

ALMA MATER STUDIORUM · UNIVERSITÀ DI BOLOGNA

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# How to get an equation named after you (Part I)

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# Abstract|EN

Scientific publications are not used nearly as much as they should be in undergraduate studies. This thesis aims to address this very simple problem, by making publications more accessible and easy to understand. For this reason, it should be treated more like the cornerstone of a project that would include the most famous papers in the history of physics.

How to get an equation named after you (Part I) covers the 1926 series of papers published by Schrödinger "*Quantisierung als Eigenwertproblem*" and the 1928 paper by Dirac "*The Quantum Theory of the Electron*", i.e. the papers where the Schrödinger and the Dirac equations were first published.

Schrödinger's series is in total over 100 pages long. He starts by showing how the basis of his wave mechanics theory can explain other phenomena e.g. hydrogen atom, Stark effect. Then, he proceeds to derive the famous complex second order differential wave equation. The maths in these papers is not straightforward and many of the proofs are unnecessarily long or simply missing.

Dirac's paper on the other hand mainly deals with the derivation of the equation, its generalisation and relativistic invariance. He also proves that his equation is compatible with previous theories. Overall Dirac's way of writing is much more systematic and compact since he uses more maths and less words than Schrödinger.

# Abstract|IT

Nelle lauree triennali le pubblicazioni scientifiche non vengono studiate. Per quanto possa rivelarsi un piccolo intervento, lo scopo di questa tesi è invece quello di rendere più accessibili ai pochi interessati alcune vecchie e polverose pubblicazioni. Inoltre, la presente tesi non dovrebbe essere trattata come un lavoro a se stante, ma come il primo mattone di un progetto che comprende tutte le maggiori pubblicazioni della storia.

How to get an equation named after you (Part I) in particolare discute la serie di pubblicazioni del 1926 di Schrödinger "*Quantisierung als Eigenwertproblem*" e l'articolo del 1928 di Dirac "*The Quantum Theory of the Electron*", ovvero quei lavori dove le equazioni di Schrödinger e Dirac vennero per prime derivate.

La serie di articoli del 1926 sommano ad un totale di oltre 100 pagine. Inizialmente Schrödinger dimostra come ciò su cui è basata la sua teoria possa spiegare correttamente fenomeni conosciuti come l'atomo di idrogeno e l'effetto Stark, per poi derivare la famosa equazione d'onda complessa del secondo ordine. I procedimenti matematici, in questi articoli, sono complicati e molte delle dimostrazioni non vengono mostrate oppure risultano inutilmente lunghe.

La pubblicazione di Dirac invece ha principalmente a che fare con la derivazione dell'equazione, la sua generalizzazione e l'invarianza relativistica. Dimostra inoltre che tale equazione è compatibile con passate teorie. La lettura di Dirac è molto più sistematica, dato il largo utilizzo di dimostrazioni matematiche laddove Schrödinger avrebbe usato parole.

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# Introduction

How did Newton come up with his theory of gravitation? Everybody knows this, he was sitting under a tree and then an apple hit him on the head. After that he wrote a 600 pages work, introducing infinitesimal calculus and deriving Kepler's law of planetary motion, stating Newton's laws of motion and Newton's law of universal gravitation. Have you ever been interested in the original works of the giants of physics? Have you ever thought:

*"Oh, I'd love to know the exact story of how Einstein came up with his theory of special relativity and how much of this theory was built upon the work of others."*

Come on, we have all had that...

Well, this thesis is for those who want easy access and recap of such monumental publications and moments of physics history. It explores papers that discussed groundbreaking physical concepts and those writers whom were the first to derive undying equations, without tweaking the concepts, notation and algebra of the author. Trying to keep their magical roots alive.

## 1.1 The idea

As you could have guessed from the title, Part I means this thesis is designed to be the first of other works. My initial idea for this thesis was to show how some of the most famous equations in physics (in my opinion) were first derived:

- Newton's law of universal gravitation
- Maxwell equations
- Lorentz transformations
- Energy-momentum relation
- Schrödinger equation

- Dirac equation.

Which would have resulted in the study of the following papers

- Philosophiæ Naturalis Principia Mathematica, 1687, by Sir Isaac Newton
- A Dynamical Theory of the Electromagnetic Field, 1865, by James Clerk Maxwell
- Electromagnetic phenomena in a system moving with any velocity smaller than that of light, 1904, by Hendrik Lorentz
- On the Electrodynamics of Moving Bodies, 1905, Albert Einstein
- Quantisation as a Problem of Proper Values Part I-IV, 1926, Erwin Schrödinger
- The Quantum Theory of the Electron, 1928, Paul Dirac.

This results in about  $10^3$  pages total, a data that quickly puts an end to my hopes of achieving this idea.

Therefore, I decided to study just Schrödinger and Dirac instead and leave the rest to someone else.

## 1.2 Schrödinger

The Schrödinger paper is divided in four parts. Here's is a quick recap of all the important steps made in each one of the parts.

Part I.

$$H\left(q, \frac{\hbar}{i} \frac{\partial \psi}{\partial q}\right) - E = 0 \implies \nabla^2 \psi - \frac{2m}{\hbar^2} \left( E + \frac{e^2}{r} \right) \psi = 0$$

$$\implies \text{Bohr energy levels} \quad -E_l = \frac{2\pi^2 m e^4}{\hbar^2 l^2}$$



## Part II.

$$\left. \begin{aligned} \frac{\partial W}{\partial t} + T(q_k, \frac{\partial W}{\partial q_k}) + V(q_k) &= 0 \\ W &= -Et + S(q_k) \\ ds^2 &= 2\bar{T}(q_k, \dot{q}_k) dt^2 \end{aligned} \right\} \begin{aligned} T + V &= E \\ ds &= \frac{E}{\sqrt{2T}} dt \\ &\downarrow \text{Wave velocity} \\ u &\equiv \frac{ds}{dt} = \frac{E}{\sqrt{2(E-V)}} \end{aligned}$$

Wave equation

$$\nabla^2 \psi - \frac{1}{u^2} \ddot{\psi} = 0 \quad \Rightarrow \quad \nabla^2 \psi - \frac{2(E-V)}{E^2} \frac{\partial^2 \psi}{\partial t^2} = 0$$

## Part III.

$$\nabla^2 \psi - \frac{8\pi^2 m}{h^2} \left( E + \frac{e^2}{r} - eFz \right) \psi = 0$$

$$\Rightarrow E_l = -\frac{2\pi^2 m e^4}{h^2 l^2} - \frac{3}{8} \frac{h^2 F l (k_z - k_i)}{\pi^2 m e}$$

## Part IV.

$$\left. \begin{aligned} \frac{\partial^2 \psi}{\partial t^2} &= -\frac{4\pi^2 E^2}{h^2} \psi \\ \nabla^2 \psi - \frac{2(E-V)}{E^2} \frac{\partial^2 \psi}{\partial t^2} &= 0 \end{aligned} \right\} \left( \nabla^2 - \frac{8\pi^2}{h^2} V \right)^2 \psi - \frac{64\pi^4}{h^4} E^2 \psi = 0$$

$$\Rightarrow \left( \nabla^2 - \frac{8\pi^2}{h^2} V + \frac{8\pi^2}{h^2} E \right) \left( \nabla^2 - \frac{8\pi^2}{h^2} V - \frac{8\pi^2}{h^2} E \right) \psi = 0$$

$$\text{for } \frac{\partial \psi}{\partial t} = \pm \frac{2\pi i}{h} \psi \Rightarrow \boxed{\nabla^2 \psi - \frac{8\pi^2}{h^2} V \psi \mp \frac{4\pi i}{h} \frac{\partial \psi}{\partial t} = 0}$$

## 1.3 Dirac

Dirac's paper is much shorter. The equation is derived in one short chapter, the second. The other chapters are proofs regarding the equation or its generalisations.

### 1.1) Previous Relativity Treatments

$$\left. \begin{array}{l} \text{Klein-Gordon equation} \\ (-p_0^2 + \underline{p}^2 + m^2 c^2) \psi = 0 \end{array} \right\} \begin{array}{l} p_0 = \frac{i\hbar}{c} \frac{\partial}{\partial t} \text{ i.e. non linear} \\ \Downarrow \\ \rho \propto W \text{ i.e. impossible to make} \\ \rho \text{ positive definite} \end{array}$$

### 1.2) The Hamiltonian for No Field

assume  $\exists \alpha_1, \alpha_2, \alpha_3, \beta : (p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta mc) \psi = 0$ ,

we'll use the K-G eq. to find these coefficients :

$$(-p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta mc)(p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta mc) \psi \equiv (-p_0^2 + \underline{p}^2 + m^2 c^2) \psi$$

$$\implies \boxed{\left( i \sum_{\mu} \gamma_{\mu} p_{\mu} + mc \right) \psi = 0} \quad \mu = 1, 2, 3, 4$$

### 1.3) Proof of Invariance under a Lorentz Transformation

### 1.4) The Hamiltonian for an Arbitrary Field

## 1.4 The future of the project is in your hands

*"I included a list of the papers I considered studying also as an idea for future students who would like to continue this project. Anyone is welcome to continue with Part II, I hope you'll keep the title format, I'm looking forward to read your work!"*

Part I

Schrödinger equation

# Collected Papers on Wave Mechanics

By E. Schrödinger, University of Berlin. [1]  
(Blackie & Son Limited, London & Glasgow, 1928.)

This book is a collection of papers published by Schrödinger in the year 1926. The papers collected were translated in English from their original language, German. In the following chapters we are going to cover the papers that proved the validity of his wave mechanics theory and showed how Schrödinger derived his famous equation, the Schrödinger equation

$$\nabla^2\psi - \frac{8\pi^2}{h^2}V\psi \mp \frac{4\pi i}{h} \frac{\partial\psi}{\partial t} = 0.$$

The original name of this series of four papers is "Quantisierung als Eigenwertproblem", published in the Annalen der Physik.

## Notation

$W$  action

$S$  Hamilton's characteristic function

Units are written with a dot after them:  $J. = kg.m.^2/s.^2$

Equations with a prime are equivalent: (1) is equivalent to (1')

# Quantisation as a Problem of Proper Values (Part I)

(Annalen der Physik (4), vol. 79, 1926)

IN this paper I wish to consider, first, the simple case of the hydrogen atom (non-relativistic and unperturbed), and show that the customary quantum conditions can be replaced by another postulate, in which the notion of "whole numbers", merely as such, is not introduced. Rather when integralness does appear, it arises in the same natural way as it does in the case of the node-numbers of a vibrating string. The new conception is capable of generalisation, and strikes, I believe, very deeply at the true nature of the quantum rules. The usual form of the latter is connected with the Hamilton-Jacobi differential equation,

$$H\left(q_k, \frac{\partial S}{\partial q_k}\right) = E \quad (1)$$

for separated variables

$$W(q_k, t) = S(q_k) - Et$$

A solution of this equation is sought such as can be represented as the sum of functions, each being a function of one only of the independent variables  $q$ .

... *Follows the relation between Hamilton-Jacobi equation, Hamiltonian, Lagrangian and Action.*

$$\begin{aligned} \text{Hamilton-Jacobi Equation} \quad \mathcal{H}\left(q_k, \frac{\partial W}{\partial q_k}, t\right) &= -\frac{\partial W}{\partial t} \\ \text{Action} \quad dW &= \sum \frac{\partial W}{\partial q_i} dq_i + \frac{\partial W}{\partial t} dt \\ \implies \frac{dW}{dt} &= \sum \frac{\partial W}{\partial q_i} \dot{q}_i + \frac{\partial W}{\partial t} = \sum p_i \dot{q}_i + \frac{\partial W}{\partial t} = \sum p_i \dot{q}_i - \mathcal{H} = \mathcal{L} \end{aligned}$$

.....  
Here we now put for  $S$  a new unknown  $\psi$  such that it will appear as a product of related functions of the single co-ordinates, *i.e.* we put

$$S = K \log \psi \quad (2).$$

The constant  $K$  must be introduced from considerations of dimensions; it has those of action. Hence we get

$$\mathcal{H}\left(q, \frac{K}{\psi} \frac{\partial \psi}{\partial q}\right) = E \quad (1')$$

Now we do not look for a solution of equation (1'), but proceed as follows. If we neglect the relativistic variation of mass, equation (1') can always be transformed so as to become a quadratic form (of  $\psi$  and its first derivatives) equated to zero. (For the one-electron problem this holds even when mass-variation is not neglected.) We now seek a function  $\psi$ , such that for any arbitrary variation of it the integral of the said quadratic form, taken over the whole co-ordinate space<sup>1</sup>, is stationary,  $\psi$  being everywhere real, single-valued, finite, and continuously differentiable up to the second order. The quantum conditions are replaced by this variation problem.

First, we will take for  $H$  the Hamilton function for Keplerian motion, and show that  $\psi$  can be so chosen for all positive, but only for a discrete set of negative values of  $E$ . That is, the above variation problem has a discrete and a continuous spectrum of proper values.

The discrete spectrum corresponds to the Balmer terms and the continuous to the energies of the hyperbolic orbits. For numerical agreement  $K$  must have the value  $h/2\pi$ .

The choice of co-ordinates in the formation of the variational equations being arbitrary, let us take rectangular Cartesians. Then (1') becomes in our case

$$\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 - \frac{2m}{K^2} \left(E + \frac{e^2}{r}\right) \psi^2 = 0 \quad (1'')$$

$$e = \text{charge, } m = \text{mass of an electron, } r^2 = x^2 + y^2 + z^2.$$

Our variation problem then reads

$$\delta J = \delta \iiint dx dy dz \left[ \left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 - \frac{2m}{K^2} \left(E + \frac{e^2}{r}\right) \psi^2 \right] = 0, \quad (3)$$

the integral being taken over all space. From this we find in the usual way usual way

$$\frac{1}{2} \delta J = \int df \delta \psi \frac{\partial \psi}{\partial n} - \iiint dx dy dz \delta \psi \left[ \nabla^2 \psi + \frac{2m}{K^2} \left(E + \frac{e^2}{r}\right) \psi \right] = 0. \quad (4)$$

Therefore we must have, firstly,

$$\nabla^2 \psi + \frac{2m}{K^2} \left(E + \frac{e^2}{r}\right) \psi = 0 \quad (5)$$

---

<sup>1</sup>I am aware this formulation is not entirely unambiguous.

and secondly,

$$\int df \delta\psi \frac{\partial\psi}{\partial n} = 0 \quad (6).$$

$df$  is an element of the infinite closed surface over which the integral is taken.

(It will turn out later that this last condition requires us to supplement our problem by a postulate as to the behaviour of  $\delta\psi$  at infinity, in order to ensure the existence of the above-mentioned continuous spectrum of proper values. See later.)

The solution of (5) can be effected, for example, in polar co-ordinates,  $r, \theta, \phi$ , if  $\psi$  be written as the product of three functions, each only of  $r$ , of  $\theta$ , or of  $\phi$ . The method is sufficiently well known. The function of the angles turns out to be a surface harmonic, and if that of  $r$  be called  $\chi$ , we get easily the differential equation,

$$\frac{d^2\chi}{dr^2} + \frac{2}{r} \frac{d\chi}{dr} + \left( \frac{2mE}{K^2} + \frac{2me^2}{K^2 r} - \frac{n(n+1)}{r^2} \right) \chi = 0 \quad (7)$$

$n = 0, 1, 2, 3$

.....  
*Proof.* Let  $\psi = \chi(r)Y(\theta, \phi)$ ,

$$\begin{aligned} \nabla^2\psi &= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial\psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial\psi}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2\psi}{\partial\phi^2} \\ &= \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial\chi}{\partial r} \right) \right) Y + \left( \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial Y}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 Y}{\partial\phi^2} \right) \chi \\ &= \frac{\chi Y}{r^2} \left( \frac{1}{\chi} \frac{\partial}{\partial r} \left( r^2 \frac{\partial\chi}{\partial r} \right) + \frac{1}{Y \sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial Y}{\partial\theta} \right) + \frac{1}{Y \sin^2\theta} \frac{\partial^2 Y}{\partial\phi^2} \right). \end{aligned}$$

Spherical harmonics are a known solution to the Laplace's equations in  $\chi$  and  $Y$ ,

$$\frac{1}{\chi} \frac{\partial}{\partial r} \left( r^2 \frac{\partial\chi}{\partial r} \right) = \lambda, \quad \frac{1}{Y \sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial Y}{\partial\theta} \right) + \frac{1}{Y \sin^2\theta} \frac{\partial^2 Y}{\partial\phi^2} = -\lambda.$$

Imposing  $Y(\theta, \phi) = \Theta(\theta)\Phi(\phi)$  and separating variables in the second differential equation we obtain

$$\begin{aligned} 0 &= \frac{1}{\Theta \sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial\Theta}{\partial\theta} \right) + \frac{1}{\Phi \sin^2\theta} \frac{\partial^2\Phi}{\partial\phi^2} + \lambda \\ 0 &= \frac{\sin\theta}{\Theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial\Theta}{\partial\theta} \right) + \lambda \sin^2\theta + \frac{1}{\Phi} \frac{\partial^2\Phi}{\partial\phi^2}, \end{aligned}$$

therefore,

$$\frac{1}{\Phi} \frac{\partial^2\Phi}{\partial\phi^2} = -m^2, \quad \frac{\sin\theta}{\Theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial\Theta}{\partial\theta} \right) + \lambda \sin^2\theta = m^2.$$

It is interesting to find out at what point of the calculations integers come into play. In other words, what gives rise to quantisation. In this case, it is due to the choice for a spherical harmonic solution. This assumption means  $\Theta$  must be a periodic function whose period divides  $2\pi$  evenly. Namely,  $m$  must be an integer. The second differential equation is a Sturm-Liouville problem that forces  $\lambda = n(n+1)$  with  $n \geq |m|$ . Substituting  $\lambda$  in the initial equation we get

$$\nabla^2\psi = \frac{\chi Y}{r^2} \left( \frac{1}{\chi} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \chi}{\partial r} \right) - \lambda \right),$$

substituting the result in equation (5),

$$\begin{aligned} 0 &= \nabla^2\psi + \frac{2m}{\bar{K}^2} \left( E + \frac{e^2}{r} \right) \psi \\ &= \frac{\chi Y}{r^2} \left( \frac{1}{\chi} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \chi}{\partial r} \right) - \lambda \right) + \frac{2m}{\bar{K}^2} \left( E + \frac{e^2}{r} \right) \chi Y, \\ 0 &= \frac{1}{r^2 \chi} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \chi}{\partial r} \right) - \frac{\lambda}{r^2} + \frac{2m}{\bar{K}^2} \left( E + \frac{e^2}{r} \right), \\ 0 &= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \chi}{\partial r} \right) + \left( \frac{2mE}{\bar{K}^2} + \frac{2me^2}{K^2 r} - \frac{\lambda}{r^2} \right) \chi. \end{aligned}$$

Which finally yields

$$0 = \frac{2}{r} \frac{\partial \chi}{\partial r} + \frac{\partial^2 \chi}{\partial r^2} + \left( \frac{2mE}{\bar{K}^2} + \frac{2me^2}{K^2 r} - \frac{n(n+1)}{r^2} \right) \chi.$$

QED

.....  
The limitation of  $n$  to integral values is necessary so that the surface harmonic may be single-valued. We require solutions of (7) that will remain finite for all non-negative real values of  $r$ . Now<sup>2</sup> equation (7) has two singularities in the complex  $r$ -plane, at  $r = 0$  and  $r = \infty$ , of which the second is an "indefinite point" (essential singularity) of all integrals, but the first on the contrary is not (for any integral). These two singularities form exactly the bounding points of our real interval. In such a case it is known now that the postulation of the finiteness of  $\chi$  at the bounding points is equivalent to a boundary condition. The equation has in general no integral which remains finite at both end points; such an integral exists only for certain special values of the constants in the equation. It is now a question of defining these special values. This is the jumping-off point of the whole investigation.<sup>3</sup>

<sup>2</sup>For guidance in the treatment of (7) I owe thanks to Hermann Weyl.

<sup>3</sup>For unproved propositions in what follows, see L. Schlesinger's Differential Equations (Collection Schubert, No. 13, Göschen, 1900, especially chapters 3 and 5).



Let us examine first the singularity at  $r = 0$ . The so-called indicial equation which defines the behaviour of the integral at this point, is

$$\rho(\rho - 1) + 2\rho - n(n + 1) = 0 \quad (8),$$

with roots,

$$\rho_1 = n, \quad \rho_2 = -(n + 1).$$

The two canonical integrals at this point have therefore the exponents  $n$  and  $-(n + 1)$ . Since  $n$  is not negative, only the first of these is of use to us. Since it belongs to the greater exponent, it can be represented by an ordinary power series, which begins with  $r^n$ . (The other integral, which does not interest us, can contain a logarithm, since the difference between the indices is an integer.)

The next singularity is at infinity, so the above power series is always convergent and represents a transcendental integral function. We therefore have established that :

The required solution is (except for a constant factor) a single-valued definite transcendental integral function, which at  $r = 0$  belongs to the exponent  $n$ .

We must now investigate the behaviour of this function at infinity on the positive real axis. To that end we simplify equation (7) by the substitution

$$\chi = r^\alpha U \quad (9),$$

where  $\alpha$  is so chosen that the term with  $1/r^2$  drops out. It is easy to verify that then  $\alpha$  must have one of the two values  $n, -(n + 1)$ . Equation (7) then takes the form,

$$\frac{d^2U}{dr^2} + \frac{2(a + 1)}{r} \frac{dU}{dr} + \frac{2m}{K^2} \left( E + \frac{e^2}{r} \right) U = 0 \quad (7').$$

.....

*Proof.*

$$\begin{aligned} 0 &= \frac{2}{r} \frac{\partial (r^\alpha U)}{\partial r} + \frac{\partial}{\partial r} \frac{\partial (r^\alpha U)}{\partial r} + \left( \frac{2mE}{\bar{K}^2} + \frac{2me^2}{K^2 r} - \frac{n(n + 1)}{r^2} \right) r^\alpha U \\ &= \frac{2\alpha r^{\alpha-1}}{r} U + \frac{2r^\alpha}{r} \frac{\partial U}{\partial r} + \frac{\partial}{\partial r} \left( r^\alpha \frac{\partial U}{\partial r} + U \alpha r^{\alpha-1} \right) + \left( \frac{2mE}{\bar{K}^2} + \frac{2me^2}{K^2 r} - \frac{n(n + 1)}{r^2} \right) r^\alpha U \\ &= 2\alpha r^{\alpha-2} U + 2r^{\alpha-1} \frac{\partial U}{\partial r} + \frac{\partial}{\partial r} \left( r^\alpha \frac{\partial U}{\partial r} + U \alpha r^{\alpha-1} \right) + \left( \frac{2mE}{\bar{K}^2} + \frac{2me^2}{K^2 r} - \frac{n(n + 1)}{r^2} \right) r^\alpha U \\ &= 2\alpha r^{\alpha-2} U + 2r^{\alpha-1} \frac{\partial U}{\partial r} + \alpha r^{\alpha-1} \frac{\partial U}{\partial r} + r^\alpha \frac{\partial^2 U}{\partial r^2} + \frac{\partial U}{\partial r} \alpha r^{\alpha-1} + U \alpha (\alpha - 1) r^{\alpha-2} \\ &\quad + \left( \frac{2mE}{\bar{K}^2} + \frac{2me^2}{K^2 r} - \frac{n(n + 1)}{r^2} \right) r^\alpha U \end{aligned}$$

$$\begin{aligned}
0 &= \frac{2\alpha}{r^2}U + \frac{2}{r}\frac{\partial U}{\partial r} + \frac{\alpha}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial r^2} + \frac{\partial U}{\partial r}\frac{\alpha}{r} + \frac{\alpha(\alpha-1)}{r^2}U + \left(\frac{2mE}{\bar{K}^2} + \frac{2me^2}{K^2r} - \frac{n(n+1)}{r^2}\right)U \\
&= \frac{2(\alpha+1)}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial r^2} + \frac{\alpha(\alpha+1)}{r^2}U + \left(\frac{2mE}{\bar{K}^2} + \frac{2me^2}{K^2r} - \frac{n(n+1)}{r^2}\right)U.
\end{aligned}$$

For  $\alpha = n$  we get (7'),

$$0 = \frac{\partial^2 U}{\partial r^2} + \frac{2(\alpha+1)}{r}\frac{\partial U}{\partial r} + \frac{2m}{K^2}\left(E + \frac{e^2}{r}\right)U$$

QED

.....  
Its integrals belong at  $r = 0$  to the exponents 0 and  $-2a-$ . For the  $\alpha$ -value,  $\alpha = n$ , the first of these integrals, and for the second  $a$ . value,  $a = -(n+1)$ , the second of these integrals is an integral function and leads, according to (9), to the desired solution, which is single-valued. We therefore lose nothing if we confine ourselves to one of the two  $a$ -values. Take, then,

$$a = n \quad (10).$$

Our solution  $U$  then, at  $r = 0$ , belongs to the exponent 0. Equation (7') is called Laplace's equation. The general type is

$$U^n + \left(\delta_0 + \frac{\delta_1}{r}\right)U' + \left(\epsilon_0 + \frac{\epsilon_1}{r}\right)U = 0 \quad (7'')$$

Here the constants have the values

$$\delta_0 = 0, \quad \delta_1 = 2(\alpha+1), \quad \epsilon_0 = \frac{2mE}{K^2}, \quad \epsilon_1 = \frac{2me^2}{\bar{K}^2} \quad (11).$$

This type of equation is comparatively simple to handle for this reason: The so-called Laplace's transformation, which in general leads again to an equation of the second order, here gives one of the first. This allows the solutions of (7'') to be represented by complex integrals. The result<sup>4</sup> only is given here. The integral

$$U = \int_L e^{zr} (z - c_1)^{\alpha_1-1} (z - c_2)^{\alpha_2-1} dz \quad (12)$$

is a solution of (7'') for a path of integration  $L$ , for which

$$\int_L \frac{d}{dz} [e^{zr} (z - c_1)^{\alpha_1} (z - c_2)^{\alpha_2}] dz = 0.$$

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<sup>4</sup>Cf. Schlesinger. The theory is due to H. Poincaré and J. Horn.

The constants  $c_1, c_2, \alpha_1, \alpha_2$  have the following values.  $c_1$  and  $c_2$  are the roots of the quadratic equation

$$z^2 + \delta_0 z + \epsilon_0 = 0$$

and

$$\alpha_1 = \frac{\epsilon_1 + \delta_1 c_1}{c_1 - c_2}, \quad \alpha_2 = -\frac{\epsilon_1 + \delta_1 c_2}{c_1 - c_2}.$$

In the case of equation (7') these become, using (11) and (10),

$$\begin{aligned} c_1 &= +\sqrt{\frac{-2mE}{K^2}}, & c_2 &= -\sqrt{\frac{-2mE}{K^2}} \\ \alpha_1 &= \frac{me^2}{K\sqrt{-2mE}} + n + 1, & \alpha_2 &= -\frac{me^2}{K\sqrt{-2mE}} + n + 1. \end{aligned}$$

The representation by the integral (12) allows us, not only to survey the asymptotic behaviour of the totality of solutions when  $r$  tends to infinity in a definite way, but also to give an account of this behaviour for one definite solution, which is always a much more difficult task.

It can be proven that

$$\frac{me^2}{K\sqrt{-2mE}} \sim \text{real integer},$$

rearranging

$$-E_l = \frac{me^4}{2K^2 l^2}.$$

Therefore the well-known Bohr energy-levels [2], corresponding to the Balmer terms, are obtained, if to the constant  $K$ , introduced into (2) for reasons of dimensions, we give the value

$$K = \frac{h}{2\pi}$$

from which comes

$$-E_l = \frac{2\pi^2 me^4}{h^2 l^2}.$$

Our  $l$  is the principal quantum number.  $n + 1$  is analogous to the azimuthal quantum number. The splitting up of this number through a closer definition of the surface harmonic can be compared with the resolution of the azimuthal quantum into an "equatorial" and a "polar" quantum. These numbers here define the system of nodelines on the sphere.

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# Quantisation as a Problem of Proper Values (Part II)

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## 2.1 The Hamiltonian Analogy between Mechanics and Optics

BEFORE we go on to consider the problem of proper values for further special systems, let us throw more light on the general correspondence which exists between the Hamilton-Jacobi differential equation of a mechanical problem and the "allied" wave equation, i.e. equation (5) of Part I, in the case of the Kepler problem. So far we have only briefly described this correspondence on its external analytical side by the transformation (2), which is in itself unintelligible, and by the equally incomprehensible transition from the equating to zero of a certain expression to the postulation that the space integral of the said expression shall be stationary.<sup>1</sup>

The inner connection between Hamilton's theory and the process of wave propagation is anything but a new idea. It was not only well known to Hamilton, but it also served him as the starting-point for his theory of mechanics, which grew<sup>2</sup> out of his Optics of Nonhomogeneous Media. Hamilton's variation principle can be shown to correspond to Fermat's Principle for a wave propagation in configuration space ( $q$ -space), and the Hamilton-Jacobi equation expresses Huygens' Principle for this wave propagation. Unfortunately this powerful and momentous conception of Hamilton is deprived, in most modern reproductions, of its beautiful raiment as a superfluous accessory, in favour of a

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<sup>1</sup>This procedure will not be pursued further in the present paper. It was only intended to give a provisional, quick survey of the external connection between the wave equation and the Hamilton-Jacobi equation.  $\psi$  is not actually the action function of a definite motion in the relation stated in (2) of part I. On the other hand the connection between the wave equation and the variation problem is of course very real; the integrand of the stationary integral is the Lagrange function for the wave process.

<sup>2</sup>Cf. e.g. 1. W. Whittaker's Anal. Dynamics, chap. xi.

more colourless representation of the analytical correspondence.<sup>3</sup>

Let us consider the general problem of conservative systems in classical mechanics. The Hamilton-Jacobi equation runs

$$\frac{\partial W}{\partial t} + T\left(q_k, \frac{\partial W}{\partial q_k}\right) + V(q_k) = 0 \quad (1).$$

$W$  is the action function, i.e. the time integral of the Lagrange function

$$W = \int (T - V) dt$$

along a path of the system as a function of the end points and the time.  $q_k$  is a representative position co-ordinate;  $T$  is the kinetic energy as function of the  $q$ 's and momenta, being a quadratic form of the latter, for which, as prescribed, the partial derivatives of  $W$  with respect to the  $q$ 's are written.  $V$  is the potential energy. To solve the equation put

$$W = -Et + S(q_k) \quad (2),$$

and obtain

$$2T\left(q_k, \frac{\partial W}{\partial q_k}\right) = 2(E - V) \quad (1').$$

... Which is just another way of saying that mechanical energy

$$T + V = E$$

is conserved in conservative systems. ....

$E$  is an arbitrary integration constant and signifies, as is known, the energy of the system. Contrary to the usual practice, we have let the function  $W$  remain itself in (1'), instead of introducing the time-free function of the co-ordinates,  $S$ . That is a mere superficiality.

Equation (1') can now be very simply expressed if we make use of the method of Heinrich Hertz. It becomes, like all geometrical assertions<sup>4</sup> in configuration space (space of the variables  $q_k$ ), especially simple and clear if we introduce into this space a non-Euclidean metric by means of the kinetic energy of the system.

<sup>3</sup>Felix Klein has since 1891 repeatedly developed the theory of Jacobi from quasioptical considerations in non-Euclidean higher space in his lectures on mechanics. Cf. F. Klein, Jahresber. d. Deutsch. Math. Ver. 1, 1891, and Zeits. f. Math. u. Phys. 46,

<sup>4</sup>1901 (Ges.-Abh. ii. pp. 601 and 603). In the second note, Klein remarks reproachfully that his discourse at Halle ten years previously, in which he had discussed this correspondence and emphasized the great significance of Hamilton's optical works, had "not obtained the general attention, which he had expected". For this allusion to  $F$ . Klein, I am indebted to a friendly communication from Prof. Sommerfeld. See also Atombau, thth ed., p. 803.

Let  $\bar{T}$  be the kinetic energy as function of the velocities  $\dot{q}_k$ , not of the momenta as above, and let us put for the line element

$$ds^2 = 2\bar{T}(q_k, \dot{q}_k) dt^2 \quad (3).$$

The right-hand side now contains  $dt$  only externally and represents (since  $\dot{q}_k dt = dq_k$ ) a quadratic form of the  $dq_k$ 's.

... One can also view this metric intuitively as

$$\bar{T} = \frac{1}{2} \left( \frac{ds}{dt} \right)^2.$$

.....  
 After this stipulation, conceptions such as angle between two line elements, perpendicularity, divergence and curl of a vector, gradient of a scalar, Laplacian operation (= div grad) of a scalar, and others, may be used in the same simple way as in three-dimensional Euclidean space, and we may use in our thinking the Euclidean three-dimensional representation with impunity, except that the analytical expressions for these ideas become a very little more complicated, as the line element (3) must everywhere replace the Euclidean line element. We stipulate, that in what follows, all geometrical statements in  $q$ -space are to be taken in this non-Euclidean sense.

... It's important to note that the line element shown in equation (3) becomes Euclidean for a free particle system. The kinetic energy would be

$$\bar{T} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

and would therefore give a line element

$$ds^2 = m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)dt^2 = m(dx^2 + dy^2 + dz^2).$$

In the general case where the system is not a free particle,  $\bar{T}$  can also explicitly be a function of position coordinates .....

One of the most important modifications for the calculation is that we must distinguish carefully between covariant and contravariant components of a vector or tensor. But this complication is not any greater than that which occurs in the case of an oblique set of Cartesian axes.

The  $dq_k$ 's are the prototype of a contravariant vector. The coefficients of the form  $2\bar{T}$ , which depend on the  $q_k$ 's, are therefore of a covariant character and form the covariant fundamental tensor.  $2T$  is the contravariant form belonging to  $2\bar{T}$ , because the momenta are known to form the covariant vector belonging to the speed vector  $\dot{q}_k$ , the momentum being the velocity vector in covariant form. The left side of (1') is now simply the contravariant fundamental form, in which the  $\partial W/\partial q_k$ 's are brought in as variables. The latter form the components of the vector,-according to its nature covariant,

$$\text{grad } W.$$

(The expressing of the kinetic energy in terms of momenta instead of speeds has then this significance, that covariant vector components can only be introduced in a contravariant form if something intelligible, i.e. invariant, is to result.)

$$2T = \mathbf{p}^2 = \left( \frac{\partial W}{\partial \mathbf{q}} \right)^2 = (\text{grad } W)^2$$

Equation (1') is equivalent thus to the simple statements

$$(\text{grad } W)^2 = 2(E - V) \quad (1''),$$

or

$$|\text{grad } W| = \sqrt{2(E - V)} \quad (1''').$$

This requirement is easily analysed. Suppose that a function  $W$ , of the form (2), has been found, which satisfies it. Then this function can be clearly represented for every definite  $t$ , if the family of surfaces  $W = \text{const.}$  be described in  $q$ -space and to each member a value of  $W$  be ascribed.

Now, on the one hand, as will be shown immediately, equation (1''') gives an exact rule for constructing all the other surfaces of the family and obtaining their  $W$ -values from any single member, if the latter and its  $W$ -value is known. On the other hand, if the sole necessary data for the construction, viz. one surface and its  $W$ -value be given quite arbitrarily, then from the rule, which presents just two alternatives, there may be completed one of the functions  $W$  fulfilling the given requirement. Provisionally, the time is regarded as constant. - The construction rule therefore exhausts the contents of the differential equation; each of its solutions can be obtained from a suitably chosen surface and  $W$ -value.

Let us consider the construction rule. Let the value  $W_0$  be given in Figure 1.1 to an arbitrary surface. In order to find the surface  $W_0 + dW_0$ , take either side of the given surface as the positive one, erect the normal at each point of it and cut off (with due regard to the sign of  $dW_0$ ) the step

$$(\text{grad } W)^2 = \left( \frac{\partial W}{\partial \mathbf{q}} \right)^2 \sim \frac{(\partial W)^2}{(\partial \mathbf{q})^2} = \frac{(dW)^2}{(ds)^2} \sim \left( \frac{dW}{ds} \right)^2$$

$$ds = \frac{dW_0}{\sqrt{2(E - V)}} \quad (4).$$

The locus of the end points of the steps is the surface  $W_0 + dW_0$ . Similarly, the family of surfaces may be constructed successively on both sides.

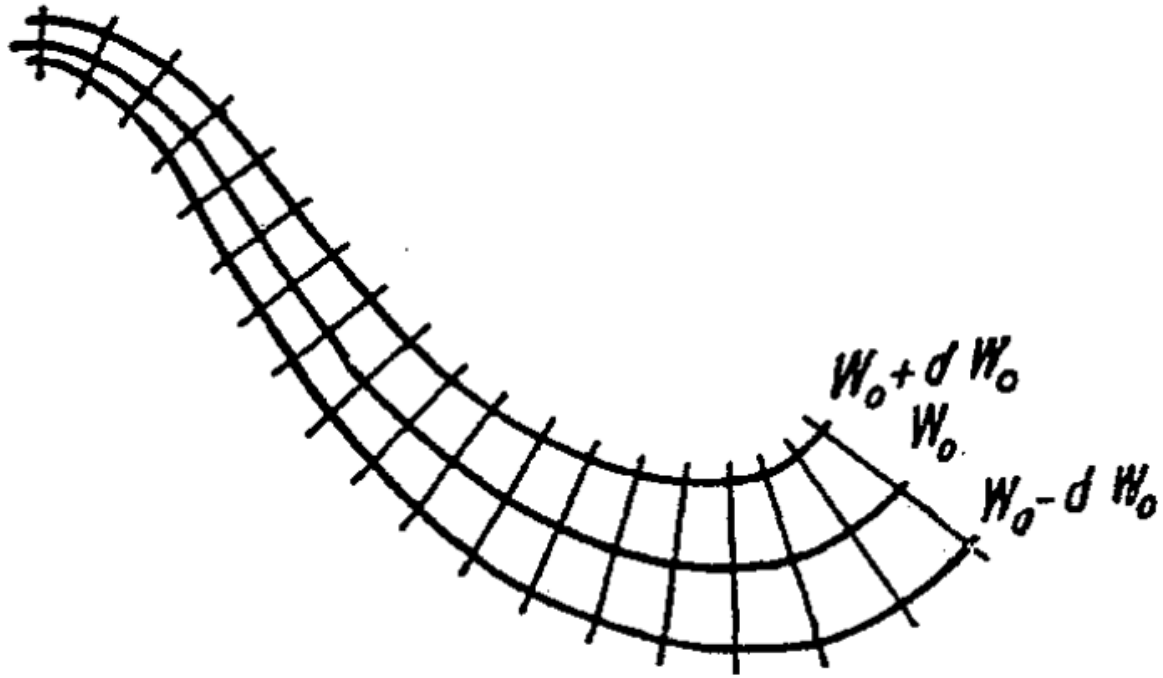


Figure 2.1: Family of surfaces on both sides of  $W_0$ .

The construction has a double interpretation, as the other side of the given surface might have been taken as positive for the first step. This ambiguity does not hold for later steps, i.e. at any later stage of the process we cannot change arbitrarily the sign of the sides of the surface, at which we have arrived, as this would involve in general a discontinuity in the first differential coefficient of  $W$ . Moreover, the two families obtained in the two cases are clearly identical; the  $W$ -values merely run in the opposite direction.

Let us consider now the very simple dependence on the time. For this, (2) shows that at any later (or earlier) instant  $t + t'$ , the same group of surfaces illustrates the  $W$ -distribution, though different  $W$ -values are associated with the individual members, namely, from each  $W$ -value ascribed at time  $t$  there must be subtracted  $Et'$ . The  $W$ -values wander, as it were, from surface to surface according to a definite, simple law, and for positive  $E$  in the direction of  $W$  increasing. Instead of this, however, we may imagine that the surfaces wander in such a way that each of them continually takes the place and exact form of the following one, and always carries its  $W$ -value with it. The rule for this wandering is given by the fact that the surface  $W_0$  at time  $t + dt$  must have reached that place, which at  $t$  was occupied by the surface  $W_0 + Edt$ . This will be attained according to (4), if each point of the surface  $W_0$  is allowed to move in the direction of the positive



normal through a distance

$$ds = \frac{E dt}{\sqrt{2(E - V)}} \quad (5).$$

That is, the surfaces move with a normal velocity

$$u = \frac{ds}{dt} = \frac{E}{\sqrt{2(E - V)}} \quad (6),$$

which, when the constant  $E$  is given, is a pure function of position.

Now it is seen that our system of surfaces  $W = \text{const.}$  can be conceived as the system of wave surfaces of a progressive but stationary wave motion in  $q$ -space, for which the value of the phase velocity at every point in the space is given by (6). For the normal construction can clearly be replaced by the construction of elementary Huygens waves (with radius (5)), and then of their envelope. The "index of refraction" is proportional to the reciprocal of (6), and is dependent on the position but not on the direction. The  $q$ -space is thus optically non-homogeneous but is isotropic. The elementary waves are "spheres", though of course -let me repeat it expressly once more- in the sense of the line-element (3).

The function of action  $W$  plays the part of the phase of our wave system. The Hamilton-Jacobi equation is the expression of Huygens' principle. If, now, Fermat's principle be formulated thus,

$$0 = \delta \int_{P_1}^{P_2} \frac{ds}{u} = \delta \int_{P_1}^{P_2} \frac{ds \sqrt{2(E - V)}}{E} = \delta \int_{t_1}^{t_2} \frac{2T}{E} dt = \frac{1}{E} \delta \int_{t_1}^{t_2} 2T dt \quad (7),$$

we are led directly to Hamilton's principle in the form given by Maupertuis (where the time integral is to be taken with the usual grain of salt, i.e.  $T + V = E = \text{constant}$ , even during the variation). The "rays", i.e. the orthogonal trajectories of the wave surfaces, are therefore the paths of the system for the value  $E$  of the energy, in agreement with the well-known system of equations

$$p_k = \frac{\partial W}{\partial q_k} \quad (8)$$

which states, that a set of system paths can be derived from each special function of action, just like a fluid motion from its velocity potential.<sup>5</sup> (The momenta  $p_k$  form the covariant velocity vector, which equations (8) assert to be equal to the gradient of the function of action.)

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<sup>5</sup>See especially. A. Einstein, Verh. d. D. Physik. Ges. 19, pp. 77, 82, 1917. Tho framing of the quantum conditions here is the most akin, out of all the older attempts, to the present one. De Broglie has returned to it.

... *Schrödinger couldn't have been more confusing.*

*Proof.* For some path that minimises the Action,

$$0 = dW = \mathcal{L}dt = (T - V)dt = (2T - E)dt.$$

We can rearrange to obtain

$$\sqrt{2T}dt = \frac{Edt}{\sqrt{2T}},$$

where  $\sqrt{2T}dt$  is in fact  $ds$ . Therefore

$$ds = \frac{Edt}{\sqrt{2T}}.$$

QED

*Alternative Proof.*

$$\frac{dW}{dt} = -E + \sum \frac{\partial S}{\partial q_k} \dot{q}_k = -E + \mathbf{p} \cdot \dot{\mathbf{q}} \quad (I.)$$

$$\frac{dW}{dt} = \frac{d}{dt} \int (T - V)dt = T - V \quad (II.)$$

$$T + V = E \quad (III.)$$

(I.) = (II.) implies

$$T - V = -E + \mathbf{p} \cdot \dot{\mathbf{q}} \quad (IV.)$$

(III.) + (IV.) implies

$$2T = \mathbf{p} \cdot \dot{\mathbf{q}} \quad (V.).$$

Using (V.) and still assuming the metric (3)  $ds = \sqrt{2T}dt$  we find that

$$\mathbf{p} \cdot d\mathbf{q} = \mathbf{p} \cdot \dot{\mathbf{q}}dt = 2Tdt = \sqrt{2T}ds = \sqrt{2(E - V)}ds.$$

Therefore

$$dW = -Edt + \mathbf{p} \cdot d\mathbf{q} = -Edt + \sqrt{2(E - V)}ds \quad (VI.).$$

Using the variational principle, for some path the action function hits a minima

$$0 = \delta W \sim dW = -E dt + \sqrt{2(E - V)} ds$$

$$ds = \frac{E dt}{\sqrt{2(E - V)}}.$$

QED

.....  
 Although in these deliberations on wave surfaces we speak of velocity of propagation and Huygens' principle, we must regard the analogy as one between mechanics and geometrical optics, and not physical or undulatory optics. For the idea of "rays", which is the essential feature in the mechanical analogy, belongs to geometrical optics; it is only clearly defined in the latter. Also Fermat's principle can be applied in geometrical optics without going beyond the idea of index of refraction. And the system of  $W$ -surfaces, regarded as wave surfaces, stands in a somewhat looser relationship to mechanical motion, inasmuch as the image point of the mechanical system in no wise moves along the ray with the wave velocity  $u$ , but, on the contrary, its velocity (for constant  $E$ ) is proportional to  $\frac{1}{u}$ . It is given directly from (3) as

$$v = \frac{ds}{dt} = \sqrt{2T} = \sqrt{2(E - V)} \quad (9).$$

This non-agreement is obvious. Firstly, according to (8), the system's point velocity is great when  $\text{grad } W$  is great, i.e. where the  $W$  surfaces are closely crowded together, i.e. where  $u$  is small. Secondly, from the definition of  $W$  as the time integral of the Lagrange function,  $W$  alters during the motion (by  $(T - V)dt$  in the time  $dt$ ), and so the image point cannot remain continuously in contact with the same  $W$ -surface.

And important ideas in wave theory, such as amplitude, wave length, and frequency -or, speaking more generally, the wave form- do not enter into the analogy at all, as there exists no mechanical parallel; even of the wave function itself there is no mention beyond that  $W$  has the meaning of the phase of the waves (and this is somewhat hazy owing to the wave form being undefined).

If we find in the whole parallel merely a satisfactory means of contemplation, then this defect is not disturbing, and we would regard any attempt to supply it as idle trifling, believing the analogy to be precisely with geometrical, or at furthest, with a very primitive form of wave optics, and not with the fully developed undulatory optics. That geometrical optics is only a rough approximation for Light makes no difference. To preserve the analogy on the further development of the optics of  $q$ -space on the lines of wave theory, we must take good care not to depart markedly from the limiting case

of geometrical optics, i.e. must choose<sup>6</sup> the wave length sufficiently small, i.e. small compared with all the path dimensions. Then the additions do not teach anything new; the picture is only draped with superfluous ornaments.

So we might think to begin with. But even the first attempt at the development of the analogy to the wave theory leads to such striking results, that a quite different suspicion arises: we know to-day, in fact, that our classical mechanics fails for very small dimensions of the path and for very great curvatures. Perhaps this failure is in strict analogy with the failure of geometrical optics, i.e. "the optics of infinitely small wave lengths", that becomes evident as soon as the obstacles or apertures are no longer great compared with the real, finite, wave length. Perhaps our classical mechanics is the complete analogy of geometrical optics and as such is wrong and not in agreement with reality; it fails whenever the radii of curvature and dimensions of the path are no longer great compared with a certain wave length, to which, in  $q$ -space, a real meaning is attached. Then it becomes a question of searching<sup>7</sup> for an undulatory mechanics, and the most obvious way is the working out of the Hamiltonian analogy on the lines of undulatory optics.

## 2.2 "Geometrical" and "Undulatory" Mechanics

We will at first assume that it is fair, in extending the analogy, to imagine the above-mentioned wave system as consisting of sine waves. This is the simplest and most obvious case, yet the arbitrariness, which arises from the fundamental significance of this assumption, must be emphasized. The wave function has thus only to contain the time in the form of a factor,  $\sin(\dots)$ , where the argument is a linear function of  $W$ . The coefficient of  $W$  must have the dimensions of the reciprocal of action, since  $W$  has those of action and the phase of a sine has zero dimensions. We assume that it is quite universal, i.e. that it is not only independent of  $E$ , but also of the nature of the mechanical system. We may then at once denote it by  $\frac{2\pi}{h}$ . The time factor then is

$$\sin\left(\frac{2\pi W}{h} + \text{const.}\right) = \sin\left(-\frac{2\pi Et}{h} + \frac{2\pi S(q_k)}{h} + \text{const.}\right) \quad (10).$$

Hence the frequency  $\nu$  of the waves is given by

$$\nu = \frac{E}{h} \quad (11),$$

---

<sup>6</sup>Cf. for the optical case, A. Sommerfeld and Iris Runge, Ann. d. Phys. 35, p. 290, 1911. There (in the working out of an oral remark of P. Debye), it is shown, how the equation of first order and second degree for the phase ("Hamiltonian equation may be accurately derived from the equation of the second order and first degree for the wave function ("wave equation"), in the limiting case of vanishing wave length.

<sup>7</sup>CA. A. Einstein, Berl. Ber. p. 9 et seq., 1925.

Thus we get the frequency of the  $q$ -space waves to be proportional to the energy of the system, in a manner which is not markedly artificial.<sup>8</sup> This is only true of course if  $E$  is absolute and not, as in classical mechanics, indefinite to the extent of an additive constant. By (6) and (11) the wave length is independent of this additive constant, being

$$\lambda = \frac{u}{\nu} = \frac{h}{\sqrt{2(E - V)}} \quad (12),$$

and we know the term under the root to be double the kinetic energy. Let us make a preliminary rough comparison of this wave length with the dimensions of the orbit of a hydrogen electron as given by classical mechanics, taking care to notice that a "step" in  $q$ -space has not the dimensions of length, but length multiplied by the square root of mass, in consequence of (3).  $\lambda$  has similar dimensions. We have therefore to divide  $\lambda$  by the dimension of the orbit,  $a$  cm., say, and by the square root of  $m$ , the mass of the electron. The quotient is of the order of magnitude of

$$\frac{h}{mva},$$

where  $v$  represents for the moment the electron's velocity (*cm./sec.*). The denominator  $mva$  is of the order of the mechanical moment of momentum, and this is at least of the order of  $10^{-27}$  for Kepler orbits, as can be calculated from the values of electronic charge and mass independently of all quantum theories.

... *In 1926 the standard units used were different. Even so,  $10^{-27}$  still doesn't mean anything without units.*

*For Kepler orbits of an H atom we have*

$$\frac{mv^2}{a} = \frac{k_e e^2}{a^2},$$

$$v = \sqrt{\frac{k_e e^2}{ma}} \sim 10^6 \times m./s.$$

$$L = mva \sim 10^{-34} \times J.s. = 10^{-27} \times g.cm.^2/s.$$

*which gives in fact*

$$L \sim \hbar \implies \frac{h}{mva} \sim 2\pi$$

.....  
 We thus obtain the correct order for the limit of the approximate region of validity of classical mechanics, if we identify our constant  $h$  with Planck's quantum of action -and this is only a preliminary attempt.

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<sup>8</sup>In Part I. this appeared merely as an approximate equation, derived from a pure speculation.

If in (6),  $E$  is expressed by means of (11) in terms of  $v$ , then we obtain

$$u = \frac{h\nu}{\sqrt{2(h\nu - V)}} \quad (6').$$

The dependence of the wave velocity on the energy thus becomes a particular kind of dependence on the frequency, i.e. it becomes a law of dispersion for the waves. This law is of great interest. We have shown in Chapter 2.1 that the wandering wave surfaces are only loosely connected with the motion of the system point, since their velocities are not equal and cannot be equal. According to (9), (11), and (6') the system's velocity  $v$  has thus also a concrete significance for the wave. We verify at once that

$$v = \frac{d\nu}{d\left(\frac{\nu}{u}\right)} \quad (13),$$

i.e. the velocity of the system point is that of a group of waves, included within a small range of frequencies (signal-velocity).

.....  
*Proof.*

$$\frac{d\nu}{d\left(\frac{\nu}{u}\right)} = \frac{d(E/h)}{d(\sqrt{2T}/h)} = \frac{dE}{d\sqrt{2T}} = \frac{d(T + V)}{d\sqrt{2T}} = \frac{1}{\sqrt{2}} \frac{dT}{d\sqrt{T}} = \sqrt{2T} = v.$$

QED

.....  
 We find here again a theorem for the "phase waves" of the electron, which M. de Broglie had derived, with essential reference to the relativity theory, in those fine researches,<sup>9</sup> to which I owe the inspiration for this work. We see that the theorem in question is of wide generality, and does not arise solely from relativity theory, but is valid for every conservative system of ordinary mechanics.

In what way now shall we have to proceed to the undulatory representation of mechanics for those cases where it is necessary? We must start, not from the fundamental equations of mechanics, but from a wave equation for  $q$ -space and consider the manifold of processes possible according to it. The wave equation has not been explicitly used or even put forward in this communication. The only datum for its construction is the wave velocity, which is given by (6) or (6') as a function of the mechanical energy parameter or frequency respectively, and by this datum the wave equation is evidently not uniquely defined. It is not even decided that it must be definitely of the second order. Only the striving for

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<sup>9</sup>L. de Broglie, *Ann. de Physique* (10) 3, p. 22, 1925. (Thèses, Paris, 1924.)

simplicity leads us to try this to begin with. We will then say that for the wave function  $\psi$  we have

$$\operatorname{div} \operatorname{grad} \psi - \frac{1}{u^2} \ddot{\psi} = 0 \quad (14),$$

valid for all processes which only depend on the time through a factor  $e^{2\pi i\nu t}$ . Therefore, considering (6), (6'), and (11), we get, respectively,

$$\operatorname{div} \operatorname{grad} \psi + \frac{8\pi^2}{h^2} (h\nu - V)\psi = 0 \quad (14'),$$

and

$$\operatorname{div} \operatorname{grad} \psi + \frac{8\pi^2}{h^2} (E - V)\psi = 0 \quad (14'')$$

The differential operations are to be understood with regard to the line element (3).

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# Quantisation as a Problem of Proper Values (Part III)

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## 3.1 Calculation of Frequencies by the Method which corresponds to that of Epstein

IF we add a potential energy  $+eFz$  to the wave equation (5), Part I., of the Kepler problem, corresponding to the influence of an electric field of strength  $F$  in the positive  $z$ -direction, on a negative electron of charge  $e$ , then we obtain the following wave equation for the Stark effect of the hydrogen atom,

$$\nabla^2\psi + \frac{8\pi^2m}{h^2} \left( E + \frac{e^2}{r} - eFz \right) \psi = 0$$

which forms the basis of the remainder of this paper. Let's introduce space parabolic co-ordinates  $\lambda_1, \lambda_2, \phi$ ,

$$\begin{cases} x = \sqrt{\lambda_1\lambda_2} \cos \phi \\ y = \sqrt{\lambda_1\lambda_2} \sin \phi \\ z = \frac{1}{2}(\lambda_1 - \lambda_2) \end{cases} \quad (33).$$

$\lambda_1$  and  $\lambda_2$  run from 0 to infinity ; the corresponding co-ordinate surfaces are the two sets of confocal paraboloids of revolution, which have the origin as focus and the positive ( $\lambda_2$ ) or negative ( $\lambda_1$ )  $z$ -axis respectively as axes.  $\phi$  runs from 0 to  $2\pi$ , and the co-ordinate surfaces belonging to it are the set of half planes limited by the  $z$ -axis. The relation of the co-ordinates is unique. For the functional determinant we get

$$\frac{\partial(x, y, z)}{\partial(\lambda_1, \lambda_2, \phi)} = \frac{1}{4}(\lambda_1 + \lambda_2) \quad (34)$$



The space element is thus

$$dxdydz = \frac{1}{4} (\lambda_1 + \lambda_2) d\lambda_1 d\lambda_2 d\phi \quad (35).$$

We notice, as consequences of (33),

$$x^2 + y^2 = \lambda_1 \lambda_2; \quad r^2 = x^2 + y^2 + z^2 = \frac{1}{4} (\lambda_1 + \lambda_2)^2 \quad (36).$$

The expression of (32) in the chosen co-ordinates gives, if we multiply by (34)<sup>1</sup> (to restore the self-adjoint form),

$$\left\{ \begin{array}{l} \frac{\partial}{\partial \lambda_1} \left( \lambda_1 \frac{\partial \psi}{\partial \lambda_1} \right) + \frac{\partial}{\partial \lambda_2} \left( \lambda_2 \frac{\partial \psi}{\partial \lambda_2} \right) + \frac{1}{4} \left( \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) \frac{\partial^2 \psi}{\partial \phi^2} \\ + \frac{2\pi^2 m}{h^2} \left[ E (\lambda_1 + \lambda_2) + 2e^2 - \frac{1}{2} eF (\lambda_1^2 - \lambda_2^2) \right] \psi = 0 \end{array} \right. \quad (32').$$

Here we can again take-and this is the why and wherefore of all "methods" of solving linear partial differential equations-the function  $\psi$  as the product of three functions, thus,

$$\psi = \Lambda_1 \Lambda_2 \Phi \quad (37),$$

each of which depends on only one co-ordinate. For these functions we get the ordinary differential equations

$$\left\{ \begin{array}{l} \frac{\partial^2 \Phi}{\partial \phi^2} = -n^2 \Phi \\ \frac{\partial}{\partial \lambda_1} \left( \lambda_1 \frac{\partial \Lambda_1}{\partial \lambda_1} \right) + \frac{2\pi^2 m}{h^2} \left( -\frac{1}{2} eF \lambda_1^2 + E \lambda_1 + e^2 - \beta - \frac{n^2 h^2}{8\pi^2 m} \frac{1}{\lambda_1} \right) \Lambda_1 = 0 \\ \frac{\partial}{\partial \lambda_2} \left( \lambda_2 \frac{\partial \Lambda_2}{\partial \lambda_2} \right) + \frac{2\pi^2 m}{h^2} \left( \frac{1}{2} eF \lambda_2^2 + E \lambda_2 + e^2 + \beta - \frac{n^2 h^2}{8\pi^2 m} \frac{1}{\lambda_2} \right) \Lambda_2 = 0 \end{array} \right. \quad (38)$$

wherein  $n$  and  $\beta$  are two further "proper value-like" constants of integration (in addition to  $E$ ), still to be defined. If we solve the system of equations we get

$$E = -\frac{2\pi^2 m e^4}{h^2 l^2} - \frac{3 h^2 F l (k_2 - k_1)}{8 \pi^2 m e} \quad (62).$$

This is our provisional conclusion; it is the well-known formula of Epstein for the term values in the Stark effect of the hydrogen spectrum.  $k_1$  and  $k_2$  correspond fully to the parabolic quantum numbers.

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<sup>1</sup>So far as the actual details of the analysis are concerned, the simplest way to get (32'), or, in general, to get the wave equation for any special co-ordinates, is to transform not the wave equation itself, but the corresponding variation problem (cf. Part I. p. 12), and thus to obtain the wave equation afresh as an Eulerian variation problem. We are thus spared the troublesome evaluation of the second derivatives. Cf. Courant-Hilbert, chap. iv. §7, p. 193.

# Quantisation as a Problem of Proper Values (Part IV)

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## 4.1 Elimination of the Energy-parameter from the Vibration Equation. THE Real Wave Equation.

The wave equation (14) or (14'') of Part II., viz.

$$\nabla^2\psi - \frac{2(E - V)}{E^2} \frac{\partial^2\psi}{\partial t^2} = 0 \quad (1)$$

or

$$\nabla^2\psi + \frac{8\pi^2}{h^2}(E - V)\psi = 0 \quad (1')$$

which forms the basis for the re-establishment of mechanics attempted in this series of papers, suffers from the disadvantage that it expresses the law of variation of the "mechanical field scalar"  $\psi$ , neither uniformly nor generally. Equation (1) contains the energy -or frequency- parameter  $E$ , and is valid, as is expressly emphasized in Part II., with a definite  $E$ -value inserted, for processes which depend on the time exclusively through a definite periodic factor:

$$\psi \sim \text{real part of } (e^{\pm 2\pi i Et/h}) \quad (2).$$

Equation (1) is thus not really any more general than equation (1'), which takes account of the circumstance just mentioned and does not contain the time at all.

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<sup>1</sup>Cf. Ann. d. Phys. 79, pp. 361, 489; 80, p. 437, 1926 (Parts I., II., III.); further, on the connection with Heisenberg's theory, *ibid.* 79, p. 734 (p. 45).

... In fact, equating (1) and (1'),

$$\begin{aligned} \nabla^2\psi - \frac{2(E - V)}{E^2} \frac{\partial^2\psi}{\partial t^2} &= \nabla^2\psi + \frac{8\pi^2}{h^2}(E - V)\psi \\ -\frac{2}{E^2} \frac{\partial^2\psi}{\partial t^2} &= \frac{8\pi^2}{h^2}\psi \\ \frac{\partial^2\psi}{\partial t^2} &= -\frac{4\pi^2 E^2}{h^2}\psi \end{aligned}$$

we obtain a differential equation that is solved for a  $\psi$  compatible with (2),

$$\psi(\mathbf{x}, t) = A(\mathbf{x})e^{\pm 2\pi i Et/h}.$$

.....  
 Thus, when we designated equation (1) or (1'), on various occasions, as "the wave equation", we were really wrong and would have been more correct if we had called it a "vibration-" or an "amplitude-" equation. However, we found it sufficient, because to it is linked the Sturm-Liouville proper value problem-just as in the mathematically strictly analogous problem of the free vibrations of strings and membranes-and not to the real wave equation.

As to this, we have always postulated up till now that the potential energy  $V$  is a pure function of the co-ordinates and does not depend explicitly on the time. There arises, however, an urgent need for the extension of the theory to non-conservative systems, because it is only in that way that we can study the behaviour of a system under the influence of prescribed external forces, e.g. a light wave, or a strange atom flying past. Whenever  $V$  contains the time explicitly, it is manifestly impossible that equation (1) or (1') should be satisfied by a function  $\psi$ , the method of dependence of which on the time is as given by (2). We then find that the amplitude equation is no longer sufficient and that we must search for the real wave equation.

For conservative systems, the latter is easily obtained. (2) is equivalent to

$$\frac{\partial^2\psi}{\partial t^2} = -\frac{4\pi^2 E^2}{h^2}\psi \quad (3).$$

We can eliminate  $E$  from (1') and (3) by differentiation, and obtain the following equation, which is written in a symbolic manner, easy to understand:

$$\left(\nabla^2 - \frac{8\pi^2}{h^2}V\right)^2 \psi + \frac{16\pi^2}{h^2} \frac{\partial^2\psi}{\partial t^2} = 0 \quad (4).$$

... I don't know how he was able to get equation (4) by differentiating (1') and (3), but it is in fact possible just by rearranging them a little.

*Proof.*

$$(1') \implies \left( \nabla^2 - \frac{8\pi^2}{h^2} V \right) \psi = -\frac{8\pi^2 E}{h^2} \psi$$

$$(3) \implies \frac{2}{E} \frac{\partial^2 \psi}{\partial t^2} = -\frac{8\pi^2 E}{h^2} \psi$$

Squaring both sides,

$$\left( \nabla^2 - \frac{8\pi^2}{h^2} V \right)^2 \psi^2 = \left( -\frac{8\pi^2 E}{h^2} \psi \right) \left( \frac{2}{E} \frac{\partial^2 \psi}{\partial t^2} \right)$$

$$\left( \nabla^2 - \frac{8\pi^2}{h^2} V \right)^2 \psi = -\frac{16\pi^2}{h^2} \frac{\partial^2 \psi}{\partial t^2}$$

QED

.....  
 This equation must be satisfied by every  $\psi$  which depends on the time as in (2), though with  $E$  arbitrary, and consequently also by every  $\psi$  which can be expanded in a Fourier series with respect to the time (naturally with functions of the co-ordinates as coefficients). Equation (4) is thus evidently the uniform and general wave equation for the field scalar  $\psi$ .

It is evidently no longer of the simple type arising for vibrating membranes, but is of the fourth order, and of a type similar to that occurring in many problems in the theory of elasticity.<sup>2</sup> However, we need not fear any excessive complication of the theory, or any necessity to revise the previous methods, associated with equation (1'). If  $V$  does not contain the time, we can, proceeding from (4), apply (2), and then split up the operator as follows :

$$\left( \nabla^2 - \frac{8\pi^2}{h^2} V + \frac{8\pi^2}{h^2} E \right) \left( \nabla^2 - \frac{8\pi^2}{h^2} V - \frac{8\pi^2}{h^2} E \right) \psi = 0 \quad (4').$$

By way of trial, we can resolve this equation into two "alternative" equations, namely, into equation (1') and into another, which only differs from (1') in that its proper value parameter will be called minus  $E$ , instead of plus  $E$ . According to (2) this does not lead to new solutions. The decomposition of (4') is not absolutely cogent, for the theorem that "a product can only vanish when at least one factor vanishes" is not valid for operators. This lack of cogency, however, is a feature common to all the methods of solution of partial differential equations. The procedure finds its subsequent justification in the fact that we can prove the completeness of the discovered proper functions, as functions of the co-ordinates. This completeness, coupled with the fact that the imaginary part as

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<sup>2</sup>E.g., for a vibrating plate,  $\nabla^2 \nabla^2 u + \frac{\partial^2 u}{\partial t^2} = 0$ . Cf. Courant-Hilbert, chap. V. §8, p. 256.

well as the real part of (2) satisfies equation (4), allows arbitrary initial conditions to be fulfilled by  $\psi$  and  $\partial\psi/\partial t$ .

Thus we see that the wave equation (4), which contains in itself the law of dispersion, can really stand as the basis of the theory previously developed for conservative systems. The generalisation for the case of a time-varying potential function nevertheless demands caution, because terms with time derivatives of  $V$  may then appear, about which no information can be given to us by equation (4), owing to the way we obtained it. In actual fact, if we attempt to apply equation (4) as it stands to non-conservative systems, we meet with complications, which seem to arise from the term in  $\partial V/\partial t$ . Therefore, in the following discussions, I have taken a somewhat different route, which is much easier for calculations, and which I consider is justified in principle.

We need not raise the order of the wave equation to four, in order to get rid of the energy-parameter. The dependence of  $\psi$  on the time, which must exist if (1') is to hold, can be expressed by

$$\frac{\partial\psi}{\partial t} = \pm \frac{2\pi i}{h} E\psi \quad (3')$$

as well as by (3). We thus arrive at one of the two equations

$$\nabla^2\psi - \frac{8\pi^2}{h^2} V\psi \mp \frac{4\pi i}{h} \frac{\partial\psi}{\partial t} = 0 \quad (4'').$$

We will require the complex wave function  $\psi$  to satisfy one of these two equations. Since the conjugate complex function  $\bar{\psi}$  will then satisfy the other equation, we may take the real part of  $\psi$  as the real wave function (if we require it). In the case of a conservative system (4'') is essentially equivalent to (4), as the real operator may be split up into the product of the two conjugate complex operators if  $V$  does not contain the time.

## 4.2 Relativistic-magnetic Generalisation of the Fundamental Equations

As an appendix to the physical problems just mentioned, in which the magnetic field, which has hitherto been completely ignored in this series of papers, plays an important part, I would like to give, briefly, the probable relativistic-magnetic generalisation of the basic equations (4''), although I can only do this meantime for the one electron problem, and only with the greatest possible reserve—the latter for two reasons. Firstly, the generalisation is provisionally based on a purely formal analogy. Secondly, as was mentioned in Part I., though it does formally lead in the Kepler problem to Sommerfeld's fine-structure formula with, in fact, the "half-integral" azimuthal and radial quantum, which is generally regarded as correct to-day, nevertheless there is still lacking the supplement, which is necessary to secure numerically correct diagrams of the splitting up of

the hydrogen lines, and which is given in Bohr's theory by Goudsmit and Uhlenbeck's electronic spin.

The Hamilton-Jacobi partial differential equation for the Lorentzian electron can readily be written :

$$\left(\frac{1}{c} \frac{\partial W}{\partial t} + \frac{e}{c} V\right)^2 - \left(\frac{\partial W}{\partial x} - \frac{e}{c} \mathfrak{U}_x\right)^2 - \left(\frac{\partial W}{\partial y} - \frac{e}{c} \mathfrak{U}_y\right)^2 - \left(\frac{\partial W}{\partial z} - \frac{e}{c} \mathfrak{U}_z\right)^2 - m^2 c^2 = 0 \quad (34).$$

Here  $e, m, c$  are the charge and mass of the electron, and the velocity of light;  $V, \mathfrak{U}$  are the electro-magnetic potentials of the external electro-magnetic field at the position of the electron, and  $W$  is the action function.

... *Expanding the squares,*

$$\begin{aligned} \frac{1}{c^2} \left(\frac{\partial W}{\partial t}\right)^2 + \frac{e^2}{c^2} V^2 + \frac{2e}{c^2} \frac{\partial W}{\partial t} V - \nabla^2 W - \frac{e^2}{c^2} \mathfrak{U}^2 + \frac{2e}{c} \text{grad } W \cdot \mathfrak{U} - m^2 c^2 = 0 \\ \frac{1}{c^2} \left(\frac{\partial W}{\partial t}\right)^2 - \nabla^2 W + \frac{2e}{c} \left(\frac{V}{c} \frac{\partial W}{\partial t} + \text{grad } W \cdot \mathfrak{U}\right) + \frac{e^2}{c^2} \left(V^2 - \mathfrak{U}^2 - \frac{m^2 c^4}{e^2}\right) = 0 \quad (34'). \end{aligned}$$

.....

From the classical (relativistic) equation (34) I am now attempting to derive the wave equation for the electron, by the following purely formal procedure, which, we can verify easily, will lead to equations (4''), if it is applied to the Hamiltonian equation of a particle moving in an arbitrary field of force in ordinary (non-relativistic) mechanics. After the squaring, in equation (34), I replace the quantities

$$\frac{\partial W}{\partial t}, \quad \frac{\partial W}{\partial x}, \quad \frac{\partial W}{\partial y}, \quad \frac{\partial W}{\partial z}$$

by the respective operators

$$\pm \frac{h}{2\pi i} \frac{\partial}{\partial t}, \quad \pm \frac{h}{2\pi i} \frac{\partial}{\partial x}, \quad \pm \frac{h}{2\pi i} \frac{\partial}{\partial y}, \quad \pm \frac{h}{2\pi i} \frac{\partial}{\partial z} \quad (35).$$

The double linear operator, so obtained, is applied to a wave function  $\psi$  and the result put equal to zero, thus:

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} \pm \frac{4\pi i e}{hc} \left(\frac{V}{c} \frac{\partial \psi}{\partial t} + \mathfrak{U} \text{ grad } \psi\right) + \frac{4\pi^2 e^2}{h^2 c^2} \left(V^2 - \mathfrak{U}^2 - \frac{m^2 c^4}{e^2}\right) \psi = 0 \quad (36).$$

(The symbols  $\nabla^2$  and grad have here their elementary three-dimensional Euclidean meaning.) The pair of equations (36) would be the possible relativistic-magnetic generalisation

of (4'') for the case of a single electron, and should likewise be understood to mean that the complex wave function has to satisfy either the one or the other equation.

From (36) the fine structure formula of Sommerfeld for the hydrogen atom may be obtained by exactly the same method as is described in Part I., and also we may derive (neglecting the term in  $\mathfrak{U}^2$ ) the normal Zeeman effect as well as the well-known selection and polarisation rules and intensity formulae. They follow from the integral relations between Legendre functions introduced at the end of Part III. For the reasons given in the first chapter of this paragraph, I withhold the detailed reproduction of these calculations meantime, and also in the following final paragraph refer to the "classical", and not to the still incomplete relativistic-magnetic version of the theory.

... Without expanding the squares, subbing  $E = -\frac{\partial W}{\partial t}$  and  $p_k = \frac{\partial W}{\partial q_k}$  into (34) will give us

$$\begin{aligned} \left(-\frac{1}{c}E + \frac{e}{c}V\right)^2 - \left(\mathbf{p} - \frac{e}{c}\mathfrak{U}\right)^2 - m^2c^2 &= 0 \\ -\left(\frac{1}{c}E - \frac{e}{c}V\right)^2 + \left(\mathbf{p} - \frac{e}{c}\mathfrak{U}\right)^2 + m^2c^2 &= 0 \end{aligned}$$

*i.e. the Klein-Gordon equation. Easily recognisable by setting the potentials to zero, just to find the energy-momentum relation. ....*

### 4.3 On the Physical Significance of the Field scalar

$\psi\bar{\psi}$  is a kind of weight-function in the system's configuration space. The wave-mechanical configuration of the system is a superposition of many, strictly speaking of all, point-mechanical configurations kinematically possible. Thus, each point-mechanical configuration contributes to the true wave-mechanical configuration with a certain weight, which is given precisely by  $\psi\bar{\psi}$ . If we like paradoxes, we may say that the system exists, as it were, simultaneously in all the positions kinematically imaginable, but not "equally strongly" in all. In macroscopic motions, the weight-function is practically concentrated in a small region of positions, which are practically indistinguishable. The centre of gravity of this region in configuration space travels over distances which are macroscopically perceptible. In problems of microscopic motions, we are in any case interested also, and in certain cases even mainly, in the varying distribution over the region.

This new interpretation may shock us at first glance, since we have often previously spoken in such an intuitive concrete way of the " $\psi$ -vibrations" as though of something quite real. But there is something tangibly real behind the present conception also, namely, the very real electro-dynamically effective fluctuations of the electric space density. The  $\psi$ -function is to do no more and no less than permit of the totality of these fluctuations being mastered and surveyed mathematically by a single partial differential equation.

We have repeatedly called attention to the fact that the  $\psi$ -function itself cannot and may not be interpreted directly in terms of three-dimensional space -however much the one-electron problem tends to mislead us on this point- because it is in general a function in configuration space, not real space.

Concerning such a weight-function in the above sense, we would wish its integral over the whole configuration space to remain constantly normalised to the same unchanging value, preferably to unity. We can easily verify that this is necessary if the total charge of the system is to remain constant on the above definitions. Even for non conservative systems, this condition must obviously be postulated. For, naturally, the charge of a system is not to be altered when, e.g., a light wave falls on it, continues for a certain length of time, and then ceases. (N.B. -This is also valid for ionisation processes. A disrupted particle is still to be included in the system, until the separation is also logically -by decomposition of configuration space- completed.)

The question now arises as to whether the postulated persistence of normalisation is actually guaranteed by equations (4''), to which  $\psi$  is subject. If this were not the case, our whole conception would practically break down. Fortunately, it is the case. Let us form

$$\frac{d}{dt} \int \psi \bar{\psi} \rho dx = \int \left( \psi \frac{\partial \bar{\psi}}{\partial t} + \bar{\psi} \frac{\partial \psi}{\partial t} \right) \rho dx \quad (37)$$

Now,  $\psi$  satisfies one of the two equations (4''), and  $\bar{\psi}$  the other. Therefore, apart from a multiplicative constant, this integral becomes

$$\int (\psi \nabla^2 \bar{\psi} - \bar{\psi} \nabla^2 \psi) \rho dx = 2i \int (J \nabla^2 R - R \nabla^2 J) \rho dx \quad (38)$$

where for the moment we put

$$\psi = R + iJ.$$

According to Green's theorem, integral (38) vanishes identically; the sole necessary condition that functions  $R$  and  $J$  must satisfy for this -vanishing in sufficient degree at infinity- means physically nothing more than that the system under consideration should practically be confined to a finite region.

We can put this in a somewhat different way, by not immediately integrating over the whole configuration space, but by merely changing the time-derivative of the weight-function into a divergence by Green's transformation. Through this we get an insight into the question of the flow of the weight-function, and thus of electricity. The two equations

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= \frac{h}{4\pi i} \left( \nabla^2 - \frac{8\pi^2}{h^2} V \right) \psi \\ \frac{\partial \bar{\psi}}{\partial t} &= -\frac{h}{4\pi i} \left( \nabla^2 - \frac{8\pi^2}{h^2} V \right) \bar{\psi} \quad (4'') \end{aligned}$$



are multiplied by  $\rho\bar{\psi}$  and  $\rho\psi$  respectively, and added. Hence

$$\frac{\partial}{\partial t}(\rho\psi\bar{\psi}) = \frac{h}{4\pi i}\rho(\bar{\psi}\nabla^2\psi - \psi\nabla^2\bar{\psi}) \quad (39)$$

To carry out in extenso the transformation of the right-hand side, we must remember the explicit form of our many-dimensional, non Euclidean, Laplacian operator<sup>3</sup>:

$$\rho\nabla^2 = \sum_k \frac{\partial}{\partial q_k} \left[ \rho T_{p_k} \left( q_l, \frac{\partial\psi}{\partial q_l} \right) \right] \quad (40).$$

By a small transformation we readily obtain

$$\frac{\partial}{\partial t}(\rho\psi\bar{\psi}) = \frac{h}{4\pi i} \sum_k \frac{\partial}{\partial q_k} \left[ \rho\bar{\psi} T_{p_k} \left( q_l, \frac{\partial\psi}{\partial q_l} \right) - \rho\psi T_{p_k} \left( q_l, \frac{\partial\bar{\psi}}{\partial q_l} \right) \right] \quad (41).$$

The right-hand side appears as the divergence of a many-dimensional real vector, which is evidently to be interpreted as the current density of the weight-function in configuration space. Equation (41) is the continuity equation of the weight-function.

From it we can obtain the equation of continuity of electricity, and, indeed, a separate equation of this sort is valid for the charge density "originating from each separate particle". Let us fix on the  $\alpha$ -th particle, say. Let its "charge" be  $e_\alpha$ , its mass  $m_\alpha$ , and let its coordinate space be described by Cartesians  $x_\alpha, y_\alpha, z_\alpha$ , for the sake of simplicity. We denote the product of the differentials of the remaining co-ordinates shortly by  $dx'$ . Over the latter, we integrate equation (41), keeping  $x_\alpha, y_\alpha, z_\alpha$ , fixed. As the result, all terms except three disappear from the right-hand side, and we obtain

$$\begin{aligned} & \frac{\partial}{\partial t} \left[ e_\alpha \int \psi\bar{\psi} dx' \right] \\ &= \frac{he_\alpha}{4\pi im_\alpha} \left\{ \frac{\partial}{\partial x_\alpha} \left[ \int \left( \bar{\psi} \frac{\partial\psi}{\partial x_\alpha} - \psi \frac{\partial\bar{\psi}}{\partial x_\alpha} \right) dx' \right] + \frac{\partial}{\partial y_\alpha} \left[ \int \left( \bar{\psi} \frac{\partial\psi}{\partial y_\alpha} - \psi \frac{\partial\bar{\psi}}{\partial y_\alpha} \right) dx' \right] + \dots \right\} \\ &= \frac{he_\alpha}{4\pi im_\alpha} \operatorname{div}_\alpha \left[ \int (\bar{\psi} \operatorname{grad}_\alpha \psi - \psi \operatorname{grad}_\alpha \bar{\psi}) dx' \right]. \end{aligned}$$

In this equation,  $\operatorname{div}$  and  $\operatorname{grad}$  have the usual three-dimensional Euclidean meaning, and  $x_\alpha, y_\alpha, z_\alpha$  are to be interpreted as Cartesian co-ordinates of real space. The equation is the continuity equation of that charge density which "originates from the  $\alpha$ -th particle".

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<sup>3</sup>Cf. paper on Heisenberg's theory, equation (31). The quantity there denoted by  $\Delta_p^{-1/2}$  is our "density function"  $\rho(x)$  (e.g.  $r^2 \sin\theta$  in spherical polars).  $T$  is the kinetic energy as function of the position co-ordinates and momenta, the suffix at  $T$  denoting differentiation with respect to a momentum. In equations (31) and (32), loc. cit., unfortunately by error the suffix  $k$  is used twice, once for the summation and then also as a representative suffix in the argument of the functions.

If we form all the others in an analogous fashion, and add them together, we obtain the total equation of continuity. Of course, we must emphasize that the interpretation of the integrals on the right-hand side as components of the current density, is, as in all such cases, not absolutely compulsory, because a divergence-free vector could be added thereto.

To give an example, in the conservative one-electron problem, if  $\psi$  is given by

$$\psi = \sum_k c_k u_k e^{2\pi i \nu_k t + i \theta_k} \quad (c_k, \theta_k \text{ real constants}),$$

we get for the current density  $J$

$$J = \frac{he_1}{2\pi m_1} \sum_{(k,l)} c_k c_l (u_l \text{grad } u_k - u_k \text{grad } u_l) \sin [2\pi (\nu_k - \nu_l) t + \theta_k - \theta_l].$$

We see, and this is valid for conservative systems generally, that, if only a single proper vibration is excited, the current components disappear and the distribution of electricity is constant in time. The latter is also immediately evident from the fact that  $\psi\bar{\psi}$  becomes constant with respect to the time. This is still the case even when several proper vibrations are excited, if they all belong to the same proper value. On the other hand, the current density then no longer needs to vanish, but there may be present, and generally is, a stationary current distribution. Since the one or the other occurs in the unperturbed normal state at any rate, we may in a certain sense speak of a return to electrostatic and magnetostatic atomic models. In this way the lack of radiation in the normal state would, indeed, find a startlingly simple explanation.

I hope and believe that the present statements will prove useful in the elucidation of the magnetic properties of atoms and molecules, and further for explaining the flow of electricity in solid bodies.

Meantime, there is no doubt a certain crudeness in the use of a complex wave function. If it were unavoidable in principle, and not merely a facilitation of the calculation, this would mean that there are in principle two wave functions, which must be used together in order to obtain information on the state of the system. This somewhat unacceptable inference admits, I believe, of the very much more congenial interpretation that the state of the system is given by a real function and its time-derivative. Our inability to give more accurate information about this is intimately connected with the fact that, in the pair of equations (4''), we have before us only the substitute -extraordinarily convenient for the calculation, to be sure- for a real wave equation of probably the fourth order, which, however, I have not succeeded in forming for the non-conservative case.

Zürich, Physical Institute of the University.  
(Received June 23, 1926.)

# Part II

## Dirac equation

# The Quantum Theory of the Electron

By P. A. M. Dirac, St. John's College, Cambridge. [3]  
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In this paper Dirac first derived the now famous Dirac equation

$$[i\Sigma\gamma_{\mu}p_{\mu} + mc]\psi = 0.$$

It also discusses its relativistic invariance, its generalisation in the presence of electromagnetic fields and compatibility with previous theories.

## Notation

$(\mathbf{a}, \mathbf{b}) \equiv a_1b_1 + a_2b_2 + a_3b_3$  dot product

$A_0$  scalar potential

$\mathbf{A}$  vector potential

$W$  energy of the electron

$\mathbf{p}$  momentum of the electron

This paper makes no reference to covariant and contravariant quantities, Dirac treats all quantities as regular covariant vectors and doesn't make use of Einstein's summation convention.

## 1.1 Previous Relativity Treatments.

... Dirac begins his paper by considering what is wrong with previous attempts to find a relativistic theory for the electron. ....

The relativity Hamiltonian for a point electron moving in an arbitrary electro-magnetic field with potential  $(A_0, \mathbf{A})$  is

$$F \equiv \left( \frac{W}{c} + \frac{e}{c} A_0 \right)^2 + \left( \mathbf{p} + \frac{e}{c} \mathbf{A} \right)^2 + m^2 c^2.$$

... Note that the dimension of this Hamiltonian is not Energy,  $[F] = [Energy]^2/[Velocity]^2$  and  $[A] = [Energy]/[Charge] = [Voltage]$ .

More importantly, a minus is missing from this equation throughout this whole chapter. There should obviously be a minus in front of the first square, otherwise the energy-momentum equation is not satisfied. Furthermore, the minus reappears in the section 1.2 equation (3). ....

Gordon<sup>1</sup> suggested that the operator of the wave equation of the quantum theory should be obtained from this F by the same procedure as in non-relativity theory, namely, by putting

$$W = ih \frac{\partial}{\partial t},$$

$$p_r = -ih \frac{\partial}{\partial x_r}, \quad r = 1, 2, 3,$$

in it. This gives the wave equation

$$F\psi \equiv \left[ \left( ih \frac{\partial}{c \partial t} + \frac{e}{c} A_0 \right)^2 + \sum_r \left( -ih \frac{\partial}{\partial x_r} + \frac{e}{c} A_r \right)^2 + m^2 c^2 \right] \psi = 0 \quad (1)$$

the wave function  $\psi$  being a function of  $x_1, x_2, x_3, t$ . This equation gives rise to two difficulties.

... By non-relativity theory is meant the wave mechanics approach introduced by Schrodinger 2 years earlier, in 1926. ....

The first is in connection with the physical interpretation of  $\psi$ . Gordon, and also independently Klein<sup>2</sup>, from considerations of the conservation theorems, make the assumption that if  $\psi$  is a solution,

$$\rho = -\frac{e}{2mc^2} \left\{ ih \left( \psi \frac{\partial \bar{\psi}}{\partial t} - \bar{\psi} \frac{\partial \psi}{\partial t} \right) + 2eA_0 \psi \bar{\psi} \right\}$$

<sup>1</sup>Gordon, 'Z. f. Physik,' vol. 40, p. 117 (1926)

<sup>2</sup>Klein, 'Z. f. Physik,' vol. 41, p. 407 (1927)

is to be interpreted as the charge associated to the wavefunction  $\psi$ .

... If such assumption was right, we could in turn assume that we are dealing with a free particle. Making use of  $W = i\hbar\partial/\partial t$  or substituting the plane-wave solution  $\psi = |\psi|\exp\{\pm i(Wt - (\mathbf{p}, \mathbf{x}))/\hbar\}$  into the charge equation, we would get

$$\rho = \pm \frac{e}{mc^2} |\psi|^2 W.$$

This makes it completely impossible to make  $\rho$  a positive definite quantity. The proportionality to the energy  $W$  is due to the presence of first order time derivatives in  $\rho$ , which is due to the second order time derivative in  $F$  or the Klein-Gordon equation. ....

The interpretation of non-relativity quantum mechanics is made possible by the wave equation being of the form

$$(\mathbf{H} - W)\psi = 0 \quad (2)$$

i.e., being linear in  $W$  or  $\partial/\partial t$ , so that the wave function at any time determines the wave function at any later time. The wave equation of the relativity theory must also be linear in  $W$ .

The second difficulty in Gordon's interpretation arises from the fact that if one takes the conjugate imaginary of equation (1), one gets

$$\left[ \left( -\frac{W}{c} + \frac{e}{c} A_0 \right)^2 + \left( -\mathbf{p} + \frac{e}{c} \mathbf{A} \right)^2 + m^2 c^2 \right] \psi = 0,$$

which is the same as one would get if one put  $-e$  for  $e$ . The wave equation (2) thus refers equally well to an electron with charge  $e$  as to one with charge  $-e$ . If one considers for definiteness the limiting case of large quantum numbers one would find that some of the solutions of the wave equation are wave packets moving in the way a particle of charge  $-e$  or  $e$  would. For this second class of solutions  $W$  has a negative value. One gets over the difficulty on the classical theory by arbitrarily excluding those solutions that have a negative  $W$ . One cannot do this on the quantum theory, since in general a perturbation will cause transitions from states with  $W$  positive to states with  $W$  negative. Such a transition would appear experimentally as the electron suddenly changing its charge from  $-e$  to  $e$ , a phenomenon which has not been observed. The true relativity wave equation should thus be such that its solutions split up into two non-combining sets, referring respectively to the charge  $-e$  and the charge  $e$ .

In the present paper we shall be concerned only with the removal of the first of these two difficulties. The resulting theory is therefore still only an approximation, but it appears to be good enough to account for all the duplexity phenomena without arbitrary assumptions.

## 1.2 The Hamiltonian for No Field.

Our problem is to obtain a wave equation of the form (2) which shall be invariant under a Lorentz transformation and shall be equivalent to (1) in the limit of large quantum numbers. We shall consider first the case of no field, when equation (1) reduces to

$$(-p_0^2 + \mathbf{p}^2 + m^2 c^2) \psi = 0 \quad (3)$$

if one puts

$$p_0 = \frac{W}{c} = i\hbar \frac{\partial}{c\partial t}.$$

... Where (3) is the relativistic energy equation  $W^2 = \mathbf{p}^2 c^2 + m^2 c^4$ . .....

The symmetry between  $p_0$  and  $p_1, p_2, p_3$  required by relativity shows that, since the Hamiltonian we want is linear in  $p_0$ , it must also be linear in  $p_1, p_2$  and  $p_3$ .

... Dirac argues that in order to be invariant under Lorentz transformation of the derivatives in time and space, the equation we seek must be linear in the space derivatives since we already require it to be linear in the time derivative. ....

Our wave equation is therefore of the form

$$(p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta) \psi = 0 \quad (4)$$

where for the present all that is known about the dynamical variables or operators  $\alpha_1, \alpha_2, \alpha_3, \beta$  is that they are independent of  $p_0, p_1, p_2, p_3$ , i.e., that they commute with  $t, x_1, x_2, x_3$ .

.....

$$\begin{aligned} & \alpha_1, \alpha_2, \alpha_3, \beta \text{ independent of } p_0, p_1, p_2, p_3 \\ \implies & \alpha_1, \alpha_2, \alpha_3, \beta \text{ independent of } \partial_t, \partial_{x_1}, \partial_{x_2}, \partial_{x_3} \\ \implies & \alpha_1, \alpha_2, \alpha_3, \beta \text{ commute with } t, x_1, x_2, x_3. \end{aligned}$$

.....

Since we are considering the case of a particle moving in empty space, so that all points in space are equivalent, we should expect the Hamiltonian not to involve  $t, x_1, x_2, x_3$ . This means that  $\alpha_1, \alpha_2, \alpha_3, \beta$  are independent of  $t, x_1, x_2, x_3$ , i.e., that they commute with  $p_0, p_1, p_2, p_3$ .

.....

*Homogeneity of empty space*  
 $\implies$  translational invariance of the Hamiltonian  
 $\implies$  Hamiltonian independent of  $t, x_1, x_2, x_3$   
 $\implies \alpha_1, \alpha_2, \alpha_3, \beta$  independent of  $t, x_1, x_2, x_3$   
 $\implies \alpha_1, \alpha_2, \alpha_3, \beta$  commute with  $\partial_t, \partial_{x_1}, \partial_{x_2}, \partial_{x_3}$ .

.....

We are therefore obliged to have other dynamical variables besides the co-ordinates and momenta of the electron, in order that  $\alpha_1, \alpha_2, \alpha_3, \beta$  may be functions of them. The wave function  $\psi$  must then involve more variables than merely  $x_1, x_2, x_3, t$ .

Multiplying equation (4) by  $(-p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta) \neq 0$  leads to

$$0 = (-p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta) (p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta) \psi$$

$$= [-p_0^2 + \Sigma \alpha_1^2 p_1^2 + \Sigma (\alpha_1 \alpha_2 + \alpha_2 \alpha_1) p_1 p_2 + \Sigma (\alpha_1 \beta + \beta \alpha_1) p_1 + \beta^2] \psi \quad (5)$$

where the  $\Sigma$  refers to cyclic permutation of the suffixes 1, 2, 3. This agrees with

$$0 = (-p_0^2 + \mathbf{p}^2 + m^2 c^2) \psi$$

$$= (-p_0^2 + \Sigma p_1^2 + m^2 c^2) \psi$$

if and only if

$$\left. \begin{aligned} \alpha_r^2 &= 1, & \alpha_r \alpha_s + \alpha_s \alpha_r &= 0 \quad (r \neq s) \\ \beta^2 &= m^2 c^2, & \alpha_r \beta + \beta \alpha_r &= 0 \end{aligned} \right\} r, s = 1, 2, 3.$$

By setting  $\beta = \alpha_4 mc$ , these conditions can be written in the simpler form

$$\alpha_\mu^2 = 1, \quad \alpha_\mu \alpha_\nu + \alpha_\nu \alpha_\mu = 0 \quad (\mu \neq \nu) \quad \mu, \nu = 1, 2, 3, 4 \quad (6)$$

We can suppose the  $\alpha_\mu$  's to be expressed as matrices in some matrix scheme.  
*... If they were scalars, it wouldn't be possible to satisfy both conditions simultaneously.*  
*Let  $\alpha_\mu$  's  $\in \mathbb{C}$ , the conditions would simplify into*

$$\alpha_\mu \alpha_\nu + \alpha_\nu \alpha_\mu = 0 \implies 2\alpha_\mu \alpha_\nu = 0 \quad (\mu \neq \nu)$$

$$\alpha_\mu^2 = 1 \implies \alpha_\mu = \pm 1,$$

*obviously mutually exclusive.* .....

Therefore, we must now find four matrices  $\alpha_\mu$  to satisfy the conditions (6). We make



use of the matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

... Now commonly known as Pauli matrices. ....  
 which Pauli introduced\* to describe the three components of spin angular momentum.  
 These matrices have just the properties

$$\sigma_r^2 = 1, \quad \sigma_r \sigma_s + \sigma_s \sigma_r = 0 \quad (r \neq s) \quad (7)$$

that we require for our  $\alpha$  's. We cannot, however, just take the  $\sigma$ 's to be three of our  $\alpha$  's, because then it would not be possible to find the fourth. We must extend the  $\sigma$  's in a diagonal manner to bring in two more rows and columns, so that we can introduce three more matrices  $\rho_1, \rho_2, \rho_3$  of the same form as  $\sigma_1, \sigma_2, \sigma_3$ , but referring to different rows and columns, thus :-

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\rho_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \rho_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \quad \rho_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

... We can formalise mathematically what Dirac meant by extending in a diagonal manner. Naming  $M([n \times n])$  the space of  $n \times n$  matrices and  $\alpha_{[n \times n]} \in M([n \times n])$  an  $n \times n$  matrix.

We can now define an operation  $\otimes$  that behaves as needed.

$$\otimes : M([2 \times 2]) \rightarrow M([4 \times 4]) \text{ with,}$$

$$\alpha_{[2 \times 2]} \otimes \mathbb{1}_{[2 \times 2]} \equiv \begin{pmatrix} \alpha_{[2 \times 2]} & 0 \\ 0 & \alpha_{[2 \times 2]} \end{pmatrix}.$$

Given this operation we can derive three useful properties

Property 1.

$$\alpha_{[2 \times 2]} \otimes \mathbb{1}_{[2 \times 2]} + \beta_{[2 \times 2]} \otimes \mathbb{1}_{[2 \times 2]} \\ = (\alpha_{[2 \times 2]} + \beta_{[2 \times 2]}) \otimes \mathbb{1}_{[2 \times 2]}.$$

*Property 2.*

$$\begin{aligned}
& (\alpha_{[2 \times 2]} \otimes \mathbb{1}_{[2 \times 2]})(\beta_{[2 \times 2]} \otimes \mathbb{1}_{[2 \times 2]}) \\
&= \begin{pmatrix} \alpha_{[2 \times 2]} & 0 \\ 0 & \alpha_{[2 \times 2]} \end{pmatrix} \begin{pmatrix} \beta_{[2 \times 2]} & 0 \\ 0 & \beta_{[2 \times 2]} \end{pmatrix} \\
&= \begin{pmatrix} \alpha_{[2 \times 2]}\beta_{[2 \times 2]} & 0 \\ 0 & \alpha_{[2 \times 2]}\beta_{[2 \times 2]} \end{pmatrix} \\
&= (\alpha_{[2 \times 2]}\beta_{[2 \times 2]}) \otimes \mathbb{1}_{[2 \times 2]}.
\end{aligned}$$

*Property 3.*

$$\begin{aligned}
& (\alpha_{[2 \times 2]} \otimes \mathbb{1}_{[2 \times 2]}) \begin{pmatrix} a_{[2 \times 2]} & b_{[2 \times 2]} \\ c_{[2 \times 2]} & d_{[2 \times 2]} \end{pmatrix} (\alpha_{[2 \times 2]} \otimes \mathbb{1}_{[2 \times 2]}) \\
&= \begin{pmatrix} \alpha_{[2 \times 2]} & 0 \\ 0 & \alpha_{[2 \times 2]} \end{pmatrix} \begin{pmatrix} a_{[2 \times 2]} & b_{[2 \times 2]} \\ c_{[2 \times 2]} & d_{[2 \times 2]} \end{pmatrix} \begin{pmatrix} \alpha_{[2 \times 2]} & 0 \\ 0 & \alpha_{[2 \times 2]} \end{pmatrix} \\
&= \begin{pmatrix} \alpha_{[2 \times 2]}a_{[2 \times 2]} & \alpha_{[2 \times 2]}b_{[2 \times 2]} \\ \alpha_{[2 \times 2]}c_{[2 \times 2]} & \alpha_{[2 \times 2]}d_{[2 \times 2]} \end{pmatrix} \begin{pmatrix} \alpha_{[2 \times 2]} & 0 \\ 0 & \alpha_{[2 \times 2]} \end{pmatrix} \\
&= \begin{pmatrix} \alpha_{[2 \times 2]}a_{[2 \times 2]}\alpha_{[2 \times 2]} & \alpha_{[2 \times 2]}b_{[2 \times 2]}\alpha_{[2 \times 2]} \\ \alpha_{[2 \times 2]}c_{[2 \times 2]}\alpha_{[2 \times 2]} & \alpha_{[2 \times 2]}d_{[2 \times 2]}\alpha_{[2 \times 2]} \end{pmatrix}.
\end{aligned}$$

*It's now trivial to prove, with the help of 1. and 2., that the  $4 \times 4$  Pauli matrices,  $\sigma_{r,[2 \times 2]} \otimes \mathbb{1}_{[2 \times 2]}$ , still satisfy equations (7),*

*Proof.*

$$\sigma_{r,[4 \times 4]}^2 = (\sigma_{r,[2 \times 2]} \otimes \mathbb{1}_{[2 \times 2]})^2 = (\sigma_{r,[2 \times 2]}^2) \otimes \mathbb{1}_{[2 \times 2]} = \mathbb{1}_{[2 \times 2]} \otimes \mathbb{1}_{[2 \times 2]} = \mathbb{1}_{[4 \times 4]}$$

QED

*Proof.*

$$\begin{aligned}
& \sigma_{r,[4 \times 4]}\sigma_{s,[4 \times 4]} + \sigma_{s,[4 \times 4]}\sigma_{r,[4 \times 4]} = (\sigma_{r,[2 \times 2]}\sigma_{s,[2 \times 2]}) \otimes \mathbb{1}_{[2 \times 2]} + (\sigma_{s,[2 \times 2]}\sigma_{r,[2 \times 2]}) \otimes \mathbb{1}_{[2 \times 2]} = \\
&= (\sigma_{r,[2 \times 2]}\sigma_{s,[2 \times 2]} + \sigma_{s,[2 \times 2]}\sigma_{r,[2 \times 2]}) \otimes \mathbb{1}_{[2 \times 2]} = 0 \otimes \mathbb{1}_{[2 \times 2]} = 0 \quad (r \neq s).
\end{aligned}$$

QED

.....  
The  $\rho$ 's are obtained from the  $\sigma$ 's by interchanging the second and third rows, and the

second and third columns.

... They can also be obtained by substituting the identity matrix  $\mathbb{1}$ , in the Pauli matrices,

$$\rho_1 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad \rho_2 = \begin{pmatrix} 0 & -i\mathbb{1} \\ i\mathbb{1} & 0 \end{pmatrix} \quad \rho_3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix},$$

therefore it is obvious that the  $\rho$ 's still satisfy the properties (7).

Finally, with the help of 3., we can prove that  $\rho$ 's and  $\sigma$ 's commute.

$$\rho \text{ and } \sigma \text{ commute} \iff \rho_r \sigma_s = \sigma_s \rho_r \iff \sigma_s \rho_r \sigma_s = \rho_r, \text{ since } \sigma_s^2 = 1.$$

*Proof.* Let a, b, c, d be scalars and

$$a_{[2 \times 2]} = a\mathbb{1}_{[2 \times 2]}, b_{[2 \times 2]} = b\mathbb{1}_{[2 \times 2]}, c_{[2 \times 2]} = c\mathbb{1}_{[2 \times 2]}, d_{[2 \times 2]} = d\mathbb{1}_{[2 \times 2]},$$

then 3. would simplify into

$$(\alpha_{[2 \times 2]} \otimes \mathbb{1}_{[2 \times 2]}) \begin{pmatrix} a_{[2 \times 2]} & b_{[2 \times 2]} \\ c_{[2 \times 2]} & d_{[2 \times 2]} \end{pmatrix} (\alpha_{[2 \times 2]} \otimes \mathbb{1}_{[2 \times 2]}) = \begin{pmatrix} a\alpha_{[2 \times 2]}^2 & b\alpha_{[2 \times 2]}^2 \\ c\alpha_{[2 \times 2]}^2 & d\alpha_{[2 \times 2]}^2 \end{pmatrix}.$$

We can now see that since the  $\rho$ 's take the form  $\begin{pmatrix} a\mathbb{1}_{[2 \times 2]} & b\mathbb{1}_{[2 \times 2]} \\ c\mathbb{1}_{[2 \times 2]} & d\mathbb{1}_{[2 \times 2]} \end{pmatrix}$  and  $\sigma_s^2 = 1$ , then

$$(\sigma_{r,[2 \times 2]} \otimes \mathbb{1}_{[2 \times 2]}) \begin{pmatrix} a\mathbb{1}_{[2 \times 2]} & b\mathbb{1}_{[2 \times 2]} \\ c\mathbb{1}_{[2 \times 2]} & d\mathbb{1}_{[2 \times 2]} \end{pmatrix} (\sigma_{r,[2 \times 2]} \otimes \mathbb{1}_{[2 \times 2]}) = \begin{pmatrix} a\mathbb{1}_{[2 \times 2]} & b\mathbb{1}_{[2 \times 2]} \\ c\mathbb{1}_{[2 \times 2]} & d\mathbb{1}_{[2 \times 2]} \end{pmatrix}.$$

which proves that  $\rho$ 's and  $\sigma$ 's commute, with the appropriate choice of a, b, c, d. QED

.....  
We now have, in addition to equations (7)

$$\rho_r^2 = 1, \quad \rho_r \rho_s + \rho_s \rho_r = 0 \quad (r \neq s), \quad \rho_r \sigma_t = \sigma_t \rho_r \quad (7').$$

If we now take

$$\alpha_1 = \rho_1 \sigma_1, \quad \alpha_2 = \rho_1 \sigma_2, \quad \alpha_3 = \rho_1 \sigma_3, \quad \alpha_4 = \rho_3,$$

all the conditions (7) are satisfied, e.g.,

$$\begin{aligned} \alpha_1^2 &= \rho_1 \sigma_1 \rho_1 \sigma_1 = p_1^2 \sigma_1^2 = 1 \\ \alpha_1 \alpha_2 &= p_1 \sigma_1 \rho_1 \sigma_2 = p_1^2 \sigma_1 \sigma_2 = -\rho_1^2 \sigma_2 \sigma_1 = -\alpha_2 \alpha_1. \end{aligned}$$

The following equations are to be noted for later reference

$$\left. \begin{aligned} \rho_1 \rho_2 &= i\rho_3 = -\rho_2 \rho_1 \\ \sigma_1 \sigma_2 &= i\sigma_3 = -\sigma_2 \sigma_1 \end{aligned} \right\} \quad (8),$$

together with the equations obtained by cyclic permutation of the suffixes.

The wave equation (4) now takes the form

$$[p_0 + \rho_1(\sigma, \mathbf{p}) + \rho_3 mc] \psi = 0 \quad (9),$$

where  $\sigma$  denotes the vector  $(\sigma_1, \sigma_2, \sigma_3)$ .

### 1.3 Proof of Invariance under a Lorentz Transformation.

Multiply equation (9) by  $\rho_3$  on the left-hand side. It becomes, with the help of (8),

$$[\rho_3 p_0 + i\rho_2(\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3) + mc] \psi = 0.$$

Putting

$$p_0 = ip_4, \quad \rho_3 = \gamma_4, \quad \rho_2 \sigma_r = \gamma_r \quad r = 1, 2, 3 \quad (10)$$

we have

$$[i\Sigma\gamma_\mu p_\mu + mc] \psi = 0 \quad \mu = 1, 2, 3, 4 \quad (11)$$

... Recall  $p_0 = \frac{W}{c}$ , which implies  $p_4 = -i\frac{W}{c}$ . From this we can recognise the "four-momentum",

$$p_\mu = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ -iW/c \end{pmatrix}.$$

It is important to note that it is not a actual four-vector because we are not using the Minkowski metric, but the Euclidean one, which is the reason why we need an  $i$  in the fourth term. Otherwise we wouldn't be able to get the energy-momentum relation back.

The  $p_\mu$  transform under a Lorentz transformation according to the law

$$p'_\mu = \Sigma_\nu \alpha_{\mu\nu} p_\nu,$$

where the coefficients  $\alpha_{\mu\nu}$  are c-numbers satisfying

$$\Sigma_\mu \alpha_{\mu\nu} \alpha_{\mu\tau} = \delta_{\nu\tau}, \quad \Sigma_\tau \alpha_{\mu\tau} \alpha_{\nu\tau} = \delta_{\mu\nu}.$$

... These are the only conditions we need the Lorentz Transformations to obey in order to prove the invariance of the Dirac equation. ....

The wave equation therefore transforms into

$$[i\Sigma\gamma'_\mu p'_\mu + mc] \psi = 0 \quad (12),$$

where

$$\gamma'_\mu = \Sigma_\nu \alpha_{\mu\nu} \gamma_\nu.$$

Now the  $\gamma_\mu$ , like the  $\alpha_\mu$ , satisfy

$$\gamma_\mu^2 = 1, \quad \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 0 \quad (\mu \neq \nu).$$

... Conditions equivalent to equations (6). .....

These relations can be summed up in the single equation

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}.$$

We have

$$\begin{aligned} \gamma_\mu' \gamma_\nu' + \gamma_\nu' \gamma_\mu' &= \sum_{\tau\lambda} a_{\mu\tau} a_{\nu\lambda} (\gamma_\tau \gamma_\lambda + \gamma_\lambda \gamma_\tau) \\ &= 2 \sum_{\tau\lambda} a_{\mu\tau} a_{\nu\lambda} \delta_{\tau\lambda} \\ &= 2 \sum_{\tau} a_{\mu\tau} a_{\nu\tau} = 2\delta_{\mu\nu}. \end{aligned}$$

Thus the  $\gamma_\mu'$  satisfy the same relations as the  $\gamma_\mu$ . Thus we can put, analogously to (10)

$$\gamma_4' = \rho_3' \quad \gamma_r' = \rho_2' \sigma_r'$$

where the  $\rho'$  's and  $\sigma'$  's are easily verified to satisfy the relations corresponding to (7), (7') and (8), if  $\rho_2'$  and  $\rho_1'$  are defined by  $\rho_2' = -i\gamma_1' \gamma_2' \gamma_3'$ ,  $\rho_1' = -i\rho_2' \rho_3'$ .

We shall now show that, by a canonical transformation, the  $\rho'$  's and  $\sigma'$  's may be brought into the form of the  $\rho$  's and  $\sigma$  's. From the equation  $(\rho_3')^2 = 1$ , it follows that the only possible characteristic values for  $\rho_3'$  are  $\pm 1$ . If one applies to  $\rho_3'$  a canonical transformation with the transformation function  $\rho_1'$ , the result is

$$\rho_1' \rho_3' (\rho_1')^{-1} = -\rho_3' \rho_1' (\rho_1')^{-1} = -\rho_3'.$$

Since characteristic values are not changed by a canonical transformation,  $\rho_3'$  must have the same characteristic values as  $-\rho_3'$ . Hence the characteristic values of  $\rho_3'$  are +1 twice and -1 twice. The same argument applies to each of the other  $\rho'$  's, and to each of the  $\sigma'$  's.

Since  $\rho_3'$  and  $\sigma_3'$  commute, they can be brought simultaneously to the diagonal form by a canonical transformation. They will then have for their diagonal elements each +1 twice and -1 twice. Thus, by suitably rearranging the rows and columns, they can be brought into the form  $\rho_3$  and  $\sigma_3$  respectively. (The possibility  $\rho_3' = \pm \sigma_3'$  is excluded by the existence of matrices that commute with one but not with the other.)

Any matrix containing four rows and columns can be expressed as

$$c + \sum_r c_r \sigma_r + \sum_r c_r' \rho_r + \sum_{rs} c_{rs} \rho_r \sigma_s \quad (13)$$

where the sixteen coefficients  $c, c_r, c'_r, c_{rs}$  are c-numbers. By expressing  $\sigma'_1$  in this way, we see, from the fact that it commutes with  $\rho'_3 = \rho_3$  and anticommutes with  $\sigma'_3 = \sigma_3$ , that it must be of the form

$$\sigma'_1 = c_1\sigma_1 + c_2\sigma_2 + c_{31}\rho_3\sigma_1 + c_{32}\rho_3\sigma_2 = \begin{Bmatrix} 0 & a_{12} & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & a_{43} & 0 \end{Bmatrix}$$

The condition  $\sigma'^2_1 = 1$  shows that  $a_{12}a_{21} = 1, a_{34}a_{43} = 1$ .

$$(\sigma'_1)^2 = \begin{Bmatrix} a_{12}a_{21} & 0 & 0 & 0 \\ 0 & a_{21}a_{12} & 0 & 0 \\ 0 & 0 & a_{34}a_{43} & 0 \\ 0 & 0 & 0 & a_{43}a_{34} \end{Bmatrix} = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix}$$

If we now apply the canonical transformation: first row to be multiplied by  $(a_{21}/a_{12})^{1/2}$  and third row to be multiplied by  $(a_{43}/a_{34})^{1/2}$ , and first and third columns to be divided by the same expressions,  $\sigma'_1$  will be brought into the form of  $\sigma_1$ , and the diagonal matrices  $\sigma'_3$  and  $\rho'_3$  will not be changed.

$$\sigma'_1 = \begin{Bmatrix} 0 & a_{12}(a_{21}/a_{12})^{1/2} & 0 & 0 \\ a_{21}/(a_{21}/a_{12})^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{34}(a_{43}/a_{34})^{1/2} \\ 0 & 0 & a_{43}/(a_{43}/a_{34})^{1/2} & 0 \end{Bmatrix} \\ = \begin{Bmatrix} 0 & (a_{12}a_{21})^{1/2} & 0 & 0 \\ (a_{21}a_{12})^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & (a_{34}a_{43})^{1/2} \\ 0 & 0 & (a_{43}a_{34})^{1/2} & 0 \end{Bmatrix} = \begin{Bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{Bmatrix}$$

If we now express  $\rho'_1$  in the form (13) and use the conditions that it commutes with  $\sigma'_1 = \sigma_1$  and  $\sigma'_3 = \sigma_3$  and anticommutes with  $\rho'_3 = \rho_3$ , we see that it must be of the form

$$\rho'_1 = c'_1\rho_1 + c'_2\rho_2.$$

The condition  $(\rho'_2)^2 = 1$  shows that  $(c'_1)^2 + (c'_2)^2 = 1$ , or  $c'_1 = \cos\theta, c'_2 = \sin\theta$ . Hence  $\rho'_1$  is of the form

$$\rho'_1 = \begin{Bmatrix} 0 & 0 & e^{-i\theta} & 0 \\ 0 & 0 & 0 & e^{-i\theta} \\ e^{i\theta} & 0 & 0 & 0 \\ 0 & e^{i\theta} & 0 & 0 \end{Bmatrix}$$

If we now apply the canonical transformation: first and second rows to be multiplied by  $e^{i\theta}$  and first and second columns to be divided by the same expression,  $\rho'_1$  will be brought into the form  $\rho_1$ , and  $\sigma_1, \sigma_3, \rho_3$  will not be altered.  $\rho'_2$  and  $\sigma'_2$  must now be of the form  $\rho_2$  and  $\sigma_2$ , on account of the relations  $i\rho'_2 = \rho_3'\rho'_1, i\sigma'_2 = \sigma_3'\sigma'_1$ .

Thus by a succession of canonical transformations, which can be combined to form a single canonical transformation, the  $\rho'$  's and  $\sigma'$  's can be brought into the form of the  $\rho$ 's and  $\sigma$ 's. The new wave equation (12) can in this way be brought back into the form of the original wave equation (11) or (9), so that the results that follow from this original wave equation must be independent of the frame of reference used.

## 1.4 The Hamiltonian for an Arbitrary Field.

To obtain the Hamiltonian for an electron in an electromagnetic field with scalar potential  $A_0$  and vector potential  $\mathbf{A}$ , we adopt the usual procedure of substituting  $p_0 + e/c A_0$  for  $p_0$  and  $\mathbf{p} + e/c \mathbf{A}$  for  $\mathbf{p}$  in the Hamiltonian for no field. From equation (9) we thus obtain

$$\left[ p_0 + \frac{e}{c} A_0 + \rho_1 \left( \boldsymbol{\sigma}, \mathbf{p} + \frac{e}{c} \mathbf{A} \right) + \rho_3 mc \right] \psi = 0 \quad (14).$$

This wave equation appears to be sufficient to account for all the duplexity phenomena. On account of the matrices  $\rho$  and  $\sigma$  containing four rows and columns, it will have four times as many solutions as the non-relativity wave equation, and twice as many as the previous relativity wave equation (1). Since half the solutions must be rejected as referring to the charge  $+e$  on the electron, the correct number will be left to account for duplexity phenomena. The proof given in the preceding section of invariance under a Lorentz transformation applies equally well to the more general wave equation (14).

We can obtain a rough idea of how (14) differs from the previous relativity wave equation (1) by multiplying it up analogously to (5). This gives, if we write  $e'$  for  $e/c$

$$\begin{aligned} 0 &= [-(p_0 + e'A_0) + \rho_1(\boldsymbol{\sigma}, \mathbf{p} + e'\mathbf{A}) + \rho_3 mc] \times [(p_0 + e'A_0) + \rho_1(\boldsymbol{\sigma}, \mathbf{p} + e'\mathbf{A}) + \rho_3 mc] \psi \\ &= [-(p_0 + e'A_0)^2 + (\boldsymbol{\sigma}, \mathbf{p} + e'\mathbf{A})^2 + m^2 c^2 \\ &\quad + \rho_1 \{(\boldsymbol{\sigma}, \mathbf{p} + e'\mathbf{A})(p_0 + e'A_0) - (p_0 + e'A_0)(\boldsymbol{\sigma}, \mathbf{p} + e'\mathbf{A})\}] \psi \quad (15). \end{aligned}$$

We now use the general formula, that if  $\mathbf{B}$  and  $\mathbf{C}$  are any two vectors that commute with  $\boldsymbol{\sigma}$ , then

$$\begin{aligned} (\boldsymbol{\sigma}, \mathbf{B})(\boldsymbol{\sigma}, \mathbf{C}) &= \Sigma \sigma_1^2 B_1 C_1 + \Sigma (\sigma_1 \sigma_2 B_1 C_2 + \sigma_2 \sigma_1 B_2 C_1) \\ &= (\mathbf{B}, \mathbf{C}) + i \Sigma \sigma_3 (B_1 C_2 - B_2 C_1) \\ &= (\mathbf{B}, \mathbf{C}) + i(\boldsymbol{\sigma}, \mathbf{B} \times \mathbf{C}). \end{aligned}$$

Taking  $\mathbf{B} = \mathbf{C} = \mathbf{p} + e'\mathbf{A}$ , we find

$$\begin{aligned} (\boldsymbol{\sigma}, \mathbf{p} + e'\mathbf{A})^2 &= (\mathbf{p} + e'\mathbf{A})^2 + i\Sigma\sigma_3[(p_1 + e'A_1)(p_2 + e'A_2) - (p_2 + e'A_2)(p_1 + e'A_1)] \\ &= (\mathbf{p} + e'\mathbf{A})^2 + he'(\boldsymbol{\sigma}, \text{curl } \mathbf{A}). \end{aligned}$$

Thus (15) becomes

$$\begin{aligned} 0 &= \left[ -(p_0 + e'A_0)^2 + (\mathbf{p} + e'\mathbf{A})^2 + m^2c^2 + e'h(\boldsymbol{\sigma}, \text{curl } \mathbf{A}) - ie'h\rho_1 \left( \boldsymbol{\sigma}, \text{grad } A_0 + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) \right] \psi \\ &= [- (p_0 + e'A_0)^2 + (\mathbf{p} + e'\mathbf{A})^2 + m^2c^2 + e'h(\boldsymbol{\sigma}, \mathbf{H}) + ie'h\rho_1(\boldsymbol{\sigma}, \mathbf{E})] \psi, \end{aligned}$$

where  $\mathbf{E}$  and  $\mathbf{H}$  are the electric and magnetic vectors of the field.

This differs from (1) by the two extra terms

$$\frac{eh}{c}(\boldsymbol{\sigma}, \mathbf{H}) + \frac{ieh}{c}\rho_1(\boldsymbol{\sigma}, \mathbf{E})$$

in F. These two terms, when divided by the factor  $2m$ , can be regarded as the additional potential energy of the electron due to its new degree of freedom. The electron will therefore behave as though it has a magnetic moment  $eh/2mc \boldsymbol{\sigma}$  and an electric moment  $ieh/2mc \rho_1 \boldsymbol{\sigma}$ . This magnetic moment is just that assumed in the spinning electron model. The electric moment, being a pure imaginary, we should not expect to appear in the model. It is doubtful whether the electric moment has any physical meaning, since the Hamiltonian in (14) that we started from is real, and the imaginary part only appeared when we multiplied it up in an artificial way in order to make it resemble the Hamiltonian of previous theories.



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