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Coordinate transformations in quantum gravity: the Schwarzschild black hole

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Sommario

Una teoria coerente della gravità quantistica deve tenere conto dell'invarianza sotto trasformazioni di coordinate della teoria classica della Relatività Generale. Questa invarianza è principalmente considerata nella teoria linearizzata intorno a un dato background e, di conseguenza, le trasformazioni che vengono prese in considerazione sono deformazioni regolari dell'identità (diffeomorfismi). Tuttavia, le soluzioni delle equazioni di Einstein sono invarianti sotto trasformazioni più generali che dipendono dalle soluzioni stesse e non possono quindi essere ricondotte all'identità. Degli esempi sono le trasformazioni utilizzate per eliminare la singolarità di coordinate sull'orizzonte dello spazio-tempo di Schwarzschild e la trasformazione tra le coordinate di Schwarzschild e quelle armoniche. Considereremo in questo lavoro quest'ultima trasformazione nel contesto di una teoria quantistica. Descriveremo la geometria classica per mezzo di uno stato coerente quantistico e costruiremo degli stati coerenti "areali" e "armonici". Inoltre, definiremo l'operatore che realizza questa trasformazione classica di coordinate a livello quantistico e studieremo alcune delle sue caratteristiche.

Abstract

A consistent quantum theory of gravity must account for the invariance under coordinate transformations of the classical theory of General Relativity. This invariance is mostly considered in the linearized theory around a given background and, as a result, the transformations that are taken into account are smooth deformations of the identity (diffeomorphisms). However, solutions of the Einstein equations are invariant under more general transformations which depend on the solutions themselves and cannot be connected to the identity. Examples are the transformations used to eliminate the coordinate singularity on the horizon of the Schwarzschild space-time and the transformation between Schwarzschild and harmonic coordinates. We consider here this latter transformation in the context of a quantum theory. We will describe the classical geometry by means of a quantum coherent state and construct "areal" and "harmonic" coherent states. Furthermore, we will define the operator which realizes this classical coordinate transformation at the quantum level, and study some of its features.

Contents

Notation and conventions

In this work, Greek indices take values $\mu, \nu, \ldots = 0, 1, 2, 3$ and Latin indices $i, j, \ldots = 1, 2, 3$.
When repeated indices appear in a formula, a summation on those indices is implied, e.g.

$$g_{\alpha\beta} u^\beta = \sum_{\beta=0}^{3} g_{\alpha\beta} u^\beta \, .$$

We will use the "mostly plus" convention for the metric. The Minkowski metric in Cartesian coordinates is then:

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1) \, . \tag{0.0.1}$$

The indices are "raised" and "lowered" using the metric tensor $g_{\mu\nu}$ or its inverse $g^{\mu\nu}$ (i.e. such that $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$):

$$\begin{aligned} u_\mu &= g_{\mu\nu} u^\nu \\ u^\mu &= g^{\mu\nu} u_\nu \, . \end{aligned}$$

In Minkowski (flat) spacetime we define differential operators as:

- the 4-derivative

$$\partial_\mu \equiv \left(\frac{1}{c} \frac{\partial}{\partial t}, \frac{\partial}{\partial x^i} \right) ,$$

- the d'Alembertian

$$\Box \equiv \partial_\mu \partial^\mu \, . \tag{0.0.2}$$

For a vector V^μ we use the convention

$$V^\mu_{\ ,\nu} = \partial_\nu V^\mu \, . \tag{0.0.3}$$

In a generic space-time equipped with a metric tensor $\boldsymbol{g}$ we define differential operators as the covariant derivative for a vector V^μ as

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\lambda\mu} V^\lambda \equiv V^\nu_{;\mu} \,,$$

and for a one-form V_μ

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda_{\mu\nu} V_\lambda \equiv V_{\nu;\mu} \,,$$

where $\Gamma^\lambda_{\mu\nu}$ are the Christoffel symbols

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}) \,.$$

In this space-time the d'Alembertian takes the form $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$.
Where otherwise indicated, we will work with units $c = 1$, while leaving $\hbar$ and G_N explicit. In this way, recalling the definitions of the Planck lenght and mass:

$$\ell_\mathrm{p} = \sqrt{\frac{\hbar G_\mathrm{N}}{c^3}}$$
$$m_\mathrm{p} = \sqrt{\frac{\hbar c}{G_\mathrm{N}}} \,,$$

we find

$$G_\mathrm{N} = \frac{\ell_\mathrm{p}}{m_\mathrm{p}}$$
$$\hbar = m_\mathrm{p} \ell_\mathrm{p} \,.$$

In addition, various notations summarized here will be used:
r: areal radial coordinate,
$\bar{r}$: harmonic radial coordinate,
R_H: horizon size,
R_S: source size,
r_H: horizon location in areal radial coordinate,
r'_H: horizon location in harmonic radial coordinate,
A: subscript relative to areal coordinate,
H: subscript relative to harmonic coordinate.

Preface

The model that most effectively describes gravity up to now is Einstein's theory of General Relativity (GR). Its fundamental equations can be formulated in geometrical terms, which makes it rather elegant, but more importantly, it is the theory we have today with the highest level of predictivity. After the experimental confirmation of the correction on the precession of the perihelion of Mercury's orbit and light deflection during last century, also in more recent times some predictions of GR have been experimentally confirmed: gravitational waves [1] and black holes [2] has been observed.
However, the prediction of the existence of black holes is also the reason why GR cannot be a fundamental theory and this has been a reason for many physicists to begin to question its completeness. Indeed, GR theory predicts that due to the gravitational collapse which leads to the formation of the black hole, the matter compacts towards the center, becoming point-like [3]. This point, which would then be at infinite density and would constitute a divergence for the gravitational field, is called *curvature singularity*. This means that it is not eliminable with a coordinate transformation. The fact that the singularity is unavoidable represents an inconsistency of classical theory and considering the scales of the region of space-time in the immediate vicinity of the singularity, we can assume that an "underlying" quantum theory can fix this inconsistency. This reason, along with the continuing search for a unified theory of fundamental interactions, has led to several attempts of formulating a consistent quantum gravity theory.

The aim of this work is to set up a possible method to study how invariance for generic coordinate changes of solutions of the Einstein's equations (classical) can be described in quantum gravity. To do so we will consider a particular model of quantum gravity and study a particular class of coordinate transformations. Let us see what those are.

Einstein's theory of gravitation is based on the principle of General Relativity, which states that the laws of physics must be the same in any frame of reference. In other words, the laws of physics must not change form by changes in coordinates. In the study of solutions of Einstein's equations, as in understanding the degrees of freedom propagating in gravitational waves, this means dealing with an invariance of the solutions under certain transformations, which can be considered a gauge invariance. Fixing the gauge, in this case, is equivalent to choosing a certain frame of reference. The transformations studied in these cases are small deviations from a background metric, therefore reducible to identity by sending a parameter to zero. There is a class of transformations, however, that remains excluded from these considerations: that of finite transformations containing parameters of the metric, in which a variation of the parameter is not possible because it would correspond to a variation of the solution itself. The transformation we want to consider belongs to the latter type. The solution on which we define the transformation is the Schwarzschild metric in vacuum, the solution of Einstein's equations outside a spherical and static source, and the transformation is the one between "Schwarzschild" coordinates and *harmonic coordinates*, defined in Ref. [4]. The investigation of this transformation in particular could be interesting because the harmonic coordinates correspond to the harmonic gauge fixing, in which the linearized Einstein theory is formulated, since it greatly simplifies the field equations.

Our approach to quantum gravity is that of a Quantum Field Theory (QFT) on flat (i.e. Minkowski) space-time, that is to say, a description of the gravitational interaction in terms of gravitons. More specifically, our work is part of a series of researches inspired by the *corpuscular model* formulated by Dvali and Gomez, who propose to describe a black hole as a system of N gravitons on the verge of a quantum phase transition to a Bose-Einstein condensate. In this context, N is presented as a measure of "classicality" for the system. Moreover, a precise estimation of N is made as a function of only the mass M of the black hole, deriving the so-called scaling law $N \simeq M^2/m_\mathrm{p}^2$. We will see this in more detail in Appendix A, where we report some of the concepts at the base of Dvali and Gomez model. In later works [5, 6], it was found a way to estimate the number of gravitons N by considering the classical gravitational field as a *coherent state* of gravitons. In [7], the coherent state of a toy scalar graviton was defined in more detail and it was shown that the scalar field expectation value on the coherent state is the classical Newtonian gravitational potential.
In this quantum description, the space-time is flat while the coordinate transformations we want to consider act on a metric describing a curved space-time. We then consider GR theory as the classical limit of this quantum description, this means that outside a macroscopic source one must be able to describe the motion of test particles in terms of geodesics in a curved metric. Since space-time is flat, this metric will be *effective*. The only quantity that can provide information about the effective metric is the gravitational potential, which in

turn is described by the coherent state. In [8] we can see how to obtain an effective metric for the potential from the Bootstrapped Theory of Newtonian gravity, which we will not discuss here, but whose development inspired this work approach. The main Refs. about this are [9–12].

The thesis structure is organized as follows. In Chapter 1 we will introduce the GR Lagrangian formalism and study the invariance of the theory under diffeomorphisms. This invariance will lead to the existence of constraints by switching to the Hamiltonian formalism. After introducing the ADM decomposition of the metric, we will then find the canonical equations of the constraints and derive the algebra of the Bergmann-Komar (BK) group, the gauge group for GR. In chapter 2 we will give some examples of coordinate transformations not generated by the algebra defined in the previous chapter, but still belonging to the BK group, for the Schwarzschild solution. In chapter 3, finally, we will try to give a quantum formulation of one of the coordinate transformations defined in the previous chapter, that from areal radial coordinate to harmonic radial coordinate. To do so, we will introduce the description of the classical potential with the coherent state and study coherent states for both coordinates, including also the black hole configuration. We will finally try to define an operator that transforms one coherent state into the other inside the quantum model.

Chapter 1

Coordinate transformations

The theory of General Relativity is invariant for generic coordinate transformations. In this chapter we will discuss the invariance of the theory under diffeomorphisms that can be smoothly connected to the identity. After analyzing the invariance of gravitational action in the Lagrangian formalism, we will see how by switching to the Hamiltonian formulation of gravity, constraints emerge due to the presence of redundant variables. To analyse the dynamics in Hamiltonian formalism, we will see how we can isolate the temporal and spatial part of the metric according to the Arnowitt-Deser-Misner procedure, called ADM decomposition. We will finally find the canonical constraint equations and the algebra of constraints. These corresponds to the algebra of the Bergmann-Komar group, the gauge group of General Relativity.

1.1 Diffeomorphisms in Lagrangian formalism

Our starting point will be the derivation of Einstein field equations from an action principle, which we will now briefly review.
The action is composed by the so called Einstein-Hilbert action with a matter source, represented by the lagrangian density $\mathcal{L}_M$ of the matter:

$$S = S_{EH} + S_M = -\int_\Omega \mathrm{d}^4x\sqrt{-g}\left(\frac{R}{16\pi G_{\mathrm{N}}} + \mathcal{L}_M\right) , \tag{1.1.1}$$

where Ω is a generic portion of the space-time and $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci tensor trace.
The equations of motion are derived varying the action with respect to the

metric tensor. Using (see [4])

$$\delta R = R_{\mu\nu}\,\delta g^{\mu\nu} \tag{1.1.2}$$

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}\,, \tag{1.1.3}$$

and defining

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = 2\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}_M\,, \tag{1.1.4}$$

with the requirement it is covariantly conserved

$$\nabla_\mu T^{\mu\nu} = 0\,, \tag{1.1.5}$$

we finally obtain, neglecting boundary terms:

$$\delta S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(\frac{-R_{\mu\nu}+\frac{1}{2}R\,g_{\mu\nu}}{8\,\pi\,G_{\mathrm{N}}} + T_{\mu\nu}\right)\delta g^{\mu\nu}\,. \tag{1.1.6}$$

Imposing that the action is not affected by variations of the metric, we then get the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} = 8\,\pi\,G_{\mathrm{N}}\,T_{\mu\nu}\,. \tag{1.1.7}$$

We will now illustrate the invariance of the action S under diffeomorphisms, starting by showing the difference between active and passive diffeomorphism and analysing the action of a passive diffeomorphism (namely a change of coordinate connected to the identity) on tensor fields of different type.
Given two n-dimensional C^∞ manifolds $\mathcal{M}$ and $\mathcal{N}$, a diffeomorphism is a C^∞ bijective map of $\mathcal{M}$ onto $\mathcal{N}$ whose inverse is also C^∞.

1.1.1 Active and passive diffeomorphism

Given a manifold $\mathcal{M}$ and a generic scalar function $f : \mathcal{M} \to \mathbb{R}$, we will illustrate passive and active diffeomorphisms acting on f and their different interpretations.
Points and operations among them are described on manifolds by means of charts, defined from the manifold to $\mathbb{R}^n$. Given a point P, we can describe it with two charts ϕ and ϕ' such that $\phi(P) = x(P) = x$ and $\phi'(P) = x'(P) = x'$. Their composition with f evaluated in P is respectively $f(P) = (f\circ\phi^{-1}) \equiv \Phi(x)$ and $f(P) = (f\circ\phi'^{-1}) \equiv \Phi'(x')$, then $\Phi(x) = \Phi'(x')$. In this case the diffeomophism we are considering acts on open subsets of $\mathbb{R}^n$ as a coordinate transformation $\phi' = \phi'(\phi)$ and is not the function f that changes (because points

remain the same) but its composition with the charts. If the coordinates change and the points remain the same, we are interpreting the diffeomorphism in a *passive* way.
Instead, let ψ be an automorphism that maps P into $P' = \psi(P)$ for every P of a given open subset of $\mathcal{M}$. If points are moved within the same open subset covered by the chart ϕ, we will have $\phi(P) = x(P) \equiv x$ and $\phi(P') = x(P') \equiv x'$. We can then define a new function f', called *Lie dragged* of f, as the function pushed forward by the automorphism ψ such that

$$f^*(P') = f^*(\psi(P)) = f(P)\,. \tag{1.1.8}$$

We can describe the composition of f^* and f with charts and the automorphism as

$$\begin{aligned} f^*(P') &= (f^* \circ \psi \circ \phi^{-1})(x) \equiv \Psi'(x)\,,\\ f(P) &= (f \circ \psi^{-1} \circ \phi^{-1})(x') \equiv \Psi(x') \end{aligned}$$

which, together with Eq. (1.1.8), implies $\Psi'(x) = \Psi(x')$. In this case, the diffeomorphism we are considering is an *active* diffeomorphism which transforms points and functions directly on the manifold, while the coordinates remains the same. The *Lie dragged* of tensor fields defined on the manifold will be invariant with respect to the Lie derivative along the direction tangent to the automorphism (for more details, see e.g. [13]).

1.1.2 Tensor fields variations

In this section we will see how "small" change of coordinates of tensor fields change their components, that is, how they modify the composition of tensors with coordinate charts. We have chosen to focus on a small change of coordinates because we will see that it can be interpreted as "dragging backwards" tensorial quantities we are considering. We will then find the Lie algebra of the group of diffeomorphisms on the manifold, which will coincide with the group of coordinate transformations that can be smoothly reconnected to the identity. We will start by considering two charts with overlapping domains in the metric manifold $\mathcal{M}$:

$$x : \mathcal{D}_x \to \mathbb{R}^n\,, \quad x = x^\mu(P) \tag{1.1.9}$$

and

$$x' : \mathcal{D}_{x'} \to \mathbb{R}^n\,, \quad x' = x'^\mu(P) \tag{1.1.10}$$

where $\mathcal{D}_x$ and $\mathcal{D}_{x'}$ are contained in $\mathcal{M}$ and $\mathcal{D}_x \cap \mathcal{D}_{x'} \neq \emptyset$. For small changes between these two charts we will thus consider only points P belonging to $\mathcal{D}_x \cap \mathcal{D}_{x'} \neq \emptyset$, for which we assume:

$$x'^{\mu}(P) = x^{\mu}(P) + \epsilon\, \xi^{\mu}(P)\,, \tag{1.1.11}$$

where we introduced ϵ to keep track of formal expansion around the identity, which is recovered for $\epsilon = 0$, and where ξ^{μ} are the components of the vector field $\vec{\xi}$ when composed with the chart x: $\vec{\xi} = \xi^{\mu} \frac{\partial}{\partial x^{\mu}}$.
Under Eq. (1.1.11), the coordinate basis $\{\partial_{\mu}\}$ transforms as:

$$\frac{\partial}{\partial x^{\mu}} = \left(\frac{\partial x'^{\nu}}{\partial x^{\mu}}\right)_P \frac{\partial}{\partial x'^{\mu}} = \left[\delta^{\nu}_{\mu} + \epsilon \left(\frac{\partial \xi^{\nu}}{\partial x^{\mu}}\right)_P\right] \frac{\partial}{\partial x'^{\mu}}\,. \tag{1.1.12}$$

We will now see how these charts are composed with tensor fields. The simplest case of a tensor field is the scalar function

$$f : \mathcal{D} \to \mathbb{R} \tag{1.1.13}$$

whose composition with charts is

$$\Phi = f \circ x^{-1} : \mathbb{R}^n \to \mathbb{R} \tag{1.1.14}$$

and

$$\Phi' = f \circ x'^{-1} : \mathbb{R}^n \to \mathbb{R}\,. \tag{1.1.15}$$

Since $f = f(P)$ is not depending on the coordinates, we must have

$$\Phi'(x'^{\mu}(P)) = \Phi(x^{\mu}(P)) \tag{1.1.16}$$

We can now substitute Eq. (1.1.11) in the previous equation and expand left handed side in ϵ to the first order, obtaining

$$\Phi'(x'^{\mu}(P)) = \Phi'(x^{\mu}(P) + \epsilon\, \xi^{\mu}) \simeq \Phi'(x^{\mu}(P)) + \epsilon\, \xi^{\mu} \left(\frac{\partial \Phi'}{\partial x^{\mu}}\right)_P \tag{1.1.17}$$

Since we must have $\Phi' = \Phi$ for vanishing ϵ, we can assume the functional change of the composit chart

$$\Phi'(x^{\mu}(P)) = \Phi(x^{\mu}(P)) + \epsilon\, \delta\Phi(x^{\mu}(P))\,, \tag{1.1.18}$$

which we can substitute in the left handed side (1.1.17) of Eq. (1.1.16), from which we finally get the variation:

$$\delta\Phi(x^\mu) = -\xi^\mu \frac{\partial\Phi}{\partial x^\mu} = -\pounds_{\vec{\xi}}\Phi = \pounds_{-\vec{\xi}}\Phi\,, \tag{1.1.19}$$

where $\pounds_{\vec{\xi}}$ denotes the Lie derivative along the vector $\vec{\xi}$. From this result, we can deduce that the passive diffeomorphism between coordinate charts x and x' induced by $\vec{\xi}$ can be seen as backward dragging of the scalar field f along the same direction on the manifold.

Similarly, we can obtain the variation of a vector field $\vec{v} = \vec{v}(P)$. From the composition with charts of its components and basis vectors we get:

$$v'^\nu(x'^\mu(P))\frac{\partial}{\partial x'^\nu} = v^\nu(x^\mu(P))\frac{\partial}{\partial x^\nu}\,. \tag{1.1.20}$$

Substituting in this latter equation Eq. (1.1.12) for the variation of basis vector in the right handed side and Eq. (1.1.11) in the left handed side, and expanding in ϵ to the first order like we did for the scalar function in eqs. (1.1.17) and (1.1.18), we finally obtain the variation:

$$\delta v^\nu = v^\mu \frac{\partial\xi^\nu}{\partial x^\mu} - \xi^\mu \frac{\partial v^\nu}{\partial x^\mu} = \left(\pounds_{-\vec{\xi}}\vec{v}\right)^\nu\,. \tag{1.1.21}$$

Thus, under a small change of coordinates, the vector fields behave like the scalar functions.

We can finally consider a particular tensor field, the metric tensor $\boldsymbol{g}$. From its composition with x and x' charts we get

$$g_{\mu\nu}(x)\,\mathrm{d}\tilde{x}^\mu \otimes \mathrm{d}\tilde{x}^\nu = g'_{\mu\nu}(x')\,\mathrm{d}\tilde{x}'^\mu \otimes \mathrm{d}\tilde{x}'^\nu\,, \tag{1.1.22}$$

where $\{\mathrm{d}\tilde{x}^\mu\}$ and $\{\mathrm{d}\tilde{x}'^\mu\}$ are the basis of 1-forms respectively dual to $\{\partial/\partial x^\mu\}$ and $\{\partial/\partial x'^\mu\}$, and for which:

$$\mathrm{d}\tilde{x}'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\mathrm{d}\tilde{x}^\nu = \left(\delta^\mu_\nu + \epsilon\frac{\partial\xi^\mu}{\partial x^\nu}\right)\mathrm{d}\tilde{x}^\nu\,. \tag{1.1.23}$$

In turn, the components of the metric tensor can be expanded to the first order in ϵ:

$$
\begin{aligned}
g'_{\mu\nu}(x') &= g'_{\mu\nu}(x) + \epsilon\, \xi^\lambda \frac{\partial g'_{\mu\nu}(x)}{\partial x^\lambda} \\
&= g_{\mu\nu}(x) + \epsilon\, \delta g_{\mu\nu}(x) + \epsilon\, \xi^\lambda \frac{\partial g'_{\mu\nu}(x)}{\partial x^\lambda} \, .
\end{aligned}
\tag{1.1.24}
$$

After substituting the expansions in eqs. (1.1.23) and (1.1.24) in the right handed side of Eq. (1.1.22), we finally get the variation:

$$
\begin{aligned}
\delta g_{\mu\nu} &= -\xi^\lambda g_{\mu\nu,\lambda} - g_{\mu\beta}\xi^\beta_{,\nu} - g_{\nu\alpha}\xi^\alpha_{,\mu} \\
&= -\xi_{\mu,\nu} - \xi_{\nu,\mu} + \xi_\alpha g^{\alpha\lambda} \left(g_{\mu\lambda,\nu} + g_{\nu\lambda,\nu} - g_{\mu\nu,\lambda}\right) \\
&= -\left(\xi_{\mu;\nu} + \xi_{\nu;\mu}\right) = \left(\pounds_{-\vec{\xi}}\, \boldsymbol{g}\right)_{\mu\nu} \, ,
\end{aligned}
\tag{1.1.25}
$$

which confirms that the (passive) diffeomorphism between the charts x and x' induce tensor field variations which are generated by Lie derivatives along the tangent vectors of the diffeomorphism, whose Lie algebra is:

$$
\left[\pounds_{-\vec{\xi}}, \pounds_{-\vec{\chi}}\right] = \pounds_{-[\vec{\xi},\vec{\chi}]} \, .
\tag{1.1.26}
$$

1.1.3 Diffeomorphism invariance

The gravitational and matter actions (1.1.1) are invariant for variations of the dynamical variables of the type of Eq. (1.1.25) (but also of eqs. (1.1.19) and (1.1.21) if other dynamical fields are involved), that is under small changes of coordinates which are smoothly connected to the identity.
We start analyzing the invariance of Einstein Hilbert action and matter action when subjected to variations of $\delta \boldsymbol{g}$.
Defining

$$
\bar{S}_{EH} \equiv -16\,\pi\, G_{\mathrm{N}}\, S_{EH} = \int \mathrm{d}^4x \sqrt{-g}\, R \, ,
\tag{1.1.27}
$$

its variation with respect to the metric is

$$
\begin{aligned}
\delta \bar{S}_{EH} &= \int \mathrm{d}^4x \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}\right) \delta g^{\mu\nu} \equiv \int \mathrm{d}^4x \sqrt{-g}\, G_{\mu\nu} \delta g^{\mu\nu} \\
&= -\int \mathrm{d}^4x \sqrt{-g}\, G^{\mu\nu} \delta g_{\mu\nu} \, .
\end{aligned}
\tag{1.1.28}
$$

If $\delta g_{\mu\nu}$ satisfies Eq. (1.1.25), we get

$$\begin{aligned}\delta_{\delta_\xi g}\bar{S}_{EH} &= \int \mathrm{d}^4x\sqrt{-g}\,G^{\mu\nu}(\nabla_\nu\xi_\mu + \nabla_\mu\xi_\nu)\\ &= \int \mathrm{d}^4x\sqrt{-g}\,(\xi_\mu\nabla_\nu G^{\mu\nu} + \xi_\nu\nabla_\mu G^{\mu\nu}) = 0\,,\end{aligned} \tag{1.1.29}$$

where we integrated by parts and neglected boundary terms. We underline that the invariance under diffeomorphism of the gravitational action is a local property valid for every point and every coordinate transformations of the type of Eq. (1.1.11). Moreover, we stress that it is a direct consequence of the Bianchi identities of the Einstein tensor

$$\nabla_\nu G^{\mu\nu} = \nabla_\mu G^{\mu\nu} = 0\,. \tag{1.1.30}$$

Under the same variation $\delta g_{\mu\nu}$, the behaviour of the matter action is:

$$\begin{aligned}\delta_{\delta_\xi g}S_M &= -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,T^{\mu\nu}\delta g_{\mu\nu}\\ &= \frac{1}{2}\int \mathrm{d}^4\sqrt{-g}\,T^{\mu\nu}(\nabla_\nu\xi_\mu + \nabla_\mu\xi_\nu)\\ &= -\frac{1}{2}\int \mathrm{d}^4\sqrt{-g}\,(\xi_\mu\nabla_\nu T^{\mu\nu} + \xi_\nu\nabla_\mu T^{\mu\nu}) = 0\,,\end{aligned} \tag{1.1.31}$$

where again we integrated by parts and neglected boundary terms. Here, the matter action invariance is due to the conservation of the energy-momentum tensor

$$\nabla_\nu T^{\mu\nu} = \nabla_\mu T^{\mu\nu} = 0 \tag{1.1.32}$$

that $\boldsymbol{T}$ satisfies by definition (and in general is enforced by Einstein equations). If other dynamical fields are involved, we need to check the invariance of matter action when these fields are subjected to the diffeomorphisms. Here we briefly analyse the case of a scalar field, for which we have:

$$\begin{aligned}\delta_{\delta\phi}S_\phi &= \int \mathrm{d}^4x\sqrt{-g}\left(\frac{\delta\mathcal{L}_\phi}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}_\phi}{\delta\partial_\mu\phi}\delta\partial_\mu\phi\right)\\ &= \int \mathrm{d}^4x\sqrt{-g}\left(\frac{\delta\mathcal{L}_\phi}{\delta\phi} - \nabla_\mu\frac{\delta\mathcal{L}_\phi}{\delta\partial_\mu\phi}\right)\delta\phi\,.\end{aligned} \tag{1.1.33}$$

The scalar field undergoes a variation equal to Eq. (1.1.19), so we obtain

$$\delta_{\delta_\xi\phi}S_\phi = -\int \mathrm{d}^4x\sqrt{-g}\left(\frac{\delta\mathcal{L}_\phi}{\delta\phi} - \nabla_\mu\frac{\delta\mathcal{L}_\phi}{\delta\partial_\mu\phi}\right)\xi^\nu\partial_\nu\phi\,, \tag{1.1.34}$$

which vanishes when the equations of motion are satisfied, having to be valid for every $\vec{\xi}$.

1.1.4 Coordinate conditions

In theories of linear perturbations, being able to distinguish between the dynamic perturbations of the fields and those due to coordinate changes, as seen above, becomes an important issue. A way to resolve this ambiguity is to choose a specific coordinate system, with a procedure that has analogies to gauge fixing in electrodynamics, for instance.

The Einstein equations consist of 10 algebraically independent equations, as many as the independent components of the Einstein tensor $G_{\mu\nu}$. Nevertheless, these are not enough to determine the 10 independent components of the metric (symmetric tensor of rank 4), since the components of $G_{\mu\nu}$ are related to each other by Bianchi identities:

$$\nabla_\mu G^{\mu\nu} = 0\,. \tag{1.1.35}$$

Thus there are $10 - 4 = 6$ independent equations, leaving 4 degrees of freedom undetermined. These 4 degrees of freedom correspond precisely to the invariance of the solutions for changes of coordinates $x^\mu \mapsto x'^\mu$, involving 4 arbitrary functions x'^μ.
This situation reminds us of Maxwell's equations

$$\partial_\mu F^{\mu\nu} = -J^\nu\,, \tag{1.1.36}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. These are 4 equations for 4 unknowns, but the independent equations are reduced to 3 because of the electromagnetic tensor property similar to Bianchi identities:

$$\partial_\mu\partial_\nu F^{\mu\nu} \equiv 0\,. \tag{1.1.37}$$

This degree of freedom corresponds to the gauge invariance of the vector potential, in fact given a solution A^μ, $A'^\mu = A^\mu + \partial^\mu\Lambda$ is also a solution, with Λ arbitrary.
This ambiguity can be resolved by fixing a particular gauge, i.e., a particular condition for Λ, which when added to the three field equations makes the solution fully determined.
Similarly, we can add to the six independent Einstein equations four conditions for the coordinates, by which the solution is determined unambiguously.
However, this analogy has limitations. The gauge conditions for the vector

potential have no effect on the choice of observers, whereas choosing a specific coordinate system corresponds to selecting specific observers.
Also, imposing gauge conditions for $g_{\mu\nu}$ to be fully determined is generally not at all straightforward. We will now see, however, for the specific case of a weak field, how this procedure greatly simplifies the field equations.

Linearised Einstein equations

If the gravitational field is weak, we can write the metric as a small perturbation around the Minkowski metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon\, h_{\mu\nu}\,, \tag{1.1.38}$$

and expand all terms of the Einstein tensor up to the first order in ϵ. For example, the Ricci scalar reduces to

$$R = \epsilon\,(\Box h - \partial^\mu\partial^\nu h_{\mu\nu}) + \mathcal{O}(\epsilon^2)\,. \tag{1.1.39}$$

Einstein equations take then the linearized form

$$\begin{aligned}\epsilon\,(-\Box h_{\mu\nu} + \eta_{\mu\nu}\Box h + \partial_\mu\partial^\lambda h_{\lambda\nu} + \partial_\nu\partial^\lambda h_{\lambda\mu} - \eta_{\mu\nu}\partial^\lambda\partial^\rho h \\ - \partial_\mu\partial_\nu h) = 16\,\pi\, G_{\rm N}\,\epsilon\, T_{\mu\nu}\,,\end{aligned} \tag{1.1.40}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$ and we also expanded the energy momentum tensor as

$$T^M_{\mu\nu} = T^{(0)}_{\mu\nu} + \epsilon\, T_{\mu\nu}\,, \tag{1.1.41}$$

with $T^{(0)}_{\mu\nu} = 0$ in Minkowski space.
We can now use the invariance for coordinate changes to arrive at a reduced and more easily solvable form of Eq. (1.1.40).
Diffeomorphisms induce a variation of the metric as in Eq. (1.1.25), so we can transform $h_{\mu\nu}$ as

$$h_{\mu\nu} \to \bar{h}_{\mu\nu} = h_{\mu\nu} - (\xi_{\mu;\nu} + \xi_{\nu;\mu})\,, \tag{1.1.42}$$

and set a condition on $\bar{h}_{\mu\nu}$. The most convenient choice is the *harmonic gauge condition* [1] (in a linearized version)

$$2\partial^\mu \bar{h}_{\mu\nu} - \partial_\nu \bar{h} = 0 \tag{1.1.43}$$

[1] Also known as de Donder gauge, we will illustrate the general versionof this condition in the next chapter.

which sets the vector ξ^μ to be such that

$$2\Box\xi_\nu = 2\partial^\mu h_{\mu\nu} - \partial_\nu h\,. \tag{1.1.44}$$

In the harmonic gauge, the field equation (1.1.40) reduces to

$$-\Box\bar{h}_{\mu\nu} = 16\,\pi\,G_{\rm N}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right)\,, \tag{1.1.45}$$

where we used $\Box\bar{h} = 16\,\pi\,G_{\rm N}\,T$ (with $T = \eta^{\mu\nu}T_{\mu\nu}$) and whose general solution can be found by means of propagator.
If, instead, we would solve the equation in **vacuum**, Eq. (1.1.40) would be reduced to

$$\Box\bar{h}_{\mu\nu} = 0\,, \tag{1.1.46}$$

which still has degrees of freedom corresponding to the transformation

$$\bar{h}_{\mu\nu} \to \tilde{h}_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_\mu\theta_\nu - \partial_\nu\theta_\mu\,, \tag{1.1.47}$$

with θ_μ such that $\Box\theta_\mu = 0$. Note that $\tilde{h}_{\mu\nu}$ is a solution and still satisfies the harmonic condition. We conclude that $\bar{h}_{\mu\nu}$ and $\tilde{h}_{\mu\nu}$ represent the same physical situation for arbitrary values of the four parameters θ_μ, so we have only 6-4=2 degrees of freedom physically relevant, that correspond to two different polarizations.

It is also worth addressing the even more specific case of the **Newtonian approximation** where, in addition to the weak field limit, we also assume all matter in the system is moving "slowly", that is, at non-relativistic speed.
The energy-momentum tensor is determined by only the contribution of energy density

$$T^{\mu\nu} \simeq T_{00} = \rho \tag{1.1.48}$$

and then only relevant component of the metric is h_{00}, that will depend solely by spacial coordinates (so that $\partial_t h_{00} = 0$). The harmonic gauge condition (1.1.43) for a static perturbation is always satisfied and then field equations Eq. (1.1.45) simply reduce to:

$$\nabla h_{00} = -8\,\pi\,G_{\rm N}\,T_{00} = -8\,\pi\,G_{\rm N}\,\rho\,. \tag{1.1.49}$$

If we identify

$$h_{00} = -2V_{\rm N}\,, \tag{1.1.50}$$

we find the Poisson equation for the Newtonian potential $V_{\rm N}$ generated by the mass density ρ:

$$\Delta V_{\mathrm{N}} = 4\,\pi\, G_{\mathrm{N}}\,\rho \tag{1.1.51}$$

(in the following, we will show how to solve this equation for a point-like density).

1.2 Gravity in Hamiltonian formalism

We will now illustrate how the diffeomorphism invariance of the Einstein-Hilbert action leads to a set of canonical constraint equations. The constraints form an algebra, which is the algebra of the so-called Bergmann-Komar group, group of gauge symmetries of General Relativity. To get these equations, we will start by seeing how to decompose the metric along foliations in Cauchy hypersurfaces of an hyperbolic 4-dimensional manifold.

1.2.1 ADM decomposition

Let $(\mathcal{M}, \boldsymbol{g})$ be a globally hyperbolic space-time metric manifold, which therefore admits foliations in spatial Cauchy hypersurfaces Σ such that there exists a scalar $t = t(x^\mu)$, constant on these hypersurfaces, so that $\Sigma = \Sigma(t)$.
We can define the chosen foliation by means of the vector $\vec{t}$ tangent to trajectories parametrised by the scalar t. Given the coordinates x^μ, the components of $\vec{t}$ with respect to them are $t^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}$. It can be noted that the vector $\vec{t}$ does not need to be orthogonal to every vector $\vec{X}$ tangent to the hypersurface Σ.
We introduce now a future directed time-like vector field

$$\vec{n} = \frac{\mathrm{d}}{\mathrm{d}\tau}\,, \tag{1.2.1}$$

which is normal to the hypersurfaces Σ and has unit norm, that is

$$X^\mu n_\mu = 0 \tag{1.2.2}$$

$$n^\mu n_\mu = -1\,. \tag{1.2.3}$$

Now we can decompose the time-like vector $\vec{t}$ as

$$\vec{t} = N\vec{n} + \vec{N}\,, \tag{1.2.4}$$

where

1. the coefficient $N > 0$ is called *lapse function*,
2. the vector $\vec{N}$ encodes three *shift functions*,

3. $n_\mu N^\mu = 0$.

There are three shift functions because $\vec{N}$ must belong to the tangent space of Σ, which is 3-dimensional, given that it is orthogonal to $\vec{n}$.
The meaning of the lapse function can be understood by recalling that, in general, the vector associated to the gradient of $\vec{t}$ is orthogonal to the tangent space of Σ, and so

$$N^\mu \, \partial_\mu t = 0 \,, \tag{1.2.5}$$

and that there is a dual relation between $\vec{t}$ and its gradient:

$$t^\mu \nabla_\mu t = 1 \,. \tag{1.2.6}$$

Substituting Eq. (1.2.4) in the latter, we obtain

$$1 = N \, n^\mu \partial_\mu t = N \frac{\mathrm{d}t}{\mathrm{d}\tau} \,,$$

and therefore

$$N = \frac{\mathrm{d}\tau}{\mathrm{d}t} \tag{1.2.7}$$

represent the rate of change of the "orthogonal" (or "synchronous") time τ between two points located on the successive hypersuperfaces $\Sigma(t)$ and $\Sigma(t + \mathrm{d}t)$ and the rate of change of the parametric time between the same two hypersurfaces.
We can write the metric tensor as

$$g_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu \,, \tag{1.2.8}$$

where $h_{\mu\nu}$ is the effectively 3-dimensional metric on the hypersurfaces.
We can immediately verify that the metric tensor takes the form

$$g_{ab} = \begin{pmatrix} -1 & \vec{0} \\ & \\ \vec{0} & h_{ij} \end{pmatrix} \tag{1.2.9}$$

with respect to the tetrad

$$\vec{e}_a = \left\{ \vec{n} = \frac{\mathrm{d}}{\mathrm{d}\tau}, \partial_i \right\} \tag{1.2.10}$$

(where now indices $a, b = \{\tau, i\}$), with respect to which the vector $\vec{t}$ has components

$$t^a \equiv \boldsymbol{g}(\vec{t}, \vec{e}_a) = t^\alpha e^a_\alpha = (N, N^i) \,. \tag{1.2.11}$$

The infinitesimal vector $\mathrm{d}\vec{t}$ has components on the tetrad $\mathrm{d}t^a = (N\mathrm{d}t, N^i\mathrm{d}t)$ and by construction connects two points having the same spacial coordinates x^i on two different hypersurfaces, respectively at constant t and $t+\mathrm{d}t$. The infinitesimal distance bewtween two point of coordinates $P(t,x^i)$ and $Q(t+\mathrm{d}t, x^i+\mathrm{d}x^i)$ is then given by the lenght of a vector $\mathrm{d}\vec{V}$ (see Fig. 1.1), which in turn has components with respect to the tetrad

$$\mathrm{d}V^a = (N\mathrm{d}t, N^i\mathrm{d}t + \mathrm{d}x^i)\,. \tag{1.2.12}$$

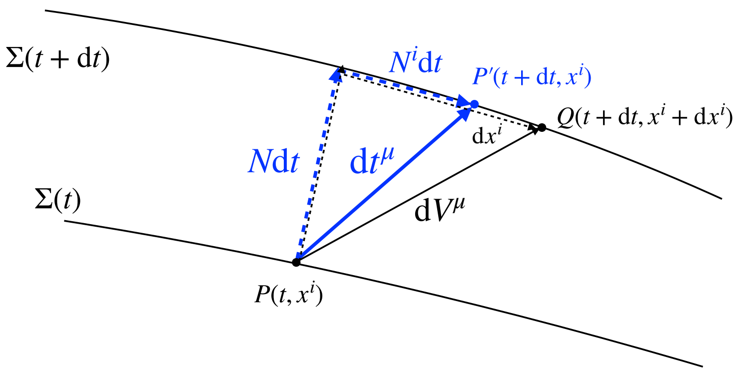


Figure 1.1: Infinitesimal distance between two points P and Q on different hypersurfaces

We can now find the metric tensor components on the coordinate basis adapted to the foliation writing:

$$\begin{aligned}
\mathrm{d}s^2 &= g_{\mu\nu}\mathrm{d}V^\mu\mathrm{d}V^\nu = g_{ab}\mathrm{d}V^a\mathrm{d}V^b \\
&= g_{\tau\tau}\mathrm{d}V^{\tau 2} + g_{ij}\mathrm{d}V^i\mathrm{d}V^j \\
&= -(N\mathrm{d}t)^2 + h_{ij}(N^i\mathrm{d}t + \mathrm{d}x^i)(N^j\mathrm{d}t + \mathrm{d}x^j) \\
&= -(N^2 - h_{ij}N^iN^j)\mathrm{d}t^2 + 2h_{ij}N^i\mathrm{d}t\mathrm{d}x^j + h_{ij}\mathrm{d}x^i\mathrm{d}x^j\,,
\end{aligned} \tag{1.2.13}$$

so that

$$g_{\mu\nu} = \begin{pmatrix} -(N^2 - N^iN_i) & N_j \\ & \\ N_i & h_{ij} \end{pmatrix}\,, \tag{1.2.14}$$

and

$$g^{\mu\nu} = \begin{pmatrix} -\dfrac{1}{N^2} & \dfrac{N^j}{N^2} \\ \\ \dfrac{N^i}{N^2} & h^{ij} - \dfrac{N^i N^j}{N^2} \end{pmatrix}, \tag{1.2.15}$$

where we defined $N_i = h_{ij}N^j$ and such that $g_{\mu\nu}g^{\nu\lambda} = \delta^\lambda_\mu$ and $h_{ij}h^{jk} = \delta^k_i$.
From $\vec{t} = \delta_0$ and its decomposition on $\vec{n}$ and $\vec{N}$ (1.2.4) we finally find

$$n^\mu = \left(\frac{1}{N}, -\frac{N^i}{N}\right) \tag{1.2.16}$$

$$n_\mu = (-N, 0)\,, \tag{1.2.17}$$

which are the components of $\vec{n}$ with respect to coordinate basis ∂_μ.
We can now stop thinking at the 4-dimensional metric manifold and consider only the evolution of the 3-dimensional metric $h_{ij} = h_{ij}(t)$ on the Cauchy surfaces Σ.
We will also need to introduce a quantity which describes the velocity of this evolution of the spacial metric, a tensor which is in turn orthogonal to $\vec{n}$. Therefore, this tensor will be the projection on the "gradient velocity" $\nabla_\mu n_\nu$ on the surface Σ, that is

$$\begin{aligned} K_{\mu\nu} &= h^\alpha_\mu \nabla_\alpha n_\nu \\ &= \frac{1}{2}\left(h^\alpha_\mu \nabla_\alpha n_\nu + h^\alpha_\nu \nabla_\alpha n_\mu\right) = \frac{1}{2}(\pounds_{\vec{n}}\boldsymbol{h})_{\mu\nu}\,, \end{aligned} \tag{1.2.18}$$

where in the second line, since $\vec{n}$ is everywhere orthogonal to Σ, the symmetry of $K_{\mu\nu}$ is ensured by the Frobenius theorem. It can be noted that if this gradient velocity tensor is zero, then $\vec{n}$ is parallelly transported along the geodesic on Σ and therefore the tensor $K_{\mu\nu}$ represents also the *extrinsic curvature* of the hypersurface Σ embedded in the 4-dimensional manifold.
Given that $h_{00} = h_{0i} = 0$, only the components K_{ij} of the velocity tensor are different from zero. We can write them explicitly using the ADM decomposition (1.2.14):

$$K_{ij} = \frac{1}{2N}\left(\dot{h}_{ij} - D_i N_j - D_j N_i\right) = \frac{1}{2N}\left(\dot{h}_{ij} - (\pounds_{\vec{N}}\boldsymbol{h})_{ij}\right)\,, \tag{1.2.19}$$

where D_i denotes the covariant derivative for the 3 metric h_{ij} and which shows how the Lie derivative on $\vec{n}$ splits in agreement to decomposition (1.2.4).

1.2.2 Canonical constraint equations

We will now proceed by writing the Einstein-Hilbert action (1.1.1) in canonical form and from its variations we will get the constraint and dynamical equations. Firstly, we project the trace of the Ricci tensor on the hypersurface Σ, obtaining up to a total derivative term [14],

$$R = K_{ij}K^{ij} - K^2 + R^{(3)}\,, \tag{1.2.20}$$

where the suffix (3) denotes quantities derived by the metric on Σ.
The lagrangian density then is

$$\begin{aligned} \mathcal{L}^G &= \frac{\sqrt{h}N}{16\pi G_{\mathrm{N}}}\left(K_{ij}K^{ij} - K^2 + R^{(3)}\right) \\ &= \frac{N}{16\pi G_{\mathrm{N}}}\left(G^{ijkl}K_{ij}K_{kl} + \sqrt{h}R^{(3)}\right) \end{aligned} \tag{1.2.21}$$

where we used

$$\sqrt{-g} = \sqrt{h}N \tag{1.2.22}$$

and we introduced the DeWitt (super-)metric

$$G_{ijkl} = \frac{\sqrt{h}}{2}\left(h^{ik}h^{jl} + h^{il}h^{jk}2h^{ij}h^{kl}\right)\,, \tag{1.2.23}$$

so that the Einstein-Hilbert action becomes

$$16\pi G_{\mathrm{N}} S_{EH} = \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma(t)} \mathrm{d}^3x\, \sqrt{h}N\left(K_{ij}K^{ij} - K^2 + R^{(3)}\right)\,, \tag{1.2.24}$$

which is also called "ADM action" and it can be seen as a kinetic energy term, since the extrinsic curvature contains the time derivative of h_{ij}, minus a (self-) potential term proportional to the intrinsic curvature $-R^{(3)}$.
Since the lagrangian density does not contain time derivative of lapse and shift functions N and N^i, their momenta vanishes and these are primary constraints in Dirac's classification. Functions N and N^i thus simply play the role of Lagrange multipliers.
The conjugated momenta of h_{ij} are

$$\begin{aligned} P^{ij} &\equiv \frac{\partial \mathcal{L}^G}{\partial \dot{h}_{ij}} = \frac{1}{16\pi G_{\mathrm{N}}} G^{ijkl}K_{kl} \\ &= \frac{\sqrt{h}}{16\pi G_{\mathrm{N}}}\,, \end{aligned} \tag{1.2.25}$$

which can be inverted to obtain

$$\dot{h}_{ij} = \frac{32\pi G_{\mathrm{N}}}{\sqrt{h}} \left(P_{ij} - \frac{1}{2} P h_{ij} \right) + D_i N_j + D_j N_i \,, \tag{1.2.26}$$

where $P \equiv h_{ij} P^{ij}$.
The full hamiltonian then is

$$\begin{aligned} H^G = \int \mathrm{d}^3 x \mathcal{H}^G &= \int \mathrm{d}^3 x \left(P^{ij} \dot{h}_{ij} - \mathcal{L}^G \right) \\ &= \int \mathrm{d}^3 x \left(N \mathcal{H}_0^G + N^i \mathcal{H}_i^G \right) \end{aligned} \tag{1.2.27}$$

where

$$\mathcal{H}_0^G \equiv 16\pi G_{\mathrm{N}} G_{ijkl} P^{ij} P^{kl} - \frac{\sqrt{h} R^{(3)}}{16\pi G_{\mathrm{N}}} \tag{1.2.28}$$

$$\mathcal{H}_i^G \equiv -2 D_i P^{ij} \,. \tag{1.2.29}$$

The latter two expression are called respectively *super-Hamiltonian* and *super-momenta*.
The Einstein Hilbert action can now be written in canonical form as

$$16\pi G_{\mathrm{N}} S_{EH} = \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma(t)} \mathrm{d}^3 x \left(P^{ij} \dot{h}_{ij} - N \mathcal{H}_0^G - N^i \mathcal{H}_i^G \right) \,. \tag{1.2.30}$$

The variations of the action with respect to lapse and shift functions will lead us to canonical constraint equations. Before doing this we must consider the corresponding projections of the energy-momentum tensor, if present, which turn out to be:

$$\mathcal{H}_0^M = \sqrt{h} \rho \tag{1.2.31}$$

and

$$\mathcal{H}_i^M = \sqrt{h} J_i \,, \tag{1.2.32}$$

where $J_i = h_i^\mu T_{\mu\nu} n^\nu$. Adding these terms to the total super-Hamiltonian and momentum density yields:

$$\mathcal{H} = \mathcal{H}^G + \mathcal{H}^M = N(\mathcal{H}_0^g + \mathcal{H}_0^M) + N^i(\mathcal{H}_i^G + \mathcal{H}_i^M) \equiv N\mathcal{H}_0 + N^i \mathcal{H}_i \,. \tag{1.2.33}$$

Primary constraints

$$P_0 \equiv \frac{\partial \mathcal{L}}{\partial \dot{N}} = 0 \tag{1.2.34}$$

$$P_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{N}^i} = 0 \tag{1.2.35}$$

must be conserved by time evolution generated by the total canonical Hamiltonian and this leads to constraint equations:

$$-\dot{P}_0 = -\{P_0, \mathcal{H}\} = \mathcal{H}_0 \approx 0\,, \tag{1.2.36}$$

called *(super-)Hamiltonian constraint*, and

$$-\dot{P}_i = -\{P_i, \mathcal{H}\} = \mathcal{H}_i \approx 0\,, \tag{1.2.37}$$

called *(super-)momentum (diffeomorphism) constraint.*
The Poisson brackets between constraints are themselves combinations of the constraints, which thus are first class constraints and they form an algebra:

$$\{\mathcal{H}_0(x), \mathcal{H}_0(y)\} = \partial_i\delta(x,y)\left[h^{ij}(x)\mathcal{H}_j(x) + h^{ij}(y)\mathcal{H}_j(y)\right] \tag{1.2.38}$$

$$\{\mathcal{H}_i(x), \mathcal{H}_0(y)\} = \partial_i\delta(x,y)\mathcal{H}_i(x) \tag{1.2.39}$$

$$\{\mathcal{H}_i(x), \mathcal{H}_j(y)\} = \partial_j\delta(x,y)\mathcal{H}_i(x) + \delta_i(x,y)\mathcal{H}_j\,, \tag{1.2.40}$$

where the derivatives are acting on the first argument of the δ's.
The meaning of this algebra can be understood if we integrate constraints as distributions on (test) lapse and shift functions, therefore we define:

$$\mathcal{H}[N] = \int_\Sigma \mathrm{d}^3x N(x)\mathcal{H}_0(x) \tag{1.2.41}$$

$$\mathcal{H}[N^i] = \int_\Sigma \mathrm{d}^3x N^i(x)\mathcal{H}_i(x)\,, \tag{1.2.42}$$

so that we can rewrite eqs. (1.2.38)–(1.2.40) as:

$$\{\mathcal{H}[N], \mathcal{H}[M]\} = \mathcal{H}[K^i] \tag{1.2.43}$$

$$\{\mathcal{H}[N^i], \mathcal{H}[N]\} = \mathcal{H}[M] \tag{1.2.44}$$

$$\{\mathcal{H}[N^i], \mathcal{H}[M^j]\} = \mathcal{H}[K^k]\,, \tag{1.2.45}$$

where

$$\begin{aligned} K^i &= h^{ij}(NM_{,j} - MN_{,j}) \\ M &= N^i N_{,i} = \pounds_{\vec{N}} N \\ \vec{K} &= \left[\vec{N}, \vec{M}\right] = \pounds_{\vec{N}} \vec{M}\,. \end{aligned}$$

The above algebra is the one of diffeomorphisms along the vector $\vec{t}$, which split into diffeomorphisms along the direction $\vec{n}$ (generated by $\mathcal{H}_0$) and those along $\vec{N}$ (generated by $\mathcal{H}_i$). This algebra represents a subalgebra of the diffeomorphism algebra (1.1.26), because only N and N^i such as the vector $\vec{t}$ is time-like are allowed as components of direction in which the diffeomorphism is performed. This naturally brings the metric in the algebra, which is thus not a Lie algebra since structure functions depends on the 3-metric h_{ij}, that is, a canonical variable and not a structure constant. The finite transformations generated by such an algebra form the so-called *Bergmann-Komar group* [15] and the algebra in eqs. (1.2.43)–(1.2.45) is the closed Bergmann-Komar sub-algebra of the whole diffeomorphisms algebra.
It can be noted from (1.2.45) that the only generators which forms a closed Lie subalgebra are the super-momenta $\mathcal{H}_i$, which generates spacial diffeomorphisms, while diffeomorphisms along $\vec{n}$ generated by the super-Hamiltonian $\mathcal{H}_0$ do not commute neither with spacial diffeomorphisms generated by $\mathcal{H}_i$ (Eq. (1.2.44)), nor with others diffeomorphisms along the synchronous time (Eq. (1.2.43)).

Chapter 2

Schwarzschild space-time

We have seen that the gauge group for General Relativity is the Bergmann-Komar group of finite transformations generated by the algebra eqs. (1.2.43)–(1.2.45). Again, it should be emphasized that the parameters of the metric, namely of the solution of Einstein equations, enter explicitly in the algebra, and so the coordinate transformations are metric-dependent. This means that the Bergmann-Komar group differs from the group of diffeomorphism because it also contains transformations that cannot be smoothly connected to the identity.

In this chapter, we will show this by considering a specific solution of the Einstein equations, namely the Schwarzschild solution. We will explicitly illustrate some examples of coordinate transformations that map this solution into itself, and for this reason belonging to the Bergmann-Komar group, but not connected to the identity: taking the parameter on which these transformations depend (i.e. the ADM mass M) to zero is not possible, because M parameterizes and uniquely determines the solution. For $M = 0$, the Schwarzschild metric in fact reduces to the Minkowski metric. One of the transformations we will illustrate is the one that maps the Schwarzschild metric written in "areal" coordinate into the metric in "harmonic" coordinate. This will be the transformation that we aim to describe from the quantum point of view, in the next chapter.

2.1 Spherical vacuum

In 1916, few months after Einstein formulation of General Relativity, Karl Schwarzschild found an exact solution of Einstein's field equations. His solution represents the spherically symmetric empty space outside a spherically symmetric source of mass M, and it is used to describe the local geometry of bodies in the solar system, stars and the gravitational collapse to good approx-

imation .
In this section, we will cover the steps to get to the Schwarzschild line element and the critical issues arising from the coordinates used to describe it.

2.1.1 General static isotropic metric

Let us consider what is the most general metric tensor that can represent a static isotropic gravitational field. Starting from a generic line element:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$$

expressed in a set of "quasi-Minkoskian" coordinates $x^0 \equiv t$ and x^i with $i = 1, 2, 3$, we assume that there are four Killing vectors $\frac{\partial}{\partial t}$, $\frac{\mathrm{d}}{\mathrm{d}\theta_i}$, where θ_i's are the rotation angles around the three spatial axes. Thus the components $g_{\mu\nu}$ will depend only on $r \equiv \sqrt{\vec{x}\cdot\vec{x}}$, and we can take as spacial coordinates the set $\{r, \theta, \varphi\}$.
A surface in a spherically symmetric space-time at constant t and r will be a 2-sphere with line element

$$\mathrm{d}l^2 = f(r)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) \equiv f(r)\mathrm{d}\Omega^2$$

and in analogy to Minkowski's spherical coordinates it is possible to decide to define r' as $r' = \sqrt{f(r)}$, since the 2-sphere would have area $4\pi f(r)$. For this reason, by dropping the $'$ symbol, the coordinate r is called *areal radius*.
We are allowed by the spherical symmetry to decide to make the coordinate basis vector $\vec{e}_r$ orthogonal to vectors $\vec{e}_\theta$ and $\vec{e}_\phi$ lying on the 2-sphere, so that diagonal terms $g_{r\theta} = 0 = g_{r\varphi}$.
Moreover, since the whole space-time has spherical symmetry, a line at constant r, θ and φ will also be orthogonal to the 2-sphere, which makes $g_{0\theta}$ and $g_{0\varphi}$ to be zero as well. Then, the line element can be written as

$$\mathrm{d}s^2 = g_{00}\mathrm{d}t^2 + 2g_{0r}\mathrm{d}t\mathrm{d}r + g_{rr}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,.$$

Finally, in addition to the existence of the Killing vector $\frac{\partial}{\partial t}$, the static nature of space-time also implies invariance by time inversion (in the absence of this second condition, the space-time would be stationary but not static). This latter condition causes g_{0r} to be null: time inversion maps g_{r0} in its opposite, leaving other components unchanged.
A general static isotropic line element then is

$$\mathrm{d}s^2 = -B(r)\mathrm{d}t^2 + A(r)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)\,, \qquad (2.1.1)$$

with functions $A(r)$ and $B(r)$ to be determined by solving Einstein's equations.

2.1.2 The Schwarzschild solution

Now, we are going to solve the field equations (1.1.7) in the esmpty space. From the vacuum hypothesis $T_{\mu\nu} = 0$ we obtain that the trace of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is zero:

$$G^{\mu}{}_{\mu} = R - \frac{1}{2} R g^{\mu}{}_{\mu} = -2R = 0\,,$$

from which we get that Einstein equations reduce to:

$$R_{\mu\nu} = 0\,. \tag{2.1.2}$$

The spherical symmetry and static nature of the metric means that the only components of the Riemann tensor computed using (2.1.1) are the diagonal ones. After some algebra [4] we find

$$A(r) = \frac{1}{B(r)} \tag{2.1.3}$$

and

$$B(r) = 1 + \frac{k}{r} \tag{2.1.4}$$

where k is a constant of integration. To fix this constant we impose that at great distances from the source the motion of a test particle is like the motion in a Newtonian potential external to the source $V_{\rm N} = -G_{\rm N}\, M/r$. We then recall that in the Newtonian approximation is possible to establish a relation between the Newtonian potential $V_{\rm N}$ and the perturbation of the Minkowski metric in the weak field limit h_{00} via Eq. (1.1.50), therefore we get

$$g_{00} = -B(r) = -1 - 2V_{\rm N}\,, \tag{2.1.5}$$

so that $k = -2G_{\rm N}\, M$.

We are now ready to write Schwarzschild's metric in its final form:

$$\mathrm{d}s^2 = -\left(1 - \frac{2G_N M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2)\,, \tag{2.1.6}$$

which, for $r \to \infty$, tends to the Minkowski metric (asymptotic flatness) and is singular for $\theta = 0, \pi$ and $r = 0$, $r = 2G_{\rm N}\, M$.

The metric is defined outside the source, so for $r > R_{\rm S}$, where $R_{\rm S}$ is the radius of the spherical massive body and it is usually greater than $2G_{\rm N}\,M$, for which the metric is singular. It is however interesting to see what happens if we consider the metric as an empty space solution for all values of r. The surfaces represented by the two radial singularities must be cutted out, so the manifold is divided by $r = 2G_{\rm N}\,M$ in two disconnected components $0 < r < \infty$ and $r > 2G_{\rm N}\,M$ and, since we must choose a connected one, it is naturally to choose $r > 2G_{\rm N}\,M$.

In Fig. 2.1 is shown the congruence of null radial geodesics, i.e. the trajectories

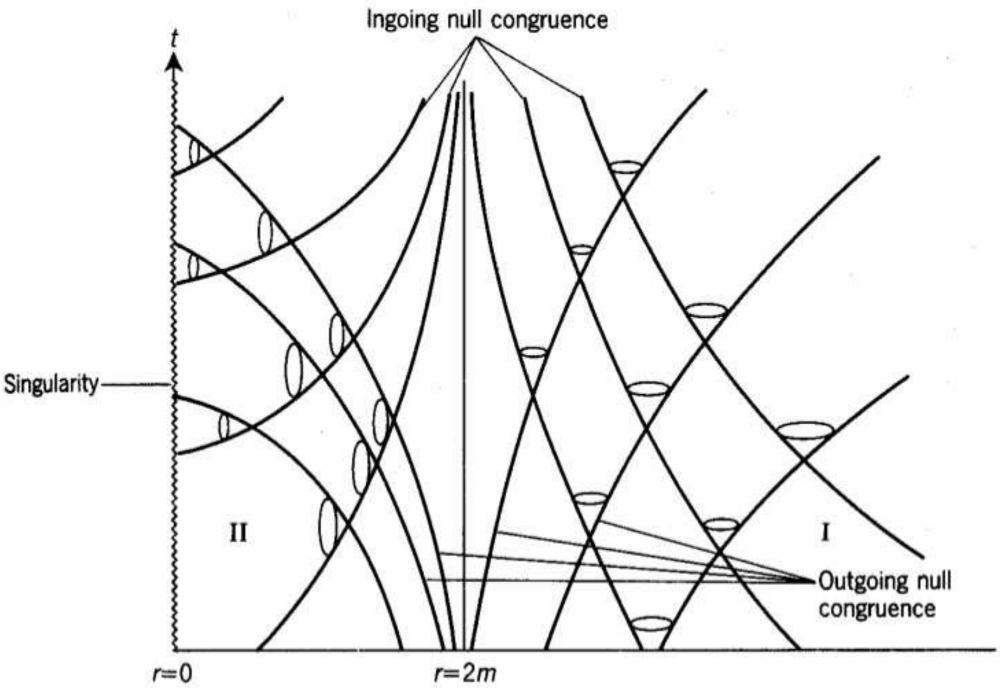


Figure 2.1: Schwarzschild solution. Adapted from [16]

of a photon for fixed θ and φ. These curves are described by $\mathrm{d}s^2 = 0$ with θ and φ constant, from which we obtain

$$\mathrm{d}t = \pm\left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r\,, \tag{2.1.7}$$

by whose integration we get

$$t = \pm(r + 2G_{\rm N}\,M \ln|r + 2G_{\rm N}\,M| + c)\,, \tag{2.1.8}$$

where c is an integration constant and the signs $+$ and $-$ denote the outgoing and incoming radial geodesics, respectively, which are exchanged by the time-

reversal transformation.
In addition to these two congruences, the two-dimensional future light cone is also shown, to emphasize the fact that a trajectory in these coordinates appears never to cross the surface $r = 2G_{\mathrm{N}}\,M$, and thus the disconnected character of this manifold.
Since switching to Cartesian coordinates the singularities in θ disappear, we can wonder if the radial singularities are real (physical) singularities or resulting from a bad choice of coordinates, and therefore eliminable by switching to another chart. We can then consider the scalar invariant of Kretschmann.

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48(G_{\mathrm{N}}\,M)^2}{r^6}\,, \tag{2.1.9}$$

which diverges for $r \to 0$. Therefore only $r = 0$ is a real singularity, while we can eliminate $r = 2G_{\mathrm{N}}\,M$ by changing coordinate system.

2.2 Coordinate extensions

The outer space-time endowed with the Schwarzschild metric $(\mathcal{M}, \boldsymbol{g})$ correspond to the connected part for $r > 2G_{\mathrm{N}}\,M$. This manifold is extendible, that is, there are larger manifolds $\mathcal{M}'$ endowed with a suitable metric $\boldsymbol{g}'$, into which $\mathcal{M}$ is embedded.
We will now look at some of the most mentioned Schwarzschild space-time charts, through which the manifold $\mathcal{M}$ can be extended.

2.2.1 Eddington-Finkelstein extension

In this solution, we will make the two disconnected parts of the Schwarzschild manifold as one connected, but we will lose the symmetry by time inversion, which will result in two different charts: one for the description of the incoming geodesics and one for the outgoing ones.
We want to extend $\mathcal{M}$ where $r \to 2G_{\mathrm{N}}\,M$, which we saw is not a real singularity.
Firstly, we define the so-called [1] *tortoise coordinate*

$$r^* \equiv \int \frac{\mathrm{d}r}{1 - \frac{2G_{\mathrm{N}}\,M}{r}} = r + 2G_{\mathrm{N}}\,M\ln(r - 2G_{\mathrm{N}}\,M) \tag{2.2.1}$$

with which we build the advanced null coordinate

$$v \equiv t + r^*\,. \tag{2.2.2}$$

[1]Introduced for the first time by Wheeler in 1955 [17].

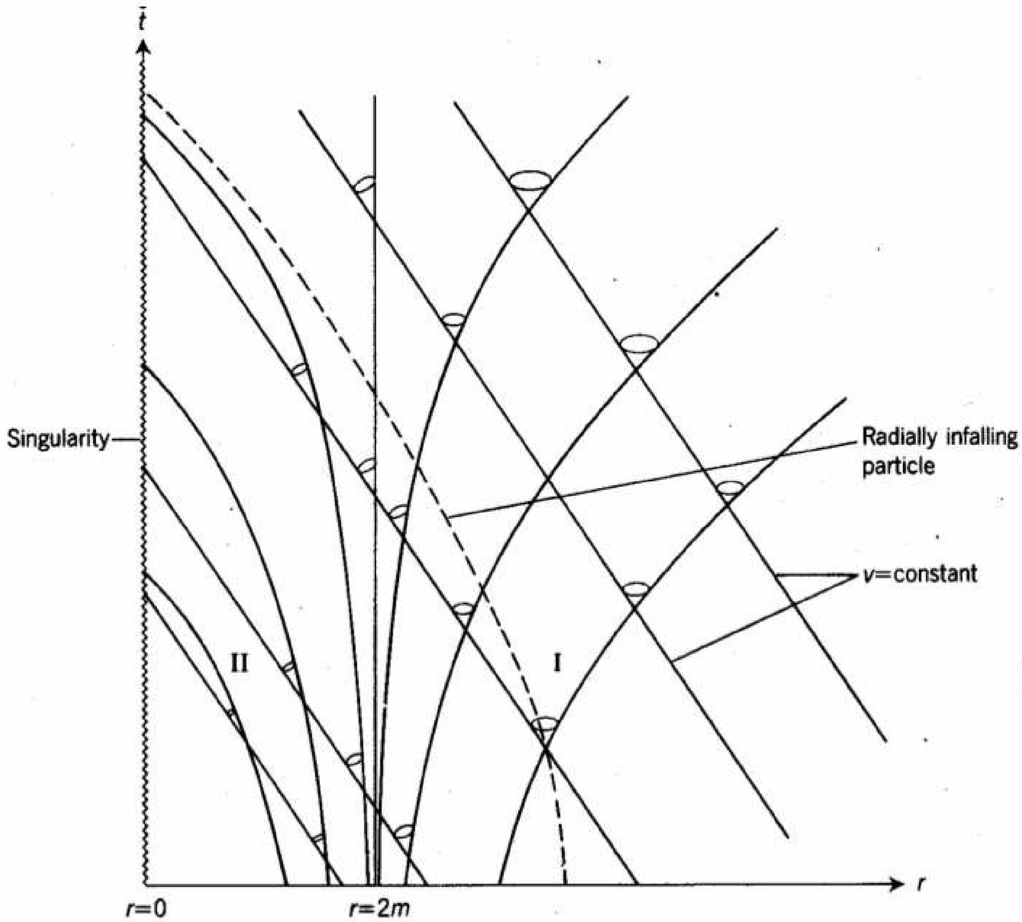


Figure 2.2: Schwarzschild solution in advanced Eddington-Finkeltastein coordinates. Adapted from [16].

Using the coordinates (v, r, θ, φ), the metric takes the *advanced Eddington-Finkelstein* form $\boldsymbol{g}'$:

$$\mathrm{d}s^2 = -\left(1 - \frac{2G_{\mathrm{N}}\, M}{r}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2)\,, \tag{2.2.3}$$

which turns out to be regular in $r = 2G_{\mathrm{N}}\, M$ and thus for all values $0 < r < \infty$ in the manifold $\mathcal{M}'$. This operation is called analytic extension because we have extended the Schwarzschild metric so that it is no longer singular in $r = 2G_{\mathrm{N}}\, M$.
Now in the manifold $\mathcal{M}'$ the surface $r = 2G_{\mathrm{N}}\, M$ is a null surface, as can be seen in fig. 2.2, where is represented the congruence of outcoming and null incoming geodesic, given by constant v. However, this solution is not symmetric with respect to the inversion of t, which can be understood by looking at the diagonal

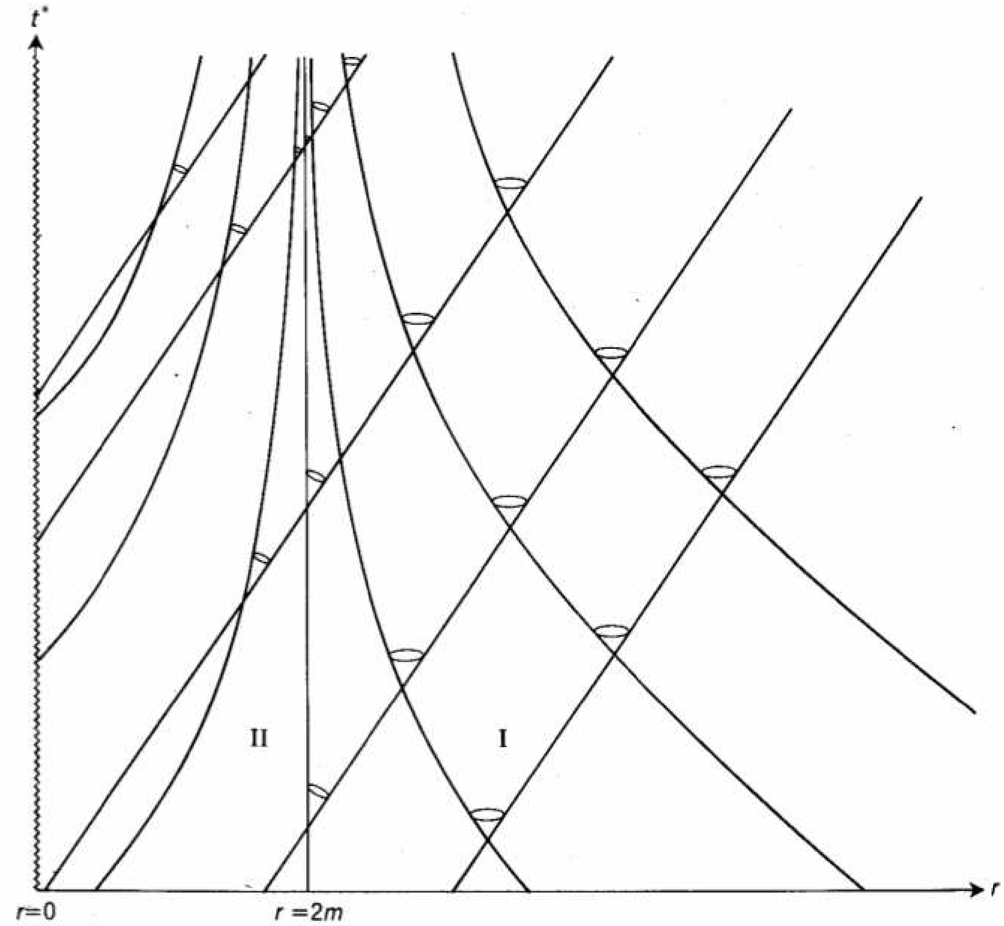


Figure 2.3: Schwarzschild solution in retarded Eddington-Finkeltastein coordinates. Adapted from [16].

terms $\mathrm{d}v\mathrm{d}r$ in Eq. (2.2.3) and at Fig. 2.2, in which the surface $r = 2G_\mathrm{N}\,M$ acts as a one way membrane for future directed curves: they can only cross from the outside to the inside.

We can therefore define the retarded null coordinate

$$w = t - r^*\,, \tag{2.2.4}$$

so that the metric now takes the form

$$\mathrm{d}s^2 = -\left(1 - \frac{2G_\mathrm{N}\,M}{r}\right)\mathrm{d}w^2 - 2\mathrm{d}w\mathrm{d}r + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)\,, \tag{2.2.5}$$

which is now analytic on the manifold $\mathcal{M}''$ defined by the coordinates (w, r, θ, ϕ) for $0 < r < \infty$. In this manifold the surface $r = 2G_\mathrm{N}\,M$ is again a null surface which acts as a one way membrane, but now it lets cross from the outside to

the inside only past-directed curves, as it can be seen in Fig. 2.3.
Considering the surface $r = 2G_{\rm N}\, M$, we have seen how it acts as a "one-way membrane" for trajectories with advanced or retarded time parameter. For this reason, this surface is often referred to as an "event horizon", in the former case it is a future horizon and in the latter a past horizon, and is denoted as

$$R_{\rm H} \equiv 2G_{\rm N}\, M\,. \tag{2.2.6}$$

2.2.2 Kruskal maximal extension

In the previous section we saw how to extend the Schwarzschild solution for $2m < r < \infty$ into the Eddington-Finkelstein solutions for $0 < r < \infty$, which can be either advanced (containing a future event horizon) or retarded (containing a past event horizon). We can now make both these extensions simultaneously and find a still larger manifold $\mathcal{M}^*$ with a metric $\boldsymbol{g}^*$ into which $\mathcal{M}'$ and $\mathcal{M}''$ can be embedded. An example of this manifold has been given by Kruskal in (1960) and it is a *maximal* metric manifold. A manifold endowed with a metric geometry is called *maximal* if every geodesic emanating from an arbitrary point is extensible to infinite values of the affine parameter from both directions or it ends in an intrinsic singularity.
Kruskal found that the maximal solution can be obtained by simultaneously straightening the incoming and outgoing radial null geodesics. Using as coordinates both v and w (advanced and retarded time coordinates of the Eddington-Finkelstein solution), the metric takes the form:

$$\mathrm{d}s^2 = -\left(1 - \frac{2G_{\rm N}\, M}{r}\right)\mathrm{d}v\mathrm{d}w + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) \tag{2.2.7}$$

where r is a function of v and w determined implicitly by the equation:

$$\frac{1}{2}(v - w) = r + 2m\ln(r - 2G_{\rm N}\, M) \tag{2.2.8}$$

If we write the line element (2.2.7) at constant θ, φ as a function of

$$t = \frac{1}{2}(v + w) \tag{2.2.9}$$

$$x = \frac{1}{2}(v - w)\,, \tag{2.2.10}$$

it becomes:

$$\mathrm{d}s^2 = -\left(1 - \frac{2G_{\rm N}\, M}{r}\right)(\mathrm{d}t^2 - \mathrm{d}x^2)\,. \tag{2.2.11}$$

The latter describes a conformally flat 2-space, equipped, however, with a singular metric in $r = 2G_{\rm N}\,M$.
The coordinate transformation that leaves the 2-space conformally flat must be of the type: $v \to v' = v'(v)$, $w \to w' = w'(w)$, leading to

$$\mathrm{d}s^2 = -\left(1 - \frac{2G_{\rm N}\,M}{r}\right)\frac{\mathrm{d}v}{\mathrm{d}v'}\frac{\mathrm{d}w}{\mathrm{d}w'}\mathrm{d}v'\mathrm{d}w' \,. \tag{2.2.12}$$

Introducing, again,

$$t' = \frac{1}{2}(v' + w') \tag{2.2.13}$$

$$x' = \frac{1}{2}(v' - w') \tag{2.2.14}$$

we can write the line element in the general form

$$\mathrm{d}s^2 = -F^2(t', x')(\mathrm{d}t'^2 - \mathrm{d}x'^2) \,. \tag{2.2.15}$$

Kruskal choice was:

$$v' = \exp\left(\frac{v}{4m}\right) \tag{2.2.16}$$

$$w' = -\exp\left(-\frac{w}{4m}\right), \tag{2.2.17}$$

such that the line element is:

$$\mathrm{d}s^2 = -\frac{16m^2}{r}\exp\left(-\frac{r}{2m}\right)(\mathrm{d}t'^2 - \mathrm{d}x'^2) + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) \,. \tag{2.2.18}$$

Now r is determined implicitly by the equation:

$$v'w' = t'^2 - x'^2 = -(r - 2G_{\rm N}\,M)\exp\left(\frac{r}{2G_{\rm N}\,M}\right) \tag{2.2.19}$$

and we can determine an implicit equation for t from the ratio:

$$\frac{v'}{w'} = -\exp\left(\frac{t}{2m}\right) = \frac{t' + x'}{t' - x'} \,. \tag{2.2.20}$$

A two-dimensional space-time diagram of the Kruskal solution is presented in Fig. (2.4).
The coordinate axes are given by $t' = 0$ and $x' = 0$.
Constant r trajectories are also drawn in the figure, which by (2.2.19) are represented by hyperbolas, except $r = 2G_{\rm N}\,M$ which represents the two null directions (asymptotes of the hyperbolas), whereas by (2.2.20) the t =constant

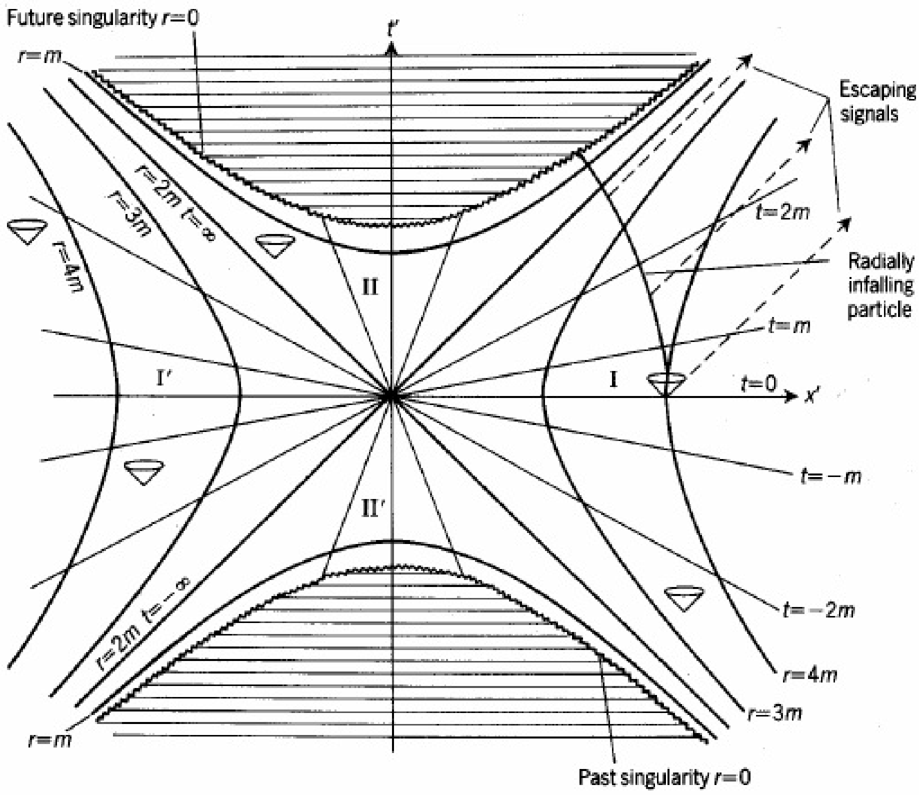


Figure 2.4: Kruskal extension. Adapted from [16].

trajectories are straight lines through the origin. The figure also shows the time-like trajectory of a particle starting from $(r = 4G_{\mathrm{N}}\, M, t = 0)$ and falling towards the event horizon $r = R_{\mathrm{H}}$, ending in the singularity $r = 0$.
Since the (2.2.19) is quadratic in t' and r', for every fixed r there will be two different hypersurfaces: in fact, space-time is now bounded by two hyperbolas, both representing the intrinsic singularity $r = 0$, namely the *past* and the *future* singularities.
Space-time is divided by the asymptotes $r = 2G_{\mathrm{N}}\, M$ into four regions I, II, I' and II'. The region I is isometric to Schwarzschild manifold $(\mathcal{M}, \boldsymbol{g})$, and so it is region I'. If we take both regions I and II, together they are isometric to the advanced Eddington-Finkelstein extension $(\mathcal{M}', \boldsymbol{g}')$, similarly, regions I and II' are isometric to the retarded Eddington-Finkelstein extension $(\mathcal{M}'', \boldsymbol{g}'')$.
Region I' can be seen as another asymptotically flat universe which cannot be connected with region I by any time-like or null curve and, combined with region II and II', is isometric to $(\mathcal{M}', \boldsymbol{g}')$ and $(\mathcal{M}'', \boldsymbol{g}'')$ respectively.

We have seen two examples of charts for Schwarzschild space-time, a solution of Einstein equation, and then transformations which leave field equation

solutions invariant and for this belonging to the Bergmann-Komar group.
We can however observe that the coordinate transformations we employed, like eqs. (2.2.2) and (2.2.4) and eqs. (2.2.16) and (2.2.17), depend explicitly on the parameter M of the metric, and hence of the solution. It is not possible to connect them to the identity by sending the parameter to zero, because variations of the parameter correspond to different solutions and therefore to different spacetimes.

2.3 Harmonic coordinates

We can now consider another coordinate system on the Schwarzschild spacetime, that is harmonic coordinates.
In general, harmonic coordinates are determined by a particular condition, called *harmonic gauge*. These coordinates are particularly useful in the context of linearized gravity theory. Indeed, the imposition of the harmonic gauge greatly simplifies the linearized field equations, as we saw in Eq. (1.1.45).
Therefore, in this section we well see, first, how the harmonic gauge condition is related to a coordinate condition, then we will see how to employ them to map the Schwarzschild space-time.

2.3.1 Harmonic gauge condition

In the previous chapter we saw that a particularly convenient gauge fixing choice is represented by the *harmonic gauge conditions*:

$$\Gamma^\lambda \equiv g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0\,. \tag{2.3.1}$$

We will now analyse more in detail the relation between these conditions and the coordinates.
First of all, we verify that it is always possible to choose a coordinate system in which these conditions are valid. The affine connection, under a generic coordinate change, transforms as [4]:

$$\Gamma'^\lambda_{\mu\nu} = \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\delta}{\partial x'^\nu}\Gamma^\rho_{\sigma\delta} - \frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma}\,,$$

thus

$$\Gamma'^\lambda = g'^{\mu\nu}\Gamma'^\lambda_{\mu\nu} = \frac{\partial x'^\lambda}{\partial x^\rho}\Gamma^\rho - g^{\rho\sigma}\frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma}\,. \tag{2.3.2}$$

If Γ^λ do not vanish, it is always possible to find a coordinate system such that

$$g^{\rho\sigma}\frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma} = \frac{\partial x'^\lambda}{\partial x^\rho}\Gamma^\rho\,, \tag{2.3.3}$$

so that $\Gamma'^\lambda = 0$ in x'^μ coordinate system.
If we want to explicitate Eq. (2.3.1) and to put it in term of the metric tensor components instead of affine connection coefficients, we can use the relations:

$$\begin{aligned} g^{\lambda\sigma}\partial_\nu g_{\sigma\mu} &= -g_{\lambda\sigma}\partial_\nu g^{\sigma\mu} \\ \frac{1}{2}g^{\mu\nu}\partial_\sigma g_{\mu\nu} &= \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\sigma}\sqrt{-g} \end{aligned}$$

and substitute them in affine connection coefficient contracted with the metric, obtaining

$$\Gamma^\lambda = \frac{1}{2}g^{\mu\nu}g^{\lambda\sigma}(\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}) = -\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\rho}\left(\sqrt{-g}g^{\rho\lambda}\right)\,,$$

so that now the harmonic condition reads

$$\frac{\partial}{\partial x^\rho}\left(\sqrt{-g}g^{\rho\lambda}\right) = 0\,. \tag{2.3.4}$$

Finally, we recall that a function ϕ is said to be "harmonic" if it satisfies

$$\Box\phi \equiv \nabla_\mu\nabla^\mu\phi = 0\,. \tag{2.3.5}$$

Given a generic function ϕ, if we write explicitly the two terms of the covariant derivative, we get

$$\Box\phi = \nabla_\mu(g^{\mu\nu}\partial_\nu\phi) = g^{\mu\nu}\partial_\mu\partial_\nu\phi - \Gamma^\lambda\partial_\lambda\phi\,. \tag{2.3.6}$$

If the condition Eq. (2.3.1) is valid, then $\Box\phi = g^{\mu\nu}\partial_\mu\partial_\nu\phi$, but if in place of a generic function ϕ we had the coordinates x^μ, remembering that $\partial_\mu x^\nu = \delta^\nu_\mu$, we get that the coordinates themselves are harmonic functions, i.e.

$$\Box x^\mu = 0\,. \tag{2.3.7}$$

Therefore, Eq. (2.3.1) is called harmonic gauge condition because it imposes the coordinates to be harmonic functions via Eq. (2.3.7).

2.3.2 Harmonic Schwarzschild coordinates

We now want to find harmonic coordinates for the Schwarzschild space-time. We have seen that given some coordinates x^μ for which $\Gamma^\lambda \neq 0$, it is always possible to find a coordinate system $\bar{x}^\mu$ for which $\bar{\Gamma}_\lambda = 0$.
Thus imposing the equation (2.3.3), where now x' is $\bar{x}$, we see that this is

$$0 = g^{\mu\rho}\partial_\mu\partial_\rho\bar{x}^\lambda - g^{\mu\rho}\Gamma^\lambda_{\mu\rho}\partial_\nu\bar{x}^\nu \equiv \Box\bar{x}^\lambda \,. \tag{2.3.8}$$

We then need to find $\bar{x}^\lambda$ such that $\Box\bar{x}^\lambda = 0$, where the derivatives are to be done with respect to the coordinate system x^μ. Since we know that Schwarzschild space-time is static and spherically symmetric, we assume that the coordinate system $\bar{x}^\mu$ is given by the quasi-Minkowskian set $(\bar{t}, \bar{x}_i)$ such that:

$$\begin{aligned} \bar{t} &= t \\ \bar{x}_1 &= \bar{r}(r)\sin\theta\cos\varphi \\ \bar{x}_2 &= \bar{r}(r)\sin\theta\sin\varphi \\ \bar{x}_3 &= \bar{r}(r)\cos\theta \,, \end{aligned} \tag{2.3.9}$$

where $\bar{r} = \bar{r}(r)$ is a smooth and invertible function.
For a generic static and isotropic metric written in the form (2.1.1), a straightforward calculation gives [4]

$$\begin{aligned} 0 = \Box\,\bar{x}_i \equiv g^{\mu\nu}\left[\frac{\partial^2\bar{x}_i}{\partial x^\mu\partial x^\nu} - \Gamma^\lambda_{\mu\nu}\frac{\partial\bar{x}_i}{\partial x^\lambda}\right] \\ = \left(\frac{\bar{x}_i}{A\bar{r}}\right)\left[\left(\frac{B'}{2B} + \frac{2}{r} - \frac{A'}{2A}\right)\bar{r}' + \bar{r}'' - \frac{2A}{r^2}\bar{r}\right] \end{aligned} \tag{2.3.10}$$

where the prime symbol $'$ is intended as a derivation with respect to r. Coordinates $\bar{x}_i$ are therefore harmonic if $\bar{r}$ satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\sqrt{\frac{A}{B}}\frac{\mathrm{d}\bar{r}}{\mathrm{d}r}\right) - 2\sqrt{AB}\bar{r} = 0\,. \tag{2.3.11}$$

The line element in harmonic coordinates $(t, \bar{x}_i)$ is then

$$\mathrm{d}s^2 = -\bar{B}(\bar{r})\mathrm{d}t^2 + \frac{r^2}{\bar{r}^2}\mathrm{d}\bar{x}_i\,\mathrm{d}\bar{x}^i + \left[\frac{\bar{A}(\bar{r})}{\bar{r}^2\,\bar{r}'^2} - \frac{r^2}{\bar{r}^4}\right](\bar{x}_i\,\mathrm{d}\bar{x}^i)^2 \tag{2.3.12}$$

where $\bar{B}(\bar{r}) = B(r)$, $\bar{A}(\bar{r}) = A(r)$ and $\mathrm{d}\bar{x}_i\,\mathrm{d}\bar{x}^i = \mathrm{d}\bar{r}^2 + \bar{r}^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)$ and $(\bar{x}_i\,\mathrm{d}\bar{x}^i)^2 = \bar{r}^2\mathrm{d}\bar{r}^2$ are rotational invariant forms.
After solving Einstein equations and substituting values for A and B, the dif-

ferential equation for $\bar{r}$ becomes:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2\left(1-\frac{2G_\mathrm{N}\,M}{r}\right)\frac{\mathrm{d}\bar{r}}{\mathrm{d}r}\right]-2\bar{r}=0\,. \tag{2.3.13}$$

It can be found that a convenient choice for integration constants is [4]:

$$\bar{r}=r-G_\mathrm{N}\,M\,, \tag{2.3.14}$$

so that the metric Eq. (2.3.12) takes the form:

$$\begin{aligned}\mathrm{d}s^2=-\left(\frac{1-G_\mathrm{N}\,M/\bar{r}}{1+G_\mathrm{N}\,M/\bar{r}}\right)\mathrm{d}t^2+\left(1+\frac{G_\mathrm{N}\,M}{\bar{r}}\right)\mathrm{d}\bar{x}_i\,\mathrm{d}\bar{x}^i+&\\+\left(\frac{1+G_\mathrm{N}\,M/\bar{r}}{1-G_\mathrm{N}\,M/\bar{r}}\right)\frac{G_\mathrm{N}^2M^2}{\bar{r}^4}(\bar{x}_i\,\mathrm{d}\bar{x}^i)^2\,.&\end{aligned} \tag{2.3.15}$$

From now on, we will refer to $\bar{r}$ as the *harmonic* radial coordinate. We must emphasize, however, that this coordinate is not harmonic; we call it so because it is the radial coordinate of the polar coordinates with which we transformed the quasi-Minkowskian harmonic coordinates $\bar{x}_i$, but it does not satisfy harmonic condition.

We can see how the transformation (2.3.14) has "shifted" the coordinate singularity for $r=R_\mathrm{H}$ of the Schwarzschild metric (2.1.6) into that for $\bar{r}=G_\mathrm{N}\,M=R_\mathrm{H}/2$ of the metric with harmonic coordinates.

It is convenient, for what we will do in the next chapter, to make a description of the metric in terms of the classical potential. In GR there is no invariant notion for the potential, however, we saw how the $\{00\}$ component of the metric and the Newtonian potential were related in the Newtonian approximation via Eq. (1.1.50). In the weak field and non relativistic limit then, a test particle moves on geodesics well described by a Newtonian-type potential, whose link to the metric tensor is

$$g_{00}=-(1+2V)=-B\,, \tag{2.3.16}$$

where we have assumed that the source we are considering is spherically symmetric and then we can use the static and isotropic metric (2.1.1).

The potential corresponding to the Schwarzschild metric of Eq. (2.1.6) described by areal coordinate is then:

$$V_A(r)=\frac{1}{2}(B-1)=-\frac{G_\mathrm{N}\,M}{r}\,, \tag{2.3.17}$$

which we will call *areal potential*.

The potential corresponding to the Schwarzschild metric of Eq. (2.3.15) described by harmonic coordinates is instead:

$$V_H(\bar{r}) = \frac{1}{2}(\bar{B} - 1) = -\frac{G_{\mathrm{N}}\, M}{\bar{r}} \left(1 + \frac{G_{\mathrm{N}}\, M}{\bar{r}}\right)^{-1} , \qquad (2.3.18)$$

which we will call *harmonic potential* [2].

[2] In this case we use the adjective "harmonic" to refer to harmonic coordinates. This is not to be confused with the usual quadratic potential in x, also called harmonic.

Chapter 3

Quantum Schwarzschild geometry

In this chapter we will see how to describe a coordinate transformation of the type discussed in the previous chapter with a simple model of gravitons as a field theory on flat space-time.
We will examine the transformation between the Schwarzschild coordinates describing the metric (2.1.6) and the harmonic coordinates describing the metric (2.3.15), realized in the General Relativity theory by the transformation between the areal radial coordinate r and the harmonic one $\bar{r}$ via Eq. (2.3.14). How to describe this coordinate transformation on Schwarzschild space-time in a quantum frame?
First, we need to find a relation between quantum theory and classical theory (i.e., the geometric description of space-time). One of the possible ways is to use the classical potential V: the metric tensor can be described in terms of the potential via Eq. (2.3.16) in the Newtonian limit, on the other hand. this potential can also be used to describe the mean field force acting on the constituents of a system as baryons and gravitons. In this way one can find the geometric description (described in terms of the metric tensor) of gravity as emerging at macroscopic scales from an underlying quantum theory.
The potential will then be given by the expectation value of the gravitonic field on a certain state. For the description of the gravitonic field for simplicity we will choose a massless free scalar field Φ.
The quantum state that best describes a classical configuration is the *coherent state*, this can be seen in Refs. [5, 6, 18] for the case of a photon field whose expectation value on the coherent state reproduces the Coulomb potential, in [19] for generic solitons and in [9] for the Newton potential, which we will next review.
The coherent state will then be the state describing the classical potential and, hence, the effective metric related to it.

Returning to the coordinate transformation (2.3.14), we will then have two different coherent states, one describing the areal potential $V_A(r)$ given by Eq. (2.3.17), and one that will describe harmonic potential $V_H(r)$ given by Eq. (2.3.18).
We must underline that in this frame both potential are dependent on the same coordinate r (see Fig. 3.1). The coordinate r is the polar radial coordinate of the Minkoswki metric describing the flat space-time in which the dynamical fields we are considering are defined, not to be confused therefore with the areal and harmonic radial coordinates describing the curved Schwarzschild space-time.

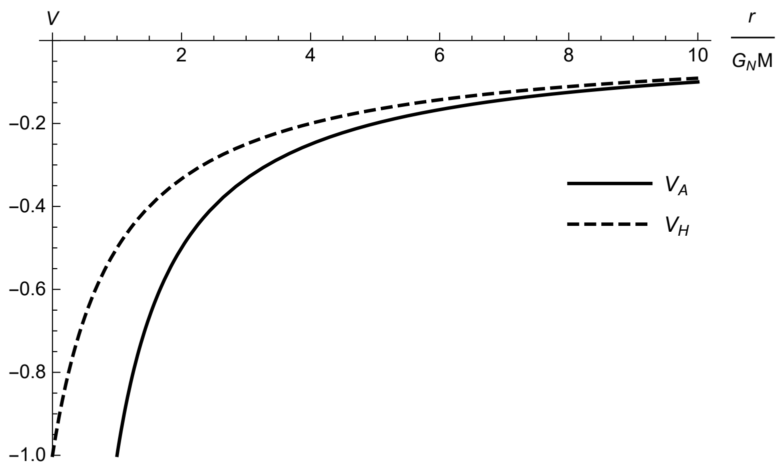


Figure 3.1: Areal (solid line) and harmonic (dashed line) potentials.

We will now see how to define the quantum coherent state starting from a generic potential for a spherical and static source and how to realize a transformation between two of these coherent states. We will then define separately the two coherent states for the two potentials of eqs. (2.3.17) and (2.3.18) and for each of them we will also consider a possible black hole configuration. Finally we will define the operator that performs the transformation (2.3.14).

3.1 Quantum state for spherical sources

A generic static potential $V = V(r)$ is dimensionless, so we first rescale it as to obtain a canonically normalised real scalar field Φ:

$$\Phi = \sqrt{\frac{m_{\mathrm{p}}}{\ell_{\mathrm{p}}}}\, V\,. \tag{3.1.1}$$

We proceed by quantising Φ as a massless field satisfying the free wave equation

$$\left[-\frac{\partial^2}{\partial t^2}+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right]\Phi(t,r)\equiv\left[-\partial_t^2+\Delta\right]\Phi=0\,. \tag{3.1.2}$$

whose solutions can be written as

$$u_k(t,r)=e^{ikt}j_0(k,r)\,, \tag{3.1.3}$$

where $k>0$ and $j_0(kr)=\frac{\sin(kr)}{kr}$ are spherical Bessel functions, which are eigenfunctions of the Laplace operator and satisfy the orthogonality relation

$$4\pi\int_0^\infty r^2\,\mathrm{d}r\,j_0(kr)\,j_0(pr)=\frac{2\,\pi^2}{k^2}\,\delta(k-p)\,. \tag{3.1.4}$$

The quantum field operator and its conjugate momentum are then given by:

$$\hat{\Phi}(t,r)=\int_0^\infty\frac{k^2\,\mathrm{d}k}{2\,\pi^2}\sqrt{\frac{\hbar}{2\,k}}\left[\hat{a}_k\,u_k(t,r)+\hat{a}_k^\dagger\,u_k^*(t,r)\right] \tag{3.1.5}$$

$$\hat{\Pi}(t,r)=i\int_0^\infty\frac{k^2\,\mathrm{d}k}{2\,\pi^2}\sqrt{\frac{\hbar\,k}{2}}\left[\hat{a}_k\,u_k(t,r)-\hat{a}_k^\dagger\,u_k^*(t,r)\right]\,, \tag{3.1.6}$$

where the creation and annihilation operators satisfy

$$\left[\hat{a}_k,\hat{a}_p^\dagger\right]=\frac{2\,\pi^2}{k^2}\,\delta(k-p) \tag{3.1.7}$$

and the factor $\sqrt{\frac{\hbar}{2\,k}}\equiv f(k)$ has been computed imposing the equal time commutation relation:

$$\left[\hat{\Phi}(t,r),\hat{\Pi}(t,s)\right]=\frac{i\,\hbar}{4\,\pi\,r^2}\delta(r-s)\,. \tag{3.1.8}$$

Quantum states in the Fock space are built from the vacuum $|0\rangle$ defined by $\hat{a}_k\,|0\rangle=0$ for all $k>0$.
The coherent state $|g\rangle$ is then defined as the state obeying:

$$\hat{a}_k\,|g\rangle=g_k e^{i\gamma_k(t)}\,|g\rangle\ . \tag{3.1.9}$$

As we want to describe a generic static spherically symmetric potential $V(r)$, we are interested in those $|g\rangle$ such that:

$$\langle g|\,\hat{\Phi}(t,r)\,|g\rangle=\sqrt{\frac{m_\mathrm{p}}{\ell_\mathrm{p}}}\,V(r)\,. \tag{3.1.10}$$

Using the expansion of $\hat{\Phi}$ (3.1.5), we obtain:

$$\begin{aligned}
\langle g|\,\hat{\Phi}(t,r)\,|g\rangle &= \langle g| \int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\sqrt{\frac{\hbar}{2k}}j_0(kr)\left(\hat{a}_k\,e^{ikt}+\hat{a}_k^\dagger\,e^{-ikt}\right)|g\rangle \\
&= \langle g|g\rangle \int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\sqrt{\frac{\hbar}{2k}}j_0(kr)\,g_k\left[e^{ikt+i\gamma_k(t)}+e^{-ikt-i\gamma_k(t)}\right] \\
&= \int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\sqrt{\frac{2\hbar}{k}}j_0(kr)\,g_k\cos\left[kt+\gamma_k(t)\right]\,.
\end{aligned}$$

If we expand $V(r)$ in Laplacian eigenfunctions, that is

$$V(r) = \int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\tilde{V}(k)\,j_0(kr)\,, \tag{3.1.11}$$

and we impose (3.1.10), we obtain $\gamma_k(t) = -kt$ and

$$g_k = \sqrt{\frac{k}{2}}\frac{\tilde{V}(k)}{\ell_\mathrm{p}}\,. \tag{3.1.12}$$

It is useful to write the coherent state in terms of the vacuum $|0\rangle$ to study its normalisation:

$$|g\rangle = e^{-\frac{N_\mathrm{G}}{2}}\exp\left\{\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\,g_k\hat{a}_k^\dagger\right\}|0\rangle\ , \tag{3.1.13}$$

where N_G is a normalisation factor that we obtain explicitly by imposing the normalisation condition.
In fact, using the commutation relations (3.1.7) and the Baker - Hausdorff - Campbell formula we get:

$$\begin{aligned}
\langle g|g\rangle &= e^{-N_\mathrm{G}}\,\langle 0|\exp\left\{\int_0^\infty \frac{p^2\,\mathrm{d}p}{2\,\pi^2}g_p\,\hat{a}_p\right\}\exp\left\{\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}g_k\,\hat{a}_k^\dagger\right\}|0\rangle \\
&= e^{-N_\mathrm{G}}\exp\left\{\frac{1}{2}\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}g_k^2\right\}\langle 0|\exp\left\{\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\,g_k\left(\hat{a}_k+\hat{a}_k^\dagger\right)\right\}|0\rangle \\
&= e^{-N_\mathrm{G}}\exp\left\{\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\,g_k^2\right\}\,,
\end{aligned}$$

where

$$
\begin{aligned}
&\langle 0|\exp\left\{\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\, g_k\left(\hat{a}_k+\hat{a}_k^\dagger\right)\right\}|0\rangle = \langle 0|\left[1+\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2} g_k\left(\hat{a}_k+\hat{a}_k^\dagger\right)+\right.\\
&\left.+\frac{1}{2!}\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2} g_k\left(\hat{a}_k+\hat{a}_k^\dagger\right)\int_0^\infty \frac{p^2\mathrm{d}p}{2\pi^2} g_p\left(\hat{a}_p+\hat{a}_p^\dagger\right)+\dots\right]|0\rangle\\
&=\exp\left\{\frac{1}{2}\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2} g_k^2\right\}.
\end{aligned}
$$

If we now impose the normalisation condition, we finally get

$$
N_{\mathrm{G}} = \int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\, g_k^2 = \langle g|\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2} \hat{a}_k^\dagger\,\hat{a}_k\,|g\rangle\ , \tag{3.1.14}
$$

which is the total occupation number of modes in the state $|g\rangle$, as we explicitly show in the latter equality, being $\hat{a}_k^\dagger\hat{a}_k = \hat{n}_k$.
The mean wavenumber $\langle k\rangle$ is then given by

$$
\langle k\rangle = \langle g|\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2} k\,\hat{a}_k\,\hat{a}_k^\dagger\,|g\rangle = \int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2} k\,g_k^2\,. \tag{3.1.15}
$$

3.1.1 Transformations between coherent states

We can write the coherent state in terms of the manifestly unitary displacement operator which, for a static and spherically symmetric source, can be written as:

$$
\hat{D}(g) = \exp\left\{\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2} g_k\left(\hat{a}_k^\dagger-\hat{a}_k\right)\right\}, \tag{3.1.16}
$$

so that the coherent state is

$$
|g\rangle = \hat{D}(g)\,|0\rangle = \exp\left\{\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2} g_k\left(\hat{a}_k^\dagger-\hat{a}_k\right)\right\}|0\rangle\ , \tag{3.1.17}
$$

in fact, after expanding the exponential, it becomes equal to Eq (3.1.13).
It can be noticed that the operator $\hat{D}(g)$ is unitary and has the properties

$$
\hat{D}(g)^\dagger = \hat{D}(g)^{-1} = \hat{D}(-g) = \exp\left\{-\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2} g_k(\hat{a}_k^\dagger-\hat{a}_k)\right\} \tag{3.1.18}
$$

and

$$
\hat{D}(g_1)\hat{D}(g_2) = \hat{D}(g_1+g_2) = \exp\left\{\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}(g_{1k}+g_{2k})(\hat{a}_k^\dagger-\hat{a}_k)\right\}. \tag{3.1.19}
$$

Now, given two coherent states $|g(a)\rangle$ and $|g'(a)\rangle$, where

$a \in A = \{M, \{M, Q\}, \dots\}$ denotes the parameter or the set of parameters by which the source is uniquely characterized (M characterizes the Schwarzschild case, M and Q the Reissner - Nordström case, and so on), we can write:

$$|g'(a)\rangle = |g'(a)\rangle \langle g(a)|g(a)\rangle \ ,$$

being the coherent state normalized.
We can then define the operator $\hat{P}_a$ which transforms the coherent state $|g(a)\rangle$ in the coherent state $|g'(a)\rangle$ as

$$\hat{P}_a \equiv |g'(a)\rangle \langle g(a)| = \hat{D}(g'(a)) |0\rangle \langle 0| \hat{D}(g(a))^\dagger \, , \tag{3.1.20}$$

such that

$$|g'(a)\rangle = \hat{P}_a |g(a)\rangle \ . \tag{3.1.21}$$

Similarly, the inverse transformation will be

$$|g(a)\rangle = |g(a)\rangle \langle g'(a)|g'(a)\rangle \equiv \hat{Q}_a |g'(a)\rangle \ ,$$

and it is straightforward to verify that

$$\hat{Q}_a \equiv |g(a)\rangle \langle g'(a)| = (|g'(a) \langle g(a)|\rangle)^\dagger = \hat{P}_a^\dagger \, , \tag{3.1.22}$$

so that the inverse transformation is given by:

$$|g(a)\rangle = \hat{P}_a^\dagger |g'(a)\rangle \ . \tag{3.1.23}$$

Finally, we calculate the square of this operator to gain a general understanding of its properties:

$$\begin{aligned} \hat{P}_a^2 &= |g'(a)\rangle \langle g(a)|g'(a)\rangle \langle g(a)| \\ &= \exp\left\{ -\frac{1}{2} \int_0^\infty \frac{k^2 \, \mathrm{d}k}{2\,\pi^2} \left(g'_k(a) - g_k(a)\right)^2 \right\} |g'(a)\rangle \langle g(a)| \\ &= \exp\left\{ -\frac{1}{2} \int_0^\infty \frac{k^2 \, \mathrm{d}k}{2\,\pi^2} \left(g'_k(a) - g_k(a)\right)^2 \right\} \hat{P}_a \, , \end{aligned} \tag{3.1.24}$$

being

$$\langle g|g'\rangle = \langle 0| \hat{D}(g)^\dagger \hat{D}(g') |0\rangle = \langle 0| \hat{D}(g' - g) |0\rangle = e^{-\frac{1}{2}\int_0^\infty \frac{k^2 \, \mathrm{d}k}{2\,\pi^2} (g'_k - g_k)^2} \ . \tag{3.1.25}$$

3.2 Quantum state in areal coordinates

We are now ready to define the coherent state corresponding to the areal potential in (2.3.17) by computing all the previous expressions explicitly from the coefficients g_k of Eq. (3.1.12), which depend on the transform $\tilde{V}$ of the potential V.
Areal potential in Eq. (2.3.17) is the Newtonian potential generated by a point-like source, whose density is given by

$$\rho = M \delta^{(3)}(\vec{x}) = \frac{M}{4\,\pi\,r^2}\delta(r)\,. \tag{3.2.1}$$

The transform $\tilde{V}$ of V can be then derived by solving the Poisson equation (1.1.51) in momentum space. In fact, by substituting there the expansions (3.1.11) and

$$\rho(r) = \int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\,\tilde{\rho}(k)\,j_0(kr)\,, \tag{3.2.2}$$

we get

$$\tilde{V}_A(k) = -\frac{4\,\pi\,G_\mathrm{N}}{k^2}\tilde{\rho}(k) = -\frac{4\,\pi\,G_\mathrm{N}\,M}{k^2}\,, \tag{3.2.3}$$

where we used $\tilde{\rho}(k) = M$ in the latter equality (see [9] for details).
The coefficients g_k are then:

$$g_{(A)k} = \sqrt{\frac{k}{2}}\frac{\tilde{V}_A(k)}{\ell_\mathrm{p}} = -\frac{4\,\pi\,M}{\sqrt{2\,k^3}\,m_\mathrm{p}}\,, \tag{3.2.4}$$

which fix the coherent state that reproduces the areal potential.
The occupation number and the mean wavenumber for such a state are then

$$N_{(A)\mathrm{G}} = \int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\,g_k^2 = \frac{4\,M^2}{m_\mathrm{p}^2}\int_0^\infty \frac{\mathrm{d}k}{k} \tag{3.2.5}$$

and

$$\langle k_{(A)} \rangle = \int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\,k\,g_k^2 = \frac{4\,M^2}{m_\mathrm{p}^2}\int_0^\infty \mathrm{d}k\,, \tag{3.2.6}$$

where it could be noticed that both N_G and $\langle k \rangle$ diverges in the ultraviolet (UV), and N_G diverges in the infrared (IR) too. The UV divergence is due to the fact that the source is point-like and usually is not present when one considers regular matter densities (see [12] for a homogeneous density source). This divergence can be also regularised by introducing a cut-off $k_\mathrm{UV} \sim 1/R_\mathrm{S}$ where R_S can be interpreted as the finite radius of a would-be regular matter

source.
The IR divergence of $N_{(A)\mathrm{G}}$ is instead due to assuming the source is eternal and its gravitational static field extends to infinite distances. Similarly to the UV divergence, we introduce an IR cut-off $k_{\mathrm{IR}} = 1/R_\infty$ to consider the finite life-time of a realistic source ([9, 12]).
With these cut-offs we rewrite

$$N_{(A)\mathrm{G}} = \frac{4\,M^2}{m_\mathrm{p}^2} \int_{k_{\mathrm{UV}}}^{k_{\mathrm{IR}}} \frac{\mathrm{d}k}{k} = \frac{4\,M^2}{m_\mathrm{p}^2} \ln\left(\frac{R_\infty}{R_\mathrm{S}}\right) \tag{3.2.7}$$

and

$$\left\langle k_{(A)} \right\rangle = \frac{4\,M^2}{m_\mathrm{p}^2} \int_{k_{\mathrm{UV}}}^{k_{\mathrm{IR}}} \mathrm{d}k = \frac{4\,M^2}{m_\mathrm{p}^2} \left(\frac{1}{R_\mathrm{S}} - \frac{1}{R_\infty}\right) . \tag{3.2.8}$$

where we recognise the scaling relation of the corpuscolar model for N_G (A.2.12).
The coherent state corresponding to the areal potential, in terms of the vacuum $|0\rangle$, then is

$$\left|g_{(A)}\right\rangle = e^{-\frac{N_{\mathrm{G}(A)}}{2}} \exp\left\{\int_{k_{\mathrm{IR}}}^{k_{\mathrm{UV}}} \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\, g_{(A)k} \hat{a}_k^\dagger\right\} |0\rangle \ . \tag{3.2.9}$$

If the source of the field is a **black hole**, the coherent state $\left|g_{\mathrm{BH}(A)}\right\rangle$ representing it cannot reproduce the classical potential everywhere, like it does in Eq. (3.1.10), but at most in the region outside the horizon r_H.
However, we need a way to define the horizon, since we have abandoned the geometric description in favour of a field theory description on flat space-time. Following what has been done in the context of the Bootstrapped Newtonian Gravity Theory (see on this, for example [8, 10]), we define the horizon for the potential as the radius where the escape velocity equals the speed of light, namely r_H such that

$$V(r_\mathrm{H}) = -\frac{1}{2} \, . \tag{3.2.10}$$

For the areal potential V_A, the horizon occurs for

$$r_\mathrm{H} = R_\mathrm{H} \equiv 2 G_\mathrm{N}\, M \, . \tag{3.2.11}$$

This means that the expectation value of the field on the coherent state must give:

$$\left\langle g_{\mathrm{BH}(A)}\right| \hat{\Phi}(t,r) \left|g_{\mathrm{BH}(A)}\right\rangle \simeq \sqrt{\frac{m_\mathrm{p}}{\ell_\mathrm{p}}}\, V_A(r) \qquad \text{for } r \gtrsim R_\mathrm{H} \, . \tag{3.2.12}$$

As a consequence of this last equation, modes of wavelength shorter than

the gravitational radius aren't needed to be integrated in the black hole coherent state $|g_{\mathrm{BH}(A)}\rangle$, including modes of infinitely short wavelength necessary to resolve the singularity at $r = 0$.
We can therefore build the coherent state of the black hole according to (3.1.13) with modes such that their wavelength is larger than some fraction of the gravitational radius R_{H}, that is

$$k^{-1} \gtrsim \frac{R_{\mathrm{H}}}{n} \,. \tag{3.2.13}$$

This fraction of R_{H} can be identified with the source size R_{S}, which we have introduced as the UV cut-off. If we exclude also wavelengths larger than the IR scale R_∞, in the coherent state $|g_{\mathrm{BH}(A)}\rangle$ are populated only the modes

$$k_{\mathrm{IR}} \lesssim k \lesssim k_{\mathrm{UV}} \,, \tag{3.2.14}$$

with $k_{\mathrm{UV}} \sim 1/R_{\mathrm{S}} = n/R_{\mathrm{H}}$ and $k_{\mathrm{IR}} = 1/R_\infty$.
If we integrate only these modes in the expansion (3.1.11), we get an effective quantum potential V_{QA}:

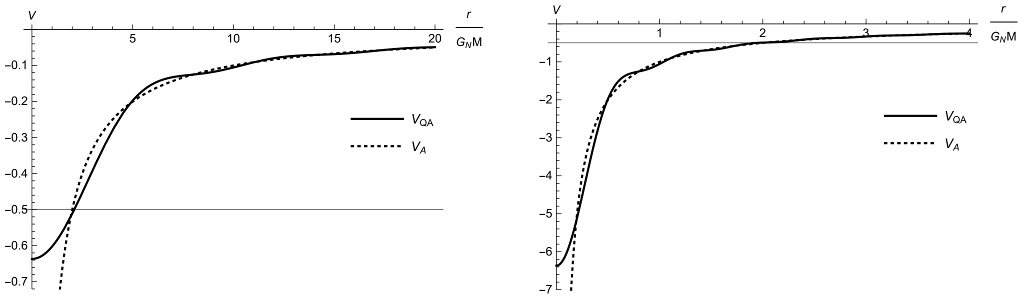
$$\begin{aligned} V_{QA} &\simeq \int_{k_{\mathrm{IR}}}^{k_{\mathrm{UV}}} \frac{k^2 \, \mathrm{d}k}{2\,\pi^2} \tilde{V}_A(k)\, j_0(k\,r) \\ &= -\frac{2}{\pi}\frac{G_{\mathrm{N}}\,M}{r} \int_{k_{\mathrm{IR}}}^{k_{\mathrm{UV}}} \mathrm{d}k \, \frac{\sin(k\,r)}{k} \\ &\simeq -\frac{2}{\pi}\frac{G_{\mathrm{N}}\,M}{r} \int_0^{r/R_{\mathrm{S}}} \mathrm{d}z \, \frac{\sin z}{z} \,, \end{aligned}$$

where in the last line we defined $z = k\,r$ and let $R_\infty \to \infty$. Going on with the calculation we find

$$\begin{aligned} V_{QA} &\simeq -\frac{2}{\pi}\frac{G_{\mathrm{N}}\,M}{r} \, \mathrm{Si}\left(\frac{r}{R_{\mathrm{S}}}\right) \\ &= V_A \left\{ 1 - \left[1 - \frac{2}{\pi} \, \mathrm{Si}\left(\frac{r}{R_{\mathrm{S}}}\right) \right] \right\} \,, \end{aligned} \tag{3.2.15}$$

where Si is the sine integral function $\mathrm{Si}(x) = \int_0^x \mathrm{d}z \frac{\sin z}{z}$.
It is possible to represent and compare the effective quantum potential V_{QA} and the areal one V_A by choosing the size of $R_{\mathrm{S}} = R_{\mathrm{H}}/n$ varying n.

Looking at their graphic representation in Fig. 3.2 we can make a few comments. Firstly, the quantum potential is regular and finite everywhere, also in the origin $r = 0$. Secondly, the potential V_{QA} is oscillating around V_A, such an effect of the oscillations on test bodies could be potentially observed at $r > R_{\mathrm{H}}$. Finally, the amplitude of these oscillations decreases with increas-



(a) $R_{\mathrm{S}} = R_{\mathrm{H}}/2$ (b) $R_{\mathrm{S}} = R_{\mathrm{H}}/20$

Figure 3.2: Quantum potential V_{Q} compared to areal potential V_A for two different values of $R_{\mathrm{S}} = R_{\mathrm{H}}/n = 2G_{\mathrm{N}}\, M/n$.

ing $n = R_{\mathrm{H}}/R_{\mathrm{S}}$, so one can always choose finite n so that the oscillations are too small to be measured by a distant observer. This effect is pointed out for $r > R_{\mathrm{H}}$ in Fig. 3.3.

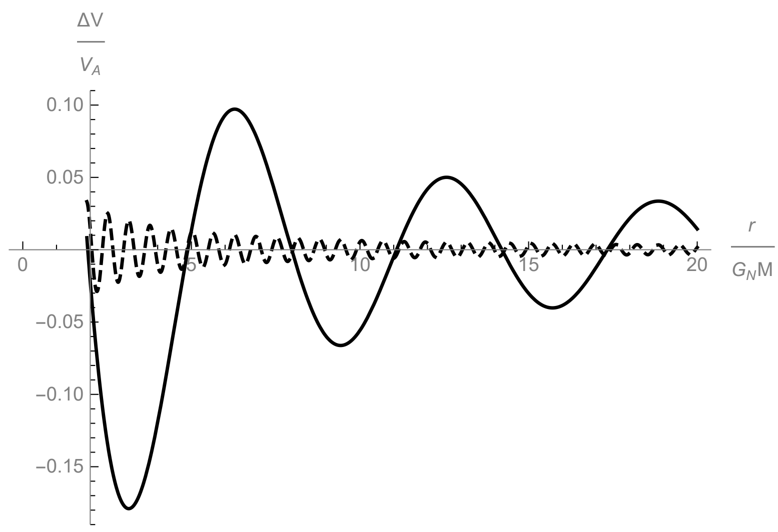


Figure 3.3: Oscillations of the quantum potential $V_{\mathrm{Q}A}$ around V_A for $n = 2$ (solid line) and $n = 20$ (dashed line) in the outer region ($r > R_{\mathrm{H}}$).

3.3 Quantum state in harmonic coordinates

We now define the coherent state for the harmonic potential V_H.
We can obtain the transform $\tilde{V}_H(k)$ of the harmonic potential (2.3.18) by pro-

jecting it on the eigenfunctions of the Laplace operator $j_0(kr)$, that is:

$$\begin{aligned}\tilde{V}_H(k) &= -\frac{4\,\pi\,G_{\rm N}\,M}{k^2}\left\{1 - G_{\rm N}\,M\,k\left[\left(\frac{\pi}{2} - \mathrm{Si}(G_{\rm N}\,M\,k)\right)\cos(G_{\rm N}\,M\,k) +\right.\right.\\ &\qquad \left.\left.+\,\mathrm{Ci}(G_{\rm N}\,M\,k)\,\sin(G_{\rm N}\,M\,k)\right]\right\}\\ &\equiv -\frac{4\,\pi\,G_{\rm N}\,M}{k^2}[(1 - G_{\rm N}\,M\,k f(G_{\rm N}\,M\,k)]\ ,\end{aligned} \tag{3.3.1}$$

as derived in appendix B, with $f(x)$ defined in Eq. (B.0.4).
The coefficients g_k are then:

$$g_{(H)k} = \sqrt{\frac{k}{2}}\frac{\tilde{V}_H(k)}{\ell_{\rm p}} = -\frac{4\pi M}{\sqrt{2\,k^3}\,m_{\rm p}}\left[1 - G_{\rm N}\,M\,k\,f(G_{\rm N}\,M\,k)\right]\ . \tag{3.3.2}$$

We can proceed by computing the occupation number $N_{\rm G}$ and the mean wavenumber $\langle k\rangle$ like we did in the areal case. So:

$$\begin{aligned}N_{{\rm G}(H)} &= \int_0^\infty \frac{k^2\,{\rm d}k}{2\,\pi^2}\,g^2_{(H)k} = \frac{4\,M^2}{m_{\rm p}^2}\int_0^\infty \frac{{\rm d}k}{k}\left[1 - G_{\rm N}\,M\,k\,f(G_{\rm N}\,M\,k)\right]^2\\ &= N_{{\rm G}(A)} - \frac{4\,M^2}{m_{\rm p}^2}\int_0^\infty \frac{{\rm d}k}{k}\big[2\,G_{\rm N}\,M\,k\,f(G_{\rm N}\,M\,k) +\\ &\quad - (G_{\rm N}\,M\,k)^2 f(G_{\rm N}\,M\,k)^2\big]\end{aligned} \tag{3.3.3}$$

and

$$\begin{aligned}\left\langle k_{(H)}\right\rangle &= \int_0^\infty \frac{k^2\,{\rm d}k}{2\,\pi^2}\,k\,g^2_{(H)k} = \frac{4\,M^2}{m_{\rm p}^2}\int_0^\infty {\rm d}k\left[1 - G_{\rm N}\,M\,k\,f(G_{\rm N}\,M\,k)\right]^2\\ &= \left\langle k_{(A)}\right\rangle - \frac{4\,M^2}{m_{\rm p}^2}\int_0^\infty {\rm d}k\big[2\,G_{\rm N}\,M\,k\,f(G_{\rm N}\,M\,k) +\\ &\quad - (G_{\rm N}\,M\,k)^2 f(G_{\rm N}\,M\,k)^2\big]\ .\end{aligned} \tag{3.3.4}$$

Despite the areal case, it is not possible to calculate these quantities in an analytical way but, in order to better understand their behaviour, we should proceed by computing them numerically. To better handle the calculation, we rewrite the two integrals by changing the integration variable to $y = G_{\rm N}\,M\,k$, that is:

$$N_{{\rm G}(H)} = \frac{4\,M^2}{m_{\rm p}^2}\int_0^\infty \frac{{\rm d}y}{y}\left[1 - y\,f(y)\right]^2 \tag{3.3.5}$$

and

$$\langle k_{(H)} \rangle = \frac{4\,M}{m_{\rm p}\ell_{\rm p}} \int_0^{\infty} {\rm d}y \left[1 - y\, f(y)\right]^2 \,. \tag{3.3.6}$$

From the numerical analysis results that the integral in the occupation number $N_{{\rm G}(H)}$ is divergent only if we leave the lower bound of integration equal to zero, which, for $M \neq 0$, corresponds to $k = 0$, so it presents an infrared divergence, while it does not diverge in the ultraviolet. The integral in the mean wavenumber is convergent. The reason for the disappearance of the UV divergence in $N_{{\rm G}(H)}$ is due to the fact that the full Schwarzschild potential (2.3.18) is not singular for $r = 0$ (it would be for negative $r = -G_{\rm N}\,M$).
We regularise the IR divergence introducing a cut-off $k_{\rm IR} = 1/R_\infty$, that is $y_{\rm IR} = G_{\rm N}\,M/R_\infty$:

$$N_{{\rm G}(H)} = \frac{4\,M^2}{m_{\rm p}^2} \int_{y_{\rm IR}}^{\infty} \frac{{\rm d}y}{y} \left[1 - y\, f(y)\right]^2 \tag{3.3.7}$$

and

$$\langle k_{(H)} \rangle = \frac{4\,M}{m_{\rm p}\ell_{\rm p}} \int_{y_{\rm IR}}^{\infty} {\rm d}y \left[1 - y\, f(y)\right]^2 \,. \tag{3.3.8}$$

The coherent state corresponding to the harmonic potential, in terms of the vacuum $|0\rangle$, then is

$$|g_{(H)}\rangle = e^{-\frac{N_{{\rm G}(H)}}{2}} \exp\left\{ \int_{k_{\rm IR}}^{\infty} \frac{k^2\,{\rm d}k}{2\,\pi^2}\, g_{(H)k} \hat{a}_k^\dagger \right\} |0\rangle \,. \tag{3.3.9}$$

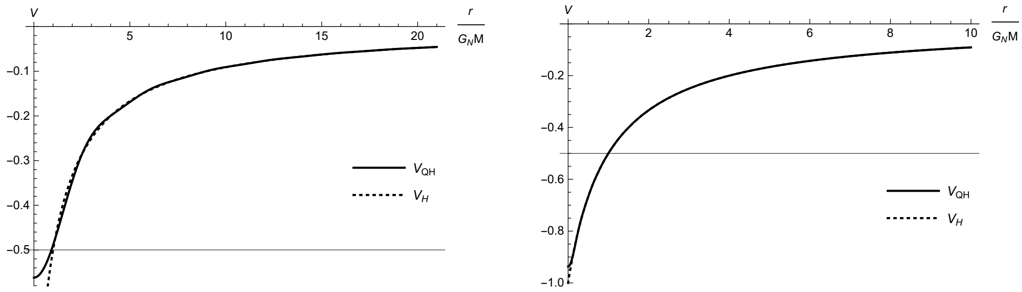
Similarly to what we did in the areal case, let us see what happens if the source of the field is a black hole.
The horizon for the harmonic potential defined as in Eq. (3.2.11) is then $r'_{\rm H}$ such that

$$V_H(r'_{\rm H}) = -\frac{1}{2} \tag{3.3.10}$$

which occurs for $r'_{\rm H} = \frac{R_{\rm H}}{2} \equiv G_{\rm N}\,M$.
In this case, the coherent state representing it can reproduce the classical potential only in the region outside the horizon $r'_{\rm H} = G_{\rm N}\,M$, that is

$$\langle g_{{\rm BH}(H)} | \, \hat{\Phi}(t,r) \, | g_{{\rm BH}(H)} \rangle \simeq \sqrt{\frac{m_{\rm p}}{\ell_{\rm p}}}\, V_H(r) \qquad \text{for } r \gtrsim \frac{R_{\rm H}}{2} \,. \tag{3.3.11}$$

Since in a black hole configuration we always have that the size of the source $R_{\rm S}$ is less than the horizon $r'_{\rm H}$, we can build a coherent state $|g_{{\rm BH}(H)}\rangle$ where only modes k such that $k^{-1} \gtrsim \frac{r'_{\rm H}}{n} = R_{\rm S}$ are populated. This leads to an effective



(a) $R_{\rm S} = r'_{\rm H}/2$ (b) $R_{\rm S} = r'_{\rm H}/20$

Figure 3.4: Quantum potential $V_{{\rm Q}H}$ compared to full Schwarzschild potential V_H for two different values of $R_{\rm S} = r'_{\rm H}/n = G_{\rm N}\, M/n$.

potential $V_{{\rm Q}H}$

$$\begin{aligned} V_{{\rm Q}H} &\simeq \int_0^{k_{\rm UV}} \frac{k^2\,{\rm d}k}{2\,\pi^2} \tilde{V}_H(k)\, j_0(k\,r) \\ &\simeq -\frac{2}{\pi}\frac{G_{\rm N}\,M}{r}\frac{1}{1-\left(\frac{G_{\rm N}\,M}{r}\right)^2}\left\{ {\rm Si}\left(\frac{r}{R_{\rm S}}\right) + \frac{G_{\rm N}\,M}{r}\left[-\frac{\pi}{2}+ \right.\right. \\ &\left.\left. \cos\left(\frac{r}{R_{\rm S}}\right) f\left(\frac{G_{\rm N}\,M}{R_{\rm S}}\right)\right] + \left(\frac{G_{\rm N}\,M}{r}\right)^2 \sin\left(\frac{r}{R_{\rm S}}\right) g\left(\frac{G_{\rm N}\,M}{R_{\rm S}}\right)\right\} , \end{aligned} \tag{3.3.12}$$

where in the second line we have substituted $k_{\rm UV} \sim 1/R_{\rm S}$ and where $f(x)$ is the function in Eq. (B.0.4) and

$$g(x) = \int_0^\infty {\rm d}t \frac{\cos t}{t+x} = \left[\frac{\pi}{2} - {\rm Si}(x)\right] \sin(x) + {\rm Ci}(x)\cos(x)\,. \tag{3.3.13}$$

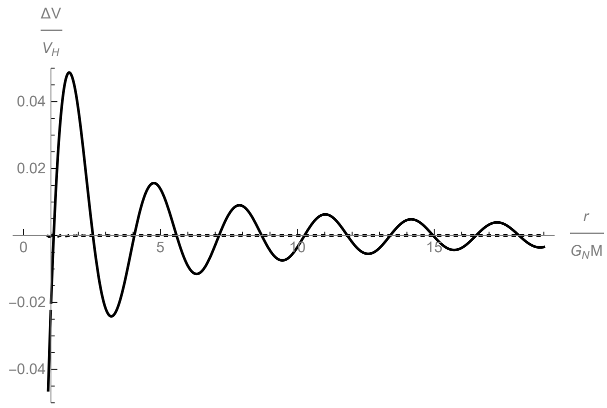


Figure 3.5: Oscillations of the quantum potential $V_{{\rm Q}H}$ around V_H for $n = 2$ (solid line) and $n = 20$ (dashed line) in the outer region ($r > r_{\rm H}$).

We can represent this effective potential varying n, similarly to what we did for the areal one (3.2.15). In Fig. 3.4 effective quantum potential $V_{\mathrm{Q}H}$ and harmonic potential V_H are compared for $n = 2$ and $n = 20$. The quantum potential $V_{\mathrm{Q}H}$ is oscillating around V_H and the amplitude of these oscillations descreas with increasing n.
This effect can be appreciated in Fig. 3.5 for $r > r_{\mathrm{H}}$.

3.4 Transformations

In this section, we will write down the operator which transforms the coherent state $\left|g_{(A)}\right\rangle$, describing the areal potential V_A, in the coherent state $\left|g_{(H)}\right\rangle$, describing the harmonic potential V_H.
The coherent states $\left|g_{(A)}\right\rangle$ and $\left|g_{(H)}\right\rangle$, as seen above, are not defined for every k but regularized with cut-off's. For every k their eigenvalues are then $\tilde{g}_{(A)k} = g_{(A)k}\,\theta(k_{\mathrm{UV}} - k)\,\theta(k - k_{\mathrm{IR}})$ and $\tilde{g}_{(H)k} = g_{(H)k}\,\theta(k - k_{\mathrm{IR}})$, so that we can write:

$$\begin{aligned} \left|g_{(A)}\right\rangle &= \hat{D}(\tilde{g}_{(A)})\left|0\right\rangle = \exp\left\{\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\tilde{g}_{(A)k}\left(\hat{a}_k^\dagger - \hat{a}_k\right)\right\}\left|0\right\rangle \\ &= \exp\left\{\int_{k_{\mathrm{IR}}}^{k_{\mathrm{UV}}} \frac{k^2\,\mathrm{d}k}{2\,\pi^2} g_{(A)k}\left(\hat{a}_k^\dagger - \hat{a}_k\right)\right\}\left|0\right\rangle \end{aligned} \tag{3.4.1}$$

and

$$\begin{aligned} \left|g_{(H)}\right\rangle &= \hat{D}(\tilde{g}_{(H)})\left|0\right\rangle = \exp\left\{\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\tilde{g}_{(H)k}\left(\hat{a}_k^\dagger - \hat{a}_k\right)\right\}\left|0\right\rangle \\ &= \exp\left\{\int_{k_{\mathrm{IR}}}^{\infty} \frac{k^2\,\mathrm{d}k}{2\,\pi^2} g_{(H)k}\left(\hat{a}_k^\dagger - \hat{a}_k\right)\right\}\left|0\right\rangle \ , \end{aligned} \tag{3.4.2}$$

with $g_{(A)k}$ and $g_{(H)k}$ equal to eqs. (3.2.4) and (3.3.2) and where, for simplicity, we used the same cut-off k_{IR} for both states.
The transformation between these states, according to (3.1.21), is then

$$\left|g_{(H)}(M)\right\rangle = \hat{P}_M\left|g_{(A)}(M)\right\rangle \tag{3.4.3}$$

with

$$\hat{P}(M) = \left|g_{(H)}(M)\right\rangle\left\langle g_{(A)}(M)\right| = \hat{D}(\tilde{g}_{(H)}(M))\left|0\right\rangle\left\langle 0\right| D(\tilde{g}_{(A)}(M))^\dagger \,, \tag{3.4.4}$$

where we made the dependence of the states and the operator on M explicit. The squared operator $\hat{P}_M$ is, using Eq. (3.1.24):

$$\begin{aligned}
\hat{P}_M^2 &= \exp\left\{-\frac{1}{2}\int_0^{\infty}\frac{k^2\,\mathrm{d}k}{2\,\pi^2}\left(\tilde{g}_{(H)k}(M)-\tilde{g}_{(A)k}(M)\right)^2\right\}\hat{P}_M \\
&= \exp\left\{-\frac{1}{2}\left[\int_{k_{\mathrm{IR}}}^{k_{\mathrm{UV}}}\frac{k^2\,\mathrm{d}k}{2\,\pi^2}\left(g_{(A)k}(M)-g_{(H)k}(M)\right)^2+\right.\right. \\
&\quad \left.\left.+\int_{k_{\mathrm{UV}}}^{\infty}\frac{k^2\,\mathrm{d}k}{2\,\pi^2}g_{(H)k}^2(M)\right]\right\}\hat{P}_M \\
&\simeq \exp\left\{-2\frac{M^2}{m_{\mathrm{p}}^2}\left[\int_0^{y_{\mathrm{UV}}}\mathrm{d}y\,y f(y)^2+\int_{y_{\mathrm{UV}}}^{\infty}\frac{\mathrm{d}y}{y}\left(1-y\,f(y)\right)^2\right]\right\}\hat{P}_M \\
&\equiv \exp\left\{-2\frac{M^2}{m_{\mathrm{p}}^2}\left[I_1(y_{\mathrm{UV}})+I_2(y_{\mathrm{UV}})\right]\right\}\hat{P}_M\,,
\end{aligned} \tag{3.4.5}$$

where we changed integration variable in $y = G_{\mathrm{N}}\,Mk$ and put k_{IR} equal to zero, which has little to no effect on the result, as hinted by explicit numerical evaluation.

The integral in square brackets $I = I_1 + 1_2$ is not computable analytically. Numerically it is possible to represent it as a function of the cut-off y_{UV} for the areal coherent state with accuracy only up to $y_{\mathrm{UV}} = G_{\mathrm{N}}\,M/k_{\mathrm{UV}} \simeq 13$. We represent this trend in Fig. 3.6, along with the single trend of the two integrals I_1 and I_2, where we can observe that I is always positive (and hence the exponent of the exponential factor in front of $\hat{P}_M$ always negative).

In particular we see that for $0 < y_{\mathrm{UV}} \lesssim 0.6$ the integral I is decreasing, while

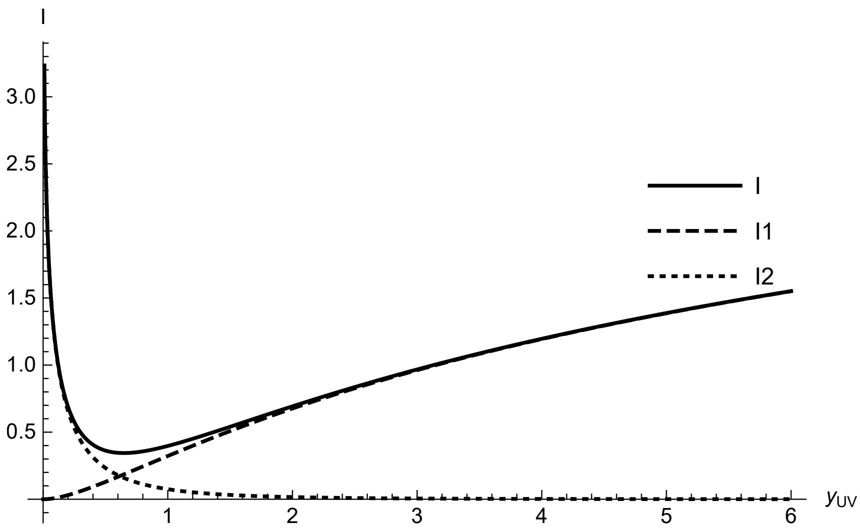


Figure 3.6: Integrals $I(y_{\mathrm{UV}})$ (solid line), $I_1(y_{\mathrm{UV}})$ (dashed line) and $I_2(y_{\mathrm{UV}})$ (dotted line).

it is slowly increasing for $y_{\mathrm{UV}} \gtrsim 0.6$.

Now we apply the operator $\hat{P}_M$ to coherent states describing different sources $M' \neq M$. We start with the particular case of state $|0\rangle$, which is the coherent state for $M' = 0$, and we obtain

$$\begin{aligned}
\hat{P}_M \left|g_{(A)}(M'=0)\right\rangle &= \left|g_{(H)}(M)\right\rangle \left\langle g_{(A)}(M)\middle|0\right\rangle = \left|g_{(H)}(M)\right\rangle \langle 0| \, \hat{D}(\tilde{g}_{(A)}(M))^\dagger \, |0\rangle \\
&= \exp\left\{-\frac{1}{2}\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\left(-\tilde{g}_{(A)k}(M)\right)^2\right\}\left|g_{(H)}(M)\right\rangle \\
&= e^{-\frac{1}{2}N_{\mathrm{G}(A)}(M)}\left|g_{(H)}(M)\right\rangle \\
&= e^{-2\frac{M^2}{m_{\mathrm{p}}^2}\ln\left(\frac{k_{\mathrm{UV}}}{k_{\mathrm{IR}}}\right)}\left|g_{(H)}(M)\right\rangle \ ,
\end{aligned} \tag{3.4.6}$$

which is approximately zero when $M \gg m_{\mathrm{p}}$, while it is non negligible only when $M \sim m_{\mathrm{p}}$.
For $M' \neq 0$ we get

$$\begin{aligned}
\hat{P}_M \left|g_{(A)}(M')\right\rangle &= \left|g_{(H)}(M)\right\rangle \left\langle g_{(A)}(M)\middle|g_{(A)}(M')\right\rangle \\
&= \left|g_{(H)}(M)\right\rangle \langle 0| \, \hat{D}(\tilde{g}_{(A)}(M))^\dagger \hat{D}(\tilde{g}_{(A)}(M')) \, |0\rangle \\
&= \exp\left\{-\frac{1}{2}\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\left(\tilde{g}_{(A)k}(M') - \tilde{g}_{(A)k}(M)\right)^2\right\}\left|g_{(H)}(M)\right\rangle \\
&= e^{-2\frac{(M-M')^2}{m_{\mathrm{p}}^2}\ln\left(\frac{k_{\mathrm{UV}}}{k_{\mathrm{IR}}}\right)}\left|g_{(H)}(M)\right\rangle \ ,
\end{aligned} \tag{3.4.7}$$

where the multiplicative coefficient is non negligible only when $|M - M'|$ is of the same order of magnitude as:

$$\sigma = \frac{m_{\mathrm{p}}}{2\sqrt{\ln\left(\frac{k_{\mathrm{UV}}}{k_{\mathrm{IR}}}\right)}} \ .$$

Note that if M and M' are $\gg m_{\mathrm{p}}$, then any difference $M - M'$ between the two (given that it has a magnitude which is not negligible with respect to the masses themselves) will be itself $\gg m_{\mathrm{p}}$. Thus, we can say that in general the exponential term will have a ~ 0 value whenever the two masses are $\gg m_{\mathrm{p}}$.

We can then observe how this operator is closely related to the source parameter M within which it is defined. In fact, when we try to apply it to coherent states describing different sources M', they are projected along the coherent states defined by the source M. The operator that realizes the coordinate transformation (2.3.14) of the classical (geometric) theory in the quantum model is strictly defined by parameters that characterize the source M, as this coordinate transformation (and in general the coordinate transformations seen in Chapter 2) are defined by the parameters that uniquely describe the metric (in this case M) and hence the solution of Einstein's equations.

Conclusions

With the aim to construct a quantum theory of gravity having the theory of General Relativity as an emergent classical description, the invariance for coordinate transformations of the theory must be taken into account and treated as a gauge symmetry, in order to consider only the physical degrees of freedom of the dynamics. In this work we have tried to set up a possible quantum treatment of the transformation between Schwarzschild coordinates and harmonic coordinates, which belongs to those transformations that, being dependent on the metric itself, cannot be misinterpreted as dynamical perturbations, but still are a symmetry for the solutions of the field equations and therefore must be included in the theory.
To summarize what we have done, starting from the invariance of the gravitational action under diffeomorphisms, in the first Chapter we have derived the algebra of the gauge group of General Relativity, the Bergmann-Komar group. This is not a Lie group and therefore its finite transformations also include transformations not reducible to the identity by sending the parameters to zero. We then saw in Chapter 2 some examples of these transformations for a particular solution of Einstein's equations, the well-known Schwarzschild metric. We decided to focus on the harmonic gauge condition for coordinates, used in linearized gravity theories to simplify the field equations, which in Schwarschild space-time results in a transformation between the areal coordinate r into the "harmonic" coordinate $\bar{r}$. Finally, in the third Chapter, we moved into the frame of the description of the classical potential (defined in the Newtonian way and linked to the metric via the Newtonian limit) as the expectation value of the (scalar) gravitonic field on the quantum coherent state (model developed, for example, in Refs. [5, 8, 9, 12, 18]). In addition to the treatment of the coherent state describing the Newtonian potential (here named areal to make the link with coordinates easily recognizable), we also treated the coherent state describing the harmonic potential. We then set up a possible method to define an operator that transforms two different coherent states, and with this we defined the operator $\hat{P}_M$ that realizes the transformation of the areal coherent state for a source of mass M into the harmonic one. We finally tried

to transform with $\hat{P}_M$ states describing different sources (i.e., with M' different from M) and saw these are still projected onto coherent states of the M source, thus confirming the strict dependence of the coordinate transformation on the parameters of the metric.

We also studied the black hole configuration for both potentials. In our description, this configuration corresponds to the definition of the coherent state only in the region outside the horizon, which is determined as the surface where the escape velocity for the considered potential equals the speed of light. Therefore, we do not need modes of any wavelength to define this state, but only those of wavelengths no smaller than of some fraction of the gravitational radius R_{H}, which we have identified as the inverse of the ultraviolet cut-off R_{S}. As shown in Ref. [20] for the areal potential, the effective geometry corresponding to this black hole state is described by a quantum (effective) potential V_{Q}. This geometry still contains the horizon, as can be observed in 3.2, but is regular at $r = 0$. In [20] an evaluation of the tidal force for $r = 0$ is also made which, contrary to the classical case, turns out to be finite since it is proportional to $(G_{\mathrm{N}}\, M)^2/R_{\mathrm{S}}^4$. This means that, in this quantum Schwarzschild description of a black hole, the matter forming it never reaches the singularity but forms a macroscopic configuration, showing to be a description compatible with the idea that gravity *classicalizes* at high energies.
Moreover, the effective quantum potential oscillates around the classical potential, and this could be interesting from a phenomenological point of view, as these oscillations can be observed by an hypothetical observer in $r > R_{\mathrm{H}}$. In this work we have verified the same behavior for the quantum harmonic potential, finding how it oscillates around the classical harmonic potential. The oscillation amplitudes decrease as the R_{S} dimension decreases, and thus as the ultraviolet cut-off increases, behaving therefore in the same way as the areal quantum potential.

Appendix A

Corpuscolar model

We will here resume the ideas at the base of the corpuscular model formulated by Dvali and Gomez, referring mainly to [21–25].

A theory of quantum gravity as a QFT of a massless spin 2 particle (i.e., the graviton), obtained by quantizing the General Relativity action in the weak field limit, leads to several problems, the largest and best known being the fact that such theory is *not renormalizable* One of the ways this can be seen is by considering Newton's constant G_{N} as a coupling constant. Indeed, we see that in Planck units $c = \hbar = 1$ this has dimension $[M]^{-2}$, and theories with coupling constant with negative powers of mass violate the renormalization principle. The problem is that the scattering amplitudes calculated by perturbative methods are composed of infinities of ever-increasing order and therefore cannot be eliminated. Because of this, gravity cannot be a predictive theory at high energies and must therefore be "completed" in other ways.
This "*UV-completion*" is what effective field theories (EFT) are all about, their purpose being to find the most suitable degrees of freedom able to describe the system using the usual perturbative theory. In the standard (Wilsonian) approach, the UV completion of a theory is achieved by introducing new degrees of freedom so that in the weak coupling regime the old theory is found, while at higher scales new physics is manifested such that predictions can again be made by calculating scattering amplitudes (an example is how Fermi's theory, which cannot be renormalized, was "cured" by introducing three bosonic fields $W^{\pm}$ and Z^0 as mediators of the weak interaction). However, the approach considered by Dvali and Gomez is non-Wilsonian and is based on the ideas of *self-completeness* [21] of gravity and *classicalization* [22].

A.1 Self-completeness of Einstein gravity by classicalization

Dvali and Gomez argue that gravity is self-completing in deep-UV. The idea behind this claim is that any new degrees of freedom introduced to complete the theory at high energy has no physical sense because it would correspond to distances that cannot be probed. The scale of strong coupling for gravity is identified in the Planck mass $m_{\rm p} \sim 10^{19}\,\text{GeV}$, and the corresponding length is therefore the Planck length $\ell_{\rm p} \sim 1/m_{\rm p} \sim 10^{-33}\,\text{cm}$. The Planck length for Einstein gravity corresponds to the absolute minimum length possible for any distance to be probed.
Suppose in fact that we want to make measurements at $l \ll \ell_{\rm p}$. The minimum amount of energy that can be located in a volume l^3 must obey Heisenberg's uncertainty principle, whereby $E \gtrsim 1/\Delta t \sim 1/l$ (in Planck units). The gravitational radius associated with this region of space would then be $R_{\rm H} \simeq 2G_{\rm N}E \sim \ell_{\rm p}^2/l$, therefore we would have $R_{\rm H} \sim \ell_{\rm p}^2/l \gg l$. This means that any attempt to resolve scales smaller than Planck's would cause us to "bounce back" at much greater distances $\ell_{\rm p}^2/l \gg l$, and the reason is the existence of the black hole of gravitational radius $R_{\rm H}$ that is created at such scales. There is therefore a correspondence between physics at sub-Planckian scales (deep-UV) and that at macroscopic scales (deep-IR):

$$l \longleftrightarrow \frac{\ell_{\rm p}^2}{l}\,. \tag{A.1.1}$$

Degrees of freedom that propagate beyond the Planckian scale therefore make no physical sense. They are instead mapped into classical states (corresponding precisely to the $\ell_{\rm p}^2/l$ scale), which are dynamically described by degrees of freedom of soft (i.e., at low energies) massless gravitons.
The theory therefore self-completes through this process of *classicalization*, producing these high multiplicity classical states of the same particles that were already in the theory, and the quanta that make up these states are *soft* and *weakly interacting* with each other.

A.2 Corpuscolar black hole

Following the classicalization scheme, Dvali and Gomez formulated the description of a black hole as a Bose-Einstein condensate of gravitons. We try here to qualitatively summarize the main points.

Gravitons self-interact; the dimensionless self-coupling constant of gravitons at low energies is [23]:

$$\alpha_{\rm G} = \frac{\hbar\, G_{\rm N}}{\lambda_{\rm G}^2} = \frac{\ell_{\rm p}^2}{\lambda_{\rm G}^2}\,, \tag{A.2.1}$$

where $\lambda_{\rm G}$ is the wavelength of the gravitons. If $\lambda_{\rm G} \gg \ell_{\rm p}$ the gravitons will interact very weakly. Now suppose we have a spherical mass M for which $R \gg R_{\rm H}$. In such a configuration the total graviton energy is the Newtonian one:

$$E \simeq \frac{M R_{\rm H}}{R}\,. \tag{A.2.2}$$

If we assume that the gravitons are weakly interacting, this energy is also equal to the sum of their individual energies $\hbar/\lambda$, therefore,

$$E = \sum_{\lambda} N_\lambda \frac{\hbar}{\lambda} \simeq \frac{N_{\rm G}\hbar}{R}\,, \tag{A.2.3}$$

where we have assumed that the peak of this distribution is in $\lambda = R$. By equating the two gravitational energies we obtain:

$$N_{\rm G} = \frac{M\, R_{\rm H}}{\hbar}\,. \tag{A.2.4}$$

Note that for $\lambda_{\rm G} = R$ the coupling constant $\alpha_{\rm G}$ (A.2.1) becomes very small.
We can now interpret the number of gravitons $N_{\rm G}$ as a measure of the *classicality* of the system. If

$$N_{\rm G} \gg 1 \tag{A.2.5}$$

we say that the system is classical.
However, if the mass M has a radius of size comparable to the gravitational radius $R \simeq R_{\rm H}$ (and is therefore in a black hole configuration), we find a condition that links the coupling constant and the number of gravitons. In fact we have:

$$\alpha_{\rm G} = \frac{\hbar\, G_{\rm N}}{R_{\rm H}^2} \simeq \frac{\hbar}{M R_{\rm H}} \simeq \frac{1}{N_{\rm G}}\,, \tag{A.2.6}$$

where in the last equality we used Eq. (A.2.4).
We can understand the meaning of this condition by giving an estimate of the collective binding potential felt by each graviton for the (weak) interaction with the other $N-1$ ones:

$$U_{\rm G} \simeq -\alpha_{\rm G} N \frac{\hbar}{\lambda_{\rm G}}\,, \tag{A.2.7}$$

with $r \simeq \lambda_{\rm G}$. Furthermore, assuming that the momentum of each graviton is

$\vec{p}_{\rm G}$, its kinetic energy will be $K \simeq |\vec{p}_{\rm G}| \simeq \hbar/\lambda_{\rm G}$. By substituting $\alpha_{\rm G} \simeq 1/N_{\rm G}$ into Eq. (A.2.7), we obtain

$$K + U \simeq 0\,. \tag{A.2.8}$$

In a black hole configuration therefore the system of gravitons is self-sustaining, because for each graviton the binding energy and the kinetic energy are compensated. Moreover, even if single gravitons interact weakly with each other, the system is globally in a strong coupling regime: indeed Eq. (A.2.6), called *marginally bound condition*, can be written as

$$g \equiv N_{\rm G}\,\alpha_{\rm G} \simeq 1\,, \tag{A.2.9}$$

where g is the collective (dimensionless) coupling. Gravitons can thus be seen as elements of a Bose-Einstein condensate at the quantum critical point [25]: although the constituents of the final state interact weakly, the effects due to the collective interaction between gravitons become extraordinarily relevant and lead to the global classical state.

From the marginally bound condition (A.2.6) we can derive all the typical quantities of a black hole in terms of the number of gravitons N. These are expressed in the so called scaling laws of corpuscular black holes:

$$\alpha \simeq \frac{1}{N_{\rm G}} \tag{A.2.10}$$

$$\lambda_{\rm G} \simeq \sqrt{N}\ell_{\rm p} \tag{A.2.11}$$

$$M \simeq \simeq \sqrt{N}m_{\rm p}\,. \tag{A.2.12}$$

Through the analogy with the Bose-Einstein condensate, this model is able to explain semiclassical black hole features as Hawking radiation and Bekenstein entropy (e.g., this is done in Refs. [23, 24]).

Appendix B

Harmonic potential transform

Let's see how to obtain the trasform of harmonic potential (2.3.18). First of all we project it on Bessel functions $j_0(kr)$:

$$\begin{aligned}
\tilde{V}_H(k) &= 4\,\pi \int_0^\infty r^2\,\mathrm{d}r\, j_0(kr)\, V_H(r) = \\
&= -4\,\pi \frac{G_\mathrm{N}\, M}{k} \int_0^\infty \mathrm{d}r\, \sin(kr) \left(1 + \frac{G_\mathrm{N}\, M}{r}\right)^{-1} = \\
&= -4\,\pi \frac{G_\mathrm{N}\, M}{k} \int_0^\infty \mathrm{d}r\, \left[\sin(kr) - G_\mathrm{N}\, M \frac{\sin(kr)}{r + G_\mathrm{N}\, M}\right] \\
&= -4\,\pi \frac{G_\mathrm{N}\, M}{k^2} \left[\int_0^\infty \mathrm{d}z\, \sin z - G_\mathrm{N}\, M k \int_0^\infty \mathrm{d}z \frac{\sin z}{k + G_\mathrm{N}\, M k}\right],
\end{aligned} \tag{B.0.1}$$

where we change integration variable in $z = kr$. The first integral gets

$$\int_0^\infty \mathrm{d}z\, \sin z = 1 - \lim_{z\to\infty} \cos z\,, \tag{B.0.2}$$

where, since $\tilde{V}_H(k)$ is to be understood as a distribution, we will ignore the second term for the Riemann-Lebesgue lemma.
For the second integral we get instead:

$$\int_0^\infty \mathrm{d}z \frac{\sin z}{k + G_\mathrm{N}\, M k} = f(G_\mathrm{N}\, M\, k) \tag{B.0.3}$$

where the function $f(x)$ is the so-called auxiliary function defined in [26]:

$$f(x) = \int_0^\infty \mathrm{d}t\, \frac{\sin t}{t + x} = \left[\frac{\pi}{2} - \mathrm{Si}(x)\right] \cos x + \mathrm{Ci}(x)\, \sin x\,. \tag{B.0.4}$$

with $\mathrm{Si}(x) = \int_0^x \mathrm{d}z \frac{\sin z}{z}$ and $\mathrm{Ci}(x) = -\int_x^\infty \mathrm{d}z \frac{\cos z}{z}$ respectively the sine and the cosine integral functions.

The harmonic potential transform finally is:

$$
\begin{aligned}
\tilde{V}_H(k) = -\frac{4\,\pi\,G_{\rm N}\,M}{k^2}\Bigg\{&1 - G_{\rm N}\,M\,k\left[\left(\frac{\pi}{2} - \mathrm{Si}(G_{\rm N}\,M\,k)\right)\cos(G_{\rm N}\,M\,k) + \right.\\
&\left.+\,\mathrm{Ci}(G_{\rm N}\,M\,k)\,\sin(G_{\rm N}\,M\,k)\right]\Bigg\}\\
\equiv -\frac{4\,\pi\,G_{\rm N}\,M}{k^2}&[(1 - G_{\rm N}\,M\,k f(G_{\rm N}\,M\,k)]\ .
\end{aligned}
\tag{B.0.5}
$$

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