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Analytic aspects of analogue Hawking radiation in Bose-Einstein condensates

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To all those who have encouraged me to always look beyond

Children are not afraid to pose basic questions that may embarrass us, as adults, to ask.

Roger Penrose,
The Emperor's New Mind
(1989)

Abstract

We present a theoretical study of analogue Hawking effect in stepwise models of Bose-Einstein condensates. We focus on subsonic-supersonic configuration, where two stationary homogeneous condensates are connected by a step-like discontinuity in the local speed of sound. We provide a detailed analysis of the scattering processes of fluctuation modes at the sonic horizon based on microscopic Bogoliubov theory of dilute BECs. Spontaneous phonon emission is predicted to occur at the horizon as a conversion of vacuum fluctuations into on-shell real particles. The condensed-matter analogue of Hawking radiation arises as a thermal Bose spectrum to low frequency order in the subsonic region. Stepwise BEC configurations with an extended sonic region are also taken into account in order to address more general and realistic velocity flow profiles.

Contents

Introduction

Acoustic black holes are the sonic analogue of gravitational black holes and are characterized by a flow transition from subsonic to supersonic regime. This analogy relies on the formal correspondence between the equation of motion for a long wavelength sound wave in a moving fluid and the wave equation for a massless scalar field in curved space-time. Therefore, sound waves couple to a fictitious curved metric, which is called *acoustic metric*. Sound waves are trapped inside the supersonic region beyond the locus where the fluid moves from subsonic to supersonic flow (which is called *sonic horizon*), similarly to light beyond the event horizon of a gravitational black hole.

One then may proceed to quantize the field describing sound waves and derive the analogue of photon Hawking radiation. The Hawking radiation is a quantum effect triggered by a collapsing black hole. According to this, off-shell vacuum fluctuations are converted into on-shell particles in the near-horizon region and detected at late-time with a steady thermal black body spectrum. Among the systems which may show an analogue version of this phenomenon for phonons, the most promising candidates are Bose-Einstein condensates. Being a BEC characterized by a macroscopic occupation of the ground state, one may assume to study quantum fluctuations above the condensate wave-function. A simple model can be developed where two semi-infinite stationary homogeneous BECs, one subsonic and the other supersonic, are separated by a steplike discontinuity in the speed of sound, which plays the role of a sonic horizon. By studying scattering processes of fluctuation modes at the discontinuity, a flux of phonons is predicted to originate in the subsonic region from the horizon, with a thermal Bose distribution to the low frequency order. This indeed is regarded as the analogue of black hole Hawking radiation.

This thesis work aims at deepening the theoretical picture of Hawking radiation in acoustic black holes. Chapter 1 is dedicated to on overview of the main reference frames for Schwarzschild metric solving the coordinate singularity at the event horizon. After reviewing null coordinates in Minkowski space-time, the corresponding Schwarzschild versions will be presented, that Eddington-Finkelstein coordinates. Then, the maximal analytic extension of Schwarzschild metric will be presented, that is the set of Kruskal coordinates. These combine the black hole and white hole pictures yielded by advanced and retarded Eddington-Finkelstein extensions, respectively. Finally, the Painlevé-Gullstrand extension will be shown, which is a regular coordinate system across the horizon based on time-like trajectories and admits a river analogy for black hole.

Chapter 2 summarizes quantum field theory in curved space-time. Firstly, the canonical quantization procedure in flat space-time will be recalled. Then, the generalization to a curved space-time will be outlined, concentrating on the particle interpretation issue. Then, this framework will be applied to a Schwarzschild black hole background, where Boulware and Unruh vacuua will be distinguished. In the last section, we will sketch the derivation of the Hawking radiation emitted by a gravitational black hole.

Chapter 3 analyzes Bose-Einstein condensates and their quantum fluctuations. After revising the ideal Bose gas and the condition for Bose-Einstein condensation, the weakly interacting dilute Bose gas and the Bogoliubov approximation for the condensate ground state will be presented. Then, the Gross-Pitaevskii equation for the condensate wave-function will be introduced, along with the Bogoliubov-de Gennes equations for elementary excitations. These will be deepened in the second quantization formalism within the microscopic Bogoliubov theory for Bose gas. At the end of this chapter, the gravitational analogy based on atomic Bose-Einstein condensates will be studied under the hydrodynamic approximation. This will yield the acoustic black hole model and predict an analogue version of Hawking radiation.

The purpose of Chapter 4 is to develop a microscopic BEC theory for acoustic black holes. This will be dealt with a stepwise model for flowing condensates undergoing a transition from subsonic to supersonic regime. By analyzing the solutions to Bogoliubov dispersion relation, we will be able to develop a scattering theory for the modes propagation at the interface between the two distinct sectors. We will concentrate on the *in* basis to expand the field operator describing quantum fluctuations and analytically calculate each scattering amplitude for propagating modes. Some details will be given in Appendix A. The last section will show how a Hawking-like phonon emission is originated from the Bogoliubov transformation between *in* and *out* vacua.

In Chapter 5, we will present a new model for acoustic black holes. It is still a stepwise model where now a sonic region is inserted between the subsonic and supersonic sectors. This will first require to study the Bogoliubov dispersion relation on the sonic horizon. Then, some predictions about the supersonic-sonic and sonic-subsonic configurations will be reported, as preliminary steps towards the complete model. In the last section, the supersonic-sonic-subsonic problem will be set-up, dealing with two interdependent systems of matching conditions.

The algebraic details will be reported in Appendix B.

Chapter 1

Analytic extensions of Schwarzschild metric

In this chapter we shall provide an overview of some coordinate systems for Schwarzschild metric which are regular at the event horizon [1],[2], [3]. Due to their property of *extending* the domain of standard Schwarzschild coordinates, they are called *analytic extensions of Schwarzschild metric.* As a matter of convenience, we set the speed of light c equal to unity along the dissertation.

1.1 Eddington-Finkelstein coordinates

1.1.1 The Schwarzschild metric

In a space-time region where the energy-momentum tensor $T_{\mu\nu}$ and the electric charge are vanishing, the Einstein's field equations reduce to the *vacuum Einstein's equation*

$$R_{\mu\nu} = 0, \tag{1.1.1}$$

where $R_{\mu\nu}$ is the Ricci tensor. If the massive source generating the gravitational field is spherically-symmetric, the solution to (1.1.1) can be expressed as

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{1.1.2}$$

known as *Schwarzschild metric.* We can observe it is both time-independent and invariant under time reversal, which constitutes the

Birkhoff Theorem *A spherically symmetric vacuum solution in the exterior region is necessarily static.*

s is the invariant proper time under general coordinate transformations, θ and ϕ are the usual angular spherical polar coordinates, while r and t are specific redefinitions of the radial and time coordinates, respectively, so as to diagonalize the metric. The m parameter is defined as

$$2m = 2GM \equiv R_H, \tag{1.1.3}$$

which is called *Schwarzschild radius.* The Schwarzschild metric presents two singularities, one at $r = 0$ and the other at $r = 2m$. While the former is a true physical singularity (since General Relativity collapses there), the latter is just a *coordinate* singularity. Indeed, the Riemann tensor contraction (*Kretschmann invariant*) $R_{\mu\nu\lambda k}R^{\mu\nu\lambda k} = 48m^2/r^6$ turn out to be *finite* around $r = 2m$.

Then, we shall look for coordinate systems which eliminate such non-physical singularity. Let us consider for a moment radial null directions in Minkowski metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{1.1.4}$$

$$\begin{cases} \text{null} \longrightarrow \mathrm{d}s^2 = 0 \\ \text{radial} \longrightarrow \theta, \phi = const \end{cases} \implies \mathrm{d}t^2 - \mathrm{d}r^2 = 0,$$

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm 1 \implies \begin{cases} u \equiv t - r = const \\ v \equiv t + r = const \end{cases},$$

where we have defined the retarded u and advanced v null coordinates. Upon varying θ and ϕ, these depict radial null directions delimiting the light cone at a certain space-time point. We can then re-express the Minkowski metric in terms of either u, v or both u and v coordinates (this last case is referred to as *double null* coordinates).

$$(u, r, \theta, \phi) : \mathrm{d}s^2 = \mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r - r^2\mathrm{d}\Omega^2, \tag{1.1.5a}$$

$$(v, r, \theta, \phi) : \mathrm{d}s^2 = \mathrm{d}v^2 - 2\mathrm{d}v\mathrm{d}r - r^2\mathrm{d}\Omega^2, \tag{1.1.5b}$$

$$(u, v, \theta, \phi) : \mathrm{d}s^2 = \mathrm{d}u\mathrm{d}v - r^2(u, v)\mathrm{d}\Omega^2, \tag{1.1.5c}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$ is the solid angle element. Notice now the Minkowski metric is no longer *static* with respect to the new "time" coordinate u or v, but rather *stationary.* As concerns their physical meaning, both $\Sigma_u : u = const$ and $\Sigma_v : v = const$ define a 2-dimensional surface with $(0, -, -)$ signature, i.e. a *null* surface.

$$\mathrm{d}s^2\Big|_{u=const} = \mathrm{d}s^2\Big|_{v=const} = -r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{1.1.6}$$

Hence, Σ_u (Σ_v) represents a spherical front of radius r expanding (contracting) at the speed of light $\frac{\mathrm{d}r}{\mathrm{d}t} = 1$ ($\frac{\mathrm{d}r}{\mathrm{d}t} = -1$). That is why u and v are said *outgoing* and *ingoing* null coordinates, respectively.

Let us go back to Schwarzschild geometry and assume our source is point-like. This permits the Schwarzschild metric to hold for all the range $r > 0$. We take radial null directions into account again:

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{1.1.7}$$

$$\begin{cases}\text{null} \longrightarrow \mathrm{d}s^2 = 0 \\ \text{radial} \longrightarrow \theta, \phi = const\end{cases} \implies \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 = 0,$$

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\left(1 - \frac{2m}{r}\right)^{-1}.$$

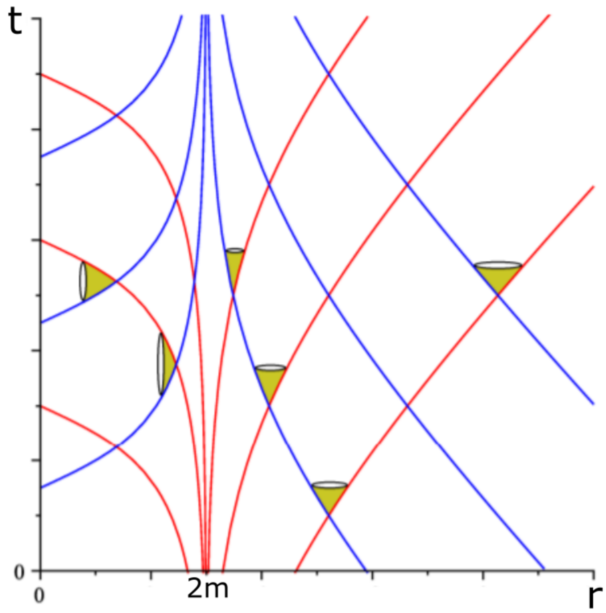

Figure 1.1.1: Null geodesics and light-cones in Schwarzschild metric according to standard time and space Schwarzschild coordinates (t, r) [4]. Red lines refer to $\frac{\mathrm{d}t}{\mathrm{d}r} = +\left(1 - \frac{2m}{r}\right)^{-1}$ solution, while blue lines to $\frac{\mathrm{d}t}{\mathrm{d}r} = -\left(1 - \frac{2m}{r}\right)^{-1}$.

By looking at Fig.(1.1.1), we note light-cones shrink at $r = 2m$ due to the metric singularity. Moreover, physical motions with $r = const$ ($t = const$) are forbidden at $r < 2m$ ($r > 2m$), since they lie outside light-cones. Analogously to the Minkowski case, we can define a pair of null coordinates. Indeed, we have:

$$\mathrm{d}t = \pm \frac{\mathrm{d}r}{1 - \frac{2m}{r}} = \pm \mathrm{d}r^*, \tag{1.1.8}$$

where we have defined the so-called *Regge-Wheeler tortoise coordinate*

$$r^* \equiv \int \frac{\mathrm{d}r}{1 - \frac{2m}{r}} = r + 2m \ln\left|\frac{r}{2m} - 1\right|. \tag{1.1.9}$$

Therefore,

$$\begin{cases} \mathrm{d}u = 0 \\ \mathrm{d}v = 0 \end{cases}, \tag{1.1.10}$$

where $u \equiv t - r^*$ and $v \equiv t + r^*$ are the new retarded (outgoing) and advanced (ingoing) null radial coordinates, respectively, which are known as *Eddington-Finkelstein coordinates*. These allow us to find extensions of Schwarzschild metric which are regular at $r = 2m$.

1.1.2 Advanced Eddington-Finkelstein extension

Let us consider the set of coordinates (v, r, θ, ϕ), i.e. move from t to $v = t + r^*$. Then, the Schwarzschild metric becomes

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\left[\mathrm{d}v^2 - \frac{2\mathrm{d}v\mathrm{d}r}{1 - \frac{2m}{r}} + \frac{\mathrm{d}r^2}{\left(1 - \frac{2m}{r}\right)^2}\right] - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{1.1.11}$$

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}v^2 - 2\mathrm{d}v\mathrm{d}r - r^2\mathrm{d}\Omega^2. \tag{1.1.12}$$

This is known as *advanced Eddington-Finkelstein extension* and results being regular at $r = 2m$. Now we shall define a new time coordinate via v and r in such a way:

$$t' \equiv v - r = t + r^* - r = t + 2m \ln\left|\frac{r}{2m} - 1\right|, \tag{1.1.13}$$

which yields

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t' - \frac{4m}{r}\mathrm{d}t'\mathrm{d}r - \left(1 + \frac{2m}{r}\right)\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{1.1.14}$$

This expression for the metric is t'-independent but not invariant under $t' \longrightarrow -t'$ transformation. Furthermore, $\Sigma_{t'} : t' = const$ surfaces have $(-, -, -)$ signature, i.e. they are *space-like*

$$\mathrm{d}s^2\Big|_{t'=const} = -\left(1 + \frac{2m}{r}\right)\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 \tag{1.1.15}$$

and we can conclude t' is a well-defined time coordinate. In the new set of coordinates (t', r, θ, ϕ), radial null directions ($\mathrm{d}s^2 = 0$ and $\theta, \phi = const$) are identified by

$$\left(1 - \frac{2m}{r}\right)\left(\frac{\mathrm{d}t'}{\mathrm{d}r}\right)^2 - \frac{4m}{r}\left(\frac{\mathrm{d}t'}{\mathrm{d}r}\right) - \left(1 + \frac{2m}{r}\right) = 0, \tag{1.1.16}$$

$$\frac{\mathrm{d}t'}{\mathrm{d}r} = \frac{\frac{2m}{r} \pm \sqrt{\frac{4m^2}{r^2} + \left(1 - \frac{2m}{r}\right)\left(1 + \frac{2m}{r}\right)}}{1 - \frac{2m}{r}} = \begin{cases} \frac{1+\frac{2m}{r}}{1-\frac{2m}{r}} \\ -1 \end{cases} . \tag{1.1.17}$$

The $\frac{\mathrm{d}t'}{\mathrm{d}r} = -1$ solution represents ingoing null directions ($t' + r = v = const$). The other solution can also be written as

$$\frac{\mathrm{d}r}{\mathrm{d}t'} = \frac{1 - \frac{2m}{r}}{1 + \frac{2m}{r}} \begin{cases} > 0 & \text{if } r > 2m \\ = 0 & \text{if } r = 2m \\ < 0 & \text{if } r < 2m \end{cases} \tag{1.1.18}$$

and defines outgoing null directions. Notice that for $r > 2m$ past light-cones never enter $r < 2m$ region, while for $r < 2m$ future light-cones never escape towards $r > 2m$ region (Fig.(1.1.2)). This entails the region $r < 2m$ is causally disconnected from the rest of the Universe: such a portion of space-time is called *black hole*. Its boundary ($r = 2m$ in our case) is called *event horizon* and is a null surface. Once light or matter crosses the event horizon, it can never go back.

1.1.3 Retarded Eddington-Finkelstein extension

The *retarded Eddington-Finkelstein extension* is defined by moving from t to $u = t - r^*$ coordinate

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r - r^2\mathrm{d}\Omega^2, \tag{1.1.19}$$

which again is regular at $r = 2m$. Analogously to the advanced extension, we shall define a new time coordinate

$$\hat{t} \equiv u + r = t - r^* + r = t - 2m \ln\left|\frac{r}{2m} - 1\right| \tag{1.1.20}$$

and in the $(\hat{t}, r, \theta, \phi)$ set of coordinates the metric becomes

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}\hat{t}^2 + \frac{4m}{r}\mathrm{d}\hat{t}\mathrm{d}r - \left(1 + \frac{2m}{r}\right)\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{1.1.21}$$

Now, radial null directions are found by

$$\left(1 - \frac{2m}{r}\right)\left(\frac{\mathrm{d}\hat{t}}{\mathrm{d}r}\right)^2 + \frac{4m}{r}\left(\frac{\mathrm{d}\hat{t}}{\mathrm{d}r}\right) - \left(1 + \frac{2m}{r}\right) = 0, \tag{1.1.22}$$

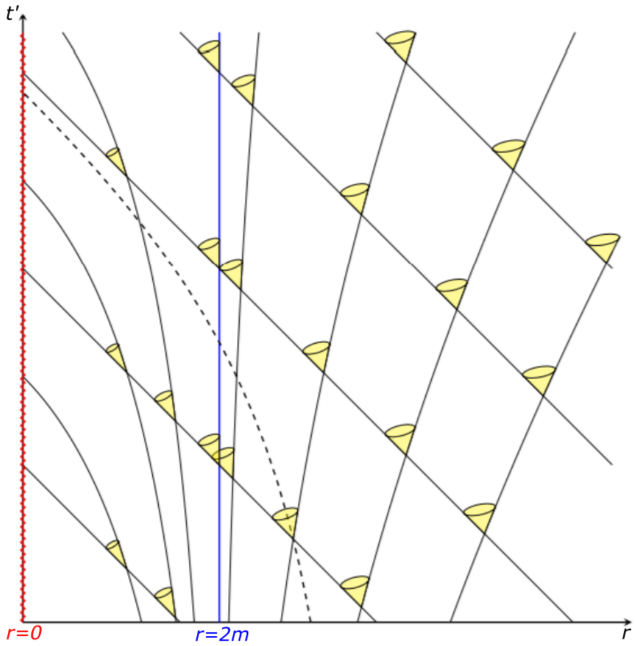

Figure 1.1.2: Null geodesics and light-cones in Schwarzschild metric according to advanced Eddington-Finkelstein time and space coordinates (t', r) [5]. The dashed line represents a time-like geodesic directed to the singularity.

$$\frac{\mathrm{d}\hat{t}}{\mathrm{d}r} = \frac{-\frac{2m}{r} \pm \sqrt{\frac{4m^2}{r^2} + \left(1 - \frac{2m}{r}\right)\left(1 + \frac{2m}{r}\right)}}{1 - \frac{2m}{r}} = \begin{cases} 1 \\ \frac{-\left(1+\frac{2m}{r}\right)}{1-\frac{2m}{r}} \end{cases} , \tag{1.1.23}$$

that is, $\frac{\mathrm{d}\hat{t}}{\mathrm{d}r} = 1$ solution represents outgoing null directions ($\hat{t} - r = u = const$), while

$$\frac{\mathrm{d}r}{\mathrm{d}\hat{t}} = \frac{1 - \frac{2m}{r}}{-\left(1 + \frac{2m}{r}\right)} = -\frac{r - 2m}{r + 2m} \begin{cases} < 0 & \text{if } r > 2m \\ = 0 & \text{if } r = 2m \\ > 0 & \text{if } r > 2m \end{cases} \tag{1.1.24}$$

defines ingoing null directions. Contrary to the advanced extension, for $r > 2m$ future light-cones never enter the $r < 2m$ region, while for $r < 2m$ past light-cones never intersects $r > 2m$ region (Fig.(1.1.3)). This entails the region $r < 2m$ can be traversed only from inside to outside. Such a portion of space-time is called *white hole*. Its boundary $r = 2m$ is again a null surface.

Both black hole and white hole horizons are found to be located at $r = 2m$. Nevertheless, as regards the effect on a light signal moving near the horizon, the former behaves as an infinite red-shift surface, whereas the latter as an infinite blue-shift one. Then, some mathematical concerns affect the white hole solution.

Contrary to the black hole, the white hole horizon is verified to be an unstable solution to Einstein's field equations. In addition, in order to deal with a proper Cauchy problem, initial conditions are supposed to be put on the $r = 0$ singularity, which are totally arbitrary. All of this, together with the empirical fact that an astrophysical white hole has never been detected, hints at the white hole not being a true physical solution.

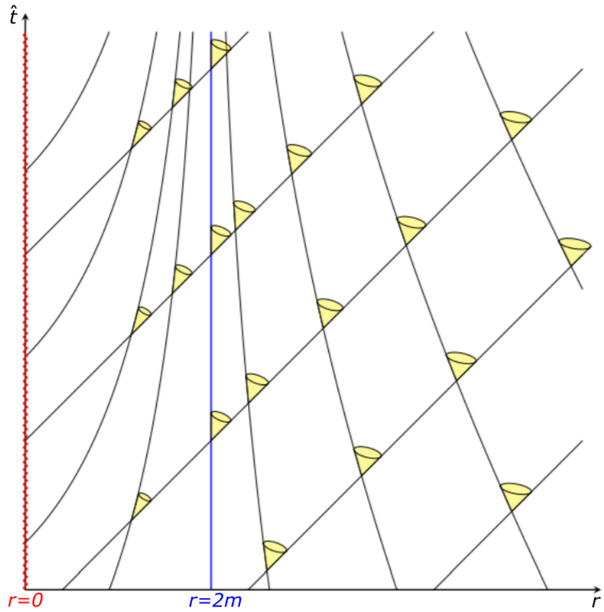

Figure 1.1.3: Null geodesics and light-cones in Schwarzschild metric according to retarded Eddington-Finkelstein time and space coordinates $(\hat{t}, r)$ [5].

1.2 Kruskal coordinates

So far, starting from null coordinates $v = t + r^*$ and $u = t - r^*$, we have got two kinds of Eddington-Finkelstein extensions which depict two completely different pictures. On the one hand, the advanced (black hole) extension owns an attractive and stable behaviour, on the other hand the retarded (white hole) extension a repulsive and unstable one. In a certain way, one is the time-reversal of the other.

Let us now have a look at time-like radial geodesics in Schwarzschild coordinates. They are defined via

$$\tilde{E}^2 = \dot{r}^2 + V_{eff}, \tag{1.2.1}$$

where $\tilde{E} \equiv u_0 = g_{00}\,\dot{t}$ ($\dot{t} \equiv u^0 = \frac{\mathrm{d}t}{\mathrm{d}s}$) stands for the energy per unit mass, $\dot{r} \equiv u^1 =$

$\frac{\mathrm{d}r}{\mathrm{d}s}$, and V_{eff} is the effective potential in Schwarzschild geometry

$$V_{eff} = \left(1 - \frac{2m}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right), \tag{1.2.2}$$

where $\tilde{L} \equiv u_3 = g_{33}\,\dot{\phi}$ ($\dot{\phi} \equiv u^3 = \frac{\mathrm{d}\phi}{\mathrm{d}s}$) is the angular momentum. In Fig.(1.2.1) we considered three possible time-like radial trajectories A, B and C:

- A comes from infinity and moves towards the singularity. It exists in the black hole solution only.
- B comes from the singularity and moves towards infinity. It exists in the white hole solution only.
- C comes from the singularity, crosses the horizon, reaches a maximum distance and finally goes back to the singularity. It exists neither in the black hole nor white hole solution.

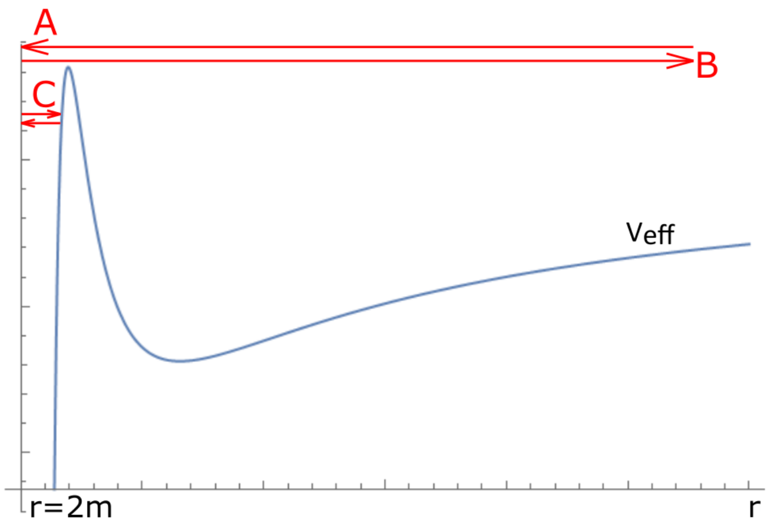

Figure 1.2.1: Effective potential in Schwarzschild metric $V_{eff} = \left(1 - \frac{2m}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right)$, with $\tilde{L}^2 > 12m^2$. A, B and C are three possible time-like radial trajectories admitted by Schwarzschild picture.

Since such a picture based on V_{eff} provides only a partial understanding of Schwarzschild space-time, we wonder whether it is possible to combine the two

Eddington-Finkelstein extensions simultaneously. First of all, we move to the double null form of the line element:

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r(u,v)}\right)\mathrm{d}u\mathrm{d}v - r^2(u,v)\mathrm{d}\Omega^2. \tag{1.2.3}$$

Actually, this is not an extension of Schwarzschild metric because it is singular at $r = 2m$ (light-cones collapse into lines). Then, in the $r > 2m$ region we perform the following coordinate transformation:

$$\begin{cases} u \longrightarrow U \equiv -e^{-u/4m} \\ v \longrightarrow V \equiv e^{v/4m} \end{cases}, \tag{1.2.4}$$

so that the double null form of Schwarzschild metric becomes

$$\begin{aligned} \mathrm{d}s^2 &= \frac{2m}{r}e^{-r/2m}e^{-u/4m}\mathrm{d}u\, e^{v/4m}\mathrm{d}v - r^2\mathrm{d}\Omega^2 \\ &= \frac{32m^3}{r}e^{-r/2m}\mathrm{d}U\mathrm{d}V - r^2\mathrm{d}\Omega^2, \end{aligned} \tag{1.2.5}$$

where now $r = r(U,V)$ and the metric is regular at $r = 2m$. U and V are called *Kruskal coordinates*. Starting from them, we can also define

$$T = \frac{V+U}{2}, \quad X = \frac{V-U}{2}, \tag{1.2.6}$$

which are indeed a time and space coordinate, respectively ($\mathrm{d}s^2\big|_{T=const}$ is a space-like surface and $\mathrm{d}s^2\big|_{X=const}$ a time-like one). With these coordinates, (1.2.5) changes into

$$\mathrm{d}s^2 = \frac{32m^3}{r}e^{-r/2m}(\mathrm{d}T^2 - \mathrm{d}X^2) - r^2\mathrm{d}\Omega^2. \tag{1.2.7}$$

Taking into account that

$$\left(\frac{r}{2m} - 1\right)e^{-r/2m} = X^2 - T^2, \tag{1.2.8}$$

we obtain a representation for Schwarzschild space-time as shown in Fig.(1.2.2).

For $r = const \geq 0$,

- if $r < 2m \implies X^2 - T^2 = const < 0 \implies$ two hyperbolae intersecting T axis,
- if $r = 2m \implies X^2 - T^2 = const = 0 \implies X = \pm T \implies$ two straight lines sloped $\pm 45°$ passing through the origin of (X,T) plane,

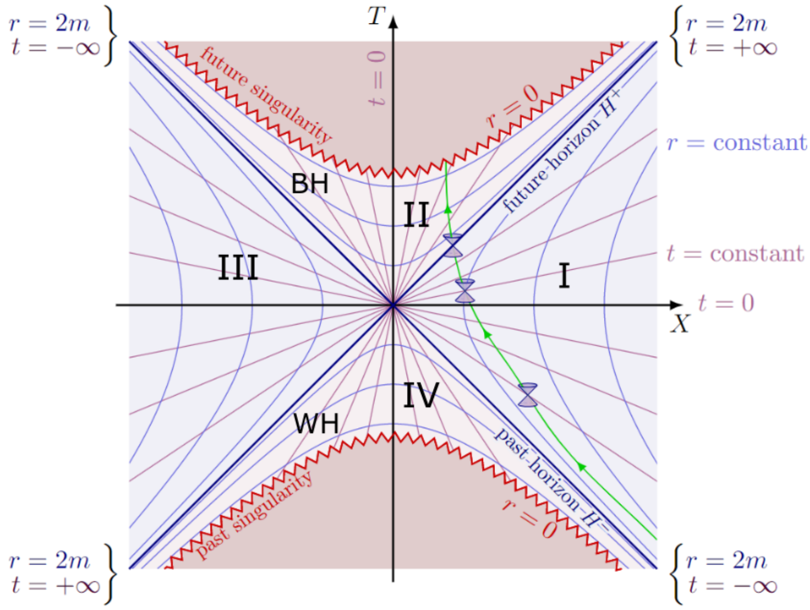

Figure 1.2.2: Kruskal diagram for Schwarzschild space-time [6]. H^+ and H^- define four distinct regions. $r = const$ trajectories are hyperbolae, while $t = const$ are straight lines passing through the origin.

- if $r > 2m \implies X^2 - T^2 = const > 0 \implies$ two hyperbolae intersecting X axis.

Fig.(1.2.2) shows the $(u, v) \longrightarrow (U, V)$ coordinate transformation has "doubled" Schwarzschild space-time. Referring to $X = \pm T$ lines, we can identify four different regions:

- region I: exterior of a black hole,
- region II: interior of a black hole,
- region III: exterior of a white hole,
- region IV: interior of a white hole.

We can also find that Schwarzschild time is given in terms of Kruskal coordinates by

$$\frac{t}{2m} = \ln\left(\frac{X+T}{X-T}\right) = 2\tanh^{-1}\left(\frac{T}{X}\right). \tag{1.2.9}$$

Then, $t = const$ trajectories correspond to straight lines passing through the origin in Kruskal diagram.

Radial null directions in Kruskal coordinates are given by

$$dT^2 - dX^2 = 0, \tag{1.2.10}$$

therefore light-cones are identified by straight lines sloped ±45°, like in Minkowski metric

$$X = \pm T + const. \tag{1.2.11}$$

Those corresponding to $r = 2m$, i.e. when $const = 0$, are said *future horizon* H^+ $(X = T)$ and *past horizon* H^- $(X = -T)$. They separate a black hole and a white hole from the rest of the Universe, respectively, as shown in Fig.(1.2.2). From this picture of light propagation, we can infer there cannot be causal connection between regions I and III.

All of this implies that Kruskal coordinates system provides a complete description of Schwarzschild space-time, therefore it is a *maximal analytic extension* of its manifold. Indeed, in the asymptotically flat regions I and III, $r = const$ trajectories are inside light-cones, while $t = const$ trajectories outside. Vice versa for regions II and IV. In particular, in region II (the black hole) all future light-cones point towards the future singularity $r = 0$ and every physical motion has decreasing values of r, whereas in region IV (the white hole) all past light-cones point towards the past singularity $r = 0$ and every physical motion has increasing values of r.

Finally, in Kruskal diagram we are now able to recover A, B and C trajectories simultaneously (Fig.(1.2.3)):

- A starts from region I and, after crossing the future horizon H^+, crushes onto the future singularity in region II,
- B starts from the past singularity in region IV and, after crossing the past horizon H^-, moves towards infinity in region I,
- C starts from the past singularity in region IV, crosses the past horizon H^-, travels along region I reaching a maximum of r, and, after crossing the future horizon H^+, crushes onto the future singularity in region II.

We point out that the transformations from Eddington-Finkelstein to Kruskal coordinates for all of the four regions are defined as follows:

$$U = \begin{cases} -e^{-u/4m} & \text{for region I and IV} \\ +e^{-u/4m} & \text{for region II and III} \end{cases} \qquad V = \begin{cases} +e^{v/4m} & \text{for region I and II} \\ -e^{v/4m} & \text{for region III and IV} \end{cases} . \tag{1.2.12}$$

Let us conclude this section by dealing with the physical meaning of Kruskal coordinates. Thanks to the Equivalence Principle, in a sufficiently small neighbourhood of a space-time point we can find a local inertial reference frame where

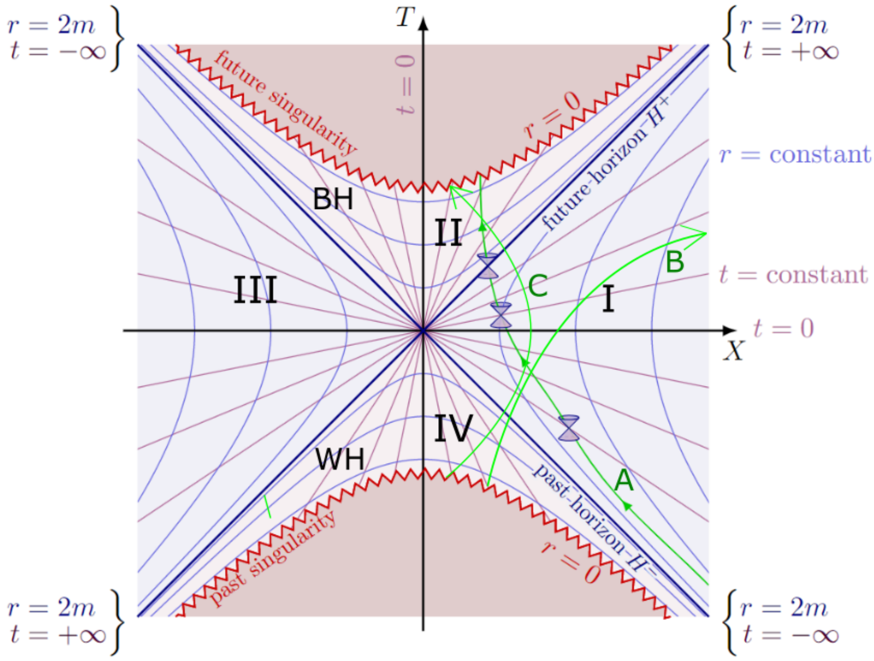

Figure 1.2.3: Kruskal diagram for Schwarzschild space-time [6]. The green curves represent A, B and C trajectories.

Special Relativity holds. If we choose our space-time point on the future horizon H^+ in terms of the advanced Eddington-Finkelstein coordinates

$$X^\mu_{v_0} = (v_0, 2m, \theta_0, \phi_0), \tag{1.2.13}$$

V Kruskal coordinate turns out to be the local inertial advanced null coordinate on H^+. Similarly, by choosing our space-time point on the past horizon H^- in terms of the retarded Eddington-Finkelstein coordinates

$$X^\mu_{u_0} = (u_0, 2m, \theta_0, \phi_0), \tag{1.2.14}$$

U Kruskal coordinate turns out to be the local inertial retarded null coordinate on H^-. U and V can be simultaneously inertial only in a neighbourhood of the origin of Kruskal diagram $(U = 0, V = 0)$, called *bifurcation point*.

1.3 Painlevé-Gullstrand coordinates

Unlike the previous systems, Painlevé-Gullstrand coordinates are an extension of Schwarzschild metric which is based on time-like geodesics [7]. Let us consider an observer moving along ingoing, radial, time-like geodesics of Schwarzschild space-time and starting from $r = +\infty$. Then, by defining the quantity

$$f \equiv 1 - \frac{2m}{r}, \tag{1.3.1}$$

the Schwarzschild metric can be rewritten, using the standard (t, r, θ, ϕ) coordinates, as

$$\mathrm{d}s^2 = f\mathrm{d}t^2 - f^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{1.3.2}$$

The geodesics equations can be expressed as (the angular momentum $\tilde{L}$ is set to zero)

$$\dot{t} = \frac{\tilde{E}}{f}, \qquad \dot{r}^2 + f = \tilde{E}^2. \tag{1.3.3}$$

Since we have chosen *ingoing* motion, r will be decreasing, therefore $\dot{r}$ solution will have negative sign

$$\dot{r} = -\sqrt{\tilde{E}^2 - f}. \tag{1.3.4}$$

At $r = +\infty$, due to the asymptotic flatness, $\tilde{E}$ is equal to the special relativistic version of u^0, that is the Lorentz factor γ

$$\tilde{E}\Big|_{r\to+\infty} = \gamma\Big|_{r\to+\infty} = \frac{1}{\sqrt{1 - v_\infty^2}}, \tag{1.3.5}$$

where v_∞ stands for the observer's initial speed at $r = +\infty$. We wish to concentrate on the case $v_\infty = 0$, which entails $\tilde{E} = 1$. The geodesics equations then reduce to

$$\dot{t} = \frac{1}{f}, \qquad \dot{r} = -\sqrt{1 - f}, \tag{1.3.6}$$

which allow us to write the observer's contravariant and covariant 4-velocity vectors as

$$u^\alpha = \left(\frac{1}{f}, -\sqrt{1-f}, 0, 0\right) \qquad u_\alpha = \left(1, \frac{\sqrt{1-f}}{f}, 0, 0\right). \tag{1.3.7}$$

We can express u_α as the gradient of a time function T

$$u_\alpha \equiv \partial_\alpha T, \tag{1.3.8}$$

where

$$T = t + \int \frac{\sqrt{1 - f(r)}}{f(r)}\mathrm{d}r = t + 4m\left(\sqrt{\frac{r}{2m}} + \frac{1}{2}\ln\left|\frac{\sqrt{\frac{r}{2m}} - 1}{\sqrt{\frac{r}{2m}} + 1}\right|\right), \tag{1.3.9}$$

except for an arbitrary integration constant. This shall be the new time coordinate, thus defining the *Painlevé-Gullstrand coordinates* as (T, r, θ, ϕ). Notice that along a time-like geodesic

$$\mathrm{d}T = \dot{t}\,\mathrm{d}s + \frac{\sqrt{1-f}}{f}\dot{r}\,\mathrm{d}s = \left[\frac{1}{f} + \frac{\sqrt{1-f}}{f}(-\sqrt{1-f})\right]\mathrm{d}s = \mathrm{d}s, \tag{1.3.10}$$

which means T corresponds to the proper time coordinate along a time-like geodesic. We now move from t to T coordinate

$$\mathrm{d}t = \mathrm{d}T - f^{-1}\sqrt{\frac{2m}{r}}\,\mathrm{d}r,$$

thus getting

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}T^2 - 2\sqrt{\frac{2m}{r}}\mathrm{d}T\mathrm{d}r - \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{1.3.11}$$

This is the *Painlevé-Gullstrand form of Schwarzschild metric*. It is manifestly regular at the horizon and still singular at $r = 0$. It has also the property that surfaces with constant T identify *flat space* sections:

$$\mathrm{d}s^2\Big|_{T=const} = -\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2,$$

which is the usual metric of flat, three-dimensional space in spherical polar coordinates. Note that Painlevé-Gullstrand coordinates cover regions I and II in a Kruskal diagram, therefore they are equivalent to the advanced Eddington-Finkelstein extension for a black hole. On the other hand, if one considers *outgoing* radial time-like geodesics, r will be increasing and $\dot{r}$ solution will have positive sign. This conveys a Painlevé-Gullstrand version of Schwarzschild metric describing a white hole. It covers regions III and IV in a Kruskal diagram, hence equivalently to the retarded Eddington-Finkelstein extension.

Another remarkable feature of the Painlevé-Gullstrand coordinates system consists of its possibility to convey a *river analogy* for black holes. Indeed, (1.3.11) can be recast as

$$\mathrm{d}s^2 = \mathrm{d}T^2 - (\mathrm{d}r + v\,\mathrm{d}T)^2 - r^2\mathrm{d}\Omega^2, \tag{1.3.12}$$

where we have defined the "velocity"

$$v \equiv \sqrt{\frac{2m}{r}}. \tag{1.3.13}$$

Let us study radial null directions in such a form of the metric: $\mathrm{d}s^2 = 0$ and $\theta, \phi = const$ entail

$$\mathrm{d}T^2 - (\mathrm{d}r + v\,\mathrm{d}T)^2 = 0,$$

$$\mathrm{d}r + v\,\mathrm{d}T = \pm \mathrm{d}T. \tag{1.3.14}$$

Therefore, there exist two solutions

$$\begin{cases} `+\text{'}: & \frac{\mathrm{d}r}{\mathrm{d}T} = 1 - v \implies c - v \\ `-\text{'}: & \frac{\mathrm{d}r}{\mathrm{d}T} = -1 - v \implies -c - v \end{cases}, \tag{1.3.15}$$

where we have reintroduced the speed of light c to make the analogy clearer. One can visualize the propagation of a light signal in the Painlevé-Gullstrand form of Schwarzschild metric as a fish swimming in a river, indeed.

Let us have a look at Fig.(1.3.1).

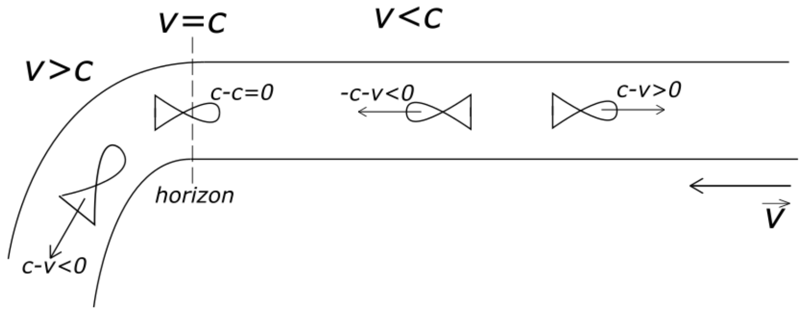

Figure 1.3.1: River analogy for a black hole.

In the river there is an increasing velocity flow $\vec{v}$ from right to left and the fish can move either *upstream* ('+' solution) or *downstream* ('−' solution). Where $v = c$ (i.e. at the horizon $r = 2m$), the fish trying to swim upstream remains stuck because its speed vanishes. On the left side of the horizon $v = c$, v is greater than c: this means the fish swimming upstream is dragged by the flow and forced to move downstream. This corresponds to the trapping behaviour of a black hole. In truth, all of this picture not only does it provide an easier way to imagine null geodesics in Schwarzschild geometry. At the end of Chapter 3, we will see a connection between Schwarzschild black holes and fluid transitions from subsonic to supersonic regime appears in an equivalent way to the river analogy.

Chapter 2

Quantum field theory in curved space-time: basics and Hawking effect

This chapter aims at introducing the reader to the Hawking effect [8], [9]. We will first review canonical quantization of a real scalar field in flat space-time [10],[11] and its extension to the curved case. Then, we will deal with Boulware and Unruh quantization schemes in black holes. Finally, a way to derive Hawking radiation will be outlined.

2.1 Canonical Quantization in flat space-time

Let us recall the quantization procedure of a real classical field $\varphi(x)$ in flat space-time, where x stands for space-time coordinates $x \equiv (t, \vec{x})$. Assuming $\varphi(x)$ to be also non-interacting and massive, it satisfies Klein-Gordon's equation

$$(\partial_\mu \partial^\mu + m^2)\varphi(x) = 0, \tag{2.1.1}$$

whose most general solution is a linear superposition of harmonic oscillators, each vibrating at a frequency with a different amplitude. The Klein-Gordon's equation derives from the application of the Principle of Least Action:

$$\delta S = \delta \int \mathrm{d}^4x\, \mathcal{L}_{KG} = 0, \tag{2.1.2}$$

where $\mathcal{L}_{KG}$ is the Klein-Gordon Lagrangian for a real scalar field

$$\mathcal{L}_{KG} = \frac{1}{2}\big(\eta^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - m^2\varphi^2\big). \tag{2.1.3}$$

Plane waves are a specific class of solutions to Klein-Gordon's equation. Conventionally, they can be either *positive frequency*

$$u_{\vec{k}}(x) \propto e^{-i\omega t+i\vec{k}\cdot\vec{x}} = e^{-ik_\alpha x^\alpha}, \tag{2.1.4}$$

or *negative frequency*

$$u^*_{\vec{k}}(x) \propto e^{i\omega t-i\vec{k}\cdot\vec{x}} = e^{ik_\alpha x^\alpha}, \tag{2.1.5}$$

where $k^\alpha = (\omega, \vec{k})$ is the 4-wavevector and fulfils such a dispersion relation:

$$\omega = \sqrt{k^2+m^2}, \quad k \equiv |\vec{k}|. \tag{2.1.6}$$

Plane waves are eigenfunctions of $\partial_t = \xi^\mu \partial_\mu$, where $\xi^\mu = (1, \vec{0})$ is the Killing vector for time translations in Minkowski space-time

$$\xi^\mu \partial_\mu u_{\vec{k}}(x) = \partial_t u_{\vec{k}}(x) = -i\omega u_{\vec{k}}(x). \tag{2.1.7}$$

Furthermore, they satisfy

$$(u_{\vec{k}}, u_{\vec{k}'}) = \delta^3(\vec{k}-\vec{k}') \equiv \delta_{kk'}, \quad (u^*_{\vec{k}}, u^*_{\vec{k}'}) = -\delta^3(\vec{k}-\vec{k}') \equiv -\delta_{kk'}, \quad (u_{\vec{k}}, u^*_{\vec{k}'}) = 0, \tag{2.1.8}$$

where the scalar product between two generic solutions f_1, f_2 to Klein-Gordon's equation is defined as follows

$$(f_1, f_2) \equiv -i \int_t \mathrm{d}^3x \, [f_1 \partial_t f_2^* - (\partial_t f_1) f_2^*]. \tag{2.1.9}$$

The subscript t in the integral denotes integration is performed on a $t = const$ surface, which is a space-like Cauchy surface in Minkowski space-time. Based on that scalar product, a positive frequency plane wave then results being normalized as

$$u_{\vec{k}}(x) = \frac{1}{(2\pi)^{3/2}\sqrt{2\omega}} e^{-ik_\alpha x^\alpha}. \tag{2.1.10}$$

Note a positive (negative) frequency mode has positive (negative) norm, in Minkoski space-time. Since the set of modes $\left\{u_{\vec{k}}, u^*_{\vec{k}}\right\}$ form a complete orthonormal basis for the space of solutions to Klein-Gordon's equation, the field $\varphi(x)$ can be expanded as

$$\varphi(x) = \sum_{\vec{k}} [a_{\vec{k}} u_{\vec{k}}(x) + b_{\vec{k}} u^*_{\vec{k}}(x)], \tag{2.1.11}$$

where $a_{\vec{k}}, b_{\vec{k}}$ coefficients are two arbitrary functions of the wave-vector $\vec{k}$. Being $\varphi(x)$ real, $\varphi^*(x) = \varphi(x)$ implies $b_{\vec{k}} = a^*_{\vec{k}}$.

We now move from classical to quantum theory via *canonical quantization*. This means the field $\varphi(x)$ is promoted to a hermitian operator valued function of space-time $\hat{\varphi}(x)$, which obeys the equal time *canonical commutation relations* in Heisenberg picture:

$$[\hat{\varphi}(t,\vec{x}),\hat{\varphi}(t,\vec{x}')] = 0, \tag{2.1.12a}$$
$$[\hat{\pi}(t,\vec{x}),\hat{\pi}(t,\vec{x}')] = 0, \tag{2.1.12b}$$
$$[\hat{\varphi}(t,\vec{x}),\hat{\pi}(t,\vec{x}')] = i\hbar\delta^3(\vec{x}-\vec{x}'). \tag{2.1.12c}$$

$\hat{\pi}(t,\vec{x})$ is the conjugate momentum of $\hat{\varphi}(t,\vec{x})$

$$\hat{\pi}(t,\vec{x}) \equiv \frac{\partial\mathcal{L}}{\partial(\partial_t\varphi)} \tag{2.1.13}$$

and $\hat{\pi} = \dot{\hat{\varphi}}$ holds in scalar theory. The quantum field now can be written as a linear combination of creation and annihilation operators

$$\hat{\varphi}(x) = \sum_{\vec{k}}[\hat{a}_{\vec{k}}u_{\vec{k}}(x) + \hat{a}^\dagger_{\vec{k}}u^*_{\vec{k}}(x)]. \tag{2.1.14}$$

Equations (2.1.12a)-(2.1.12c) induce equivalent commutation relations for $\hat{a}_k$ and $\hat{a}^\dagger_k$:

$$\left[\hat{a}_{\vec{k}},\hat{a}_{\vec{k}'}\right] = 0, \tag{2.1.15a}$$
$$\left[\hat{a}^\dagger_{\vec{k}},\hat{a}^\dagger_{\vec{k}'}\right] = 0, \tag{2.1.15b}$$
$$\left[\hat{a}_{\vec{k}},\hat{a}^\dagger_{\vec{k}'}\right] = \hbar\delta^3_{kk'}. \tag{2.1.15c}$$

The state $|0\rangle$, which produces the zero-ket when any annihilation operator acts on it,

$$\hat{a}_{\vec{k}}\,|0\rangle = 0, \quad \forall\vec{k} \tag{2.1.16}$$

is called *vacuum*. The Fock basis is constructed by acting with creation operators on the vacuum state

$$\left|n^{(1)}_{\vec{k}_1}, n^{(2)}_{\vec{k}_2}, ..., n^{(j)}_{\vec{k}_j}\right\rangle \propto (a^\dagger_{\vec{k}_1})^{n^{(1)}}(a^\dagger_{\vec{k}_2})^{n^{(2)}}...(a^\dagger_{\vec{k}_j})^{n^{(j)}}\,|0\rangle\,, \tag{2.1.17}$$

where $n^{(1)}, n^{(2)}, ..., n^{(j)}$ denote the number of particles in the state with wave-vector $\vec{k}_1, \vec{k}_2, ..., \vec{k}_j$, respectively. Particles then emerge as excitations of the quantum field above the ground state, whose role is played by the vacuum $|0\rangle$. What allows us to interpret (2.1.17) as a *many-particle state* is *Nöther theorem*. The Minkowski symmetry under space-time translations gives rise to energy-momentum conservation, and, upon quantization, (2.1.17) is an eigenstate of the 4-momentum operator. Its eigenvalue coincides with the sum of 4-wave vectors $\sum_j (k^{\alpha_j})^{n(j)}$ such that $\omega_j^2 = k_j^2 + m^2$. We then recognize the *mass-shell relation* for a particle in Special Relativity with mass m, if we interpret the frequency ω_j as the energy E_j and the wave-vector $\vec{k}$ as the momentum $\vec{p}$

$$\omega_j^2 = k_j^2 + m^2 \implies E_j^2 = p_j^2 + m^2. \tag{2.1.18}$$

In addition, when (2.1.17) is subject to a unitary transformation build out of a Lorentz one, nothing changes but each 4-wavevector, which is Lorentz-transformed

$$\begin{cases} U \left| n_{\vec{k}_1}^{(1)}, n_{\vec{k}_2}^{(2)}, ..., n_{\vec{k}_j}^{(j)} \right\rangle = \left| n_{\vec{k}'_1}^{(1)}, n_{\vec{k}'_2}^{(2)}, ..., n_{\vec{k}'_j}^{(j)} \right\rangle \\ U \equiv e^{-\frac{i}{2}\varepsilon_\mu \hat{L}^{\mu\nu}}, \ \hat{L}^{\mu\nu} \equiv \text{ generators of a Lorentz transformation}, \ \varepsilon_\mu \equiv \text{ Lie parameters} \\ k^{\alpha_j} \longrightarrow k'^{\alpha_j} = \Lambda^{\alpha_j}{}_{\beta_j} k^{\beta_j}, \ \Lambda^{\alpha_j}{}_{\beta_j} \equiv \text{ Lorentz transformation} \end{cases} . \tag{2.1.19}$$

From a mathematical point of view, a particle is then defined as an irreducible representation of Poincaré group, within quantum field theory in *flat* space-time.

2.2 Quantization in curved space-time

Once the space-time is *curved*, Poincaré symmetry is lost and we need to generalize the previous steps for the construction of a quantum field theory. According to the Principle of General Covariance, the Klein-Gordon's equation and Lagrangian modify into

$$[\nabla_\mu \nabla^\mu + m^2]\varphi(x) = 0, \tag{2.2.1}$$

$$\mathcal{L}_{KG} = \frac{1}{2}\sqrt{g}[g^{\mu\nu}\partial_\mu\varphi(x)\partial_\nu\varphi(x) - m^2\varphi^2(x)], \tag{2.2.2}$$

where $g \equiv |\det g_{\mu\nu}|$ and the covariant d'Alembertian operator is found to be equal to

$$\nabla_\mu \nabla^\mu = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu). \tag{2.2.3}$$

The scalar product between two generic solutions to (2.2.1) is now defined as

$$(f_1, f_2) \equiv -i \int_\Sigma \mathrm{d}\Sigma^\mu \sqrt{g_\Sigma}[f_1 \partial_\mu f_2^* - (\partial_\mu f_1) f_2^*], \tag{2.2.4}$$

where Σ is an "initial data" Cauchy surface, $\mathrm{d}\Sigma^\mu \equiv u^\mu \mathrm{d}\Sigma$, with $\mathrm{d}\Sigma$ being the surface element and u^μ a future directed unit normal vector to Σ, and $g_\Sigma \equiv \left|\det \left(g_{\mu\nu}\big|_\Sigma\right)\right|$, with $g_{\mu\nu}\big|_\Sigma$ being the induced metric on Σ. The canonical commutation relations are given on a Cauchy surface as well

$$[\hat{\varphi}(x), \hat{\varphi}(y)]_\Sigma = 0, \tag{2.2.5a}$$
$$[u^\mu \partial_\mu \hat{\varphi}(x), u^\mu \partial_\mu \hat{\varphi}(y)]_\Sigma = 0, \tag{2.2.5b}$$
$$[\hat{\varphi}(x), u^\mu \partial_\mu \hat{\varphi}(y)]_\Sigma = \frac{i}{\sqrt{g_\Sigma}} \hbar \delta^3(\vec{x} - \vec{y}), \tag{2.2.5c}$$

where now the conjugate momentum to $\hat{\varphi}$ is $u^\mu \partial_\mu \hat{\varphi}$. At this point, one can proceed to expand $\hat{\varphi}$ in terms of an orthonormal basis of modes $\{u_i, u_i^*\}$ (with i continuous or discrete index characterizing the modes).

$$\hat{\varphi}(x) = \sum_i [\hat{a}_i u_i(x) + \hat{a}_i^\dagger u_i^*(x)], \tag{2.2.6}$$

and construct the Fock space by acting with creation operators on the *vacuum* state. Nevertheless, in a curved space-time, different choices of positive frequency modes lead, in general, to different definitions of the vacuum state and therefore of the corresponding Fock space. A way to extend the standard particle interpretation of Fock states is to consider a *stationary* space-time. In that case, there exists a Killing vector ξ^μ for time translations which allows to define positive frequency modes u_i as

$$\xi^\mu \nabla_\mu u_i = -i\omega_i u_i, \quad \omega_i > 0, \tag{2.2.7}$$

thus generalizing the condition (2.1.7). This permits to recover a particle interpretation for those space-times which are not necessarily stationary overall but at least asymptotically in the past and in the future. We can then construct an orthonormal basis of modes such as to be positive (negative) frequency solutions with respect to the inertial time in the *past* and call them $u_i^{in} (u_i^{in*})$. The same can be done for the *future* region, thus getting an orthonormal basis of positive (negative) frequency modes $u_j^{out} (u_j^{out*})$. Since both $\{u_i^{in}, u_i^{in*}\}$ and $\{u_j^{out}, u_j^{out*}\}$ are complete, we can expand one set of modes in terms of the other, e.g.

$$u_j^{out} = \sum_i (\alpha_{ji} u_i^{in} + \beta_{ji} u_i^{in*}). \tag{2.2.8}$$

This kind of relations is called *Bogoliubov transformation* and the matrix elements α_{ji}, β_{ji} *Bogoliubov coefficients*. Notice this transformation mixes positive and negative *in* norm modes to yield a positive *out* norm mode, in general. The inverse relation results being

$$u_i^{in} = \sum_j (\alpha_{ji}^* u_j^{out} - \beta_{ji} u_j^{out*}). \tag{2.2.9}$$

By taking into account that

$$\hat{a}_{i(j)}^{in(out)} = (\hat{\varphi}, u_{i(j)}^{in(out)}), \qquad \hat{a}_{i(j)}^{in(out)\dagger} = -(\hat{\varphi}, u_{i(j)}^{in(out)^*}), \tag{2.2.10}$$

we obtain Bogoliubov transformations for creation and annihilation operators as well, e.g.

$$\hat{a}_i^{in} = \sum_j (\alpha_{ji} \hat{a}_j^{out} + \beta_{ji}^* \hat{a}_j^{out\dagger}), \tag{2.2.11}$$

$$\hat{a}_j^{out} = \sum_i (\alpha_{ji}^* \hat{a}_i^{in} - \beta_{ji}^* \hat{a}_i^{in\dagger}). \tag{2.2.12}$$

A striking consequence of Bogoliubov transformations is the *in* and *out* vacua

$$\hat{a}_i^{in} |in\rangle = 0 \quad \forall i, \tag{2.2.13a}$$
$$\hat{a}_j^{out} |out\rangle = 0 \quad \forall j. \tag{2.2.13b}$$

are not equivalent. Indeed, if we compute the expectation value of the *out* particle number operator for the j^{th} mode in the *in* vacuum, we obtain

$$\langle in|\hat{N}_j^{out}|in\rangle = \langle in|\hbar^{-1}\hat{a}_j^{out\dagger}\hat{a}_j^{out}|in\rangle = \sum_i |\beta_{ij}|^2, \tag{2.2.14}$$

which in general is non-vanishing. More precisely, the particle content of $|in\rangle$ in terms of *out* Fock states is the following:

$$|in\rangle = \langle out|in\rangle \exp\left(\frac{1}{2\hbar}\sum_{ij} V_{ij}\hat{a}_i^{out\dagger}\hat{a}_j^{out}\right)|out\rangle, \tag{2.2.15}$$

where $V_{ij} \equiv -\sum_i \beta_{ji}^* \alpha_{ik}^{-1}$ is a symmetric matrix element. This implies couples of particles are produced in the future region, starting from a vacuum state in the past region. As we will see further, this is the mechanism underlying Hawking radiation.

2.3 Quantum field theory in a Schwarzschild black hole

We now wish to apply the quantization procedure in a curved space-time to a *massless* scalar field $\hat{\varphi}$ in a Schwarzschild black hole. The Klein-Gordon's equation reduces to the wave equation

$$\nabla_\mu \nabla^\mu \hat{\varphi} = 0 \tag{2.3.1}$$

and, in order to expand $\hat{\varphi}(x)$, we can take advantage of the asymptotic flatness in Schwarzschild metric. Firstly, we shall recall how the various regions at infinity in Minkowski space-time are defined (Fig.(2.3.1)):

- past time-like infinity i^-: $t \longrightarrow -\infty$ at fixed r or, equivalently, $v \longrightarrow -\infty, u \longrightarrow -\infty$
- future time-like infinity i^+: $t \longrightarrow +\infty$ at fixed r, or $v \longrightarrow +\infty, u \longrightarrow +\infty$
- space-like infinity i^0: $r \longrightarrow +\infty$ at fixed t, or $v \longrightarrow +\infty, u \longrightarrow -\infty$
- past null infinity I^-: $t \longrightarrow -\infty,\ r \longrightarrow +\infty$ with fixed $t + r$, or $u \longrightarrow -\infty$ with fixed v
- future null infinity I^+: $t \longrightarrow +\infty,\ r \longrightarrow +\infty$ with fixed $t - r$, or $v \longrightarrow +\infty$ with fixed u.

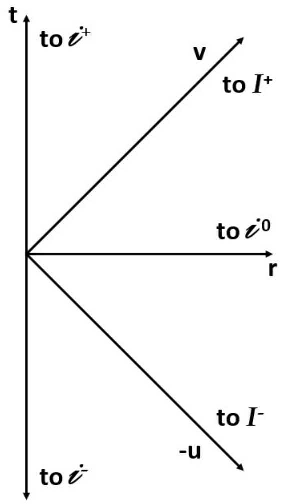

Figure 2.3.1: Directions towards the various regions at infinity in Minkowski space-time.

Via a proper conformal transformation of the metric, points at infinity can be mapped into a finite diagram such as to conserve the causal properties of the original geometry. This kind of graphical representation of a space-time manifold is called *Penrose diagram.* Some example will be shown in the following subsections.

Boulware vacuum

Let us concentrate on regions I and II – which we will now call R (right) and L (left), respectively – of the maximally extended Schwarzschild space-time, i.e. those covered by the advanced Eddington-Finkelstein coordinates (Fig.(2.3.2)). We recall the Schwarzschild line element is given by

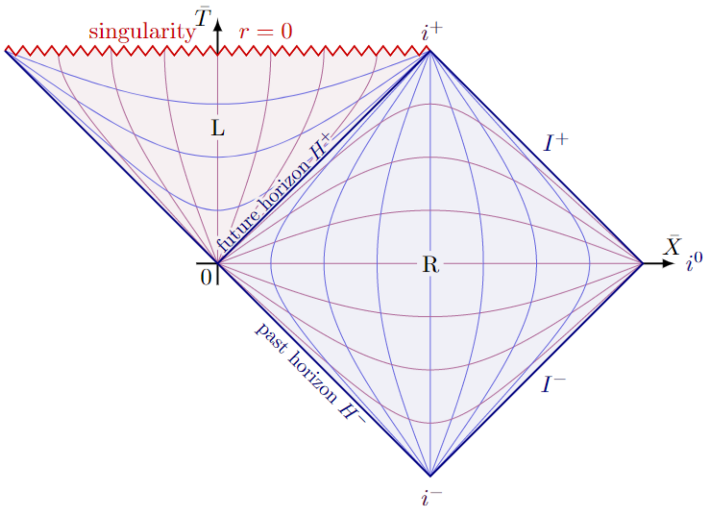

Figure 2.3.2: Penrose diagram for Schwarzschild black hole [6]. $\bar{X}$ and $\bar{T}$ denote space and time Kruskal coordinates, respectively, after conformal compactification.

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right) \mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2). \qquad (2.3.2)$$

Based on the spherical symmetry of Schwarzschild metric, we are led to make an ansatz for the structure of the classical field φ: we assume it can be expressed as

$$\varphi = \sum_{l,m} \frac{f_l(r,t)}{r} Y_l^m(\theta, \phi), \qquad (2.3.3)$$

where $\frac{f_l(r,t)}{r}$ is a radial function, $Y_l^m(\theta, \phi)$ a spherical harmonic, l and m the orbital and magnetic quantum numbers, respectively. Once it is plugged into the massless Klein-Gordon's equation (2.3.1), we obtain

$$[\partial_t^2 - \partial_{r^*}^2 + V_l(r)] f_l(r,t) = 0, \qquad (2.3.4)$$

where

$$V_l(r) \equiv \left(1 - \frac{2m}{r}\right) \left[\frac{2m}{r^3} + \frac{l(l+1)}{r^2}\right] \qquad (2.3.5)$$

is an effective potential induced by the gravitational field (Fig.(2.3.3)) and r^* is the Regge-Wheeler tortoise coordinate (1.1.9).

We shall now make some approximations. First of all, let us restrict to outgoing light rays emitted near the horizon. Because of the gravitational red-shift, an observer at *late* time will detect them only if their initial frequency is very high. This means their propagation is not supposed to be affected by the curvature of

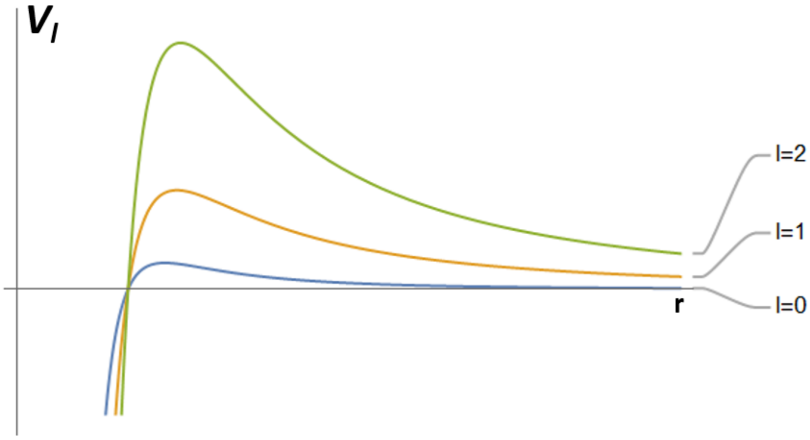

Figure 2.3.3: Graphical representation of the effective potential V_l for the three lowest values of l.

space-time and the late time behaviour of such a signal will turn out to be independent of the specific space-time model. The effective potential $V_l(r)$, which causes modes scattering, vanishes at $r = 2m$ and $r \longrightarrow +\infty$ and its pick increases along with l (Fig.(2.3.3)). Therefore, in order to maximize the transmission probability of outgoing null rays from the near-horizon region towards I^+, we set $l = 0$ and $V_l(r) = 0$. This means we will restrict to "s-wave component", which is the one less affected by the potential, and avoid backscattering effects.

Within these approximations, (2.3.4) reduces to

$$(\partial_t^2 - \partial_{r^*}^2) f(r,t) = 0, \tag{2.3.6}$$

$$\partial_u \partial_v f(u,v) = 0, \tag{2.3.7}$$

where $u = t - r^*$ and $v = t + r^*$ are the Eddington-Finkelstein null coordinates and we have set $f \equiv f_0$. The most general solution to (2.3.7) is

$$f = F(u) + G(v), \tag{2.3.8}$$

thus decupling the ingoing and outgoing wave contributions. We know the physical interpretation of a quantum field theory in curved space-time strictly depends on the choice of the modes basis. Being Schwarzschild metric asymptotically flat, one possibility to quantize f consists of choosing a set of modes such as to approach Minkowski ones at both I^- and I^+, i.e.

$$F(u) = e^{-i\omega u} = e^{-i\omega(t-r^*)} \xrightarrow{I^+} e^{-i\omega(t-r)}, \tag{2.3.9}$$

$$G(v) = e^{-i\omega v} = e^{-i\omega(t+r^*)} \xrightarrow{I^-} e^{-i\omega(t+r)}. \tag{2.3.10}$$

This way of selecting modes is called *Boulware choice*. Notice $\{e^{-i\omega u}, e^{-i\omega v}\}$ are indeed positive frequency eigenfunctions of $\xi^\mu \nabla_\mu$, where $\xi^\mu = (1, \vec{0})$ is the Killing vector associated to staticity in Schwarzschild metric. However, we recall in Schwarzschild space-time t and r feature inverted roles depending on region R or L. This has a relevant impact on the proper selection of modes. Since they require to be normalized on a Cauchy surface, we will choose a $t = const$ surface (Fig.(2.3.4)) in region R ($2m < r < +\infty$) and calculate the scalar product between two radial solutions f_1 and f_2

$$\begin{aligned}(f_1, f_2) &= -i \int_{\Sigma:\, t=const} \mathrm{d}\Sigma^\mu [f_1 \partial_\mu f_2^* - (\partial_\mu f_1) f_2^*] = \\ &= -i4\pi \int_{2m}^{+\infty} \frac{\mathrm{d}r}{\left(1 - \frac{2m}{r}\right)} r^2 [f_1 \partial_t f_2^* - (\partial_t f_1) f_2^*] = \\ &= -i4\pi \int_{-\infty}^{+\infty} \mathrm{d}r^* r^2 [f_1 \partial_t f_2^* - (\partial_t f_1) f_2^*].\end{aligned} \tag{2.3.11}$$

For both*outgoing* and *ingoing* modes

$$f \equiv f_\omega = \begin{cases} A\frac{e^{-i\omega u}}{r} \\ B\frac{e^{-i\omega v}}{r} \end{cases}, \tag{2.3.12}$$

we obtain

$$A = B = \frac{1}{4\pi\sqrt{\omega}}. \tag{2.3.13}$$

Therefore, in region R the Boulware basis is composed of $\frac{1}{4\pi\sqrt{\omega}}\frac{e^{-i\omega u}}{r}$, $\frac{1}{4\pi\sqrt{\omega}}\frac{e^{-i\omega v}}{r}$ and their complex conjugates. We highlight that, as long as we are in region R, positive (negative) frequency modes have positive (negative) norm, like in Minkowski space-time. On the other hand, we will choose $r = const$ (Fig.(2.3.4)) as a Cauchy surface in region L ($0 < r < 2m$). In that case, we have

$$\begin{aligned}(f_1, f_2) &= -i \int_{\Sigma:\, r=const} \mathrm{d}\Sigma^\mu [f_1 \partial_\mu f_2^* - (\partial_\mu f_1) f_2^*] = \\ &= -i4\pi \int_{-\infty}^{+\infty} \mathrm{d}t\; r^2 [f_1 \partial_{r^*} f_2^* - (\partial_{r^*} f_1) f_2^*],\end{aligned} \tag{2.3.14}$$

where we recognize the same formula as (2.3.11) upon exchanging t and r^*. This leads to the same result as before for both ingoing and outgoing modes

$$A = B = \frac{1}{4\pi\sqrt{\omega}}, \tag{2.3.15}$$

but with a major difference: in region L a positive (negative) frequency *outgoing* mode has negative (positive) norm, thus losing the sign correspondence between frequency and norm. For this reason, the *Boulware basis* for the whole space-time (Fig.(2.3.4)) is defined as the set of *positive* norm modes

$$u_R \equiv \begin{cases} \frac{1}{4\pi\sqrt{\omega}} \frac{e^{-i\omega u}}{r} & \text{if } r > 2m \\ 0 & \text{if } r < 2m \end{cases} \tag{2.3.16}$$

$$u_L \equiv \begin{cases} 0 & \text{if } r > 2m \\ \frac{1}{4\pi\sqrt{\omega}} \frac{e^{+i\omega u}}{r} & \text{if } r < 2m \end{cases} \tag{2.3.17}$$

$$u_I \equiv \frac{1}{4\pi\sqrt{\omega}} \frac{e^{-i\omega v}}{r} \quad \text{both in } R \text{ and } L \tag{2.3.18}$$

and their complex conjugates (*negative* norm modes).

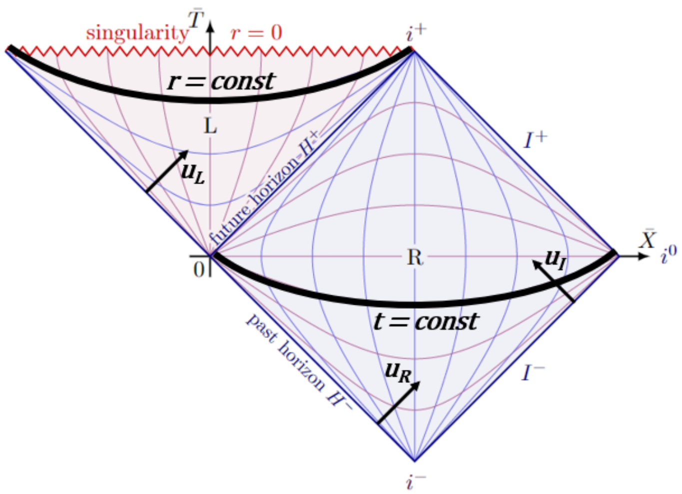

Figure 2.3.4: Boulware modes and Cauchy surfaces in regions R and L [6].

Finally, we can expand the field $\hat{\varphi}$ as

$$\hat{\varphi} = \sum_{\omega} (\hat{a}^I_\omega u_I + \hat{a}^L_\omega u_L + \hat{a}^R_\omega u_R + \hat{a}^{I\dagger}_\omega u^*_I + \hat{a}^{L\dagger}_\omega u^*_L + \hat{a}^{R\dagger}_\omega u^*_R), \tag{2.3.19}$$

where $\hat{a}^{I,L,R}_\omega$ and $\hat{a}^{I,L,R\dagger}_\omega$ are the annihilation and creation operators in each sector. The Fock space is built out of the *Boulware vacuum* $|B\rangle$, which is defined as

$$\hat{a}^i_\omega |B\rangle = 0, \quad \forall\omega, \quad i = I, L, R \tag{2.3.20}$$

and corresponds to a state with no particles both at I^- and I^+, thus reproducing Minkowski vacuum $|0_M\rangle$ asymptotically

$$|B\rangle \xrightarrow{I^\pm} |0_M\rangle \,. \tag{2.3.21}$$

This quantization procedure provides a real physical realization as well. Indeed, if we restrict to region R, the Boulware vacuum describes the state for a field outside a static star (Fig.(2.3.5)).

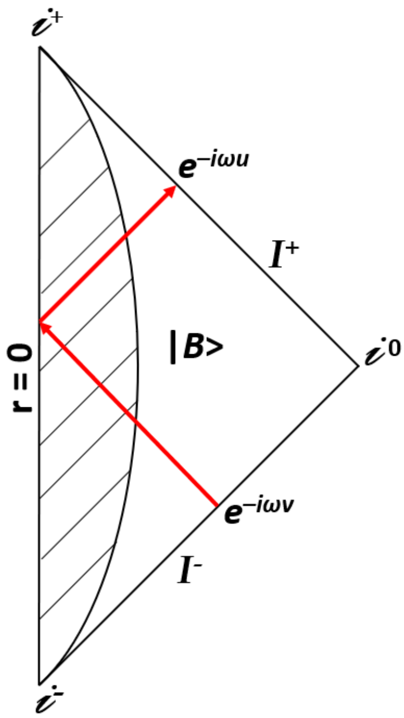

Figure 2.3.5: Ingoing and outgoing modes in a Penrose diagram for a static star.

Unruh vacuum

The modes constructed out of u and v we saw in the previous subsection fail to be regular on H^+ and H^-, respectively. Therefore, on H^+ (H^-) outgoing (ingoing) modes $\sim e^{-i\omega u}$ ($\sim e^{-i\omega v}$) oscillate indefinitely, which produces a diverging vacuum expectation value for some componenents of the energy-momentum tensor operator on the horizon [9]. When we analyzed the maximal analytic extension of Schwarzschild metric in Chapter 1, we found out Kruskal coordinate U and

V played the role of local inertial coordinates on H^- and H^+, respectively. One possible quantization scheme considers Kruskal coordinate U to build *outgoing* modes both in region R and L, while again Eddington-Finkelstein v coordinate for the *ingoing* modes[1]. As concerns the normalization, we perform the scalar product on H^-, being a Cauchy surface for outgoing modes[2],

$$\begin{aligned}(f_1, f_2) &= -i \int_{H^-} \mathrm{d}\Sigma^\mu [f_1 \partial_\mu f_2^* - (\partial_\mu f_1) f_2^*] = \\ &= -i4\pi \int_{-\infty}^{+\infty} \mathrm{d}U \; r^2 [f_1 \partial_U f_2^* - (\partial_U f_1) f_2^*],\end{aligned} \tag{2.3.22}$$

where we take f_1, f_2 to be outgoing modes of the form $A\frac{e^{-i\omega_k U}}{r}$ (the subscript k labels Kruskal frequencies). The end result is the same as for Boulware normalization, thus getting the *Unruh basis*: this is given by

$$u^I_\omega \equiv \frac{1}{4\pi\sqrt{\omega}} \frac{e^{-i\omega v}}{r}, \quad u_{\omega_k} \equiv \frac{1}{4\pi\sqrt{\omega_k}} \frac{e^{-i\omega_k U}}{r} \tag{2.3.23}$$

and their complex conjugates. Hence, we can now expand $\hat{\varphi}$ as

$$\hat{\varphi} = \sum_\omega (\hat{a}^I_\omega u_I + \hat{a}^{I\,\dagger}_\omega u_I^*) + \sum_{\omega_k} (\hat{a}_{\omega_k} u_{\omega_k} + \hat{a}^\dagger_{\omega_k} u^*_{\omega_k}), \tag{2.3.24}$$

where again $\hat{a}^I_\omega, \hat{a}_{\omega_k}$ and their hermitian conjugates are the annihilation and creation operators in each sector. The *Unruh vacuum* is defined to be the state $|U\rangle$ such that

$$\hat{a}^I_\omega |U\rangle = 0, \;\; \hat{a}_{\omega_k} |U\rangle = 0, \quad \forall \omega, \omega_k. \tag{2.3.25}$$

Not only have we got a quantization scheme which convey regular modes at H^+. Its physical application is a black hole formed by gravitational collapse. In that background, the outgoing modes originate from ingoing modes reflection on $r = 0$ and behave like $e^{-i\omega F(u)}$, where $F(u)$ is a complicated function of u depending on the interior metric of the collapsing matter, in general (Fig.(2.3.6)). Yet, for outgoing modes emitted from the near-horizon region, i.e. at $u \longrightarrow +\infty$, $F(u)$ approaches Kruskal coordinate U and the Unruh vacuum "mimics" the *late time* behaviour of the *in* vacuum. By "mimicking" we mean the *in* vacuum expectation

[1]There also exists another choice, according to which ingoing modes are built out of Kruskal coordinate V. This defines the so-called *Hartle-Hawking vacuum* $|HH\rangle$.

[2]It can be proven the scalar product on null surfaces is defined analogously to space-like case.

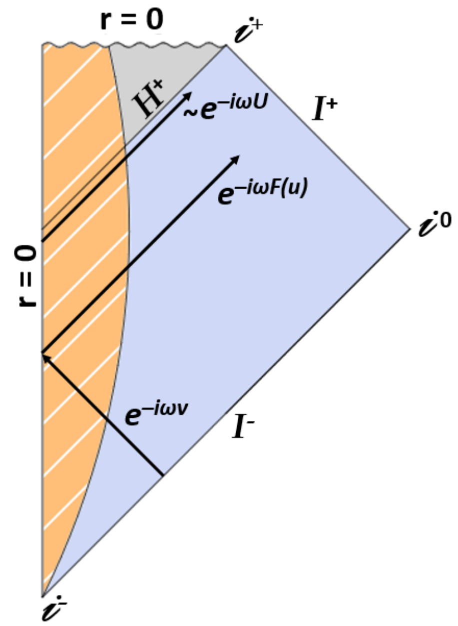

Figure 2.3.6: Ingoing and outgoing modes in a Penrose diagram for gravitational collapse [12]. In the region near the future horizon, the outgoing mode $e^{-i\omega F(u)}$ approaches Unruh mode $e^{-i\omega U}$.

value of an observable $\hat{O}(\hat{\varphi})$ at late time can be assessed performing the calculation with respect to the Unruh vacuum

$$\langle in|\hat{O}(\hat{\varphi})|in\rangle \overset{u\to+\infty}{\longrightarrow} \langle U|\hat{O}(\hat{\varphi})|U\rangle \,. \tag{2.3.26}$$

In the end, according to Unruh choice, $|U\rangle$ reproduces Minkowski vacuum $|0_M\rangle$ asymptotically at I^- but not at I^+

$$|U\rangle \overset{I^-}{\longrightarrow} |0_M\rangle \,, \qquad |U\rangle \overset{I^+}{\not\longrightarrow} |0_M\rangle \,. \tag{2.3.27}$$

2.4 Hawking effect

Let us compute the number of outgoing particles at I^+ starting from an initial vacuum state $|in\rangle$. We are assuming they are emitted from the near-horizon region of a black hole formed by gravitational collapse. As a consequence of the Unruh construction, we are allowed to perform the calculation independently of the collapse details as

$$N_\omega = \langle in|\hat{a}^{R\dagger}_\omega \hat{a}^R_\omega|in\rangle \overset{u\to+\infty}{\longrightarrow} \langle U|\hat{a}^{R\dagger}_\omega \hat{a}^R_\omega|U\rangle \,. \tag{2.4.1}$$

To begin with, we need the Bogoliubov transformation between outgoing Boulware operators in the right sector and outgoing Unruh operators

$$\hat{a}^R_\omega = \sum_{\omega_k} \big(\alpha^R_{\omega_k\omega}\hat{a}_{\omega_k} + \beta^{R^*}_{\omega_k\omega}\hat{a}^\dagger_{\omega_k}\big). \tag{2.4.2}$$

Then, by means of the Bogoliubov transformation from Boulware to Unruh outgoing modes

$$u_{\omega_k} = \sum_\omega (\alpha^L_{\omega_k\omega}u^L_\omega + \beta^L_{\omega_k\omega}u^{L^*}_\omega + \alpha^R_{\omega_k\omega}u^R_\omega + \beta^R_{\omega_k\omega}u^{R^*}_\omega), \tag{2.4.3}$$

we can evaluate the Bogoliubov coefficients of Eq.(2.4.2)

$$\alpha^R_{\omega_k\omega} = (u_{\omega_k}, u^R_\omega), \quad \beta^R_{\omega_k\omega} = -(u_{\omega_k}, u^{R^*}_\omega). \tag{2.4.4}$$

At the end, we obtain

$$\langle U|\hat{a}^{R\dagger}_\omega \hat{a}^R_\omega|U\rangle = \sum_{\omega_k} \left|\beta^R_{\omega_k\omega}\right|^2 = \frac{1}{e^{8\pi m\omega} - 1}. \tag{2.4.5}$$

If we define

$$T_H \equiv \frac{\hbar}{8\pi m k_B} = \frac{\hbar\kappa}{2\pi k_B}, \tag{2.4.6}$$

where $\kappa = 1/4m$ is the surface gravity for a Schwarzschild black hole and k_B is Boltzmann's constant, the number of outgoing particles at late time N_ω becomes

$$N_\omega = \frac{1}{e^{\hbar\omega/k_B T_H} - 1}, \tag{2.4.7}$$

We recognize (2.4.7) to be a Planck's distribution of black-body radiation for bosons, if T_H is interpreted as a temperature. This means the black hole formation induced by gravitational collapse excites the vacuum of $\hat{\varphi}$, up to produce a steady thermal particle emission at late time. This effect is called *Hawking radiation* and T_H *Hawking temperature*. In particular, by comparing Boulware and Unruh vacua for outgoing modes, we find that [8]

$$|U\rangle_{out} \propto \exp\left(\sum_\omega e^{-4\pi m\omega}\hat{a}^{L\dagger}_\omega \hat{a}^{R\dagger}_\omega\right)|B\rangle_{out}\,, \tag{2.4.8}$$

which recalls that, when the Bogoliubov transformation between two modes bases is non-trivial, vacuum fluctuations are converted into *pairs* of particles. In the case at present, Hawking process produces couples of $\hat{a}^{R\dagger}_\omega|B\rangle$ and $\hat{a}^{L\dagger}_\omega|B\rangle$ 1-particle states, which are called *Hawking particle* and *partner*, respectively. The Hawking particle is emitted outside the horizon and has positive energy, whereas the partner pops out inside the black hole and has negative energy, thus conserving the total energy.

In truth, the values predicted for the Hawking temperature are extremely small ($T_H \sim 10^{-7}\frac{M_\odot}{M}\ K$), which makes the Hawking effect still hard to be detected in astrophysical objects. Moreover, a late-time light signal would encounter an infinite blue-shift while it is traced back to the near-horizon region, thus exceeding Planck energy scale, theoretically. Then, one could argue Hawking radiation is an artifact showing limits of validity. This issue is called *transplanckian problem*. However, its derivation shows the Hawking effect is a *kinematical* phenomenon dealing with light rays propagation near a black hole-like horizon, thus not exclusively bound to a gravitational background. This argument has boosted the effort to investigate Hawking effect in *analogue models* of black holes, e.g. acoustic black holes. At the end of Chapter 3 we will introduce such systems.

We would also like to emphasize Hawking effect accomplishes a synthesis between gravity and thermodynamics via quantum mechanics, which is unlikely to be just an artifact of its theoretical derivation. This achievement is claimed to be another aspect in favour of its validity.

Chapter 3

Quantum fluctuations in Bose-Einstein condensates

In this chapter on overview of Bose-Einstein condensation will be given [13], [14]. In particular, we will present the Gross-Pitaevskii equation for the condensate wave function and the Bogoliubov-de Gennes equation for its quantum fluctuations. In the last part, we will show how density-phase parametrization for the field operator leads to a gravitational analogy [15].

3.1 The ideal Bose gas

Let us consider a system of bosons described by the independent particle Hamiltonian

$$\hat{H} = \sum_i \hat{H}_i^{(1)}. \tag{3.1.1}$$

Its eigenstates $|q\rangle$ are defined by specifying the set n_i of microscopic occupation numbers of single-particle states. They are obtained by solving the Schrodinger equation

$$\hat{H}_i^{(1)} \varphi_i(\vec{r}) = \epsilon_i \varphi_i(\vec{r}), \tag{3.1.2}$$

where $\varphi_i(\vec{r})$ is the single-particle wave function for the i-th state. In fact, using the formalism of second quantization, the state

$$|q\rangle \propto (\hat{a}_0^\dagger)^{n_0} (\hat{a}_1^\dagger)^{n_1} ... |0\rangle \tag{3.1.3}$$

specifies the many-body eigenstate of the Hamiltonian (3.1.2) in a complete way. $\hat{a}_i^\dagger$ and $\hat{a}_i$ are the particle creation and annihilation operators relative to the i-th single-particle state, while $|0\rangle$ is the vacuum

$$\hat{a}_i |0\rangle = 0, \quad \forall i. \tag{3.1.4}$$

Creation and annihilation operators obey the canonical commutation rules for bosons

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \quad [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0. \tag{3.1.5}$$

The total number of particles N can be written as the sum of the average occupation numbers $\bar{n}_i$, whose value is given by the Bose-Einstein distribution

$$N = \sum_i \bar{n}_i = \sum_i \frac{1}{e^{[\beta(\epsilon_i - \mu)]} - 1} \tag{3.1.6}$$

$\beta = \frac{1}{k_B T}$ and μ is the chemical potential. This provides the physical constraint $\mu < \epsilon_0$ for the ideal Bose gas, where ϵ_0 is the lowest eigenvalue of the single-particle Hamiltonian $\hat{H}^{(1)}$. The violation of this inequality would result in a negative value for the occupation number of the states with energy smaller than μ. When $\mu \to \epsilon_0$, the occupation number of the lowest energy state becomes larger and larger

$$N_0 \equiv \bar{n}_0 = \frac{1}{e^{[\beta(\epsilon_0 - \mu)]} - 1} \longrightarrow +\infty. \tag{3.1.7}$$

which is the mechanism at the origin of Bose–Einstein condensation. Let us write the total number of particles as

$$N = N_0 + N_T, \tag{3.1.8}$$

where

$$N_T(T, \mu) = \sum_{i \neq 0} \bar{n}_i(T, \mu) \tag{3.1.9}$$

is the number of particles out of the condensate, also called the *thermal depletion* of the condensate. For a fixed value of T, N_T has a smooth behaviour as a function of μ and reaches its maximum N_c at $\mu = \epsilon_0$ (Fig. (3.1.1)). The behaviour of N_0 is very different. In fact, N_0 is always of order 1, except when μ is close to ϵ_0, where N_0 diverges. If $N_c = N_T(T, \mu = \epsilon_0) > N$, (3.1.8) is satisfied for $\mu < \epsilon_0$ values and N_0 is negligible with respect to N. Since $N_c(T)$ is an increasing function of T, this scenario takes place at temperatures T higher than the critical temperature T_c, which is defined by

$$N_T(T_c, \mu = \epsilon_0) = N. \tag{3.1.10}$$

On the other hand, if $N_c(T) < N$ (or, equivalently, $T < T_c$), the condensate contribution is crucial in order to satisfy (3.1.8), and μ will approach ϵ_0 in the thermodynamic limit (i.e. $N, V \longrightarrow +\infty$, in such a way that the particle density N/V remains constant). T_c then defines the critical temperature below which N_0/N remains finite in the thermodynamic limit, which means there exists a macroscopic occupation of a single-particle state. This phenomenon is called *Bose–Einstein condensation.*

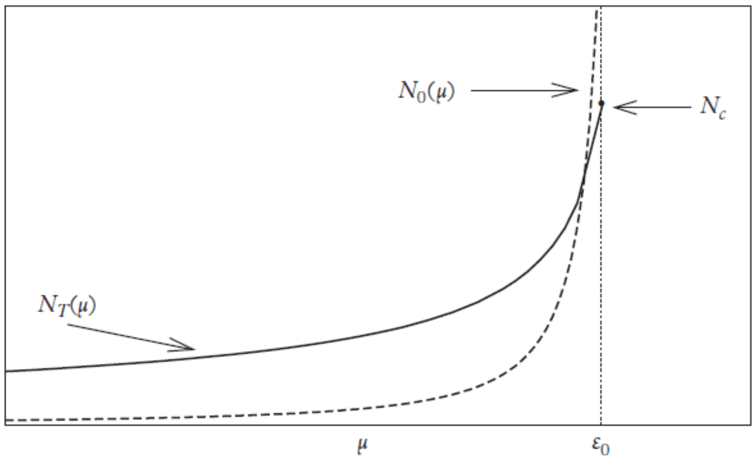

Figure 3.1.1: Condensate N_0 and thermal N_T components of the total number of particles N as functions of the chemical potential μ [13]. $N_T(\mu)$ reaches its maximum N_c at $\mu = \epsilon_0$, whereas $N_0(\mu) \longrightarrow +\infty$ as $\mu \longrightarrow \epsilon_0$.

The ideal 3D Bose gas in a box

For an ideal Bose gas confined in a box of volume V, the single-particle Hamiltonian contains only the kinetic energy operator

$$\hat{H}^{(1)} = \frac{\hat{p}^2}{2m}, \tag{3.1.11}$$

(m is the particle mass and $\hat{p}$ the momentum operator). Its eigenfunctions, using cyclic boundary conditions, are plane waves

$$\varphi_{\vec{p}} = \frac{1}{\sqrt{V}} e^{i\vec{p}\cdot\vec{r}/\hbar}, \tag{3.1.12}$$

with energy eigenvalue $\epsilon_p = \frac{p^2}{2m}$. The plane wave momentum is

$$\vec{p} = \frac{2\pi\hbar}{L}\vec{n}, \tag{3.1.13}$$

where

$$\vec{n} \equiv (n_x, n_y, n_z), \quad n_x, n_y, n_z \in \mathbb{Z}. \tag{3.1.14}$$

The lowest energy eigenvalue ϵ_0 is zero, therefore the chemical potential μ must be always negative. The thermal depletion of the condensate can be written in the form

$$N_T = \sum_{\vec{p}\neq 0} \frac{1}{e^{\beta(p^2/2m-\mu)} - 1} = \frac{V}{\lambda_T^3} g_{3/2}(z), \tag{3.1.15}$$

where $z = e^{\beta\mu}$ is the *fugacity*, $\lambda_T = \sqrt{\frac{2\pi\hbar^2}{mk_BT}}$ the *thermal wavelength* and

$$g_{3/2}(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{d}x\, x^{1/2} \frac{1}{z^{-1}e^x - 1} \tag{3.1.16}$$

a special case of the more general class of Bose functions

$$g_l(z) = \frac{1}{(l-1)!} \int_0^\infty \mathrm{d}x\, x^{l-1} \frac{1}{z^{-1}e^x - 1} = \sum_{a=1}^{\infty} \frac{z^a}{a^l}. \tag{3.1.17}$$

The criterion for BEC (3.1.10) yields the critical temperature for Bose–Einstein condensation

$$T_c = \frac{2\pi\hbar^2}{mk_B} \left(\frac{N/V}{g_{3/2}(1)} \right)^{2/3}. \tag{3.1.18}$$

For $T > T_c$, μ is obtained by setting $N_T = N$ in (3.1.15), which becomes

$$N_T = N \quad \longrightarrow \quad g_{3/2}(z) = \lambda_T^3 \frac{N}{V}. \tag{3.1.19}$$

For $T < T_c$, the normalization condition (3.1.8) is instead solved by setting $\mu = 0$ inside N_T

$$N_T = N_T(T, \mu = 0) = \left(\frac{mk_BT}{2\pi\hbar^2} \right)^{3/2} g_{3/2}(1)V = \left(\frac{T}{T_c} \right)^{3/2} N. \tag{3.1.20}$$

This entails that the *condensate fraction* N_0/N becomes *macroscopic* below T_c indeed (Fig. (3.1.2))

$$\frac{N_0(T)}{N} = 1 - \left(\frac{T}{T_c} \right)^{3/2}. \tag{3.1.21}$$

3.2 Weakly Interacting Bose gas

We shall now concentrate on a weakly interacting dilute atomic Bose gas. The condition defining a *dilute* gas is

$$r_0 \ll d, \tag{3.2.1}$$

where r_0 is the range of interatomic forces and $d = (N/V)^{-1/3}$ is the average distance between particles, fixed by the particle density N/V of the gas. (3.2.1) allows to consider only configurations involving pairs of interacting particles and replace the scattering amplitude with its low energy value. According to standard

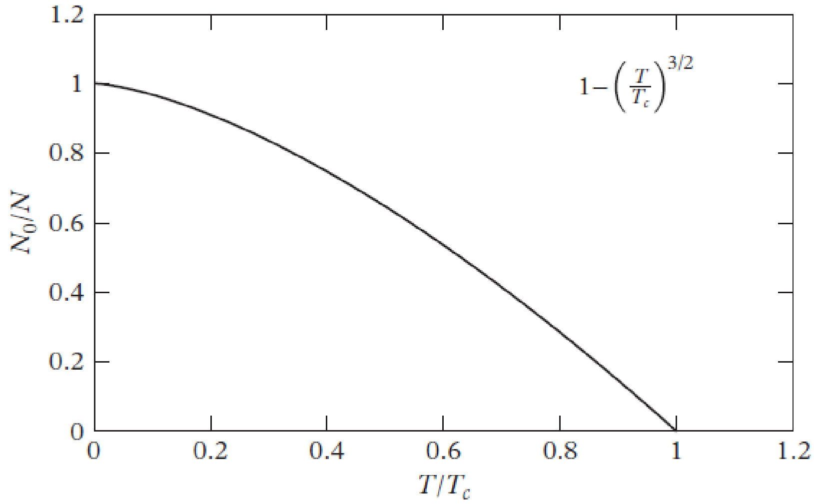

Figure 3.1.2: Condensate fraction N_0/N as function of the adimensional ratio T/T_c [13]. The emerging picture is that typical of an order parameter.

scattering theory, this will be determined by the s-wave scattering length a only. The gas can be considered *weakly interacting* if the condition

$$a \ll d \tag{3.2.2}$$

holds.

Then, a quantum system of N weakly interacting bosons leading to a BEC can be suitably described by a many-body Hamiltonian of the type

$$\begin{aligned}\hat{H} = &\int \hat{\Psi}^\dagger(\vec{r},t)\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}\right)\hat{\Psi}(\vec{r},t)\mathrm{d}^3r+\\ &+\frac{1}{2}\int \hat{\Psi}^\dagger(\vec{r},t)\hat{\Psi}^\dagger(\vec{r}',t)V(\vec{r}'-\vec{r})\hat{\Psi}(\vec{r}',t)\hat{\Psi}(\vec{r},t)\mathrm{d}^3r'\mathrm{d}^3r,\end{aligned} \tag{3.2.3}$$

where $\hat{\Psi}^\dagger$ and $\hat{\Psi}$ are the creation and annihilation field operators for a boson, V_{ext} is the external *trapping* potential and $V(\vec{r})$ is the two-body potential. In the Heisenberg representation, $\hat{\Psi}^\dagger$ and $\hat{\Psi}$ satisfy bosonic equal-time commutation relations

$$[\hat{\Psi}(\vec{r},t), \hat{\Psi}(\vec{r}',t)] = 0 \tag{3.2.4a}$$

$$[\hat{\Psi}^\dagger(\vec{r},t), \hat{\Psi}^\dagger(\vec{r}',t)] = 0 \tag{3.2.4b}$$

$$[\hat{\Psi}(\vec{r},t), \hat{\Psi}^\dagger(\vec{r}',t)] = \delta(\vec{r}-\vec{r}') \tag{3.2.4c}$$

Since we are assuming our system is extremely dilute, $V(\vec{r}\,' - \vec{r})$ represents short-range interactions and can be approximated by a local term

$$V(\vec{r}\,' - \vec{r}) \simeq g\delta(\vec{r}\,' - \vec{r}), \tag{3.2.5}$$

where, according to *Born approximation*,

$$g = \frac{4\pi\hbar^2 a}{m} \tag{3.2.6}$$

plays the role of *effective coupling constant*. With such a choice, the Hamiltonian becomes

$$\hat{H} = \int \left[\hat{\Psi}^\dagger \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} \right) \hat{\Psi} + \frac{g}{2} \hat{\Psi}^\dagger \hat{\Psi}^\dagger \hat{\Psi} \hat{\Psi} \right] \mathrm{d}^3 r. \tag{3.2.7}$$

$\hat{\Psi}(\vec{r}, t)$ fulfils the *Heisenberg equation*

$$i\hbar \frac{\partial \hat{\Psi}(\vec{r}, t)}{\partial t} = \left[\hat{\Psi}(\vec{r}, t), \hat{H} \right] = \left(-\frac{\hbar^2 \nabla^2}{2m} + V_{ext} + g\hat{\Psi}^\dagger \hat{\Psi} \right) \hat{\Psi} \tag{3.2.8}$$

which is a non-linear Schrödinger equation. In the Schrödinger representation, we can write the field operator in terms of single-particle wave functions φ_k and annihilation operators $\hat{a}_k$

$$\hat{\Psi}(\vec{r}, t) = e^{i\hat{H}t/\hbar}\ \hat{\Psi}(\vec{r})\ e^{-i\hat{H}t/\hbar}, \quad \hat{\Psi}(\vec{r}) = \sum_k \varphi_k(\vec{r}) \hat{a}_k, \tag{3.2.9}$$

To account for Bose-Einstein condensation, we shall separate the condensate term $k = 0$ from the other components

$$\hat{\Psi}(\vec{r}) = \varphi_0 \hat{a}_0 + \sum_{k \neq 0} \varphi_k(\vec{r}) \hat{a}_k. \tag{3.2.10}$$

Since in BEC we have a macroscopic occupation for the $k = 0$ state, i.e. the expectation value of the number operator on the ground state far exceeds the unity

$$N_0 = \langle \hat{a}_0^\dagger \hat{a}_0 \rangle = \langle 0 | \hat{a}_0^\dagger \hat{a}_0 | 0 \rangle \gg 1, \tag{3.2.11}$$

we can apply the *Bogoliubov approximation*, which consists of replacing the operators $\hat{a}_0$ and $\hat{a}_0^\dagger$ with the c-number $\sqrt{N_0}$. This is equivalent to ignoring the non-commutativity of the two operators and treating the macroscopic component $\varphi_0 \hat{a}_0$ of the field operator (3.2.10) as a classical field. Therefore, (3.2.10) can be rewritten as

$$\hat{\Psi}(\vec{r}) = \Psi_0(\vec{r}) + \delta\hat{\Psi}(\vec{r}), \tag{3.2.12}$$

where we have defined

$$\Psi_0(\vec{r}) = \sqrt{N_0}\varphi_0(\vec{r}), \quad \delta\hat{\Psi}(\vec{r}) = \sum_{k\neq 0} \varphi_k(\vec{r})\hat{a}_k. \tag{3.2.13}$$

$\Psi_0(\vec{r})$ is called the *condensate wave function* and plays the role of an *order parameter*. Indeed, as shown in Fig.(3.1.2), the condensate fraction N_0/N features such a behaviour

$$\frac{N_0(T)}{N} = \begin{cases} 0 & \text{if } T \geq T_c \\ 1 - \left(\frac{T}{T_c}\right)^{3/2} \neq 0 & \text{if } T < T_c \end{cases}. \tag{3.2.14}$$

This betrays the fact that Bose-Einstein condensation is a phase transition occurring at T_c which corresponds to a *spontaneous symmetry breaking*. In the formalism of second quantization, the free hamiltonian reads

$$\hat{H} = \sum_k \frac{\hbar^2 k^2}{2m} \hat{a}_k^\dagger \hat{a}_k, \tag{3.2.15}$$

which is invariant under the $U(1)$ symmetry generated by the number operator $\hat{N}$

$$\hat{U}(\theta) = e^{i\theta\hat{N}}, \quad \hat{N} = \sum_k \hat{a}_k^\dagger \hat{a}_k, \quad \left[\hat{H}, \hat{N}\right] = 0, \tag{3.2.16}$$

such that

$$\hat{a}_k \xrightarrow{\hat{U}} \hat{U}(\theta)\hat{a}_k\hat{U}(\theta)^\dagger = e^{i\theta}\hat{a}_k, \tag{3.2.17a}$$

$$\hat{a}_k^\dagger \xrightarrow{\hat{U}} \hat{U}(\theta)\hat{a}_k^\dagger\hat{U}(\theta)^\dagger = e^{-i\theta}\hat{a}_k^\dagger. \tag{3.2.17b}$$

Nevertheless, when $T < T_c$, this symmetry is said to be *spontaneously broken* due to the macroscopic occupation of the ground state $N_0 = \langle \hat{a}_0^\dagger \hat{a}_0 \rangle \gg 1$. This implies $\langle \hat{a}_0 \rangle$ is different from zero and no longer invariant under $U(1)$ symmetry

$$\langle \hat{a}_0 \rangle \xrightarrow{\hat{U}} e^{i\theta} \langle \hat{a}_0 \rangle, \tag{3.2.18}$$

which means different choices for the Lie phase θ correspond to the same (ground state) energy but represent different physical states.

The Bogoliubov ansatz (3.2.12) for the field operator can then be interpreted by stating that its expectation value $\langle \hat{\Psi} \rangle$ is non-vanishing. That would not be possible if the states on the left and on the right had exactly the same number of particles. Therefore, the spontaneous $U(1)$-symmetry breaking means the ground state is a superposition of an indefinite number of particle states and the condensate plays the role of a reservoir. Finally, we can set $\Psi_0 = \langle \hat{\Psi} \rangle$ (and similarly $\Psi_0^* = \langle \hat{\Psi}^\dagger \rangle$), having in mind that the number of particles in the left and right states differ by one. If we calculate this expectation value over stationary states − whose time dependence is governed by the law $e^{-iEt/\hbar}$ − the condensate wave function evolves in time as

$$\Psi_0(\vec{r},t) = \Psi_0(\vec{r})e^{-i\mu t/\hbar}, \tag{3.2.19}$$

where $\mu = E(N) - E(N-1) \approx \frac{\partial E}{\partial N}$.

3.3 The Gross-Pitaevskii equation

Let us consider a weakly interacting *non-uniform* gas. Upon Bogoliubov prescription for the field operator, to the lowest-order approximation and at very low temperature

$$T \ll T_c \quad \Rightarrow N_0 \gg 1 \tag{3.3.1}$$

the *quantum* field operator $\hat{\Psi}(\vec{r},t)$ can be replaced by a *classical* field $\Psi(\vec{r},t)$. Hence, (3.2.8) turns into

$$i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\vec{r},t) + g|\Psi(\vec{r},t)|^2\right)\Psi(\vec{r},t), \tag{3.3.2}$$

which is called the *Gross-Pitaevskii equation* for the condensate wave function. It has the typical form of a mean-field equation and its non-linearity arises from the interaction among particles.

In case of stationary solutions, the time dependence of the condensate wave function is fixed by the chemical potential (Eq.(3.2.19)), while the Gross-Pitaevskii equation reduces to

$$\left[-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\vec{r}) + g|\Psi_0(\vec{r})|^2\right]\Psi_0(\vec{r}) = \mu\Psi_0(\vec{r}), \tag{3.3.3}$$

where we have assumed the external potential does not depend on time. It is a non-linear Schrödinger-like equation with eigenvalue μ, which is fixed by the normalization condition for the number of particles

$$\int |\Psi_0(\vec{r})|^2 \mathrm{d}\vec{r} = N. \tag{3.3.4}$$

Then, the solution to the time-independent Gross-Pitaevskii equation (3.3.3) at the lowest energy conveys the condensate wave function for the ground state. For a *uniform* gas, i.e. in absence of an external potential, it gives

$$\mu = gn \tag{3.3.5}$$

with $n = |\Psi_0|^2$, and the energy functional

$$E[\Psi] = \int \left(\frac{\hbar^2}{2m} |\nabla\Psi|^2 + V_{ext}(\vec{r})|\Psi|^2 + \frac{g}{2}|\Psi|^4 \right) \mathrm{d}\vec{r} \tag{3.3.6}$$

reduces to

$$E = \frac{1}{2} N g n. \tag{3.3.7}$$

Consistently, as will be elucidated further, these expressions for μ and $E[\Psi]$ agree with the lowest-order approximation of microscopic Bogoliubov theory for uniform gases.

Elementary excitations

An important class of time-dependent solutions to the Gross-Pitaevskii equation is that of *small-amplitude oscillations*, i.e. perturbative solutions where changes in space and time of the condensate wave function with respect to the stationary configuration are taken small. These solutions can be interpreted as elementary excitations of the system and they admit a natural quantum description which we will see in the next subsection. The small-amplitude oscillations of the system around equilibrium can then be investigated by writing the condensate wave function as

$$\Psi(\vec{r}, t) = [\Psi_0(\vec{r}) + \varepsilon(\vec{r}, t)] e^{-i\mu t/\hbar} \tag{3.3.8}$$

$\varepsilon(\vec{r}, t)$ is a small quantity for which we look for solutions in the form

$$\varepsilon(\vec{r}, t) = \sum_i \left[u_i(\vec{r}) e^{-i\omega_i t} + v_i^*(\vec{r}) e^{+i\omega_i t} \right], \tag{3.3.9}$$

where ω_i is the frequency of the i-th oscillation.

The spatial functions $u_i(\vec{r})$ and $v_i(\vec{r})$ are determined by linearizing the Gross-Pitaevskii equation. By collecting all terms evolving in time like $e^{-i\omega_i t}$ and $e^{+i\omega_i t}$, separately, we obtain

$$\hbar\omega_i u_i(\vec{r}) = \left(\hat{H}_0 - \mu + 2gn(\vec{r}) \right) u_i(\vec{r}) + g(\Psi_0(\vec{r}))^2 v_i(\vec{r}), \tag{3.3.10a}$$

$$-\hbar\omega_i v_i(\vec{r}) = \left(\hat{H}_0 - \mu + 2gn(\vec{r}) \right) v_i(\vec{r}) + g(\Psi_0^*(\vec{r}))^2 u_i(\vec{r}), \tag{3.3.10b}$$

where $\hat{H}_0 = -\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\vec{r})$ and the index i labels the $i - th$ solution. This pair of differential equations are called *Bogoliubov-de Gennes equations*. These yield the eigenfrequencies ω_i and the amplitudes $u_i(\vec{r})$ and $v_i(\vec{r})$ for the normal modes of the system. In general, they must be solved numerically, but an analytically soluble example is provided by the collective oscillations around the ground state of a *uniform* gas ($V_{ext} = 0$). In that case, $\mu = gn$ and Ψ_0, being independent of $\vec{r}$, can be set equal to $\sqrt{n} \in \mathbb{R}$. As a consequence, we find that the amplitudes can be written as (let us omit the index i)

$$u(\vec{r}) = ue^{i\vec{k}\cdot\vec{r}}, \quad v(\vec{r}) = ve^{i\vec{k}\cdot\vec{r}}, \tag{3.3.11}$$

and Bogoliubov-de Gennes equations reduce to

$$\hbar\omega u = \frac{\hbar^2 k^2}{2m} u + gn(u+v), \tag{3.3.12a}$$

$$-\hbar\omega v = \frac{\hbar^2 k^2}{2m} v + gn(u+v). \tag{3.3.12b}$$

This last pair of equations can be recast as an eigenvalue equation for the eigenvector $\begin{pmatrix} u \\ v \end{pmatrix}$, yielding the analytic solution

$$(\hbar\omega)^2 = \left(\frac{\hbar^2 k^2}{2m}\right)^2 + \frac{\hbar^2 k^2}{m} gn, \tag{3.3.13}$$

which is called *Bogoliubov dispersion relation.*

Microscopic Bogoliubov theory of Bose gas

To deepen the quantum nature of excitations, we need to develop a microscopic theory of the Bose gas. Upon expressing the field operator in Schrödinger picture, we take quantum fluctuations about the condensed equilibrium state into account by recurring to the decomposition (3.2.12). Let us consider a uniform gas of N interacting bosons contained in a box of volume V. We shall write the field operator occurring in the Hamiltonian (3.2.7) in terms of an operator that destroys particles in momentum states via the transformation

$$\hat{\Psi}(\vec{r}) = \frac{1}{V^{1/2}} \sum_{\vec{p}} e^{i\vec{p}\cdot\vec{r}/\hbar} \hat{a}_{\vec{p}} = \frac{V^{1/2}}{(2\pi\hbar)^3} \int \mathrm{d}\vec{p} e^{i\vec{p}\cdot\vec{r}/\hbar} \hat{a}_{\vec{p}}, \tag{3.3.14}$$

whose inverse is

$$\hat{a}_{\vec{p}} = \frac{1}{V^{1/2}} \int \mathrm{d}\vec{r} e^{-i\vec{p}\cdot\vec{r}/\hbar} \hat{\Psi}(\vec{r}), \tag{3.3.15}$$

and similarly for $\hat{\Psi}^\dagger(\vec{r})$ and $\hat{a}^\dagger_{\vec{p}}$. The Hamiltonian then reads

$$\hat{H} = \sum_{\vec{p}} \epsilon^0_p \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} + \frac{g}{2V} \sum_{\vec{p},\vec{p}',\vec{q}} \hat{a}^\dagger_{\vec{p}+\vec{q}} \hat{a}^\dagger_{\vec{p}'-\vec{q}} \hat{a}_{\vec{p}'} \hat{a}_{\vec{p}}, \tag{3.3.16}$$

where $\epsilon_p^0 \equiv p^2/2m$ is the free-particle energy and the operators $\hat{a}^\dagger_{\vec{p}}$ and $\hat{a}_{\vec{p}}$ that create and destroy bosons in the $\vec{p}$-momentum state obey bosonic commutation relations.

Within Bose-Einstein condensation, we assume that in the interacting system the lowest-lying single-particle state is macroscopically occupied. In the unperturbed system, we have

$$\hat{a}_0^\dagger \left|N_0\right\rangle = \sqrt{N_0+1}\left|N_0+1\right\rangle \sim \sqrt{N_0}\left|N_0\right\rangle, \tag{3.3.17a}$$

$$\hat{a}_0 \left|N_0\right\rangle = \sqrt{N_0}\left|N_0-1\right\rangle \sim \sqrt{N_0}\left|N_0\right\rangle. \tag{3.3.17b}$$

Therefore, based on Bogoliubov prescription, we will replace $\hat{a}_0$ and $\hat{a}_0^\dagger$ with $\sqrt{N_0}$ in the Hamiltonian. Since we assume $\delta\Psi(\vec{r})$ to be a small fluctuation, all terms which have (at least) two powers of $\Psi(\vec{r})$ or $\Psi^*(\vec{r})$ are retained. This is equivalent to including terms which are no more than quadratic in $\delta\Psi(\vec{r})$ and $\delta\Psi^\dagger(\vec{r})$ − i.e. in $\hat{a}_{\vec{p}}$ and $\hat{a}^\dagger_{\vec{p}}$, for $\vec{p}\neq 0$ − in the Hamiltonian. thereafter, we find

$$\hat{H} = \frac{N_0^2 g}{2V} + \sum_{\vec{p}(\vec{p}\neq\vec{0})} (\epsilon_p^0 + 2n_0 g)\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} + \frac{n_0 g}{2}\sum_{\vec{p}(\vec{p}\neq\vec{0})} (\hat{a}^\dagger_{\vec{p}}\hat{a}^\dagger_{-\vec{p}} + \hat{a}_{\vec{p}}\hat{a}_{-\vec{p}}), \tag{3.3.18}$$

where $n_0 = N_0/V$ is the particle density in the zero-momentum state. The first term is the energy of N_0 particles in the zero-momentum state; the second one is that of independent excitations with energy $\epsilon_p^0 + 2n_0 g$; the final terms correspond to the scattering of two atoms in the condensate to states with momenta $\pm\vec{p}$ and the inverse process in which two atoms with momenta $\pm\vec{p}$ are scattered into the condensate.

The task now is to find the Hamiltonian eigenvalues. Since the total number of particles is conserved, we wish to find the energy eigenvalues for a fixed average particle number. The total particle number operator is given by

$$\hat{N} = N_0 + \sum_{\vec{p}(\vec{p}\neq 0)} \hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} \tag{3.3.19}$$

on treating the zero-momentum-state operators as c-numbers. The Hamiltonian may be written in terms of $\hat{N}$, thus getting

$$\hat{H} = \frac{N^2 g}{2V} + \sum_{\vec{p}(\vec{p}\neq 0)} \left[(\epsilon_p^0 + n_0 g)\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} + \frac{n_0 g}{2}(\hat{a}^\dagger_{\vec{p}}\hat{a}^\dagger_{-\vec{p}} + \hat{a}_{\vec{p}}\hat{a}_{-\vec{p}})\right], \tag{3.3.20}$$

where in the first term we have replaced $\hat{N}$ with its expectation value. This is allowed by the fluctuation in the particle number being small. The energy $\epsilon_0 + n_0 g$

does not depend on the direction of $\vec{p}$, which leads to write the Hamiltonian in a symmetrical form

$$\hat{H} = \frac{N^2 g}{2V} + \sum_{\vec{p}(\vec{p}\neq\vec{0})}{}' \Big[(\epsilon_p^0 + n_0 g)(\hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}}^\dagger + \hat{a}_{-\vec{p}}\hat{a}_{-\vec{p}}) + n_0 g(\hat{a}_{\vec{p}}^\dagger \hat{a}_{-\vec{p}}^\dagger + \hat{a}_{\vec{p}}\hat{a}_{-\vec{p}})\Big], \quad (3.3.21)$$

The prime ($'$) symbol indicates the sum is to be taken only over one half of momentum space, since the terms corresponding to $\vec{p}$ and $-\vec{p}$ must be counted only once.

The structure of the Hamiltonian now consists of a sum of independent terms of the form

$$\hat{h} = \epsilon_0(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}) + \epsilon_1(\hat{a}^\dagger\hat{b}^\dagger + \hat{b}\hat{a}). \quad (3.3.22)$$

Here ϵ_0 and ϵ_1 are c-numbers. The operators $\hat{a}^\dagger$ and $\hat{a}$ create and annihilate bosons in the state with momentum $\vec{p}$, and $\hat{b}^\dagger$ and $\hat{b}$ are the corresponding operators for the state with momentum $-\vec{p}$. The eigenvalues and eigenstates of this Hamiltonian may be obtained by performing a canonical transformation, that is a *Bogoliubov transformation*. Hence, we introduce a new set of bosonic operators $\hat{\alpha}, \hat{\alpha}^\dagger$ and $\hat{\beta}, \hat{\beta}^\dagger$ via the transformation

$$\hat{\alpha} = u\hat{a} + v\hat{b}^\dagger, \quad (3.3.23a)$$

$$\hat{\beta} = u\hat{b} + v\hat{a}^\dagger, \quad (3.3.23b)$$

such that the Hamiltonian has only terms proportional to $\hat{\alpha}^\dagger\hat{\alpha}$ and $\hat{\beta}^\dagger\hat{\beta}$. Since the phases of u and v coefficients are arbitrary, we may take u and v to be real. Inserting the definitions for $\hat{\alpha}$ and $\hat{\beta}$ into their commutation relations, together with using the commutation relations for $\hat{a}$ and $\hat{b}$, conveys the normalization condition for u and v

$$u^2 - v^2 = 1. \quad (3.3.24)$$

Inverting (3.3.23a) and (3.3.23b) yields

$$\hat{a} = u\hat{\alpha} - v\hat{\beta}^\dagger, \quad \hat{b} = u\hat{\beta} - v\hat{\alpha}^\dagger. \quad (3.3.25)$$

Plugging these last expressions into (3.3.22), we obtain

$$\begin{aligned}\hat{h} = 2v^2\epsilon_0 - 2uv\epsilon_1 + [\epsilon_0(u^2+v^2) - 2uv\epsilon_1](\hat{\alpha}^\dagger\hat{\alpha} + \hat{\beta}^\dagger\hat{\beta}) \\ + [\epsilon_1(u^2+v^2) - 2uv\epsilon_0](\hat{\alpha}\hat{\beta} + \hat{\beta}^\dagger\hat{\alpha}^\dagger).\end{aligned} \quad (3.3.26)$$

In the end, the term proportional to $\hat{\alpha}\hat{\beta} + \hat{\beta}^\dagger\hat{\alpha}^\dagger$ can be made to vanish by a proper selection of u and v. Adopting the convention of positive sign for u, the normalization condition (3.3.24) is satisfied by the following parametrization of u and v

$$u = \cosh t, \quad v = \sinh t. \tag{3.3.27}$$

Consequently, setting the coefficient of $\hat{\alpha}\hat{\beta} + \hat{\beta}^\dagger\hat{\alpha}^\dagger$ equal to zero

$$\epsilon_1(u^2 + v^2) - 2uv\epsilon_0 = 0 \tag{3.3.28}$$

is equivalent to

$$\tanh{(2t)} = \frac{\epsilon_1}{\epsilon_0}. \tag{3.3.29}$$

From this result, we find

$$u^2 = \frac{1}{2}\Big(\frac{\epsilon_0}{\epsilon} + 1\Big) \text{ and } v^2 = \frac{1}{2}\Big(\frac{\epsilon_0}{\epsilon} - 1\Big), \tag{3.3.30}$$

where $\epsilon = \sqrt{\epsilon_0^2 - \epsilon_1^2}$. Solving for $u^2 + v^2$ and $2uv$ in terms of the ratio ϵ_1/ϵ_0 leads to

$$\hat{h} = (\epsilon - \epsilon_0) + \epsilon(\hat{\alpha}^\dagger\hat{\alpha} + \hat{\beta}^\dagger\hat{\beta}). \tag{3.3.31}$$

Notice the ground-state energy is $\epsilon - \epsilon_0$, which is negative, and the excited states correspond to the addition of two independent kinds of bosons with energy ϵ, created by the operators $\hat{\alpha}^\dagger$ and $\hat{\beta}^\dagger$. For ϵ to be real, there must hold $|\epsilon_0| \geq |\epsilon_1|$.

Now, via the transformation

$$\hat{a}_{\vec{p}} = u_p\hat{\alpha}_{\vec{p}} - v_p\hat{\alpha}^\dagger_{-\vec{p}}, \quad \hat{a}_{-\vec{p}} = u_p\hat{\alpha}_{-\vec{p}} - v_p\hat{\alpha}^\dagger_{\vec{p}}, \tag{3.3.32}$$

we may bring the initial Hamiltonian into diagonal form, where we intend the following correspondencies

$$\hat{a}_{\vec{p}} \longleftrightarrow \hat{a}, \quad \hat{a}_{-\vec{p}} \longleftrightarrow \hat{b}, \quad \hat{\alpha}_{\vec{p}} \longleftrightarrow \hat{\alpha}, \quad \hat{\alpha}_{-\vec{p}} \longleftrightarrow \hat{\beta}. \tag{3.3.33}$$

Finally, we get

$$\hat{H} = \frac{N^2 g}{2V} + \sum_{\vec{p}(\vec{p}\neq\vec{0})} \epsilon_p \hat{\alpha}^\dagger_{\vec{p}}\hat{\alpha}_{\vec{p}} - \frac{1}{2}\sum_{\vec{p}(\vec{p}\neq\vec{0})} (\epsilon_p^0 + n_0 g - \epsilon_p), \tag{3.3.34}$$

where

$$\epsilon_p = \sqrt{(\epsilon_p^0 + n_0 g)^2 - (n_0 g)^2} = \sqrt{(\epsilon_p^0)^2 + 2\epsilon_p^0 n_0 g}. \tag{3.3.35}$$

Notice that consistently, via the quantum identifications

$$\hbar\omega_p = \epsilon_p, \quad \hbar\vec{k} = \vec{p}, \tag{3.3.36}$$

the energy spectrum (3.3.35) agrees precisely with the Bogoliubov dispersion relation (3.3.13). Creation and annihilation operators for elementary excitations are given by

$$\hat{\alpha}^\dagger_{\vec{p}} = u_p \hat{a}^\dagger_{\vec{p}} + v_p \hat{a}_{-\vec{p}}, \quad \hat{\alpha}_{\vec{p}} = u^*_p \hat{a}_{\vec{p}} + v^*_p \hat{a}^\dagger_{-\vec{p}}. \tag{3.3.37}$$

u_p and v_p coefficients, which are also said *Bogoliubov weights*, satisfy the normalization condition

$$u_p^2 - v_p^2 = 1, \tag{3.3.38}$$

where

$$u_p^2 = \frac{1}{2}\left(\frac{\xi_p}{\epsilon_p} + 1\right) \text{ and } v_p^2 = \frac{1}{2}\left(\frac{\xi_p}{\epsilon_p} - 1\right). \tag{3.3.39}$$

$\xi_q \equiv \epsilon^0_p + gn$ is the energy of an excitation if we neglect the coupling between u and v, that is the mean-field term gn

$$(\hbar\omega_p)^2 = \epsilon_p^2 = (\epsilon^0_p + gn)^2 - (gn)^2. \tag{3.3.40}$$

In Fig.(3.3.1) we can observe u_p and v_p tend to infinity when p tends to zero, while they tend to 1 an 0, respectively, when p tends to infinity. Therefore, at short wavelength this corresponds to addition of a single particle with momentum p, and the removal of a particle in the zero-momentum state; at longer wavelength, excitations are linear superpositions of the state in which a particle with momentum p is added (and a particle in the condensate is removed) and the state in which a particle with momentum $-p$ is removed (and a particle added to the condensate). At long wavelengths, Bogoliubov weights diverge as $1/\sqrt{p}$, and the two components of the wave function are essentially equal in magnitude.

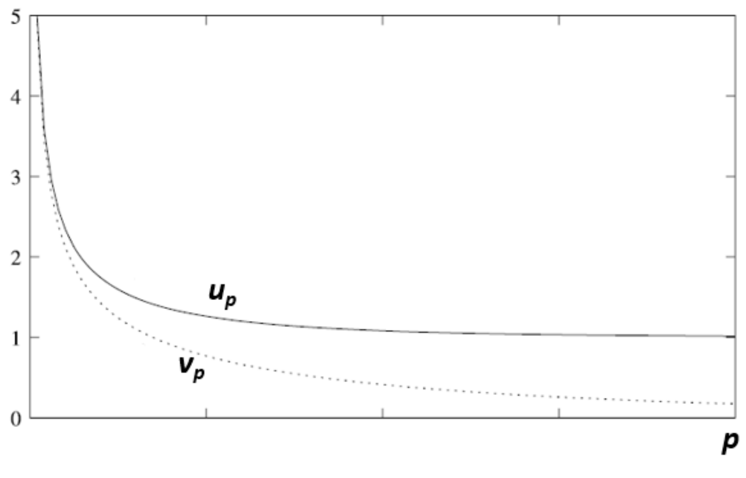

Figure 3.3.1: Bogoliubov weights u_p and v_p as functions of momentum p [14].

In conclusion, the excited states of a weakly interacting Bose gas can be described in terms of a gas of *non-interacting quasi-particles*, with energies according to Bogoliubov spectrum previously derived from classical considerations. The ground state $|0\rangle$, where there no excitations, is identified by

$$\hat{\alpha}_{\vec{p}}\,|0\rangle = 0, \quad \forall \vec{p}. \tag{3.3.41}$$

- At small momenta, the energy spectrum of quasi-particles takes the *phonon*-like form
$$\epsilon_p = cp, \tag{3.3.42}$$
where $c = \sqrt{n_0 g/m}$ is the sound velocity (in agreement with the hydrodynamic relation $\frac{\partial n}{\partial P} = \frac{1}{mc^2}$, where $P =$ pressure). The Bogoliubov theory then predicts that the long-wavelength excitations of a weakly interacting Bose gas are sound waves[1].

- At big momenta, the Bogoliubov dispersion relation approaches the free-particle law plus a mean-field term
$$\epsilon_p = \epsilon_p^0 + gn. \tag{3.3.43}$$

The transition between the phonon and particle regimes takes place when $\epsilon_p^0 \sim gn = mc^2$ i.e. for $p \sim mc$ (Fig.(3.3.2)). By setting $\epsilon_p^0 = gn$, with $p = \hbar/\xi$, we can define a characteristic interaction length

$$\xi = \sqrt{\frac{\hbar^2}{2mgn}} = \frac{1}{\sqrt{2}}\frac{\hbar}{mc}, \tag{3.3.44}$$

also called the *healing length*.

3.4 Analogue gravity in atomic BECs

We wish to introduce a different parametrization of the field operator, which leads to a reinterpretation of the above equations in a hydrodynamic language and then to a gravitational analogy. Let us again consider a non-uniform dilute gas of N weakly interacting bosonic atoms, which we saw is described by the many-body Hamiltonian (3.2.7)

$$\hat{H} = \int \left[\hat{\Psi}^\dagger \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}\right)\hat{\Psi} + \frac{g}{2}\hat{\Psi}^\dagger\hat{\Psi}^\dagger\hat{\Psi}\hat{\Psi}\right] \mathrm{d}^3 r.$$

[1] These excitations can be also regarded as the Goldstone modes associated to $U(1)$-symmetry breaking in BEC.

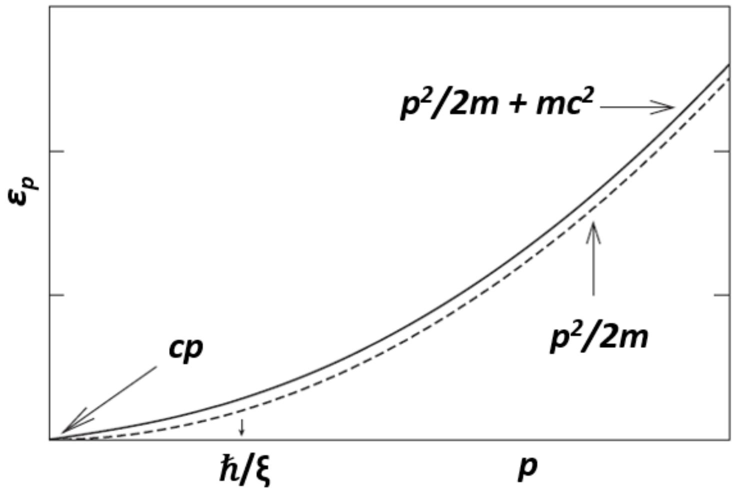

Figure 3.3.2: Bogoliubov energy ϵ_p as a function of momentum p. $p \sim \hbar/\xi$ delimits phonon and particle regimes [13].

To study quantum fluctuations above a classical mean-field within the Bogoliubov approximation, we will now decompose the field operator with a slight a change of notation with respect to the previous section

$$\hat{\Psi}(\vec{r}, t) = \Psi_0(\vec{r})[1 + \hat{\phi}(\vec{r}, t)]e^{-i\mu t/\hbar}, \tag{3.4.1}$$

where $\hat{\Psi}$ is taken in Heisenberg picture. The steady-state $\Psi_0(\vec{r})$ is the classical mean-field, while the field operator $\hat{\phi}(\vec{r}, t)$ describes quantum fluctuations. By linearizing the Heisenberg equation (3.2.8)

$$i\hbar\frac{\partial\hat{\Psi}(\vec{r}, t)}{\partial t} = \left[\hat{\Psi}(\vec{r}, t), \hat{H}\right] = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{ext} + g\hat{\Psi}^\dagger\hat{\Psi}\right]\hat{\Psi},$$

we find that the condensate wave function

$$\Psi_0'(\vec{r}, t) = \Psi_0(\vec{r})e^{-i\mu t/\hbar} \tag{3.4.2}$$

satisfies the Time-Dependent Gross-Pitaevskii equation

$$i\hbar\frac{\partial\Psi_0'(\vec{r}, t)}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\vec{r}, t) + g|\Psi_0'(\vec{r}, t)|^2\right)\Psi_0'(\vec{r}, t), \tag{3.4.3}$$

while the quantum field operator $\hat{\phi}(\vec{r}, t)$ satisfies the Bogoliubov-de Gennes equation

$$i\hbar\frac{\partial\hat{\phi}}{\partial t} = -\left(\frac{\hbar\nabla^2}{2m} + \frac{\hbar^2}{m}\frac{\nabla\Psi_0}{\Psi_0}\nabla\right)\hat{\phi} + ng(\hat{\phi} + \hat{\phi}^\dagger), \tag{3.4.4}$$

where $n = |\Psi_0|^2$.

In the *density-phase representation*, the condensate wave function appears as

$$\Psi_0(\vec{r}, t) = \sqrt{n(\vec{r}, t)} e^{i\theta(\vec{r}, t)}. \tag{3.4.5}$$

If we now multiply (3.4.3) by $\Psi_0'^*$ and subtract the complex conjugate, we obtain the following *continuity equation* for the density

$$\frac{\partial n}{\partial t} + \nabla \cdot \vec{j} = 0, \tag{3.4.6}$$

where the *current density* $\vec{j}(\vec{r}, t)$ defines the *condensate flow velocity* $\vec{v}$

$$\vec{j} = -\frac{i\hbar}{2m}(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) = n\frac{\hbar}{m}\nabla\theta = n\vec{v}. \tag{3.4.7}$$

The conservation the total number of particles follows from the continuity equation (3.4.6)

$$\frac{\mathrm{d}N}{\mathrm{d}t} = 0, \quad N = \int |\Psi|^2 \mathrm{d}^3 r. \tag{3.4.8}$$

Notice that $\vec{v}$, being the gradient of a scalar function, turns out to be irrotational ($\nabla \times \vec{v} = 0$).

We can also derive an explicit equation for the phase θ by inserting the density-phase representation into the Gross-Pitaevskii equation, which yields

$$\hbar\frac{\partial \theta}{\partial t} = -\frac{\hbar^2}{2m}(\nabla\theta)^2 - gn - V_{ext} - V_q. \tag{3.4.9}$$

In hydrodynamic framework, (3.4.9) corresponds to the *Euler-Bernoulli's equation* for an irrotational inviscid fluid with an additional "quantum pressure" term

$$V_q \equiv \frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n}. \tag{3.4.10}$$

The Planck constant inside V_q reveals that the importance of quantum effects is emphasized in non-uniform gases. It is worth pointing out that the continuity equation (3.4.6) and the Euler's equation (3.4.9) provide a closed set of coupled equations exactly equivalent to the original Gross-Pitaevskii equation.

In this density-phase representation, the Bogoliubov decomposition (3.4.1) for the field operator is rewritten as

$$\hat{\Psi} = \sqrt{n + \hat{n}_1} e^{i(\theta + \hat{\theta}_1)} e^{-i\mu t/\hbar} \simeq \Psi_0 \left(1 + \frac{\hat{n}_1}{2n} + i\hat{\theta}_1\right) e^{-i\mu t/\hbar}, \tag{3.4.11}$$

so that we can express $\hat{\phi}$ as

$$\hat{\phi} = \frac{\hat{n}_1}{2n} + i\hat{\theta}_1 \quad \longleftrightarrow \quad \hat{n}_1 = n(\hat{\phi} + \hat{\phi}^\dagger),\ \hat{\theta}_1 = \frac{\hat{\phi} - \hat{\phi}^\dagger}{2i}. \tag{3.4.12}$$

Therefore, upon linearizing the two hydrodynamic equations (3.4.6) and (3.4.9) encoding Gross-Pitaevskii equation, the Bogoliubov-de Gennes equation (3.4.4) reduces to a pair of equations of motion for density $\hat{n}_1$ and phase $\hat{\theta}_1$ fluctuations.

$$\hbar\frac{\partial\hat{\theta}_1}{\partial t} = -\hbar\vec{v}\cdot\nabla\hat{\theta}_1 - \frac{mc^2}{n}\hat{n}_1 + \frac{mc^2}{4n}\xi^2\nabla\cdot\left[n\nabla\left(\frac{\hat{n}_1}{n}\right)\right], \tag{3.4.13}$$

$$\frac{\partial\hat{n}_1}{\partial t} = -\nabla\cdot\left(\vec{v}\hat{n}_1 + \frac{\hbar n}{m}\nabla\hat{\theta}_1\right). \tag{3.4.14}$$

Here, a fundamental length scale is set by the healing length[2] $\xi \equiv \hbar/mc$ in terms of the local speed of sound $c = \sqrt{ng/m}$. On length scales much larger than ξ (*hydrodynamic approximation*), the last term in (3.4.13) can be neglected. As a result, the density fluctuations can be decoupled as

$$\hat{n}_1 = -\frac{\hbar n}{mc^2}\left[\vec{v}\cdot\nabla\hat{\theta}_1 + \frac{\partial\hat{\theta}_1}{\partial t}\right]. \tag{3.4.15}$$

When this form is inserted in (3.4.14), the equation of motion for the phase perturbation reads

$$-\left(\frac{\partial}{\partial t} + \nabla\cdot\vec{v}\right)\frac{n}{mc^2}\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right)\hat{\theta}_1 + \nabla\cdot\left(\frac{n}{m}\nabla\hat{\theta}_1\right) = 0. \tag{3.4.16}$$

Notice this last equation can be rewritten in a matrix form

$$\partial_\mu(f^{\mu\nu}\partial_\nu\hat{\theta}_1) = 0, \tag{3.4.17}$$

where the matrix elements $f^{\mu\nu}$ are defined as[3]

$$f^{00} = \frac{n}{c^2}, \quad f^{0i} = f^{i0} = \frac{n}{c^2}v^i, \quad f^{ij} = -\frac{n}{c^2}(c^2\delta^{ij} - v^i v^j). \tag{3.4.18}$$

in terms of the condensate density n and local flow velocity $\vec{v}$. As we have already mentioned in Chapter 2, in any Lorentzian manifold the covariant d'Alembertian operator results being

$$\nabla_\mu\nabla^\mu = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu), \tag{3.4.19}$$

[2]Except for a factor $1/\sqrt{2}$, it coincides with (3.3.44).

[3]The adopted convention is that Greek indices $\mu,\nu = 0,1,2,3$ indicate 4-dimensional objects, while latin ones $i = 1,2,3$ indicate the space coordinates.

where $g^{\mu\nu}$ is the inverse metric and $g = |\det(g_{\mu\nu})|$. Consequently, the equation of motion (3.4.16) can be recast as a wave equation in curved space-time

$$\nabla_\mu \nabla^\mu \hat{\theta}_1 = 0 \tag{3.4.20}$$

provided the identification

$$\sqrt{g} g^{\mu\nu} = f^{\mu\nu}. \tag{3.4.21}$$

This can be inverted, thus leading to the effective metric

$$g_{\mu\nu} = \frac{n}{mc} \begin{pmatrix} (c^2 - v^2) & v^i \\ v^j & -\delta_{ij} \end{pmatrix} \tag{3.4.22}$$

which is called *acoustic metric*. This corresponds to the line element

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \frac{n}{mc}\big[c^2\mathrm{d}t^2 - (\mathrm{d}x^i - v^i\mathrm{d}t)\delta_{ij}(\mathrm{d}x^j - v^j\mathrm{d}t)\big]. \tag{3.4.23}$$

Let us move to spherical polar coordinates. Eq.(3.4.23) then becomes

$$\begin{aligned} \mathrm{d}s^2 = \frac{n}{mc}[&(c^2 - v^2)\mathrm{d}t^2 + 2\mathrm{d}t\Big(v^{\hat{r}}\mathrm{d}r + v^{\hat{\theta}} r\mathrm{d}\theta + v^{\hat{\phi}} r\sin\theta \mathrm{d}\phi\Big) \\ &- \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)], \end{aligned} \tag{3.4.24}$$

where $v^{\hat{r}}, v^{\hat{\theta}}, v^{\hat{\phi}}$ denote the fluid velocity components along r, θ, ϕ directions, respectively. If we restrict to a spherically-symmetric flow, $v^{\hat{\theta}}$ and $v^{\hat{\phi}}$ vanish and Eq.(3.4.24) reduces to

$$\mathrm{d}s^2 = \frac{n}{mc}[(c^2 - v^2)\mathrm{d}t^2 + 2v^{\hat{r}}\mathrm{d}t\mathrm{d}r - \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)], \tag{3.4.25}$$

which can be recast as

$$\mathrm{d}s^2 = \frac{n}{mc}[c^2\mathrm{d}t^2 - (\mathrm{d}r + v^{\hat{r}}\mathrm{d}t)^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)]. \tag{3.4.26}$$

This last form for the line element recalls the Schwarzschild metric written in Painlevé-Gullstrand coordinates (1.3.12) we saw in Sect.1.3

$$\mathrm{d}s^2 = c^2\mathrm{d}T^2 - (\mathrm{d}r + v\,\mathrm{d}T)^2 - r^2\mathrm{d}\Omega^2, \tag{3.4.27}$$

except for a constant conformal factor $\frac{n}{mc}$. In Eq.(3.4.27) we have reintroduced the speed of light c explicitly. Recall we had defined the "velocity"

$$v \equiv \sqrt{\frac{2m}{r}}, \tag{3.4.28}$$

and T and r played the role of time and space coordinates, respectively. As a consequence, we may establish a remarkable analogy between Schwarzschild black holes and atomic BECs undergoing a spherically-symmetric flow, provided that we identify the speed of light with that of sound and the condensate flow velocity $v = v^{\hat{r}}$ with "Schwarzschild velocity" $\sqrt{\frac{2m}{r}}$.

Hence, sound waves propagation along radial directions in a spherically-symmetric atomic BEC can be equivalently inferred from radial null directions in the acoustic metric (3.4.26):

$$\mathrm{d}s^2 = 0, \quad \theta, \phi = const, \tag{3.4.29}$$

which yields

$$\begin{aligned} c^2\mathrm{d}t^2 - (\mathrm{d}r + v\mathrm{d}t)^2 = 0, \\ \mathrm{d}r + v\mathrm{d}t = \pm c^2\mathrm{d}t. \end{aligned} \tag{3.4.30}$$

Notice the constant conformal factor $\frac{n}{mc}$ is not influential in studying radial null directions. There exist two solutions to Eq.(3.4.30)

$$\begin{cases} `+': & \frac{\mathrm{d}r}{\mathrm{d}t} = c - v \\ `-': & \frac{\mathrm{d}r}{\mathrm{d}t} = -c - v \end{cases} . \tag{3.4.31}$$

Therefore, we have obtained an equivalent result to the river analogy for Schwarzschild black hole presented in Chapter 1.

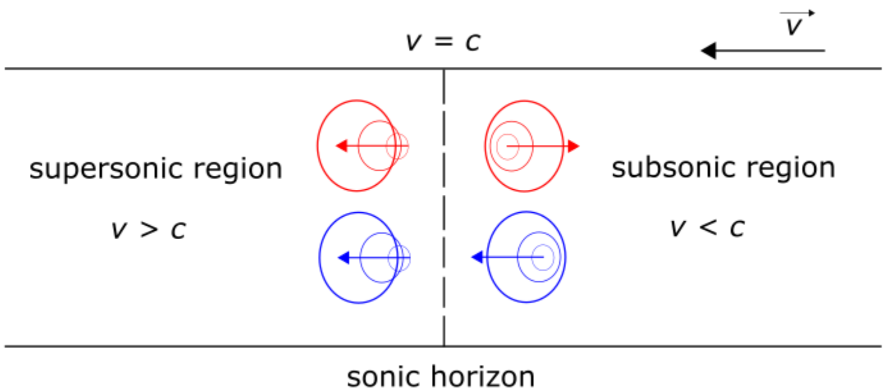

Figure 3.4.1: Scheme of an acoustic black hole. The right to left direction corresponds to radial ingoing motion. Red sound wave fronts show the upstream solution, while blue sound wave fronts the downstream one. In the supersonic region, even an upstream front is forced to propagate downstream.

Let us assume the flow velocity $\vec{v}$ of our atomic BEC increases along radial ingoing direction (Fig.(1.3.1)). According to Eq.(3.4.31), a sound wave can move either *upstream* ('+' solution) or *downstream* ('−' solution). Where $v = c$, the sound wave trying to propagate upstream remains stuck because its speed vanishes. On the left side of the locus where $v = c$, v is greater than c: this means the sound wave propagating upstream is dragged by the flow and forced to move downstream. This corresponds to the trapping behaviour of a Schwarzschild black hole. Therefore, the role of light in a Schwarzschild black hole is played by sound in such fluid systems, that is why they are called *acoustic black holes*. The surface where $v = c$ is called *sonic horizon*, since it separates the subsonic from the supersonic region, inside which any sound perturbation remains trapped.

Moreover, within a classical context, by linearizing the continuity and Euler-Bernoulli's hydrodynamic equations around a certain background of fluid variables, the following theorem can be proven [16]

Theorem *If a fluid is barotropic and inviscid, and the flow is irrotational (though possibly time-dependent), then the equation of motion for the velocity potential ψ describing an acoustic disturbance is identical to the d'Alembertian equation of motion for a minimally coupled massless scalar field propagating in a (3+1)-dimensional Lorentzian geometry*

$$\Delta\psi \equiv \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\, \partial_\nu\psi) = 0. \tag{3.4.32}$$

$g_{\mu\nu}$ stands for the hydrodynamic version of acoustic metric

$$g_{\mu\nu} = \frac{\rho_0}{c}\begin{pmatrix} (c^2 - v^2) & v^i \\ v^j & -\delta_{ij} \end{pmatrix} \tag{3.4.33}$$

and depends on the background fluid density ρ_0, the fluid velocity $\vec{v}$ and the local speed of sound c. Notice it differs from the BEC acoustic metric (3.4.22) just by a conformal factor (though still encoding the density feature of the system). Therefore, radial null directions for the acoustic disturbance velocity potential ψ in a classical hydrodynamic system are identical to those for long-wavelength sound waves in a BEC.

We can observe Eqs.(3.4.32) and (3.4.20) have the same form. The latter, which describes the phase dynamics in a BEC under the hydrodynamic approximation, is the Klein-Gordon's equation for a massless scalar field propagating in a fictitious curved space-time (3.4.22). Indeed, Eq.(3.4.20) extends the *classical* gravitational analogy in flowing systems to a *quantum* level. At this point, the same computational steps we reviewed in Chapter 2 to derive the Hawking effect would lead to its analogue in a BEC. By expanding the field $\hat{\theta}_1$ in modes and proceeding with quantization, *ingoing* and *outgoing* vacuum states are shown to be non-equivalent, since their Bogoliubov transformation mixes the corresponding creation and annihilation operators in a non-trivial way. As a result, we would expect an emission of Bogoliubov phonons by the horizon in the subsonic region, with a thermal distribution related to the surface gravity κ of the sonic horizon defined as [15]

$$\kappa = \frac{1}{2c}\frac{\mathrm{d}(c^2 - v_0^2)}{\mathrm{d}n}\bigg|_{hor}, \tag{3.4.34}$$

where n here stands for the spatial coordinate normal to the horizon.

However, in Chapter 2 we pointed out that the modes responsible for the Hawking emission experience an infinite blue-shift when traced back to the near-horizon region. In an atomic BEC, their wavelength in this region decreases below the healing length, which then plays the role of a Planck length. Therefore, the derivation of Hawking effect in atomic BECs under the long wavelength approximation produces the analogue of transplanckian problem, which could make the procedure outlined above questionable. Yet, an alternative strategy based on the original microscopic BEC theory may show that the emission of Hawking radiation in BEC supersonic configurations emerges "naturally" from the underlying quantum theory without any hydrodynamic approximation. This shall be the purpose of the next chapter.

To sum up, atomic BEC transonic configurations may be regarded as models of *analogue gravity*, a branch of modern physics which pursues the theoretical description and experimental investigation of gravitational issues by means of analogue systems.

Chapter 4

Stepwise Bose-Einstein condensates: subsonic-supersonic configuration

In this chapter a stepwise BEC model reproducing an acoustic black hole is presented [15]. Unlike the analogue gravity approach, this system will be studied by means of the underlying microscopic quantum theory. The scattering processes of quantum fluctuations at the interface between the subsonic and supersonic region will be developed analytically and will show the prediction of analogue Hawking radiation.

4.1 Microscopic BEC theory for acoustic black holes

We wish to show how the creation of Bogoliubov phonons with the same feature as Hawking radiation may occur in atomic Bose-Einstein condensates. To this purpose, we will consider a simple model consisting of two semi-infinite stationary homogeneous one dimensional condensates. In both left and right sectors the condensate is assumed to have uniform density $n = |\Psi_0|^2$ and flow with constant velocity v along the negative direction of x axis. Moreover, the external potential V_{ext} and the repulsive atom-atom interaction coupling g are set constant within each sector but in such a way that

$$V^l_{ext} + g^l n = V^r_{ext} + g^r n. \tag{4.1.1}$$

As a matter of convention, we denote physical quantities regarding either the right or left sector via the indices r and l, respectively. The two sections are connected by a step-like discontinuity in the speed of sound located at $x = 0$, so that the left sector coincides with $x < 0$ region, while the right sector with $x > 0$. Since $mc^2 = gn$, we have two different values for the local speed of sound c^l and c^r. Thanks to the condition (4.1.1), the Gross-Pitaevskii equation for the background field $\Psi_0(t,x)$ admits plane-wave solutions for the whole x and t intervals. Indeed,

$$\Psi_0(t,x) = \sqrt{n}e^{ik_0x - i\omega_0 t} \tag{4.1.2}$$

satisfies

$$i\hbar\frac{\partial\Psi_0(t,x)}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + \underbrace{V_{ext} + gn}_{const}\right)\Psi_0(t,x), \tag{4.1.3}$$

with

$$\hbar\omega_0 = \frac{\hbar^2 k_0^2}{2m} + V_{ext} + gn, \tag{4.1.4}$$

that is, the energy spectrum for the condensate wave function is given by the free-particle law plus a mean-field contribution. The wavenumber k_0 is related to the fluid velocity v by $v = \hbar k_0/m$.

Now we would like to study quantum fluctuations above the equilibrium configuration Ψ_0. Then, within each sector, we will look for solutions to Bogoliubov-de Gennes equation (3.4.4). The fluctuation field operator $\hat{\phi}$ shall be decomposed into its positive and negative frequency components

$$\hat{\phi} = \sum_j [\hat{a}_j\phi_j(t,x) + \hat{a}_j^\dagger\varphi_j^*(t,x)], \tag{4.1.5}$$

where $\hat{a}_j$ and $\hat{a}_j^\dagger$ are the annihilation and creation operators for phonons, respectively, while ϕ_j and φ_j are mode functions oscillating at a frequency ω_j. Eq.(3.4.4) and its hermitian conjugate can be rewritten as a couple of equations of motion for the mode functions

$$\left[i(\partial_t + v\partial_x) + \frac{\xi c}{2}\partial_x^2 - \frac{c}{\xi}\right]\phi_j = \frac{c}{\xi}\varphi_j, \tag{4.1.6a}$$

$$\left[-i(\partial_t + v\partial_x) + \frac{\xi c}{2}\partial_x^2 - \frac{c}{\xi}\right]\varphi_j = \frac{c}{\xi}\phi_j. \tag{4.1.6b}$$

Imposing the operator $\hat{\phi}$ satisfies the following equal-time commutation relation

$$\left[\hat{\phi}(t,x), \hat{\phi}^\dagger(t,x')\right] = \frac{1}{n}\delta(x - x'), \tag{4.1.7}$$

we can define the *Bogoliubov scalar product* by

$$\int \mathrm{d}x[\phi_j\phi_{j'}^* - \varphi_j^*\varphi_{j'}] = \pm\frac{\delta_{jj'}}{\hbar n}. \tag{4.1.8}$$

In each of the spatially uniform sectors we can make the ansatz of plane wave form for the mode functions

$$\phi_\omega = D(\omega)e^{-i\omega t + ikx}, \quad \varphi_\omega = E(\omega)e^{-i\omega t + ikx}, \tag{4.1.9}$$

where the normalization coefficients $D(\omega)$ and $E(\omega)$ are the *Bogoliubov weights.* Inserting plane wave ansatz (4.1.9) into the equations of motions (4.1.6a) and (4.1.6b) for the modes yields

$$\left[(\omega - vk) - \frac{\xi c k^2}{2} - \frac{c}{\xi}\right] D(\omega) = \frac{c}{\xi} E(\omega), \tag{4.1.10a}$$

$$\left[-(\omega - vk) - \frac{\xi c k^2}{2} - \frac{c}{\xi}\right] E(\omega) = \frac{c}{\xi} D(\omega), \tag{4.1.10b}$$

which expresses the couple of equations (3.3.12a) and (3.3.12b) for weak excitations on top of a spatially uniform condensate ground state in an alternative way. Indeed, non-trivial solutions to the homogeneous system of equations above can be found by requiring its determinant vanishes, which yields

$$(\omega - vk)^2 = c^2\left(k^2 + \frac{\xi^2 k^4}{4}\right). \tag{4.1.11}$$

This coincides exactly with Bogoliubov dispersion relation (3.3.13), provided one takes the frequency Doppler shift induced by the condensate flow into account

$$\omega \longrightarrow \Omega(k) \equiv \omega - vk. \tag{4.1.12}$$

Hence,

$$\Omega(k) = \pm c\sqrt{k^2 + \frac{\xi^2 k^4}{4}} \tag{4.1.13}$$

is the excitation frequency as measured in the frame co-moving with the fluid. For small k such that $k\xi \ll 1$ (i.e. in the hydrodynamic regime), $\Omega(k)$ approaches a linear profile

$$\omega - vk = \pm ck, \tag{4.1.14}$$

thus yielding the Doppler-shifted version of the gravitational analogy between photons and phonons. For big k such that $k\xi \gg 1$, $\Omega(k)$ tends to the quadratic dispersion for free particles plus a mean-field term

$$\omega - vk = \frac{\hbar^2 k^2}{2m} + mc^2. \tag{4.1.15}$$

The Bogoliubov norm in terms of $D(\omega)$ and $E(\omega)$ appears as

$$|D(\omega)|^2 - |E(\omega)|^2 = \pm\frac{1}{2\pi\hbar n}\left|\frac{\mathrm{d}k}{\mathrm{d}\omega}\right|, \tag{4.1.16}$$

and together with Eqs. (4.1.10a) and (4.1.10b) yields a couple of expressions for Bogoliubov weights

$$D(\omega) = \frac{\omega - vk + \frac{c\xi k^2}{2}}{\sqrt{4\pi\hbar n c\xi k^2 \left|(\omega - vk)\left(\frac{\mathrm{d}k}{\mathrm{d}\omega}\right)^{-1}\right|}}, \tag{4.1.17a}$$

$$E(\omega) = -\frac{\omega - vk - \frac{c\xi k^2}{2}}{\sqrt{4\pi\hbar n c\xi k^2 \left|(\omega - vk)\left(\frac{\mathrm{d}k}{\mathrm{d}\omega}\right)^{-1}\right|}}. \tag{4.1.17b}$$

The sign of the frequency in the co-moving frame indicates whether the corresponding mode belongs to either the positive or negative norm branch. Indeed, thanks to Eqs. (4.1.17a) and (4.1.17b), the Bogoliubov norm can be also expressed as

$$\begin{aligned} |D(\omega)|^2 - |E(\omega)|^2 &= \frac{\left(\omega - vk + \frac{c\xi k^2}{2}\right)^2 - \left(\omega - vk - \frac{c\xi k^2}{2}\right)^2}{4\pi\hbar n c\xi k^2 \left|(\omega - vk)\left(\frac{\mathrm{d}k}{\mathrm{d}\omega}\right)^{-1}\right|} \\ &= \frac{2\Omega c\xi k^2}{4\pi\hbar n c\xi k^2 \left|(\omega - vk)\left(\frac{\mathrm{d}k}{\mathrm{d}\omega}\right)^{-1}\right|} \\ &= \frac{\Omega}{2\pi\hbar n \left|(\omega - vk)\left(\frac{\mathrm{d}k}{\mathrm{d}\omega}\right)^{-1}\right|} \end{aligned} \tag{4.1.18}$$

Since the denominator is always positive, the sign of $|D(\omega)|^2 - |E(\omega)|^2$ is completely determined by Ω. Furthermore, there exists a duality between the two norm branches: for any positive norm state of frequency ω and wavenumber k, there is a corresponding negative norm state of opposite frequency $-\omega$ and wavenumber $-k$. By means of this duality, we may replace the sum over j in Eq.(4.1.5) with an integral over ω and restrict to $\omega > 0$ values.

At fixed $\omega > 0$, the dispersion relation (4.1.11) admits four solutions, being a fourth order equation in k. Consequently, the mode functions ϕ_ω and φ_ω are linear combinations of four plane waves constructed from these solutions

$$\phi_\omega(t,x) = e^{-i\omega t} \sum_{i=1}^{4} A_i(\omega) D_i(\omega) e^{ik_\omega^{(i)} x}, \tag{4.1.19a}$$

$$\varphi_\omega(t,x) = e^{-i\omega t} \sum_{i=1}^{4} A_i(\omega) E_i(\omega) e^{ik_\omega^{(i)} x}, \tag{4.1.19b}$$

where $A_i(\omega)$ are the amplitudes of the modes. The equations of motion for the mode functions (4.1.6a) and (4.1.6b) imply that the solutions in the left and right regions satisfy a set of four matching conditions at the locus of step-like discontinuity in the speed of sound ($x = 0$). These are

$$[\phi] = 0, \quad [\phi'] = 0, \quad [\varphi] = 0, \quad [\varphi'] = 0, \tag{4.1.20}$$

where we have assumed the following notation

$$[f(x)] = \lim_{\epsilon \to 0}[f(x+\epsilon) - f(x-\epsilon)], \quad ' = \frac{\mathrm{d}}{\mathrm{d}x}. \tag{4.1.21}$$

These four conditions allow to establish a linear relation between the left and right amplitudes

$$A_i^l = M_{ij} A_j^r, \tag{4.1.22}$$

where M is a 4×4 matrix called *matching matrix*.

The four roots $k_\omega^{(i)}$ of the Bogoliubov dispersion relation have different positions in the complex plane according to whether the flow is subsonic or supersonic.

Subsonic case

If the flow is subsonic ($|v| < c$), for any $\omega > 0$ there exist two real solutions k_u and k_v belonging to the positive norm branch (Fig.(4.1.1)):

- k_u has a positive group velocity $\left(v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} > 0\right)$ and propagates upstream,
- k_v has a negative group velocity $\left(v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} < 0\right)$ and propagates downstream

where u, v labels have been used in analogy to Eddington-Finkelstein null coordinates in General Relativity. These solutions admit a perturbative expansion in the dimensionless parameter $z \equiv \frac{\xi\omega}{c}$

$$k_u = \frac{\omega}{v-c}\left[1 + \frac{c^3 z^2}{8(v-c)^3} + \mathcal{O}\big(z^4\big)\right], \tag{4.1.23a}$$

$$k_v = \frac{\omega}{v+c}\left[1 - \frac{c^3 z^2}{8(v-c)^3} + \mathcal{O}\big(z^4\big)\right], \tag{4.1.23b}$$

where the lowest order in z corresponds to the hydrodynamic results.

The other two solutions are a pair of complex conjugate roots $k_\pm$, which admit a perturbative expansion in z as well

$$\begin{aligned} k_\pm = &\frac{\omega v}{c^2 - v^2}\left[1 - \frac{(c^2+v^2)c^4 z^2}{4(c^2-v^2)^3} + \mathcal{O}\big(z^4\big)\right] \\ &\pm \frac{2i\sqrt{c^2-v^2}}{c\xi}\left[1 + \frac{(c^2+2v^2)c^4 z^2}{8(c^2-v^2)^3} + \mathcal{O}\big(z^4\big)\right]. \end{aligned} \tag{4.1.24}$$

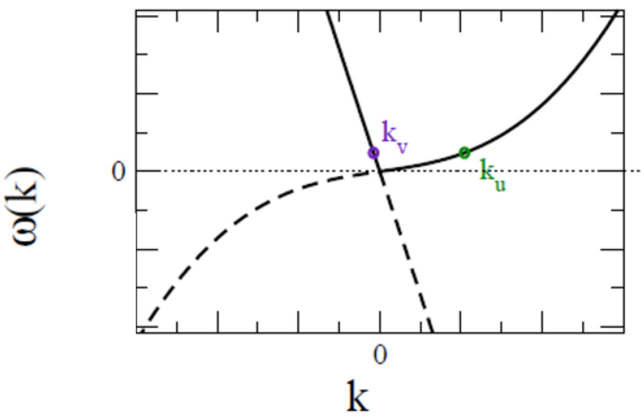

Figure 4.1.1: Dispersion relation of Bogoliubov modes in subsonic flow [15]. The solid (dashed) lines correspond to positive (negative) norm branches.

Therefore, the mode functions ϕ_ω and φ_ω are decomposed as follows

$$\phi_\omega(t,x) = e^{-i\omega t}\big[A_v D_v e^{ik_v x} + A_u D_u e^{ik_u x} + A_+ D_+ e^{ik_+ x} + A_- D_- e^{ik_- x}\big], \quad (4.1.25a)$$

$$\varphi_\omega(t,x) = e^{-i\omega t}\big[A_v E_v e^{ik_v x} + A_u E_u e^{ik_u x} + A_+ E_+ e^{ik_+ x} + A_- E_- e^{ik_- x}\big]. \quad (4.1.25b)$$

Supersonic case

If the flow is supersonic ($|v| > c$), the shape of the dispersion relation is distinguished by a threshold frequency ω_{max}, which is given by the maximum frequency of the negative norm Bogoliubov mode (Fig.(4.1.2))

$$\omega_{max} \equiv \omega(k_{max}), \quad (4.1.26)$$

where

$$k_{max} = -\frac{1}{\xi}\left[-2 + \frac{v^2}{2c^2} + \frac{|v|}{2c}\sqrt{8 + \frac{v^2}{c^2}}\right]^{1/2} \quad (4.1.27)$$

- $\omega > \omega_{max}$: the plot resembles the subsonic regime one, i.e. there exist two positive norm oscillatory modes, one propagating downstream and the other upstream[1].

[1] Despite of the supersonic flow, the upstream propagation is not trapped, because the high-k limit of Bogoliubov dispersion relation (4.1.11) predicts a free-particle behaviour.

- $0 < \omega < \omega_{max}$: the solutions to the dispersion relation are four *real* oscillatory modes, two on the *positive* norm branch and two on the *negative* norm one. Two modes lie in the small-k region and coincide with the real solutions k_u and k_v (Eqs. (4.1.23a) and (4.1.23b)), showing a hydrodynamic character. Unlike the subsonic case, both of them have a negative group velocity $v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} < 0$: this means also the u mode, which propagates upstream in the reference frame with the fluid, is dragged by the flow and forced to propagate downstream like the v mode. In addition, now the u mode has a negative norm. The other two roots k_3 and k_4 lie outside the hydrodynamic region and correspond to the analytic continuation for supersonic flow of the complex roots (4.1.24) in the subsonic regime
$$\begin{aligned} k_{3,4} =& \frac{\omega v}{c^2 - v^2}\left[1 - \frac{(c^2+v^2)c^4 z^2}{4(c^2-v^2)^3} + \mathcal{O}\left(z^4\right)\right] \\ &\pm \frac{2\sqrt{c^2-v^2}}{c\xi}\left[1 + \frac{(c^2+2v^2)c^4 z^2}{8(c^2-v^2)^3} + \mathcal{O}\left(z^4\right)\right]. \end{aligned} \tag{4.1.28}$$
The k_3 mode belongs to the positive norm branch, while k_4 to the negative one; both of them have a positive group velocity $v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} > 0$, thus propagating in the upstream direction.

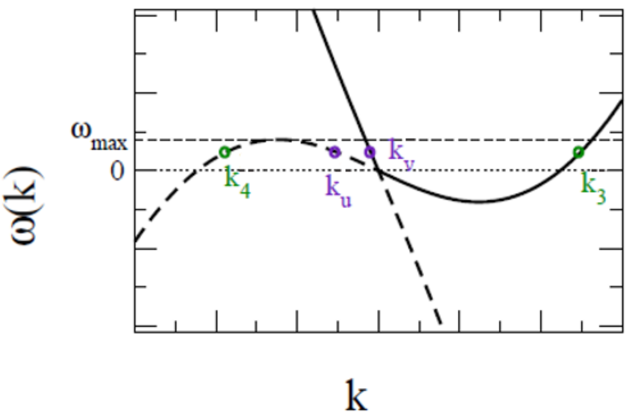

Figure 4.1.2: Dispersion relation of Bogoliubov modes in supersonic flow [15]. The solid (dashed) lines correspond to positive (negative) norm branches.

Therefore, for $0 < \omega < \omega_{max}$, the mode functions ϕ_ω and φ_ω are decomposed as follows
$$\phi_\omega(t,x) = e^{-i\omega t}\left[A_v D_v e^{ik_v x} + A_u D_u e^{ik_u x} + A_3 D_3 e^{ik_3 x} + A_4 D_4 e^{ik_4 x}\right], \tag{4.1.29a}$$
$$\varphi_\omega(t,x) = e^{-i\omega t}\left[A_v E_v e^{ik_v x} + A_u E_u e^{ik_u x} + A_3 E_3 e^{ik_3 x} + A_4 E_4 e^{ik_4 x}\right]. \tag{4.1.29b}$$

At this point, we are ready to build the *acoustic black hole* configuration. To this purpose, as we mentioned in the context of analogue gravity at the end of Chapter 3, we need a flowing BEC which undergoes a transition from subsonic to supersonic regime. Then, in our model we shall assume the flow to be subsonic in the right ($x > 0$) sector, while supersonic in the left ($x < 0$) one, with fluid velocity $v = const$. This entails two different speeds of sound c_r and c_l such that $|v| < c_r$ and $|v| > c_l$. The analogy with a gravitational black hole still lies in the dynam-

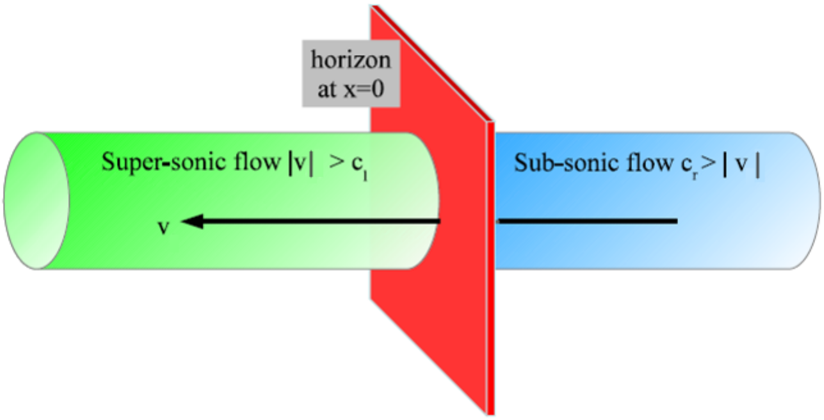

Figure 4.1.3: Sketch of the subsonic-supersonic flow configuration [15]. The fluid velocity is directed from right to left and the locus of step-like discontinuity in the sound speed (the *(sonic) horizon* is located at $x = 0$.

ics of long-wavelength Bogoliubov waves. Small-k limit of Bogoliubov dispersion relation (4.1.11) yields the sound wave linear profile (4.1.14). The emerging picture provides that long-wavelength sound waves can cross the locus separating the subsonic and the supersonic regions ($x = 0$) only in the direction of the flow and eventually get trapped beyond it. Then, the supersonic sector is called *acoustic black hole* and its outer boundary *sonic horizon*. Nevertheless, the full Bogoliubov dispersion relation introduces two more supersonic modes (k_3 and k_4), whose effects will be elucidated by studying the scattering process at the horizon.

Let us restrict to $0 < \omega < \omega_{max}$. In the supersonic (left) region, the mode functions are expanded as (Eqs.(4.1.29a) and (4.1.29b))

$$\phi^l_\omega(t,x) = e^{-i\omega t}\Big[A^l_v D^l_v e^{ik^l_v x} + A^l_u D^l_u e^{ik^l_u x} + A^l_3 D^l_3 e^{ik^l_3 x} + A^l_4 D^l_4 e^{ik^l_4 x}\Big], \quad (4.1.30a)$$

$$\varphi^l_\omega(t,x) = e^{-i\omega t}\Big[A^l_v E^l_v e^{ik^l_v x} + A^l_u E^l_u e^{ik^l_u x} + A^l_3 E^l_3 e^{ik^l_3 x} + A^l_4 E^l_4 e^{ik^l_4 x}\Big], \quad (4.1.30b)$$

while in the subsonic (right) region as (Eqs. (4.1.25a) and (4.1.25b))

$$\phi^r_\omega(t,x) = e^{-i\omega t}\big[A^r_v D^r_v e^{ik^r_v x} + A^r_u D^r_u e^{ik^r_u x} + A^r_+ D^r_+ e^{ik^r_+ x}\big], \tag{4.1.31a}$$

$$\varphi^r_\omega(t,x) = e^{-i\omega t}\big[A^r_v E^r_v e^{ik^r_v x} + A^r_u E^r_u e^{ik^r_u x} + A^r_+ E^r_+ e^{ik^r_+ x}\big]. \tag{4.1.31b}$$

Notice in the subsonic region the k^r_--mode has not been included because it is growing (thus non-renormalizable) at $x = +\infty$, whereas the k^r_+-mode is decaying at $x = +\infty$. The opposite would happen if the subsonic region went from $x = 0$ to $x = -\infty$.

4.2 Scattering states at the sonic horizon

Let us analyze the scattering states of the quantum fluctuations operator $\hat{\phi}$ at the sonic horizon. In order to construct a complete and orthonormal basis, we can either choose a *in* basis built out of incoming states (i.e. propagating from $x = \pm\infty$ towards $x = 0$) or an *out* basis built out of outgoing states (i.e. from $x = 0$ towards $x = \pm\infty$).

The *in* basis is composed of three possible incoming modes (Fig.(4.2.1) and similarly for φ^{in} modes):

- $\phi^{in}_{v,r}$

  Left-moving, unit amplitude, positive norm v-wave, which originates in the subsonic region.

- $\phi^{in}_{3,l}$

  Right-moving, unit amplitude, positive norm k^l_3-wave, which originates in the supersonic region.

- $\phi^{in}_{4,l}$

  Right-moving, unit amplitude, negative norm k^l_4-wave, which originates in the supersonic region.

After scattering, each of them generates a reflected right-moving u-wave of amplitude A^r_u and a decaying wave of amplitude A^r_+ in the right sector, and two transmitted left-moving waves in the left sector, the former a positive norm v-wave of amplitude A^l_v and the latter a negative norm u-wave of amplitude A^l_u (the so-called *anomalous transmitted* wave).

The construction of the *out* basis proceeds along similar lines and yields three possible outgoing scattering modes $\phi^{out}_{v,l}$,$\phi^{out}_{u,r}$, $\phi^{out}_{u,l}$. Now the field operator can be equivalently expanded in either of the two bases. We shall consider the *in* one, thus obtaining

$$\hat{\phi} = \int_0^{\omega_{\max}} \mathrm{d}\omega [\hat{a}_\omega^{vr,in} \phi_{v,r}^{in} + \hat{a}_\omega^{3l,in} \phi_{3,l}^{in} + \hat{a}_\omega^{4l,in\dagger} \phi_{4,l}^{in} + \hat{a}_\omega^{vr,in\dagger} \varphi_{v,r}^{in\,*} + \hat{a}_\omega^{3l,in\dagger} \varphi_{3,l}^{in\,*} + \hat{a}_\omega^{4l,in} \varphi_{4,l}^{in\,*}]. \tag{4.2.1}$$

A peculiar feature of this expansion is the third term of the integrand: since the corresponding k_4^l-wave has a negative norm, $\phi_{4,l}^{in}$ is multiplied by a *creation* operator $\hat{a}_\omega^{4l,in\dagger}$ (and, consistently, $\varphi_{4,l}^{in\,*}$ is multiplied by a *annihilation* operator $\hat{a}_\omega^{4l,in}$). This will turn out to be the key element leading to the emission of analogue Hawking radiation by the horizon.

4.3 Scattering amplitudes for the *in* basis modes

We shall now provide an analytic derivation for the scattering amplitudes for each *in* basis mode. Since the $\phi_{4,l}^{in}$ mode will result being the incoming channel responsible for the analogue Hawking radiation, here we will show the computational steps for its amplitudes explicitly. Calculations concerning the $\phi_{3,l}^{in}$ and $\phi_{v,r}^{in}$ mode are reported in Appendix A.

The scattering amplitudes for the $\phi_{4,l}^{in}$ modes are determined by the matching conditions (4.1.20), which in the case at present ($A_4^l = 1$) read

$$D_u^l A_u^l + D_v^l A_v^l + D_4^l = D_u^r A_u^r + D_+^r A_+^r \tag{4.3.1}$$

$$k_u^l D_u^l A_u^l + k_v^l D_v^l A_v^l + k_4^l D_4^l = k_u^r D_u^r A_u^r + k_+^r D_+^r A_+^r \tag{4.3.2}$$

$$E_u^l A_u^l + E_v^l A_v^l + E_4^l = E_u^r A_u^r + E_+^r A_+^r \tag{4.3.3}$$

$$k_u^l E_u^l A_u^l + k_v^l E_v^l A_v^l + k_4^l E_4^l = k_u^r E_u^r A_u^r + k_+^r E_+^r A_+^r, \tag{4.3.4}$$

It is a linear system of four equations in the four unknowns A_u^l,A_v^l,A_u^r,A_+^r, thus completely solvable. Our aim is to obtain implicit expressions for each amplitude that will be approximated to leading order in ω. To that purpose, an algorithm to simplify systems of matching equations shall be introduced.

A_u^l amplitude

Let us start by calculating A_u^l:

1) Step 1 isolates $D_+^r A_+^r$ and $E_+^r A_+^r$ from Eq.(4.3.1) and (4.3.3), respectively

$$D_+^r A_+^r = D_u^l A_u^l + D_v^l A_v^l + D_4^l - D_u^r A_u^r, \tag{4.3.5}$$

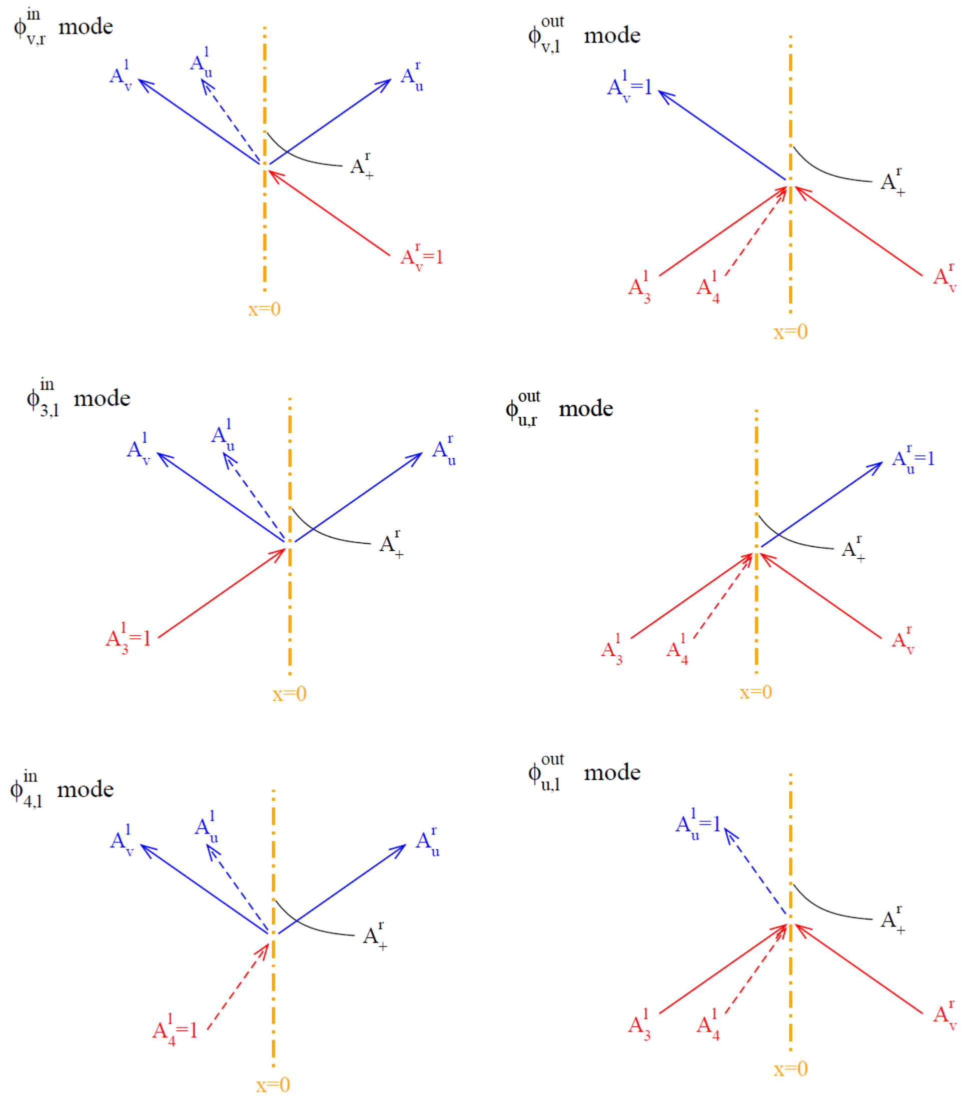

Figure 4.2.1: Sketch of the Bogoliubov modes involved in the *in* (*left panels*) and *out* basis (*right panels*) for the subsonic-supersonic configuration [15]. The first subscript refers to the wavenumber k, while the second one to the sector where the mode originates.

$$E^r_+ A^r_+ = E^l_u A^l_u + E^l_v A^l_v + E^l_4 - E^r_u A^r_u. \tag{4.3.6}$$

2) Step 2 plugs-in (4.3.5) and (4.3.6) into (4.3.2) and (4.3.4), respectively

$$(k^l_u - k^r_+) D^l_u A^l_u + (k^l_v - k^r_+) D^l_v A^l_v + (k^l_4 - k^r_+) D^l_4 = (k^r_u - k^r_+) D^r_u A^r_u, \tag{4.3.7}$$

$$(k^l_u - k^r_+) E^l_u A^l_u + (k^l_v - k^r_+) E^l_v A^l_v + (k^l_4 - k^r_+) E^l_4 = (k^r_u - k^r_+) E^r_u A^r_u. \tag{4.3.8}$$

3) Step 3 combines Eqs.(4.3.5) and (4.3.6) via E^r_+ and D^r_+

$$E^r_+ \times (4.3.5) - D^r_+ \times (4.3.6):$$

$$(E^r_+ D^l_u - D^r_+ E^l_u) A^l_u + (E^r_+ D^l_v - D^r_+ E^l_v) A^l_v + (E^r_+ D^l_4 - D^r_+ E^l_4) = (E^r_+ D^r_u - D^r_+ E^r_u) A^r_u. \tag{4.3.9}$$

4) Step 4 eliminates A^r_u contribution by means of a combination of Eqs.(4.3.7) and (4.3.8) via E^r_u and D^r_u

$$E^r_u \times (4.3.7) - D^r_u \times (4.3.8):$$

$$(k^l_u - k^r_+)(E^r_u D^l_u - D^r_u E^l_u) A^l_u + (k^l_v - k^r_+)(E^r_u D^l_v - D^r_u E^l_v) A^l_v = (k^r_+ - k^l_4)(E^r_u D^l_4 - D^r_u E^l_4). \tag{4.3.10}$$

5) Step 5 eliminates A^r_u contribution by means of a combination of Eqs.(4.3.7), (4.3.8), (4.3.9) via E^r_+, D^r_+, $(k^r_u - k^r_+)$

$$E^r_+ \times (4.3.7) - D^r_+ \times (4.3.8) - (k^r_u - k^r_+) \times (4.3.9):$$

$$(k^l_u - k^r_u)(E^r_+ D^l_u - D^r_+ E^l_u) A^l_u + (k^l_v - k^r_u)(E^r_+ D^l_v - D^r_+ E^l_v) A^l_v = (k^r_u - k^l_4)(E^r_+ D^l_4 - D^r_+ E^l_4). \tag{4.3.11}$$

6) Step 6 combines Eq.(4.3.10) from Step 4 and Eq.(4.3.11) from Step 5 as

$$(k^l_v - k^r_u)(E^r_+ D^l_v - D^r_+ E^l_v) \times (4.3.10) - (k^l_v - k^r_+)(E^r_u D^l_v - D^r_u E^l_v) \times (4.3.11).$$

The last step yields the solution

$$A^l_u = \frac{N_{ul}}{D_{ul}}, \tag{4.3.12}$$

where

$$\begin{aligned} N_{ul} = &(k^r_+ - k^l_4)(k^l_v - k^r_u)(E^r_+ D^l_v - D^r_+ E^l_v)(E^r_u D^l_4 - D^r_u E^l_4) \\ &+ (k^l_4 - k^r_u)(k^l_v - k^r_+)(E^r_u D^l_v - D^r_u E^l_v)(E^r_+ D^l_4 - D^r_+ E^l_4), \end{aligned} \tag{4.3.13}$$

and

$$\begin{aligned} D_{ul} = &(k^l_u - k^r_+)(k^l_v - k^r_u)(E^r_+ D^l_v - D^r_+ E^l_v)(E^r_u D^l_u - D^r_u E^l_u) \\ &+ (k^r_u - k^l_u)(k^l_v - k^r_+)(E^r_u D^l_v - D^r_u E^l_v)(E^r_+ D^l_u - D^r_+ E^l_u). \end{aligned} \tag{4.3.14}$$

At this point, we wish to series-expand A^l_u to leading order in ω. For that reason, we need small-ω approximation for the quantities entering its definition:

- k_u^l-wave

$$k_u^l \sim \frac{\omega}{v+c_l} < 0, \quad D_u^l \sim \frac{-1-\frac{\xi_l}{2}\frac{\omega}{v+c_l}}{\sqrt{2\xi_l\omega}}, \quad E_u^l \sim \frac{1-\frac{\xi_l}{2}\frac{\omega}{v+c_l}}{\sqrt{2\xi_l\omega}}. \tag{4.3.15}$$

- k_v^l-wave

$$k_v^l \sim \frac{\omega}{v-c_l} < 0, \quad D_v^l \sim \frac{1-\frac{\xi_l}{2}\frac{\omega}{v-c_l}}{\sqrt{2\xi_l\omega}}, \quad E_v^l \sim \frac{-1-\frac{\xi_l}{2}\frac{\omega}{v-c_l}}{\sqrt{2\xi_l\omega}}. \tag{4.3.16}$$

- k_u^r-wave

$$k_u^r \sim \frac{\omega}{v+c_r} > 0, \quad D_u^r \sim \frac{1+\frac{\xi_r}{2}\frac{\omega}{v+c_r}}{\sqrt{2\xi_r\omega}}, \quad E_u^r \sim \frac{-1+\frac{\xi_r}{2}\frac{\omega}{v+c_r}}{\sqrt{2\xi_r\omega}}. \tag{4.3.17}$$

- With the previous expansions, we have

$$E_u^r D_v^l - D_u^r E_v^l \sim \frac{\sqrt{c_r c_l}}{2}\left(\frac{1}{c_r(v+c_r)} + \frac{1}{c_l(v-c_l)}\right), \tag{4.3.18}$$

$$E_u^r D_u^l - D_u^r E_u^l \sim \frac{\sqrt{c_r c_l}}{2}\left(\frac{1}{c_l(v+c_l)} - \frac{1}{c_r(v+c_r)}\right). \tag{4.3.19}$$

- k_4^l-wave

$$k_4^l \sim {k_4^l}^0 + \mathcal{O}(\omega), \tag{4.3.20}$$

with

$${k_4^l}^0 = -\frac{2\sqrt{v^2-c_l^2}}{c_l\xi_l}, \qquad \left|\frac{\mathrm{d}k_4^l}{\mathrm{d}\omega}\right| \sim \frac{|v|}{v^2-c_l^2}. \tag{4.3.21}$$

Then,

$$D_4^l \sim -\frac{|v|-\sqrt{v^2-c_l^2}}{2(v^2-c_l^2)^{3/4}}, \quad E_4^l \sim \frac{|v|+\sqrt{v^2-c_l^2}}{2(v^2-c_l^2)^{3/4}}. \tag{4.3.22}$$

- k_+^r-wave

$$k_+^r \sim k_+^{r^0} + \mathcal{O}(\omega), \tag{4.3.23}$$

with

$$k_+^{r^0} = \frac{2i\sqrt{c_r^2-v^2}}{c_r\xi_r}, \qquad \left|\frac{\mathrm{d}k_+^r}{\mathrm{d}\omega}\right| \sim \frac{|v|}{c_r^2-v^2}. \tag{4.3.24}$$

Then[2]

$$\frac{D_+^r}{E_+^r} \sim -\frac{|v| + i\sqrt{c_r^2 - v^2}}{|v| - i\sqrt{c_r^2 - v^2}}. \tag{4.3.25}$$

- With the previous expansions, we have ($c_r\xi_r = c_l\xi_l \equiv c\xi$)

$$(k_+^{r^0} - k_4^{l\,0})(D_4^l + E_4^l) \sim \frac{2i}{c\xi}\left(\sqrt{c_r^2 - v^2} - i\sqrt{v^2 - c_l^2}\right)\frac{1}{(v^2 - c_l^2)^{1/4}}. \tag{4.3.26}$$

Let us consider the leading order in ω of the numerator (4.3.13), following from the approximated expressions above:

$$\begin{aligned} N_{ul} \sim\ & (k_+^{r^0} - k_4^{l\,0})\left(\frac{1}{v - c_l} - \frac{1}{v + c_r}\right)\left[E_+^r \frac{(1 - \frac{\xi_l}{2}\frac{\omega}{v - c_l})}{\sqrt{2\xi_l\omega}} - D_+^r \frac{(-1 - \frac{\xi_l}{2}\frac{\omega}{v - c_l})}{\sqrt{2\xi_l\omega}}\right] \times \\ & \times \left[\frac{\left(-1 + \frac{\xi_r}{2}\frac{\omega}{v + c_r}\right)}{\sqrt{2\xi_r\omega}} D_4^l - \frac{\left(1 + \frac{\xi_r}{2}\frac{\omega}{v + c_r}\right)}{\sqrt{2\xi_r\omega}} E_4^l\right] \\ & + (k_4^{l\,0} - k_u^r)(k_v^l - k_+^{r^0})\frac{\sqrt{c_r c_l}}{2}\left[\frac{1}{c_r(v + c_r)} + \frac{1}{c_l(v - c_l)}\right](E_+^r D_4^l - D_+^r E_4^l). \end{aligned} \tag{4.3.27}$$

Once divided by $(E_+^r + D_+^r)$, it becomes

$$\begin{aligned} \frac{N_{ul}}{E_+^r + D_+^r} \sim\ & \omega\left(\frac{1}{v - c_l} - \frac{1}{v + c_r}\right)(k_+^{r^0} - k_3^{l\,0})\left(-\frac{1}{2\omega\sqrt{\xi_l\xi_r}}\right)(D_3^l + E_3^l) \\ & - k_3^{l\,0} k_+^{r^0}\frac{\sqrt{c_r c_l}}{2}\left[\frac{1}{c_r(v + c_r)} + \frac{1}{c_l(v - c_l)}\right]\frac{(E_+^r D_3^l - D_+^r E_3^l)}{(E_+^r + D_+^r)}. \end{aligned} \tag{4.3.28}$$

Since

$$\frac{(E_+^r D_3^l - D_+^r E_3^l)}{(E_+^r + D_+^r)} = \frac{\left(D_3^l - \frac{D_+^r}{E_+^r}E_3^l\right)}{\left(1 + \frac{D_+^r}{E_+^r}\right)} = \left(i\sqrt{v^2 - c_l^2} - \sqrt{c_r^2 - v^2}\right)\frac{|v|}{2\sqrt{c_r^2 - v^2}(v^2 - c_l^2)^{3/4}}, \tag{4.3.29}$$

[2] It is not completely clear how to extend the definition for the co-moving frequency modulus $|\Omega| = |\omega - vk|$ to a decaying wave. Consequently, it may be questionable how to express Bogoliubov weights for the k_+^r-wave. Yet, being the denominators in both weights equal, we can calculate the ratio between weights unambiguously, and this will be the actual quantity entering the series-expansion of all amplitudes.

and

$$k_4^{l\,0} k_+^{r^0} = -\frac{4i\sqrt{v^2-c_l^2}\sqrt{c_r^2-v^2}}{c^2\xi^2}, \tag{4.3.30}$$

we obtain

$$\frac{N_{ul}}{E_+^r+D_+^r} \sim -i\frac{\left(\sqrt{c_r^2-v^2}-i\sqrt{v^2-c_l^2}\right)}{(v^2-c_l^2)^{1/4}} \left[\left(\frac{1}{v-c_l}-\frac{1}{v+c_r}\right)\frac{1}{\sqrt{\xi_l\xi_r}}\frac{1}{\sqrt{c_rc_l\xi_l\xi_r}}+\frac{|v|\sqrt{c_rc_l}}{c_rc_l\xi_l\xi_r}\left(\frac{1}{c_r(v+c_r)}+\frac{1}{c_l(v-c_l)}\right)\right], \tag{4.3.31}$$

$$\frac{N_{ul}}{E_+^r+D_+^r} \sim -i\frac{\left(\sqrt{c_r^2-v^2}-i\sqrt{v^2-c_l^2}\right)}{(v^2-c_l^2)^{1/4}}\frac{1}{\xi_l\xi_r\sqrt{c_rc_l}}\left[\frac{-(c_r+c_l)}{c_rc_l}\right]. \tag{4.3.32}$$

The denominator (4.3.14) to leading order in ω reduces to

$$D_{ul} \sim \omega\left(\frac{1}{v-c_l}-\frac{1}{v+c_r}\right)(-k_+^{r^0})\frac{1}{\sqrt{2\xi_l\omega}}(E_+^r+D_+^r)\frac{\sqrt{c_rc_l}}{2}\left(\frac{1}{c_l(v+c_l)}-\frac{1}{c_r(v+c_r)}\right) - k_+^{r^0}\omega\left(\frac{1}{v+c_r}-\frac{1}{v+c_l}\right)\frac{\sqrt{c_rc_l}}{2}\left(\frac{1}{c_r(v+c_r)}+\frac{1}{c_l(v-c_l)}\right)\left(-\frac{1}{\sqrt{2\xi_l\omega}}\right)(E_+^r+D_+^r). \tag{4.3.33}$$

which becomes, once divided by $(E_+^r+D_+^r)$,

$$\frac{D_{ul}}{E_+^r+D_+^r} \sim -i\frac{\sqrt{c_r^2-v^2}}{c\xi}\sqrt{\frac{2\omega}{\xi_lc_rc_l}}\frac{(c_r^2-c_l^2)}{(v+c_r)(v+c_l)(v-c_l)}. \tag{4.3.34}$$

Finally, A_u^l to leading order in ω is

$$\begin{aligned} A_u^l &= \frac{N_{ul}}{D_{ul}} = \frac{\frac{N_u^l}{E_+^r+D_+^r}}{\frac{D_u^l}{E_+^r+D_+^r}} = \\ &= \frac{(v^2-c_l^2)^{3/4}(v+c_r)}{(c_l-c_r)\sqrt{c_r^2-v^2}}\frac{1}{\sqrt{2z_l}c_l^{3/2}}\left(\sqrt{c_r^2-v^2}-i\sqrt{v^2-c_l^2}\right), \end{aligned} \tag{4.3.35}$$

where $z_l \equiv \omega\frac{\xi_l}{c_l}$.

A_v^l amplitude

Let us now move to the calculation of A_v^l. We can combine Eq.(4.3.10) from Step 4 and Eq.(4.3.11) from Step 5 as

$$(k_u^l-k_u^r)(E_+^rD_u^l-D_+^rE_u^l)\times(4.3.10)-(k_u^l-k_+^r)(E_u^rD_u^l-D_u^rE_u^l)\times(4.3.11). \tag{4.3.36}$$

This yields

$$A_v^l = \frac{N_{vl}}{D_{vl}}, \tag{4.3.37}$$

where

$$\begin{aligned} N_{vl} = & (k_+^r - k_4^l)(k_u^l - k_u^r)(E_+^r D_u^l - D_+^r E_u^l)(E_u^r D_4^l - D_u^r E_4^l) \\ & + (k_4^l - k_u^r)(k_u^l - k_+^r)(E_u^r D_u^l - D_u^r E_u^l)(E_+^r D_4^l - D_+^r E_4^l), \end{aligned} \tag{4.3.38}$$

and D_{vl} is the same as D_{ul}. Keeping only leading terms in ω, for N_{vl} we shall write

$$\begin{aligned} N_{vl} \sim & \, \omega(k_+^{r^0} - k_4^{l^0})\left(\frac{1}{v+c_l} - \frac{1}{v+c_r}\right)\times \\ & \times \frac{1}{\sqrt{2\xi_l\omega}}\left[E_+^r\left(-1-\frac{\xi_l}{2}\frac{\omega}{v+c_l}\right) - D_+^r\left(1-\frac{\xi_l}{2}\frac{\omega}{v+c_l}\right)\right]\times \\ & \times \frac{1}{\sqrt{2\xi_r\omega}}\left[\left(-1+\frac{\xi_r}{2}\frac{\omega}{v+c_r}\right)D_4^l - \left(1+\frac{\xi_r}{2}\frac{\omega}{v+c_r}\right)E_4^l\right] \\ & + \left(k_4^{l^0} - \frac{\omega}{v+c_r}\right)\left(\frac{\omega}{v+c_l} - k_+^{r^0}\right)\frac{\sqrt{c_r c_l}}{2}\left[\frac{1}{c_l(v+c_l)} - \frac{1}{c_r(v+c_r)}\right](E_+^r D_4^l - D_+^r E_4^l). \end{aligned} \tag{4.3.39}$$

Once divided by $(E_+^r + D_+^r)$, it becomes

$$\begin{aligned} \frac{N_{vl}}{E_+^r + D_+^r} \sim & \, \omega\left(\frac{1}{v+c_l} - \frac{1}{v+c_r}\right)(k_+^{r^0} - k_4^{l\,0})\frac{1}{2\omega\sqrt{\xi_l\xi_r}}(D_4^l + E_4^l) \\ & - k_4^{l\,0} k_+^{r^0}\frac{\sqrt{c_r c_l}}{2}\left[\frac{1}{c_l(v+c_l)} - \frac{1}{c_r(v+c_r)}\right]\frac{(E_+^r D_4^l - D_+^r E_4^l)}{(E_+^r + D_+^r)}. \end{aligned} \tag{4.3.40}$$

Since

$$\frac{(E_+^r D_4^l - D_+^r E_4^l)}{(E_+^r + D_+^r)} = \frac{\left(D_4^l - \frac{D_+^r}{E_+^r}E_4^l\right)}{\left(1 + \frac{D_+^r}{E_+^r}\right)} = \left(i\sqrt{v^2 - c_l^2} - \sqrt{c_r^2 - v^2}\right)\frac{|v|}{2\sqrt{c_r^2 - v^2}(v^2 - c_l^2)^{3/4}}, \tag{4.3.41}$$

and

$$k_4^{l\,0} k_+^{r^0} = -\frac{4i\sqrt{v^2 - c_l^2}\sqrt{c_r^2 - v^2}}{c^2\xi^2}, \tag{4.3.42}$$

we obtain

$$\begin{aligned} \frac{N_{vl}}{E_+^r + D_+^r} \sim & -i\frac{(\sqrt{c_r^2 - v^2} - i\sqrt{v^2 - c_l^2})}{(v^2 - c_l^2)^{1/4}}\times \\ & \times\left[\left(\frac{1}{v+c_r} - \frac{1}{v+c_l}\right)\frac{1}{\sqrt{\xi_l\xi_r}}\frac{1}{\sqrt{c_r c_l \xi_l \xi_r}} + \frac{|v|\sqrt{c_r c_l}}{c_r c_l \xi_l \xi_r}\left(\frac{1}{c_l(v+c_l)} - \frac{1}{c_r(v+c_r)}\right)\right], \end{aligned} \tag{4.3.43}$$

$$\frac{N_{vl}}{E_+^r + D_+^r} \sim -i\frac{(\sqrt{c_r^2 - v^2} - i\sqrt{v^2 - c_l^2})}{(v^2 - c_l^2)^{1/4}} \frac{1}{\xi_l \xi_r \sqrt{c_r c_l}} \left(\frac{c_l - c_r}{c_r c_l}\right). \tag{4.3.44}$$

Finally, A_v^l to leading order in ω is

$$\begin{aligned} A_v^l &= \frac{N_{vl}}{D_{vl}} = \frac{\frac{N_{vl}}{E_+^r + D_+^r}}{\frac{D_{vl}}{E_+^r + D_+^r}} = \\ &= \frac{(v^2 - c_l^2)^{3/4}(v + c_r)}{(c_r + c_l)\sqrt{c_r^2 - v^2}} \frac{1}{\sqrt{2 z_l} c_l^{3/2}} \left(\sqrt{c_r^2 - v^2} - i\sqrt{v^2 - c_l^2}\right). \end{aligned} \tag{4.3.45}$$

4.3.1 A_u^r amplitude

For the calculation of A_u^r, we shall combine the first two matching equations (4.3.1) and (4.3.2) via k_+^r

$$k_+^r \times (4.3.1) - (4.3.2), \tag{4.3.46}$$

where now A_u^l and A_v^l are substituted by the solutions (4.3.35) and (4.3.45) we have just obtained. We then have

$$(k_+^r - k_u^l) D_u^l A_u^l + (k_+^r - k_v^l) D_v^l A_v^l + (k_+^r - k_4^l) D_4^l = (k_+^r - k_u^r) D_u^r A_u^r. \tag{4.3.47}$$

This yields

$$A_u^r = \frac{N_{ur}}{D_{ur}}, \tag{4.3.48}$$

where

$$N_{ur} = (k_+^r - k_u^l) D_u^l A_u^l + (k_+^r - k_v^l) D_v^l A_v^l + (k_+^r - k_4^l) D_4^l, \tag{4.3.49}$$

$$D_{ur} = (k_+^r - k_u^r) D_u^r. \tag{4.3.50}$$

Keeping only leading terms in ω, we shall write

$$N_{ur} \sim -\frac{k_+^{r^0}}{\sqrt{2\xi_l \omega}} A_u^l + \frac{k_+^{r^0}}{\sqrt{2\xi_l \omega}} A_v^l + (k_+^{r^0} - k_4^{l^0}) D_4^l, \tag{4.3.51}$$

while for D_{ur} we have

$$D_{ur} \sim \frac{k_+^{r^0}}{\sqrt{2\xi_r \omega}}. \tag{4.3.52}$$

We divide both N_{ur} and D_{ur} by $k_+^{r^0}$

$$\frac{N_{ur}}{k_+^{r^0}} \sim -\frac{1}{\sqrt{2\xi_l \omega}} A_u^l + \frac{1}{\sqrt{2\xi_l \omega}} A_v^l + \frac{(k_+^{r^0} - k_4^{l^0})}{k_+^{r^0}} D_4^l, \tag{4.3.53}$$

$$\frac{D_{ur}}{k_+^{r^0}} \sim \frac{k_+^{r^0}}{\sqrt{2\xi_r\omega}}. \tag{4.3.54}$$

Since

$$\frac{(k_+^{r^0} - k_4^{l^0})}{k_+^{r^0}} D_4^l = \frac{\left(\sqrt{c_r^2 - v^2} - i\sqrt{v^2 - c_l^2}\right)}{\sqrt{c_r^2 - v^2}} \left(-\frac{|v| - \sqrt{v^2 - c_l^2}}{2(v^2 - c_l^2)^{3/4}}\right) \sim \mathcal{O}(1), \tag{4.3.55}$$

this term is negligible with respect to $\left(-\frac{A_u^l}{\sqrt{2\xi_l\omega}} + \frac{A_v^l}{\sqrt{2\xi_l\omega}}\right)$, which is $\sim \mathcal{O}(1/\omega)$. Finally, A_u^r to leading order in ω is

$$\begin{aligned} A_u^r &= \frac{N_{ur}}{D_{ur}} = \frac{N_{ur}/k_+^{r^0}}{D_{ur}/k_+^{r^0}} = \\ &= \frac{\sqrt{2c_r}}{(c_r^2 - c_l^2)c_l\sqrt{z_l}} \frac{(v^2 - c_l^2)^{3/4}(v + c_r)}{\sqrt{c_r^2 - v^2}} \left(\sqrt{c_r^2 - v^2} - i\sqrt{v^2 - c_l^2}\right). \end{aligned} \tag{4.3.56}$$

The solution for the decaying wave amplitude A_+^r to leading order in ω is reported in literature to be

$$A_+^r = \frac{(v^2 - c_l^2)^{1/4}(v^2 - c_l^2 + v\sqrt{v^2 - c_l^2})}{2D_+^r(c_r^2 - v^2)(c_l^2 - v^2 + v\sqrt{v^2 - c_l^2})} \left(v - i\sqrt{c_r^2 - v^2}\right). \tag{4.3.57}$$

The conservation of the Bogoliubov norm determines a unitarity condition involving the amplitudes of the propagating waves. For $\phi_{4,l}^{in}$ mode, it reads

$$\left|A_v^l\right|^2 + |A_u^r|^2 - \left|A_u^l\right|^2 = -1, \tag{4.3.58}$$

where the minus signs come from the fact that k_u^l and k_4^l modes have negative norm. Since the amplitudes entering this equation are $\sim \mathcal{O}(1/\sqrt{\omega})$, the left hand side should vanish at leading order. This is verified with the amplitude solutions we have obtained, indeed.

4.4 Bogoliubov transformation

As both the *in* and *out* bases are complete, the *in* and *out* scattering modes can be related by a 3×3 *scattering matrix* S built out of the scattering amplitudes

$$\begin{pmatrix} \phi_{v,r}^{in} \\ \phi_{3,l}^{in} \\ \phi_{4,l}^{in} \end{pmatrix} = \begin{pmatrix} S_{vl,vr} & S_{ur,vr} & S_{ul,vr} \\ S_{vl,3l} & S_{ur,3l} & S_{ul,3l} \\ S_{vl,4l} & S_{ur,4l} & S_{ul,4l} \end{pmatrix} \begin{pmatrix} \phi_{v,l}^{out} \\ \phi_{u,r}^{out} \\ \phi_{u,l}^{out} \end{pmatrix}. \tag{4.4.1}$$

The first and second index in S matrix elements indicate the outgoing and incoming channel, respectively. Since $\phi_{u,l}^{out}$ and $\phi_{4,l}^{in}$ have negative norm, the conservation of Bogoliubov norm imposes a *modified unitarity transformation* [17]

$$S^{\dagger}\eta S = \eta = S\eta S^{\dagger}, \tag{4.4.2}$$

with $\eta = \text{diag}(1, 1, -1)$. This implies S mixes positive and negative norm modes and induces a non-trivial Bogoliubov transformation between creation and annihilation operators of the *in* and *out* scattering states

$$\begin{pmatrix} \hat{a}_{\omega}^{vl,out} \\ \hat{a}_{\omega}^{ur,out} \\ \hat{a}_{\omega}^{ul,out\dagger} \end{pmatrix} = \begin{pmatrix} S_{vl,vr} & S_{vl,3l} & S_{vl,4l} \\ S_{ur,vr} & S_{ur,3l} & S_{ur,4l} \\ S_{ul,vr} & S_{ul,3l} & S_{ul,4l} \end{pmatrix} \begin{pmatrix} \hat{a}_{\omega}^{vr,in} \\ \hat{a}_{\omega}^{3l,in} \\ \hat{a}_{\omega}^{4l,in\dagger} \end{pmatrix}. \tag{4.4.3}$$

Let us consider the two kinds of vacuum implied by the *in* and *out* basis. The *in* vacuum $|0, in\rangle$ is defined as the state annihilated by $\hat{a}_{\omega}^{in}$ operators, while the *out* vacuum state $|0, out\rangle$ as the one annihilated by $\hat{a}_{\omega}^{out}$ operators. The non-triviality of Bogoliubov transformation (4.4.3) generates a parametric conversion at the horizon which makes $|0, in\rangle$ and $|0, out\rangle$ non-equivalent. Indeed, let us calculate the expectation values for the number of *outgoing* phonons per unit time and frequency with respect to the *in* vacuum

$$n_{\omega}^{u,r} = \langle 0, in|\hat{a}_{\omega}^{ur,out\dagger}\hat{a}_{\omega}^{ur,out}|0, in\rangle = |S_{ur,4l}|^2, \tag{4.4.4a}$$

$$n_{\omega}^{u,l} = \langle 0, in|\hat{a}_{\omega}^{ul,out\dagger}\hat{a}_{\omega}^{ul,out}|0, in\rangle = |S_{ul,vr}|^2 + |S_{ul,3l}|^2, \tag{4.4.4b}$$

$$n_{\omega}^{v,l} = \langle 0, in|\hat{a}_{\omega}^{vl,out\dagger}\hat{a}_{\omega}^{vl,out}|0, in\rangle = |S_{vl,4l}|^2. \tag{4.4.4c}$$

By equating the $(3, 3)$ elements in $S^{\dagger}\eta S$ and $S\eta S^{\dagger}$, we obtain the following remarkable relation

$$n_{\omega}^{u,l} = |S_{ul,vr}|^2 + |S_{ul,3l}|^2 = |S_{ur,4l}|^2 + |S_{vl,4l}|^2 = n_{\omega}^{u,r} + n_{\omega}^{v,l}. \tag{4.4.5}$$

This means that in $|0, in\rangle$ (which holds at all time in Heisenberg picture and does not contain initial incoming phonons at $t = -\infty$) will present outgoing quanta *at late time* on both sides of the sonic horizon. This happens as conversion of $\phi_{4,l}^{in}$ mode vacuum fluctuations into real on-shell Bogoliubov phonons (u, r) and (v, l). Furthermore, Eq.(4.4.5) ensures energy conservation: the number of positive energy (u, r) and (v, l) phonons popping-out from the vacuum is equal to that of negative energy (u, l) phonons. The latter are equivalent to the *partners* we saw in Chapter 2 and propagate downstream in the supersonic region. Pair production of opposite $\pm\omega$ frequencies then leads to total energy conservation.

This particle creation from vacuum as an effect of Bogoliubov transformation recalls the Hawking effect within Quantum Field Theory in curved space-time.

Indeed, Eq.(4.4.4a) hints at the possibility to observe a flux of phonons from the sonic horizon at late time in the subsonic region, which is the analogue phenomenon of black hole Hawking radiation. More precisely, Eq.(4.4.4a) predicts the number of particles emitted per unit time and bandwidth

$$n_{\omega}^{u,r} = \frac{\partial^2 N_{\omega}^{u,r}}{\partial t \partial \omega} = |S_{ur,4l}|^2 \sim \frac{(c_r + v)}{(c_r - v)} \frac{(v^2 - c_l^2)^{3/2}}{(c_r^2 - c_l^2)} \frac{2c_r}{c_l \xi_l \omega}, \tag{4.4.6}$$

whose $\sim 1/\omega$ profile is consistent with the low frequency expansion of black-body Planck distribution for bosons

$$n_T(\omega) = \frac{1}{e^{\hbar\omega/k_B T} - 1} \sim \frac{k_B T}{\hbar \omega}. \tag{4.4.7}$$

Therefore, we may extrapolate the analogue Hawking temperature from the $1/\omega$ coefficient in Eq.(4.4.6)

$$T = \frac{\hbar}{k_B} \frac{(c_r + v)}{(c_r - v)} \frac{(v^2 - c_l^2)^{3/2}}{(c_r^2 - c_l^2)} \frac{2c_r}{c_l \xi_l}. \tag{4.4.8}$$

Yet, the abrupt discontinuity in the speed of sound at the horizon makes the surface gravity (Eq.(3.4.34)) of our model diverge, while the expected temperature T remains finite. This is a matter of concern for the connection to the original gravitational framework. More general and realistic velocity profiles should be investigated, so as to deal with a smooth enough transition between subsonic and supersonic regime to justify the hydrodynamic approximation. In the next chapter, we will introduce an attempt to develop a new acoustic black hole configuration by "broadening" the sonic horizon. For the moment, we highlight that the above derivation of Hawking effect based on the original microscopic quantum theory of BECs does not depend on the hydrodynamic approximation, therefore no transplanckian problem is present.

Chapter 5

Broadening the horizon: the sonic region

This chapter has the purpose of introducing a toy model for acoustic black holes where a sonic region is present. We will first analyze the Bogoliubov dispersion relation on the sonic horizon. Then, systems of two semi-infinite connected BECs, where one is sonic, will be studied. Finally, the subsonic-sonic-supersonic configuration will be depicted.

5.1 Dispersion relation on the horizon

One first way of generalizing the stepwise BEC model we have dealt with so far may be to consider a broadened version for the sonic horizon. This is also supposed to help better understand the effect of a sonic horizon on quantum field phenomenology.

To begin with, in a more realistic model of acoustic black hole, we should consider that the fluid velocity modulus equals the local speed of sound on the horizon, being the locus where the transition from subsonic to supersonic regime occurs

$$|v| = c, \tag{5.1.1}$$

where v is taken negative. Then, in order to analyze the Bogoliubov dispersion relation (4.1.11) on the horizon, we will set $v = -c$

$$(\omega + ck)^2 = c^2\left(k^2 + \frac{\xi^2 k^4}{4}\right), \tag{5.1.2}$$

i.e.

$$\frac{c^2\xi^2}{4}k^4 - 2\omega ck - \omega^2 = 0. \tag{5.1.3}$$

This equation, being fourth order in k, admits four solutions, which we shall series-expand up to leading order in ω. Then, we obtain two real solutions

$$k_u \sim \frac{2}{\xi}\left(\frac{\xi\omega}{c}\right)^{1/3}, \tag{5.1.4a}$$

$$k_v \sim -\frac{\omega}{2c} \tag{5.1.4b}$$

and two complex conjugate solutions

$$k_\pm \sim \frac{1}{\xi}\left(\frac{\xi\omega}{c}\right)^{1/3}(1 \pm i\sqrt{3}). \tag{5.1.5}$$

Notice that in the hydrodynamic approximation (i.e. $\xi \to 0$), Eq.(5.1.3) reduces to a linear equation in k, whose unique solution coincides with $k_v \sim -\omega/2c$. Indeed, k_v hydrodynamic solution to the general Bogoliubov dispersion relation

$$k_v \sim \frac{\omega}{v - c} \tag{5.1.6}$$

reduces to Eq.(5.1.4b) when $v = -c$, while the other hydrodynamic solution

$$k_u \sim \frac{\omega}{v + c} \tag{5.1.7}$$

diverges in the same limit. This means u-waves oscillate indefinitely on the horizon, thus recovering the behaviour of outgoing modes $\sim e^{-i\omega u}$ on the future horizon H^+ in a Schwarzschild black hole (see Chapter 2). However, the dispersion relation on the sonic horizon yields a k_u solution regularizing the corresponding hydrodynamic mode. Qualitatively, the two real k_u and k_v solutions provide the same picture as the dispersion relation in the subsonic regime (Fig.(5.1.1)): both k_u and k_v have positive norm for $\omega > 0$ (and, according to the duality between norm branches, negative norm for $\omega < 0$), the former propagating upstream $\left(v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} > 0\right)$ while the latter downstream $\left(v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} < 0\right)$.

Now we wish to extend the subsonic-supersonic configuration of Chapter 4 by inserting a *sonic region* (i.e. where $|v| = c$) in between. This means modelling the horizon of our BEC system as a *finite* sector, instead of just a surface. Therefore, we will get two discontinuity surfaces in the speed of sound, one separating the supersonic from the sonic region and the other the sonic from the subsonic region. As preliminary steps, we will first concentrate on the two scattering processes occurring at each discontinuity surface separately, assuming the sonic region to be semi-infinite. From now on, we will conventionally refer to quantities regarding the sonic region with the index s.

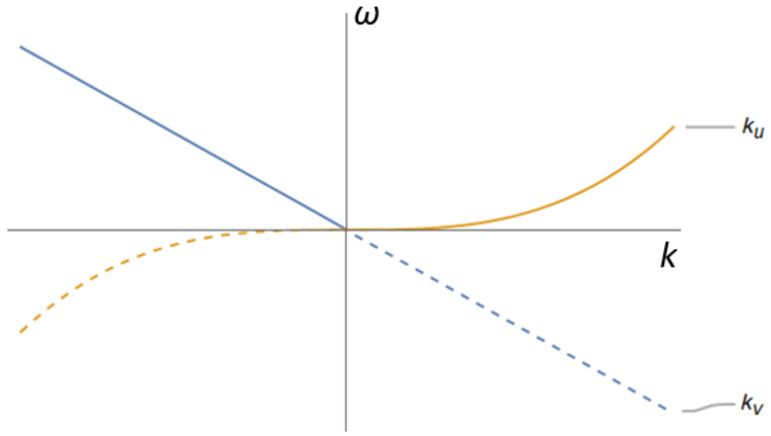

Figure 5.1.1: Dispersion relation of Bogoliubov modes in sonic flow. The solid (dashed) lines correspond to positive (negative) norm branches.

5.2 Supersonic-sonic configuration

The supersonic-sonic configuration is an ideal model consisting of two semi-infinite 1D BECs connected at $x = 0$, the left one with $|v| > c_l$ and the right one with $|v| = c_s$. Again, the fluid velocity v is assumed to be constant and directed along the negative x axis. The *in* basis for the quantum fluctuations operator $\hat{\phi}$ is composed of three possible incoming modes, analogously to the subsonic-supersonic case (Fig.(5.2.1)). This provides qualitatively the same picture and matching conditions depicted in Sect.4.2, upon substituting subsonic with sonic quantities (i.e. by replacing index r with s).

Theoretically, one could make the same replacements in the implicit expressions for A^l_u, A^l_v, and A^r_u solutions and insert the approximate expressions for each term entering formulas. Nevertheless, by keeping only the leading order in ω, the numerator and denominator defining each amplitude may vanish. This entails one should take the next-to-leading order term in each expansion into account, which would turn out to be quite an involved calculation. Therefore, in order to estimate such amplitudes, we evaluated each implicit solution by inserting the full expressions for Bogoliubov weights[1], while again we used the leading order approximations for

[1]The absolute value of the co-moving frequency $\left|\omega - vk^s_+\right|$ for the decaying k^s_+-mode has been intended as the modulus of a complex number. As we have already mentioned, at present this possible interpretation may be questionable and needs some further insights, though.

wavenumbers. By means of a code written in *Wolfram Mathematica*, we observe all scattering amplitudes for $\phi_{4,l}^{in}$ and $\phi_{3,l}^{in}$ are $\sim \mathcal{O}\big(\omega^{-1/6}\big)$, while all those for $\phi_{v,s}^{in}$ mode are $\sim \mathcal{O}(1)$, to leading order in ω. The coefficients multiplying each ω power depend on v, c_l, ξ_l, upon considering $\xi_s = \frac{c_l}{c_s}\xi_l$. Therefore, not only does the decaying amplitude seem to behave like the others of its corresponding incoming mode, but $\omega^{-1/2}$ profile – which characterizes the propagating wave amplitudes for $\phi_{4,l}^{in}$ and $\phi_{3,l}^{in}$ modes in the subsonic-supersonic configuration – modifies into $\omega^{-1/6}$.

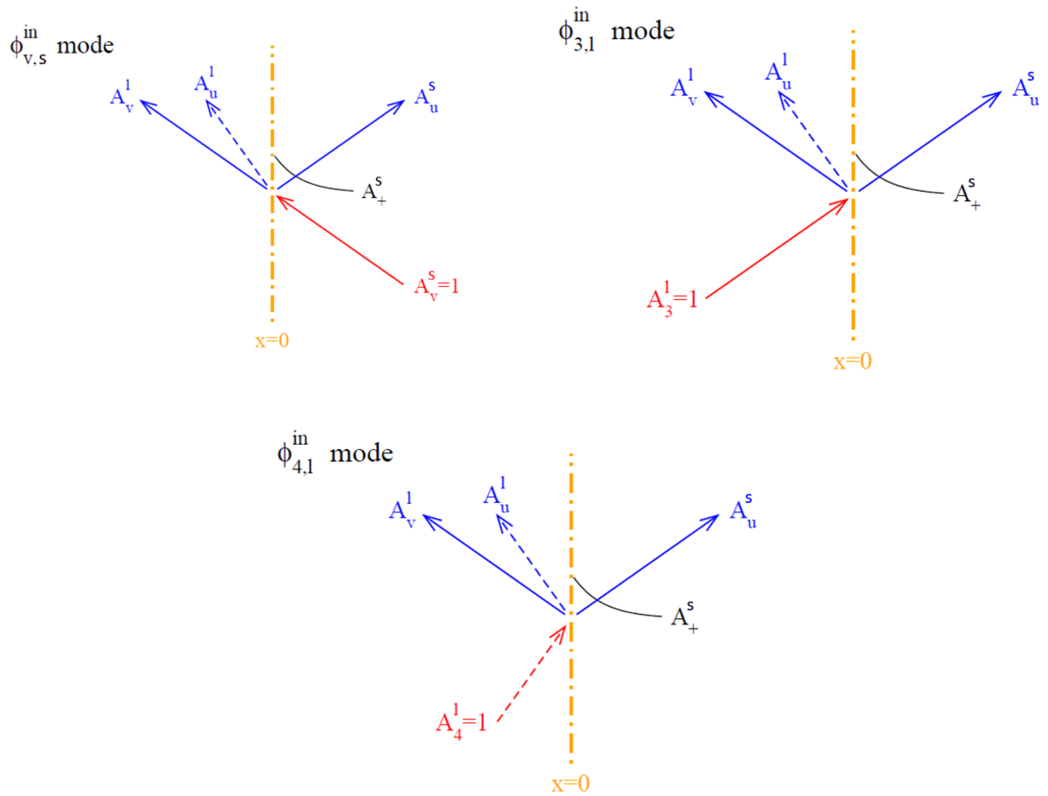

Figure 5.2.1: Sketch of the Bogoliubov modes involved in the *in* basis for the supersonic-sonic configuration [15]. The first subscript refers to the wavenumber k, while the second one to the sector where the mode originates.

5.3 Sonic-subsonic configuration

Let us consider the sonic-subsonic configuration. Again, it consists of two semi-infinite 1D BECs connected at $x = 0$, with constant fluid velocity v directed along the negative x axis. Now the left region has $|v| = c_s$, while the right one

$|v| < c_r$, and the *in* basis for $\hat{\phi}$ is composed of two possible incoming modes (Fig.(5.3.1)). Then, it provides qualitatively the same picture as depicted in the subsonic-subsonic configuration (see [15]). The matching conditions for $\phi^{in}_{u,s}$ ($A^s_u = 1$) read

$$D^s_v A^s_v + D^s_- A^s_- + D^s_u = D^r_u A^r_u + D^r_+ A^r_+ \tag{5.3.1}$$

$$k^s_v D^s_v A^s_v + k^s_- D^s_- A^s_- + k^s_u D^s_u = k^r_u D^r_u A^r_u + k^r_+ D^r_+ A^r_+ \tag{5.3.2}$$

$$E^s_v A^s_v + E^s_- A^s_- + E^s_u = E^r_u A^r_u + E^r_+ A^r_+ \tag{5.3.3}$$

$$k^s_v E^s_v A^s_v + k^s_- E^s_- A^s_- + k^s_u E^s_u = k^r_u E^r_u A^r_u + k^r_+ E^r_+ A^r_+. \tag{5.3.4}$$

Let us compare them with the matching equations for $\phi^{in}_{4,l}$ mode in the subsonic-supersonic configuration

$$D^l_u A^l_u + D^l_v A^l_v + D^l_4 = D^r_u A^r_u + D^r_+ A^r_+$$

$$k^l_u D^l_u A^l_u + k^l_v D^l_v A^l_v + k^l_4 D^l_4 = k^r_u D^r_u A^r_u + k^r_+ D^r_+ A^r_+$$

$$E^l_u A^l_u + E^l_v A^l_v + E^l_4 = E^r_u A^r_u + E^r_+ A^r_+$$

$$k^l_u E^l_u A^l_u + k^l_v E^l_v A^l_v + k^l_4 E^l_4 = k^r_u E^r_u A^r_u + k^r_+ E^r_+ A^r_+.$$

Notice the two systems of equations are equivalent, upon the following substitutions for quantities in the left hand sides

$$u, s \longleftrightarrow 4, l \qquad v, s \longleftrightarrow v, l \qquad -, s \longleftrightarrow u, l.$$

The matching conditions for $\phi^{in}_{v,r}$ ($A^r_v = 1$) instead are

$$D^s_v A^s_v + D^s_- A^s_- = D^r_u A^r_u + D^r_+ A^r_+ + D^r_v \tag{5.3.5}$$

$$k^s_v D^s_v A^s_v + k^s_- D^s_- A^s_- = k^r_u D^r_u A^r_u + k^r_+ D^r_+ A^r_+ + k^r_v D^r_v \tag{5.3.6}$$

$$E^s_v A^s_v + E^s_- A^s_- = E^r_u A^r_u + E^r_+ A^r_+ + E^r_v \tag{5.3.7}$$

$$k^s_v E^s_v A^s_v + k^s_- E^s_- A^s_- = k^r_u E^r_u A^r_u + k^r_+ E^r_+ A^r_+ + k^r_v E^r_v, \tag{5.3.8}$$

which can be obtained from those for $\phi^{in}_{u,s}$ mode upon substituting

$$D^s_u \longrightarrow -D^r_v, \qquad E^s_u \longrightarrow -E^r_v, \qquad k^s_u \longrightarrow k^r_v.$$

Such correspondences between different systems of equations allow us to reconnect to previous derivations and obtain implicit expressions for scattering amplitudes in the form $A = N/D$. By inserting approximate expressions for each term entering formulas, N and D may vanish when keeping only the leading order in ω. Therefore, one should take the next-to-leading order term into account. To overcome such a computational effort, amplitudes have been estimated again by means of a code written in *Wolfram Mathematica*: the full expressions for Bogoliubov weights[2] and the leading order approximations for wavenumbers have been inserted into the equations for N's and D's, then each N/D has been series-expanded. Eventually, we observe the following profiles to leading order in ω

$$\phi^{in}_{u,s}: \quad A^s_v \sim \mathcal{O}(1), \quad A^s_- \sim \mathcal{O}(1), \quad A^r_u \sim \mathcal{O}(1), \quad A^r_+ \sim \mathcal{O}\big(\omega^{1/3}\big). \tag{5.3.9}$$

$$\phi^{in}_{v,r}: \quad A^s_v \sim \mathcal{O}(1), \quad A^s_- \sim \mathcal{O}\big(\omega^{2/3}\big), \quad A^r_u \sim \mathcal{O}(1), \quad A^r_+ \sim \mathcal{O}\big(\omega^{1/2}\big). \tag{5.3.10}$$

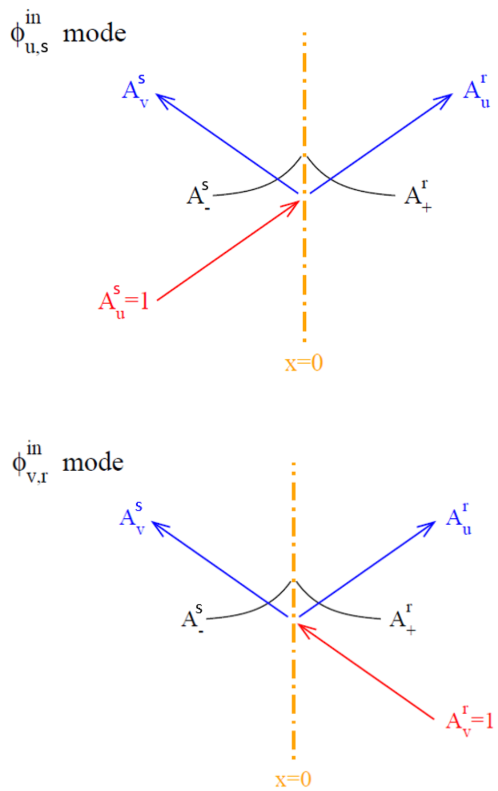

Figure 5.3.1: Sketch of the Bogoliubov modes involved in the *in* basis for the sonic-subsonic configuration [15]. The first subscript refers to the wavenumber k, while the second one to the sector where the mode originates.

[2]For the decaying k^s_- and k^r_+ waves, again we have interpreted the absolute value of their co-moving frequencies as the modulus of a complex number.

5.4 Subsonic-sonic-supersonic configuration

In this final section we will introduce a stepwise model for acoustic black holes with a broadened horizon (Fig.(5.4.1)). It consists of *three* stationary homogeneous 1D BEC regions, each with a different local speed of sound c, and a constant fluid velocity v along the negative direction of x axis:

- left: supersonic sector ($|v| > c_l$),
- centre: sonic sector ($|v| = c_s$),
- right: subsonic sector ($|v| < c_r$).

The left and right regions will be again semi-infinite, whereas the central region will be *finite*, which corresponds to extending the sonic horizon by a length L. Now we are facing *two* discontinuity surfaces, separating the sonic region from the supersonic and subsonic ones, respectively. We assume these separations are located symmetrically with respect to the origin of x axis at $x = \pm L/2$.

Let us study the scattering processes. Being the sonic region *finite*, there will be no infinitely-growing modes we can neglect in that sector, therefore we will take both k_-^s and k_+^s waves into account. As concerns the other two regions, the wave-expansion for each mode function depends on the specific mode considered. Here we will deal only with $\phi_{4,l}^{in}$ ($A_4^l = 1$) channel (Fig.(5.4.1)), since it is the one responsible for Hawking radiation. For this reason, we shall try and find the A_u^r scattering amplitude in this new set-up, which is supposed to yield a more refined definition of analogue Hawking temperature. The waves considered are the ones associated to the following amplitudes

- left region: ingoing $A_4^l = 1$, outgoing A_u^l and A_v^l,
- centre region: all sonic amplitudes $A_u^s, A_v^s, A_+^s, A_-^s$,
- right region: outgoing A_u^r and decaying A_+^r.

The two systems of matching conditions then read

- Supersonic-sonic discontinuity at $x = -L/2$:

$$\begin{aligned} &A_v^l D_v^l e^{-ik_v^l L/2} + A_u^l D_u^l e^{-ik_u^l L/2} + D_4^l e^{-ik_4^l L/2} = \\ &A_v^s D_v^s e^{-ik_v^s L/2} + A_u^s D_u^s e^{-ik_u^s L/2} + A_-^s D_-^s e^{-ik_-^s L/2} + A_+^s D_+^s e^{-ik_+^s L/2}, \end{aligned} \tag{5.4.1}$$

$$\begin{aligned} &k_v^l A_v^l D_v^l e^{-ik_v^l L/2} + k_u^l A_u^l D_u^l e^{-ik_u^l L/2} + k_4^l D_4^l e^{-ik_4^l L/2} = \\ &k_v^s A_v^s D_v^s e^{-ik_v^s L/2} + k_u^s A_u^s D_u^s e^{-ik_u^s L/2} + k_-^s A_-^s D_-^s e^{-ik_-^s L/2} + k_+^s A_+^s D_+^s e^{-ik_+^s L/2}, \end{aligned} \tag{5.4.2}$$

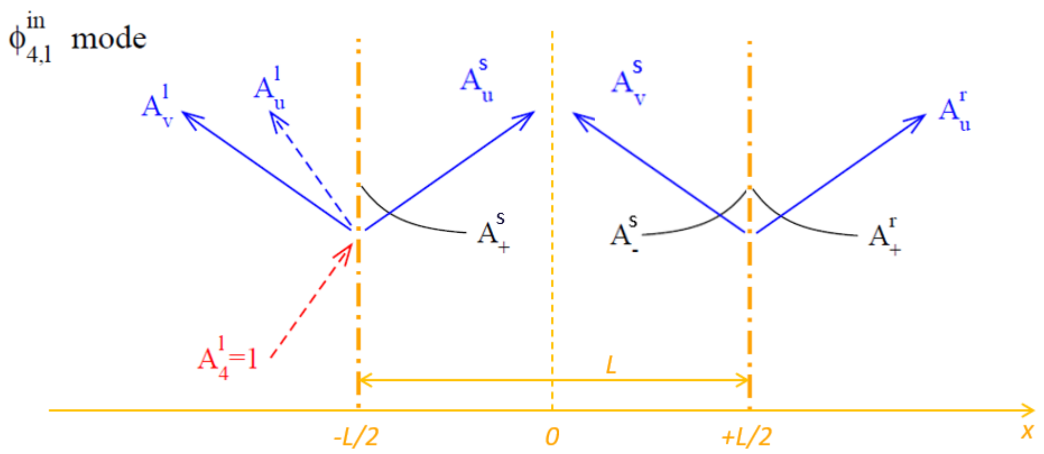

Figure 5.4.1: Sketch of $\phi^{in}_{4,l}$ Bogoliubov mode for the subsonic-sonic-supersonic configuration [15]. The x axis origin is located at the centre of the sonic region, whose length is denoted by L.

$$A^l_v E^l_v e^{-ik^l_v L/2} + A^l_u E^l_u e^{-ik^l_u L/2} + E^l_4 e^{-ik^l_4 L/2} = \\ A^s_v E^s_v e^{-ik^s_v L/2} + A^s_u E^s_u e^{-ik^s_u L/2} + A^s_- E^s_- e^{-ik^s_- L/2} + A^s_+ E^s_+ e^{-ik^s_+ L/2}, \tag{5.4.3}$$

$$k^l_v A^l_v E^l_v e^{-ik^l_v L/2} + k^l_u A^l_u E^l_u e^{-ik^l_u L/2} + k^l_4 E^l_4 e^{-ik^l_4 L/2} = \\ k^s_v A^s_v E^s_v e^{-ik^s_v L/2} + k^s_u A^s_u E^s_u e^{-ik^s_u L/2} + k^s_- A^s_- E^s_- e^{-ik^s_- L/2} + k^s_+ A^s_+ E^s_+ e^{-ik^s_+ L/2}; \tag{5.4.4}$$

- Sonic-subsonic discontinuity at $x = +L/2$:

$$A^s_v D^s_v e^{ik^s_v L/2} + A^s_u D^s_u e^{ik^s_u L/2} + A^s_- D^s_- e^{ik^s_- L/2} + A^s_+ D^s_+ e^{ik^s_+ L/2} \\ = A^r_u D^r_u e^{ik^r_u L/2} + A^r_+ D^r_+ e^{ik^r_+ L/2} \tag{5.4.5}$$

$$k^s_v A^s_v D^s_v e^{ik^s_v L/2} + k^s_u A^s_u D^s_u e^{ik^s_u L/2} + k^s_- A^s_- D^s_- e^{ik^s_- L/2} + k^s_+ A^s_+ D^s_+ e^{ik^s_+ L/2} \\ = k^r_u A^r_u D^r_u e^{ik^r_u L/2} + k^r_+ A^r_+ D^r_+ e^{ik^r_+ L/2}, \tag{5.4.6}$$

$$A^s_v E^s_v e^{ik^s_v L/2} + A^s_u E^s_u e^{ik^s_u L/2} + A^s_- E^s_- e^{ik^s_- L/2} + A^s_+ E^s_+ e^{ik^s_+ L/2} \\ = A^r_u E^r_u e^{ik^r_u L/2} + A^r_+ E^r_+ e^{ik^r_+ L/2} \tag{5.4.7}$$

$$k^s_v A^s_v E^s_v e^{ik^s_v L/2} + k^s_u A^s_u E^s_u e^{ik^s_u L/2} + k^s_- A^s_- E^s_- e^{ik^s_- L/2} + k^s_+ A^s_+ E^s_+ e^{ik^s_+ L/2} \\ = k^r_u A^r_u E^r_u e^{ik^r_u L/2} + k^r_+ A^r_+ E^r_+ e^{ik^r_+ L/2}. \tag{5.4.8}$$

They both are systems of 4 equations in 6 unknowns. One way to obtain A_u^r amplitude from these is to solve the left-side system for A_-^s and A_+^s both in terms of A_u^s and A_v^s, and then plug these expressions into the right-side system. This will turn out to be composed of 4 equations in 4 unknowns, thus completely solvable. The algebraic procedure is quite laborious and is reported in Appendix B. Finally, we find an implicit solution for A_u^r in terms of a ratio

$$A_u^r = \frac{N_{ur}}{D_{ur}}, \tag{5.4.9}$$

like in previous calculations.

At this point, we could insert approximate expressions for each term entering A_u^r definition and keep the A_u^r leading order in ω. However, this would result in N_{ur} and D_{ur} vanishing and we should properly consider next-to-leading order terms for wavenumbers or even the full definitions for Bogoliubov weights. Unlike scattering amplitude computations we showed previously, this A_u^r definition is the consequence of *two* scattering processes, thus appearing particularly involved. At the moment we still have not been able to achieve a convergent result, neither with the help of a *Wolfram Mathematica* code. Furthermore, stability conditions for the finite sonic region need to be verified, in order for this model to be regarded as an equilibrium configuration. In conclusion, we expect the subsonic-sonic-supersonic acoustic black hole configuration to provide an $A_u^r \sim \mathcal{O}(\omega^{-1/2})$ amplitude, to leading order in ω. This would give a correction to Hawking temperature (4.4.8) and elucidate the effect of the sonic horizon on modes propagation.

Conclusions

This dissertation has shown how to derive analogue Hawking radiation in atomic condensed systems by applying microscopic Bogoliubov theory of BECs. Our attention has been focused on a stepwise toy model where a subsonic and a supersonic uniform flow are connected by an sharp discontinuity surface in the speed of sound. By analyzing the scattering processes occurring at the interface between the two sectors, we have predicted vacuum fluctuations of the $\phi_{4,l}^{in}$ mode are converted into real on-shell k_u^r-phonons, which are emitted from the sonic horizon in the subsonic region. The total energy is shown to be conserved indeed, since negative energy partner k_u^l-phonons are produced inside the supersonic region in couple with positive energy Bogoliubov phonons. Moreover, the number of k_u^r-phonons per unit time and bandwidth shows a $1/\omega$ dependence, which is consistent with the low frequency expansion of a thermal Bose distribution. As a consequence, the study of quantum fluctuations above stationary (non homogeneous) transonic configurations yields an analogue picture of Hawking effect. Indeed, the kinematic nature of Hawking radiation does not constrain this kind of phenomenon to a gravitational background, but rather evokes the chance to search for analogue Hawking processes in different physical systems.

Dealing with Bose-Einstein condensates by means of the microscopic Bogoliubov theory provides an *ab initio* approach. This does not recur to the hydrodynamic approximation, according to which the healing length ξ is regarded as the fundamental length scale of a BEC and is sent to zero [18]. In this limit, the field describing phase perturbations of the condensate wave function satisfies the wave equation for a massless scalar field propagating in an effective curved spacetime, thus yielding an analogue gravity model based on a BEC. However, within this approach, the modes responsible for Hawking-like radiation are affected by an analogue version of the transplanckian problem. In the same way as Hawking particles experience an infinite blue shift when traced back towards the near-horizon region, phonons leading to analogue Hawking radiation are subject to an infinite Doppler-shift at the sonic horizon, thus oscillating at a wavelength much smaller than ξ. Therefore, ξ plays the same role as the Planck length ℓ_{Pl}, which is assumed to be the limit of validity for the semi-classical framework of quantum field theory in curved space-time. This may cast some doubts about the derivation of analogue Hawking radiation under the hydrodynamic approximation. On the other hand, an approach based on the microscopic Bogoliubov theory yields a robust result for analogue Hawking radiation with respect to the microscopic details of a BEC, showing the analogue transplanckian problem emerges as an artifact of the hydrodynamic approximation. Indeed, starting from the emission spectrum of Bogoliubov phonons, we have been able to extrapolate an effective Hawking temperature with a $\sim 1/\xi$ dependence, which is regular in the $\xi \to \infty$ limit. In a certain way, upon the correspondence between ξ and ℓ_{Pl}, such an approach might be regarded as the counterpart of dealing with quantum gravity issues.

The analogy between gravitational and condensed matter systems not only is restricted to deepening theoretical issues, but has also opened an actual experimental activity. In atomic BECs, Hawking temperature is estimated to be of order of 10 nK, about one order of magnitude lower than the thermal bath of a typical condensate. In order to overcome this difficulty and find a possible signature for Hawking emission, a method based on density-density correlation function has been applied [19]. A Hawking particle is expected to pop-out from the sonic horizon in a couple with a partner, thus yielding correlated pairs of quanta propagating in opposite directions with respect to the horizon. This results into a peculiar long-range density correlation, which was confirmed by numerical simulations. Indeed, the first experimental observation of such a signal was presented by Jeff Steinhauer [20], who then repeated this kind of measurements in a more advanced apparatus [21]. In both cases, he extracted the Hawking temperature via Fourier transforming the density-density correlation function between the Hawking and partner modes in a ^{87}Rb atomic BEC, confirming the thermal spectrum of Hawking radiation. Spontaneous Hawking radiation has also been observed to be stationary by studying the time evolution of an analogue black hole [22], which ensures the correspondence with black-body radiation. Such results achieved in the last few years have boosted the *experimental gravity* branch, and ongoing projects look promising towards refined confirmation of analogue Hawking radiation phenomenology.

In the last part of this thesis we proposed a new model for acoustic black holes composed of three regions, where the central one is sonic. Indeed, it has been shown that, when one takes short distance dispersion into account, Hawking temperature is no longer simply fixed by the surface gravity. Rather, it appears to be determined by an *average* surface gravity over a finite interval across the horizon [23]. This modified version of surface gravity is defined as the average of the flow gradient across the horizon over an interval fixed by the healing length, as if the horizon were broadened. Therefore, an acoustic black hole configuration with an extended sonic sector may be addressed as an attempt to build a toy model to assess the sensitivity of black hole spectrum to short distance dispersion.

Overall, analogue black hole models turn out to be a powerful tool to investigate theoretical and experimental aspects regarding both gravitational and condensed matter phenomena. When quantum field theory is taken into consideration, such a remarkable crossover is achieved through Hawking radiation, which then connects gravity and thermodynamics.

Acknowledgments I warmly thank Profs. Roberto Balbinot and Alessandro Fabbri for their mentorship and constant helpfulness in supervising this thesis work. I also thank the members of the Department of Theoretical Physics at the University of Valencia, where I have had the opportunity to conduct a research period in the past few months. In particular, continuous precious advice from Drs. Flavio Bombacigno, Roberto Bruschini, Leonardo Coito Pereyra, Silvia Pla Garcia and Arsenii Titov is acknowledged.

Appendix A

Analytic derivation of scattering amplitudes for $\phi_{3,l}^{in}$ and $\phi_{v,r}^{in}$ modes in subsonic-supersonic configuration

In this appendix, we will provide an analytic derivation of the scattering amplitudes for $\phi_{3,l}^{in}$ and $\phi_{v,r}^{in}$ modes in the subsonic-supersonic BEC configuration, thus completing the calculation treated in Sect.4.3.

A.1 $\phi_{3,l}^{in}$ mode

The matching conditions determining the scattering amplitudes for the $\phi_{3,l}^{in}$ mode ($A_3^l = 1$) are

$$D_u^l A_u^l + D_v^l A_v^l + D_3^l = D_u^r A_u^r + D_+^r A_+^r \tag{A.1.1}$$

$$k_u^l D_u^l A_u^l + k_v^l D_v^l A_v^l + k_3^l D_3^l = k_u^r D_u^r A_u^r + k_+^r D_+^r A_+^r \tag{A.1.2}$$

$$E_u^l A_u^l + E_v^l A_v^l + E_3^l = E_u^r A_u^r + E_+^r A_+^r \tag{A.1.3}$$

$$k_u^l E_u^l A_u^l + k_v^l E_v^l A_v^l + k_3^l E_3^l = k_u^r E_u^r A_u^r + k_+^r E_+^r A_+^r. \tag{A.1.4}$$

We notice they are equivalent to those for the $\phi_{4,l}^{in}$ mode ($A_4^l = 1$), upon substituting D_3^l, E_3^l and k_3^l with the corresponding Bogoliubov weights and momentum of the k_4^l-wave. This enables to follow the same steps of the algorithm we introduced in Sect 4.3 for the $\phi_{4,l}^{in}$ mode and substitute

$$D_4^l \longrightarrow D_3^l, \quad E_4^l \longrightarrow E_3^l, \quad k_4^l \longrightarrow k_3^l \tag{A.1.5}$$

in the implicit expressions for A_u^l, A_v^l and A_u^r.

A_u^l amplitude

For

$$A_u^l = \frac{N_{ul}}{D_{ul}}, \tag{A.1.6}$$

we have

$$\begin{aligned} N_{ul} = &(k_+^r - k_3^l)(k_v^l - k_u^r)(E_+^r D_v^l - D_+^r E_v^l)(E_u^r D_3^l - D_u^r E_3^l) \\ &+ (k_3^l - k_u^r)(k_v^l - k_+^r)(E_u^r D_v^l - D_u^r E_v^l)(E_+^r D_3^l - D_+^r E_3^l), \end{aligned} \tag{A.1.7}$$

and

$$\begin{aligned} D_{ul} = &(k_u^l - k_+^r)(k_v^l - k_u^r)(E_+^r D_v^l - D_+^r E_v^l)(E_u^r D_u^l - D_u^r E_u^l) \\ &+ (k_u^r - k_u^l)(k_v^l - k_+^r)(E_u^r D_v^l - D_u^r E_v^l)(E_+^r D_u^l - D_+^r E_u^l). \end{aligned} \tag{A.1.8}$$

Let us proceed with the series-expansion of A_u^l to leading order in ω. For that reason, among the small-ω approximations we need for the quantities entering its definition, we miss the ones relative to the k_3^l-wave:

$$k_3^l \sim {k_3^l}^0 + \mathcal{O}(\omega), \tag{A.1.9}$$

with

$${k_3^l}^0 = \frac{2\sqrt{v^2 - c_l^2}}{c_l \xi_l}, \qquad \left|\frac{\mathrm{d}k_3^l}{\mathrm{d}\omega}\right| \sim \frac{|v|}{v^2 - c_l^2}. \tag{A.1.10}$$

Then,

$$D_3^l \sim \frac{|v| + \sqrt{v^2 - c_l^2}}{2(v^2 - c_l^2)^{3/4}}, \quad E_3^l \sim -\frac{|v| - \sqrt{v^2 - c_l^2}}{2(v^2 - c_l^2)^{3/4}} \tag{A.1.11}$$

and ($c_r \xi_r = c_l \xi_l \equiv c\xi$)

$$({k_+^r}^0 - {k_3^l}^0)(D_3^l + E_3^l) \sim \frac{2i}{c\xi}\left(\sqrt{c_r^2 - v^2} + i\sqrt{v^2 - c_l^2}\right)\frac{1}{(v^2 - c_l^2)^{1/4}}. \tag{A.1.12}$$

Inserting the leading order expansions, the numerator (A.1.7) reads

$$\begin{aligned} N_{ul} \sim &({k_+^r}^0 - {k_3^l}^0)\left(\frac{1}{v - c_l} - \frac{1}{v + c_r}\right)\left[E_+^r \frac{(1 - \frac{\xi_l}{2}\frac{\omega}{v - c_l})}{\sqrt{2\xi_l \omega}} - D_+^r \frac{(-1 - \frac{\xi_l}{2}\frac{\omega}{v - c_l})}{\sqrt{2\xi_l \omega}}\right] \times \\ &\times \left[\frac{\left(-1 + \frac{\xi_r}{2}\frac{\omega}{v + c_r}\right)}{\sqrt{2\xi_r \omega}} D_3^l - \frac{\left(1 + \frac{\xi_r}{2}\frac{\omega}{v + c_r}\right)}{\sqrt{2\xi_r \omega}} E_3^l\right] \\ &+ ({k_3^l}^0 - k_u^r)(k_v^l - {k_3^l}^0)\frac{\sqrt{c_r c_l}}{2}\left[\frac{1}{c_r(v + c_r)} + \frac{1}{c_l(v - c_l)}\right](E_+^r D_3^l - D_+^r E_3^l). \end{aligned} \tag{A.1.13}$$

Once divided by $(E_+^r + D_+^r)$, it becomes

$$\begin{aligned}\frac{N_{ul}}{E_+^r + D_+^r} \sim \omega\left(\frac{1}{v-c_l} - \frac{1}{v+c_r}\right)(k_+^{r^0} - k_4^{l\,0})\left(-\frac{1}{2\omega\sqrt{\xi_l\xi_r}}\right)(D_4^l + E_4^l) \\ - k_4^{l\,0} k_+^{r^0}\frac{\sqrt{c_r c_l}}{2}\left[\frac{1}{c_r(v+c_r)} + \frac{1}{c_l(v-c_l)}\right]\frac{(E_+^r D_4^l - D_+^r E_4^l)}{(E_+^r + D_+^r)}.\end{aligned} \tag{A.1.14}$$

Since

$$\frac{(E_+^r D_3^l - D_+^r E_3^l)}{(E_+^r + D_+^r)} = \frac{\left(D_3^l - \frac{D_+^r}{E_+^r}E_3^l\right)}{\left(1 + \frac{D_+^r}{E_+^r}\right)} = \left(i\sqrt{v^2 - c_l^2} + \sqrt{c_r^2 - v^2}\right)\frac{|v|}{2\sqrt{c_r^2 - v^2}(v^2 - c_l^2)^{3/4}}, \tag{A.1.15}$$

and

$$k_3^{l\,0} k_+^{r^0} = \frac{4i\sqrt{v^2 - c_l^2}\sqrt{c_r^2 - v^2}}{c^2\xi^2}, \tag{A.1.16}$$

we obtain

$$\begin{aligned}\frac{N_{ul}}{E_+^r + D_+^r} \sim -i\frac{\left(\sqrt{c_r^2 - v^2} + i\sqrt{v^2 - c_l^2}\right)}{(v^2 - c_l^2)^{1/4}} \\ \left[\left(\frac{1}{v-c_l} - \frac{1}{v+c_r}\right)\frac{1}{\sqrt{\xi_l\xi_r}}\frac{1}{\sqrt{c_r c_l \xi_l \xi_r}} + \frac{|v|\sqrt{c_r c_l}}{c_r c_l \xi_l \xi_r}\left(\frac{1}{c_r(v+c_r)} + \frac{1}{c_l(v-c_l)}\right)\right],\end{aligned} \tag{A.1.17}$$

$$\frac{N_{ul}}{E_+^r + D_+^r} \sim -i\frac{\left(\sqrt{c_r^2 - v^2} + i\sqrt{v^2 - c_l^2}\right)}{(v^2 - c_l^2)^{1/4}}\frac{1}{\xi_l\xi_r\sqrt{c_r c_l}}\left[\frac{-(c_r + c_l)}{c_r c_l}\right]. \tag{A.1.18}$$

The denominator (A.1.8) to leading order in ω is the same as D_{ul} in the $\phi_{4,l}^{in}$ case, therefore

$$\frac{D_{ul}}{E_+^r + D_+^r} \sim -i\frac{\sqrt{c_r^2 - v^2}}{c\xi}\sqrt{\frac{2\omega}{\xi_l c_r c_l}}\frac{(c_r^2 - c_l^2)}{(v+c_r)(v+c_l)(v-c_l)}. \tag{A.1.19}$$

Finally, A_u^l to leading order in ω is

$$\begin{aligned}A_u^l = \frac{N_{ul}}{D_{ul}} = \frac{\frac{N_{ul}}{E_+^r + D_+^r}}{\frac{D_{ul}}{E_+^r + D_+^r}} = \\ = \frac{(v^2 - c_l^2)^{3/4}(v+c_r)}{(c_l - c_r)\sqrt{c_r^2 - v^2}}\frac{1}{\sqrt{2z_l}c_l^{3/2}}\left(\sqrt{c_r^2 - v^2} + i\sqrt{v^2 - c_l^2}\right),\end{aligned} \tag{A.1.20}$$

where $z_l \equiv \omega\frac{\xi_l}{c_l}$.

A_v^l amplitude

For

$$A_v^l = \frac{N_{vl}}{D_{vl}}, \tag{A.1.21}$$

we have

$$\begin{aligned} N_{vl} &= (k_+^r - k_3^l)(k_u^l - k_u^r)(E_+^r D_u^l - D_+^r E_u^l)(E_u^r D_3^l - D_u^r E_3^l) \\ &+ (k_3^l - k_u^r)(k_u^l - k_+^r)(E_u^r D_u^l - D_u^r E_u^l)(E_+^r D_3^l - D_+^r E_3^l), \end{aligned} \tag{A.1.22}$$

and

$$\begin{aligned} D_{vl} &= (k_u^l - k_+^r)(k_v^l - k_u^r)(E_+^r D_v^l - D_+^r E_v^l)(E_u^r D_u^l - D_u^r E_u^l) \\ &+ (k_u^r - k_u^l)(k_v^l - k_+^r)(E_u^r D_v^l - D_u^r E_v^l)(E_+^r D_u^l - D_+^r E_u^l). \end{aligned} \tag{A.1.23}$$

Inserting the leading order expansions, the numerator (A.1.22) reads

$$\begin{aligned} N_{vl} \sim \omega &\left(k_+^{r^0} - k_3^{l^0}\right)\left(\frac{1}{v+c_l} - \frac{1}{v+c_r}\right)\times \\ &\times \frac{1}{\sqrt{2\xi_l\omega}}\left[E_+^r\left(-1 - \frac{\xi_l}{2}\frac{\omega}{v+c_l}\right) - D_+^r\left(1 - \frac{\xi_l}{2}\frac{\omega}{v+c_l}\right)\right]\times \\ &\times \frac{1}{\sqrt{2\xi_r\omega}}\left[\left(-1 + \frac{\xi_r}{2}\frac{\omega}{v+c_r}\right)D_3^l - \left(1 + \frac{\xi_r}{2}\frac{\omega}{v+c_r}\right)E_3^l\right] \\ &+ \left(k_3^{l^0} - \frac{\omega}{v+c_r}\right)\left(\frac{\omega}{v+c_l} - k_+^{r^0}\right)\frac{\sqrt{c_r c_l}}{2}\left(\frac{1}{c_l(v+c_l)} - \frac{1}{c_r(v+c_r)}\right)(E_+^r D_3^l - D_+^r E_3^l). \end{aligned} \tag{A.1.24}$$

Once divided by $(E_+^r + D_+^r)$, it becomes

$$\begin{aligned} \frac{N_{vl}}{E_+^r + D_+^r} &\sim \omega\left(\frac{1}{v+c_l} - \frac{1}{v+c_r}\right)(k_+^{r^0} - k_3^{l\,0})\frac{1}{2\omega\sqrt{\xi_l\xi_r}}(D_3^l + E_3^l) \\ &- k_3^{l\,0} k_+^{r^0}\frac{\sqrt{c_r c_l}}{2}\left[\frac{1}{c_l(v+c_l)} - \frac{1}{c_r(v+c_r)}\right]\frac{(E_+^r D_3^l - D_+^r E_3^l)}{(E_+^r + D_+^r)}. \end{aligned} \tag{A.1.25}$$

Since

$$\frac{(E_+^r D_3^l - D_+^r E_3^l)}{(E_+^r + D_+^r)} = \frac{\left(D_3^l - \frac{D_+^r}{E_+^r}E_3^l\right)}{\left(1 + \frac{D_+^r}{E_+^r}\right)} = \left(i\sqrt{v^2 - c_l^2} + \sqrt{c_r^2 - v^2}\right)\frac{|v|}{2\sqrt{c_r^2 - v^2}(v^2 - c_l^2)^{3/4}}, \tag{A.1.26}$$

and

$$k_3^{l\,0} k_+^{r^0} = \frac{4i\sqrt{v^2 - c_l^2}\sqrt{c_r^2 - v^2}}{c^2\xi^2}, \tag{A.1.27}$$

we obtain

$$\frac{N_{vl}}{E_+^r + D_+^r} \sim -i\frac{(\sqrt{c_r^2 - v^2} + i\sqrt{v^2 - c_l^2})}{(v^2 - c_l^2)^{1/4}}$$
$$\left[\left(\frac{1}{v+c_r} - \frac{1}{v+c_l}\right)\frac{1}{\sqrt{\xi_l \xi_r}}\frac{1}{\sqrt{c_r c_l \xi_l \xi_r}} + \frac{|v|\sqrt{c_r c_l}}{c_r c_l \xi_l \xi_r}\left(\frac{1}{c_l(v+c_l)} - \frac{1}{c_r(v+c_r)}\right)\right], \tag{A.1.28}$$

$$\frac{N_{vl}}{E_+^r + D_+^r} \sim -i\frac{(\sqrt{c_r^2 - v^2} + i\sqrt{v^2 - c_l^2})}{(v^2 - c_l^2)^{1/4}}\frac{1}{\xi_l \xi_r \sqrt{c_r c_l}}\left(\frac{c_l - c_r}{c_r c_l}\right). \tag{A.1.29}$$

Finally, A_v^l to leading order in ω is

$$\begin{aligned} A_v^l &= \frac{N_{vl}}{D_{vl}} = \frac{\frac{N_{vl}}{E_+^r + D_+^r}}{\frac{D_{vl}}{E_+^r + D_+^r}} \\ &= \frac{(v^2 - c_l^2)^{3/4}(v+c_r)}{(c_r + c_l)\sqrt{c_r^2 - v^2}}\frac{1}{\sqrt{2z_l}c_l^{3/2}}\left(\sqrt{c_r^2 - v^2} + i\sqrt{v^2 - c_l^2}\right). \end{aligned} \tag{A.1.30}$$

A_u^r amplitude

For

$$A_u^r = \frac{N_{ur}}{D_{ur}}, \tag{A.1.31}$$

we have

$$N_{ur} = (k_+^r - k_u^l)D_u^l A_u^l + (k_+^r - k_v^l)D_v^l A_v^l + (k_+^r - k_3^l)D_3^l, \tag{A.1.32}$$

$$D_{ur} = (k_+^r - k_u^r)D_u^r. \tag{A.1.33}$$

Inserting the leading order expansions, the numerator (A.1.32) reads

$$N_{ur} \sim -\frac{k_+^{r^0}}{\sqrt{2\xi_l \omega}}A_u^l + \frac{k_+^{r^0}}{\sqrt{2\xi_l \omega}}A_v^l + (k_+^{r^0} - k_3^{l^0})D_3^l, \tag{A.1.34}$$

Once divided by $k_+^{r^0}$, it becomes

$$\frac{N_{ur}}{k_+^{r^0}} \sim -\frac{1}{\sqrt{2\xi_l \omega}}A_u^l + \frac{1}{\sqrt{2\xi_l \omega}}A_v^l + \frac{(k_+^{r^0} - k_3^{l^0})}{k_+^{r^0}}D_3^l, \tag{A.1.35}$$

Since

$$\frac{(k_+^{r^0} - k_3^{l^0})}{k_+^{r^0}}D_3^l = \frac{\left(\sqrt{c_r^2 - v^2} + i\sqrt{v^2 - c_l^2}\right)}{\sqrt{c_r^2 - v^2}}\left(\frac{|v| + \sqrt{v^2 - c_l^2}}{2(v^2 - c_l^2)^{3/4}}\right) \sim \mathcal{O}(1), \tag{A.1.36}$$

this term is negligible with respect to $\left(-\frac{A^l_u}{\sqrt{2\xi_l\omega}}+\frac{A^l_v}{\sqrt{2\xi_l\omega}}\right)$, which is $\sim \mathcal{O}(1/\omega)$. The denominator (A.1.33) to leading order in ω is the same as D_{ur} in the $\phi^{in}_{4,l}$ case, therefore

$$\sim \frac{k_+^{r^0}}{\sqrt{2\xi_r\omega}}. \tag{A.1.37}$$

Finally, A^r_u to leading order in ω is

$$\begin{aligned} A^r_u &= \frac{N_{ur}}{D_{ur}} = \frac{N_{ur}/k_+^{r^0}}{D_{ur}/k_+^{r^0}} = \\ &= \frac{\sqrt{2c_r}}{(c_r^2-c_l^2)c_l\sqrt{z_l}}\frac{(v^2-c_l^2)^{3/4}(v+c_r)}{\sqrt{c_r^2-v^2}}\left(\sqrt{c_r^2-v^2}+i\sqrt{v^2-c_l^2}\right). \end{aligned} \tag{A.1.38}$$

The solution for the decaying wave amplitude A^r_+ to leading order in ω is reported in literature to be

$$A^r_+ = \frac{(v^2-c_l^2)^{1/4}}{2D^r_+(v^2-c_r^2)}\left(v-i\sqrt{c_r^2-v^2}\right). \tag{A.1.39}$$

The unitarity condition deriving from the conservation of the Bogoliubov norm reads

$$\left|A^l_v\right|^2+|A^r_u|^2-\left|A^l_u\right|^2 = 1, \tag{A.1.40}$$

where the minus sign comes from the negative norm k^l_u-wave. Since the amplitudes entering this equation are $\sim \mathcal{O}(1/\sqrt{\omega})$, the left hand side should vanish at leading order. This is verified with the amplitude solutions we have obtained, indeed.

A.2 $\phi^{in}_{v,r}$ mode

The matching conditions determining the scattering amplitudes for the $\phi^{in}_{v,r}$ mode ($A^r_v = 1$) are

$$D^l_uA^l_u + D^l_vA^l_v = D^r_uA^r_u + D^r_v + D^r_+A^r_+ \tag{A.2.1}$$

$$k^l_uD^l_uA^l_u + k^l_vD^l_vA^l_v = k^r_uD^r_uA^r_u + k^r_vD^r_v + k^r_+D^r_+A^r_+ \tag{A.2.2}$$

$$E^l_uA^l_u + E^l_vA^l_v = E^r_uA^r_u + E^r_v + E^r_+A^r_+ \tag{A.2.3}$$

$$k^l_uE^l_uA^l_u + k^l_vE^l_vA^l_v = k^r_uE^r_uA^r_u + k^r_vE^r_v + k^r_+E^r_+A^r_+. \tag{A.2.4}$$

We notice they are equivalent to those for the $\phi^{in}_{4,l}$ mode ($A^l_4 = 1$), upon substituting D^r_v, E^r_v and k^r_v with the corresponding Bogoliubov weights (with opposite sign, though) and momentum of the k^l_4-wave. This enables to follow the same steps of the algorithm we introduced in Sect 4.3 for the $\phi^{in}_{4,l}$ mode and substitute

$$D^l_4 \longrightarrow -D^r_v, \quad E^l_4 \longrightarrow -E^r_v, \quad k^l_4 \longrightarrow k^r_v \tag{A.2.5}$$

in the implicit expressions for A^l_u, A^l_v and A^r_u.

A^l_u amplitude

For

$$A^l_u = \frac{N_{ul}}{D_{ul}}, \tag{A.2.6}$$

we have

$$\begin{aligned} N_{ul} = &- (k^r_+ - k^r_v)(k^l_v - k^r_u)(E^r_+ D^l_v - D^r_+ E^l_v)(E^r_u D^r_v - D^r_u E^r_v) \\ &- (k^r_v - k^r_u)(k^l_v - k^r_+)(E^r_u D^l_v - D^r_u E^l_v)(E^r_+ D^r_v - D^r_+ E^r_v), \end{aligned} \tag{A.2.7}$$

and

$$\begin{aligned} D_{ul} = &(k^l_u - k^r_+)(k^l_v - k^r_u)(E^r_+ D^l_v - D^r_+ E^l_v)(E^r_u D^l_u - D^r_u E^l_u) \\ &+ (k^r_u - k^l_u)(k^l_v - k^r_+)(E^r_u D^l_v - D^r_u E^l_v)(E^r_+ D^l_u - D^r_+ E^l_u). \end{aligned} \tag{A.2.8}$$

Let us proceed with the series-expansion of A^l_u to leading order in ω. For that reason, among the small-ω approximations we need for the quantities entering its definition, we miss the ones relative to the k^r_v-wave:

$$k^r_v \sim \frac{\omega}{v - c_r} > 0, \quad D^r_v \sim \frac{-1 + \frac{\xi_r}{2}\frac{\omega}{v-c_r}}{\sqrt{2\xi_r \omega}}, \quad E^r_v \sim \frac{1 + \frac{\xi_r}{2}\frac{\omega}{v-c_r}}{\sqrt{2\xi_r \omega}}. \tag{A.2.9}$$

Inserting the leading order expansions, the numerator (A.2.7) reads

$$\begin{aligned} N_{ul} \sim &(k^{r^0}_+ - \frac{\omega}{v - c_r})\left(\frac{\omega}{v - c_l} - \frac{\omega}{v + c_r}\right)\left[E^r_+ \frac{(1 - \frac{\xi_l}{2}\frac{\omega}{v-c_l})}{\sqrt{2\xi_l \omega}} - D^r_+ \frac{(-1 - \frac{\xi_l}{2}\frac{\omega}{v-c_l})}{\sqrt{2\xi_l \omega}}\right]\times \\ &\times \left[\frac{\left(-1 + \frac{\xi_r}{2}\frac{\omega}{v+c_r}\right)}{\sqrt{2\xi_r\omega}}\frac{\left(-1 + \frac{\xi_r}{2}\frac{\omega}{v-c_r}\right)}{\sqrt{2\xi_r\omega}} - \frac{\left(-1 + \frac{\xi_r}{2}\frac{\omega}{v+c_r}\right)}{\sqrt{2\xi_r\omega}}\frac{\left(-1 + \frac{\xi_r}{2}\frac{\omega}{v-c_r}\right)}{\sqrt{2\xi_r\omega}}\right] \\ &+ \left(\frac{\omega}{v - c_r} - \frac{\omega}{v + c_r}\right)\left(\frac{\omega}{v - c_l} - k^{r^0}_+\right)\frac{\sqrt{c_r c_l}}{2}\left[\frac{1}{c_r(v + c_r)} + \frac{1}{c_l(v - c_l)}\right]\times \\ &\times \left[D^r_+ \frac{(1 + \frac{\xi_r}{2}\frac{\omega}{v-c_r})}{\sqrt{2\xi_r\omega}} - E^r_+ \frac{(-1 - \frac{\xi_r}{2}\frac{\omega}{v-c_r})}{\sqrt{2\xi_r\omega}}\right]. \end{aligned} \tag{A.2.10}$$

Keeping the leading order in ω, we obtain

$$\begin{aligned} \frac{N_{ul}}{E^r_+ + D^r_+} \sim &\frac{k^{r^0}_+}{\sqrt{2\xi_l\omega}}\left(\frac{1}{v - c_l} - \frac{1}{v + c_r}\right)\frac{1}{2\xi_r}\frac{\xi_r\omega}{2}\left(\frac{2}{v + c_r} + \frac{2}{v - c_r}\right) \\ &+ \omega\left(\frac{1}{v - c_r} - \frac{1}{v + c_r}\right)\left(-\frac{k^{r^0}_+\sqrt{c_r c_l}}{2}\right)\left(\frac{1}{c_r(v + c_r)} + \frac{1}{c_l(v - c_l)}\right)\frac{1}{\sqrt{2\xi_r\omega}} \\ = &\frac{k^{r^0}_+\sqrt{\omega}}{\sqrt{2\xi_l}}\frac{vc_l^2 + c_r c_l^2 - vc_r^2 - c_r^3}{(v - c_l)(v - c_r)(v + c_r)^2 c_l}. \end{aligned} \tag{A.2.11}$$

The denominator (A.2.8) to leading order in ω is the same as D^l_u in the $\phi^{in}_{4,l}$ case, therefore

$$\frac{D_{ul}}{E^r_+ + D^r_+} \sim -i\frac{\sqrt{c_r^2 - v^2}}{c\xi}\sqrt{\frac{2\omega}{\xi_l c_r c_l}}\frac{(c_r^2 - c_l^2)}{(v+c_r)(v+c_l)(v-c_l)}. \tag{A.2.12}$$

Finally, A^l_u to leading order in ω is

$$A^l_u = \frac{N_{ul}}{D_{ul}} = \frac{\frac{N_{ul}}{E^r_+ + D^r_+}}{\frac{D_{ul}}{E^r_+ + D^r_+}} = \sqrt{\frac{c_r}{c_l}}\frac{v+c_l}{c_r - v}. \tag{A.2.13}$$

A^l_v amplitude

For

$$A^l_v = \frac{N_{vl}}{D_{vl}} \tag{A.2.14}$$

we have

$$\begin{aligned} N_{vl} = &-(k^r_+ - k^r_v)(k^l_u - k^r_u)(E^r_+ D^l_u - D^r_+ E^l_u)(E^r_u D^r_v - D^r_u E^r_v) \\ &-(k^r_v - k^r_u)(k^l_u - k^r_+)(E^r_u D^l_u - D^r_u E^l_u)(E^r_+ D^r_v - D^r_+ E^r_v), \end{aligned} \tag{A.2.15}$$

and

$$\begin{aligned} D_{vl} = &(k^l_u - k^r_+)(k^l_v - k^r_u)(E^r_+ D^l_v - D^r_+ E^l_v)(E^r_u D^l_u - D^r_u E^l_u) \\ &+(k^r_u - k^l_u)(k^l_v - k^r_+)(E^r_u D^l_v - D^r_u E^l_v)(E^r_+ D^l_u - D^r_+ E^l_u). \end{aligned} \tag{A.2.16}$$

Inserting the leading order expansions, the numerator (A.2.15) reads

$$\begin{aligned} N_{vl} \sim &\left(\frac{\omega}{v+c_l} - \frac{\omega}{v+c_r}\right)\left[E^r_+\frac{\left(-1 - \frac{\xi_l}{2}\frac{\omega}{v+c_l}\right)}{\sqrt{2\xi_l\omega}} - D^r_+\frac{\left(1 - \frac{\xi_l}{2}\frac{\omega}{v+c_l}\right)}{\sqrt{2\xi_l\omega}}\right]\left(k^{r0}_+ - \frac{\omega}{v-c_r}\right)\times \\ &\times\left[\frac{\left(1 + \frac{\xi_r}{2}\frac{\omega}{v+c_r}\right)}{\sqrt{2\xi_r\omega}}\frac{\left(1 + \frac{\xi_r}{2}\frac{\omega}{v-c_r}\right)}{\sqrt{2\xi_r\omega}} - \frac{\left(-1 + \frac{\xi_r}{2}\frac{\omega}{v+c_r}\right)}{\sqrt{2\xi_r\omega}}\frac{\left(-1 + \frac{\xi_r}{2}\frac{\omega}{v-c_r}\right)}{\sqrt{2\xi_r\omega}}\right] \\ &+\left(\frac{\omega}{v+c_r} - \frac{\omega}{v-c_r}\right)\left(k^{r0}_+ - \frac{\omega}{v+c_l}\right)\frac{\sqrt{c_r c_l}}{2}\left[\frac{1}{c_l(v+c_l)} - \frac{1}{c_r(v+c_r)}\right]\times \\ &\times\left[D^r_+\frac{\left(1 + \frac{\xi_r}{2}\frac{\omega}{v-c_r}\right)}{\sqrt{2\xi_r\omega}} - E^r_+\frac{\left(-1 + \frac{\xi_r}{2}\frac{\omega}{v-c_r}\right)}{\sqrt{2\xi_r\omega}}\right]. \end{aligned} \tag{A.2.17}$$

Keeping the leading order in ω, we obtain

$$\begin{aligned}\frac{N_{vl}}{E^r_+ + D^r_+} \sim & -\frac{k^{r^0}_+}{\sqrt{2\xi_l\omega}}\left(\frac{1}{v+c_l}-\frac{1}{v+c_r}\right)\frac{1}{2\xi_r}\frac{\xi_r\omega}{2}\left(\frac{2}{v+c_r}+\frac{2}{v-c_r}\right)\\ &+\frac{k^{r^0}_+\sqrt{c_r c_l}}{2}\omega\left(\frac{1}{v+c_r}-\frac{1}{v-c_r}\right)\left(\frac{1}{c_l(v+c_l)}-\frac{1}{c_r(v+c_r)}\right)\frac{1}{\sqrt{2\xi_r\omega}}\\ &=-\frac{k^{r^0}_+\sqrt{\omega}}{\sqrt{2\xi_l}}\frac{vc_l(c_r-c_l)+c_r[c_r(v+c_r)-c_l(v+c_l)]}{(v+c_l)(v-c_r)(v+c_r)^2 c_l}.\end{aligned} \tag{A.2.18}$$

The denominator (A.2.16) to leading order in ω is the same as D^l_v in the $\phi^{in}_{4,l}$ case, therefore

$$\frac{D_{ul}}{E^r_+ + D^r_+} \sim -i\frac{\sqrt{c_r^2-v^2}}{c\xi}\sqrt{\frac{2\omega}{\xi_l c_r c_l}}\frac{(c_r^2-c_l^2)}{(v+c_r)(v+c_l)(v-c_l)}. \tag{A.2.19}$$

Finally, A^l_v to leading order in ω is

$$A^l_v = \frac{N_{vl}}{D_{vl}} = \frac{\frac{N_{vl}}{E^r_+ + D^r_+}}{\frac{D_{vl}}{E^r_+ + D^r_+}} = \sqrt{\frac{c_r}{c_l}}\frac{v-c_l}{v-c_r}. \tag{A.2.20}$$

A^r_u amplitude

For

$$A^r_u = \frac{N_{ur}}{D_{ur}} \tag{A.2.21}$$

we have

$$N_{ur} = (k^r_+ - k^l_u)D^l_u A^l_u + (k^r_+ - k^l_v)D^l_v A^l_v - (k^r_+ - k^r_v)D^r_v, \tag{A.2.22}$$

$$D_{ur} = (k^r_+ - k^r_u)D^r_u. \tag{A.2.23}$$

Inserting the leading order expansions, the numerator (A.2.22) reads

$$N_{ur} \sim -\frac{k^{r^0}_+}{\sqrt{2\xi_l\omega}}A^l_u + \frac{k^{r^0}_+}{\sqrt{2\xi_l\omega}}A^l_v - \frac{k^{r^0}_+}{\sqrt{2\xi_r\omega}}, \tag{A.2.24}$$

Once divided by $k^{r^0}_+$, it becomes

$$\frac{N_{vr}}{k^{r^0}_+} \sim -\frac{1}{\sqrt{2\xi_l\omega}}A^l_u + \frac{1}{\sqrt{2\xi_l\omega}}A^l_v - \frac{1}{\sqrt{2\xi_r\omega}}. \tag{A.2.25}$$

The denominator (A.2.23) is the same as $D_u r$ in the $\phi^{in}_{4,l}$ case, therefore

$$\frac{D_{ur}}{k^{r^0}_+} \sim \frac{1}{\sqrt{2\xi_r\omega}}. \tag{A.2.26}$$

Finally, A^r_u to leading order in ω is

$$A^r_u = \frac{N_{ur}}{D_{ur}} = \frac{N_{ur}/k^{r0}_+}{D_{ur}/k^{r0}_+} = \frac{v + c_r}{v - c_r}. \tag{A.2.27}$$

The solution for the decaying wave amplitude A^r_+ to leading order in ω is reported in literature to be

$$\begin{aligned} A^r_+ = {} & \frac{c_l\sqrt{z_l}\sqrt{c_r(v^2 - c_l^2)}}{\sqrt{2}D^r_+(v - c_l)(c_r^2 - v^2)^{3/2}(c_r + c_l)} \times \\ & \times \left[\sqrt{c_r^2 - v^2}\left(v + \sqrt{v^2 - c_l^2}\right) + i\left(v\sqrt{v^2 - c_l^2} + v^2 - c_r^2\right)\right]. \end{aligned} \tag{A.2.28}$$

The unitarity condition deriving from the conservation of the Bogoliubov norm reads

$$\left|A^l_v\right|^2 + \left|A^r_u\right|^2 - \left|A^l_u\right|^2 = 1, \tag{A.2.29}$$

where the minus sign again comes from the negative norm k^l_u-wave. Since the amplitudes entering this equation are $\sim \mathcal{O}(1)$, the left hand side should be equal to 1 exactly. This is verified with the amplitude solutions we have obtained, indeed.

Appendix B

Analytic derivation of implicit solution A_u^r for $\phi_{4,l}^{in}$ mode in subsonic-sonic-supersonic configuration

In this appendix we will provide the algebraic steps yielding A_u^r scattering amplitude for $\phi_{4,l}^{in}$ mode in subsonic-sonic-supersonic configuration. Therefore, we will need to solve the two sets of matching conditions shown in Chapter 5 together. Since they both are systems of 4 equations in 6 unknowns, firstly we will solve the left-side system for A_-^s and A_+^s in terms of A_u^s and A_v^s. Secondly, we will plug these expressions into the right-side system. At the end, we will obtain an implicit solution which further will require to be series-expanded in ω.

B.1 Supersonic-sonic side

The plane-waves expansion for mode functions ϕ_ω and φ_ω in the supersonic region is

$$\phi_\omega^l = e^{-i\omega t}[A_v^l D_v^l e^{ik_v^l x} + A_u^l D_u^l e^{ik_u^l x} + \overbrace{A_3^l}^{=0} D_3^l e^{ik_3^l x} + \overbrace{A_4^l}^{=1} D_4^l e^{ik_4^l x}], \tag{B.1.1}$$

$$\varphi_\omega^l = e^{-i\omega t}[A_v^l E_v^l e^{ik_v^l x} + A_u^l E_u^l e^{ik_u^l x} + \overbrace{A_3^l}^{=0} E_3^l e^{ik_3^l x} + \overbrace{A_4^l}^{=1} E_4^l e^{ik_4^l x}], \tag{B.1.2}$$

while in the sonic region

$$\phi_\omega^s = e^{-i\omega t}[A_v^s D_v^s e^{ik_v^s x} + A_u^s D_u^s e^{ik_u^s x} + A_-^s D_-^s e^{ik_-^s x} + A_+^s D_+^s e^{ik_+^s x}], \tag{B.1.3}$$

$$\varphi_\omega^s = e^{-i\omega t}[A_v^s E_v^s e^{ik_v^s x} + A_u^s E_u^s e^{ik_u^s x} + A_-^s E_-^s e^{ik_-^s x} + A_+^s E_+^s e^{ik_+^s x}]. \tag{B.1.4}$$

The $\phi^{in}_{4,l}$ mode is characterized by

$$A^l_4 = 1, A^l_3 = 0.$$

The matching conditions at the left discontinuity surface − i.e. at $x = -L/2$, where $L =$ length of the sonic region − are

$$\begin{aligned} &A^l_v D^l_v e^{-ik^l_v L/2} + A^l_u D^l_u e^{-ik^l_u L/2} + D^l_4 e^{-ik^l_4 L/2} = \\ &A^s_v D^s_v e^{-ik^s_v L/2} + A^s_u D^s_u e^{-ik^s_u L/2} + A^s_- D^s_- e^{-ik^s_- L/2} + A^s_+ D^s_+ e^{-ik^s_+ L/2}, \end{aligned} \tag{B.1.5}$$

$$\begin{aligned} &k^l_v A^l_v D^l_v e^{-ik^l_v L/2} + k^l_u A^l_u D^l_u e^{-ik^l_u L/2} + k^l_4 D^l_4 e^{-ik^l_4 L/2} = \\ &k^s_v A^s_v D^s_v e^{-ik^s_v L/2} + k^s_u A^s_u D^s_u e^{-ik^s_u L/2} + k^s_- A^s_- D^s_- e^{-ik^s_- L/2} + k^s_+ A^s_+ D^s_+ e^{-ik^s_+ L/2}, \end{aligned} \tag{B.1.6}$$

$$\begin{aligned} &A^l_v E^l_v e^{-ik^l_v L/2} + A^l_u E^l_u e^{-ik^l_u L/2} + E^l_4 e^{-ik^l_4 L/2} = \\ &A^s_v E^s_v e^{-ik^s_v L/2} + A^s_u E^s_u e^{-ik^s_u L/2} + A^s_- E^s_- e^{-ik^s_- L/2} + A^s_+ E^s_+ e^{-ik^s_+ L/2}, \end{aligned} \tag{B.1.7}$$

$$\begin{aligned} &k^l_v A^l_v E^l_v e^{-ik^l_v L/2} + k^l_u A^l_u E^l_u e^{-ik^l_u L/2} + k^l_4 E^l_4 e^{-ik^l_4 L/2} = \\ &k^s_v A^s_v E^s_v e^{-ik^s_v L/2} + k^s_u A^s_u E^s_u e^{-ik^s_u L/2} + k^s_- A^s_- E^s_- e^{-ik^s_- L/2} + k^s_+ A^s_+ E^s_+ e^{-ik^s_+ L/2}; \end{aligned} \tag{B.1.8}$$

To solve for A^s_u, A^s_v as functions of A^s_-, A^s_+, we will follow algebraic steps which are qualitatively similar to those shown in Chapter 4.

1) Isolation
From (B.1.5),

$$\begin{aligned} A^l_u D^l_u e^{-ik^l_u L/2} = &A^s_v D^s_v e^{-ik^s_v L/2} + A^s_u D^s_u e^{-ik^s_u L/2} + \\ &A^s_- D^s_- e^{-ik^s_- L/2} + A^s_+ D^s_+ e^{-ik^s_+ L/2} - A^l_v D^l_v e^{-ik^l_v L/2} - D^l_4 e^{-ik^l_4 L/2}, \end{aligned} \tag{B.1.9}$$

and from (B.1.7),

$$\begin{aligned} A^l_u E^l_u e^{-ik^l_u L/2} = &A^s_v E^s_v e^{-ik^s_v L/2} + A^s_u E^s_u e^{-ik^s_u L/2} \\ &+ A^s_- E^s_- e^{-ik^s_- L/2} + A^s_+ E^s_+ e^{-ik^s_+ L/2} - A^l_v E^l_v e^{-ik^l_v L/2} - E^l_4 e^{-ik^l_4 L/2}. \end{aligned} \tag{B.1.10}$$

2) Plug-in
From (B.1.6),

$$\begin{aligned} &A^l_v D^l_v e^{-ik^l_v L/2}(k^l_v - k^l_u) + D^l_4 e^{-ik^l_4 L/2}(k^l_4 - k^l_u) = \\ &A^s_v D^s_v e^{-ik^s_v L/2}(k^s_v - k^l_u) + A^s_u D^s_u e^{-ik^s_u L/2}(k^s_u - k^l_u) + A^s_- D^s_- e^{-ik^s_- L/2}(k^s_- - k^l_u) + A^s_+ D^s_+ e^{-ik^s_+ L/2}(k^s_+ - k^l_u), \end{aligned} \tag{B.1.11}$$

and from (B.1.8),

$$A_v^l E_v^l e^{-ik_v^l L/2}(k_v^l - k_u^l) + E_4^l e^{-ik_4^l L/2}(k_4^l - k_u^l) =$$
$$A_v^s E_v^s e^{-ik_v^s L/2}(k_v^s - k_u^l) + A_u^s E_u^s e^{-ik_u^s L/2}(k_u^s - k_u^l) + A_-^s E_-^s e^{-ik_-^s L/2}(k_-^s - k_u^l) + A_+^s E_+^s e^{-ik_+^s L/2}(k_+^s - k_u^l). \quad \text{(B.1.12)}$$

3) <u>Combination via E_u^l and D_u^l</u>

$$E_u^l \times \text{(B.1.9)} - D_u^l \times \text{(B.1.10)} :$$

$$\begin{aligned} 0 = &(E_u^l D_v^s - D_u^l E_v^s) A_v^s e^{-ik_v^s L/2} + (E_u^l D_u^s - D_u^l E_u^s) A_u^s e^{-ik_u^s L/2} \\ &+ (E_u^l D_-^s - D_u^l E_-^s) A_-^s e^{-ik_-^s L/2} + (E_u^l D_+^s - D_u^l E_+^s) A_+^s e^{-ik_+^s L/2} \\ &- (E_u^l D_v^l - D_u^l E_v^l) A_v^l e^{-ik_v^l L/2} - (E_u^l D_4^l - D_u^l E_4^l) e^{-ik_4^l L/2}. \end{aligned} \quad \text{(B.1.13)}$$

4) <u>Combination via E_v^l and D_v^l (first final equation)</u>

$$E_v^l \times \text{(B.1.11)} - D_v^l \times \text{(B.1.12)} :$$

$$\begin{aligned} 0 + (E_v^l D_4^l - D_v^l E_4^l)(k_4^l - k_u^l) e^{-ik_4^l L/2} = &(E_v^l D_v^s - D_v^l E_v^s)(k_v^s - k_u^l) A_v^s e^{-ik_v^s L/2} \\ &+ (E_v^l D_u^s - D_v^l E_u^s)(k_u^s - k_u^l) A_u^s e^{-ik_u^s L/2} + (E_v^l D_-^s - D_v^l E_-^s)(k_-^s - k_u^l) A_-^s e^{-ik_-^s L/2} \\ &+ (E_v^l D_+^s - D_v^l E_+^s)(k_+^s - k_u^l) A_+^s e^{-ik_+^s L/2}. \end{aligned} \quad \text{(B.1.14)}$$

5) <u>Eliminate A_v^l contribution (second final equation)</u>

$$E_u^l \times \text{(B.1.11)} - D_u^l \times \text{(B.1.12)} - (k_v^l - k_u^l) \times \text{(B.1.13)} :$$

$$\begin{aligned} &\cancel{(E_u^l D_v^l - D_u^l E_v^l)(k_v^l - k_u^l) A_v^l e^{-ik_v^l L/2}} - \cancel{(k_v^l - k_u^l)(E_u^l D_v^l - D_u^l E_v^l) A_v^l e^{-ik_v^l L/2}} \\ &+ (E_u^l D_4^l - D_u^l E_4^l)(k_4^l - k_u^l) e^{-ik_4^l L/2} - (k_v^l - k_u^l)(E_u^l D_4^l - D_u^l E_4^l) e^{-ik_4^l L/2} \\ &= (E_u^l D_v^s - D_u^l E_v^s)(k_v^s - k_u^l) A_v^s e^{-ik_v^s L/2} - (k_v^l - k_u^l)(E_u^l D_v^s - D_u^l E_v^s) A_v^s e^{-ik_v^s L/2} \\ &+ (E_u^l D_u^s - D_u^l E_u^s)(k_u^s - k_u^l) A_u^s e^{-ik_u^s L/2} - (k_v^l - k_u^l)(E_u^l D_u^s - D_u^l E_u^s) A_u^s e^{-ik_u^s L/2} \\ &+ (E_u^l D_-^s - D_u^l E_-^s)(k_-^s - k_u^l) A_-^s e^{-ik_-^s L/2} - (k_v^l - k_u^l)(E_u^l D_-^s - D_u^l E_-^s) A_-^s e^{-ik_-^s L/2} \\ &+ (E_u^l D_+^s - D_u^l E_+^s)(k_+^s - k_u^l) A_+^s e^{-ik_+^s L/2} - (k_v^l - k_u^l)(E_u^l D_+^s - D_u^l E_+^s) A_+^s e^{-ik_+^s L/2}, \end{aligned}$$

$$\begin{aligned} \Longrightarrow &(k_4^l - k_v^l)(E_u^l D_4^l - D_u^l E_4^l) e^{-ik_4^l L/2} \\ &= (k_v^s - k_v^l)(E_u^l D_v^s - D_u^l E_v^s) A_v^s e^{-ik_v^s L/2} + (k_u^s - k_v^l)(E_u^l D_u^s - D_u^l E_u^s) A_u^s e^{-ik_u^s L/2} \\ &+ (k_-^s - k_v^l)(E_u^l D_-^s - D_u^l E_-^s) A_-^s e^{-ik_-^s L/2} + (k_+^s - k_v^l)(E_u^l D_+^s - D_u^l E_+^s) A_+^s e^{-ik_+^s L/2}. \end{aligned} \quad \text{(B.1.15)}$$

6) Calculate A^s_- via combination of the final equations
Define

$$F_1 \equiv (k^s_+ - k^l_v)(E^l_u D^s_+ - D^l_u E^s_+), \tag{B.1.16}$$

$$F_2 \equiv (k^s_+ - k^l_u)(E^l_v D^s_+ - D^l_v E^s_+). \tag{B.1.17}$$

Then,

$$\begin{aligned}
&F_1 \times (\text{B.1.14}) - F_2 \times (\text{B.1.15}) : \\
&F_1(E^l_v D^l_4 - D^l_v E^l_4)(k^l_4 - k^l_u)e^{-ik^l_4 L/2} - F_2(E^l_u D^l_4 - D^l_u E^l_4)(k^l_4 - k^l_v)e^{-ik^l_4 L/2} \\
&= F_1(E^l_v D^s_v - D^l_v E^s_v)(k^s_v - k^l_u)A^s_v e^{-ik^s_v L/2} - F_2(E^l_u D^s_v - D^l_u E^s_v)(k^s_v - k^l_v)A^s_v e^{-ik^s_v L/2} \\
&+ F_1(E^l_v D^s_u - D^l_v E^s_u)(k^s_u - k^l_u)A^s_u e^{-ik^s_u L/2} - F_2(E^l_u D^s_u - D^l_u E^s_u)(k^s_u - k^l_v)A^s_u e^{-ik^s_u L/2} \\
&+ F_1(E^l_v D^s_- - D^l_v E^s_-)(k^s_- - k^l_u)A^s_- e^{-ik^s_- L/2} - F_2(E^l_u D^s_- - D^l_u E^s_-)(k^s_- - k^l_v)A^s_- e^{-ik^s_- L/2},
\end{aligned} \tag{B.1.18}$$

$$\begin{aligned}
&\overbrace{[F_1(E^l_v D^s_- - D^l_v E^s_-)(k^s_- - k^l_u) - F_2(E^l_u D^s_- - D^l_u E^s_-)(k^s_- - k^l_v)]e^{-ik^s_- L/2}}^{D_3} A^s_- \\
&= \underbrace{[F_1(E^l_v D^l_4 - D^l_v E^l_4)(k^l_4 - k^l_u) - F_2(E^l_u D^l_4 - D^l_u E^l_4)(k^l_4 - k^l_v)]e^{-ik^l_4 L/2}}_{N^1_3} \\
&\underbrace{-[F_1(E^l_v D^s_v - D^l_v E^s_v)(k^s_v - k^l_u) - F_2(E^l_u D^s_v - D^l_u E^s_v)(k^s_v - k^l_v)]e^{-ik^s_v L/2}}_{N^2_3} A^s_v \\
&\underbrace{-[F_1(E^l_v D^s_u - D^l_v E^s_u)(k^s_u - k^l_u) - F_2(E^l_u D^s_u - D^l_u E^s_u)(k^s_u - k^l_v)]e^{-ik^s_u L/2}}_{N^3_3} A^s_u,
\end{aligned} \tag{B.1.19}$$

thus getting an expression for A^s_- as function of A^s_v and A^s_u

$$A^s_- = \frac{N^1_3 + N^2_3 A^s_v + N^3_3 A^s_u}{D_3}. \tag{B.1.20}$$

7) Analogously, calculate A^s_+ via combination of the final equations
Define

$$F_3 \equiv (k^s_- - k^l_v)(E^l_u D^s_- - D^l_u E^s_-), \tag{B.1.21}$$

$$F_4 \equiv (k^s_- - k^l_u)(E^l_v D^s_- - D^l_v E^s_-). \tag{B.1.22}$$

Then,

$$\begin{aligned}
&F_3 \times (\text{B.1.14}) - F_4 \times (\text{B.1.15}) : \\
&F_3(E^l_v D^l_4 - D^l_v E^l_4)(k^l_4 - k^l_u)e^{-ik^l_4 L/2} - F_4(E^l_u D^l_4 - D^l_u E^l_4)(k^l_4 - k^l_v)e^{-ik^l_4 L/2} \\
&= F_3(E^l_v D^s_v - D^l_v E^s_v)(k^s_v - k^l_u)A^s_v e^{-ik^s_v L/2} - F_4(E^l_u D^s_v - D^l_u E^s_v)(k^s_v - k^l_v)A^s_v e^{-ik^s_v L/2} \\
&+ F_3(E^l_v D^s_u - D^l_v E^s_u)(k^s_u - k^l_u)A^s_u e^{-ik^s_u L/2} - F_4(E^l_u D^s_u - D^l_u E^s_u)(k^s_u - k^l_v)A^s_u e^{-ik^s_u L/2} \\
&+ F_3(E^l_v D^s_+ - D^l_v E^s_+)(k^s_+ - k^l_u)A^s_+ e^{-ik^s_+ L/2} - F_4(E^l_u D^s_+ - D^l_u E^s_+)(k^s_+ - k^l_v)A^s_+ e^{-ik^s_+ L/2},
\end{aligned} \tag{B.1.23}$$

$$\begin{aligned}
&\overbrace{[F_3(E^l_v D^s_+ - D^l_v E^s_+)(k^s_+ - k^l_u) - F_4(E^l_u D^s_+ - D^l_u E^s_+)(k^s_+ - k^l_v)]e^{-ik^s_+ L/2}}^{D_4} A^s_+ \\
=&\underbrace{[F_3(E^l_v D^l_4 - D^l_v E^l_4)(k^l_4 - k^l_u) - F_4(E^l_u D^l_4 - D^l_u E^l_4)(k^l_4 - k^l_v)]e^{-ik^l_4 L/2}}_{N^1_4} \\
&\underbrace{-[F_3(E^l_v D^s_v - D^l_v E^s_v)(k^s_v - k^l_u) - F_4(E^l_u D^s_v - D^l_u E^s_v)(k^s_v - k^l_v)]e^{-ik^s_v L/2}}_{N^2_4} A^s_v + \\
&\underbrace{-[F_3(E^l_v D^s_u - D^l_v E^s_u)(k^s_u - k^l_u) - F_4(E^l_u D^s_u - D^l_u E^s_u)(k^s_u - k^l_v)]e^{-ik^s_u L/2}}_{N^3_4} A^s_u,
\end{aligned} \tag{B.1.24}$$

thus getting an expression for A^s_+ as function of A^s_v and A^s_u:

$$A^s_+ = \frac{N^1_4 + N^2_4 A^s_v + N^3_4 A^s_u}{D_4}. \tag{B.1.25}$$

B.2 Sonic-subsonic side

The plane-waves expansion for mode functions ϕ_ω and φ_ω in the sonic region is

$$\phi^s_\omega = e^{-i\omega t}[A^s_v D^s_v e^{ik^s_v x} + A^s_u D^s_u e^{ik^s_u x} + A^s_- D^s_- e^{ik^s_- x} + A^s_+ D^s_+ e^{ik^s_+ x}], \tag{B.2.1}$$

$$\varphi^s_\omega = e^{-i\omega t}[A^s_v E^s_v e^{ik^s_v x} + A^s_u E^s_u e^{ik^s_u x} + A^s_- E^s_- e^{ik^s_- x} + A^s_+ E^s_+ e^{ik^s_+ x}]. \tag{B.2.2}$$

while in the subsonic region

$$\phi^r_\omega = e^{-i\omega t}[\overbrace{A^r_v}^{=0} D^r_v e^{ik^r_v x} + A^r_u D^r_u e^{ik^r_u x} + A^r_+ D^r_+ e^{ik^r_+ x} + \overbrace{A^r_-}^{=0} D^r_- e^{ik^r_- x}], \tag{B.2.3}$$

$$\varphi^r_\omega = e^{-i\omega t}[\overbrace{A^r_v}^{=0} E^r_v e^{ik^r_v x} + A^r_u E^r_u e^{ik^r_u x} + A^r_+ E^r_+ e^{ik^r_+ x} + \overbrace{A^r_-}^{=0} E^r_- e^{ik^r_- x}]. \tag{B.2.4}$$

For $\phi^{in}_{4,l}$ mode, we set $A^r_v = 0$ (there are no other incoming waves besides k^l_4 one) and $A^r_- = 0$ (we neglect growing waves at infinity). The matching conditions at the right discontinuity surface − i.e. at $x = +L/2$, where $L =$ length of the sonic region − are

$$\begin{aligned}
&A^s_v D^s_v e^{ik^s_v L/2} + A^s_u D^s_u e^{ik^s_u L/2} + A^s_- D^s_- e^{ik^s_- L/2} + A^s_+ D^s_+ e^{ik^s_+ L/2} \\
&= A^r_u D^r_u e^{ik^r_u L/2} + A^r_+ D^r_+ e^{ik^r_+ L/2}
\end{aligned} \tag{B.2.5}$$

$$\begin{aligned}
&k^s_v A^s_v D^s_v e^{ik^s_v L/2} + k^s_u A^s_u D^s_u e^{ik^s_u L/2} + k^s_- A^s_- D^s_- e^{ik^s_- L/2} + k^s_+ A^s_+ D^s_+ e^{ik^s_+ L/2} \\
&= k^r_u A^r_u D^r_u e^{ik^r_u L/2} + k^r_+ A^r_+ D^r_+ e^{ik^r_+ L/2},
\end{aligned} \tag{B.2.6}$$

$$\begin{aligned}
&A^s_v E^s_v e^{ik^s_v L/2} + A^s_u E^s_u e^{ik^s_u L/2} + A^s_- E^s_- e^{ik^s_- L/2} + A^s_+ E^s_+ e^{ik^s_+ L/2} \\
&= A^r_u E^r_u e^{ik^r_u L/2} + A^r_+ E^r_+ e^{ik^r_+ L/2}
\end{aligned} \tag{B.2.7}$$

$$\begin{aligned} &k_v^s A_v^s E_v^s e^{ik_v^s L/2} + k_u^s A_u^s E_u^s e^{ik_u^s L/2} + k_-^s A_-^s E_-^s e^{ik_-^s L/2} + k_+^s A_+^s E_+^s e^{ik_+^s L/2} \\ &= k_u^r A_u^r E_u^r e^{ik_u^r L/2} + k_+^r A_+^r E_+^r e^{ik_+^r L/2}. \end{aligned} \tag{B.2.8}$$

From the left-side matching equations, we have got

$$\begin{cases} A_-^s = \frac{N_3^1 + N_3^2 A_v^s + N_3^3 A_u^s}{D_3}, \\ A_+^s = \frac{N_4^1 + N_4^2 A_v^s + N_4^3 A_u^s}{D_4}. \end{cases}$$

Therefore, substituting into the right-side system, we have

$$\alpha A_v^s + \beta A_u^s + \gamma + \delta = A_u^r D_u^r e^{ik_u^r L/2} + A_+^r D_+^r e^{ik_+^r L/2} \tag{B.2.9}$$

$$\alpha' A_v^s + \beta' A_u^s + \gamma' + \delta' = k_u^r A_u^r D_u^r e^{ik_u^r L/2} + k_+^r A_+^r D_+^r e^{ik_+^r L/2} \tag{B.2.10}$$

$$\tilde{\alpha} A_v^s + \tilde{\beta} A_u^s + \tilde{\gamma} + \tilde{\delta} = A_u^r E_u^r e^{ik_u^r L/2} + A_+^r E_+^r e^{ik_+^r L/2} \tag{B.2.11}$$

$$\tilde{\alpha}' A_v^s + \tilde{\beta}' A_u^s + \tilde{\gamma}' + \tilde{\delta}' = k_u^r A_u^r E_u^r e^{ik_u^r L/2} + k_+^r A_+^r E_+^r e^{ik_+^r L/2}, \tag{B.2.12}$$

where the following quantities have been defined

$$\begin{aligned} \alpha &\equiv D_v^s e^{ik_v^s L/2} + \frac{N_3^2}{D_3} D_-^s e^{ik_-^s L/2} + \frac{N_4^2}{D_4} D_+^s e^{ik_+^s L/2}, \\ \beta &\equiv D_u^s e^{ik_u^s L/2} + \frac{N_3^3}{D_3} D_-^s e^{ik_-^s L/2} + \frac{N_4^3}{D_4} D_+^s e^{ik_+^s L/2}, \\ \gamma &\equiv \frac{N_3^1}{D_3} D_-^s e^{ik_-^s L/2}, \quad \delta \equiv \frac{N_4^1}{D_4} D_+^s e^{ik_+^s L/2}, \end{aligned}$$

and

$$\begin{aligned} \alpha' &\equiv k_v^s D_v^s e^{ik_v^s L/2} + k_-^s \frac{N_3^2}{D_3} D_-^s e^{ik_-^s L/2} + k_+^s \frac{N_4^2}{D_4} D_+^s e^{ik_+^s L/2}, \\ \beta' &\equiv k_u^s D_u^s e^{ik_u^s L/2} + k_-^s \frac{N_3^3}{D_3} D_-^s e^{ik_-^s L/2} + k_+^s \frac{N_4^3}{D_4} D_+^s e^{ik_+^s L/2}, \\ \gamma' &\equiv k_-^s \frac{N_3^1}{D_3} D_-^s e^{ik_-^s L/2}, \quad \delta' \equiv k_+^s \frac{N_4^1}{D_4} D_+^s e^{ik_+^s L/2}. \end{aligned}$$

The corresponding quantities with ' $\sim$ ' on top contain E Bogoliubov weights in place of D ones.

Notice that now we have 4 equations in 4 unknowns, therefore we can completely solve the system. We shall concentrate on A_u^r amplitude solution, which yields the number of created phonons outside the sonic horizon and the analogue Hawking temperature. We will follow a similar algorithm as the one for the left-side system.

1) Isolation
From (B.2.9),

$$A_+^r D_+^r e^{ik_+^r L/2} = \alpha A_v^s + \beta A_u^s + \gamma + \delta - A_u^r D_u^r e^{ik_u^r L/2}, \tag{B.2.13}$$

and from (B.2.11)

$$A_+^r E_+^r e^{ik_+^r L/2} = \tilde{\alpha} A_v^s + \tilde{\beta} A_u^s + \tilde{\gamma} + \tilde{\delta} - A_u^r E_u^r e^{ik_u^r L/2}. \tag{B.2.14}$$

2) Plug-in
From (B.2.10)

$$\begin{aligned} &A_v^s(\alpha' - k_+^r \alpha) + A_u^s(\beta' - k_+^r \beta) + [(\gamma' - k_+^r \gamma) + (\delta' - k_+^r \delta)] \\ &= (k_u^r - k_+^r) A_u^r D_u^r e^{ik_u^r L/2}, \end{aligned} \tag{B.2.15}$$

and from (B.2.12),

$$\begin{aligned} &A_v^s(\tilde{\alpha}' - k_+^r \tilde{\alpha}) + A_u^s(\tilde{\beta}' - k_+^r \tilde{\beta}) + [(\tilde{\gamma}' - k_+^r \tilde{\gamma}) + (\tilde{\delta}' - k_+^r \tilde{\delta})] \\ &= (k_u^r - k_+^r) A_u^r E_u^r e^{ik_u^r L/2}. \end{aligned} \tag{B.2.16}$$

3) Combination via D_+^r and E_+^r

$$E_+^r \times \text{(B.2.13)} - D_+^r \times \text{(B.2.14)} :$$

$$\begin{aligned} 0 = &(E_+^r \alpha - D_+^r \tilde{\alpha}) A_v^s + (E_+^r \beta - D_+^r \tilde{\beta}) A_u^s \\ &+ (E_+^r \gamma - D_+^r \tilde{\gamma}) + (E_+^r \delta - D_+^r \tilde{\delta}) \\ &- (E_+^r D_u^r - D_+^r E_u^r) A_u^r e^{ik_u^r L/2}. \end{aligned} \tag{B.2.17}$$

4) Eliminate A_v^s contribution (first final equation)

$$(\tilde{\alpha}' - k_+^r \tilde{\alpha}) \times \text{(B.2.15)} - (\alpha' - k_+^r \alpha) \times \text{(B.2.16)} :$$

$$F_{us1} A_u^s + F_{x1} = F_{ur1} A_u^r, \tag{B.2.18}$$

where we have defined

$$F_{us1} \equiv (\tilde{\alpha}' - k_+^r \tilde{\alpha})(\beta' - k_+^r \beta) - (\alpha' - k_+^r \alpha)(\tilde{\beta}' - k_+^r \tilde{\beta}), \tag{B.2.19}$$

$$\begin{aligned} F_{x1} \equiv &(\tilde{\alpha}' - k_+^r \tilde{\alpha})[(\gamma' - k_+^r \gamma) + (\delta' - k_+^r \delta)] \\ &- (\alpha' - k_+^r \alpha)[(\tilde{\gamma}' - k_+^r \tilde{\gamma}) + (\tilde{\delta}' - k_+^r \tilde{\delta})], \end{aligned} \tag{B.2.20}$$

$$F_{ur1} \equiv [(\tilde{\alpha}' - k_+^r \tilde{\alpha}) D_u^r - (\alpha' - k_+^r \alpha) E_u^r](k_u^r - k_+^r) e^{ik_u^r L/2}. \tag{B.2.21}$$

5) Eliminate A_v^s contribution (second final equation)

$$(E_+^r \alpha - D_+^r \tilde{\alpha}) \times [\text{(B.2.15)} + \text{(B.2.16)}] + [(\alpha' - k_+^r \alpha) + (\tilde{\alpha}' - k_+^r \tilde{\alpha})] \times \text{(B.2.17)} :$$

$$F_{us2} A_u^s + F_{x2} = F_{ur2} A_u^r, \tag{B.2.22}$$

where we have defined

$$F_{us2} \equiv (E_+^r \alpha - D_+^r \tilde{\alpha})[(\beta' + \tilde{\beta}') - k_+^r(\beta + \tilde{\beta})] \\ - [(\alpha' - k_+^r \alpha) + (\tilde{\alpha}' - k_+^r \tilde{\alpha})](E_+^r \beta - D_+^r \tilde{\beta}), \tag{B.2.23}$$

$$F_{x2} \equiv (E_+^r \alpha - D_+^r \tilde{\alpha})[(\gamma' + \delta' + \tilde{\gamma}' + \tilde{\delta}') - k_+^r(\gamma + \delta + \tilde{\gamma} + \tilde{\delta})] \\ - [(\alpha' - k_+^r \alpha) + (\tilde{\alpha}' - k_+^r \tilde{\alpha})](E_+^r(\gamma + \delta) - D_+^r(\gamma' + \delta')), \tag{B.2.24}$$

$$F_{ur2} \equiv \{(E_+^r \alpha - D_+^r \tilde{\alpha})(D_u^r + E_u^r)(k_u^r - k_+^r) - [(\alpha' - k_+^r \alpha) + (\tilde{\alpha}' - k_+^r \tilde{\alpha})](E_+^r D_u^r - D_+^r E_u^r)\} e^{ik_u^r L/2}. \tag{B.2.25}$$

6) <u>Calculate A_u^r via combination of the final equations</u>

$$F_{us2} \times (\text{B.2.18}) - F_{us1} \times (\text{B.2.22}) :$$

$$\cancel{F_{us2} F_{us1} A_u^s} + F_{us2} F_{x1} - \cancel{F_{us1} F_{us2} A_u^s} - F_{us1} F_{x2} = F_{us2} F_{ur1} A_u^r - F_{us1} F_{ur2} A_u^r$$

$$\implies A_u^r \underbrace{(F_{us2} F_{ur1} - F_{us1} F_{ur2})}_{D_{ur}} = \underbrace{F_{us2} F_{x1} - F_{us1} F_{x2}}_{N_{ur}}$$

$$\implies A_u^r = \frac{N_{ur}}{D_{ur}}. \tag{B.2.26}$$

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