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Minimal Length Scale and Generalized Uncertainty Principle

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Abstract

L'esistenza di una lunghezza minima, dell'ordine della lunghezza di Planck $l_{Pl} = 1.6 \times 10^{-35}$ m, pone un limite alla piccolezza delle distanze che possiamo misurare. Tale lunghezza minima può essere ottenuta con una modificazione del principio di indeterminazione di Heisenberg in un principio di indeterminazione generalizzato (generalized uncertainty principle o GUP). In ciò che segue, dapprima vengono analizzate le diverse motivazioni che suggeriscono l'esistenza di una lunghezza minima: esperimenti mentali, geometria non-commutativa, teorie della gravità quantistica. Successivamente viene mostrato come il GUP possa essere implementato a partire da una modifica delle relazioni di commutazione canoniche e si indaga come esso vada ad agire sul limite Newtoniano della relatività generale. Infine, esaminando l'influenza del GUP sull'Hamiltoniana di una particella e sulla temperatura della radiazione di Hawking, che a sua volta implica una deformazione della metrica di Schwarzschild, si osserva come tali risultati possano essere utilizzati per stimare il parametro di deformazione del principio di indeterminazione e quindi la scala di lunghezza minima.

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Chapter 1

Introduction

Matter is made of atoms. Nowadays this is common knowledge but it took a long time since this was first postulated till it actually gained scientific value: the history of atomism has its roots in ancient Greece and it was only in the 19th century that evidence showed that atoms were real, physical objects. Atomism as a philosophy was first introduced by a group of Greek philosophers, called atomists, who proposed, in different ways, that matter is made of fundamental entities, the *atoms*, from the prefix ”a-” which is a negation and "tomos" which means "cut", thus atoms are indivisible, not composed of anything smaller. The most famous atomist is Democritus (5th century B.C.), according to which atoms of different shapes and sizes but made of the same substance combined together to form all different materials and bodies.

We currently know that although matter is, in fact, made of atoms of different shapes and sizes (and even made of the same substances), these aren't indivisible: atoms are made of electrons orbiting around a nucleus, nuclei are made of protons and neutrons, protons and neutrons are made of quarks bound together by the strong interaction and, as far as we know, quarks and electrons are elementary particles, i.e. they are point particle with no substructure. Atoms have a typical dimension of an Ångstrom 1 Å$= 10^{-10}$ m, nuclei are of the order of a femtometer 1 fm $= 10^{-15}$ m, protons and neutrons of a fraction of a femtometer. With the current technology we can put an upper bound on the dimension of electrons and quarks of about 10^{-18} m.

However, we could wonder if, with an improvement of our equipment which would allow us to investigate smaller distances, also electrons and quarks would reveal a smaller substructure. Naturally it arises the question if the process of probing smaller and smaller distances is virtually infinite or if there is a limit below which nothing exists and our currently understanding of physics ceases to make sense.

Evidences for the existence of a minimal length, of the order of the Planck length l_{Pl}, are multiple and emerging from different arguments in physics. In 1899 Max Planck [1] came up with a set of natural units, called Planck units, in an attempt to find units of measurement that were universal, that even an extraterrestrial civilization could use, stripped of the anthropocentrism inherent to the definition of the meter, the second, the kilogram, the Kelvin. These units are formulated in terms of universal physical constants: the speed of light c, the reduced Planck constant $\hbar$, the gravitational constant G and the Boltzmann constant k_B. By means of a dimensional analysis, using only these constants, he built units of mass, length, time and temperature:

$$\begin{aligned} m_{Pl} &= \sqrt{\frac{\hbar c}{G}} \approx 10^{-8} \text{ kg}, \\ l_{Pl} &= \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-35} \text{ m}, \\ t_{Pl} &= \sqrt{\frac{\hbar G}{c^5}} \approx 10^{-43} \text{ s}, \\ T_{Pl} &= \sqrt{\frac{\hbar c^5}{G k_B^2}} \approx 10^{32} \text{ K}. \end{aligned} \tag{1.1}$$

The first concerns about the necessity of a minimal length arose with the advent of quantum field theory, in which divergences showed up. It was believed that a minimal length of the order of a femtometer could be used as a cut-off to resolve these divergences. In particular, W. K. Heisenberg thought that spacetime could be quantized, thus leading to the cut-off. He even speculated that this quantization could come from letting the position operators be non-commuting $[\hat{x}^\mu, \hat{x}^\nu] \neq 0$, a possibility we will investigate in Section 3.2. However, these divergences were solved with renormalizations techniques and thus the hypothesis of a cut-off was abandoned.

In the meantime M. Bronstein, in 1935, was the first to understand that because condensing more and more energy inside a region leads eventually to the formation of a black hole, it follows that there exists an uncertainty on position even greater than the usual one coming from the Heisenberg uncertainty principle. Unfortunately his work got little attention and few years later, in 1938, during the Stalinian Great Purge, he was executed.

It was only in 1964 that the idea of a minimal length showed up again. C. A. Mead, with a series of thought experiments that will be analyzed in Section 3.1, in which quantum and gravitational effects were considered [2], found out that there is a minimal uncertainty of the order of the Planck length associated with distance measurements. Mead's work was met with skepticism, as himself stated in a letter he wrote in 2001 [3]: "years of referee trouble, eventual publication, a cold shoulder from the physics community". It took five years from when he first submitted its paper in 1959 till it was finally published.

The presence of a minimal length scale has been more and more studied over the years. This length could mark the ultimate limit we could push our understanding of the quantum realm. It could also be used as a cut-off that would cure the non-renormalization of the gravitational field. A particular attention has been put on the *generalized uncertainty principle* or GUP, which is a modification of the Heisenberg uncertainty principle of the form:

$$\Delta x \Delta p \geq \frac{\hbar}{2}(1 + \beta \Delta p^2), \tag{1.2}$$

where $\beta = \beta_0 / m_{Pl}^2$ is called the deformation parameter. Minimizing Δx we find a minimal length of $l_{Pl}\sqrt{\beta_0}$. We expect β_0 to be of the order of the unity, so that the minimal length is of the order of the Planck length. However this is a free parameter whose value has to be constrained by experiments. A first estimation of this parameter can already be made: as we said earlier, the shortest distance currently measured is of the order of 10^{-18} m, thus β_0 has to be smaller than 10^{34}.

In Section 2 we introduce shortly the concepts of state, observable and measurement in quantum mechanics, which we need to understand the Heisenberg uncertainty principle. In Section 3 we review some of the motivations that lead to the GUP and the minimal length scale: thought experiments, non-commutative geometry and a short mention of quantum gravity theories. In Section 4 we see how the GUP can be obtained by a modification of the canonical commutation relations and how the classical limit is affected. Then in Section 5 we analyze different approaches to the estimation of the deformation parameter appearing in the GUP. We reserve Section 6 for conclusions.

Chapter 2

Basics of quantum mechanics

2.1 States, observables and measurements

We start by giving a rather intuitive definition for the concepts of state, observable and measurement that constitute the fundamental framework of quantum mechanics and then see how these can be mathematically modeled. We don't pretend this review to be exhaustive, we will only focus on those concepts that we must know in order to derive and understand the Heisenberg uncertainty principle.

- The *states* of a system are the physical conditions or modes of being of the system, for any state s of the system there exists a device that prepares the system in that state;
- the *observables* of a system are the numerically quantifiable physical properties of the system, for every observable Q of the system there exists an apparatus that furnishes the value of the observable;
- a *measurement* is the combination of instruments and experimental procedures that returns the value of an observable of the system in a certain state, a measurement procedure $P(s, Q)$ consists in preparing the system in a state s and then detecting the value x of the observable Q with a suitable apparatus.

Given a system in a state s and an observable Q, repeating the measurement procedure $P(s, Q)$ we get a series of values x_i for Q. In general these values will be different from each other and we can only know the probability $p(s, Q, I)$ that a value of the observable falls in a certain range I: for a generic state s the outcome of the measurement procedure $P(s, Q)$ is a *statistical process*. If instead all the values are the same then the state s' for which this happens is called an *eigenstate*: there exists a function that assigns to each eigenstate s' of the observable Q the value $x(s', Q)$ of Q in that state, called *eigenvalue*.

We note, however, that the measurement procedure always returns a value for the observable, even if the system is not in an eigenstate. Because a value can be assigned to an observable only if the system is in an eigenstate, we thus infer that the measurement procedure first takes the system in an eigenstate s' of the observable Q and then gives the value $x = x(s', Q)$ of the observable. This process is called *state reduction*.

A generic state s can then be seen as a *superposition* of eigenstates s_i with different weights, which determine the probability that a measurement procedure $P(s, Q)$ gives the value $x_i(s_i, Q)$ of Q. The states of a collection of states s_i are said *independent* if none of them is a superposition of the others. A collection of independent states is said to be *complete*.

Recalling that the measurement procedure $P(s, Q)$ is a statistical process we can define the *expectation value* of Q in the state s as:

$$\langle Q \rangle_s = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} x_i, \tag{2.1}$$

and the associated *uncertainty* is:

$$\Delta Q_s = \left[\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} (x_i - \langle Q \rangle_s)^2 \right]^{1/2}. \tag{2.2}$$

If we try to carry out two measurements $P(s, Q), P(s, R)$ for two different observables Q, R simultaneously we force the reduction of the state of the system to a common eigenstate of the two observables. Generally such a common eigenstate does not exist and the two measurements *interfere* with each other: this implies that the product of the uncertainties $\Delta Q_s \Delta R_s$ cannot vanish, it results to be bounded from below.

The observables of a set of observables Q_i are said *compatible* if they can be measured simultaneously without interference. If, repeating the simultaneous measurement procedures on the system in the state s we obtain always the same values for each compatible observable then s is said to be a *simultaneous eigenstate* of all the observables.

2.2 Mathematical formalism

This physical phenomenology has to be modelled by a suitable mathematical apparatus. The states s of a system are represented by kets $|\psi\rangle$ or, equivalently, by bras $\langle\psi|$, which can be seen as vectors defined, up to a phase factor, in a Hilbert space $\mathcal{H}$ with inner product $\langle \psi, \psi \rangle = \langle \psi | \psi \rangle = \int d^3x \psi^* \psi < \infty$. More precisely, because we can multiply such vectors for a complex number with absolute value of 1, there is a one–to–one correspondence between the states of the system and 1-dimensional subspaces of the Hilbert space called "rays". A superposition of states s_i is a linear combination of kets $|\psi_i\rangle$. A complete collection of linearly independent kets constitutes an orthonormal basis of kets. The observables Q are represented by selfadjoint operators $\hat{A}$. Eigenstates and eigenvalues are determined by the *eigenvalue problem*:

$$\hat{A} |\psi\rangle = x |\psi\rangle . \tag{2.3}$$

Generally this problem has few solutions as long as we require that the eigenkets belong to the Hilbert space $\mathcal{H}$. We must then allow the kets to belong to a vector space $\mathcal{H}'$ containing $\mathcal{H}$ such that $\langle\psi|\psi\rangle = \infty$. However there is a restriction on vectors of $\mathcal{H}'$, they have to be approximable by a sequence of vectors of $\mathcal{H}$:

$$|\psi\rangle = \lim_{n\to\infty} |\psi_n\rangle \,, \tag{2.4}$$

with $\langle\psi_n|\psi_n\rangle < \infty$.

The mean value of an observable Q in the state s is:

$$\langle Q\rangle_s = \langle\psi|\hat{A}|\psi\rangle \,, \tag{2.5}$$

and the uncertainty is:

$$\Delta Q_s = \Big[\, \langle\psi|(\hat{A} - \langle\psi|\hat{A}|\psi\rangle\, \hat{1})^2|\psi\rangle \Big]^{1/2} . \tag{2.6}$$

Two observables are compatible if the commutator of the corresponding operators vanishes: $[\hat{A},\hat{B}] = 0$. If the commutator doesn't vanish then the product of the uncertainties is bounded from below, leading to the Heisenberg uncertainty principle.

2.3 The Heisenberg uncertainty principle

The *Heisenberg uncertainty principle* states that when we try to measure simultaneously two observables A and B, represented by the selfadjoint operators $\hat{A}$ and $\hat{B}$, the product of their uncertainties is bounded from below:

$$\Delta A \Delta B \geq \left| \frac{1}{2} \langle\psi|i[\hat{A},\hat{B}]|\psi\rangle \right|, \tag{2.7}$$

where the quantity $\langle\psi|i[\hat{A},\hat{B}]|\psi\rangle$ is real. If the observables are compatible than the commutator $[\hat{A},\hat{B}]$ vanishes and so does the product of the uncertainties.

We can demonstrate this result as follows. Let us define two operators $\hat{U},\hat{V}$ as:

$$\hat{U} = \hat{A} - \langle\psi|\hat{A}|\psi\rangle\, \hat{1}, \tag{2.8}$$

$$\hat{V} = \hat{B} - \langle\psi|\hat{B}|\psi\rangle\, \hat{1}. \tag{2.9}$$

Combining them we define the ket $|\phi\rangle$ as:

$$|\phi\rangle = (\hat{U} + i\lambda\hat{V})\,|\psi\rangle \,, \tag{2.10}$$

where λ is a real parameter that can be obtained using the property of the bra-ket product $\langle\phi|\phi\rangle \geq 0$ and by noting that $[\hat{U}, \hat{V}] = [\hat{A}, \hat{B}]$:

$$\begin{aligned}
0 \leq \langle\phi|\phi\rangle &= \langle\psi|(\hat{U} - i\lambda\hat{V})(\hat{U} + i\lambda\hat{V})|\psi\rangle \\
&= \langle\psi|\hat{U}^2 + i\lambda\hat{U}\hat{V} - i\lambda\hat{V}\hat{U} + \lambda^2\hat{V}^2|\psi\rangle \\
&= \langle\psi|(\hat{A} - \langle\psi|\hat{A}|\psi\rangle\,\hat{1})^2|\psi\rangle + \lambda\,\langle\psi|i[\hat{U}, \hat{V}]|\psi\rangle + \lambda^2\,\langle\psi|(\hat{B} - \langle\psi|\hat{B}|\psi\rangle\,\hat{1})^2|\psi\rangle \\
&= \Delta A^2 + \lambda\,\langle\psi|i[\hat{A}, \hat{B}]|\psi\rangle + \lambda^2\Delta B^2.
\end{aligned} \tag{2.11}$$

Deriving with respect to lambda:

$$0 = \frac{\partial}{\partial\lambda}\langle\phi|\phi\rangle = \langle\psi|i[\hat{A}, \hat{B}]|\psi\rangle + 2\lambda\Delta B^2, \tag{2.12}$$

we find a minimal value for:

$$\lambda = -\frac{\langle\psi|i[\hat{A}, \hat{B}]|\psi\rangle}{2\Delta B^2}. \tag{2.13}$$

This parameter is real because the operator $i[\hat{A}, \hat{B}]$ is selfadjoint:

$$\begin{aligned}
(i[\hat{A}, \hat{B}])^\dagger &= -i(\hat{A}\hat{B} - \hat{B}\hat{A})^\dagger \\
&= -i(\hat{B}^\dagger\hat{A}^\dagger - \hat{A}^\dagger\hat{B}^\dagger) \\
&= -i(\hat{B}\hat{A} - \hat{A}\hat{B}) \\
&= -i[\hat{B}, \hat{A}] \\
&= i[\hat{A}, \hat{B}].
\end{aligned} \tag{2.14}$$

Upon inserting (2.13) in (2.11) we obtain:

$$\Delta A^2\Delta B^2 \geq -\frac{1}{4}(\langle\psi|[\hat{A}, \hat{B}]|\psi\rangle)^2, \tag{2.15}$$

and finally we recover the relation:

$$\Delta A\Delta B \geq \frac{1}{2}\Big|\langle\psi|i[\hat{A}, \hat{B}]|\psi\rangle\Big|. \tag{2.16}$$

Using relation (2.16) for position and momentum operators $\hat{q}, \hat{p}$ and knowing that their canonical commutation relation is $[\hat{q}, \hat{p}] = i\hbar\hat{1}$, the uncertainty principle takes the simple form:

$$\Delta q\Delta p \geq \frac{1}{2}|\langle\psi|i[\hat{q}, \hat{p}]|\psi\rangle| = \frac{\hbar}{2}, \tag{2.17}$$

which states that if we decrease the uncertainty on the position of the particle then we increase the uncertainty on its momentum and vice versa. It is important to underline that the uncertainty principle is not due to the interaction between the system and the experimental setup we use to investigate it but is an intrinsic property of the system itself.

Chapter 3

Minimal length scale and GUP

A minimal length scale and the generalized uncertainty principle arise from various arguments in physics. From thought experiments to more formal topics, for example, non-commutative geometry and theories of quantum gravity, the presence of a minimal length scale seems to be inevitable. In what follows we provide a series of arguments that show how and when the GUP and the minimal length scale fall into the picture. This analysis is largely based on the review given by S. Hossenfelder in [4], occasionally referring to the original papers when deemed appropriate.

3.1 Thought experiments

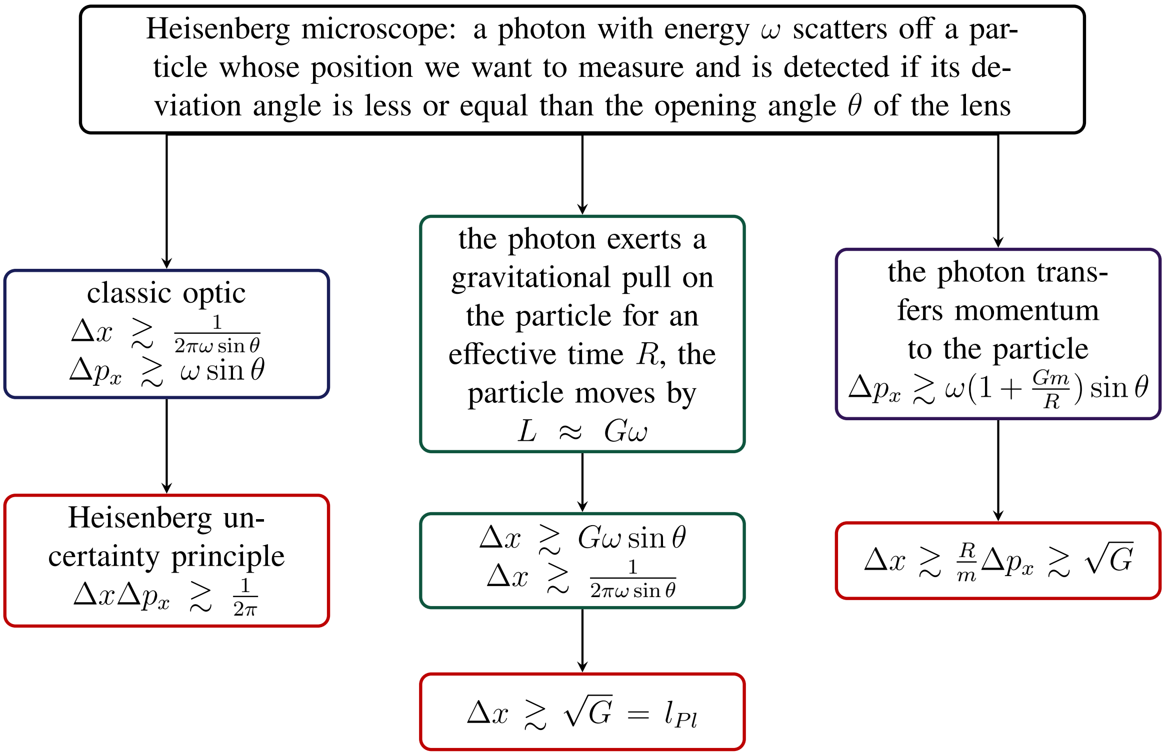
3.1.1 Heisenberg microscope in Newtonian gravity

The Heisenberg microscope is a thought experiment whose aim was to provide an intuitive explanation of the uncertainty principle. In this approach the gravitational interaction between the particles in exam is usually neglected. However, if we instead include gravitational effects, we will also be able to obtain a form of a generalized uncertainty principle, which modifies the resolution Δx we can achieve. In this section we use the convention $c = \hbar = 1$, show explicitly the gravitational constant G and use the Minkowski metric with signature $(+1, -1, -1, -1)$. We want to measure the position on the x-axis of a particle, in order to do so we send a photon with angular frequency ω against the measured particle, the photon scatters and then reaches the lens of the microscope if its deviation angle is less or equal than the opening angle θ (see Fig. 3.2). From classical optics we know that the resolution Δx of the position is limited by the wavelength of the photon $\lambda = 2\pi/\omega$:

$$\Delta x \gtrsim \frac{\lambda}{\sin\theta} > \frac{1}{2\pi\omega\sin\theta}. \tag{3.1}$$

The photon transfers a momentum to the measured particle. Since we do not know the direction of the photon better than θ, the momentum of the particle in the x direction will have

Figure 3.1: Heisenberg microscope in Newtonian gravity: workflow



an uncertainty of:

$$\Delta p_x \gtrsim \omega\sin\theta. \tag{3.2}$$

By multiplying the two uncertainties we finally obtain Heisenberg's principle:

$$\Delta x \Delta p_x \gtrsim \frac{1}{2\pi}. \tag{3.3}$$

Not knowing precisely the position of the measured particle, it is more accurate to consider its interaction with the photon not in a particular point but rather in a region of an appropriate size R. The photon cannot be focused while it is interacting with the particle in the region R, so the time between the scattering of the photon and the subsequent measurement has to be at least $\tau \gtrsim R$. We now take into account gravity and consider the measured particle to be non-relativistic: the photon carries a certain amount of energy and thus exerts a gravitational pull on the particle, resulting in an acceleration which is, considering the Newtonian formula for the gravitational force:

$$a \approx \frac{G\omega}{R^2}. \tag{3.4}$$

Assuming the particle is much slower than the photon, this acceleration lasts for the time the

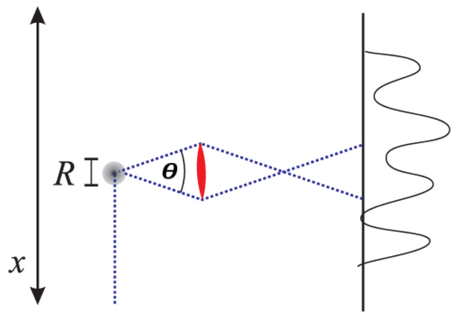


Figure 3.2: Heisenberg microscope. A photon moving along the x-axis scatters off a particle within an interaction region of size R and is detected by a microscope (the lens and screen) with an opening angle of θ.

photon is inside the interaction region, thus the velocity acquired by the particle is:

$$v \approx aR \approx \frac{G\omega}{R}, \tag{3.5}$$

and the distance travelled is:

$$L \approx vR \approx G\omega. \tag{3.6}$$

Since the direction of the photon was unknown within the angle θ then also the direction of the acceleration and thus of the motion of the particle are unknown. The projection of L on the x-axis gives then the additional uncertainty:

$$\Delta x \gtrsim G\omega \sin\theta. \tag{3.7}$$

Comparing the two expressions (3.1) and (3.7) we obtain:

$$\Delta x \gtrsim \sqrt{G} = l_{Pl}, \tag{3.8}$$

the uncertainty on the position of the particle is then comparable with the Planck length l_{Pl}.

We omitted to take also into account an increase in the particle's momentum $Gm\omega/R$ due to the interaction with the photon. Projecting it on the x-axis the uncertainty in the particle's momentum increases to:

$$\Delta p_x \gtrsim \omega \left(1 + \frac{Gm}{R}\right) \sin\theta, \tag{3.9}$$

which translates in a position uncertainty $\Delta x \gtrsim \tau \Delta v \gtrsim R\Delta p_x/m$:

$$\Delta x \gtrsim \omega \left(\frac{R}{m} + G\right) \sin\theta, \tag{3.10}$$

which is however larger than the uncertainty (3.7) and thus we once again find that:

$$\Delta x \gtrsim \sqrt{G} = l_{Pl}. \tag{3.11}$$

R. J. Adler and D. I. Santiago proposed a very similar argument [5] but sustained that the particle's momentum uncertainty Δp should be of the order of the photon's momentum ω, leading to:

$$\Delta x \gtrsim G \Delta p. \tag{3.12}$$

Upon assuming that the usual and the gravitational uncertainty add linearly we find:

$$\Delta x \gtrsim \frac{1}{\Delta p} + G \Delta p. \tag{3.13}$$

which is a generalized uncertainty principle.

3.1.2 Heisenberg microscope in General Relativity

We now review Heisenberg's microscope using general relativity following the approach of Mead [2]. We consider a generic test particle instead of a photon, with four-momentum $(\omega, \boldsymbol{k})$ and rest mass μ. The particle moves along the x-axis with velocity v:

$$v = \frac{p}{\omega} = \frac{k}{\sqrt{\mu^2 + k^2}}, \tag{3.14}$$

with $k^2 = \omega^2 - \mu^2$ following from the relativistic energy-momentum relation. We want to estimate how much the measured particle moves due to the gravitational pull of the test particle. In order to do so we move to the rest frame of the test particle by boosting in the x-direction: the measured particle now travels towards the test particle in the direction $-x'$, and we can use the Schwarzschild metric. Being on the x-axis we have $y = z = 0$ and then:

$$g'_{00} = 1 + 2\phi', \tag{3.15}$$

$$g'_{11} = -\frac{1}{g'_{00}}, \tag{3.16}$$

$$g'_{22} = g'_{33} = -1, \tag{3.17}$$

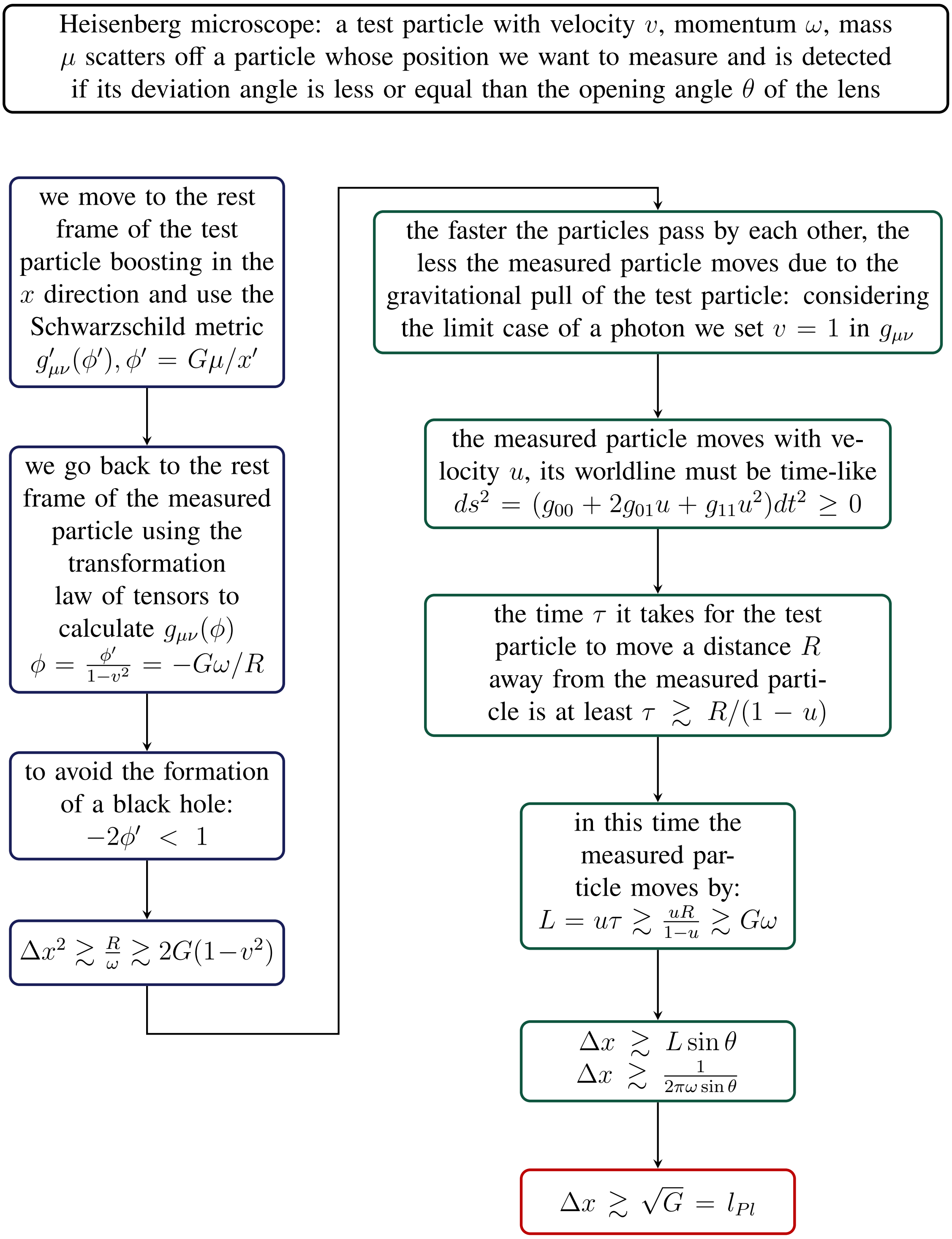
where:

$$\phi' = -\frac{G\mu}{|x'|} = \frac{G\mu}{x'}. \tag{3.18}$$

We now return to the rest frame of the measured particle using the general transformation law for tensors:

$$g_{\mu\nu} = \frac{\partial (x')^\alpha}{\partial x^\mu} \frac{\partial (x')^\beta}{\partial x^\nu} g'_{\alpha\beta}, \tag{3.19}$$

Figure 3.3: Heisenberg microscope in General Relativity: workflow



and so the metric becomes:

$$
\begin{aligned}
g_{00} &= \frac{1+2\phi}{1+2\phi(1-v^2)} + 2\phi, \\
g_{11} &= \frac{-1+2\phi v^2}{1+2\phi(1-v^2)} + 2v^2\phi, \\
g_{01} = g_{10} &= -\frac{2v\phi}{1+2\phi(1-v^2)} - 2v\phi, \\
g_{22} = g_{33} &= -1,
\end{aligned}
\tag{3.20}
$$

where:

$$
\phi = \frac{\phi'}{1-v^2} = -\frac{G\omega}{R}, \tag{3.21}
$$

$R = vt - x$ is the mean distance between the two particles. In order to avoid the collapse of the particle into a black hole we must impose that its mass is never compacted at any given time in a region with radius less than its Schwarzschild radius $R_H = 2G\mu$ (we assumed the particle has spherical symmetry), thus we must have:

$$
-2\phi' = 2\frac{G\omega}{R}(1-v^2) < 1, \tag{3.22}
$$

from which we find:

$$
\frac{1}{\omega} > \frac{2G}{R}(1-v^2). \tag{3.23}
$$

We know from (3.1) that $\Delta x \gtrsim 1/\omega$ but we also know that the uncertainty must be greater than the dimension of the interaction region $\Delta x \gtrsim R$ and thus:

$$
\Delta x^2 \gtrsim \frac{R}{\omega} \gtrsim 2G(1-v^2). \tag{3.24}
$$

If $v^2 \ll 1$, that is if the velocity is non-relativistic, we recover relation (3.8). We then analyze the case in which $1 - v^2 \ll 1$, that is when the test particle has relativistic velocity, and see that from (3.21) this means $-\phi \gg 1$.

We note that the faster the particles pass by each other, the shorter the interaction time will be and thus the less the measured particle moves due to the gravitational pull the test particle exerts on it. If we consider a photon as the test particle, we are in the case of least influence and thus if we find a minimal length in this case, it must be present for all other cases. We set then $v = 1$ in (3.20) and find:

$$
g_{00} = \frac{1+2\phi(1+\alpha)}{\alpha}, \tag{3.25}
$$

$$
g_{11} = \frac{-1+2\phi(1+\alpha)}{\alpha}, \tag{3.26}
$$

$$
g_{01} = g_{10} = -\frac{2\phi(1+\alpha)}{\alpha}, \tag{3.27}
$$

where:

$$\alpha = 1 + 2\phi(1 - v^2) = 1 - 2\frac{G\omega}{R}(1 - v^2). \tag{3.28}$$

From (3.22) follows that $0 < \alpha < 1$. We observe that the worldline of the measured particle must be *time-like*. Denoting its velocity along the x-axis as u we have:

$$ds^2 = g_{00}dt^2 + g_{01}dtdx + g_{10}dxdt + g_{11}dx^2 = (g_{00} + 2g_{01}u + g_{11}u^2)dt^2 \geq 0. \tag{3.29}$$

We then get [2]:

$$u \geq \frac{\eta - 1}{\eta + 1}, \tag{3.30}$$

where:

$$\eta = -2\phi(1 + \alpha). \tag{3.31}$$

The two particles interact until the test particle moves a distance R away from the measured particle. The velocity of the test particle cannot be greater than 1, thus the time τ it takes for the test particle to move a distance R away from the measured particle is at least $\tau \gtrsim R/(1 - u)$. In this time, the measured particle moves by:

$$L = u\tau \gtrsim \frac{uR}{1 - u} \geq \frac{R(\eta - 1)}{2} \sim \frac{R\eta}{2}, \tag{3.32}$$

since, considering the restrictions on ϕ and α, it results $\eta \gg 1$. We finally get:

$$L \gtrsim G\omega. \tag{3.33}$$

Projecting on the x-axis we have:

$$\Delta x \gtrsim G\omega \sin\theta, \tag{3.34}$$

and again, combining it with (3.1), we recover:

$$\Delta x \gtrsim \sqrt{G} = l_{Pl}. \tag{3.35}$$

Therefore, for every test particle we might consider, a minimal length of the order of the Planck length is present.

3.1.3 Limit to distance measurements

Consider we want to measure a length D by sending photons to a mirror and using a clock that detects them when they travel back: knowing the speed of light is universal we can measure the distance the photons travelled by measuring their travel-time. The position of the clock is known up to an uncertainty Δx, and thus the uncertainty on its velocity is:

$$\Delta v \sim \frac{1}{2M\Delta x}, \tag{3.36}$$

with M the mass of the clock. A photon takes the time $T = 2D$ to travel to the mirror and back, during which the clock moves by $T\Delta v$ and so the uncertainty on its position increases to:

$$\Delta x + \frac{T}{2M\Delta x}. \tag{3.37}$$

Varying with respect to Δx we find that the expression has a minimum for:

$$\Delta x_{min} = \sqrt{\frac{T}{2M}} = \sqrt{\frac{D}{M}}. \tag{3.38}$$

The distance D has to be greater than the Schwarzschild radius of the clock $D > 2GM$, otherwise it would collapse into a black hole, thus losing causal connection with the rest of the world, hence we recover:

$$\Delta x_{min} \gtrsim l_{Pl}. \tag{3.39}$$

3.1.4 Limit to clock synchronization

Consider we want to synchronize two clocks by sending photons from one of them to the other. The energy of the photon is known up to an uncertainty $\Delta\omega$, and thus the uncertainty on the time will be:

$$\Delta T \sim \frac{1}{2\Delta\omega}. \tag{3.40}$$

The clock and the photon interact in a region R for a time $\tau \gtrsim R$. If the clock remains stationary, the time it measures is $T = \tau\sqrt{g_{00}}$. From (3.20), setting $v = \alpha = 1$, we get:

$$T = \tau\sqrt{1 - \frac{4G\omega}{r}}. \tag{3.41}$$

The metric depends on the energy ω of the photon which has an uncertainty $\Delta\omega$, this error propagates into T by:

$$\Delta T = \left|\frac{\partial T}{\partial\omega}\right|\Delta\omega, \tag{3.42}$$

thus:

$$\Delta T = \frac{2G\tau}{r\sqrt{1 - 4G\omega/r}}\Delta\omega. \tag{3.43}$$

Considering that the interaction takes place within the region R we have $\tau \gtrsim R \gtrsim r$, then:

$$\Delta T \gtrsim \frac{2G}{\sqrt{1 - 4G\omega/R}}\Delta\omega \gtrsim 2G\Delta\omega. \tag{3.44}$$

Combining it with (3.40) we find:

$$\Delta T \gtrsim l_{Pl}. \tag{3.45}$$

We see in this way that the precision by which clocks can be synchronized is bound from below by the Planck length.

In general, however, the clock will move towards the photon due to the gravitational pull the latter exerts on it with a velocity u. The time it records is then:

$$T = \int ds \sim \tau \sqrt{g_{00} + 2g_{01}u + g_{11}u^2}. \tag{3.46}$$

Using $v = 1$ and $u \leq 1$ we estimate [4]:

$$\left| \frac{\partial T}{\partial \omega} \right| \gtrsim \tau \frac{8G}{r} \frac{1}{\sqrt{1 + 4G\omega/r}}, \tag{3.47}$$

and recalling that $\tau \gtrsim R \gtrsim r$, we get:

$$\Delta T \gtrsim \tau \frac{G}{R} \Delta\omega \gtrsim G\Delta\omega. \tag{3.48}$$

Combining once again this relation with (3.40) yields:

$$\Delta T \gtrsim l_{Pl}, \tag{3.49}$$

even in the general case the limitation on ΔT stands the same.

3.1.5 Device independent limit for non-relativistic particles

So far we have worked with a particular measurement apparatus. We could thus wonder if the limitations we found are due to the technological inefficiencies of the apparatus itself instead of being an intrinsic feature of the phenomena under exam. X. Calmet, M. L. Graesser and S. D. H. Hsu [6] found the limitation imposed by the Planck length using a device-independent argument, following just from the uncertainty principle and the formation of black holes as predicted by general relativity.

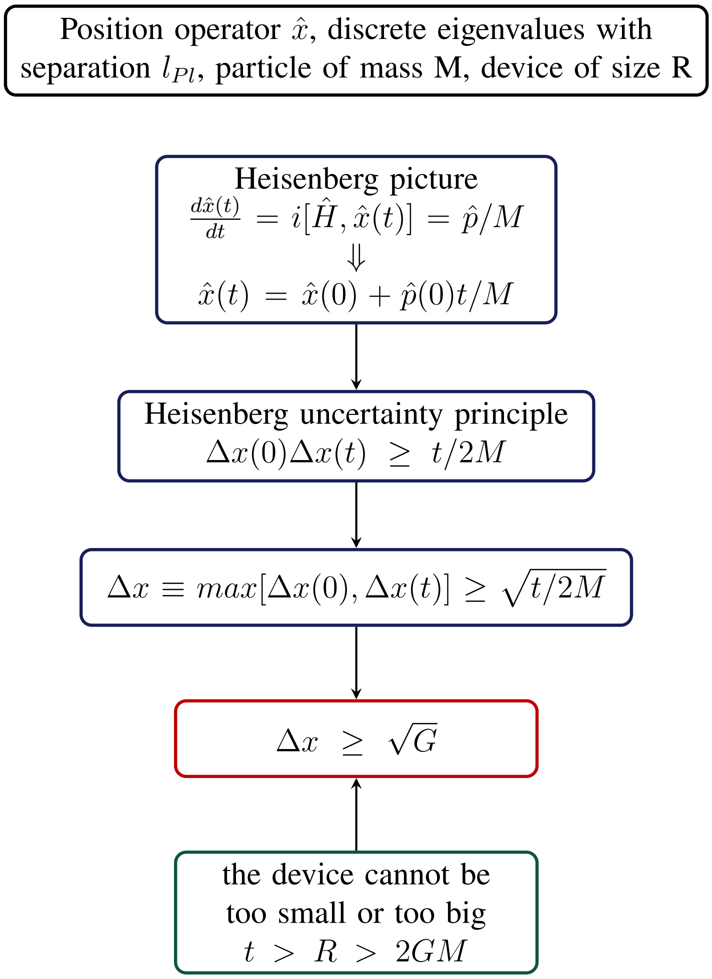
Let us consider a position operator $\hat{x}$ with discrete eigenvalues x, which have a separation of order l_{Pl} (equivalent to a spatial lattice). To rule out this model we would have to measure the distance between two subsequent eigenvalues x, x' of some test particle with mass M and find it to be smaller than Planck's length: $|x - x'| < l_{Pl}$. Using the non-relativistic Schrödinger equation, the time-evolution of the position operator is given by (in the Heisenberg picture):

$$\frac{d\hat{x}(t)}{dt} = i[\hat{H}, \hat{x}(t)] = \frac{\hat{p}}{M}, \tag{3.50}$$

and thus:

$$\hat{x}(t) = \hat{x}(0) + \hat{p}(0)\frac{t}{M}, \tag{3.51}$$

Figure 3.4: Device independent limit for non-relativistic particles: workflow



where t is the time over which the measurement occurs. The two operators are subject to the uncertainty principle (2.7):

$$\Delta x(0)\Delta x(t) \geq \frac{1}{2i} \langle[\hat{x}(0), \hat{x}(t)]\rangle . \tag{3.52}$$

From (3.51) we find:

$$[\hat{x}(0), \hat{x}(t)] = [\hat{x}(0)\hat{p}(0) - \hat{p}(0)\hat{x}(0)]\frac{t}{M} = [\hat{x}(0), \hat{p}(0)]\frac{t}{M} = i\frac{t}{M}, \tag{3.53}$$

and thus:

$$\Delta x(0)\Delta x(t) \geq \frac{t}{2M}. \tag{3.54}$$

We note that at least one of the two uncertainties must be bigger than $\sqrt{t/2M}$, the measurement of the discreteness of the position operator is limited by the greater of the two uncertainties and thus:

$$\Delta x \equiv max[\Delta x(0), \Delta x(t)] \geq \sqrt{\frac{t}{2M}}. \tag{3.55}$$

We point out the analogy of this expression with (3.38) when we discussed how to measure distances by help of a clock, but in the current case we arrived at the same result without considering a particular apparatus.

Let us now take into account gravity. Suppose the size of our apparatus is R, then in order to avoid gravitational collapse, R must be greater than the Schwarzschild radius of the test particle:

$$R > 2GM. \tag{3.56}$$

The apparatus cannot be made arbitrarily small, but it also cannot be made arbitrarily large, in fact, because nothing can exceed the speed of light, it must be $t \geq R$, thus we find:

$$\Delta x \geq \sqrt{\frac{t}{2M}} \geq \sqrt{\frac{R}{2M}} \geq \sqrt{G}. \tag{3.57}$$

We arrived once again to the result (3.8), but this time we found it using just general principles, without considering a particular experimental setup, for non-relativistic particles.

3.2 Non-commutative geometry

Spacetime coordinates x^μ are represented quantically by the selfadjoint operator $\hat{x}^\mu$, which satisfies the commutation relation:

$$[\hat{x}^\mu, \hat{x}^\nu] = 0, \tag{3.58}$$

leading to an uncertainty between spacetime coordinates equal to zero:

$$\Delta x^\mu \Delta x^\nu = 0. \tag{3.59}$$

The key concept of non-commutative geometry [7, 8] is that position operators don't commute, satisfying instead the commutation relation:

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}. \tag{3.60}$$

The tensor $\theta^{\mu\nu}$ appearing in the right side of the equation is a real-valued and antisymmetric two-tensor of dimension length squared, known as the *Poisson tensor*, which is the deformation parameter in this modification of the canonical commutation relation. We note that such tensor is not a dynamical field and defines a preferred frame, thus violating Lorentz invariance. The uncertainty relation between the position operators is then:

$$\Delta x^\mu \Delta x^\nu \gtrsim \frac{1}{2}|\theta^{\mu\nu}|. \tag{3.61}$$

From this equation we see that as the reduced Planck constant $\hbar$ enters in the relation between the uncertainties on position and momentum so does the deformation parameter $|\theta^{\mu\nu}|$ in the relation between the uncertainties on coordinates, thus its physical interpretation is that of a smallest observable area in the $\mu\nu$-plane. We expect that the entries of $\theta^{\mu\nu}$ are of the order of the Planck length squared, though these are free parameters which have to be determined experimentally.

The non-commutation of spacetime coordinates x can be extended to the algebra of functions $f(x)$. Because in quantum physics observables are represented by selfadjoint operators, we need a procedure W that to each element $f(x)$ in the algebra of functions $\mathcal{A}$ assigns a selfadjoint operator $\hat{f} = W(f)$ in the algebra of operators $\hat{\mathcal{A}}$. We thus have to choose a suitable basis for each algebra and then find an isomorphism between them. The most common choice for a basis of the algebra of functions $\mathcal{A}$ is a Fourier transform of the function $f(x)$ [4]:

$$\tilde{f}(k) = \frac{1}{(2\pi)^4} \int d^4x e^{-ik_\mu x^\mu} f(x). \tag{3.62}$$

The corresponding basis of the algebra of operators $\hat{\mathcal{A}}$ is the inverse transform with the non-commutative operator $\hat{x}^\mu$:

$$\hat{f} = W(f) = \frac{1}{(2\pi)^4} \int d^4k e^{ik_\mu \hat{x}^\mu} \tilde{f}(k). \tag{3.63}$$

Given a vector space A defined over a field K equipped with an additional binary operation:

$$* : A \times A \to A, \tag{3.64}$$

A is an *algebra* over K if the binary operation $*$ is bilinear. Two algebras $\mathcal{A}, \mathcal{B}$ defined over the same field are isomorphic if there exists a linear bijective map $W : \mathcal{A} \to \mathcal{B}$ such that $W(f * g) = W(f) * W(g)$, with $f, g \in \mathcal{A}$. We thus construct a new product, the star product $\star$, that defines an isomorphism between the algebra of functions and the algebra of operators, such that:

$$W(f \star g) = W(f) \star W(g) = W(f) \cdot W(g) = \hat{f} \cdot \hat{g}, \tag{3.65}$$

where $f, g \in \mathcal{A}$ and $\hat{f}, \hat{g} \in \hat{\mathcal{A}}$. We thus find the explicit expression:

$$W(f \star g) = \frac{1}{(2\pi)^4} \int d^4k d^4p e^{ik_\mu \hat{x}^\mu} e^{ip_\mu \hat{x}^\mu} \tilde{f}(k) \tilde{g}(p). \tag{3.66}$$

Using the Campbell-Baker-Hausdorff formula:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]-[B,[A,B]])+\dots}, \tag{3.67}$$

we have:

$$e^{ik_\mu \hat{x}^\mu} e^{ip_\mu \hat{x}^\mu} \simeq e^{i(k_\mu + p_\mu)\hat{x}^\mu - \frac{i}{2} k_\mu \theta^{\mu\nu} p_\nu}, \tag{3.68}$$

and so:

$$W(f \star g) = \int \frac{d^4k}{(2\pi)^4} d^4p \tilde{f}(k)\tilde{g}(p) e^{i(k_\mu + p_\mu)\hat{x}^\mu - \frac{i}{2}k_\mu \theta^{\mu\nu} p_\nu}. \tag{3.69}$$

This map can be inverted to:

$$f(x) \star g(x) = \int \frac{d^4k}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} f(k)g(p) e^{-i(k_\nu + p_\nu)x^\nu - \frac{i}{2}k_\nu \theta^{\nu\mu} p_\mu}. \tag{3.70}$$

Equipped with this product we can continue to work with functions as usual, we just have to keep in mind that they obey a modified product rule.

One of the main consequences of non-commutative geometry is non-locality. We can see this by considering the star product with a delta function. Let us first rewrite (3.70) as [4]:

$$\begin{aligned} f(x) \star g(x) &= \int \frac{d^4p}{(2\pi)^4} d^4y f(x + \frac{1}{2}\theta p) g(x+y) e^{-ik_\nu y^\nu} \\ &= \frac{1}{\pi^4 |det\theta|} \int d^4z d^4y f(x+z) g(x+y) e^{-2iz^\mu \theta^{-1}_{\mu\nu} y^\nu}, \end{aligned} \tag{3.71}$$

then the star product with the delta function is:

$$\delta(x) \star g(x) = \frac{1}{|det\theta|} \int d^4y e^{2ix^\mu \theta^{-1}_{\mu\nu} y^\nu} g(x). \tag{3.72}$$

In contrast to the normal product of functions, the star product describes a highly non-local operation. Another way to see how the non-vanishing commutator (3.60) requires some minimal resolution is by considering the product of two Gaussians centered around zero [7]. A normalized Gaussian in position space centered around zero with variance σ^2:

$$\psi_\sigma(x) = \frac{1}{\pi\sigma} e^{-x^2/\sigma^2}, \tag{3.73}$$

has the Fourier transform:

$$\tilde{\psi}_\sigma(k) = \int d^2x e^{ikx} \psi_\sigma(x) = e^{-\pi^2 k^2 \sigma^2}. \tag{3.74}$$

Given two Gaussians with variance σ_1^2 and σ_2^2, their star product is, in the momentum space:

$$\begin{aligned} \tilde{\psi}_{\sigma_1} \star \tilde{\psi}_{\sigma_2}(k) &= \int d^2p \psi_{\sigma_1}(k-p) \psi_{\sigma_2}(p) e^{\frac{i}{2}k^i \theta_{ij} p^j} \\ &= \frac{\pi}{(4\sigma_1^2 + \sigma_2^2)^2} e^{-k^2 \sigma_{12}^2/4}, \end{aligned} \tag{3.75}$$

with:

$$\sigma_{12}^2 = \frac{\sigma_1^2 \sigma_2^2 + \theta^2}{\sigma_1^2 + \sigma_2^2}. \tag{3.76}$$

In the position space this yields:

$$\tilde{\psi}_{\sigma_1} \star \tilde{\psi}_{\sigma_2}(x) = \frac{1}{\pi\sigma_{12}(4\sigma_1^2 + \sigma_2^2)^2} e^{-x^2/\sigma_{12}^2}. \tag{3.77}$$

Multiplying two Gaussians with $\sigma_1, \sigma_2 < \theta$, the resulting width σ_{12} is larger than θ. Inserting $\sigma_1 = \sigma_2 = \sigma_{12} = \sigma$ in (3.76), we find $\sigma^2 = \theta$: a Gaussian with width θ squares to itself. Thus the Gaussian with width θ can be thought of as having a minimum effective size.

3.3 Other motivations

On a final note, a minimal length emerges, in a more theoretical and formal way, in the two main theories that attempt to describe quantum gravity: *string theory* and *loop quantum gravity* (LQG).

In string theory is the string scale l_S that plays the role of a minimal length scale: studying the scattering of strings and D-branes, it has been shown that at high energies the strings grow in size, thus leading to a form of generalized uncertainty principle.

In LQG some operators, e.g. the area and volume operators acting on the spin network ψ_s representing quantum states of the gravitational field, have a discrete spectrum and thus a minimal value is identifiable. For example, the eigenvalues of the area operator $\hat{A}$ of a two-dimensional surface Σ are given by [9]:

$$\hat{A}\psi_s = 8\pi l_{Pl}^2 \gamma \sum_i \sqrt{j_i(j_i+1)}\psi_s, \tag{3.78}$$

where j_i is a positive, half-integer number representing the spin of the link i of the spin network, the sum is taken over all the intersections of the surface Σ with the spin network, and γ is the Immirzi parameter, of order of the unity. Clearly the smallest area eigenvalue is:

$$A_{min} = 4\pi\sqrt{3}\gamma l_{Pl}^2. \tag{3.79}$$

From these considerations emerges that rather than speaking of a precise minimal length, we should speak more generally of a minimal length scale. This is expected to be of the order of the Planck length but it is, in principle, a free parameter whose value has to be constrained by observations.

Chapter 4

GUP

In this section we study the connection between the GUP and the canonical commutation relations and analyze briefly some of the problems arising from this method. We then proceed to see how the Newtonian gravitational potential is influenced, critically reviewing some of the approaches to this issue that are fundamentally fallacious and make, indeed, wrong predictions.

4.1 Modified commutators

We can obtain the GUP from a modified commutation relation for position and momentum operators. This modification could also imply a modification of the commutator of these operators with themselves, thus the geometry in position or momentum space would become non-commuting.

Let us consider the canonical commutation relations for the operators representing the wave vector and the spacetime coordinates $k^\mu = (\omega, \boldsymbol{k})$ and $x^\mu = (t, \boldsymbol{x})$ (we drop the hat for operators in order to lighten the notation), with the three vector components labeled with small Latin indices:

$$[x_\mu, x_\nu] = 0, \quad [x^\mu, k_\nu] = i\delta^\mu_\nu, \quad [k_\mu, k_\nu] = 0. \tag{4.1}$$

We define the momentum as $p^\mu = (E, \boldsymbol{p}) = f(k^\mu)$, with f an injective function, so that $f^{-1}(p^\mu) = k^\mu$ is well defined. The commutation relations associated with these variables are:

$$[x_\mu, x_\nu] = 0, \quad [x_\mu, p_\nu] = i\frac{\partial f_\nu}{\partial k_\mu}, \quad [p_\mu, p_\nu] = 0. \tag{4.2}$$

The uncertainty relation between x_i and p_i is then:

$$\Delta x_i \Delta p_i \geq \frac{1}{2}\left\langle \frac{\partial f_i}{\partial k_i} \right\rangle. \tag{4.3}$$

Let us say that the expression defining $\boldsymbol{p} = f(\boldsymbol{k})$ is expandable as:

$$\boldsymbol{p} \approx \boldsymbol{k}(1 + \beta_0 k^2/m_{Pl}^2), \tag{4.4}$$

plus higher orders in k/m_{Pl}, with β_0 a dimensionless parameter; the inverse relation will then be:

$$\boldsymbol{k} \approx \boldsymbol{p}(1 - \beta_0 p^2/m_{Pl}^2). \tag{4.5}$$

Hence we find:

$$\frac{\partial f_i}{\partial k_j} \approx \delta_{ij}\left(1 + \beta_0 \frac{p^2}{m_{Pl}^2}\right) + 2\beta_0 \frac{p_i p_j}{m_{Pl}^2}, \tag{4.6}$$

and the commutator of x_i and p_j becomes:

$$[x_i, p_j] \approx i\left(\delta_{ij} + \beta\delta_{ij}p^2 + 2\beta p_i p_j\right), \tag{4.7}$$

where $\beta = \beta_0/m_{Pl}^2$. We can now write (4.3) as:

$$\begin{aligned}\Delta x_i \Delta p_i &\geq \frac{1}{2}\left(1 + \beta_0 \frac{\langle p^2\rangle}{m_{Pl}^2} + 2\beta_0 \frac{\langle p_i^2\rangle}{m_{Pl}^2}\right) \\ &\geq \frac{1}{2}\left(1 + \frac{\beta_0}{m_{Pl}^2}\Delta p_i^2\right),\end{aligned} \tag{4.8}$$

recalling that $\Delta p^2 = \langle p^2\rangle - \langle p\rangle^2$. We thus reproduced the GUP we saw emerging in Section 3:

$$\Delta x_i \geq \frac{1}{2}\left(\frac{1}{\Delta p_i} + \beta\Delta p_i\right). \tag{4.9}$$

However, this determination for p^μ and k^μ doesn't come without complications. For example, it arises the problem of how these quantities transform if we change reference frame. If we assume that k^μ obeys the normal Lorentz transformation Λ and perform such a transformation on it, obtaining $k^{\mu\prime} = \Lambda^{\mu'}_{\nu} k^\nu$, then we have:

$$p^{\mu\prime} = f(k^{\mu\prime}) = f(\Lambda^{\mu'}_{\nu} k^\nu) = f(\Lambda^{\mu'}_{\nu} f^{-1}(p^\nu)), \tag{4.10}$$

and we can construct the modified Lorentz transformation obeyed by the momentum: $p^{\mu\prime} = \tilde{\Lambda}^{\mu'}_{\nu}(p^\nu)$. In particular, we can choose a function f such that it maps an infinite value of k^μ in a finite value of p^μ at the Planck energy, thus making the Planck length Lorentz invariant ($E_{Pl} \sim 1/l_{Pl}$). This is the main concept of *Doubly Special Relativity* or DSR (sometimes referred to as Deformed Special Relativity), "doubly" because in this theory there are now an observer-independent maximum speed, the speed of light, and an observer-independent maximum energy or mass, the Planck energy or mass ($E_{Pl} = m_{Pl}$) [10, 11].

Another issue is how momenta must be summed. The function f must be non linear in k^μ if we want the Planck energy to be a Lorentz invariant, thus the modified Lorentz transformation $\tilde{\Lambda}$ has also to be non linear in k^μ. Hence, the sum of the transformations of momenta is different from the transformation of the sum of momenta:

$$\tilde{\Lambda}^{\mu'}_{\nu}(p_1^\nu + p_2^\nu) \neq \tilde{\Lambda}^{\mu'}_{\nu}(p_1^\nu) + \tilde{\Lambda}^{\mu'}_{\nu}(p_2^\nu). \tag{4.11}$$

We can obviate this problem by defining a new, non-linear addition operator $\oplus$, making use of the normal behaviour under Lorentz transformation of the wave vector. The sum $k_1^\mu + k_2^\mu$ is invariant under the normal Lorentz transformation, thus we define the new addition operator as:

$$p_1^\mu \oplus p_2^\mu = f(k_1^\mu + k_2^\mu) = f(f^{-1}(p_1^\mu) + f^{-1}(p_2^\mu)). \tag{4.12}$$

We now note that, having chosen f in such a way that it has a maximum value for the Planck energy (or equivalently the Planck mass $m_{Pl} \approx 10^{-8}$ kg), then the sum of momenta will never exceed this value. However, while this doesn't represent a problem in the realm of particle physics, it is an issue at bigger scales, where this mass is easily exceeded. This takes the name of *soccer ball problem*, because it arises for macroscopic objects, for example a soccer ball [12]. Anyway, solutions to this problem are beyond our scope as we already dwelled enough on implications of the commutation relations (4.2). We just pointed out this and others issues to stress that the investigation of these relations is still under development and much effort is being put in it. The main point we must take away from this section is that the GUP can be obtained, formally, from the modified commutation relations (4.2).

4.2 Newtonian limit

We know that a freely-falling object in a gravitational field moves along geodesics defined by:

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0, \tag{4.13}$$

where x^μ are the coordinates of the particle, τ its proper time and $\Gamma^\alpha_{\mu\nu}$ the Christoffel symbols which are related to the metric $g_{\mu\nu}$ by:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}), \tag{4.14}$$

where the comma denotes partial derivative w.r.t. the coordinates x^μ. We want to tie the metric to the Newtonian gravitational potential. In order to do so we put ourselves in the *weak field limit* of the geodesic equation, that is we consider a metric $g_{\mu\nu}$ static and very close to the Minkowski metric $\eta_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}, \tag{4.15}$$

where $|\epsilon| \leq 1$ is the deformation parameter, and consider the *non-relativistic limit*, that is the particle's speed is far from the speed of light and the four-velocity of the particle can be written as:

$$u^\mu = (\gamma, \gamma\vec{u}) = (\gamma, \gamma\epsilon\vec{v})\,, \tag{4.16}$$

where $v \ll c = 1$ and $\gamma = 1/\sqrt{1-u^2}$. We now expand to first-order in ϵ:

$$u^\mu = \left(1 + \mathcal{O}(\epsilon^2), \epsilon\vec{v} + \mathcal{O}(\epsilon^2)\right), \tag{4.17}$$

thus:

$$\frac{d^2x^\alpha}{d\tau^2} = \epsilon\left(0, \frac{d\vec{v}}{dt}\right) + \mathcal{O}(\epsilon^2), \tag{4.18}$$

and:

$$\begin{aligned}\Gamma^\alpha_{\mu\nu} &= \frac{1}{2}g^{\alpha\beta}(g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}) \\ &= \frac{\epsilon}{2}g^{\alpha\beta}(h_{\mu\beta,\nu} + h_{\nu\beta,\mu} - h_{\mu\nu,\beta}) \\ &= \frac{\epsilon}{2}\eta^{\alpha\beta}(h_{\mu\beta,\nu} + h_{\nu\beta,\mu} - h_{\mu\nu,\beta}) + \mathcal{O}(\epsilon^2).\end{aligned} \tag{4.19}$$

Recalling that $t = \gamma\tau$, the geodesic equation (4.13) becomes, to first-order in ϵ:

$$\begin{aligned}0 = \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} &\simeq \epsilon\frac{d^2x^i}{dt^2} + \Gamma^i_{\mu\nu}\delta^\mu{}_0\delta^\nu{}_0 \\ &\simeq \left(\frac{d^2x^i}{dt^2} - \frac{1}{2}\eta^{ii}h_{00,i}\right),\end{aligned} \tag{4.20}$$

in which we used:

$$\Gamma^i_{\mu\nu}\delta^\mu{}_0\delta^\nu{}_0 = \Gamma^i_{00} \simeq \frac{\epsilon}{2}\eta^{ii}(h_{0i,0} + h_{0i,0} - h_{00,i}) = -\frac{\epsilon}{2}\eta^{ii}h_{00,i}\ , \tag{4.21}$$

remembering that being $h_{\mu\nu}$ time-independent, its derivatives are different from zero only when not taken w.r.t the time. From (4.20) we see that:

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}h_{00,i}\ , \tag{4.22}$$

and defining the gravitational potential V as:

$$V = -\frac{1}{2}h_{00}, \tag{4.23}$$

from (4.22) we find:

$$\frac{d^2x^i}{dt^2} = -\frac{\partial V}{\partial x^i}. \tag{4.24}$$

The Newtonian limit is recovered for spherically symmetric body using the Schwarzschild metric:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 - r^2d\Omega^2, \tag{4.25}$$

where M is the mass of the body and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. From (4.23), we find:

$$V(r) = \frac{g_{00} - 1}{2} = -\frac{GM}{r}, \tag{4.26}$$

which is the Newtonian potential.

Now we see what are the effects of the GUP. In Section 5.2 we will show how the GUP modifies the Hawking radiation temperature and how the Schwarzschild metric can be then deformed in order to recover this modified temperature. The deformation of the metric we will later make use of is obtained by expanding the first component as [13]:

$$g_{00} \simeq 1 - \frac{2GM}{r} + \epsilon \frac{G^2M^2}{r^2}, \tag{4.27}$$

with $|\epsilon| \leq 1$ the deformation parameter. Once again, according to (4.23), we have:

$$V(r) = -\frac{GM}{r} + \epsilon \frac{G^2M^2}{2r^2} \equiv V_N(r) + V_{GUP}(r). \tag{4.28}$$

The deformation parameter ϵ and β_0 can be shown to be related as:

$$\beta_0(\epsilon) \simeq -\pi^2 \frac{GM^2}{\hbar} \epsilon^2, \tag{4.29}$$

V_{GUP} can then be approximated as:

$$V_{GUP} = \epsilon \frac{G^2M^2}{2r^2} \simeq \sqrt{|\beta_0|} \frac{m_{Pl}}{M} V_N^2. \tag{4.30}$$

We see that the Equivalence Principle is satisfied, in fact:

$$\ddot{r} = -\frac{dV(r)}{dr} = -\frac{GM}{r^2} + \epsilon \frac{G^2M^2}{r^3}. \tag{4.31}$$

This correction to the Newtonian potential can be used to investigate non-relativistic phenomena.

4.3 Other approaches

We know that the classical "counterpart" of the commutator is the Poisson bracket. In some works [14, 15] in order to analyze the classical limit, the modified commutation relations (4.7) have been used to construct modified Poisson brackets:

$$[\hat{q}, \hat{p}] = i\hbar(1 + \beta \hat{p}^2) \rightarrow \{q, p\} = (1 + \beta p^2), \tag{4.32}$$

with q, p canonical variables satisfying:

$$\{q, q\} = \{p, p\} = 0. \tag{4.33}$$

Given the Hamiltonian $H(q,p)$ we have:

$$\begin{aligned}\dot{q} &= \{q,H\} = (1+\beta p^2)\frac{\partial H}{\partial p},\\ \dot{p} &= \{p,H\} = -(1+\beta p^2)\frac{\partial H}{\partial q}.\end{aligned} \tag{4.34}$$

Considering now a particle of mass m moving in a Newtonian potential, its Hamiltonian is:

$$H = \frac{p^2}{2m} - \frac{GMm}{q}, \tag{4.35}$$

and (4.34) becomes:

$$\dot{q} = \{q,H\} = (1+\beta p^2)\frac{p}{m}, \tag{4.36}$$

$$\dot{p} = \{p,H\} = -(1+\beta p^2)\frac{GMm}{q^2}. \tag{4.37}$$

From (4.36) we have:

$$m\dot{q} = p + \beta p^3, \tag{4.38}$$

and then:

$$m\ddot{q} = (1+3\beta p^2)\dot{p}. \tag{4.39}$$

Inserting (4.37), to first-order in β we find:

$$\ddot{q} \simeq -(1+4\beta p^2)\frac{GM}{q^2}. \tag{4.40}$$

Solving (4.38) for p, to first-order in β we get:

$$p \simeq m\dot{q} - \frac{\beta(m\dot{q}^3)}{1+3\beta(m\dot{q})^2}, \tag{4.41}$$

and thus we find, again in first-order in β:

$$\ddot{q} \simeq -[1+4\beta(m\dot{q})^2]\frac{GM}{q^2}. \tag{4.42}$$

This clearly violates the Equivalence Principle, because the acceleration of the body depends on its mass and velocity and thus the results obtained using the modified Poisson brackets (4.34) aren't valid.

In the approach followed by S. Benczik et al. in [15], one starts from the conservation of the energy $E = m\mathcal{E}$, from which:

$$p^2 = 2m^2(\mathcal{E} - V_N). \tag{4.43}$$

Inserting this expression in (4.36) we get, to first-order in β:

$$\dot{q}^2 \simeq 2(\mathcal{E} - V_N)[1 + 4\beta m^2(\mathcal{E} - V_N)]. \tag{4.44}$$

Here the term in β depends on the mass of the particle and on its velocity, roughly $\dot{q} \sim (\mathcal{E} - V_N)^{1/2}$ and thus violates the Equivalence Principle. In fact in [15], the deformation parameter is estimated as being very small:

$$\beta_0 \leq 10^{-66}, \tag{4.45}$$

which makes perturbations induced by the GUP too small to be relevant even in the quantum realm, thus contradicting the very reason why the GUP is introduced.

As pointed out in [16], this error comes from the limit (4.32). In fact, the semiclassical limit of the commutator for a generic state ψ with $\langle \hat{p} \rangle \neq 0$ is formally given by:

$$\{q, p\} = \lim_{\hbar \to 0} \frac{\langle \psi | [\hat{q}, \hat{p}] | \psi \rangle}{i\hbar} = \lim_{\hbar \to 0} \left[1 + \beta_0 \frac{G}{\hbar} \left(\langle \hat{p} \rangle^2 + \Delta p^2 \right) \right]. \tag{4.46}$$

Macroscopic objects with non vanishing momentum are better represented by semiclassical states ψ_{cl} defined by:

$$\lim_{\hbar \to 0} \langle \hat{p} \rangle = p, \tag{4.47}$$

$$\lim_{\hbar \to 0} \Delta p^2 = \lim_{\hbar \to 0} \left(\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2 \right) = 0. \tag{4.48}$$

Thus the limit (4.46) becomes:

$$\{q, p\} = \lim_{\hbar \to 0} \left(1 + \beta_0 \frac{G p^2}{\hbar} \right), \tag{4.49}$$

which diverges badly like $\hbar^{-1}$.

Chapter 5

GUP phenomenology

The phenomenology of the GUP is vast and has been largely investigated. We will focus on its consequences on two main areas: the Hamiltonian of a free particle, applied to the 1-dimensional potential barrier, the Lamb shift, the Landau levels and the Hawking radiation temperature, which can be reproduced starting from a deformed Schwarzschild metric.

5.1 Single particle Hamiltonian

In this section we will analyze how the GUP affects some well understood low energy systems and how the corrections it determines can be used to experimentally estimate the deformation parameter β_0, based on the work of S. Das and E. C. Vagenas [17].

We showed in Section 4.1 that the commutator of position and momentum operators can be modified (4.7) in such a way that leads to the GUP (4.8). If we now define two operators x_i, p_i as:

$$x_i = x_{0i}, \tag{5.1}$$

$$p_i = p_{0i}(1 + \beta p_0^2), \tag{5.2}$$

where:

$$p_0^2 = \sum_{j=1}^{3} p_{0j}^2, \tag{5.3}$$

and with x_{0i}, p_{0j} satisfying the canonical commutation relation:

$$[x_{0i}, p_{0j}] = i\hbar\delta_{ij}, \tag{5.4}$$

we recover (4.7) to first-order in β, thus in the following we neglect terms of order β^2 and higher.

Any Hamiltonian H of the form:

$$H = \frac{p^2}{2m} + V(\boldsymbol{r}), \quad \boldsymbol{r} = (x_1, x_2, x_3), \tag{5.5}$$

can be written, using (5.2), as:

$$H = \frac{p_0^2}{2m} + V(\boldsymbol{r}) + \frac{\beta}{m} p_0^4 + O(\beta^2) \equiv H_0 + H_1 + O(\beta^2), \tag{5.6}$$

where:

$$H_0 = \frac{p_0^2}{2m} + V(\boldsymbol{r}), \tag{5.7}$$

$$H_1 = \frac{\beta}{m} p_0^4. \tag{5.8}$$

It is remarkable that using the modified Hamiltonian in the time-dependent Schrödinger equation:

$$H\psi(\boldsymbol{r}, t) = i\hbar \frac{\partial \psi(\boldsymbol{r}, t)}{\partial t} \tag{5.9}$$

the continuity equation is still fulfilled [17]:

$$\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0, \tag{5.10}$$

with ρ the usual probability density:

$$\rho = |\psi|^2, \tag{5.11}$$

and $\boldsymbol{J}$ the probability current:

$$\begin{aligned} \boldsymbol{J} &= \frac{\hbar}{2mi}(\psi^* \nabla \psi - \psi \nabla \psi^*) - \frac{\beta \hbar^3}{mi}[(\psi^* \nabla \nabla^2 \psi - \psi \nabla \nabla^2 \psi^*) + (\nabla^2 \psi^* \nabla \psi - \nabla^2 \psi \nabla \psi^*)] \\ &\equiv \boldsymbol{J}_0 + \boldsymbol{J}_1. \end{aligned} \tag{5.12}$$

In what follows, for the different phenomena in exam we first review the standard approach with the usual Hamiltonian H_0 and then proceed taking into account the modified Hamiltonian H.

5.1.1 STM and 1-dimensional potential barrier

The *scanning tunneling microscope* or STM is a type of microscope which allows to image and manipulate single atoms. The STM consists of an extremely sharp and conductive tip, which moves upon the surface one wants to investigate (Fig. 5.1). A voltage is applied between the

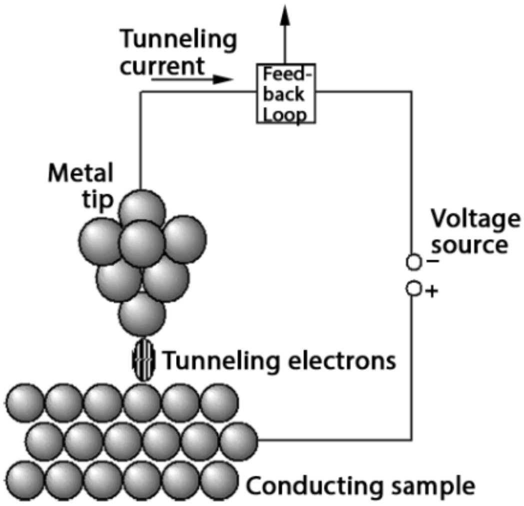


Figure 5.1: Schematic picture of an STM. The sharp conductive tip is located on top of an atom of the surface to analyze. A voltage is applied between them. Electrons move due to the tunnelling effect, thus creating a current.

two. When the tip is located upon an atom, electrons move due to quantum tunnelling and thus create a current. The voltage applied can be modeled by a 1-dimensional potential barrier :

$$V(x) = 0 \quad \text{for } x < 0 \text{ and } x > a, \tag{5.13a}$$

$$V(x) = V_0 \quad \text{for } 0 < x < a, \tag{5.13b}$$

where V_0 is the height of the potential barrier (Fig. 5.2). The time-independent Schrödinger

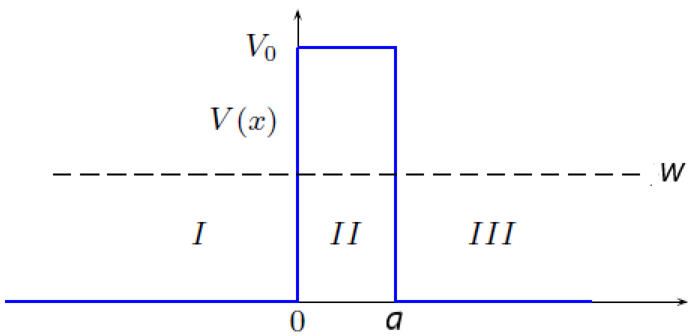


Figure 5.2: Potential barrier of height V_0 for $0 < x < a$ and 0 for $x < 0, x > a$.

equation for electrons with potential energy $V(x)$ is, in the position representation:

$$H\phi(x) = -\frac{\hbar^2}{2m}\frac{d^2\phi(x)}{dx^2} + V(x)\phi(x) = w\phi(x), \tag{5.14}$$

where w is the energy eigenvalue and we consider the case $w < V_0$. This equation can be written as:

$$\frac{d^2\phi}{dx^2} + k^2\phi = 0 \quad \text{for } x < 0 \text{ and } x > a, \tag{5.15a}$$

$$\frac{d^2\phi}{dx^2} + k'^2\phi = 0 \quad \text{for } 0 < x < a, \tag{5.15b}$$

where:

$$k^2 = \frac{2mw}{\hbar^2}, \tag{5.16}$$

$$k'^2 = \frac{2m(w - V_0)}{\hbar^2}. \tag{5.17}$$

We are interested in solutions of (5.15) of the form:

$$\phi(x) = e^{(ikx)} + Ae^{(-ikx)} \quad \text{for } x < 0, \tag{5.18a}$$

$$\phi(x) = Be^{(ik'x)} + Ce^{(-ik'x)} \quad \text{for } 0 < x < a, \tag{5.18b}$$

$$\phi(x) = De^{(ikx)} \quad \text{for } x > a. \tag{5.18c}$$

The coefficients A, B, C, D are determined by imposing that $\phi(x)$ and its first derivative $d\phi(x)/dx$ are continuous at $x = 0, a$:

$$\phi(0 + 0) = \phi(0 - 0), \tag{5.19a}$$

$$\phi(a + 0) = \phi(a - 0), \tag{5.19b}$$

$$\frac{d\phi(0 + 0)}{dx} = \frac{d\phi(0 - 0)}{dx}, \tag{5.19c}$$

$$\frac{d\phi(a + 0)}{dx} = \frac{d\phi(a - 0)}{dx}, \tag{5.19d}$$

where $f(x \pm 0)$ is an abbreviation for $\lim_{\epsilon \to 0^+} f(x \pm \epsilon)$. The resulting linear system is:

$$B + C = 1 + A, \tag{5.20a}$$

$$De^{ika} = Be^{ik'a} + Ce^{-ik'a}, \tag{5.20b}$$

$$k'(B - C) = k(1 - A), \tag{5.20c}$$

$$kDe^{ika} = k'(Be^{ik'a} - Ce^{-ik'a}). \tag{5.20d}$$

Solving the system we find:

$$A = \frac{-i(k^2 - k'^2)\sin(k'a)}{2kk'\cos(k'a) - i(k^2 + k'^2)\sin(k'a)}, \tag{5.21a}$$

$$B = \frac{k(k + k')e^{-ik'a}}{2kk'\cos(k'a) - i(k^2 + k'^2)\sin(k'a)}, \tag{5.21b}$$

$$C = \frac{-k(k - k')e^{ik'a}}{2kk'\cos(k'a) - i(k^2 + k'^2)\sin(k'a)}, \tag{5.21c}$$

$$D = \frac{2kk'}{e^{ika}[2kk'\cos(k'a) - i(k^2 + k'^2)\sin(k'a)]}. \tag{5.21d}$$

From collision theory we know that the reflection and transmission amplitudes are respectively:

$$r = A, \quad t = D, \tag{5.22}$$

thus the reflection and transmission coefficients are:

$$R = \frac{\delta^2 \sinh^2(\tilde{k}'a)}{(2k\tilde{k}'a^2)^2 + \delta^2 \sinh^2(\tilde{k}'a)}, \tag{5.23}$$

$$T = \frac{(2k\tilde{k}'a^2)^2}{(2k\tilde{k}'a^2)^2 + \delta^2 \sinh^2(\tilde{k}'a)}, \tag{5.24}$$

where:

$$\delta = \frac{2mV_0a^2}{\hbar^2}, \tag{5.25}$$

$$\tilde{k}'^2 = \frac{\delta}{a^2} - k^2, \tag{5.26}$$

and of course $R + T = 1$. It can be shown that when $k'a \gg 1$, the transmission coefficient can be approximated with:

$$T_0 = \frac{16w(V_0 - w)}{V_0^2} e^{-2k'a}. \tag{5.27}$$

We now take into account the GUP. The time-independent Schrödinger equation for electrons with potential energy $V(x)$ and with the modified Hamiltonian (5.6) is, in the position representation:

$$H\phi(x) = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) + \frac{\beta\hbar^4}{m}\frac{d^4}{dx^4}\right]\phi(x) = w\phi(x), \tag{5.28}$$

where w is the energy eigenvalue and we consider the case $w < V_0$. This equation can be written as:

$$\frac{d^2\phi}{dx^2} + k^2\phi - l_{Pl}^2\frac{d^4\phi}{dx^4} = 0 \quad \text{for } x < 0 \text{ and } x > a, \tag{5.29a}$$

$$\frac{d^2\phi}{dx^2} + k'^2\phi - l_{Pl}^2\frac{d^4\phi}{dx^4} = 0 \quad \text{for } 0 < x < a, \tag{5.29b}$$

where k, k' given by (5.16), (5.17) and $l_{Pl}^2 = 2\beta\hbar^2$. We are interested in solutions of (5.29) of the form:

$$\phi(x) = Ae^{(ik_1 x)} + Be^{(-ik_1 x)} + A_1 e^{(x/l_{Pl})} \quad \text{for } x < 0, \tag{5.30a}$$
$$\phi(x) = Fe^{(ik_1' x)} + Ge^{(-ik_1' x)} + H_1 e^{(x/l_{Pl})} + L_1 e^{(-x/l_{Pl})} \quad \text{for } 0 < x < a, \tag{5.30b}$$
$$\phi(x) = Ce^{(ik_1 x)} + D_1 e^{(-x/l_{Pl})} \quad \text{for } x > a. \tag{5.30c}$$

where:

$$k_1 = k(1 - \beta\hbar^2 k^2), \tag{5.31}$$
$$k_1' = k'(1 - \beta\hbar^2 k'^2). \tag{5.32}$$

The coefficients are determined by imposing that $\phi(x)$ and its derivatives of order n, $d^n\phi(x)/dx^n$, with $n = 1, 2, 3$, are continuous at $x = 0, a$:

$$\phi(0+0) = \phi(0-0), \tag{5.33a}$$
$$\phi(a+0) = \phi(a-0), \tag{5.33b}$$
$$\frac{d^n\phi(0+0)}{dx^n} = \frac{d^n\phi(0-0)}{dx^n}, \tag{5.33c}$$
$$\frac{d^n\phi(a+0)}{dx^n} = \frac{d^n\phi(a-0)}{dx^n}. \tag{5.33d}$$

The resulting linear system is:

$$A + B + A_1 = F + G + H_1 + L_1, \tag{5.34a}$$
$$ik_1(A - B) + \frac{A_1}{l_{Pl}} = k_1'(F - G) + \frac{H_1 - L_1}{l_{Pl}}, \tag{5.34b}$$
$$-k_1'^2(A + B) + \frac{A_1}{l_{Pl}^2} = k_1'^2(F + G) + \frac{H_1 - L_1}{l_{Pl}^2}, \tag{5.34c}$$
$$-ik_1{}^3(A - B) + \frac{A_1}{l_{Pl}^3} = k_1'^3(F - G) + \frac{H_1 - L_1}{l_{Pl}^3}, \tag{5.34d}$$
$$Fe^{k_1' a} + Ge^{-k_1' a} + H_1 e^{a/l_{Pl}} + L_1 e^{-a/l_{Pl}} = Ce^{ik_1 a} + D_1 e^{-a/l_{Pl}}, \tag{5.34e}$$
$$k_1'(Fe^{k_1' a} + Ge^{-k_1' a}) + \frac{H_1}{l_{Pl}} e^{a/l_{Pl}} - \frac{L_1}{l_{Pl}} e^{-a/l_{Pl}} = ik_1 Ce^{ik_1 a} - \frac{D_1}{l_{Pl}} e^{-a/l_{Pl}}, \tag{5.34f}$$
$$k_1'^2(Fe^{k_1' a} + Ge^{-k_1' a}) + \frac{H_1}{l_{Pl}^2} e^{a/l_{Pl}} + \frac{L_1}{l_{Pl}^2} e^{-a/l_{Pl}} = -k_1^2 Ce^{ik_1 a} + \frac{D_1}{l_{Pl}^2} e^{-a/l_{Pl}}, \tag{5.34g}$$
$$k_1'^3(Fe^{k_1' a} - Ge^{-k_1' a}) + \frac{H_1}{l_{Pl}^3} e^{a/l_{Pl}} - \frac{L_1}{l_{Pl}^3} e^{-a/l_{Pl}} = -ik_1^3 Ce^{ik_1 a} - \frac{D_1}{l_{Pl}^3} e^{-a/l_{Pl}}. \tag{5.34h}$$

Solving the system we find:

$$\frac{B}{A} = \frac{(k_1^2 + k_1'^2)(e^{2k_1'^2 a} - 1)}{e^{2k_1' a}(k_1 + ik_1')^2 - (k_1 - ik_1')^2}, \tag{5.35a}$$

$$\frac{C}{A} = \frac{4ik_1 k_1' e^{-ik_1 a} e^{-ik_1' a}}{e^{2k_1' a}(k_1 + ik_1')^2 - (k_1 - ik_1')^2}, \tag{5.35b}$$

$$\frac{F}{A} = \frac{-2k_1(k_1 - ik_1')}{e^{2k_1' a}(k_1 + ik_1')^2 - (k_1 - ik_1')^2}, \tag{5.35c}$$

$$\frac{G}{A} = \frac{2e^{2k_1' a} k_1(k_1 + ik_1')}{e^{2k_1' a}(k_1 + ik_1')^2 - (k_1 - ik_1')^2}. \tag{5.35d}$$

We can now compute the conserved probability currents using (5.12):

$$J = k_1 \left(|A|^2 - |B|^2\right) \quad \text{for } x < 0, \tag{5.36}$$

$$J = k_1 |C|^2 \quad \text{for } x > a. \tag{5.37}$$

The reflection and transmission coefficients are respectively:

$$R = \left|\frac{B}{A}\right|^2 = \left[1 + \frac{(2k_1 k_1')^2}{(k_1^2 + k_1'^2)^2 \sinh^2(k_1' a)}\right]^{-1}, \tag{5.38}$$

$$T = \left|\frac{C}{A}\right|^2 = \left[1 + \frac{(k_1^2 + k_1'^2)^2 \sinh^2(k_1' a)}{(2k_1 k_1')^2}\right]^{-1}, \tag{5.39}$$

with $R + T = 1$. It can be shown that when $k'a \gg 1$, T can be approximated with:

$$T = T_0 \left[1 + \frac{4m\beta(2w - V_0)^2}{V_0} + \frac{2\beta a[2m(V_0 - w)]^{3/2}}{\hbar}\right], \tag{5.40}$$

where T_0 as given by (5.27).

The transmission coefficient T is proportional to the current I flowing between the tip and the sample in the STM. This current is usually amplified by an amplifier of gain $\mathcal{G}$. The gain in current given by the GUP is:

$$\begin{aligned} \frac{\delta I}{I_0} = \frac{\delta T}{T_0} &= \frac{4\beta m}{V_0}(2w - V_0)^2 + \frac{2\beta a}{\hbar}[2m(V_0 - w)]^{3/2} \\ &= \frac{4\beta_0 m}{m_{Pl} E_{Pl}} \frac{(2w - V_0)^2}{V_0} + \frac{2\sqrt{2}\beta_0 a}{l_{Pl}} \left(\frac{m}{m_{Pl}}\right)^{3/2} \left(\frac{V_0 - w}{E_{Pl}}\right)^{3/2}. \end{aligned} \tag{5.41}$$

Assuming the following approximate values:

$$\begin{aligned} &m = m_e = 0.5 \text{ MeV}, \\ &w, V_0 = 10 \text{ eV}, \\ &a = 10^{-10} \text{ m}, \\ &I = 10^{-9} \text{ A}, \\ &\mathcal{G} = 10^9, \end{aligned} \tag{5.42}$$

we obtain:

$$k' = 10^{10}\ \text{m}^{-1}, \tag{5.43a}$$

$$\frac{\delta I}{I_0} = \frac{\delta T}{T_0} = 10^{-48}\beta_0, \tag{5.43b}$$

$$\delta\mathcal{I} \equiv \mathcal{G}\delta I = 10^{-48}\beta_0\ \text{A}. \tag{5.43c}$$

The time τ it would take for the excess current $\delta\mathcal{I}$ to add the charge of just one electron, $e = 10^{-19}$ C, is:

$$\tau = \frac{e}{\delta\mathcal{I}} = 10^{29}\beta_0^{-1}\ \text{s}. \tag{5.44}$$

If we assume $\beta_0 \approx 1$, this is a time much bigger than the age of the universe (10^{18} s), by that time Earth would have been already wiped out by the Sun and we wouldn't be able to conclude our measurement. However, if we manage to increase $\delta\mathcal{I}$ by a factor 10^{21}, by a combination of increase in I, G and β_0, the above time reduces to $\tau = 10^8$ s, i.e. a year, and we can hope to measure the excess current. If instead the excess current cannot be measured in such a time scale, this puts an upper bound on β_0 of:

$$\beta_0 < 10^{21}. \tag{5.45}$$

5.1.2 Lamb shift

The Hamiltonian of an hydrogen atom is:

$$H_0 = \frac{p_0^2}{2m} - \frac{k}{r}, \tag{5.46}$$

where $k = e^2/4\pi\epsilon_0$. Using $p_0^2 = 2m(H_0 + k/r)$, the deformation term (5.8) can be written as:

$$H_1 = 4\beta m\left[H_0^2 + k\left(\frac{1}{r}H_0 + H_0\frac{1}{r}\right) + \left(\frac{k}{r}\right)^2\right]. \tag{5.47}$$

We can compute how this term affects the energy eigenfunctions and eigenvalues of the system by using the *time-independent perturbation theory* [18].

We suppose that the time-independent Hamiltonian H of the system can be expressed as:

$$H = H_0 + \lambda H', \tag{5.48}$$

where the unperturbed Hamiltonian H_0 is sufficiently simple so that the corresponding time-independent Schrödinger problem:

$$H_0\psi_n^{(0)} = E_n^{(0)}\psi_n^{(0)}, \tag{5.49}$$

can be solved exactly and the perturbation energy H' is small compared to H_0. The parameter λ is real and allows us to distinguish between the various orders of perturbation. The eigenfunctions $\psi_n^{(0)}$ corresponding to eigenvalues $E_n^{(0)}$ form a complete orthonormal set:

$$\langle \psi_i^{(0)} | \psi_j^{(0)} \rangle = \delta_{ij}. \tag{5.50}$$

We want to solve the perturbed Schrödinger problem:

$$H\psi_n = E_n \psi_n. \tag{5.51}$$

The basic assumption of perturbation theory is that the eigenfunctions ψ_n corresponding to eigenvalues E_n can be expanded as a power series in λ:

$$\psi_n = \sum_{j=0}^{\infty} \lambda^j \psi_n^{(j)}, \tag{5.52}$$

$$E_n = \sum_{j=0}^{\infty} \lambda^j E_n^{(j)}, \tag{5.53}$$

where the index j labels the order of the perturbation. Inserting (5.52), (5.53) into the Schrödinger equation (5.51) we have:

$$\begin{aligned}(H_0 + \lambda H')(\psi_n^{(0)} + \lambda \psi_n^{(1)} + \lambda^2 \psi_n^{(2)} + \dots) = (E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots) \\ \times (\psi_n^{(0)} + \lambda \psi_n^{(1)} + \lambda^2 \psi_n^{(2)} + \dots).\end{aligned} \tag{5.54}$$

Now we equate the terms in the same order of λ on both sides of the equation:

$$\begin{aligned}
&H_0 \psi_n^{(0)} = E_n^{(0)} \psi_n^{(0)}, \\
&\lambda(H_0 \psi_n^{(1)} + H' \psi_n^{(0)}) = \lambda(E_n^{(0)} \psi_n^{(1)} + E_n^{(1)} \psi_n^{(0)}), \\
&\lambda^2(H_0 \psi_n^{(2)} + H' \psi_n^{(1)}) = \lambda^2(E_n^{(0)} \psi_n^{(2)} + E_n^{(1)} \psi_n^{(1)} + E_n^{(2)} \psi_n^{0)}), \\
&\vdots \\
&\lambda^j(H_0 \psi_n^{(j)} + H' \psi_n^{(j-1)}) = \lambda^j(E_n^{(0)} \psi_n^{(j)} + E_n^{(1)} \psi_n^{(j-1)} + \dots + E_n^{(j)} \psi_n^{(0)}).
\end{aligned} \tag{5.55}$$

We note that to zero-order of perturbation we recover the unperturbed Schrödinger problem. To obtain the first-order energy correction $E_n^{(1)}$, we multiply the first-order equation in λ by $\psi_n^{(0)*}$ and, using bra-ket notation, we have:

$$\left\langle \psi_n^{(0)} \middle| H_0 - E_n^{(0)} \middle| \psi_n^{(1)} \right\rangle + \left\langle \psi_n^{(0)} \middle| H' - E_n^{(1)} \middle| \psi_n^{(0)} \right\rangle = 0. \tag{5.56}$$

Recalling that the unperturbed Hamiltonian operator is selfadjoint:

$$\left\langle \psi_n^{(0)} \middle| H_0 \middle| \psi_n^{(1)} \right\rangle = E_n^{(0)} \left\langle \psi_n^{(0)} \middle| \psi_n^{(1)} \right\rangle, \tag{5.57}$$

we see that the first term in (5.56) vanishes. Being the eigenfunctions orthonormal we have $\langle\psi_n^{(0)}|\psi_n^{(0)}\rangle = 1$ and then:

$$E_n^{(1)} = \left\langle\psi_n^{(0)}\middle|H'\middle|\psi_n^{(0)}\right\rangle , \tag{5.58}$$

thus the first-order correction is just the average value of the perturbation energy over the unperturbed state.

We are now interested in finding $\psi_n^{(1)}$. In order to do so we expand $\psi_n^{(1)}$ in the basis of the unperturbed eigenfunctions $\psi_j^{(0)}$:

$$\psi_n^{(1)} = \sum_j c_{nj}\psi_j^{(0)}. \tag{5.59}$$

Upon inserting this expression in the first-order equation in λ of (5.55) we have:

$$(H_0 - E_n^{(0)})\sum_j c_{nj}\psi_j^{(0)} + (H' - E_n^{(1)})\psi_n^{(0)} = 0. \tag{5.60}$$

We now multiply this expression for $\psi_l^{(0)*}$ and, using bra-ket notation, we have:

$$c_{nl}^{(1)}(E_l^{(0)} - E_n^{(0)}) + H'_{ln} - E_n^{(1)}\delta_{nl} = 0, \tag{5.61}$$

where $H'_{nl} \equiv \langle\psi_l^{(0)}|H'|\psi_n^{(0)}\rangle$. If $l = n$, (5.61) reduces to $E_n^{(1)} = H'_{nn} = \langle\psi_n^{(0)}|H'|\psi_n^{(0)}\rangle$, which is the result (5.58). If $l \neq n$ we have:

$$c_{nl}^{(1)} = \frac{H'_{nl}}{E_n^{(0)} - E_l^{(0)}}. \tag{5.62}$$

Combining (5.59) and (5.62) we see that $\psi_n^{(1)}$ is given by:

$$\psi_n^{(1)} = \sum_{l\neq n} \frac{H'_{nl}}{E_n^{(0)} - E_l^{(0)}}\psi_l^{(0)}. \tag{5.63}$$

To first-order of perturbation, the perturbed eigenfunction ψ_n is:

$$\psi_n = \psi_n^{(0)} + \psi_n^{(1)}. \tag{5.64}$$

We now return to the hydrogen atom. To first-order of perturbation, the eigenkets of the system are:

$$|\psi_{nlm}\rangle' = |\psi_{nlm}\rangle + \sum_{n'l'm'\neq nlm} \frac{H'_{n'l'm',nlm}}{E_n - E_{n'}}\,|\psi_{n'l'm'}\rangle , \tag{5.65}$$

where n, l, m are respectively the principal, orbital and magnetic quantum numbers and $H'_{n'l'm',nlm} \equiv \langle n'l'm'|H'|nlm\rangle$. Using (5.47) we find:

$$\frac{H'_{n'l'm',nlm}}{4\beta m} = \left[(E_n)^2\delta_{nn'} + k(E_n + E_{n'})\,\langle n'l'm'|\frac{1}{r}|nlm\rangle + k^2\,\langle n'l'm'|\frac{1}{r^2}|nlm\rangle\right]. \tag{5.66}$$

Hence the first-order shift in the ground state wavefunction is [17]:

$$\Delta\psi_{100}(\boldsymbol{r}) \equiv \psi'_{100}(\boldsymbol{r}) - \psi_{100}(\boldsymbol{r}) = \frac{H'_{200,100}}{E_1 - E_2}\psi_{200}(\boldsymbol{r}). \tag{5.67}$$

We recall that the wavefunctions of the hydrogen atom are factorized in a spatial and an angular part:

$$\psi_{nlm}(\boldsymbol{r}) = \frac{\chi_{nl}(r)}{r} Y_{lm}(\theta, \phi), \tag{5.68}$$

with:

$$\chi_{10} = \frac{2}{a_0^{3/2}} r \exp\left\{-\frac{r}{a_0}\right\}, \tag{5.69}$$

$$\chi_{20} = \frac{1}{\sqrt{2}a_0^{3/2}} r \exp\left\{-\frac{r}{2a_0}\right\}\left(1 - \frac{r}{2a_0}\right), \tag{5.70}$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}, \tag{5.71}$$

and the energy eigenvalues are:

$$E_n = -\frac{E_0}{n^2}, \tag{5.72}$$

where $E_0 = e^2/8\pi\epsilon_0 a_0 = k/2a_0 = 13.6$ eV, $a_0 = 4\pi\epsilon_0\hbar^2/m_e e^2 = 5.3 \times 10^{-11}$ m, $m_e = 0.5$ Mev/c^2. We can then compute the first-order shift in the ground state wavefunction (5.67) finding:

$$\Delta\psi_{100}(\boldsymbol{r}) = \frac{928\sqrt{2}\beta m E_0}{81}\psi_{200}(\boldsymbol{r}). \tag{5.73}$$

We now see how this calculations can help us estimate the parameter β_0 with the Lamb shift. According to Dirac equation, for a given n the energy levels ${}^2S_{1/2}$ and ${}^2P_{1/2}$ of the hydrogen atom should have the same energy. However Lamb and Retherford, in 1947, first discovered in their experiment that this was not the case and there was, indeed, a difference between the levels $2^2S_{1/2}$ and $2^2P_{1/2}$, the former having a greater energy than the latter [19]. The Lamb shift arises from the interaction between the electron of the atom and the fluctuations of the quantized electromagnetic field: even when no external field is applied, there are still vacuum fluctuations. Although these fluctuations average to zero, their mean-square value does not and thus the position coordinate of the electron has a non-vanishing mean-square fluctuation. The expression for the Lamb shift of the n-th level of the hydrogen atom is [20]:

$$\Delta E_n = \frac{4\alpha^2}{3m^2}\ln\left(\frac{1}{\alpha}\right)|\psi_{nlm}(0)|^2, \tag{5.74}$$

where $\alpha = e^2/4\pi\epsilon_0\hbar c \approx 1/137$ is the fine structure constant. Varying $\psi_{nlm}(0)$, the additional contribution to the Lamb shift due to the GUP in proportion to its original value is given by:

$$\frac{\Delta E_{n(GUP)}}{\Delta E_n} = 2\frac{\Delta\psi_{nlm}(0)}{\psi_{nlm}(0)}. \tag{5.75}$$

For the ground state, using $\psi_{100}(0) = a_0^{-3/2}\pi^{-1/2}$ and $\psi_{200}(0) = a_0^{-3/2}(8\pi)^{-1/2}$, we have:

$$\begin{aligned}\frac{\Delta E_{n(GUP)}}{\Delta E_n} &= 2\frac{\Delta\psi_{nlm}(0)}{\psi_{nlm}(0)} = \frac{928\beta m E_0}{81} \\ &\approx 10\beta_0 \frac{m}{m_{Pl}}\frac{E_0}{m_{Pl}c^2} \\ &\approx 10 \times (0.42 \times 10^{-22}) \times (1.12 \times 10^{-27})\beta_0 \\ &\approx (0.47 \times 10^{-48})\beta_0.\end{aligned} \tag{5.76}$$

If we assume $\beta_0 \approx 1$, this predicts an additional energy that is too small to be measurable. If instead we consider β_0 as a free parameter, an accuracy in the measurement of the Lamb shift of 1 part in 10^{12} puts an upper bound on β_0 of:

$$\beta_0 < 10^{36}. \tag{5.77}$$

Of course this constraint can be weakened by increasing accuracy of measurements.

5.1.3 Landau levels

We consider a particle of mass m and electric charge e in a constant magnetic field $\boldsymbol{B} = B\hat{z}$, whose vector potential is $\boldsymbol{A} = Bx\hat{y}$ using the Landau gauge, where $\hat{z}, \hat{y}$ are, respectively, the versors along the z-axis and the y-axis. The particle then moves in the xy-plane with momentum $\boldsymbol{p} = (p_x, p_y, 0)$. The Hamiltonian is:

$$\begin{aligned}H_0 &= \frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A})^2 \\ &= \frac{p_x^2}{2m} + \frac{p_y^2}{2m} - \frac{eB}{m}xp_y + \frac{e^2B^2}{2m}x^2.\end{aligned} \tag{5.78}$$

Since p_y commutes with H_0 we can replace it with its eigenvalue $\hbar k$:

$$H_0 = \frac{p_x^2}{2m} + \frac{1}{2}m\omega_c^2\left(x - \frac{\hbar k}{m\omega_c}\right)^2, \tag{5.79}$$

where $\omega_c = eB/m$ is the cyclotron frequency. We see that this is the Hamiltonian of an harmonic oscillator in the x direction with equilibrium position $x_0 = \hbar k/m\omega_c$. Because p_y commutes with the Hamiltonian, the eigenfunctions factorize into a product between eigenfunctions e^{iky} of the momentum along the y direction and eigenfunctions ϕ_n of the harmonic oscillator shifted by x_0 in the x-axis:

$$\phi_{k,n}(x, y) = e^{iky}\phi_n(x - x_0), \tag{5.80}$$

and the energy eivengalues are:

$$E_n = \hbar\omega_c \left(n + \frac{1}{2}\right), \quad n \in \mathbb{N}. \tag{5.81}$$

Such energy levels take the name of Landau levels.

Now we consider the GUP modified Hamiltonian of the system, which can be shown to be (see Appendix A of [17]):

$$H = \frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A})^2 + \frac{\beta}{m}(\boldsymbol{p} - e\boldsymbol{A})^4 = H_0 + 4\beta m H_0^2. \tag{5.82}$$

The eigenfunctions remain the same while the eigenvalues are increased by the term:

$$\begin{aligned} \Delta E_{n(GUP)} &= 4\beta m \langle \phi_n | H_0^2 | \phi_n \rangle \\ &= 4\beta m (\hbar\omega_c)^2 \left(n + \frac{1}{2}\right)^2 . \end{aligned} \tag{5.83}$$

The additional contribution to the energy of Landau levels due to the GUP in proportion to its original value is given by:

$$\begin{aligned} \frac{\Delta E_{n(GUP)}}{E_n} &= 4\beta m(\hbar\omega_c) \left(n + \frac{1}{2}\right) \\ &\approx \beta_0 \frac{m}{m_{Pl}} \frac{\hbar\omega_c}{m_{Pl}c^2}. \end{aligned} \tag{5.84}$$

For an electron in a magnetic field of $B = 10$ T, $\omega_c \approx 10^3$ GHz and we have:

$$\frac{\Delta E_{n(GUP)}}{E_n} \approx (0.42 \times 10^{-22}) \times (5.48 \times 10^{-32})\beta_0 = (2.3 \times 10^{-54})\beta_0. \tag{5.85}$$

If we assume $\beta_0 \approx 1$, this predicts an additional energy that is too small to be measured. If instead we consider β_0 as a free parameter, an accuracy of 1 part in 10^3 in direct measurements of Landau levels using a STM puts an upper bound on β_0 of:

$$\beta_0 < 10^{50}. \tag{5.86}$$

This constraint can be weakened by increasing accuracy of measurements.

5.2 Hawking radiation and Schwarzschild metric

Another way we have to estimate β_0 is by investigating how the GUP affects gravity and thus the motion of bodies. We see how the GUP modifies the Hawking radiation temperature and, knowing the connection between Hawking's temperature and the Schwarzschild metric, we find a deformed metric that can reproduce the modified Hawking temperature. These results, obtained by F. Scardigli and R. Casadio in [13] have been used to compute corrections for the deflection of light passing near the Sun, for the perihelion precession of Mercury and for the periastron precession in the binary pulsars system *Pulsar PRS B 1913+16*, thus allowing to estimate β_0 using astronomical data collected over the years.

In what follows we set $c = k_B = 1$ and show explicitly the gravitational constant G and the Planck constant $\hbar$. The Planck lenght is defined as $l_{Pl}^2 = G\hbar/c^3$, the Planck energy as $E_{Pl} = \hbar c/2l_{Pl}$ and the Planck mass as $m_{Pl} = E_{Pl}/c^2$. Thus $G = l_{Pl}^2/\hbar = l_{Pl}/2m_{Pl}$ and $\hbar = 2l_{Pl}m_{Pl}$.

5.2.1 Hawking temperature

We consider a generic metric of the form:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = F(r)dt^2 - F(r)^{-1}dr^2 - r^2 d\Omega^2, \tag{5.87}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. At the moment we don't impose any specific expression for $F(r)$. The horizons are given by the positive zeros of $F(r)$. Denoting as r_H the position of an horizon we consider the region with $r \geq r_H$. Performing the Wick rotation $t \to i\tau$, passing from the Minkowski time to the Euclidean time, the metric becomes:

$$ds^2 = -[F(r)d\tau^2 + F(r)^{-1}dr^2 + r^2 d\Omega^2]. \tag{5.88}$$

Using now two new variables α, R defined by:

$$\begin{aligned} R d\alpha &= F(r)^{1/2} d\tau, \\ dR &= F(r)^{-1/2} dr, \end{aligned} \tag{5.89}$$

the metric takes the form:

$$ds^2 = -[R^2 d\alpha^2 + dR^2 + r^2(R) d\Omega^2]. \tag{5.90}$$

We note that the first two terms in ds^2 are the squared length element of the 2-dimensional Euclidean plane in polar coordinates, thus the Euclidean time τ is proportional to the polar angle α, whose period are respectively Θ and 2π. We see from the second equation of (5.89) that R depends only on r, we can then integrate the first of the two equations over a full period both for α and Θ:

$$R(r) \int_0^{2\pi} d\alpha = F(r)^{1/2} \int_0^{\Theta} d\tau, \tag{5.91}$$

and thus:

$$2\pi R(r) = \Theta\sqrt{F(r)}. \tag{5.92}$$

Focusing on what happens very near to the horizon, we can expand $F(r)$ around r_H:

$$F(r)^{1/2}\Big|_{r=r_H} = [F(r_H) + F'(r_H)(r - r_H) + \dots]^{1/2} \simeq \sqrt{F'(r_H)}\sqrt{r - r_H}, \tag{5.93}$$

recalling that r_H is a zero for $F(r)$. Then (5.92) becomes:

$$2\pi R(r) \simeq \Theta\sqrt{F'(r_H)}\sqrt{r - r_H}, \tag{5.94}$$

and the second of the equations (5.89) is now:

$$dR(r) \simeq \frac{dr}{\sqrt{F'(r_H)}\sqrt{r - r_H}}, \tag{5.95}$$

which, upon integration, yields:

$$R(r) \simeq 2\frac{\sqrt{r - r_H}}{\sqrt{F'(r_H)}}. \tag{5.96}$$

Combining (5.94) and (5.96) we find:

$$\Theta = \frac{4\pi}{F'(r_H)}. \tag{5.97}$$

According to QFT, the temperature of the radiation near the black hole horizon, as seen by a distant observer, is in general:

$$T = \hbar\Theta^{-1} = \hbar\frac{F'(r_H)}{4\pi}. \tag{5.98}$$

For a Schwarzschild metric describing static, non-rotating and electrically neutral black holes, $F(r)$ is given by:

$$F(r) = 1 - \frac{2GM}{r}, \tag{5.99}$$

where M is the mass of the black hole. The horizon is at $r_H = 2GM$ and thus (5.98) becomes:

$$T_H = \frac{\hbar}{8\pi GM}, \tag{5.100}$$

which is the standard Hawking radiation temperature.

5.2.2 GUP modified temperature

We use now a GUP of the form:

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left(1 + \beta_0\frac{\Delta p^2}{m_{Pl}^2}\right) = \frac{\hbar}{2}\left(1 + \beta_0\frac{4l_{Pl}^2}{\hbar^2}\Delta p^2\right). \tag{5.101}$$

From classical optics we know that the smallest part δx of an object resolvable with a beam of photons of energy E is roughly:

$$\delta x \simeq \frac{\hbar}{2E}. \tag{5.102}$$

Taking into account the GUP (5.101) we have:

$$\delta x \simeq \frac{\hbar}{2E} + 2\beta_0 l_{Pl}^2 \frac{E}{\hbar}. \tag{5.103}$$

We now consider an ensemble of unpolarized photons of Hawking radiation near the event horizon of a Schwarzschild black hole. The uncertainty on the position of these photons can be estimated as:

$$\delta x \simeq \mu R_H, \tag{5.104}$$

where μ is a constant who will be later constrained and $R_H = 2GM$ is the Schwarzschild radius. Being the photons unpolarized they have two degrees of freedom and thus, from the equipartition principle, their average energy is related to the temperature as:

$$E = T. \tag{5.105}$$

Inserting (5.104) and (5.105) in (5.103) we have:

$$2\mu GM \simeq \frac{\hbar}{2T} + 2\beta_0 GT. \tag{5.106}$$

To fix μ we impose that, upon considering the semiclassical limit $\beta_0 \to 0$, (5.106) must reduce to the usual Hawking temperature (5.100). Thus we find $\mu = 2\pi$ and solving for the mass we get:

$$M = \frac{\hbar}{8\pi GT} + \beta_0 \frac{T}{2\pi}. \tag{5.107}$$

Inverting in order to find T we have:

$$T = \frac{\pi}{\beta_0}\left(M - \sqrt{M^2 - \frac{\beta_0 \hbar}{4\pi^2 G}}\right) = \frac{\pi}{\beta_0}\left(M - M\sqrt{1 - \frac{\beta_0 m_{Pl}^2}{M^2\pi^2}}\right), \tag{5.108}$$

where $\hbar/4G = m_{Pl}^2$. The term proportional to β_0 is small because of the fraction m_{Pl}/M, hence we can expand in powers of β_0:

$$\begin{aligned} T &= \frac{\pi}{\beta_0}\left[M - M\left(1 - \beta_0 \frac{m_{Pl}^2}{2\pi^2 M^2} - \beta_0^2 \frac{1}{8}\frac{m_{Pl}^4}{\pi^4 M^4} + \dots\right)\right] \\ &\simeq \frac{m_{Pl}^2}{2\pi M} + \beta_0 \frac{m_{Pl}^4}{8\pi^3 M^3} \\ &= \frac{\hbar}{8\pi GM}\left(1 + \beta_0 \frac{\hbar}{16\pi^2 GM^2}\right) \\ &= T_H \left(1 + \beta_0 \frac{m_{Pl}^2}{4\pi^2 M^2}\right). \end{aligned} \tag{5.109}$$

We note that to zero-order in β_0 we recover the standard Hawking temperature (5.100). For this expansion to be valid, the term of first-order in β_0 has to be little:

$$\beta_0 \frac{m_{Pl}^2}{4\pi^2 M^2} \ll 1, \tag{5.110}$$

thus a first, rough estimation of β_0 for a black hole of one solar mass, for which $M \simeq 10^{38} m_{Pl}$, is in order:

$$\beta_0 \leq 1.3 \times 10^{78}. \tag{5.111}$$

5.2.3 GUP modified metric

Having just seen how the GUP affects the Hawking temperature and knowing how the latter is related to the element $F(r)$ of the generic metric (5.88) we can wonder how the modified Hawking temperature (5.109) could be predicted by a suitably deformed metric. Since we are interested only in small corrections, we consider a modification of the Schwarzschild metric of the form:

$$F(r) = 1 - \frac{2GM}{r} + \epsilon \frac{G^2 M^2}{r^2}. \tag{5.112}$$

The horizon is located at the modified Schwarzschild radius defined by:

$$r^2 - 2GMr + \epsilon G^2 M^2 = 0, \tag{5.113}$$

thus, keeping only the solution closest to the Schwarzschild radius R_H, we find the position of the horizon to be:

$$r_H = R_H \frac{1 + \sqrt{1-\epsilon}}{2}, \tag{5.114}$$

with $\epsilon \leq 1$. Then:

$$F'(r) = \frac{2GM}{r^2} - 2\epsilon \frac{G^2 M^2}{r^3}, \tag{5.115}$$

and evaluating it at r_H we get:

$$\begin{aligned} F'(r_H) &= \frac{2GM}{R_H^2} \frac{4}{(1+\sqrt{1-\epsilon})^2} - 2\epsilon \frac{G^2 M^2}{R_H^3} \frac{8}{(1+\sqrt{1-\epsilon})^3} \\ &= \frac{2}{GM} \frac{\sqrt{1-\epsilon}}{(1+\sqrt{1-\epsilon})^2} \\ &\simeq \frac{1}{R_H} \left(1 - \frac{\epsilon^2}{16} + \dots \right). \end{aligned} \tag{5.116}$$

The modified Hawking radiation temperature is thus:

$$T(\epsilon) = \hbar \frac{F'(r_H)}{4\pi} = \frac{\hbar}{2\pi GM} \frac{\sqrt{1-\epsilon}}{(1+\sqrt{1-\epsilon})^2}, \tag{5.117}$$

which has to return the same result of the temperature $T(\beta_0)$ given by (5.108), that is $T(\epsilon)$ has to fulfill (5.107):

$$M = \frac{\hbar}{8\pi G T(\epsilon)} + \beta_0 \frac{T(\epsilon)}{2\pi}. \tag{5.118}$$

We can now relate the parameters β_0 and ϵ:

$$\beta_0(\epsilon) = -\pi^2 \frac{GM^2}{\hbar} \frac{\epsilon^2}{1-\epsilon}. \tag{5.119}$$

Expanding for $|\epsilon| \ll 1$ we have:

$$\beta_0(\epsilon) = -\pi^2 \frac{GM^2}{\hbar} \epsilon^2 \left(1 + \epsilon + \epsilon^2 + \dots\right) \simeq -\pi^2 \frac{GM^2}{\hbar} \epsilon^2. \tag{5.120}$$

We note that from this relation β_0 results negative. Modification caused by the GUP have been investigated in [13] for the following phenomena and the value of β_0 has been estimated using astronomical data:

- the deflection of light passing near the surface of the Sun, from which is estimated: $|\beta_0| \leq 5.3 \times 10^{78}$;
- the perihelion precession of Mercury, from which is estimated: $|\beta_0| \leq 3 \times 10^{72}$;
- the periastron precession in the bynary pulsars system *Pulsar PRS B 1913+16*, from which is estimated: $|\beta_0| \leq 2 \times 10^{71}$.

Chapter 6

Conclusions

We have seen that evidences for the existence of a minimal length scale, which we expect to be of the order of the Planck length l_{Pl}, are multiple and quite convincing. The presence of a minimal length scale can be obtained from thought experiments considering just the Heisenberg uncertainty principle and the formation of black holes expected from general relativity, it is an inevitable feature of non-commutative geometries and shows also both in quantum string theory and loop quantum gravity. The main model implementing a minimal length is the generalized uncertainty principle, using which is simple to show that distance measurements have a smaller achievable resolution of $l_{Pl}\sqrt{\beta_0}$. The GUP itself emerges from geometries based on a modification of the canonical commutation relations. Much effort has been put into finding phenomenological implications of the GUP, both in the microscopic and macroscopic field, thus opening a way to experimentally test this theory and determine the value of the deformation parameter β_0 although, unfortunately, there is still not a generally agreed upon value for it. We should also mention here that there is, of course, evidence against the need of a minimal length scale found, for example, in scenarios of emergent gravity. We stress this out in order to show that the debate is far from being settled. However, the study of the minimal length scale and of the GUP is a very promising field that we hope can lead us to a better understanding of the quantum realm and the structure of spacetime itself.

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