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Growth of cosmological structures in an interacting vacuum scenario

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Abstract

The widely accepted standard model of cosmology consists of a cosmological constant, Λ, responsible for the present expansion of the Universe, and cold dark matter, (CDM), accountable for structure formation. This famous model, also known as ΛCDM, suffers from both theoretical and observational problems, which motivate the study of alternative models.
In this thesis I present an alternative scenario: the interacting vacuum. In this class of models, unlike in ΛCDM, vacuum energy, a dark energy with $w = -1$ equation of state, and CDM are allowed to exchange energy and momentum.
I focus, in particular, on the geodesic CDM case, where only a pure energy exchange is allowed. Within this subclass, three relevant models are considered: the linear vacuum model, whose energy exchange is proportional to the vacuum energy density, the generalised Chaplygin gas (GCG) interacting model, obtained decomposing the original GCG into the two interacting dark components, and the Shan-Chen (SC) interacting model obtained as well from a decomposition of the original SC model.
The aim of this elaborate is, using the covariant gauge-invariant (CGI) theory of cosmological perturbations, to show, for the aforementioned models, how the vacuum-CDM interaction affects the growth of cosmic structures. It is found that a growing/decaying vacuum energy always leads, for fixed initial conditions, to a less/more decreasing growth rate f w.r.t. the non-interacting limit, namely the ΛCDM model.

Sommario

Il modello cosmologico standard comprende la costante cosmologica, Λ, responsabile per la presente espansione dell'Universo, e la materia oscura fredda, dall'inglese (CDM), responsibile quest'ultima per la formazione di strutture cosmiche. Questo famoso modello cosmologico, riconosciuto in letteratura come ΛCDM, presenta tuttavia problemi sia dal punto di vista teorico che da quello osservativo, fatto che motiva lo studio di modelli alternativi.
Nell'elaborato presento uno scenario alternativo; quello di un vuoto interagente. In questa classe di modelli, a differenza del modello ΛCDM; il vuoto, ovvero una energia oscura con equazione di stato $w = -1$, e la materia oscura fredda possono interagire liberamente scambiandosi energia e momento.
In particolare, mi sono concentrato sul caso di CDM geodetica, per il qual caso è concesso solo un trasferimento di energia e non di momento. All'interno di questa sottoclasse di modelli ne sono stati presi in considerazione tre: un vuoto linearmente interagente, dove il trasferimento di energia è proporzionale al vuoto stesso, il modello 'interacting generalised Chaplygin gas' (GCG) ricavato separando le due componenti oscure dall'omonimo modello GCG, e infine il modello interagente Shan-Chen (SC) ottenuto, anche questo, attraverso uno splitting del suo omonimo SC non interagente.
L'obiettivo di questo elaborato è, all'interno del formalismo della teoria covariante gauge invariante delle perturbazioni cosmologiche, quello di mostrare, per i modelli sopra citati, come l'interazione vuoto-CDM modifichi la crescita di strutture cosmiche. Si è trovato che la crescita/decrescita del vuoto implica, per le condizioni iniziali che sono state imposte, una minore/maggiore decrescita del 'growth rate' f rispetto al limite non interagente, ossia al modello ΛCDM.

Ai miei genitori

Non piove mai... E piove sempre.

Maurizio Ferro

Contents

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Introduction

The relatively recent discovery of the accelerated expansion of the Universe indicates that ($\sim 95\%$) of the total energy of the Universe is in the form of unknown dark fluid components, namely, dark energy ($\sim 70\%$) and dark matter ($\sim 25\%$). The remaining components are in the form of baryonic matter and radiation, the latter being, at present time, negligible compared to the former. The clustering dark matter with zero pressure (cold dark matter) is concentrated in structures and plays crucial role for forming galaxies and clusters of galaxies; dark energy instead possesses the negative pressure that drives the recent accelerated expansion. The simplest way to describe dark energy is to associate it with a constant vacuum energy density V characterised by the equation of state parameter $w = -1$. In this manner, V is equivalent to a cosmological constant in GR. The cosmological model that incorporates both a constant vacuum energy, denoted by the Greek letter Λ, plus cold dark matter is known as ΛCDM. This model is the standard model of cosmology. However, the cosmological constant explanation of dark energy is deficient in many aspects. This has motivated many authors to work on alternative dark energy models, e.g. quintessence, beyond GR theories, etc. In this thesis we will focus on the so-called interacting vacuum scenario, an alternative type of dark energy in which the vacuum is free to interact with cold dark matter.
The aim of the paper is to show, for different interacting vacuum models, how the aforementioned interaction affects the growth of cosmological structures w.r.t. ΛCDM case; which, as stated above, corresponds to a non interacting limit.
We begin by laying the foundations of the standard model of cosmology in Chapter 1, discussing the Friedmann-Lemaître-Robertson-Walker (FLRW) solution to the Einstein equations and its implications. Later on in this chapter, we have presented the standard approach to cosmological perturbation theory focusing, in particular, on its differential geometry foundations. Next, in Chapter 2, we discuss an alternative way for dealing with cosmological perturbations, based on the covariant approach to GR. This is the covariant gauge invariant (CGI) approach to cosmological perturbations. Following this path, in Chapter 3, we present in detail the interacting vacuum scenario. Then, within the CGI framework, we derive the dynamical equations for the scalar part of the comoving matter density gradient. We conclude the exposition of this thesis with Chapter 4, where we show how the the growth rate of cosmic structures f behaves, compared to ΛCDM,

for three different interacting vacuum models: a linear vacuum model, the generalised Chaplygin gas (GCG) and the Shan-Chen (SC) dark energy models both cast into the framework of the interacting vacuum.

Chapter 1

Fundamentals of cosmological perturbation theory

The present view of the Universe divides its history into two parts: an early stage behind the curtain of the surface of last scattering, which is most commonly modelled within the paradigm of inflation, and a later stage during which the visible large scale structures came into being and is described by the phenomenological ΛCDM model (frequently referred to as the standard model of cosmology). This model is based upon the spatially homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime. It successfully describes the average expansion of the universe on large scales according to Einstein's theory of General Relativity (GR), and the evolution from a hot, dense initial state dominated by radiation to the cool, low density state dominated by non-relativistic matter and, evidently, vacuum energy at the present day.
To a high degree of accuracy the large scale structure of the Universe can be described by a homogeneous isotropic cosmology, the aforementioned FLRW spacetime. However, the Universe we observe on sufficiently small scales is neither homogeneous nor isotropic, otherwise we would not observe structures such as galaxies and clusters of galaxies. This shows that FLRW spacetimes cannot be the whole picture and we have to modify them if we want to incorporate inhomogeneities and anisotropies.
A very successful solution is to view FLRW spacetimes as a 0-th order term in a perturbation expansion and derive higher order perturbation terms using perturbation theory. Perturbation theory is used in many other areas in physics and the idea to employ these techniques in GR, cosmology in particular, originates from the work of Lifshitz in 1946 [20], who was mainly interested in the dynamical stability of FLRW spacetimes against perturbations. However, as Lifshitz himself pointed out, perturbation theory within GR faces some additional problems w.r.t. other areas in physics, the so-called 'gauge issues'. Even if one fixes a particular gauge (coordinate system) and work within that, spurious (or fictitious) gauge modes may appear during the derivation of the dynamical equations. A way to circumvent the problem is to use a gauge invariant (coordinate choice indepen-

dent) theory of cosmological perturbations. This theory was for developed by Bardeen in 1980 [4], who introduced a set of GI quantities describing perturbations in both the matter and geometry sectors, thus avoiding the appearance of spurious gauge modes. However, the physical meaning of these variables is recovered only in some particular gauge. This is why Ellis & Bruni in 1989 [1, 2] proposed an alternative path. They developed the covariant gauge invariant (CGI) approach to cosmological perturbations, where they defined a set of physically meaningful CGI variables, exploiting the Stewart & Walker Lemma [6]. In this chapter, we shall present, in a rigorous way, what we have just discussed.

1.1 The FLRW background spacetime

The form of a cosmological metric can be partly fixed by assuming the existence of Killing vectors. In particular, we will have three space-like Killing vectors generating spatial translations (which mathematically defines homogeneity), and three space-like Killing vectors generating rotations (which mathematically defines isotropy). It is just worth pointing out that we have no time-like Killing vector since we want to describe an evolving Universe.[†]

In GR this translates into the statement that the Universe can be foliated into spacelike slices such that each three-dimensional slice is maximally symmetric, i.e. it posses the maximum number of Killing vectors [15], which for a d-dimensional space is:

$$N = \frac{d(d+1)}{2} \, . \tag{1.1}$$

We therefore consider our spacetime to be $\mathbb{R} \times \Sigma$, where $\mathbb{R}$ represents the time direction and Σ is a maximally symmetric spatial manifold. The spacetime metric thus takes the form:

$$ds^2 = -dt^2 + \gamma_{ij} dx^i dx^j \qquad i, j = 1, 2, 3 \, . \tag{1.2}$$

where t is the timelike coordinate and γ_{ij} is the metric on the three-dimensional surface Σ, where for convenience we have considered a synchronous frame. For a maximally symmetric space the spatial Riemann tensor is given by:

$${}^{(3)}R_{ijkl} = \kappa(\gamma_{ik}\gamma_{jl} - \gamma_{jk}\gamma_{il}) \, , \tag{1.3}$$

where κ is a constant that can assume any real value. In particular, throughout our work we will focus on the FLRW metric, where $\gamma_{ij} = \gamma_{ij}(t) = a(t)^2 h_{ij}$. In this case the spatial Riemann tensor can be written as:

$${}^{(3)}R_{ijkl} = \frac{6K}{a^2}(h_{ik}h_{jl} - h_{jk}h_{il}) \, , \tag{1.4}$$

[†]In 1929 E. Hubble and M. Humason observed that farer galaxies recedes from us faster and faster.

where ${}^{(3)}R = 6K/a^2$ is the spatial Ricci curvature. From the above, one can then derive the FLRW metric in his most known form as:

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \right] . \tag{1.5}$$

The origin $r = 0$ is totally arbitrary, t is the proper time of an observer moving along with the homogenous and isotropic cosmic fluid (the idealized representation of galaxies) at r, θ and ϕ constant. k is instead the constant curvature of the Universe, whose sign depends on κ, and whose values are re-scaled from the ones of κ to $k = 0, +1, -1$. These corresponds respectively to a flat (Σ_t is flat), closed (Σ_t is a 3-sphere) and open (Σ_t is a 3-hyperboloid) spacetime:

$$ds^2 = -dt^2 + a^2 \begin{cases} d\psi^2 + \psi^2(d\theta^2 + \sin^2\theta \, d\phi^2) \\ d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta \, d\phi^2) \\ d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta \, d\phi^2) \end{cases} \tag{1.6}$$

To sum up, the relevant coordinates and quantities in the FLRW metric are:

$$\begin{aligned} \{t, r, \theta, \phi\} \qquad & \text{comoving coordinates} \\ a(t) \qquad & \text{(cosmic) scale factor} \\ k = 0, \pm 1 \qquad & \text{curvature constant .} \end{aligned} \tag{1.7}$$

Observations are consistent with the Universe being spatially flat, to a high degree of accuracy. In the following we will therefore restrict our attention to a spatially flat FLRW metric, $K = 0$. The FLRW metric then reads:

$$ds^2 = -dt^2 + a(t)^2 d\mathbf{x}^2 . \tag{1.8}$$

Physical coordinates are defined w.r.t. the comoving ones as $\mathbf{x}_{ph} = a(t)\mathbf{x}$. For a spatially flat FLRW metric the overall normalization of the scale factor can be chosen at will, since it amounts to a rescaling of the coordinates $\mathbf{x}$. The standard choice is $a(t_0) = 1$, where t_0 is the present value of cosmic time. A relevant quantity in cosmology is the Hubble parameter which is defined by:

$$H = \frac{\dot{a}}{a} , \tag{1.9}$$

its present value, denoted by H_0, is:

$$H_0 = h_0 \, 100 \, \text{km s}^{-1}\text{Mpc}^{-1} , \tag{1.10}$$

where the measured value of h_0 is close to ~ 0.7, with differences at the level of a few percent, depending on the cosmological measurements and datasets used (CMB+BAO+SNe).

The Hubble parameter $H(t)$ has unit of inverse time and is positive for an expanding Universe. It sets the fundamental scale of the FLRW spacetime, i.e. the characteristic time-scale of this Universe is given by $t_H \sim H^{-1}$, while the characteristic length-scale is given by $\lambda_H \sim H^{-1}$(in units where c = 1). The former sets the scale for the age of the Universe, while the latter sets its current observable size.

1.1.1 Particle horizon

Let's introduce the conformal time η, defined by:

$$d\eta \equiv \frac{dt}{a(t)}\,, \tag{1.11}$$

so that

$$\eta = \int^t \frac{dt'}{a(t')}\,, \tag{1.12}$$

where the lower limit of integration can in principle be chosen arbitrarily. Adopting the conformal coordinates (η, $\mathbf{x}$), the FLRW metric takes the form:

$$ds^2 = a(\eta)^2(-d\eta^2 + d\mathbf{x}^2)\,. \tag{1.13}$$

The maximum comoving distance light can propagate between an initial time t_i and some later time t is:

$$\mathrm{r_{PH}}(t) = \eta - \eta_i = \int_{t_i}^t \frac{dt'}{a(t')}\,, \tag{1.14}$$

where we have replaced $\mathbf{x}_{\mathrm{PH}}$ with $\mathrm{r_{PH}}$ just for notation convenience. Since light travels along the curve $ds^2 = 0$, in the FLRW model (without an earlier inflationary phase) $\mathrm{r_{PH}}(t)$ is equal to the comoving distance traveled by light from time $t_i = 0$ up to time t. The initial time t_i is often taken to be the 'origin of the universe', $t_i \equiv 0$, defined by the initial singularity, $a(t_i = 0) \equiv 0$.
This comoving distance therefore represents the maximum size of a region (measured in comoving coordinates) that can be causally connected at time t, and it goes under the name of comoving particle horizon. The corresponding physical particle horizon at time t is:

$$\mathrm{R_{PH}}(t) = a(t)\mathrm{r_{PH}}(t)\,. \tag{1.15}$$

In an accelerating Universe, if two particles are separated by a comoving distance greater than the comoving Hubble scale, $\lambda > (aH)^{-1}$, they cannot talk to each other from the time considered on. However, if they are outside of each other's comoving particle horizon, this means they could have never communicated.

It is useful to remind the reader that each comoving (or coordinate) wavelength λ has an associated comoving wave number $k \equiv |\vec{k}|$ following from $\lambda = 2\pi/k$. These quantities are related to the physical wavelength and wave number of the Fourier mode by:

$$k_{\rm ph} = \frac{2\pi}{\lambda_{\rm ph}} = \frac{2\pi}{a\lambda} = a^{-1}k\,. \tag{1.16}$$

Thus the wavelength $\lambda_{\rm ph}$ corresponding to a given Fourier mode $\vec{k}$ grows in time as the Universe expands. Since fluctuations are created on all length scales, i.e. with a spectrum of wavenumbers k, we say that a fluctuation is inside/outside the horizon if:

$$\text{sub - Horizon:} \quad k \gg (aH)\,, \tag{1.17}$$

$$\text{super - Horizon:} \quad k \ll (aH)\,. \tag{1.18}$$

1.1.2 Cosmic fluids and dynamics

As we wrote above, we assume the Universe is filled with a (perfect) fluid of matter and energy. If this fluid, that is isotropic in some frame, leads to a metric that is isotropic in some frame, then the two frames have to coincide.
This means that the fluid is at rest in comoving coordinates. The four-velocity is then $u^\mu = (1, \bar{0})$ and the corresponding energy-momentum tensor takes the simple form:

$$T^\mu_{\ \nu} = \text{diag}(-\rho, P, P, P)\,, \tag{1.19}$$

where $\rho = \rho(t)$ and $P = P(t)$ are the density and pressure of the cosmic fluid. The Einstein equations $G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$, relating the geometry of spacetime to the distribution of energy and matter within it, then take the form of two coupled, non-linear ordinary differential equations, called the Friedmann equations:

$$G_{00} = 8\pi G_N T_{00} \quad \Rightarrow \quad \left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] = \frac{8\pi G_N}{3}\rho\,, \tag{1.20}$$

$$G_{ii} = 8\pi G_N T_{ii} \quad \Rightarrow \quad \frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\rho + 3P)\,. \tag{1.21}$$

Notice that in an expanding Universe (i.e. $\dot{a} > 0$) filled with ordinary matter (i.e. matter satisfying the strong energy condition: $\rho + 3P > 0$) Eqn. (1.21) implies $\ddot{a} < 0$. This indicates, as we have previously anticipated, the existence of a singularity in the finite past: $a(t \equiv 0) = 0$. Of course, this conclusion relies on the assumption that GR and the Friedmann equations are applicable up to arbitrary high energies and that no exotic forms of matter become relevant at high energies. More likely the singularity simply signals the breakdown of GR.

Using either the Bianchi identities for $T_{\mu\nu}$ or alternatively by combining Eqns. (1.20) and (1.21) we arrive at the continuity equation, which can be expressed in the form:

$$\frac{\dot{\rho}}{\rho} = -3\left(1 + \frac{P}{\rho}\right)\frac{\dot{a}}{a}\,. \tag{1.22}$$

If we now assume a barotropic fluid, namely a fluid whose pressure depends only on the density

$$P = w\rho\,, \tag{1.23}$$

with w being a constant, we then have:

$$\rho \propto a^{-3(1+w)}\,. \tag{1.24}$$

The simplest components of cosmic fluids are:

- **Dust**: in this case no force is present, beside gravity, and $w = 0$ (so that $P = 0$). This corresponds to pressure-less or non-relativistic matter.
- **Radiation**: since mass is totally negligible, so is the trace of the energy-momentum tensor:[†]

  $$T = (-\rho + 3P) = 0\,, \tag{1.25}$$

  we then find

  $$P = \frac{1}{3}\rho \qquad \Rightarrow \qquad w = \frac{1}{3}\,, \tag{1.26}$$

  This case corresponds to massless particles or highly-relativistic matter.

For a long time it was thought that we now live in a matter (dust) -dominated Universe, whereas in the early stages, the Universe dynamics was controlled by radiation, since the density of the latter increases faster (going backward in time, see Fig. (1.1)). We now know that the Universe expansion is presently accelerating ($\ddot{a} > 0$), which is not compatible with the effect of dust. In order to give rise to the cosmic acceleration in Eq. (1.21) we need,

$$P < -\frac{1}{3}\rho \qquad \Rightarrow \qquad w < -\frac{1}{3}\,, \tag{1.27}$$

where ρ is assumed to be positive. From this requirement we introduce the following additional component:

[†]Recall that the trace of a (0, 2) tensor is invariant under rotations and this result is lifted to any reference frame in GR. For a plane wave moving along the x-axis, the energy-momentum tensor is $T^{\mu}{}_{\nu} = \mathrm{diag}(-E, p, 0, 0)$, where E is the energy and p the momentum, and the "mass-shell" condition $E = p$ implies $T = 0$.

- **Vacuum or dark energy**: When $w = -1$, i.e. $P = -\rho$, it follows from Eq. (1.24) that ρ is constant:

$$\rho = \frac{\Lambda}{8\pi G_N}\,, \tag{1.28}$$

where Λ is the famous cosmological constant representing the vacuum energy of the Universe. This is the simplest component that fulfils the criteria in Eq. (1.27), explaining so the current accelerated expansion of the Universe.

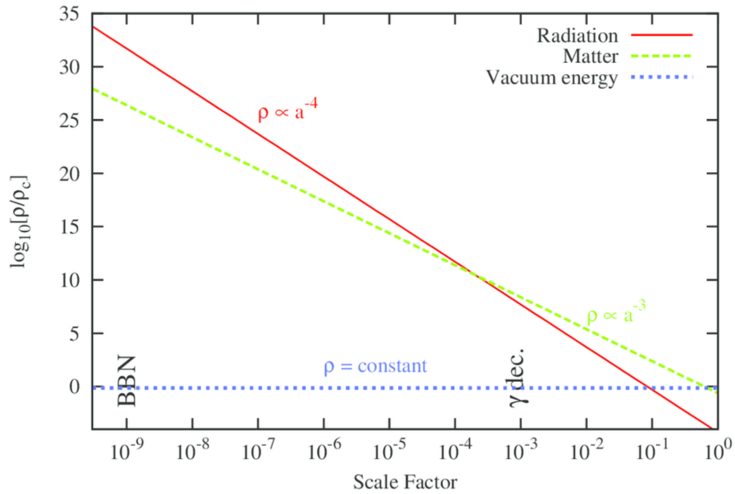

Figure 1.1: Simple sketch of the different epochs in the history of the Universe. In this picture, the Hot Big Bang (HBB) is the initial moment at the onset of the radiation phase, when the Universe was very small, hot and dense.

Eq. (1.24) along with the Friedmann Eq. (1.20) lead to the time evolution of the scale factor:

$$a(t) \propto \begin{cases} t^{\frac{2}{3(1+w)}} & w \neq -1 \\ e^{Ht} & w = -1\,. \end{cases} \tag{1.29}$$

For a flat ($K = 0$) FLRW Universe we have:

$$\begin{aligned} a(t) &\propto t^{\frac{2}{3}} && \Rightarrow && \text{Matter dominated Universe,} \\ a(t) &\propto t^{\frac{1}{2}} && \Rightarrow && \text{Radiation dominated Universe,} \\ a(t) &\propto e^{Ht} && \Rightarrow && \text{Vaccum energy dominated Universe.} \end{aligned} \tag{1.30}$$

Where in the vacuum energy dominated Universe, most commonly known as de Sitter Universe, H is just a constant factor. For such a multi-component model Eq. (1.20) can be rewritten using the density parameters, in the following manner:

$$H^2 = \sum_i \frac{8\pi G_N}{3}\rho_i - \frac{k}{a^2} \quad \Rightarrow \quad 1 = \sum_i \Omega_i + \Omega_k\,, \tag{1.31}$$

where

$$\Omega_i \equiv \frac{\rho_i}{\rho_*}, \qquad \Omega_k \equiv -\frac{k}{(aH)^2} \qquad \rho_* \equiv \frac{3H^2}{8\pi G_N}. \tag{1.32}$$

We often refer to the present values of the density parameters for radiation, non-relativistic matter (baryons + cold-dark-matter), dark energy, and curvature respectively as:

$$\Omega_r^{(0)} \equiv \frac{\rho_r^{(0)}}{\rho_*}, \quad \Omega_m^{(0)} \equiv \frac{\rho_m^{(0)}}{\rho_*}, \quad \Omega_{de}^{(0)} \equiv \frac{\rho_{de}^{(0)}}{\rho_*}, \quad \Omega_k^{(0)} \equiv -\frac{k}{H_0^2}. \tag{1.33}$$

If the expansion of the Universe is decelerating (i.e. $\ddot{a} < 0$) then the curvature term $|\Omega_k|$ continues to increase (because the term $(aH)^2 = \dot{a}^2$ decreases), apart from the case where the Universe is exactly flat ($K = 0$) from the very beginning.
Latest observations, as we mentioned above, point towards $\Omega_k^{(0)} \simeq 0$. This means we need a phase of cosmic acceleration ($\ddot{a} > 0$) to reduce $|\Omega_k|$ in the past cosmic expansion history, unless the initial state of the Universe is extremely close to the flat one. In order to realize the present level of flatness of the Universe, we require, prior to the radiation-dominated epoch, a phase of cosmic inflation during which the scale factor increases by more than e^{70} times.
From Eq. (1.31), assuming zero spatial curvature we have:

$$1 = \Omega_r + \Omega_m + \Omega_{de} = \Omega_{tot}\,. \tag{1.34}$$

A spatially flat Universe, with $\Omega_{tot} \simeq 1$, corresponds to a current average density

$$\rho_0 = \rho_* \simeq 10^{-29} \mathrm{g/cm}^3\,, \tag{1.35}$$

equivalent to about 6 protons per square cubic meter. In particular, three different sources have been identified to contribute to ρ_0:

- **Baryonic matter**, well approximated by a dust fluid, and estimated through the luminosity of galaxies in the cosmos to be $\Omega_b^{(0)} \simeq 0.05$.
- **Dark matter**, which again behaves like dust but is not directly detected, amounting to $\Omega_{dm}^{(0)} \simeq 0.25$.
- **Radiation**, negligible w.r.t. the other components, $\Omega_r^{(0)} \simeq 10^{-5}$.
- **Dark energy**, described by the above condition (1.27) $\Omega_{de}^{(0)} \simeq 0.70$, meaning that a total of 95% of the Universe content is nowadays still unknown.

A description of the Universe based on a flat FLRW spacetime, whose components are the ones listed above, corresponds to the famous ΛCDM model. To a high degree of accuracy the large scale structure of the Universe can be described by a homogeneous isotropic cosmology, the aforementioned FLRW spacetime. However, the Universe we observe on sufficiently small scales is neither homogeneous nor isotropic (there are galaxies and clusters of galaxies). This shows that FLRW spacetimes cannot be the whole picture and we have to modify them if we want to incorporate these inhomogeneities and anisotropies. We thus present, in the following section, the theory of cosmological perturbations.

1.2 Standard approach to cosmological perturbation theory

In GR perturbation theory one has a real perturbed spacetime that is close to a simple known symmetric background (BG) spacetime. This means that there exists a coordinate system on the perturbed spacetime, where its metric can be written as

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x)\,, \tag{1.36}$$

$\bar{g}_{\mu\nu}$ being the metric of the background spacetime[†], and $\delta g_{\mu\nu}$ a small deviation from it. In addition, we require first and second partial derivatives, $\delta g_{\mu\nu,\rho}$ and $\delta g_{\mu\nu,\rho\sigma}$ to be small as well. The Einstein tensor and the energy-momentum tensor of the perturbed spacetime can then be written as:

$$G_{\mu\nu}(x) = \bar{G}_{\mu\nu}(x) + \delta G_{\mu\nu}(x)\;, \tag{1.37}$$

$$T_{\mu\nu}(x) = \bar{T}_{\mu\nu}(x) + \delta T_{\mu\nu}(x)\;, \tag{1.38}$$

where $\delta G_{\mu\nu}$ and $\delta T_{\mu\nu}$ are again a small deviation from their respective BG quantities. From the above and using Einstein equations:

$$G_{\mu\nu}(x) = 8\pi G_N T_{\mu\nu}(x) \qquad \text{and} \qquad \bar{G}_{\mu\nu}(x) = 8\pi G_N \bar{T}_{\mu\nu}(x)\,, \tag{1.39}$$

we the obtain:

$$\delta G_{\mu\nu}(x) = 8\pi G_N \delta T_{\mu\nu}(x)\,. \tag{1.40}$$

In particular, in order to write the above, a pointwise correspondence between the two spacetimes is needed; so that we can compare different tensorial quantities belonging to different manifolds. We will return to this point much more in detail in Chapter 2.

In first-order (or linear) perturbation theory, we drop all terms from our equations which contain products of the small quantities $\delta g_{\mu\nu}$, $\delta g_{\mu\nu,\rho}$ and $\delta g_{\mu\nu,\rho\sigma}$. The field equation (1.40) becomes then a linear differential equation for $\delta g_{\mu\nu}$, making things much easier than in full GR. In second-order perturbation theory instead, one keeps also terms with a product of two (but not more) small quantities. This makes the perturbation approach

[†]Throughout this work, we will denote background quantities with either an overbar or a '0' subscript. However, from Chapter 2 on, we will use a 'lighter' notation where BG quantities will be understood.

to GR much more involved, see for example [17]. In cosmological perturbation theory the observable Universe can approximately be described by a homogeneous and isotropic FLRW spacetime. Since observations suggest we live in a spatially flat ($K = 0$) Universe, throughout this work we shall only consider this case. It is much simpler than the open and closed ones, since now the $t = const$ time slices have Euclidean geometry.
Following this assumption, all physical quantities can be decomposed into a homogeneous BG, where quantities depend solely on cosmic time, and inhomogeneous perturbations. The perturbations thus 'live' on the BG spacetime and it is this BG which is used to split the four-dimensional spacetime into spatial three-hypersurfaces as we will later see. Any tensorial quantity can then be split in this manner:

$$T(t, x^i) = T_0 + \delta T(t, x^i) \ . \tag{1.41}$$

The background part is a time-dependent quantity $T_0 \equiv T_0(t)$, whereas the perturbations depend on time and space coordinates $x^\mu = (t, x^i)$. The above perturbation can be further expanded as a power series:

$$\delta T(t, x^i) = \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \delta T_n(t, x^i) \, , \tag{1.42}$$

where the subscript n denotes the order of the perturbation, and ϵ is a small parameter. In linear perturbation theory, for example, we only consider first-order terms, i.e. ϵ^1, and can neglect terms resulting from the product of two perturbations, which would necessarily be of order ϵ^2 or higher.

1.2.1 Metric perturbations

Given the spatially flat ($K = 0$) FLRW background metric

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j \, , \tag{1.43}$$

the most general first-order perturbation is:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -(1+2\phi) dt^2 + 2a(t) B_i dx^i dt + a^2(t) [(1-2\psi)\delta_{ij} + 2E_{ij}] dx^i dx^j \ . \tag{1.44}$$

From the 3+1 (ADM) point of view the almost FLRW spacetime is described by means of spacetime foliations where spatial hypersurfaces of constant time t are called slices and curves of constant spatial coordinates x^i but varying time t are called threads. Using this formalism, E_{ij} is a spatial symmetric and traceless† tensor $E^i{}_i = \delta^{ij} E_{ij} = 0$, while ψ is the curvature perturbation, describing the intrinsic scalar curvature of spatial hypersurfaces. These are both perturbations of the metric on the slices.

†Depending on the notation E_{ij} can have also $E^i{}_i \neq 0$

The term B_i is the shift vector; it specifies the relative velocity between the threading and the worldlines orthogonal to the slicing. ϕ is the lapse function, it specifies the relation between the coordinate time t and the proper time τ along the threading. To first order:

$$\frac{d\tau}{dt} = \sqrt{1+2\phi} \simeq 1+\phi\,. \tag{1.45}$$

In GR perturbation theory there are two kinds of coordinate transformations of interest. One is the gauge transformation, to be later discussed, where the coordinates of the background are kept fixed, but the coordinates in the perturbed spacetime are changed, changing so the correspondence between the points in the background and the perturbed spacetime.

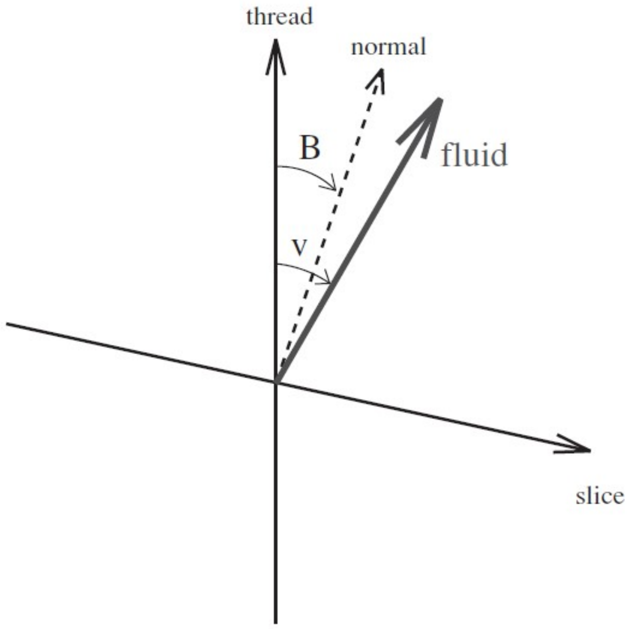

Figure 1.2: The coordinates (t, x^i) define a threading and a slicing (corresponding, respectively, to fixed x^i and fixed t). As indicated, the slicing typically isn't orthogonal to the threading. Depicted in the drawing are: the fluid velocity of a comoving observer $\mathbf{v}$, and the velocity of the worldline orthogonal to the slices $\mathbf{B}$.

The other kind is one where we keep the gauge, i.e the correspondence between the background and perturbed spacetime points, fixed, but do a coordinate transformation in the background spacetime. This, in turn, induces a corresponding coordinate transformation in the perturbed spacetime.
The homogeneity and isotropy of the background allows us to separate the time from the space dependence; i.e. allows us to foliate the spacetime into $t = \text{const}$ spacelike slices. This leaves us with the following transformations:

- Homogeneous transformations of the time coordinate, i.e., reparameterizations of time, e.g. going from cosmic time t to conformal time η: $dt \to d\eta = \frac{dt}{a(t)}$.

- Transformations in the space coordinates: $x^{i'} = X^{i'}{}_j x^j$.

It is by means of the latter one that perturbations in various quantities can be classified.† In particular, according to how they transform under spatial rotations in the background spacetime, we are able to distinguish among spatial scalars, vectors and tensors. The full transformation matrices are:

$$X^{\mu'}_{\ \nu} = \begin{bmatrix} 1 & 0 \\ 0 & X^{i'}_{\ j} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & R^{i'}_{\ j} \end{bmatrix} \qquad \text{and} \qquad X^{\mu}_{\ \nu'} = \begin{bmatrix} 1 & 0 \\ 0 & R^{i}_{\ j'} \, , \end{bmatrix} \tag{1.46}$$

where $R^{i'}_{\ j}$ is a rotation matrix,† with the property $RR^T = I$, or $R^{i'}{}_j R^k{}_{i'} = (RR^T)_{jk} = \delta_{jk}$. This BG coordinate transformation induces the corresponding transformation in the perturbed spacetime $x^{\mu'} = X^{\mu'}_{\ \nu} x^\nu$ where the metric is:

$$g_{\mu\nu} = \begin{bmatrix} -(1+2\phi) & aB_i \\ aB_i & a^2[(1-2\psi)\delta_{ij} + 2E_{ij}] \end{bmatrix} \tag{1.47}$$

Transforming the metric $g_{\rho'\sigma'} = X^\mu{}_{\rho'} X^\nu{}_{\sigma'} g_{\mu\nu}$, we get for the different components:

$$\begin{aligned} g_{0'0'} &= -(1+2\phi) \\ g_{0'i'} &= aR^k_{\ i'} B_k \\ g_{i'j'} &= a^2[(1-2\psi)\delta_{kl} + E_{kl}] R^k_{\ i'} R^l{}_{j'} \\ &= a^2[(1-2\psi)\delta_{ij} + E_{kl} R^k_{\ i'} R^l{}_{j'}] \, , \end{aligned} \tag{1.48}$$

from which we identify the perturbations in the new coordinates:

$$\begin{aligned} \phi' &= \phi \\ \psi' &= \psi \\ B_{i'} &= R^j_{\ i'} B_j \\ E_{i'j'} &= R^k_{\ i'} R^l{}_{j'} E_{kl} \, . \end{aligned} \tag{1.49}$$

Thus ϕ and ψ transform as scalars under rotations in the background spacetime coordinates, B_i transforms as a 3-vector, and E_{ij} as a spatial tensor. While staying in a fixed gauge, we can think of them as scalar, vector and tensor fields on the flat background

†In the literature this is known as the SVT decomposition of metric perturbations.

†In this notation $R^{i'}_{\ j}$ and $R^i{}_{j'}$ are two different matrices, corresponding to opposite rotations. The position of the $'$ indicates the direction of rotation. Notice that we have put the first index up to follow the Einstein summation convention even if it is not needed.

space. However, we are not yet satisfied; we can extract two more scalar quantities and one more vector quantity from B_i and E_{ij}. We know from the Helmholtz theorem of standard vector calculus, that a vector field can be decomposed into two parts: one with zero curl (longitudinal) and one with zero divergence (transverse):

$$B_i = B_i^{\parallel} + B_i^{\perp} \,, \tag{1.50}$$

Remember that a transverse vector is divergenceless and a longitudinal vector is curl-free, namely:

$$\partial^i B_i^{\perp} = 0 \,, \quad \epsilon^{ijk}\partial_j B_k^{\parallel} = 0 \,. \tag{1.51}$$

The curl-free part can be written as the gradient of a scalar $B_i^{\parallel} = B_{,i}$; further, we rename the divergenceless part as $B_i^{\perp} = -S_i$. In this manner B_i can be written as:

$$B_i = B_{,i} - S_i \,. \tag{1.52}$$

A similar procedure applies to the symmetric traceless spatial tensor E_{ij}, which can be divided into a transverse part $E_{ij}^{\perp}$, a solenoidal part E_{ij}^{S} and a longitudinal part $E_{ij}^{\parallel}$:

$$E_{ij} = E_{ij}^{\perp} + E_{ij}^{S} + E_{ij}^{\parallel} \,. \tag{1.53}$$

$E_{ij}^{\perp}$ is divergenceless, while the divergence of E_{ij}^{S} is a transverse (divergenceless) vector, and the divergence of $E_{ij}^{\parallel}$ is a longitudinal (curl-free) vector:

$$\partial^i E_{ij}^{\perp} = 0 \,, \qquad \partial^i \partial^j E_{ij}^{S} = 0 \,, \qquad \epsilon^{lkj}\partial_k \partial^i E_{ij}^{\parallel} = 0 \,. \tag{1.54}$$

This means that the $E_{ij}^{\parallel}$ can be derived from a scalar field E, and E_{ij}^{S} can be derived from a transverse vector F_i:

$$\begin{aligned} E_{ij}^{\parallel} &= (\partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2)E \\ E_{ij}^{S} &= F_{(i,j)} \quad \text{with} \quad F^i_{\ ,i} = 0 \\ E_{ij}^{\perp} &= \frac{1}{2}h_{ij} \quad \text{with} \quad \partial^i h_{ij} = 0 \,, \quad h_i^{\ i} = 0 \\ E_{ij} &= (\partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2)E + F_{(i,j)} + \frac{1}{2}h_{ij} \,. \end{aligned} \tag{1.55}$$

The longitudinal part describes a single degree of freedom, the solenoidal part describes two degrees of freedom; the transverse part h_{ij} cannot be further decomposed. It describes the remaining two degrees of freedom of the symmetric traceless spatial tensor E_{ij}.

The metric perturbation can thus be divided into:

- a scalar part: consisting of ϕ, B, ψ, and E;
- a vector part: consisting of B_i and F_i;
- a tensor part h_{ij}.

Notice that we have imposed one constraint on both B_i and F_i, and $3+1=4$ constraints on h_{ij} leaving each of them 2 independent components. Thus, the 10 degrees of freedom corresponding to the 10 components of the metric perturbation $\delta g_{\mu\nu}$ are divided into:

$$\begin{aligned} 1+1+1+1=4 &\qquad \text{scalar} \\ 2+2=4 &\qquad \text{vector} \\ 2=2 &\qquad \text{tensor} \end{aligned}$$

degrees of freedom.
The Einstein tensor perturbation $\delta G_{\mu\nu}$ and the energy tensor perturbation $\delta T_{\mu\nu}$ can likewise be divided into SVT components.
The important thing about this division is that the scalar, vector and tensor parts at first-order do not couple to each other and evolve so independently. This allows us to treat them separately, i.e. we can study, for example, scalar perturbations as if the vector and tensor perturbations were absent.
Throughout this work we are going to focus on scalar perturbations only, which, from the cosmological perspective, are the most important. They couple to density and pressure perturbations and exhibit gravitational instability (namely overdense regions grow more overdense). In other words, these type of perturbations causes structure formation in the Universe from small initial perturbations.
Vector perturbations instead couple to the rotational velocity perturbations of the cosmic fluid. They tend to decay, and are therefore not important in cosmology.
The two degrees of freedom of the tensor perturbations correspond instead to the two polarizations of gravitational waves. They do have cosmological importance since, if strong enough, they have an observable effect on the anisotropy of the CMB. However, as mentioned above, in the following we are going to focus just on scalar perturbations.

1.2.2 Energy-momentum perturbations

The energy-momentum tensor, as we have seen, can be split into an homogeneous BG plus inhomogeneous perturbation $T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}$, where the homogeneous part $\bar{T}_{\mu\nu}$ is necessarily of the perfect fluid form:†

$$\bar{T}^{\mu}_{\ \nu} = (\bar{\rho} + \bar{P})\bar{u}^{\mu}\bar{u}_{\nu} + \bar{P}\delta^{\mu}_{\ \nu}\,. \tag{1.56}$$

†The 'imperfections' can only show up in the energy-momentum tensor if there is inhomogeneity.

Because of homogeneity $\bar{\rho} = \bar{\rho(t)}$ and $\bar{P} = \bar{P(t)}$. Because of isotropy we cannot have preferred spatial directions, implying so that $u^\mu = (1, \bar{0})$ in the BG. The inhomogeneous perturbation $\delta T_{\mu\nu}(x)$ is:

$$\delta T^\mu_{\ \nu} = (\delta\rho + \delta P)\bar{u}^\mu \bar{u}_\nu + (\bar{\rho} + \bar{P})\delta u^\mu \delta u_\nu + \delta P \delta^\mu_{\ \nu} + \Pi^\mu_{\ \nu} \,, \tag{1.57}$$

where the anisotropic stress tensor, deviating from the perfect fluid form, is subject to the constraints

$$\Pi_{\mu\nu} u^\nu = 0 \,, \qquad \Pi_\mu^{\ \mu} = 0 \,. \tag{1.58}$$

The orthogonality with u^μ implies $\Pi_{00} = \Pi_{0i} = 0$, i.e. only the spatial components $\Pi_{ij} \neq 0$. The trace condition then implies $\Pi_i^{\ i} = 0$; hence the anisotropic stress is a traceless symmetric spatial tensor. Density and pressure perturbations can be decomposed in the following manner:

$$\rho(t, x^i) = \bar{\rho}(t) + \delta\rho(t, x^i) \,, \qquad \text{and} \qquad P(t, x^i) = \bar{P}(t) + \delta P(t, x^i) \,. \tag{1.59}$$

From the constraint $u^\mu u_\mu = -1$ on the 4-velocity u^μ we get:

$$u^\mu = [1 - \phi, a^{-1}\partial^i v] \,, \qquad u_\mu = [-1 - \phi, \partial_i \theta] \,, \tag{1.60}$$

where we have freely defined $\delta u^i = a^{-1}\partial^i v$ with $\partial^i v = a(\partial x^i / \partial t)$ and $\theta = a(v + B)$. We can now write the energy-momentum tensor as:

$$\begin{aligned} T^\mu_{\ \nu} &= \bar{T}^\mu_{\ \nu} + \delta T^\mu_{\ \nu} \\ &= \begin{bmatrix} -\bar{\rho} & 0 \\ 0 & \bar{P}\delta^i_{\ j} \end{bmatrix} + \begin{bmatrix} -\delta\rho & (\bar{\rho} + \bar{P})\partial_i\theta \\ -(\bar{\rho} + \bar{P})a^{-1}\partial^i v & \delta P \delta^i_{\ j} + \Pi^i_{\ j} \end{bmatrix} \,. \end{aligned} \tag{1.61}$$

From the above we notice that the perturbation $\delta T_{\mu\nu}$ can also be divided into perfect plus non-perfect fluid, with $5 + 5$ degrees of freedom. Further, the energy-momentum tensor $T_{\mu\nu}$, just like the metric tensor $g_{\mu\nu}$, can be decomposed into SVT parts, with $4 + 4 + 2$ degrees of freedom.

1.3 The gauge problem

In contrast to the homogeneous and isotropic FLRW Universe, where the preferable coordinate system is fixed by the symmetry properties of the BG, there are no obvious preferable coordinates for analyzing perturbations. The freedom in the coordinate choice, or gauge freedom, leads to the appearance of fictitious perturbation modes. These fictitious modes do not describe any real inhomogeneities, but reflect only the properties of the coordinate system used.

To demonstrate this point let us consider an undisturbed homogeneous isotropic Universe, where, as a reference quantity, we take the energy density $\rho(t, x^i) = \rho(t)$. In

GR any coordinate system is allowed, we can in principle decide to use a 'new' time coordinate $\tilde{t}$, related to the 'old' time t via $\tilde{t} = t + \delta t(t, x^i)$. Then the energy density $\tilde{\rho}(\tilde{t}, x^i) \equiv \rho(t(\tilde{t}, x^i))$ on the hypersurface $\tilde{t} = \text{const}$ depends, in general, on the spatial coordinates x^i,see Fig. (1.3). Assuming that $\delta t \ll t$, we have

$$\rho(t) = \rho(\tilde{t} - \delta t(t, \mathbf{x})) \simeq \rho(\tilde{t}) - \frac{\partial \rho}{\partial t}\delta t \equiv \rho(\tilde{t}) + \widetilde{\delta}\rho(t, \mathbf{x})\,. \tag{1.62}$$

The first term on the right hand side must be interpreted as the background energy density in the new coordinate system, while the second describes a linear perturbation. This perturbation is nonphysical and entirely due to the choice of the new 'disturbed' time. Thus we can 'produce' fictitious perturbations simply by perturbing the coordinates. Moreover, we can 'remove' a real perturbation in the energy density by choosing the hypersurfaces of constant time to be the same as the hypersurfaces of constant energy: in this case $\delta\rho(t, \mathbf{x}) = 0$ in spite of the presence of a real inhomogeneity.

$$\begin{aligned}\rho(t, \mathbf{x}) &= \rho(t) + \delta\rho(t, \mathbf{x}) \\ &= \rho(\tilde{t}) + \widetilde{\delta}\rho(t, \mathbf{x}) + \delta\rho(\tilde{t}, \mathbf{x}) \\ &= \rho(\tilde{t})\end{aligned} \tag{1.63}$$

Choosing the hypersurfaces in such a way, the last two terms in Eq. (1.63) cancel and we are left with an unperturbed energy density $\rho(\tilde{t})$.

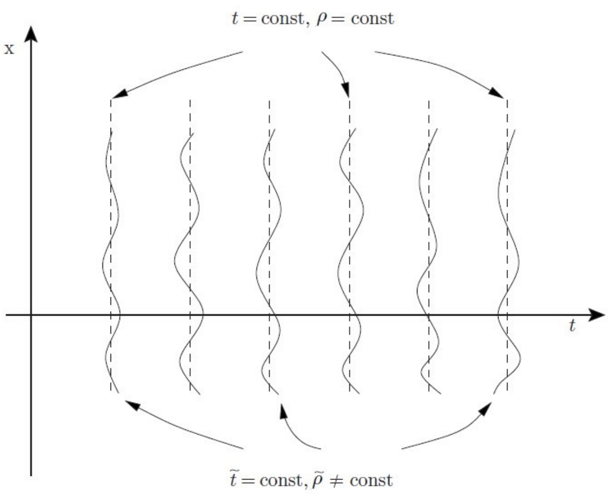

Figure 1.3: A graphical illustration of the gauge issue in GR. Real perturbations can easily disappear while fictitious ones can likewise be created.

The above was just a simple but meaningful example conveying the main idea about the gauge issue affecting GR. Spurious coordinates artifacts, or gauge modes, in the calculations will always arise in any approach to GR that splits quantities into a background

plus a perturbation. Although GR is covariant, i.e. manifestly coordinate choice independent, splitting variables into a background part and a perturbation is not a covariant procedure, and therefore introduces a coordinate or gauge dependence.
To see what the gauge problem is in a more rigorous way, let us consider an idealised Universe model $\bar{\mathcal{S}}$. Each quantity in this model will, as usual, be indicated by an overbar, then the spacetime $\bar{\mathcal{S}}$ will be given by the metric $\bar{g}_{\mu\nu}$ and the manifold $\bar{\mathcal{M}}$, namely $\bar{\mathcal{S}} \equiv \{\bar{g}_{\mu\nu}, \bar{\mathcal{M}}\}$. We perturb this model to obtain a 'realistic' Universe $\mathcal{S} \equiv \{g_{\mu\nu}, \mathcal{M}\}$. The perturbation in each quantity is then the difference between the value the quantity has at a given point in the physical spacetime $\mathcal{S}$ and the value at the corresponding point in the background $\bar{\mathcal{S}}$. Considering all points, the perturbation field is determined. For example, the metric perturbation is given by:

$$\delta g_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu} \,, \tag{1.64}$$

while for the perturbation in the energy momentum tensor we have:

$$\delta T_{\mu\nu} = T_{\mu\nu} - \bar{T}_{\mu\nu} \,. \tag{1.65}$$

Two assumptions are implicit in writing the above relations: one is obvious, while the other is rather obscure. The first one is that the unperturbed metric is a solution of the Einstein equations with the unperturbed energy-momentum tensor as the source term (we shall call this the 0-order solution). The second one is that the perturbations are 'small'. These two assumptions are typical of any perturbation theory: the fact that is

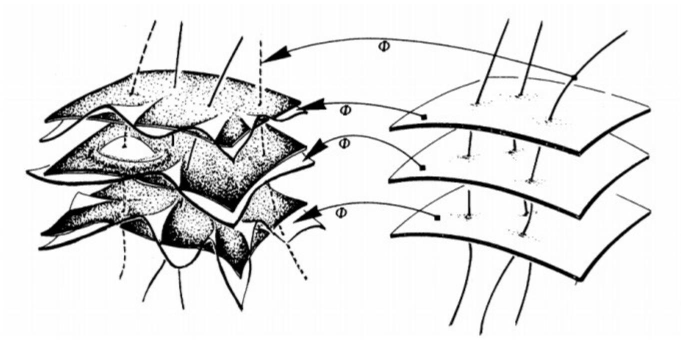

Figure 1.4: Illustration of the mapping Φ from the idealised spacetime $\bar{\mathcal{S}}$ into the real spacetime $\mathcal{S}$.

peculiar to GR is that we must perturb spacetime itself, so that the barred and unbarred fields, as we previously sketched, are actually defined in different manifolds $\bar{\mathcal{M}}$ and $\mathcal{M}$. This implies that the procedure outlined above makes sense only if the correspondence between points in $\bar{\mathcal{M}}$ and $\mathcal{M}$ is fixed, i.e. if we have a point identification map, so that points in $\bar{\mathcal{M}}$ and $\mathcal{M}$ are 'the same', and operations such as subtraction of tensors (e.g. Eqns. (1.64) and (1.65)) are well defined. Otherwise, even if we embed $\bar{\mathcal{M}}$ and $\mathcal{M}$ in

a higher dimensional manifold $\mathcal{N}$, we would be trying to subtract tensors defined at different points, an ill-defined operation.
The choice of a particular map Φ between the background $\bar{\mathcal{S}}$ and the perturbed $\mathcal{S}$ spacetimes is usually referred to as a gauge choice. Such a map is, in general, completely arbitrary, although particular ones may be suitable for some purposes; this arbitrariness is the gauge freedom of perturbation theory. Equivalently, a choice of coordinates in $\mathcal{S}$ determines a map from $\bar{\mathcal{S}}$ into $\mathcal{S}$. Thus the gauge freedom is represented also as a freedom of coordinate choice in $\mathcal{S}$.
Hence there are two ways to outline this freedom, usually referred to as the active and passive approaches. In the next section we are going to discuss both of them from a differential geometry perspective trying to introduce only the essential mathematical tools that we need; we are going to follow [18, 19].

1.3.1 Geometrical aspects of gauge transformations

Consider a one-parameter family of 4-manifolds $\mathcal{M}_\epsilon$ embedded in a 5-manifold $\mathcal{N}$. Each manifold in the family represents a perturbed spacetime with the base or unperturbed spacetime manifold represented by $\mathcal{M}_0$.
We define a point identification map $P_\epsilon : \mathcal{M}_0 \to \mathcal{M}_\epsilon$ which identifies points in the unperturbed manifolds with points in the perturbed manifold. This correspondence specifies a vector field $\mathbf{X}$ upon $\mathcal{N}$ which is transverse to $\mathcal{M}_\epsilon$ at all points. The points which lie on the same integral curve γ of $\mathbf{X}$ are to be regarded as the same point, as in Fig. (1.5).

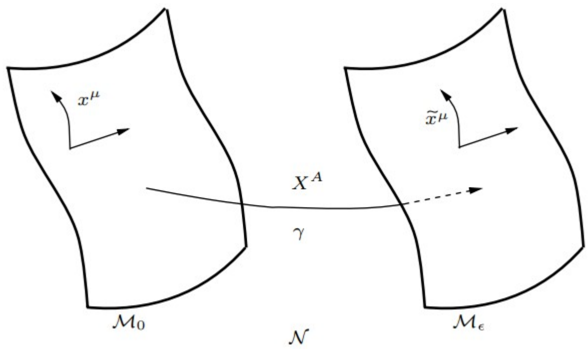

Figure 1.5: The vector field X^A generates a point identification map between the manifolds $\mathcal{M}_0$ and $\mathcal{M}_\epsilon$. This in turn yields a diffeomorphism ϕ_ϵ between coordinate neighbourhoods on the manifolds.

In terms of coordinates, take x^μ on $\mathcal{M}_0$ and extend them to $\mathcal{N}$ by requiring that $x^\mu =$ const along each of the curves γ. This, in turn, induces coordinates $x^A = (x^\mu, \epsilon)$ with $A = 0, 1, ..., 4$ and $\mu, \nu = 0, 1, ..., 3$ on $\mathcal{N}$. We parametrise the curves γ by ϵ implying

$dx^A/d\epsilon = X^A$; we further choose the scaling of ϵ such that:

$$\phi_\epsilon : \mathcal{M}_0 \to \mathcal{M}_\epsilon \,. \tag{1.66}$$

In this way the vector field $\mathbf{X}$ generates a one-to-one, invertible, differentiable mapping between $\mathcal{M}_0$ and $\mathcal{M}_\epsilon$, i.e. a one-parameter group of diffeomorphisms and it follows that $\phi_{\epsilon+\lambda} = \phi_\epsilon \circ \phi_\lambda$. In particular the inverse map from $\mathcal{M}_\epsilon$ to $\mathcal{M}_0$ will be denoted by $\phi_\epsilon^{-1} = \phi_{-\epsilon}$, the identity map by $\phi_{\epsilon=0}$.
Given a geometric quantity $\mathbf{T}$ defined on $\mathcal{N}$ the simplest way to produce a perturbation expansion of $\mathbf{T}$ is to expand it as a Taylor series along γ. This yields a covariant power series for $\mathbf{T}$ along the curve, see [19] for a proof. To first order the series has the form

$$\phi_\epsilon^*\mathbf{T}|_0 = \mathbf{T}_0 + \epsilon(\pounds_X \mathbf{T})\big|_0 + O(\epsilon^2)\,, \tag{1.67}$$

where ϕ^* stands for the pullback and $\phi_\epsilon^*\mathbf{T}$ for the tensor $\mathbf{T}$ evaluated at the point where $\epsilon = 0$. Lie derivatives are used instead of partial derivatives so that the series is covariant. At higher orders the Taylor expansion is given as in [19]:

$$\phi_\epsilon^*\mathbf{T}|_0 = \mathbf{T}_0 + \sum_{k=1}^{\infty} \frac{\epsilon^k}{k!}\big(\pounds_X^k \mathbf{T}\big)\Big|_0 = (\mathrm{e}^{(\epsilon\pounds_X)}\mathbf{T})\big|_0\,. \tag{1.68}$$

The expansion automatically provides the covariant perturbation expansion we want. Each term in the series is proportional to a power of ϵ. The first term $\mathbf{T}_0$ is proportional to ϵ^0, the background value, $\epsilon(\pounds_X\mathbf{T})|_0$ is proportional to ϵ^1, the linear value, the n-th order term to $\frac{\epsilon^n}{n!}\big(\pounds_X^n\mathbf{T}\big)\big|_0$. Once again $\phi_*\mathbf{T}_\epsilon|_0$ is the perturbed value of $\mathbf{T}$ pulled back to $\mathcal{M}_0$ and so the perturbed value of $\mathbf{T}$ is given by

$$\sum_{k=1}^{\infty} \frac{\epsilon^k}{k!}\big(\pounds_X^k \mathbf{T}\big)\Big|_0 = \Delta\mathbf{T}_\epsilon\big|_0 := \phi_\epsilon^*\mathbf{T}|_0 - \mathbf{T}_0\,, \tag{1.69}$$

where we note that we could have not done the subtraction if we had not pulled $\mathbf{T}$ back to $\mathcal{M}_0$. In a commonly used notation, including ϵ in $\mathbf{T}$ we write:

$$\begin{aligned}\mathbf{T} &= \mathbf{T}_0 + \sum_{k=1}^{\infty}\delta^k\mathbf{T}\\ &= \mathbf{T}_0 + \delta^1\mathbf{T} + \delta^2\mathbf{T} + \delta^3\mathbf{T} + \ldots\,,\end{aligned} \tag{1.70}$$

where $\delta^n\mathbf{T} = \frac{\epsilon^n}{n!}\big(\pounds_X^n\mathbf{T}\big)\big|_0$. Notice that $\Delta\mathbf{T}_\epsilon\big|_0$ and $\delta^k\mathbf{T}$ are defined on $\mathcal{M}_0$; this formalizes the statement one commonly finds in the literature that 'perturbations are fields living in the background'.

1.3.2 Active and passive gauge transformations

We define a gauge choice as the choice of correspondence between points on $\mathcal{M}_0$ and points on $\mathcal{M}_\epsilon$ or, equivalently, the choice of a vector field $\mathbf{X}$, generator of the gauge. Let us now turn to defining gauge dependence in a clearer way. Consider a point p in $\mathcal{M}_0$ and the generators $\mathbf{X}$ and $\mathbf{Y}$ corresponding to two different gauge choices, as in Fig. (1.6); the choice $\mathbf{X}$ will identify the point p on $\mathcal{M}_0$ with a point q on $\mathcal{M}_\epsilon$ and will assign to q the same x^μ coordinates as at point p.

On the other hand, the gauge choice $\mathbf{Y}$ will identify p with a different point u on $\mathcal{M}_\epsilon$ assigning in its turn the coordinates of p to u. Clearly, the choice of gauge induces a coordinate change (a gauge transformation) on $\mathcal{M}_\epsilon$. This is the passive interpretation. In the active one, we choose a point u on $\mathcal{M}_\epsilon$ and find the point p on $\mathcal{M}_0$ which maps to u under the gauge choice $\mathbf{X}$ and the point q, also on $\mathcal{M}_0$, which maps to u under the gauge choice $\mathbf{Y}$, see again Fig. (1.6). The gauge transformation this time is defined on $\mathcal{M}_0$ and takes the coordinates of q to those of p in one of the two choices of gauge.

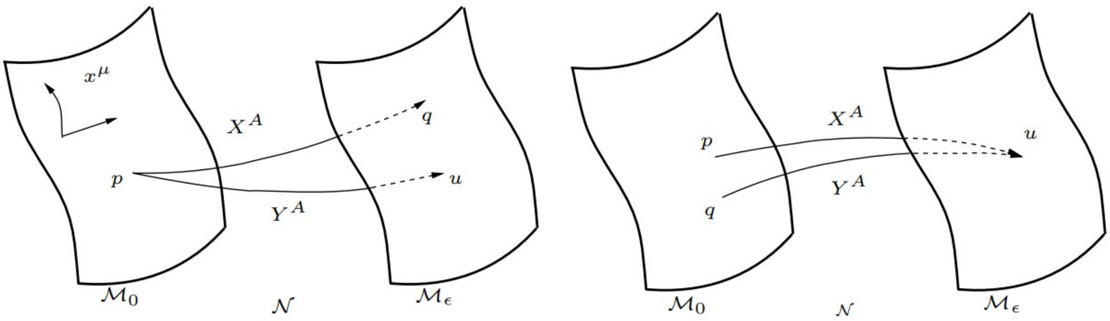

Figure 1.6: The **left panel** represents the *passive view*, while the **right panel** the *active view*. The vector fields generate the gauge choice. A change in gauge from X^A to Y^A produces a gauge transformation.

To sum up, in the active approach the transformation of the perturbed quantities is evaluated at the same coordinate point, whereas in the passive approach the transformation is taken at the same physical point.

The gauge dependence in perturbation theory stems from the fact that we separate quantities into a background and a perturbed part, a operation not covariant in general, which introduces additional, unphysical degrees of freedom. However, as we shall see in Sec. (1.3.3), by choosing and combining suitable matter and metric variables the gauge dependencies can be made to cancel out (the quantities so constructed will not change under a gauge transformation).†

†An alternative way, as we will see in Chapter 2, is to consider CGI variables.

Active point of view

Let's now focus on the active interpretation of the gauge transformation. Corresponding to the gauge choice $\mathbf{X}$, we have a diffeomorphism $\phi_\epsilon : \mathcal{M}_0 \to \mathcal{M}_\epsilon$ and corresponding to the vector field $\mathbf{Y}$, we have a diffeomorphism $\psi_\epsilon : \mathcal{M}_0 \to \mathcal{M}_\epsilon$. The fields $\mathbf{X}$ and $\mathbf{Y}$ can both be used to pull a generic tensor field $\mathbf{T}$ and therefore to construct two other tensor fields $\phi_\epsilon^* \mathbf{T}$ and $\phi_\epsilon^* \mathbf{T}$, for any value of ϵ. In particular, on $\mathcal{M}_0$ we now have three tensor fields: $\mathbf{T}_0$ and

$$\mathbf{T}_\epsilon^X := \phi_\epsilon^* \mathbf{T}|_0 , \qquad \mathbf{T}_\epsilon^X := \phi_\epsilon^* \mathbf{T}|_0 . \tag{1.71}$$

Since $\mathbf{X}$ and $\mathbf{Y}$ represent gauge choices for mapping a perturbed manifold $\mathcal{M}_\epsilon$ into the unperturbed one $\mathcal{M}_0$, $\mathbf{T}_\epsilon^X$ and $\mathbf{T}_\epsilon^Y$ are the representations in $\mathcal{M}_0$ of the perturbed tensor according to the two gauges. We can write, using Eq. (1.69):

$$\mathbf{T}_\epsilon^X = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \pounds_X \mathbf{T}\Big|_0 = \mathbf{T}_0 + \delta^1 \mathbf{T}^X + \delta^2 \mathbf{T}^X , \tag{1.72}$$

$$\mathbf{T}_\epsilon^Y = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \pounds_Y \mathbf{T}\Big|_0 = \mathbf{T}_0 + \delta^1 \mathbf{T}^Y + \delta^2 \mathbf{T}^Y , \tag{1.73}$$

where $\delta^0 \mathbf{T}^X = \delta^0 \mathbf{T}^Y = \mathbf{T}_0$ by definition. For each ϵ the two vector fields $\mathbf{X}$ and $\mathbf{Y}$ induce a composite diffeomorphism (gauge transformation) Φ_ϵ on $\mathcal{M}_0$ represented in Fig. (1.7) and given by:

$$\Phi_\epsilon : \mathcal{M}_0 \to \mathcal{M}_0 , \tag{1.74}$$

where Φ_ϵ is made up of two parts: a map ψ_ϵ from $\mathcal{M}_0$ to $\mathcal{M}_\epsilon$ and a map $\phi_{-\epsilon}$ from $\mathcal{M}_\epsilon$ to $\mathcal{M}_0$:

$$\Phi_\epsilon := \phi_{-\epsilon} \circ \psi_\epsilon . \tag{1.75}$$

It is easy to notice that the tensor fields $\mathbf{T}_\epsilon^X$ and $\mathbf{T}_\epsilon^Y$, defined by the gauges ϕ_ϵ and ψ_ϵ, are connected by the linear map Φ_ϵ^*:

$$\mathbf{T}_\epsilon^Y = \psi_\epsilon^* \mathbf{T}\Big|_0 = (\psi_\epsilon^* \circ \phi_{-\epsilon}^* \circ \phi_\epsilon^* \mathbf{T})\Big|_0 = (\phi_{-\epsilon} \circ \psi_\epsilon)^* (\phi_\epsilon^* \mathbf{T})\Big|_0 = \Phi_\epsilon^* (\phi_\epsilon^* \mathbf{T})\Big|_0 = \Phi_\epsilon^* \mathbf{T}_\epsilon^X . \tag{1.76}$$

We can now expand the pull-back $\Phi_\epsilon^* \mathbf{T}_\epsilon^X$ of a tensor field $\mathbf{T}_\epsilon^X$ applying the generating formula for a generic n-th order gauge transformation, see *Lemma 2* in [19]:

$$\mathbf{T}_\epsilon^Y = \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} \cdots \frac{\epsilon^{l_1 + 2l_2 + \cdots + kl_k + \cdots}}{2^{l_2} \cdots (k!)^{l_k} \cdots l_1! l_2! \cdots l_k! \cdots} \pounds_{\xi_{(1)}}^{l_1} \pounds_{\xi_{(k)}}^{l_2} \cdots \pounds_{\xi_{(k)}}^{l_1} \cdots \mathbf{T}_\epsilon^X ; \tag{1.77}$$

doing so we can write $\mathbf{T}_\epsilon^Y$ as:

$$\mathbf{T}_\epsilon^Y = \mathbf{T}_\epsilon^X + \epsilon \pounds_{\xi_{(1)}} \mathbf{T}_\epsilon^X + \frac{\epsilon^2}{2} (\pounds_{\xi_{(1)}}^2 + \pounds_{\xi_{(2)}}) \mathbf{T}_\epsilon^X + \cdots , \tag{1.78}$$

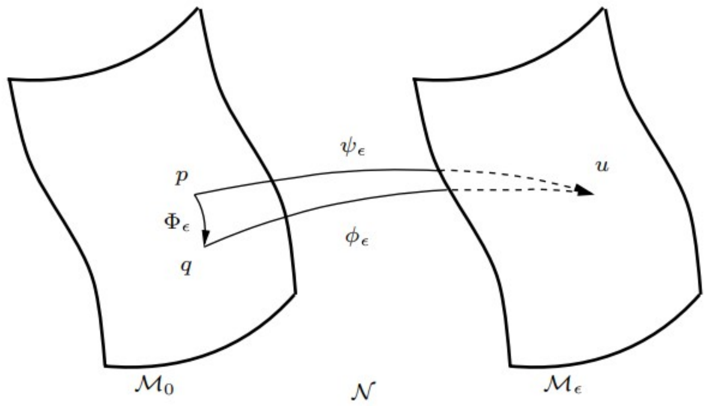

Figure 1.7: The map Φ_ϵ which maps the point p to the point q is formed by first mapping p to u using the map ψ_ϵ and then mapping u to q using the map $\phi_{-\epsilon}$, i.e. $\Phi_\epsilon = \phi_{-\epsilon} \circ \psi_\epsilon$.

where $\xi_{(1)}$ and $\xi_{(2)}$ are now the first two generators of Φ_ϵ or of the gauge transformation, if one prefers.
Now, that we have enough geometrical tools, let us finally discuss the concept of gauge invariance. If $\mathbf{T}_\epsilon^X = \mathbf{T}_\epsilon^Y$, for any pair of gauges $\mathbf{X}$ and $\mathbf{Y}$, we say that $\mathbf{T}$ is totally gauge invariant. This is a very strong condition, because then (1.72) and (1.73) imply that $\delta^k \mathbf{T}^X = \delta^k \mathbf{T}^Y$, for all gauges $\mathbf{X}$ and $\mathbf{Y}$ and for any k.
In any practical case, however, one is interested in perturbations to a fixed order n; it is thus convenient to weaken the definition above, saying that $\mathbf{T}$ is gauge invariant to order n iff $\delta^k \mathbf{T}^X = \delta^k \mathbf{T}^Y$ for any two gauges $\mathbf{X}$ and $\mathbf{Y}$, and $\forall k \leq n$. We have then the following result, directly quoting M. Bruni [19]:

"**Proposition.** A tensor field $\mathbf{T}$ is gauge invariant to order $n \geq 1$ iff $\pounds_\xi \delta^k \mathbf{T} = 0$, for any vector field ξ on $\mathcal{M}$ and $\forall k < n$.

Proof. Let us first show that the statement is true for $n = 1$. In fact, if $\delta^1 \mathbf{T}^X = \delta^1 \mathbf{T}^Y$, we have $\pounds_{X-Y} \mathbf{T}|_0 = 0$. But since $\mathbf{X}$ and $\mathbf{Y}$ define arbitrary gauges, it follows that $\mathbf{X} - \mathbf{Y}$ is an arbitrary vector field ξ with $\xi^{A=4} = 0$, i.e. tangent to $\mathcal{M}_\epsilon \ \forall \epsilon$. Let us now suppose that the statement is true for some n. Then, if one also has $\delta^{n+1} \mathbf{T}^X|_0 = \delta^{n+1} \mathbf{T}^Y|_0$, it follows that $\pounds_{X-Y} \delta^n \mathbf{T}^X = 0$ and we establish the result by induction over n."

As a consequence of the above, we have the so-called Stewart & Walker Lemma, quoting again from their original paper [6]:

"**Lemma.** $\mathbf{T}$ is gauge invariant to order n iff $\mathbf{T}_0$ and all its perturbations of order lower than n are, in any gauge, either:

(i) vanishing;

(ii) or constant scalars;

(iii) or a combination of Kronecker deltas with constant coefficients."

Going back to Eq. (1.78) we can now relate the perturbations in two different gauges. To the two lowest orders, this is easy to do explicitly:

$$\delta\mathbf{T}^Y - \delta\mathbf{T}^X = \pounds_{\xi_{(1)}}\mathbf{T}_0\,, \tag{1.79}$$
$$\delta^2\mathbf{T}^Y - \delta^2\mathbf{T}^X = (\pounds_{\xi_{(2)}} + \pounds^2_{\xi_{(1)}})\mathbf{T}_0 + 2\pounds_{\xi_{(1)}}\delta\mathbf{T}^X\,. \tag{1.80}$$

Further, substituting (1.72) and (1.73) into (1.78), using the fact that $\pounds_{\xi_{(k)}}\epsilon = 0$, and identifying terms of the same order in ϵ, we can obtain the expressions for $\xi_{(k)}$:

$$\xi_1 = Y - X\,, \tag{1.81}$$
$$\xi_2 = [X, Y]\,. \tag{1.82}$$

The same principle applies for higher order terms.

The map (1.78) generated by Φ_ϵ, see Fig. (1.5), enables us to relate two coordinate systems: $(U, x) \xrightarrow{\Phi_\epsilon} (U', \tilde{x})$ under an infinitesimal transformation generated by $\epsilon\,\xi^\mu$.
In the active view, this transformation takes the point p with coordinates $x^\mu(p)$ to the point $q = \Phi_\epsilon(p)$ with coordinates $x^\mu(q)$.
Note that in the active view it is the point that changes. Applying the map as in (1.78) we get:

$$x^\mu(q) = x^\mu(p) + \epsilon\xi^\mu_{(1)}(x(p)) + \frac{1}{2}\epsilon^2\Big(\xi^\mu_{(1)\,,\nu}(x(p))\xi^\nu_{(1)}(x(p)) + \xi^\mu_{(1)}(x(p))\Big) + O(\epsilon^3)\,. \tag{1.83}$$

Notice that, as stated above, the left-hand side and the right-hand side of (1.83) are evaluated at different points.

Passive point of view

In the passive approach we specify the relation between two coordinate systems directly, hence all quantities in the passive approach have to be evaluated at the same physical point.† In order to make contact with the active approach we take directly Eqn. (1.83) as our starting point. To take the passive approach further, we therefore need to rewrite the left-hand side and the right-hand side of (1.83), since they are evaluated at two

†For a more mathematically rigorous approach, see once again [19]. We have instead followed for a lighter approach [18].

different coordinate points. Referring to Fig. (1.6), we choose p and q to be points such that the coordinates of q in the new coordinates are the same as the coordinates of p in the old coordinates: i.e. $\widetilde{x^\mu}(q) = x^\mu(p)$, then from Eqn. (1.83):

$$\widetilde{x^\mu}(q) := x^\mu(p) = x^\mu(q) - \epsilon\xi^\mu_{(1)}(x(p)) - \frac{1}{2}\epsilon^2\Big(\xi^\mu_{(1),\nu}(x(p))\xi^\nu_{(1)}(x(p)) + \xi^\mu_{(1)}(x(p))\Big) + O(\epsilon^3)\,. \tag{1.84}$$

Using the first two terms of Eqn. (1.83):

$$x^\mu(q) = x^\mu(p) + \epsilon\xi^\mu_{(1)}(x(p))\,, \tag{1.85}$$

to get a Taylor expansion for $\xi^\mu_{(1)}(x(p))$, namely:

$$\begin{aligned}\xi^\mu_{(1)}(x(p)) &= \xi^\mu_1\big(x^\mu(q) - \epsilon\xi^\mu_{(1)}(x(p))\big)\\ &= \xi^\mu_{(1)}(x(q)) - \epsilon\xi^\mu_{(1),\nu}(x(q))\xi^\nu_{(1)}(x(q))\,,\end{aligned} \tag{1.86}$$

where in the very last term we have replaced $\xi^\nu_{(1)}(x(p))$ by $\xi^\nu_{(1)}(x(q))$, the correction being third-order small. Substituting Eq. (1.86) into (1.84) finally gives the desired result, namely a relation between the 'old' x^μ and the 'new' $\widetilde{x^\mu}$ coordinate systems:

$$\widetilde{x^\mu}(q) = x^\mu(q) - \epsilon\xi^\mu_{(1)}(x(q)) + \frac{1}{2}\epsilon^2\Big(\xi^\mu_{(1),\nu}(x(q))\xi^\nu_{(1)}(x(q)) - \xi^\mu_{(2)}(x(q))\Big) + O(\epsilon^3)\,, \tag{1.87}$$

all evaluated at the same point q. Eqns. (1.83) and (1.87) express the relationship, in the language of coordinates, between the active and the passive views. While (1.83) provides us with the coordinates, in the same chart (U, x), of the different points p and $q = \Phi_\epsilon(p)$, Eqn. (1.87) gives the transformation law between the coordinates of the same point q in the two different charts (U, x) and $(\Phi_\epsilon(U), \tilde{x})$.

1.3.3 Application to cosmology

As an application of the above, we now derive the gauge transformation laws, at first-order, for relevant cosmological quantities; see [17] for a second-order treatment. In the following we are going to adopt the active approach as it is simply more direct w.r.t. the passive one. At first order we have:

$$x^\mu(q) = x^\mu(p) + \epsilon\xi^\mu_1(p)\,, \tag{1.88}$$

Before studying the transformation behaviour of the perturbations at first order, we split the generating vector $\xi^\mu_{(1)}$ into a scalar temporal part α and a spatial scalar and vector part, β and γ^i, according to:

$$\xi^\mu_{(1)} = (\alpha, \beta\,,^i + \gamma^i)\,, \tag{1.89}$$

where the vector part is divergence free $\partial_k \gamma^k = 0$. We remind the reader that Lie derivatives on different tensorial quantites are given by:

$$\pounds_\xi f \equiv f_{;\lambda}\xi^\lambda \,, \tag{1.90}$$
$$\pounds_\xi V_\mu \equiv V_{\mu;\lambda}\xi^\lambda + \xi_{\lambda;\mu}V^\lambda \,, \tag{1.91}$$
$$\pounds_\xi g_{\mu\nu} \equiv g_{\mu\nu;\lambda}\xi^\lambda + g_{\mu\lambda}\xi^\lambda{}_{;\nu} + g_{\lambda\nu}\xi^\lambda{}_{;\mu} \,. \tag{1.92}$$

It follows immediately from Eqn. (1.79) and (1.90) that, to first-order, a scalar quantity such as the energy density transform as:†

$$\widetilde{\delta\rho} = \delta\rho + \bar{\rho}'\alpha \,. \tag{1.93}$$

We now adopt an alternative definition, w.r.t. Eqn. (1.60), of the 4-velocity u^μ:

$$u^\mu = a^{-1}[1-\phi, v^i] \,, \qquad u_\mu = a[-1-\phi, v_i + B_i] \,, \tag{1.94}$$

and write using the Lie derivative in Eqn. (1.91):

$$\widetilde{\delta u_\mu} = \delta u_\mu + u^{(0)}_{\mu,0}\,\xi^0 + u^{(0)}_\lambda \xi^\lambda{}_{,\mu} \,. \tag{1.95}$$

For the above defined 4-velocity we end up with:

$$\widetilde{v}_i + \widetilde{B}_i = v_i + B_i - \alpha_{,i} \,. \tag{1.96}$$

Now, anticipating the transformation law of the metric perturbation B_i, we get the transformations for the scalar and vector parts, respectively:

$$\widetilde{v} = v - \beta' \,, \tag{1.97}$$
$$\widetilde{v^i} = v^i - \gamma^{i'} \,. \tag{1.98}$$

Considering metric perturbations instead, from Eqns. (1.79) and (1.92) we find:

$$\widetilde{\delta g_{\mu\nu}} = \delta g_{\mu\nu} + g^{(0)}_{\mu\nu,\lambda}\xi^\lambda_{(1)} + g^{(0)}_{\mu\lambda}\xi^\lambda_{(1),\,\nu} + g^{(0)}_{\lambda\nu}\xi^\lambda_{(1),\,\mu} \,. \tag{1.99}$$

The metric we are adopting here is:

$$ds^2 = a^2\big[-(1+2\phi)d\eta^2 + 2(B_{,i} - S_i)d\eta\, dx^i + [(1-2\psi)\delta_{ij} + 2E_{ij}]dx^i dx^j\big] \,. \tag{1.100}$$

Notice that E_{ij} is this time given by:‡

$$E_{ij} = (E_{,ij} + F_{(i,j)} + \frac{1}{2}h_{ij}) \,, \tag{1.101}$$

†Remember that tensorial quantities in the FLRW background have no spatial dependence. Furthermore, the FLRW BG we are considering is flat ($K = 0$); this allows us to replace covariant with partial derivatives.

‡Where E_{ij} is not traceless anymore. In this manner we have a simpler metric, i.e. we don't have to carry around the laplacian term $-\frac{1}{3}\delta_{ij}\nabla^2 E$ as in Eqn. (1.44).

additionally we group the tensor perturbations in:

$$C_{ij} = -\psi\delta_{ij} + E_{,ij} + F_{(i,j)} + \frac{1}{2}h_{ij}\,. \tag{1.102}$$

Using the previously discussed conditions on metric perturbations:

$$S^k{}_{,k} = 0\,, \qquad F^k{}_{,k} = 0\,, \qquad h^{ik}{}_{,k} = 0\,, \qquad h^k{}_k = 0\,, \tag{1.103}$$

we can write the gauge transformations for the metric perturbations using Eqn. (1.99). The transformations for the perturbations in $\delta g^{(1)}_{00}$ and $\delta g^{(1)}_{0i}$ are quite straightforward, leading to Eqn. (1.110) and

$$\widetilde{B_i} = B_i + \xi_i' - \alpha_{,i}\,, \tag{1.104}$$

whose scalar part is obtained performing the divergence of the above, see Eqn. (1.112). Focusing on $\delta g^{(1)}_{ij}$ and starting from Eqn. (1.99) we have:

$$2\widetilde{C}_{ij} = 2C_{ij} + 2\mathcal{H}\alpha\delta_{ij} + \xi_{i,j} + \xi_{j,i}\,. \tag{1.105}$$

Taking the trace of the above and substituting it into (1.3.3) we get:

$$-3\widetilde{\psi} + \nabla^2\widetilde{E} = -3\psi + \nabla^2 E + 3\mathcal{H}\alpha + \nabla^2\beta\,. \tag{1.106}$$

Taking the spatial trace of (1.105) we get a second equation relating the scalar perturbation ψ and E:

$$-3\nabla^2\widetilde{\psi} + \nabla^2\nabla^2\widetilde{E} = -3\nabla^2\psi + \nabla^2\nabla^2 E + 3\mathcal{H}\nabla^2\alpha + \nabla^2\nabla^2\beta\,, \tag{1.107}$$

from which we can obtain the transformation laws for $\widetilde{\psi}$ and $\widetilde{E}$. Taking now the divergence of Eqn. (1.105) we get:

$$2\widetilde{C}_{ij,}{}^{j} = 2C_{ij,}{}^{j} + 2\mathcal{H}\alpha_{,i} + \nabla^2\xi_i + \nabla^2\beta_{,i}\,. \tag{1.108}$$

Substituting the obtained results for $\widetilde{\psi}$ and $\widetilde{E}$ we then arrive at:

$$\nabla^2\widetilde{F}_i = \nabla^2 F_i + \nabla^2\gamma_i\,. \tag{1.109}$$

We can sum up the transformations of the first-order metric perturbations we have from the above, first for scalars as:

$$\widetilde{\phi} = \phi + \mathcal{H}\alpha + \alpha'\,, \tag{1.110}$$
$$\widetilde{\psi} = \psi - \mathcal{H}\alpha\,, \tag{1.111}$$
$$\widetilde{B} = B - \alpha + \beta'\,, \tag{1.112}$$
$$\widetilde{E} = E + \beta\,, \tag{1.113}$$

where $\mathcal{H} = \frac{a'}{a}$, for the vector perturbations as:

$$\widetilde{S}^i = S^i - \gamma^{i'}\,, \tag{1.114}$$

$$\widetilde{F}^i = F^i + \gamma^i\,. \tag{1.115}$$

The first-order tensor perturbation is found to be gauge invariant:

$$\widetilde{h}_{ij} = h_{ij}\,, \tag{1.116}$$

by substituting the above transformations laws into Eqn. (1.105). This can also be understood from the Stewart–Walker lemma [6]: at first order, quantities that are identically zero in the background are manifestly gauge invariant, and there is no tensor part in the background.

1.4 Gauge invariant variables and gauge choices

It is important to understand that physical observables are not dependent on the choice of coordinates, though physical observables may be different for different observers. This is what led Bardeen in [4] to find a way to specify quantities unambiguously, such that they have a GI definition. Notice that this is not the same as gauge independence.
In fact, a quantity like the tensor metric perturbation, h_{ij}, is truly gauge independent at first order in that the tensor part of the metric perturbation is the same in all gauges. The scalar curvature perturbation, ψ, on the other hand is intrinsically gauge dependent. It is different under different time slicings, see Eqn. (1.111). As we have previously anticipated, one can construct a GI combination, which may be referred to as the GI curvature perturbation, but they only correspond with the curvature perturbation ψ in one particular gauge.
As a result one can find in the literature many different GI curvature perturbations corresponding to the many different choices of gauge, such as Ψ, ζ and $\mathcal{R}$, corresponding to the curvature perturbations in the longitudinal, uniform density and the comoving gauge respectively, to name just three. Let's now give a more rigorous description of what we have just sketched above.

1.4.1 Specific gauges

As stated at the beginning of Section 1.3, the conformal time η gives the slicing of the perturbed spacetime into $\eta =$ const hypersurfaces; while the spatial coordinates x^i give the threading of the perturbed spacetime into $x^i =$ const threads (i.e. timelike curves). Slicing and threading are orthogonal to each other iff the shift vector vanishes $B_i = 0$.

In the gauge transformation,

$$\tilde{\eta} = \eta - \xi^0 \tag{1.117}$$

$$\tilde{x}^i = x^i - \xi^i\,, \tag{1.118}$$

$\xi^0 = \alpha$ changes the slicing and $\xi^i = (\beta,^i + \gamma^i)$ changes the threading. From the 4-scalar transformation law we see that perturbations in 4-scalars, e.g. $\delta\rho$, depend only on the slicing. The dependence of metric quantities can be immediately noticed from Eqns. (1.110)-(1.116).
s Focusing now only on scalar perturbations, we notice that the two scalar gauge functions in ξ^μ allow two of the metric scalar perturbations to be eliminated. This is the meaning of gauge fixing (defining a gauge choice); we use the freedom we have in α and β to eliminate two d.o.f. in the perturbed metric. Let's some examples starting from what is called variously orthogonal zero-shear, longitudinal and conformal Newtonian gauge.

Longitudinal gauge

If we choose to work on spatial hypersurfaces with vanishing shear ($\sigma \equiv E' - B$), we find from Eqs. (1.112) and (1.113) that the shear scalar transforms as:

$$\widetilde{\sigma} = \sigma + \alpha\,. \tag{1.119}$$

This implies that to obtain perturbations in the longitudinal gauge starting from arbitrary coordinates we should perform the transformation:

$$\alpha_\ell = -\sigma = B - E'\,. \tag{1.120}$$

In addition, the longitudinal gauge is completely determined by the spatial gauge choice $\widetilde{E}_\ell = 0$, which from $\widetilde{\sigma}_\ell = \widetilde{E}'_\ell - \widetilde{B}_\ell = 0$, implies $\widetilde{B}_\ell = 0$. This, in turn, from Eqn. (1.113), leads to:

$$\beta_\ell = -E\,. \tag{1.121}$$

The remaining scalar metric perturbations, ϕ and ψ, are given from Eqns. (1.110) and (1.111), they are:

$$\widetilde{\phi}_\ell = \phi + \mathcal{H}(B - E') + (B - E')'\,, \tag{1.122}$$

$$\widetilde{\psi}_\ell = \psi + \mathcal{H}(B - E')\,. \tag{1.123}$$

the expressions for ϕ_ℓ and ψ_ℓ are identical to the so called Bardeen potentials Φ and Ψ. These variables are defined by hand in such a way to be invariant under the gauge transformations in Eqns. (1.110 - 1.113):

$$\Phi \equiv \phi + \mathcal{H}(B - E') + (B - E')'\,, \tag{1.124}$$

$$\Psi \equiv \psi - \mathcal{H}(B - E')\,. \tag{1.125}$$

In this particular gauge we are able to assign to (1.124) and (1.125) a precise physical meaning via their equality to $\widetilde{\phi}_\ell$ and $\widetilde{\psi}_\ell$. Moreover, in many cases of physical interest such as in the absence of anisotropic stress ($\Pi_{ij} = 0$) one finds $\Phi = \Psi$; i.e. there is only one variable required to describe all scalar metric perturbations:

$$ds^2 = a^2\big[-(1+2\Phi)d\eta^2 + (1-2\Phi)\delta_{ij}dx^i dx^j\big]\,. \tag{1.126}$$

We can, in general, obtain GI expressions for the energy density perturbation $\delta\rho$, exploiting the transformation law in Eqn. (1.93):

$$\Delta = \delta + \frac{\bar{\rho}'}{\bar{\rho}}(v+B)\,. \tag{1.127}$$

Other gauge-invariant density perturbations are:

$$\delta\rho_\sigma = \delta\rho + \bar{\rho}'(B-E')\,, \qquad \delta\rho_\psi = \delta\rho + \frac{\bar{\rho}'}{\mathcal{H}}\psi\,. \tag{1.128}$$

Notice that in the longitudinal gauge $\delta\rho_\sigma \equiv \widetilde{\delta\rho}_\ell$. The intrinsic spatial curvature on $\eta =$ const hypersurfaces is:

$${}^{(3)}R = {}^{(3)}R_0 + {}^{(3)}\delta R = \frac{6K}{a^2} + \frac{12K}{a^2}\psi + \frac{4}{a^2}\nabla^2\psi\,. \tag{1.129}$$

Thus, the metric perturbation ψ determines the curvature of perturbed $\eta =$ const hypersurfaces.[†]. However, there are also other useful GI curvature perturbations, especially those which are conserved under certain broad conditions. Two such quantities are:

$$\mathcal{R} = \psi - \mathcal{H}(v+B)\,, \tag{1.130}$$

$$\zeta = -\psi - \mathcal{H}\frac{\delta\rho}{\bar{\rho}'}\,, \tag{1.131}$$

The relation between the two GI quantities can be obtained using Eqn. (1.127) and is given by:

$$\zeta = -\mathcal{R} - \frac{\mathcal{H}\bar{\rho}}{\bar{\rho}'}\Delta\,. \tag{1.132}$$

The below generalized Poisson equation (1.154) shows that $\mathcal{R} = -\zeta$ on super-Hubble scales. In particular, using the perturbed energy conservation equation, $u_\nu\nabla_\mu\delta T^{\mu\nu} = 0$, one can see that $\zeta' = \mathcal{R}' = 0$ on super-Hubble scales for adiabatic modes.
Cosmological inhomogeneity is characterized by the intrinsic curvature of spatial hypersurfaces, ζ or $\mathcal{R}$. Both ζ and $\mathcal{R}$ have the aforementioned attractive feature that they

[†] ${}^{(3)}R$ is gauge invariant for a flat FLRW background, since in that case this quantity vanishes in the background, see Chapter 2.

remain constant outside the horizon, i.e. when $k < (aH)$ (during inflation). An important fact is that their amplitude is not affected by the unknown physical properties of the Universe shortly after inflation (we remind the reader that we know next to nothing about the details of reheating; it is the constancy of ζ and $\mathcal{R}$ outside the horizon that allows us to nevertheless predict cosmological observables). After inflation, the comoving Hubble horizon grows, so eventually all fluctuations will re-enter the horizon. After horizon re-entry, i.e. $k > (aH)$, ζ or $\mathcal{R}$ determine the perturbations of the cosmic fluid resulting in the observed CMB anisotropies and the observed LSS. For a detailed review of how quantum fluctuations from inflation affects the LSS of the Universe see [22].

Comoving orthogonal gauge

We say that the slicing is orhogonal, if the $\eta = \text{const}$ slices are orthogonal to the fluid 4-velocity. This condition turns out to be equivalent to the condition that the fluid velocity perturbation equals the shift vector. For scalar perturbations:

$$\text{Orthogonal slicing} \;\Leftrightarrow\; v = B\,. \tag{1.133}$$

From the gauge transformation Eqns. (1.97) and (1.112),

$$\widetilde{v} = v - \beta'\,, \tag{1.134}$$
$$\widetilde{B} = B - \alpha + \beta'\,, \tag{1.135}$$

we see that we get to the comoving slicing by $\alpha = B - v - 2\beta'$. Moreover, we say that the threading is comoving if the threads are world lines of fluid elements, i.e. the velocity perturbation vanishes, $\delta u^i = 0$:

$$\text{Comoving threading} \;\Leftrightarrow\; v = 0\,. \tag{1.136}$$

We obtain a comoving threading from the gauge transformation $\beta' = v$. Finally, the comoving orthogonal gauge is defined by requiring both orthogonal slicing and comoving threading:

$$\text{Comoving orthogonal gauge} \;\Leftrightarrow\; v_c = B_c = 0\,. \tag{1.137}$$

The threading is now orthogonal to the slicing. The gauge transformations allowing to place us in this gauge are:

$$\beta'_c = v\,, \tag{1.138}$$
$$\alpha_c = B + v\,. \tag{1.139}$$

However, this does not fully specify the coordinate system in the perturbed spacetime, since only β' is specified, not β. Thus we remain free to do time-independent transformations:

$$\widetilde{x}^i = x^i + \xi(\vec{x}),^{\,i} \tag{1.140}$$

while staying in the comoving gauge. This does not change the way the spacetime is sliced and threaded by the coordinate system, it just relabels the threads with different coordinate values x^i. In this gauge we have the following identities for $\mathcal{R}$ and Δ:

$$\mathcal{R} \equiv \widetilde{\psi}_c \,, \qquad \Delta \equiv \widetilde{\delta}_c = \frac{\widetilde{\delta\rho_c}}{\bar{\rho}} \,. \tag{1.141}$$

Synchronous gauge

The synchronous gauge was the first one to be used in cosmological perturbation theory, by Lifshitz in [20]. The synchronous gauge is defined for scalar perturbations by the requirement $\widetilde{\phi} = \widetilde{B} = 0$; in this manner the proper time for observers at fixed spatial coordinates coincides with cosmic time in the FLRW background, i.e. $d\tau = dt$. This implies the following gauge transformations:

$$\alpha'_{\text{sync}} + \alpha_{\text{sync}}\mathcal{H} = -\phi \,, \tag{1.142}$$

$$\beta'_{\text{sync}} = \alpha_{\text{sync}} - B \,. \tag{1.143}$$

We see that, like for the comoving gauge, only the derivative of β_{sync} is determined. In addition, α_{sync} is determined only up to solutions of the homogeneous equation $\alpha'_{\text{sync}} + \mathcal{H}\alpha_{\text{sync}} = 0$.†
Thus, the gauge is not fully specified by the synchronous condition, leading to the impossibility of introducing GI quantities. In turn, if one is not careful enough, this leads to the presence of spurious gauge modes inside the dynamical equations. Fact remarked by Lifshitz himself.

Spatially flat gauge

In this gauge one selects spatial hypersurfaces on which the induced metric is left unperturbed by scalar perturbations, condition that requires $\widetilde{\psi} = \widetilde{E} = 0$. This corresponds to a gauge transformation where:

$$\alpha_{\text{flat}} = \frac{\psi}{\mathcal{H}} \,, \qquad \beta_{\text{flat}} = -E \,. \tag{1.144}$$

The density perturbation has a gauge-invariant definition in this gauge; from Eqn. (1.128):

$$\delta\rho_\psi \equiv \widetilde{\delta\rho_{\text{flat}}} = \delta\rho + \bar{\rho}'\frac{\psi}{\mathcal{H}} \,. \tag{1.145}$$

†By that, we mean that the solution of the particular equation implies the presence of a constant spatially dependent term $\mathcal{C}(x^i)$.

Uniform density gauge

The uniform energy density gauge is defined by the condition $\widetilde{\delta\rho} = 0$, which implies a temporal gauge transformation:

$$\alpha_{\delta\rho} = -\frac{\delta\rho}{\bar{\rho}}\,. \tag{1.146}$$

In this gauge $\zeta \equiv -\widetilde{\psi_{\delta\rho}}$.

Dynamics in the longitudinal gauge

This is one of the more convenient gauges to use for deriving dynamical equations. Let's briefly sketch its dynamics.
The first-order perturbed Einstein equations $\delta G_{\mu\nu} = 8\pi G_N \delta T_{\mu\nu}$ for scalar modes give two constraints and two evolution equations. In a general gauge, the (00) (energy) and $(0i)$ (momentum) constraints are:

$$(\nabla^2 + 3K)\psi - 3\mathcal{H}(\psi' + \mathcal{H}\phi) + \mathcal{H}\nabla^2\sigma = 4\pi G_N a^2 \delta\rho\,, \tag{1.147}$$

$$\psi' + \mathcal{H}\phi + K\sigma = -4\pi G_N a^2(\rho + P)(v + B)\,. \tag{1.148}$$

The (ij) evolution equations are:

$$\psi'' + 2\mathcal{H}\psi' - K\psi + \mathcal{H}\phi' + (2\mathcal{H}' + \mathcal{H}^2)\phi = 4\pi G_N a^2(\delta P + \frac{2}{3}\nabla^2\Pi)\,, \tag{1.149}$$

$$\sigma' + 2\mathcal{H}\sigma - \phi + \psi = 8\pi G_N a^2 \Pi\,, \tag{1.150}$$

where the last one using the Bardeen potentials become:

$$\Psi - \Phi = 8\pi G_N a^2 \Pi\,. \tag{1.151}$$

Hence, in absence of anisotropic stress we have $\Psi = \Phi$. In addition, for a flat $(K = 0)$ Universe, Eq. (1.149) provides a second-order evolution equation for the metric perturbation in the longitudinal gauge driven by isotropic pressure:

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G_N a^2 \delta P\,. \tag{1.152}$$

For adiabatic perturbations we can relate the pressure to the density, $\delta P = c_s^2\delta\rho$, in which case (1.147) and (1.152) yield a closed second-order differential equation:

$$\Phi'' + 3(1 + c_s^2)\mathcal{H}\Phi' + [2\mathcal{H}' + (1 + 3c_s^2)\mathcal{H}^2 - c_s^2\nabla^2]\Phi = 0\,, \tag{1.153}$$

from which explicit solutions may be obtained. Writing the energy and momentum constraints in terms of GI variables we can further arrive at a GI generalization of the Newtonian-Poisson equation:

$$\nabla^2\Psi = 4\pi G_N a^2 \Delta\,\bar{\rho} \equiv 4\pi G_N a^2 \widetilde{\delta\rho}_c\,. \tag{1.154}$$

Chapter 2

Covariant gauge invariant ΛCDM perturbations

This chapter is divided in two main sections: in Section 2.1 we present the covariant fluid approach to GR, where, for obvious reasons, we focus on the case of a FLRW spacetime [21]. In cosmology we assume that the description of the matter content of the Universe is approximated by a continuous fluid. At each point in spacetime we can assign a 4-velocity vector u^a representing the velocity of the volume element of the fluid surrounding that point.[†] It follows that, at each point p, we can foliate our FLRW spacetime into a time direction parallel to the 4-velocity u^a of the fluid and a 3-dimensional slice orthogonal to u^a. This splitting is achieved by means of the three spatial metric (or projector) h_{ab}.
In Section 2.2 we present an alternative approach to the theory of cosmological perturbations. This GI approach, w.r.t. the one of Bardeen presented in Chapter 1, is based on the covariant approach to cosmology and is thus CGI (covariant and GI). This alternative has been proposed by Ellis & Bruni [1, 2] and is based on the Stewart & Walker Lemma discussed in Section 1.3. In particular, by means of this Lemma, we can define CGI variables which have the advantage of having an explicit physical meaning independently of any chosen gauge.

2.1 Covariant approach to Cosmology

2.1.1 Kinematics

Let u^a be a future directed timelike vector field ($u_a u^a = -1$); this vector field can be regarded as the 4-velocity of the corresponding observer $\mathcal{O}_u$. At each point p of the spacetime we have a subspace H_p of the tangent space T_p at p which is orthogonal to u^a.

[†]Notice that we will here use the indices a, b, c... and μ, ν, σ... interchangeably.

We can then define the projection tensor in this subspace as:

$$h_{ab} \equiv g_{ab} + u_a u_b \,, \tag{2.1}$$

where we have the following identities:

$$h_{ab} \equiv g_{ab} + u_a u_b \quad \Rightarrow \quad h^a_{\ b} h^b_{\ c} = h^a_{\ c}, \quad h^a_{\ a} = 3\,, \quad h_a^{\ b} u_b = 0\,. \tag{2.2}$$

This defines the spatial part of the local rest frame of the observer. Hence, the tensor is the metric in the subspace H_p of T_p, orthogonal to the corresponding 4-velocity u^a.
In cosmology, there exists a preferred family of world-lines, named fundamental world-lines, representing the motion of fundamental observers in the Universe, i.e. observers which are at rest w.r.t. a given volume element of fluid. Let us now define some relevant kinematical quantities for the the normalized 4-velocity vector tangent to the flow lines:

$$u^a = \frac{dx^a}{d\tau} \quad \Rightarrow \quad u^a u_a = -1, \tag{2.3}$$

where τ, the proper time along the fluid flow lines, coincide with the cosmic time t. In a FLRW Universe we have $u^\mu = \delta^\mu_0$ and $u_\mu = -\delta^0_\mu$. The time derivative of any tensor $T^{a\cdots}{}_{b\cdots}$ along the fluid flow lines is simply the covariant derivative along u^a:

$$\dot{T}^{a\cdots}{}_{b\cdots} \equiv u^c \nabla_c T^{a\cdots}{}_{b\cdots}\,, \tag{2.4}$$

where the above is the rate of change of $T^{a\cdots}{}_{b\cdots}$ as measured by a fundamental observer. Using $h^a_{\ b}$ we can also define the spatial derivative in the local reference frame $\mathcal{O}_u$ as:

$${}^{(3)}\nabla_c T^{a\cdots}{}_{b\cdots} \equiv h_c^{\ f} h^a_{\ d} \cdots h_b^{\ e} \cdots \nabla_f T^{d\cdots}{}_{e\cdots}\,. \tag{2.5}$$

It follows from the above definition that ${}^{(3)}\nabla_a$ preserves the orthogonal metric h_{bc}: that is, ${}^{(3)}\nabla_a h_{bc} = 0$. Consequently, we can raise and lower indices through equations acted on by ${}^{(3)}\nabla_a$ using h_{ab}, h^{ab}.
The 4-acceleration is defined as $a^a \equiv \dot{u}^a = u^b \nabla_b u^a$ and from Eqn. (2.3) it follows that $a^a u_a = 0$. The **expansion scalar** (volume expansion) Θ is the trace of $\nabla_b u_a$:

$$\Theta \equiv \nabla_a u^a\,, \tag{2.6}$$

which represents the isotropic part of the expansion of the fluid. For instance, the action of Θ alone during a small time interval on a sphere of fluid changes the latter in a larger or smaller sphere with the same orientation. The **shear tensor** is instead the spatial trace-free symmetric part of $\nabla_b u_a$:

$$\sigma_{ab} \equiv h_a^{\ c} h_b^{\ d} \nabla_{(d} u_{c)} - \frac{1}{3}\Theta h_{ab} \quad \Rightarrow \quad \sigma_{ab} u^b = 0, \quad \sigma^a_{\ a} = 0\,. \tag{2.7}$$

Since the shear tensor is symmetric, we can choose an orthonormal basis of shear eigenvectors, so the components of σ_{ab} become $\sigma_{ab} = \mathrm{diag}(0, \sigma_1, \sigma_2, \sigma_3)$, where $\sigma_1 + \sigma_2 + \sigma_3 = 0$ (since the tensor is trace-free). Then if there is an expansion in the 1-direction ($\sigma_1 > 0$), there must be a contraction in at least one other direction (say $\sigma_2 < 0$). Thus, the previous sphere becomes an ellipsoid, expanded in the 1-direction but contracted in the 2-direction, with the same volume as before. Hence the shear tensor describes a pure distortion, without rotation or change of volume. Its magnitude is given by

$$\sigma^2 \equiv \frac{1}{2}\sigma_{ab}\sigma^{ab} \geq 0\,, \quad \sigma = 0 \quad \Leftrightarrow \quad \sigma_{ab} = 0\,. \tag{2.8}$$

The **vorticity tensor** ω_{ab} is the skew-symmetric spatial part of $\nabla_b u_a$ defined as:

$$\omega_{ab} \equiv h_a^{\ c} h_b^{\ d} \nabla_{[d} u_{c]} \quad \Rightarrow \quad \omega_{ab} u^b = 0\,, \tag{2.9}$$

with magnitude

$$\omega^2 \equiv \frac{1}{2}\omega_{ab}\omega^{ab} \geq 0\,. \tag{2.10}$$

Since ω_{ab} is skew-symmetric, all the information contained in it can be put in a vector, the vorticity vector ω^a:

$$\omega_a \equiv \frac{1}{2}\eta_{abc}\omega^{bc} \quad \Leftrightarrow \quad \omega_{ab} = \eta_{abc}\omega^c\,, \tag{2.11}$$

obviously $\omega_a u^a = 0$, where η_{abc} is the 3-dimensional totally skew-symmetric tensor:

$$\eta_{abc} \equiv \eta_{abcd} u^d\,, \quad \eta^{abcd} = \eta^{[abcd]} = -\sqrt{|g|}\epsilon^{abcd}\,, \quad g \equiv \det(g_{ab})\,, \tag{2.12}$$

where ϵ^{abcd} is the totally skew-symmetric Levi Civita tensor. The action of ω^a rotates the sphere, leaving its shape and volume unchanged. The quantities we have now defined characterize the kinematic features of the fluid flow. In fact, splitting $\nabla_b u_a$ as:

$$\nabla_b u_a = {}^{(3)}\nabla_b u_a - a_a u_b\,, \quad {}^{(3)}\nabla_b u_a = \omega_{ab} + \Theta_{ab}\,, \quad \Theta_{ab} = \frac{1}{3}\Theta h_{ab} + \sigma_{ab}\,; \tag{2.13}$$

where Θ_{ab} is the symmetric part of ${}^{(3)}\nabla_b u_a$, we so obtain:

$$\nabla_b u_a = \omega_{ab} + \sigma_{ab} + \frac{1}{3}\Theta h_{ab} - a_a u_b\,, \tag{2.14}$$

which shows that this derivative is completely determined by the previously defined kinematic quantities. In addition, they can also be used to characterize some simple Universe models. For example, in an Einstein static Universe $\omega = \sigma = a^a = \Theta = 0$; while in all other FLRW Universes $\omega = \sigma = a^a = 0$; $\Theta \neq 0$.
It is convenient to define a representative length scale $a(t)$ by the relation:

$$\frac{\dot{a}}{a} = \frac{1}{3}\Theta\,, \tag{2.15}$$

which, in a FLRW Universe, corresponds to the Hubble parameter H along the flow.

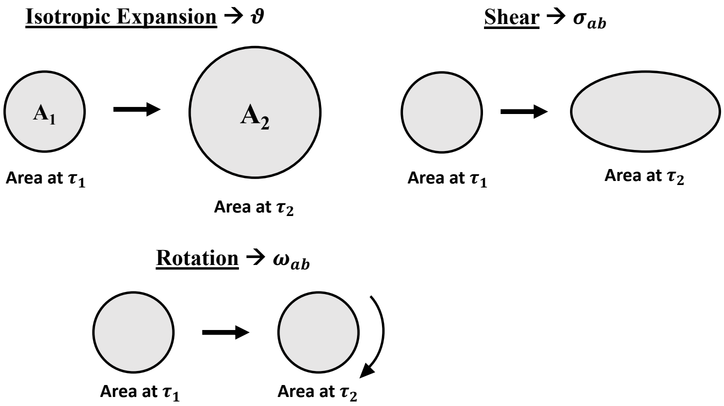

Figure 2.1: Geometrical representation of the expansion Θ, shear σ_{ab} and rotation ω_{ab}.

2.1.2 Geometry and matter

The Riemann tensor R_{abcd} is defined by the commutation relation satisfied by the covariant derivatives of any arbitrary 4-vector; for the 4-velocity u^a we have:

$$(\nabla_c \nabla_d - \nabla_d \nabla_c) u_a = R_{abcd} u^b \,. \tag{2.16}$$

On contracting the above with u^c we obtain a propagation equation for $(\nabla_d u_a)$ along the fluid flow lines:

$$(\nabla_d u_a)^{\cdot} - \nabla_d \dot{u}_a + (\nabla_d u^c)(\nabla_c u_a) = R_{abcd} u^b u^c \,, \tag{2.17}$$

Contracting $a - d$ indices we get:

$$(\nabla_a u^a)^{\cdot} - \nabla_a \dot{u}^a + (\nabla^a u^c)(\nabla_c u_a) = -R_{bc} u^b u^c \,. \tag{2.18}$$

In terms of the kinematic quantities, this is

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2(\sigma^2 - \omega^2) - A = -R_{bc} u^b u^c \,, \tag{2.19}$$

where we have defined $\nabla_a \dot{u}^a = A$ and used symmetry properties of the kinematical quantities.

Riemann tensor

Just for sake of completeness we mention that the Riemann tensor can be decomposed into its ‘trace’, i.e. the Ricci tensor and its trace-less part, the Weyl tensor:

$$R^a{}_{bad} \equiv R_{bd} \,, \qquad C^{ab}{}_{cd} \equiv R^{ab}{}_{cd} - 2 g^{[a}{}_{[c} R^{b]}{}_{d]} + \frac{1}{3} R g^{[a}{}_{[c} g^{b]}{}_{d]} \quad \Rightarrow \quad C^a{}_{bad} = 0 \,. \tag{2.20}$$

We can further split the Weyl tensor into its 'electric' and 'magnetic' components, respectively defined by:

$$E_{ac} \equiv C_{acbd} u^b u^d \,, \qquad H_{ac} \equiv \frac{1}{2} \eta_{acd} C^{cd}{}_{be} u^e \,; \tag{2.21}$$

$$E_{ab} = E_{(ab)} \,, \quad H_{ab} = H_{(ab)} \,, \quad E^a{}_a = H^a{}_a = 0 \,, \quad E_{ab} u^b = H_{ab} u^b = 0 \,. \tag{2.22}$$

The reason for this terminology is that E_{ab} and H_{ab} satisfy a 'Maxwellian' form of the Bianchi's identity, see [21, 8] for a complete explanation.

Energy-momentum tensor

Contracting the Bianchi identities $\nabla_{[e} R_{|ab|cd]} = 0$ we get:

$$\nabla_b G^{ab} = 0 \,, \qquad G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab} \,, \tag{2.23}$$

thus from Einstein equations $G_{ab} = \kappa T_{ab}$:†

$$\nabla_b T^{ab} = 0 \,, \tag{2.24}$$

where T_{ab}, as measured by an observer moving with 4-velocity u^a, can be split into its parts parallel and orthogonal to u^a as follows:

$$T_{ab} = \rho u_a b_b + P h_{ab} + 2 q_{(a} u_{b)} + \pi_{ab} \,. \tag{2.25}$$

The observer measures that:

- $\rho = T_{ab} u^a u^b$ is the relativistic energy density (rest mass density plus total internal energy);
- $P = \frac{1}{3} h^{ab} T_{ab}$ is the relativistic pressure;
- $q_a = -h_a{}^b T_{bc} u^c$ is the relativistic energy-flux due to processes such as diffusion and heat conduction;
- $\pi_{ab} \equiv h_a{}^c h_b{}^d T_{cd} - \frac{1}{3}(h^{cd} T_{cd}) h_{ab}$ is the relativistic anisotropic (trace-free) stress tensor due to effects such as viscosity or free-streaming or magnetic fields;

where in general ρ and P will be related through an equation of state. For a perfect fluid the energy-momentum tensor becomes:

$$T^{ab} = \rho u^a u^b + P h^{ab} = (\rho + P) u^a u^b + P g^{ab} \,. \tag{2.26}$$

The conservation of energy-momentum can be projected either along u^a leading to energy conservation, i.e. $u_b \nabla_a T^{ab} = 0$, or along h_{ab} leading to momentum conservation, i.e. $h_{cb} \nabla_a T^{ab} = 0$.

†Throughout this work we have adopted the notation: $8\pi G_N = 1 = c$.

Raychaudhuri equation

The evolution of Θ along the fluid flow lines is given by the Raychaudhuri equation. Inserting the expression for $R_{bc}u^b u^c$, rephrased in terms of energy-momentum quantities, in Eqn. (2.19) we get:

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2(\sigma^2 - \omega^2) - A + \frac{1}{2}\kappa(\rho + 3P - 2V) = 0\,, \tag{2.27}$$

where $V = \Lambda\kappa^{-1}$.

The Raychaudhuri equation is the fundamental equation of gravitational attraction that establishes that in GR, $(\rho + 3P)$ is the active gravitational mass of the fluid.

For $\rho + 3P > 0$, a.k.a. strong energy condition (SEC), we have from Eqn. (2.27) a volume contraction; we also see that V contributes as a constant repulsive force, and a similar repulsive role is played by the acceleration divergence A and by the vorticity ω. On the other hand, the shear σ tends to shrink the volume.

Notice that, within this approach to cosmology, the above Raychaudhuri equation reduces to the dynamical Friedmann equation in case we consider a homogeneous and isotropic Universe, i.e. setting $\sigma = \omega = A = 0$:

$$3(\dot{H} + H^2) = -\frac{1}{2}\kappa(\rho + 3P - 2V)\,. \tag{2.28}$$

2.1.3 Intrinsic 3-curvature with zero vorticity

Only when the fluid vorticity vanishes there exists a family of spatial hypersurfaces $\Sigma_\perp$ everywhere orthogonal to the fluid flow vector u^a. Indeed it is possible to show that:

$$\begin{aligned} &\omega_a = 0 \;\Leftrightarrow\; u_{[b}\nabla_c u_{d]} = 0 \;\Leftrightarrow\; u_{[b}u_{c,d]} = 0 \\ &\Leftrightarrow\; \exists \text{ local functions } r,\, t\text{:} \quad u_a = -r t_{,a}\,, \end{aligned} \tag{2.29}$$

Analytically, t is a potential function for the direction of u^a. Geometrically, this means u^a is orthogonal to the surfaces $t = \text{const}$, for $X^a u_a = 0 \Leftrightarrow X^a t_{,a} = 0$. This means that the derivative of t in the direction X^a is zero for every vector X^a orthogonal to u^a.

In other words, for the hypersurfaces $\Sigma_\perp$ to exist, it must be possible to write u_a as a 4-gradient. Then the hypersurfaces $\Sigma_\perp \equiv t = \text{const}$ are instantaneous hypersurfaces of simultaneity for all the fundamental observers, i.e. the hypersurfaces $\Sigma_\perp$ define a cosmic time t. However, the function t does not necessarily measure proper time along the world lines. Indeed, the derivative of t along the world lines with respect to proper time is $\dot{t} = t_{,a}u^a = -r^{-1}u_a u^a = r^{-1}$. Thus t can be chosen to measure proper time along the world lines only if $r = r(t)$, for only then can we choose $r = 1$ by rescaling $t \to t'(t)$; such a t is a normalized cosmic time, which both determines the rest space of each fundamental observer and measures proper time along all the fundamental world lines. However one can normalize the cosmic time to measure also the proper time along each flow line only if $a_a = 0$. In fact, in the case $a_a \neq 0$, one can normalize the cosmic time to measure proper time along one world line but then, even though it synchronizes instantaneous events on the different world lines, it will not measure proper time along other world lines.[†]

Spatial Ricci curvature

In a general fluid flow, we can define the quantity:

$$\mathcal{K} = 2\big(-\frac{1}{3}\Theta^2 + \sigma^2 + \kappa(\rho + V)\big)\,, \tag{2.30}$$

which turns out to satisfy the relation:

$${}^{(3)}R = \mathcal{K} - 2\omega^2\,. \tag{2.31}$$

When $\omega = 0$, $\mathcal{K}$ acquires a special significance: it is the Ricci scalar ${}^{(3)}R$ of the 3-dimensional spatial hypersurfaces $\Sigma_{\perp}$; that is: $\omega = 0 \Rightarrow {}^{(3)}R = \mathcal{K}$. In this context, we have:

$${}^{(3)}R = \mathcal{K} = \frac{6K}{a^2}\,, \quad \rightarrow \quad \frac{K}{a^2} = \frac{1}{3}\bigg(-\frac{1}{3}\Theta^2 + \sigma^2 + \kappa\big(\rho + V\big)\bigg)\,, \tag{2.32}$$

where σ^2 obviously vanishes if we are in a FLRW background (where $K = 0, \pm 1$). In such a case the above equation is just the constraint Friedmann equation:

$$H^2 + \frac{K}{a^2} = \frac{1}{3}\kappa(\rho + V)\,. \tag{2.33}$$

2.2 Covariant gauge invariant cosmological perturbations

In Chapter 1 we have discussed the standard perturbative formalism of cosmological perturbations as developed by Bardeen [4]. His theory is based on the definition of linear GI variables built from GI linear combinations of gauge-dependent perturbations. However, this method could lead to some interpretation issues, since the physical and geometrical meaning of the resulting quantities is often obscure unless a gauge is specified.
This has led Ellis & Bruni [1, 2] to develop a different approach to GI cosmological perturbations, which often provides a clearer picture of the almost FLRW model describing

[†]For example, the comoving time coordinate t in the FLRW universes is a fundamental cosmic time that measures proper time along each world line. The standard time coordinate t in a Schwarzschild solution is a cosmic time for static observers, but does not measure proper time along their world lines. Their acceleration is in fact non-zero, following from $d\tau^2 = (1 - 2m/r)dt^2$.

the real Universe. Following the aforementioned authors, we here study what is called the covariant gauge invariant (CGI) approach, based on what we have presented in Section 2.1.
Central to the CGI approach is the Stewart & Walker Lemma of Section 1.3. The basic idea is to introduce covariantly defined exact variables $\mathbf{T}$ (i.e. meaningful in any spacetime) such that $\mathbf{T}_0$ in a FLRW BG Universe vanish. In this way, from the aforementioned Lemma, the quantity itself is a GI perturbation in the almost FLRW Universe; its physical significance being apparent via the covariant definition.
Given the symmetries of FLRW models, any tensor that describes spatial inhomogeneity or anisotropy must vanish in the background and therefore its linear perturbation will remain invariant under gauge transformations.

2.2.1 CGI variables

A FLRW Universe model, as stated in Section 2.1, is characterized by the below conditions:

$$\sigma = \omega = 0 = A\,, \tag{2.34}$$

with

$$\rho = \rho(t)\,, \quad P = P(t)\,, \quad \Theta = \Theta(t)\,. \tag{2.35}$$

where t is the normalized cosmic time defined by the FLRW fluid flow vector: $u_a = -t_{,a}$ and

$$E_{ab} = 0\,, \quad H_{ab} = 0\,, \tag{2.36}$$

i.e. the Weyl tensor vanishes implying that these spacetimes are conformally flat [16]. From this characterization plus the Stewart & Walker lemma of Section 1.3, the basic GI quantities for an almost FLRW Universe are as follows:

$$\sigma_{ab} \equiv h_a{}^c h_b{}^d \nabla_{(d} u_{c)} - \frac{1}{3}\Theta h_{ab}\,, \tag{2.37}$$

$$\omega_{ab} \equiv h_a{}^c h_b{}^d \nabla_{[d} u_{c]}\,, \tag{2.38}$$

$$a^a \equiv u^b \nabla_b u^a\,, \tag{2.39}$$

$$E_{ab} \equiv C_{abcd} u^b u^d\,, \tag{2.40}$$

$$H_{ab} \equiv \frac{1}{2}\eta_{abd} C^{bd}{}_{ce} u^e\,, \tag{2.41}$$

$$q_a \equiv -h_a{}^b T_{bc} u^c\,, \tag{2.42}$$

$$\pi_{ab} \equiv h_a{}^c h_b{}^d T_{cd} - \frac{1}{3}(h^{cd} T_{cd}) h_{ab}\,. \tag{2.43}$$

The last two, in particular, vanish identically in the perfect fluid case and are not going to be discussed further in this work.

These are the simplest covariantly defined quantities which vanish in FLRW models, and so are GI. The problem consists in the fact that the list so far discussed does not contain quantities characterizing the variation of the 0-order variables (energy density ρ, pressure P and fluid expansion Θ) which are in general non-zero in FLRW and so are not GI. However, we can find associated GI quantities: namely, the orthogonal spatial gradients of these variables:

$$X_a \equiv h_a{}^b \nabla_b \, \rho \, , \qquad Y_a \equiv h_a{}^b \nabla_b P \, , \qquad Z_a \equiv h_a{}^b \nabla_b \Theta \, . \tag{2.44}$$

Notice that we have replaced the covariant derivative denoted by ';' with the partial one ',' since ρ, P and Θ are all scalar quantities. Each quantity is GI as they all vanish in the FLRW BG; due to Eqns. (2.2) and (2.35). The spatial gradient can, in a more convenient notation, be denoted as:

$$h_a{}^b \nabla_b(\ ) \equiv {}^{(3)}\nabla_a(\ ) \, . \tag{2.45}$$

We can define two other important GI quantities, namely the divergence of the acceleration, and its spatial gradient:

$$A \equiv \nabla_a a^a \, , \qquad A_a \equiv {}^{(3)}\nabla_a A \, . \tag{2.46}$$

The last GI quantity we are going to define stems from the following consideration: in the case of vanishing vorticity, the Ricci scalar ${}^{(3)}R$ is GI iff $K = 0$, see Eqn. (2.32). However, its spatial gradient is always GI. Thus, for a general fluid flow, it is interesting to define from $\mathcal{K}$ the GI quantity:

$$\mathcal{K}_a \equiv {}^{(3)}\nabla_a \mathcal{K} = -\frac{4}{3}\Theta Z_a + 2X_a + 2\,{}^{(3)}\nabla_a(\sigma^2) \, . \tag{2.47}$$

Then isocurvature fluctuations can be defined as the zero-vorticity perturbations for which $\mathcal{K}_a = 0$.

2.2.2 Key CGI variables

As we mentioned at the beginning of Chapter 1, the quantity we are mostly interested in is the density perturbation $\delta\rho$ which causes the formation of cosmic structures. The point of this discussion is that instead of considering $\delta\rho$ with the arbitrariness that implies (due to gauge transformations), we can find three simple GI quantities suitable for describing the evolution of density perturbations, without the complexity of the Bardeen variables' interpretation. The analogue of the density perturbation in the CGI formalism is the spatial projection of the energy density gradient: X_a. It describes the density inhomogeneities which we wish to investigate, for if there is an overdensity which is a viable protogalaxy (i.e. a cloud of gas forming a galaxy), this will be evidenced by a

non-zero value of X_a (its magnitude directly indicating how rapid the spatial variation of density is).
However, we normally wish to compare the density gradient with the existing density, to characterize its significance. Hence we define a second quantity, the fractional density gradient:

$$\chi_a \equiv \frac{X_a}{\rho} = \frac{1}{\rho}\, {}^{(3)}\nabla_a \rho \,, \tag{2.48}$$

which is also GI, and represents the relative importance of the density gradient. However there is still a problem with χ_a; it is not dimensionless. In fact, when we consider the time evolution of the fluid, both X_a and χ_a represent the change in density to a fixed distance, whereas in the context of considering the growth of protogalaxy fluctuations we want to consider density variations at a fixed comoving scale. Thus the third quantity of interest is the comoving fractional density gradient obtained simply multiplying χ_a by the scale factor $a(t)$:

$$\mathcal{D}_a \equiv a\chi_a = \frac{a}{\rho}\, {}^{(3)}\nabla_a \rho \,, \tag{2.49}$$

which is GI and dimensionless. The time variation of this quantity precisely reflects the relative growth of density in neighboring fluid comoving volumes; this is what we wish to investigate. The vector $\mathcal{D}_a$ can be separated into a direction e_a and a magnitude $\mathcal{D}$:

$$\mathcal{D}_a = \mathcal{D} e_a \quad e_a e^a = 1 \quad e_a u^a = 0 \quad \Rightarrow \quad \mathcal{D} = (\mathcal{D}_a \mathcal{D}^a)^2 \,, \tag{2.50}$$

where the magnitude $\mathcal{D}$ is the GI variable that most closely corresponds to the intention of the usual $\delta\rho/\rho$ in representing the fractional density increase in a comoving density fluctuation. The crucial difference from the usual Bardeen definition is that $\mathcal{D}$ represents a (real) spatial fluctuation, rather than a (fictitious) time fluctuation, see B.1.
Since we have defined the comoving density gradient $\mathcal{D}_a$ its useful to introduce also the comoving expansion gradient:

$$\mathcal{Z}_a \equiv a Z_a \,. \tag{2.51}$$

In the following section we will derive dynamical equations for these CGI variables, with the aim of obtaining a CGI dynamical equation for the comoving density gradient $\mathcal{D}_a$.

2.2.3 Exact CGI dynamical equations

We now derive the propagation equations along arbitrary fluid flow lines for the previously defined quantities. We obtain dynamic equations for the zero-order quantities ρ, P and Θ on the one hand, and for the exact quantities X_a, Y_a and Z_a, that are GI in an almost FLRW Universe, on the other.

0-th order dynamical equations

$$u_b\nabla_a T^{ab} = 0 \quad \Rightarrow \quad \dot{\rho} + \Theta(\rho + P) = 0\,, \tag{2.52}$$

$$h_{ac}\nabla_b T^{bc} = 0 \quad \Rightarrow \quad a_a = -(\rho + P)^{-1} Y_a\,. \tag{2.53}$$

The time evolution of P is determined from the above when we specify an equation of state determining P from ρ. Following a standard notation we define:

$$w \equiv \frac{P}{\rho}\,, \qquad c_s^2 \equiv \frac{\mathrm{d}P}{\mathrm{d}\rho} = \frac{\dot{P}}{\dot{\rho}}\,, \tag{2.54}$$

$$\dot{w} = -(1 + w)(c_s^2 - w)\Theta\,, \tag{2.55}$$

where c_s^2 is the adiabatic speed of sound, defined starting from the thermodynamic relation $P \equiv P(\rho, S)$:

$$\delta P = \frac{\partial P}{\partial S}\delta S + \frac{\partial P}{\partial \rho}\delta\rho\,, \tag{2.56}$$

as

$$c_s^2 \equiv \left.\frac{\partial P}{\partial \rho}\right|_S\,, \tag{2.57}$$

For a barotropic fluid $P = w\rho$ and $c_s^2 = w$.
Notice that the evolution equation for Θ has already been obtained, see Eqn. (2.27).
Finally the time derivative of $\mathcal{K}$ along the fluid flow lines obeys the equation:

$$(\mathcal{K} - 2\sigma^2)^{\cdot} = \frac{2}{3}\Theta(6\sigma^2 - \mathcal{K} - 4\omega^2 - 2A)\,. \tag{2.58}$$

Remember that when $\omega = 0$ this is an equation for the evolution of ${}^{(3)}R$. Let's now focus on the exact GI variables.

1-st order dynamical equations

To obtain the propagation equation for X_a we proceed directly from Eq. (2.52), performing the spatial gradient of the energy conservation equation we get:

$$h_a{}^b\nabla_b\big(\dot{\rho}\big) + (\rho + P)Z_a + \Theta(X_a + Y_a) = 0\,. \tag{2.59}$$

After some calculations one arrives at:

$$(X_a)^{\cdot} + (X_b)(\sigma^b{}_a + \omega^b{}_a) + \frac{4}{3}\Theta(X_a) + (\rho + P)Z_a = 0\,. \tag{2.60}$$

Similarly we start from the spatial variation of the Raychaudhuri equation (2.27) to obtain the propagation equation for Z_a:

$$h_a{}^b\nabla_b\big(u^c\nabla_c\Theta\big) + \frac{2}{3}\Theta Z_a + h_a{}^b\Big(2\nabla_b(\sigma^2 - \omega^2) - A_b\Big) + \frac{1}{2}(X_a + 3Y_a) = 0\,. \tag{2.61}$$

After some algebra we obtain:

$$h_a{}^b(Z_b)^{\cdot} + \Theta Z_a - a_a\mathcal{R} + \left(\frac{1}{2}X_a + 2\,{}^{(3)}\nabla_a(\sigma^2-\omega^2) - A_a\right) + Z_b(\sigma^b{}_a + \omega^b{}_a) = 0\,, \quad (2.62)$$

where $\mathcal{R}$ is given by:

$$\mathcal{R} \equiv -\frac{1}{3}\Theta^2 - 2(\sigma^2-\omega^2) + A + \rho + V \quad (2.63)$$

$$= \frac{1}{2}\mathcal{K} + A - 3\sigma^3 + 2\omega^2\,. \quad (2.64)$$

When the e.o.s. of the fluid is known, the evolution of Y_a will follow from the one for X_a. Let us now compute the time evolution of the comoving fractional density gradient $\mathcal{D}_a$. In order to do so, we exploit its definition as in the following:

$$\mathcal{D}_a = \frac{aX_a}{\rho} \qquad \Rightarrow \qquad (\mathcal{D}_a)^{\cdot} = \frac{a}{\rho}(X_a)^{\cdot} + \frac{\dot{a}}{\rho}X_a - \frac{a}{\rho^2}\dot{\rho}X_a \quad (2.65)$$

$$= \frac{a}{\rho}(X_a)^{\cdot} + \left(\frac{4}{3}+w\right)\Theta\mathcal{D}_a\,, \quad (2.66)$$

from the definition of $\mathcal{Z}_a = aZ_a$ we get:

$$(\mathcal{D}_a)^{\cdot} = w\Theta\mathcal{D}_a - (1+w)\mathcal{Z}_a - \mathcal{D}_b(\sigma^b{}_a + \omega^b{}_a)\,, \quad (2.67)$$

where $\mathcal{Z}_a$ is:

$$(\mathcal{Z}_a)^{\cdot} = -\frac{2}{3}\Theta\mathcal{Z}_a - \frac{1}{2}\rho\mathcal{D}_a - \mathcal{Z}_b(\sigma^b_a + \omega^b_a) + a\Big(a_a\mathcal{R} + A_a - 2\,{}^{(3)}\nabla_a\big(\sigma^2-\omega^2\big)\Big)\,. \quad (2.68)$$

We notice, from the above equations, that the comoving gradients $\mathcal{D}_a$ and $\mathcal{Z}_a$ are coupled. Their equations contain also non-linear terms coupling these quantities with $\sigma^a{}_b$, $\omega^a{}_b$, a_a, A and A_a, too.

Therefore, to consider a closed non-linear system of equations, one should take into account the evolution of all these quantities (including also the Maxwell-like evolution equations for E_{ab} and H_{ab}).

It is not surprising that to consider the fully non-linear equation for the density gradient one has to take into account so many other quantities. After all, the fully non-linear system is equivalent to the complete content of the Einstein equations. We have only chosen new variables, more suitable for the study of the growth in time of spatial density inhomogeneities.

In order to solve the equations and determine this growth, one has therefore to adopt some restrictive and physically motivated hypothesis. The first step in this direction is the linear approximation.

2.2.4 Linearization procedure

The variables we have been considering so far are exactly defined CGI quantities, thus they have a physical or geometrical meaning in an arbitrary spacetime. We want now to restrict our attention to a real physical spacetime which is close to a FLRW Universe. Instead of starting from an exact FLRW model and perturb it in the standard way, we want to approach this Universe from a general spacetime.
Following the very mentioned Stewart & Walker Lemma from Section 1.3 and the covariant characterization of the FLRW spacetime, we know that variables that vanish in the FLRW background are GI. We can therefore consider two sets of variables:

- **0-th order variables:** $\rho(t)$, $P(t)$, $\Theta(t) = 3H$; these are variables that do not vanish in the FLRW background, i.e. in the spacetime that is the zero-order approximation to the physical almost FLRW Universe.
- **1-st order GI variables:** these are the variables that do vanish in the FLRW background: in considering them as first-order quantities, we automatically define the almost FLRW Universe as the spacetime in which these variables are non-vanishing. Terms quadratic in these variables are negligible, e.g. $\mathcal{D}_a\, \sigma^a{}_b$, $\nabla_a(\sigma^2)$.

Our aim is now to derive linear equations for these GI (and covariant) variables, therefore we have to establish a linearization procedure for the aforementioned exact non-linear equations.
However, given the above characterization of our variables, such a procedure is trivial: the variables ρ, P and Θ, that always appear in the above exact equations as coefficients of the GI variables are needed only at zero-order, i.e. they are treated as known functions in the equations. The quadratic and higher order GI variables are simply dropped from the equations.

2.2.5 Linearized CGI dynamical equations

Before deriving the linear dynamical equations for our CGI variables let us introduce a new exact quantity. In Secction 2.2.1, we have defined $\mathcal{K} = {}^{(3)}R + 2\omega^2$, which reduces to the Ricci 3-curvature scalar corresponding to the hypersurfaces orthogonal to u^a when $\omega = 0$. We now define the new CGI variable $\mathcal{C}_a$:

$$\mathcal{C}_a \equiv a^3\, {}^{(3)}\nabla_a\, {}^{(3)}R = a^3\mathcal{K}_a - 2a^3\, {}^{(3)}\nabla_a(\omega^2)\,, \tag{2.69}$$

where $\mathcal{K}_a = {}^{(3)}\nabla_a\mathcal{K}$ has been defined previously. When $\omega = 0$, $\mathcal{C}_a$ is the exact curvature gradient of the u^a orthogonal hypersurfaces. Since the last term in $\mathcal{C}_a$ is second-order, it follows that at linear order, $\mathcal{C}_a$ is the comoving curvature gradient, i.e. $\mathcal{C}_a = a^3\mathcal{K}_a$. Expanding such an expression at linear order we obtain:

$$\mathcal{C}_a = -\frac{4}{3}\Theta a^2 \mathcal{Z}_a + 2a^2\rho\mathcal{D}_a\,. \tag{2.70}$$

We thus have an additional physical quantity that we can exploit to obtain the linearized dynamical equations.
Linearizing Eqns. (2.67) and (2.68) and inserting them in (2.70) we obtain the following system of coupled differential equations:

$$\dot{\mathcal{D}}_a = w\Theta\mathcal{D}_a - (1+w)\mathcal{Z}_a\,, \tag{2.71}$$

$$\dot{\mathcal{Z}}_a = -\frac{2}{3}\Theta\mathcal{Z}_a - \frac{1}{2}\rho\mathcal{D}_a + a\Big(\frac{3K}{a^2}a_a + A_a\Big)\,, \tag{2.72}$$

$$\dot{\mathcal{C}}_a = \frac{6K}{a^2}\Theta^{-1}\Big(\frac{1}{2}\mathcal{C}_a - \rho a^2\mathcal{D}_a\Big) - \frac{4}{3}\Theta a^3\Big(\frac{3K}{a^2}a_a + A_a\Big)\,. \tag{2.73}$$

From the definition of A_a and the momentum conservation equation (2.53) we see that to first order we have:

$$A_a = -\frac{{}^{(3)}\nabla_a\,{}^{(3)}\nabla^2 P}{(\rho+P)} = -c_s^2\frac{{}^{(3)}\nabla_a\,{}^{(3)}\nabla^b\mathcal{D}_b}{a(1+w)}\,, \tag{2.74}$$

where ${}^{(3)}\nabla^2 \equiv {}^{(3)}\nabla_a\,{}^{(3)}\nabla^a$. The second equality follows from the assumption of adiabatic evolution, used throughout this thesis. Notice, from Eqn. (2.73), that for a flat Universe $(K=0)$ we have:

$$(\mathcal{C}_a)^{\cdot} = \frac{4}{3}\frac{a^3}{\rho+P}\Theta\,{}^{(3)}\nabla_a\,{}^{(3)}\nabla^2 P\,, \tag{2.75}$$

Thus on large scales $(\mathcal{C}_a)^{\cdot} = 0$. Further in the case of dust $(P=0)$, $\mathcal{C}_a$ is conserved no matter the scale.
From the above system of coupled differential equations we can then obtain the below second-order differential equation for $\mathcal{D}_a$:

$$\ddot{\mathcal{D}}_a + \Big(2-6w+3c_s^2\Big)H\dot{\mathcal{D}}_a - \Big[\Big(\frac{1}{2}+4w-3c_s^2-\frac{3}{2}w^2\Big)\rho - 12(w-c_s^2)\frac{K}{a^2} + c_s^2\,{}^{(3)}\nabla^2\Big]\mathcal{D}_a = 0\,. \tag{2.76}$$

We now adopt the local decomposition of $\mathcal{D}_a$ as in [2] and we write:

$$a\,{}^{(3)}\nabla_b\mathcal{D}_a \equiv \Delta_{ab} = W_{ab} + \Sigma_{ab} + \frac{1}{3}\Delta h_{ab}\,, \tag{2.77}$$

where

$$W_{ab} \equiv \Delta_{[ab]}\,, \qquad \Sigma_{ab} \equiv \Delta_{(ab)} - \frac{1}{3}\Delta h_{ab}\,, \qquad \Sigma_{ab} = \Sigma_{(ab)}\,, \qquad \Sigma^a{}_a = 0\,. \tag{2.78}$$

In analogy with Fig. (2.1), W_{ab} represents a spatial variation of $\mathcal{D}_a$ with no modification of its magnitude (i.e. a rotation of the vector $\mathcal{D}_a$), Σ_{ab} represents a change in the spatial anisotropy of $\mathcal{D}_a$, and Δ is related to spherically symmetric spatial variation of $\mathcal{D}_a$ where density is accumulated. Namely Δ is associated with spatial aggregation of matter that we might expect to reflect the existence of high density structures in the Universe. Using the linear identities in Appendix A.2 we can write the scalar part of Eqn. (2.76) as:

$$\ddot{\Delta}+\left(2-6w+3c_s^2\right)H\dot{\Delta}-\left[\left(\frac{1}{2}+4w-3c_s^2-\frac{3}{2}w^2\right)\rho-12(w-c_s^2)\frac{K}{a^2}+c_s^2\,{}^{(3)}\nabla^2\right]\Delta=0\,. \quad (2.79)$$

This is the covariant analogue of the standard equation for the density contrast $\delta = \delta\rho/\rho$, which is obtained from the above in the case we fix the comoving-orthogonal gauge. The last term on the right demonstrates the competing effects of gravitational attraction and pressure support, with collapse occurring when the quantity within the braces is positive. The physical wavelength of the mode is $\lambda = 2\pi a/k$, so that gravitational contraction will take place only on scales larger than the critical Jeans length:

$$\lambda_J = \frac{2\pi a}{k_J} \approx \frac{2\pi c_s}{\sqrt{(1/2+4w-3c_s^2-3/2w^2)\rho+12(w-c_s^2)K/a^2}}\,. \quad (2.80)$$

Gravitational collapse will occur for $\lambda > \lambda_J$ only. For $\lambda < \lambda_J$ pressure gradients are large enough to resist the collapse and lead to oscillations instead.
For a flat ($K = 0$) and matter dominated ($\Lambda = 0$, $w = 0 = c_s^2$) Universe Eqn. (2.79) becomes scale independent:

$$\ddot{\Delta}+2H\dot{\Delta}-\frac{1}{2}\rho\Delta=0\,, \quad (2.81)$$

and a simple solution can be obtained:

$$\Delta(a) = \Delta_1 a^{-3/2} + \Delta_2 a\,. \quad (2.82)$$

Matter density perturbations in the matter era grow as $\Delta \propto a$ on all scales. In this case, what we will later define as the growth rate f is simply:

$$f = \frac{\mathrm{dlog}\Delta_+}{\mathrm{dlog}a} = 1\,. \quad (2.83)$$

In the next chapter we will extend the CGI formalism to the so-called interacting vacuum scenario. We will, in the same manner, derive a second-order differential equation for Δ_+.

Chapter 3

Covariant gauge invariant interacting vacuum perturbations

The recent accelerated expansion of the universe detected by the precision measurements of type Ia supernovae (SNe Ia), anisotropies in the cosmic microwave background radiation (CMB) and observations of large-scale structures (LSS) indicates that about 95% of the total energy of our Universe is in the form of unknown dark fluid components, namely, dark energy and dark matter. The remaining components being in the form of baryonic matter and radiation.
Clustering dark matter with zero pressure (cold dark matter) concentrates in local structures and plays crucial role for the formation of galaxies and clusters of galaxies, while dark energy possesses a negative pressure that drives the recent accelerated expansion. The simplest way to describe dark energy is to associate it with a constant vacuum energy density V characterised by the equation of state parameter $w = -1$, equivalent to a cosmological constant in Einstein gravity $\Lambda = 8\pi G_N V$. In fact, if spacetime retains a non-zero energy density even in the absence of any particles, then this energy density would be undiluted by the cosmological expansion and could drive an accelerated expansion as the density of ordinary matter and radiation become sub-dominant.
As we stated in the Chapter 1, cosmology with Λ and CDM has become the standard model of the Universe, known as ΛCDM. However, there are a few problems even within the model that best describes our Universe.
These problems manifest in both the discrepancy between the predicted and observed values of the cosmological constant, and in the tensions that exist between low-redshift probes of the expansion rate and structure growth and the corresponding values inferred from CMB measurements (for which a cosmological model must be assumed).
The former is known as the cosmological constant problem, namely a 120 orders of magnitude discrepancy between the current vacuum energy density, $\rho_\Lambda \sim 10^{-29}\mathrm{g/cm}^3$, and the theoretical value predicted by quantum field theories $\rho_V \sim 10^{92}\mathrm{g/cm}^3$.
The latter involves the tensions measured in the cosmological parameters H_0 and σ_8.

In recent years, the precision of surveys has improved and these tensions have become more apparent, especially in the value of the present Hubble parameter H_0. The most recent CMB measurement, from the Planck satellite, is $H_0 = 67.4 \pm 0.5$ km $\mathrm{s}^{-1}\mathrm{Mpc}^{-1}$, whereas the most recent local determination, from the Hubble Space Telescope, is $H_0 = 73.45 \pm 1.66$ km $\mathrm{s}^{-1}\mathrm{Mpc}^{-1}$, a discrepancy of 3.7σ. The tension in σ_8, the amplitude of the linear matter power spectrum on a scale of 8 h^{-1}Mpc, is less severe than the H_0 one, but is yet another indicator of problems within ΛCDM.

Motivated by the aforementioned issues of ΛCDM cosmology, alternative models should be considered to explain the current accelerated expansion of the Universe, where now the origin of dark energy is not a cosmological constant anymore.
Among the many alternative approaches present in the literature, we adopt the so-called interacting vacuum scenario, in which we consider an interacting vacuum energy, whose present value is dependent on energy-momentum transfer with existing matter fields [24, 25]. Since the physics underlying the dark sector is still unknown, it could be that vacuum energy and dark matter interact directly and exchange energy. In particular this dark energy class of models, unlike many others, does not introduce any new dynamical degrees of freedom and it reduces to ΛCDM when the interaction vanishes, i.e. when the vacuum energy density $V = \Lambda/8\pi G_N = \text{const}$.

3.1 Interacting vacuum scenario

We define a vacuum energy V to have an energy-momentum tensor proportional to the metric:

$$\check{T}^{\mu}{}_{\nu} = -V\delta^{\mu}{}_{\nu}\,. \tag{3.1}$$

By comparison with the energy-momentum tensor of a perfect fluid:

$$T^{\mu}{}_{\nu} = (\rho + P)u^{\mu}u_{\nu} + P\delta^{\mu}{}_{\nu}\,, \tag{3.2}$$

we identify the vacuum energy density and pressure with $\check{\rho} = -\check{P} = V$. However, since there is no particle flow the 4-velocity of the vacuum remains undefined. Note that the energy density ρ and 4-velocity of a fluid u^{μ} can be identified with the eigenvalue and eigenvector of the energy–momentum tensor:

$$T^{\mu}{}_{\nu}u^{\nu} = -\rho u^{\mu}\,. \tag{3.3}$$

Because the vacuum energy–momentum tensor (3.1) is proportional to the metric tensor, any 4-velocity u^{μ} is an eigenvector, i.e.:

$$\check{T}^{\mu}_{\nu}u^{\nu} = -Vu^{\mu} \quad \forall\, u^{\mu}\,, \tag{3.4}$$

and all observers see the same vacuum energy density V, i.e. the vacuum energy is boost invariant.
We will now consider the possibility of a time and space dependent vacuum energy, i.e. an inhomogeneous vacuum energy. Then, from Eqn. (3.1), we have:

$$\nabla_\mu \check{T}^\mu_\nu = Q_\nu \,, \tag{3.5}$$

where the energy flow is given by:

$$Q_\nu \equiv -\nabla_\nu V \,. \tag{3.6}$$

We see that, if the vacuum energy is covariantly conserved $Q_\nu = 0$, then V must be homogeneous in spacetime, i.e. $\nabla_\nu V = 0$. Thus a non-interacting vacuum energy is equivalent to a cosmological constant in Einstein gravity, i.e. $V = \Lambda/8\pi G_N$.
Conversely, an interacting vacuum $Q_\nu \neq 0$, is inhomogeneous in spacetime, i.e. $\nabla_\nu V \neq 0$, which implies $V = V(t, x^i)$. Although the vacuum does not have a unique 4-velocity, we can use the energy flow Q_ν to define a preferred unit 4-vector in the interacting vacuum case:

$$\check{u}^\mu = \frac{-\nabla^\mu V}{\sqrt{|\nabla_\nu V \nabla^\nu V|}} \,, \tag{3.7}$$

normalized such that $\check{u}_\mu \check{u}^\mu = \pm 1$, for a spacelike or timelike flow. Note however that $\check{u}^\mu$ defines a potential flow, i.e. we have no vorticity $\nabla_{[\mu} \check{u}_{\nu]} = 0$. This is not a negligible fact, as it leads to some physical issues. We will discuss these issues later in Section 3.4.
The conservation of the total energy-momentum (including matter fields and vacuum energy) in GR is:

$$\nabla_\mu T^{\mu\ (\mathrm{tot})}_\nu = \nabla_\mu (T^\mu_\nu + \check{T}^\mu_\nu) = 0 \,, \tag{3.8}$$

and implies that the vacuum transfers energy-momentum to or from the matter fields:

$$\nabla_\mu T^\mu_\nu = -Q_\nu \,. \tag{3.9}$$

If we consider the dark sector, the above implies an energy-momentum transfer between the two dark components, i.e. CDM and vacuum.
We now consider the interaction 4-vector Q^μ and project it in two parts, parallel and orthogonal to the cold dark matter 4-velocity u^μ:

$$Q^\mu = Q u^\mu + f^\mu \,. \tag{3.10}$$

In the frame of observers comoving with cold dark matter (i.e. for a vanishing f^μ), Q represents the energy flow; while in a generic frame, f^μ is the momentum exchange between CDM and vacuum. In particular, f^μ is defined to be orthogonal to u^μ, i.e. $f^\mu u_\mu = 0$.

3.1.1 Spatially homogeneous background

The symmetries of a spatially homogeneous and isotropic FLRW metric require the vacuum to be spatially homogeneous and isotropic, too; hence $V = \bar{V}(t)$. This time dependence, due to a non-zero energy transfer $Q \neq 0$, is what distinguishes the present background from the ΛCDM case. In this context, vacuum and matter are both homo-

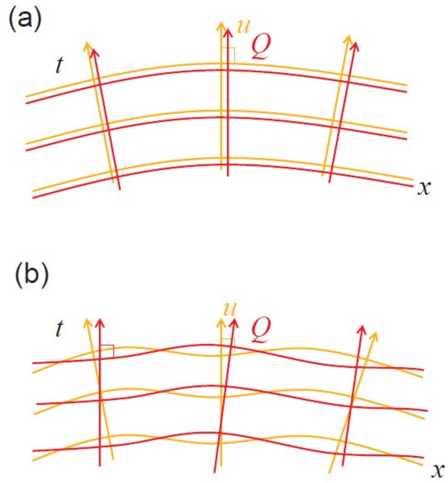

Figure 3.1: **(a)** In an FLRW cosmology the homogeneous spatial hypersurfaces are orthogonal to both the fluid 4-velocity u^μ, and the vacuum energy flow Q^μ. **(b)** In an inhomogeneous cosmology the spatial hypersurfaces orthogonal to the fluid 4-velocity (light orange) and the vacuum energy flow (dark red) do not necessarily coincide.

geneous on spatial hypersurfaces orthogonal to the matter four-velocity $u^\mu = (1, \bar{0})$. We remark the fact that the energy flow $\check{u}^\mu$, and the matter velocity u^μ, necessarily coincide in FLRW cosmology due to the assumption of isotropy.
We have then the following dynamical equations for a flat ($K = 0$) FLRW Universe:

- continuity equation for matter fields:

$$\dot{\rho} + 3H(\rho + P) = -Q\,, \tag{3.11}$$

- continuity equation for vacuum:

$$\dot{V} = Q\,, \tag{3.12}$$

- Friedmann constraint equation:

$$H^2 = \frac{8\pi G_N}{3}(\rho + V)\,, \tag{3.13}$$

- Friedmann Raychaudhuri equation:

$$\dot{H} + H^2 = -\frac{4\pi G_N}{3}(\rho - 2V)\,, \tag{3.14}$$

As one can see from Eqn. (3.14), in both cases: the non-interacting one $V = \Lambda/8\pi G_N$ and the interacting one with $V > 0$, the vacuum energy density always contributes positively to $\ddot{a}$, eventually accelerating the expansion if and when it becomes the dominant component.

3.2 CGI interacting vacuum

This is a rather central section of the thesis; we 'translate' D. Wands papers [24, 25] on vacuum perturbations into the previously seen CGI formalism adopted by Ellis & Bruni in [1, 2].
Let's then apply the CGI formalism of Chapter 2 to the interacting vacuum scenario. In order to do that, we need to define a new additional CGI quantity, the vacuum energy spatial gradient V_a, that adds to our set of exact variables, see Section 2.2.2:

$$V_a \equiv {}^{(3)}\nabla_a V = h_a{}^b \nabla_b V = -h_a{}^b Q_b = -f_a \,, \tag{3.15}$$

where the last equality follows from the splitting of Q_b. This covariant gauge invariant quantity, at first order, corresponds to the comoving vacuum perturbation of [24, 25]. It goes without saying that the above follows from the definition of the $h_a{}^b$ projector, it being orthogonal to the fluid 4-velocity u^a.
If we now consider the $\check{h}_a{}^b$ projector, orthogonal to the energy flow 4-velocity $\check{u}^a$ in Eqn. (3.7), we obtain the vacuum energy spatial gradient $\check{V}_a$, which is defined as:

$$\check{V}_a = \check{h}_a{}^b \nabla_b V = -\check{h}_a{}^b \check{u}_b \, |\nabla_c V \nabla^c V|^{1/2} = 0 \,, \tag{3.16}$$

and vanishes so by construction. The above CGI notation is clearly equivalent to what appears in [24, 25]. However, using the covariant approach, we notice that in the $\check{u}^\mu$ frame of reference the vacuum is now exactly unperturbed. This being valid at all orders not just linearly as highlighted in the aforementioned papers. In Appendix B we show explicitly that both, the covariant and standard approaches are, as expected, totally equivalent at linear order.

3.2.1 Exact CGI interacting dynamical equations

From the conservation of the total energy-momentum tensor in Eqn. (3.8), using the (1+3) splitting, we project $\nabla_\mu T^{\mu\nu}_{(\mathrm{tot})} = 0$ along the time and space directions in the following manner:

$$u_\nu \nabla_\mu T^{\mu\nu}_{(\mathrm{tot})} = 0 \quad \Rightarrow \quad \dot{\rho} + \Theta(\rho + P) = -\dot{V} \,, \tag{3.17}$$

$$h_{a\nu} \nabla_\mu T^{\mu\nu}_{(\mathrm{tot})} = 0 \quad \Rightarrow \quad a_a = -(\rho + P)^{-1}(Y_a - V_a) \,, \tag{3.18}$$

obtaining so energy and momentum conservation, respectively. We also have from Eqns. (3.6) and (3.15) that:

$$\dot{V} = Q \qquad \text{and} \qquad V_a = -f_a \,. \tag{3.19}$$

To obtain the propagation equation for X_a in an interacting vacuum scenario we proceed directly from Eqn. (3.17). Performing the spatial gradient of the energy conservation equation we so obtain:

$$h_a{}^b \nabla_b\big(\dot{\rho} + \dot{V}\big) + (\rho + P)Z_a + \Theta(X_a + Y_a) = 0 \,. \tag{3.20}$$

Now we can write:

$$h_a{}^b \nabla_b\big(\dot{\rho} + \dot{V}\big) = h_a{}^b \nabla_b\big(\nabla_c(\rho + V)u^c\big) \tag{3.21}$$

$$= h_a{}^b \nabla_c \nabla_b(\rho + V)u^c + h_a{}^b \nabla_c(\rho + V)(\sigma^c{}_b + \omega^c{}_b + \frac{1}{3}\Theta h^c{}_b) \tag{3.22}$$

where we have used for a generic scalar $\nabla_b \nabla_c f = \nabla_c \nabla_b f$. It is useful to express the first term in the RHS of the previous equality as:

$$h_a{}^b \nabla_c \nabla_b(\rho + V)u^c = h_a{}^d \nabla_c\big[h_d{}^b \nabla_b(\rho + V)\big]u^c - h_a{}^d \nabla_c\big(h_d{}^b\big)u^c \nabla_b(\rho + V) \tag{3.23}$$

$$= h_a{}^b (X_b + V_b)^{\cdot} - h_a{}^d\big(a_d u^b + a^b u_d\big)\nabla_b(\rho + V) \tag{3.24}$$

$$= h_a{}^b (X_b + V_b)^{\cdot} + (Y_a - V_a)(\rho + P)^{-1}(\dot{\rho} + \dot{V}) \tag{3.25}$$

$$= h_a{}^b (X_b + V_b)^{\cdot} - \Theta(Y_a - V_a) \,, \tag{3.26}$$

where we have used $\nabla_a h_{bc} = 0$ for the spatial projector. From the above relation we can now write the exact covariant expression:

$$(X_a + V_a)^{\cdot} + (X_b + V_b)(\sigma^b{}_a + \omega^b{}_a) + \frac{4}{3}\Theta(X_a + V_a) + (\rho + P)Z_a = 0 \,. \tag{3.27}$$

This is the CGI equivalent of $\delta\rho$ in [24, 25], see B.2. Following Chapter 2, let us explicitly compute the expressions for the three relevant CGI quantities; now in an interacting vacuum scenario:

$$\mathcal{D}_a = \frac{aX_a}{\rho} \,, \qquad \mathcal{Z}_a = aZ_a \,, \qquad \mathcal{C}_a = a^3 \, {}^{(3)}\nabla_a \, {}^{(3)}R \,, \tag{3.28}$$

where, unlike before, V in ${}^{(3)}R$ is not a constant term:

$${}^{(3)}R = 2\bigg(-\frac{1}{3}\Theta^2 + \rho + V\bigg) + 2(\sigma^2 - \omega^2) \,. \tag{3.29}$$

We begin with the derivation of the evolution equation for $\mathcal{Z}_a$. Computing the spatial variation of the Raychaudhuri equation (2.27) we get:

$$h_a{}^b \nabla_b\big(u^c \nabla_c \Theta\big) + \frac{2}{3}\Theta Z_a + h_a{}^b\Big(2\nabla_b(\sigma^2 - \omega^2) - A_b\Big) + \frac{1}{2}(X_a + 3Y_a - 2V_a) = 0 \,. \tag{3.30}$$

The first term of this equation can be rephrased as:

$$h_a{}^b\nabla_b\big(u^c\nabla_c\Theta\big) = h_a{}^d h_d{}^b u^c(\nabla_b\nabla_c\Theta) + h_a{}^b u^c\nabla_c\Theta\nabla_b \tag{3.31}$$

$$= h_a{}^d\, u^c\nabla_c(h_d{}^b\nabla_b\Theta) - h_a{}^d(\nabla_b\Theta)u^c\nabla_c h_d{}^b + Z_b(\sigma_a^b+\omega_a^b) + \frac{1}{3}\Theta Z_a \tag{3.32}$$

$$= h_a{}^b(Z_b)^{\cdot} - \dot{\Theta}a_a + Z_b(\sigma_a^b+\omega_a^b) + \frac{1}{3}\Theta Z_a\,, \tag{3.33}$$

where we have used the same method as before. If we now substitute the last expression in Eqn. (3.30), we obtain:

$$h_a{}^b(Z_b)^{\cdot} + a_a\left(\frac{1}{3}\Theta^2 + 2\big(\sigma^2-\omega^2\big) - A + \frac{1}{2}(\rho+3P-2V)\right) + Z_b(\sigma^b{}_a+\omega^b{}_a) + \Theta Z_a +$$
$$+ h_a{}^b\Big(2\nabla_b(\sigma^2-\omega^2) - A_b\Big) + \frac{1}{2}(X_a+3Y_a-2V_a) = 0\,. \tag{3.34}$$

Expanding the time derivative of $\mathcal{Z}_a$:

$$h_a{}^b(\mathcal{Z}_a)^{\cdot} = h_a{}^b\Big(\dot{a}Z_b + a\big(Z_b\big)^{\cdot}\Big)\,, \tag{3.35}$$

and substituting in the above we arrive at:

$$(\mathcal{Z}_a)^{\cdot} = -\frac{2}{3}\Theta\mathcal{Z}_a - a a_a\left(\frac{1}{3}\Theta^2 + 2\big(\sigma^2-\omega^2\big) - A + \frac{1}{2}(\rho+3P-2V)\right) - \mathcal{Z}_b(\sigma^b{}_a+\omega^b{}_a) +$$
$$- a\Big(2\,{}^{(3)}\nabla_a(\sigma^2-\omega^2) - A_a\Big) - \frac{1}{2}(\rho\mathcal{D}_a - 3aY_a + 2aV_a)\,, \tag{3.36}$$

from which we finally obtain:

$$(\mathcal{Z}_a)^{\cdot} = -\frac{2}{3}\Theta\mathcal{Z}_a - \frac{1}{2}(\rho\mathcal{D}_a + aV_a) - \mathcal{Z}_b(\sigma_a^b+\omega_a^b) +$$
$$+ a\Big(a_a\mathcal{R}^* + A_a - 2\,{}^{(3)}\nabla_a\big(\sigma^2-\omega^2\big)\Big)\,. \tag{3.37}$$

The difference between the quantity $\mathcal{R}$ in Eqn. (2.63) and $\mathcal{R}^*$ is the replacement of the constant Λ with the varying vacuum energy V.[†]

Let us now compute the time evolution of the comoving fractional density gradient $\mathcal{D}_a$. In order to do so, we exploit its definition as in the following:

$$\mathcal{D}_a = \frac{aX_a}{\rho} \qquad \Rightarrow \qquad (\mathcal{D}_a)^{\cdot} = \frac{a}{\rho}(X_a)^{\cdot} + \frac{\dot{a}}{\rho}X_a - \frac{a}{\rho^2}\dot{\rho}X_a \tag{3.38}$$

$$= \frac{a}{\rho}(X_a)^{\cdot} + \left(\frac{4}{3}+w\right)\Theta\mathcal{D}_a + \frac{Q}{\rho}\mathcal{D}_a\,, \tag{3.39}$$

[†]Notice that if the vacuum energy V is constant, $V = \Lambda/8\pi G_N$, then the interacting vacuum equations reduce, as expected, to the expressions of Chapter 2.

where we now have, w.r.t. to the ΛCDM case, an additional term Q coming from the background interaction. Projecting the above with ${h_c}^a$ and inserting Eqn. (3.27) in place of the time derivative of X_a we arrive at:

$$(\mathcal{D}_a)^{\cdot} = -(1+w)\mathcal{Z}_a + w\Theta\mathcal{D}_a + \frac{\dot{V}}{\rho}\mathcal{D}_a - \frac{4}{3}\frac{a}{\rho}\Theta V_a - \frac{a}{\rho}(V_a)^{\cdot} \\ - \mathcal{D}_b({\sigma^b}_a + {\omega^b}_a) - \frac{a}{\rho}V_b({\sigma^b}_a + {\omega^b}_a)\,. \quad (3.40)$$

Finally one can compute the time evolution of $\mathcal{C}_a$ from Eqn. (3.28):†

$$\mathcal{C}_a = -\frac{4}{3}\Theta a^2\mathcal{Z}_a + 2a^2\rho\mathcal{D}_a + 2a^3 V_a + 2a^3\, {}^{(3)}\nabla_a(\sigma^2 - \omega^2)\,. \quad (3.41)$$

Notice that all the above equations are written in terms of CGI variables, hence they are exact. In the following sections, we are going to linearise the above equations and consider two cases of physical interest: the inhomogeneous vacuum where $f^\mu \neq 0$, and the homogeneous vacuum where $f^\mu = 0$.

3.3 Inhomogeneous vacuum model

In this general case we keep vacuum inhomogeneities $V_a = -f_a \neq 0$, i.e. we have non-zero momentum transfer between vacuum and CDM; as we will later discuss in Section 3.4, neglecting it may lead to false interpretations of cosmological observations.
At the very base of the interacting vacuum scenario is the aim to give a description of the possible interaction between the two dark components of our Universe, namely cold dark matter (CDM) and vacuum. In this context, we are going to expand the above equations at first-order following the same procedure used in Chapter 2. In particular, since we are dealing with pressure-less CDM in a flat FLRW background the following simplifications are in order:

$$P = 0\,, \qquad K = 0\,. \quad (3.42)$$

We now go on with the linearisation procedure, from Section 3.2.1 we then have:

$$(\mathcal{D}_a)^{\cdot} = \frac{Q}{\rho}\mathcal{D}_a - \mathcal{Z}_a - \frac{4}{3}\frac{a}{\rho}\Theta V_a - \frac{a}{\rho}(V_a)^{\cdot} \quad (3.43)$$

$$(\mathcal{Z}_a)^{\cdot} = -\frac{2}{3}\Theta\mathcal{Z}_a - \frac{1}{2}\rho\mathcal{D}_a - \frac{a}{2}V_a + \frac{a}{\rho}\,{}^{(3)}\nabla_a\,{}^{(3)}\nabla^2 V\,, \quad (3.44)$$

†The expression for $(\mathcal{C}_a)^{\cdot}$ is not so enlightening; hence we will not present its exact form here. However, in the linear case and under some particular conditions, it does acquire some physical insight, as we will later see.

where the spatial laplacian term in $\mathcal{Z}_a$ comes from the following linearisation in the $P = 0$ case:

$$a_a = \frac{V_a}{\rho} \quad \Rightarrow \quad A = \frac{\nabla_a V^a}{\rho} \quad \Rightarrow \quad A_a = \frac{1}{\rho}\, {}^{(3)}\nabla_a \, {}^{(3)}\nabla^2 V \,. \tag{3.45}$$

Taking the expression for A_a and comparing it with the one in Eqn. (2.74), we immediately realise that the two equations have the same form with opposite signs. The presence of vacuum causes a positive acceleration whose action, unlike pressure, is to counteract gravity.
The time evolution of $\mathcal{C}_a$ can now be explicitly derived starting from Eqn. (3.41). After some calculations we end up with:

$$(\mathcal{C}_a)^{\cdot} = -\frac{4}{3}\frac{a^3}{\rho}\Theta \, {}^{(3)}\nabla_a \, {}^{(3)}\nabla^2 V \,. \tag{3.46}$$

Notice that the above has the same form of the ΛCDM Eqn. (2.73), in the case we there set $K = 0$.[†] Notice further that $\mathcal{C}_a$ is conserved on large scales. Using the linearized version of Eqn. (3.41) we can write $\mathcal{Z}_a$ as:

$$\mathcal{Z}_a = -\frac{3}{4}\Theta^{-1}a^{-2}\mathcal{C}_a + \frac{3}{2}\Theta^{-1}\rho\mathcal{D}_a + \frac{3}{2}\Theta^{-1}aV_a \,, \tag{3.47}$$

from which we finally obtain the time evolution of $\mathcal{D}_a$:

$$\begin{aligned}
(\mathcal{D}_a)^{\cdot} &= \left(\frac{Q}{\rho} - \frac{3}{2}\Theta^{-1}\rho\right)\mathcal{D}_a + \frac{3}{4}\Theta^{-1}a^{-2}\mathcal{C}_a - \left(\frac{3}{2}\Theta^{-1}a + \frac{4}{3}\frac{a}{\rho}\Theta\right)V_a - \frac{a}{\rho}(V_a)^{\cdot} \,, \\
&= \left(\frac{Q}{\rho} - \frac{3}{2}\Theta^{-1}\rho\right)\mathcal{D}_a + \frac{3}{4}\Theta^{-1}a^{-2}\mathcal{C}_a + \\
&\qquad - \left(\frac{3}{2}\Theta^{-1}a + \frac{4}{3}\frac{a}{\rho}\Theta + a\frac{Q}{\rho^2} - \frac{1}{3}\frac{a}{\rho}\Theta\right)V_a - \frac{a}{\rho}Q_a \,,
\end{aligned} \tag{3.48}$$

where in the above we have used for the time derivative of V_a:

$$\begin{aligned}
(V_a)^{\cdot} &= a_a\dot{V} + Q_a - \frac{1}{3}\Theta V_a \\
&= \frac{V_a}{\rho}Q + Q_a - \frac{1}{3}\Theta V_a \,.
\end{aligned} \tag{3.49}$$

We have defined the energy transfer spatial gradient as $Q_a \equiv {}^{(3)}\nabla_a Q = {}^{(3)}\nabla_a \dot{V}$. What we have here presented is the first order expansion of the CGI interacting vacuum dynamics. This corresponds to a CGI 'translation' of the work done by Borges et al. in [32].

[†]This again suggests that the action of vacuum on the curvature is the same as a negative pressure.

The main result of his paper is the derivation of a second order equation for the dark matter density contrast δ, in the comoving gauge. Such a result is obtained assuming a particular ansatz for the vacuum interaction $\delta V \sim V_a$, in the case of a decomposed generalized Chaplygin gas. In our treatment we have remained completely generic and we have assumed no explicit form for the momentum transfer V_a. However, an ansatz for the latter quantity is obviously required if one wants to obtain a solution for $\mathcal{D}_a$.
In the following we are going to discuss the geodesic CDM model, where we have no vacuum perturbations, i.e. $V_a = -f_a = 0$.

3.4 Homogeneous vacuum model

In this case, like in the above, we consider pressure-less cold dark matter (CDM) interacting with vacuum. However, this time we describe the interaction in a simplified manner; namely we consider a pure energy exchange in the CDM frame wherein $f^a = 0$ and so $Q^a = Qu^a$.†
Notice that the momentum transfer a.k.a. 4-force f^a is related to the 4-acceleration $a^a = u^b \nabla_b u^a$ by:

$$f^a = a^a \rho \, . \tag{3.50}$$

Since we set $f^a = 0$, it follows that $a^a = 0$ by construction (since also $P = 0$), meaning there is no acceleration in a frame comoving with CDM due to both, pressure gradients and the interaction with vacuum, as can be seen from Eqn. (3.18). Hence CDM remains geodesic. We may call this interacting scenario the geodesic CDM scenario, see [28, 27]. A remarkable fact, concerning geodesic CDM models, e.g. ΛCDM, is that they have, for matter perturbations, a vanishing speed of sound. This fact implies that the dynamical equations are scale independent, see Eqn. (2.79).

Irrotational cold dark matter

This case, as we have previously mentioned, has a very important consequence following from the assumption of pure energy exchange, $Q_a = Qu_a$ or equivalently $\check{u}_a = u_a$. In fact, from Eqn. (3.7) the cold dark matter 4-velocity u^a defines a potential flow $u_a \propto \nabla_a V$. This means that CDM is described by an irrotational fluid. We expect this to be a good description of cold dark matter at early times and on large scales where the initial density field is set by primordial scalar perturbations.
This is sufficient for a linear perturbative treatment where only scalar perturbations are relevant for the initial growth of structures, but at late times we would expected the non-linear growth of structures to develop vorticity and indeed to develop rotationally supported dark matter haloes. Thus we expect the geodesic approximation to break

†Remember that the energy-momentum tensor of a pure matter fluid is characterized by $P = 0$, thus $T^a{}_b = \rho u^a u_b$.

down below some length scale. Otherwise truly irrotational CDM would have distinctive observational consequences. In fact, if $f^a = 0$, i.e. for vanishing vorticity, CDM collapses too easily leading to problems in the process of structure formation [30].

In this context, we are going to expand the above equations at first-order following the same procedure outlined in Chapter 2. In particular, since we are now dealing with geodesic CDM we have the additional simplification w.r.t. the previous case:

$$P = 0\,, \qquad K = 0\,, \qquad f^a = 0\,. \tag{3.51}$$

From the inhomogeneous case we now obtain:

$$(\mathcal{D}_a)^{\cdot} = \frac{Q}{\rho}\mathcal{D}_a - \mathcal{Z}_a\,, \tag{3.52}$$

$$(\mathcal{Z}_a)^{\cdot} = -\frac{2}{3}\Theta\mathcal{Z}_a - \frac{1}{2}\rho\mathcal{D}_a\,, \tag{3.53}$$

$$(\mathcal{C}_a)^{\cdot} = 0\,. \tag{3.54}$$

The scalar part of Eqn. (3.52) is the CGI analogue of the equation for $\dot{\delta}$ in [27]; where, in the mentioned paper the author has chosen a comoving and synchronous gauge, see Appendix B.2. Taking also the linearised geodesic CDM case of Eqn. (3.27) we notice two things:

- For $f^a = 0$, i.e. if we consider a frame comoving with CDM, then vacuum interactions do no explicitly appear. This is immediate in the CGI approach, in accordance with [24, 25].

- To describe matter perturbations one usually considers the density contrast $\delta = \delta\rho/\rho$. Using this quantity, the interaction is then re-introduced via the evolution of the background ρ, which, now, is that of an interacting vacuum cosmology [27].

The last point cannot be stressed enough, as it shows that the interaction has an effect also on the perturbations and not just on the background. This, in turn, has important implications for the cosmological growth of structures. We will explicitly discuss this in the upcoming Chapter 4.
Going back to our set of Eqns. (3.52) - (3.54), using the linearized version of Eqn. (3.41) we can write $\mathcal{Z}_a$ as:

$$\mathcal{Z}_a = -\frac{3}{4}\Theta^{-1}a^{-2}\mathcal{C}_a + \frac{3}{2}\Theta^{-1}\rho\mathcal{D}_a\,, \tag{3.55}$$

which can be used to write the time derivative of $\mathcal{D}_a$ in the following manner:

$$(\mathcal{D}_a)^{\cdot} = \left(\frac{Q}{\rho} - \frac{3\rho}{2\Theta}\right)\mathcal{D}_a + \frac{3}{4a^2\Theta}\mathcal{C}_a\,. \tag{3.56}$$

Considering the corresponding scalar quantities $\mathcal{D}_a \to \Delta$, $\mathcal{Z}_a \to \mathcal{Z}$ and $\mathcal{C}_a \to \mathcal{C}$ as done in Chapter 2 we obtain:

$$\dot{\Delta} = \left(\frac{Q}{\rho} - \frac{3\rho}{2\Theta}\right)\Delta + \frac{3}{4a^2\Theta}\mathcal{C}\,. \tag{3.57}$$

We now adopt Borges & Wands notation [31] and introduce the dimensionless interaction parameter g defined as:

$$g \equiv -\frac{aQ}{\mathcal{H}\rho}\,, \tag{3.58}$$

in this way we can write the above as:

$$\mathcal{H}\Delta' + \left(\frac{1}{2}a^2\rho + g\mathcal{H}^2\right)\Delta - \frac{1}{4}\mathcal{C} = 0\,, \tag{3.59}$$

from which, performing a time derivative, we finally obtain:

$$\Delta'' + (1+g)\mathcal{H}\Delta' + \left((g\mathcal{H})' + g\mathcal{H}^2 - \frac{1}{2}a^2\rho\right)\Delta = 0\,. \tag{3.60}$$

It is standard practice to write the general solution of the above equation as a linear combination of a growing mode and a decaying mode:

$$\Delta(\eta, \mathbf{x}) = C_1(\mathbf{x})\Delta_+(\eta) + C_2(\mathbf{x})\Delta_-(\eta)\,. \tag{3.61}$$

We can relate these solutions with $\mathcal{C}$ through the first integral in Eqn. (3.59). The decaying mode Δ_- is the solution of the homogeneous part while the growing mode Δ_+ is the solution corresponding to the particular part with source term $\mathcal{C} \neq 0$. Hereafter, we discard the decaying mode and in what follows we are left with the growing mode driven by the nonzero CGI scalar curvature $\mathcal{C}$:

$$\Delta(\eta, \mathbf{x}) = C_1(\mathbf{x})\Delta_+(\eta)\,. \tag{3.62}$$

Fixing the initial conditions at a given initial moment from (3.59) we have:

$$C_1(\mathbf{x})\left[\frac{\mathrm{d}\Delta_+}{\mathrm{d}a}4a\mathcal{H}^2 + 4a\mathcal{H}^2\Delta_+\left(\frac{3}{2}a^{-1}\Omega_m + g\,a^{-1}\right)\right]_I = \mathcal{C}\,, \tag{3.63}$$

we so obtain:

$$C_1(\mathbf{x}) = \frac{\mathcal{C}}{4\mathcal{H}_I^2\Delta_I}\left(f_I + \frac{3}{2}\Omega_{m,I} + g_I\right)^{-1}, \tag{3.64}$$

where we have introduced the linear growth rate defined as:

$$f = \frac{\Delta'_+}{\mathcal{H}\Delta_+}\,. \tag{3.65}$$

From the above we can now write $\Delta(\eta, \mathbf{x})$ in (3.61) as:

$$\Delta(\eta, \mathbf{x}) = \frac{\mathcal{C}}{4\mathcal{H}^2}\left(f + \frac{3}{2}\Omega_m + g\right)^{-1} . \tag{3.66}$$

We remind the reader that up to this point we still haven't fixed any gauge. Let's now proceed with the choice of a synchronous gauge comoving with the fluid 4-velocity of CDM u^a, fixed by setting $\phi = v = B = 0$. Let us now expand at, first-order, the exact matter continuity Eqn. (3.17):†

$$\delta\rho' + a\,\delta\Theta\rho + 3\mathcal{H}\delta\rho = 0\,, \tag{3.67}$$

using the following identity,

$$\left[\frac{\delta\rho}{\rho}\right]' = \frac{\delta\rho'}{\rho} - \frac{\rho'}{\rho}\delta\,, \tag{3.68}$$

we can then write:

$$\delta' + g\mathcal{H}\delta + a\delta\Theta = 0\,. \tag{3.69}$$

Substituting the growing mode solution (3.62) and (3.65) in the continuity equation (3.69) we obtain the following expression for the expansion perturbation $\delta\Theta$:

$$\delta\Theta = -(f+g)H\delta\,. \tag{3.70}$$

In Section 4.5.2, we will use the latter expression to study redshift-space distortions.
In order to find explicit solutions for the growing mode $\Delta_+(a)$ from Eqn. (3.60), we need a physical model for vacuum interactions, i.e. a given form of the energy transfer Q. This is precisely what we are going to discuss in the next and final chapter.

†Note that we are here fixing the gauge just for coherence with the literature in [31, 27]. One could have equivalently kept our covariant notation describing peculiar velocities by means of the scalar CGI variable $\mathcal{Z}$.

Chapter 4

Interacting vacuum models

Before going into the details of what has been obtained in this final chapter, let's have a look back at what we have done so far in this thesis. In Chapter 1, we have presented the standard theory of cosmological perturbations, following Bardeen approach, for the ΛCDM model. In Chapter 2 we have used the Ellis & Bruni CGI one based on the covariant approach to cosmology. In Chapter 3 we have presented a possible alternative to the ΛCDM model. In the interacting vacuum scenario we allow an interaction between the two dark components, namely CDM and vacuum energy. As done for Chapter 2, we have then 'translated' the standard cosmological perturbation theory into the CGI one, showing the equivalence of the two approaches in Appendix B.
We now apply all the previously discussed theoretical background in the present chapter. We consider as our starting point the second order dynamical differential equation for $\Delta_+(a)$ in Chapter 3 and solve it numerically for three different geodesic CDM interacting vacuum models. The first one is the linear vacuum model, where the energy transfer Q is proportional to the vacuum energy V. The second and the third models come from the decomposition into interacting CDM and vacuum of preexisting unified dark matter models: the generalized Chaplygin gas and the Shan-Chen dark energy models, respectively. The aim being to see how the growth rate $f(a)$, describing how much cosmological structures grow, is affected by these different types of interactions. We further confront the three interacting vacuum models with the ΛCDM case, using relevant values for the parameters Ω_{m0}, q, α and q_* from [27, 29, 37]. In the final Section 4.5.2 we, additionally, present an analytical approximation for the growth rate $f(a)$ proposed by Borges & Wands [31].

4.1 Growth of structures in an interacting vacuum scenario

In this section we recover and elaborate the linear geodesic CDM results obtained at the end of Chapter 3; we start from Eqn. (3.60):

$$\Delta'' + (1+g)\mathcal{H}\Delta' + \Big((g\mathcal{H})' + g\mathcal{H}^2 - \frac{1}{2}a^2\rho\Big)\Delta = 0\,. \tag{4.1}$$

One can rephrase the latter equation as a function of the scale factor $a(t)$ by means of the following relation between η and a derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}\eta} \equiv a\mathcal{H}\frac{\mathrm{d}}{\mathrm{d}a}\,. \tag{4.2}$$

Inserting the Friedmann Eqns. (3.13) and (3.14) in the above, we get:

$$\frac{\mathrm{d}^2}{\mathrm{d}a^2}\Delta + a^{-1}\Big(3 + g - \frac{3}{2}\Omega_m\Big)\frac{\mathrm{d}}{\mathrm{d}a}\Delta + \Big((a\mathcal{H})^{-1}\frac{\mathrm{d}}{\mathrm{d}a}(g\mathcal{H}) + a^{-2}(g - \frac{3}{2}\Omega_m)\Big)\Delta = 0\,, \tag{4.3}$$

where we have used $\Omega_m(a) \equiv a^2\rho/3\mathcal{H}^2$. The expressions for g, Ω_m and $\mathcal{H}$ can be derived using the continuity and Raychaudhuri equations; obviously they will depend on the chosen form for the energy transfer Q.
Let's derive these background expressions first for the non interacting case. In such a framework, $Q = 0 \to g = 0$, vacuum energy neither decays nor grows, in fact since $\dot{V} = Q = 0$, it remains constant both in time and space, i.e. $V = \Lambda/8\pi G_N$. This case is equivalent to a cosmological constant; hence we go back to ΛCDM. In this context, from the Raychaudhuri background equation:

$$\mathcal{H}' = \frac{1}{2}(2 - 3\Omega_m)\mathcal{H}^2\,, \tag{4.4}$$

we get

$$\mathcal{H}(a) = a\mathcal{H}_0\left[1 - \Omega_{m0} + \frac{\Omega_{m0}}{a^3}\right]^{1/2}. \tag{4.5}$$

While from the continuity equation for the density parameter Ω_m:

$$\Omega_m' = [-3(1-\Omega_m) + g]\mathcal{H}\Omega_m\,, \tag{4.6}$$

we obtain

$$\Omega_m(a) = \frac{\Omega_{m0}}{\Omega_{m0} + (1-\Omega_{m0})a^3}\,. \tag{4.7}$$

The subscript '0' refers to the present value ($a_0 = 1$) and the density parameters obey the relation $\Omega_m + \Omega_V = 1$.

For early times, $a \ll 1$, we have a matter-dominated epoch, i.e. a Einstein de Sitter cosmology with $\Omega_m \approx 1$. At late times, $a \gg 1$, a de Sitter vacuum dominated epoch is obtained, in which case $\Omega_V \approx 1$.
In the following we consider three geodesic cold dark matter models where $g \neq 0$, i.e. we take different parameterisations for the energy transfer Q and we impose the geodesic condition, i.e. we neglect the momentum exchange by setting $f^a = 0$, as in Section (3.4). In particular, the models we are going to discuss are: linear vacuum interacting model, generalised Chaplygin gas interacting model and Shan Chen interacting model.

4.2 Linear vacuum interacting model

4.2.1 Linear vacuum interacting background

This geodesic CDM model is characterized by a simple parameterisation of the energy transfer Q, i.e. $Q = qHV$ as in [26, 27, 31]. The constant q is a dimensionless parameter that describes the strength of the CDM-vacuum energy coupling. Such a form for Q allows us to obtain simple analytical solutions for both V and ρ.
In fact, from the two continuity Eqns. (3.11) and (3.12) we obtain for the two background energy densities ρ and V:

$$\rho = \frac{3\mathcal{H}_0^2}{3+q}\Big((q+3\Omega_{m0})a^{-3} - q\Omega_{V0}a^q\Big)\,, \tag{4.8}$$

$$V = 3\mathcal{H}_0^2\Omega_{V0}a^q\,. \tag{4.9}$$

In particular, from the latter expression of the vacuum energy density V it is immediate to realise that:

- for $q > 0$ the vacuum grows, i.e. we have an energy transfer from matter to vacuum.
- Conversely, for $q < 0$ the vacuum decays, hence we have an energy transfer from vacuum to matter.

Eqns. (4.8) and (4.9) imply the following for the Hubble parameter $\mathcal{H}(a)$, the matter density parameter $\Omega_m(a)$ and the dimensionless interaction parameter $g(a)$:

$$\mathcal{H}(a) = \mathcal{H}_0\sqrt{\frac{3(1-\Omega_{m0})a^{3+q} + 3\Omega_{m0} + q}{(3+q)a}}\,, \tag{4.10}$$

$$\Omega_m(a) = \frac{3\Omega_{m0} + q - q(1-\Omega_{m0})a^{3+q}}{3\Omega_{m0} + q + 3(1-\Omega_{m0})a^{3+q}}\,, \tag{4.11}$$

$$g(a) = -\left(\frac{1-\Omega_m}{\Omega_m}\right)q\,. \tag{4.12}$$

The standard matter dominated era, recovered at early times, i.e. $a \ll 1$ implies $\Omega_m \approx 1$ and $g \approx 0$.[†] Again, the ΛCDM model corresponds to the $q = 0$ case.
From these background solutions we can then plot the scale factor $a(t)$ as a function of the cosmic time t for different values of the coupling strength q. From Fig. (4.1) we immediately understand what is the effect of this interaction on the background:

[†]Notice that $\Omega_m(a)$ becomes negative for values $q > 0$ at large times ($a \gg 1$).

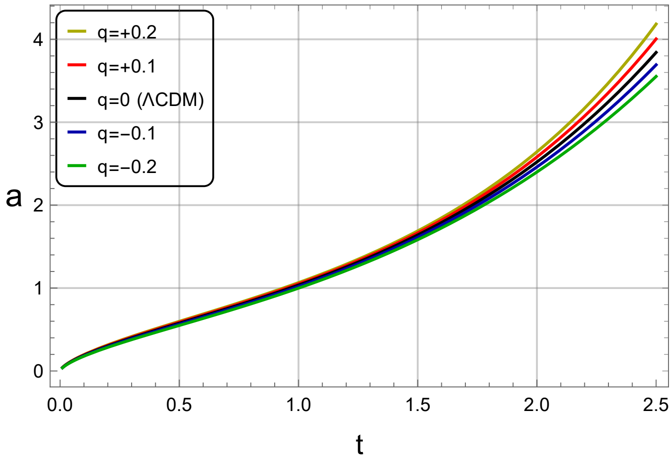

Figure 4.1: Scale factor $a(t)$ as a function of the adimensionless cosmic time, $H_0 = 1$, for a two component cosmology with a linear vacuum-CDM interaction. Five different q values are shown, for all of them $\Omega_{m0} = 0.3$. Notice that, for $a \ll 1$ (matter domination) $a \sim t^{2/3}$, while for $a \gg 1$ (vacuum domination) $a \sim \mathrm{e}^t$.

- for $q > 0$ we have an enhancement of $a(t)$ w.r.t. ΛCDM.
- The opposite for $q < 0$.

In other words the vacuum-CDM interaction contributes, depending on the sign of q, to increase or diminish the accelerated expansion of the Universe. In particular, for a growing/decaying V, we obtain a Universe which expands more/less rapidly compared to the non-interacting case (ΛCDM).

4.2.2 Numerical solutions for the linear vacuum model

The above expressions for $\mathcal{H}(a)$, $\Omega_m(a)$ and $g(a)$ are all we need in order to solve the second order differential equation (4.3).
As one might have guessed, it is not possible to obtain an analytical result for the growing mode $\Delta_+(a)$. Hence, we have used *Mathematica* to find a numerical solution. In order to obtain the below plots, the initial conditions have been set at very early times: $a_I \sim 10^{-3}$, i.e. during a standard matter dominated era, where $\Omega_{m\,I} = 1$, $f_I = 1$ and $g_I = 0$:

$$\Delta_{+I}(a_I = 10^{-3}) = 10^{-3}\,, \tag{4.13}$$

$$\Delta'_{+I}(a_I = 10^{-3}) = 1\,. \tag{4.14}$$

The initial amplitude Δ_{+I} and Δ'_{+I} come from the behaviour of the growing mode and the growth rate in an Einstein de Sitter cosmology, see Section 2.2.5. The effect these initial conditions have on structures' growth is evident: at early times ($a \ll 1$) we have

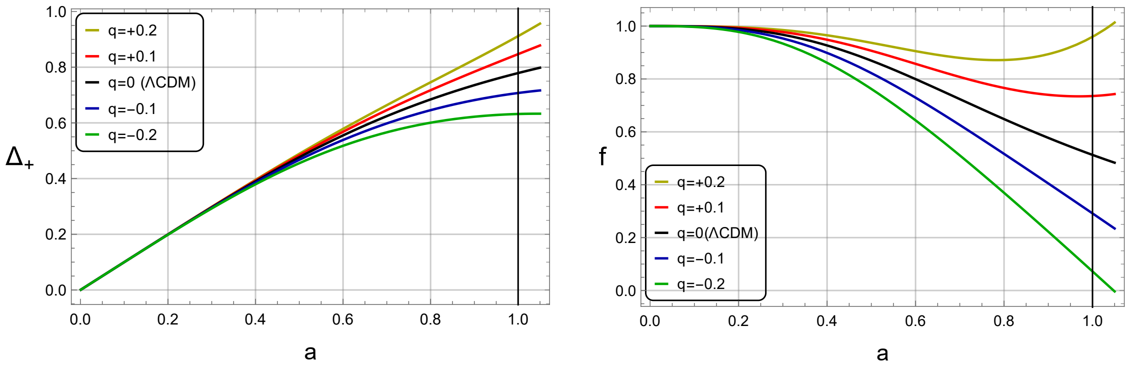

Figure 4.2: **Left Panel:** First order growing mode $\Delta_+(a)$ as a function of scale factor a for the linear vacuum model. We have five different curves corresponding to five different values of the coupling strength: $q = +0.2$ (yellow curve, top), $q = +0.1$ (red curve), $q = 0$ (Λ black curve), $q = -0.1$ (blue curve) and $q = -0.2$ (green curve, bottom). For all of them $\Omega_{m0} = 0.3$. **Right Panel:** First order growth rate $f(a)$, as a function of scale factor a for the aforementioned coupling strength values of the linear vacuum model.

a matter dominated behaviour, i.e. $\Delta_+(a) \sim a$; only at later times ($a \simeq 1$) we start having a significant departure from the Einstein de Sitter case. This is precisely what we expect, given the very recent domination of the dark energy component. The solution for both, the first order growing mode $\Delta_+(a)$ and the first order growth rate $f(a)$ have been plotted in Fig. (4.2). In this plot we have considered five different values for the dimensionless coupling q.[†]

In the $q < 0$ case, vacuum decays into CDM. In this context, the first order growing mode is suppressed w.r.t. the ΛCDM case, for a given value of the nowadays Ω_{m0}, see Fig. (4.2). This is because CDM is lower at early times when we fix its density today. In other words, since V decays into CDM, we need less matter in the past in order to reach the same value of Ω_{m0} today. As a consequence, since the growth of structures depends mainly on $\Omega_m(a)$ [41], this leads to a suppression in the growing mode $\Delta_+(a)$ w.r.t. ΛCDM case.

Conversely, when we have an energy flow from cold dark matter to the vacuum, i.e. when $q > 0$, cold dark matter is annihilated. In this case there is enhancement in both the first order growing mode $\Delta_+(a)$ and the first order growth rate $f(a)$. The reason for this following from the same principle that has led to an enhancement of $\Delta_+(a)$ and $f(a)$ in the $q < 0$ case. We remark that, this fact that may appear counterintuitive, if one looks at the BG evolution in Fig. (4.1), is instead simply a consequence of the initial conditions, set during matter domination.

[†]Notice that in the $q = 0$ case we go back to the ΛCDM model.

Constraints on the linear vacuum growth rate

What we do now, is to use the constraints imposed on the parameters Ω_{m0} and q, from the *Cfix* case in Martinelli et al. [27].
In particular, we want to see how both, the first order growing mode $\Delta_+(a)$ and the first order growth rate $f(a)$ behave w.r.t. the ΛCDM model. We have taken the two parameters in the below one σ range, values obtained in [27] using *Planck+Low-z* data.†
We have then plotted $\Delta_+(a)$ and $f(a)$ for both the interacting $Q = qHV$, and the non

q	Ω_{m0}
0.4 ± 0.10	0.2674 ± 0.0250

interacting $Q = 0$ (i.e. ΛCDM) cases, see Fig. (4.3). Considering the largest possible intervals for both the quantities $\Delta_+(a)$ and $f(a)$, we see from Fig. (4.3) that the ΛCDM values are well inside the linear vacuum model range. From the below numerical results

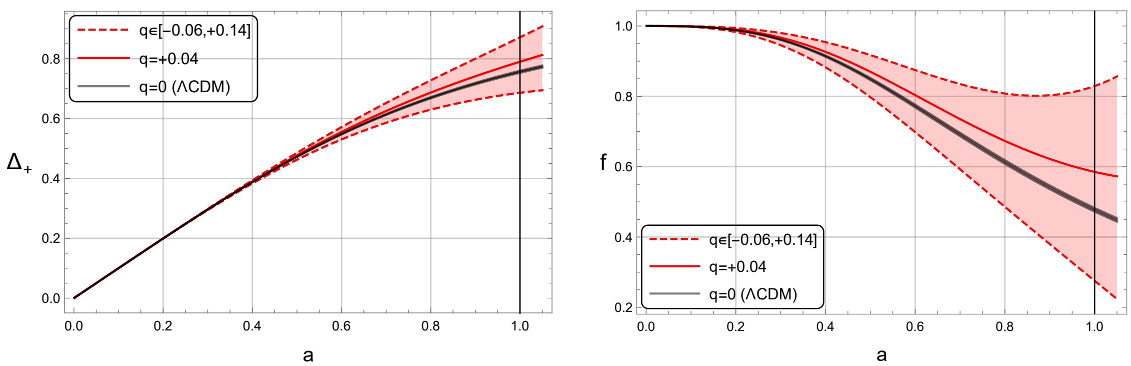

Figure 4.3: **Left Panel:** First order growing mode $\Delta_+(a)$ as a function of scale factor a for the linear vacuum model. **Right Panel:** First order growth rate $f(a)$ as a function of scale factor a for the linear vacuum model. In both graphs we have considered the largest possible intervals coming from the previous constraints on the parameters. For ΛCDM, even if they are not distinguishable, we have plotted 3 curves corresponding to the one σ range of the Ω_{m0} parameter.

we extrapolate the values in Tab. (4.1) for $a_{eq}(q)$; i.e. the value of the scale factor corresponding to CDM-vacuum energy equality, and the growth rate $f(a)$ calculated respectively at a_{eq} and a_0.
We notice from Tab. (4.1) that a_{eq} grows with q: let's explain this feature for the $q > 0$ case ($q < 0$ being exactly the opposite). As a consequence of taking a positive q

†More precisely Planck 2015 measurements of the CMB temperature and polarization and the BAO scale measurement from the SDSS DR7 Main Galaxy Sample, see [27] and references therein.
The constraint placed on the ΛCDM parameter Ω_{m0} comes instead from the more recent *Planck2018* data, see Aghanim et al. [36].

	$q_+ = +0.14$	$q_{\rm bf} = +0.04$	$q_- = -0.06$	ΛCDM
a_{eq}	0.769	0.723	0.668	0.711
f_{eq}	0.815	0.720	0.626	0.683
f_0	0.829	0.585	0.276	0.478

Table 4.1: a_{eq}, f_{eq} and f_0 as a function of the coupling strength q for the linear vacuum model.

(following the same reasoning as before) we have that CDM is annihilated into vacuum energy; this means that, in order to have the same amount of CDM today Ω_{m0} (our fixed initial condition), we should have had more CDM in the past. This, in turn, implies a longer matter dominated phase w.r.t. the standard $q = 0$ case.
By the same token, one can see that the growth rate $f(a)$ grows with the coupling strength q as well. In particular, for the best fit values, we notice that the growth rate decreases less rapidly w.r.t. the ΛCDM case.
Let us now focus on another interacting vacuum model, the generalised Chaplygin gas interacting model.

4.3 Generalised Chaplygin gas interacting model

The generalised Chaplygin gas interacting vacuum (geodesic CDM) model is characterized by a simple parameterisation of the energy transfer Q:

$$Q = 3\alpha H\Big(\frac{\rho V}{\rho + V}\Big)\,. \tag{4.15}$$

as in [24, 28, 31, 32]. Again, ρ and V are the CDM and vacuum energy densities, respectively; H is the Hubble parameter. The constant α is a dimensionless parameter that describes the strength of the vacuum-CDM energy coupling, just like the previous case.

However, before going on with the interacting vacuum scenario, let us briefly discuss the generalised Chaplygin gas model, from which the above energy transfer has been obtained.

4.3.1 Generalised Chaplygin Gas

The discovery that the expansion of the Universe is accelerating [33], and the dominance of dark energy and dark matter in the present cosmological energy density has led many people to seek a unified description in terms of a single dark component, referred to as unified dark matter. This dark component should explain both the current accelerated expansion of the Universe and the role of non-baryonic dark matter in structure formation.

One candidate from this kind of models is the generalized Chaplygin gas (GCG) [34]. The GCG model is described by a perfect fluid, defined by an exotic equation of state:

$$P = -\frac{A}{\rho^{\alpha}}\,, \tag{4.16}$$

with parameters α and A.[†] Inserting the above equation of state, for the Chaplygin gas ($\alpha = 1$ case), into the relativistic energy conservation equation leads to the following energy density:

$$\rho = \sqrt{A + \frac{B}{a^6}}\,, \tag{4.17}$$

where a is the scale factor and B is an integration constant. By choosing a positive value for B we see that for small a (i.e. $a^6 \ll B/A$) Eqn. (4.17) is approximated by:

$$\rho \sim \frac{\sqrt{B}}{a^3}\,, \tag{4.18}$$

[†] The GCG model with $\alpha = 1$ reduces to the original Chaplygin gas (CG), firstly proposed by Kamenshchik et al. [35].

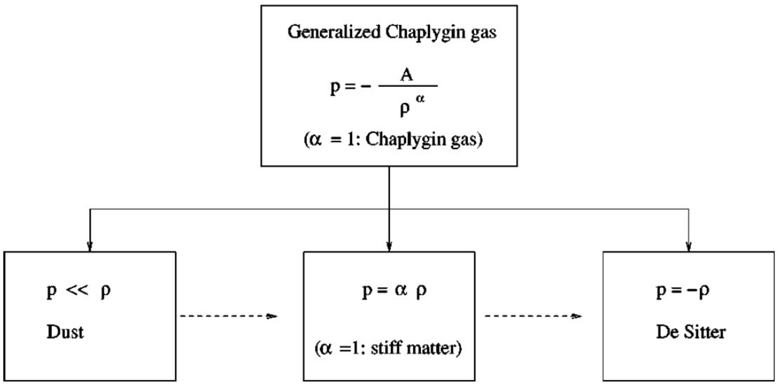

Figure 4.4: Cosmological evolution of a Universe described by a generalized Chaplygin gas equation of state.

that corresponds to a Universe dominated by dust-like matter. For large values of a we instead have:

$$\rho \sim \sqrt{A}\,, \quad P \sim -\sqrt{A}\,, \tag{4.19}$$

which, in turn, corresponds to a Universe dominated by a cosmological constant $\sqrt{A}$. We have just showed that this simple and elegant model smoothly interpolates between an Einstein de Sitter phase where $\rho \simeq \sqrt{B}a^{-3}$, and a de Sitter phase where $P = -\rho = -\sqrt{A}$. This transition is not discrete but it takes place gradually via an intermediate regime. In fact, if one considers the subleading terms in Eqn. (4.17) for large values of a, then one obtains the following expressions for ρ and P:

$$\rho \simeq \sqrt{A} + \sqrt{\frac{B^2}{4A}}a^{-6}\,, \tag{4.20}$$

$$P \simeq -\sqrt{A} + \sqrt{\frac{B^2}{4A}}a^{-6}\,. \tag{4.21}$$

The above equations describe a mixture between a cosmological constant $\sqrt{A}$ and a type of matter known as "stiff" matter, described by the equation of state $P = \rho$, see [35]. Let's see now the GCG case, where α is now unfixed. Solving the continuity equation we have:

$$\rho = \left(A + \frac{B}{a^{3(1+\alpha)}}\right)^{\frac{1}{1+\alpha}}\,, \tag{4.22}$$

with limiting cases given by:

$$a \ll 1 \qquad \Rightarrow \qquad \rho \sim \frac{B^{\frac{1}{1+\alpha}}}{a^3}\,, \tag{4.23}$$

$$a \gg 1 \qquad \Rightarrow \qquad \rho \sim A^{\frac{1}{1+\alpha}}\,. \tag{4.24}$$

The effective equation of state in the intermediate regime, between the dust dominated phase and the de Sitter phase, can be obtained expanding Eqn. (4.22) at subleading order in a, like the CG case:

$$\rho \simeq A^{\frac{1}{1+\alpha}} + \left(\frac{1}{1+\alpha}\right)\frac{B}{A^{\frac{1}{1+\alpha}}}a^{-3(1+\alpha)}\,, \tag{4.25}$$

$$P \simeq -A^{\frac{1}{1+\alpha}} + \left(\frac{1}{1+\alpha}\right)\frac{B}{A^{\frac{1}{1+\alpha}}}a^{-3(1+\alpha)}\,. \tag{4.26}$$

The above corresponds to a mixture of vacuum energy density and matter described this time by the "soft" equation of state, $P = \alpha\rho$ [34].
As mentioned above, this unified dark matter model is able to describe both a matter dominated phase and a vacuum energy dominated phase. What we are going to do now is to decompose this perfect fluid into two interacting components: a pressureless cold dark matter with energy density ρ and a vacuum term V with equation of state $\check{P} = -V$, as in [24, 25].

4.3.2 GCG interacting background

In general, any dark energy fluid energy–momentum tensor:

$$T^{\mu}_{\ \nu} = (\rho_{\rm de} + P_{\rm de})u^{\mu}u_{\nu} + P_{\rm de}\delta^{\mu}_{\nu}\,, \tag{4.27}$$

can be described by pressureless matter, with density ρ and velocity u^{μ}, interacting with the vacuum V, such that $\rho_{\rm de} = \rho + V$. The corresponding matter and vacuum densities are given by

$$\rho = \rho_{\rm de} + P_{\rm de} \qquad V = -P_{\rm de}\,, \tag{4.28}$$

while the energy flow is $Q_{\mu} = \nabla_{\mu}P_{\rm de}$. In an FLRW cosmology this corresponds to $Q = -\dot{P}_{\rm de}$. For a generic decomposition into two interacting barotropic fluids such that $\rho_{\rm de} = \rho_1 + \rho_2$, one would end up doubling the d.o.f. unless one of these fluids is the vacuum.
In the GCG we have:

$$P_{gCg} = -\frac{A}{\rho^{\alpha}} \qquad \Rightarrow \qquad \rho_{gCg} = \left(A + \frac{B}{a^{3(1+\alpha)}}\right)^{\frac{1}{1+\alpha}}\,, \tag{4.29}$$

from which, following the above decomposition, one obtains:

$$\rho = \rho_{gCg} - V\,, \tag{4.30}$$

$$V = A\rho_{gCg}^{-\alpha}\,. \tag{4.31}$$

Expanding the latter expression we get:

$$V = A(A + Ba^{-3(1+\alpha)})^{-\frac{\alpha}{1+\alpha}} \quad \Rightarrow \quad A = (\rho + V)^{\alpha} V \,. \tag{4.32}$$

The form of the FLRW solution suggests then a simple interaction:

$$Q = 3\alpha H\Big(\frac{\rho V}{\rho + V}\Big) \,. \tag{4.33}$$

In the vacuum or matter dominated limits this reduces to an interaction of the form $Q \propto H\rho$ or $Q \propto HV$ where the latter one corresponds to the studied linear vacuum case. As for the previous model, from $\dot{V} = Q$ we notice that:

- for $\alpha > 0$ the vacuum grows, i.e. we have an energy transfer from matter to vacuum.
- Conversely, for $\alpha < 0$ the vacuum decays, hence we have an energy transfer from vacuum to matter.

Fixing the parameters A and B at present time:

$$A = (3H_0^2)^{\alpha} V_0 \,, \tag{4.34}$$

$$B = (3H_0^2)^{1+\alpha}\Big[1 - \frac{V_0}{3H_0^2}\Big] \,, \tag{4.35}$$

we can now obtain, solving the background Eqn. (4.4) (using *Mathematica*), the following expressions for the Hubble parameter $\mathcal{H}(a)$, the matter density parameter $\Omega_m(a)$ and the dimensionless interaction parameter $g(a)$:

$$\mathcal{H}(a) = aH_0\Big[1 - \Omega_{m0} + \frac{\Omega_{m0}}{a^{3(1+\alpha)}}\Big]^{\frac{1}{2(1+\alpha)}} \,, \tag{4.36}$$

$$\Omega_m(a) = \frac{\Omega_{m0}}{\Omega_{m0} + (1 - \Omega_{m0})a^{3(1+\alpha)}} \,, \tag{4.37}$$

$$g(a) = -3\alpha(1 - \Omega_m) \,. \tag{4.38}$$

The standard matter era is recovered at early times, i.e. $a \ll 1$ with $\Omega_m \approx 1$ and $g \approx 0$. As usual, the ΛCDM model corresponds to taking $\alpha = 0$ in the above equations. In the special case $\alpha = -1/2$ we have from Eqn. (4.36) the following Hubble rate:

$$H(a) = aH_0\Big[1 - \Omega_{m0} + \frac{\Omega_{m0}}{a^{3/2}}\Big] \,, \tag{4.39}$$

and thus from Eqn. (4.37):

$$\frac{H'}{H} = -\frac{3}{2}\mathcal{H}\Omega_m\,. \tag{4.40}$$

Comparing the above with $V' = aQ$, for Q as in Eqn. (4.33), we see that $V'/V = H'/H$. Thus the vacuum density decays linearly with the Hubble rate, e.g. $V = 2\Gamma H$, and matter is produced at a constant rate $\dot{\rho} + (3H - \Gamma)\rho = 0$.

From these background solutions we can then plot the scale factor $a(t)$ as a function of the cosmic time t as done for the previous case. Fig. (4.5) shows seven different α values. From the plot we see that vacuum-CDM interaction leads, depending on the sign of α, to an enhanced or reduced acceleration for the scale factor, just like the previous linear vacuum model. In particular, notice the behaviour of the two cyan lines: the dashed

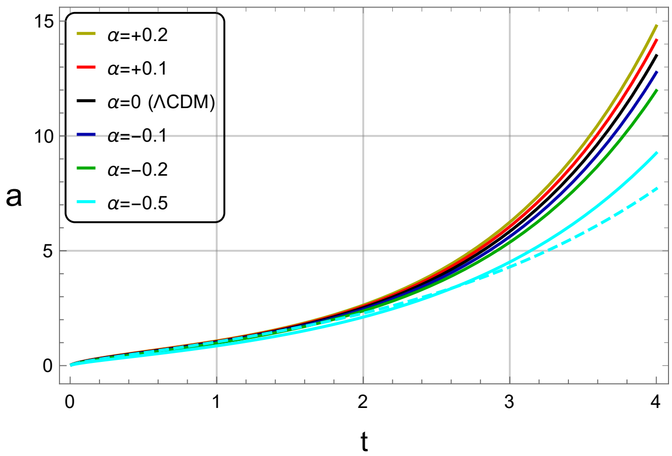

Figure 4.5: Scale factor $a(t)$ as a function of the adimensionless cosmic time, $H_0 = 1$, for a two component cosmology with a GCG vacuum-CDM interaction. Seven different α values are shown, where for all the solid curves $\Omega_{m0} = 0.3$. The dashed curve corresponds instead to $\Omega_{m0} = 0.45$.

one $\Omega_{m0} = 0.45$, grows more w.r.t. to the solid one $\Omega_{m0} = 0.3$, at early times (matter domination); but later on (vacuum domination), it is this latter that 'wins', thanks to a lower value of Ω_{m0}.

4.3.3 Numerical solutions for the generalised Chaplygin gas

Inserting the expressions for $\mathcal{H}(a)$, $\Omega_m(a)$ and $g(a)$ in the second order differential equation (4.3) and using *Mathematica*, we find different numerical solutions for $\Delta_+(a)$ depending on the given α coupling values. Once again, the initial conditions have been set at very early times. The solution for both, the first order growing mode $\Delta_+(a)$ and the first order growth rate $f(a)$ have been plotted in Fig. (4.6). In this plot we have considered seven different values for the dimensionless coupling α.[†]

[†]Notice that for $\alpha = 0$ we go back to the ΛCDM model.

In the $\alpha < 0$ case, vacuum decays into CDM. In this context, the first order growing mode $\Delta_+(a)$ and the first order growth rate $f(a)$ are both suppressed w.r.t. the ΛCDM case, for a given value of the present Ω_{m0}. The explanation of the behaviours of $\Delta_+(a)$

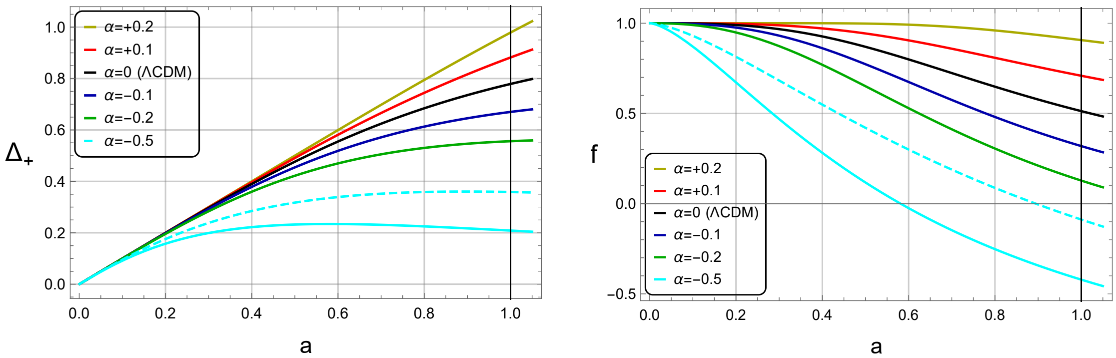

Figure 4.6: **Left Panel:** First order growing mode $\Delta_+(a)$ as a function of scale factor a for the GCG interacting model. We have seven different curves corresponding to seven different values of the coupling strength: $\alpha = +0.2$ (yellow curve, top), $\alpha = +0.1$ (red curve), $\alpha = 0$ (Λ black curve), $\alpha = -0.1$ (blue curve), $\alpha = -0.2$ (green curve), where we have used $\Omega_{m0} = 0.3$. For $\alpha = -0.5$ we adopted $\Omega_{m0} = 0.45$ (cyan dashed curve) and $\Omega_{m0} = 0.3$ (solid curve, bottom). **Right Panel:** First order growth rate $f(a)$, as a function of scale factor a for the aforementioned coupling strength values of the GCG interacting model.

and $f(a)$ for different values of α is the same as in the linear vacuum model, we refer so to Section 4.2.2. However, it is worth explaining why the dashed cyan curve ($\Omega_{m0} = 0.45$) grows more w.r.t. the solid one ($\Omega_{m0} = 0.3$). Once more, the explanation follows the same reasoning of the aforementioned section. In this case α is fixed, i.e. $\alpha = -0.5$ and we have two different Ω_m; where a bigger Ω_{m0} (dotted cyan curve) simply requires more matter in the past to be reached w.r.t a smaller Ω_{m0} (solid cyan curve).
Hence, since the growth of structures depends mainly on $\Omega_m(a)$ [41], $\Delta_+(a)$ and $f(a)$ are enhanced for a higher present matter density parameter Ω_{m0}.
In Section 4.5.2 we will compare the present numerical results with an analytical approximation proposed by Borges % Wands in [31].

Constraints on the interacting GCG growth rate

Once again the aim is to see how both, the first order growing mode $\Delta_+(a)$ and the first order growth rate $f(a)$ behave w.r.t. the ΛCDM model using some constraints imposed on the parameters Ω_{m0} and α.[†]
We take the two parameters in the below one σ range, taken from Wang et al. [29]

[†]The constraint placed on the ΛCDM parameter $\Omega_{m0} = 0.2645 \pm 0.0052$ comes from the *Planck2018* data, see Aghanim et al. [36].

constraints using *Planck2015+WP+BAO* data.[†] We have so plotted $\Delta_+(a)$ and $f(a)$ for

α	Ω_{m0}
0.011 ± 0.054	0.2533 ± 0.0179

both the interacting $Q = 3\alpha H(\rho V)/(\rho + V)$ and the non-interacting $Q = 0$ (i.e. ΛCDM) cases, see Fig. (4.7). Considering the largest possible intervals for both the quantities $\Delta_+(a)$ and $f(a)$, coming from α and Ω_{m0} one σ limits, we see from the below plots that the ΛCDM values are well inside the interacting GCG range. From the below plots we

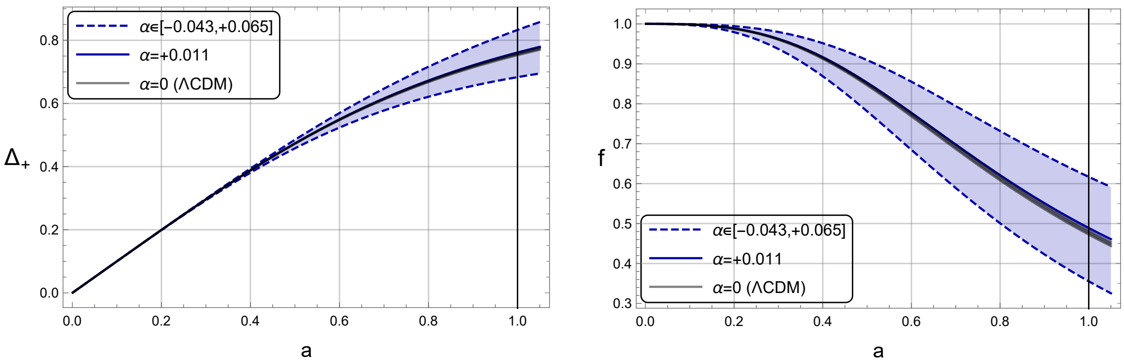

Figure 4.7: **Left Panel:** First order growing mode $\Delta_+(a)$ as a function of scale factor a for the GCG interacting model. **Right Panel:** First order growth rate f as a function of scale factor a for the GCG interacting model. In both graphs we have considered the largest possible intervals coming from the previous constraints on the parameters. For ΛCDM, even if they are not distinguishable, we have plotted 3 curves corresponding to the one σ range of the Ω_{m0} parameter.

extrapolate the following values for $a_{eq}(\alpha)$; i.e. the value of the scale factor corresponding to vacuum-CDM energy equality, and the growth rate $f(a)$ calculated respectively at a_{eq} and a_0. We notice from Tab. (4.2) that both a_{eq} and f grow with α as in the linear

	$\alpha_+ = +0.065$	$\alpha_{\rm bf} = +0.011$	$\alpha_- = -0.043$	ΛCDM
a_{eq}	0.734	0.700	0.663	0.711
f_{eq}	0.773	0.699	0.623	0.683
f_0	0.617	0.489	0.355	0.478

Table 4.2: a_{eq}, f_{eq} and f_0 as a function of the coupling strength α for the GCG interacting model.

vacuum model. In particular, unlike the linear vacuum model, we can see from Fig. (4.7) that the best fit $f(a)$, i.e. for $\alpha = \alpha_{\rm bf}$, is basically indistinguishable from ΛCDM.

[†]For a more detailed description about how the constraints have been imposed check Wang papers [28, 29].

4.4 Shan-Chen interacting model

As we have already seen, the validity of the cosmological constant's use as the dark energy has long been questioned, and many alternatives have been proposed. In the previous sections, we have considered two interacting vacuum models: the linear vacuum [27], and the GCG [28, 29].

We now consider a more exotic case, namely the interacting Shan-Chen (SC) model following the recent work of Hogg & Bruni [37]. As for the GCG, before presenting its interacting vacuum analogue, we consider the original unified dark matter SC model. This model describes a Universe filled with a cosmological fluid characterized by the SC equation of state, i.e. a non-ideal e.o.s. with 'asymptotic freedom'. This equation of state corresponds to an ideal gas behavior (i.e. we have no interaction among the gas constituents) at both low and high density regimes, with a liquid-gas coexistence phase in between (i.e. a region of temperature and pressure in which the fluid can be in both the liquid and gas state).

The idea of an 'asymptotic free' non-ideal equation of state was first proposed by Shan & Chen (SC) in the context of lattice kinetic theory [38], this is why in the above we have used some atypical terms. In particular, the term 'asymptotic freedom' refers to the fact that attraction among particles becomes vanishingly small beyond a given density threshold; and since high density implies short spatial separation we then understand the use of such term.

For a detailed discussion we refer to [38, 39, 40]; what we need to know is that this model was successfully applied in the cosmological context by Bini et al. [39, 40]. In fact, they have found that a dark energy fluid with a Shan–Chen equation of state naturally evolves towards a Universe with a late time accelerating expansion without the presence of a cosmological constant.

4.4.1 The Shan-Chen equation of state

The nonlinear Shan-Chen equation of state is:

$$P = w\rho_* \left[\frac{\rho}{\rho_*} + \frac{g}{2}\psi^2 \right] , \tag{4.41}$$

where ρ is the energy density of the fluid and ψ is given by:

$$\psi = 1 - \mathrm{e}^{-\alpha \frac{\rho}{\rho_*}} , \tag{4.42}$$

where ρ_* is a characteristic energy scale and w, g and α are free (dimensionless) parameters of the model.

In the Shan-Chen model of dark energy introduced by Bini et al. [39, 40], the matter-energy content of the Universe is assumed to be a perfect fluid which obeys the equation

of state (4.41), ρ_* being identified with the critical density today: $\rho_* = \rho_{\rm crit,0} = 3H_0^2$, where H_0 is the value of the present Hubble parameter.
We can then define the effective equation of state parameter $w_{\rm eff}$ as:

$$w_{\rm eff} = \frac{P(\rho)}{\rho} = w + \frac{wg\rho_*}{2\rho}\left(1 - {\rm e}^{-\alpha\frac{\rho}{\rho_*}}\right)^2 . \tag{4.43}$$

Clearly, this only depends on $x = \rho/\rho_*$. Although the energy scale ρ_* is physically meaningful, in that it determines at which energies the nonlinear term in (4.42) becomes relevant, it has no influence on the qualitative behaviour of the Shan–Chen effective equation of state, see [37].
Around ρ_* the fluid is not ideal, while in the limits $\rho \gg \rho_*$ and $\rho \ll \rho_*$ we get $P = w\rho$ and the usual linear equation of state is thus recovered. In general, several different dark energy models can be obtained from Eqn. (4.42); for a more complete description of such an equation of state see once again [37, 40].

SC interacting model

Let's present now, as we have done for the GCG case, the (geodesic CDM) interacting SC model. We can recast the above SC model as a parameterisation of the coupling Q between the vacuum and cold dark matter, starting from:

$$\dot{\rho} = -3H(\rho + P(\rho)) , \tag{4.44}$$

we insert P from Eqn. (4.41) and replace ρ with the vacuum energy density V. In this way we obtain an expression for the energy transfer Q between vacuum and cold dark matter which is given by:

$$Q = \dot{V} = -3Hq\left[(1+\beta)V + \frac{\beta g}{2}\rho_*\left(1 - {\rm e}^{-\alpha\frac{V}{\rho_*}}\right)^2\right] . \tag{4.45}$$

To avoid confusion with the dark energy case, we have renamed the parameter w as β. Additionally, we have also introduced the dimensionless parameter q which controls the overall strength of the interaction. When $q = 0$, there is no interaction and we therefore return to ΛCDM.
Notice that we have a number of additional free parameters in the SC interacting vacuum model with respect to ΛCDM: q, α, g, β and ρ_*. In the simplified case we are going to examine we will fix α, g, β to the best fit values of Bini et al. [39]: $\alpha = 2.7$, $g = -8.0$, $\beta = 1/3$.
q is instead free to vary and ρ_*, the characteristic energy scale, is fixed at the critical density today in ΛCDM, i.e. $\rho_* = \rho_{\rm crit,0}$, as we are interested in the effects the interaction may have at late times.

For later convenience we define an effective equation of state for the Shan-Chen interacting vacuum, which we call $w_{\rm int}$. In analogy with Eqn. (4.43) we can rewrite (4.45) as

$$\hat{Q} = (\hat{V})^{\cdot} = Hx(1 + w_{\rm int})\,, \tag{4.46}$$

where we define

$$w_{\rm int} = q_* \left[(1+\beta) + \frac{\beta g}{2x}\left(1 - \mathrm{e}^{-\alpha x}\right)^2\right] - 1\,. \tag{4.47}$$

Where now $x = V/\rho_*$ and the hatted quantities have been divided by ρ_*, i.e. $\hat{Q} = Q/\rho_*$ and $\hat{V} = V/\rho_*$. In addition, we have scaled the parameter q such that $q_* \equiv -q/3$. In this manner, for $\beta = 0$, we go back to the notation in [31]. In the case $w_{\rm int} = -1$, i.e. $\dot{V} = 0$, we return once again to the ΛCDM case. In the following we have plotted $w_{\rm int}(x)$, where have kept fixed, at their best fit values, all the parameters except q_*. From Fig. (4.8) we

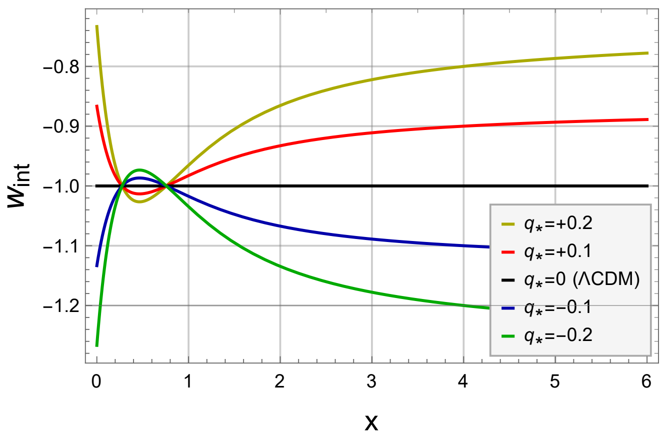

Figure 4.8: Behaviour of $w_{\rm int}$ as a function of $x = V/\rho_*$ for different values of q_* for the interacting SC model. The black $w_{\rm int} = -1$ line separates growing vacuum models from the decaying ones.

can see the effect of changing both the magnitude and sign of the interaction strength q_*. In particular, changing its sign simply mirrors curves with same parameter values about the $w_{\rm int} = -1$ line. In all cases shown there are two cosmological constants where the curves cross the $w_{\rm int} = -1$ line. In general, at high energies (i.e. $x \gg 1$) $w_{\rm int}$ behaves as a constant, then the nonlinear term in Eqn. (4.47) becomes dominant for $x \simeq 1$.
For models above the $w_{\rm int} = -1$ line (dark yellow and red lines) V grows from high energies to the cosmological constant on the right, or to zero from the cosmological constant on the left. For models below the $w_{\rm int} = -1$ line (dark blue and dark green lines) V decays from the cosmological constant on the right to high energies, or from zero to the cosmological constant on the left.
Models in between the two cosmological constants, grow or decay between the two for $w_{\rm int} > -1$ and $w_{\rm int} < -1$ respectively. For a complete analysis of $w_{\rm int}$ where also β, g and α are left free, see [37].

4.4.2 Numerical solutions for the Shan Chen model

We now focus on the Id case in Hogg & Bruni [37] where we fix α, g and β to the Bini et al. best fit values: $\alpha = 2.7$, $g = -8.0$, $\beta = 1/3$ with $\rho_* = \rho_{\text{crit},0} = 3H_0^2$. The only free parameter being the adimensional coupling strenght q_*. Unlike the previous cases, where we had a linear equation of state, we are now facing a model with the more complicated interaction parameter $\hat{Q}$:

$$\hat{Q} = q_* H x \left[(1+\beta) + \frac{\beta g}{2x} \left(1 - \mathrm{e}^{-\alpha x} \right)^2 \right]. \tag{4.48}$$

The non-linear term in $\hat{Q}$, makes even the simpler dynamical equations for $\hat{\rho}$ and x analytically unsolvable. Therefore, unlike the linear vacuum and GCG interacting models, we had to solve everything numerically (always using *Mathematica*) from the beginning. Inserting the numerical results for $\mathcal{H}(a)$, $\Omega_m(a)$ and $g(a)$ in the second-order differential equation (4.3), we then find different numerical solutions for the first order growing mode $\Delta_+(a)$ depending on the given coupling strength values q_*.
The initial conditions have been set at very early times as for the previous cases; the solution for both, the first order growing mode $\Delta_+(a)$ and the first order growth rate $f(a)$ have been plotted in Fig. (4.9). Notice that the behaviour of both $\Delta_+(a)$ and $f(a)$

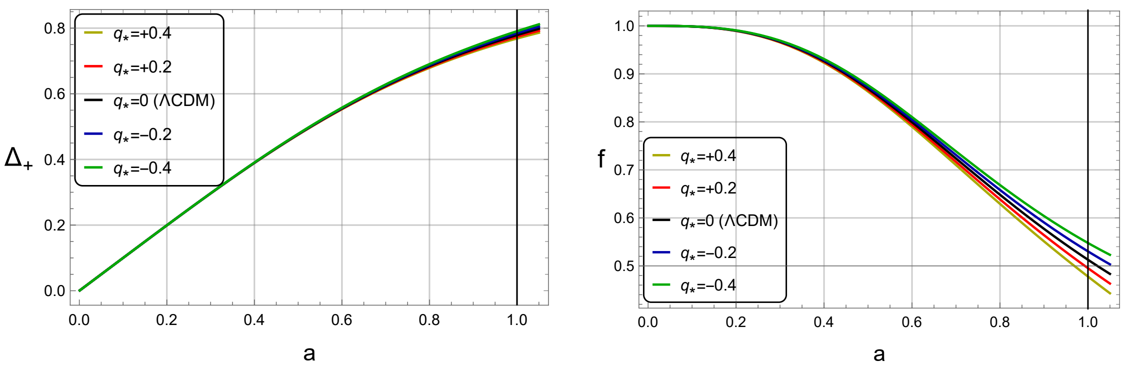

Figure 4.9: **Left Panel:** First order growing mode $\Delta_+(a)$ as a function of scale factor a for the SC interacting model. We have five different curves corresponding to five different values of the coupling strength: $q_* = 0.2$ (yellow curve, bottom), $q_* = 0.1$ (red curve), $q_* = 0$ (Λ black curve), $q_* = -0.1$ (blue curve), $q_* = -0.2$ (green curve, top), where we have used $\Omega_{m0} = 0.3$. **Right Panel:** First order growth rate $f(a)$, as a function of scale factor a for the SC interacting model. The coupling strength values being the same as before.

is the opposite w.r.t. to the previous two cases. In fact for $q_* > 0$ we have this time a suppression of $\Delta_+(a)$ and $f(a)$ w.r.t. the ΛCDM case. The reason being the negative value of the square bracket term in Eqn. (4.48), due to the best fit parameters.

This, in turn, implies an energy transfer:

- from CDM to vacuum, $\hat{Q} > 0$ (growing vacuum) for $q_* < 0$,
- and from vacuum to CDM $\hat{Q} < 0$ (decaying vacuum) for $q_* > 0$.

Constraints on the interacting SC growth rate

We now use the constraints imposed on the parameters Ω_{m0} and q_* to plot the first order growing mode $\Delta_+(a)$ and the first order growth rate $f(a)$. The aim being to compare the present SC case to the ΛCDM model.† We take the two parameters in the below one σ range, obtained by Hogg & Bruni [37] using *Planck2018+BAO+SDSS* data.‡ Considering the largest possible intervals for both $\Delta_+(a)$ and $f(a)$, coming from

q_*	Ω_{m0}
0.017 ± 0.02	0.2408 ± 0.0202

q_* and Ω_{m0} one σ limits, we see from Fig. (4.10) that the ΛCDM values are slightly above the interacting SC range. This time the explanation for this has nothing to do with the

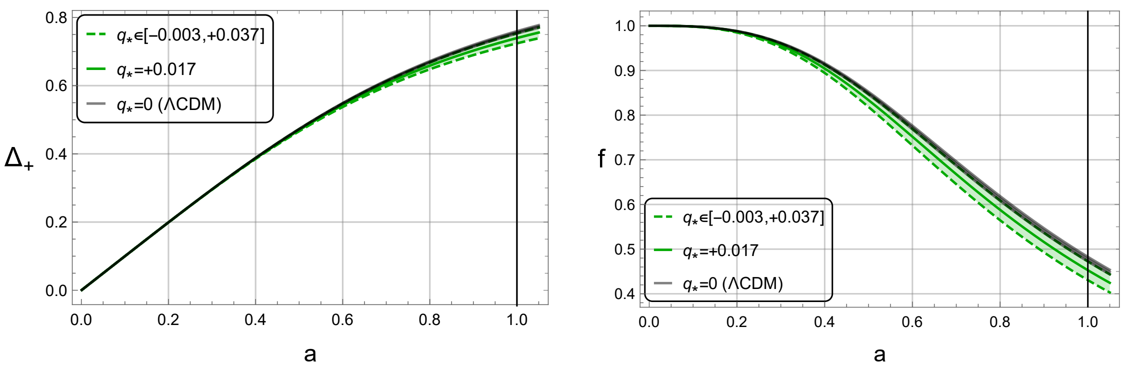

Figure 4.10: **Left Panel:** First order growing mode $\Delta_+(a)$ as a function of scale factor a for the SC interacting model. **Right Panel:** First order growth rate $f(a)$ as a function of scale factor a for the SC interacting model. In both graphs we have considered the largest possible intervals coming from the previous constraints on the parameters. For ΛCDM, even if they are not distinguishable, we have plotted 3 curves corresponding to the one σ range of the Ω_{m0} parameter.

adimensional interacting parameter q_*, whose range includes also the non-interacting

†The constraint placed on the ΛCDM parameter $\Omega_{m0} = 0.2645 \pm 0.0052$ comes from the *Planck2018* data, see Aghanim et al. [36].

‡For a more detailed description about how the constraints have been imposed check [37].

$q_* = 0$ case. The motivation comes instead from the almost total non-overlapping of the Ω_{m0} ΛCDM and SC parameters. This is also the reason why, in this best fit plot, we have the $q_* > 0$ dashed line above the $q_* < 0$ one, unlike Fig. (4.9) and like the two previously discussed interacting cases in Sections 4.2.2, 4.3.3.
From the plots in Fig. (4.10) we extrapolate the following values for $a_{eq}(q_*)$; i.e. the value of the scale factor corresponding to CDM-vacuum energy equality, and the growth rate $f(a)$ calculated respectively at a_{eq} and a_0. We notice from Tab. (4.3) that both a_{eq}

	$q_{*,+} = +0.037$	$q_{*\mathrm{bf}} = +0.017$	$q_{*,-} = -0.003$	ΛCDM
a_{eq}	0.707	0.682	0.657	0.711
f_{eq}	0.682	0.683	0.683	0.683
f_0	0.473	0.453	0.431	0.478

Table 4.3: a_{eq}, $f_{eq} = f(a_{eq})$ and $f_0 = f(a_0)$ as a function of the coupling strength q_* for the SC interacting case.

and f_0 grow with q_* as in the previous models.[†] The almost non-overlapping one σ ranges of the ΛCDM and SC parameters Ω_{m0} and the non-linear term in $\hat{Q}$ make the range of the growth rate interacting SC values to only slightly overlap with ΛCDM, unlike the two previous cases. From the plots in Fig. (4.10), we can state that the whole range of growth rate values $f(a)$, coming from the constraints on the parameters, for this SC interacting model, decreases more rapidly than the ΛCDM one.

[†]Pay attention to the fact that f_{eq} is actually $f(a_{eq})$, a_{eq} changing with q_* as in Tab. (4.3). This means that, unlike for f_0, we are calculating $f(a)$ at different scale factor values.

4.5 An analytical approximation

In this section we are going to present an analytical approximation proposed by Borges & Wands in [31]. Since the approximation involves the quantity $f_{\rm rsd}$, i.e. the redshift-space distortion growth rate, before going into further details of the approximation, let us briefly discuss redshift-space distortions.

4.5.1 Redshift-space distortions

Redshift-space distortions (RSD) arise from peculiar velocities of galaxies. If we use redshift as distance, the inferred distance to a galaxy is related to its redshift z, by:

$$s = cz\,. \tag{4.49}$$

This distance is usually referred to as the redshift distance of the galaxy. The true distance expressed in velocity units is:

$$r = H_0 d\,, \tag{4.50}$$

which we will refer to as the real distance of the galaxy.[†] s and r are related by:

$$s = r + v_r\,, \tag{4.51}$$

where $v_r = \mathbf{v}\cdot\hat{r}$ is the peculiar velocity along the line-of-sight. Peculiar velocities introduce a radial anisotropic distortion in redshift-space via a Doppler effect. This observed anisotropy provides information about the formation of large-scale structures. In the linear regime (i.e. on sufficiently large scales), the distortion is a 'squashing' in the radial (line of sight) direction, while in the nonlinear regime there is a stretching ('finger of god') effect.

On large scales, the peculiar velocity of an infalling shell is small compared to its radius, and the shell appears squashed. On smaller scales, not only is the radius of a shell smaller, but also its peculiar infall velocity tends to be larger. For the shell that is just at turnaround, its peculiar velocity cancels the Hubble expansion, and it appears collapsed to a single velocity in redshift space. On even smaller scales, shells that are collapsing in proper coordinates appear inside out in redshift space. The combination of collapsing shells with previously collapsed, virialised shells, gives rise to the 'finger-of-god' shape, see Fig. (4.11).

In the interacting model RSDs constrain structure growth; in fact the divergence of the peculiar velocity field, $\nabla\cdot v_r \sim a\,\delta\Theta$, is related to the density contrast δ by Eqn. (3.70) that we here report:[‡]

$$\delta\Theta = -(f+g)H\delta\,. \tag{4.52}$$

[†]We assume that all galaxies are local enough that a linear mean Hubble relation is appropriate.

[‡]Notice that we are now in a synchronous gauge comoving with the CDM 4-velocity u^a (as at the end of Chapter 3). The scalar part of the CGI variable Δ then reduces simply to $\Delta = a^2\delta$.

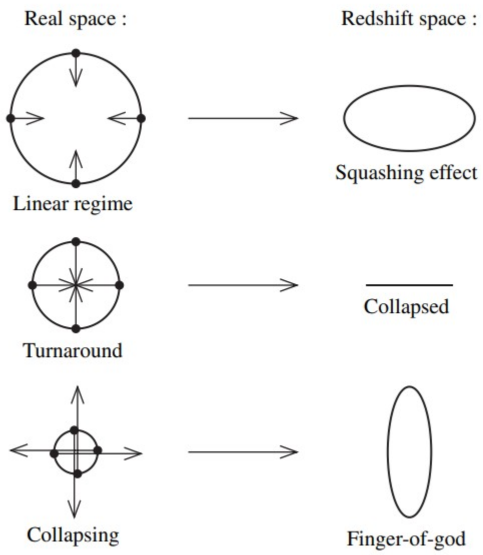

Figure 4.11: An illustration of how peculiar velocities distort the galaxy distribution in redshift-space in different regimes.

In a standard ΛCDM cosmology, where the dimensionless interaction parameter $g = 0$, the variance of the expansion is characterized by:

$$\left\langle \frac{\delta\Theta^2}{H^2} \right\rangle^{1/2} = f(a)\sigma_8(a)\,, \tag{4.53}$$

where $f(a)$ is the linear growth rate and $\sigma_8(a) = \langle \delta(a)^2 \rangle^{1/2}$ is the rms mass fluctuation in a sphere with comoving radius $8\mathrm{h}^{-1}$Mpc which is used to describe the amplitude of density perturbations.†

More generally, for interacting models, the dimensionless interaction parameter $g(a)$ contributes explicitly in (4.52) for redshift-space distortions. If we assume that galaxies still trace the motion of the underlying dark matter then the variance of the expansion (4.52) is given by:

$$\left\langle \frac{\delta\Theta^2}{H^2} \right\rangle^{1/2} = f_{\mathrm{rsd}}(a)\sigma_8(a)\,, \tag{4.54}$$

where

$$f_{\mathrm{rsd}}(a) = f(a) + g(a)\,. \tag{4.55}$$

We remark the fact that, in principle, independent measurements of the RSD parameter f_{rsd} and the linear growth rate for the density contrast f could reveal the effective dark

†The $8\mathrm{h}^{-1}$Mpc value for the comoving radius is due to the fact that galaxy distributions are strongly inhomogeneous on scales approximately $\leq 8\mathrm{h}^{-1}$Mpc, but starts to approach homogeneity on significantly larger scales.

matter interaction, namely if the dimensionless interacting parameter g vanishes or not. Now that we have sketched the effect that RDSs have on the growth of structures, we can discuss the analytical approximation of Borges & Wands [31].

4.5.2 An analytical approximation for the growth rate

The second order differential equation (4.1) for the density contrast:

$$\delta'' + (1+g)\mathcal{H}\delta' + \Big((g\mathcal{H})' + g\mathcal{H}^2 - \frac{1}{2}a^2\rho\Big)\delta = 0\,, \tag{4.56}$$

can be written as a first-order differential equation for the redshift-space distortion parameter $f_{\rm rsd}$, substituting in the above equation, Eqn. (4.55). After some lengthy calculations we get:

$$2\mathcal{H}^{-1}f'_{\rm rsd} + (2f_{\rm rsd} + 4 - 3\Omega_m - 2g)f_{\rm rsd} = 3\Omega_m\,. \tag{4.57}$$

In the conventional matter-dominated era for $a \ll 1$ with $\Omega_m = 1$ and the dimensionless interaction parameter $g = 0$, we have a solution corresponding to the standard growing mode with $f_{\rm rsd} = f = 1$ and the linear growing mode is proportional to the scale factor, $\delta_+ \propto a$.[†] This describes the early growing mode at high redshifts as $g \to 0$ and $\Omega_m \to 1$ in the linear vacuum and GCG interacting models, as well as ΛCDM.
More generally, when vacuum energy contributes to the total density ($\Omega_m < 1$) we can express the first-order equation (4.57) for the RSD parameter as a function of the density parameter, written in terms of $\Omega_V = 1 - \Omega_m$.

$$2(3\Omega_V - g)(1-\Omega_V)\frac{\mathrm{d}}{\mathrm{d}\Omega_V}f_{\rm rsd} + (2f_{\rm rsd} + 1 + 3\Omega_V - 2g)f_{\rm rsd} = 3(1-\Omega_V)\,, \tag{4.58}$$

where we have used

$$\mathcal{H}^{-1}\frac{\mathrm{d}}{\mathrm{d}\eta} = (3\Omega_V - g)(1-\Omega_V)\frac{\mathrm{d}}{\mathrm{d}\Omega_V}\,. \tag{4.59}$$

Note that $g(\Omega_V)$ is a given function of the density parameter Ω_V, in both the linear vacuum and GCG interaction models. In the case $\Omega_m = 1$ (also $\Omega_V = 0$), from Eqns. (4.12) and (4.38) it follows that $g = 0$.
We then expand the dimensionless interaction parameter $g = 0$ as a Taylor series about the standard matter-dominated ($\Omega_m = 1$, $\Omega_V = 0$) solution, namely:

$$g(\Omega_V) \simeq g(0) + \left.\frac{\mathrm{d}g}{\mathrm{d}\Omega_V}\right|_{\Omega_V=0}\Omega_V + \cdots \tag{4.60}$$

$$g(\Omega_V) \simeq g_1\Omega_V + \cdots\,. \tag{4.61}$$

[†]There is also a solution $f_{\rm rsd} = f = -3/2$ corresponding to the standard decaying mode.

We can do the same thing for the RSD parameter:

$$f_{\rm rsd}(\Omega_V) = f(\Omega_V) + g(\Omega_V) \tag{4.62}$$

$$f_{\rm rsd}(\Omega_V) \simeq f(0) + g(0) + \left.\frac{\mathrm{d}(f+g)}{\mathrm{d}\Omega_V}\right|_{\Omega_V=0} \Omega_V + \cdots \tag{4.63}$$

$$f_{\rm rsd}(\Omega_V) \simeq f_{\rm rsd,0} + f_{\rm rsd,1}\Omega_V + \cdots . \tag{4.64}$$

This approximation is justified by the fact that, as found by Lahav & Lilje in [41], $f_{\rm rsd}$ depends mainly on $\Omega_m(a)$ at any epochs.[†]
Let us now apply the above approximation to Eqn. (4.58). Order by order in Ω_V we get:

$$\text{0-th order} \quad \Rightarrow \quad (1+2f_{\rm rsd,0})f_{\rm rsd,0} = 3\,, \tag{4.65}$$

$$\begin{aligned}\text{1-st order} \quad \Rightarrow \quad & (3-2g_1+2f_{\rm rsd,1})f_{\rm rsd,0}+ \\ & + (1+2f_{\rm rsd,0})f_{\rm rsd,1} + 2(3-g_1)f_{\rm rsd,1} = -3\,.\end{aligned} \tag{4.66}$$

For the ΛCDM case with $g=0$ we have from the above:

$$(1+2f_{\rm rsd,0})f_{\rm rsd,0} = 3\,, \tag{4.67}$$

$$(3+2f_{\rm rsd,1})f_{\rm rsd,0} + (1+2f_{\rm rsd,0})f_{\rm rsd,1} + 6f_{\rm rsd,1} = -3\,. \tag{4.68}$$

This gives either $f_{\rm rsd,0} = -3/2$ (decaying mode) or $f_{\rm rsd,0} = 1$ (growing mode) and then, for the first order growing mode, $f_{\rm rsd,1} = -6/11$, corresponding to the well known approximation in [41, 42]:

$$f = f_{\rm rsd} \simeq \Omega_m^{6/11}\,, \tag{4.69}$$

More generally, we can give a similar approximation for the RSD parameter in terms of Ω_m when $g \neq 0$. For the models in Sections 4.2, 4.3 we can as well write:

$$f + g = f_{\rm rsd} \simeq \Omega_m^{\gamma}\,. \tag{4.70}$$

Linear vacuum case

For the linear vacuum case we have $g = -q\Omega_V(1-\Omega_V)^{-1}$ hence $g_1 = -q$, then from Eqns. (4.65)-(4.66) we obtain $f_{\rm rsd,0} = 1$ and $f_{\rm rsd,0} = -\gamma$ where γ is:

$$\gamma = \frac{6+2q}{11+2q}\,. \tag{4.71}$$

[†]Intuitively, the growth rate of matter density perturbations has to depend on the matter content of the Universe Ω_m.

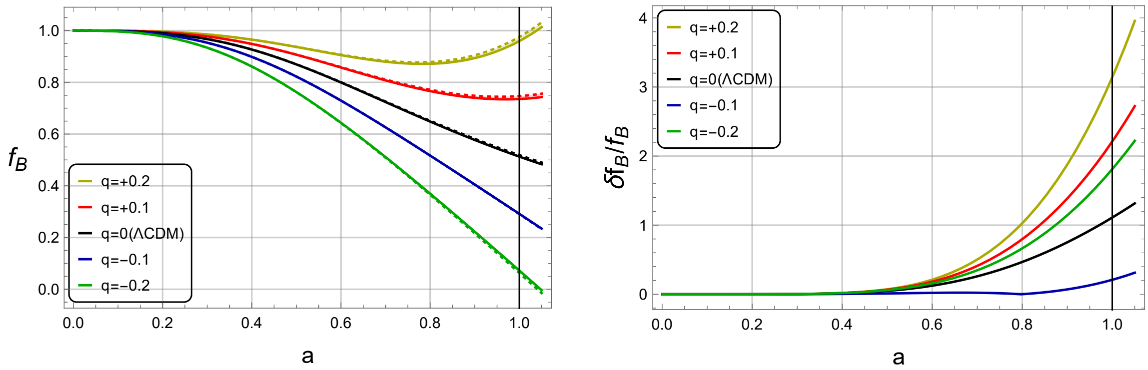

Figure 4.12: **Left Panel:** First order growth rate $f_B(a)$ as a function of scale factor a for the illustrated (linear vacuum model) q values. Dotted lines stand for the analytical approximation while the solid ones for numerical solutions. **Right Panel:** Relative percentage difference between the analytical approximation f_B and its numerical solutions for the linear vacuum model. Notice that the errors remain under 3.5% for $|q| \leq 0.2$.

Generalised Chaplygin gas

For the GCG case we have $g = -3\alpha\Omega_V$ hence $g_1 = -3\alpha$, then from Eqns. (4.65)-(4.66) we obtain $f_{\rm rsd,0} = 1$ and $f_{\rm rsd,0} = -\gamma$ where γ this time is:

$$\gamma = \frac{6+6\alpha}{11+6\alpha}\,. \tag{4.72}$$

As shown in Figs. (4.12) and (4.13), the analytical formula (4.70) for the RSD parameter

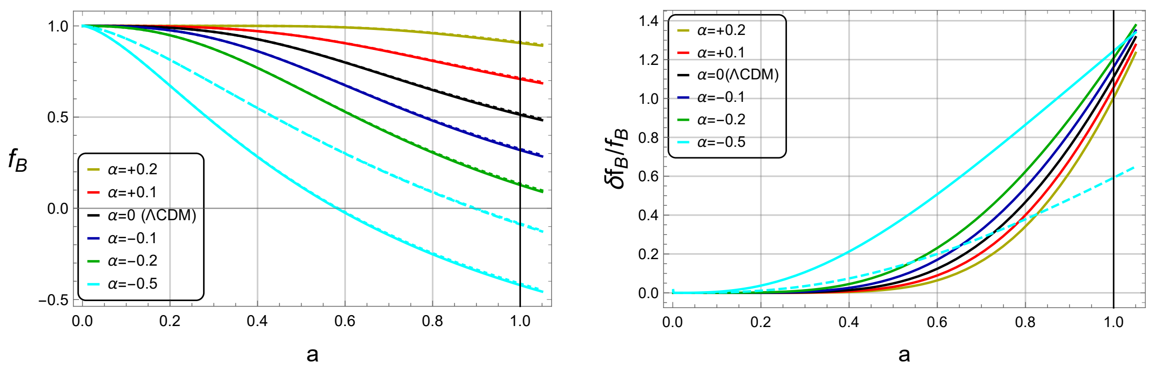

Figure 4.13: **Left Panel:** First order growth rate $f_B(a)$ as a function of scale factor a for the illustrated (GCG interacting model) α values. Dotted lines stand for the analytical approximation while the solid ones for numerical solutions. **Right Panel:** Relative percentage difference between the analytical approximation f_B and its numerical solutions for the GCG interacting model. Notice that the errors remain under 1.5% for $|\alpha| \leq 0.5$.

can be used as a good approximation for both the interacting models. For the linear vacuum case, shown in Fig. (4.12) the expression (4.70) with the growth index (4.71) is a good approximation with errors below 3.5% for $|q| \leq 0.2$.
For the GCG case, shown in Fig. (4.13) the expression (4.70) with the growth index (4.72) is an even better approximation with errors below 1.5% for $|\alpha| \leq 0.5$.
In both cases, the approximations for $f_{\rm rsd}$ become extremely accurate at early times where $(1 - \Omega_m) = \Omega_V \ll 1$. In accordance with the fact that we have initially expanded $f_{\rm rsd}$ around $\Omega_m = 1$.

Conclusions

In this elaborate we have shown, for three different models, how the growth rate of cosmic structures, $f(a)$, is affected in the case we allow an interaction in the dark sector, namely between CDM and vacuum energy. But let's make a step back and sum up what we have done from the beginning. In Chapter 1 we have lied the foundations of the ΛCDM model, which is based on the homogeneous and isotropic FLRW spacetimes. Later in this chapter, we have analysed, from both the differential geometry point of view and the physical point of view, the standard theory of cosmological perturbations; which has been developed perturbing the aforementioned symmetric spacetimes. In Chapter 2, we have instead discussed an alternative way, w.r.t. the Bardeen one, of dealing with cosmological perturbations, which is both covariant and gauge-invariant. This is the CGI approach to cosmological perturbations developed by Ellis & Bruni. Adopting this latter formalism, we have presented, in Chapter 3, an alternative to the ΛCDM model generically known as the interacting vacuum scenario. This class of models is based on the possible existence of a dark sector interaction, namely an energy-momentum exchange between the two unknown dark components: CDM and a dark energy with $w = -1$ e.o.s., i.e. a vacuum energy. This as many other alternative models are motivated by the lack of physical interpretation of the cosmological constant Λ, and the both theoretical and observational issues of ΛCDM.
In the final Chapter 4, we have obtained numerical solutions for the growing mode of the scalar comoving density contrast $\Delta_+(a)$, this being the part that best reflects the formation of cosmic structures. The numerical solutions have been obtained for three relevant geodesic CDM (no momentum exchange allowed) interacting vacuum models: a linear vacuum, the generalised Chaplygin gas (GCG) and the Shan-Chen (SC) interacting models. All of them, in the case we fix the matter density parameter today Ω_{m0}, display the same behaviour: for a growing vacuum, i.e. $Q > 0$, the growth rate $f(a)$ decreases less rapidly than the non-interacting limit (ΛCDM); while for a decaying vacuum, i.e. $Q < 0$, the growth rate $f(a)$ decrease faster than ΛCDM. We have then considered the constrained parameters of these models and found that the best fit range of the growth rate $f(a)$ of the first two mentioned interacting models overlaps with the entire set of values of ΛCDM. However, in the SC interacting case, due to its non-linear equation of state, we have found a very small overlap between the SC and ΛCDM best fit growth

rate range.
In conclusion we want to remark the fact that, as stated in Section 4.5.2, the existence of this vacuum-CDM interaction could, in principle, be detected through independent measurements of the redshift-space-distortions parameter $f_{\rm rsd}$ and the linear growth rate of the density contrast f.

Appendix A

Some useful CGI relations

A.1 Scalar relations

For a generic scalar f we have the following relation:

$$
\begin{aligned}
\left({}^{(3)}\nabla_a f\right)^{\cdot} &= \nabla_c\left({}^{(3)}\nabla_a f\right)u^c && \text{(A.1)}\\
&= \nabla_c\left(h_a{}^b\nabla_b f\right)u^c && \text{(A.2)}\\
&= \nabla_c h_a{}^b u^c \nabla_b f + h_a{}^b \nabla_c\nabla_b f u^c && \text{(A.3)}\\
&= \left(\nabla_c u_a u^c u^b + u_a \nabla_c u^b u^c\right)\nabla_b f + {}^{(3)}\nabla_a \dot{f} && \text{(A.4)}\\
&= a_a \dot{f} + u_a\, a^b\, {}^{(3)}\nabla_b f + {}^{(3)}\nabla_a \dot{f} - {}^{(3)}\nabla_b(\sigma^b{}_a + \omega^b{}_a) - \frac{1}{3}\Theta\, {}^{(3)}\nabla_a f\,. && \text{(A.5)}
\end{aligned}
$$

At linear order:

$$
\left({}^{(3)}\nabla_a f\right)^{\cdot} = a_a \dot{\bar{f}} + {}^{(3)}\nabla_a \dot{f} - H\, {}^{(3)}\nabla_a f\,. \qquad \text{(A.6)}
$$

A.2 Vector relations

For a generic tensor X_b we have the following relation at linear order:

$$
\begin{aligned}
a\, {}^{(3)}\nabla_a(\dot{X}_b) &= a\, {}^{(3)}\nabla_a(u_c\nabla^c X_b) && \text{(A.7)}\\
&= a\, {}^{(3)}\nabla_a u_c(\nabla^c X_b) + a\, {}^{(3)}\nabla_a(\nabla^c X_b)u_c && \text{(A.8)}\\
&= a\Big(h_a{}^d h_c{}^e(\nabla_d u_e)\nabla^c X_b + {}^{(3)}\nabla_a(\nabla^c X_b)u_c\Big) && \text{(A.9)}\\
&= a\Big(H\, {}^{(3)}\nabla_a X_b + ({}^{(3)}\nabla_a X_b)^{\cdot}\Big) && \text{(A.10)}\\
&= \Big(\dot{a}\, {}^{(3)}\nabla_a X_b + a({}^{(3)}\nabla_a X_b)^{\cdot}\Big) && \text{(A.11)}\\
&= (a\, {}^{(3)}\nabla_a X_b)^{\cdot}\,, && \text{(A.12)}
\end{aligned}
$$

where the second term in (A.8) has been written at first order:

$$
\begin{aligned}
{}^{(3)}\nabla_a(\nabla^c X_b)u_c &= {}^{(3)}\nabla_a(\nabla^c)X_b u_c + \nabla^c\, {}^{(3)}\nabla_a(X_b)u_c && \text{(A.13)}\\
&= ({}^{(3)}\nabla_a X_b)^{\cdot}\,. && \text{(A.14)}
\end{aligned}
$$

We then see that

$$
\begin{aligned}
a\, {}^{(3)}\nabla_a(\dot{X}^a) &= a\Big(H\, {}^{(3)}\nabla_a X^a + ({}^{(3)}\nabla_a X^a)^{\cdot}\Big) && \text{(A.15)}\\
&= HX + a(Xa^{-1})^{\cdot} && \text{(A.16)}\\
&= \dot{X}\,. && \text{(A.17)}
\end{aligned}
$$

The same goes for the second time derivative of X_b

$$
\begin{aligned}
a\, {}^{(3)}\nabla_a(\ddot{X}_b) &= a\, {}^{(3)}\nabla_a(u_c\nabla^c\dot{X}_b) && \text{(A.18)}\\
&= \dot{a}\, {}^{(3)}\nabla_a\dot{X}_b + a({}^{(3)}\nabla_a\dot{X}_b)^{\cdot}\,, && \text{(A.19)}
\end{aligned}
$$

implying thus

$$
\begin{aligned}
a\, {}^{(3)}\nabla_a(\ddot{X}^a) &= \dot{a}\, {}^{(3)}\nabla_a\dot{X}^a + a({}^{(3)}\nabla_a\dot{X}^a)^{\cdot} && \text{(A.20)}\\
&= H\dot{X} + a(\dot{X}a^{-1})^{\cdot} && \text{(A.21)}\\
&= \ddot{X}\,. && \text{(A.22)}
\end{aligned}
$$

Appendix B

Equivalence between the standard and covariant approaches

B.1 From Ellis & Bruni to Bardeen GI variables

The standard approach to cosmological perturbation theory is based on the Bardeen GI variables [4]. These variables are nothing else than linear combination of gauge-dependent first-order quantities constructed ad hoc to be GI first-order variables. This approach, unlike the CGI one is based on coordinates; therefore most of these Bardeen GI variables acquire a physical or geometrical significance only once a specific gauge is chosen. Expanding the E&B CGI variables [1, 2], at first-order, we expect to recover Bardeen first-order variables; this is indeed the case. We show this equivalence for our key GI quantity $\mathcal{D}_a$, we refer to [2] for a complete list. In the following calculations we have used the metric:

$$ds^2 = -(1+2\phi)dt^2 + 2a\partial_i B\, dt\, dx^i + a^2(t)[(1-2\psi)\delta_{ij} + 2\partial_i\partial_j E]dx^i dx^j \; . \tag{B.1}$$

The inhomogeneous linear, scalar perturbations are given by

$$\rho(t,x^i) = \bar{\rho}(t) + \delta\rho(t,x^i) \tag{B.2}$$

$$P(t,x^i) = \bar{P}(t) + \delta P(t,x^i) \tag{B.3}$$

$$V(t,x^i) = \bar{V}(t) + \delta V(t,x^i)\,, \tag{B.4}$$

the four-velocity of matter is given by:

$$u^\mu = [1-\phi, a^{-1}\partial^i v]\,, \qquad u_\mu = [-1-\phi, \partial_i\theta]\,, \tag{B.5}$$

with $\theta = a(v+B)$.

Spatial metric

$$h_a{}^b = \delta_a{}^b + u_a u_b \tag{B.6}$$
$$\bar{h}_a{}^b + \delta h_a{}^b = \delta_a{}^b + \bar{u}_a\bar{u}^b + \bar{u}_a\delta u^b + \delta u_a\bar{u}^b \tag{B.7}$$
$$\delta h_a{}^b = \bar{u}_a\delta u^b + \delta u_a\bar{u}^b\,. \tag{B.8}$$

Spatial gradient of a scalar quantity

$$^{(3)}\nabla_a f \equiv h_a{}^b\nabla_b f \tag{B.9}$$
$$= \Big(\delta_a{}^b + \bar{u}_a\bar{u}^b + \bar{u}_a\delta u^b + \delta u_a\bar{u}^b\Big)\nabla_b(\bar{f} + \delta f) \tag{B.10}$$
$$= \bar{h}_a{}^b\nabla_b(\bar{f} + \delta f) + \bar{u}_a\delta u^b\nabla_b\bar{f} + \bar{u}^b\delta u_a\nabla_b\bar{f} \tag{B.11}$$
$$= \bar{h}_a{}^0(\bar{f} + \delta f)^{\cdot} + \bar{h}_a{}^j\partial_j(\bar{f} + \delta f) + \bar{u}_a\delta u^0\dot{\bar{f}} + \delta u_a\dot{\bar{f}} \tag{B.12}$$
$$= \bar{h}_a{}^j\partial_j\delta f + \bar{u}_a\delta u^0\dot{\bar{f}} + \delta u_a\dot{\bar{f}} \tag{B.13}$$
$$= \left[\phi\dot{\bar{f}} - \phi\dot{\bar{f}}, \Big(\delta_i{}^j + \bar{u}_i\bar{u}^j\Big)\partial_j\delta f + \partial_i\theta\phi\dot{\bar{f}}\right] \tag{B.14}$$
$$= \left[0, \partial_i\Big(\delta f + \theta\dot{\bar{f}}\Big)\right]. \tag{B.15}$$

From this generic expression we can write:

$$\mathcal{D}_a \equiv \frac{a}{\rho}\,^{(3)}\nabla_a\rho = \left[0, a\,\partial_i\Big(\delta + \theta\frac{\dot{\rho}}{\rho}\Big)\right] \tag{B.16}$$
$$= \left[0, \partial_i\Big(a\,\delta - 3\mathcal{H}\theta(1+w) - \frac{a\theta Q}{\rho}\Big)\right] \tag{B.17}$$

If we now consider the Bardeen GI density contrast defined in Section 1.4:

$$\Delta_B = \delta + \frac{\rho'}{\rho}(v + B)\,, \tag{B.18}$$

we immediately realize the below relation:

$$\mathcal{D}_a = a\,\chi_a = a\,^{(3)}\nabla_a\Delta_B\,. \tag{B.19}$$

Namely, the GI Bardeen variable Δ_B is the first order 'scalar' potential for the CGI comoving density gradient $\mathcal{D}_a$. Notice that $\mathcal{D}_a = [0, \partial_i(a\,\delta)] = a\,[0, \partial_i\,\Delta_B]$ in the case we choose the comoving orthogonal gauge, where $v = B = \theta = 0$.

B.2 Standard and CGI interacting vacuum scenario

Let's show explicitly the equivalence between our equations from Section 3.2.1 and the ones in [24, 25, 27]. Be begin by expanding some relevant kinematical quantities:

Expansion scalar

$$
\begin{aligned}
\Theta \equiv \nabla_a u^a &= \partial_a u^a + \Gamma^a_{ab} u^b && \text{(B.20)}\\
&= \bar{u}^a{}_a + \bar{\Gamma}^a_{ab}\bar{u}^b + \delta u^a{}_a + \Gamma^a_{ab}\delta u^b + \delta\Gamma^a_{ab}\bar{u}^b && \text{(B.21)}\\
&= \bar{\Theta} - \dot{\phi} + a^{-1}\nabla^2 v - \phi\bar{\Gamma}^a_{a0} + a^{-1}\partial^i v \Gamma^a_{ai} + \delta\Gamma^a_{a0} && \text{(B.22)}\\
&= \bar{\Theta} + a^{-1}\nabla^2 v - \bar{\Theta}\phi + \nabla^2\dot{E} - 3\dot{\psi} && \text{(B.23)}\\
&= \bar{\Theta} + \delta\Theta\,. && \text{(B.24)}
\end{aligned}
$$

where ∇^2 is a spatial laplacian, i.e. $\nabla^2 = \partial^i\partial_i$.

Acceleration

$$
\begin{aligned}
a_a \equiv u^b\nabla_b u_a &= \Big(\nabla_b u_a - \Gamma^c_{ab}u_c\Big)u_b && \text{(B.25)}\\
&= \Big(\partial_b\bar{u}_a + \partial_b\delta u_a - \bar{\Gamma}^c_{ab}\bar{u}_c - \bar{\Gamma}^c_{ab}\delta u_c - \delta\Gamma^c_{ab}\bar{u}_c\Big)(\bar{u}^b + \delta u^b) && \text{(B.26)}\\
&= \partial_b\bar{u}_a\delta u^b + \partial_b\delta u_a\bar{u}^b - \bar{\Gamma}^c_{ab}\bar{u}_c\delta u^b - \bar{\Gamma}^c_{ab}\delta u_c\bar{u}^b - \delta\Gamma^c_{ab}\bar{u}_c\bar{u}^b && \text{(B.27)}\\
&= \dot{\delta u}_a + a^{-1}\partial^i v\bar{\Gamma}^0_{ai} - \partial_i\theta\bar{\Gamma}^i_{a0} + \delta\Gamma^0_{a0} && \text{(B.28)}\\
&= \Big[-\dot{\phi} + \dot{\phi}, \partial_i\dot{\theta} + a^{-1}\partial^j v\, a^2 H\delta_{ij} - \partial_j\theta H\delta^j{}_i + \partial_i\phi + \partial_i B\dot{a}\Big] && \text{(B.29)}\\
&= \Big[0, \partial_i(\dot{\theta} + \phi)\Big]\,. && \text{(B.30)}
\end{aligned}
$$

From momentum conservation we can also write the above as:

$$
a_a = \Big[0, (\bar{\rho} + \bar{P})^{-1}\partial_i\Big(-\delta P - \theta\dot{\bar{P}} + \delta V + \theta\dot{\bar{V}}\Big)\Big]\,. \tag{B.31}
$$

B.2.1 Interacting vacuum equations

We can now expand our CGI energy-momentum conservation equations and show the equivalence at first order with [24, 25, 27]. From the energy conservation Eqn. (3.27) at linear order:

$$
(X_a + V_a)^{\cdot} + \frac{4}{3}\Theta(X_a + V_a) + (\rho + P)Z_a = 0\,, \tag{B.32}
$$

using the above relations we obtain:

$$
\begin{aligned}
a_a\dot{\rho} + {}^{(3)}\nabla_a\dot{\rho} - H\,{}^{(3)}\nabla_a\rho + a_a\dot{V} + {}^{(3)}\nabla_a\dot{V} - H\,{}^{(3)}\nabla_a V + \\
+ 4H\,{}^{(3)}\nabla_a\rho + 4H\,{}^{(3)}\nabla_a V + (\rho+P)\,{}^{(3)}\nabla_a\Theta = 0 \quad \text{(B.33)}
\end{aligned}
$$

$$
\begin{aligned}
\Big[-\delta P + \delta V + \theta(-\dot{P}+\dot{V})\Big](\rho+P)^{-1}(\dot{\rho}+\dot{V}) + \Big[\dot{\delta\rho} + \dot{\delta V} + \theta(\ddot{\rho}+\ddot{V})\Big] + \\
+ 3H\Big[\delta\rho + \delta V + \theta(\dot{\rho}+\dot{V})\Big] + (\rho+P)(\delta\Theta + \theta\dot{\Theta}) = 0 \quad \text{(B.34)}
\end{aligned}
$$

$$
\dot{\delta\rho} + 3H(\delta\rho+\delta P) - 3(\rho+P)\dot{\psi} + (\rho+P)\frac{\nabla^2}{a^2}(\theta + a^2\dot{E} - aB) = -\dot{\delta V}\,. \quad \text{(B.35)}
$$

From $Q = \dot{V}$ we have:

$$
\begin{aligned}
\bar{Q} + \delta Q &= u^\mu\nabla_\mu V && \text{(B.36)}\\
&= (\bar{u}^\mu + \delta u^\mu)\nabla_\mu(\bar{V} + \delta V) && \text{(B.37)}\\
&= \dot{\bar{V}} + \dot{\delta V} - \phi\dot{\bar{V}} + a^{-1}\partial^i v\,\partial_i\bar{V}\,, && \text{(B.38)}
\end{aligned}
$$

thus at first order:

$$
\delta Q = \dot{\delta V} - \phi Q\,. \quad \text{(B.39)}
$$

We remark the fact that the time derivative of V along the fluid flow, Eqn. (B.36), is valid in general for every scalar quantity:

$$
\begin{aligned}
\dot{f} &\equiv u^\mu\nabla_\mu f && \text{(B.40)}\\
&= (\bar{u}^\mu + \delta u^\mu)\nabla_\mu(\bar{f} + \delta f) && \text{(B.41)}\\
&= \dot{\bar{f}} + \dot{\delta f} - \phi\dot{\bar{f}}\,. && \text{(B.42)}
\end{aligned}
$$

We remark once more the fact that the splitting of a given tensorial quantity (a scalar in this case) into a background plus a small perturbation is not a covariant procedure, i.e. coordinate independent. This leads to a gauge (coordinate) dependence which can be noticed from the $\phi\dot{\bar{f}}$ term. Only once we have fixed a particular gauge, e.g. a synchronous gauge ($\phi = 0 = B$), the latter term vanishes and we obtain the expected splitting:

$$
\dot{f} = \dot{\bar{f}} + \dot{\delta f}\,. \quad \text{(B.43)}
$$

From the momentum conservation Eqn. (3.18) at linear order:

$$
\begin{aligned}
a_a(\rho+P) + Y_a - V_a = 0 && \text{(B.44)}\\
\Big[(\dot{\theta}+\phi)(\rho+P) + \delta P + \theta\dot{P} - \delta V - \theta\dot{V}\Big] = 0 && \text{(B.45)}\\
(\rho+P)\dot{\theta} + \theta\dot{\rho}c_s^2 - \theta\dot{V} + (\rho+P)\phi + \delta P = \delta V\,. && \text{(B.46)}
\end{aligned}
$$

From $V_a = -f_a$ we have:

$$V_a = -f_a \tag{B.47}$$

$$\left[0, \partial_i(\delta V + \theta \dot{V})\right] = -\left[0, \partial_i f\right] \tag{B.48}$$

thus at first order:

$$\delta V = -f - \theta Q\,, \tag{B.49}$$

from which we recover

$$(\rho + P)\dot{\theta} - 3c_s^2 H(\rho + P)\theta + (\rho + P)\phi + \delta P = -f + c_s^2 Q\theta\,. \tag{B.50}$$

Following the same procedure for the homogeneous vacuum model ($f^\mu = 0$) in Section 3.4, starting from:

$$(\mathcal{D}_a)^{\cdot} = \frac{Q}{\rho}\mathcal{D}_a - \mathcal{Z}_a\,, \tag{B.51}$$

we obtain:

$$\dot{\delta} = \frac{Q}{\rho}\delta + 3\dot{\psi} - \frac{\nabla^2}{a^2}(\theta + a^2\dot{E} - aB)\,. \tag{B.52}$$

B.2.2 Remark on the CGI approach

In the CGI approach scalars like ρ, that do not vanish in the background, cannot be gauge-invariantly split into a perturbation and a background value. In this approach, we avoid this gauge problem by not directly using perturbations of scalars (e.g. $\delta\rho$), but instead by using their spatial gradients (e.g. ${}^{(3)}\nabla_a\rho$). Since these vanish in the background, they are automatically GI. However, the CGI approach is not frame independent, since it depends on a choice of fundamental 4-velocity u^a. A change of fundamental 4-velocity (a linearized Lorentz boost):

$$u^a \quad \rightarrow \quad \tilde{u}^a = u^a + v^a\,, \qquad u^a v_a = 0 \tag{B.53}$$

leads to the transformations:

$$\tilde{\rho} = \rho\,, \quad \tilde{P} = P\,, \quad \tilde{\Theta} = \Theta + {}^{(3)}\nabla_a v^a \tag{B.54}$$

$$\tilde{\omega}_{ab} = \omega_{ab} + {}^{(3)}\nabla_{[a}v_{b]}\,, \quad \tilde{\sigma}_{ab} = \sigma_{ab} + {}^{(3)}\nabla_{\langle a}v_{b\rangle} \tag{B.55}$$

where the angle brackets in ${}^{(3)}\nabla_{\langle a}v_{b\rangle}$ stands for its symmetric and traceless part. This is the CGI analogue of a gauge transformation in the standard formalism.

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