot with the reverse orientation to K. Then $M_{-K} \overset{S}{\sim} M_K^T$, where M_K^T is the transpose matrix of M_K.*

Proof. If we suppose that D is a diagram of K, we may take as a diagram D' for $-K$, the diagram D with all the orientations reversed. Therefore, the orientations of the subsequent Seifert surface are completely opposite. Hence, the under and over relations for a_i and a_j^* are completely reversed. The Seifert

matrix obtained from D' is therefore the transpose of that from D. It follows from theorem that $M_{-K} \overset{S}{\sim} M_K^T$. □

Proposition 3.6. *Suppose that K^* is the mirror image of a knot K, then $M_{K^*} \overset{S}{\sim} -M_K^T$.*

Proof. We can obtain a diagram D^* of K^* from K by changing the under and over crossing segment at each of the crossing points. Therefore, since the under and over relations for the closed curves that follow from D and D^* are completely reversed, $M_{K^*} \overset{S}{\sim} -M_K^T$. □

3.2 The Alexander polynomial

In this section we introduce the Alexander polynomial.
To compute this polynomial let us consider a Seifert matrix M, its transpose and the polynomial $det(M - tM^T)$ with indeterminate t. Now, we examine how this polynomial changes when we apply Λ_1 and $\Lambda_2^{\pm 1}$. Firstly, since $detP = detP^T = \pm 1$,

$$det(\Lambda_1(M - tM^T)) = det[P(M - tM^T)P^T] = det(M - tM^T).$$

Therefore, it is not affected by the operation Λ_1. However, if we apply Λ_2,

$$\begin{aligned} det(\Lambda_2(M - tM^T)) &= det \begin{bmatrix} & & & b_1 & 0 \\ & M - tM^T & & \dots & \dots \\ & & & b_m & 0 \\ -b_1 t & \dots & -b_m t & 0 & 1 \\ 0 & \dots & 0 & -t & 0 \end{bmatrix} \\ &= det \begin{bmatrix} & & & b_1 & 0 \\ & M - tM^T & & \dots & \dots \\ & & & b_m & 0 \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & -t & 0 \end{bmatrix} \\ &= tdet(M - tM^T). \end{aligned}$$

Similarly, we can obtain $det(\Lambda_2^{-1}(M_2 - tM_2^T)) = t^{-1}det(M_1 - tM_1^T)$.

Theorem 3.7. *Suppose that M_1 and M_2 are the Seifert matrices for a knot K. Further, if r and s are, respectively, the orders of M_1 and M_2, then the following equality holds:*

$$t^{-\frac{r}{2}} det(M_1 - tM_1^T) = t^{-\frac{s}{2}} det(M_2 - tM_2^T). \tag{2.1}$$

Therefore, if M is a Seifert matrix of K and its order is k, then

$$t^{-\frac{k}{2}} det(M - tM^T)$$

is an invariant of K. This invariant is known as the *Alexander polynomial normalized* of K and its denoted by $\Delta_K(t)$. Note that $k = 2g(F) + n - 1$, where F is the Seifert surface, $g(F)$ is the genus of the surface and n is the number of components of the link. Sometimes it is preferable to work with such an interpretation of $\Delta_K(t)$, like:

$$\Delta(K)(t) = t^{\frac{k}{2}} det(M - tM^T). \tag{2.2}$$

Let us see the Alexander polynomial in some relevant examples.

Example 2. *If K is a trivial knot, then $\Delta_K(t) = 1$.*

Example 3. *Let K be the figure-eight knot. As we found above, the Seifert matrix, M, is:*

$$M = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

Therefore,

$$\Delta_K(t) = t^{-1}(M - tM^T) = t^{-1} det \begin{bmatrix} t-1 & 1 \\ -t & 1-t \end{bmatrix} = -t + 3 - t^{-1}$$

Example 4. *Let us consider the trefoil knot. Let us begin with its knot diagram, from which we can construct the Seifert circles and then the Seifert surface.*
We find the closed curves as we did above for the figure-eight knot.
In the following figure, the pairs of curves are display.
We can compute now the linking numbers for the Seifert matrix.
Thus we obtain:

$$M = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$$

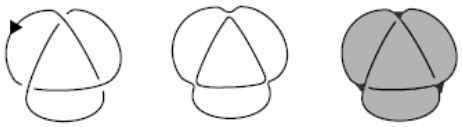


Figure 2.21: The Seifert surface of the trefoil knot.

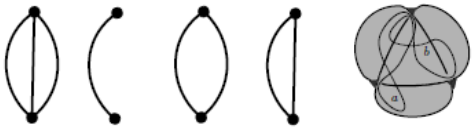


Figure 2.22: The process of obtaining the closed curves from the Seifert graph of the trefoil knot.

We can finally compute the Alexander polynomial of the trefoil knot:

$$\Delta_K(t) = t^{-1}(M - tM^T) = t^{-1}det \begin{bmatrix} t-1 & -1 \\ t & t-1 \end{bmatrix} = t - 1 + t^{-1}. \quad (2.3)$$

3.3 The Alexander-Conway polynomial

Definition 2.13. Given an oriented knot K, then we may assign to it a polynomial $\nabla_K(z)$, with a fixed indeterminate z, by means of the following two axioms:

- If K is the trivial knot, then $\nabla_K(z) = 1$.
- Suppose that D_+, D_-, D_0 are diagrams, respectively of the three knots K_+, K_-, K_0, where these knots are the same as K, except at a neighborhood of one crossing point. The different projections are shown is the figure.

Then the polynomials of these three knots are related by the following equality:

$$\nabla_{K_+}(z) - \nabla_{K_-}(z) = z\nabla_{K_0}(z). \quad (2.4)$$

The three diagrams D_+, D_-, D_0 are called *skein diagrams*, and the relation between the polynomial of the three knots is called *skein relation*. Also, an operation that replace one of by the other two is called *skein operation*.
The polynomial $\nabla_K(z)$ is called the *Conway polynomial*.

Figure 2.23: How these closed curves link with themselves and each other.

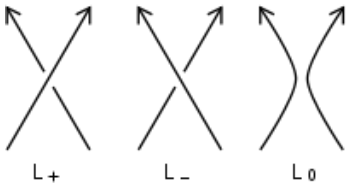


Figure 2.24: Skein diagrams.

Theorem 3.8. *[15]*

$$\Delta_K(t) = \nabla_K(\sqrt{t} - \frac{1}{\sqrt{t}}). \tag{2.5}$$

That means that if we replace z by $\sqrt{t} - \frac{1}{\sqrt{t}}$ in the Conway polynomial, the result will be the Alexander polynomial. Due to this relationship, $\nabla_K(z)$ is also called the *Alexander-Conway polynomial*.
For calculating the Conway polynomial of a given knot, we need firsts to state the following proposition

Proposition 3.9. *The Conway polynomial of the trivial link with m ($m \geq 2$) components is* 0.

Proof. The skein relation corresponding to the skein diagram in the figure 2.25 is:

$$\nabla_{D_+}(z) - \nabla_{D_-}(z) = z\nabla_{D_0}(z). \tag{2.6}$$

Since both D_+ and D_- are the trivial knot, $\nabla_{D_+}(z) = \nabla_{D_-}(z)$, therefore $z\nabla_{D_0}(z) = 0$, i.e. $\nabla_{D_0}(z) = 0$. □

Usually the most effective way to calculate the Conway polynomial is to make use of the *skein tree diagram*.

Example 5. *Let K be the right-hand trefoil knot and D is its diagram.*

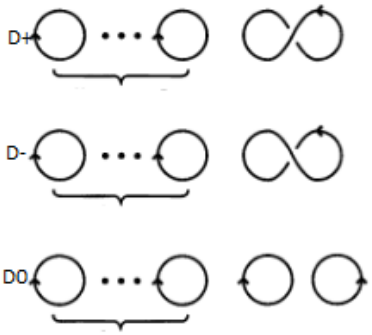


Figure 2.25: Skein diagrams for the unlink.

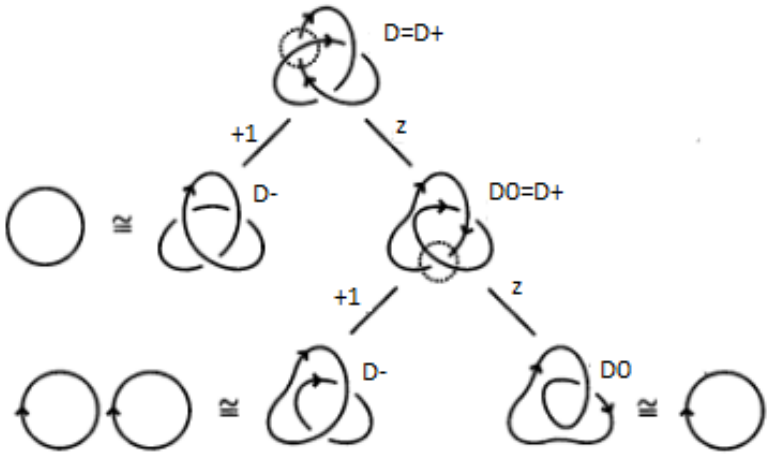


Figure 2.26: Skein diagrams for the trefoil knot.

At one crossing of D we will perform a skein operation, as shown in the figure. Since that particular crossing is positive, it is better to rename it D_+. By performing the skein operation, D_+ is transformed into diagrams: D_-, obtained by changing the crossing, and D_0, obtained by resolving the crossing. It is straightforward to see that D_- is equivalent to the trivial knot, and hence $\nabla_{D_-}(z) = 1$. Therefore D_- will not produce any further branches. Now for the first pair of branches, we can evaluate $\nabla_{D_+}(z) = 1\nabla_{D_-}(z) + z\nabla_{D_0}(z)$ We can now consider D_0. We perform the skein operation on the positive crossing of D_0 (again it is better to rename it D_+). We will obtain others two projections as before, D_- and D_0. D_0 is equivalent to the trivial knot, and again $\nabla_{D_0}(z) = 1$. D_- is equivalent to the trivial link of two component. Hence no further branches may be formed. $\nabla_K(z)$ can now be calculated as the

sum of the Conway polynomial of the each terminating trivial knot multiply by the coefficients along the branch paths that begins with the diagram, D, of K, and terminates with the projection of the trivial knot. The coefficients follow from the skein relation:

$$\nabla_{D_+}(z) = \nabla_{D_-}(z) + z\nabla_{D_0}(z)\nabla_{D_-}(z) = \nabla_{D_+}(z) - z\nabla_{D_0}(z) \tag{2.7}$$

Therefore we get:

$$\nabla_K(z) = 1\nabla_{\bigcirc}(z) + z\nabla_{\bigcirc\bigcirc}(z) + z^2\nabla_{\bigcirc}(z). \tag{2.8}$$

Since $\nabla_{\bigcirc}(z) = 1$ and $\nabla_{\bigcirc\bigcirc}(z) = 0$, $\nabla_K(z) = 1 + z^2$. Applying the relation with the Alexander polynomial:

$$\Delta_K(t) = 1 + (\sqrt{t} - \frac{1}{\sqrt{t}})^2 = t^{-1} - 1 + t. \tag{2.9}$$

Example 6. *The skein diagram for the Conway polynomial for the figure-eight knot is the following:*

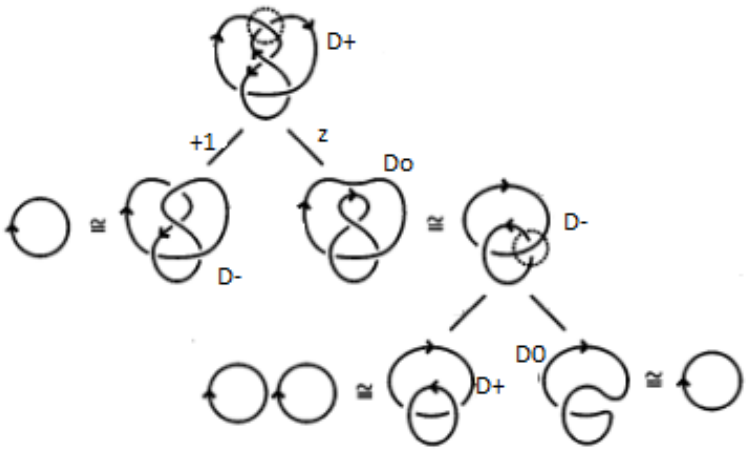


Figure 2.27: Skein diagrams for the figure-eight knot.

Using the same calculation as above, we get:

$$\nabla_K(z) = 1\nabla_{\bigcirc}(z) + z\nabla_{\bigcirc\bigcirc}(z) - z^2\nabla_{\bigcirc}(z) = 1 - z^2.$$

Therefore:

$$\Delta_K(t) = 1 + (\sqrt{t} - \frac{1}{\sqrt{t}})^2 = -t^{-1} + 3 - t.$$

4 The Jones revolution

4.1 Braids theory

In this section, we introduce some aspects of the theory of braids that will be useful in explaining some developments in knot theory.

Definition 2.14. A braid is a set of n strings, all of which are attached to an horizontal bar at the top and at the bottom, such that each strings intersects any horizontal plane between the two bars exactly once.

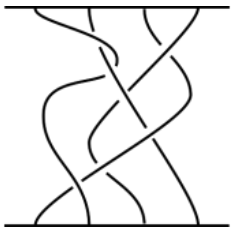


Figure 2.28: Example of braid.

Now, suppose that a n-braid, A has its strings connected as follows: A_1 to A'_{i_1}, A_2 to A'_{i_2}, ..., A_n to A'_{i_n}. Then we can assign to α a permutation,

$$\begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}$$

We call this permutation the *braid permutation*. Note that different braids may correspond to same permutation.

Example 7. *The following figure represents the trivial braid*

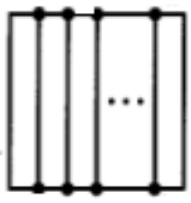


Figure 2.29: Trivial braid.

The trivial braid corresponds to the identity permutation,

$$\begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}$$

Intuitively, two braids are said to be equivalent(in $\mathbb{R}^3$) if we can continuously deform one to the other without causing any strings to intersect each other.

Proposition 4.1. *If two braids are equivalent, they have the same permutation.*

Example 8. *In the figure 2.30 we can see an example of two equivalent braids.*

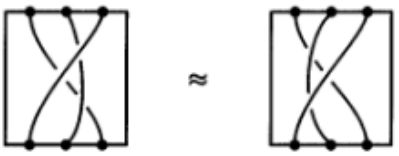


Figure 2.30: Two equivalent braids.

The braid permutation is

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

The two braids are equivalent, then they have the same permutation. Therefore the braid permutation is a braid invariant. This is the simplest braid invariant.

The main result of this section is showing that existence of the *braid group.* Suppose that $\mathcal{B}$ is the set of all braids with n strings. For two elements in $\mathcal{B}$, α and β, we can define a *product* by attaching the bottom bar of braid to the upper bar of the other.
The resultant braid is called the product of α and β and is denoted by $\alpha\beta$. In general, it is not true that $\alpha\beta = \beta\alpha$, i.e. $\alpha\beta$ and $\beta\alpha$ need not to be equivalent braids.
Although not necessarily commutative, the product of braids is associative, i.e.

$$(\alpha\beta)\gamma = \alpha(\beta\gamma).$$

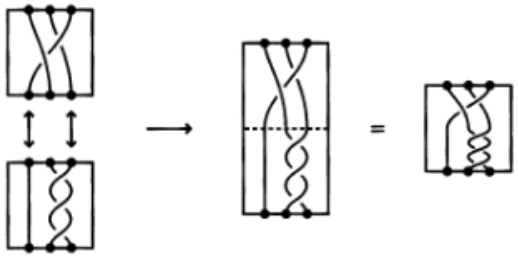


Figure 2.31: The product of two braids.

Figure 2.32: We can check the product of the braids permutation of $\alpha\beta$ and $\beta\alpha$. These are respectively $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. Therefore the two products of braids are not equivalent.

In order to show the $\mathcal{B}$ is a group under the action of the product, we must find a unit element and an inverse element. The *unit*, ϵ is simply the trivial braid. It follows that $\alpha\epsilon = \alpha$ and $\epsilon\alpha = \alpha$.

In order to find an *inverse* for an arbitrary α, let us consider the mirror image, α^*, of α. If we consider the bottom bar to be a mirror, then the mirror image is the image of α reflected in this mirror. It follows that $\alpha\alpha^* = \epsilon$ and $\alpha^*\alpha = \epsilon$. The inverse element is denoted by α^{-1}.

Therefore, we now all the essentials for $\mathcal{B}$ to be a group. This group is called the *n-braid group*.

Among the n-braids, we can create certain specific n-braids by connecting A_i to A'_{i+1} and A_{i+1} to A'_i, and then connecting the remaining A_j and A'_j $(j \neq i, i+1)$ by line segments.

We shall denote these types of n-braids by σ_i. In this way we can create $n-1$ special n-braids σ_1, σ_2, $\ldots$, σ_{n-1}, called the *elementary transposition*. Also we have an inverse element of σ_i, the n-braids σ_i^{-1}.

We may now use the elements to express any element in the braid group. First we divide the braid by lines parallel to the bottom bar, so that in each rectangle we have only one crossing point. In each rectangle we have a

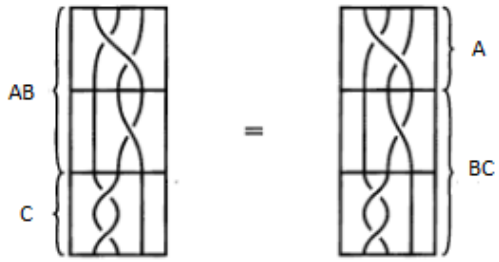


Figure 2.33: The product of two braids is associative.

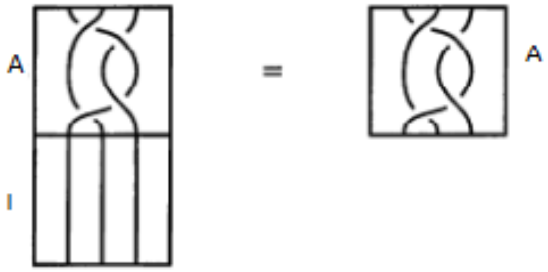


Figure 2.34: The unit of braid group.

braid that is of the form σ_i or σ_i^{-1}. By definition of product of braids we can decompose the braid into the product of these σ_i and σ_i^{-1}. For example, in the 2.37 we have the braid $\alpha = \sigma_3^{-1}\sigma_1\sigma_2\sigma_3\sigma_2^{-1}$.
Therefore, given any braid, we can express it as the finite product of the σ_i and σ_i^{-1}. For this reason, the braids $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ are said to generate the braid group $\mathcal{B}_n$, and so we call $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ the *generators* of $\mathcal{B}_n$.
From the above, we have a way of describing algebraically a braid as a product of σ_i and σ_i^{-1}. However, these algebraic representations are not unique. For example, the two braids $\sigma_1\sigma_3$ and $\sigma_3\sigma_1$ in the figure are equivalent 4-braids. Further, since $\sigma_1\sigma_2\sigma_1$ and $\sigma_2\sigma_1\sigma_2$ are equivalent 3-braids, the following relation holds:

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$

.
This equality holds even if this braid is considered as a general n-braid ($n \geq 3$).
These equalities are called *braid relations* of the braid group. In fact, if two n-braids are equivalent, then we can change one to other by using these equalities several times. A fundamental result on the braid group $\mathcal{B}$, is that

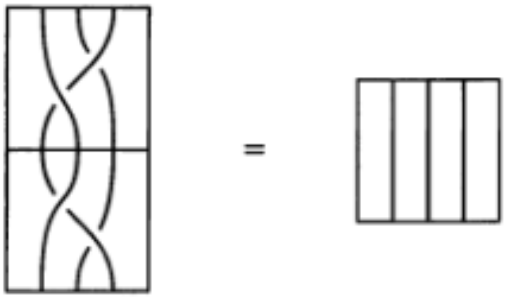


Figure 2.35: The inverse of braid group.

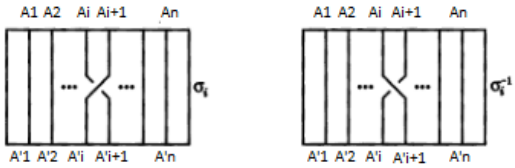


Figure 2.36: The generators of the braid group.

it only has the following two type of relationship called the *fundamental relations*:

- $\sigma_i \sigma_i^{-1} = \epsilon$
- $\sigma_i \sigma_j = \sigma_j \sigma_i \quad (|i-j| \geq 2)$;
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad (i = 1, 2, \ldots, n-2)$.

In conclusion, we may write $\mathcal{B}_n$ in terms of its generators σ_1, σ_2, $\ldots$, σ_{n-1} and these fundamental relations,

$$\mathcal{B}_n = \left(\sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{cc} \sigma_i \sigma_j = \sigma_j \sigma_i & (|i-j| \geq 2) \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & (i = 1, 2, \ldots, n-2) \end{array} \right)$$

where the right-hand side is said to be a *presentation* of $\mathcal{B}_n$
For example:
$\mathcal{B}_1 = (\sigma_1)$ [1]
$\mathcal{B}_2 = (\sigma_1, \sigma_2 | \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2)$
$\mathcal{B}_3 = (\sigma_1, \sigma_2, \sigma_3 | \sigma 1 \sigma_3 = \sigma_3 \sigma_1, \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2, \sigma_2 \sigma_3 \sigma_2 = \sigma_3 \sigma_2 \sigma_3)$.
Now, connecting the points A_1, A_2, $\ldots$, A_n of a braid α, to the points A_1', A_2', $\ldots$, A_n'.

[1] Expect for the trivial relation $\sigma_i \sigma_i^{-1} = \epsilon$, $\mathcal{B}_1$ does not have any relations.

Figure 2.37: A braid expresses in terms of its generators.

Figure 2.38: Two equivalent braids with different representation.

In a natural way we form a regular diagram of knots or link from a braid. A knot, K that has been created in this way is said to be a knot *created* from a braid α. Conversely, we say that K is the *closure* of α.

Theorem 4.2 (Alexander's theorem)**.** *Given an arbitrary knot, then it is equivalent to a knot that has been formed from a braid.*

In order to prove the theorem, we start by considering the following definition.

Definition 2.15. An oriented knot diagram is said to possess a center $x \in \mathbb{R}^3$ if every arc in the knot diagram has the same orientation with respect to the point x.

Theorem 4.3. *Any knot has a diagram which possess a center point.*

Proof. We describe an algorithm that transforms every knot to an equivalent knot with a center. Given an arbitrary link, firsts we fix a center O and choose a specific orientation. Then we consider any arc with opposite orientation to the chosen one. We then modify the knot diagram by imagining an axis through O, and by passing this arc over the axis so as to obtain a new diagram. If the chosen arc contains an overcrossing(undercrossing), we push it under(over) every other arc in the knot diagram. We continue this operation until we obtain a diagram with O as its center. Every time we do apply this

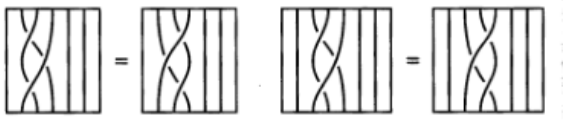


Figure 2.39: Two equivalent braids with different representation.

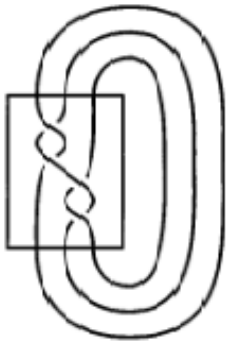


Figure 2.40: The closure of braid.

move, the number of the of arcs that have opposite orientation with respect to O is reduced by one. So this algorithm terminates, and since it terminates only when there are no arcs with opposite orientation with respect to O, we know that the knot obtained has a diagram with a center. □

Now we can prove the Alexander's theorem.

Proof. Given any knot, we can use the above theorem to modify it to a link with a center O. We consider a perpendicular axis to the plane of the diagram through the center O, and thus, the knot appears to be winding around this axis. In fact, the knot can be viewed as being contained inside a tube or solid torus T centered at O. Now we cut the torus along the ray from O through its meridian and open it out, and obtain a cylinder which clearly contains a braid. □

Now, if two braids are equivalent, their knots, are also equivalent. However, it is possible to obtain equivalent knots from the closure of nonequivalent braids. For example the following braids are not equivalent, but their closure are equivalent, to the trivial knot.
Having said that, we need to find out from which braids we can form equivalent knots. Thus, we shall introduce the concept of M-equivalence between

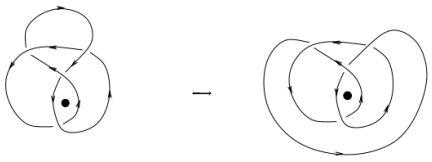


Figure 2.41: An algorithm for creating center of knots.

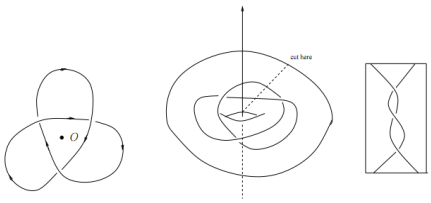


Figure 2.42: An algorithm for creating center of knots.

two braids.

Definition 2.16. Suppose that $\mathcal{B}_\infty$ is the union of the groups $\mathcal{B}_1$, $\mathcal{B}_2$, ..., $\mathcal{B}_n$, ..., i.e. $\mathcal{B}_\infty = \bigcup_{k\geq 1} \mathcal{B}_k$. We may performing two operations in $\mathcal{B}_\infty$; these operation are called the *Markov moves*:

- If β is an element of the braid group $\mathcal{B}_n$, then M_1 is the operation that transforms of β into the n-braid, $\gamma\beta\gamma^{-1}$, where γ is some element of $\mathcal{B}_n$. The element $\gamma\beta\gamma^{-1}$ is the *conjugate* of β.
- M_2 is the operation that transforms a n-braid, β, into either of the two $(n+1)$-braids, $\beta\sigma_n$ or $\beta\sigma_n^{-1}$, where σ_n is a generator of B_{n+1}, the $(n+1)$-braid group.

Definition 2.17. Suppose that α and β are elements of $\mathcal{B}_\infty$. If we can transform α into β by performing the Markov moves M_1, M_2, and their inverse M_1^{-1}, M_2^{-1} a finite number of times, then α is said to be *Markov equivalent* or *M-equivalent* to β and its denoted by $\alpha \sim_M \beta$. If also $\beta \sim_M \alpha$, then α and β are said to be Markov equivalent.

The following theorem shows that Markov equivalence is the fundamental concept that connects a knot to a braid.

Theorem 4.4. *Suppose that K_1 and K_2 are two oriented knots, which can be formed from the braids β_1 and β_2, respectively. Then*

$$K_1 \cong K_2 \Leftrightarrow \beta_1 \sim_M \beta_2.$$

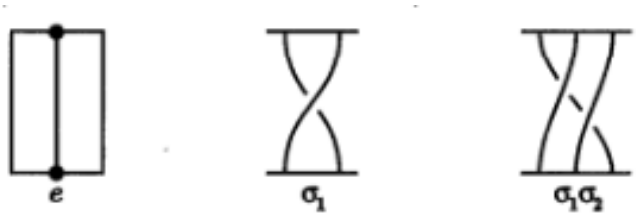


Figure 2.43: These three braids non-equivalent, but their closure is equivalent to the trivial knot.

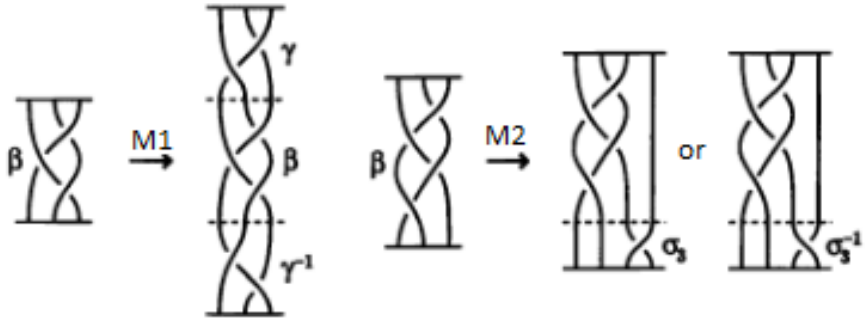


Figure 2.44: The Markov moves.

4.2 The Jones polynomial

From the previous section we have the following correspondence:

$$\text{knot} \Rightarrow \text{braid} \Rightarrow \text{braid group}\, \mathcal{B}_n$$

Suppose we can map the braid group $\mathcal{B}_n$ into some sort of algebraic system, say, A, whose structure we understand (for example, the group of invertible matrices, or more generally, an algebra such as a group ring in which the sum and product have been defined). The aim is to be able to represent an arbitrary knot by an element of A. However, we find an initial block in the correspondence

$$\text{knot} \Rightarrow \text{braid}.$$

In fact, this correspondence is not $1-1$. To a single knot we may assign an infinite number of braids. Likely, we have seen that due to Markov's theorem, each knot corresponds to only one M-equivalence class. Therefore, when a braid α corresponds to a certain value, say $\phi(\alpha)$, then if this value $\phi(\alpha)$ is the same for any other M-equivalent braid, β, it follows that this $\phi(\alpha)$ is

an invariant of the knot K_α, formed from the braid α. So, from the first condition of M-equivalence ϕ must have the same value for α and $\gamma\alpha\gamma^{-1}$. Now, if we want to represent the braid group by some algebraic structure, A, it must have a similar structure to $\mathcal{B}_n$. Further, A should have a simpler or more restricted algebraic structure than $\mathcal{B}_n$. Jones, accidentally, found that one of the algebras, which he was studying for other purpose, had a structure that resembled that of the braid group. He was able to define a function that was invariant under both the Markov moves, M_1 and M_2. This function could be written in terms of complex number q, and from this it was possible to associate to each knot a complex polynomial. This polynomial is known as the *Jones polynomial*. In this section we will study the Jones polynomial from the point of view of knot theory, leaving the perspective of operator algebra and quantum groups as application of knot theory.

Definition 2.18. Suppose K is a knot and D is a regular diagram for K. Then the Jones polynomial of K, $V_K(t)$, can be define, uniquely, from the following two axioms. The polynomial $V_K(t)$ is an invariant of K.

- If K is the trivial knot, then $V_K(t) = 1$.
- Suppose that D_+,D_- and D_0 are skein diagram (see fig.2.24), then the following skein relation holds:

$$\frac{1}{t}V_{D_+}(t) - tV_{D_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{D_0}(t). \tag{2.10}$$

The algorithm to calculate the Jones polynomial is completely analogous to the one for the Alexander-Conway polynomial. We can write out the Jones polynomial as the sum of the Jones polynomial of the trivial m-component links, $\bigcirc_m$ (the result of using the skein tree diagram),

$$V_K(t) = f_1(t)V_{\bigcirc}(t) + f_2(t)V_{\bigcirc\bigcirc}(t) + \cdots + f_m(t)V_{\bigcirc_m}(t). \tag{2.11}$$

In the Alexander case, the above expression is superfluous, since the $\nabla_{\bigcirc_m}(t) = 0$. But this is not the case for the Jones polynomial.

Theorem 4.5. *For the trivial m-component link $\bigcirc_m$,*

$$V_{\bigcirc_m}(t) = (-1)^{m-1}(\sqrt{t} + \frac{1}{\sqrt{t}})^{m-1} \tag{2.12}$$

Proof. We will proof the theorem by induction on m. If $m = 1$, then we have the axiom 1. So let us assume for our induction hypothesis that the following holds:

$$V_{\bigcirc_{m-1}}(t) = (-1)^{m-2}(\sqrt{t} + \frac{1}{\sqrt{t}})^{m-2}$$

Now consider the skein diagram as in figure, then since $D_+ \cong D_- \cong \bigcirc_{m-1}$ and $D_0 \cong \bigcirc_m$, by the above induction hypothesis and the skein relation (in the Jones definition),

$$(-1)^{m-2}(\sqrt{t} + \sqrt{\frac{1}{\sqrt{t}}})^{m-2} - t(-1)^{m-2}(\sqrt{t} + \frac{1}{\sqrt{t}})^{m-2} =$$

$$= (\sqrt{t} - \frac{1}{\sqrt{t}})V_{\bigcirc_m}(t).$$

Since the left-hand side of the above formula is

$$(-1)^{m-2}(\sqrt{t} + \frac{1}{\sqrt{t}})^{m-2}(\frac{1}{t} - t) =$$

$$= (-1)^{m-1}(\sqrt{t} + \frac{1}{\sqrt{t}})^{m-2}(\sqrt{t} + \frac{1}{\sqrt{t}})(\sqrt{t} - \frac{1}{\sqrt{t}}),$$

the result follows. □

Now, let us calculate the Jones polynomial of some knots. For helping in the calculation, let us write down the following equalities:

$$V_{D_+}(t) = t^2 V_{D_-}(t) + tzV_{D_0}(t)$$

$$V_{D_-}(t) = t^{-2}V_{D_+}(t) + t^{-1}zV_{D_0}(t)$$

where $z = (\sqrt{t} - \frac{1}{\sqrt{t}})$.

Example 9. .
The skein tree diagram of the trefoil knot is:
It follows from the skein diagram that:

$$V_K(t) = t^2 V_{\bigcirc}(t) + t^3 z V_{\bigcirc\bigcirc}(t) + t^2 z^2 V_{\bigcirc}(t) = t + t^3 - t^{-4}.$$

Example 10. *It follows from the skein diagram of the figure-eight knot that:*

$$V_K(t) = t^2 V_{\bigcirc}(t) + tz[t^{-2}V_{\bigcirc\bigcirc}(t) + t^{-1}zV_{\bigcirc}(t)] = t^2 + t^{-2} - t - t^{-1} + 1.$$

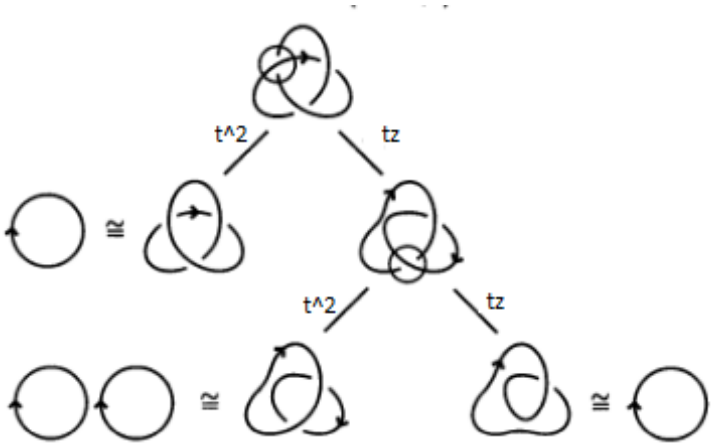


Figure 2.45: The Jones polynomial of the trefoil knot.

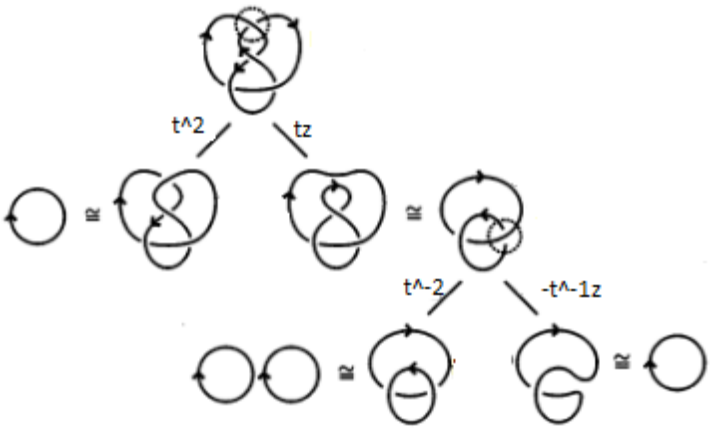


Figure 2.46: The Jones polynomial of the figure-eight knot.

The Jones polynomial is a powerful tool in the study of the achirality of a knot.

Theorem 4.6. *Suppose $K*$ is the mirror image of a knot K, then*

$$V_{K^*}(t) = V_K(t^{-1}).$$

Therefore , if a knot K is achiral, then $V_K(t) = V_K(t^{-1})$.

Proof. Suppose D is a diagram of K and D^* is its mirror image. If the skein diagram of D is, say, R, then we can form the skein tree diagram of D^*, R^* as follows: when we perform a skein operation at a crossing point, say c, of D to make R, at the equivalent crossing point of D^* also perform a skein

operation, so forming R^*. Since the signs of the crossing point c of D and the equivalent crossing point in D^* are opposite, the coefficients assigned at this juncture to R and R^* differ. If at a certain segment α of R we assigned t^2 (or t^{-2}), then at the equivalent segment, α^*. On the other hand if we have assigned tz (or $-t^{-1}z$), then equivalently at α^* we assign $-t^{-1}z$ (or tz). Since this change

$$t^2 \rightleftharpoons t^{-2} \qquad \text{and} \qquad tz \rightleftharpoons -t^{-1}z$$

is nothing but the replacement of t by t^{-1}, it follows that $V_{K^*}(t) = V_K(t^{-1})$ □

The Jones polynomial of the right-hand trefoil knot K is $V_K(t) = t + t^3 - t^4$, and since is not symmetric by the above theorem, K is not achiral. Instead, the figure-eight knot is achiral, in fact its Jones polynomial is $t^2 + t^{-2} - t - t{-}1 + 1$ that is symmetric. Although the Jones polynomial is a strong invariant, it is not a complete invariant. That is to say, there exist an infinite number of non-equivalent knots that have the same Jones polynomial.

Example 11. *The Jones polynomial of the two knots K_1 and K_2 (in fig.2.47) are the same, namely, are equal to $(t^{-2} - t^{-1} + 1 - t + t^2)^2$. However, their Alexander polynomials are respectively:*

$$\Delta_{K_1}(t) = (t^{-1} - 3 + t)^2$$

$$\Delta_{K_2}(t) = -t^{-3} + 3t^{-2} - 5t^{-1} + 7 - 5t + 3t^2 - t^3$$

Therefore, K_1 and K_2 are not equivalent.

Figure 2.47: Nonequivalent knots with the same Jones Polynomial.

This example shows that even though there are cases when knots can be distinguished by the Jones polynomial and not by the Alexander polynomial, the reverse is also true.

5 The Kauffman polynomial

In this section we show how another approach to find a knot invariant is to show that it remains unchanged under the Reidemeister moves. Hence, we must investigate in what way all three Reidemeister moves affect the function. For example, we know that if we apply ω_3 or ω_3^{-1}, then the number of crossing points of the diagram D remains unchanged.

Definition 2.19. Suppose that D is an oriented diagram of an oriented knot, then the *writhe number* (or *Tait number*), $w(D)$, is the sum of the signs of all the crossing points of D.

Also the writhe number remains unchanged when we apply the ω_2 or ω_3, or their inverse. However the writhe number is not itself a knot invariant, therefore a further consideration would be to check how far we are from a knot invariant if we restrict ourselves to just to ω_2 or ω_3 and its inverse. The main result is due to L. Kauffman.

Definition 2.20. Let us call the Reidemeister move ω_2 or ω_3 and their inverse *regular moves*. Then, if we can obtain a diagram D' by applying these moves a finite number of times to a regular diagram D of some knot, we say that D and D' are *regular equivalent*.

Theorem 5.1 (Kauffmann's principle)**.** *Suppose that a function, f, with indeterminate t is invariant under the regular moves. If we choose m suitably, then $t^m f$ is an invariant of knots.*

The Kauffman bracket polynomial is an example of the use of this theorem. Suppose K is an unoriented knot and D is its diagram. Splice each crossing point of D in the two ways showed in the figure 2.48:

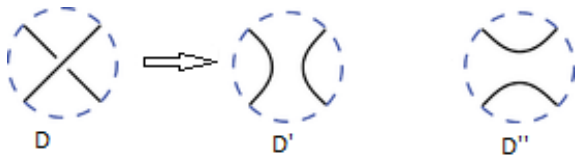


Figure 2.48: Splicing of a knot.

In order to define the Kauffman bracket polynomial we will use this process of splicing a knot.

Theorem 5.2. *Let D be an unoriented regular diagram o a knot or link K. Then there exists a unique one-variable integer polynomial $P_D(A)$ that satisfies the following conditions:*

- *$P_D(A)$ is invariant under regular equivalence.*
- *If D is the trivial projection ($\bigcirc$) of the trivial knot, then*

$$P_{\bigcirc}(A) = 1. \tag{2.13}$$

- *If D consists of two split diagram D_1, D_2, then*

$$P_D(A) = -(A^2 + A^{-2})P_{D_1}(A)P_{D_2}(A). \tag{2.14}$$

- *Let D, D' and D'' be the skein diagrams given in the figure. Then the following equality holds:*

$$P_D(A) = AP_{D'}(A) + A^{-1}P_{D''}(A) \tag{2.15}$$

$P_D(A)$ is called the *Kauffman's bracket polynomial* Is important to understand that this polynomial is defined on the knot diagram D. For example, the first condition does not mean that $P_K(A) = 1$, where K is the trivial knot. Consider the trivial knot T as in the figure. In order to find $P_T(A) = -(A^{-3})$,

Figure 2.49: Two equivalent trivial knot.

we need to use the third condition to evaluate. Therefore, $P_D(A)$ is not invariant under the first Reidemeister move, ω_1. However, according to the Kauffman's principle, it is possible to define an invariant from $P_D(A)$ that is also invariant under ω_1.

Theorem 5.3. *Suppose D is an oriented diagram of an oriented knot K. If $P_D(A)$ is the Kauffman polynomial of the unoriented projection D, and $w(D)$ is the writhe number of D, then define*

$$\hat{P}_D(A) = (-A^{-3})^{w(D)} P_D(A) \tag{2.16}$$

Then $\hat{P}_D(A)$ is an invariant of an oriented knot, denoted by $\hat{P}_K(A)$.

Proof. Suppose that D' is a diagram of K that has been obtained by performing a single Reidemeister move on D. Then it is enough to show that $\hat{P}_D(A) = \hat{P}_{D'}(A)$. Firstly let us suppose that D' has been obtained by performing ω_2, ω_2 or their inverse. Then D and D' are regular equivalent and so by theorem it follows that $P_D(A) = P'_D(A)$. Now, also the writhe number is invariant under ω_2 and ω_3, in fact the third moves does not effect the number of crossings, while we apply ω_2, we are adding just a negative and a positive crossings, that is does not affect the writhe number. It follows that:

$$\hat{P}_D(A) = \hat{P}'_D(A)$$

. This leave the case of D' obtained by applying ω_1 or its inverse. Since D'

Figure 2.50: Applying the first Reidemeister move to a knot.

has an extra crossing point, to evaluate $P_{D'}(A)$ we need to use the following skein tree diagram.
Then, by the formulas , we have

$$P_{D'}(A) = AP_D(A) + A^{-1}P_D(A)(-(A^2 + A^{-2})) = -A^{-3}P_D(A).$$

Irrespectively of how we assign the orientation to D, the sign of the new crossing is -1. Therefore, $w(D') = w(D) - 1$. This fact allows us to say:

$$\hat{P}_{D'}(A) = (-A^{-3})^{w(D')} P_{D'}(A) = (-A^{-3})^{w(D)} P_D(A) = \hat{P}_D(A).$$

□

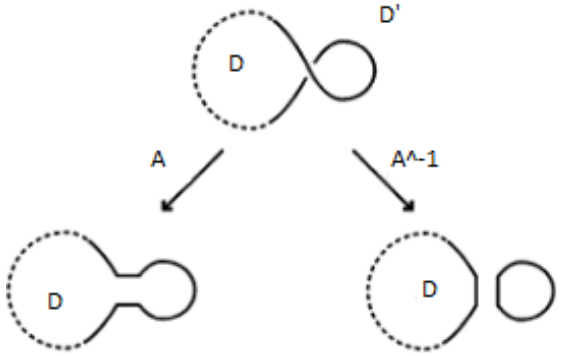


Figure 2.51: Skein tree diagram of the unknot.

Remark 2. If we substitute $A = t^{-\frac{1}{4}}$, then $\hat{P}_K(A)$ coincides with the Jones polynomial $V_K(t)$ of K.

Example 12. *Consider the following skein diagram of the Hopf link, say L.*

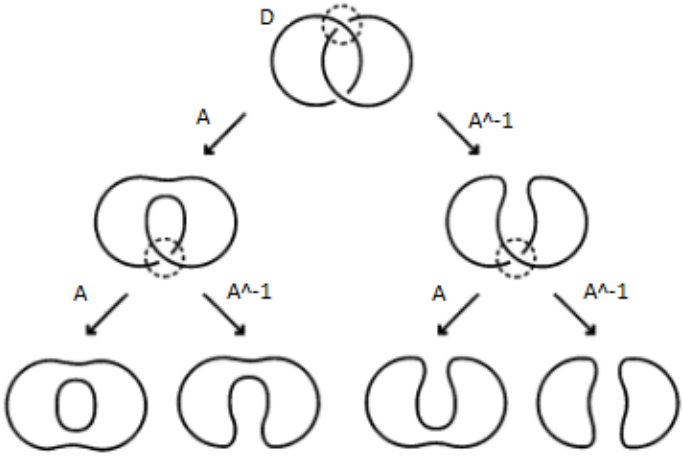


Figure 2.52: Skein tree diagram of the Hopf link.

Now, let us evaluate the Kauffman bracket polynomial, reading the coefficient from the skein tree diagram.

$$P_D(A) = A^2(-(A^2 + A^{-2})) + 1 + 1 + A^{-2}(-(A^2 + A^{-2})) = -A^4 - A^{-4}.$$

Since $w(D) = 2$, *we have*

$$\hat{P}_D(A) = (-A^{-3})^2(-A^4 - A^{-4}) = -A^{-2} - A^{-10}.$$

Therefore $\hat{P}_D(t^{-\frac{1}{4}}) = -t^{\frac{1}{2}} - t^{\frac{5}{2}}$ *which is the same as the Jones polynomial of* L.

Example 13. *Consider the skein relation diagram of the trefoil knot. Then, evaluating the Kauffman bracket polynomial we get:*

$$P_D(A) = A(-(A^4 + A^{-4})) + A^{-2}(-A^2 - A^{-2})(-A^{-3}) - A^{-3} =$$

$$-A^5 - A^{-3} + (-1 - A^{-4})(-A^{-3}) - A^{-3} =$$

$$A^{-7} - A^5 - A^{-3}.$$

Since $w(D) = 3$*, we have*

$$\hat{P}_D(A) = (-A^{-3})^3(A^{-7} - A^5 - A^{-3}) = A^{-4} + A^{-12} - A^{-16}.$$

Chapter 3

Knot theory: Application

1 Knot theory in chemistry

In this chapter the different application of knot theory are introduced. First, let us have a look of what means molecular chirality and how the knot theory can be useful in determining when a molecular is chiral or not. Later on we will see how the Yang-Baxter equation is related to the braids group, and so with the knots. In the end another important knot theory's application is given, this time in biology. In fact, it was discovered that the DNA-recombination can be described as a mathematical model, know as tangles model.

1.1 The Molecular chirality

> How would you like to live in Looking-glass House, Kitty? I wonder if they'd give you milk, there? Perhaps Looking-glass milk isn't good to drink...
>
> Lewis Carroll, *Through the looking-glass*

One of the most important characteristics of a knot is its *chirality*. During the years, all the knots theorists have tried to find a way to determinate it. As we said previously, the Jones polynomial was the key for this problem.

The word *chiral* derives from the ancient greek word $\chi\epsilon\iota\rho$ which means "hand". Since one hand is always different from its mirror image, this word is used to indicate an object who cannot be superposable onto its image in the mirror. The chirality is really important in chemistry, since a molecule and its mirror image can have different properties. Two mirror images of a chiral molecule are called *enantiomers*.

An example is the *limonene*, $C_{10}H_{16}$. The L-limonene (L stands for *levo*, the left-handed enantiomer) has a harsh lemon scent, while the D-limonene (D stands for *dextro* the right-handed enantiomer) has a more pleasing orange scent.
In the pharmaceutical industry this matter can make the difference. A chiral drug could have two different effects: one of the enantiomer produces the desired effect while the other one has a side effect that could outweigh the benefit of the wanted effect. In 1960's the drug Thalidomide was given to pregnant women to avoid the morning sickness. It causes, besides the reducing of sickness, also birth defect. This happens because, while the L-Thalidomide was making the mother feeling better, the D-Thalidomide was causing damage to the fetus. Therefore if a drug is known to be chiral, the pharmaceutical industry can choose to manufacture just the preferred form in order to minimize the side effect and maximize effectiveness. This is costly since most methods of molecular synthesis do not distingue between different chirality. Sadly, as the Thalidomide, in most of the cases the wrong chirality produces negative effect on human body.

Definition 3.1. A molecule is said to be *chemically achiral* if it can be changed into its mirror image. Otherwise is said to be *chemically chiral*.

This is the definition given in chemistry. In mathematics there are other two definitions, *geometrical chirality* and *topological chirality* according to the characteristics of the molecule, respectively rigid or flexible.

Definition 3.2. A molecule is said to be *geometrical achiral* if, as a rigid object, can be superposable on its mirror image. Otherwise is said to be *geometric chiral*.

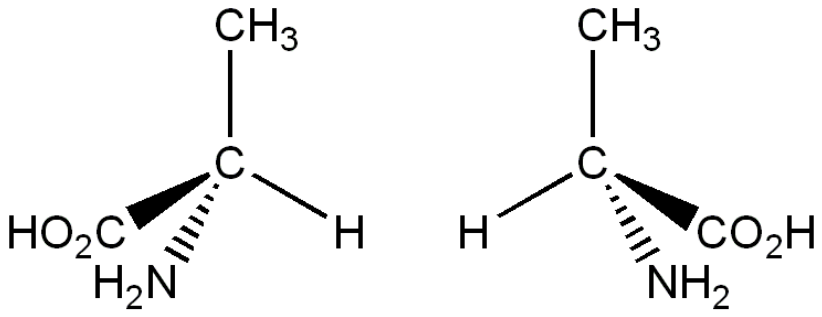


Figure 3.1: A chiral molecule.

Obviously the geometric achirality implies the chemical one and chemical chirality implies the geometric chirality. But is not true that the two definitions coincide.

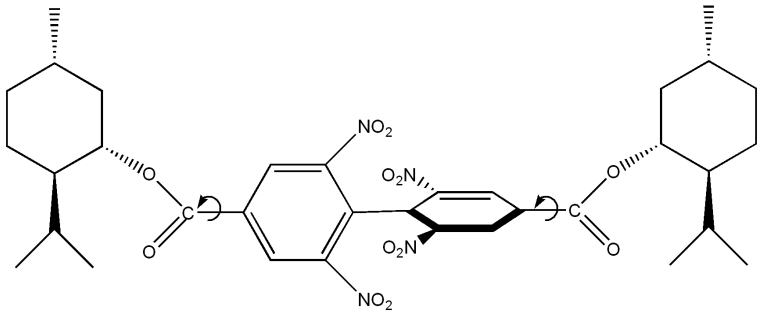


Figure 3.2: Example of a molecule chemically achiral but geometrically chiral. This molecule was synthesized for the first times by Mislow in 1954.

Now consider a right-handed rubber glove. It is easy to imagine that, taken a left-handed rubber glove and turning it inside out, we obtain again the right-handed glove. Note that at no point of this turning inside out, the glove become superposable to its mirror image. That is, is not geometrical achiral. From this we get another definition.

Definition 3.3. A molecule is said to be a *Euclidean rubber glove* if is chemically achiral but it cannot attain a position which can be rigidly superimposed to its mirror image.

According to this definition, the molecule in fig.3.2 is our Euclidean rubber glove as it's chemical achiral but at no stage of the rotation it is superimposed to its mirror image. But if the molecule was completely flexible it could lie

in the plane and become superimposed to its mirror image. Similarly for the rubber glove. However the natural characteristic of this object does not allow a completely flexibility. Since the possibility that a completely flexible molecule exists another definition is required.

Definition 3.4. A molecule is said to be a *topologically rubber glove* if is chemically achiral but it cannot be deformed to a position in which is superimposed to its mirror image.

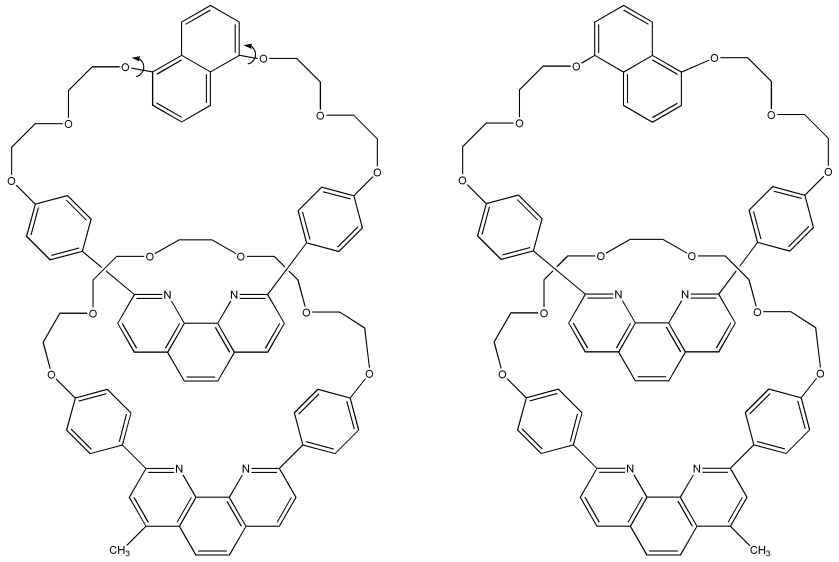


Figure 3.3: This molecule and its mirror image are example of topological rubber glove molecule.

Summarizing, the Euclidean and topological rubber gloves are examples of molecules that are chemically achiral but not geometrically achiral.

Since the geometrical chirality considers molecules as rigid objects, and in nature molecules are not rigid objects it is worth to consider the case of molecules as completely flexible objects.

Definition 3.5. A molecule is said to be *topologically achiral* if, assuming complete flexibility, there is a deformation which takes it to its mirror image. Otherwise it is said to be *topologically chiral*.

If a molecule is topologically chiral no deformation can be made to make it become its mirror image, indeed is also chemically chiral. Otherwise if it is chemical achiral, then it is also topologically achiral.
The following scheme summarizes the relationship between the previous three definition:

topologically chirality $\rightarrow$ *chemical chirality* $\rightarrow$ *geometrical chirality.*

geometrical achirality $\rightarrow$ *chemical achirality* $\rightarrow$ *topologically achirality.*

1.2 Graphs

Let V be a finite set, and denote by

$$E(V) = \{\{u, v\} | u, v \in V, u \neq v\}$$

. the subsets of V with two distinct elements.

Definition 3.6. A pair $G = (V, E)$ with $E = E(V)$ is called a *graph* (on V). The elements of V are the *vertices* of G, and those of E the *edges* of G. The vertex set of a graph G is denoted by V_G and its edge set by E_G. Therefore $G = (V_G, E_G)$.

Sometimes graphs are also called *simple graphs*; vertices are called *nodes* or *points*; edges are called *lines* or *links*. A pair $\{u, v\}$ is usually written simply as uv. Notice that then $uv = vu$. In order to simplify notations, we also write v_G and e_G instead of $v \in V_G$ and $e \in E_G$.
For a graph G, we denote

$$\nu_G = |V_G| \qquad \text{and} \qquad \epsilon_G = |E_G|.$$

The number ν_G of the vertices is called the *order* of G, and ϵ_G is called the *size* of G. For an edge $e = uv \in G$, the vertices u and v are its *ends*. Vertices u and v are *adjacent* or *neighbors*, if $uv \in G$. Two edges $e_1 = uv$ and $e_2 = uw$ having a common end, are adjacent with each other.
A graph G can be represented as a plane figure by drawing a line (or a curve) between the points u and v (representing vertices) if $e = uv$ is an edge of G. The figure 3.4 is a geometric representation of the graph G with $V_G = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E_G = \{v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_5v_6\}$.

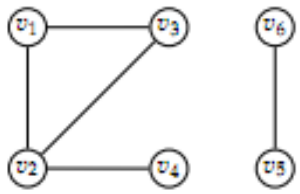


Figure 3.4: Geometric representation of a graph.

Definition 3.7. Two graphs G and H are *homeomorphic*, denoted by $G \cong H$, if there exists a bijection $\alpha : V_G \longrightarrow V_H$ such that

$$uv \in E_G \Leftrightarrow a(u)a(v) \in E_H$$

for all $u, v \in G$.

Hence G and H are homeomorphic if the vertices of one graph can be rearranged so that the two graph are identical, ignoring the labels of the vertices.

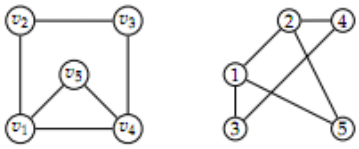


Figure 3.5: The following graphs are isomorphic. Indeed, the required isomorphism is given by $v_1 \to 1, v_2 \to 3, v_3 \to 4, v_4 \to 2, v_5 \to 5$.

Definition 3.8.

- An *automorphism* of a graph is a permutation of the vertices of the graphs that takes adjacent vertices to adjacent vertices. For a molecular graph is also required that an automorphism takes atoms of given type to atoms to the same type.
- The *order* of an automorphism is the smallest number of time an automorphism needs to be performed to get every vertices back to its original position.
- The *valence* of a vertex in a graph is the number of edge that contain it.
- The *distance* between the vertex in a graph is the fewest number of edge contained in a path from one to the other.

Note some properties of automorphisms:

1. An automorphism takes a vertex of a particular valence to a vertex of the same valence:
2. An automorphism takes a pair of vertices which are a certain distance apart in the graph to a pair of vertices which are the same distance apart.

Definition 3.9. A *planar graph* is a graph that can be embedded in the plane, i.e. it can be drawn in the plane with its edges only intersecting at vertices of G.

The following are planar graphs:

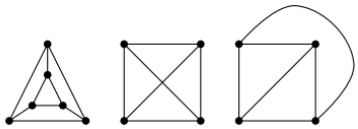


Figure 3.6: Examples of planar graphs.

Definition 3.10. A *complete graph* is a graph in which every pair of distinct vertices is connected by a unique edge. We denote it by K_n.

Theorem 1.1. *The complete graph K_n is planar for $n = 1, 2, 3, 4$.*

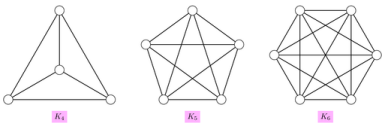


Figure 3.7: Complete graph K_n, $n = 1, 2, 3, 4$..

Theorem 1.2. *The complete graph K_5 is non-planar.*

Proof. We attempt to draw K_5 in the plane. We first start with a pentagon as in fig. 3.8
A complete graph contains an edge between every pair of vertices, so there is an edge between a and c.(see fig. 3.9 This may as well be inside the pentagon (as if it is outside then we just adjust the following argument appropriately).

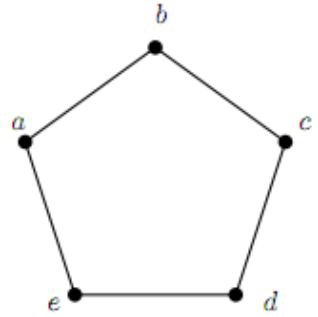


Figure 3.8: A pentagon

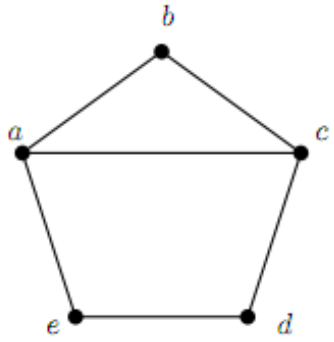


Figure 3.9: Make an edge between a and c.

Now we add the edge between b and e (this must be outside the pentagon as it cannot cross a, c), the edge between a and d (inside so as to not cross b, e), and then between c and e as in fig.3.10. (outside so as to not cross a, d).
All these edges were forced into position and we have no choice. It remains to add an edge between b and d. We cannot add it inside (since it would cross a, c) nor can we add it outside (since it would cross c, e). Consequently K_5 is non-planar. □

Definition 3.11. A graph $G = (V, E)$ is called *bipartite* if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \varnothing$ and every edge $e = ab \in G$ is such that one of the vertices a and b is in V_1 and the other in V_2.

If every vertex in V_1 is joined to every vertex in V_2 we obtain a *complete bipartite graph*. We write $K_{m,n}$ for the complete bipartite graph with $|V_1| = m$ and $|V_2| = n$. Here $|E| = mn$.

Theorem 1.3. *The complete bipartite graph $K_{3,3}$ is non-planar.*

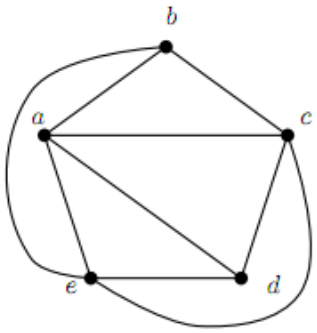


Figure 3.10: The edge betwenn b and d cannot be added it inside (since it would cross a, c) nor can we add it outside (since it would cross c, e).

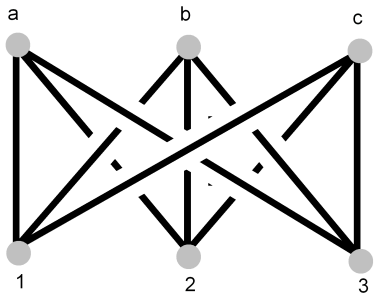
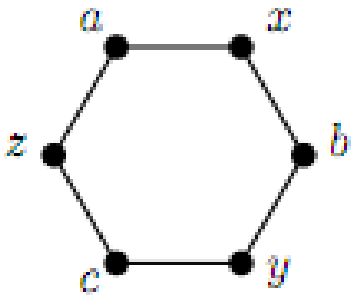


Figure 3.11: The complete bipartite graph $K_{3,3}$

Proof. Let $V_1 = a, b, c$ and $V_2 = x, y, z$ and draw a hexagonal circuit: $a \to x \to b \to y \to c \to z \to a$



The proof is completed by observing that two of the edges a, y, b, z or c, x must both lie inside or both outside the hexagon and hence must cross. □

Theorem 1.4. *(Kuratowski 1930) A graph is non-planar if and only if it contains a subgraph that is homeomorphic to either K_5 or $K_{3,3}$.*

This result is fundamental in graph theory since it allows us to see whether or

not a graph can be embedded in a plane. Furthermore this theorem will allow us to translate the problem about the topology of how graph are embedded in $\mathbb{R}^{\not\#}$ into a problem of abstract graphs.

1.3 Establishing the topological chirality of a molecule

The following mathematical methods are used to find out if a molecule is topological chiral or not.

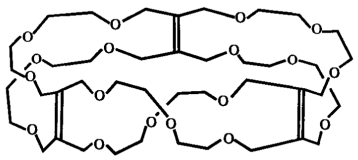


Figure 3.12: A molecular Möbius ladder.

Method 1: Knot polynomial . This method can be used when a molecule is knotted. As we have seen in the previous chapter when we described history of knot theory, different theorists tried to find ways to distinguish whether a knot is chiral or not. The Jones polynomial is the only method which can actually determinate such difference. Indeed every chiral knot and its mirror image have different Jones polynomials. However, if the Jones polynomial is the same it doesn't necessarily mean that the knot is achiral. That is, the Jones polynomial is useful to establish topological chirality, but not for proving topological achirality. The only disadvantage of this method is that not all the molecules are knotted. For example the Möbius ladder (fig.3.12) doesn't contain any link or knot. So another method is needed in order to establish if this molecule is topologically chiral.

Method 2: 2-fold cover This method was first used to prove that the Möbius ladder is topologically chiral.

Definition 3.12. Let G be a graph. Another graph C is said to be a covering graph for G if there is a covering map f from the vertices of C to the vertex of G that is surjective and a local isomorphism, that is for every $v \in V_C$ the restriction of f to a neighborhood of v is a bijection

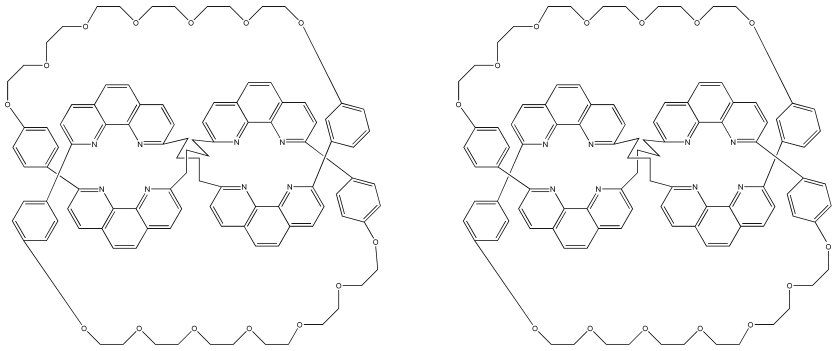


Figure 3.13: The two molecules have respectively Jones polynomial $t+t^3-t^4$ and $t^{-1}+t^{-3}-t^{-4}$. Note that the two polynomials are equal, except that the signs of the exponents are changed from positive to negative.

onto the neighborhood of V_G. In particular a 2-fold cover is when each verteces of G has exactly two preimages.

Consider the molecule as in fig.3.14 and color with different colors the three rings. Then unknot the two central circle into a single one. It will give a single figure as in 3.14.

Now consider the 2-fold cover as in 3.15. Eliminating the black circle in the middle (this is possible for chemical reason) we obtain a link. Using the linking number is now easy to prove that the Möbius ladder is topologically chiral.

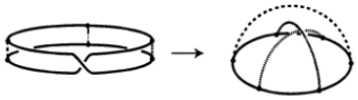


Figure 3.14: We color the rings of a Möbius ladder and then deform the sides of the ladder into a planar circle.

Method 3: Using another chiral molecules . The idea of this method is to show, that if some particular molecule is topologically chiral then also another molecule containing it is topologically chiral. This is really useful especially when a molecule contains the Möbius ladder. It is shown in the fig.3.17.

If the TLN were to be deformed to its mirror image, then also the graph on the right in fig.3.17 could. But that graph can be deformed to the

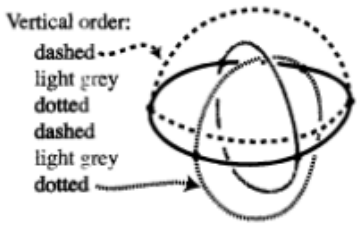


Figure 3.15: The 2-fold branched cover of a Möbius ladder, branched over the side of the ladder. The vertical descending order of the rungs is: dashed, light grey, dotted, dashed, light grey, dotted.

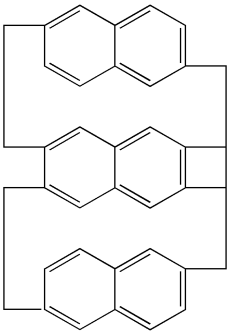


Figure 3.16: Triple layered naphthalenophane (TLN).

Möbius ladder. Hence it follows that the TLN is chiral.

Method 4: Combinatorics The center idea of this method is to translate the question of showing if a molecular is topologically chiral to how an abstract graph is embedded in a plane. We already discuss this issue in the previous section. The method can be summarized in the following theorem:

Theorem 1.5. *If a graph contains either K_5 or $K_{3,3}$ (fig.3.18) and has no order 2 automorphism, then any embedding of this graph in $\mathbb{R}^{\not\vdash}$ is topologically chiral.*

The ferrocenophane molecule gives a perfect example of this method.

In this molecule all the atoms are carbons, except for the iron in the center and the oxygen at the bottom. Every automorphism takes atoms of some type to atoms of the same type, then every automorphism in

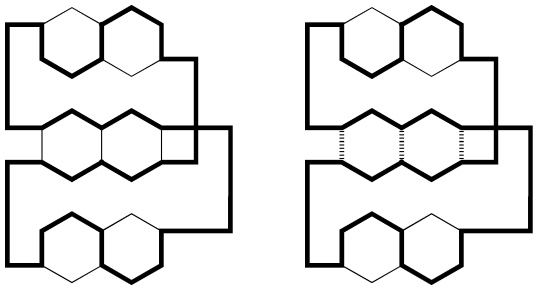


Figure 3.17: The TLN contains a long circle C, and only three segment with endpoints on C.

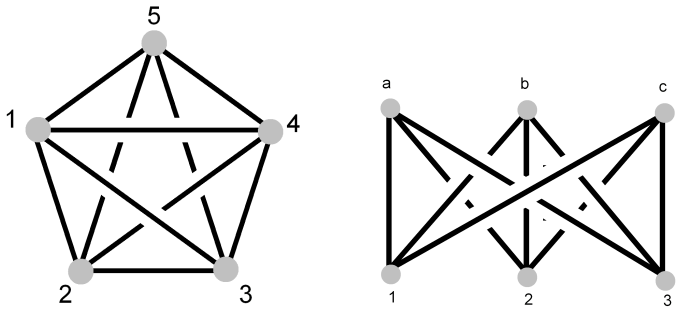


Figure 3.18: K_5 and $K_{3,3}$.

this graph has to fix the iron atom and the oxygen. Note that the oxygen atom is fixed, it follows that also the carbon attached to the oxygen has to be fixed by any automorphism. Since valence and distance must be preserved, also the two adjacent carbons to that fixed carbon must must be fixed. Continuing in this way we obtain that the only automorphism is the identity.

Also notice that K_5 is contained in this graph.

This proves, by the previous theorem, that this molecule is topologically chiral.

The theorem 3.2.1 states that every embedding in $\mathbb{R}^3$ is topologically chiral. But of course it is not always like this, in particular, an embedding of a molecular graph in $\mathbb{R}^3$ is not necessary a molecular graph. This suggests the following definition.

Definition 3.13. A molecular graph is said to be *intrinsically chiral* if every embedding in $\mathbb{R}^3$ is topologically chiral.

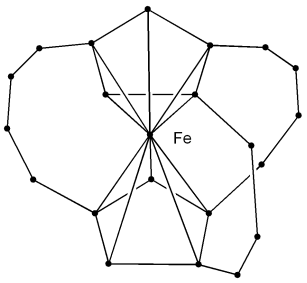


Figure 3.19: The ferrocenophane molecule.

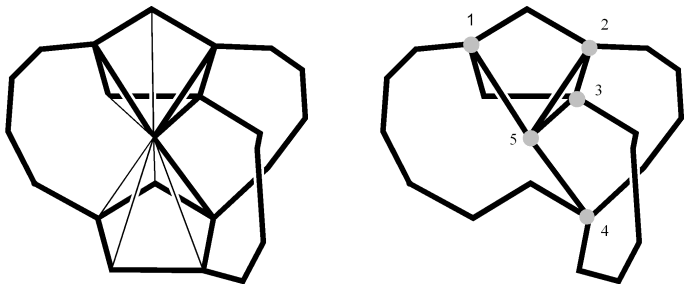


Figure 3.20: K_5 is contained in the ferrocenophane.

The adjective "intrinsically" shows that the chirality is intrinsic in the graph itself and does not depend on how the graph is embedded in $\mathbb{R}^3$.
Different embeddings of the same molecule are called *stereoisomers*.

2 Knots and Physics

Statistic mechanics is a branch of physics that tries to understand, using statistical methods, macroscopic properties by looking at microscopic properties. The easiest example is trying to understand what happens to water in a kettle when we bring it to a boil by looking at how the molecules interact. Statistical mechanics together with quantum mechanics formed the base for the study of physics of matter. In general, the constituent of matter, even if we assume they obey the principles of dynamics, have extremely complicated means of motion, so modeling these motion is the most difficult problem in physics. In order to find a reasonable successful method around this problem, an ideal realization of matter, a model, has been formed. In order to give a mathematical meaning to these realization we define a function Z called the *partition function*,

$$Z = \sum_{\sigma} \exp(\frac{-E(\sigma)}{kT})$$

in which σ is a state of the particular model, $E(\sigma)$ is the total energy of this state, T is the absolute temperature and k is the Boltzmann's constant. If the partition function of a model can be derived exactly, then this model is said to be *exactly solvable*. The idea of the statistical mechanical approach to knot invariants is to construct lattice models whose partition functions are taken to be knot invariants. Specifically, starting from a given knot projection, we can construct a two-dimensional lattice and a lattice model on this lattice. If model parameters are chosen such that the partition function of the model remains invariant when the knot (and the lattice) is deformed, the partition function is a knot invariant by its very definition.

There are generally three different kinds of lattice models: The vertex models, the closely related IRF (interaction round a face) models and the spin models. The firsts two models are the most general and have been widely studied in connection with quantum groups. Instead very few knot invariants seem to admit a spin model description, and no clear connection with quantum groups is known in general.

2.1 The Yang Baxter equation and knots invariants

A central role in the theory of exactly solvable models is the *Yang Baxter equation*(YBE). This equation, appeared for the first time independently in this paper by C.N. Yang and R.J. Baxter in the late 1960's, early 1970's. The YBE represents a sufficient condition for a statistical mechanical model to be solvable.

Notation. In what that follows, let Q be an arbitrary ring, $N \geq 2$ a positive integer and R an $N^2 \times N^2$ invertible matrix on Q. We may denote R as $\| R_{i,j}^{k,l} \|$, where $(i,j),(k,l)$ are chosen from the N^2 sets of pairs

$$(1,1),(1,2),\ldots,(1,N),(2,1),\ldots,(N,N)$$

and (i,j) represents the appropriate row of R while (k,l) the appropriate column of R.

Now we can use the matrix R in order to form $r-1$ $N^2 \times N^2$ matrices $R_i(r)$ $(i = 1,2,\ldots,r-1)$, where r is a positive integer, in the following way:

$$R_i = \underbrace{I \otimes \ldots I}_{(i-1)\,times} \otimes R \otimes \underbrace{I \otimes \cdots \otimes I}_{(r-i-1) \quad times}$$

where I is the identity matrix. The tensor product of two matrices $A \otimes B$ is defined as follows. Suppose A is a $p \times p$ matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1p} \\ a_{21} & a_{22} & \ldots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \ldots & a_{pp} \end{bmatrix}$$

and similarly let B be a $q \times q$ matrix. Then $A \otimes B$ is a $pq \times pq$ matrix of the form

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \ldots & a_{1p}B \\ a_{21}B & a_{22}B & \ldots & a_{2p}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1}B & a_{p2}B & \ldots & a_{pp}B \end{bmatrix}.$$

Definition 3.14. For every $i = 1,2,\ldots,r-2$ if $R_i(r)$ satisfies the following condition, called the *Yang-Baxter equation*

$$R_i(r)R_{i-1}(r)R_i(r) = R_{i+1}(r)R_i(r)R_{i+1}(r)$$

then $R_1(r), R_2(r), \ldots, R_{r-1}(r)$ are called the *Yang-Baxter operators.*

Definition 3.15. Let $a, b \in Q*$. Let μ be a $N^2 \times N^2$ diagonal matrix with trace $\mu_1, \mu_2, \ldots, \mu_N$. A set $\{R, \mu, a, b\}$ is called an *enhanced Yang Baxter operators*[19] if it satisfies:

- For any $i, j, k, l \in 1, \ldots, N$, $(\mu_i\mu_j - \mu_j\mu_i)R^{k,l}_{i,j} = 0$
- 1. $\sum_{j=1}^{N} R^{k,l}_{i,j}\mu_j = ab\delta^k_i$
  2. $\sum_{j=1}^{N} R^{-1}{}^{k,l}_{i,j}\mu_j = a^{-1}b\delta^k_i$

(Here δ^k_i is the Kronecker symbol: $\delta^k_i = 1, \delta^k_i = 0$ for $k \neq j$)

$\mu \otimes \mu$ commute with R if and only if $\mu \otimes \mu$ commute with R^{-1}. Therefore, the first condition in def.3.18 implies that for i, j, k, l $\mu_i\mu_j - \mu_k\mu_l)R^{-1}{}^{k,l}_{i,j} = 0$. The secondo condition implies that the product of the square matrix $R^{k,l}_{i,j}$ with the column

$$\begin{bmatrix} \mu_1 \\ \ldots \\ \mu_N \end{bmatrix}$$

is equal to the constant column

$$\begin{bmatrix} ab \\ \ldots \\ ab \end{bmatrix}$$

Recall that the braid group $\mathcal{B}_r$ (see chapter 2.4.1) has the following presentation:

$$\mathcal{B}_r = \left(\sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{cc} \sigma_i\sigma_j = \sigma_j\sigma_i & (|i-j| \geq 2) \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & (i = 1, 2, \ldots, n-2) \end{array} \right)$$

Notice that, by definition

$$R_i(r)R_j(r) = R_j(r)R_i(r) \qquad for \qquad |i - | \geq 2.$$

Therefore, there is a unique homomorphism

$$b_R : \mathcal{B}_n \to M_{N^2 \times N^2}$$

which transforms $\sigma_i \in \mathcal{B}_r$ into R_i for all i. Then, every element of $\mathcal{B}_r$, say $\beta = \sigma^{\epsilon_1}_{j_1}(r)\sigma^{\epsilon_2}_{j_2}(r)\ldots\sigma^{\epsilon_m}_{j_m}(r)$, can be represented by an $N^r \times N^r$ matrix, i.e.

$$b_R(\beta) = R^{\epsilon_1}_{j_1}(r)R^{\epsilon_2}_{j_2}(r)\ldots R^{\epsilon_m}_{j_m}(r).$$

Every enhanced Yang-Baxter operator $S = \{R, \mu, a, b\}$ determines a mapping $T_S : \coprod_{r\geq 1} \mathcal{B}_r \to Q$ as follows: Denote the $N^r \times N^r$ matrix $\underbrace{\mu \otimes \mu \otimes \cdots \otimes \mu}_{\text{r times}}$ as $\mu^{(r)}$. And let w be the homomorphism from $\mathcal{B}_r$ to the additive group of integers, which sends $\sigma_1, \sigma_2, \ldots, \sigma_{r-1}$ to 1. Then, for a braid β:

$$T_S = a^{-w(\beta)} b^{-r} tr(b_R(\beta)\mu^{(r)})$$

Theorem 2.1. *For any* $\beta, \gamma \in \mathcal{B}_r$

$$T_S(\beta^{-1}\gamma\beta) = T_S(\gamma\sigma_r) = T_S(\gamma\sigma_r^{-1}) = T_S(\gamma)$$

Now, in order to find a knot invariant, recall briefly the relationship between braids and knots.(see chap.2). The Alexander's theorem (2.4.1) asserts that any oriented link is isotopic to the closure of some braid. A theorem of A. Markov (2.4.3) asserts that the closures of two braids are isotopic (and so also their respective closure, i.e. knots) if and only if these braids are equivalent with respect to the equivalence relation generated by Markov moves $\gamma \to \beta^{-1}\gamma\beta$ and $\gamma \to \gamma\sigma_r^{\pm 1}$.
By the previous theorem we know that T_S induces a map from the set of isotopic classes of knots into Q. Therefore T_S is a knot invariant. Let us denote T_S as T_K. Also, if if $T_\bigcirc = b^{-1}tr(\mu)$ is not zero, we can normalize T_K:

$$\hat{T_K} = \frac{T_K}{T_\bigcirc}.$$

3 Knots and biology

The relationship between mathematics and DNA began in the 1950s when F. H. C. Crick and J. D. Watson unraveled the basic structure of DNA. The DNA is made of two linear strands intertwined in the form a double helix.

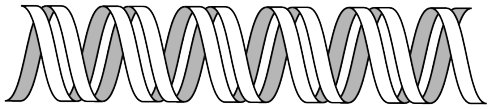


Figure 3.21: The DNA.

Anyway this is not the only shape it can have. Indeed it may take the form of a ring, and so it can become tangled or knotted. Furtherly a piece of

DNA can break temporarily and then recombine. In fact, in the 1970's it was discovered that a single enzyme called a *topoisomerase* can facilitate this process, from the initial break to the recombination. And in particular the DNA recombination model is well described by knot theory. De Witt Sumners was the first to introduce such a mathematical model, the *tangle model* for site-specific recombination.

3.1 Tangles and 4-Plats

A 2-string tangle is a pair (B, t), where B is a 3-ball and t is a pair of strings properly embedded in B so that the end points of the arcs go to a specific set of 4 points on the equator of the ball. A tangle diagram is the projection of the tangle on the plane of the equator as in fig. 3.22. We will label the endpoints in the diagram NW, NE, SW, SE.

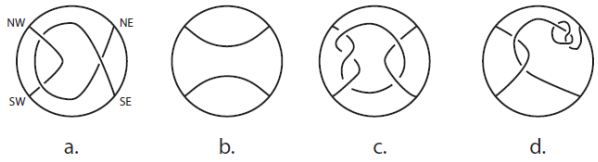


Figure 3.22: Example of tangles: a. rational, b. trivial, c. prime, d.locally knotted

Rational tangles are defined as the family of tangles that can be transformed into the trivial tangle by a sequence of twisting of the endpoints. Other examples are the locally knotted tangle, that is the ones which contain a knotted strand. While tangles which are not rational or locally knotted are said to be *prime*.

Given two tangles A and B, the tangle addition $A+B$ is defined in the figure below. The resulting object A + B is obtained by gluing NE of A to NW of B, and SE of A to SW of B. Note that the sum of two tangles is not always a tangle since the strands of (A + B) can include a simple closed curve.

Given a tangle A, others two operations are defined: the *numerator* and *denominator*, denoted $N(A)$ and $D(A)$ respectively. The numerator is formed by connecting the NW and NE endpoints and the SW and SE endpoints, while the denominator is formed by connecting the NW and SW endpoints and connecting the NE and SE endpoints.

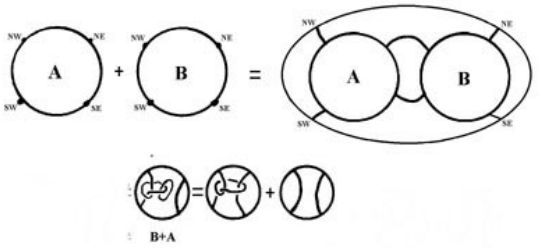


Figure 3.23: Addition between two tangles

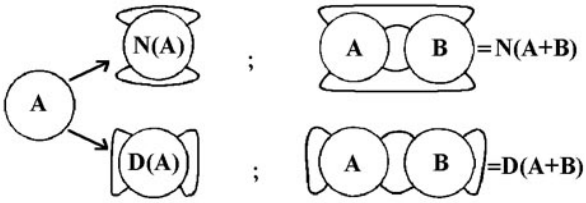


Figure 3.24: Numerator and Denominator operations

These operation produce knots or 2-component links. For example the numerator of the tangle in figure 3.25 is the Hopf link.

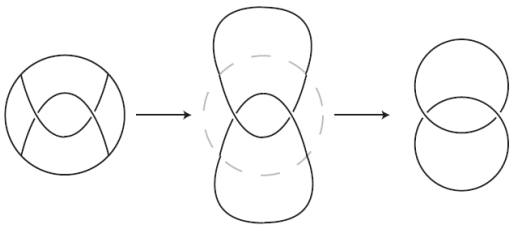


Figure 3.25: The Hopf link as the numerator of a tangle

Note that if $A + B$ is not a 2-string tangle then the result of $N(A + B)$ or $D(A + B)$ can be a link of more than two components.
In the following we will discuss rational tangles, since they are of special importance in tangle analysis of site-specific recombination.
We will now see that exists a correspondence between classes of rational tangles and the extended rational numbers. Let's start by noticing that every rational tangle can be represented by a vector $(a_1, a_2, ..., a_m)$ in the following way: we start with a circle with points labeled NW, NE, SW, SE and we connect the arcs. If m is even, start at the bottom (SW and SE) and do a_1 half-twists (using the convention of right-hand twists for positive a_1 and

left-hand twists for negative a_1). Next, do a_2 half-twists of the NE-SE side of the diagram. Then go back to the bottom, etc. If n is odd, start on the right and repeat the procedure as before.
From each vector we can construct a fraction $\frac{\beta}{\alpha} \in \mathbb{Q} \cup \infty$. The rational number $\frac{\beta}{\alpha}$ is called the *fraction* of a tangle and its very construction gives:

$$\frac{\beta}{\alpha} = a_m + \cfrac{1}{a_{m-1} + \cfrac{1}{a_{m-2} + \cfrac{1}{\cdots + \cfrac{1}{a_1}}}}$$

Definition 3.16. Two tangles are isotopic or equivalent if there is a mapping which deforms one tangle to the other without moving the endpoints, breaking a string, or passing one string through another.

The following theorem says that two tangles have the same fraction representation if and only if they are equivalent.

Theorem 3.1. *Two rational tangles are isotopic if and only if they have the same fraction.*

We constructed the fraction of a tangle starting from its vector notation. Also it is possible to go the other way around, however the fraction expansion of a rational number is not unique, so the same fraction of tangle can give us in principle two different vectors. For example, the vectors $(3, -2, 2)$ and $(2, 2, 1)$ represent the same tangles. This can be seen by computing the rational number that corresponds to $(3, -2, 2)$, which is $2 + \frac{1}{-2+\frac{1}{3}} = \frac{7}{5}$. Then by expanding $\frac{7}{5}$ into a fraction, we have $1 + \frac{1}{2+\frac{1}{2}}$. So that the vector $(2, 2, 1)$ is also obtained from the same tangle. The theorem (3.3.1) tells us that both vectors represent the same tangle.
To avoid this problem, Conway introduces a unique canonical vector representation called the Conway symbol.

Definition 3.17. A *Conway symbol* is a unique vector representation $(a_1, a_2, \ldots, a_m)$ where $a_i \in \mathbb{Z}$ for all i, are such that:

- $a_1 \neq \pm 1$
- $a_i \neq 0$ for $1 \leq i < m$
- a_i are of the same sign.

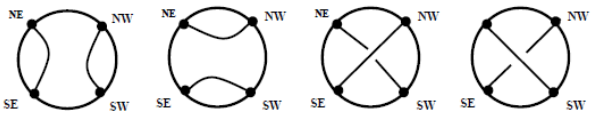


Figure 3.26: Four exceptional rational tangles in the Conway symbols

These Conway symbols can be applied to all but four exceptional rational tangles, as in figure 3.26.
The Conway symbol of the example above is $(2, 2, 1)$.
Using the Conway symbol we finally showed that exists a bijection between equivalence classes of rational tangles and the extended rational numbers.
A 4-plat is a knot made by braiding four strings and connecting the ends as shown below in Figure 3.27. 4-plats can be represented by a vector, like rational tangles. The 4-plat vector representative is an odd-length vector $< c_1, ..., c_{2k+1} >$ where $c_i \geq 1$ for all i and where each integer represents a half-twist between strings. Also, like rational tangles, the vector can be used to draw the 4-plat diagram. Start with four strings do c_1 half-twists between the middle two strings, bringing the bottom string on top. Next, do c_2 half-twists between the top and second string, this time bring the top string down. Go back to the middle two strings and repeat this process until you have completed the twists for all integers in the vector. Last, connect the ends. In fig. 3.27 is showed an example of the 4-plat $< 2, 1, 1 >$.

Figure 3.27: 4-plat $< 2, 1, 1 >$

This vector representation is called the Conway symbol for the 4-plat. Two 4-plats are the same if and only if they have the same Conway symbol or if one is the same as the reverse of the other, i.e. $< c_1, \ldots, c_{2k+1} >$ is the same 4-plat as $< c_{2k+1}, \ldots, c_1 >$. With the exception of the unknot $< 1 >$ and the unlink of 2-components $< 0 >$, the Conway symbol can be used to compute a classifying rational number $\frac{\beta}{\alpha} = \cfrac{1}{c_1+\cfrac{1}{c_2+\cfrac{1}{\ldots+\cfrac{1}{c_{2k+1}}}}}$. The 4-plat $\frac{\beta}{\alpha}$ is denoted by $b(\alpha, \beta)$.

Theorem 3.2. *Two 4-plats $b(\alpha, \beta)$ and $b(\alpha', \beta')$ are equivalent iff $\alpha = \alpha'$*

and $\beta^{\pm 1} \equiv \beta'(mod\alpha)$.

For example, let us look at the two 4-plats $b(17,5)$ and $b(17,7)$. The Conway symbol corresponding to $b(17,5)$ is $(3,2,2)$, and the Conway symbol corresponding to $b(17,7)$ is $(2,2,3)$. Therefore, these are equivalent 4-plats. Notice $17 = 17$ and $5^{-1} \equiv 7(mod17)$.
Rational tangles and 4-plats are closely related by means of the rational number representation. Given a rational number $\frac{\beta}{\alpha}$, if $0 < \frac{\beta}{\alpha} < 1$, the denominator of the rational tangle $\frac{\beta}{\alpha}$ gives the 4-plat $b(\alpha,\beta)$ and if $\frac{\beta}{\alpha} \geq 1$ the numerator of the tangle $\frac{\beta}{\alpha}$ gives the 4-plat $b(\beta,-\alpha)$.
It follows the next theorem:

Theorem 3.3. *Suppose the rational tangles with reduced fractions* $\frac{\beta}{\alpha}$ *and* $\frac{\beta'}{\alpha'}$ *are given. If* $N(\frac{\beta}{\alpha})$ *and* $N(\frac{\beta'}{\alpha'})$ *denote the corresponding rational knots obtained by taking numerator of those tangles, then* $N(\frac{\beta}{\alpha})$ *and* $N(\frac{\beta'}{\alpha'})$ *are topologically equivalent iff* $\alpha = \alpha'$ *and* $\beta^{\pm 1} \equiv \beta'(mod\alpha)$.

3.2 The site-specific recombination

As discussed in the introduction of this section, DNA must be topologically manipulated by enzymes in order for vital life processes to occur. One of these enzymatic actions is called *site-specific recombination*. Site-specific recombination is a process by which a piece of DNA is moved to another position on the molecule or to import a foreign piece of a DNA molecule into it. Recombination is used for gene rearrangement, gene regulation, copy number control, and gene therapy. This process is mediated by an enzyme called a *recombinase*. A small segment of the genetic sequence of the DNA that is recognized by the recombinase is called a *recombination site*. A pair of sites on the same molecule or different molecules, once recognized, are aligned and then bound by the enzyme. This is the stage of the reaction called *synapsis*. The DNA molecule and the enzyme itself are called the *synaptic complexes*. Before recombination the DNA molecule is called the *substrate* and after recombination it is called the *product*. Once bound to the DNA, the enzyme breaks the DNA at the two sites and then recombines the ends by exchanging them. Each of the recombination sites is oriented by the order in which the bases appear as one reads around the DNA strand in some predetermined

order. If the orientations of the sites agree, the site configuration is called *direct repeat*. If they disagree, this is called *inverted repeat*.

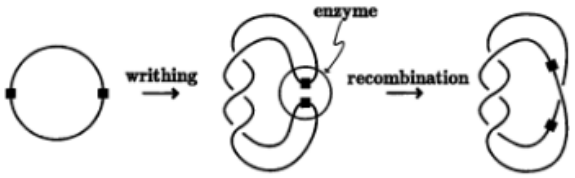


Figure 3.28: A simple site-specific recombination

3.3 The tangle method for site-specific recombination

The tangle model was introduced by DeWitt Sumners in 1980, in order to deduce mathematically what happens during recombination. From empirical evidence we have the following:

- Most observed products of recombination experiments are 4-plats.
- The part of the synaptic complex acted on by an enzyme is a $(2, 2)$-tangle.

Therefore, the product is just the replacement of one $(2, 2)$-tangle by another. This process may be express by means of the definition of sum of tangle. For example, the $(2, 2)$-tangle within the circle T is replaced by the $(2, 2)$-tangle to form the product, as in fig.3.29.

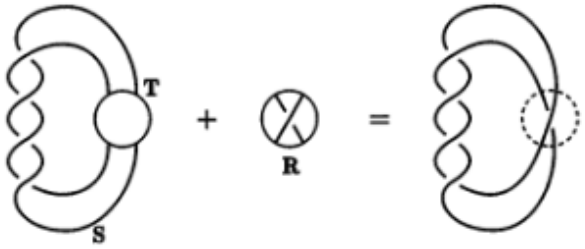


Figure 3.29: The synaptic complex as a result of a tangles operation.

Mathematically we can consider S to be the $(2, 2)$-tangles in T. The numerator of the sum of S and R is then the product.
So the following tangle-equation holds:

$$N(R + S) = \text{the product}$$

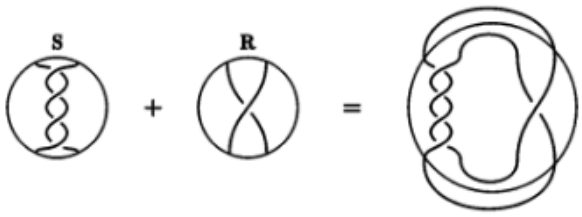


Figure 3.30: $N(S+R)$ =the product.

Furthermore we may divide the substrate into the external tangle S and the internal tangle E, since the substrate is then the numerator of the sum of S and E. Again we have the tangle-equation:

$$N(S+E) = \text{the substrate}$$

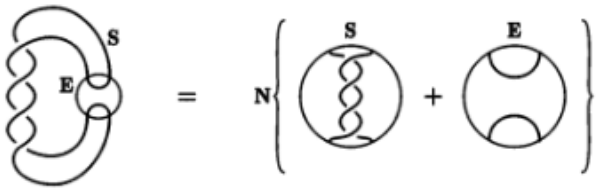


Figure 3.31: $N(S+E)$ = the substrate.

If it possible to observe the substrate and the product, then the ideal situation would be to determinate R, E, S from the above tangle-equations. Mathematically, however, since there are only two equations but three unknowns it is not possible in general to solve the system.
So, in order to solve these tangle-equations, we should consider the following two assumption, coming from experimental data.

Assumption A. *1. The enzymatic mechanism is constant and independent of the geometry and topology of the substrate population.*

2. The product of a series of site-specific recombinations can be expressed as the numerator of the sum of tangles, namely it is of the form

$$N(S+R+R+\cdots+R).$$

With these new assumption we are now able to solve the tangle equations, as the next example shows.

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