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Corso di Laurea Magistrale in Matematica

Curriculum Generale ed Applicativo

Lipschitz regularity for weak solutions of parabolic p-Laplacian type equations in certain subriemannian structures

Tesi di Laurea Magistrale in Analisi Matematica

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Introduzione

L'obiettivo principale della tesi è di mostrare la locale regolarità Lipschitz delle soluzioni deboli di una classe di PDE paraboliche modellate sul $p-$Laplaciano parabolico. Consideriamo l'equazione

$$\partial_t u = \sum_{i=1}^{2n} X_i(A_i(x, \nabla_0 u)) \tag{1}$$

su un cilindro $Q = \Omega \times (0, T)$ dove Ω è un sottoinsieme aperto di una varietà sub-riemanniana connessa M di dimensione $2n + 1$ e $T > 0$. $X_1, ..., X_{2n}$ sono dei campi vettoriali che soddisfano la condizione di Hörmander e ∇_0 denota il gradiente orizzontale $\nabla_0 u = (X_1 u, ..., X_{2n} u)$. Assumiamo le seguenti condizioni strutturali sulle $A_i(x, \xi)$: supponiamo che esistano $2 \leq p \leq 4$ e $0 < \lambda \leq \Lambda < \infty$ tali che per quasi ogni $x \in \Omega$, $\xi \in \mathbb{R}^{2n+1}$ e per ogni $\eta \in \mathbb{R}^{2n}$ si abbia che

$$\begin{cases} \lambda|\xi|^{p-2}|\eta|^2 \leq \partial_{\xi_j} A_i(x, \xi)\eta_i\eta_j \leq \Lambda|\xi|^{p-2}|\eta|^2 \\ |A_i(x, \xi)| + |\partial_{x_j} A_i(x, \xi)| \leq \Lambda|\xi|^{p-1} \end{cases} \tag{2}$$

(Il più semplice esempio di tali A_i è $A_i(x, \xi) = |\xi|^{p-2}\xi_i$).

Questo risultato è ben noto nel caso euclideo sia nel caso stazionario (si veda [D1] e [L]) sia nel caso dipendente dal tempo (si veda [LSU] e [D2]). Recentemente il risultato è stato esteso nel gruppo di Heisenberg nel caso stazionario in [Z] e nel caso che dipende dal tempo in [CCG] e [CCZ]. Notiamo esplicitamente che il gradiente orizzontale ha solo $2n$ componenti in una varietà di dimensione $2n+1$, quindi l'operatore è totalmente degenere.

Nel caso Heisenberg $X_1, ..., X_{2n}$ assieme con $[X_1, X_{n+1}]$ genera l'intero spazio tangente in ogni punto, permettendo di ottenere regolarità anche nella direzione non esplicitamente presente nel gradiente. In questa tesi considereremo dei campi vettoriali più generali, in particolare i commutatori sono differenti dal caso Heisenberg, introducendo diverse difficoltà tecniche nell'estensione del risultato. Per ottenere il nostro risultato principale useremo un'iterazione alla Moser.
Scegliamo come campi vettoriali (per $1 \leq i \leq n$)

$$X_i = \partial_i - \frac{\mathcal{K}_{n+i}}{2}\partial_{2n+1} \qquad X_{n+i} = \partial_{n+i} + \frac{\mathcal{K}_i}{2}\partial_{2n+1} \tag{3}$$

dove le $\mathcal{K}_i$ sono funzioni sufficientemente regolari tali che i campi vettoriali $X_1, ..., X_{2n}$ soddisfino la condizione di Hörmander. Useremo la notazione Z per denotare il campo vettoriale $\partial_{x_{2n+1}}$ per enfatizzare il fatto che tecnicamente ha il ruolo di una derivata seconda (visto che è ottenuto mediante un commutatore).
Il teorema principale che vogliamo mostrare è il seguente:

Teorema 0.0.1. *Sia u una soluzione debole di* (1). *Allora esiste una costante positiva $C = C(n, p, \lambda, \Lambda, \mathcal{K}_i)$ tale che*

$$\sup_{Q_{\mu,r}} |\nabla_0 u| \leq C \left(⨍⨍_{Q_{\mu,2r}} |\nabla_0 u|^p dxdt \right)^{\frac{1}{p}}$$

sui sottocilindri $Q_{\mu,r}, Q_{\mu,2r} \subset Q$ definiti come $Q_{\mu,r} = B(x,r) \times [t_0 - \mu r, t_0]$ dove $x \in \Omega$, $t_0 \in (0,T)$, $r > 0$ e la costante μ è scelta in modo che si abbia

$$\mu = \left(\frac{1}{r^{2n+4}} \int\int_{Q_{\mu,2r}} |\nabla_0 u|^p dxdt \right)^{\frac{2-p}{p}}$$

Per ottenere questo risultato prima approssimeremo una soluzione debole della PDE (1) tramite una successione di soluzioni deboli $u^{\delta,\varepsilon}$ di PDE meno degeneri (dipendenti da due parametri $\delta, \varepsilon > 0$). Poi useremo un'iterazione alla Moser su $\nabla_\varepsilon u^{\delta,\varepsilon}$ ottenendo stime che sono indipendenti da δ e da ε. Poi mandando $\delta, \varepsilon \to 0$ otterremo il Teorema 0.0.1. Per usare l'iterazione di Moser avremo bisogno di due stime fondamentali: una disuguaglianza di interpolazione alla Poincarè e una disuguaglianza alla Caccioppoli. A causa della struttura subriemanniana del problema quando cercheremo di ottenere una disuguaglianza di Caccioppoli per $\nabla_\varepsilon u^{\delta,\varepsilon}$ otteremo una disuguaglianza che dipende anche da $Zu^{\delta,\varepsilon}$.

Proposizione 0.0.2. *Per ogni $\beta \geq 0$, per ogni $\eta \in C^1([0,T], C_0^\infty(\Omega))$ e per ogni $T > t_2 \geq t_1 \geq 0$ esiste una costante $C = C(\lambda, \Lambda, n, p, \mathcal{K}_i) \geq 0$ tale che*

$$\frac{1}{\beta+2}\int_\Omega [(\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}+1}\eta^2]\Big|_{t_2}^{t_1} dx + \int_{t_1}^{t_2}\int_\Omega \eta^2(\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2+\beta}{2}}|\nabla_\varepsilon^2 u|^2 dxdt \leq$$

$$\leq C\int_{t_1}^{t_2}\int_\Omega (\eta^2 + |\nabla_\varepsilon \eta|^2 + |\eta Z\eta|)(\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}} dxdt +$$

$$+ C(\beta+1)^4\int_{t_1}^{t_2}\int_\Omega \eta^2(\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta-2}{2}}|Zu^{\delta,\varepsilon}|^2 dxdt +$$

$$+ \frac{C}{\beta+2}\int_{t_1}^{t_2}\int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}+1}|\partial_t \eta|\eta dxdt$$

Ciò ci costringe a provare una disuguaglianza di Caccioppoli anche per $Zu^{\delta,\varepsilon}$. Tramite l'uso di una disugalglianza pesata di iterpolazione alla Poincarè per $Zu^{\delta,\varepsilon}$ possiamo combinare le disuguaglianze di Caccippoli per $\nabla_\varepsilon u^{\delta,\varepsilon}$ e $Zu^{\delta,\varepsilon}$ per ottenere una disuguaglianza di Caccioppoli per $\nabla_\varepsilon u^{\delta,\varepsilon}$ che non dipende da $Zu^{\delta,\varepsilon}$.

Proposizione 0.0.3. *Esiste una costante $C = C(n, p, \lambda, \Lambda, \mathcal{K}_i) > 0$ tale che per ogni $\beta \geq 0$ e per ogni funzione non negativa $\eta \in C^1([0,T], C_0^\infty(\Omega))$ nulla sul bordo parabolico di Q si ha che*

$$\sup_{t_1<t<t_2}\int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta+2}{2}}\eta^2 dx + \int_{t_1}^{t_2}\int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2+\beta}{2}}|\nabla_\varepsilon^2 u^{\delta,\varepsilon}|^2\eta^2 dxdt \leq$$

$$\leq C(p+\beta)^7(||\eta||^2_{L^\infty} + ||\nabla_\varepsilon \eta||^2_{L^\infty} + ||\eta Z\eta||_{L^\infty}) \int\int_{spt(\eta)} (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}} dxdt +$$

$$+ C(p+\beta)^7 ||\eta \partial_t \eta||_{L^\infty} |spt(\eta)|^{\frac{p-2}{p+\beta}} \left(\int\int_{spt(\eta)} (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}} dxdt \right)^{\frac{\beta+2}{p+\beta}}$$

A questo punto usiamo delle funzioni cut-off η che dipendono da un parametro μ per stimare il termine non omogeneo $||\eta\partial_t\eta||_{L^\infty}$ che deriva dalla non omogeneità della PDE (1). Ora abbiamo una disuguaglianza di Caccioppoli "classica" per $\nabla_\varepsilon u^{\delta,\varepsilon}$. A questo punto, nonostante il problema sia parabolico, usiamo una versione semplificata del metodo di Moser nel caso ellittico. Infatti anzichè usare la disuguaglianza di Sobolev parabolica (usata nel metodo di Moser parabolico) usiamo la disuguaglianza di Sobolev ellittica. Ciò ci dà un incremento di regolarità ridotto per igni iterazione ma ci permette comunque di provare la limitatezza del gradiente orizzontale $\nabla_\varepsilon u^{\delta,\varepsilon}$ e conseguentemente di ottenere il nostro risultato principale ovvero il Teorema 0.0.1. La tesi è organizzata in quattro capitoli: nel primo capitolo richiamiamo la definizione di varietà subiriemanniana e definiamo la nostra scelta dei campi vettoriali. Nel secondo capitolo richiamiamo la definizone di spazi di Sobolev degeneri associati alla nostra scelta di campi vettoriali. Nel terzo capitolo richiamiamo l'interazione di Moser nel caso ellittico e nell'ultimo capitolo presentiamo la parte più originale della tesi, ovvero la locale regolarità Lipschitz per le soluzioni deboli nel caso dei campi vettoriali scelti, in un contesto di maggior generalità rispetto al caso Heisenberg.

Introduction

The main aim of the thesis is to prove the local Lipschitz regularity of the weak solutions to a class of parabolic PDEs modeled on the parabolic $p-$Laplacian. We consider the equation

$$\partial_t u = \sum_{i=1}^{2n} X_i(A_i(x, \nabla_0 u)) \tag{1}$$

in a cylinder $Q = \Omega \times (0, T)$ where Ω is an open subset of a connected subriemannian manifold M of dimension $2n+1$ and $T > 0$. The $X_1, ..., X_{2n}$ are vector fields that satisfy the Hörmander condition and ∇_0 denotes the horizontal gradient $\nabla_0 u = (X_1 u, ..., X_{2n} u)$. We assume the following structural conditions on the $A_i(x, \xi)$: we suppose that there exists $2 \leq p \leq 4$ and $0 < \lambda \leq \Lambda < \infty$ such that for a.e. $x \in \Omega, \xi \in \mathbb{R}^{2n}$ and for all $\eta \in \mathbb{R}^{2n}$ one has

$$\begin{cases} \lambda |\xi|^{p-2} |\eta|^2 \leq \partial_{\xi_j} A_i(x, \xi) \eta_i \eta_j \leq \Lambda |\xi|^{p-2} |\eta|^2 \\ |A_i(x, \xi)| + |\partial_{x_j} A_i(x, \xi)| \leq \Lambda |\xi|^{p-1} \end{cases} \tag{2}$$

(The simplest example of such A_i is $A_i(x, \xi) = |\xi|^{p-2} \xi_i$).
This result is well known in the euclidean case both in the stationary case (see [D1] and [L]) and in the time dependent case (see [LSU] and [D2]). Recently the result has been extended in the Heisenberg group in the stationary case in [Z] and in the time dependent case in [CCG] and [CCZ]. Let us note explicitly that the horizontal gradient has only $2n$ components in a manifold of dimension $2n+1$, so that the operator is totally degenerate. In the Heisenberg setting $X_1, ..., X_{2n}$ toghether with $[X_1, X_{n+1}]$ span the whole tangent

space at every point, allowing to obtain regularity even in the direction not explicitly present in the gradient. In this thesis we will consider more general vector fields, in particular the commutators rule is different from the Heisenberg one, introducing some technical difficulties in the extension. To obtain our main result we will use a Moser-like iteration.

We chose as vector fields (for $1 \leq i \leq n$)

$$X_i = \partial_i - \frac{\mathcal{K}_{n+i}}{2}\partial_{2n+1} \qquad X_{n+i} = \partial_{n+i} + \frac{\mathcal{K}_i}{2}\partial_{2n+1} \tag{3}$$

where the $\mathcal{K}_i$ are sufficiently regular functions such that the vector fields $X_1, ..., X_{2n}$ satisfy the Hörmander condition. We will use the notation Z to denote the vector field $\partial_{x_{2n+1}}$ to emphasize the fact that it has technically the role of a second derivative (since it is obtained through a commutator).

The main theorem that we will prove is the following:

Theorem 0.0.1. *Let u be a weak solution of* (1)*. Then there exists a positive constant $C = C(n, p, \lambda, \Lambda, \mathcal{K}_i)$ such that*

$$\sup_{Q_{\mu,r}} |\nabla_0 u| \leq C \left(\fint \fint_{Q_{\mu,2r}} |\nabla_0 u|^p dxdt \right)^{\frac{1}{p}}$$

on some subcylinders $Q_{\mu,r}, Q_{\mu,2r} \subset Q$ defined as $Q_{\mu,r} = B(x,r) \times [t_0 - \mu r, t_0]$ where $x \in \Omega$, $t_0 \in (0,T)$, $r > 0$ and the constant μ is chosen as

$$\mu = \left(\frac{1}{r^{2n+4}} \int \int_{Q_{\mu,2r}} |\nabla_0 u|^p dxdt \right)^{\frac{2-p}{p}}$$

In order to obtain this result we will first approximate a weak solution u of the PDE (1) through a sequence of weak solutions $u^{\delta,\varepsilon}$ of less degenerated PDEs (depending on two parameters $\delta, \varepsilon > 0$). Then we will use a Moser-like iteration on $\nabla_\varepsilon u^{\delta,\varepsilon}$ obtaining estimates that are independent of δ and ε. Then letting $\delta, \varepsilon \to 0$ we will get Theorem 0.0.1. In order to use the Moser iteration we will need two fundamental estimates: a Poincarè-like interpolation inequality and a Caccioppoli-like inequality. By the subriemannian structure of the problem when we try to obtain a Caccioppoli-like inequality for $\nabla_\varepsilon u^{\delta,\varepsilon}$ we will obtain an inequality that depends also on $Zu^{\delta,\varepsilon}$.

Proposition 0.0.2. *For all $\beta \geq 0$, for all $\eta \in C^1([0,T], C_0^\infty(\Omega))$ and for all $T > t_2 \geq t_1 \geq 0$ we have that there exists a constant $C = C(\lambda, \Lambda, n, p, \mathcal{K}_i) \geq 0$ such that*

$$\frac{1}{\beta+2}\int_\Omega [(\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}+1}\eta^2]\Big|_{t_2}^{t_1} dx + \int_{t_1}^{t_2}\int_\Omega \eta^2(\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2+\beta}{2}}|\nabla_\varepsilon^2 u|^2 dxdt \leq$$

$$\leq C\int_{t_1}^{t_2}\int_\Omega (\eta^2 + |\nabla_\varepsilon \eta|^2 + |\eta Z\eta|)(\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}} dxdt +$$

$$+ C(\beta+1)^4 \int_{t_1}^{t_2}\int_\Omega \eta^2(\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta-2}{2}}|Zu^{\delta,\varepsilon}|^2 dxdt +$$

$$+ \frac{C}{\beta+2}\int_{t_1}^{t_2}\int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}+1}|\partial_t \eta|\eta dxdt$$

This forces us to prove a Caccioppoli-like inequality also for $Zu^{\delta,\varepsilon}$. By the use of a weighted Poincarè-like interpolation inequality for $Zu^{\delta,\varepsilon}$ we can combine the Caccioppoli-like inequalities for $\nabla_\varepsilon u^{\delta,\varepsilon}$ and $Zu^{\delta,\varepsilon}$ to obtain a Caccioppoli inequality for $\nabla_\varepsilon u^{\delta,\varepsilon}$ which does not depend on $Zu^{\delta,\varepsilon}$.

Proposition 0.0.3. *There exists a constant $C = C(n, p, \lambda, \Lambda, \mathcal{K}_i) > 0$ such that for all $\beta \geq 0$ and for all non negative functions $\eta \in C^1([0,T], C_0^\infty(\Omega))$ vanishing on the parabolic boundary of Q we have that*

$$\sup_{t_1<t<t_2}\int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta+2}{2}}\eta^2 dx + \int_{t_1}^{t_2}\int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2+\beta}{2}}|\nabla_\varepsilon^2 u^{\delta,\varepsilon}|^2\eta^2 dxdt \leq$$

$$\leq C(p+\beta)^7(||\eta||_{L^\infty}^2 + ||\nabla_\varepsilon \eta||_{L^\infty}^2 + ||\eta Z\eta||_{L^\infty})\int\int_{spt(\eta)} (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}} dxdt +$$

$$+C(p+\beta)^7||\eta\partial_t\eta||_{L^\infty}|spt(\eta)|^{\frac{p-2}{p+\beta}}\left(\int\int_{spt(\eta)}(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}}dxdt\right)^{\frac{\beta+2}{p+\beta}}$$

At this point we use cut-off functions η that depend on a parameter μ to estimate the non-homogeneous term $||\eta\partial_t\eta||_{L^\infty}$ which cames from the non-homogeneity of the PDE (1). Now we have a "classical" Caccioppoli inequality for $\nabla_\varepsilon u^{\delta,\varepsilon}$. At this point, although the problem is parabolic, we use a simplified verson of the Moser's method in the elliptic case. In fact we replace the use of the parabolic Sobolev inequality (used in the parabolic Moser's method) with the use of the elliptic Sobolev inequality. This will give us a smaller increase in regularity for each iteration but it will allow us anyway to prove the boundendness of the horizontal gradient $\nabla_\varepsilon u^{\delta,\varepsilon}$ and consequently to obtain our main Theorem 0.0.1.
The thesis is organized in four chapters: in the first chapter we recall the definition of subriemannian manifold and define our choice of vector fields. In the second chapter we recall the definition of degenerate Sobolev spaces associated to the choice of vector fields. In the third chapter we recall the Moser iteration in the elliptic case and in the last we present the most original part of the thesis, which is the local Lipschitz regularity of the weak solutions in the case of the chosen vector fields, more general than the Heisenberg ones.

Chapter 1

Some topics on Subriemannian Geometry

1.1 Riemannian manifolds

Definition 1.1.1. Let M be a topological space. A **local chart** (U, φ) on M is a couple where U is an open subset of M and φ is an homeomorphism from U to a certain open subset of $\mathbb{R}^n$.

Definition 1.1.2. Let M be a topological space. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of M (i.e. a collection of subset U_i such that every $U_i \subset M$ is open and $\bigcup_{i\in I} U_i = M$). If for each $i \in I$ there exists a local chart (U_i, φ_i) then $\{(U_i, \varphi_i)\}_{i\in I}$ is called **atlas** for M.

Definition 1.1.3. Let M be a topological space and $(U_1, \varphi_1), (U_2, \varphi_2)$ two local charts of an atlas $\{(U_i, \varphi_i)\}_{i\in I}$ for M. Then the isomorphism

$$\varphi_2 \circ \varphi_1^{-1}\Big|_{\varphi_1(U_1\cap U_2)} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is called **transition function** between the two local charts (U_1, φ_1) and (U_2, φ_2).

Definition 1.1.4. Let M be a topological space and $\{(U_i, \varphi_i)\}_{i\in I}$ be an atlas for M such that all the transiction functions are differentiable. Then M is a **differentiable manifold**. More generally we say that M is a C^k-**manifold** if all the transiction functions are C^k.

Definition 1.1.5. Let M be a differentiable manifold with an atlas $\{(U_i, \varphi_i)\}_{i\in I}$. Then we say that the **dimension** of M is the dimension of the $\mathbb{R}^n$ where the homeorphisms φ_i arrive.

Definition 1.1.6. Let M be a differentiable manifold and $p \in M$. Let (U, φ) be a local chart such that $p \in U$. Now we consider the set of the smooth curves $\Omega_p = \{\gamma : (-1, 1) \to M$ such that $\gamma(0) = p\}$ and we define an equivalence relation $\sim$ on this set: we say that γ_1 is equivalent to γ_2 if $(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$. Then we can define the **tangent space** of M in p as

$$T_pM = \frac{\Omega_p}{\sim}$$

and the **tangent bundle** of M as

$$TM = \bigcup_{p\in M} T_pM$$

Observation 1.1.7. If M is a differential manifold and $p \in M$ then T_pM is a real vector space.

Theorem 1.1.8. *Let M be a differential manifold of dimension n. Then the dimension of T_pM as a real vector space is n for all $p \in M$.*

Proof. See [T] chapter 3 section 8. □

Definition 1.1.9. Let M be a differentiable manifold. A **vector field** is an application $X : M \to TM$ such that $\pi \circ X = id_M$ where $\pi : TM \to M$ is the projection map from TM to M.

Observation 1.1.10. Let M be a differential manifold of dimension n, (U, φ) a local chart and X a vector field. Then locally on U we can identify $X(p) = (a_1(p), ..., a_n(p))$ as a differential operator of the first order as

$$X(p) = \sum_{i=1}^{n} a_i(p)\partial_{x_i}$$

where $x_1, ..., x_n$ are the local coordinates given by the local chart (U, φ) and ∂_{x_i} denotes the directional derivatives. For further details see [T] chapter 3 section 12. The inverse is also true: if we have a directional derivative $X(p) = \sum_{i=1}^{n} a_i(p)\partial_{x_i}$ we can find the corresponding vector field "applying" X to $I = (x_1, ..., x_n)$ i.e.

$$XI(p) = (a_1(p), ..., a_n(p))$$

In the rest of this chapter we will use XI when we want to denote the vector field and X when we want to denote the differential operator of the first order. When there is no risk of confunsion we will indistinctly use X to denote both.

Definition 1.1.11. Let M be a differentiable manifold. A differential function g defined on M is called **riemannian metric** on M if g_p is an inner product on T_pM for all $p \in M$.

Definition 1.1.12. Let M be a differentiable manifold and g a riemannian metric. Then (M, g) is called **riemannian manifold**.

Observation 1.1.13. Let (M, g) be a riemannian manifold of dimension n and (U, φ) a local chart. Then in the local coordinates given by (U, φ) we have that

$$g_p(u, v) = \sum_{i,j=1}^{n} g_{ij}(p)u_iv_j \qquad \forall u, v \in T_pM$$

where $p \in U$ and $G(p) = (g_{ij}(p))_{ij}$ is a $n \times n$ positive-definite matrix. In the following we will sometimes use the following notations:

$$\langle u, v\rangle_{g(p)} = g_p(u, v)$$

$$||u||_{g(p)} = \sqrt{\langle u, u\rangle_{g(p)}}$$

Definition 1.1.14. Let (M, g) a riemannian manifold and γ:$[a, b] \to M$, $\gamma \in C^1$. Then we define the **lenght** of γ as

$$l(\gamma) = \int_a^b ||\gamma'(t)||_{g(\gamma(t))}dt$$

We can define also the length of a piecewise C^1 curve: if $\gamma : [a, b] \to M$, γ piecewise C^1 (i.e. if exists $a = x_0 < x_1 < ... < x_i < x_{i+1} < ... < x_m = b$ and $\gamma|_{(x_i,x_{i+1})} \in C^1$ for all $1 \leq i \leq m-1$) we define $l(\gamma)$ as

$$l(\gamma) = \sum_i l\left(\gamma|_{(x_i,x_{i+1})}\right)$$

Definition 1.1.15. Let (M, g) be a connected riemannian manifold (at least C^1). We can define a **distance** on M in the following way: let $p, q \in M$, their distance is defined as

$$d(p, q) = \inf\{l(\gamma) \text{ with } \gamma \text{ piecewise } C^1 \text{ s.t. } \gamma(0) = p, \gamma(1) = q\}$$

1.2 Subriemannian manifolds

Definition 1.2.1. Let M be a differentiable manifold. We say that $\Delta = (\Delta_p)_{p\in M}$ is a **distribution** if for all $p \in M$ Δ_p is a subspace of T_pM.

Observation 1.2.2. If we have a family of differential operators of the first order

$(X_1, ..., X_m)$ we can consider the corresponding distribution given by

$$\Delta_p = \text{span}(X_1 I|_p, ..., X_m I|_p)$$

Definition 1.2.3. Let M be a differentiable manifold, Δ a distribution and g a subriemannian metric (i.e a riemannian metric only defined on Δ). Then we say that (M, Δ, g) is a **subriemannian manifold**.

Definition 1.2.4. Let M be a differential manifold and Δ a distribution. We say that a curve $\gamma : [a, b] \to M$, $\gamma \in C^1$ is **admissible** if

$$\gamma'(t) \in \Delta_{\gamma(t)} \qquad \forall t \in [a, b]$$

Definition 1.2.5. Let (M, Δ, g) be a subriemannian manifold and $\gamma : [a, b] \to M$ an admissimble curve; we can define the **lenght** of γ as

$$l(\gamma) = \int_a^b ||\gamma'(t)||_{g(\gamma(t))} dt$$

Note that we can generalize the definitions of adimissible curve and lenght of a curve for piecewise C^1 curve as in the riemannian case.

Definition 1.2.6. Let (M, Δ, g) be a subriemannian manifold and $x, y \in M$ we define the **distance** between x and y as

$$d(x, y) = \begin{cases} +\infty \text{ if there are no admissible curves that join } x \text{ and } y \\ \inf\{l(\gamma) \text{ with } \gamma \text{ admissible s.t. } \gamma(0) = x, \gamma(1) = y\} \text{ otherwise} \end{cases}$$

Definition 1.2.7. Let $X = \sum_i a_i \partial_{x_i}$, $Y = \sum_j b_j \partial_{x_j}$ two differential operators of the first order. We define the **commutator** between X and Y as

$$[X, Y] = XY - YX$$

Observation 1.2.8. If $X = \sum_i a_i \partial_{x_i}$ and $Y = \sum_j b_j \partial_{x_j}$ are two differential operators of the first order then also $[X, Y]$ is a differential operator of the first order. In fact we have

$$[X, Y] = \sum_{i,j} \left(X b_j \partial_{x_j} - Y a_i \partial_{x_i} \right)$$

and so if $f \in C^\infty$ we get

$$[X, Y]f = \sum_{i,j} \left(a_i \partial_{x_i} (b_j \partial_{x_j} f) - b_j \partial_{x_j} (a_i \partial_{x_i} f) \right) =$$

$$= \sum_{i,j} \left(a_i \partial_{x_i} b_j \partial_{x_j} f + a_i b_j \partial^2_{x_i x_j} f - b_j \partial_{x_j} a_i \partial_i f - a_i b_j \partial^2_{x_i x_j} f \right) = \sum_i \left(X(b_i \partial_{x_i} f) - Y(a_i \partial_{x_i} f) \right)$$

Definition 1.2.9. Let $X_1, ..., X_m$ be differential operators of the first order. We define the **Lie algebra** generated by $X_1, ..., X_m$ as the closure of $\mathrm{span}(X_1 I, ..., X_m I)$ through the commutator. We denote such space with $Lie(X_1, ..., X_m)$.

Definition 1.2.10. Let $X_1, ..., X_m$ be differential opeators of the first order. We say that $X_1, ..., X_m$ satisfy the **Hörmander condition** if $Lie(X_1, ..., X_m)$ has maximum rank in every point.

Definition 1.2.11. Let $X_1, ..., X_m$ be differential operators of the first order and $\Delta = \mathrm{span}(X_1 I, ..., X_m I)$ the corresponding distribution. We say that $X_1, ..., X_m$ have **degree** 1 (and we write $deg(X_1) = ... = deg(X_m) = 1$). Then we denote with $\Delta^{(2)} = \mathrm{span}(\Delta, [X_i, X_j])_{1 \leq i,j \leq m}$ and we say that $deg(X) = 2$ if $X \in \Delta^{(2)} \setminus \Delta$. More generally we denote with $\Delta^{(j)} = \mathrm{span}(\Delta^{(j-1)}, [X_i, Y_k])$ with $X_i \in \Delta$ and $Y_k \in \Delta^{(j-1)}$ and we say that $deg(X) = j$ if $X \in \Delta^{(j)} \setminus \Delta^{(j-1)}$.

Observation 1.2.12. If we have a subriemannian manifold (M, Δ, g) and $X_1, ..., X_m$ differential operators of the first order satisfying the Hörmander condition we can construct a basis of T_pM using the method described in the previous definition. A basis of this type is said to be **adapted to the stratification**.

Definition 1.2.13. Let (M, Δ, g) be a subriemannian manifold, X a differential operator (with C^∞ coefficients) of the first order and $\xi_0 \in M$. We denote with $\exp(tX)(\xi_0)$ the unique solution of the following Cauchy problem

$$\begin{cases} \gamma'(t) = (XI)(\gamma(t)) \\ \gamma(0) = \xi_0 \end{cases}$$

Proposition 1.2.14. *Let (M, Δ, g) be a subriemannian manifold, $\xi \in M$ and X a differential operator (with C^∞ coeffiecients) of the first order. Then for $t \to 0$ we have that*

$$\exp(tX)(\xi) = \xi + t(XI)(\xi) + \cdots + \frac{t^k}{k!}(X^k I)(\xi) + o(t^k)$$

Proof. For convenience we denote $\gamma(t) := \exp(tX)(\xi)$. We have that

$$\gamma(0) = \xi$$

$$\gamma'(t) = (XI)(\gamma(t)) \Rightarrow \gamma'(0) = (XI)(\xi)$$

$$\vdots$$

$$\gamma^{(j)}(t) = (X^j I)(\gamma(t)) \Rightarrow \gamma^{(j)}(0) = (X^j I)(\xi)$$

So if we use the Taylor expansion for $t \to \gamma(t)$ in $t = 0$ we get

$$\gamma(t) = \gamma(0) + t\gamma'(0) + \cdots \frac{t^k}{k!}\gamma^{(k)}(0) + o(t^k)$$

and consequently

$$\exp(tX)(\xi) = \xi + t(XI)(\xi) + \cdots + \frac{t^k}{k!}(X^k I)(\xi) + o(t^k)$$

□

Theorem 1.2.15. *Let* (M, Δ, g) *be a subriemannian manifold,* $\xi \in M$ *and* X, Y *two differential operators (with* C^∞ *coeffiecients) of the first order. Then we have that for* $t \to 0$

$$\exp(-tY)(-\exp(-tX)(\exp(tY)(\exp(tX)(\xi)))) = \xi + t^2([X,Y]I)(\xi) + o(t^2)$$

Proof. Using the previous proposition we have (for $t \to 0$)

$$\exp(tX)(\xi) = \xi + t(XI)(\xi) + \frac{t^2}{2}(X^2 I)(\xi) + o(t^2)$$

and so

$$\exp(tY)(\exp(tX)(\xi)) = \exp(tX)(\xi) + t(YI)(\exp(tX)(\xi)) + \frac{t^2}{2}(Y^2 I)(\exp(tX)(\xi)) + o(t^2) =$$

$$= \xi + t(XI)(\xi) + \frac{t^2}{2}(X^2 I)(\xi) + t(YI)(\exp(tX)(\xi)) + \frac{t^2}{2}(Y^2 I)(\exp(tX)(\xi)) + o(t^2) =$$

$$= \xi + t(XI)(\xi) + \frac{t^2}{2}(X^2 I)(\xi) + t((YI)(\xi) + (XYI)(\xi) + o(t)) + \frac{t^2}{2}(Y^2 I)(\xi) + o(t^2) =$$

$$= \xi + t((X+Y)I)(\xi) + t^2\left(\frac{X^2}{2}I + XYI + \frac{Y^2}{2}I\right)(\xi) + o(t^2)$$

In a similar way we get

$$\exp(-tX)(\exp(tY)(\exp(tX)(\xi))) =$$

$$= \exp(tY)(\exp(tX)(\xi)) - t(XI)(\exp(tY)(\exp(tX)(\xi))) + \frac{t^2}{2}(X^2 I)(\exp(tY)(\exp(tX)(\xi))) + o(t^2) =$$

$$= \xi + t((X+Y)I(\xi)) + t^2\left(\frac{X^2}{2}I + XYI + \frac{Y^2}{2}I\right)(\xi) +$$

$$-t(XI)(\xi + t(X+Y)I(\xi) + o(t)) + \frac{t^2}{2}(X^2I)(\xi + o(1)) + o(t^2) =$$

$$= \xi + t(YI)(\xi) + \frac{t^2}{2}(X^2I)(\xi) + t^2(XYI)(\xi) + \frac{t^2}{2}(Y^2I)(\xi) + t(XI)(\xi) +$$

$$-t(XI)(\exp(t(X+Y))(\xi)) + \frac{t^2}{2}(X^2I)(\xi) + o(t^2) =$$

$$= \xi + t(XI)(\xi) + t(YI)(\xi) + t^2(X^2I)(\xi) + t^2(XYI)(\xi) + \frac{t^2}{2}(Y^2I)(\xi) +$$

$$-t((XI)(\xi) + t((X+Y)(XI))(\xi) + o(t)) + o(t^2) =$$

$$= \xi + t(YI)(\xi) + t^2(XYI)(\xi) + \frac{t^2}{2}(Y^2I)(\xi) + t^2(X^2I)(\xi) - t^2(X^2I)(\xi) - t^2(YXI)(\xi) + o(t^2) =$$

$$= \xi + t(YI)(\xi) + \frac{t^2}{2}(Y^2I)(\xi) + t^2([X,Y]I)(\xi) + o(t^2)$$

Finally we get

$$\exp(-tY)(-\exp(-tX)(\exp(tY)(\exp(tX)(\xi)))) =$$

$$= \exp(-tX)(\exp(tY)(\exp(tX)(\xi))) - t(YI)(\exp(-tX)(\exp(tY)(\exp(tX)(\xi)))) +$$

$$+\frac{t^2}{2}(Y^2I)(\exp(-tX)(\exp(tY)(\exp(tX)(\xi)))) + o(t^2) =$$

$$= \xi + t(YI)(\xi) + \frac{t^2}{2}(Y^2I)(\xi) + t^2([X,Y]I)(\xi) - t(YI)(\xi + t(YI)(\xi) + o(t)) +$$

$$+\frac{t^2}{2}(Y^2I)(\xi + o(1)) + o(t^2) =$$

$$= \xi + t(YI)(\xi) + \frac{t^2}{2}(Y^2I)(\xi) + t^2([X,Y]I)(\xi) +$$

$$-t((YI)(\xi) + t(Y^2I)(\xi) + o(t)) + \frac{t^2}{2}(Y^2I)(\xi) + o(t^2) =$$

$$= \xi + t^2([X,Y]I)(\xi) + o(t^2)$$

□

Definition 1.2.16. Let (M, Δ, g) be a subriemannian manifold where $\Delta = \text{span}(X_1 I, ..., X_m I)$ and $X_1, ..., X_m$ are differential operators of the first order with C^∞ coefficients. Let $\xi \in M$ and $X \in \Delta$. We denote with $C_X(t)$ the following application

$$C_X(t) = \exp(tX)(\xi)$$

More generally, if X is a differential operator of the first order with C^∞ coefficients such that $X \in \Delta^{(j)}$ (with $j \geq 2$) we have that exist $X_1 \in \Delta$ and $X_2 \in \Delta^{(j-1)}$ such that $X = [X_1, X_2]$ and we denote with $C_X(t)$ the following application

$$C_X(t) = \begin{cases} C_{X_2}(-|t|^{\frac{1}{j}})C_{X_1}(-|t|^{\frac{1}{j}})C_{X_2}(|t|^{\frac{1}{j}})C_{X_1}(|t|^{\frac{1}{j}}) \text{ if t} \geq 0 \\ C_{X_2}(-|t|^{\frac{1}{j}})C_{-X_1}(-|t|^{\frac{1}{j}})C_{X_2}(|t|^{\frac{1}{j}})C_{-X_1}(|t|^{\frac{1}{j}}) \text{ if t} < 0 \end{cases}$$

Theorem 1.2.17. *Let (M, Δ, g) be a subriemannian manifold where $\Delta = \text{span}(X_1 I, ..., X_m I)$ and $X_1, ..., X_m$ are differential operators of the first order with C^∞ coefficients. Let $\xi \in M$ and $X \in \Delta^{(j)}$ for some $j \geq 1$. Then C_X is an admissible curve, $C_X(0) = \xi$ and $C'_X(0) = (XI)(\xi)$.*

Proof. If $j = 1$ then $X \in \Delta$ and C_X is trivially an admissible curve since $C_X(t) = \exp(tX)(\xi)$ is the solution of the Cauchy problem

$$\begin{cases} \gamma'(t) = (XI)(\gamma(t)) \\ \gamma(0) = \xi \end{cases}$$

and $\exp(tX)(0) = \xi$, $\exp(tX)'(0) = (XI)(\xi)$. If $j = 2$ then there exist $X_1, X_2 \in \Delta$ such that $X = [X_1, X_2]$ and C_X is defined as

$$C_X(t) = \begin{cases} C_{X_2}(-|t|^{\frac{1}{2}})C_{X_1}(-|t|^{\frac{1}{2}})C_{X_2}(|t|^{\frac{1}{2}})C_{X_1}(|t|^{\frac{1}{2}}) \text{ if t} \geq 0 \\ C_{X_2}(-|t|^{\frac{1}{2}})C_{-X_1}(-|t|^{\frac{1}{2}})C_{X_2}(|t|^{\frac{1}{2}})C_{-X_1}(|t|^{\frac{1}{2}}) \text{ if t} < 0 \end{cases}$$

Now since $C_{\pm X_1}$ and $C_{\pm X_2}$ are admissible curves we have that also C_X is an admissible curve since the composition of admissible curves is an admissible curve. Now using theorem 1.2.15 we have (for $t \to 0$) that

$$C_X(t) = \begin{cases} \xi + |t|[X_1, X_2]I(\xi) + o(|t|) \text{ if } t \geq 0 \\ \xi + |t|[-X_1, X_2]I(\xi) + o(|t|) \text{ if } t > 0 \end{cases} = \xi + t[X_1, X_2]I(\xi) + o(t)$$

and so $C_X(0) = \xi$ and $C'_X(0) = ([X_1, X_2]I)(\xi) = (XI)(\xi)$. For $j > 2$ we just need to reiterate the previous reasoning. □

Proposition 1.2.18. *Let (M, Δ, g) be a subriemannian manifold of dimension n where $\Delta = \text{span}(X_1I, ..., X_mI)$, $X_1, ..., X_m$ are differential operators of the first order with C^∞ coefficients satisfying the Hörmander condition and $X_1, ..., X_m, ..., X_n$ is the corresponding basis adapted to the stratification. Let $\xi \in M$ and $t = (t_1, ..., t_n) \in \mathbb{R}^n$. We define $\tilde{C}$ as*

$$\tilde{C}(t)(\xi) = C_{X_1}(t_1)C_{X_2}(t_2)\cdots C_{X_n}(t_n)$$

then $\tilde{C}$ is a local diffeomorphism in a neighborhood of $t = 0$.

Proof. We just have to check that the derminant of the Jacobian of $\tilde{C}$ is non zero in $t = 0$. We calculate $\frac{\partial \tilde{C}}{\partial t_j}(0)$ remembering that

$$\tilde{C}(0, ..., 0, t_j, 0, ..., 0) = C_{X_j}(t_j)$$

and so

$$\frac{\partial \tilde{C}}{\partial t_j}(0) = \frac{d}{dt_j} C_{X_j}(0) = (X_j I)(\xi)$$

consequently we have

$$J_{\tilde{C}}(0) = ((X_1 I)(\xi), ..., (X_n I)(\xi))$$

but since $X_1, ..., X_n$ is a basis we have that $det(J_{\tilde{C}}(0)) \neq 0$. □

Notice that by the definition of $\tilde{C}$ we have that $\tilde{C}$ is a composition of admissible curves and consequently also $\tilde{C}$ is an admissible curve. The following corollary is an immediate consequence of the definition of local diffeomorphism.

Corollary 1.2.19. *Let (M, Δ, g) be a subriemannian manifold of dimension n where $\Delta = \text{span}(X_1 I, ..., X_m I)$, $X_1, ..., X_m$ are differential operators of the first order with C^∞ coefficients satisfying the Hörmander condition and $X_1, ..., X_m, ..., X_n$ is the corresponding basis adapted to the stratification. Let $\xi \in M$. Then there exist a neighborhood U of 0 in $\mathbb{R}^n$ and a neighborhood V of ξ in M such that for all $\eta \in V$ there exists a unique $t \in U$ such that $\eta = \tilde{C}(t)(\xi)$.*

Theorem 1.2.20. *Let (M, Δ, g) be a connected subriemannian manifold of dimension n where $\Delta = \text{span}(X_1 I, ..., X_m I)$ and $X_1, ..., X_m$ are differential operators of the first order with C^∞ coefficients satisfying the Hörmander condition. Then for each $\xi, \xi_0 \in M$ there exist at least an admissible curve of extremes ξ and ξ_0.*

Proof. We fix $\xi_0 \in M$ and we consider the following set

$$A = \{\xi \in M \text{ s.t. } \exists \gamma \text{ admissible curve of extremes } \xi \text{ and } \xi_0\}$$

Since M is connected if we prove that A is open,closed and $A \neq \emptyset$ we proved the theorem. $A \neq \emptyset$ since $\xi_0 \in A$. Now we prove that A is open. If $\delta \in A$ then by the previous corollary there exist a neighborhood U of 0 in $\mathbb{R}^n$ and a neighborhood V of δ in M such that for all $\eta \in V$ there exists a unique $t \in U$ such that $\eta = \tilde{C}(t)(\delta)$. Since $\tilde{C}$ is an admissible curve we have that there exist an admissible curve γ_1 between η and δ for all $\eta \in V$. But $\delta \in A$ so by the definition of A there exist an admissible curve γ_2 between δ and ξ_0. But if we compose γ_1 and γ_2 we get an admissible curve between ξ_0 and η for all $\eta \in V$ i.e. $V \subset A$ i.e. A is open. Now we prove that A is closed. Let $(\xi_n)_{n\geq 1} \subset A$ be a sequence that converges to a certain $\overline{\xi}$. If we prove that $\xi \in A$ then A is closed. By the previous corollary there exist a neighborhood U of 0 in $\mathbb{R}^n$ and a neighborhood V of $\overline{\xi}$ in M such that for all $\eta \in V$ there exists a unique $t \in U$ such that $\eta = \tilde{C}(t)(\overline{\xi})$. But since $\xi_n \to \overline{\xi}$ and V is a neighborhood of $\overline{\xi}$ there exist $\overline{n} \in \mathbb{N}$ such that for all $n \geq \overline{n}$ we have that $\xi_n \in V$. So (since $\tilde{C}$ is an admissible curve) we have that there exists an admissible curve γ_1 between $\xi_{\overline{n}}$ and $\overline{\xi}$. But $\xi_{\overline{n}} \in A$ so there exists an admissible curve γ_2 between $\xi_{\overline{n}}$ and ξ_0. So if we compose γ_1 and γ_2 we get an admissible curve between ξ_0 and $\overline{\xi}$ i.e. $\overline{\xi} \in A$ i.e. A is closed. □

An immediate consequence of this theorem is that if we consider a connected subriemannian manifold (M, Δ, g) where $\Delta = \text{span}(X_1 I, ..., X_m I)$ and $X_1, ..., X_m$ are differential operators of the first order with C^∞ coefficients satisfying the Hörmander condition then for all $p, q \in M$ we have that $d(p, q) < +\infty$.

1.3 Our choice of vector fields

We now introduce the vector fields that we wil use in the Chapter 4. Let M be a connected subriemannian manifold of dimension $2n + 1$. We define the following $2n$ vector fields. Let $1 \leq i \leq n$ (for convenience for $1 \leq j \leq 2n + 1$ we denote ∂_{x_j} with ∂_j)

$$X_i = \partial_i - \frac{\mathcal{K}_{n+i}}{2}\partial_{2n+1} \qquad X_{n+i} = \partial_{n+i} + \frac{\mathcal{K}_i}{2}\partial_{2n+1} \tag{1.1}$$

We now define (for $1 \leq i, j \leq 2n$) the application h_{ij} as the unique function such that

$$[X_i, X_j] = h_{ij}\partial_{2n+1}$$

In order to obtain the Hörmander condition we will choose as $\mathcal{K}_i$ C^∞ functions such that for each point p of M there exists at least one h_{ij} such that $h_{ij}(p) \neq 0$.
We also define (for $1 \leq i \leq 2n$) the application k_i as the unique function such that

$$[Z, X_i] = k_i Z$$

where we used Z to denote ∂_{2n+1}.
This choice of vector field is modeled on the vector fields usually considered on the Heisenberg group which is the case obtained when we choose $\mathcal{K}_i = x_i$.

Chapter 2

Parabolic L^p spaces and Sobolev subriemannian spaces

2.1 Some basic facts on $L^p(\Omega)$ and $W^{k,p}(\Omega)$ spaces

Definition 2.1.1. A triple $(\Omega, \mathcal{M}, \mu)$ is a **measure space** if Ω is a set and

1. $\mathcal{M}$ is a $\sigma-$algebra in Ω i.e. $\mathcal{M}$ is a collection of subsets of Ω such that:

    (a) $\emptyset \in \mathcal{M}$,

    (b) $A \in \mathcal{M} \Rightarrow A^C \in \mathcal{M}$,

    (c) $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$ whenever $A_n \in \mathcal{M}$ $\forall n$,

2. μ is a measure i.e. $\mu : \mathcal{M} \to [0, +\infty]$ satisfies

    (a) $\mu(\emptyset) = 0$,

    (b) $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} \mu(A_n)$ whenever $(A_n)_{n\geq 1}$ is a disjoint countable family of members of $\mathcal{M}$.

3. Ω is $\sigma-$finite i.e. there exists a countable family $(\Omega_n)_{n\geq 1}$ in $\mathcal{M}$ such that $\bigcup_{n=1}^{\infty} \Omega_n$ and $\mu(\Omega_n) < +\infty$ $\forall n$.

Definition 2.1.2. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space; we denote by $L^1(\Omega, \mu)$ (or simply $L^1(\Omega)$ or L^1) the space of integrable functions from Ω to $\mathbb{R}$ (where we identify two functions that coincide a.e.). We also denote with $||\cdot||_{L^1(\Omega)}$ the following quantity

$$||f||_{L^1(\Omega)} = \int_\Omega |f| d\mu$$

We now recall some important results about integration.

Theorem 2.1.3 (Monotone convergence theorem)**.** *Let $(f_n)_{n\geq 1}$ be a sequence of functions in L^1 that satisfy*

1. $f_1 \leq f_2 \leq \cdots \leq f_n \leq f_{n+1} \leq \cdots$ *a.e. on* Ω,
2. $\sup_n \int f_n < \infty$.

Then $f_n(x)$ converges a.e. on Ω to a finite limit, which we denote by $f(x)$; the function f belongs to L^1 and $||f_n - f||_{L^1} \to 0$.

Proof. See [R] p.21. □

Theorem 2.1.4 (Dominated convergence theorem)**.** *Let $(f_n)_{n\geq 1}$ be a sequence of functions in L^1 that satisfy*

1. $f_n(x) \to f(x)$ *a.e. on* Ω,
2. *there is a function $g \in L^1$ such that for all n, $|f_n(x)| \leq g(x)$ a.e. on Ω.*

Then $f \in L^1$ and $||f_n - f||_{L^1} \to 0$.

Proof. See [R] p.26. □

Theorem 2.1.5 (Fatou's lemma)**.** *Let $(f_n)_{n\geq 1}$ be a sequence of functions in L^1 that satisfy*

1. *for all n, $f_n \geq 0$ a.e.*

2. $\sup_n \int f_n < \infty$.

For almost all $x \in \Omega$ *we set* $f(x) = \liminf_{n\to+\infty} f_n(x) \leq +\infty$. *Then* $f \in L^1$ *and*

$$\int f \leq \liminf_{n\to+\infty} \int f_n$$

Proof. See [R] p.23. □

Definition 2.1.6. Let $p \in \mathbb{R}$ with $1 < p < +\infty$; we set

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R} \text{ s.t. } f \text{ is measurable and } |f|^p \in L^1(\Omega)\}$$

with

$$||f||_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p d\mu\right)^{\frac{1}{p}}$$

and we set

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{R} \text{ s.t. } f \text{ is measurable and } \exists C \in \mathbb{R} \text{ s.t. } |f(x)| \leq C \text{ a.e. on } \Omega\}$$

with

$$||f||_{L^\infty(\Omega)} = \text{ess sup}_\Omega |f|$$

where

$$\text{ess sup}_\Omega |f| = \inf\{C \text{ s.t. } |f(x)| \leq C \text{ a.e. on } \Omega\}$$

Definition 2.1.7. Let $1 \leq p \leq +\infty$; we denote by p' the **conjugate exponent** which is the $1 \leq p' \leq +\infty$ such that

$$\frac{1}{p} + \frac{1}{p'} = 1$$

Theorem 2.1.8 (Hölder's inequality)**.** *Assume that* $f \in L^p(\Omega)$ *and* $g \in L^{p'}(\Omega)$ *with* $1 \leq p \leq +\infty$. *Then* $fg \in L^1(\Omega)$ *and*

$$||fg||_{L^1(\Omega)} \leq ||f||_{L^p(\Omega)}||g||_{L^{p'}(\Omega)}$$

Proof. See [Bre] p.92. □

Theorem 2.1.9 (Minkowski's inequality)**.** *Let* $1 \le p \le +\infty$ *and* $f, g \in L^p(\Omega)$. *Then*

$$||f+g||_{L^p(\Omega)} \le ||f||_{L^p(\Omega)} + ||g||_{L^p(\Omega)}$$

Proof. See [Bre] p.93. □

Theorem 2.1.10. *For any* $1 \le p \le +\infty$ $L^p(\Omega)$ *is a Banach space when equipped with the norm* $||\cdot||_{L^p(\Omega)}$.

Proof. See [Bre] p.93. □

Definition 2.1.11. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in L^1(\Omega)$; a function $g \in L^1(\Omega)$ is said to be the α**-weak derivative** of u (and denoted with $\partial_\alpha u$) if for any $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega u\partial_\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega g\varphi dx$$

Definition 2.1.12. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $p \in [1, +\infty]$, $k \in \mathbb{N}$, $k \ge 1$. The **Sobolev space** $W^{k,p}(\Omega)$ is the set of all the functions $u \in L^p(\Omega)$ such that their weak derivatives $\partial_\alpha u \in L^p(\Omega)$ for all $|\alpha| \le k$. The space $W^{k,p}(\Omega)$ with $p \in [1, +\infty)$ is equipped with the norm

$$||u||_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \le k} ||\partial_\alpha u||^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

and the space $W^{k,\infty}(\Omega)$ is equipped with the norm

$$||u||_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| \le k} \text{ess sup}_\Omega |\partial_\alpha u|$$

Theorem 2.1.13. *The Sobolev spaces are Banach spaces.*

Proof. See [Bre] p.203. □

2.2 Subriemannian Sobolev spaces

Definition 2.2.1. Let (M, Δ, g) be a subriemannian manifold where $\Delta = \text{span}(X_1 I, ..., X_m I)$, $X_1, ..., X_m$ are differential operators of the first order with C^∞ coefficients satisfying the Hörmander condition. Let $\Omega \subset M$ be an open subset and $p \in [1, +\infty]$. We define the **subriemannian Sobolev space** $W^{1,p}(\Omega)$ as the set of all the functions $u \in L^p(\Omega)$ such that all the distributional derivatives $X_i u$ for $1 \leq i \leq m$ are also in $L^p(\Omega)$. The subriemannian Sobolev space $W^{1,p}(\Omega)$ is a Banach space when equipped with the norm

$$||u||^p_{W^{1,p}(\Omega)} = ||u||_{L^p(\Omega)} + ||\nabla_0 u||_{L^p(\Omega)}$$

Where we used $\nabla_0 u$ to denote $(X_1 u, ..., X_m u)$. We use the notation $W^{1,p}_{loc}(\Omega)$ to denote the local version of these spaces.

We have the following useful reformulation of the Sobolev embedding theorem

Theorem 2.2.2. *Let v be a Lipschitz function in $Q = \Omega \times (0, T)$ and assume that for all $0 < t < T$ $v(\cdot, t)$ has compact support in $\Omega \times \{t\}$. Let $N = 2n + 2$. Then there exists $C > 0$ such that*

$$||v||_{L^{\frac{2N}{N-2},2}(Q)} \leq C||\nabla_0 u||_{L^{2,2}(Q)}$$

Proof. See [FS]. □

For completeness we recall also a proof of the "classical" Sobolev inequality

Theorem 2.2.3. *Let $1 \leq p < n$. Then there exists a constant C depending only on n and p such that for all $u \in C^1_c(\mathbb{R}^n)$ we have*

$$||u||_{L^{p^*}} \leq C||\nabla u||_{L^p}$$

where $p^* = \frac{np}{n-p}$.

Proof. First we prove the theorem for $p = 1$. Since u has compact support in $\mathbb{R}^n$ we have for each $1 \leq i \leq n$ that

$$u(x) = \int_{-\infty}^{x_i} \partial_{x_i} u(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n) dy_i$$

and so

$$|u(x)| \leq \int_{-\infty}^{+\infty} |\nabla u(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)| dy_i$$

Consequently we get

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{+\infty} |\nabla u(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)| dy_i \right)^{\frac{1}{n-1}}$$

Now we integrate this inequality with respect to x_1 and by the use of Hölder's inequality we get

$$\int_{-\infty}^{+\infty} |u(x)|^{\frac{n}{n-1}} dx_1 \leq \int_{-\infty}^{+\infty} \prod_{i=1}^{n} \left(\int_{-\infty}^{+\infty} |\nabla u(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)| dy_i \right)^{\frac{1}{n-1}} dx_1 =$$

$$= \left(\int_{-\infty}^{+\infty} |\nabla u| dy_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{+\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{+\infty} |\nabla u| dy_i \right)^{\frac{1}{n-1}} dx_1 \leq$$

$$\leq \left(\int_{-\infty}^{+\infty} |\nabla u| dy_1 \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\nabla u| dx_1 dy_i \right)^{\frac{1}{n-1}}$$

Now we integrate this inequality with respect to x_2 and we get

$$\int_{-\infty}^{+\infty} |u|^{\frac{n}{n-1}} dx_1 dx_2 \leq \left(\int_{-\infty}^{+\infty} |\nabla u| dx_1 dy_2 \right)^{\frac{1}{n-1}} \int_{-\infty}^{+\infty} \prod_{i=1, i\neq 2}^{n} I_i^{\frac{1}{n-1}} dx_2$$

where

$$I_i := \begin{cases} \int_{-\infty}^{+\infty} |\nabla u| dy_1 \text{ if } i = 1 \\ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\nabla u| dx_1 dy_i \text{ if } i > 1 \end{cases}$$

Applying the Hölder's inequalty again we obtain

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|u|^{\frac{n}{n-1}}dx_1dx_2 \leq \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|\nabla u|dx_1dy_2\right)^{\frac{1}{n-1}}\cdot$$

$$\cdot\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|\nabla u|dy_1dx_2\right)^{\frac{1}{n-1}}\prod_{i=3}^{n}\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|\nabla u|dx_1dx_2dy_i\right)^{\frac{1}{n-1}}$$

We continue to integrate with respect to $x_3, ..., x_n$ finally obtaining

$$\int_{\mathbb{R}^n}|u(x)|^{\frac{n}{n-1}}dx \leq \prod_{i=1}^{n}\left(\int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty}|\nabla u|dx_1...dy_i...dx_n\right)^{\frac{1}{n-1}} = \int_{\mathbb{R}^n}|\nabla u(x)|dx \quad (2.1)$$

i.e. we proved the theorem for $p = 1$. Now we consider the case $1 < p < n$. We use the estimate (2.1) on the function $v = |u|^\gamma$ where $\gamma > 1$ is to be selected. So we get

$$\left(\int_{\mathbb{R}^n}|u(x)|^{\frac{\gamma n}{n-1}}dx\right)^{\frac{n-1}{1}} \leq \int_{\mathbb{R}^n}|\nabla|u(x)|^\gamma|dx = \gamma\int_{\mathbb{R}^n}|u(x)|^{\gamma-1}|\nabla u(x)|dx \leq$$

$$\leq \gamma\left(\int_{\mathbb{R}^n}|u(x)|^{(\gamma-1)\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^n}|\nabla u(x)|^p dx\right)^{\frac{1}{p}}$$

If we choose γ such that

$$\frac{\gamma n}{n-1} = (\gamma-1)\frac{p}{p-1}$$

i.e. we set

$$\gamma := \frac{p(n-1)}{n-p} > 1$$

we obtain that

$$\frac{\gamma n}{n-1} = (\gamma-1)\frac{p}{p-1} = \frac{np}{n-p} = p^*$$

and so our last estimate becomes

$$||u||_{L^{p^*}(\mathbb{R}^n)} \leq \frac{p(n-1)}{n-p}||\nabla u||_{L^p(\mathbb{R}^n)}$$

i.e. we proved the theorem for $1 < p < n$. □

2.3 $L^p(0,T;X)$ spaces

Definition 2.3.1. Let X be a Banach space and $T > 0$, a function $s : [0,T] \to X$ is called **simple** if it has the form

$$s(t) = \sum_{i=1}^{m} \kappa_{E_i}(t)u_i \qquad (0 \leq t \leq T)$$

where each E_i is a Lebesque measurable subset of $[0,T]$ and $u_i \in X$ $(i = 1, \dots, m)$.

Definition 2.3.2. Let X be a Banach space and $T > 0$. A function $u : [0,T] \to X$ is **strongly measurable** if there exist simple functions $s_k : [0,T] \to X$ such that

$$s_k(t) \to u(t) \text{ for a.e. } 0 \leq t \leq T$$

Definition 2.3.3. Let X be a Banach space, $T > 0$ and $p \in [1, +\infty]$. Let $u : [0,T] \to X$. We say that $u \in L^p(0,T;X)$

1. u is strongly measurable,

2. $||u||_{L^p(0,T;X)} < +\infty$ where $||\cdot||_{L^p(0,T;X)}$ is defined as follows

$$||u||_{L^p(0,T;X)} := \begin{cases} \left(\int_0^T ||u(t)||_X^p dt\right)^{\frac{1}{p}} \text{ if } p \in [1, +\infty) \\ \text{ess sup}_{0 \leq t \leq T} ||u(t)||_X \text{ if } p = +\infty \end{cases}$$

Theorem 2.3.4 (Minkowski's inequality for $L^p(0,T;X)$ spaces)**.** *Let X be a Banach space, $T > 0$ and $p \in [1, +\infty]$. Let $f, g \in L^p(0,T;X)$. Then*

$$||f + g||_{L^p(0,T;X)} \leq ||f||_{L^p(0,T;X)} + ||g||_{L^p(0,T;X)}$$

Proof. Since $||\cdot||_X$ is a norm we have for $t \in [0,T]$ that

$$||f(t)+g(t)||_X \leq ||f(t)||_X + ||g(t)||_X$$

and so

$$\left(\int_0^T ||f(t)+g(t)||_X^p dt\right)^{\frac{1}{p}} \leq \left(\int_0^T (||f(t)||_X + ||g(t)||_X)^p \, dt\right)^{\frac{1}{p}}$$

but using the Minkowski's inequality for scalar functions on the RHS we get

$$\left(\int_0^T (||f(t)||_X + ||g(t)||_X)^p \, dt\right)^{\frac{1}{p}} \leq \left(\int_0^T ||f(t)||_X^p dt\right)^{\frac{1}{p}} + \left(\int_0^T ||g(t)||_X^p dt\right)^{\frac{1}{p}}$$

and so

$$\left(\int_0^T ||f(t)+g(t)||_X^p dt\right)^{\frac{1}{p}} \leq \left(\int_0^T ||f(t)||_X^p dt\right)^{\frac{1}{p}} + \left(\int_0^T ||g(t)||_X^p dt\right)^{\frac{1}{p}}$$

i.e.

$$||f+g||_{L^p(0,T;X)} \leq ||f||_{L^p(0,T;X)} + ||g||_{L^p(0,T;X)}$$

□

Lemma 2.3.5. *Let X be a Banach space and $T > 0$. If $(f_n)_{n\geq 0}$, $f_n : [0,T] \to X$ is a sequence such that*

1. *f_n is strongly measurable for each $n \geq 1$,*
2. *$(f_n)_{n\geq 1}$ converges a.e. on $[0,T]$ to a certain f*

then f is strongly measurable.

Proof. Since every f_n is strongly measurable we have that for each $n \in \mathbb{N}$ exists a sequence $(s_{n_k})_{k\geq 1}$, $s_{n_k} : [0,T] \to X$ simple functions such that

$$||s_{n_k}(t) - f_n(t)||_X \xrightarrow{k\to+\infty} 0 \text{ a.e. on } [0,T]$$

So for each $n \in \mathbb{N}$ we can choose a simple function s_n in the previous sequence such that

$$||s_n(t) - f_n(t)||_X \leq \frac{1}{n} \text{ a.e. on } [0,T]$$

If we use this new sequence of simple functions to approximate f we get a.e. on $[0,T]$

$$||s_n(t) - f(t)||_X = ||s_n(t) - f_n(t) + f_n(t) - f(t)||_X \leq ||s_n(t) - f_n(t)||_X + ||f_n(t) - f(t)||_X \leq$$

$$\leq \frac{1}{n} + ||f_n(t) - f(t)||_X$$

letting $n \to +\infty$ we obtain the thesis since $\lim_{n\to+\infty} ||f_n(t) - f(t)||_X \to 0$ a.e on $[0,T]$. $\square$

The following theorem proves the completeness of the $L^p(0,T;X)$ spaces. The proof is based on the proof of the completeness of the $L^p(\Omega)$ spaces in [Bre].

Theorem 2.3.6. *Let X be a Banach space, $T > 0$ and $p \in [1, +\infty]$. Then $L^p(0,T;X)$ is a Banach space.*

Proof. Case $p \in [1, +\infty)$: Let $(f_n)_{n\geq 1} \subset L^p(0,T;X)$ be a Cauchy sequence. We can now extract a subsequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$ such that for each $k \geq 1$ we have

$$||f_{n_{k+1}} - f_{n_k}||_{L^p(0,T;X)} \leq \frac{1}{2^k}$$

We can construct this subsequence simply using the definiton of Cauchy sequence: first we choose n_1 such that $||f_m - f_n||_{L^p(0,T;X)} \leq \frac{1}{2}$ for each $m, n \geq n_1$, then we choose $n_2 > n_1$ such that $||f_m - f_n||_{L^p(0,T;X)} \leq \frac{1}{2^2}$ for each $m, n \geq n_2$ and so on... . For convenience we write f_k instead of f_{n_k}. Now we define the function $g_n : [0,T] \to \mathbb{R}$ as

$$g_n(t) = \sum_{k=1}^{n} ||f_{k+1}(t) - f_k(t)||_X$$

So we have that

$$||g_n||_{L^p([0,T])} = \left(\int_0^T |g_n(t)|^p dt\right)^{\frac{1}{p}} = \left(\int_0^T \left|\sum_{k=1}^{n} ||f_{k+1}(t) - f_k(t)||_X\right|^p dt\right)^{\frac{1}{p}} \leq$$

$$\leq \sum_{k=1}^{n} \left(\int_0^T ||f_{k+1}(t) - f_k(t)||_X^p dt\right)^{\frac{1}{p}} = \sum_{k=1}^{n} ||f_{k+1} - f_k||_{L^p(0,T;X)} \leq$$

$$\leq \sum_{k\geq 1} ||f_{k+1} - f_k||_{L^p(0,T;X)} \leq \sum_{k\geq 1} \frac{1}{2^k} = 1$$

i.e. $g_n \in L^p([0,T])$. As a consequence of the monotone convergence theorem we have that there exists a function $g \in L^p([0,T])$ such that $g_n \to g$ a.e. on $[0,T]$. Then we have for $m \geq n \geq 2$ that

$$||f_m(t) - f_n(t)||_X \leq ||f_m(t) - f_{m-1}(t)||_X + \cdots + ||f_{n+1}(t) - f_n(t)||_X \leq g(t) - g_{n-1}(t)$$

It follows that a.e. on $[0,T]$ $(f_n(t))_{n\geq 1}$ is a Cauchy sequence in X. Since X is a Banach space we get that a.e. on $[0,T]$ $(f_n(t))_{n\geq 1}$ converges to a certain $f(t) \in X$. Now we observe that

$$f_n = f_1 + \sum_{k=1}^{n-1}(f_{k+1} - f_k)$$

and so a.e. on $[0,T]$ we get

$$f(t) = f_1(t) + \lim_{n\to+\infty} \sum_{k=1}^{n}(f_{k+1}(t) - f_k(t))$$

and so

$$||f||_{L^p(0,T;X)} = \left|\left|f_1 + \lim_{n\to+\infty} \sum_{k=1}^{n}(f_{k+1} - f_k)\right|\right|_{L^p(0,T;X)} \leq$$

$$\leq ||f_1||_{L^p(0,T;X)} + \sum_{k\geq 1} ||f_{k+1} - f_k||_{L^p(0,T;X)} \leq ||f_1||_{L^p(0,T;X)} + 1 < +\infty$$

Using also the lemma (2.3.5) we get that $f \in L^p(0,T;X)$. Now we want to prove that $f_n \to f$ in $L^p(0,T;X)$. We have that (by Fatou's lemma)

$$||f - f_n||_{L^p(0,T;X)} \leq \liminf_{k \to +\infty} ||f_k - f_n||_{L^p(0,T;X)}$$

but this concludes the proof for $p \in [1, +\infty)$ since $(f_n)_{n \geq 1}$ was a Cauchy sequence.

Case $p = +\infty$: Let $(f_n)_{n \geq 1} \subset L^\infty(0,T;X)$ be a Cauchy sequence. For each $k \in \mathbb{N}$ exists $N_k \in \mathbb{N}$ such that $||f_m - f_n||_{L^\infty(0,T;X)} \leq \frac{1}{k}$ for $m, n \geq N_k$. So there exists a null set $E_k \subset [0,T]$ such that

$$||f_m(t) - f_n(t)||_X \leq \frac{1}{k} \ \forall t \in [0,T] \setminus E_k \ \forall m, n \geq N_k$$

Then we define $E = \bigcup_{k \in \mathbb{N}} E_k$, which is still a null set, and we observe that for all $t \in [0,T] \setminus E$ the sequence $(f_n(t))_{n \geq 1}$ is a Cauchy sequence in X. But X is a Banach space so $(f_n(t))_{n \geq 1}$ converges to a certain $f(t)$ for all $t \in [0,T] \setminus E$. Passing to the limit in the previous inequality as $m \to +\infty$ we get

$$||f(t) - f_n(t)||_X \leq \frac{1}{k} \ \forall t \in [0,T] \setminus E \ \forall n \geq N_k$$

In the same way as we discussed the case p finite we can get that $||f||_{L^\infty(0,T;X)} < +\infty$ and $f \in L^\infty(0,T;X)$. Remembering the definition of $|| \cdot ||_{L^\infty(0,T;X)}$ we get finally that $f_n \to f$ in $L^\infty(0,T;X)$. □

Corollary 2.3.7. *Let $\Omega \subseteq \mathbb{R}^n$, $p, q \in [0, +\infty]$ and $T > 0$. Then the space*

$$L^p(0,T;L^q(\Omega))$$

is a Banach space when equipped with the norm

$$||f||_{L^p(0,T;L^q(\Omega))} = \left(\int_0^T \left(\int_\Omega |f(x,t)|^q dx\right)^{\frac{p}{q}} dt\right)^{\frac{1}{p}} \quad \textit{if } p,q \in [1,+\infty)$$

$$||f||_{L^p(0,T;L^q(\Omega))} = \left(\int_0^T (\text{ess sup}_{x\in\Omega} |f(t,x)|)^p\, dt\right)^{\frac{1}{p}} \quad \textit{if } p \in [1,+\infty), q = +\infty$$

$$||f||_{L^p(0,T;L^q(\Omega))} = \text{ess sup}_{t\in[0,T]} \left(\left(\int_\Omega |f(x,t)|^q dx\right)^{\frac{1}{q}}\right) \quad \textit{if } p = +\infty, q \in [1,+\infty)$$

$$||f||_{L^p(0,T;L^q(\Omega))} = \text{ess sup}_{t\in[0,T]} (\text{ess sup}_{x\in\Omega} |f(x,t)|) \quad \textit{if } p = q + \infty$$

In the following we will use the notation $L^{p,q}(Q)$ (with $Q = \Omega \times (0,T)$) to denote the space $L^q(0,T;L^p(\Omega))$.

Chapter 3

The Moser iteration in the elliptic case

We consider the following PDE

$$\sum_{i,j=1}^{n} \partial_j(a_{ij}\partial_i u) = 0 \tag{3.1}$$

on the open ball $B_{2r} = B(0, 2r) \subset \mathbb{R}^n$. We assume the following structural conditions on the $a_{ij} \in L^\infty(B_{2r})$: we suppose that for any $x \in B_{2r}$ and for any $\xi \in \mathbb{R}^n$ there exist $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \tag{3.2}$$

We say that a function $u \in H^1(B_{2r}) = W^{1,2}(B_{2r})$ is a weak solution of (3.1) if

$$\sum_{i,j=1}^{n} \int_{B_{2r}} a_{ij}\partial_i u \partial_j \phi dx = 0$$

for any $\phi \in H_0^1(B_{2r})$. The aim of this chapter will be to prove the following Theorem 3.0.1. In other words we will prove the local boundedness of the weak solutions of the PDE (3.1). We will mainly follow the presentation given in [HL].

Theorem 3.0.1. *Let $u \in H^1(B_{2r})$ be a weak solution of* (3.1)*. Then we have that for any $p \geq 2$ there exists a constant $C = C(n, \lambda, \Lambda, p) > 0$ such that*

$$\sup_{B_r} |u| \leq C \left(\fint_{B_{2r}} |u|^p dx \right)^{\frac{1}{p}}$$

In order to prove the theorem we will need a Sobolev inequality (Theorem (2.2.3)) and the following Caccioppoli inequality.

Proposition 3.0.2. *Let u be a weak solution of* (3.1)*, $\beta \geq 0$ and $\eta \in C_0^1(B_{2r})$ a non negative function. Then there exists a constant $C = C(n, \lambda, \Lambda) > 0$ such that*

$$\int_{B_{2r}} |\nabla(|u|^{\frac{\beta+2}{2}} \eta)|^2 dx \leq C(1+\beta) \int_{B_{2r}} |\nabla \eta|^2 |u|^{\beta+2} dx$$

Proof. We choose the test function ϕ as

$$\phi = \eta^2 |u|^\beta u \in H_0^1(B_1)$$

where u is a weak solution of the (3.1), $\beta \geq 0$ and some non negative function $\eta \in C_0^1(B_{2r})$. By direct calculations we get that

$$\partial_l \phi = \eta^2 |u|^\beta \partial_l u(1+\beta) + 2\eta \partial_l \eta |u|^\beta u \tag{3.3}$$

Now we integrate on B_{2r} and by (3.3), the structural assumptions, the fact that u is a weak solution and the Young's inequality we get

$$
\begin{aligned}
0 = \sum_{i,j=1}^{n} \int_{B_{2r}} a_{ij}\partial_i u \partial_j \phi dx = (1+\beta) \sum_{i,j=1}^{n} \int_{B_{2r}} a_{ij}\partial_i u \partial_j u \eta^2 |u|^\beta dx + \\
+ 2 \sum_{i,j=1}^{n} \int_{B_{2r}} a_{ij}\partial_i u \partial_j \eta |u|^\beta u \eta dx \geq \\
\geq \lambda(1+\beta) \int_{B_{2r}} \eta^2 |u|^\beta |\nabla u|^2 dx - \Lambda \int_{B_{2r}} |\nabla u||\nabla \eta||u|^\beta u \eta dx \geq \\
\geq \lambda \left(\frac{1}{2}+\beta\right) \int_{B_{2r}} \eta^2 |u|^\beta |\nabla u|^2 dx - \frac{2\Lambda^2}{\lambda} \int_{B_{2r}} |\nabla \eta|^2 |u|^\beta u^2 dx \quad (3.4)
\end{aligned}
$$

i.e.

$$
\int_{B_{2r}} \eta^2 |u|^\beta |\nabla u|^2 dx \leq \frac{2\Lambda^2}{\lambda^2 \left(\frac{1}{2}+\beta\right)} \int_{B_{2r}} |\nabla \eta|^2 |u|^\beta u^2 dx \qquad (3.5)
$$

Now by direct calculations we get

$$
|\nabla(|u|^{\frac{\beta+2}{2}}\eta)|^2 \leq C(1+\beta)|u|^\beta |\nabla u|^2 |\eta|^2 + |u|^{\beta+2}|\nabla \eta|^2 \qquad (3.6)
$$

Now integrating (3.6) on B_{2r} and using the previous estimate (3.5) we can finally get the following Caccioppoli inequality.

$$
\int_{B_{2r}} |\nabla(|u|^{\frac{\beta+2}{2}}\eta)|^2 dx \leq C(1+\beta) \int_{B_{2r}} |\nabla \eta|^2 |u|^{\beta+2} dx \qquad (3.7)
$$

□

Now we have all the ingredients to prove Theorem 3.0.1.

Proof. The idea is to establish a reversed Hölder inequality

$$
\left(\fint_{B_{r_1}} |u|^{p_1} dx \right)^{\frac{1}{p_1}} \leq C \left(\fint_{B_{r_2}} |u|^{p_2} dx \right)^{\frac{1}{p_2}}
$$

for $p_1 > p_2$ and $r_1 < r_2$ by choosing an appropriate test function. Then by iteration and by choosing carefully the sequences of $\{r_i\}$ and $\{p_i\}$ we will get the result.

Using the Sobolev's inequality on the left hand side of (3.7) we get that

$$\left(\int_{B_{2r}} (\eta|u|^{\frac{\beta+2}{2}})^{2\kappa} dx\right)^{\frac{1}{\kappa}} \leq C \int_{B_{2r}} |\nabla(|u|^{\frac{\beta+2}{2}}\eta)|^2 dx \tag{3.8}$$

where $\kappa = \frac{n}{n-2}$. Now combining (3.7) and (3.8) we obtain

$$\left(\int_{B_{2r}} \eta^{2\kappa}|u|^{(\beta+2)\kappa} dx\right)^{\frac{1}{\kappa}} \leq C(1+\beta) \int_{B_{2r}} |\nabla\eta|^2|u|^{\beta+2} dx \tag{3.9}$$

Now we choose the cut-off function η in (3.9) in the following way: for any $0 < r' < 2r$ we set $\eta \in C_0^1(B_{2r})$ such that

$$\eta \equiv 1 \text{ in } B_{r'}$$

and

$$|\nabla\eta| \leq \frac{2}{2r - r'}$$

Putting such function in (3.9) we get

$$\left(\fint_{B_{r'}} |u|^{(\beta+2)\kappa} dx\right)^{\frac{1}{\kappa}} \leq C\frac{(1+\beta)}{(2r-r')^2} \fint_{B_{2r}} |u|^{\beta+2} dx \tag{3.10}$$

and setting $\gamma = \beta + 2$ we obtain

$$\left(\fint_{B_{r'}} |u|^{\gamma\kappa} dx\right)^{\frac{1}{\kappa}} \leq C\frac{(\gamma-1)}{(2r-r')^2} \fint_{B_{2r}} |u|^{\gamma} dx \tag{3.11}$$

i.e.

$$\left(\fint_{B_{r'}} |u|^{\gamma\kappa} dx\right)^{\frac{1}{\gamma\kappa}} \leq \left(C\frac{(\gamma-1)}{(2r-r')^2}\right)^{\frac{1}{\gamma}} \left(\fint_{B_{2r}} |u|^{\gamma} dx\right)^{\frac{1}{\gamma}} \tag{3.12}$$

Now we are finally ready to start the Moser iteration and conclude the proof of the theorem. We choose two sequences of $\{\gamma_i\}$ and $\{r_i\}$ (for $i \geq 0$) in the following way

$$\gamma_i = p\kappa^i$$

and

$$r_i = (1 + 2^{-i})r$$

Now we observe that

$$\gamma_i = \kappa\gamma_{i-1}$$

and

$$r_{i-1} - r_i = \frac{r}{2^i}$$

Using these sequences of $\{\gamma_i\}$ and $\{r_i\}$ in (3.12) we get

$$\left(\fint_{B_{r_i}} |u|^{\gamma_i} dx \right)^{\frac{1}{\gamma_i}} \leq C^{\frac{i}{\kappa^i}} \left(\fint_{B_{r_{i-1}}} |u|^{\gamma_{i-1}} dx \right)^{\frac{1}{\gamma_{i-1}}} \tag{3.13}$$

Iterating (3.13) we finally get

$$\left(\fint_{B_{r_i}} |u|^{\gamma_i} dx \right)^{\frac{1}{\gamma_i}} \leq C^{\sum_i \frac{i}{\kappa^i}} \left(\fint_{B_{2r}} |u|^p dx \right)^{\frac{1}{p}} \tag{3.14}$$

and in particular we get

$$\left(\fint_{B_r} |u|^{p\kappa^i} dx \right)^{\frac{1}{p\kappa^i}} \leq C \left(\fint_{B_{2r}} |u|^p dx \right)^{\frac{1}{p}}$$

Letting $i \to +\infty$ we finallly get the thesis i.e.

$$\sup_{B_r} |u| \leq C \left(\fint_{B_{2r}} |u|^p dx \right)^{\frac{1}{p}}$$

□

Chapter 4

Lipschitz regularity for weak solutions

4.1 Introduction of the problem

We want to prove the local Lipschitz regularity of the weak solutions to a class of parabolic PDEs modeled on the parabolic $p-$Laplacian. We consider the equation

$$\partial_t u = \sum_{i=1}^{2n} X_i(A_i(x, \nabla_0 u)) \tag{4.1}$$

in a cylinder $Q = \Omega \times (0, T)$ where Ω is an open subset of a connected subriemannian manifold M of dimension $2n+1$ and $T > 0$. The $X_1, ..., X_{2n}$ are the vector fields definded in Section 1.3 and $\nabla_0 u = (X_1 u, ..., X_{2n} u)$ denotes the horizontal gradient. We assume the following structural conditions on the $A_i(x, \xi)$: we suppose that there exists $2 \leq p \leq 4$ and $0 < \lambda' \leq \Lambda' < \infty$ such that for a.e. $x \in \Omega, \xi \in \mathbb{R}^{2n}$ and for all $\eta \in \mathbb{R}^{2n}$ one has

$$\begin{cases} \lambda' |\xi|^{p-2} |\eta|^2 \leq \partial_{\xi_j} A_i(x, \xi) \eta_i \eta_j \leq \Lambda' |\xi|^{p-2} |\eta|^2 \\ |A_i(x, \xi)| + |\partial_{x_j} A_i(x, \xi)| \leq \Lambda' |\xi|^{p-1} \end{cases} \tag{4.2}$$

The simplest example of such A_i is $A_i(x,\xi) = |\xi|^{p-2}\xi_i$.

We say that a function $u \in L^p(0,T;W^{1,p}_{loc}(\Omega))$ is a weak solution of (4.1) if

$$\int_0^T \int_\Omega \left(u\phi_t - \sum_{i=1}^{2n} A_i(x,\nabla_0 u)X_i\phi \right) dxdt = 0$$

for all $\phi \in C_0^\infty(Q)$.

4.2 Approximation of the weak solutions

In order to obtain our results we will use a riemannian approximation scheme. We consider the vector fields $X_1, ..., X_{2n}$ defined in Section 1.3 and the corresponding subriemannian metric g_0 defined by $\langle X_i, X_j\rangle = \delta_{ij}$. We now define the following sets of vector fields $X_1^\varepsilon, ..., X_{2n}^\varepsilon, X_{2n}^\varepsilon$ in the following way: $X_i^\varepsilon := X_i$ if $1 \leq i \leq 2n$ and $X_{2n+1}^\varepsilon := \varepsilon Z$ and the corresponding riemannian metric g_ε defined by $\langle X_i^\varepsilon, X_j^\varepsilon\rangle = \delta_{ij}$. We observe that the corresponding gradient ∇_ε has the obvious property that $\nabla_\varepsilon f \to (\nabla_0 f, 0)$ as $\varepsilon \to 0$; in fact we have that

$$\nabla_\varepsilon f = \sum_{i=1}^{2n} X_i f X_i + \varepsilon^2 Z f Z$$

We note explicitly that

$$|\nabla_\varepsilon f|^2_{g_\varepsilon} = \sum_{i=1}^{2n} (X_i f)^2 + \varepsilon^2 (Zf)^2 \xrightarrow{\varepsilon\to 0} |\nabla_0 f|^2_{g_0}$$

Now we prove the regolarization result that will allow us to work with smooth functions.

Lemma 4.2.1. *Let u be a weak solution of* (4.1) *in $Q = \Omega \times (0,T)$ with the structure conditions* (4.2)*. Then for any sub-cylinder $Q_1 = \Omega_1 \times (t_1,t_2) \subset\subset \Omega \times (0,T)$, there exists* *a sequence* $\{u^{\delta,\varepsilon}\}$ *of smooth solutions of the regolarized problem*

$$\partial_t u^{\delta,\varepsilon} = \sum_{i=1}^{2n+1} X_i^{\varepsilon}(A_i^{\delta,\varepsilon}(x, \nabla_\varepsilon u^{\delta,\varepsilon})) \text{ in } Q_1 \text{ and } u^{\delta,\varepsilon} = u \text{ on } \partial_p Q_1 \tag{4.3}$$

converging to u*, as* $\delta \to 0^+$ *and* $\varepsilon \to 0^+$*, uniformly on compact subsets of* Q_1 *and weakly in the the* $W^{1,p}-$*norm. Here we have denoted by* $\partial_p Q_1 = \Omega_1 \times \{t = t_1\} \cup \partial\Omega_1 \times (t_1, t_2)$ *the parabolic boundary of* Q_1*. The functions* $A^{\delta,\varepsilon}$ *satisfy*

$$(A_1^{\delta,\varepsilon}(x,\xi), ..., A_{2n}^{\delta,\varepsilon}(x,\xi), A_{2n+1}^{\delta,\varepsilon}(x,\xi)) \xrightarrow{\varepsilon \to 0} (A_1^{\delta}(x,\xi), ..., A_{2n}^{\delta}(x,\xi), 0)$$

and

$$A^{\delta}(x,\xi) \xrightarrow{\delta \to 0} A(x,\xi)$$

and

$$\begin{cases} \lambda(\delta + |\xi|_{g_\varepsilon}^2)^{\frac{p-2}{2}} |\eta|_{g_\varepsilon}^2 \le \partial_{\xi_j} A_i^{\delta,\varepsilon}(x,\xi)\eta_i\eta_j \le \Lambda(\delta + |\xi|_{g_\varepsilon}^2)^{\frac{p-2}{2}} |\eta|_{g_\varepsilon}^2 \\ |A_i^{\delta,\varepsilon}(x,\xi)|_{g_\varepsilon} + |\partial_{x_j} A_i^{\delta,\varepsilon}(x,\xi)|_{g_\varepsilon} \le \Lambda(\delta + |\xi|_{g_\varepsilon}^2)^{\frac{p-1}{2}} \end{cases}$$

with $0 < \lambda \le \Lambda < \infty$ *depending only on the original* λ', Λ'*.*

Proof. For each $\xi = \sum_{i=1}^{2n+1} \xi_i X_i^{\varepsilon} \in \mathbb{R}^{2n+1}$ and $\delta, \varepsilon > 0$ we define (for $1 \le i \le 2n+1$)

$$A_i^{\delta,\varepsilon}(x,\xi) = \tilde{A}_i(x, \xi_H) + \lambda(\delta + |\xi|_{g_\varepsilon}^2)^{\frac{p-2}{2}} \xi_i$$

where $\xi_H = (\xi_1, ..., \xi_{2n})$ and $\tilde{A} = (A, 0) \in \mathbb{R}^{2n+1}$. It is clear that the $A^{\delta,\varepsilon}$ satisfy the requested conditions. Moreover the regularized PDE

$$\partial_t u^{\delta,\varepsilon} = \sum_{i=1}^{2n+1} X_i^{\varepsilon}(A_i^{\delta,\varepsilon}(x, \nabla_\varepsilon u^{\delta,\varepsilon})) \text{ in } Q_1 \text{ and } u^{\delta,\varepsilon} = u \text{ on } \partial_p Q_1$$

is strongly parabolic for every $\delta, \varepsilon > 0$ and so the solutions are smooth in every subcylinder $Q_1 \subset\subset Q$. □

Notice that in the following we will drop the subscript g_ε in $|\cdot|_{g_\varepsilon}$ and so we will simply write $|\cdot|$ instead of $|\cdot|_{g_\varepsilon}$ in order to avoid too heavy notation.

4.3 Poincarè-type interpolation inequality

In Chapter 3 we use the Sobolev inequality (Theorem 2.2.3), here we can not use the analogous result because of the degeneracy of the PDE (4.1): we are searching for an inequality that is independent of the choice of δ, if the gradient vanish then this would not be the case. The following inequality is not the "usual" Poincarè inequality because of the presence of the "weight" $(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)$. In the following this will allow us, together with a "weighted" Caccioppoli inequality for $Zu^{\delta,\varepsilon}$, to obtain an estimate of $Zu^{\delta,\varepsilon}$ which depends on the weight i.e. depends on $\nabla_\varepsilon u^{\delta,\varepsilon}$.

Lemma 4.3.1. *Let $2 \le p \le 4$, $u^{\delta,\varepsilon} \in C^2(Q)$. Then there exists a constant $C = C(n,p,\mathcal{K}_i) > 0$ such that for all $\beta \ge 0$ and for all non negative function $\eta \in C^1([0,T], C_0^\infty(\Omega))$ vanishing on the parabolic boundary of Q we have that*

$$\int_{t_1}^{t_2}\int_\Omega |Zu^{\delta,\varepsilon}|^{p+\beta}\eta^{p+\beta}dxdt \le C(p+\beta)||\nabla_\varepsilon\eta||_{L^\infty}||1+\eta||_{L^\infty}\int\int_{spt(\eta)}(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}}dxdt+$$

$$+C(p+\beta)\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}}|Zu^{\delta,\varepsilon}|^\beta|\nabla_\varepsilon Zu^{\delta,\varepsilon}|^2\eta^{4+\beta}dxdt$$

Proof. We denote with L, R and M the following quantities:

$$L = \int_{t_1}^{t_2}\int_\Omega |Zu^{\delta,\varepsilon}|^{p+\beta}\eta^{p+\beta}dxdt$$

$$R = \int\int_{spt(\eta)}(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}}dxdt$$

$$M = \int_{t_1}^{t_2}\int_\Omega |\nabla_\varepsilon u^{\delta,\varepsilon}|^{p-2}|Zu^{\delta,\varepsilon}|^\beta|\nabla_\varepsilon Zu^{\delta,\varepsilon}|^2\eta^{4+\beta}dxdt$$

We already saw that we can choose $i, j \in \{1, ..., 2n\}$ such that $h_{ij} \neq 0$ and write

$$Zu^{\delta,\varepsilon} = \frac{1}{h_{ij}} \left(X_i^\varepsilon X_j^\varepsilon u^{\delta,\varepsilon} - X_j^\varepsilon X_i^\varepsilon u^{\delta,\varepsilon} \right)$$

consequently we can write

$$|Zu^{\delta,\varepsilon}|^{p+\beta} = |Zu^{\delta,\varepsilon}|^{p-2+\beta} Zu^{\delta,\varepsilon} \frac{1}{h_{ij}} \left(X_i^\varepsilon X_j^\varepsilon u^{\delta,\varepsilon} - X_j^\varepsilon X_i^\varepsilon u^{\delta,\varepsilon} \right)$$

Now we put this equation into L and we integrate by parts with respect to X_i^ε and X_j^ε obtaining

$$L = \int_{t_1}^{t_2} \int_\Omega |Zu^{\delta,\varepsilon}|^{p-2+\beta} Zu^{\delta,\varepsilon} \frac{1}{h_{ij}} (X_i^\varepsilon X_j^\varepsilon u^{\delta,\varepsilon} - X_j^\varepsilon X_i^\varepsilon u^{\delta,\varepsilon}) \eta^{p+\beta} dxdt =$$

$$= \int_{t_1}^{t_2} \int_\Omega |Zu^{\delta,\varepsilon}|^{p-2+\beta} Zu^{\delta,\varepsilon} \frac{1}{h_{ij}} X_i^\varepsilon X_j^\varepsilon u^{\delta,\varepsilon} \eta^{p+\beta} dxdt +$$

$$- \int_{t_1}^{t_2} \int_\Omega |Zu^{\delta,\varepsilon}|^{p-2+\beta} Zu^{\delta,\varepsilon} \frac{1}{h_{ij}} X_j^\varepsilon X_i^\varepsilon u^{\delta,\varepsilon} \eta^{p+\beta} dxdt =$$

$$= - \int_{t_1}^{t_2} \int_\Omega X_i^\varepsilon \left(|Zu^{\delta,\varepsilon}|^{p-2+\beta} Zu^{\delta,\varepsilon} \frac{1}{h_{ij}} \eta^{p+\beta} \right) X_j^\varepsilon u^{\delta,\varepsilon} dxdt +$$

$$+ \int_{t_1}^{t_2} \int_\Omega X_j^\varepsilon \left(|Zu^{\delta,\varepsilon}|^{p-2+\beta} Zu^{\delta,\varepsilon} \frac{1}{h_{ij}} \eta^{p+\beta} \right) X_i^\varepsilon u^{\delta,\varepsilon} dxdt$$

Now we apply the chain rule and we get

$$L = - \int_{t_1}^{t_2} \int_\Omega \Bigg[(p-2+\beta) |Zu^{\delta,\varepsilon}|^{p-3+\beta} X_i^\varepsilon Zu^{\delta,\varepsilon} Zu^{\delta,\varepsilon} \frac{1}{h_{ij}} \eta^{p+\beta} + |Zu^{\delta,\varepsilon}|^{p-2+\beta} X_i^\varepsilon Zu^{\delta,\varepsilon} \frac{1}{h_{ij}} \eta^{p+\beta} +$$

$$+ |Zu^{\delta,\varepsilon}|^{p-2+\beta} Zu^{\delta,\varepsilon} \frac{1}{h_{ij}} (p+\beta) \eta^{p+\beta-1} X_i^\varepsilon \eta + |Zu^{\delta,\varepsilon}|^{p-2+\beta} Zu^{\delta,\varepsilon} \eta^{p+\beta} X_i^\varepsilon \left(\frac{1}{h_{ij}} \right) \Bigg] X_j^\varepsilon u^{\delta,\varepsilon} dxdt +$$

$$+ \int_{t_1}^{t_2} \int_\Omega \Bigg[(p-2+\beta) |Zu^{\delta,\varepsilon}|^{p-3+\beta} X_j^\varepsilon Zu^{\delta,\varepsilon} Zu^{\delta,\varepsilon} \frac{1}{h_{ij}} \eta^{p+\beta} + |Zu^{\delta,\varepsilon}|^{p-2+\beta} X_j^\varepsilon Zu^{\delta,\varepsilon} \frac{1}{h_{ij}} \eta^{p+\beta}$$

$$+|Zu|^{p-2+\beta}Zu^{\delta,\varepsilon}\frac{1}{h_{ij}}(p+\beta)\eta^{p+\beta-1}X_j^\varepsilon\eta+|Zu^{\delta,\varepsilon}|^{p-2+\beta}Zu^{\delta,\varepsilon}\eta^{p+\beta}X_j^\varepsilon\left(\frac{1}{h_{ij}}\right)\Big]X_i^\varepsilon u^{\delta,\varepsilon}dxdt$$

Now we rearrange the terms and we get

$$\begin{aligned}
L=&-(p-1+\beta)\int_{t_1}^{t_2}\int_\Omega|Zu^{\delta,\varepsilon}|^{p-2+\beta}X_i^\varepsilon Zu^{\delta,\varepsilon}\frac{1}{h_{ij}}\eta^{p+\beta}X_j^\varepsilon u^{\delta,\varepsilon}dxdt+\\
&-(p+\beta)\int_{t_1}^{t_2}\int_\Omega|Zu^{\delta,\varepsilon}|^{p-2+\beta}Zu^{\delta,\varepsilon}\frac{1}{h_{ij}}\eta^{p+\beta-1}X_i^\varepsilon\eta X_j^\varepsilon u^{\delta,\varepsilon}dxdt+\\
&-\int_{t_1}^{t_2}\int_\Omega|Zu^{\delta,\varepsilon}|^{p-2+\beta}Zu^{\delta,\varepsilon}\eta^{p+\beta}X_i^\varepsilon\left(\frac{1}{h_{ij}}\right)X_j^\varepsilon u^{\delta,\varepsilon}dxdt+\\
&+(p-1+\beta)\int_{t_1}^{t_2}\int_\Omega|Zu^{\delta,\varepsilon}|^{p-2+\beta}X_j^\varepsilon Zu^{\delta,\varepsilon}\frac{1}{h_{ij}}\eta^{p+\beta}X_i^\varepsilon u^{\delta,\varepsilon}dxdt+\\
&+(p+\beta)\int_{t_1}^{t_2}\int_\Omega|Zu^{\delta,\varepsilon}|^{p-2+\beta}Zu^{\delta,\varepsilon}\frac{1}{h_{ij}}\eta^{p+\beta-1}X_j^\varepsilon\eta X_i^\varepsilon u^{\delta,\varepsilon}dxdt+\\
&+\int_{t_1}^{t_2}\int_\Omega|Zu^{\delta,\varepsilon}|^{p-2+\beta}Zu^{\delta,\varepsilon}\eta^{p+\beta}X_j^\varepsilon\left(\frac{1}{h_{ij}}\right)X_i^\varepsilon u^{\delta,\varepsilon}dxdt=\\
=&-(p-1-\beta)\int_{t_1}^{t_2}\int_\Omega\frac{1}{h_{ij}}|Zu^{\delta,\varepsilon}|^{p-2+\beta}(X_i^\varepsilon Zu^{\delta,\varepsilon}X_j^\varepsilon u^{\delta,\varepsilon}-X_j^\varepsilon Zu^{\delta,\varepsilon}X_i^\varepsilon u^{\delta,\varepsilon})\eta^{p+\beta}dxdt+\\
&-(p+\beta)\int_{t_1}^{t_2}\int_\Omega\frac{1}{h_{ij}}|Zu^{\delta,\varepsilon}|^{p-2+\beta}Zu^{\delta,\varepsilon}(X_j^\varepsilon u^{\delta,\varepsilon}X_i^\varepsilon\eta-X_i^\varepsilon u^{\delta,\varepsilon}X_j^\varepsilon\eta)\eta^{p-1+\beta}dxdt+\\
&-\int_{t_1}^{t_2}\int_\Omega|Zu^{\delta,\varepsilon}|^{p-2+\beta}Zu^{\delta,\varepsilon}\left(X_i^\varepsilon\left(\frac{1}{h_{ij}}\right)X_j^\varepsilon u^{\delta,\varepsilon}-X_j^\varepsilon\left(\frac{1}{h_{ij}}\right)X_i^\varepsilon u^{\delta,\varepsilon}\right)\eta^{p+\beta}dxdt=:I_1+I_2+I_3
\end{aligned}\tag{4.4}$$

Now we have the following estimates on the terms I_1, I_2 and I_3

$$I_1\le 2(p+\beta)C\int_{t_1}^{t_2}\int_\Omega|\nabla_\varepsilon u^{\delta,\varepsilon}||Zu^{\delta,\varepsilon}|^{p-2+\beta}|\nabla_\varepsilon Zu^{\delta,\varepsilon}|\eta^{p+\beta}dxdt$$

$$I_2\le 2(p+\beta)C\int_{t_1}^{t_2}\int_\Omega|\nabla_\varepsilon u^{\delta,\varepsilon}||Zu^{\delta,\varepsilon}|^{p-1+\beta}|\nabla_\varepsilon\eta|\eta^{p-1+\beta}dxdt$$

$$I_3\le\int_{t_1}^{t_2}\int_\Omega|\nabla_\varepsilon u^{\delta,\varepsilon}||Zu^{\delta,\varepsilon}|^{p-1+\beta}\left|\nabla_\varepsilon\left(\frac{1}{h_{ij}}\right)\right|\eta^{p+\beta}dxdt\le$$

$$\leq C \int_{t_1}^{t_2} \int_\Omega |\nabla_\varepsilon u^{\delta,\varepsilon}||Zu^{\delta,\varepsilon}|^{p-1+\beta} \eta^{p+\beta} dxdt$$

We now observe that

$$\begin{aligned} I_2 + I_3 \leq C(p+\beta)||\nabla_\varepsilon \eta||_{L^\infty} \int_{t_1}^{t_2} \int_\Omega |\nabla_\varepsilon u^{\delta,\varepsilon}||Zu^{\delta,\varepsilon}|^{p-1+\beta} \eta^{p-1+\beta} dxdt + \\ + C \int_{t_1}^{t_2} \int_\Omega |\nabla_\varepsilon u^{\delta,\varepsilon}||Zu^{\delta,\varepsilon}|^{p-1+\beta} \eta^{p+\beta} dxdt \leq \\ \leq C(p+\beta)||\nabla_\varepsilon \eta||_{L^\infty} \int_{t_1}^{t_2} \int_\Omega |\nabla_\varepsilon u^{\delta,\varepsilon}||Zu^{\delta,\varepsilon}|^{p-1+\beta} \eta^{p-1+\beta} (1+\eta) dxdt \leq \\ \leq C(p+\beta)||\nabla_\varepsilon \eta||_{L^\infty} ||1+\eta||_{L^\infty} \int_{t_1}^{t_2} \int_\Omega |\nabla_\varepsilon u^{\delta,\varepsilon}||Zu^{\delta,\varepsilon}|^{p-1+\beta} \eta^{p-1+\beta} dxdt \end{aligned} \quad (4.5)$$

Now by the use of the Hölder's inequality on the estimates of I_1 and $I_2 + I_3$ we get

$$I_1 \leq 2(p+\beta) C M^{\frac{1}{2}} R^{\frac{4-p}{2(p+\beta)}} L^{\frac{2p-4+\beta}{2(p+\beta)}} \quad (4.6)$$

and

$$I_2 + I_3 \leq 2(p+\beta) C ||\nabla_\varepsilon \eta||_{L^\infty} ||1+\eta||_{L^\infty} R^{\frac{1}{p+\beta}} L^{\frac{p-1+\beta}{p+\beta}} \quad (4.7)$$

Combining (4.6) and (4.7) with (4.4) we get

$$L \leq C(p+\beta)||\nabla_\varepsilon \eta||_{L^\infty} ||1+\eta||_{L^\infty} R^{\frac{1}{p+\beta}} L^{\frac{p-1+\beta}{p+\beta}} + C(p+\beta) M^{\frac{1}{2}} R^{\frac{4-p}{2(p+\beta)}} L^{\frac{2p-4+\beta}{2(p+\beta)}}$$

and finally using the Young's inequality we get the thesis.

$$L \leq C(p+\beta)||\nabla_\varepsilon \eta||_{L^\infty} ||1+\eta||_{L^\infty} \int \int_{spt(\eta)} (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}} dxdt +$$

$$+ C(p+\beta) \int_{t_1}^{t_2} \int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|)^{\frac{p-2}{2}} |Zu^{\delta,\varepsilon}|^\beta |\nabla_\varepsilon Z u^{\delta,\varepsilon}|^2 \eta^{4+\beta} dxdt$$

□

4.4 Caccioppoli-type inequalities

Now we want to obtain an estimate that could play the role of the Caccioppoli inequality (Proposition 3.0.2) in Chapter 3. Here the problem is harder becouse of the subriemannian structure of the problem. From the two following lemmas we find the PDEs solved by $X_l^\varepsilon u^{\delta,\varepsilon}$ (for $1 \leq l \leq 2n$) and $Zu^{\delta,\varepsilon}$ where $u^{\delta,\varepsilon}$ is a solution of (4.3). We recall that we obtain Z through a commutator so technically it has the role of a second derivative but we will see in the following that in order to obtain an estimate of the gradient we need to obtain an estimate also for Zu^δ.

Lemma 4.4.1. *Let $u^{\delta,\varepsilon}$ be a solution of* (4.3)*. Then $v_l^{\delta,\varepsilon} = X_l^\varepsilon u^{\delta,\varepsilon}$ (for $1 \leq l \leq 2n$) is a solution of*

$$\partial_t v_l^{\delta,\varepsilon} = \sum_{i,j=1}^{2n+1} X_i^\varepsilon (A_{i,\xi_j}^{\delta,\varepsilon}(x, \nabla_\varepsilon u^{\delta,\varepsilon}) X_l^\varepsilon X_j^\varepsilon u^{\delta,\varepsilon}) +$$

$$+ \sum_{i=1}^{2n+1} X_i^\varepsilon \left(A_{i,x_l}^{\delta,\varepsilon}(x, \nabla_\varepsilon u^{\delta,\varepsilon}) - \frac{s_l \mathcal{K}_{l+s_l n}}{2} A_{i,x_{2n+1}}^{\delta,\varepsilon}(x, \nabla_\varepsilon u^{\delta,\varepsilon}) \right) + \sum_{i=1}^{2n+1} h_{li} Z(A_i^{\delta,\varepsilon}(x, \nabla_\varepsilon u^{\delta,\varepsilon}))$$

where $s_l = (-1)^{[l/n]}$.

Proof. We apply X_l^ε to $\partial_t u^{\delta,\varepsilon} = \sum_{i=1}^{2n+1} X_i^\varepsilon A_i^{\delta,\varepsilon}(x, \nabla_\varepsilon u^{\delta,\varepsilon})$ when $l \leq n$: we get that

$$\partial_t (X_l^\varepsilon u^{\delta,\varepsilon}) = \partial_t v_l^{\delta,\varepsilon} = \sum_{i=1}^{2n+1} X_l^\varepsilon X_i^\varepsilon A_i^{\delta,\varepsilon}(x, \nabla_\varepsilon u^{\delta,\varepsilon}) =$$

$$= \sum_{i=1}^{2n+1} X_i^\varepsilon (X_l^\varepsilon A_i^{\delta,\varepsilon}(x, \nabla_\varepsilon u^{\delta,\varepsilon})) + \sum_{i=1}^{2n+1} [X_l^\varepsilon, X_i^\varepsilon] A_i^{\delta,\varepsilon}(x, \nabla_\varepsilon u^{\delta,\varepsilon})$$

Applying the chain rule we get

$$\partial_t v_l^{\delta,\varepsilon} = \sum_{i,j=1}^{2n+1} X_i^\varepsilon (A_{i,\xi_j}^{\delta,\varepsilon}(x, \nabla_\varepsilon u^{\delta,\varepsilon}) X_l^\varepsilon X_j^\varepsilon u^{\delta,\varepsilon}) +$$

$$+\sum_{i=1}^{2n+1} X_i^\varepsilon\left(A^{\delta,\varepsilon}_{i,x_l}(x,\nabla_\varepsilon u^{\delta,\varepsilon})-\frac{\mathcal{K}_{l+n}}{2}A^{\delta,\varepsilon}_{i,x_{2n+1}}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\right)+\sum_{i=1}^{2n+1}[X_l^\varepsilon,X_i^\varepsilon]A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})$$

If instead we have $l\geq n+1$ we have that

$$\partial_t(X_l^\varepsilon u^{\delta,\varepsilon})=\partial_t v_l^{\delta,\varepsilon}=\sum_{i=1}^{2n+1}X_l^\varepsilon X_i^\varepsilon A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})=$$

$$=\sum_{i=1}^{2n+1}X_i^\varepsilon(X_l^\varepsilon A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}))+\sum_{i=1}^{2n+1}[X_l^\varepsilon,X_i^\varepsilon]A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})$$

Applying the chain rule we get

$$\partial_t v_l^{\delta,\varepsilon}=\sum_{i,j=1}^{2n+1}X_i^\varepsilon(A^{\delta,\varepsilon}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon})X_l^\varepsilon X_j^\varepsilon u^{\delta,\varepsilon})+$$

$$+\sum_{i=1}^{2n+1} X_i^\varepsilon\left(A^{\delta,\varepsilon}_{i,x_l}(x,\nabla_\varepsilon u^{\delta,\varepsilon})+\frac{\mathcal{K}_{l-n}}{2}A^{\delta,\varepsilon}_{i,x_{2n+1}}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\right)+\sum_{i=1}^{2n+1}[X_l^\varepsilon,X_i^\varepsilon]A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})$$

and so we get the thesis. □

In Lemma 4.4.1 we start to see the typical difficulties that appear in the subriemannian case: in the euclidean case all the commutators are zero. The last term in the right hand side contains the derivative $[X_l^\varepsilon,X_i^\varepsilon]=h_{li}Z$ and forces us to consider also $Zu^{\delta,\varepsilon}$ that, as we said precedently, is technically a second derivative.

Lemma 4.4.2. *Let $u^{\delta,\varepsilon}$ be a solution of* (4.3). *Then $Zu^{\delta,\varepsilon}$ is a solution of*

$$\partial_t Zu^{\delta,\varepsilon}=\sum_{i,j=1}^{2n+1}(X_i^\varepsilon+k_i)(A^{\delta,\varepsilon}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon})(k_j+X_j^\varepsilon)Zu^{\delta,\varepsilon})+\sum_{i=1}^{2n+1}(X_i^\varepsilon+k_i)(A^{\delta,\varepsilon}_{i,x_{2n+1}}(x,\nabla_\varepsilon u^{\delta,\varepsilon}))$$

where the k_i are defined as in section 1.3.

Proof. We apply Z to $\partial_t u^{\delta,\varepsilon} = \sum_{i=1}^{2n+1} X_i^\varepsilon A_i^{\delta,\varepsilon}(x, \nabla_\varepsilon u^{\delta,\varepsilon})$: we get that

$$\partial_t Z u^{\delta,\varepsilon} = \sum_{i=1}^{2n+1} Z(X_i^\varepsilon A_i^{\delta,\varepsilon}(x, \nabla_\varepsilon u^{\delta,\varepsilon})) = \sum_{i=1}^{2n+1} X_i^\varepsilon (Z A_i^{\delta,\varepsilon}(x, \nabla_\varepsilon u^{\delta,\varepsilon})) + \sum_{i=1}^{2n+1} k_i Z A_i^{\delta,\varepsilon}(x, \nabla_\varepsilon u^{\delta,\varepsilon}) =$$

$$= \sum_{i=1}^{2n+1} (X_i^\varepsilon + k_i) \left(\sum_{j=1}^{2n+1} A_{i,\xi_j}^{\delta,\varepsilon}(x, \nabla_\varepsilon u^{\delta,\varepsilon}) Z X_j^\varepsilon u^{\delta,\varepsilon} + A_{i,x_{2n+1}}^{\delta,\varepsilon}(x, \nabla_\varepsilon u^{\delta,\varepsilon}) \right) =$$

$$= \sum_{i,j=1}^{2n+1} (X_i^\varepsilon + k_i)(A_{i,\xi_j}^{\delta,\varepsilon}(x, \nabla_\varepsilon u^{\delta,\varepsilon}) \{[Z, X_j^\varepsilon] + X_j^\varepsilon Z\} u^{\delta,\varepsilon}) + \sum_{i=1}^{2n+1} (X_i^\varepsilon + k_i)(A_{i,x_{2n+1}}^{\delta,\varepsilon}(x, \nabla_\varepsilon u^{\delta,\varepsilon})) =$$

$$= \sum_{i,j=1}^{2n+1} (X_i^\varepsilon + k_i)(A_{i,\xi_j}^{\delta,\varepsilon}(x, \nabla_\varepsilon u^{\delta,\varepsilon})(k_j + X_j^\varepsilon) Z u^{\delta,\varepsilon}) + \sum_{i=1}^{2n+1} (X_i^\varepsilon + k_i)(A_{i,x_{2n+1}}^{\delta,\varepsilon}(x, \nabla_\varepsilon u^{\delta,\varepsilon}))$$

□

Now we want to establish some estimates in order to obtain a Caccioppoli-like inequality to start our Moser type iteration. The analogous in the euclidean case is the Caccioppoli inequality in term of the L^p norm of the gradient.

Proposition 4.4.3. *Let $u^{\delta,\varepsilon}$ be a solution of (4.3). Then for all $\beta \geq 0$, for all $\eta \in C^1([0,T], C_0^\infty(\Omega))$ and for all $T > t_2 \geq t_1 \geq 0$ we have that there exists a constant $C = C(\lambda, \Lambda, n, p, \mathcal{K}_i) \geq 0$ such that*

$$\frac{1}{\beta+2} \int_\Omega [(\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}+1} \eta^2] \Big|_{t_2}^{t_1} dx + \int_{t_1}^{t_2} \int_\Omega \eta^2 (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2+\beta}{2}} \sum_{i,j=1}^{2n+1} |X_i^\varepsilon X_j^\varepsilon u^{\delta,\varepsilon}|^2 dx dt \leq$$

$$\leq C \int_{t_1}^{t_2} \int_\Omega (\eta^2 + |\nabla_\varepsilon \eta|^2 + |\eta Z \eta|)(\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}} dx dt +$$

$$+ C(\beta+1)^4 \int_{t_1}^{t_2} \int_\Omega \eta^2 (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta-2}{2}} |Z u^{\delta,\varepsilon}|^2 dx dt +$$

$$+ \frac{C}{\beta+2} \int_{t_1}^{t_2} \int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}+1} |\partial_t \eta| \eta \, dx dt$$

Proof. From Lemma 4.4.1 we have

$$\partial_t v_l^{\delta,\varepsilon} = \sum_{i,j=1}^{2n+1} X_i^\varepsilon (A_{i,\xi_j}^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) X_l^\varepsilon X_j^\varepsilon u^{\delta,\varepsilon}) +$$
$$+ \sum_{i=1}^{2n+1} X_i^\varepsilon \left(A_{i,x_l}^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) - \frac{s_l \mathcal{K}_{l+s_l n}}{2} A_{i,x_{2n+1}}^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) \right) + \sum_{i=1}^{2n+1} h_{li} Z(A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})) \tag{4.8}$$

Now we observe that

$$X_l^\varepsilon X_j^\varepsilon u^{\delta,\varepsilon} = X_j^\varepsilon X_l^\varepsilon u^{\delta,\varepsilon} + [X_l^\varepsilon, X_j^\varepsilon] u^{\delta,\varepsilon} = X_j^\varepsilon v_l^{\delta,\varepsilon} + h_{lj} Z u^{\delta,\varepsilon}$$

Substituting in the first addendum of the right hand side of (4.8) we have that

$$\partial_t v_l^{\delta,\varepsilon} = \sum_{i,j=1}^{2n+1} X_i^\varepsilon (A_{i,\xi_j}^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) X_j^\varepsilon v_l^{\delta,\varepsilon}) + \sum_{i,j=1}^{2n+1} X_i^\varepsilon (A_{i,\xi_j}^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) h_{lj} Z u^{\delta,\varepsilon}) +$$

$$+ \sum_{i=1}^{2n+1} X_i^\varepsilon \left(A_{i,x_l}^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) - \frac{s_l \mathcal{K}_{l+s_l n}}{2} A_{i,x_{2n+1}}^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) \right) + \sum_{i=1}^{2n+1} h_{li} Z(A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}))$$

Now we use as test function $\phi = \eta^2(\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}} X_l^\varepsilon u^{\delta,\varepsilon}$ with $\eta \in C^1([0,T], C_0^\infty(\Omega))$ and integrating by parts we get

$$\frac{1}{2}\int_{t_1}^{t_2}\int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|)^{\frac{\beta}{2}} \partial_t [X_l^\varepsilon u^{\delta,\varepsilon}]^2 \eta^2 dxdt +$$

$$+ \sum_{i,j=1}^{2n+1} \int_{t_1}^{t_2}\int_\Omega A_{i,\xi_j}^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) X_j^\varepsilon v_l^{\delta,\varepsilon} X_i^\varepsilon (\eta^2(\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}} X_l^\varepsilon u^{\delta,\varepsilon}) dxdt =$$

$$- \sum_{i,j=1}^{2n+1} \int_{t_1}^{t_2}\int_\Omega A_{i,\xi_j}^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) h_{lj} Z u^{\delta,\varepsilon} X_i^\varepsilon (\eta^2(\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}} X_l^\varepsilon u^{\delta,\varepsilon}) dxdt +$$

$$+\int_{t_1}^{t_2}\int_\Omega \Big[\sum_{i=1}^{2n+1} X_i^\varepsilon \Big(A^{\delta,\varepsilon}_{i,x_l}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) - \frac{s_l\mathcal{K}_{l+s_l n}}{2} A^{\delta,\varepsilon}_{i,x_{2n+1}}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\Big)\cdot$$

$$\cdot\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}} X_l^\varepsilon u^{\delta,\varepsilon}\Big] dxdt+$$

$$+\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega h_{li} Z(A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}))\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}} X_l^\varepsilon u^{\delta,\varepsilon} dxdt$$

Integrating by parts we get that

$$\begin{aligned}
&\frac{1}{2}\int_{t_1}^{t_2}\int_\Omega (\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|)^{\frac{\beta}{2}}\partial_t[X_l^\varepsilon u^{\delta,\varepsilon}]^2\eta^2 dxdt+\\
&\quad+\sum_{i,j=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega A^{\delta,\varepsilon}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) X_j^\varepsilon X_l^\varepsilon u^{\delta,\varepsilon} X_i^\varepsilon X_l^\varepsilon u^{\delta,\varepsilon}\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}} dxdt+\\
&+\sum_{i,j=1}^{2n+1}\frac{\beta}{2}\int_{t_1}^{t_2}\int_\Omega A^{\delta,\varepsilon}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) X_j^\varepsilon X_l^\varepsilon u^{\delta,\varepsilon} X_l^\varepsilon u^{\delta,\varepsilon} X_i^\varepsilon(|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta-2}{2}} dxdt=\\
&\quad=-\sum_{i,j=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega A^{\delta,\varepsilon}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) X_j^\varepsilon X_l^\varepsilon u^{\delta,\varepsilon} X_l^\varepsilon u^{\delta,\varepsilon} X_i^\varepsilon(\eta^2)(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}} dxdt+\\
&\quad-\sum_{i,j=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega A^{\delta,\varepsilon}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) h_{lj} Z u^{\delta,\varepsilon} X_i^\varepsilon(\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}} X_l^\varepsilon u^{\delta,\varepsilon}) dxdt+\\
&\qquad-\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega \Big[(A^{\delta,\varepsilon}_{i,x_l}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) - \frac{s_l\mathcal{K}_{l+s_l n}}{2} A^{\delta,\varepsilon}_{i,x_{2n+1}}(x,\nabla_\varepsilon u^{\delta,\varepsilon}))\cdot\\
&\qquad\qquad\cdot X_i^\varepsilon(\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}} X_l^\varepsilon u^{\delta,\varepsilon})\Big] dxdt+\\
&+\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega h_{li} Z(A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}))\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}} X_l^\varepsilon u^{\delta,\varepsilon} dxdt =: I_{l,1}+I_{l,2}+I_{l,3}+I_{l,4}
\end{aligned} \tag{4.9}$$

Now we concentrate on the left hand side of (4.9) and adding for l from 1 to $2n$ we get, using the structural assumptions, that

$$\frac{1}{\beta+2}\int_{t_1}^{t_2}\int_\Omega \partial_t[(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}+1}]\eta^2 dxdt+$$

$$+\sum_{l=1}^{2n}\sum_{i,j=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega A^{\delta,\varepsilon}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon})X_j^\varepsilon v_l^{\delta,\varepsilon}X_i^\varepsilon(\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}X_l^\varepsilon u^{\delta,\varepsilon})dxdt+$$

$$+\sum_{l=1}^{2n}\sum_{i,j=1}^{2n+1}\frac{\beta}{2}\int_{t_1}^{t_2}\int_\Omega A^{\delta,\varepsilon}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon})X_j^\varepsilon X_l^\varepsilon u^{\delta,\varepsilon}X_l^\varepsilon u^{\delta,\varepsilon}X_i^\varepsilon(|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta-2}{2}}dxdt \geq$$

$$\geq \frac{1}{\beta+2}\int_{t_1}^{t_2}\int_\Omega \partial_t[(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}+1}]\eta^2 dxdt+$$

$$+\lambda\int_{t_1}^{t_2}\int_\Omega \eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2+\beta}{2}}\sum_{i,j=1}^{2n+1}|X_i^\varepsilon X_j^\varepsilon u^{\delta,\varepsilon}|^2 dxdt+$$

$$+\frac{\lambda\beta}{4}\int_{t_1}^{t_2}\int_\Omega \eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta-4}{2}}|\nabla_\varepsilon(|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)|^2 dxdt$$

the last addendum is always non negative so our equality (4.9) becomes the following inequality

$$\frac{1}{\beta+2}\int_\Omega [(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|)^{\frac{\beta}{2}+1}\eta^2]\Big|_{t_2}^{t_1} dx+$$

$$+\lambda\int_{t_1}^{t_2}\int_\Omega \eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2+\beta}{2}}\sum_{i,j=1}^{2n+1}|X_i^\varepsilon X_j^\varepsilon u^{\delta,\varepsilon}|^2 dxdt+$$

$$-\frac{C}{\beta+2}\int_{t_1}^{t_2}\int_\Omega (\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}+1}|\partial_t\eta|\eta dxdt \leq$$

$$\leq \sum_{l=1}^{2n}(I_{l,1}+I_{l,2}+I_{l,3}+I_{l,4}) \quad (4.10)$$

For $I_{l,1}$ we use the structural assumptions and the Young's inequality and we get the estimate

$$\sum_{l=1}^{2n} I_{l,1} = -\sum_{l=1}^{2n}\sum_{i,j=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega A^{\delta,\varepsilon}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon})X^\varepsilon_j X^\varepsilon_l u^{\delta,\varepsilon} X^\varepsilon_l u^{\delta,\varepsilon} X^\varepsilon_i(\eta^2)(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}dxdt \leq$$
$$\leq 2\sum_{i,j=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega |\eta|(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}}|X^\varepsilon_j X^\varepsilon_i u^{\delta,\varepsilon}||\nabla_\varepsilon u^{\delta,\varepsilon}||\nabla_\varepsilon\eta|(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}dxdt \leq$$
$$\leq \alpha\sum_{i,j=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega \eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta-2}{2}}|X^\varepsilon_j X^\varepsilon_i u^{\delta,\varepsilon}|^2 dxdt+$$
$$+ C_\alpha\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}}|\nabla_\varepsilon\eta|^2 dxdt \quad (4.11)$$

where α is any real number and C_α a constant which depend only from α, p, n.
We define $C_h = \sup_{i,j}||h_{ij}||_{L^\infty}$. In the same way as we did for $I_{l,1}$ we get the following estimate for $I_{l,2}$:

$$\sum_{l=1}^{2n} I_{l,2} = -\sum_{l=1}^{2n}\sum_{i,j=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega A^{\delta,\varepsilon}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon})h_{lj}Zu^{\delta,\varepsilon}X^\varepsilon_i(\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}X^\varepsilon_l u^{\delta,\varepsilon})dxdt \leq$$
$$\leq \alpha\sum_{i,j=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega \eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta-2}{2}}|X^\varepsilon_i X^\varepsilon_j u^{\delta,\varepsilon}|^2 dxdt+$$
$$+ C\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}}|\nabla_\varepsilon\eta|^2 dxdt+$$
$$+ C_\alpha C_h^2(\beta+1)^2\int_{t_1}^{t_2}\int_\Omega \eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta-2}{2}}|Zu^{\delta,\varepsilon}|^2 dxdt \quad (4.12)$$

We define $C_{\mathcal{K}} = \sup_j ||\mathcal{K}_j||_{L^\infty}$. As we did for $I_{l,1}$ and $I_{l,2}$ we get the following estimate for $I_{l,3}$:

$$\sum_{l=1}^{2n} I_{l,3} = -\sum_{l=1}^{2n}\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega \left(A^\delta_{i,x_l}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) - \frac{s_l\mathcal{K}_{l+s_l n}}{2}A^{\delta,\varepsilon}_{i,x_{2n+1}}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\right)\cdot$$
$$\cdot X_i^\varepsilon(\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}X_l u^{\delta,\varepsilon})dxdt \le$$
$$\le \alpha\sum_{i,j=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta-2}{2}}|X_i^\varepsilon X_j^\varepsilon u^{\delta,\varepsilon}|^2\eta^2 dxdt+$$
$$+ C_\alpha C_{\mathcal{K}}^2(\beta+1)^2\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}}(|\nabla_\varepsilon\eta|^2+|\eta|^2)dxdt \quad (4.13)$$

For $I_{l,4}$ we integrate by parts with respect to Z and we get

$$\sum_{l=1}^{2n} I_{l,4} = \sum_{l=1}^{2n}\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega h_{li}Z(A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}))\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}X_l^\varepsilon u^{\delta,\varepsilon}dxdt =$$
$$= -2\sum_{l=1}^{2n}\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega h_{li}A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\eta Z\eta(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}X_l^\varepsilon u^{\delta,\varepsilon}dxdt+$$
$$-\beta\sum_{l=1}^{2n}\sum_{i,j=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega h_{li}(A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}))\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta-2}{2}}X_j^\varepsilon u^{\delta,\varepsilon}ZX_j^\varepsilon u^{\delta,\varepsilon}X_l^\varepsilon u^{\delta,\varepsilon}dxdt+$$
$$-\sum_{l=1}^{2n}\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega h_{li}A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}ZX_l^\varepsilon u^{\delta,\varepsilon}dxdt+$$
$$-\sum_{l=1}^{2n}\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega Z(h_{li})A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}X_l^\varepsilon u^{\delta,\varepsilon}dxdt \quad (4.14)$$

Now we recall that $ZX_j^\varepsilon = X_j^\varepsilon Z + k_j Z$ and putting this equation into (4.14) we get

$$\sum_{l=1}^{2n} I_{l,4} = -2\sum_{l=1}^{2n}\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega h_{li}A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\eta Z\eta(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}X_l^\varepsilon u^{\delta,\varepsilon}dxdt+$$
$$-\beta\sum_{l=1}^{2n}\sum_{i,j=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega h_{li}(A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}))\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta-2}{2}}X_j^\varepsilon u^{\delta,\varepsilon}X_j^\varepsilon Zu^{\delta,\varepsilon}X_l^\varepsilon u^{\delta,\varepsilon}dxdt+$$
$$-\beta\sum_{l=1}^{2n}\sum_{i,j=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega h_{li}(A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}))\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta-2}{2}}X_j^\varepsilon u^{\delta,\varepsilon}k_jZu^{\delta,\varepsilon}X_l^\varepsilon u^{\delta,\varepsilon}dxdt+$$
$$-\sum_{l=1}^{2n}\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega h_{li}A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}X_l^\varepsilon Zu^{\delta,\varepsilon}dxdt+$$
$$-\sum_{l=1}^{2n}\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega h_{li}A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}k_lZu^{\delta,\varepsilon}dxdt+$$
$$-\sum_{l=1}^{2n}\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega Z(h_{li})A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}X_l^\varepsilon u^{\delta,\varepsilon}dxdt \quad (4.15)$$

Now we recall that, using Hörmander condition, we have that for each point of the manifold there exists at least one $h_{st}\neq 0$ and so we can write $Z=\frac{1}{h_{st}}(X_s^\varepsilon X_t^\varepsilon - X_t^\varepsilon X_s^\varepsilon)$

and if we put this equation in (4.15) we obtain

$$\sum_{l=1}^{2n} I_{l,4} = -2\sum_{l=1}^{2n}\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega h_{li}A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\eta Z\eta(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}X_l^\varepsilon u^{\delta,\varepsilon}dxdt+$$
$$-\beta\sum_{l=1}^{2n}\sum_{i,j=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega h_{li}(A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}))\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta-2}{2}}X_j^\varepsilon u^{\delta,\varepsilon}X_j^\varepsilon Zu^{\delta,\varepsilon}X_l^\varepsilon u^{\delta,\varepsilon}dxdt+$$
$$-\beta\sum_{l=1}^{2n}\sum_{i,j=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega h_{li}(A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}))\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta-2}{2}}X_j^\varepsilon u^{\delta,\varepsilon}k_j\frac{1}{h_{st}}X_s^\varepsilon X_t^\varepsilon u^{\delta,\varepsilon}X_l^\varepsilon u^{\delta,\varepsilon}dxdt+$$
$$+\beta\sum_{l=1}^{2n}\sum_{i,j=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega h_{li}(A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}))\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta-2}{2}}X_j^\varepsilon u^{\delta,\varepsilon}k_j\frac{1}{h_{st}}X_t^\varepsilon X_s^\varepsilon u^{\delta,\varepsilon}X_l^\varepsilon u^{\delta,\varepsilon}dxdt+$$
$$-\sum_{l=1}^{2n}\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega h_{li}A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}X_l^\varepsilon Zu^{\delta,\varepsilon}dxdt+$$
$$-\sum_{l=1}^{2n}\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega h_{li}A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}k_l\frac{1}{h_{st}}X_s^\varepsilon X_t^\varepsilon u^{\delta,\varepsilon}dxdt+$$
$$+\sum_{l=1}^{2n}\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega h_{li}A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}k_l\frac{1}{h_{st}}X_t^\varepsilon X_s^\varepsilon u^{\delta,\varepsilon}dxdt+$$
$$-\sum_{l=1}^{2n}\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega Z(h_{li})A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}X_l^\varepsilon u^{\delta,\varepsilon}dxdt \quad (4.16)$$

Now integrating by parts we get that

$$
\begin{aligned}
\sum_{l=1}^{2n} I_{l,4} = & -2\sum_{l=1}^{2n}\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega h_{li}A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\eta Z\eta(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}X_l^\varepsilon u^{\delta,\varepsilon}dxdt+ \\
& +\beta\sum_{l=1}^{2n}\sum_{i,j=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega \Bigg[X_j^\varepsilon\Bigg(h_{li}(A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}))\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta-2}{2}}\cdot \\
& \qquad \cdot X_j^\varepsilon u^{\delta,\varepsilon}X_l^\varepsilon u^{\delta,\varepsilon}\Bigg)Zu^{\delta,\varepsilon}\Bigg]dxdt+ \\
& +\beta\sum_{l=1}^{2n}\sum_{i,j=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega \Bigg[X_s^\varepsilon\Bigg(h_{li}(A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}))\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta-2}{2}}\cdot \\
& \qquad \cdot X_j^\varepsilon u^{\delta,\varepsilon}k_j\frac{1}{h_{st}}X_l^\varepsilon u^{\delta,\varepsilon}\Bigg)X_t^\varepsilon u^{\delta,\varepsilon}\Bigg]dxdt+ \\
& -\beta\sum_{l=1}^{2n}\sum_{i,j=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega \Bigg[X_t^\varepsilon\Bigg(h_{li}(A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}))\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta-2}{2}}\cdot \\
& \qquad \cdot X_j^\varepsilon u^{\delta,\varepsilon}k_j\frac{1}{h_{st}}X_l^\varepsilon u^{\delta,\varepsilon}\Bigg)X_s^\varepsilon u^{\delta,\varepsilon}\Bigg]dxdt+ \\
& +\sum_{l=1}^{2n}\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega X_l^\varepsilon\left(h_{li}A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}\right)Zu^{\delta,\varepsilon}dxdt+ \\
& +\sum_{l=1}^{2n}\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega X_s^\varepsilon\left(h_{li}A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}k_l\frac{1}{h_{st}}\right)X_t^\varepsilon u^{\delta,\varepsilon}dxdt+ \\
& -\sum_{l=1}^{2n}\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega X_t^\varepsilon\left(h_{li}A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}k_l\frac{1}{h_{st}}\right)X_s^\varepsilon u^{\delta,\varepsilon}dxdt+ \\
& -\sum_{l=1}^{2n}\sum_{i=1}^{2n+1}\int_{t_1}^{t_2}\int_\Omega Z(h_{li})A_i^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta}{2}}X_l^\varepsilon u^{\delta,\varepsilon}dxdt \qquad (4.17)
\end{aligned}
$$

and using the structural assumptions and the Young's inequality we get that

$$\sum_{l=1}^{2n} I_{l,4} \leq C_h\alpha \sum_{i,j=1}^{2n+1} \int_{t_1}^{t_2} \int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta-2}{2}} |X_i^\varepsilon X_j^\varepsilon u^{\delta,\varepsilon}|^2 \eta^2 dxdt +$$
$$+ C_h(\beta+1)C' \int_{t_1}^{t_2} \int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}} (\eta^2 + |\nabla_\varepsilon \eta|^2 + |\eta Z \eta|) dxdt$$
$$+ C_h C_\alpha (\beta+1)^4 \int_{t_1}^{t_2} \int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta-2}{2}} |Z u^{\delta,\varepsilon}|^2 \eta^2 dxdt \quad (4.18)$$

Combining (4.11), (4.12), (4.13), (4.18) with (4.10) we get the thesis. □

From Proposition 4.4.3 we can see that the presence of the second term of the right hand side (which is absent in the euclidean case) in the Caccioppoli inequality for the horizontal derivatives force us to prove a Caccioppoli inequality also for $Zu^{\delta,\varepsilon}$.

Proposition 4.4.4. *Let $u^{\delta,\varepsilon}$ be a solution of (4.3). Then for all $\beta \geq 0$, for all $\eta \in C^1([0,T], C_0^\infty(\Omega))$ and for all $T > t_2 \geq t_1 \geq 0$ we have that there exists a constant $C = C(\lambda, \Lambda, \mathcal{K}_i) \geq 0$ such that*

$$\frac{1}{\beta+2} \int_\Omega |Zu^{\delta,\varepsilon}|^{\beta+2} \eta^2 \Big|_{t_1}^{t_2} dx + C \int_{t_1}^{t_2} \int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}} |\nabla_\varepsilon Z u^{\delta,\varepsilon}|^2 |Zu^{\delta,\varepsilon}|^\beta |\eta|^2 dxdt$$

$$\leq C \int_{t_1}^{t_2} \int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}} |\nabla_\varepsilon \eta|^2 |Zu^{\delta,\varepsilon}|^{\beta+2} dxdt +$$

$$+ \frac{2}{\beta+2} \int_{t_1}^{t_2} \int_\Omega |Zu^{\delta,\varepsilon}|^{\beta+2} \eta \partial_t \eta dxdt + C \int_{t_1}^{t_2} \int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p}{2}} \eta^2 |Zu^{\delta,\varepsilon}|^\beta dxdt$$

Proof. From Lemma 4.4.2 we have

$$\partial_t Z u^{\delta,\varepsilon} = \sum_{i,j=1}^{2n+1} (X_i^\varepsilon + k_i)(A^{\delta,\varepsilon}_{i,\xi_j}(x, \nabla_\varepsilon u^{\delta,\varepsilon})(k_j + X_j^\varepsilon) Z u^{\delta,\varepsilon}) + \sum_{i=1}^{2n+1} (X_i^\varepsilon + k_i)(A^{\delta,\varepsilon}_{i,x_{2n+1}}(x, \nabla_\varepsilon u^{\delta,\varepsilon}))$$

We now use as test function $\phi = \eta^2 |Zu^{\delta,\varepsilon}|^\beta Z u^{\delta,\varepsilon}$ where $\beta \geq 0$ and $\eta \in C^1([0,T], C_0^\infty(\Omega))$

and we get

$$\int_{t_1}^{t_2}\int_\Omega \partial_t Zu^{\delta,\varepsilon}\eta^2|Zu^{\delta,\varepsilon}|^\beta Zu^{\delta,\varepsilon}dxdt =$$

$$= \int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{2n+1}(X_i^\varepsilon + k_i)(A_{i,\xi_j}^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})(X_j^\varepsilon + k_j)Zu^{\delta,\varepsilon})\eta^2|Zu^{\delta,\varepsilon}|^\beta Zu^{\delta,\varepsilon}dxdt +$$

$$+ \int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^{2n+1}(X_i^\varepsilon + k_i)(A_{i,x_{2n+1}}^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon}))\eta^2|Zu^{\delta,\varepsilon}|^\beta Zu^{\delta,\varepsilon}dxdt =: I_1 + I_2 + I_3$$

For I_1 we have that

$$I_1 = \frac{1}{\beta+2}\int_{t_1}^{t_2}\int_\Omega \partial_t|Zu^{\delta,\varepsilon}|^{\beta+2}\eta^2 dxdt$$

If we expand I_2 and we integrate by parts with respect with X_i^ε and X_j^ε we get

$$I_2 = \int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{2n+1} k_i(A_{i,\xi_j}^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})(X_j^\varepsilon + k_j)Zu^{\delta,\varepsilon})\eta^2|Zu^{\delta,\varepsilon}|^\beta Zu^{\delta,\varepsilon}dxdt +$$

$$+ \int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{2n+1} X_i^\varepsilon(A_{i,\xi_j}^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})(X_j^\varepsilon + k_j)Zu^{\delta,\varepsilon})\eta^2|Zu^{\delta,\varepsilon}|^\beta Zu^{\delta,\varepsilon}dxdt =$$

$$= \int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{2n+1} k_i(A_{i,\xi_j}^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})(X_j^\varepsilon + k_j)u^{\delta,\varepsilon})\eta^2|Zu^{\delta,\varepsilon}|^\beta Zu^{\delta,\varepsilon}dxdt +$$

$$- \int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{2n+1} A_{i,\xi_j}^{\delta}(x,\nabla_\varepsilon u^{\delta,\varepsilon})(X_j^\varepsilon + k_j)Zu^{\delta,\varepsilon}X_i^\varepsilon(\eta^2|Zu^{\delta,\varepsilon}|^\beta Zu^{\delta,\varepsilon})dxdt =$$

$$= \int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{2n+1} k_i(A_{i,\xi_j}^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})(X_j^\varepsilon + k_j)Zu^{\delta,\varepsilon})\eta^2|Zu^{\delta,\varepsilon}|^\beta Zu^{\delta,\varepsilon}dxdt +$$

$$-2\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{2n+1} A_{i,\xi_j}^{\delta,\varepsilon}(x,\nabla_\varepsilon u^{\delta,\varepsilon})(X_j^\varepsilon + k_j)Zu^{\delta,\varepsilon}\eta X_i^\varepsilon\eta|Zu^{\delta,\varepsilon}|^\beta Zu^{\delta,\varepsilon}dxdt +$$

$$-(\beta+1)\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{2n+1} A_{i,\xi_j}^{\delta}(x,\nabla_\varepsilon u^{\delta})(X_j^\varepsilon + k_j)Zu^{\delta,\varepsilon}\eta^2|Zu^{\delta,\varepsilon}|^\beta X_i^\varepsilon Zu^{\delta,\varepsilon}dxdt =$$

$$= \int_{t_1}^{t_2} \int_\Omega \sum_{i,j=1}^{2n+1} k_i (A^{\delta,\varepsilon}_{i,\xi_j}(x, \nabla_\varepsilon u^{\delta,\varepsilon}) k_j Z u^{\delta,\varepsilon}) \eta^2 |Zu^{\delta,\varepsilon}|^\beta Z u^{\delta,\varepsilon} dxdt+$$

$$+ \int_{t_1}^{t_2} \int_\Omega \sum_{i,j=1}^{2n+1} k_i (A^{\delta,\varepsilon}_{i,\xi_j}(x, \nabla_\varepsilon u^{\delta,\varepsilon}) X^\varepsilon_j Z u^{\delta,\varepsilon}) \eta^2 |Zu^{\delta,\varepsilon}|^\beta Z u^{\delta,\varepsilon} dxdt+$$

$$-2 \int_{t_1}^{t_2} \int_\Omega \sum_{i,j=1}^{2n+1} A^{\delta,\varepsilon}_{i,\xi_j}(x, \nabla_\varepsilon u^{\delta,\varepsilon}) X^\varepsilon_j Z u^{\delta,\varepsilon} \eta X^\varepsilon_i \eta |Zu^{\delta,\varepsilon}|^\beta Z u^{\delta,\varepsilon} dxdt+$$

$$-2 \int_{t_1}^{t_2} \int_\Omega \sum_{i,j=1}^{2n+1} A^{\delta,\varepsilon}_{i,\xi_j}(x, \nabla_\varepsilon u^{\delta,\varepsilon}) k_j Z u^{\delta,\varepsilon} \eta X^\varepsilon_i \eta |Zu^{\delta,\varepsilon}|^\beta Z u^{\delta,\varepsilon} dxdt+$$

$$-(\beta+1) \int_{t_1}^{t_2} \int_\Omega \sum_{i,j=1}^{2n+1} A^{\delta,\varepsilon}_{i,\xi_j}(x, \nabla_\varepsilon u^{\delta,\varepsilon}) X^\varepsilon_j Z u^{\delta,\varepsilon} \eta^2 |Zu^{\delta,\varepsilon}|^\beta X^\varepsilon_i Z u^{\delta,\varepsilon} dxdt+$$

$$-(\beta+1) \int_{t_1}^{t_2} \int_\Omega \sum_{i,j=1}^{2n+1} A^{\delta,\varepsilon}_{i,\xi_j}(x, \nabla_\varepsilon u^{\delta,\varepsilon}) k_j Z u^{\delta,\varepsilon} \eta^2 |Zu^{\delta,\varepsilon}|^\beta X^\varepsilon_i Z u^{\delta,\varepsilon} dxdt$$

For I_3 we have (as we did for I_2) that

$$I_3 = \int_{t_1}^{t_2} \int_\Omega \sum_{i,j=1}^{2n+1} k_i (A^{\delta,\varepsilon}_{i,x_{2n+1}}(x, \nabla_\varepsilon u^{\delta,\varepsilon})) \eta^2 |Zu^{\delta,\varepsilon}|^\beta Z u^{\delta,\varepsilon} dxdt+$$

$$+ \int_{t_1}^{t_2} \int_\Omega \sum_{i=1}^{2n+1} X^\varepsilon_i (A^{\delta,\varepsilon}_{i,x_{2n+1}}(x, \nabla_\varepsilon u^{\delta,\varepsilon})) \eta^2 |Zu^{\delta,\varepsilon}|^\beta Z u^{\delta,\varepsilon} dxdt =$$

$$= \int_{t_1}^{t_2} \int_\Omega \sum_{i=1}^{2n+1} k_i (A^{\delta,\varepsilon}_{i,x_{2n+1}}(x, \nabla_\varepsilon u^{\delta,\varepsilon})) \eta^2 |Zu^{\delta,\varepsilon}|^\beta Z u^{\delta,\varepsilon} dxdt+$$

$$-2 \int_{t_1}^{t_2} \int_\Omega \sum_{i=1}^{2n+1} A^{\delta,\varepsilon}_{i,x_{2n+1}}(x, \nabla_\varepsilon u^{\delta,\varepsilon}) X^\varepsilon_i \eta |Zu^{\delta,\varepsilon}|^\beta Z u^{\delta,\varepsilon} dxdt+$$

$$-(\beta+1) \int_{t_1}^{t_2} \int_\Omega \sum_{i=1}^{2n+1} A^{\delta,\varepsilon}_{i,x_{2n+1}}(x, \nabla_\varepsilon u^{\delta,\varepsilon}) \eta^2 |Zu^{\delta,\varepsilon}|^\beta X^\varepsilon_i Z u^{\delta,\varepsilon} dxdt$$

Putting togheter I_1, I_2, I_3 we get the following equality

$$
\begin{aligned}
\frac{1}{\beta+2}\int_{t_1}^{t_2}\int_\Omega \partial_t |Zu^{\delta,\varepsilon}|^{\beta+2}\eta^2 dxdt = \\
+\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{2n+1} k_i (A^{\delta,\varepsilon}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) k_j Z u^{\delta,\varepsilon})\eta^2 |Zu^{\delta,\varepsilon}|^\beta Zu^{\delta,\varepsilon} dxdt + \\
+\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{2n+1} k_i (A^{\delta,\varepsilon}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) X^\varepsilon_j Z u^{\delta,\varepsilon})\eta^2 |Zu^{\delta,\varepsilon}|^\beta Zu^{\delta,\varepsilon} dxdt + \\
-2\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{2n+1} A^{\delta,\varepsilon}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) X^\varepsilon_j Z u^{\delta,\varepsilon}\eta X^\varepsilon_i\eta |Zu^{\delta,\varepsilon}|^\beta Zu^{\delta,\varepsilon} dxdt + \\
-2\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{2n+1} A^{\delta,\varepsilon}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) k_j Z u^{\delta,\varepsilon}\eta X^\varepsilon_i\eta |Zu^{\delta,\varepsilon}|^\beta Zu^{\delta,\varepsilon} dxdt + \\
-(\beta+1)\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{2n+1} A^{\delta,\varepsilon}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) X^\varepsilon_j Z u^{\delta,\varepsilon}\eta^2 |Zu^{\delta,\varepsilon}|^\beta X^\varepsilon_i Zu^{\delta,\varepsilon} dxdt + \\
-(\beta+1)\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{2n+1} A^{\delta}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) k_j Z u^{\delta,\varepsilon}\eta^2 |Zu^{\delta,\varepsilon}|^\beta X^\varepsilon_i Zu^{\delta,\varepsilon} dxdt + \\
+\int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^{2n+1} k_i (A^{\delta,\varepsilon}_{i,x_{2n+1}}(x,\nabla_\varepsilon u^{\delta,\varepsilon}))\eta^2 |Zu^{\delta,\varepsilon}|^\beta Zu^{\delta,\varepsilon} dxdt + \\
-2\int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^{2n+1} A^{\delta,\varepsilon}_{i,x_{2n+1}}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) X^\varepsilon_i\eta |Zu^{\delta,\varepsilon}|^\beta Zu^{\delta,\varepsilon} dxdt + \\
-(\beta+1)\int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^{2n+1} A^{\delta,\varepsilon}_{i,x_{2n+1}}(x,\nabla_\varepsilon u^{\delta,\varepsilon})\eta^2 |Zu^{\delta,\varepsilon}|^\beta X^\varepsilon_i Zu^{\delta,\varepsilon} dxdt \qquad (4.19)
\end{aligned}
$$

We now define $C_k = \sup_j ||k_j||_{L^\infty}$ and using the structural assumptions we have the following sequence of inequalities

$$
\begin{aligned}
-2\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{2n+1} A^{\delta,\varepsilon}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) X^\varepsilon_j Z u^{\delta,\varepsilon}\eta X^\varepsilon_i\eta |Zu^{\delta,\varepsilon}|^\beta Zu^{\delta,\varepsilon} dxdt \le \\
\le 2\Lambda \int_{t_1}^{t_2}\int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}} |\nabla_\varepsilon Z u^{\delta,\varepsilon}|\eta|\nabla_\varepsilon\eta||Zu^{\delta,\varepsilon}|^{\beta+1} dxdt \qquad (4.20)
\end{aligned}
$$

and

$$- (\beta+1)\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{2n+1} A^{\delta,\varepsilon}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) X^\varepsilon_j Z u^{\delta,\varepsilon}\eta^2 |Zu^{\delta,\varepsilon}|^\beta X^\varepsilon_i Z u^{\delta,\varepsilon} dxdt \leq$$
$$\leq -\lambda(\beta+1)\int_{t_1}^{t_2}\int_\Omega (\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}} |\nabla_\varepsilon Z u^{\delta,\varepsilon}|^2 |Zu^{\delta,\varepsilon}|^\beta |\eta|^2 dxdt \quad (4.21)$$

and

$$- 2\int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^{2n+1} A^{\delta,\varepsilon}_{i,x_{2n+1}}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) X^\varepsilon_i \eta |Zu^{\delta,\varepsilon}|^\beta Z u^{\delta,\varepsilon} dxdt \leq$$
$$\leq 2\Lambda\int_{t_1}^{t_2}\int_\Omega (\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-1}{2}} \eta |\nabla_\varepsilon \eta| |Zu^{\delta,\varepsilon}|^{\beta+1} dxdt \quad (4.22)$$

and

$$- (\beta+1)\int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^{2n+1} A^{\delta,\varepsilon}_{i,x_{2n+1}}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) \eta^2 |Zu^{\delta,\varepsilon}|^\beta X^\varepsilon_i Z u^{\delta,\varepsilon} dxdt \leq$$
$$\leq (\beta+1)\Lambda\int_{t_1}^{t_2}\int_\Omega (\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-1}{2}} \eta^2 |Zu^{\delta,\varepsilon}|^\beta |\nabla_\varepsilon Z u^{\delta,\varepsilon}| dxdt \quad (4.23)$$

and

$$- 2\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{2n+1} A^{\delta,\varepsilon}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) k_j Z u^{\delta,\varepsilon} \eta X^\varepsilon_i \eta |Zu^{\delta,\varepsilon}|^\beta Z u^{\delta,\varepsilon} dxdt \leq$$
$$\leq 2\Lambda\int_{t_1}^{t_2}\int_\Omega (\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}} C_k \eta |\nabla_\varepsilon \eta| |Zu^{\delta,\varepsilon}|^{\beta+2} dxdt \quad (4.24)$$

and

$$- (\beta+1)\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{2n+1} A^{\delta,\varepsilon}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon}) k_j Z u^{\delta,\varepsilon} \eta^2 |Zu^{\delta,\varepsilon}|^\beta X^\varepsilon_i Z u^{\delta,\varepsilon} dxdt \leq$$
$$\leq -\lambda(\beta+1)\int_{t_1}^{t_2}\int_\Omega (\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}} C_k |\eta|^2 |\nabla_\varepsilon Z u^{\delta,\varepsilon}| |Zu^{\delta,\varepsilon}|^{\beta+1} dxdt \quad (4.25)$$

and

$$\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{2n+1} k_i(A^{\delta,\varepsilon}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon})k_j Zu^{\delta,\varepsilon})\eta^2|Zu^{\delta,\varepsilon}|^\beta Zu^{\delta,\varepsilon}dxdt \leq$$
$$\leq \Lambda\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}}C_k^2|Zu^{\delta,\varepsilon}|^{\beta+2}|\eta|^2dxdt \quad (4.26)$$

and

$$\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^{2n+1} k_i(A^{\delta,\varepsilon}_{i,\xi_j}(x,\nabla_\varepsilon u^{\delta,\varepsilon})X^\varepsilon_j Zu^{\delta,\varepsilon})\eta^2|Zu^{\delta,\varepsilon}|^\beta Zu^{\delta,\varepsilon}dxdt \leq$$
$$\leq \Lambda\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}}C_k|\nabla_\varepsilon Zu^{\delta,\varepsilon}||\eta|^2|Zu^{\delta,\varepsilon}|^{\beta+1}dxdt \quad (4.27)$$

and

$$\int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^{2n+1} k_i(A^{\delta,\varepsilon}_{i,x_{2n+1}}(x,\nabla_\varepsilon u^{\delta,\varepsilon}))\eta^2|Zu^{\delta,\varepsilon}|^\beta Zu^{\delta,\varepsilon}dxdt \leq$$
$$\leq \Lambda\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-1}{2}}C_k\eta^2|Zu^{\delta,\varepsilon}|^{\beta+1}dxdt \quad (4.28)$$

Combining (4.20) − (4.28) with (4.19) we obtain the following inequality

$$\frac{1}{\beta+2}\int_\Omega |Zu^{\delta,\varepsilon}|^{\beta+2}\eta^2\Big|_{t_1}^{t_2}dx + C\int_{t_1}^{t_2}\int_\Omega (\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}}|\nabla_\varepsilon Zu^{\delta,\varepsilon}|^2|Zu^{\delta,\varepsilon}|^\beta|\eta|^2dxdt+$$
$$+\lambda(\beta+1)\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}}C_k|\eta|^2|\nabla_\varepsilon Zu^{\delta,\varepsilon}||Zu^{\delta,\varepsilon}|^{\beta+1}dxdt \le$$
$$\le C\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}}|\nabla_\varepsilon\eta|^2|Zu^{\delta,\varepsilon}|^{\beta+2}dxdt + \frac{2}{\beta+2}\int_{t_1}^{t_2}\int_\Omega |Zu^{\delta,\varepsilon}|^{\beta+2}\eta\partial_t\eta dxdt+$$
$$+C\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p}{2}}\eta^2|Zu^{\delta,\varepsilon}|^\beta dxdt+$$
$$+2\Lambda\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}}C_k\eta|\nabla_\varepsilon\eta||Zu^{\delta,\varepsilon}|^{\beta+2}dxdt+$$
$$+\Lambda\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}}C_k^2|Zu^{\delta,\varepsilon}|^{\beta+2}|\eta|^2dxdt+$$
$$+\Lambda\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-1}{2}}C_k\eta^2|Zu^{\delta,\varepsilon}|^{\beta+1}dxdt$$
$$+\Lambda\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}}C_k|\nabla_\varepsilon Zu^{\delta,\varepsilon}||\eta|^2|Zu^{\delta,\varepsilon}|^{\beta+1}dxdt \quad (4.29)$$

Now we apply several times the Young's inequality: we get that

$$\lambda(\beta+1)\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}}C_k|\eta|^2|\nabla_\varepsilon Zu^{\delta,\varepsilon}||Zu^{\delta,\varepsilon}|^{\beta+1}dxdt \le$$
$$\le C\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}}\eta^2|\nabla_\varepsilon Zu^{\delta,\varepsilon}|^2|Zu^{\delta,\varepsilon}|^\beta dxdt+$$
$$+C\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}}\eta^2|Zu^{\delta,\varepsilon}|^{\beta+2}dxdt \quad (4.30)$$

and

$$2\Lambda \int_{t_1}^{t_2} \int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}} C_k \eta |\nabla_\varepsilon \eta| |Zu^{\delta,\varepsilon}|^{\beta+2} dxdt \leq$$

$$\leq C \int_{t_1}^{t_2} \int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}} \eta^2 |Zu^{\delta,\varepsilon}|^{\beta+2} +$$

$$+ C \int_{t_1}^{t_2} \int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}} |\nabla_\varepsilon \eta|^2 |Zu^{\delta,\varepsilon}|^{\beta+2} dxdt \quad (4.31)$$

and finally

$$\Lambda \int_{t_1}^{t_2} \int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}} C_k |\nabla_\varepsilon Z u^{\delta,\varepsilon}| |\eta|^2 |Zu^{\delta,\varepsilon}|^{\beta+1} dxdt \leq$$

$$\leq C \int_{t_1}^{t_2} \int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}} \eta^2 |\nabla_\varepsilon Z u^{\delta,\varepsilon}|^2 |Zu^{\delta,\varepsilon}|^{\beta} +$$

$$+ C \int_{t_1}^{t_2} \int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}} \eta^2 |Zu^{\delta,\varepsilon}|^{\beta+2} dxdt \quad (4.32)$$

Combining (4.30), (4.31), (4.32) with (4.29) we get the thesis. □

4.5 Main Caccioppoli inequality

Using the Poincaré-like interpolation inequality (Lemma 4.3.1) we can get the following estimate which will be the key to prove the main Caccioppoli inequality.

Lemma 4.5.1. *Let $u^{\delta,\varepsilon}$ be a solution of (4.3), $2 \leq p \leq 4$. Then there exists a constant $C = C(n, p, \lambda, \Lambda, \mathcal{K}_i) > 0$ such that for all $\beta \geq 0$ and for all non negative function $\eta \in C^1([0,T], C_0^\infty(\Omega))$ vanishing on the parabolic boundary of Q we have that*

$$\left(\int_{t_1}^{t_2} \int_\Omega |Zu^{\delta,\varepsilon}|^{p+\beta} \eta^{p+\beta} dxdt \right)^{\frac{1}{p+\beta}} \leq$$

$$\leq C(p+\beta) ||\nabla_\varepsilon \eta||_{L^\infty} ||1+\eta||_{L^\infty} \left(\int \int_{spt(\eta)} (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}} dxdt \right)^{\frac{1}{p+\beta}} +$$

$$+C(p+\beta)||\eta\partial_t\eta||_{L^\infty}^{\frac{1}{2}}|spt(\eta)|^{\frac{p-2}{2(p+\beta)}}\left(\int\int_{spt(\eta)}(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}}dxdt\right)^{\frac{4-p}{2(p+\beta)}}$$

Proof. Using Proposition 4.4.4 and Lemma 4.3.1 we get the following estimate for M (where M is defined as in Lemma 4.3.1):

$$M \leq C\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2}{2}}|\nabla_\varepsilon\eta|^2\eta^{2+\beta}|Zu^{\delta,\varepsilon}|^{\beta+2}dxdt+$$

$$+C\int_{t_1}^{t_2}\int_\Omega|Zu^{\delta,\varepsilon}|^{\beta+2}\eta^{3+\beta}|\partial_t\eta|dxdt$$

Now using the Hölder's inequality we obtain the estimate

$$M \leq C||\nabla_\varepsilon\eta||_{L^\infty}^2 R^{\frac{p-2}{p+\beta}}L^{\frac{\beta+2}{p+\beta}}+C||\eta\partial_t\eta||_{L^\infty}|spt(\eta)|^{\frac{p-2}{p+\beta}}L^{\frac{\beta+2}{p+\beta}} \tag{4.33}$$

where L and R are defined as in Lemma 4.3.1. In the proof of Lemma 4.3.1 we also saw that

$$I_1 \leq 2(p+\beta)CM^{\frac{1}{2}}R^{\frac{4-p}{2(p+\beta)}}L^{\frac{2p-4+\beta}{2(p+\beta)}} \tag{4.34}$$

Combining (4.33) with (4.34) we get

$$I_1 \leq 2(p+\beta)C\left(C||\nabla_\varepsilon\eta||_{L^\infty}^2 R^{\frac{p-2}{p+\beta}}L^{\frac{\beta+2}{p+\beta}}+C||\eta\partial_t\eta||_{L^\infty}|spt(\eta)|^{\frac{p-2}{p+\beta}}L^{\frac{\beta+2}{p+\beta}}\right)^{\frac{1}{2}}R^{\frac{4-p}{2(p+\beta)}}L^{\frac{2p-4+\beta}{2(p+\beta)}} \leq$$

$$\leq C(p+\beta)||\nabla_\varepsilon\eta||_{L^\infty}R^{\frac{1}{p+\beta}}L^{\frac{p-1+\beta}{p+\beta}}+C(p+\beta)||\eta\partial_t\eta||_{L^\infty}^{\frac{1}{2}}|spt(\eta)|^{\frac{p-2}{2(p+\beta)}}R^{\frac{4-p}{2(p+\beta)}}L^{\frac{p-1+\beta}{2(p+\beta)}}$$

and adding I_2+I_3 to both sides of this inequality we obtain

$$L \leq C(p+\beta)||\nabla_\varepsilon\eta||_{L^\infty}||1+\eta||_{L^\infty}R^{\frac{1}{p+\beta}}L^{\frac{p-1+\beta}{p+\beta}}+C(p+\beta)||\eta\partial_t\eta||_{L^\infty}^{\frac{1}{2}}|spt(\eta)|^{\frac{p-2}{2(p+\beta)}}R^{\frac{4-p}{2(p+\beta)}}L^{\frac{p-1+\beta}{2(p+\beta)}}$$

and therefore the thesis. □

The following Proposition is our main Caccioppoli inequality and play the role of the

Caccioppoli inequality (Proposition 3.0.2) in Chapter 3.

Proposition 4.5.2. *Let $u^{\delta,\varepsilon}$ be a weak solution of (4.3) and $2 \leq p \leq 4$. Then there exists a constant $C = C(n, p, \lambda, \Lambda, \mathcal{K}_i) > 0$ such that for all $\beta \geq 0$ and for all non negative function $\eta \in C^1([0,T], C_0^\infty(\Omega))$ vanishing on the parabolic boundary of Q we have that*

$$
\begin{aligned}
\sup_{t_1<t<t_2} \int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta+2}{2}} \eta^2 dx + \int_{t_1}^{t_2} \int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2+\beta}{2}} |\nabla_\varepsilon^2 u^{\delta,\varepsilon}|^2 \eta^2 dxdt \leq \\
\leq C(p+\beta)^7 (||\eta||^2_{L^\infty} + ||\nabla_\varepsilon \eta||^2_{L^\infty} + ||\eta Z \eta||_{L^\infty}) \int\int_{spt(\eta)} (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}} dxdt + \\
+ C(p+\beta)^7 ||\eta \partial_t \eta||_{L^\infty} |spt(\eta)|^{\frac{p-2}{p+\beta}} \left(\int\int_{spt(\eta)} (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}} dxdt \right)^{\frac{\beta+2}{p+\beta}}
\end{aligned} \tag{4.35}
$$

Proof. From Proposition 4.4.3 we get the following estimate for the left hand side of (4.35)

$$
\begin{aligned}
\sup_{t_1<t<t_2} \int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta+2}{2}} \eta^2 dx + \int_{t_1}^{t_2} \int_\Omega \eta^2 (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p-2+\beta}{2}} |\nabla_0^2 u^{\delta,\varepsilon}|^2 dxdt \leq \\
\leq C(p+\beta) \int_{t_1}^{t_2} \int_\Omega (\eta^2 + |\nabla_\varepsilon \eta|^2 + \eta |Z\eta|)(\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}} dxdt + \\
+ C(\beta+1)^5 \int_{t_1}^{t_2} \int_\Omega \eta^2 (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta-2}{2}} |Zu^{\delta,\varepsilon}|^2 dxdt + \\
+ C \int_{t_1}^{t_2} \int_\Omega (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta+2}{2}} |\partial_t \eta| \eta dxdt
\end{aligned} \tag{4.36}
$$

To obtain the thesis we prove that each integral on the right hand side of (4.36) can be bounded from above from the right hand side of (4.35). For the first term of the right hand side of (4.36) we have the obvious inequality

$$
\begin{aligned}
C(p+\beta) \int_{t_1}^{t_2} \int_\Omega (\eta^2 + |\nabla_\varepsilon \eta|^2 + \eta |Z\eta|)(\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}} dxdt \leq \\
\leq C(p+\beta)^7 (||\eta||^2_{L^\infty} + ||\nabla_\varepsilon \eta||^2_{L^\infty} + ||\eta Z \eta||_{L^\infty}) \int\int_{spt(\eta)} (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}} dxdt
\end{aligned} \tag{4.37}
$$

For the third term of the right hand side of (4.36) we get, using the Hölder's inequality:

$$C\int_{t_1}^{t_2}\int_{\Omega}(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\beta+2}{2}}|\partial_t\eta|\eta dxdt\leq$$
$$\leq C||\eta\partial_t\eta||_{L^\infty}|spt(\eta)|^{\frac{p-2}{p+\beta}}\left(\int\int_{spt(\eta)}(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}}dxdt\right)^{\frac{\beta+2}{p+\beta}}\leq$$
$$\leq C(p+\beta)^7||\eta\partial_t\eta||_{L^\infty}|spt(\eta)|^{\frac{p-2}{p+\beta}}\left(\int\int_{spt(\eta)}(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}}dxdt\right)^{\frac{\beta+2}{p+\beta}} \quad (4.38)$$

For the second term of the right hand side of (4.36) we get, using before the Hölder inequality and then the Lemma 4.5.1

$$\int_{t_1}^{t_2}\int_{\Omega}\eta^2(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta-2}{2}}|Zu^{\delta,\varepsilon}|^2dxdt\leq$$
$$\leq\left(\int\int_{spt(\eta)}(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}}dxdt\right)^{\frac{p-2+\beta}{p+\beta}}\left(\int_{t_1}^{t_2}\int_{\Omega}|Zu^{\delta,\varepsilon}|^{p+\beta}\eta^{p+\beta}dxdt\right)^{\frac{2}{p+\beta}}\leq$$
$$\leq C(p+\beta)^2||\nabla_\varepsilon\eta||^2_{L^\infty}||1+\eta||^2_{L^\infty}\left(\int\int_{spt(\eta)}(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}}dxdt\right)^{\frac{2}{p+\beta}}+$$
$$+C(p+\beta)^2||\eta\partial_t\eta||_{L^\infty}|spt(\eta)|^{\frac{p-2}{p+\beta}}\left(\int\int_{spt(\eta)}(\delta+|\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{2}}dxdt\right)^{\frac{\beta+2}{p+\beta}} \quad (4.39)$$

Combining (4.37), (4.38) and (4.39) with (4.36) we get the thesis. □

4.6 Boundedness of the horizontal gradient

Now we have all the ingredients to start the Moser type iteration i.e. we can obtain the following theorem which proves the local boundnedness of the horizontal gradient.

Theorem 4.6.1. *Let $u^{\delta,\varepsilon}$ be a weak solution of* (4.3)*,* $2\leq p\leq 4$ *and* $N=2n+2$*. Then for all* $Q_{\mu,r}=B(x,r)\times[t_0-\mu r,t_0], Q_{\mu,2r}\subset\subset Q$ *there exists a constant* $C=$

$C(n, p, \lambda, \Lambda, \mathcal{K}_i) > 0$ *such that*

$$\sup_{Q_{\mu,r}} |\nabla_\varepsilon u^{\delta,\varepsilon}| \leq C\mu^{\frac{1}{2}} \max\left(\left(\frac{1}{\mu r^{N+2}} \int\int_{Q_{\mu,2r}} (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p}{2}} dxdt\right)^{\frac{1}{2}}, \mu^{\frac{p}{2(2-p)}}\right)$$

Proof. Let $\eta \in C^1([0,T], C_0^\infty(\Omega))$ be a non negative function vanishing on the parabolic boundary of Q and such that $|\eta| \leq 1$ in Q. For $\beta \geq 0$ we consider the function

$$v = (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p+\beta}{4}} \eta^2$$

From Proposition 4.5.2 we get the following estimate

$$\begin{aligned} \sup_{t_1<t<t_2} \int_\Omega v^m dx + \int_{t_1}^{t_2} \int_\Omega |\nabla_\varepsilon v|^2 \leq \\ \leq C(p+\beta)^7 (||\eta||^2_{L^\infty} + ||\nabla_\varepsilon \eta||^2_{L^\infty} + ||\eta Z\eta||_{L^\infty}) \int\int_{spt(\eta)} v^2 dxdt + \\ + C(p+\beta)^7 ||\eta\partial_t\eta||_{L^\infty} |spt(\eta)|^{\frac{p-2}{p+\beta}} \left(\int\int_{spt(\eta)} v^2\right)^{\frac{\beta+2}{p+\beta}} \end{aligned} \tag{4.40}$$

where $m = \frac{2(\beta+2)}{p+\beta}$. Now let $q = \frac{2(m+N)}{N}$. Using before the Hölder inequality and then the Sobolev inequality in the space variables we get the following estimate:

$$\begin{aligned} \int_{t_1}^{t_2} \int_\Omega v^q dxdt \leq \int_{t_1}^{t_2} \left(\left(\int_\Omega v^m dx\right)^{\frac{2}{N}} \left(\int_\Omega v^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}}\right) dt \leq \\ \leq C\left(\sup_{t_1<t<t_2} \int_\Omega v^m dx\right)^{\frac{2}{N}} \left(\int_{t_1}^{t_2} \int_\Omega |\nabla_\varepsilon v|^2 dxdt\right) \end{aligned} \tag{4.41}$$

Combining (4.40) with (4.41) we get

$$\left(\int_{t_1}^{t_2}\int_{\Omega} v^q dxdt\right)^{\frac{N}{N+2}} \leq C(p+\beta)^7(||\eta||^2_{L^\infty}+||\nabla_\varepsilon\eta||^2_{L^\infty}+||\eta Z\eta||_{L^\infty})\int\int_{spt(\eta)} v^2 dxdt+$$
$$+C(p+\beta)^7||\eta\partial_t\eta||_{L^\infty}|spt(\eta)|^{\frac{p-2}{p+\beta}}\left(\int\int_{spt(\eta)} v^2 dxdt\right)^{\frac{\beta+2}{p+\beta}} \quad (4.42)$$

The inequality (4.42) is the inequality on which our Moser-type iteration is based. Now let $Q_{\mu,r} = B(x,r)\times[t_0-\mu r, t_0]\subset Q$. We define for $i\geq 0$ a sequence of radius $r_i = (1+2^{-i})r$ and a sequence of exponents β_i in the following way:

$$\beta_i = \begin{cases} 0 \text{ if } i=0 \\ p+(p+\beta_i)\left(1+\frac{2(\beta_i+2)}{N(p+\beta_i)}\right) \text{ if } i>0 \end{cases}$$

which can be written in a more compact form as

$$\beta_i = 2(\kappa^i - 1)$$

where $\kappa = \frac{N+2}{N}$. We denote $Q_i = Q_{\mu,r_i}$. Note that $Q_0 = Q_{\mu,2r}$ and $Q_\infty = Q_{\mu,r}$. Now we choose a standard parabolic cut-off function $\eta_i \in C^\infty(Q_i)$ such that $\eta_i = 1$ in Q_{i+1} and such that in Q_i we have

$$|\nabla_\varepsilon \eta_i| \leq \frac{2^{i+8}}{r}$$
$$|Z\eta_i| \leq \frac{2^{i+8}}{r^2}$$
$$|\partial_t\eta_i| \leq \frac{2^{2i+8}}{\mu r^2}$$

Now we write $\eta = \eta_i$ and $\beta = \beta_i$ and we get from (4.42) that for $i = 0, 1, 2, ...$ we have

$$\left(\int\int_{Q_{i+1}} (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\alpha_{i+1}}{2}} dxdt\right)^{\frac{N}{N+2}} \leq$$
$$\leq C2^{2i}\alpha_i^7 r^{-2} \left[\left(\int\int_{Q_i} (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\alpha_i}{2}} dxdt\right)^{\frac{p-2}{\alpha_i}} + \right.$$
$$\left. + \mu^{-1}(\mu r^{N+2})^{\frac{p-2}{\alpha_i}}\right] \left(\int\int_{Q_i} (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\alpha_i}{2}} dxdt\right)^{\frac{\alpha_i - p+2}{\alpha_i}} \quad (4.43)$$

whete $\alpha_i = p + \beta_i = p - 2 + 2\kappa^i$. Now we define M_i as

$$M_i = \left(\frac{1}{\mu r^{N+2}} \int\int_{Q_i} (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{\alpha_i}{2}} dxdt\right)^{\frac{1}{\alpha_i}}$$

and we can write (4.43) as

$$M_{i+1}^{\frac{\alpha_{i+1}}{\kappa}} \leq C\mu^{\frac{2}{N+2}} 2^{2i}\alpha_i^7 (M_i^{p-2} + \mu^{-1}) M_i^{\alpha_i - p+2}$$

We now define $\overline{M_i}$ as

$$\overline{M_i} = \max(M_i, \mu^{\frac{1}{2-p}})$$

and we immediatly get that

$$\overline{M_{i+1}}^{\frac{\alpha_{i+1}}{\kappa}} \leq C\mu^{\frac{2}{N+2}} 2^{2i}\alpha_i^7 \overline{M_i}^{\alpha_i} \quad (4.44)$$

Iterating (4.44) we obtain that

$$\overline{M_{i+1}} \leq \left(\prod_{j=0}^{i} K_j^{\frac{\kappa^{i+1-j}}{\alpha_{i+1}}}\right) \overline{M_0}^{\frac{\alpha_0 \kappa^{i+1}}{\alpha_{i+1}}} \quad (4.45)$$

where K_j is defined as

$$K_j = C\mu^{\frac{2}{N+2}} 2^{2j}\alpha_j^7$$

Letting $i \to +\infty$ in (4.45) we get

$$\overline{M_\infty} = \limsup_{i \to +\infty} \overline{M_i} \leq C\mu^{\frac{1}{2}} \overline{M_0}^{\frac{p}{2}} \tag{4.46}$$

but from the definition of $\overline{M_0}$ we also know that

$$\overline{M_0} = \max \left(\left(\frac{1}{\mu r^{N+2}} \int\int_{Q_{\mu,2r}} (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p}{2}} dxdt \right)^{\frac{1}{p}}, \mu^{\frac{1}{2-p}} \right) \tag{4.47}$$

but since

$$\overline{M_\infty} \geq \sup_{Q_{\mu,r}} |\nabla_\varepsilon u^{\delta,\varepsilon}| \tag{4.48}$$

combining (4.46), (4.47) and (4.48) we finally get the thesis. □

Corollary 4.6.2. *In the hypotesis of Theorem* 4.6.1 *we have that*

$$\sup_{Q_{\mu,r}} |\nabla_\varepsilon u^{\delta,\varepsilon}| \leq C \left(\fint\fint_{Q_{\mu,2r}} (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p}{2}} dxdt \right)^{\frac{1}{p}}$$

Proof. It suffices to choose μ in Theorem 4.6.1 as

$$\mu = \left(\frac{1}{r^{N+2}} \int\int_{Q_{\mu,2r}} (\delta + |\nabla_\varepsilon u^{\delta,\varepsilon}|^2)^{\frac{p}{2}} dxdt \right)^{\frac{2-p}{p}}$$

□

Now since this estimate continues to be true when $\delta, \varepsilon \to 0$ we can obtain the main Theorem which was stated in the introduction i.e. we have the following Theorem.

Theorem 4.6.3. *Let u be a weak solution of* (4.1). *Then for all $Q_{\mu,r} = B(x,r) \times [t_0 -$*

$\mu r, t_0], Q_{\mu,2r} \subset\subset Q$ *there exists a positive constant* $C = C(n, p, \lambda', \Lambda', \mathcal{K}_i)$ *such that*

$$\sup_{Q_{\mu,r}} |\nabla_0 u| \leq C \left(\fint \fint_{Q_{\mu,2r}} |\nabla_0 u|^p dxdt \right)^{\frac{1}{p}}$$

Proof. It suffices to send δ and ε to 0 in Corollary 4.6.2. □

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