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Heat kernel and worldline path integrals for the Proca theory

Supervisor:
Prof. Fiorenzo Bastianelli

Submitted by:
Marta Picciau

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Abstract

In this thesis we study the heat kernel, a useful tool to analyze various properties of different quantum field theories. In particular, we focus on the study of the one-loop effective action and the application of worldline path integrals to derive perturbatively the heat kernel coefficients for the Proca theory of massive vector fields. It turns out that the worldline path integral method encounters some difficulties if the differential operator of the heat kernel is of non-minimal kind. More precisely, a direct recasting of the differential operator in terms of worldline path integrals, produces in the classical action a non-perturbative vertex and the path integral cannot be solved. In this work we wish to find ways to circumvent this issue and to give a suggestion to solve similar problems in other contexts.

Contents

Introduction

The heat kernel has been a useful method to investigate different properties of quantum field theories, among which there is the study of one-loop effective actions. The most familiar method to compute its expansion coefficients is the DeWitt iterative procedure contained in [1]. An alternative method was proposed by Gilkey [2], which happens to be particularly convenient when dealing with manifolds without boundaries. They both show a direct application only for simple second order partial differential operators of "minimal" type, namely the ones where the covariant derivatives appear only in the usual Laplacian form. The present work focuses instead on the study of a particular case of "non-minimal" operator, which comes from the quantum field theory of the massive vector field. The latter represents a relevant example useful to understand how the heat kernel for such operators can be computed. Historically DeWitt's technique has been generalized to cover the more general cases of higher order non-minimal operators by Barvinsky and Vilkovisky [3]. In their work, they provided a useful trace operator identity for the Proca theory, which reduces the Proca non-minimal operator to two simple and minimal ones. By starting from this identity we compute the heat kernel coefficients for the Proca theory, which also identify the one-loop divergences of the effective action.

An alternative method to evaluate the one-loop effective action concerns first quantization approaches together with the computation of path integrals. The one discussed and applied in this thesis is based on the construction of a worldline representation of the operator of interest. It consists in the study of a first quantized particle whose Hamiltonian correctly reproduces the operator for which we want to identify the heat kernel coefficients. The latter are indeed computed by evaluating the one-loop effective action in terms of a worldline path integral. This method presents a problem with non-minimal operators when the perturbative approach is used to evaluate the path integral. A solutions consists in reformulating the QFT Proca action by reinstating the gauge symmetry broken by the mass term.

We structure the thesis as follows. In chapter 1 we introduce the heat kernel as a topic with a wide range of applications, we focus on its applications to quantum field theory and its relation to the one-loop effective action. The heat kernel expansion is thus illustrated for manifolds without boundaries and the related Seeley-DeWitt coefficients are defined. The general formulae for the first three coefficients are written in terms of geometric invariants of the background metric, the gauge connection and the scalar potential.

In the second chapter we describe the action for the Proca massive vector field and identify the associated kinetic operator. Since the latter belongs to the class of the so-called “non-minimal” operators, some manipulations are in order. For this purpose we present the reduction method suggested by Barvinsky and Vilkovisky in [3] and apply it to the case at hand. Finally, we compute the first three heat kernel coefficients: a_0, a_1, a_2.

Chapter 3 is devoted to the study of an alternative method to compute the Seeley-DeWitt coefficients for the Proca theory. This new procedure is based on the connection between the heat kernel and path integrals, which is explained with a simple example at the beginning of the chapter. We then proceed with the construction of a worldline model, whose Hamiltonian is able to reproduce the kinetic operator found in chapter 2. Nevertheless, we encounter a problem in the computation of the path integral, characterized by a non-perturbative vertex which is rather difficult to treat.

In chapter 4 we reformulate the Proca action in QFT with the introduction of a gauge symmetry via a Stückelberg scalar field, whose usefulness emerged also in the worldline treatment of [4] with the study of an $N = 2$ spinning particle for the worldline path integral representation of massive antisymmetric tensor fields. The gauge fixing is achieved by means of two scalar ghosts, so that the total action is given by the sum of three actions, i.e. the ones for the vector field, the Stückelberg field and the ghost fields, which only contain minimal operators. In this way, the application of the path integral method, together with a corresponding worldline model, is possible for each operator. The perturbative approach is employed to compute the path integrals and the correct coefficients are rederived.

Chapter 1

Heat kernel

In this chapter, we present the heat kernel approach in quantum field theory. After some historical background, the heat kernel is introduced as the solution of the heat equation. We then describe our conventions on covariant derivatives and define the geometric invariants needed for the computation of the heat kernel coefficients. Later in the chapter, the relation between the heat kernel and the effective action is made explicit for an Euclidean path integral. The most popular technique for the coefficients' calculation is ascribed to B. DeWitt, but in this thesis we follow the simpler one proposed by Gilkey [2]. Finally, the coefficients are reported for the general case of a second order differential operator of Laplace type.

1.1 A brief introduction

The heat kernel is a powerful tool in mathematical physics, which has a large variety of applications. Not only it is a classical subject in mathematics, but it can also be used to study one-loop divergences, anomalies and asymptotics of the effective action. Moreover, the heat kernel is employed to study the index theorem of Atiyah and Singer, for calculations of the vacuum polarization and the Casimir effect. One of the most important applications in physics is Fock's [5], who introduced the heat kernel to quantum theory. He noted that one can write Green's functions as integrals over an auxiliary coordinate, the proper time, of a kernel satisfying the heat equation. A further study in that direction came from Schwinger [6], who applied this representation of Green's functions to the analysis of renormalization and gauge invariance. Other remarkable works using the heat kernel are DeWitt's [1, 7, 8, 9], on his manifestly covariant approach to quantum field theory and quantum gravity. Some more recent applications regard string theory and other connected areas, as for example the study of logarithmic corrections to black holes' entropy [10].

One can formally introduce the heat kernel as the matrix element of the evolution operator between position eigenstates in the following way

$$K(t;x,y;\hat{F}) = \langle x|\exp{(-t\hat{F})}|y\rangle\,, \tag{1.1}$$

where $\hat{F}$ is a second order partial differential operator and t is an auxiliary coordinate called "proper time". It can be seen as the solution of the heat conduction equation

$$\left(\frac{\partial}{\partial t} + \hat{F}\right)K(t;x,y;\hat{F}) = 0, \tag{1.2}$$

which is related to the Schrödinger equation by means of a Wick rotation of the time variable, with initial conditions

$$K(0;x,y;\hat{F}) = \delta^d(x-y). \tag{1.3}$$

In the case of a simple second order partial differential operator of Laplace type with a mass term

$$\hat{F}_0 = -\partial^2 + m^2, \tag{1.4}$$

the solution of the heat equation takes the form

$$K(t;x,y;\hat{F}_0) = \frac{1}{(4\pi t)^{d/2}}\exp\left(-\frac{(x-y)^2}{4t} - tm^2\right). \tag{1.5}$$

Our interest will be directed towards a more general form of a differential operator which contains also an arbitrary smooth potential term, i.e.

$$\hat{F} = \hat{F}_0 + V(\hat{x}). \tag{1.6}$$

In the above case it is common to expand perturbatively the heat kernel in powers of t:

$$K(t;x,y;\hat{F}) = K(t;x,y;\hat{F}_0)(c_0(x,y) + c_1(x,y)t + c_2(x,y)t^2 + \cdots). \tag{1.7}$$

The coefficients $c_k(x,y)$ are the so-called heat kernel or Seeley-DeWitt coefficients. In the following chapters we will study their value at coinciding points $c_k(x,x)$. The heat kernel coefficients are given in terms of a few geometric invariants constructed out of the background fields of the space-time. This is the main advantage of the heat kernel approach, since a single calculation can be suitable for different applications and/or because calculations in some particular cases give information on the general structure of the heat kernel, which may be used then when dealing with more complicated geometries.

1.2 Local invariants and covariant derivatives

As we stated previously, the Seeley-DeWitt coefficients are related to geometric invariants constructed out of the background fields of the space-time. In order to introduce such invariants, some remarks about differential geometry and covariant derivatives are needed.

First, we introduce our space-time as a smooth d-dimensional Riemannian manifold $\mathcal{M}$ without boundary equipped with a space-time dependent metric $g_{\mu\nu}(x)$, where x^μ are the space-time coordinates. Consider a vector bundle V over $\mathcal{M}$, i.e. a vector space is attached to each point of the manifold. The sections of the vector bundle V are called fields, that are locally represented by a set of smooth functions, which carry a discrete index related to internal or spin degrees of freedom.

For the aim of this thesis, we will focus on second order differential operators of Laplace type. For this kind of operators, the heat kernel expansion is rather simple and well known. In this respect we will follow the precise and complete report on the heat kernel by Vassilevich [11].

Second order Laplace type operators can be expressed in the following way

$$\hat{F} = -(g^{\mu\nu}\nabla_\mu\nabla_\nu + V). \tag{1.8}$$

Here V is a matrix valued potential and $g^{\mu\nu}$ the inverse of the metric tensor. Furthermore, ∇_μ is the covariant derivative which contains both the Riemannian part and the "gauge" part, related respectively to the Christoffel connection $\Gamma^\lambda_{\mu\nu}$ and to the gauge connection ω_μ. Thus, we may write the covariant derivative in the following manner:

$$\nabla_\mu = \nabla^R_\mu + \omega_\mu. \tag{1.9}$$

The action of ∇^R_μ on a generic vector field is

$$\nabla^R_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda_{\mu\nu} V_\lambda, \tag{1.10}$$

where the Christoffel connection is defined as

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}), \tag{1.11}$$

where with the comma we denote the action of the ordinary derivative $g_{\mu\nu,\sigma} \equiv \partial_\sigma g_{\mu\nu}$.

The "gauge" part ω_μ is a matrix valued gauge field, such that the covariant derivative acting on tensors produces new tensors. It can be seen as a connection, similarly to the Christoffel one, since it acts as a sort of parallel transport. This is the part of the covariant derivative acting on the internal indices, also said "color" indices, via a gauge field.

We can now write the action of the full covariant derivative on a generic vector V_ν as

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda_{\mu\nu} V_\lambda + \omega_\mu V_\nu \tag{1.12}$$

where V_ν has hidden color indices on which the matrix ω_μ acts.

On top of the metric, one uses very often the concept of a "vielbein", a local orthonormal frame with basis vectors $\vec{e}_i = e_i^\mu \partial_\mu$, defined in all tangent spaces to the manifold $\mathcal{M}$. Then, one finds that the vielbein and the metric are related by

$$e_i^\mu e_j^\nu g_{\mu\nu} = \eta_{ij} \tag{1.13}$$

and

$$e_\mu^i e_\nu^j \eta_{ij} = g_{\mu\nu}. \tag{1.14}$$

The components e_i^μ and e_ν^j are therefore used to transform "curved" indices in "flat" ones and vice-versa. Moreover the vielbein indices i, j are raised and lowered by the Minkowski metric η_{ij}, while the space-time indices μ, ν are raised and lowered by means of the space-time metric $g_{\mu\nu}$. Taking advantage of the vielbein basis, we can extend the definition of covariant derivative when applied to vectors with a flat index, by means of a "spin" connection:

$$\nabla_\mu v^i = \partial_\mu v^i + \sigma_\mu^{ij} v_j. \tag{1.15}$$

The vielbein is required to be covariantly constant, so that the condition $\nabla_\mu e_\nu^i = 0$ yields the expression for the spin connection:

$$\sigma_\mu^{ij} = e_j^\nu \Gamma_{\mu\nu}^\rho e_\rho^i - e_j^\nu \partial_\mu e_\nu^i. \tag{1.16}$$

For the construction of the local invariants that appear in the heat kernel coefficients we need to introduce a few objects. Let the Riemann curvature tensor be defined as

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\sigma\nu} - \partial_\sigma \Gamma^\mu_{\rho\nu} + \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\rho\nu}, \tag{1.17}$$

the Ricci tensor as $R_{\mu\nu} \equiv R^\rho{}_{\mu\rho\nu}$ and the Ricci scalar as $R \equiv R^\mu_\mu$. We also define the field strength tensor $\Omega_{\mu\nu}$ of the connection ω as

$$\Omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + \omega_\mu \omega_\nu - \omega_\nu \omega_\mu. \tag{1.18}$$

The only invariants that we will encounter in the Seeley-DeWitt coefficients are R, R^2, $\nabla^2 R$, $R^2_{\mu\nu}$, $R^2_{\mu\nu\rho\sigma}$ and $\mathrm{tr}(\Omega_{\mu\nu}\Omega^{\mu\nu})$, on top of the potential V and its covariant derivatives. Notice that in the last term, the trace is taken over the internal indices.

1.3 Relation with the effective action

Let us consider the generating functional of correlation functions of the field ϕ in the Euclidean path integral representation

$$Z[J] = \int \mathcal{D}\phi \; e^{-S[\phi,J]}, \tag{1.19}$$

where the action functional can be written as an expansion up to quadratic order in the quantum fluctuations of the field ϕ, since this happens to be enough for our purposes, in the following way

$$S[\phi, J] = S_{cl} + \langle \phi, J \rangle + \langle \phi, \hat{F}\phi \rangle + \cdots . \tag{1.20}$$

Here with S_{cl} we denote the classical action on a classical background, $\hat{F}$ is our differential operator and $\langle ., . \rangle$ represents an inner product on the space of quantum fields. This inner product is usually just an integral over the d-dimensional space-time

$$\langle \phi_1, \phi_2 \rangle = \int_{\mathcal{M}} d^d x \, \sqrt{g} \, \phi_1(x)\phi_2(x), \tag{1.21}$$

with $g = |\det g_{\mu\nu}|$. The integral (1.19) can be approximated perturbatively by a Gaussian integral as follows

$$Z[J] = e^{-S_{cl}} \det{}^{-1/2}(\hat{F}) \exp\left(\frac{1}{4} J \hat{F}^{-1} J\right). \tag{1.22}$$

It is important to stress that in general the classical background field, which gives the action S_{cl}, and the quantum field fluctuations can be different. A common example is the one of quantum scalar fields on a classical gravitational background.

At this stage we can relate the heat kernel, introduced in the previous section, to the effective action. Let's start from the definition of the heat kernel given in equation (1.1), i.e.

$$K(t; x, y; \hat{F}) = \langle x | \exp(-t\hat{F}) | y \rangle .$$

Then, the propagator $\hat{F}^{-1}$ can be defined through the heat kernel by the integral representation

$$\hat{F}^{-1}(x, y) = \int_0^\infty dt \, K(t; x, y; \hat{F}), \tag{1.23}$$

if we assume that the heat kernel vanishes sufficiently fast as $t \to \infty$.

The generating functional of correlation functions in the case of null sources can be related to the effective action Γ by

$$Z[0] = e^{-\Gamma}, \tag{1.24}$$

so that, by comparing with equation (1.22), it is easy to see that

$$\Gamma = S_{cl} + \frac{1}{2} \ln \mathrm{Det}(\hat{F}) \tag{1.25}$$

and one can write the one-loop approximation of the effective action by means of the functional determinant of the operator $\hat{F}$

$$\Gamma_{1-\mathrm{loop}} = \frac{1}{2} \ln \mathrm{Det}(\hat{F}). \tag{1.26}$$

It represents the quantum effects due to the background fields at one-loop.

Let us consider the positive definite eigenvalues λ and λ_0 associated to the operators $\hat{F}$ and $\hat{F}_0$ respectively. A commonly used identity is the following

$$\ln\frac{\lambda}{\lambda_0} = -\int_0^\infty \frac{dt}{t}(e^{-t\lambda} - e^{-t\lambda_0}). \tag{1.27}$$

This is known as proper time representation of the logarithm, originally introduced by Schwinger [6], and provides the starting point for the heat kernel method.

The above identity can be extended to the operator $\hat{F}$, by removing an additional infinite constant and using the identity $\ln \mathrm{Det}(\hat{F}) = \mathrm{Tr}\ln(\hat{F})$, in order to connect the one-loop effective action with the heat kernel as written below:

$$\Gamma_{1-\text{loop}} = -\frac{1}{2}\int_0^\infty \frac{dt}{t}K(t,\hat{F}), \tag{1.28}$$

where

$$K(t,\hat{F}) = \mathrm{Tr}(e^{-t\hat{F}}) = \int d^dx\sqrt{g}K(t;x,x;\hat{F}). \tag{1.29}$$

This means that the computation of the Seeley-DeWitt coefficients allows us to study the one-loop effective action.

1.4 The heat kernel coefficients

The evolution operator $\exp(-t\hat{F})$, with $t > 0$, admits a trace on the space of square-integrable functions $L^2(V)$, that for an arbitrary smooth function $f(x)$ is

$$K(t,f,\hat{F}) = \mathrm{Tr}_{L^2}(f\exp(-t\hat{F})). \tag{1.30}$$

The above expression represents the functional trace that we would like to expand. Here the quantity $K(t,f,\hat{F})$ is related to the heat kernel at coinciding points limit $(y \to x)$ by

$$K(t,f,\hat{F}) = \int_{\mathcal{M}} d^dx\sqrt{g}\,\mathrm{tr}[K(t;x,x;\hat{F})f(x)], \tag{1.31}$$

where the trace is over the internal (bundle) indices.

It is possible, as showed by Gilkey [12], to identify a complete orthonormal set of eigenvectors $\{\phi_\lambda\} \in L^2(V)$ of the operator $\hat{F}$, which correspond to smooth sections of the vector bundle V, with associated eigenvalues λ. Then the heat kernel can be written in the following fashion

$$K(t;x,y;\hat{F}) = \sum_\lambda \phi_\lambda^\dagger(x)\phi_\lambda(y)e^{-t\lambda}. \tag{1.32}$$

It is possible to write the right hand side of equation (1.30) as an asymptotic expansion for $t \to 0$:

$$\mathrm{Tr}_{L^2}(f \exp(-t\hat{F})) \approx \sum_{k\geq 0} t^{k-(d/2)} a_k(f, \hat{F}). \tag{1.33}$$

The coefficients a_k are related to the coefficients c_k of equation (1.7) in the following way

$$a_k(f, \hat{F}) = \frac{1}{(4\pi)^{d/2}} \int_{\mathcal{M}} d^d x \sqrt{g} c_k(x, x) f(x). \tag{1.34}$$

This expansion that contains only integer powers of t is valid only on manifolds without boundaries. On manifolds with boundaries also half-integer powers of t appear in the expansion, which introduce extra coefficients.

In this thesis, for the calculation of the Seeley-DeWitt coefficients, we will follow the procedure developed by Gilkey [2] as illustrated by Vassilevich in his report [11], stressing only the main steps that are useful for our purposes.

As we previously mentioned, the coefficients which enter the expansion (1.33) can be computed in terms of local invariants, constructed from R_{ijkl}, Ω_{ij}, V and their derivatives (we use here flat indices by employing the vielbein basis described previously). Denoting with $\mathcal{A}_k^I(\hat{F})$ these invariants, we have

$$a_k(f, \hat{F}) = \mathrm{tr} \int_{\mathcal{M}} d^d x \sqrt{g} [f(x) a_k(x; \hat{F})] = \sum_J \mathrm{tr} \int_{\mathcal{M}} d^d x \sqrt{g} [f(x) b^J \mathcal{A}_k^I(\hat{F})], \tag{1.35}$$

with b^J some constants.

Below we present the explicit expressions for the first three heat kernel coefficients, that will be a central topic of this work:

$$\begin{aligned}
a_0(f, \hat{F}) &= (4\pi)^{-d/2} \int_{\mathcal{M}} d^d x \sqrt{g}\, \mathrm{tr}[f(x) I] \\
a_1(f, \hat{F}) &= (4\pi)^{-d/2} \int_{\mathcal{M}} d^d x \sqrt{g}\, \mathrm{tr}\left[f(x) \left(\frac{RI}{6} + V \right) \right] \\
a_2(f, \hat{F}) &= (4\pi)^{-d/2} \int_{\mathcal{M}} d^d x \sqrt{g}\, \mathrm{tr}\left\{ f(x) \left[\frac{1}{6} \left(\frac{RI}{5} + V \right)_{;kk} + \frac{1}{2} \left(\frac{RI}{6} + V \right)^2 \right.\right. \\
&\quad \left.\left. + \frac{1}{180} \left(R_{ijkl}^2 - R_{ij}^2 \right) I + \frac{1}{12} \Omega_{ij}^2 \right] \right\}.
\end{aligned} \tag{1.36}$$

In the latter we explicitly inserted the identity matrix I, to clarify that the trace has to be taken over the internal indices.

Chapter 2

The Proca theory: a particular case of non-minimal operator

In what follows, we want to find the Seeley-DeWitt coefficients for the Proca field. We start by introducing the Proca action and the corresponding equations of motion and we proceed by identifying the kinetic operator for the massive vector field. The latter operator is of non-minimal kind and the Gilkey procedure for the computation of the heat kernel coefficients is not directly applicable. A method to reduce the calculation of the effective action for a non-minimal operator to the ones for minimal operators was developed by Barvinsky and Vilkovisky [3]. In the literature some papers treat the case of a general non-minimal operator, with a focus on the Proca case as well [13, 14, 15]. We therefore apply the method introduced in [3] to the present case. The heat kernel coefficients can be then computed by evaluating the one-loop effective action of two simpler minimal operators.

2.1 The Proca field

Consider a massive spin one field $A_\mu(x)$ with mass m on a curved d-dimensional spacetime $\mathcal{M}$ endowed with a metric $g_{\mu\nu}(x)$. The evolution of this field is described by the so-called Proca equations, which read

$$\nabla^\mu \mathcal{F}_{\mu\nu} - m^2 A_\nu = 0, \tag{2.1}$$

which imply as a consequence the constraint

$$\nabla^\mu A_\mu = 0. \tag{2.2}$$

Here $\mathcal{F}_{\mu\nu}$ is the field strength tensor, antisymmetric in $\mu \leftrightarrow \nu$ defined as

$$\mathcal{F}_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu, \tag{2.3}$$

where ∇_μ is the covariant derivative which action on an arbitrary vector field V_ν is given by

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda_{\mu\nu} V_\lambda \tag{2.4}$$

with $\Gamma^\lambda_{\mu\nu}$ being the Christoffel connection defined in (1.11). Notice that in equation (2.3) the connections drop out and one could have used usual derivatives as well.

The Proca equations can be derived from an action, which takes the form

$$S_P[A_\mu] = \int_{\mathcal{M}} d^d x \sqrt{g} \mathcal{L}_P, \tag{2.5}$$

where $\mathcal{L}_P$ is the Lagrangian density for the Proca field which reads

$$\mathcal{L}_P = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu. \tag{2.6}$$

Our convention for the signature of the metric is $(-+++)$ and $g = |\det g_{\mu\nu}|$.

Taking the functional derivative $\delta S_P[A_\mu]/\delta A_\nu = 0$ one obtains equation (2.1), while equation (2.2) comes from making use of the "Bianchi-like" identity for the field strength tensor $\nabla_\mu \nabla_\nu \mathcal{F}^{\mu\nu} = 0$.

In $d = 4$ the field A_μ contains four components, but a massive spin 1 field has only three degrees of freedom. In fact, equation (2.2), sometimes referred to as "transversality condition", acts as a contraint that removes the unphysical component, thus A_μ correctly describes the three polarizations of a massive spin 1 field.

The path integral for the massive vector field, in the absence of sources, is then given by

$$Z[0] = \int DA\ e^{iS_P[A_\mu]} = \int DA\ e^{-i\int_{\mathcal{M}} d^d x \sqrt{g}(\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu)}. \tag{2.7}$$

Let us perform a Wick rotation on the path integral, in order to obtain the Euclidean path integral that has been used in the definitions of the first chapter. By making an analytic continuation of the time variable as $t' \to -it$, one finds that the action written in "Minkowskian" time t', turns into the "Euclidean" action S_E in the "Euclidean" time t. Therefore, the path integral, after the Wick rotation, takes the form

$$Z[0] = \int DA\ e^{-S_{P,E}[A_\mu]}. \tag{2.8}$$

The Proca action in Euclidean time is explicitly given by

$$S_P[A_\mu] = \int_{\mathcal{M}} d^d x \sqrt{g} \left(\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu \right) \tag{2.9}$$

For simplicity we renamed the new action as $S_{P,E} \equiv S_P$.

The kinetic operator, whose inverse represents the propagator of the theory, can be formally obtained from the action by taking functional derivatives

$$F^{\mu}{}_{\nu}(\nabla)\delta(x,y) = \frac{1}{\sqrt{g}}\frac{\delta^2 S[A]}{\delta A_{\mu}(x)\delta A^{\nu}(y)}. \tag{2.10}$$

However, it is simpler to extract the operator from the action by casting the latter in the following form

$$S_P[A_\mu] = \int_{\mathcal{M}} d^dx\sqrt{g}\frac{1}{2}A_\mu F^{\mu}{}_{\nu}A^{\nu}, \tag{2.11}$$

By expanding $\mathcal{F}_{\mu\nu}$ as in equation (2.3), and by using the commutation relation between covariant derivatives acting on a vector field, i.e.

$$[\nabla_\mu, \nabla_\nu]A^\rho = R_{\mu\nu}{}^{\rho}{}_{\sigma}A^\sigma, \tag{2.12}$$

it is possible to rewrite the Proca action in the following manner

$$S_P[A_\mu] = \int_{\mathcal{M}} d^dx\sqrt{g}\frac{1}{2}A_\mu\left(-\Box\delta^\mu_\nu + \nabla^\mu\nabla_\nu + R^{\mu}{}_{\nu} + m^2\delta^\mu_\nu\right)A^\nu, \tag{2.13}$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$.

From the action written in this form, we can see that the operator

$$F^{\mu}{}_{\nu} = -\Box\delta^\mu_\nu + \nabla^\mu\nabla_\nu + R^{\mu}{}_{\nu} + m^2\delta^\mu_\nu \tag{2.14}$$

is not of Laplace type (i.e. “minimal”), like the one written in equation (1.8), therefore the Gilkey procedure for the computation of the heat kernel coefficients is not valid for the Proca differential operator. Such operator is instead said to be “non-minimal”, which means that its principal part has a non trivial matrix structure, i.e. is not simply given by the Laplacian. A necessary generalization for the calculation of the Seeley-DeWitt coefficients in the presence of non-minimal operators was suggested by Barvinsky and Vilkovisky [3], which is what we will employ for our computations.

2.2 The reduction of the Proca non-minimal operator

In what follows we describe a method to reduce the study of a non-minimal operator to the one of simpler minimal operators. Here we go through the main steps of this procedure, but if the reader is interested in deepening the subject and in the technical details we suggest the consultation of [3].

Consider the action $S[\phi]$ for an arbitrary set of fields $\phi^A(x)$. The propagation of small perturbations $\delta\phi^A(x) \equiv \varphi^A(x)$ is described by the equation

$$F_{AB}(\nabla)|_{\phi=\phi_0}\varphi^B(x) = 0 \tag{2.15}$$

with ϕ_0 a stationary configuration of the field and F_{AB} the differential operator found by taking left and right functional derivatives of the action as follows

$$F_{AB}(\nabla)\delta(x,y) = \frac{\delta_l}{\delta\phi^A(x)}\frac{\delta_r}{\delta\phi^B(y)}S[\phi]. \tag{2.16}$$

We are interested in the effective action for the operator $F_{AB}(\nabla)$, defined as

$$\Gamma[\phi] = \frac{1}{2}\ln \mathrm{Det}\, F_{AB}(\nabla). \tag{2.17}$$

Suppose to take the operator $F_{AB}(\nabla)$ to be of even order $2k$ and let's split the leading derivative term from the lower derivative terms as follows:

$$F_{AB}(\nabla) = D_{AB}(\nabla) + \Pi_{AB}(\nabla), \tag{2.18}$$

where $D_{AB}(\nabla)$ represents the principal part, i.e. the leading derivative term, which takes the form

$$D_{AB}(\nabla) = D_{AB}^{\mu_1\ldots\mu_{2k}}\nabla_{\mu_1}\ldots\nabla_{\mu_{2k}}. \tag{2.19}$$

∇_μ is the covariant derivative acting on the full set of fields $\varphi^A(x)$ defined with respect to any connection and $\Pi_{AB}(\nabla)$ is a differential operator of order $2k-1$. The covariant derivative satisfies the commutation relation

$$[\nabla_\mu, \nabla_\nu]\varphi^A = R^A_{B\mu\nu}\varphi^B, \tag{2.20}$$

which defines the quantity $R^A_{B\mu\nu}$. The latter becomes the Riemann tensor when the commutator acts on a quantity carrying only space-time indices.

Let's define the so-called principal symbol of the principal part of the operator $F_{AB}(\nabla)$:

$$D_{AB}(n) = D_{AB}^{\mu_1\ldots\mu_{2k}}n_{\mu_1}\ldots n_{\mu_{2k}}, \tag{2.21}$$

formally obtained by replacing the covariant derivatives in (2.19) with an arbitrary vector n_μ. We shall assume that the operator $D_{AB}(\nabla)$ is non-degenerate in the sense that

$$\det D_{AB}(n) \neq 0. \tag{2.22}$$

We call the full operator (2.18) minimal if its principal part takes the following form:

$$\begin{aligned} D_{AB}(n) &= C_{AB}(g^{\mu\nu}n_\mu n_\nu)^k \\ D_{AB}(\nabla) &= C_{AB}\Box^k, \end{aligned} \tag{2.23}$$

where the operator $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$, $g^{\mu\nu}$ is the inverse of the space-time metric and C_{AB} is a matrix independent of n_μ with nonzero determinant.

In order to generalize the Schwinger-DeWitt technique to non-minimal operators we shall further assume that the operator (2.18) can be included in a one-parameter family

$$F_{AB}(\nabla|\lambda) = D_{AB}(\nabla|\lambda) + \Pi_{AB}(\nabla|\lambda), \tag{2.24}$$

with $0 \leq \lambda < \lambda_0$, such that the operator is minimal at $\lambda = 0$. Thus we have:

$$D_{AB}(\nabla|0) = \gamma_{AB}\Box^k. \tag{2.25}$$

The matrix γ_{AB} serves as configuration space metric in the sense that

$$\gamma^{-1AC}F_{CB}(\nabla|\lambda) = F^A_B(\nabla|\lambda) \equiv \hat{F}(\nabla|\lambda) \tag{2.26}$$

and similarly for the other operators.

The key point in obtaining the effective action for a non-minimal operator is inverting the matrix $\hat{D}(n)$. Because of (2.22), the inverse can always be found as

$$\hat{D}^{-1}(n) = \frac{\hat{K}(n)}{(n^2)^m} \quad n^2 \equiv g^{\mu\nu}n_\mu n_\nu, \tag{2.27}$$

where the matrix $\hat{K}(n)$ is a polynomial in n_μ of power $(2m - 2k)$. The latter equation can be cast as

$$\hat{D}(n)\hat{K}(n) = (n^2)^m\hat{1}. \tag{2.28}$$

The relation found by replacing here the n_μ's with ∇_μ, is valid only for the terms of highest order derivative, whereas it doesn't hold for the lower derivative terms, since the covariant derivatives do not commute with each other and with the background fields contained in $\hat{K}(n)$.

Let's call $\hat{K}(\nabla)$ the quantity obtained by replacing n_μ in $\hat{K}(n)$ with ∇_μ. Equation (2.27) will then become

$$\hat{D}(\nabla)\hat{K}(\nabla) = \Box^m + \hat{K}_1(\nabla), \tag{2.29}$$

where $\hat{K}_1(\nabla)$ is a differential operator of order $(2m-1)$. A similar relation can be found for the full operator $\hat{F}$:

$$\hat{F}(\nabla)\hat{K}(\nabla) = \Box^m + \hat{M}(\nabla), \tag{2.30}$$

with $\hat{M}(\nabla) = \hat{K}_1(\nabla) + \hat{\Pi}(\nabla)\hat{K}(\nabla)$.

We shall now find relation (2.30) for the Proca operator in (2.14). We start by noticing that the highest derivative term of our operator is degenerate, because the longitudinal component of the field A_μ enters the Lagrangian algebraically. In this case one should not only include the highest derivative terms in the definition of the principal symbol

$\hat{D}(n)$ but also the lower derivative term. We therefore define the principal symbol in the following manner

$$\hat{D}(n) = -n^2\delta^\mu_\nu + n^\mu n_\nu + m^2\delta^\mu_\nu. \tag{2.31}$$

By means of the matrix determinant lemma

$$\det(A + uv^T) = (1 + v^T A^{-1} u) \det A, \tag{2.32}$$

for an invertible square matrix A and u,v column vectors, one gets the determinant of the principal symbol

$$\det \hat{D}(n) = -m^2(n^2 - m^2)^3. \tag{2.33}$$

It is possible to compute the inverse of $\hat{D}(n)$, which provides the following identity

$$\hat{D}(n)\hat{K}(n) = (m^2 - n^2)\delta^\mu_\nu, \tag{2.34}$$

where $\hat{K}(n)$ is given by

$$\hat{K}(n) = \delta^\mu_\nu - \frac{n^\mu n_\nu}{m^2}. \tag{2.35}$$

We can finally write the relation (2.30) for the Proca operator:

$$\hat{F}(\nabla)\hat{K}(\nabla) = (m^2 - \Box)\delta^\mu_\nu + R^\mu{}_\nu, \tag{2.36}$$

where $\hat{F}(\nabla)$ is the Proca kinetic operator (2.14) and $\hat{K}(\nabla)$ is formally obtained by replacing the n^μ's in $\hat{K}(n)$ with the covariant derivatives, i.e.

$$\hat{K}(\nabla) = \delta^\mu_\nu - \frac{\nabla^\mu \nabla_\nu}{m^2}. \tag{2.37}$$

Lastly, (2.36) can be cast in the following fashion

$$F^\mu{}_\alpha \left(\delta^\alpha_\nu - \frac{\nabla^\alpha \nabla_\nu}{m^2} \right) = -\Box\delta^\mu_\nu + R^\mu{}_\nu + m^2\delta^\mu_\nu. \tag{2.38}$$

The latter equation represents a powerful operator identity, in that it relates the non-minimal Proca kinetic operator $F^\mu{}_\nu$ in the left hand side, to the minimal and simpler operator in the right hand side, whose leading derivative is simply the Laplacian. This is indeed what we were looking for in order to compute the heat kernel coefficients for the massive vector field, that identify the one-loop divergences of the effective action.

Before doing that we still need to manipulate this relation in order to transform the second term in the left hand side in a minimal operator. Let us perform the functional trace of the logarithm of relation (2.38) as follows

$$\mathrm{Tr}_1 \ln F^\mu{}_\nu(\nabla) = \mathrm{Tr}_1 \ln(-\Box\delta^\mu_\nu + R^\mu{}_\nu + m^2\delta^\mu_\nu) - \mathrm{Tr}_1 \ln\left(\delta^\mu_\nu - \frac{\nabla^\mu \nabla_\nu}{m^2} \right). \tag{2.39}$$

Here the subscript 1 indicates that the functional trace is performed over the functional space of vector fields.

Utilizing the cyclic property of the trace, we can transform the second term of the right hand side such that a minimal operator appears, i.e.

$$\mathrm{Tr}_1(\nabla^\mu\nabla_\nu)^n = \mathrm{Tr}_1(\nabla^\mu\Box^{n-1}\nabla_\nu) = \mathrm{Tr}_0(\Box_{\mathrm{scalar}})^n, \tag{2.40}$$

where the subscript 0 indicates the functional space of the scalar fields. Let us use the following property for the logarithm of a matrix A

$$\ln[A] = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(A-I)^n}{n} \tag{2.41}$$

and let us apply it to the present case as follows

$$\ln\left[\delta^\mu_\nu - \frac{\nabla^\mu\nabla_\nu}{m^2}\right] = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(-\frac{\nabla^\mu\nabla_\nu}{m^2}\right)^n. \tag{2.42}$$

By inserting the functional trace in the latter relation, using equation (2.40), we get

$$\begin{aligned}\mathrm{Tr}_1\ln\left[\delta^\mu_\nu - \frac{\nabla^\mu\nabla_\nu}{m^2}\right] &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\,\mathrm{Tr}_1\left[\left(-\frac{\nabla^\mu\nabla_\nu}{m^2}\right)^n\right]\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\,\mathrm{Tr}_0\left[\left(-\frac{\Box_{\mathrm{scalar}}}{m^2}\right)^n\right] = \mathrm{Tr}_0\ln\left[1-\frac{\Box_{\mathrm{scalar}}}{m^2}\right].\end{aligned} \tag{2.43}$$

Equation (2.39) can be finally written in the following way

$$\mathrm{Tr}_1\ln F^\mu{}_\nu(\nabla) = \mathrm{Tr}_1\ln(-\Box\delta^\mu_\nu + R^\mu{}_\nu + m^2\delta^\mu_\nu) - \mathrm{Tr}_0\ln(-\Box_{\mathrm{scalar}} + m^2) + \delta(0)(\dots).^1 \tag{2.44}$$

This is a rather important result, since it allows us to compute the effective action for the Proca field, which is a massive vector field, as the effective action for the four-component vector field minus the one for the one-component scalar field, the latter being equivalent to the non-dynamical longitudinal mode. As we anticipated the Proca field in $d = 4$ is characterized by three propagating degrees of freedom, which correspond to the four components of the vector field where one is unphysical and gets removed by the transversality condition (2.2). This is indeed verified by equation (2.44), where in arbitrary dimension the number of degrees of freedom of the massive vector field is computed by subtracting the one of the scalar field to the ones of the generic vector field, as follows

$$N^{\mathrm{Proca}}_{\mathrm{d.o.f.}} = d-1 \xrightarrow{d=4} 4-1=3, \tag{2.45}$$

which correctly reproduce the three dynamical components of the Proca field.

[1] The term proportional to $\delta(0)$ corresponds to the trace of a constant in the functional space and it is usually neglected in QFT (it can be easily eliminated via a renormalization).

2.3 Proca heat kernel coefficients

We would now like to compute the heat kernel coefficients for the Proca operator, by applying the methods introduced in the above sections. The problem has completely been reduced to the study of minimal operators, therefore it is possible to straightforwardly apply the procedure developed by Gilkey, described in the first chapter.

On manifolds without boundaries one can write the one-loop effective action as an expansion in powers of the proper time t:

$$\Gamma_{1-\mathrm{loop}} = -\frac{1}{2}\int_0^\infty \frac{dt}{t} t^{-d/2}(a_0 + a_1 t + a_2 t^2 + \mathcal{O}(t^3)), \tag{2.46}$$

where the coefficients a_0, a_1, a_2 identify the heat kernel coefficients (1.36) for the case $f(x) = 1$ we wish to compute.

The identity (2.44) shows how the Proca Seeley-DeWitt coefficients are given by two contributions as follows

$$a_k(\hat{F}) = a_k(\hat{F}_\mathrm{v}) - a_k(\hat{F}_\mathrm{s}), \tag{2.47}$$

where $a_k(\hat{F}_\mathrm{v})$ and $a_k(\hat{F}_\mathrm{s})$ represent the coefficients for the vector and scalar operators respectively, which appear in the right hand side of equation (2.44).

Let us start with the vector field operator

$$\hat{F}_\mathrm{v} = -\Box\delta^\mu_\nu + R^\mu{}_\nu + m^2\delta^\mu_\nu. \tag{2.48}$$

For the computation of the coefficients (1.36) we start by identifying the quantities I, V and Ω_{ij} that appear in their formulae.

The identity I, in this context, corresponds to the Kronecker delta, δ^μ_ν.

The potential term V can be simply deduced from (2.48) together with (1.8):

$$V = -R^\mu{}_\nu - m^2\delta^\mu_\nu \tag{2.49}$$

The field strength tensor Ω_{ij} can be formally defined by means of the following relation for the commutator of covariant derivatives, when acting on a scalar field carrying color indices:

$$[\nabla_\mu, \nabla_\nu]\phi = \Omega_{\mu\nu}\phi. \tag{2.50}$$

Since in the present case we are dealing with uncharged vector fields, the field strength tensor simply corresponds to the Riemann tensor $R_{\mu\nu}{}^\rho{}_\sigma$, from (2.12). This can be thought of as a set of $d \times d$ matrices labelled by μ and ν.

After simple algebra the Seeley-DeWitt coefficients for the vector field operator $\hat{F}_\mathrm{v}$

read

$$\begin{aligned}
a_0(\hat{F}_{\rm v}) &= (4\pi)^{-d/2} \int_{\mathcal{M}} d^dx\sqrt{g}\ d \\
a_1(\hat{F}_{\rm v}) &= (4\pi)^{-d/2} \int_{\mathcal{M}} d^dx\sqrt{g}\ \left[\frac{d-6}{6}R - dm^2\right] \\
a_2(\hat{F}_{\rm v}) &= (4\pi)^{-d/2} \int_{\mathcal{M}} d^dx\sqrt{g}\ \left[\frac{d-15}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{d-90}{180}R_{\mu\nu}R^{\mu\nu} + \frac{d-12}{72}R^2 \right.\\
&\qquad \left. - \frac{d-6}{6}m^2R + \frac{d-5}{30}\nabla^2R + \frac{d}{2}m^4\right].
\end{aligned} \tag{2.51}$$

We proceed in a similar fashion for the scalar operator

$$\hat{F}_{\rm s} = -\Box_{\rm scalar} + m^2, \tag{2.52}$$

where now we identify $I = 1$, $V = -m^2$ and $\Omega_{\mu\nu} = 0$, from $[\partial_\mu, \partial_\nu]\phi = 0$ for an uncharged scalar field ϕ. The corresponding coefficients are thus given by

$$\begin{aligned}
a_0(\hat{F}_{\rm s}) &= (4\pi)^{-d/2} \int_{\mathcal{M}} d^dx\sqrt{g} \\
a_1(\hat{F}_{\rm s}) &= (4\pi)^{-d/2} \int_{\mathcal{M}} d^dx\sqrt{g}\ \left[\frac{1}{6}R - m^2\right] \\
a_2(\hat{F}_{\rm s}) &= (4\pi)^{-d/2} \int_{\mathcal{M}} d^dx\sqrt{g}\ \left[\frac{1}{180}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu}) + \frac{1}{30}\nabla^2R \right.\\
&\qquad \left. + \frac{1}{2}\left(\frac{1}{36}R^2 + m^4 - \frac{1}{3}Rm^2\right)\right].
\end{aligned} \tag{2.53}$$

Finally, we can write the heat kernel coefficients for the Proca operator in arbitrary dimensions by means of equation (2.47):

$$\begin{aligned}
a_0(\hat{F}) &= (4\pi)^{-d/2} \int_{\mathcal{M}} d^dx\sqrt{g}(d-1) \\
a_1(\hat{F}) &= (4\pi)^{-d/2} \int_{\mathcal{M}} d^dx\sqrt{g}\ \left[\frac{d-7}{6}R - (d-1)m^2\right] \\
a_2(\hat{F}) &= (4\pi)^{-d/2} \int_{\mathcal{M}} d^dx\sqrt{g}\ \left[\frac{d-16}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{d-91}{180}R_{\mu\nu}R^{\mu\nu} + \frac{d-13}{72}R^2 \right.\\
&\qquad \left. - \frac{d-7}{6}m^2R + \frac{d-6}{30}\nabla^2R + \frac{d-1}{2}m^4\right].
\end{aligned} \tag{2.54}$$

The one-loop effective action for the Proca field finally reads

$$\begin{aligned}\Gamma_{1-\text{loop}} = -\frac{1}{2}\int_0^\infty \frac{dt}{t}\int_{\mathcal{M}} \frac{d^dx\sqrt{g}}{(4\pi t)^{d/2}}\Bigg\{&(d-1) + \left[\frac{d-7}{6}R - (d-1)m^2\right]t \\ &+ \left(\frac{d-16}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{d-91}{180}R_{\mu\nu}R^{\mu\nu} + \frac{d-13}{72}R^2\right. \\ &\left. - \frac{d-7}{6}m^2R + \frac{d-6}{30}\nabla^2 R + \frac{d-1}{2}m^4\right)t^2 + \mathcal{O}(t^3)\Bigg\},\end{aligned} \tag{2.55}$$

which for $d=4$ reduces to the following

$$\begin{aligned}\Gamma_{1-\text{loop}} = -\frac{1}{2}\int_0^\infty \frac{dt}{t}\int_{\mathcal{M}} \frac{d^4x\sqrt{g}}{(4\pi t)^{2}}\Bigg\{&3 - \left(\frac{1}{2}R + 3m^2\right)t + \left(-\frac{1}{15}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right. \\ &\left.+ \frac{29}{60}R_{\mu\nu}R^{\mu\nu} - \frac{1}{8}R^2 + \frac{1}{2}m^2R - \frac{1}{15}\nabla^2 R + \frac{3}{2}m^4\right)t^2 + \mathcal{O}(t^3)\Bigg\}.\end{aligned} \tag{2.56}$$

One can notice that one-loop effective action can also be written without inserting the mass term inside the potential V and keeping it outside of the expansion as an exponential factor, i.e.

$$\begin{aligned}\Gamma_{1-\text{loop}} = -\frac{1}{2}\int_0^\infty \frac{dt}{t}\int_{\mathcal{M}} \frac{d^dx\sqrt{g}}{(4\pi t)^{d/2}}e^{-tm^2}\Bigg\{&(d-1) + \left(\frac{d-7}{6}R\right)t \\ &+ \left(\frac{d-16}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{d-91}{180}R_{\mu\nu}R^{\mu\nu} + \frac{d-13}{72}R^2\right. \\ &\left. + \frac{d-6}{30}\nabla^2 R\right)t^2 + \mathcal{O}(t^3)\Bigg\}.\end{aligned} \tag{2.57}$$

This allows to highlight the fact that the mass term acts as a cut-off for infrared divergences, making the integral over the proper time convergent at the upper limit.

The result in (2.56) has some peculiar features we wish to discuss.

The logarithmic divergent part of the one-loop effective action in $d=4$, identified by the a_2 coefficient in (2.56), is in agreement with [3, 13].

The coefficient a_0 in (2.56) represents the propagating degrees of freedom for the massive vector field in $d=4$, which as we expected is correctly 3. Indeed, we computed the effective action for the Proca field as the one of a four-component vector field minus one scalar mode. One may think that by taking the limit $m \to 0$, the one-loop effective action for the Abelian gauge field is recovered, reproducing the two expected degrees of freedom. However the coefficient a_0 remains unaffected. This means that it is not possible to obtain the effective action for the massless vector field from the massive one by simply taking the limit $m \to 0$. As showed in [13], the Abelian gauge field, being degenerate due to the gauge symmetry, must be gauge fixed. This introduces a complex anti-commuting scalar ghost operator, whose trace has to be subtracted twice because of the presence of the anti-ghost. This reduces the number of propagating degrees of freedom exactly to 2.

The coefficients found for generic dimension d, present in (2.55), agree with those calculated in Bastianelli, Benincasa, Giombi's [4]. In this paper a $N = 2$ massive spinning particle model allows to study massive antisymmetric tensor fields of rank p (massive p-forms) in first quantization, which correspond to a generalization of the Proca field. Indeed, the Seeley-DeWitt coefficients for the 1-form in this paper, coincide with the ones we found in (2.55).

Chapter 3

The path integral method and worldline formalism

In the previous chapters we have introduced the heat kernel expansion for the computation of the first three Seeley-DeWitt coefficients for the Proca (massive vector) field that enter the one-loop effective action. The results were achieved by means of the standard heat kernel approach and of the reduction method for non-minimal operators developed by Barvinsky and Vilkovisky. In this chapter, we want to propose an alternative approach to address the problem, which makes use of path integrals as well as a worldline model instead. In the first section, we present the path integral method able to compute the heat kernel coefficients for the simple case of a particle subject to a smooth scalar potential $V(x)$. In this way, we show how by employing path integrals and by means of a perturbative expansion of the interactive part of the action, one is in principle able to rederive the heat kernel coefficients. For the case of interest to this thesis work, however, we first need to build a worldline model. The latter is based on the study of a first quantized particle model whose Hamiltonian acts on a Hilbert space that, with proper constraints, contains only the vector field. The above mentioned Hamiltonian, when quantized with a well defined ordering prescription, provides a worldline representation of the differential kinetic operator of the Proca field. The classical action which derives from this Hamiltonian, is then inserted in a path integral which in turn should reproduce the transition amplitude, related to the heat kernel and its trace, when quantized on the circle. The worldline model so described has been employed in [16] for the computation of the one-loop divergences of the gauge fixed graviton in 4 dimensions.

3.1 The path integral method

In the first chapter we have introduced the heat kernel as the solution of the heat equation (1.2), which can be obtained from the Schrödinger equation

$$i\frac{\partial}{\partial t'}\psi = \hat{H}\psi \tag{3.1}$$

by analytic continuation of the time variable $t' \to -it$, i.e. by performing a Wick rotation, as follows

$$-\frac{\partial}{\partial t}\psi = \hat{H}\psi. \tag{3.2}$$

Indeed, we can identify the fundamental solution ψ as the heat kernel $K(t; x, y; \hat{H})$.

Let us consider the simple case of a particle subject to a smooth scalar potential $V(x)$, whose Hamiltonian is given by

$$\hat{H} = -\frac{1}{2m}\partial^2 + V(x) \tag{3.3}$$

As anticipated, the heat kernel can be defined as the following transition amplitude between position eigenstates

$$K(t; x, y; \hat{H}) = \langle x| \exp{(-t\hat{H})}|y\rangle\,, \tag{3.4}$$

which under the boundary conditions

$$K(0; x, y; \hat{H}) = \delta^d(x - y), \tag{3.5}$$

satisfies equation (3.2). The solution in the null potential case can be explicitly found and takes the simple form

$$K(t; x, y; \hat{H}_0) = \frac{m}{(2\pi t)^{d/2}} \exp\left(-\frac{m(x-y)^2}{2t}\right), \tag{3.6}$$

where $\hat{H}_0 = -\frac{1}{2m}\partial^2$ is the free Hamiltonian.

At this stage, it is possible to show that the transition amplitude (3.4) can be computed by means of the Euclidean path integral written below

$$K(t; x, y; \hat{H}) = \int_{x(0)=x}^{x(\beta)=y} Dx\ e^{-S[x]}. \tag{3.7}$$

where the measure represents the sum over all paths $x^\mu(t)$ between initial $x^\mu(0) = x^\mu$ and final $x^\mu(\beta) = y^\mu$ points. The Euclidean action $S[x]$ given by

$$S[x] = \int_0^\beta dt\left(\frac{m}{2}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + V(x)\right) \tag{3.8}$$

is the one associated to the Hamiltonian introduced above. It is simple to check how the path integral so constructed, correctly reproduces the transition amplitude for the free particle ($V(x) = 0$). In the case of an arbitrary non vanishing potential, instead, the path integral cannot be solved exactly, but it can be evaluated perturbatively for small propagation times β. The procedure we present in what follows provides an alternative approach to the calculation of the heat kernel in the presence of an arbitrary potential.

Let us first rescale the time variable as $t = \beta\tau$, in such a way that β becomes the parameter which will be used to control the order of the perturbative expansion for small values of β. Therefore, the action becomes

$$S[x] = \frac{1}{\beta}\int_0^1 d\tau \left(\frac{m}{2}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \beta^2 V(x)\right) \tag{3.9}$$

One can split the particle's trajectory into a classical path $x^\mu_{bg}(\tau)$, representing the background, and a quantum fluctuation $q^\mu(\tau)$, i.e.

$$x^\mu(\tau) = x^\mu_{bg}(\tau) + q^\mu(\tau). \tag{3.10}$$

The classical part given by

$$x^\mu_{bg}(\tau) = x^\mu + \xi^\mu\tau \tag{3.11}$$

satisfies the classical equation of motion $\ddot{x}^\mu(\tau) = 0$ under boundary conditions $x^\mu_{bg}(0) = x^\mu$, $x^\mu_{bg}(1) = y^\mu$. Here $\xi^\mu = (y^\mu - x^\mu)$ indicates the displacement between the initial and the final positions. The quantum fluctuations instead have vanishing boundary conditions $q^\mu(0) = q^\mu(1) = 0$. Under this splitting the action can be divided into a free part and an interaction part as

$$S[x] = S_0[x] + S_{\rm int}[x] \tag{3.12}$$

where

$$S_0[x] = \frac{m}{\beta}\int_0^1 d\tau \frac{1}{2}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \tag{3.13}$$

$$S_{\rm int}[x] = \beta\int_0^1 d\tau V(x(\tau)). \tag{3.14}$$

We define the average of an arbitrary functional $F[q]$ by means of the free path integral, i.e.

$$\langle F[q]\rangle = \frac{1}{A}\int Dq\ F[q]\, e^{-S_0[q]} \tag{3.15}$$

where A is the path integral normalization given by

$$A = \int Dq\ e^{-S_0[q]} = \left(\frac{m}{2\pi\beta}\right)^{d/2} \tag{3.16}$$

In a similar fashion the generic N-point correlation function can be written by using (3.15), i.e.

$$\langle q^{\mu_1}(\tau_1)\dots q^{\mu_N}(\tau_N)\rangle = \frac{1}{A}\int Dq\; q^{\mu_1}(\tau_1)\dots q^{\mu_N}(\tau_N)e^{-S_0[q]} \tag{3.17}$$

where the ones below are of particular interest for our computations

$$\begin{aligned}\langle q^{\mu}(\tau)\rangle &= \frac{1}{A}\int Dq\; q^{\mu}(\tau)e^{-S_0[q]} = 0\\ \langle q^{\mu}(\tau)q^{\nu}(\sigma)\rangle &= \frac{1}{A}\int Dq\; q^{\mu}(\tau)q^{\nu}(\sigma)e^{-S_0[q]} = \frac{\beta}{m}\eta^{\mu\nu}\Delta(\tau,\sigma)\end{aligned} \tag{3.18}$$

The 1-point function is null as all the odd-point functions, while the 2-point function represents the propagator of the free theory, written by means of the Green function $\Delta(\tau,\sigma)$ that satisfies the equation

$$\frac{\partial^2}{\partial\tau^2}\Delta(\tau,\sigma) = \delta(\tau-\sigma) \tag{3.19}$$

and its explicit expression is given by

$$\Delta(\tau,\sigma) = (\tau-1)\sigma\Theta(\tau-\sigma) + (\sigma-1)\tau\Theta(\sigma-\tau). \tag{3.20}$$

Here $\Theta(\tau-\sigma)$ is the Heaviside step function which assumes the values $\Theta(\tau-\sigma)=1$ for $\tau>\sigma$, $\Theta(\tau-\sigma)=1/2$ for $\tau=\sigma$ and $\Theta(\tau-\sigma)=0$ for $\tau<\sigma$. The quantities $\Delta(\tau,\sigma)$ and $\Theta(\tau-\sigma)$ are distributions acting on the space of functions with vanishing Dirichlet boundary conditions in the time interval $I=[0,1]$.

By employing the splitting (3.12) and the translational invariance of the path integral measure ($Dx = D(x_{bg}+q) = Dq$), it is possible to cast the path integral in the following fashion

$$\begin{aligned}\int_{x(0)=x}^{x(1)=y} Dx e^{-S[x]} &= e^{-S_0[x_{bg}]}\int_{q(0)=0}^{q(1)=0} Dq e^{-S_{\text{int}}[x_{bg}+q]}e^{-S_0[q]} = Ae^{-S_0[x_{bg}]}\left\langle e^{-S_{\text{int}}[x_{bg}+q]}\right\rangle\\ &= \left(\frac{m}{2\pi\beta}\right)^{d/2} e^{-\frac{m(x-y)^2}{2\beta}}\left\langle 1 - S_{\text{int}}[x_{bg}+q] + \frac{1}{2}S_{\text{int}}^2[x_{bg}+q] + \cdots\right\rangle.\end{aligned} \tag{3.21}$$

In the latter we exploited the definition (3.15) for the average of an arbitrary functional, we substituted the path integral normalization (3.16) and we have Taylor expanded the exponential of the interacting part. Higher powers of S_{int} in the expansion will provide terms of order β^3 which become negligible for small values of β.

Let us perform a Taylor expansion of the potential $V(x)$ around the point x^μ

$$V(x_{bg}+q) = V(x) + (\xi^\mu \tau + q^\mu(\tau))\partial_\mu V(x) + \frac{1}{2}(\xi^\mu \tau + q^\mu(\tau)(\xi^\nu \tau + q^\nu(\tau))\partial_\mu \partial_\nu V(x) + \cdots \tag{3.22}$$

that we can plug in S_{int}. The average $\langle 1 \rangle = 1$ is trivial while the second term in the expansion reads

$$\begin{aligned} \langle -S_{\text{int}}[x_{bg}+q] \rangle = -\beta V(x) - \frac{\beta}{2}\xi^\mu \partial_\mu V(x) - \frac{\beta}{6}\xi^\mu \xi^\nu \partial_\mu \partial_\nu V(x) \\ - \frac{\beta}{2}\partial_\mu \partial_\nu V(x) \int_0^1 d\tau \langle q^\mu(\tau) q^\nu(\tau) \rangle + \cdots \end{aligned} \tag{3.23}$$

By using the propagator in (3.18) we solve the integral as

$$\int_0^1 d\tau \langle q^\mu(\tau) q^\nu(\tau) \rangle = -\frac{\beta}{m}\eta^{\mu\nu} \int_0^1 d\tau \Delta(\tau,\tau) = -\frac{\beta}{m}\eta^{\mu\nu} \int_0^1 d\tau \tau(\tau-1) = \frac{\beta}{6m}\eta^{\mu\nu} \tag{3.24}$$

to get

$$\begin{aligned} \langle -S_{\text{int}}[x_{bg}+q] \rangle = -\beta V(x) - \frac{\beta}{2}\xi^\mu \partial_\mu V(x) - \frac{\beta}{6}\xi^\mu \xi^\nu \partial_\mu \partial_\nu V(x) \\ - \frac{\beta^2}{12m}\partial^2 V(x) + \cdots . \end{aligned} \tag{3.25}$$

We can operate similarly for the quadratic term in the expansion to get, at lowest order in β,

$$\left\langle \frac{1}{2} S_{\text{int}}^2[x_{bg}+q] \right\rangle = \frac{\beta^2}{2} V^2(x) + \cdots . \tag{3.26}$$

By inserting everything back into (3.21), we can finally write the expression for the heat kernel as follows

$$\begin{aligned} K(x,y;\beta) = \left(\frac{m}{2\pi\beta}\right)^{d/2} e^{-\frac{m(x-y)^2}{2\beta}} \bigg[1 - \beta V(x) - \frac{\beta}{2}\xi^\mu \partial_\mu V(x) \\ - \frac{\beta}{6}\xi^\mu \xi^\nu \partial_\mu \partial_\nu V(x) - \frac{\beta^2}{12m}\partial^2 V(x) + \frac{\beta^2}{2} V^2(x) \bigg], \end{aligned} \tag{3.27}$$

from which one can identify the heat kernel coefficients c_0, c_1 and c_2, which at coinciding points limit (i.e. for $\xi^\mu = 0$) read

$$\begin{aligned} c_0(x,x) &= 1 \\ c_1(x,x) &= -V(x) \\ c_2(x,x) &= \frac{1}{2}V^2(x) - \frac{1}{12m}\partial^2 V(x). \end{aligned} \tag{3.28}$$

Therefore, the use of path integrals provides an alternative method to evaluate the coefficients of the heat kernel expansion.

3.2 The worldline vector model

In the previous chapter we were able to compute the Seeley-DeWitt coefficients for the Proca field by employing the reduction method for non-minimal operators originally introduced by Barvinsky and Vilkovisky. This allowed to reduce the problem of a non-minimal operator to the study of minimal operators, for which the Gilkey procedure for the computation of the heat kernel coefficients is straightforwardly applicable. The next sections are devoted to the study of a worldline model for the Proca vector field that, together with the path integral method discussed above, should be able to compute the heat kernel coefficients without necessarily manipulating the non-minimal operator.

The basic idea of our model is to provide a worldline representation of the kinetic operator (2.14) used in the standard heat kernel approach. For simplicity, we first reduce ourselves to the case of a flat d-dimensional space-time, with metric $\eta_{\mu\nu}$. We expect that the only terms surviving in the coefficients (2.54) are the ones which do not involve the Riemann curvature tensor, the Ricci tensor and the Ricci scalar. The reduction to the simpler flat space-time case is useful to verify the first coefficient a_0, which provides the number of propagating degrees of freedom of the Proca field. Under this condition, the differential operator (2.14) becomes

$$F^{\mu}{}_{\nu} = -\partial^2 \delta^{\mu}_{\nu} + \partial^{\mu}\partial_{\nu} + m^2 \delta^{\mu}_{\nu}, \tag{3.29}$$

as covariant derivatives became usual derivative when using Cartesian coordinates and flat metric $\eta_{\mu\nu}$. Clearly, the curvature term is null.

By using the flat metric it is possible to write the differential operator as follows

$$F_{\mu\nu} = \eta_{\mu\lambda} F^{\lambda}{}_{\nu} = (-\partial^2 + m^2)\eta_{\mu\nu} + \partial_{\mu}\partial_{\nu}. \tag{3.30}$$

Its action on an arbitrary vector field $V^{\mu}(x)$ reads

$$(\hat{F}V)_{\mu} = F_{\mu\nu}V^{\nu} = (-\partial^2 + m^2)V_{\mu} + \partial_{\mu}\partial_{\nu}V^{\nu}. \tag{3.31}$$

The coordinates and related conjugate momenta of the flat d-dimensional target space, with metric $\eta_{\mu\nu}$, are the usual real phase space variables $x^{\mu}(t)$ and $p_{\mu}(t)$. Let us also introduce additional fermionic variables given by the worldline complex fermions $\bar{\lambda}^{\mu}(x)$ and their conjugate momenta $\lambda_{\mu}(x)$. The real bosonic and complex fermionic variables define a graded phase space. We proceed with the canonical quantization by promoting these variables to operators which satisfy (anti)commutation relations, i.e.

$$[\hat{x}^{\mu}, \hat{p}_{\nu}] = i\delta^{\mu}_{\nu} \tag{3.32}$$

$$\{\hat{\lambda}_{\mu}, \hat{\lambda}^{\dagger\nu}\} = \delta^{\nu}_{\mu} \tag{3.33}$$

where $\hbar = 1$.

We call $|\Psi\rangle$ the generic state of the Hilbert space defined by the phase space variables. The bosonic phase space variables x^μ are the eigenvalues of the operator $\hat{x}^\mu$ when acting on position eigenstates $|x\rangle$ with eigenvalue equation $\hat{x}^\mu |x\rangle = x^\mu |x\rangle$. It acts multiplicatively on wave functions of the form $\Psi(x) = \langle x|\Psi\rangle$, while its conjugate momenta $\hat{p}_\mu$ act by means of the derivative: $\hat{p}_\mu = i\partial_\mu$.

Similarly we can introduce "bra" coherent states $\langle\bar{\lambda}|$, eigenstates of the operator $\hat{\lambda}^{\dagger\mu}$, with eigenvalue equation $\langle\bar{\lambda}|\,\hat{\lambda}^{\dagger\mu} = \langle\bar{\lambda}|\,\bar{\lambda}^\mu$, where $\bar{\lambda}$ is an anticommuting number with Grassmann parity.[1] As before, when acting on wave functions of the form $\Psi(\bar{\lambda}) = \langle\bar{\lambda}|\Psi\rangle$, the operator $\hat{\lambda}^{\dagger\mu}$ acts multiplicatively as $\hat{\lambda}^{\dagger\mu} \sim \bar{\lambda}^\mu$, while for its conjugate momenta we have $\hat{\lambda}_\mu \sim \frac{\partial}{\partial\bar{\lambda}^\mu}$, so that their anticommutator realizes the algebra in (3.33)

$$\left\{ \frac{\partial}{\partial\bar{\lambda}^\mu}, \bar{\lambda}^\nu \right\} = \delta_\mu^\nu. \tag{3.34}$$

By projecting the generic state of the Hilbert space $|\Psi\rangle$ on the position eigenstates together with the coherent states we obtain the wave function

$$\begin{aligned} |\Psi\rangle \sim \Psi(x,\bar{\lambda}) &= (\langle x| \otimes \langle\bar{\lambda}|)\,|\Psi\rangle \\ &= \Psi(x) + \Psi_\mu(x)\bar{\lambda}^\mu + \frac{1}{2}\Psi_{\mu\nu}(x)\bar{\lambda}^\mu\bar{\lambda}^\nu + ... + \frac{1}{d!}\Psi_{\mu_1...\mu_d}(x)\bar{\lambda}^{\mu_1}...\bar{\lambda}^{\mu_d}, \end{aligned} \tag{3.35}$$

which has been Taylor expanded by using the Grassmannian property of the λ's. As one can see, the real bosonic variables provide the functional dependence on the space-time points, while the complex fermionic ones are used to introduce a discrete index $\mu = 0 \dots d-1$.

The above wave function contains different antisymmetric tensor fields among which we need to select the wanted vector field $\Psi_\mu(x)$. This is done by means of a constraint acting on the wave function in the following way:

$$\hat{C}\,|\Psi\rangle = 0 \tag{3.36}$$

where the form of $\hat{C}$ will be discussed later on.

Let us assume that we have successfully reduced the wave function to the subsector containing only the vector field, for which the state of the Hilbert space is given by $|V\rangle \sim V_\mu(x)\bar{\lambda}^\mu$. How to do so is better described later in this section.

We want to find the Hamiltonian that acting on the wave function provides the action of the differential operator (3.30) on the vector field as showed below

$$\hat{H}(V_\mu(x)\bar{\lambda}^\mu) = (\hat{F}V_\mu(x))\bar{\lambda}^\mu \tag{3.37}$$

For this purpose we define the abstract operator $\hat{\partial}_\mu = i\hat{p}_\mu$ so that when acting on wave functions we have

$$\hat{\partial}_\mu \to \partial_\mu, \quad \hat{\partial}^2 \to \partial^2. \tag{3.38}$$

[1] For a digression on coherent states see Appendix A.

The action of the Laplacian operator on the wave function will be then given by

$$\hat{\partial}^2(V_\mu(x)\bar{\lambda}^\mu) = (\partial^2 V_\mu(x))\bar{\lambda}^\mu. \tag{3.39}$$

For the mass term we trivially have

$$m^2(V_\mu(x)\bar{\lambda}^\mu) = (m^2 V_\mu(x))\bar{\lambda}^\mu. \tag{3.40}$$

In order to reproduce the action of the last term in equation (3.30) we need to make the following construction:

$$\begin{aligned}\hat{\partial}_\mu\hat{\partial}_\nu\bar{\lambda}^\mu\lambda^\nu(V_\sigma(x)\bar{\lambda}^\sigma) &= \partial_\mu\partial_\nu V_\sigma\bar{\lambda}^\mu\lambda^\nu\bar{\lambda}^\sigma \\ &= \partial_\mu\partial_\nu V_\sigma\bar{\lambda}^\mu\delta^{\nu\sigma} = (\partial_\mu\partial_\nu V^\nu)\bar{\lambda}^\mu\ .\end{aligned} \tag{3.41}$$

By collecting all the terms, we can finally write the wanted Hamiltonian operator of the model as follows:

$$\hat{H} = -\hat{\partial}^2 + m^2 + \hat{\partial}_\mu\hat{\partial}_\nu\hat{\lambda}^{\dagger\mu}\hat{\lambda}^\nu = \hat{p}_\mu\hat{p}_\nu\eta^{\mu\nu} + m^2 - \hat{p}_\mu\hat{p}_\nu\hat{\lambda}^{\dagger\mu}\hat{\lambda}^\nu. \tag{3.42}$$

This is the correct quantum Hamiltonian that reproduces the Proca differential operator (3.29) acting on the vector wave function that satisfies the constraint (3.36).

Now, we have to write the classical particle action related to this quantum Hamiltonian $\hat{H}$, which will then be inserted in the path integral. The classical action will contain a classical Hamiltonian H, that upon quantization must give rise to the quantum version $\hat{H}$. In the procedure of canonical quantization, however, one has to deal with ordering ambiguities when moving from the classical Hamiltonian to the quantum one, which arise from the non vanishing (anti)commutation relations (3.32) and (3.33) of the bosonic and fermionic operators. The latter commute in the classical theory, therefore a single classical Hamiltonian would give rise to many different quantum extensions with the same classical limit. A common method to take care of such ambiguities relies on requesting the preservation of the symmetries present in the classical theory also at the quantum level. For this purpose a specific ordering prescription is needed. We will focus on the path integral quantization, where the equivalent of the ordering prescription is contained in the regularization adopted in properly defining the path integral, which may also include a counterterm to be added to the classical action.

From (3.42), one may expect that the classical Hamiltonian is of the form

$$H = p^2 + m^2 - p_\mu p_\nu\bar{\lambda}^\mu\lambda^\nu + \alpha p^2, \tag{3.43}$$

with α parametrizing a possible conterterm[2]. Indeed, one can choose the Weyl ordering

[2]The classical term $p_\mu p_\nu\bar{\lambda}^\mu\lambda^\nu$ can be written in many ways, e.g. $p_\mu p_\nu\bar{\lambda}^\mu\lambda^\nu = -p_\mu p_\nu\lambda^\nu\bar{\lambda}^\mu$ and equivalent classical versions lead to different quantum operators that differ by a term proportional to p^2.

prescription in which the canonical variables appear symmetrized (bosonic) or antisymmetrized (fermionic), i.e.

$$
\begin{aligned}
\hat{p}_\mu \hat{p}_\nu \hat{\lambda}^{\dagger\mu} \hat{\lambda}^\nu &= \hat{p}_\mu \hat{p}_\nu \left(\frac{1}{2} \hat{\lambda}^{\dagger\mu} \hat{\lambda}^\nu + \frac{1}{2} \{ \hat{\lambda}^{\dagger\mu}, \hat{\lambda}^\nu \} - \frac{1}{2} \hat{\lambda}^\nu \hat{\lambda}^{\dagger\mu} \right) \\
&= \hat{p}_\mu \hat{p}_\nu \frac{1}{2} (\hat{\lambda}^{\dagger\mu} \hat{\lambda}^\nu - \hat{\lambda}^\nu \hat{\lambda}^{\dagger\mu}) + \frac{1}{2} p^2.
\end{aligned}
\tag{3.44}
$$

This will introduce in the classical action a term of the form

$$
p_\mu p_\nu \frac{1}{2} (\bar{\lambda}^\mu \lambda^\nu) + \frac{1}{2} p^2, \tag{3.45}
$$

which corresponds to the choice $\alpha = 1/2$ in (3.43). At this point, we can write the classical action in phase space which in Euclidean time takes the form

$$
S[x, p; \bar{\lambda}, \lambda] = \int_0^\beta dt (-i p_\mu \dot{x}^\mu + \bar{\lambda}^\mu \dot{\lambda}_\mu + H), \tag{3.46}
$$

where the dot denotes the derivative with respect to time. However, one also needs to implement the reduction to a subsector of the Hilbert space corresponding to the constraint in (3.36). To achieve this, we need to add a coupling between the fermionic fields and an auxiliary worldline gauge field a, with an extra Chern-Simons coupling fixed in order to select the subsector of the Hilbert space that contains only the vector field. We briefly explain this procedure below.

The kinetic term of the fermionic variables in the above action reads

$$
S[\bar{\lambda}, \lambda]_{\text{free}} = \int_0^\beta dt \bar{\lambda}^\mu \dot{\lambda}_\mu \tag{3.47}
$$

and enjoys a $U(1)$ global symmetry under the transformations

$$
\begin{aligned}
\bar{\lambda}^\mu &\xrightarrow{U(1)_g} \bar{\lambda}'^\mu = e^{i\phi} \bar{\lambda}^\mu \\
\lambda_\mu &\xrightarrow{U(1)_g} \lambda'_\mu = e^{-i\phi} \lambda_\mu
\end{aligned}
\tag{3.48}
$$

where ϕ is the constant angle of the transformation. However, the action is not invariant under the local $U(1)$ transformation

$$
\begin{aligned}
\bar{\lambda}^\mu &\xrightarrow{U(1)_l} \bar{\lambda}'^\mu = e^{i\phi(t)} \bar{\lambda}^\mu \\
\lambda_\mu &\xrightarrow{U(1)_l} \lambda'_\mu = e^{-i\phi(t)} \lambda_\mu
\end{aligned}
\tag{3.49}
$$

in which we sent $\phi \to \phi(t)$. In order to do so, we introduce an auxiliary worldline gauge field $a(t)$ whose transformation under the local $U(1)$ symmetry group is given by

$$
a(t) \xrightarrow{U(1)_l} a'(t) = a(t) + \partial_t \phi(t). \tag{3.50}
$$

This auxiliary gauge field enters the action via a coupling with the fermionic variables and makes it invariant. By adding an extra Chern-Simons term $-isa$, with quantized coupling constant s, the gauge field a can be seen as a Lagrange multiplier whose equations of motion provide the constraint

$$C = \bar{\lambda}^\mu \lambda_\mu - s. \tag{3.51}$$

The latter, upon canonical quantization, gives rise to some ordering ambiguities, which may be resolved by a graded symmetrization of the quantum constraint, i.e.

$$\hat{C} = \frac{1}{2}(\hat{\lambda}^{\dagger\mu}\hat{\lambda}_\mu - \hat{\lambda}_\mu\hat{\lambda}^{\dagger\mu}) - s. \tag{3.52}$$

In this way, the quantum constraint acting on the wave function $\Psi(x, \bar{\lambda})$ gives

$$\begin{aligned}\hat{C}\Psi(x,\bar{\lambda}) &= \left[\frac{1}{2}\left(\bar{\lambda}^\mu \frac{\partial}{\partial\bar{\lambda}^\mu} - \frac{\partial}{\partial\bar{\lambda}^\mu}\bar{\lambda}^\mu\right) - s\right]\Psi(x,\bar{\lambda}) \\ &= \left(\bar{\lambda}^\mu \frac{\partial}{\partial\bar{\lambda}^\mu} - \frac{1}{2}\left\{\bar{\lambda}^\mu, \frac{\partial}{\partial\bar{\lambda}^\mu}\right\} - s\right)\Psi(x,\bar{\lambda}) = \left(\bar{\lambda}^\mu \frac{\partial}{\partial\bar{\lambda}^\mu} - \frac{d}{2} - s\right)\Psi(x,\bar{\lambda}) = 0.\end{aligned} \tag{3.53}$$

If we define the number operator $\hat{N} \equiv \bar{\lambda}^\mu \frac{\partial}{\partial\bar{\lambda}^\mu}$, that acting on wave functions gives rise to the occupation number n, we are able to select the subsector of the Hilbert space containing only vector fields, i.e. the one with occupation number $n = 1$, by choosing the Chern-Simons coupling constant to be $s = 1 - d/2$. In this way the constraint becomes

$$(\hat{N} - 1)\Psi(x, \bar{\lambda}) = 0. \tag{3.54}$$

The reader can find further explanations of the origin and the use of this projection mechanism in [16, 17, 18].

The full classical action in phase space, together with the additional gauge field, reads

$$S[x,p;\bar{\lambda},\lambda;a] = \int_0^\beta dt(-ip_\mu\dot{x}^\mu + \frac{1}{2}\eta^{\mu\nu}p_\mu p_\nu + \bar{\lambda}^\mu(\partial_t + ia)\lambda_\mu - p_\mu p_\nu\bar{\lambda}^\mu\lambda^\nu - isa + m^2) \tag{3.55}$$

3.3 Computing the path integral

Let us test if the path integral quantization of the classical action (3.55) reproduces the expected properties of the Proca model. In particular, we see that the action contains a quartic term $\sim p_\mu p_\nu \bar{\lambda}^\mu \lambda^\nu$, which we may interpret as an interaction on the worldline. The next step is to understand if we can treat this term perturbatively or not. For this purpose, as anticipated in 3.1, it is rather useful to rescale the time coordinate as follows

$$t \to \tau = \frac{t}{\beta} \tag{3.56}$$

where β is interpreted as the small parameter that will control the order of the perturbative expansion. In order to get a factor $1/\beta$ in front of the action, we also rescale the momenta, the fermions and the auxiliary gauge field as

$$p_\mu \to \frac{p_\mu}{\beta},\ \lambda_\mu \to \frac{\lambda_\mu}{\sqrt{\beta}},\ \bar{\lambda}_\mu \to \frac{\bar{\lambda}_\mu}{\sqrt{\beta}},\ a \to \frac{a}{\beta}. \tag{3.57}$$

The classical phase space action thus becomes

$$\begin{aligned} S[x,p;\bar{\lambda},\lambda;a] = &\frac{1}{\beta}\int_0^1 d\tau(-ip_\mu\dot{x}^\mu + \frac{1}{2}\eta^{\mu\nu}p_\mu p_\nu + \bar{\lambda}^\mu(\partial_t + ia)\lambda_\mu - \frac{1}{\beta}p_\mu p_\nu\bar{\lambda}^\mu\lambda^\nu) \\ &+ \frac{1}{\beta}\int_0^1 d\tau(\beta^2 m^2 - isa) \end{aligned} \tag{3.58}$$

In principle one could integrate out the momenta by means of their equations of motion. However, this would lead to an expression which is not amenable for the computations, therefore we work in phase space.

One can notice that the only gauge invariant quantity that can be constructed from the gauge field $a(\tau)$ is the Wilson loop

$$\omega = e^{i\int_0^1 d\tau a(\tau)}. \tag{3.59}$$

By means of "small" gauge transformations, continuously connected to the identity, it is possible to bring $a(\tau)$ to a constant value θ

$$\theta = \int_0^1 d\tau a(\tau). \tag{3.60}$$

Thus "large" gauge transformations with $\phi(\tau) = 2\pi n\tau$ lead to

$$\theta \sim \theta + 2\pi n, \quad n \text{ integer}. \tag{3.61}$$

Therefore θ represents a modular parameter ranging from 0 to 2π. After gauge fixing, one is left with an integral over θ corresponding to the Wilson loop $\omega = e^{i\theta}$. Since the $U(1)$ gauge group is Abelian, the Faddeev-Popov determinant is just a constant that can be factorized out and absorbed in the overall normalization.

The one-loop effective action can be written in terms of the path integral as follows

$$\Gamma = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta}\int d^dx \int_0^{2\pi}\frac{d\theta}{2\pi}e^{is\theta}e^{-\beta m^2}\int_P DxDp\int_A D\bar{\lambda}D\lambda e^{-S[x,p;\bar{\lambda},\lambda]}, \tag{3.62}$$

where the subscripts P and A indicate periodic and antiperiodic boundary conditions for bosons and fermions respectively, i.e.

$$x^\mu(0) = x^\mu(1),\ \ \bar{\lambda}^\mu(0) = -\bar{\lambda}^\mu(1) \tag{3.63}$$

and the action $S[x,p;\bar{\lambda},\lambda]$ is the one written in (3.58) where we have factorized out the constant values and the modular integral in θ. We also remind that s is the Chern-Simons coupling constant given by $s = 1 - d/2$.

We proceed by splitting the action into free part and interacting part as follows

$$S_{\text{free}} = \frac{1}{\beta}\int_0^1 d\tau(-ip_\mu \dot{x}^\mu + \frac{1}{2}\eta^{\mu\nu}p_\mu p_\nu + \bar{\lambda}^\mu(\partial_\tau + i\theta)\lambda_\mu) \tag{3.64}$$

$$S_{\text{int}} = \frac{1}{\beta}\int_0^1 d\tau(-\frac{1}{\beta}p_\mu p_\nu \bar{\lambda}^\mu \lambda^\nu). \tag{3.65}$$

At this stage, one can decompose all the bosonic trajectories into a fixed path (called classical or background) and quantum fluctuations by

$$x^\mu(\tau) = x^\mu_{bg}(\tau) + q^\mu(\tau), \tag{3.66}$$

where the background (classical) paths satisfy the classical equation of motion and are given by

$$x^\mu_{bg}(\tau) = \xi^\mu \tau \tag{3.67}$$

with boundary conditions $x^\mu_{bg}(0) = 0$ and $x^\mu_{bg}(1) = \xi^\mu$. In order to compute the transition amplitude at coinciding points $\langle x|e^{-\beta\hat{H}}|x\rangle$ we need to set $\xi^\mu = 0$. In this way the background parts vanish in our coordinate system, and we are left with the quantum fluctuations $q^\mu(\tau)$. The latter, as usual, have vanishing boundary conditions $q^\mu(0) = q^\mu(1) = 0$. Similarly we decompose the momenta

$$p_\mu(\tau) = p^{bg}_\mu(\tau) + \pi_\mu(\tau) \tag{3.68}$$

where the background is given by

$$p^{bg}_\mu(\tau) = i\eta_{\mu\nu}\xi^\nu \tag{3.69}$$

so that it vanishes due to our boundary conditions and we are left with the quantum part $\pi_\mu(\tau)$ entering the action.

The normalization of the bosonic path integral gives

$$A = \int DqD\pi e^{-S_{\text{free}}[q^\mu,\pi_\mu]} = \left(\frac{1}{2\pi\beta}\right)^{d/2}. \tag{3.70}$$

It is used to compute the correlation functions that will appear in the perturbative expansion of the interacting term, with the procedure discussed for the simple case in section 3.1. The exponential of the free action in the background is trivally given by

$$e^{-S_{\text{free}}[x^\mu_{bg},p^{bg}_\mu]} = 1. \tag{3.71}$$

In what follows we will also need the fermionic path integral normalization, which is directly computable from the free action by Gaussian integration as described in [19], i.e.

$$\begin{aligned}\int_A D\bar{\lambda}D\lambda e^{-S_{\text{free}}[\bar{\lambda}^\mu,\lambda_\mu]} &= \int_A D\bar{\lambda}D\lambda \exp\left(-\frac{1}{\beta}\int_0^1 d\tau\bar{\lambda}^\mu(\partial_\tau+i\theta)\lambda_\mu\right) \\ &= \det{}^d(\partial_\tau+i\theta).\end{aligned} \tag{3.72}$$

By using the operator formalism, the free fermionic path integral can be traced back to the computation of the functional trace of the evolution operator. Hence the latter determinant is given by

$$\det{}^d(\partial_\tau+i\theta) = \mathrm{Tr}\, e^{-\hat{H}_\theta} = \mathrm{Tr}\, e^{-i\theta(\hat{\lambda}^{\dagger\mu}\hat{\lambda}_\mu-\frac{d}{2})} = e^{i\theta\frac{d}{2}}(1+e^{-i\theta})^d = \left(2\cos\frac{\theta}{2}\right)^d, \tag{3.73}$$

where we exploited the fact that in one dimension the eigenvalues of the fermionic number operator $\hat{\lambda}^{\dagger\mu}\hat{\lambda}_\mu$ are either 0 or 1.

The one-loop effective action can be therefore written in the following fashion

$$\begin{aligned}\Gamma &= -\frac{1}{2}\int_0^\infty\frac{d\beta}{\beta}\int d^dx\int_0^{2\pi}\frac{d\theta}{2\pi}e^{is\theta}e^{-\beta m^2} \\ &\qquad \int_D DqD\pi\int_A D\bar{\lambda}D\lambda\; e^{-S_{\text{int}}[\pi_\mu,\bar{\lambda}^\mu,\lambda_\mu]}e^{-S_{\text{free}}[q^\mu,\pi_\mu,\bar{\lambda}^\mu,\lambda_\mu]} \\ &= -\frac{1}{2}\int_0^\infty\frac{d\beta}{\beta}\int\frac{d^dx}{(2\pi\beta)^{d/2}}\int_0^{2\pi}\frac{d\theta}{2\pi}e^{is\theta}e^{-\beta m^2}\left(2\cos\frac{\theta}{2}\right)^d\langle e^{-S_{int}}\rangle,\end{aligned} \tag{3.74}$$

where the subscript D stands for Dirichlet boundary conditions and $-S_{\text{int}}[\pi_\mu,\bar{\lambda}^\mu,\lambda_\mu]$ is now given by

$$-S_{\text{int}}[\pi_\mu,\bar{\lambda}^\mu,\lambda_\mu] = \frac{1}{\beta^2}\int_0^1 d\tau(\pi_\mu\pi_\nu\bar{\lambda}^\mu\lambda^\nu). \tag{3.75}$$

In (3.74) we also used the definition of the average of a generic functional given in (3.15) and the bosonic and fermionic path integral normalizations (3.70) and (3.73). The perturbative expansion is generated by Taylor expanding the euclidean exponential of the interaction part in powers of β, i.e.

$$\langle e^{-S_{\text{int}}}\rangle = \langle 1 - S_{\text{int}} + \frac{1}{2}S_{\text{int}}^2 + \ldots\rangle. \tag{3.76}$$

It is possible to extract the propagators of the theory from the free part of the action with quantum fluctuations, namely

$$S_{\text{free}}[q^\mu,\pi_\mu,\bar{\lambda}^\mu,\lambda_\mu] = \frac{1}{\beta}\int_0^1 d\tau(-i\pi_\mu\dot{q}^\mu + \frac{1}{2}\eta^{\mu\nu}\pi_\mu\pi_\nu + \bar{\lambda}^\mu(\partial_\tau+i\theta)\lambda_\mu). \tag{3.77}$$

Let us first focus on the bosonic part, i.e.

$$S_{\text{free}}[q^\mu, \pi_\mu, \bar{\lambda}^\mu, \lambda_\mu] \supset S_{\text{free}}[q^\mu, \pi_\mu] = \frac{1}{\beta}\int_0^1 d\tau(-i\pi_\mu \dot{q}^\mu + \frac{1}{2}\eta^{\mu\nu}\pi_\mu\pi_\nu). \tag{3.78}$$

The propagators are easily extracted from this action by identifying the matrix kinetic operator K acting on $\psi(\tau) = \begin{pmatrix} q^\mu(\tau) \\ p_\mu(\tau) \end{pmatrix}$ as

$$K(\tau,\sigma) = \frac{1}{\beta}\begin{pmatrix} 0 & i\delta^\nu_\mu \partial_\tau \\ -i\delta^\mu_\nu \partial_\tau & \eta^{\mu\nu} \end{pmatrix}\delta(\tau-\sigma) \tag{3.79}$$

from which one obtains its inverse

$$G(\tau,\sigma) = \beta\begin{pmatrix} -\eta^{\mu\nu}\Delta(\tau,\sigma) & i\delta^\nu_\mu \Delta^\bullet(\tau,\sigma) \\ -i\delta^\mu_\nu {}^\bullet\!\Delta(\tau,\sigma) & \eta_{\mu\nu} \end{pmatrix}\delta(\tau-\sigma). \tag{3.80}$$

This gives rise to the 2-point functions

$$\begin{aligned} \langle q^\mu(\tau) q^\nu(\sigma)\rangle &= -\beta\eta^{\mu\nu}\Delta(\tau,\sigma) \\ \langle q^\mu(\tau)\pi_\nu(\sigma)\rangle &= -i\beta\delta^\mu_\nu \Delta^\bullet(\tau,\sigma) \\ \langle \pi_\mu(\tau)\pi_\nu(\sigma)\rangle &= \beta\eta_{\mu\nu} \end{aligned} \tag{3.81}$$

where $\Delta(\tau,\sigma)$ is the same as (3.20) and the left-right dots indicate derivatives with respect to the left-right variable:

$${}^\bullet\!\Delta(\tau,\sigma) = \sigma - \Theta(\sigma-\tau), \quad \Delta^\bullet(\tau,\sigma) = \tau - \Theta(\tau-\sigma). \tag{3.82}$$

The propagator and its derivatives at equal time are

$$\Delta(\tau,\tau) = \tau(\tau-1), \quad {}^\bullet\!\Delta(\tau,\sigma)|_{\tau=\sigma} = \Delta^\bullet(\tau,\sigma)|_{\tau=\sigma} = \tau - \frac{1}{2}. \tag{3.83}$$

The next step is to compute the fermionic propagator. For this purpose it is useful to expand the fermionic variables in half-integer modes as

$$\bar{\lambda}^\mu(\tau) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \bar{\lambda}^\mu_r e^{-2\pi i r\tau}, \quad \lambda^\mu(\tau) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \lambda^\mu_r e^{2\pi i r\tau}. \tag{3.84}$$

Plugging this expansion in the free action we get

$$S_{\text{free}}[\bar{\lambda}^\mu, \lambda_\mu] = \frac{1}{\beta}\int_0^1 d\tau \bar{\lambda}^\mu(\partial_\tau + i\theta)\lambda_\mu = \frac{i}{\beta}\sum_{r\in\mathbb{Z}+\frac{1}{2}}(2\pi r + \theta)\bar{\lambda}^\mu_r \lambda_{r\mu}. \tag{3.85}$$

The propagator is given as usual by the 2-point function, computed as follows

$$\langle \lambda_\mu(\tau)\bar{\lambda}^\nu(\sigma)\rangle = \sum_{r,s\in\mathbb{Z}+\frac{1}{2}} \langle \lambda^r_\mu \bar{\lambda}^\nu_s\rangle e^{2\pi i r\tau} e^{-2\pi i s\sigma} = \beta\delta^\nu_\mu \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{-i}{2\pi r+\theta} e^{2\pi i r(\tau-\sigma)}, \tag{3.86}$$

where the last equality follows from the fact that

$$\langle \lambda^r_\mu \bar{\lambda}^\nu_s\rangle = -i\frac{\beta}{2\pi r+\theta}\delta^\nu_\mu \delta^r_s. \tag{3.87}$$

Therefore one finds the propagator for the antiperiodic fermions as

$$\langle \lambda_\mu(\tau)\bar{\lambda}^\nu(\sigma)\rangle = \beta\delta^\nu_\mu \Delta_F(\tau-\sigma,\theta), \quad \Delta_F(\tau-\sigma,\theta) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{-i}{2\pi r+\theta} e^{2\pi i r(\tau-\sigma)}, \tag{3.88}$$

where Δ_F satisfies the Green equation

$$(\partial_x + i\theta)\Delta_F(x,\theta) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} e^{2\pi i r x} = \delta_F(x). \tag{3.89}$$

In the last expression $\delta_F(x)$ is the Dirac delta distribution acting on functions on the time segment $I = [0, 1]$ with antiperiodic boundary conditions.

Under the condition $(\tau-\sigma) \in]-1, 1[$, the propagator (3.88) can be written, after computing the sum, as below

$$\Delta_F(\tau-\sigma,\theta) = \frac{e^{-i\theta(\tau-\sigma)}}{2\cos\frac{\theta}{2}}\left[e^{i\frac{\theta}{2}}\Theta(\tau-\sigma) - e^{-i\frac{\theta}{2}}\Theta(\sigma-\tau)\right]. \tag{3.90}$$

In the computations of the perturbative expansion we will need the following identities:

$$\Delta_F(0,\theta) = \frac{i}{2}\tan\frac{\theta}{2}, \tag{3.91}$$

$$\Delta_F(\tau-\sigma,\theta)\Delta_F(\sigma-\tau,\theta) = -\frac{1}{4}\cos^{-2}\frac{\theta}{2}, \tag{3.92}$$

where the first one comes from summing up (3.88) using the symmetry in the modes $+r$ and $-r$.

At this stage, it is worthwhile to keep track of the small parameter β when we perform the expansion (3.76). As one can see, every propagator carries a factor of β. Therefore when we consider $\langle S_{\text{int}}\rangle$ as in (3.75), it turns out that all the β factors cancel out and $\langle S_{\text{int}}\rangle$ will contribute to the order β^0. The same happens for all the higher orders of the expansion. For example we can already see that when we consider the order $\langle S^2_{\text{int}}\rangle$, we will get a factor of $1/\beta^4$ in front of the integral, that gets cancelled by the factor of β^4

coming from the propagators. It seems that a perturbative treatment of the path integral is not possible for this interaction vertex $\pi_\mu\pi_\nu\bar\lambda^\mu\lambda^\nu$, as a matter of fact this calculation looks rather non-perturbative. Of course it is not possible to compute an infinite amount of terms, which is made even harder by the rapidly increasing amount of diagrams that appear at every higher order of the expansion. However, one can try to compute the first orders of the expansion to see if a recognizable and known series occurs. Indeed, since we already know from (2.54) that at order β^0 one should get the first heat kernel coefficient a_0, we expect from this series to eventually get the result $d-1$, which provides the number of degrees of freedom of the massive vector field in d dimensions.

Let us proceed with the computation of $\langle -S_{\text{int}}\rangle$ by using the propagators (3.81) and (3.88)

$$\langle -S_{\text{int}}\rangle = \frac{1}{\beta^2}\int_0^1 d\tau\langle\pi_\mu\pi_\nu\bar\lambda^\mu\lambda^\nu\rangle = \frac{1}{\beta^2}\int_0^1 d\tau\langle\pi_\mu\pi_\nu\rangle\langle\bar\lambda^\mu\lambda^\nu\rangle = -d\frac{i}{2}\tan\frac{\theta}{2} \tag{3.93}$$

where the 4-point function $\langle\pi_\mu\pi_\nu\bar\lambda^\mu\lambda^\nu\rangle$ has been divided into products of 2-point functions by means of the Wick theorem, with an extra minus sign coming from the anti-commuting character of the Grassmann numbers.

In a similar way we compute $\frac{1}{2}\langle S^2_{\text{int}}\rangle$ by considering only connected diagrams and then adding the disconnected one given by 1/2 times the square of (3.93), i.e.

$$\begin{aligned}
\frac{1}{2}\langle S^2_{\text{int}}\rangle_c &= \frac{1}{2\beta^4}\int_0^1 d\tau\int_0^1 d\sigma\langle\pi_\mu(\tau)\pi_\nu(\tau)\bar\lambda^\mu(\tau)\lambda^\nu(\tau)\pi_\alpha(\sigma)\pi_\beta(\sigma)\bar\lambda^\alpha(\sigma)\lambda^\beta(\sigma)\rangle_c \\
&= \frac{1}{2\beta^4}\int_0^1 d\tau\int_0^1 d\sigma\Bigg[\Big(\langle\pi_\mu(\tau)\pi_\alpha(\sigma)\rangle\langle\pi_\nu(\tau)\pi_\beta(\sigma)\rangle + \langle\pi_\mu(\tau)\pi_\beta(\sigma)\rangle\langle\pi_\nu(\tau)\pi_\alpha(\sigma)\rangle\Big) \\
&\qquad\Big(\langle\lambda^\nu(\tau)\bar\lambda^\mu(\tau)\rangle\langle\lambda^\beta(\sigma)\bar\lambda^\alpha(\sigma)\rangle - \langle\lambda^\beta(\sigma)\bar\lambda^\mu(\tau)\rangle\langle\lambda^\nu(\tau)\bar\lambda^\alpha(\sigma)\rangle\Big) \\
&\qquad - \Big(\langle\pi_\mu(\tau)\pi_\nu(\tau)\rangle\langle\pi_\alpha(\sigma)\pi_\beta(\sigma)\rangle\Big)\Big(\langle\lambda^\beta(\sigma)\bar\lambda^\mu(\tau)\rangle\langle\lambda^\nu(\tau)\bar\lambda^\alpha(\sigma)\rangle\Big)\Bigg] \\
&= \frac{1}{2}\int_0^1 d\tau\int_0^1 d\sigma\Big(2d\Delta_F(0)\Delta_F(0) - 2d\Delta_F(\sigma-\tau)\Delta_F(\tau-\sigma) \\
&\qquad - d^2\Delta_F(\sigma-\tau)\Delta_F(\tau-\sigma)\Big) \\
&= \frac{1}{2}\int_0^1 d\tau\int_0^1 d\sigma\left(\frac{d}{2}\cos^{-2}\frac{\theta}{2} + \frac{d^2}{4}\cos^{-2}\frac{\theta}{2} - \frac{d}{2}\tan^2\frac{\theta}{2}\right) \\
&= \frac{d}{4}\left(\frac{d}{2}\cos^{-2}\frac{\theta}{2} + 1\right).
\end{aligned} \tag{3.94}$$

Here we performed as usual all the possible Wick contractions that give rise to connected diagrams only and we used the identity $\tan^2\frac{\theta}{2} = \cos^{-2}\frac{\theta}{2} - 1$. The expansion of the average of the interaction part is thus given by

$$\langle e^{-S_{\rm int}}\rangle = 1 - d\frac{i}{2}\tan\frac{\theta}{2} + \frac{d}{4} + \frac{d^2}{8}\cdots . \tag{3.95}$$

The last effort regards the computation of the modular integrals present in (3.74) [3].

$$\begin{aligned} I_1 &\equiv \int_0^{2\pi}\frac{d\theta}{2\pi}e^{is\theta}\left(2\cos\frac{\theta}{2}\right)^d = d, \\ I_2 &\equiv \int_0^{2\pi}\frac{d\theta}{2\pi}e^{is\theta}\left(2\cos\frac{\theta}{2}\right)^d\tan\frac{\theta}{2} = -i(d-2), \\ I_3 &\equiv \int_0^{2\pi}\frac{d\theta}{2\pi}e^{is\theta}\left(2\cos\frac{\theta}{2}\right)^d\cos^{-2}\frac{\theta}{2} = 4, \end{aligned} \tag{3.96}$$

where I_3 has been computed because it will be useful later on in the thesis.

By using these results one could write the first terms of the effective action, i.e.

$$\begin{aligned} \Gamma &= -\frac{1}{2}\int_0^\infty\frac{d\beta}{\beta}\int\frac{d^dx}{(2\pi\beta)^{d/2}}e^{-\beta m^2}\left[d - \frac{d(d-2)}{2} + \frac{d^2(d+2)}{8} + \cdots\right] \\ &= -\frac{1}{2}\int_0^\infty\frac{d\beta}{\beta}\int\frac{d^dx}{(2\pi\beta)^{d/2}}e^{-\beta m^2}\left[2d - \frac{d^2}{4} + \frac{d^3}{8}\cdots\right] \end{aligned} \tag{3.97}$$

At a first glimpse it seems that there is no recursive relation that could induce to guess a possible convergent series as a result of the expansion.

[3]For a complete description of the computation of the modular integrals see Appendix B.

Chapter 4

The worldline model for a new gauge fixed action

In the previous chapter, we faced the problem of a non-perturbative vertex $(\pi_\mu \pi_\nu \bar{\lambda}^\mu \lambda^\nu)$ during the computation of the path integral that should reproduce the one-loop effective action reported in equation (2.55) for the Proca vector field. This hindrance, which at first glimpse seems not easy to overcome, makes the perturbative treatment non feasible. However, a solution to the problem, or better an alternative way of computing that bothersome vertex, that will make the path integral method, together with the worldline formalism, directly applicable to the Proca differential operator (2.14) without relying on alternative procedures such as the Barvinsky-Vilkovisky reduction method, is left for future research.

Nevertheless, in the present chapter, we are going to present a different way to manipulate the Proca action and to reduce the problem to the study of only minimal operators. By following a procedure suggested in [4], we introduce a scalar field, commonly known as Stückelberg field, to reinstate the gauge symmetry enjoyed by the Maxwell Lagrangian $-1/4\ \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$, broken by the mass term. This procedure, with the introduction of anti-commuting scalar ghosts necessary for the gauge-fixing, will reduce the analysis of the Proca operator to the study of minimal operators acting on vectors and scalars.

By generalizing the scheme described in 3.2 to these operators in a curved target space-time, we construct the action whose path integral is able to reproduce the wanted one-loop effective action. The mentioned path integral is completely free of non-perturbative vertices, therefore the perturbative approach is now doable.

The quantization of a particle moving in a curved space-time requires the analysis of ordering ambiguities, arising from the process of canonical quantization, with the choice of a particular ordering prescription for the Hamiltonian operator that must reproduce the operator of interest. This also translates into the choice of a specific regularization scheme for the evaluation of the path integral, with extra finite local counterterms associated to the above mentioned ordering.

4.1 Introducing a Stückelberg field to restore gauge invariance

The Proca action in Minkowskian time

$$S_P[A_\mu] = \int_{\mathcal{M}} d^dx\sqrt{g}\Big(-\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu\Big), \tag{4.1}$$

does not enjoy the same gauge symmetry associated to its massless counterpart, i.e. the Maxwell action describing massless photons, identified by the following gauge transformation rule

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \nabla_\mu\Lambda(x), \tag{4.2}$$

where $\Lambda(x)$ indicates an arbitrary function. This is a consequence of the non vanishing mass term $\propto m^2 A_\mu A^\mu$ which is not invariant under the inhomogeneous transformation (4.2) of the field $A_\mu(x)$.

However, there exists a way, originally introduced by Stückelberg in 1938, that allows to restore gauge invariance in the Proca action by introducing an extra scalar field $\phi(x)$, in addition to the four components vector field $A_\mu(x)$, in such a way that the new action for the now five fields not only presents a manifest Lorentz covariance, but also a manifest gauge invariance. Therefore, the Stückelberg field $\phi(x)$ restores the gauge symmetry present in the Maxwell theory, which had been broken by the mass term. The Stückelberg field $\phi(x)$ can thus be introduced in the action (4.1) in the following way

$$S[A_\mu, \phi] = \int_{\mathcal{M}} d^dx\sqrt{g}\Big[-\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{m^2}{2}\Big(A_\mu - \frac{1}{m}\nabla_\mu\phi\Big)^2\Big]. \tag{4.3}$$

We can clearly see that this new action presents a gauge symmetry under the transformations rules

$$\begin{aligned} A_\mu(x) \to A'_\mu(x) &= A_\mu(x) + \nabla_\mu\Lambda(x) \\ \phi(x) \to \phi'(x) &= \phi(x) + m\Lambda(x). \end{aligned} \tag{4.4}$$

Of course the new scalar field $\phi(x)$ could be completely gauged away from the action by means of this symmetry, but we shall keep it in order to study the action with a different gauge fixing, performed with the so-called BRST gauge fixing procedure, which is more useful for our purposes. The latter is an algebraic method that allows one to find the complete gauge-fixed action entering the path integral. The BRST quantization method is widely used and can be employed to get the gauge-fixed action even for more general non-abelian gauge theories. For example, it is commonly applied to Yang-Mills theories or to quantize gravity starting from the Einstein-Hilbert action. It consists in a rather universal approach that is applicable to all the cases where the gauge algebra, related to the gauge symmetry, has constant structure functions and closes “off-shell”, i.e. without using the equations of motion.

In this thesis we will use it to remove the two unphysical degrees of freedom, now represented by the longitudinal component of $A_\mu(x)$ and the Stückelberg field $\phi(x)$, via the introduction of two scalar ghost fields. Let us start by introducing a complex scalar anticommuting ghost $c(x)$, the corresponding complex conjugate, namely the antighost $\bar{c}(x)$ and a scalar auxiliary bosonic field $B(x)$. The BRST transformation laws read

$$\begin{aligned} \delta_B A_\mu(x) &= \eta \ \nabla_\mu c(x) \\ \delta_B \phi(x) &= \eta \ m \ c(x) \\ \delta_B c(x) &= 0 \\ \delta_B \bar{c}(x) &= \eta \ B(x) \\ \delta_B B(x) &= 0 \end{aligned} \tag{4.5}$$

where η is the anticommuting parameter of the transformation. The BRST variations of $A_\mu(x)$ and $\phi(x)$ are easily obtained by replacing the arbitrary function $\Lambda(x)$ in (4.4) with $\eta c(x)$. The transformation laws (4.5) can be used to introduce in the action a gauge fermion $\mathbf{\Psi}$ of the form

$$\mathbf{\Psi} = \bar{c} \ (\nabla_\mu A^\mu - m\phi + \frac{B}{2}), \tag{4.6}$$

where $\nabla_\mu A^\mu - m\phi + \frac{B}{2}$ plays the role of a gauge-fixing function.

In fact, the most important property of the BRST symmetry is that it is nilpotent, i.e. it satisfies

$$[\delta_B(\eta_1), \delta_B(\eta_2)] = 0. \tag{4.7}$$

Equivalently, by defining the Slavnov variation as the BRST variation with the anticommuting parameter η removed from the left, that is

$$\delta_B(\eta) = \eta s, \tag{4.8}$$

the nilpotency of the BRST symmetry means that

$$s^2 = 0. \tag{4.9}$$

One can then take advantage of this property to modify the action (4.3) by adding to it the Slavnov variation of the gauge fermion, which is manifestly BRST invariant thanks precisely to the nilpotency, i.e.

$$S_{\text{tot}}[A_\mu, \phi] = S[A_\mu, \phi] + s\mathbf{\Psi} = \int_{\mathcal{M}} d^d x \sqrt{g} \left[-\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{m^2}{2}\left(A_\mu - \frac{1}{m}\nabla_\mu\phi\right)^2 + s\mathbf{\Psi}\right] \tag{4.10}$$

The new gauge-fixed action is then manifestly BRST invariant by construction. The crucial observation here is that the nilpontecy of the BRST symmetry introduces the concept of cohomology, which allows to work with equivalence classes. Indeed, physical observables are identified as cohomology classes: two BRST invariant quantities differing by the BRST variation of something are equivalent and therefore belong to the same class, identifying the same physical observable.

By using the (4.5) one can compute $s\boldsymbol{\Psi}$ that, replaced in the gauge-fixed action, gives

$$\begin{aligned} S_{\text{tot}}[A_\mu, \phi, \bar{c}, c, B] = \int_{\mathcal{M}} d^d x \sqrt{g} \Big[& -\frac{1}{2} \nabla_\mu A_\nu \nabla^\mu A^\nu + \frac{1}{2} \nabla_\mu A_\nu \nabla^\nu A^\mu - \frac{m^2}{2} \left(A_\mu - \frac{1}{m} \nabla_\mu \phi \right)^2 \\ & + B \left(\nabla_\mu A^\mu - m\phi + \frac{B}{2} \right) - \bar{c}\, (\nabla^2 - m^2) c \Big]. \end{aligned} \tag{4.11}$$

At this stage we can integrate out the auxiliary field $B(x)$ by means of its equation of motion, i.e.

$$\frac{\delta S_{\text{tot}}}{\delta B} = 0 \to B = m\phi - \nabla_\mu A^\mu. \tag{4.12}$$

The last equation, together with the commutator between covariant derivatives (2.12), can be used in (4.11) to write the total action in a rather useful form

$$S_{\text{tot}}[A_\mu, \phi, \bar{c}, c] = S_{\text{vector}}[A_\mu] + S_{\text{scalar}}[\phi] - S_{\text{ghost}}[\bar{c}, c], \tag{4.13}$$

where the vector, scalar and ghost actions are respectively given by

$$\begin{aligned} S_{\text{vector}}[A_\mu] &= \int_{\mathcal{M}} d^d x \sqrt{g} \frac{1}{2} A_\mu (\nabla^2 \delta^\mu_\nu - m^2 \delta^\mu_\nu - R^\mu{}_\nu) A^\nu, \\ S_{\text{scalar}}[\phi] &= \int_{\mathcal{M}} d^d x \sqrt{g}\; \frac{1}{2} \phi (\nabla^2 - m^2) \phi, \\ S_{\text{ghost}}[\bar{c}, c] &= \int_{\mathcal{M}} d^d x \sqrt{g}\; \bar{c} (\nabla^2 - m^2) c. \end{aligned} \tag{4.14}$$

The gauge-fixed action for the Proca field in curved space-time is thus described by the sum of three actions, that is the ones for the vector field $A_\mu(x)$, the scalar bosonic field $\phi(x)$ and the two anticommuting scalar ghosts $\bar{c}(x)$ and $c(x)$. The latter carry two unphysical degrees of freedom that cancel the ones coming from the Stuckelberg field and the longitudinal mode of the vector $A_\mu(x)$.

We observe that this method represents an alternative approach to the reduction formula (2.44) of Barvinsky and Vilkovisky used to compute the one-loop effective action of the non-minimal Proca operator (2.14). One may also infer that the two methods are formally equivalent. In fact, the Proca field is described by vector and scalar fields whose actions, properly summed, provide the correct number of physical degrees of freedom. This means that one could equally start from the action (4.13) and apply the heat kernel approach separately to the vector and scalars operators, summing the results of the one-loop effective actions at the end.

Moreover, the introduction of the Stückelberg field and the application of the BRST quantization method, allowed to remove the problematic term in the path integral approach discussed in the previous chapter. Indeed the kinetic operator for the vector field A_μ, which can be read off from the action, is of minimal type, i.e. without terms $\sim \nabla_\mu \nabla_\nu A^\mu$. This is exactly the term that in the worldline construction gave rise to a non-perturbative vertex in the path integral computation. In this sense, we will start from this gauge-fixed action to apply the path integral method for the calculation of the heat kernel coefficients, after the construction of a proper model for the worldline representation of the kinetic operators present in the action, that is the ones for the vector field $A_\mu(x)$, the scalar field $\phi(x)$ and the ghost fields $c(x)$ and $\bar{c}(x)$.

Before we proceed with the formulation of this new model, let us rewrite the action (4.13) in Euclidean time by performing the usual Wick rotation. The Euclidean action is then given by

$$\begin{aligned} S_{\mathrm{E}}^{\mathrm{tot}}[A_\mu, \phi, \bar{c}, c] = \int_{\mathcal{M}} d^d x \sqrt{g} \bigg[&\frac{1}{2} A_\mu (-\nabla^2 \delta^\mu_\nu + m^2 \delta^\mu_\nu + R^\mu{}_\nu) A^\nu \\ &+ \frac{1}{2} \phi (-\nabla^2 + m^2) \phi - \bar{c}(-\nabla^2 + m^2) c \bigg]. \end{aligned} \tag{4.15}$$

4.2 The worldline model for the vector field

In the following we want to construct a worldline representation of the operator acting on the space of vector fields $A_\mu(x)$. To do so we proceed similarly to what we did in section 3.2 by performing a straightforward generalization to a curved d-dimensional spacetime with metric tensor $g_{\mu\nu}$. There are however some subtleties in this particular case represented by ordering ambiguities in the quantum Hamiltonian which must reproduce the wanted vector operator. The choice of a precise ordering prescription must be taken care of in the path integral construction as well, via a proper regularization scheme, that we will discuss later on with much more details.

Let us start from the Euclidean action for the vector field $A_\mu(x)$, which is given by

$$S^{\mathrm{E}}_{\mathrm{vector}}[A_\mu] = \int_{\mathcal{M}} d^d x \sqrt{g} \frac{1}{2} A_\mu (-\nabla^2 \delta^\mu_\nu + m^2 \delta^\mu_\nu + R^\mu{}_\nu) A^\nu. \tag{4.16}$$

From the latter we can easily extract the differential operator

$$F_{\mu\nu} = -g_{\mu\nu} \nabla^2 + g_{\mu\nu} m^2 + R_{\mu\nu} \tag{4.17}$$

whose action on a generic vector field $V_\mu(x)$ reads

$$F_{\mu\nu} V^\nu = -\nabla^2 V_\mu + m^2 V_\mu + R_{\mu\nu} V^\nu. \tag{4.18}$$

We now proceed with the construction of the Hilbert space. The coordinates and related conjugate momenta are again given by the phase space variables $x^\mu(t)$ and $p_\mu(t)$. As usual they provide the functional dependence on the space-time points of the wave function representing the generic state of the Hilbert space. The latter needs also discrete indices, introduced by means of worldline complex fermions that are represented by the fermionic variables $\bar{\lambda}^a$ with associated conjugate momenta λ_a. These indices are indeed needed to identify the vector field among the other fields contained in the Hilbert space. One could use curved indices as well, i.e. $\bar{\lambda}^\mu$ and λ_μ, but this would lead to various issues related to the ordering of fermionic bilinears $\sim \bar{\lambda}\lambda$. Flat indices are employed to overcome such problems and can be easily implemented by means of the vielbein basis introduced in section 1.2. The bosonic and fermionic variables altogether define a graded phase space.

We promote these variables to operators with the usual canonical quantization via the following (anti)commutation relations

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu \tag{4.19}$$

$$\{\hat{\lambda}_a, \hat{\lambda}^{\dagger b}\} = \delta_a^b \tag{4.20}$$

where we set $\hbar = 1$. As before the above mentioned bosonic variables correspond to the eigeinvalues of the position operator $\hat{x}^\mu$ when acting on position eigenstates $|x\rangle$, while the anticommuting numbers $\bar{\lambda}^a$ are the eigenvalues of the operator $\hat{\lambda}^{\dagger a}$ when acting on the fermionic coherent states $\langle\bar{\lambda}|$.

Since we are considering a curved target space-time with metric $g_{\mu\nu}(x)$, we have to keep track of the possible factors arising in the definition of the momentum due to the form of the covariant measure, i.e. $d^dx\sqrt{g}$, entering the scalar product between two generic wave functions $\Psi_1(x)$ and $\Psi_2(x)$

$$\langle\Psi_1|\Psi_2\rangle = \int_{\mathcal{M}} d^dx\sqrt{g}\Psi_1^*(x)\Psi_2(x), \tag{4.21}$$

with $g = |\det g_{\mu\nu}|$. The Hermitian momentum operator is therefore given by

$$\hat{p}_\mu = -ig^{-1/4}\partial_\mu g^{1/4}, \tag{4.22}$$

where the derivative acts through. Integration by parts allow to verify its hermiticity property and the precise powers of g arising in its definition.

By proceeding like in section 3.2, it is now possible to write the generic wave function by projecting the generic state of the Hilbert space on the position eigenstates $\langle x|$ together with the coherent states $\langle\bar{\lambda}|$, to get

$$\begin{aligned} |\Psi\rangle \sim \Psi(x,\bar{\lambda}) &= (\langle x| \otimes \langle\bar{\lambda}|)\,|\Psi\rangle \\ &= \Psi(x) + \Psi_a(x)\bar{\lambda}^a + \frac{1}{2}\Psi_{a_1a_2}(x)\bar{\lambda}^{a_1}\bar{\lambda}^{a_2} + ... + \frac{1}{d!}\Psi_{a_1...a_d}(x)\bar{\lambda}^{a_1}...\bar{\lambda}^{a_d}. \end{aligned} \tag{4.23}$$

Once more, we want to project the full Hilbert space onto the subsector containing only vector fields. This can be done by following the same procedure discussed above, by means of the constraint (3.54), which relies on the introduction of the auxiliary worldline gauge field $a(t)$, taking the role of a Lagrange multiplier, and the addition of an extra Chern-Simons coupling $-isa$, with $s = 1 - d/2$.

Since we want to find the Hamiltonian that acting on the wave function provides the action of the differential operator (4.17) on the vector field, we need a worldline representation of the covariant derivative. In order to do so, we introduce the generators of the Lorentz group $SO(d)$, namely

$$M^{ab} = -M^{ba} \equiv \bar{\lambda}^a \lambda^b - \bar{\lambda}^b \lambda^a, \tag{4.24}$$

which obey the $\mathfrak{so}(d)$ algebra

$$[M^{ab}, M^{cd}] = \eta^{bc} M^{ad} + \eta^{ad} M^{bc} - \eta^{ac} M^{bd} - \eta^{bd} M^{ac}. \tag{4.25}$$

We can now define the covariant derivative operator acting on wave functions in the following way

$$\hat{\nabla}_\mu \equiv \partial_\mu + \frac{1}{2}\sigma_{\mu ab} M^{ab} = \partial_\mu + \sigma_{\mu ab} \bar{\lambda}^a \lambda^b, \tag{4.26}$$

where $\sigma_{\mu ab}$ is the spin connection defined in (1.16). However we need the action of the covariant derivative on vector fields, that can be extracted by acting with the latter operator on wave functions of the type $V(x, \lambda) \sim V_a(x)\bar{\lambda}^a$, i.e.

$$\hat{\nabla}_\mu V(x, \lambda) = \hat{\nabla}_\mu (V_a \bar{\lambda}^a) = (\partial_\mu V_a + \sigma_{\mu a}{}^b V_b)\bar{\lambda}^a = (\nabla_\mu V_a)\bar{\lambda}^a. \tag{4.27}$$

In order to write the Laplacian operator $\hat{\nabla}^2$ it is useful to define the covariant momentum π_μ in terms of the momentum p_μ and the spin connection $\sigma_{\mu ab}$ as follows

$$\pi_\mu \equiv p_\mu - i\sigma_{\mu ab} \bar{\lambda}^a \lambda^b. \tag{4.28}$$

In this way, the covariant derivative operator can be rewritten as

$$\hat{\nabla}_\mu = i g^{1/4} \pi_\mu g^{-1/4} = i g^{1/4} (p_\mu - i\sigma_{\mu ab} \bar{\lambda}^a \lambda^b) g^{-1/4}. \tag{4.29}$$

Prior to writing the expression of the Laplacian operator that will be used for the correct worldline representation of the differential operator (4.17), we have to face the ordering issues brought up earlier. A few words are now in order. As we know, the process of canonical quantization gives rise to ordering ambiguities due to the non vanishing commutator (4.19), with the consequence that many different Hamiltonians at the quantum level correspond to the same classical counterpart. As we stated previously, the preservation of the classical symmetries at the quantum level, is what is used to build the correct quantum Hamiltonian. For example, we could impose covariance under general change of coordinates at the quantum level by choosing the following ordering of the Laplacian operator

$$\hat{\nabla}^2 \equiv \frac{1}{\sqrt{g}}\hat{\nabla}_\mu g^{\mu\nu}\sqrt{g}\hat{\nabla}_\nu = -g^{-1/4}\pi_\mu g^{\mu\nu}\sqrt{g}\pi_\nu g^{-1/4}. \tag{4.30}$$

Nonetheless, it occurs that the above requirement fixes the quantum Hamiltonian only up to an arbitrary term proportional to the Ricci scalar R, i.e. it contains a term of the type ξR. This is because the Ricci scalar represents the only covariant scalar object that can be constructed out of the metric up to its second order partial derivatives. All the ambiguities that are left after the imposition of invariance under general change of coordinates are then parametrized by the coupling constant ξ. The simplest and most common choice is to set $\xi = 0$ in the quantum Hamiltonian, because other values of this coupling can always be introduced later with the addition of a proper scalar potential of the form $V(x) \sim \xi R$.

We can finally write the Hamiltonian operator for the vector field as follows

$$\hat{H} = g^{-1/4}\pi_\mu g^{\mu\nu}\sqrt{g}\pi_\nu g^{-1/4} + m^2 + R_{ab}\bar{\lambda}^a\lambda^b. \tag{4.31}$$

We can verify that this is indeed the correct Hamiltonian that reproduces the operator (4.17) acting on the vector wave function that satisfies the constraint (3.54). In fact, the last term in this expression, acting on the constrained wave function, provides a worldline representation of the term proportional to the Ricci tensor in the differential operator (4.17), i.e.

$$R_{ab}\bar{\lambda}^a\lambda^b(V_c\bar{\lambda}^c) = R_{ab}V_c\bar{\lambda}^a\eta^{bc} = (R_{ab}V^b)\bar{\lambda}^a. \tag{4.32}$$

The classical action in phase space can be written by means of the classical version of the Hamiltonian operator (4.31) as below

$$S[x,p,\bar{\lambda},\lambda,a] = \int_0^\beta dt[-ip_\mu\dot{x}^\mu + \bar{\lambda}^a(\partial_t + ia)\lambda_a + g_{\mu\nu}\pi^\mu\pi^\nu + R_{ab}\bar{\lambda}^a\lambda^b + m^2 - isa]. \tag{4.33}$$

The momentum p_μ can be integrated out via its equations of motion, which give

$$\frac{\delta S}{\delta p_\mu} = 0 \rightarrow p^\mu = \frac{i}{2}\dot{x}^\mu + i\sigma^\mu{}_{ab}\bar{\lambda}^a\lambda^b \tag{4.34}$$

to get the classical action in configuration space

$$S[x,\bar{\lambda},\lambda,a] = \int_0^\beta dt\left[\frac{1}{4}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \bar{\lambda}^a(D_t + ia)\lambda_a + R_{ab}\bar{\lambda}^a\lambda^b + m^2 - isa\right], \tag{4.35}$$

where the covariant derivative D_t has been defined as

$$D_t\lambda_a = \partial_t\lambda_a + \dot{x}^\mu\sigma_{\mu ab}\lambda^b. \tag{4.36}$$

4.3 Path integral regularization and counterterms

The classical action written in equation (4.35) can be used to construct the path integral in a similar fashion as described in section 3.1, that provides the transition amplitude at coinciding points $\langle x| e^{-t\hat{H}} |x\rangle$ representing the heat kernel. The foregoing discussion on ordering ambiguities affecting the canonical quantization of the Hamiltonian returns in the path integral definition. Such issues, in this instance, are treated by introducing a proper regularization scheme.

In particular, the model we are treating, as represented by the action (4.35), belongs to the class of "non-linear sigma models" for which the construction of path integrals and their computation in a perturbative approach can be very problematic because of ultraviolet divergences and ill-defined products of distributions like the Dirac delta or the Heaviside step function. As described in [20], with a power counting procedure, double derivatives interactions in non-linear sigma models, read off from the term $\sim g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$, give rise to Feynman diagrams that are linearly divergent. However, the covariant measure of the path integral contains a factor proportional to $\sqrt{g(x(t))}$, i.e.

$$\mathcal{D}x = \prod_{0<t<\beta} \sqrt{g(x(t))}\; d^dx(t), \tag{4.37}$$

that can be exponentiated by making use of auxiliary commuting ($a^\mu(t)$) and anticommuting ($b^\mu(t)$, $c^\mu(t)$) ghosts as discussed below. The metric dependent factor can be first written as follows

$$\sqrt{g(x(t))} = \sqrt{\det g_{\mu\nu}(x(t))} = \frac{\det g_{\mu\nu}(x(t))}{\sqrt{\det g_{\mu\nu}(x(t))}}, \tag{4.38}$$

where at fixed time the numerator and the denominator can be exponentiated taking advantage of the fermionic ghosts $b^\mu(t)$, $c^\mu(t)$ and the bosonic ghosts $a^\mu(t)$ respectively, i.e.

$$\mathcal{D}x = \prod_{0<t<\beta} \sqrt{g(x(t))}\; d^dx(t) = \prod_{0<t<\beta} d^dx(t) \int DaDbDc\; e^{-S_{\mathrm{gh}}}, \tag{4.39}$$

with

$$S_{\mathrm{gh}} = \int_0^\beta dt\; \frac{1}{2} g_{\mu\nu}(x)(a^\mu a^\nu + b^\mu c^\nu). \tag{4.40}$$

It is easy to check how the path integral over these auxiliary ghosts produces the mentioned $\sqrt{g}$ factor by means of fermionic and bosonic Gaussian integrations. The introduction of this type of ghosts in the classical action constitutes a rather simple way to underline the contribution of the metric factor to the path integral, whose treatment is not clear in its initial form. Indeed, the metric dependence of the covariant path integral measure written in terms of these auxiliary ghosts, provides extra linear divergent Feynman graphs that cancel exactly with the ones mentioned above.

However, this particular and rather convenient cancellation can be achieved carefully only after a proper regularization of individual diagrams that involve the previously mentioned products of distributions. The latter appear when double derivatives of 2-point functions are multiplied together or with 2-point functions themselves and derive from the double derivative term $\sim g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ present in the action, which is typical of this kind of models. The regularization schemes precisely concern the way in which these type of products coming from correlation functions are treated and computed. A finite local counterterm, to be added to the classical action in configuration space, is associated to each regularization procedure in such a way that the final result is regularization independent. These counterterms are actually related to the so-called renormalization conditions of the path integral that regard the ordering prescription of the quantum Hamiltonian, that in our case was dictated by preserving general coordinate invariance.

Different regularization schemes can be employed for this purpose. To be specific, three different regularization schemes have been refined and applied to path integrals of non-linear sigma models: mode regularization (MR), time slicing (TS) and worldline dimensional regularization (DR)[1].

Mode regularization is based on the Taylor expansion of the paths in Fourier sine series, regularized thanks to a fixed cut-off M that dictates which modes are allowed to enter the path integral. The latter, being finite, makes the computation no longer ambiguous or problematic and one could in principle perform all the integrals. At the end the continuum limit $M \to \infty$ can be taken without any troubles, obtaining a finite and well defined result. However, this is not the simplest way to use this regularization scheme. An alternative approach can be instead to introduce the regulator (cut-off) M, recast the ambiguous integrals in a form that is no longer problematic and take the $M \to \infty$ limit to compute the integrals by using now simpler forms of propagators. The required counterterm in this case must reinstate general coordinate invariance which is broken by the regularization procedure. This finite local counterterm is proportional to the Ricci curvature scalar plus a term proportional to the product of Christoffel connections $\Gamma\Gamma$, the latter not being covariant.

Time slicing regularization scheme starts by constructing the path integral from the operatorial expression of the transition amplitude $\langle x| e^{-\hat{H}(t_f-t_i)} |y\rangle$. Then one can discretize the time interval $t_f - t_i$ in N equally spaced points and insert $N-1$ completeness relations of position and momentum eigenstates in the transition element, using the "mid-point rule" which corresponds to the Weyl ordering of the Hamiltonian operator $\hat{H}$. The resulting path integral is so discretized in momentum space and taking the continuum limit $N \to \infty$ (i.e. removing the regulator N) one gets the Feynman rules to compute the ambiguous diagrams. As mentioned this regularization procedure corresponds to the choice of a Weyl ordered Hamiltonian, that breaks the general coordinate invariance. Therefore, also in the present case, a local counterterm is needed and it must

[1]Not to be confused with its homonym used in QFT.

contain a $\Gamma\Gamma$ term to restore the symmetry.

In the present thesis we will only use the third regularization scheme, namely dimensional regularization. This procedure is valid only in the regime of the perturbative approach, exactly the one used for our purposes. It is based on the analytic continuation of the compact time interval $I = [0, \beta]$ of our one-dimensional worldline model to a non-compact space $\Omega = I \times R^d$ by introducing d extra non-compact dimensions. As a consequence, the measure of the classical action in configuration space is now $(d+1)$-dimensional, i.e. $d^{d+1}t$. As usual, one can take the way of calculating the regularized integrals in the non-compact space and, after removing the singularities by proper counterterms, remove the regulator at the end (namely taking the $d \to 0$ limit). Similarly to what we stated for the mode regularization, the most practical way to take advantage of DR is actually to introduce the regulator only to cast the ill-defined integrals in a no longer troublesome form. This is done by taking advantage of manipulations valid in the non-compact space, like partial integrations and/or exploiting the Green equations satisfied by the propagators. The regulator is then removed to compute the integrals directly in the $d \to 0$ limit in a simpler and safe way. This procedure may seem complicated at first sight, but it is actually simpler than it looks and it will be described more in detail with practical examples in the following computations. The advantage of DR resides in its property of preserving the general coordinate invariance, which means that the only counterterm required is proportional to the Ricci curvature scalar R, while the non-covariant $\sim \Gamma\Gamma$ counterterm, present in the other two regularization schemes, is absent.

The counterterm we just mentioned is the one associated to the Laplacian term, ∇^2, present in the Hamiltonian, but it is not the only one needed for the construction of a well defined path integral. In fact, we still need to consider ordering issues of the fermions. The construction of the path integral usually produces Weyl ordered fermionic polynomials. This doesn't affect the term of the type $\sigma\bar{\lambda}\lambda$ present in the action (4.35), whereas for the term of the type $R_{ab}\bar{\lambda}^a\lambda^b$ there is again an extra contribution proportional to the Ricci scalar curvature.

We decide to implement the overall counterterm to be added in the classical action (4.35) by using an arbitrary coefficient α in front of R. Its value is defined by requesting the coefficient a_1 to agree with (2.51) and later verified by the correctness of a_2. The classical action in configuration space is thus given by

$$S = \int_0^\beta dt \left[\frac{1}{2} g_{\mu\nu} \left(\frac{1}{2} \dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu \right) + \bar{\lambda}^a (D_t + ia) \lambda_a + R_{ab} \bar{\lambda}^a \lambda^b + \alpha R + m^2 - isa \right] \quad (4.41)$$

4.4 The one-loop effective action for the vector operator

Once we have found the classical action by constructing the Hamiltonian operator that provides the correct worldline representation of the differential operator (4.17), we can proceed with the path integral quantization and therefore recover the heat kernel coefficients (2.51).

We begin by rescaling the time variable t as we previously did:

$$t \to \tau = \frac{t}{\beta}. \tag{4.42}$$

Consequently also the fermions and the auxiliary gauge field must be rescaled in order to get the usual factor of $1/\beta$ in front of the action, which will serve to control the order of the perturbative expansion, i.e.

$$\lambda_\mu \to \frac{\lambda_\mu}{\sqrt{\beta}},\ \bar{\lambda}_\mu \to \frac{\bar{\lambda}_\mu}{\sqrt{\beta}},\ a \to \frac{a}{\beta}. \tag{4.43}$$

As before, we bring the gauge field $a(\tau)$ to the constant value θ, like in equation (3.60). Hence, the classical action becomes

$$S = \frac{1}{\beta}\int_0^1 d\tau \left[\frac{1}{2}g_{\mu\nu}\left(\frac{1}{2}\dot{x}^\mu\dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu\right) + \bar{\lambda}^a(D_t + i\theta)\lambda_a + \beta R_{ab}\bar{\lambda}^a\lambda^b + \beta^2\alpha R\right], \tag{4.44}$$

where the terms proportional to m^2 and to the Chern-Simons coupling have been factorized out.

We go ahead by performing the familiar background-quantum splitting of the bosonic paths as follows

$$x^\mu(\tau) = x^\mu_{bg}(\tau) + q^\mu(\tau), \tag{4.45}$$

with periodic boundary conditions

$$x^\mu(1) = x^\mu(0). \tag{4.46}$$

By fixing the origin of our coordinate system to be given by $x^\mu(0) = 0$, the background part vanishes and we are left only with the quantum fluctuations $q^\mu(\tau)$, which satisfy vanishing Dirichlet boundary conditions. The fermionic variables enjoy the usual antiperiodic boundary conditions

$$\bar{\lambda}^\mu(0) = -\bar{\lambda}^\mu(1), \tag{4.47}$$

while the ghosts a^μ, b^μ and c^μ do not need any boundary conditions, being auxiliary fields.

By employing the background-quantum splitting (4.45) in the action (4.44) we get

$$S = \frac{1}{\beta}\int_0^1 d\tau \left[\frac{1}{2}g_{\mu\nu}(x_{bg}+q)\left(\frac{1}{2}\dot{q}^\mu\dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu\right) + \bar{\lambda}^a(\partial_t + i\theta)\lambda_a \right. \\ \left. + \sigma_{\mu ab}(x_{bg}+q)\dot{q}^\mu\bar{\lambda}^a\lambda^b + \beta R_{ab}(x_{bg}+q)\bar{\lambda}^a\lambda^b + \beta^2\alpha R(x_{bg}+q)\right], \tag{4.48}$$

which can be in turn splitted into a free part and an interacting part as below

$$S_{\text{free}} = \frac{1}{\beta}\int_0^1 d\tau \left[\frac{1}{2}g_{\mu\nu}(x_{bg})\left(\frac{1}{2}\dot{q}^\mu\dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu\right) + \bar{\lambda}^a(\partial_t + i\theta)\lambda_a\right], \tag{4.49}$$

$$S_{\text{int}} = \frac{1}{\beta}\int_0^1 d\tau \left[\frac{1}{2}[g_{\mu\nu}(x_{bg}+q) - g_{\mu\nu}(x_{bg})]\left(\frac{1}{2}\dot{q}^\mu\dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu\right) + \bar{\lambda}^a(\partial_t + i\theta)\lambda_a \right. \\ \left. + \sigma_{\mu ab}(x_{bg}+q)\dot{q}^\mu\bar{\lambda}^a\lambda^b + \beta R_{ab}(x_{bg}+q)\bar{\lambda}^a\lambda^b + \beta^2\alpha R(x_{bg}+q)\right]. \tag{4.50}$$

The free action is thus used to recover the propagators of the free theory. We start by expanding the quantum fluctuations $q^\mu(\tau)$ and the auxiliary ghosts in Fourier sine series as described below

$$q^\mu(\tau) = \sum_{m=1}^{\infty} q^\mu_m \sin(\pi m\tau), \qquad a^\mu(\tau) = \sum_{m=1}^{\infty} a^\mu_m \sin(\pi m\tau), \\ b^\mu(\tau) = \sum_{m=1}^{\infty} b^\mu_m \sin(\pi m\tau), \qquad c^\mu(\tau) = \sum_{m=1}^{\infty} c^\mu_m \sin(\pi m\tau). \tag{4.51}$$

The Fourier sine series is the correct one to preserve the boundary conditions mentioned above since $\sin(\pi m) = \sin(0) = 0$, being m a positive integer. Thus the path integral is defined as an integration over the Fourier coefficients q^μ_m, a^μ_m, b^μ_m and c^μ_m. The free action part containing these fields gets modified as

$$S_{\text{free}}[q^\mu, a^\mu, b^\mu, c^\mu] = \frac{1}{2\beta}\int_0^1 d\tau g_{\mu\nu}(x_{bg})\left(\frac{1}{2}\dot{q}^\mu\dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu\right) \\ = \frac{1}{4\beta}g_{\mu\nu}(x_{bg})\sum_{m=1}^{\infty}\left(\frac{(\pi m)^2}{2}\dot{q}^\mu_m\dot{q}^\nu_m + a^\mu_m a^\nu_m + b^\mu_m c^\nu_m\right), \tag{4.52}$$

where we have performed the integrals in $d\tau$.

The correlation function $\langle q^\mu_m q^\nu_n\rangle$ can be easily found by casting the exponential $e^{-S_{\text{free}}[q^\mu]}$ in the form $e^{-\frac{1}{2}\phi K\phi}$ and by inverting the kinetic operator K, i.e.

$$\langle q^\mu_m q^\nu_n\rangle = \beta g^{\mu\nu}\delta_{mn}\frac{4}{\pi^2 m^2}. \tag{4.53}$$

In a similar way we can find the correlator for the bosonic ghosts

$$\langle a_m^\mu a_n^\nu \rangle = 2\beta g^{\mu\nu}\delta_{mn}. \tag{4.54}$$

For the fermionic ghosts, we cast the above mentioned exponential in the form $e^{-\bar{\eta}K\eta}$, where $\bar{\eta}$ is the antighost and η is the ghost, obtaining

$$\langle b_m^\mu c_n^\nu \rangle = -4\beta g^{\mu\nu}\delta_{mn}. \tag{4.55}$$

The full propagator for the bosonic fluctuations is then given by

$$\begin{aligned}\langle q^\mu(\tau) q^\nu(\sigma)\rangle &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\langle q_m^\mu q_n^\nu\rangle \sin(\pi m\tau)\sin(\pi n\tau) \\ &= -2\beta g^{\mu\nu}\sum_{m=1}^{\infty} -\frac{2}{(\pi m)^2}\sin(\pi m\tau)\sin(\pi n\tau) \\ &= -2\beta g^{\mu\nu}\Delta(\tau,\sigma),\end{aligned} \tag{4.56}$$

where

$$\Delta(\tau,\sigma) = (\tau-1)\sigma\Theta(\tau-\sigma) + (\sigma-1)\tau\Theta(\sigma-\tau). \tag{4.57}$$

The full propagator for the bosonic ghosts is

$$\langle a^\mu(\tau) a^\nu(\sigma)\rangle = \beta g^{\mu\nu}\Delta_{gh}(\tau,\sigma), \tag{4.58}$$

while for the fermionic ghosts we have

$$\langle b^\mu(\tau) c^\nu(\sigma)\rangle = -2\beta g^{\mu\nu}\Delta_{gh}(\tau,\sigma), \tag{4.59}$$

where $\Delta_{gh}(\tau,\sigma)$ is given by

$$\Delta_{gh}(\tau,\sigma) = \sum_{m=1}^{\infty} 2\,\sin(\pi m\tau)\sin(\pi m\sigma) = \partial_\tau^2\Delta(\tau,\sigma) = \delta(\tau,\sigma). \tag{4.60}$$

We now report some identities, useful for the calculations that will follow, for derivatives and equal time expressions:

$$\begin{aligned} &{}^\bullet\!\Delta(\tau,\sigma) = \sigma - \Theta(\sigma-\tau)\,, && \Delta^\bullet(\tau,\sigma) = \tau - \Theta(\tau-\sigma)\,, \\ &{}^\bullet\!\Delta^\bullet(\tau,\sigma) = 1-\delta(\tau,\sigma)\,, && \Delta_{gh}(\tau,\sigma) = {}^{\bullet\bullet}\!\Delta(\tau,\sigma) = \delta(\tau,\sigma) \\ &\Delta(\tau,\tau) = \tau(\tau-1)\,, && {}^\bullet\!\Delta(\tau,\sigma)|_{\tau=\sigma} = \Delta^\bullet(\tau,\sigma)|_{\tau=\sigma} = \tau - \frac{1}{2}\,, \end{aligned} \tag{4.61}$$

To find the propagator for the fermionic variables we follow the same procedure of section 3.3 and here we just recall the result

$$\langle \lambda_a(\tau)\bar{\lambda}^b(\sigma)\rangle = \beta\delta_a^b\Delta_F(\tau-\sigma,\theta), \tag{4.62}$$

where

$$\Delta_F(\tau-\sigma,\theta) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{-i}{2\pi r+\theta} e^{2\pi i r(\tau-\sigma)} = \frac{e^{-i\theta(\tau-\sigma)}}{2\cos\frac{\theta}{2}}\left[e^{i\frac{\theta}{2}}\Theta(\tau-\sigma) - e^{-i\frac{\theta}{2}}\Theta(\sigma-\tau)\right]. \quad (4.63)$$

Once the propagators have been found we can proceed with the perturbative calculation of the path integral. The one-loop effective action for the so constructed vector model is indeed given by

$$\Gamma = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta}\int \frac{d^dx\sqrt{g}}{(4\pi\beta)^{d/2}}\int_0^{2\pi}\frac{d\theta}{2\pi} e^{is\theta} e^{-\beta m^2}\left(2\cos\frac{\theta}{2}\right)^d \langle e^{-S_{int}}\rangle, \quad (4.64)$$

where we made explicit the correct normalization for the bosonic and fermionic path integrals.

As already explored in the previous section, the computation of the path integral, or more precisely of the average of the interacting action $\langle e^{-S_{int}}\rangle$, involves products of distributions that have to be treated by means of a proper regularization scheme. As we said, in this thesis we use worldline dimensional regularization.

Let us introduce d extra non-compact time dimensions via the variables t^i, where the index $i = 0,\ldots,d$ and let $t^0 = \tau$ coincide with the time taking values in the compact interval $I = [0,1]$, i.e. in a compact notation $t^i \equiv (\tau, \mathbf{t})$. The action in the extendend non compact space $\Omega = I \times R^d$ is accordingly given by

$$\begin{aligned} S = \frac{1}{\beta}\int_\Omega d^{d+1}t\Bigg[&\frac{1}{2}g_{\mu\nu}(x)\left(\frac{1}{2}\partial_i q^\mu \partial_i q^\nu + a^\mu a^\nu + b^\mu c^\nu\right) + \bar\lambda^a(\gamma^i\partial_i + i\theta)\lambda_a \\ &+ \bar\lambda^a\gamma^i\partial_i q^\mu \sigma_{\mu ab}\lambda^b + \beta R_{ab}\bar\lambda^a\lambda^b + \alpha\beta^2 R\Bigg], \end{aligned} \quad (4.65)$$

where we made use of the Dirac gamma matrices γ^i in $d+1$ dimensions and of the shorthand notation $\partial_i = \frac{\partial}{\partial t^i}$. The free action becomes

$$S_{\text{free}} = \frac{1}{\beta}\int_\Omega d^{d+1}t\left[\frac{1}{2}g_{\mu\nu}(x_{bg})\left(\frac{1}{2}\partial_i q^\mu\partial_i q^\nu + a^\mu a^\nu + b^\mu c^\nu\right) + \bar\lambda^a(\gamma^i\partial_i + i\theta)\lambda_a\right]. \quad (4.66)$$

The latter provides the propagators in the extended non-compact space Ω which are much more complicated than those written in equations (4.57), (4.60) and (4.63), i.e.

$$\begin{aligned} \Delta(t,s) &= \int \frac{d^dk}{(2\pi)^d}\sum_{m=1}^\infty \frac{-2}{(\pi m)^2 + \mathbf{k}^2}\sin(\pi m\tau)\,\sin(\pi m\sigma) e^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})}, \\ \Delta_{gh}(t,s) &= \int \frac{d^dk}{(2\pi)^d}\sum_{m=1}^\infty 2\sin(\pi m\tau)\sin(\pi m\sigma)e^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})} = \delta(\tau,\sigma)\delta^d(\mathbf{t}-\mathbf{s}), \\ \Delta_F(t-s,\theta) &= -\,i\int\frac{d^dk}{(2\pi)^d}\sum_{r\in\mathbb{Z}+1/2}\frac{2\pi r\gamma^0 + \mathbf{k}\cdot\gamma - \theta}{(2\pi r)^2 + \mathbf{k}^2 - \theta^2} e^{2\pi i r(\tau-\sigma)} e^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})}. \end{aligned} \quad (4.67)$$

Nonetheless, the latter will not be used since the advantage of DR is to use the following useful identities to manipulate ambiguous integrals

$$\partial^i \partial_i \Delta(t, s) = \Delta_{gh}(t, s) = \delta(\tau, \sigma)\delta^d(\mathbf{t} - \mathbf{s}), \tag{4.68}$$

$$\left(\gamma^i \frac{\partial}{\partial t^i} + i\theta\right)\Delta_F(t - s, \theta) = \delta_A(\tau - \sigma)\delta^d(\mathbf{t} - \mathbf{s}), \tag{4.69}$$

$$\left[\left(\frac{\partial^2}{\partial t^i \partial s_i} + \frac{\partial^2}{\partial t^i \partial t_i}\right)\Delta(t, s)\right]\Bigg|_{t=s} = \frac{\partial}{\partial \tau}\left[\left(\frac{\partial}{\partial \tau}\Delta(t, s)\right)\Bigg|_{t=s}\right]. \tag{4.70}$$

The notation $\mathbf{t} = (t^1, \ldots, t^d)$ has been employed in the last few equations. It is fairly immediate to understand how the last propagators reduce to (4.57), (4.60) and (4.63) in the $d \to 0$ limit.

We can now go ahead with the perturbative path integral computation by expanding the average of the exponential of the interacting action as follows

$$\langle e^{-S_{\text{int}}}\rangle = 1 - \langle S_{\text{int}}\rangle + \frac{1}{2}\langle S^2_{\text{int}}\rangle + \cdots . \tag{4.71}$$

Since the path integral can be computed in any coordinate system, we prefer to work with the so-called Riemann normal coordinates. The latter are the closest analogue to Cartesian coordinates in curved space. As described in [21], in gauge theory the Fock-Schwinger gauge is commonly used for the calculation of effective actions and anomalies. There is a counterpart in gravity, which is represented by the choice of the Riemann normal coordinates. Geometrically they are defined in the neighbourhood of a chosen point called the origin, that in our case is x^μ_{bg}. The metric tensor, the spin connection and the curvatures can be expanded in the following manner

$$\begin{aligned}
g_{\mu\nu}(x_{bg} + q) =& g_{\mu\nu} + \frac{1}{3}q^\lambda q^\sigma R_{\lambda\mu\nu\sigma} + \mathcal{O}(q^3) + q^\lambda q^\sigma q^\alpha q^\beta \left[\frac{1}{20}\nabla_\lambda \nabla_\sigma R_{\alpha\mu\nu\beta} + \frac{2}{45}R_{\tau\lambda\sigma\mu}R^\tau{}_{\alpha\beta\nu}\right], \\
\omega_{\mu ab}(x_{bg} + q) =& \frac{1}{2}q^\nu R_{\nu\mu ab} + \mathcal{O}(q^2) + q^\nu q^\lambda q^\sigma \left[\frac{1}{8}\nabla_\lambda \nabla_\sigma R_{\nu\mu ab} + \frac{1}{24}R^\tau{}_{\nu\lambda\mu}R_{\sigma\tau ab}\right], \\
R_{abcd}(x_{bg} + q) =& R_{abcd} + \mathcal{O}(q) + \frac{1}{2}q^\mu q^\nu \nabla_\mu \nabla_\nu R_{abcd}\,, \\
R_{ab}(x_{bg} + q) =& R_{ab} + \mathcal{O}(q) + \frac{1}{2}q^\mu q^\nu \nabla_\mu \nabla_\nu R_{ab}\,, \\
R(x_{bg} + q) =& R + \mathcal{O}(q) + \frac{1}{2}q^\mu q^\nu \nabla_\mu \nabla_\nu R\,,
\end{aligned} \tag{4.72}$$

where we considered only the terms that will give rise to contributions in the correlation functions only up to order β^2 and we neglected terms that give rise to odd-point functions, since those give a null result. The latter expressions can be replaced in the interacting action (4.50). It is useful to rewrite (4.50) as the sum of two contributions of different order in β, i.e.

$$S_{\text{int}} = S_4 + S_6 + \mathcal{O}(\beta^3), \tag{4.73}$$

where S_4 contributes to the order β and is given by

$$\begin{aligned} S_4 =& \frac{1}{6\beta} R_{\lambda\mu\nu\sigma} \int_0^1 d\tau q^\lambda q^\sigma \left(\frac{1}{2} \dot{q}^\mu \dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu \right) + \frac{1}{2\beta} R_{\nu\mu ab} \int_0^1 d\tau \dot{q}^\mu q^\nu \bar{\lambda}^a \lambda^b \\ &+ R_{ab} \int_0^1 d\tau \bar{\lambda}^a \lambda^b + \alpha\beta R, \end{aligned} \tag{4.74}$$

while S_6 contributes to the order β^2 and reads

$$\begin{aligned} S_6 =& \frac{1}{\beta} \left(\frac{1}{40} \nabla_\lambda \nabla_\sigma R_{\alpha\mu\nu\beta} + \frac{1}{45} R_{\tau\lambda\sigma\mu} R^\tau{}_{\alpha\beta\nu} \right) \int_0^1 d\tau q^\lambda q^\sigma q^\alpha q^\beta \left(\frac{1}{2} \dot{q}^\mu \dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu \right) \\ &+ \frac{1}{\beta} \left[\frac{1}{8} \nabla_\lambda \nabla_\sigma R_{\nu\mu ab} + \frac{1}{24} R^\tau{}_{\nu\lambda\mu} R_{\sigma\tau ab} \right] \int_0^1 d\tau \dot{q}^\mu q^\nu q^\lambda q^\sigma \bar{\lambda}^a \lambda^b \\ &+ \frac{1}{2} \nabla_\mu \nabla_\nu R_{ab} \int_0^1 d\tau q^\mu q^\nu \bar{\lambda}^a \lambda^b + \frac{1}{2} \beta \alpha \nabla_\mu \nabla_\nu R \int_0^1 d\tau q^\mu q^\nu. \end{aligned} \tag{4.75}$$

where the subscripts 4 and 6 represent the number of quantum fields present in each term with a factor $1/\beta$ in front of the integral. The average (4.71) therefore becomes

$$\langle e^{-S_{\text{int}}} \rangle = 1 - \langle S_4 \rangle - \langle S_6 \rangle + \frac{1}{2} \langle S_4^2 \rangle + \mathcal{O}(\beta^3). \tag{4.76}$$

In what follows we present directly the result of (4.76), which computations will be discussed with all the subtleties and the details in Appendix C. Therefore, by means of the Wick theorem for the evaluation of all the non-vanishing correlation functions and by exploiting the propagator identities previously mentioned, we get

$$\begin{aligned} \langle e^{-S_{\text{int}}} \rangle =& 1 - \beta R \left(\frac{1}{3} - \frac{i}{2} \tan \frac{\theta}{2} \right) \\ & - \beta^2 \nabla^2 R \left(\frac{1}{20} - \frac{i}{12} \tan \frac{\theta}{2} \right) + \beta^2 R^2 \left(\frac{13}{72} - \frac{1}{8} \cos^{-2} \frac{\theta}{2} - \frac{i}{6} \tan \frac{\theta}{2} \right) \\ & - \beta^2 R_{\mu\nu} R^{\mu\nu} \left(\frac{1}{180} - \frac{1}{8} \cos^{-2} \frac{\theta}{2} \right) + \beta^2 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \left(\frac{1}{180} - \frac{1}{48} \cos^{-2} \frac{\theta}{2} \right) \\ & + \mathcal{O}(\beta^3). \end{aligned} \tag{4.77}$$

Here we fixed $\alpha = \frac{1}{4}$ in such a way that the coefficient a_1 is in agreement with (2.51). This value has been checked by verifying the consistency of a_2, as written below[2].

Finally, by performing all the modular integrals, reported in (3.96), we can write the one-loop effective action for the vector field operator (4.17) to be

$$\Gamma^{\text{vector}}_{1-\text{loop}} = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta}\int_{\mathcal{M}} \frac{d^dx\sqrt{g}}{(4\pi\beta)^{d/2}} e^{-\beta m^2}\bigg(d + \beta\frac{d-6}{6}R + \beta^2\frac{d-5}{30}\nabla^2 R + \beta^2\frac{d-12}{72}R^2 - \beta^2\frac{d-90}{180}R_{\mu\nu}R^{\mu\nu} + \beta^2\frac{d-15}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \mathcal{O}(\beta^3)\bigg). \tag{4.78}$$

By expanding the exponential $e^{-\beta m^2}$ we obtain the final result

$$\begin{aligned}\Gamma^{\text{vector}}_{1-\text{loop}} = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta}\int_{\mathcal{M}} \frac{d^dx\sqrt{g}}{(4\pi\beta)^{d/2}} \bigg[& d + \beta\bigg(\frac{d-6}{6}R - dm^2\bigg) \\ & + \beta^2\bigg(\frac{d-5}{30}\nabla^2 R + \frac{d-12}{72}R^2 - \frac{d-6}{6}m^2 R \\ & - \frac{d-90}{180}R_{\mu\nu}R^{\mu\nu} + \frac{d-15}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \frac{d}{2}m^4\bigg) \\ & + \mathcal{O}(\beta^3)\bigg].\end{aligned} \tag{4.79}$$

The above result is in agreement with the heat kernel coefficients (2.51) for the vector operator obtained by following the standard heat kernel approach in chapter 2. It must be so, since the vector operators (2.48) and (4.17), the former obtained by means of the Barvinsky and Vilkovisky reduction method and the latter by reinstating the gauge invariance via a Stückelberg scalar field, are indeed the same. We can already see from this result how the path integral approach discussed in this thesis, starting from a well constructed worldline representation of the operator, provides an alternative method to evaluate the heat kernel.

[2] The value $\alpha = \frac{1}{4}$ was somewhat expected. In fact, one has a counterterm amounting to $-\frac{1}{8}R$ in dimensional regularization of non-linear sigma models when the action has a term of the type $\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$, coming from the term $-\frac{1}{2}\nabla^2$ in the Hamiltonian. For this reason, we expect a counterterm of the value $-\frac{1}{4}R$ in the present case, since our action contains a term of the form $\frac{1}{4}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$, again from the term $-\nabla^2$ present in the Hamiltonian. The second counterterm we need, as previously mentioned, is the one related to the ordering of fermions in the term $R_{ab}\bar{\lambda}^a\lambda^b$. The counterterm used for actions that contain a term given by $-\frac{1}{2}R_{ab}\bar{\lambda}^a\lambda^b$ is equal to $-\frac{1}{4}R$. By a comparison with the present case we expect a term of the type $\frac{1}{2}R$ to be added instead. Therefore the total counterterm to be added to the action amounts to $-\frac{1}{4}R + \frac{1}{2}R = \frac{1}{4}R$, which is exactly the value we used. The reader can find the above mentioned counterterms and more detailed discussions in [16, 22].

The following section is dedicated to the construction of a similar but simpler worldline model for the scalar operator $-\nabla^2 + m^2$ associated to the Stückelberg field ϕ and to the ghosts c and $\bar{c}$. Altogether they carry the unphysical degree of freedom that represents the scalar longitudinal mode of the Proca field.

4.5 The worldline model for the scalar field

Let us consider the action in Euclidean time, after the usual Wick rotation, for the Stückelberg field ϕ. Its discussion and the corresponding worldline model construction will work also for the case of the scalar ghosts c and $\bar{c}$.

The Stückelberg action reads

$$S^{\mathrm{E}}_{\mathrm{scalar}}[\phi] = \int_{\mathcal{M}} d^d x \sqrt{g}\, \frac{1}{2}\phi(\nabla^2 - m^2)\phi. \tag{4.80}$$

The associated differential kinetic operator is given by

$$\hat{F}_s = -\nabla^2 + m^2, \tag{4.81}$$

whose action on the scalar field ϕ is

$$\hat{F}_s \phi = (-\nabla^2 + m^2)\phi. \tag{4.82}$$

The Hilbert space can be constructed as usual by means of the phase space variables x^μ and p_μ that upon canonical quantization satisfy the commutation relation (4.19). In the present case the generic state of the Hilbert space is simply given by the wave function $\Psi(x)$ without the necessity of introducing the worldline fermionic variables for the description of the discrete index. Thus, the Hilbert space contains only the scalar function and the procedure of projection into a specific subsector via the introduction of the auxiliary gauge field $a(t)$ and a Chern-Simons coupling is no longer required. For simplicity, let us further notice that the Laplacian ∇^2, when acting on scalars, reduces to ∂^2. Hence, the correct Hamiltonian operator now is

$$\hat{H} = -\frac{1}{\sqrt{g}}\partial_\mu g^{\mu\nu}\sqrt{g}\;\partial_\nu + m^2 = g^{-1/4} p_\mu g^{\mu\nu}\sqrt{g}\; p_\nu g^{-1/4} + m^2. \tag{4.83}$$

The classical action in phase space is written exploiting the classical Hamiltonian in the following manner

$$S[x,p] = \int_0^\beta dt(-ip_\mu \dot{x}^\mu + g^{\mu\nu} p_\mu p_\nu + m^2), \tag{4.84}$$

which can be in turn written in configuration space by integrating out the momentum p_μ as

$$S[x] = \int_0^\beta dt\left(\frac{1}{4} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + m^2 + \rho R\right), \tag{4.85}$$

where we introduced a the generic counterterm ρR.

By rescaling the time as usual and introducing auxiliary ghosts for the metric dependent factor in the path integral measure we get

$$S = \frac{1}{\beta}\int_0^1 d\tau \left[\frac{1}{2}g_{\mu\nu}\left(\frac{1}{2}\dot{x}^\mu\dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu\right) + \rho\beta^2 R\right], \tag{4.86}$$

where the term $\propto m^2$ has been factorized out. We can now perform the familiar background-quantum splitting as in (4.45) for the bosonic paths, to write the action as follows

$$S = \frac{1}{\beta}\int_0^1 d\tau \left[\frac{1}{2}g_{\mu\nu}(x_{bg}+q)\left(\frac{1}{2}\dot{q}^\mu\dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu\right) + \rho\beta^2 R(x_{bg}+q)\right]. \tag{4.87}$$

We can once more split the action into a free part and an interacting part, obtaining

$$S_{\text{free}} = \frac{1}{\beta}\int_0^1 d\tau \left[\frac{1}{2}g_{\mu\nu}(x_{bg})\left(\frac{1}{2}\dot{q}^\mu\dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu\right)\right], \tag{4.88}$$

$$S_{\text{int}} = \frac{1}{\beta}\int_0^1 d\tau \left[\frac{1}{2}[g_{\mu\nu}(x_{bg}+q) - g_{\mu\nu}(x_{bg})]\left(\frac{1}{2}\dot{q}^\mu\dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu\right) + \rho\beta^2 R(x_{bg}+q)\right]. \tag{4.89}$$

Once again we can extract the propagators of the theory from the free action, which are the same as the ones reported in equations (4.56) - (4.60). By taking advantage of the usual propagator identities (4.61) and by expanding the interacting action in Riemann normal coordinates as before, we get

$$S_{\text{int}} = S_4 + S_6 + \mathcal{O}(\beta^3), \tag{4.90}$$

where the contribution to the order β is given by S_4 which reads

$$S_4 = \frac{1}{6\beta}R_{\lambda\mu\nu\sigma}\int_0^1 d\tau q^\lambda q^\sigma \left(\frac{1}{2}\dot{q}^\mu\dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu\right) + \alpha\beta R, \tag{4.91}$$

while at the order β^2 we have S_6 which is given by

$$\begin{aligned} S_6 =& \frac{1}{\beta}\left(\frac{1}{40}\nabla_\lambda\nabla_\sigma R_{\alpha\mu\nu\beta} + \frac{1}{45}R_{\tau\lambda\sigma\mu}R^\tau{}_{\alpha\beta\nu}\right)\int_0^1 d\tau q^\lambda q^\sigma q^\alpha q^\beta \left(\frac{1}{2}\dot{q}^\mu\dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu\right) \\ &+ \frac{1}{2}\beta\alpha\nabla_\mu\nabla_\nu R\int_0^1 d\tau q^\mu q^\nu. \end{aligned} \tag{4.92}$$

Therefore, the expansion (4.76) becomes

$$\begin{aligned}\langle e^{-S_{\text{int}}}\rangle = & 1+\frac{1}{6}\beta R+\frac{1}{30}\beta^2\nabla^2 R+\frac{1}{72}\beta^2 R^2-\frac{1}{180}\beta^2 R_{\mu\nu}R^{\mu\nu} \\ & +\frac{1}{180}\beta^2 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}+\mathcal{O}(\beta^3),\end{aligned} \tag{4.93}$$

where the counterterm has been fixed to the value $\rho = -\frac{1}{4}$ with the same argument as before. The one-loop effective action for the bosonic Stückelberg field operator (4.81) takes the form

$$\begin{aligned}\Gamma^{\text{scalar}}_{1-\text{loop}} = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta}\int_{\mathcal{M}}\frac{d^d x\sqrt{g}}{(4\pi\beta)^{d/2}}\bigg[& 1+\beta\left(\frac{1}{6}R-m^2\right) \\ & +\beta^2\bigg(\frac{1}{30}\nabla^2 R+\frac{1}{72}R^2-\frac{1}{6}m^2 R \\ & -\frac{1}{180}R_{\mu\nu}R^{\mu\nu}+\frac{1}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}+\frac{1}{2}m^4\bigg) \\ & +\mathcal{O}(\beta^3)\bigg].\end{aligned} \tag{4.94}$$

The result we just obtained is in agreement with the heat kernel coefficients for the scalar field obtained in (2.53). The procedure we just followed is valid also to obtain the heat kernel coefficients associated to the scalar operator (4.81) acting on the ghosts c and $\bar{c}$.

4.6 The final result

We recall here the Euclidean action that describes the Proca field (4.15), i.e.

$$\begin{aligned}S^{\text{tot}}_{\text{E}}[A_\mu,\phi,\bar{c},c] = \int_{\mathcal{M}} d^d x\sqrt{g}\bigg[& \frac{1}{2}A_\mu(-\nabla^2\delta^\mu_\nu+m^2\delta^\mu_\nu+R^\mu{}_\nu)A^\nu \\ & +\frac{1}{2}\phi(-\nabla^2+m^2)\phi-\bar{c}(-\nabla^2+m^2)c\bigg].\end{aligned} \tag{4.95}$$

The full one-loop effective action is given by the contributions coming from the vector operator for the vector field A^μ, the scalar operator for the scalar field ϕ and the one for the ghost fields c and $\bar{c}$. By noticing the factor -1 present in front of the ghost term in the above action, we can write the expression for the total one-loop effective action as

$$\Gamma^{\text{tot}}_{1-\text{loop}} = \Gamma^{\text{vector}}_{1-\text{loop}}+\Gamma^{\text{scalar}}_{1-\text{loop}}-2\Gamma^{\text{scalar}}_{1-\text{loop}} = \Gamma^{\text{vector}}_{1-\text{loop}}-\Gamma^{\text{scalar}}_{1-\text{loop}}. \tag{4.96}$$

Therefore, the final result is

$$
\begin{aligned}
\Gamma^{\text{tot}}_{1-\text{loop}} = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta}\int_{\mathcal{M}} \frac{d^dx\sqrt{g}}{(4\pi\beta)^{d/2}} \bigg[& (d-1) + \beta\bigg(\frac{d-7}{6}R - (d-1)m^2\bigg) \\
& + \beta^2\bigg(\frac{d-6}{30}\nabla^2 R + \frac{d-13}{72}R^2 - \frac{d-7}{6}m^2 R \\
& - \frac{d-91}{180}R_{\mu\nu}R^{\mu\nu} + \frac{d-16}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \frac{d-1}{2}m^4\bigg) \\
& + \mathcal{O}(\beta^3)\bigg].
\end{aligned}
\tag{4.97}
$$

The above results represents the one-loop effective action for the Proca field operator from which one can read off the associated first three Seeley-DeWitt coefficients a_0, a_1 and a_2. The coefficient a_0 correctly provides the number N of degrees of freedom of the massive vector field in a curved d-dimensional space-time, namely $N = d - 1$. This value precisely reduces to $N = 3$ in a 4-dimensional space-time, as expected. As one can immediately check, these coefficients agree with those in equation (2.54) computed with the standard heat kernel approach and clearly agree also with the results found in [4].

The introduction of a Stückelberg scalar field to restore the gauge symmetry, broken by the mass term, and the addition of auxiliary ghosts for the gauge fixing made the application of the path integral method, discussed in chapter 3, possible without having to face the problem of a non-perturbative vertex. Moreover, the procedure discussed in section 4.1 can be generalized to antisymmetric tensor fields of higher rank, as suggested in [4].

This could hint the fact that this perspective might be useful to treat similar problems in QFT or in the evaluation of worldline path integrals with analogous non-perturbative vertices. In the present work, the latter has been removed by introducing a gauge symmetry that has been properly fixed for our pourposes. The introduction of such gauge symmetries with a related gauge fixing procedure could solve similar problems in other contexts. For instance, a case traceable to the above description is contained in [23], which examines a worldline model able to reproduce the gauge-fixed graviton action of perturbative quantum gravity. Similarly to the situation studied in 3.3, the worldline phase space action for the above mentioned model presents a non-perturbative vertex of the type

$$
\frac{1}{2}g^{\mu\nu}\pi_\mu\pi_\nu\bigg(1 - \frac{1}{4}\psi\bar{\psi}\bigg), \tag{4.98}
$$

where $\psi = \eta_{ab}\psi^{ab}$ and $\bar{\psi} = \eta_{ab}\bar{\psi}^{ab}$, with ψ^{ab} and $\bar{\psi}^{ab}$ rank-2 symmetric tensors with non-vanishing trace taking the role of the complex fermionic variables of the graded phase space.

Conclusions

The aim of this thesis was to compute the first three heat kernel coefficients a_0, a_1 and a_2 for the Proca theory of massive vector fields.

The first part of the thesis is dedicated to the introduction to the topic, i.e. the heat kernel, and to the derivation of the above mentioned coefficients with the standard approach. In particular we followed the procedure proposed by Gilkey in [2]. However, this method is not directly applicable to the case of interest, since it can be employed only in those theories where the kinetic operator coming from the action is of minimal kind. Therefore we first needed to manipulate the Proca differential operator so that we could reduce ourselves to the study of a sum of minimal operators. For this purpose we considered the reduction method for non-minimal operators originally introduced by Barvinsky and Vilkovisky in [3]. In this way we were able to compute the first three Seeley-DeWitt coefficients in arbitrary dimensions d, which turned out to be in agreement with those present in the literature. Specifically, the logarithmic divergent part of the one-loop effective action, identified by the coefficient a_2, in $d = 4$ is in agreement with the one found in [3, 13]. The first coefficient, namely a_0, correctly reproduces the degrees of freedom of the theory, as one would expect. Moreover, the coefficients found in arbitrary dimensions coincide with the ones present in [4], which studied a $N = 2$ spinning particle model to reproduce the heat kernel coefficients for massive antisymmetric tensor fields of generic rank p, via a worldline path integral representation of the one-loop effective action.

The second part of the thesis is devoted to the derivation of the first three heat kernel coefficients for the Proca theory by means of an analogous first quantization procedure, based on the use of path integrals as well as a worldline model. Our first idea was to rederive the wanted coefficients without manipulating the Proca differential operator. For this goal we built a first quantized particle model, whose Hamiltonian acting on the Hilbert space correclty reproduced the action of the Proca differential operator on vector fields. However, we came across a non-perturbative vertex in the path integral expansion, which made the calculations unachievable.

For this reason, we thought that maybe the problem could have been solved by recasting the Proca operator in such a way that the cumbersome interaction vertex would disappear. Since the origin of this vertex is precisely the non-minimal nature of the initial operator, we decided to implement a Stückelberg field with additional ghost scalars to eliminate the non-perturbative vertex and work only with minimal operators. In this way, we were able to build a worldline model in curved space-time and to compute the wanted heat kernel coefficients with the path integral method. We think that this trick of introducing one or more gauge symmetries via additional fields, might be useful to solve similar problems that arose in the application of the worldline model to other theories. The latter are indeed characterized by operators which are non-minimal or in which the Laplacian doesn't take the standard form. The case reported in [23], where the worldline model of the graviton in perturbative quantum gravity presents a similar non-perturbative vertex, could be such an example.

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Appendix A

Fermionic coherent states

Let us review the fermionic coherent states by first recalling the oscillator algebra satisfied by the operators $\hat{\lambda}^{\dagger\mu}$ and $\hat{\lambda}_\nu$ (3.33), i.e.

$$\{\hat{\lambda}_\mu, \hat{\lambda}^{\dagger\nu}\} = \delta_\mu^\nu, \tag{A.1}$$

where $\mu = 0 \dots d-1$. This is exactly the algebra for fermionic creation ($\hat{\lambda}^{\dagger\mu}$) and annihilation ($\hat{\lambda}_\nu$) operators, whose action on the Fock vacuum state is defined as follows

$$|\lambda\rangle \equiv e^{\hat{\lambda}^{\dagger\mu}\lambda_\mu} |0\rangle\,, \quad \langle\bar{\lambda}| \equiv \langle 0| \, e^{\bar{\lambda}^\mu \hat{\lambda}_\mu}. \tag{A.2}$$

These states are called coherent states and they obey the following eigenvalue equations

$$\hat{\lambda}_\mu |\lambda\rangle = \lambda_\mu |\lambda\rangle\,, \quad \langle\bar{\lambda}| \, \hat{\lambda}^{\dagger\mu} = \langle\bar{\lambda}| \, \bar{\lambda}^\mu. \tag{A.3}$$

They are normalized as

$$\langle\bar{\lambda}|\lambda\rangle = e^{\bar{\lambda}^\mu \lambda_\mu}. \tag{A.4}$$

Moreover, they satisfy the following properties, i.e. the resolution of the identity and the trace for an operator respectively:

$$\mathbb{1} = \int d^d\bar{\lambda} d^d\lambda \; e^{-\bar{\lambda}^\mu \lambda_\mu} |\lambda\rangle \langle\bar{\lambda}| \tag{A.5}$$

$$\mathrm{Tr}\,\hat{A} = \int d^d\bar{\lambda} d^d\lambda e^{-\bar{\lambda}^\mu \lambda_\mu} \langle\bar{\lambda}| \, \hat{A} \, |\lambda\rangle\,, \tag{A.6}$$

where we employed the following shorthand for the measure

$$d^d\bar{\lambda} d^d\lambda = d\bar{\lambda}^0 d\lambda_0 \dots d\bar{\lambda}^{d-1} d\lambda_{d-1}. \tag{A.7}$$

Appendix B

Computation of modular integrals

Here we follow step by step the computation of the modular integrals whose results are written in (3.96).

Let us start with the simplest one, namely I_1, which gives the example of the procedure followed for the calculation of the others as well. By restoring the value of the quantized Chern-Simons coupling constant $s = 1 - \frac{d}{2}$ and by using the complex definition of the cosine we have

$$\begin{aligned} I_1 &\equiv \int_0^{2\pi} \frac{d\theta}{2\pi} e^{is\theta} \left(2\cos\frac{\theta}{2}\right)^d = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i(1-d/2)\theta} \left(2\cos\frac{\theta}{2}\right)^d \\ &= \int_0^{2\pi} \frac{d\theta}{2\pi} (1+e^{-i\theta})^d \, e^{i\theta}. \end{aligned} \tag{B.1}$$

Let us now perform the change of variable $z = e^{-i\theta}$ so that $d\theta = i\frac{dz}{z}$ to get the contour integral, easily solved by employing the Cauchy's residue theorem

$$I_1 = i \int_C \frac{dz}{2\pi} \frac{(1+z)^d}{z^2} = -2\pi i \ \mathrm{Res}_{z=0} f(z) = d, \tag{B.2}$$

where $f(z) = \frac{i}{2\pi} \frac{(1+z)^d}{z^2}$ and the minus comes from the -1 winding number due to the clockwise contour integration.

Let's now compute the second modular integral I_2 by performing the same change of variable as before

$$\begin{aligned} I_2 &\equiv \int_0^{2\pi} \frac{d\theta}{2\pi} e^{is\theta} \left(2\cos\frac{\theta}{2}\right)^d \tan\frac{\theta}{2} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i(1-d/2)\theta} \left(2\cos\frac{\theta}{2}\right)^d \tan\frac{\theta}{2} \\ &= i \int_0^{2\pi} \frac{d\theta}{2\pi} (1+e^{-i\theta})^{d-1} (1-e^{i\theta}) = -\int_C \frac{dz}{2\pi} \frac{(1+z)^{d-1}(z-1)}{z^2} \\ &= -2\pi i \ \mathrm{Res}_{z=0} f(z) = -i(d-2), \end{aligned} \tag{B.3}$$

where $f(z) = \frac{-(1+z)^{d-1}(z-1)}{2\pi z^2}$.

Finally we proceed with the computation of the third modular integral in a similar fashion, i.e.

$$
\begin{aligned}
I_3 &\equiv \int_0^{2\pi} \frac{d\theta}{2\pi} e^{is\theta} \left(2\cos\frac{\theta}{2}\right)^d \cos^{-2}\frac{\theta}{2} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i(1-d/2)\theta} \left(2\cos\frac{\theta}{2}\right)^d \cos^{-2}\frac{\theta}{2} \\
&= 4i \int_0^{2\pi} \frac{d\theta}{2\pi} (1+e^{-i\theta})^{d-2} = 4i \int_C \frac{dz}{2\pi} \frac{(1+z)^{d-2}}{z} \\
&= -2\pi i \ \mathrm{Res}_{z=0} f(z) = 4,
\end{aligned}
\tag{B.4}
$$

where $f(z) = \frac{4i(1+z)^{d-2}}{2\pi z}$.

Appendix C

Averages computations and dimensional regularization

In what follows we will explain the main steps for the evaluation of the averages $\langle S_4\rangle$, $\langle S_6\rangle$ and $\langle S_4^2\rangle$ needed for the calculation of $\langle e^{-S_{\text{int}}}\rangle$ whose result is reported in (4.77). We will also show how to concretely apply dimensional regularization to evaluate problematic integrals when products of distributions are present.

Let us start with the simplest one, i.e. the evaluation of $\langle S_4\rangle$. By taking the average of (4.74) we obtain

$$\begin{aligned}
\langle S_4\rangle =& \frac{1}{6\beta}R_{\lambda\mu\nu\sigma}\int_0^1 d\tau \langle q^\lambda q^\sigma\left(\frac{1}{2}\dot{q}^\mu\dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu\right)\rangle + \frac{1}{2\beta}R_{\nu\mu ab}\int_0^1 d\tau\langle \dot{q}^\mu q^\nu\bar{\lambda}^a\lambda^b\rangle \\
&+R_{ab}\int_0^1 d\tau\langle\bar{\lambda}^a\lambda^b\rangle + \alpha\beta R \\
=& \frac{1}{6\beta}R_{\lambda\mu\nu\sigma}\int_0^1 d\tau\left[\frac{1}{2}\left(\langle q^\lambda q^\sigma\rangle\langle\dot{q}^\mu\dot{q}^\nu\rangle + \langle q^\lambda\dot{q}^\mu\rangle\langle q^\sigma\dot{q}^\nu\rangle + \langle q^\lambda\dot{q}^\nu\rangle\langle q^\sigma\dot{q}^\mu\rangle\right) + \langle q^\lambda q^\sigma\rangle\langle a^\mu a^\nu\rangle\right. \\
&\left. + \langle q^\lambda q^\sigma\rangle\langle b^\mu c^\nu\rangle\right] - \frac{1}{2\beta}R_{\nu\mu ab}\int_0^1 d\tau\langle\dot{q}^\mu q^\nu\rangle\langle\lambda^b\bar{\lambda}^a\rangle - R_{ab}\int_0^1 d\tau\langle\lambda^b\bar{\lambda}^a\rangle + \alpha\beta R \\
=& \frac{1}{6\beta}R_{\lambda\mu\nu\sigma}\int_0^1 d\tau\left[\frac{1}{2}\left(\langle q^\lambda q^\sigma\rangle\langle\dot{q}^\mu\dot{q}^\nu\rangle + \langle q^\lambda\dot{q}^\nu\rangle\langle q^\sigma\dot{q}^\mu\rangle\right) + \langle q^\lambda q^\sigma\rangle\langle a^\mu a^\nu\rangle\right. \\
&\left. + \langle q^\lambda q^\sigma\rangle\langle b^\mu c^\nu\rangle\right] - R_{ab}\int_0^1 d\tau\langle\lambda^b\bar{\lambda}^a\rangle + \alpha\beta R.
\end{aligned} \tag{C.1}$$

In the latter we used the Wick theorem to perform all the possible Wick contractions and removed the propagators that give a null contribution when multiplied with the Riemann tensor.

By means of the propagators formulae (4.56), (4.58), (4.59) and (4.62) we can write

$$\langle S_4 \rangle = \frac{\beta}{3} R \int_0^1 d\tau [-\Delta|_\tau {}^\bullet\Delta^\bullet|_\tau + \Delta^\bullet|_\tau \Delta^\bullet|_\tau - \Delta|_\tau {}^{\bullet\bullet}\Delta|_\tau] - i\frac{\beta}{2} R \tan\frac{\theta}{2} + \alpha\beta R, \tag{C.2}$$

where we also exploited the identity $\Delta_{gh}(\tau, \sigma) = {}^{\bullet\bullet}\Delta(\tau, \sigma)$ in (4.61) and used (3.91) for the propagator between fermionic variables at equal times. In equation (C.2) one can identify two terms that constitute ambiguous integrals because of the double derivative of the propagator that multiplied with the propagator itself give rise to ill-defined products of Dirac delta distributions and Heaviside theta distributions, as we can see from (4.57) and (4.61). Hence, the manipulations permitted by dimensional regularization, anticipated in Chapter 4, are needed. First we isolate the problematic term and then we perform useful manipulations as follows

$$\begin{aligned}
&\int_0^1 d\tau \Delta|_\tau ({}^\bullet\Delta^\bullet|_\tau + {}^{\bullet\bullet}\Delta|_\tau) \xrightarrow{d+1} \int d^{d+1}t\, \Delta|_t ({}_i\Delta_i + {}_{ii}\Delta)|_t \\
&= \int d^{d+1}t\, \Delta|_t [{}_0({}_0\Delta|_t)] = -\int d^{d+1}t\, \partial_0(\Delta|_t) {}_0\Delta|_t \xrightarrow{d\to 0} -\int_0^1 d\tau \partial_\tau(\Delta|_\tau) {}^\bullet\Delta|_\tau \\
&= -\frac{1}{2}\int_0^1 d\tau (2\tau - 1)^2 = -\frac{1}{6},
\end{aligned} \tag{C.3}$$

where in the first line we extended the compact space $I = [0, 1]$ to the non-compact one $\Omega = I \times R^d$ In the second line we employed the identity (4.70) and, after an integration by parts, we used the expressions for the bosonic propagator and its derivative reported in equations (4.61). The remaining integral is trivial and non troublesome

$$\int_0^1 d\tau \Delta^\bullet|_\tau \Delta^\bullet|_\tau = \int_0^1 d\tau \left(\tau - \frac{1}{2}\right)^2 = \frac{1}{12}. \tag{C.4}$$

Replacing the above results in (C.2) we get

$$\langle S_4 \rangle = \frac{1}{12}\beta R - \frac{i}{2}\beta R \tan\frac{\theta}{2} + \beta\alpha R. \tag{C.5}$$

Let us go ahead with the evaluation of the average $\langle S_6 \rangle$, from (4.75):

$$\begin{aligned}
\langle S_6 \rangle =& \frac{1}{\beta}\left(\frac{1}{40}\nabla_\lambda \nabla_\sigma R_{\alpha\mu\nu\beta} + \frac{1}{45} R_{\tau\lambda\sigma\mu} R^\tau{}_{\alpha\beta\nu}\right) \int_0^1 d\tau \bigg(\frac{1}{2}\langle q^\lambda q^\sigma q^\alpha q^\beta \dot{q}^\mu \dot{q}^\nu\rangle \\
&+ \langle q^\lambda q^\sigma q^\alpha q^\beta a^\mu a^\nu\rangle + \langle q^\lambda q^\sigma q^\alpha q^\beta b^\mu c^\nu\rangle\bigg) \\
&- \frac{1}{\beta}\left[\frac{1}{8}\nabla_\lambda \nabla_\sigma R_{\nu\mu ab} + \frac{1}{24} R^\tau{}_{\nu\lambda\mu} R_{\sigma\tau ab}\right] \int_0^1 d\tau \langle \dot{q}^\mu q^\nu q^\lambda q^\sigma\rangle \langle \lambda^b \bar{\lambda}^a\rangle \\
&- \frac{1}{2}\nabla_\mu \nabla_\nu R_{ab} \int_0^1 d\tau \langle q^\mu q^\nu\rangle \langle \lambda^b \bar{\lambda}^a\rangle + \frac{1}{2}\beta\alpha \nabla_\mu \nabla_\nu R \int_0^1 d\tau \langle q^\mu q^\nu\rangle.
\end{aligned} \tag{C.6}$$

Let us write all the possible (and useful) Wick contractions below:

$$
\begin{aligned}
\langle q^\lambda q^\sigma q^\alpha q^\beta \dot{q}^\mu \dot{q}^\nu\rangle =& \langle q^\lambda q^\sigma\rangle[\langle q^\alpha q^\beta\rangle\langle \dot{q}^\mu \dot{q}^\nu\rangle + \langle q^\alpha \dot{q}^\mu\rangle\langle q^\beta \dot{q}^\nu\rangle + \langle q^\alpha \dot{q}^\nu\rangle\langle q^\beta \dot{q}^\mu\rangle] \\
&+ \langle q^\lambda q^\alpha\rangle[\langle q^\sigma q^\beta\rangle\langle \dot{q}^\mu \dot{q}^\nu\rangle + \langle q^\sigma \dot{q}^\mu\rangle\langle q^\beta \dot{q}^\nu\rangle + \langle q^\sigma \dot{q}^\nu\rangle\langle q^\beta \dot{q}^\mu\rangle] \\
&+ \langle q^\lambda q^\beta\rangle[\langle q^\sigma q^\alpha\rangle\langle \dot{q}^\mu \dot{q}^\nu\rangle + \langle q^\sigma \dot{q}^\mu\rangle\langle q^\alpha \dot{q}^\nu\rangle + \langle q^\sigma \dot{q}^\nu\rangle\langle q^\alpha \dot{q}^\mu\rangle] \qquad (C.7) \\
&+ \langle q^\lambda \dot{q}^\mu\rangle[\langle q^\sigma q^\alpha\rangle\langle q^\beta \dot{q}^\nu\rangle + \langle q^\sigma q^\beta\rangle\langle q^\alpha \dot{q}^\nu\rangle + \langle q^\sigma \dot{q}^\nu\rangle\langle q^\alpha q^\beta\rangle] \\
&+ \langle q^\lambda \dot{q}^\nu\rangle[\langle q^\sigma q^\alpha\rangle\langle q^\beta \dot{q}^\mu\rangle + \langle q^\sigma q^\beta\rangle\langle q^\alpha \dot{q}^\mu\rangle + \langle q^\sigma \dot{q}^\mu\rangle\langle q^\alpha q^\beta\rangle]\,;
\end{aligned}
$$

$$
\langle q^\lambda q^\sigma q^\alpha q^\beta a^\mu a^\nu\rangle = (\langle q^\lambda q^\sigma\rangle\langle q^\alpha q^\beta\rangle + \langle q^\lambda q^\alpha\rangle\langle q^\sigma q^\beta\rangle + \langle q^\lambda q^\beta\rangle\langle q^\sigma q^\alpha\rangle)\langle a^\mu a^\nu\rangle\,; \qquad (C.8)
$$

$$
\langle q^\lambda q^\sigma q^\alpha q^\beta b^\mu c^\nu\rangle = (\langle q^\lambda q^\sigma\rangle\langle q^\alpha q^\beta\rangle + \langle q^\lambda q^\alpha\rangle\langle q^\sigma q^\beta\rangle + \langle q^\lambda q^\beta\rangle\langle q^\sigma q^\alpha\rangle)\langle b^\mu c^\nu\rangle\,. \qquad (C.9)
$$

Let's now compute all the integrals, working backwards.

- $$
\begin{aligned}
&\frac{1}{2}\beta\alpha\nabla_\mu\nabla_\nu R\int_0^1 d\tau\langle q^\mu q^\nu\rangle = -\beta^2\alpha\nabla^2 R\int_0^1 d\tau\Delta|_\tau \\
&= -\beta^2\alpha\nabla^2 R\int_0^1 d\tau\tau(\tau-1) = \frac{\beta^2\alpha}{6}\nabla^2 R;
\end{aligned} \qquad (C.10)
$$
- $$
\begin{aligned}
&-\frac{1}{2}\nabla_\mu\nabla_\nu R_{ab}\int_0^1 d\tau\langle q^\mu q^\nu\rangle\langle\lambda^b\bar{\lambda}^a\rangle = \frac{i}{2}\beta^2\nabla^2 R\tan\frac{\theta}{2}\int_0^1 d\tau\Delta|_\tau \\
&= \frac{i}{2}\beta^2\nabla^2 R\tan\frac{\theta}{2}\int_0^1 d\tau\tau(\tau-1) = -\frac{i\beta^2}{12}\nabla^2 R\tan\frac{\theta}{2};
\end{aligned} \qquad (C.11)
$$
- $$
-\frac{1}{\beta}\left[\frac{1}{8}\nabla_\lambda\nabla_\sigma R_{\nu\mu ab} + \frac{1}{24}R^\tau{}_{\nu\lambda\mu}R_{\sigma\tau ab}\right]\int_0^1 d\tau\langle\dot{q}^\mu q^\nu q^\lambda q^\sigma\rangle\langle\lambda^b\bar{\lambda}^a\rangle = 0. \qquad (C.12)
$$

To compute the first two integrals we used the identities (4.56), (4.61), (4.62) and (3.91). The third integral gives a zero result from contracting η^{ba} coming from the fermionic propagator with the last two indices of the Riemann tensor.

Proceeding with the calculation of the integral for the first term in $\langle S_6\rangle$ we obtain:

$$
\begin{aligned}
\bullet\ \ & \frac{1}{\beta}\bigg(\frac{1}{40}\nabla_\lambda\nabla_\sigma R_{\alpha\mu\nu\beta}+\frac{1}{45}R_{\tau\lambda\sigma\mu}R^\tau{}_{\alpha\beta\nu}\bigg)\int_0^1 d\tau\langle q^\lambda q^\sigma q^\alpha q^\beta b^\mu c^\nu\rangle\\
&=\frac{1}{\beta}(\dots)\int_0^1 d\tau(\langle q^\lambda q^\sigma\rangle\langle q^\alpha q^\beta\rangle+\langle q^\lambda q^\alpha\rangle\langle q^\sigma q^\beta\rangle+\langle q^\lambda q^\beta\rangle\langle q^\sigma q^\alpha\rangle)\langle b^\mu c^\nu\rangle\\
&=\beta^2\bigg(\frac{1}{5}\nabla^2 R-\frac{8}{45}R_{\mu\nu}R^{\mu\nu}+\frac{2}{5}\nabla_\mu\nabla_\nu R^{\mu\nu}-\frac{4}{15}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\bigg)\int_0^1 d\tau\Delta|_\tau\Delta|_\tau\Delta_{gh}|_\tau\\
&=\beta^2(\dots)\,\mathcal{I}_1;
\end{aligned}
\tag{C.13}
$$

$$
\begin{aligned}
\bullet\ \ & \frac{1}{\beta}\bigg(\frac{1}{40}\nabla_\lambda\nabla_\sigma R_{\alpha\mu\nu\beta}+\frac{1}{45}R_{\tau\lambda\sigma\mu}R^\tau{}_{\alpha\beta\nu}\bigg)\int_0^1 d\tau\langle q^\lambda q^\sigma q^\alpha q^\beta a^\mu a^\nu\rangle\\
&=\frac{1}{\beta}(\dots)\int_0^1 d\tau(\langle q^\lambda q^\sigma\rangle\langle q^\alpha q^\beta\rangle+\langle q^\lambda q^\alpha\rangle\langle q^\sigma q^\beta\rangle+\langle q^\lambda q^\beta\rangle\langle q^\sigma q^\alpha\rangle)\langle a^\mu a^\nu\rangle\\
&=\beta^2\bigg(-\frac{1}{10}\nabla^2 R+\frac{4}{45}R_{\mu\nu}R^{\mu\nu}-\frac{1}{5}\nabla_\mu\nabla_\nu R^{\mu\nu}+\frac{2}{15}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\bigg)\int_0^1 d\tau\Delta|_\tau\Delta|_\tau\Delta_{gh}|_\tau\\
&=\beta^2(\dots)\,\mathcal{I}_1;
\end{aligned}
\tag{C.14}
$$

$$
\begin{aligned}
\bullet\ \ & \frac{1}{\beta}\bigg(\frac{1}{40}\nabla_\lambda\nabla_\sigma R_{\alpha\mu\nu\beta}+\frac{1}{45}R_{\tau\lambda\sigma\mu}R^\tau{}_{\alpha\beta\nu}\bigg)\int_0^1 d\tau\langle q^\lambda q^\sigma q^\alpha q^\beta \dot q^\mu \dot q^\nu\rangle\\
&=\beta^2\bigg(\frac{1}{10}\nabla^2 R-\frac{4}{45}R_{\mu\nu}R^{\mu\nu}+\frac{1}{5}\nabla_\mu\nabla_\nu R^{\mu\nu}-\frac{2}{15}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\bigg)\\
&\int_0^1 d\tau(\Delta|_\tau\Delta|_\tau{}^{\bullet}\Delta^{\bullet}|_\tau-\Delta|_\tau\Delta^{\bullet}|_\tau\Delta^{\bullet}|_\tau)\\
&=\beta^2(\dots)\,(\mathcal{I}_2-\mathcal{I}_3).
\end{aligned}
\tag{C.15}
$$

Summing together the results of these last three integrals we get

$$
\beta^2\bigg(\frac{1}{10}\nabla^2 R-\frac{4}{45}R_{\mu\nu}R^{\mu\nu}+\frac{1}{5}\nabla_\mu\nabla_\nu R^{\mu\nu}-\frac{2}{15}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\bigg)(\mathcal{I}_1+\mathcal{I}_2-\mathcal{I}_3). \tag{C.16}
$$

$\mathcal{I}_3$ is the simplest one to compute and, using (4.61), it gives

$$
\mathcal{I}_3=\int_0^1 d\tau\Delta|_\tau\Delta^{\bullet}|_\tau\Delta^{\bullet}|_\tau=-\frac{1}{120}. \tag{C.17}
$$

For the computation of $\mathcal{I}_1$ and $\mathcal{I}_2$ we need once again to apply dimensional regularization to cast the integrals in a nicer form, i.e.

$$\begin{aligned}\mathcal{I}_2 &= \int_0^1 d\tau \Delta|_\tau \Delta|_\tau {}^\bullet\Delta^\bullet|_\tau \xrightarrow{d+1} \int d^{d+1}t\, \Delta|_t\ \Delta|_t\ {}_i\Delta_i|_t \\ &= -\int d^{d+1}t({}_i\Delta|_t\ \Delta|_t\ \Delta_i|_t + \Delta|_t\ {}_i\Delta|_t\ \Delta_i|_t) \xrightarrow{d\to 0} -2\int_0^1 d\tau {}^\bullet\Delta|_\tau \Delta|_\tau \Delta^\bullet|_\tau = \frac{1}{60};\end{aligned} \tag{C.18}$$

$$\begin{aligned}\mathcal{I}_1 &= \int_0^1 d\tau \Delta|_\tau \Delta|_\tau {}^{\bullet\bullet}\Delta|_\tau \xrightarrow{d+1} \int d^{d+1}t\, \Delta|_t\ \Delta|_t\ {}_{ii}\Delta|_t \\ &= -\int d^{d+1}t({}_i\Delta|_t\ \Delta|_t\ {}_i\Delta|_t + {}_i\Delta|_t\ \Delta|_t\ {}_i\Delta|_t) \xrightarrow{d\to 0} -2\int_0^1 d\tau \Delta^\bullet|_\tau \Delta|_\tau \Delta^\bullet|_\tau = \frac{1}{60}.\end{aligned} \tag{C.19}$$

Putting together all the above results we finally obtain

$$\langle S_6 \rangle = \beta^2 \left[\left(\frac{1}{120} + \frac{\alpha}{6} - \frac{i}{12}\tan\frac{\theta}{2} \right) \nabla^2 R - \frac{1}{270} R_{\mu\nu} R^{\mu\nu} - \frac{1}{180} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right] \tag{C.20}$$

Lastly we need to compute $\langle S_4^2 \rangle$. For this purpose it is very convenient to write S_4 as the sum of four pieces, i.e.

$$S_4 = \mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D}, \tag{C.21}$$

where

$$\mathcal{A} = \frac{1}{6\beta} R_{\lambda\mu\nu\sigma} \int_0^1 d\tau q^\lambda q^\sigma \left(\frac{1}{2} \dot{q}^\mu \dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu \right); \tag{C.22}$$

$$\mathcal{B} = \frac{1}{2\beta} R_{\nu\mu ab} \int_0^1 d\tau \dot{q}^\mu q^\nu \bar{\lambda}^a \lambda^b; \tag{C.23}$$

$$\mathcal{C} = R_{ab} \int_0^1 d\tau \bar{\lambda}^a \lambda^b; \tag{C.24}$$

$$\mathcal{D} = \alpha\beta R. \tag{C.25}$$

In this way it is possible to write the square as

$$S_4^2 = \mathcal{A}^2 + \mathcal{B}^2 + \mathcal{C}^2 + \mathcal{D}^2 + 2\mathcal{AB} + 2\mathcal{AC} + 2\mathcal{AD} + 2\mathcal{BC} + 2\mathcal{BD} + 2\mathcal{CD}. \tag{C.26}$$

We now proceed with the computation of all the terms.

$$\bullet\langle\mathcal{D}^2\rangle = \beta^2\alpha^2 R^2 \tag{C.27}$$

$$\bullet\langle 2\mathcal{C}\mathcal{D}\rangle = -2\beta\alpha R R_{ab}\int_0^1 d\tau\langle\lambda^b\bar{\lambda}^a\rangle = -\beta^2\alpha R^2 i\tan\frac{\theta}{2} \tag{C.28}$$

$$\bullet\langle 2\mathcal{B}\mathcal{D}\rangle = 0 \tag{C.29}$$

$$\bullet\langle 2\mathcal{A}\mathcal{D}\rangle = 2\mathcal{D}\langle\mathcal{A}\rangle = 2\beta\alpha R\frac{1}{12}\beta R = \frac{\beta^2}{6}\alpha R^2 \tag{C.30}$$

$$\begin{aligned}
\bullet\langle\mathcal{C}^2\rangle = & R_{ab}R^{cd}\int_0^1 d\tau\int_0^1 d\sigma(\langle\lambda^b(\tau)\bar{\lambda}^a(\tau)\rangle\langle\lambda_d(\sigma)\bar{\lambda}_c(\sigma)\rangle - \langle\lambda_d(\sigma)\bar{\lambda}^a(\tau)\rangle\langle\lambda^b(\tau)\bar{\lambda}_c(\sigma)\rangle) \\
&= \beta^2 R_{ab}R^{cd}\int_0^1 d\tau\int_0^1 d\sigma[\eta^{ba}\eta_{dc}\Delta_F^2(0) - \delta_d^a\delta_c^b\Delta_F(\sigma-\tau)\Delta_F(\tau-\sigma)] \\
&= \beta^2 R^2(\frac{i}{2}\tan\frac{\theta}{2})^2 - \beta^2 R_{ab}R^{ab}(-\frac{1}{4}\cos^{-2}\frac{\theta}{2}) \\
&= -\frac{\beta^2}{4}R^2(\cos^{-2}\frac{\theta}{2} - 1) + \frac{\beta^2}{4}R_{ab}R^{ab}\cos^{-2}\frac{\theta}{2} \\
&= \frac{\beta^2}{4}[(R_{ab}R^{ab} - R^2)\cos^{-2}\frac{\theta}{2} + R^2]
\end{aligned} \tag{C.31}$$

$$\bullet\langle 2\mathcal{B}\mathcal{D}\rangle = 0 \tag{C.32}$$

$$\bullet\langle 2\mathcal{A}\mathcal{C}\rangle = -2\langle\mathcal{A}\rangle R_{ab}\int_0^1 d\sigma\langle\lambda^b\bar{\lambda}^a\rangle = -\frac{i\beta^2}{12}R^2\tan\frac{\theta}{2} \tag{C.33}$$

$$\bullet\langle 2\mathcal{A}\mathcal{B}\rangle = 0 \tag{C.34}$$

In the latter we applied the usual propagator identities and the zeros are due to vanishing contractions of the Riemann tensor.

We now consider the computation of $\langle\mathcal{B}^2\rangle$ which requires a bit more algebra. Taking into account only the Wick contractions that give a non vanishing contribution when multiplied with the Riemann tensors, we get

$$\begin{aligned}
\langle\mathcal{B}^2\rangle =& \frac{1}{4\beta^2}R_{\mu\nu ab}R^{\rho\sigma cd}\int_0^1 d\tau\int_0^1 d\sigma\langle\dot{q}^\mu(\tau)q^\nu(\tau)\bar{\lambda}^a(\tau)\lambda^b(\tau)\dot{q}^\rho(\sigma)q^\sigma(\sigma)\bar{\lambda}^c(\sigma)\lambda^d(\sigma)\rangle \\
=&(\dots)\langle\dot{q}^\mu(\tau)q^\nu(\tau)\dot{q}^\rho(\sigma)q^\sigma(\sigma)\rangle\langle\bar{\lambda}^a(\tau)\lambda^b(\tau)\bar{\lambda}^c(\sigma)\lambda^d(\sigma)\rangle \\
=&(\dots)(\langle\dot{q}^\mu(\tau)\dot{q}^\rho(\sigma)\rangle\langle q^\nu(\tau)q^\sigma(\sigma)\rangle + \langle\dot{q}^\mu(\tau)q^\sigma(\sigma)\rangle\langle q^\nu(\tau)\dot{q}^\rho(\sigma)\rangle) \\
&(\langle\lambda^b(\tau)\bar{\lambda}^a(\tau)\rangle\langle\lambda^d(\sigma)\bar{\lambda}^c(\sigma)\rangle - \langle\lambda^d(\sigma)\bar{\lambda}^a(\tau)\rangle\langle\lambda^b(\tau)\bar{\lambda}^c(\sigma)\rangle).
\end{aligned} \tag{C.35}$$

By means of the familiar propagator identities (4.61) and (4.62) we can write

$$\langle\mathcal{B}^2\rangle = \beta^2 R_{\mu\nu ab}R^{\mu\nu ab}\int_0^1 d\tau\int_0^1 d\sigma({}^\bullet\!\Delta^\bullet\, {}^\bullet\!\Delta - {}^\bullet\!\Delta\Delta^\bullet)\Delta_F(\tau-\sigma)\Delta_F(\sigma-\tau). \tag{C.36}$$

Dimensional regularization is again used to regulate ill-defined products of distributions, i.e. the ones coming from the Dirac delta contained in ${}^\bullet\!\Delta^\bullet$ times the Heaviside step function contained in the Δ_F's. In the analytic extension of the interval $I = [0,1]$ to the non-compact space with $d+1$ dimensions also some traces and Dirac gamma matrices will appear, necessary for the contraction of the extended indices of the derivatives.

$$
\begin{aligned}
&\int_0^1 \int_0^1 d\tau d\sigma (\Delta^\bullet\Delta^\bullet - {}^\bullet\!\Delta\Delta^\bullet)\Delta_F(\tau-\sigma)\Delta_F(\sigma-\tau) \\
&\xrightarrow{d+1} \int d^{d+1}t \int d^{d+1}s [{}_\alpha\Delta_\beta(t,s)\Delta(t,s) - {}_\alpha\Delta(t,s)\Delta_\beta(t,s)] \operatorname{tr}[\gamma^\alpha \Delta_F(t-s)\gamma^\beta\Delta_F(s-t)] \\
&= -2\int d^{d+1}t \int d^{d+1}s [{}_\alpha\Delta(t,s)\Delta_\beta(t,s)] \operatorname{tr}[\gamma^\alpha \Delta_F(t-s)\gamma^\beta\Delta_F(s-t)] \\
&+ \int d^{d+1}t \int d^{d+1}s \Delta_\beta(t,s)\Delta(t,s) \operatorname{tr}\Bigg[\bigg(\gamma^\alpha \frac{\partial}{\partial t^\alpha}\Delta_F(t-s)\bigg)\gamma^\beta\Delta_F(s-t) \\
&+ \Delta_F(t-s)\gamma^\beta\bigg(\Delta_F(s-t)\frac{\overleftarrow{\partial}}{\partial t^\alpha}\gamma^\alpha\bigg)\Bigg].
\end{aligned}
\tag{C.37}
$$

Now, a "mass term" $i\theta$ can be added freely, in order to obtain the Dirac equations:

$$
\begin{aligned}
\bigg(\gamma^\alpha \frac{\partial}{\partial t^\alpha} + i\theta\bigg)\Delta_F(t-s) &= \delta_F(\tau-\sigma)\delta^d(\mathbf{t}-\mathbf{s}), \\
\Delta_F(s-t)\bigg(\gamma^\beta \frac{\overleftarrow{\partial}}{\partial s^\beta} - i\theta\bigg) &= \delta_F(\sigma-\tau)\delta^d(\mathbf{s}-\mathbf{t}).
\end{aligned}
\tag{C.38}
$$

so that we have

$$
\begin{aligned}
&\int d^{d+1}t \int d^{d+1}s \Delta_\beta(t,s)\Delta(t,s)\ \operatorname{tr}\Bigg[\bigg(\gamma^\alpha \frac{\partial}{\partial t^\alpha}\Delta_F(t-s)\bigg)\gamma^\beta\Delta_F(s-t) \\
&+ \Delta_F(t-s)\gamma^\beta\bigg(\Delta_F(s-t)\frac{\overleftarrow{\partial}}{\partial t^\alpha}\gamma^\alpha\bigg)\Bigg] \\
&= 2\int d^{d+1}t \Delta_\beta|_t\ \Delta|_t \operatorname{tr}[\gamma^\beta\Delta_F|_{t=s}] \xrightarrow{d\to 0} 2\int d\tau \Delta^\bullet|_\tau \Delta|_\tau \bigg(\frac{i}{2}\tan\frac{\theta}{2}\bigg) = 0.
\end{aligned}
\tag{C.39}
$$

Hence we only need to compute the following term

$$
\begin{aligned}
&-2\int d^{d+1}t \int d^{d+1}s [{}_\alpha\Delta(t,s)\Delta_\beta(t,s)] \operatorname{tr}[\gamma^\alpha \Delta_F(t-s)\gamma^\beta\Delta_F(s-t)] \\
&\xrightarrow{d\to 0} -2\int_0^1 d\tau \int_0^1 d\sigma\, {}^\bullet\!\Delta\Delta^\bullet \Delta_F(\tau-\sigma)\Delta_F(\sigma-\tau) = -\frac{1}{24}\cos^{-2}\frac{\theta}{2}.
\end{aligned}
\tag{C.40}
$$

The final result is

$$\langle \mathcal{B}^2 \rangle = -\frac{\beta^2}{24} R_{\mu\nu ab} R^{\mu\nu ab} \cos^{-2}\frac{\theta}{2}. \tag{C.41}$$

The last step is the computation of $\langle \mathcal{A}^2 \rangle$, which involves rather substantial and long algebra.

$$\begin{aligned}
\langle \mathbf{A}^2 \rangle = \frac{1}{36\beta^2} R_{\lambda\mu\nu\sigma} R_{\alpha\beta\gamma\delta} \int_0^1 d\tau \int_0^1 d\sigma & \frac{1}{4} \langle q^\lambda(\tau) q^\sigma(\tau) \dot{q}^\mu(\tau) \dot{q}^\nu(\tau) q^\alpha(\sigma) q^\delta(\sigma) \dot{q}^\beta(\sigma) \dot{q}^\gamma(\sigma) \rangle \\
& + \frac{1}{2} \langle q^\lambda(\tau) q^\sigma(\tau) \dot{q}^\mu(\tau) \dot{q}^\nu(\tau) q^\alpha(\sigma) q^\delta(\sigma) a^\beta(\sigma) a^\gamma(\sigma) \rangle \\
& + \frac{1}{2} \langle q^\lambda(\tau) q^\sigma(\tau) \dot{q}^\mu(\tau) \dot{q}^\nu(\tau) q^\alpha(\sigma) q^\delta(\sigma) b^\beta(\sigma) c^\gamma(\sigma) \rangle \\
& + \frac{1}{2} \langle q^\lambda(\tau) q^\sigma(\tau) a^\mu(\tau) a^\nu(\tau) q^\alpha(\sigma) q^\delta(\sigma) \dot{q}^\beta(\sigma) \dot{q}^\gamma(\sigma) \rangle \\
& + \langle q^\lambda(\tau) q^\sigma(\tau) a^\mu(\tau) a^\nu(\tau) q^\alpha(\sigma) q^\delta(\sigma) a^\beta(\sigma) a^\gamma(\sigma) \rangle \\
& + \langle q^\lambda(\tau) q^\sigma(\tau) a^\mu(\tau) a^\nu(\tau) q^\alpha(\sigma) q^\delta(\sigma) b^\beta(\sigma) c^\gamma(\sigma) \rangle \\
& + \frac{1}{2} \langle q^\lambda(\tau) q^\sigma(\tau) b^\mu(\tau) c^\nu(\tau) q^\alpha(\sigma) q^\delta(\sigma) \dot{q}^\beta(\sigma) \dot{q}^\gamma(\sigma) \rangle \\
& + \langle q^\lambda(\tau) q^\sigma(\tau) b^\mu(\tau) c^\nu(\tau) q^\alpha(\sigma) q^\delta(\sigma) a^\beta(\sigma) a^\gamma(\sigma) \rangle \\
& + \langle q^\lambda(\tau) q^\sigma(\tau) b^\mu(\tau) c^\nu(\tau) q^\alpha(\sigma) q^\delta(\sigma) b^\beta(\sigma) c^\gamma(\sigma) \rangle .
\end{aligned} \tag{C.42}$$

As we can see the first term is an 8-point correlation function, which means that all the possible Wick contractions give rise to $7!! = 105$ terms. For this reason, the computation of this term is quite tricky and laborious. Once the evaluation of these 105 terms is done, the computation of the other 8 correlation functions results much simpler. After really careful algebra we obtain

$$\begin{aligned}
\langle \mathbf{A}^2 \rangle = \frac{\beta^2}{9} \int_0^1 \int_0^1 d\tau d\sigma \{ & R^2 [({}^\bullet\Delta^\bullet + \Delta_{gh})|_\tau \Delta|_\tau \Delta|_\sigma ({}^\bullet\Delta^\bullet + \Delta_{gh})|_\sigma + \Delta^\bullet|_\tau^2 \Delta^\bullet|_\sigma^2 \\
& - 2(\Delta^\bullet|_\tau^2 \Delta|_\sigma \Delta_{gh}|_\sigma + \Delta^\bullet|_\sigma^2 \Delta|_\tau {}^\bullet\Delta^\bullet|_\tau)] \\
& + R_{\mu\nu} R^{\mu\nu} [2\Delta|_\tau (({}^\bullet\Delta^\bullet)^2 - \Delta_{gh}^2) \Delta|_\sigma + 2({}^\bullet\Delta^\bullet + \Delta_{gh})|_\tau \Delta^2 ({}^\bullet\Delta^\bullet + \Delta_{gh})|_\sigma \\
& + 2\Delta|_\tau {}^\bullet\Delta^2 ({}^\bullet\Delta^\bullet + \Delta_{gh})|_\sigma + 2({}^\bullet\Delta^\bullet + \Delta_{gh})|_\tau (\Delta^\bullet)^2 \Delta|_\sigma \\
& - 4\Delta^\bullet|_\tau \Delta ({}^\bullet\Delta)({}^\bullet\Delta^\bullet + \Delta_{gh})|_\sigma - 4({}^\bullet\Delta^\bullet + \Delta_{gh})|_\tau \Delta (\Delta^\bullet) \Delta^\bullet|_\sigma \\
& - 4\Delta|_\tau ({}^\bullet\Delta)({}^\bullet\Delta^\bullet) \Delta^\bullet|_\sigma - 4\Delta^\bullet|_\tau \Delta^\bullet ({}^\bullet\Delta^\bullet) \Delta|_\sigma \\
& + 4\Delta^\bullet|_\tau \Delta ({}^\bullet\Delta^\bullet) \Delta^\bullet|_\sigma + 4\Delta^\bullet|_\tau \Delta^\bullet ({}^\bullet\Delta) \Delta^\bullet|_\sigma] \\
& + R_{\mu\nu\alpha\beta}^2 [-3\Delta^2 \Delta_{gh}^2 + 3\Delta^2 ({}^\bullet\Delta^\bullet)^2 - 6(\Delta^\bullet) \Delta ({}^\bullet\Delta^\bullet) {}^\bullet\Delta + 3(\Delta^\bullet)^2 ({}^\bullet\Delta)^2] \}.
\end{aligned} \tag{C.43}$$

The solution is found by applying all the techniques we already mentioned, like dimensional regularization and integration by parts, and by exploiting all the necessary propagator identities. The final result is then given by

$$\langle \mathbf{A}^2 \rangle = \frac{\beta^2}{9}\left(\frac{1}{16}R^2 - \frac{1}{6}R_{\mu\nu}R^{\mu\nu}\right). \tag{C.44}$$

We can now sum up all the results obtained above to write $\langle S_4^2 \rangle$ in the following form

$$\begin{aligned}\langle S_4^2 \rangle = \beta^2 \Bigg[&\left(\frac{37}{144} + \alpha^2 + \frac{\alpha}{6} - i \tan\frac{\theta}{2}\left(\frac{1}{12} + \alpha\right) - \frac{1}{4}\cos^{-2}\frac{\theta}{2}\right) R^2 \\ &+ \left(-\frac{1}{54} + \frac{1}{4}\cos^{-2}\frac{\theta}{2}\right) R_{\mu\nu}R^{\mu\nu} \\ &- \frac{1}{24}\cos^{-2}\frac{\theta}{2} R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\Bigg]. \end{aligned} \tag{C.45}$$

The last step is to add all the results together to get $\langle e^{-S_{int}} \rangle$ as the sum $1 - \langle S_4 \rangle - \langle S_6 \rangle + \frac{1}{2}\langle S_4^2 \rangle$, which can be checked to be the one reported in equation (4.93).

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