ALMA MATER STUDIORUM · UNIVERSITÀ DI BOLOGNA

FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI

Corso di Laurea Magistrale in Matematica

THE WIENER CRITERION
FOR THE LAPLACIAN AND THE HEAT OPERATOR

Tesi di Laurea in

Analisi Matematica

Relatore:
Chiar.mo Prof.
Ermanno Lanconelli

Presentata da:
Paola Elefante

II Sessione

Anno Accademico 2010-2011

Sommario

La tesi presenta il criterio di regolaritá di Wiener nell'ambito classico dell'operatore di Laplace ed in seguito alcune nozioni di teoria del potenziale e la dimostrazione del criterio nel caso dell'operatore del calore; in questa seconda sezione viene dedicata particolare attenzione alle formule di media e ad una diseguaglianza forte di Harnack, che risultano fondamentali nella trattazione dell'argomento centrale.

A mamma Daniela
e babbo Pino.

Introduction

The classical Dirichlet problem offers two sides, the first is the determination of the harmonic function corresponding to given boundary values, while the second is the analysis of the behaviour of this map near the boundary. Both Norbert Wiener and Henri Lebesgue pointed out this aspect of the problem in 1924 and in the same year the first one came up with a brand new characterization of regular points ([21]), i.e. a point $x_0 \in \partial\Omega$ ($\Omega \subset \mathbb{R}^n$ open set) is regular if and only if the series

$$\sum_{k=1}^{\infty} \lambda^k \operatorname{cap}(\Omega^c \cap \{\lambda^k \le |x - x_0|^{2-n} \le \lambda^{k+1}\})$$

diverges for some $\lambda > 1$, where $\operatorname{cap}(\cdot)$ is the Choquet capacity. The first chapter of this thesis shows the basic definitions and the main concepts of potential theory for the Laplace operator Δ, some properties of the Choquet capacity, then goes through some other characterizations of regular points and ends with the already mentioned Wiener's criterion. In the second chapter some more recent results are presented: the heat operator (and, in general operators of parabolic type) shows some basic features that are deeply different from the laplacian and needs a different approach. A whole potential theory was developed and the corresponding Wiener's criterion was stated. The works Bruno Pini were fundamental in the development of this field: in his work [14] of 1951,for instance, he proved a mean formula for the solutions of some parabolic operators by using the level surfaces of the fundamental solution. These formulas were used to characterize temperatures in a new way (see [16]). The formula proved by Pini was later extended to the case of several spatial dimensions later by Montaldo (1955), Fulks (1961) and Watson (1971); in this thesis we will show that it has a

fundamental role in the proof of the criterion. Another pillar of the heat operator theory is the Harnack inequality, proved indipendently by Pini and Hadarmard in 1954.
The criterion for the heat equation takes the following form: a point $z_0 \in \partial\Omega \subset \mathbb{R}^{n+1}$, ($\Omega$ bounded open set) is regular if and only if the series

$$\sum_{k=1}^{\infty} \lambda^k \operatorname{cap}_H(\Omega^c \cap \{\lambda^{k+1} \geq K(z_0 - z) \geq \lambda^k\}) = +\infty \quad \text{for some } \lambda > 1$$

where K is the fundamental solution with pole at $(0,0)$ and cap_H is the thermal capacity, analogous to the Choquet capacity of the theory of harmonic functions.
In his investigations, Pini was able to give several sufficient conditions of regularity and irregularity for some open subsets of $\mathbb{R}^{n+1}$, but it was in 1973 that Ermanno Lanconelli ([11]) proved the necessity of the criterion for any bounded open set $\Omega \subset \mathbb{R}^{n+1}$; the proof of the criterion was completed later in 1980 by Lawrence C. Evans and Ronald F. Gariepy ([4]), who used some mean formulas and a kind of strong Harnack inequality.

Contents

Chapter 1

Wiener's criterion for the Laplace operator

1.1 The Perron-Wiener-Brelot method

Let $\Omega \subset \mathbb{R}^n (n > 2)$ be an open set and f a continuous function on $\partial\Omega$; we denote by $\mathcal{U}(\Omega)$ the set of all superharmonic functions on Ω and by $\mathcal{L}(\Omega)$ the set of subharmonic functions on Ω[1]. Call the objects

$$\overline{H}_f = \inf\{v \in \mathcal{U}(\Omega), \liminf_{\Omega \ni y \to x} v(y) \geq f(x) \text{ for any } x \in \partial\Omega\}$$

$$\underline{H}_f = \sup\{v \in \mathcal{L}(\Omega), \limsup_{\Omega \ni y \to x} v(y) \leq f(x) \text{ for any } x \in \partial\Omega\}$$

respectively the *upper solution* and the *lower solution* to the generalized Dirichlet problem with boundary value f:

$$\begin{cases} \Delta u = 0 & \text{on } \Omega \\ u = f & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

It happens that $\underline{H}_f \leq \overline{H}_f$ if Ω is a bounded open set (see [9], lemma (8.2)). A function $f \in C(\partial\Omega)$ is called *resolutive* if $\overline{H}_f = \underline{H}_f =: H_f$ and they are finite-valued. One can prove H_f is harmonic (see for example [1], chapter 6); H_f is called the Perron-Wiener-Brelot solution to the Dirichlet problem (1.1).

[1]We ask superharmonic functions not to be identically $+\infty$ on any component of Ω.

Wiener proved that every continuous function defined on the boundary of a bounded open set is resolutive (see [9], theorem (8.11) for a proof).

1.2 Preliminaries

In all our discussion we will be working in $\mathbb{R}^n$ with $n > 2$. Let $F \subset \mathbb{R}^n$ be a compact subset of Ω. Pick $u \in \mathfrak{U}(\Omega), u \geq 0$ and put:

$$\begin{cases} \Phi_F^u := \{v \in \mathfrak{U}(\mathbb{R}^n) : v \geq 0 \text{ on } \mathbb{R}^n, v \geq u \text{ on } F\} \\ \mathbf{R}_F^u(x) := \inf\limits_{v \in \Phi_F^u} v(x) \end{cases}$$

$\mathbf{R}_F^u$ is called the *reduced function* (or *réduite*) of u relative to F in $\mathbb{R}^n$.
In general, $\mathbf{R}_F^u$ is not lower semicontinuous (l.s.c.): consider the fundamental solution $U_y(x) = |x-y|^{2-n}$ with pole at y, take $\Omega = \mathbb{R}^n, F = \{y\}$. Then one has

$$\mathbf{R}_F^{U_y}(x) = \begin{cases} 0 & x \neq y \\ +\infty & x = y \end{cases}$$

and hence it is not l.s.c..
This is why we define the *regularized reduced function* (or *balayage*) of u relative to F on $\mathbb{R}^n$ as:

$$\widehat{\mathbf{R}}_F^u(x) = \liminf_{y \to x} \mathbf{R}_F^u(y) \qquad x \in \mathbb{R}^n$$

From now on we will denote by $W_F = \mathbf{R}_F^1, V_F = \widehat{\mathbf{R}}_F^1$ where F is a compact set as above. Let us state some basic properties of these new objects.

Lemma 1.2.1. *With the same notations as before:*

(i) $0 \leq V_F \leq W_F \leq 1$.

(ii) $W_F = 1$ *on* F.

(iii) $V_F = W_F$ *on* $\operatorname{int} F \cup F^c$.

(iv) One has

$$\lim_{\Omega \ni |y| \to \infty} V_F(y) = \lim_{\Omega \ni |y| \to \infty} W_F(y) = 0$$

except for a polar set (see the definiton later).

(v) V_F *is superharmonic on* Ω *and harmonic on* F^c.

Proof. Since $1 \in \Phi_F^1$, by definition of V_F and W_F one sees immediately that $V_F \leq W_F \leq 1$. Plus, as the elements of Φ_F^1 are non-negative, one must have $V_F \geq 0$. By definition, it is obvious that $W_F = 1$ on F, as $v \geq 1$ on F for any $v \in \Phi_F^1$; as W_F is constantly 1 on $\operatorname{int} F$, V_F must be identically 1 there as well.
Helms proves the rest of this theorem in [9], chapter 7. □

Denote by $\mathcal{M}^+(F)$ the set of all non-negative Radon measures with support contained in F. For any $\mu \in \mathcal{M}^+(F)$ we define the μ-*Green potential*:

$$G_\mu(x) = \int_{\mathbb{R}^n} |x-y|^{2-n} d\mu(y)$$

Now set:

$$\operatorname{cap}(F) = \sup\{\mu(\mathbb{R}^n) : \mu \in \mathcal{M}^+(F), G_\mu \leq 1 \text{ on } \mathbb{R}^n\}$$

Theorem 1.2.2. *For any compact sets* $F, F' \subset \mathbb{R}^n$*, one has:*

(i) $\operatorname{cap}(F) < \infty$.

(ii) $\operatorname{cap}(F \cup F') \leq \operatorname{cap}(F) + \operatorname{cap}(F')$.

(iii) If $F \subset F'$ *then* $\operatorname{cap}(F) \leq \operatorname{cap}(F')$.

(iv) If $\{F_l\}_{l\in\mathbb{N}}$ *is a decreasing sequence of compact sets and* $F = \bigcap_{l=1}^{\infty} F_l$*, one has* $\lim_{l\to\infty} \operatorname{cap}(F_l) = \operatorname{cap}(F)$.

Proof. (i): as F is compact, we can pick $x_0 \in F^c$. Then $x \mapsto G(x_0, x) \geq 0$ is harmonic on a neighbourhood of F and hence by maximum principle $G(x_0, x) > \epsilon > 0$ on F for some ϵ. Set $\alpha := \inf_{x\in F} G(x_0, x) > 0$; for any $\mu \in \mathcal{M}^+(F)$ such that $G_\mu \leq 1$ one has:

$$\alpha\mu(F) = \inf_{x\in F} G(x_0,x) \int_{\mathbb{R}^n} d\mu(y) \leq \int_{\mathbb{R}^n} G(x_0,y)d\mu(y) = G_\mu(x_0) \leq 1 \Rightarrow \mu(F) \leq \frac{1}{\alpha}$$

and this proves the finiteness of $\operatorname{cap}(F)$.
(ii): pick one $\mu \in \mathcal{M}^+(F \cup F')$ such that $\mu(\mathbb{R}^n) = \operatorname{cap}(F \cup F')$. Obviously $\mu_{|F} \in \mathcal{M}^+(F), G_{\mu_{|F}} \leq G_\mu \leq 1$ and the same happens for $\mu_{|F'}$; hence we have

$$\operatorname{cap}(F \cup F') = \mu(F \cup F') \leq \mu(F) + \mu(F') \leq \operatorname{cap}(F) + \operatorname{cap}(F')$$

(iii): this holds because

$$\left\{\mu \in \mathcal{M}^+(F), G_\mu \leq 1 \text{ on } \mathbb{R}^n\right\} \subset \left\{\mu \in \mathcal{M}^+(F'), G_\mu \leq 1 \text{ on } \mathbb{R}^n\right\}$$

(iv): see (**??**), theorem (5.4.2) for a proof. □

We now state a very important result.

Theorem 1.2.3. *Let $F \subset \mathbb{R}^n$ be compact. There exists a unique $\mu^\star \in \mathcal{M}^+(F)$ such that:*

(i) $G_{\mu^\star} = V_F$.

(ii) $\mathrm{cap}(F) = \mu^\star(\mathbb{R}^n)$.

(iii) $G_\mu \leq G_{\mu^\star}$ *for any* $\mu \in \mathcal{M}^+(F)$ *such that* $G_\mu \leq 1$.

We call $\mu^\star$ the *capacitary distribution* of F and V_F the *capacitary potential* for F.
It is possibile to define a capacity for any subset E of $\mathbb{R}^n$; we call *inner capacity* of E the quantity:

$$\mathrm{cap}_\star(E) = \sup\{\mathrm{cap}(F) : F \subset E, F \text{ compact}\}$$

Similarly, the *outer capacity* of E is defined as

$$\mathrm{cap}^\star(E) = \inf\{\mathrm{cap}_\star(U) : E \subset U, U \text{ open}\}$$

A set $E \subset \mathbb{R}^n$ is said to be *capacitable* if $\mathrm{cap}_\star(E) = \mathrm{cap}^\star(E)$ and we call *capacity* of E the quantity $\mathrm{cap}(E) := \mathrm{cap}_\star(E)$.

Lemma 1.2.4. *All open and compact subsets of $\mathbb{R}^n$ are capacitable.*

Proof. If $U \subset \mathbb{R}^n$ is open, $\mathrm{cap}^\star(U) = \mathrm{cap}_\star(U)$ by definition. Let $F \subset \mathbb{R}^n$ be compact; obviously $\mathrm{cap}_\star(F) = \mathrm{cap}(F)$ accordingly to the definition of capacity of compact sets given at first. For any $\epsilon > 0$, there exist U so that if F' is compact and $F \subset F' \subset U$, one has $\mathrm{cap}(F') \leq \mathrm{cap}(F) + \epsilon$. Hence:

$$\mathrm{cap}(F') = \mathrm{cap}_\star(F') \leq \mathrm{cap}_\star(U) = \sup_{F'' \subset U} \mathrm{cap}(F'') \leq \mathrm{cap}(F) + \epsilon$$

For any open set V such that $F \subset V \subset U$ one gets $\text{cap}_\star(F) \le \text{cap}_\star(V) \le \text{cap}_\star(U)$ and therefore

$$\text{cap}(F) = \text{cap}_\star(F) \le \inf_{V \supset F} \text{cap}_\star(V) = \text{cap}^\star(F) \le \text{cap}_\star(U) \le \text{cap}(F) + \epsilon$$

This is true for any $\epsilon > 0$, so $\text{cap}(F) = \text{cap}^\star(F)$. □

If $\omega \subset \mathbb{R}^n$ is such that $\text{cap}^\star(\omega) = 0$ we call it a *polar set*.

Lemma 1.2.5. *If $\omega \subset \mathbb{R}^n$ and $u \ge 0$ is a superharmonic function on $\mathbb{R}^n$ it occurs that*

$$\mathbf{R}^u_\omega = \widehat{\mathbf{R}}^u_\omega$$

except for a polar set $Z \subset \partial\omega$.

Proof. See [9], corollary (7.40). □

1.3 Characterization of regular points

Let $\Omega \subset \mathbb{R}^n$ be an open set. A function w is a barrier at $x_0 \in \partial\Omega$ if it is defined on $W \cap \Omega$ for some neighbourhood W of x_0 and has the following properties:

1. w is superharmonic on $W \cap \Omega$.

2. $w > 0$ on $W \cap \Omega$.

3. $\displaystyle\lim_{W \cap \Omega \ni x \to x_0} w(x) = 0$.

If x_0 is a limit point of Ω, we say Ω is *thin at x_0* if there exist a superharmonic function u on a neighbourhood of x_0 such that $0 < u(x_0) < \displaystyle\liminf_{\Omega \ni x \to x_0, x \ne x_0} u(x) = +\infty$.

Lemma 1.3.1. *Let Ω be thin at the limit point x_0; then*

$$V_{F(r)}(x_0) \to 0 \qquad \textit{for } r \to 0^+$$

where $F(r) = \Omega \cap B(x_0, r)$.

Proof. By definition we know there exists a superharmonic function u such that

$$0 < u(x_0) < +\infty = \liminf_{\Omega \ni x \to x_0, x \neq x_0} u(x)$$

Choose $\epsilon > 0$; we can pick $r > 0$ so that $u(x) > \dfrac{u(x_0)}{\epsilon}$ for all $x \in F(r) \setminus \{x_0\}$. The set $\{x_0\}$ has capacity zero, hence $V_{F(r)} = V_{F(r)\setminus\{x_0\}}$ (see [1], theorem (5.3.4)). By lemma (1.2.1)-(i) we know $V_{F(r)}(x) \leq \epsilon \dfrac{u(x)}{u(x_0)}$ on $F(r)$ and in particular $V_{F(r')}(x_0) \leq V_{F(r)}(x_0) \leq \epsilon$ for any $0 < r' < r$, as $F(r)$ is a decreasing sequence. □

Lemma 1.3.2. *If Ω is not thin at $x_0 \in \partial\Omega$, then $V_{F(r)}(x_0) = 1$ for all $r > 0$.*

Proof. If Ω is not thin at x_0, by definition this means that for any superharmonic function v defined on a neighbourhood of x_0, at least one of the following statements holds:

(i) $u(x_0) = +\infty$.

(ii) $u(x_0) \geq \liminf_{\Omega \ni x \to x_0, x \neq x_0} u(x)$.

(iii) $\liminf_{\Omega \ni x \to x_0, x \neq x_0} u(x) < +\infty$

If we pick $u = V_{F(r)}$, we know the only possible case is $1 \geq u(x_0) \geq \liminf_{\Omega \ni x \to x_0, x \neq x_0} u(x) = 1$, hence $V_{F(r)}(x_0) = 1$ for any r. □

We say that the function $u : D \to [-\infty, +\infty]$ *peaks at* $x_0 \in D$ if

$$\sup\{u(x) : x \in D \setminus B(x_0, r)\} < u(x_0)$$

for all $r > 0$ such that $D \setminus B(x_0, r) \neq \emptyset$.

Theorem 1.3.3. *Let $F \subset \Omega$ and $u \geq 0, u \in \mathcal{U}(\Omega)$. Assume u peaks at $x_0 \in \Omega$ and $u(x_0) < \infty$. Then F is thin at x_0 if and only if $\widehat{\mathbf{R}}_u^F(y) < u(y)$.*

Proof. See [1], theorem (7.3.4). □

Theorem 1.3.4. *Let Ω be a bounded open set that has a Green function and $x_0 \in \partial\Omega$. Then the following statements are equivalent:*

(i) x_0 is a regular point.

(ii) There exists a barrier at x_0.

(iii) Ω^c is not thin at x_0.

Proof. (i)$\Leftrightarrow$(ii): suppose at first x_0 is regular and consider the map $w(x) = |x_0 - x|, x \in \partial\Omega$; then by maximum principle the harmonic solution H_w is such that $H_w(x) \geq |x - x_0| \geq 0$ for any $x \in \Omega$ and if we prove $\lim_{\Omega \ni x \to x_0} H_w(x) = 0$, we can say it is a barrier at x_0, but this is true as by assumption, we know $\lim_{\Omega \ni x \to x_0} H_w(x) = w(x_0) = 0$, because w is a continuous function on $\partial\Omega$. Now suppose there is a barrier at x_0; for any bounded f on $\partial\Omega$ one has

$$\limsup_{\Omega \ni x \to x_0} \overline{H}_f(x) \leq \limsup_{\partial\Omega \ni x \to x_0} f(x)$$
$$\liminf_{\Omega \ni x \to x_0} \underline{H}_f(x) \geq \liminf_{\partial\Omega \ni x \to x_0} f(x)$$

(see [9], lemma (8.20) for a proof). If f is continuous, by applying this last statement we get

$$\liminf_{\Omega \ni x \to x_0} \overline{H}_f(x) \leq \limsup_{\Omega \ni x \to x_0} \overline{H}_f(x) \leq f(x_0) \leq \liminf_{\Omega \ni x \to x_0} \underline{H}_f(x) \leq \liminf_{\Omega \ni x \to x_0} \overline{H}_f(x)$$

hence $\lim_{\Omega \ni x \to x_0} \overline{H}_f(x) = f(x_0)$; as $\liminf_{\Omega \ni x \to x_0} \underline{H}_f(x) \leq \liminf_{\Omega \ni x \to x_0} \overline{H}_f(x)$ one can conclude that also $\lim_{\Omega \ni x \to x_0} \underline{H}_f(x) = f(x_0)$, hence x_0 is a regular point.
(i)$\Leftrightarrow$(iii): assume Ω^c is not thin at x_0 and consider the ball $B = B(x_0, 1)$. Define the positive superharmonic function $u(x) = 1 - |x - x_0|^2$ on B ($\Delta u = -2n < 0$); now build the positive superharmonic function $w := u - \widehat{\mathbf{R}}^u_{\Omega^c \cap B}$ where the balayage is with respect to B. w has to be strictly positive on $B \cap \Omega$; in fact, if there is $x \in B \cap \Omega$ such that $w(x) = 0$, by maximum principle $w = 0$ on all $B \cap \Omega$, hence $u = \widehat{\mathbf{R}}^u_{\Omega^c \cap B}$ there and by lemma (1.2.1)-(v) u would be harmonic on $B \cap \Omega$, but this would be a contradiction since $\Delta u < 0$ there. As u peaks in x_0, theorem (1.3.3) and the assumption imply that $w(x_0) = 0$, therefore w is a barrier at x_0 and this implies x_0 is regular. Suppose now x_0 is regular. Pick r' small enough so that $\Omega' := \Omega \cup B(x_0, r')$ still has a Green function; for each $0 < r < r'$ define

$$f_r(x) = \begin{cases} 1 & x \in \partial\Omega \cup B(x_0, r) \\ 0 & x \in \partial\Omega \setminus B(x_0, r) \end{cases}$$

Call $E(r)$ the compact set $\overline{B(x_0, r)} \setminus \Omega$. Taking the reductions with respect to Ω' and recalling lemma (1.2.1):

$$1 = V_{E(r)}(x) = W_{E(r)}(x) \geq H_{f_r}(x) \qquad x \in \mathrm{int}(E(r))$$

As x_0 is regular, we have $\lim_{\Omega \ni x \to x_0} H_{f_r}(x) = f_r(x_0) = 1$ hence $V_{E(r)}(x_0) = 1$ for all $0 < r < r'$; by lemma (1.3.1) we know Ω^c is not thin at x_0. □

We can now state a very important result that has an analogy in the theory of the heat operator (see lemma (2.3.2), chapter 2) that is a fundamental tool for various proofs.

Lemma 1.3.5. *If Ω is a bounded open set with Green function, $x_0 \in \partial\Omega$ is regular if and only if*

$$V_{F(r)}(x_0) = 1 \qquad \forall r > 0$$

Proof. It is enough to put together the results of lemma (1.3.1), lemma (1.3.2) and theorem (1.3.4). □

Theorem 1.3.6 (Wiener's criterion)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $x_0 \in \partial\Omega$. Then x_0 is a regular boundary point of Ω if and only if*

$$\sum_{k=1}^{\infty} \lambda^k \operatorname{cap}(\Omega^c \cap \{\lambda^k \leq |x - x_0|^{2-n} \leq \lambda^{k+1}\}) = +\infty \qquad \textit{for some } \lambda > 1$$

Proof. Suppose at first that

$$\sum_{k=1}^{\infty} \lambda^k \operatorname{cap}(C_k) < +\infty$$

where we call $\Omega^c \cap \{\lambda^k \leq |x - x_0|^{2-n} \leq \lambda^{k+1}\} =: C_k$. As the regularity is a local property, we can assume $\Omega \subset B := B(x_0, 1)$. Let $\{\epsilon_l\}_l$ be a sequence of positive numbers such that $\sum_{k=1}^{\infty} \lambda^k \epsilon_k < \infty$; for any $k \in \mathbb{N}$ call μ_k the capacitary distribution of the compact set C_k and $V_k = G_{\mu_k}$ its capacitary potential; recall that $\mu_k(\mathbb{R}^n) = \operatorname{cap}(C_k)$. Then

$$V_k(x_0) = \int_{C_k} |y - x_0|^{2-n} d\mu_k(y) \leq \lambda^{k+1} \operatorname{cap}(C_k)$$

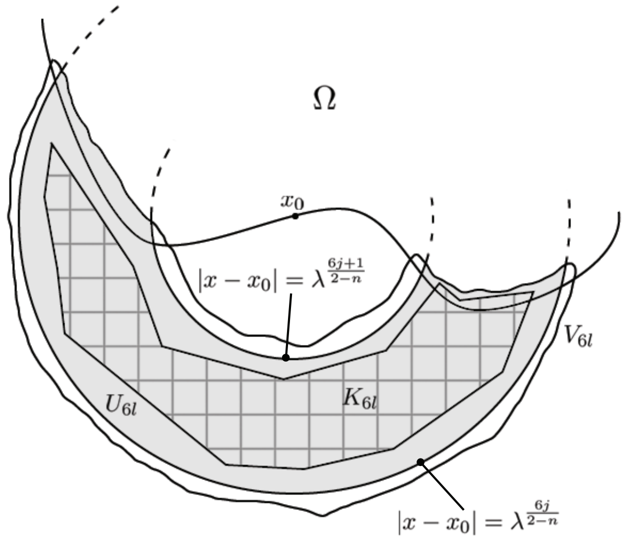

Figure 1.1: Proof of theorem (1.3.6).

As by assumption the series converges, we can choose k so that

$$V_k(x_0) \le \lambda^2 \sum_{j \ge k} \operatorname{cap}(C_j) < 1$$

and by lemma (1.3.4) this makes us conclude that x_0 is irregular.
Assume x_0 is irregular; by thereom (1.3.4) this tells us there is a superharmonic function u such that $0 < u(x_0) < +\infty = \liminf_{\Omega \ni x \to x_0, x \neq x_0} u(x)$; call $\alpha_k = \inf_{C_k} u$ and fix arbitrarily a positive number α. Set

$$\begin{aligned} V_k &= \{x : v(x) > \alpha_k - \alpha\} \supset C_k \\ U_k &= V_k \cap \{\lambda^k \le |x - x_0|^{2-n} \le \lambda^{k+1}\} \supset C_k \end{aligned}$$

If we prove $\sum_{k=1}^{\infty} \lambda^k \operatorname{cap}^\star(U_k) < \infty$ we are done, because $C_k \subset U_k, \forall k$. Let $\{\epsilon_l\}_l$ be a sequence of positive numbers such that $\sum_{k=1}^{\infty} \lambda^k \epsilon_k < \infty$ and for any $l \in \mathbb{N}$ pick a compact set $K_l \subset U_l$ such that $\operatorname{cap}^\star(U_l) < \operatorname{cap}(K_l) + \epsilon_l$ (see figure 1.3.6). Then we need to prove $\sum_{l=1}^{\infty} \lambda^l \operatorname{cap}(K_l) < \infty$. To do that, consider the six series:

$$\sum_{l=1}^{\infty} \lambda^{6l+j} \operatorname{cap}(K_{6l+j}) \qquad j = 0, 1, \cdots, 5$$

If they all converge, we are done; pick, for instance, $j = 0$. Note that

$$\begin{aligned} K_{6l} \subset U_{6l} \subset \{(\lambda^3)^{2l} \leq |x - x_0|^{2-n} \leq (\lambda^3)^{2l}\lambda\} \\ \subset \{\lambda^{-1}(\lambda^3)^{2l} \leq |x - x_0|^{2-n} \leq \lambda^2(\lambda^3)^{2l}\} = \{(\lambda^3)^{2l} \leq \lambda|x - x_0|^{2-n} \leq (\lambda^3)^{2l+1}\} \end{aligned} \tag{1.2}$$

Let $l, j \in \mathbb{N}$ and consider the three cases:

$$\begin{cases} l = j & K_{6l} \subset \{(\lambda^3)^{2j} \leq \lambda|x - x_0|^{2-n} \leq (\lambda^3)^{2j+1}\} \\ l > j \Rightarrow (\lambda^3)^{2l} > (\lambda^3)^{2j+1} & K_{6l} \subset \{(\lambda^3)^{2j} \leq \lambda|x - x_0|^{2-n} \leq (\lambda^3)^{2j+1}\}^c \\ l < j \Rightarrow (\lambda^3)^{2l+1} < (\lambda^3)^{2j} & K_{6l} \subset \{(\lambda^3)^{2j} \leq \lambda|x - x_0|^{2-n} \leq (\lambda^3)^{2j+1}\}^c \end{cases}$$

In particular one sees that the sets $K_{6l}, l \geq 1$ are disjoint; call $K := \{x_0\} \cup \Big(\bigcup_{l \geq 1} K_l\Big)$. K is cleary bounded and it contains all its limit points, therefore it is compact. Call $w = \widehat{\mathbf{R}}_K^u$; as $u(x_0) < \infty$ and $w \leq u$, we have $w(x_0) < \infty$. Let μ be such that $w = G_\mu$; μ has support in K and $\mu(B \setminus \bigcup_{l \geq 1} K_l) = 0$, hence we can write

$$w(x) = \int_{K_{6j}} |x - y|^{2-n} d\mu(y) + \int_{\bigcup_{l \neq j} K_l} |x - y|^{2-n} d\mu(y)$$

Since $\bigcup_{l \neq j} K_l \subset B \cap \{(\lambda^3)^{2j} \leq \lambda|x - x_0|^{2-n} \leq (\lambda^3)^{2j+1}\}^c$ there is a constant $\beta = \beta(\lambda, \mu)$ so that

$$\int_{\bigcup_{l \neq j} K_l} |x - y|^{2-n} d\mu(y) \leq \beta$$

(see lemma (10.20) in [9]). This implies

$$w(x) \leq \beta + \int_{K_{6j}} |x - y|^{2-n} d\mu(y) \quad x \in B \cap \{(\lambda^3)^{2j} \leq |x - x_0|^{2-n} \leq (\lambda^3)^{2j}\lambda\}$$

By lemma (1.2.5) we know $w = u$ except for a polar set $Z \subset \partial K$; recall that by definition $u \geq \alpha_j - \alpha$ on K_j and $\alpha_j \overset{j \to \infty}{\longrightarrow} +\infty$, so $\lim_{(K \setminus Z) \ni x \to x_0} w(x) = +\infty$. Pick a positive integer q so that $q - \beta \geq \epsilon > 0$ for some ϵ. By what we have proved above, there is j_0 such that $w(x) \geq q$ for $x \in K_{6j}, j \geq j_0$ except for a polar set. Therefore

$$\int_{K_{6j}} |x-y|^{2-n} d\mu(y) \geq \epsilon \qquad x \in K_{6j}, j \geq j_0$$

Claim: $\mu(K_{6j}) \geq \epsilon \operatorname{cap}(K_{6j})$ for all $j \geq j_0$. Call $\mu^\star$ the capacitary distribution of K_{6j}; then

$$\mu(K_{6j}) = \int_{K_{6j}} d\mu \geq \int_{K_{6j}} G_{\mu^\star} d\mu = \int_{K_{6j}} G_\mu d\mu^\star \geq \epsilon \operatorname{cap}(K_{6j})$$

therefore

$$\sum_{j \geq j_0} \lambda^{6j} \mu(K_{6j}) \geq \epsilon \sum_{j \geq j_0} \lambda^{6j} \operatorname{cap}(K_{6j}) \tag{1.3}$$

Notice that

$$\sum_{j \geq j_0} \lambda^{6j} \mu(K_{6j}) \leq \sum_{j \geq j_0} \int_{K_{6j}} |x_0 - x|^{2-n} d\mu(x) \leq \int_K |x_0 - x|^{2-n} d\mu(x)$$

and

$$\infty > w(x_0) = \int_K |x - x_0|^{2-n} d\mu(x)$$

therefore the original series converges. $\square$

Chapter 2

Wiener's criterion for the heat equation

2.1 Preliminaries

In this chapter we would like to show the proof of the Wiener's test in the case of the heat operator:

$$H = \partial_t - \Delta_x$$

in $\mathbb{R}^n_x \times \mathbb{R}_t$. We denote by

$$K(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

the fundamental solution of the heat operator with pole at $(0,0)$.
The next definitions and notions are parallel to the ones of the potential theory of the Laplacian. Fix a bounded open set $\Omega \subset \mathbb{R}^{n+1}$.
A function $u \in C^2(\Omega)$ is said to be a *temperature* in Ω if $Hu = 0$ on Ω.
A bounded open set $U \subset \mathbb{R}^{n+1}$ is *H-regular* if for each $f \in C(\partial U)$ there exists a unique temperature H^U_f such that

$$\lim_{z \to z_0} H^U_f(z) = f(z_0) \qquad \forall z_0 \in \partial U$$

Any u such that

(i) $-\infty < u \leq +\infty$, $\{u \neq +\infty\}$ is a dense subset of Ω.

(ii) u is lower semicontinuous.

(iii) If $U \subset \overline{U} \subset \Omega$ is a regular open set and $f \in C(\partial U)$ is such that $f \leq u$ on ∂U, then $H_f^U \leq u$ on U.

is called *supertemperature* on Ω. We denote by $\mathcal{U}_T(\Omega)$ the set of all supertemperatures on Ω.
u is a *subtemperature* if $-u$ is a supertemperature and we denote by $\mathcal{L}_T(\Omega)$ the set of subtemperatures on Ω.
Fix $f \in C(\partial\Omega)$. We define the *generalized solution* in the sense of Perron-Wiener-Brelot-Bauer of the Dirichlet problem

$$\begin{cases} Hu = 0 & \text{on } \Omega \\ u = f & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

to be

$$H_f^\Omega = \inf\{u : u \in \mathcal{U}_T(\Omega), \liminf_{\Omega \ni \zeta \to z_0} u(\zeta) \geq f(z_0) \text{ for any } z_0 \in \partial\Omega\}$$

The function H_f^Ω is a temperature but it can happen that it does not take continuously the given boundary values; for this reason we introduce the following notion: a point $z_0 \in \partial\Omega$ is said to be *regular* for Ω if

$$\lim_{\Omega \ni \zeta \to z_0} H_f^\Omega(\zeta) = f(z_0) \quad \forall f \in C(\partial\Omega)$$

Pick a closed $K \subset \Omega$; then we denote by

$$\mathcal{M}^+(K) = \{\mu : \mu \text{ is a nonnegative Radon measure on } \mathbb{R}^{n+1}, \operatorname{supp}\mu \subset K\}$$

For any $\mu \in \mathcal{M}^+(\mathbb{R}^{n+1})$ we define the *μ-potential*:

$$K_\mu(z) = \int_{\mathbb{R}^{n+1}} K(z-\zeta)d\mu(\zeta)$$

where $K(z) = K(x,t)$ is the fundamental solution with pole at $(0,0)$. Now let F be a compact subset of $\mathbb{R}^{n+1}$; the *thermal capacity* of F is

$$\text{cap}_H(F) = \sup\{\mu(\mathbb{R}^{n+1}) : \mu \in \mathcal{M}^+(F), K_\mu \leq 1 \text{ on } \mathbb{R}^{n+1}\}$$

For $z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$ and $c > 0$, we define respectively the parabolic ball and the parabolic sphere centered at z_0 and with radius c to be:

$$\Omega(z_0, c) = \{z \in \mathbb{R}^{n+1} : K(z_0 - z) > (4\pi c)^{-n/2}\} \tag{2.2}$$

$$\Psi(z_0, c) = \{z \in \mathbb{R}^{n+1} : K(z_0 - z) = (4\pi c)^{-n/2}\} \tag{2.3}$$

and build the analogous anulus as in the case of the laplacian:

$$A(z_0, c) = \overline{\Omega(z_0, c) \setminus \Omega(z_0, c/2)}$$

The aim of this chapter is to prove the following result.

Theorem 2.1.1 (Wiener's criterion for heat equation)**.** *[Lanconelli - Evans - Gariepy] A point $z_0 \in \partial\Omega$ is regular if and only if*

$$\sum_{k=1}^{\infty} 2^{kn/2} \operatorname{cap}_H(\Omega^c \cap A(z_0, 2^{-k})) = +\infty \tag{2.4}$$

or, equivalently

$$\sum_{k=1}^{\infty} \lambda^k \operatorname{cap}_H(\Omega^c \cap \{\lambda^{k+1} \geq K(z_0 - z) \geq \lambda^k\}) = +\infty \quad \textit{for any } \lambda > 1 \tag{2.5}$$

One of the key ideas was to notice that the anuli used in the criterion for the laplacian were the level surfaces of the fundamental solution; for this reason we have defined the $A(z_0, c)$'s.

2.2 Properties of thermal capacity and potentials

We want to show some useful properties of these two important concepts.

Lemma 2.2.1. *For any compact sets $F, F' \subset \mathbb{R}^{n+1}$ and $\lambda > 0$, we have:*

(i) $\operatorname{cap}_H(F) < \infty$.

(ii) (subadditivity) $\operatorname{cap}_H(F \cup F') \leq \operatorname{cap}_H(F) + \operatorname{cap}_H(F')$.

(iii) If $F \subset F' \Rightarrow \operatorname{cap}_H(F) \leq \operatorname{cap}_H(F')$.

(iv) Denote $\lambda F = \{(\lambda x, \lambda^2 t) : (x, t) \in F\}$*; then* $\operatorname{cap}_H(\lambda F) = \lambda^n \operatorname{cap}_H(F)$.

(v) If $\{F_l\}_{l\in\mathbb{N}}$ is a decreasing sequence of compact sets and $F = \bigcap_{l=1}^{\infty} F_l$, one has $\lim_{l\to\infty} \operatorname{cap}_H(F_l) = \operatorname{cap}_H(F)$.

(vi) Denote by $\widehat{F} = \{(x, t) \in \mathbb{R}^{n+1} : (x, -t) \in F\}$; hence $\operatorname{cap}_H(\widehat{F}) = \operatorname{cap}_H(F)$.

(vii) $\text{cap}_H(z_0 + F) = \text{cap}_H(F) \quad \forall z_0 \in \mathbb{R}^{n+1}$.

(viii) $\text{cap}_H(\{z\}) = 0 \quad \forall z \in \mathbb{R}^{n+1}$.

Proof. (i): as F is compact, we can pick $z_0 = (x_0, t_0)$ so that $t_0 > t \ \forall t : (x,t) \in F$; $\alpha := \inf_F K(z_0 - z)$ is strictly positive. Therefore for any $\mu \in \mathcal{M}^+(F)$ with $K_\mu \leq 1$ one has

$$\alpha\mu(F) = \int_{\mathbb{R}^{n+1}} \inf_F K(z_0 - z) d\mu(\zeta) \leq K_\mu(z_0) \leq 1 \Rightarrow \text{cap}_H(F) \leq \frac{1}{\alpha}$$

(ii): the proof is the same of theorem (1.2.2)-(ii).
(iii): the statement follows from:

$$\Big\{\mu \in \mathcal{M}^+(F), K_\mu \leq 1\Big\} \subset \Big\{\mu \in \mathcal{M}^+(F'), K_\mu \leq 1\Big\}$$

(iv): it is easy to check that $K(\lambda x, \lambda^2 t) = \lambda^{-n} K(x,t)$ for any $(x,t) \in \mathbb{R}^{n+1}, \lambda > 0$.
(v)-(vi): see [**?**] for the proof on these statements.
(vii): this is obvious as every $\mu \in \mathcal{M}^+(F)$ is invariant under translation.
(viii): by (vii) it is enough to prove that $\text{cap}_H(\{0\}) = 0$. Consider the decreasing sequence $\{F_n\}_n$ where $F_n = \overline{\Omega(0, c_n)}$ and $c_n = 2^{-n}$; then $\bigcap_{n=0}^{\infty} F_n = \{0\}$ and by (v) we have

$$\text{cap}_H(\{0\}) = \lim_{n\to\infty} \text{cap}_H(F_n) = \lim_{n\to\infty} 2^{-n^2} \text{cap}_H(F_0) = 0$$

$\square$

Let $F \subset \mathbb{R}^{n+1}$ be a compact set; we define

$$\begin{cases} \Phi_F := \{v \in \mathcal{U}_T(\mathbb{R}^{n+1}) : v \geq 0, v \geq 1 \text{ on } F\} \\ \eta_F(z) := \inf_{v \in \Phi_F} v(z) \end{cases}$$

We also denote the *lower semicontinuous regularization of* η_F in $\mathbb{R}^{n+1}$ by:

$$\zeta_F(z) := \lim_{\epsilon \downarrow 0} \Big(\inf_{w \in B(z,\epsilon)} \eta_F(w) \Big) = \liminf_{w \to z} \eta_F(w)$$

The theorem that follows collects several properties that will be fundamental in the proof of the final result.

Theorem 2.2.2. *For any compact set $F \subset \mathbb{R}^{n+1}$ one has:*

(i) $0 \le \zeta_F \le \eta_F \le 1$ *on* $\mathbb{R}^{n+1}$.

(ii) $\eta_F = \zeta_F$ *on* $(\partial F)^c$.

(iii) $\eta_F = 1$ *on* F.

(iv) $\lim_{|z|\to\infty} \eta_F(z) = \lim_{|z|\to\infty} \zeta_F(z) = 0$.

(v) ζ_F *is a supertemperature on* $\mathbb{R}^{n+1}$ *and a temperature on* F^c.

(vi) There exists a unique $\widetilde{\mu} \in \mathcal{M}^+(F)$, *called equilibrium measure, such that:*

$$\begin{cases} \zeta_F = K_{\widetilde{\mu}} \quad \textit{(equilibrium potential)} \\ \widetilde{\mu}(\mathbb{R}^{n+1}) = \mathrm{cap}_H(F) \\ H\zeta_F = \widetilde{\mu} \quad \textit{in the sense of distributions on } \mathbb{R}^{n+1} \\ K_\mu \le K_{\widetilde{\mu}} \quad \textit{for all } \mu \in \mathcal{M}^+(F) \textit{ so that } K_\mu \le 1 \end{cases}$$

Proof. For (i) and (iii) we can refer to the proof of lemma (1.2.1) in Chapter 1.
The other statements are proved in (**??**), p. 86-88. □

Remark 2.2.3. *Thanks to lemma* (2.2.1)-(vi) *we can state the existence and uniqueness of* $\hat{\mu} \in \mathcal{M}^+(F)$ *and a lower semicontinuous function* $\hat{\zeta}_F, 0 \le \hat{\zeta}_F \le 1$, *such that*

$$\begin{cases} \hat{\mu}(\mathbb{R}^{n+1}) = \mathrm{cap}_H(\widehat{F}) = \mathrm{cap}_H(F) \\ \widehat{H}u = \hat{\mu} \quad \textit{in the sense of distributions} \end{cases}$$

where $\widehat{H} = -\partial_t - \Delta$ *is the backwards heat operator. We call* $\hat{\mu}, \hat{\zeta}_F$ *respectively the backwards equilibrium measure and the backwards equilibrium potential.*

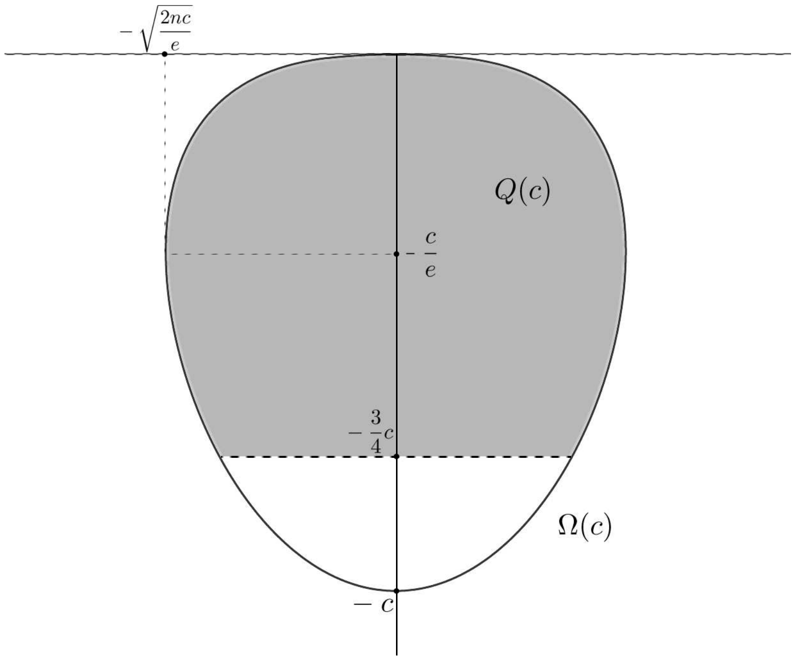

Figure 2.1: Section 2.3: the generic set $\Omega(c)$.

2.3 Properties of the sets $\Omega(c)$

It is not restrictive to suppose $z_0 = (0,0)$ in our discussion. Recalling definition (2.2), we can rewrite:

$$\Omega(c) := \Omega(0,c) = \{(x,t) : -c < t < 0, |x|^2 < R_c(t)^2\}$$

$R_c(t) = \sqrt{2nt \log\left(\frac{-t}{c}\right)}$ being the radius of the spherical cross section of $\Omega(c)$ at t. Notice that:

$$\begin{cases} R_c(0) = R_c(-c) = 0 \\ R_c \text{ attains its maximum at } \left(\frac{-c}{e}, \sqrt{\frac{2nc}{e}}\right) \end{cases}$$

Remark 2.3.1. *For any $c_1 < c_2$ one has $\Omega(c_1) \subset \Omega(c_2)$.*

Proof. This is a simple calculation: there is no $-c_1 < t < 0$ such that $R_{c_1}(t) = R_{c_2}(t)$. Indeed

$$2nt \log\left(-\frac{t}{c_1}\right) = 2nt \log\left(-\frac{t}{c_2}\right) \Leftrightarrow c_1 = c_2$$

□

Lemma 2.3.2. *The point $0 \in \partial\Omega$ is regular for Ω if and only if*

$$\zeta_{\Omega^c \cap C_R}(0) = 1 \qquad \forall R > 0$$

where $C_R := \{(x,t) : |x| \leq R, -R^2 \leq t \leq 0\}$.

Proof. See lemma (1.3) in [11]. □

We now want to report a deep result concerning temperatures that will be generalized in the next section by theorem (2.4.1).

Theorem 2.3.3. *Let u be a continuous function on the open set $D \subset \mathbb{R}^{n+1}$; the following are equivalent:*

(i) u is a temperature on D.

(ii) For each $z_0 = (x_0, t_0) \in D$ one has

$$u(z_0) = \frac{1}{4(4\pi c)^{n/2}} \int_{\Omega(z_0,c)} u(x,t) \frac{|x - x_0|^2}{(t - t_0)^2} dxdt$$

whenever $\overline{\Omega(z_0,c)} \subset D$.

2.4 Mean-value formulas and strong Harnack inequality

Before stating the main results of this sections, we would like to premise a very important theorem that is a generalization of the classical mean formula for temperatures.

Theorem 2.4.1 (Pini - Fulks - Watson). *Let $u \in C^\infty(\mathbb{R}^{n+1})$ and let $z_0 \in \mathbb{R}^{n+1}$. For almost any $c > 0$ one has*

$$\begin{aligned} -\int_{\Psi(z_0,c)} u(\zeta)(\nabla_\zeta K(z_0-\zeta)) \cdot \overrightarrow{N}_\xi(\zeta) dH_n(\zeta) \\ = u(z_0) + \int_{\Omega(z_0,c)} Hu(z)(K(z_0-z) - (4\pi c)^{-n/2}) dz \end{aligned} \tag{2.6}$$

where $(\overrightarrow{N}_\xi, N_\tau)$ is the outer normal to the surface $\Psi(z_0,c)$.
For every $c > 0$ we have

$$\begin{aligned} u_c(z_0) &:= \frac{1}{(4\pi c)^{n/2}} \int_{\Omega(z_0,c)} u(z) \frac{|x_0-x|^2}{4(t_0-t)^2} dz \\ &= u(z_0) + \frac{n}{2} c^{-n/2} \int_0^c l^{n/2} \int_{\Omega(z_0,l)} Hu(z)(K(z_0-z) - (4\pi l)^{-n/2}) dz \frac{dl}{l} \end{aligned} \tag{2.7}$$

Proof. First note that for every $u, v \in C^\infty$

$$\begin{aligned} vHu - u\hat{H}v &= v(\partial_t u - \mathrm{div}(\nabla u)) - u(-\partial_t v - \mathrm{div}(\nabla v)) \\ &= \mathrm{div}(u\nabla v - v\nabla u) + \partial_t(uv) \end{aligned} \tag{2.8}$$

Consider a generic bounded piecewise smooth domain in $\mathbb{R}^{n+1}_- := \mathbb{R}^n \times \mathbb{R}^-$ and denote by $N(z) = (\overrightarrow{N}_\xi(z), N_\tau(z)))$ the outer normal to ∂D at the point z; pick the $(n+1)$-dimensional vector field $\mathbf{F} = (u\nabla v - v\nabla u, uv)$ and apply both (2.8) and the divergence theorem:

$$\begin{aligned} \int_D \mathrm{div}\, \mathbf{F}(z) dz &= \int_D (v(z)Hu(z) - u(z)\hat{H}v(z)) dz \\ &= \int_{\partial D} \langle \mathbf{F}(z), N(z) \rangle dH_n(z) \\ &= \int_{\partial D} \Big(\langle (u\nabla v - v\nabla u)(z), N_\xi(z) \rangle + u(z)v(z)N_\tau(z) \Big) dH_n(z) \end{aligned} \tag{2.9}$$

As $K(z_0 - \cdot) \in C^\infty(\mathbb{R}^{n+1} \setminus \{z_0\})$, by applying Sard's theorem (see [17] for a reference) we can state that for a.e. sufficiently small $c > 0$, $\Psi(z_0, c)$ is a smooth n-dimensional manifold in $\mathbb{R}^{n+1}_-$ and it is the boundary of $\Omega(z_0, c)$; fix such c. For each $s \in (t_0 - c, t_0)$ set

$$\begin{aligned}
\Omega_s(z_0, c) &= \{(x,t) \in \Omega(z_0, c) : t < s\} \\
I_s(z_0, c) &= \{(x,t) \in \overline{\Omega(z_0, c)} : t = s\} \\
\Psi_s(z_0, c) &= \{(x,t) \in \Psi(z_0, c) : t < s\}
\end{aligned}$$

If we take $D = \Omega_s(z_0, c), v(z) = K(z_0 - z) - (4\pi c)^{-n/2}$ and $u \in C^\infty(\mathbb{R}^{n+1})$ in formula (2.9), recalling that $\hat{H}(K(z_0 - z)) = 0$ for all $z \in \mathbb{R}^{n+1}_- \setminus \{z_0\}$, we get:

$$\begin{aligned}
&\int_{\Omega_s(z_0,c)} (K(z_0 - z) - (4\pi c)^{-n/2}) H u(z) dz \\
&= \int_{\Psi_s(z_0,c) \cup I_s(z_0,c)} (u \nabla (K(z_0 - \cdot)) - (K(z_0 - \cdot) - (4\pi c)^{-n/2}) \nabla u) \cdot N_\xi dH_n \\
&+ \int_{\Psi_s(z_0,c) \cup I_s(z_0,c)} u (K(z_0 - \cdot) - (4\pi c)^{-n/2}) N_\tau dH_n \\
&\text{as on } I_s(z_0, c) : N_\xi = 0, N_\tau = -1 \text{ and on } \Psi_s(z_0, c) : K(z_0 - z) = (4\pi c)^{n/2} \\
&= - \int_{\Psi_s(z_0,c)} u(z) \nabla_z K(z_0 - z) \cdot N_\xi(z) dH_n(z) \\
&- \int_{I_s(z_0,c)} u(z) (K(z_0 - z) - (4\pi c)^{-n/2}) dH_n(z)
\end{aligned}$$

Now let $s \to t_0^-$:

$$\int_{\Omega(z_0,c)} (K(z_0 - z) - (4\pi c)^{-n/2}) H u(z) dz = -u(z_0) - \int_{\Psi(z_0,c)} u(z) \nabla_\xi K(z_0 - z) \cdot N_\xi(z) dH_n(z)$$

Now we want to obtain (2.7) from (2.6). To do that we will need the coarea formula by Federer (see [7], theorem (3.2.12)), that states the following: if $f \in L^1(\mathbb{R}^{n+1}), g \in \mathrm{Lip}(\mathbb{R}^{n+1})$, then

$$\int_{\mathbb{R}^{n+1}} f(z) |\nabla_x g, \partial_t g| dz = \int_{-\infty}^{+\infty} \Big(\int_{g^{-1}(\alpha)} f(z) dH_n(z) \Big) d\alpha \tag{2.10}$$

or, in an equivalent form,

$$\int_{\mathbb{R}^{n+1}} f(z) dz = \int_{-\infty}^{+\infty} \Big(\int_{g=\alpha} \frac{f(z)}{|\nabla_x g, \partial_t g|} dH_n(z) \Big) d\alpha \tag{2.11}$$

Now take (2.6), multiply both sides by $l^{-1+n/2}$ and integrate between 0 and c:

$$\begin{aligned}&\int_0^c l^{n/2}\int_{K=(4\pi l)^{n/2}} u(\zeta)(\nabla_\zeta K(z_0-\zeta))\cdot\overrightarrow{N}_\xi(\zeta)dH_n(\zeta)\frac{dl}{l}\\&=\frac{2}{n}u(z_0)c^{n/2}+\int_0^c l^{n/2}\int_{K>(4\pi l)^{n/2}} Hu(z)(K(z_0-z)-(4\pi c)^{-n/2})dz\frac{dl}{l}\end{aligned}\tag{2.12}$$

where we denote by $K=(4\pi l)^{n/2}$ and $K>(4\pi l)^{n/2}$ the sets $\Psi(z_0,c)$ and $\Omega(z_0,c)$ respectively. We would like to apply (2.11) to the left-hand side: set $\alpha=(4\pi l)^{-n/2}$; then $d\alpha=-\dfrac{n}{2}\dfrac{dl}{l}(4\pi l)^{-n/2}$. Putting this in (2.12) and applying the coarea formula:

$$\begin{aligned}&\int_0^c l^{n/2}\int_{K=(4\pi l)^{n/2}} u(\zeta)(\nabla_\zeta K(z_0-\zeta))\cdot\overrightarrow{N}_\xi(\zeta)dH_n(\zeta)\frac{dl}{l}\\&=\frac{2}{n}\Big(\frac{1}{4\pi}\Big)^{n/2}\int_{(4\pi c)^{-n/2}}^{\infty}\frac{1}{\alpha^2}\int_{K=\alpha}u(\zeta)\frac{|\nabla_\zeta K(z_0-\zeta)|^2}{|\nabla_x K(z_0-\zeta),\partial_t K(z_0-\zeta)|}dH_n(\zeta)d\alpha\\&=\frac{2}{n}\Big(\frac{1}{4\pi}\Big)^{n/2}\int_{K>(4\pi c)^{-n/2}}u(z)\frac{|x_0-x|^2}{4(t_0-t)^2}dz\end{aligned}$$

By putting this last form in (2.12) we prove the thesis. □

We are not ready to state one of the main results of this chapter; indeed, the proof of the Wiener's criterion for the heat equation is mainly based on two lemmas: the following and one strong Harnack inequality that we will show later.

Lemma 2.4.2. *If $u:\mathbb{R}^{n+1}\to\mathbb{R}$ is smooth, then*

(i) the average

$$\phi(c)=\frac{1}{c^{n/2}}\int_{\Omega(c)}u(x,t)\frac{|x|^2}{t^2}dxdt$$

is differentiable for $c>0$.

(ii) Explicitly:

$$\phi'(c)=\frac{n}{2c^{(n+2)/2}}\int_{\Omega(c)}Hu(x,t)\frac{R_c(t)^2-|x|^2}{t}dxdt \qquad c>0$$

(iii) There exists $C_1 > 0$ that only depends on n such that if $Hu \leq 0$ in $\Omega(2c), c > 0$ then

$$\phi(2c) - \phi(c) \geq \frac{C_1}{c^{n/2}} \int_{\Omega(c/2)} (-Hu(x,t))dxdt$$

Proof. If we consider the change of variable $(x,t) \overset{D}{\mapsto} (\sqrt{c}x, ct)$, we notice that $D(\Omega(c)) = \Omega(1)$ and hence:

$$\phi(c) = \frac{1}{c^{n/2}} \int_{\Omega(1)} u(\sqrt{c}\xi, c\tau) \frac{1}{c} \frac{|\xi|^2}{\tau^2} c^{1+n/2} d\xi d\tau = \int_{\Omega(1)} u(\sqrt{c}\xi, c\tau) \frac{|\xi|^2}{\tau^2} d\xi d\tau$$

By applying theorem (2.3.3) to the temperature $v = 1$ (with $z_0 = 0$) one sees that $\displaystyle\int_{\Omega(1)} \frac{|\xi|^2}{\tau^2} d\xi d\tau = 2^{n+2}\pi^{n/2}$; by this and the fact that u is smooth on $\mathbb{R}^{n+1}$ and $\overline{\Omega(1)}$ is compact, we know the integrand function is integrable on $\Omega(1)$ with respect to variable (ξ, τ), hence we can differentiate under the sign of integral. Notice that $\phi(c) = 4(4\pi)^{n/2} u_c(0)$; if we differentiate (2.7) with respect to c we get:

$$\begin{aligned} \frac{du_c(0)}{dc} = -\left(\frac{n}{2}\right)^2 c^{-1-n/2} \int_0^c l^{n/2} \int_{\Omega(l)} Hu(z)(K(-z) - (4\pi l)^{-n/2}) dz \frac{dl}{l} \\ + \frac{n}{2c} \int_{\Omega(c)} Hu(z)(K(-z) - (4\pi c)^{-n/2}) dz \quad (2.13) \end{aligned}$$

By Tonelli's theorem one has

$$\begin{aligned} &\int_0^c l^{n/2} \int_{\Omega(l)} Hu(z)(K(-z) - (4\pi l)^{-n/2}) dz \frac{dl}{l} \\ &= \int_{\Omega(c)} Hu(z) \int_{((4\pi)^{n/2}K(-z))^{-2/n}}^{c} l^{n/2}(K(-z) - (4\pi l)^{-n/2}) \frac{dl}{l} dz \\ &= \int_{\Omega(c)} Hu(z) \Big[K(-z) \frac{2}{n} l^{n/2} - \frac{1}{(4\pi)^{n/2}} \log l \Big]_{l=((4\pi)^{n/2}K(-z))^{-2/n}}^{l=c} dz \\ &= \frac{2}{n} c^{n/2} \int_{\Omega(c)} Hu(z)K(-z)dz - \frac{2}{n(4\pi)^{n/2}} \int_{\Omega(c)} Hu(z)dz \\ &\quad - \frac{2}{n(4\pi)^{n/2}} \int_{\Omega(c)} Hu(z) \log((4\pi c)^{n/2} K(-z)) dz \quad (2.14) \end{aligned}$$

By putting this in (2.13) we finally get:

$$\frac{1}{4(4\pi c)^{n/2}} \phi'(c) = \frac{du_c(0)}{dc} = \frac{n}{2c}(4\pi c)^{-n/2} \int_{\Omega(c)} Hu(z) \frac{R_c(t)^2 - |x|^2}{4t} dz$$

We just need to prove (iii): assume $Hu \leq 0$ in $\Omega(2c)$. Hence

$$\phi(2c)-\phi(c) = \int_c^{2c} \phi'(s)ds = \frac{n}{2}\int_c^{2c} \frac{1}{s^{1+n/2}} \int_{\Omega(s)} (-Hu(z)) \frac{R_s(t)^2 - |x|^2}{-t} dzds$$

As the integrand function is positive and $\Omega(c/2) \subset \Omega(c) \subset \Omega(s)$, we get

$$\begin{aligned}\phi(2c)-\phi(c) &\geq \frac{n}{2}\int_c^{2c} \frac{1}{s^{1+n/2}} \int_{\Omega(c/2)} (-Hu(z)) \frac{R_c(t)^2 - |x|^2}{-t} dzds \\ &\geq \frac{n}{2}\int_c^{2c} \frac{1}{s^{1+n/2}} \int_{\Omega(c/2)} (-Hu(z)) \frac{R_c(t)^2 - R_{\frac{c}{2}}(t)^2}{-t} dzds\end{aligned}$$

By simple calculation one can see that $\dfrac{R_c(t)^2 - R_{\frac{c}{2}}(t)^2}{-t} = 2n \log 2$, therefore:

$$\begin{aligned}\phi(2c)-\phi(c) \geq \frac{n}{2}(2n\log 2)\frac{1}{(2c)^{n/2}}\Big(\int_{\Omega(c/2)} (-Hu(z))dz\Big)\Big(\int_c^{2c} \frac{1}{s}ds\Big) \\ = \frac{C_1}{c^{n/2}} \int_{\Omega(c/2)} (-Hu(z))dz\end{aligned}$$

□

Before going further, it is useful to remind the classical Harnack inequality for the heat equation, for which one can refer to [3].

Theorem 2.4.3 (Pini-Hadamard). *Let*

$$\begin{aligned}&\rho > \rho' > 0 \\ &t_1 > t_2 > t_3 > t_4 > 0\end{aligned}$$

and set

$$\begin{aligned}R &= \{(x,t) : |x_j| < \rho, j = 1, \cdots, n, 0 < t < t_1\} \\ R^- &= \{(x,t) : |x_j| < \rho', j = 1, \cdots, n, t_4 < t < t_3\} \\ R^+ &= \{(x,t) : |x_j| < \rho', j = 1, \cdots, n, t_2 < t < t_1\}\end{aligned}$$

Then there is a positive constant C such that, for every non-negative temperature u on R, one has:

$$\sup_{R^-} u \leq C \inf_{R^+} u$$

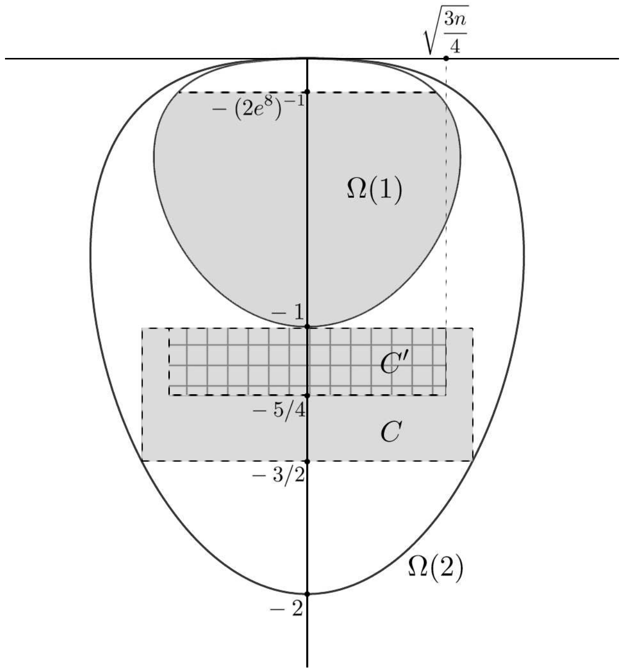

Figure 2.2: Proof of lemma (2.4.4): the cylinders C, C'.

Now set

$$Q(c) := \{(x,t) \in \Omega(c) : -\frac{3}{4}c < t\} = \Big\{(x,t) : -\frac{3}{4}c < t < 0, |x|^2 < 2nt \log\Big(-\frac{t}{c}\Big)\Big\}$$

We are now ready to state the other important result that will be crucial for the final proof.

Lemma 2.4.4 (strong Harnack inequality)**.** *[Evans - Gariepy] Let $u \geq 0$ a solution of $Hu = 0$ in $Q(2c), c > 0$ and suppose $u \in C(\partial Q(2c) - \{0\})$; then there exists a constant $C_2 > 0$, depending only on n, such that*

$$\fint_{|x|^2 \leq \frac{3nc}{4}} u\Big(x, -\frac{3c}{2}\Big) dx \leq C_2 \inf_{\Omega(3c/4)} u \tag{2.15}$$

(here $\fint$ denotes the n-dimensional integral average.)

Proof. Consider the change of variable $(x,t) \overset{D}{\mapsto} \Big(\frac{x}{\sqrt{c}}, \frac{t}{c}\Big)$. Then $D(Q(1)) = Q(c)$ and if u is such that $Hu = 0$ in $Q(2)$, then $H(u \circ D) = 0$ in $D(Q(2)) = Q(2c)$. The conclusion is unchanged over the mapping D, therefore it is enough to prove the theorem in the case $c = 1$. Thanks to homogeneity it is not restrictive to assume

$$\fint_{|x|^2 \leq \frac{3n}{4}} u\Big(x, -\frac{3}{2}\Big) dx = 1 \tag{2.16}$$

We will prove that there is C_2 as in the statement, so that $1 \leq C_2 \inf_{\Omega(3/4)} u$.
The proof is divided in two parts: at first we estimate $\inf_{\Omega(3/4)} u$ on $P \cap \Omega(1)$, where P is a paraboloid we will build, using the classical Harnack inequality, then we find a lower bound in $\Omega(1) \setminus P$ by building a particular subsolution v and using the maximum principle on u/v.
Consider at first the cylinder $C \subset Q(2)$:

$$C := \{(x,t) : -\frac{3}{2} < t < -1, |x|^2 < -3n \log(3/4)\}$$

We can state there exists a constant $\alpha_1 > 0$ so that

$$u(x,t) \geq \alpha_1 > 0 \qquad (x,t) \in C'$$
$$C' = \{(x,t) : -\frac{5}{4} < t < -1, |x|^2 \leq \frac{3n}{4}\} \subset C$$

In fact, suppose this last statement is not true and that there is one point $z' = (x', t') \in C'$ such that $u(z') = 0$; thanks to Nierenberg maximum principle we know $u = 0$ on $C \cap \{(x,t) : t \leq t'\}$. This would be a contradiction of the assumption (2.16).
In the very same way, one can conclude there is $\alpha_2 > 0$ so that

$$u(x,t) \geq \alpha_2 > 0 \qquad (x,t) \in D := \Omega(1) \cap \Big\{(x,t) : t < -\frac{1}{2e^8}\Big\} \tag{2.17}$$

By theorem (2.3.3), for any $(x,t) \in L := \{(0,t) : -1 < t < 0\}$, one can write:

$$u(0,t) = \frac{1}{4\pi^{n/2}} \int_{\Omega((0,t),1/4)} u(x,s) \frac{|x|^2}{(t-s)^2} dxds$$

and since for any $t \in \Big(-\frac{1}{2e^8}, 0\Big)$, the Lebesgue measure of $D \cap \Omega((0,t), 1/4)$ is strictly positive, by (2.17) we have

$$u(0,t) \geq \alpha_3 > 0 \qquad (0,t) \in L$$

Define the truncated paraboloids:

$$P_1 = \Big\{(x,t) : |x|^2 < -8nt, -\frac{1}{e^8} < t < 0\Big\}$$
$$P_2 = \Big\{(x,t) : |x|^2 < -16nt, -\frac{1}{e^8} < t < 0\Big\}$$

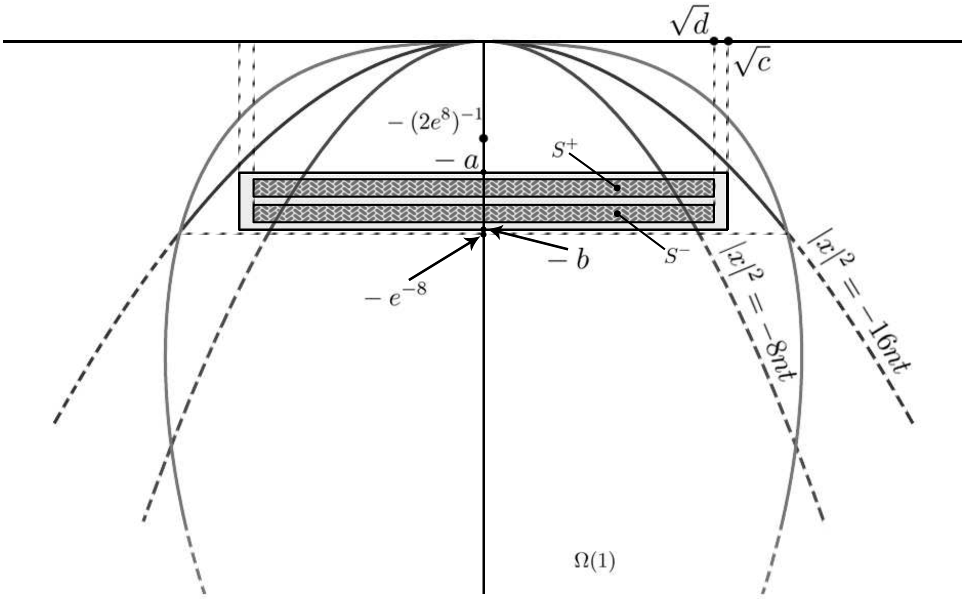

Figure 2.3: Proof of lemma (2.4.4): the paraboloids and the cylinders.

As $2nt\log(-t) = -16nt$ just for $t = 0, t = -e^{-8}$, we can conclude $P_1 \subset P_2 \subset \Omega(1)$.
Denote by S the closed cylinder

$$S := \{(x,t) : -b \leq t \leq -a, |x|^2 \leq c\}$$

with $a, b, c > 0$ such that $\dfrac{1}{2e^8} < a < b < \dfrac{1}{e^8}$ and

$$\begin{aligned} P_1 \cap \{(x,t) : -b \leq t \leq -a\} \subset \{(x,t) : |x|^2 \leq d, -b \leq t \leq -a\} \\ \subset S \subset P_2 \cap \{(x,t) : -b \leq t \leq -a\} \end{aligned}$$

for some $d \in (0,c)$(see figure 2.4.4). Now call $h := b - a > 0$ and set

$$S^+ = \left\{(x,t) : -a - \frac{3}{8}h \leq t \leq -a - \frac{1}{8}h, |x|^2 \leq d\right\} \subset S$$
$$S^- = \left\{(x,t) : -a - \frac{7}{8}h \leq t \leq -a - \frac{5}{8}h, |x|^2 \leq d\right\} \subset S$$

Theorem (2.4.3) tells us that there exists $C_3 > 0$ so that

$$\max_{S^-} u \leq C_3 \min_{S^+} u$$

The constant C_3 does not change under the dilatation $(x,t) \mapsto (\lambda x, \lambda^2 t), \lambda > 0$, hence:

$$\max_{\lambda S^-} u \leq C_3 \min_{\lambda S^+} u \qquad 0 < \lambda \leq 1$$

But for $0 < \lambda \leq 1$ one has $L \cap \lambda S^- \neq \emptyset$ and by what we have proved before:

$$\alpha_3 \leq C_3 \min_{\lambda S^+} u \qquad 0 < \lambda \leq 1 \tag{2.18}$$

For any $(x', t') \in P_1 \cap \Big\{(x,t) : -\dfrac{1}{2e^8} < t < 0\Big\}$ there exists $\lambda \in (0,1]$ so that $(x',t') \in \lambda S^+$, so by (2.18) one has $u(x',t') \geq \dfrac{\alpha_3}{C_3}$. By putting this last remark together with (2.17) we get

$$u(x',t') \geq \alpha_4 > 0 \qquad (x',t') \in \{(x,t) \in \Omega(1), |x|^2 \leq -8nt\} \tag{2.19}$$

Now we need an estimate on $W := \Omega(1) \setminus \{(x,t) \leq -8nt\}$ to complete the proof. Consider the function $\varphi : \mathbb{R}^+ \to \mathbb{R}, \varphi(x) = \arctan(x+16) - \arctan(16)$; then for any $x > 0$:

$$\begin{cases} \varphi(0) = 0 \\ 0 < \varphi(x) < \dfrac{\pi}{2} \\ \varphi'(x) > 0 \\ 0 \leq -\varphi''(x) \leq \dfrac{1}{8}\varphi'(x) \end{cases}$$

The last inequality comes from the following fact: call $x + 16 = t (\Rightarrow t \geq 16)$ and differentiate:

$$\varphi'(x) = \frac{1}{1+t^2} \qquad -\varphi''(x) = \frac{2t}{(1+t^2)^2}$$

One has to prove $\dfrac{2t}{1+t^2} \leq \dfrac{1}{8} \Leftrightarrow t^2 - 16t + 1 \geq 0$ and this is always true when $t \geq 16$. Now we can build the subsolution v as we anticipated at the beginning of this proof:

$$v(x,t) = \varphi\Big(\frac{|x|^2}{t} - 2n\log(-t)\Big) \qquad (x,t) \in \Omega(1) \tag{2.20}$$

Claim: $Hv \leq 0$ on W.

$$
\begin{aligned}
Hv &= \Big(-\frac{|x|^2}{t^2} - \frac{4n}{t}\Big)\varphi' - \frac{4|x|^2}{t^2}\varphi'' \\
&\le \Big(-\frac{|x|^2}{t^2} - \frac{4n}{t} + \frac{1}{8}\frac{4|x|^2}{t^2}\Big)\varphi' \\
&= \Big(-\frac{|x|^2}{2t^2} - \frac{4n}{t}\Big)\varphi' \le 0
\end{aligned}
$$

since $|x|^2 \ge -8nt$ in W. As $\varphi(0) = 0$, we have that

$$
v = 0 \qquad \text{on } \partial\Omega(1) \setminus \{0\}
$$

and

$$
v \le \frac{\pi}{2} \qquad \text{on } \{(x,t) : |x|^2 = -8nt\}
$$

This and (2.19) imply

$$
\frac{u}{v} \ge \frac{2\alpha_4}{\pi} =: \alpha \qquad \text{on } \{(x,t) : |x|^2 = -8nt\}
$$

and by maximum principle we can say

$$
u \ge \alpha v \qquad \text{on all } W \tag{2.21}
$$

Now let $\Omega(3/4) \cap W$ and use the monotonicity of φ:

$$
\begin{aligned}
v(x,t) &= \varphi\Big(\frac{|x|^2}{t} - 2n\log(-t)\Big) \overset{t<0}{\ge} \varphi\Big(\frac{2nt\log(-4t/3)}{t} - 2n\log(-t)\Big) \\
&= \varphi\Big(2n\log\Big(\frac{4}{3}\Big)\Big) =: \beta > 0
\end{aligned}
$$

Putting this last inequality, (2.17) and (2.21) we get:

$$
u(x,t) \ge \max\Big\{\alpha_4, \alpha\beta\Big\} > 0 \qquad (x,t) \in \Omega(3/4)
$$

and this proves the assertion. $\square$

2.5 Necessity of (2.4)

This part of the proof is taken from Lanconelli's work (see [11]). Assume

$$
\sum_{k=1}^{\infty} 2^{kn/2}\, \mathrm{cap}_H(\Omega^c \cap A(2^{-k})) < \infty
$$

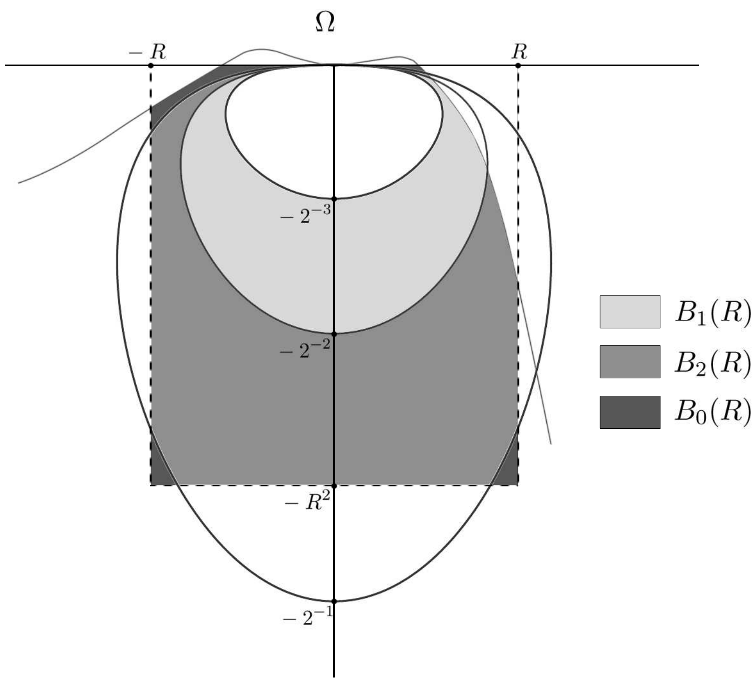

Figure 2.4: Section 2.5: the sequence $B_l(R)$.

where $A(2^{-k}) = A(0, 2^{-k})$. Thanks to lemma (2.3.2) we need to find $R > 0$ such that $\zeta_{\Omega^c \cap C_R}(0) < 1$. Fix $R > 0$ and build the following sequence:

$$\begin{cases} B_k(R) = A(2^{-k}) \cap \Omega^c \cap C_R & k \geq 1 \\ B_0(R) = (\Omega^c \cap C_R) \setminus \bigcup_{k=1}^{\infty} B_k(R) \end{cases}$$

Denote by $\widetilde{\mu}$ the equilibrium measure of $\Omega^c \cap C_R$ and set

$$\nu_k = \widetilde{\mu}_{|B_k(R)} \qquad k \geq 0$$

Now let ν'_k be the equilibrium measure of $B_k(R)$; by theorem (2.2.2)-(vi) we have that $K_{\nu_k} \leq K_{\nu'_k}$ for any k since $\nu_k \in \mathcal{M}^+(B_k(R))$ and $K_{\nu_k} \leq K_{\widetilde{\mu}} \leq 1$. As

$$\Omega^c \cap C_R = \bigcup_{k=0}^{\infty} B_k(R)$$

using the sub-additivity property we get

$$\zeta_{\Omega^c \cap C_R} = P_{\widetilde{\mu}} \leq \sum_{k=0}^{\infty} K_{\nu_k} \leq \sum_{k=0}^{\infty} K_{\nu'_k} \quad \text{on } \mathbb{R}^{n+1}$$

In particular

$$\begin{aligned} K_{\nu_k'}(0) &= \int_{\mathbb{R}^{n+1}} K(-z)d\nu_k'(z) = \int_{B_k(R)} K(-z)d\nu_k'(z) \quad (B_k(R) \subset A(2^{-k})) \\ &\leq \Big(\frac{2^k}{2\pi}\Big)^{n/2} \nu_k'(B_k(R)) = \Big(\frac{2^k}{2\pi}\Big)^{n/2} \operatorname{cap}_H(B_k(R)) \end{aligned}$$

Putting the last two remarks together we get

$$\begin{aligned} \zeta_{\Omega^c \cap C_R}(0) &\leq \frac{1}{(2\pi)^{n/2}} \sum_{k=0}^{\infty} 2^{kn/2} \operatorname{cap}_H(B_k(R)) \\ &\leq \frac{1}{(2\pi)^{n/2}} \operatorname{cap}_H(C_R) + \frac{1}{(2\pi)^{n/2}} \sum_{k=1}^{\infty} 2^{kn/2} \operatorname{cap}_H(\Omega^c \cap A(2^{-k}) \cap C_R) \\ &=: c(R) \overset{hyp.}{<} \infty \end{aligned}$$

As $c(R)$ is increasing, by properties of the thermal capacity we can find R so that $c(R) < 1$.

2.6 Sufficiency of (2.4)

The proof of the sufficiency was firstly given by L. C. Evans and R. F. Gariepy (see [4]) in 1982. The proof is divided in several parts and strongly relies on the strong Harnack inequality we showed before (theorem (2.4.4)) and on the average formulas of lemma (2.4.2) as we will soon see; this proof was completed in details also following the work of N. Garofalo and E. Lanconelli (see [8]).

Suppose that

$$\sum_{k=1}^{\infty} 2^{kn/2} \operatorname{cap}_H(\Omega^c \cap A(2^{-k})) = +\infty \tag{2.22}$$

We want to prove 0 is a regular point for Ω.

Modification of Ω.

Thanks to (2.22) we know at least one of the following four series diverges:

$$\sum_{k=1}^{\infty} 2^{\frac{n}{2}(4k+j)} \operatorname{cap}_H(\Omega^c \cap A(2^{-(4k+j)})) \qquad j = 0, 1, 2, 3$$

therefore it is not restrictive to assume

$$\sum_{k=1}^{\infty} 2^{2kn} \operatorname{cap}_H(\Omega^c \cap A(2^{-4k})) = +\infty \tag{2.23}$$

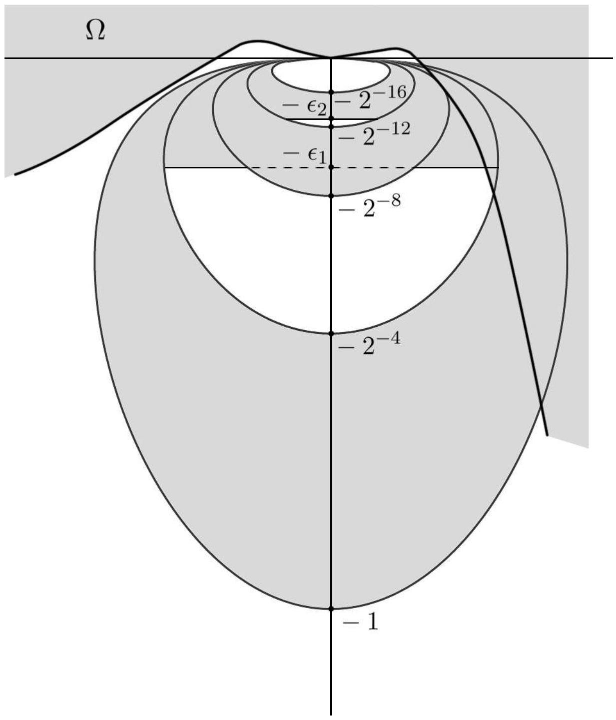

Figure 2.5: Section 2.6: modification of Ω.

By sub-additivity for any ϵ:

$$\begin{aligned}\operatorname{cap}_H(\Omega^c \cap A(2^{-4k})) \leq \operatorname{cap}_H(\Omega^c \cap A(2^{-4k}) \cap \{(x,t) : t \leq -\epsilon\}) \\ + \operatorname{cap}_H(\Omega^c \cap A(2^{-4k}) \cap \{(x,t) : -\epsilon \leq t \leq 0\}) \quad (2.24)\end{aligned}$$

By lemma (2.2.1)-(v),(viii) we can say

$$\begin{aligned}&\lim_{\epsilon \downarrow 0} \operatorname{cap}_H(\Omega^c \cap A(2^{-4k}) \cap \{(x,t) : -\epsilon \leq t \leq 0\}) \\ &\qquad \leq \lim_{\epsilon \downarrow 0} \operatorname{cap}_H(A(2^{-4k}) \cap \{(x,t) : -\epsilon \leq t \leq 0\}) = \operatorname{cap}_H(\{0\}) = 0\end{aligned}$$

hence by (2.22) there exist a subsequence $\{\epsilon_l\}_l$ so that

$$\sum_{l=1}^{\infty} 2^{2ln} \operatorname{cap}_H(\Omega^c \cap A(2^{-4l}) \cap \{(x,t) : t \leq -\epsilon_l\}) = +\infty \qquad (2.25)$$

Define

$$\widetilde{\Omega}^c = \{0\} \cup \Big(\bigcup_{l=1}^{\infty} \Omega^c \cap A(2^{-4l}) \cap \{(x,t) : t \leq -\epsilon_l\}\Big) \cup (\Omega^c \setminus \Omega(1))$$

Then $\Omega \subset \widetilde{\Omega}, \widetilde{\Omega}$ is bounded and $0 \in \partial\widetilde{\Omega}$. If 0 is a regular point for $\widetilde{\Omega}$, it is regular also for Ω; in view of this, we can assume $\Omega = \widetilde{\Omega}$; hence we can say

$$\Omega^c = \{0\} \cup \Big(\bigcup_{l=0}^{\infty} B_l\Big)$$

where

$$\begin{cases} B_l \subset A(2^{-4l}) \cap \{(x,t) : t \leq -\epsilon_l\} & l = 1, 2, \cdots \\ \Omega(1) \subset B_0^c & B_0^c \text{ bounded} \end{cases}$$

and

$$\sum_{l=0}^{\infty} 2^{2nl} \operatorname{cap}_H(B_l) = +\infty$$

Approximation.

The aim of this part is to regularize some equilibrium potentials; fix $R > 0$ and consider the compact set $F = \Omega^c \cap C_R$. Let ζ_F denote its equilibrium potential and define

$$u := 1 - \zeta_F$$

Thanks to the arbitrary choice of R, we need to prove $u(0) = 0$ to apply (2.3.2) and state that 0 is regular for Ω. Observe that by theorem (2.2.2):

$$\begin{cases} 0 \leq u \leq 1 \\ u \text{ is a subtemperature on } \mathbb{R}^{n+1} \text{ and a temperature on } \Omega \\ Hu = -\widetilde{\mu} \text{ in the sense of distributions} \end{cases}$$

where $\widetilde{\mu}$ is the equilibrium measure of F.
For each $\epsilon > 0$ choose a compact set F^ϵ so that

$$F^\epsilon = \{0\} \cup \Big(\bigcup_{l=1}^{\infty} B_l^\epsilon \Big) \subset C_{R+1}$$

where

$$\begin{cases} B_l^\epsilon \text{ is compact} & \epsilon > 0, l = 1, 2, \cdots \\ B_l \cap C_R \subset \operatorname{int}(B_l^\epsilon) \\ B_l^\epsilon \subset \Omega(2^{-4l+1}) \setminus \overline{\Omega(2^{-4l-2})} \\ F^\epsilon \subset F^{\epsilon'} & \text{if } 0 < \epsilon < \epsilon' \\ F = \bigcap\limits_{\epsilon > 0} F^\epsilon \end{cases}$$

Denote by $u^\epsilon = 1 - \zeta_\epsilon$, where ζ_ϵ is the equilibrium potential of F^ϵ. *Claim*: $u^\epsilon(z) \uparrow \widetilde{u}(z) := 1 - \eta_F(z)$ for $\epsilon \downarrow 0, z \neq 0$ As $F_{\epsilon'} \subset F_\epsilon$ for $\epsilon' < \epsilon$, we have $\Phi_{\epsilon'} \subset \Phi_\epsilon$ (see lemma (2.2.1)-(iii)), hence $\eta_{\epsilon'} \leq \eta_\epsilon$ and consequently $\zeta_{\epsilon'} \leq \zeta_\epsilon$; this monotonicity guarantees the existence of $\xi(z) := \lim\limits_{\epsilon \to 0} \zeta_\epsilon(z)$ for any $z \in \mathbb{R}^{n+1}$. As $F \subset F_\epsilon$ for any ϵ, we also have $\zeta_F \leq \eta_\epsilon$. Recall (theorem (2.2.2)) that $\zeta_F = \eta_F$ on $(\partial F)^c$ and $\eta_F = 1 = \zeta_\epsilon$ on $F \setminus \{0\}$ as $F \setminus \{0\} \subset \operatorname{int}(F_\epsilon)$. This forces $\eta_F \leq \zeta_\epsilon$ except possibly at 0, hence

$$\eta_F \leq \xi \quad \text{ on } \mathbb{R}^{n+1}$$

On the other hand, ξ is a temperature on F^c, $0 \leq \xi \leq 1$ and $\lim\limits_{|z| \to +\infty} \xi(z) = 0$, hence the weak maximum principle for supertemperatures (see for example [18]) implies $\xi \leq v$ for any $v \in \Phi_F$, thus $\xi \leq \eta_F$. Concluding, $\xi = \eta_F$ on $\mathbb{R}^{n+1} \setminus \{0\}$.
As $B_l \subset A(2^{-4l}) \subset \overline{\Omega(2^{-4l})}$ for any l and the sequence $\{\overline{\Omega(2^{-4l})}\}_l$ is decreasing, we can find $N = N(R)$ so that $B_l \subset C_R$ for all $l \geq N$; fix such l and call

$\hat{\zeta}, \hat{\mu}$ respectively the backwards equilibrium potential and the corresponding equilibrium measure for B_l (hence $\hat{\mu} \in \mathcal{M}^+(B_l), H\hat{\zeta} = \hat{\mu}, \mathrm{cap}_H(B_l) = \hat{\mu}(\mathbb{R}^{n+1})$).
Now we will realize what we anticipated at the beginning of this part and approximate $\hat{\zeta}$ and u^ϵ by smooth functions. Pick the usual $\varphi \in C_0^\infty(\mathbb{R}^{n+1}), 0 \leq \varphi \leq 1, \int_{\mathbb{R}^{n+1}} \varphi(z)dz = 1$ and define

$$\varphi_\delta = \frac{1}{\delta^{n+1}}\varphi\Big(\frac{z}{\delta}\Big) \qquad \delta > 0$$

Set for any $\delta > 0$

$$\begin{aligned} u_\delta^\epsilon &= \varphi_\delta * u^\epsilon \\ \hat{\zeta}_\delta &= \varphi_\delta * \hat{\zeta} \end{aligned}$$

u_δ^ϵ and $\hat{\zeta}_\delta$ are smooth, nonnegative and bounded above by 1; moreover, $Hu_\delta^\epsilon = H(\varphi_\delta * u^\epsilon) = \varphi_\delta * Hu^\epsilon = \varphi_\delta * (-\widetilde{\mu}_\epsilon) \leq 0$.
If we define a measure on Borel sets by $\hat{\mu}_\delta(A) = \int_A \hat{H}\hat{\zeta}_\delta(z)dz$ ($\hat{H} = -\Delta - \partial_t$ is the backwards heat operator), we notice that it converges weakly to $\hat{\mu}$ for $\delta \downarrow 0$.
Now set $c := 2^{-4l+1}$ for simplicity; we know by the previous part that

$$B_l \subset A(c/2), B_l^\epsilon \subset \Omega(c) \setminus \overline{\Omega(c/8)}$$

If we take δ to be small enough we can have

$$\begin{aligned} &\mathrm{supp}\,\hat{\mu}_\delta \subset \mathrm{int}(B_l^\epsilon) \cap \Omega\Big(\frac{3}{4}c\Big) \\ &u_\delta^\epsilon = 0 \text{ on } \mathrm{supp}\,\hat{\mu}_\delta \end{aligned}$$

as $u^\epsilon = 0$ on $\mathrm{int}(B_l^\epsilon)$. To apply lemma (2.4.4) we need to write $u_\delta^\epsilon = v_\delta^\epsilon - w_\delta^\epsilon$ where

$$\begin{cases} Hv_\delta^\epsilon = 0 & \text{on } Q(2c) \\ v_\delta^\epsilon = u_\delta^\epsilon & \text{on } \partial Q(2c) \setminus \{0\} \end{cases}$$

$$\begin{cases} Hw_\delta^\epsilon = -Hu_\delta^\epsilon \geq 0 & \text{on } Q(2c) \\ w_\delta^\epsilon = 0 & \text{on } \partial Q(2c) \setminus \{0\} \end{cases}$$

Estimates.

By direct calculation we are going to get an useful estimate:

$$
\begin{aligned}
&(\inf_{\Omega(3c/4)} v_\delta^\epsilon)^2 \hat{\mu}_\delta\Big(\Omega\Big(\frac{3}{4}c\Big)\Big) \le \int_{\Omega(3c/4)} (v_\delta^\epsilon)^2 d\hat{\mu}_\delta \overset{\Omega(3c/4)\subset Q(2c)}{\le} \int_{Q(2c)} (v_\delta^\epsilon)^2 d\hat{\mu}_\delta \\
&= \int_{Q(2c)} (w_\delta^\epsilon)^2 d\hat{\mu}_\delta \quad (\text{as for small enough } \delta, u_\delta^\epsilon = 0 \text{ on } \operatorname{supp}\hat{\mu}_\delta) \\
&= \int_{Q(2c)} (w_\delta^\epsilon(z))^2 (\hat{H}\hat{\zeta}_\delta(z))dz \\
&= \int_{Q(2c)} H(w_\delta^\epsilon(z))^2 \hat{\zeta}_\delta(z)dz \quad (\text{because } w_\delta^\epsilon = 0 \text{ on } \partial\Omega\setminus\{0\}) \\
&2\int_{Q(2c)} w_\delta^\epsilon(z)(Hw_\delta^\epsilon(z))\hat{\zeta}_\delta(z)dz - 2\int_{Q(2c)} \hat{\zeta}_\delta(z)|Dw_\delta^\epsilon(z)|^2 dz \\
&\overset{w_\delta^\epsilon\hat{\zeta}_\delta\in C^\infty}{\le} C\int_{Q(2c)} (-Hu_\delta^\epsilon(z))dz \quad (\text{as } \int_{Q(2c)} \hat{\zeta}_\delta(z)|Dw_\delta^\epsilon(z)|^2 dz \ge 0) \\
&\overset{Q(2c)\subset\Omega(2c)}{\le} C\int_{\Omega(2c)} (-Hu_\delta^\epsilon(z))dz
\end{aligned}
$$

Summing up

$$(\inf_{\Omega(3c/4)} v_\delta^\epsilon)^2 \hat{\mu}_\delta\Big(\Omega\Big(\frac{3}{4}c\Big)\Big) \le C\int_{\Omega(2c)} (-Hu_\delta^\epsilon(z))dz$$

for small δ. By applying lemma (2.4.4) and (2.4.2)-(iii) we get

$$\frac{1}{c^{n/2}}\Big(\fint_{|x|^2\le 3nc/4} u_\delta^\epsilon\Big(x, -\frac{3}{2}c\Big)dx\Big)^2 \hat{\mu}_\delta\Big(\Omega\Big(\frac{3}{4}c\Big)\Big) \le C(\phi(u_\delta^\epsilon, 8c) - \phi(u_\delta^\epsilon, 4c)) \tag{2.26}$$

where

$$\phi(f,s) = \frac{1}{s^{n/2}}\int_{\Omega(s)} f(x,t)\frac{|x|^2}{t^2}dxdt$$

Passage to limits.

Remember that

$$\hat{\mu}_\delta \overset{weak}{\longrightarrow} \hat{\mu}$$

$$\operatorname{supp}\hat{\mu}_\delta \subset \Omega(3c/4)$$

$$u_\delta^\epsilon \overset{\delta\to 0}{\longrightarrow} u^\epsilon \quad \text{pointwise a.e. and uniformly on compact subsets of } (F^\epsilon)^c$$

Hence

$$\hat{\mu}_\delta\Big(\Omega\Big(\frac{3}{4}c\Big)\Big) = \hat{\mu}_\delta(\mathbb{R}^{n+1}) \overset{\delta\to 0}{\longrightarrow} \hat{\mu}(\mathbb{R}^{n+1}) = \operatorname{cap}_H(B_l)$$

From (2.26) we get

$$\frac{1}{c^{n/2}}\Big(\fint_{|x|^2\le 3nc/4} u^\epsilon\Big(x,-\frac{3}{2}c\Big)dx\Big)^2 \operatorname{cap}_H(B_l) \le C(\phi(u^\epsilon,8c)-\phi(u^\epsilon,4c))$$

and letting $\epsilon \downarrow 0$

$$\frac{1}{c^{n/2}}\Big(\fint_{|x|^2\le 3nc/4} u\Big(x,-\frac{3}{2}c\Big)dx\Big)^2 \operatorname{cap}_H(B_l) \le C(\phi(\widetilde{u},8c)-\phi(\widetilde{u},4c)) \quad (2.27)$$

as $\widetilde{u}=u$ on F^c. By lemma (2.4.2)-(iii), as $Hu^\epsilon_\delta \le 0$, we have that

$$\phi(u^\epsilon_\delta,2^jc)-\phi(u^\epsilon_\delta,2^{j-1}c) \qquad j=0,1,2$$

By letting $\delta,\epsilon\downarrow 0$ in the last expression, we get

$$\phi(\widetilde{u},2^jc)-\phi(\widetilde{u},2^{j-1}c) \qquad j=0,1,2$$

As these are positive constants, we can modify (2.27) as follows

$$\frac{1}{c^{n/2}}\Big(\fint_{|x|^2\le 3nc/4} u\Big(x,-\frac{3}{2}c\Big)dx\Big)^2 \operatorname{cap}_H(B_l) \le C\sum_{j=0}^{3}(\phi(\widetilde{u},2^jc)-\phi(\widetilde{u},2^{j-1}c))$$

Recalling that $c=2^{-4l+1}$, $B_l\subset C_R, \forall l\ge N$ and observing that the right hand side is a telescopic series, we get

$$\sum_{l\ge N}(\beta_l)^2 2^{2nl}\operatorname{cap}_H(B_l)<\infty \qquad (2.28)$$

where we have called

$$\beta_l := \fint_{|x|^2\le 3nc/4} u\Big(x,-\frac{3}{2}c\Big)dx$$

therefore by (2.23) we can conclude

$$\liminf_{l\to\infty}\beta_l = 0 \qquad (2.29)$$

Estimates in cylinders.

Build the descreasing sequence of cylinders

$$S_l=\Big\{(x,t):|x|^2\le\frac{3}{2}n2^{-4l},|t|\le 3\cdot 2^{-4l}\Big\} \qquad l=0,1,\cdots$$

and set $M_l := \sup_{S_l} u$ and remember that $0\le M_l\le 1$; we want to prove $M_l \overset{l\to\infty}{\longrightarrow} 0$.

Claim: there exist $C_4 > 0$ and $0 < \nu < 1$ such that if $Hv = 0$ on S_0, v is continuous on the parabolic boundary $\partial_p S_0$ and $0 \leq v \leq M$ in S_0, then

$$\sup_{S_1} v \leq \nu M + C_4 \fint_{|x|^2 \leq n} v(x, -3)dx \tag{2.30}$$

Consider the representation of v in terms of the Green function for S_0:

$$\begin{aligned} v(x,t) = &\int_{|y|^2 \leq 3n/2} G(x,t;y,-3)v(y,-3)dy \\ &- \int_{-3}^{3} \int_{|y|^2 = 3n/2} \frac{\partial G}{\partial n}(x,t;y,s)v(y,s)dH^{n-1}yds \end{aligned}$$

By taking $v = 1$ we notice that for any $(x,t) \in S_1$:

$$\int_{|y|^2 \leq 3n/2} G(x,t;y,-3)dy - \int_{-3}^{3} \int_{|y|^2 = 3n/2} \frac{\partial G}{\partial n}(x,t;y,s)dH^{n-1}yds = 1 \tag{2.31}$$

Recall the properties of the Green function:

(i) $\frac{\partial G}{\partial n} \leq 0$

(ii) $G \geq 0$

(iii) $\inf_{(x,t) \in S_1, |y|^2 \leq n} G(x,t;y,-3) \geq \gamma > 0$ for some γ

By these and (2.31), we get

$$0 < \int_{|y|^2 \leq 3n/2} G(x,t;y,-3)dy < 1$$

$$0 < \gamma_2 := -\int_{-3}^{3} \int_{|y|^2 = 3n/2} \frac{\partial G}{\partial n}(x,t;y,s)dH^{n-1}yds < 1$$

Call $\gamma_1 := \int_{n < |y|^2 \leq 3n/2} G(x,t;y,-3)dy$; by property (iii) of G, there must be $\epsilon > 0$ so that $\gamma_1 < 1 - \epsilon$. Hence for any $(x,t) \in S_1$:

$$\begin{aligned} v(x,t) &\leq M \int_{n < |y|^2 \leq 3n/2} G(x,t,y,-3)dy \\ &\quad + \sup_{|y|^2 \leq n} G(x,t;y,-3) \int_{|y|^2 \leq n} v(y,-3)dy + M\gamma_2 \\ &\leq M(\gamma_1 + \gamma_2) + C_4 \int_{|y|^2 \leq n} v(y,-3)dy \\ &= M\nu + C_4 \int_{|y|^2 \leq n} v(y,-3)dy \end{aligned}$$

The use the parabolic dilatation $(x,t) \mapsto \left(\frac{x}{\sqrt{c}}, \frac{t}{c}\right)$ allows us to conclude that

$$\sup_{S_{l+1}} v \leq \nu M_l + C_4 \fint_{|x|^2 \leq n \cdot 2^{-4l}} v(x, -3 \cdot 2^{-4l}) dx \tag{2.32}$$

for any v such that $Hv = 0, 0 \leq v \leq M_l$ in S_0, v continuous on $\partial_p S_l$.
Now fix some number $0 < \Theta < \nu$ and fix S_l. Recalling who is β_l and (2.29), by using this last inequality one can state there is $m > l$ such that

$$\nu M_l + C_4 \fint_{|x|^2 \leq n \cdot 2^{-4m}} u(x, -3 \cdot 2^{-4m}) dx \leq \Theta M_l \tag{2.33}$$

Since u is bounded on $\mathbb{R}^{n+1}$ there is a continuous function f defined on $\partial_p S_l$ so that

$$u \leq f \leq M_l \qquad \text{on } \partial_p S_l$$

with $u = f$ on $\{(x,t) : |x|^2 \leq n \cdot 2^{-4m}, t = -3 \cdot 2^{-4m}\}$.
Set $v = H_f^{S_m}$; obviously v is a temperature and $v \leq f$ on $\partial_p S_m$, hence $u \leq v$ on S_m. We have

$$\begin{aligned}
M_{m+1} &= \sup_{S_{m+1}} u \leq \sup_{S_{m+1}} v \\
&\overset{(2.32)}{\leq} \nu M_l + C_4 \fint_{|x|^2 \leq n \cdot 2^{-4m}} v(x, -3 \cdot 2^{-4m}) dx \\
&\overset{v = H_f^{S_m}}{\leq} \nu M_l + C_4 \fint_{|x|^2 \leq n \cdot 2^{-4m}} f(x, -3 \cdot 2^{-4m}) dx \\
&= \nu M_l + C_4 \fint_{|x|^2 \leq n \cdot 2^{-4m}} u(x, -3 \cdot 2^{-4m}) dx = \leq \Theta M_l
\end{aligned}$$

Hence we have proved that for any l there exists $m > l$ such that

$$M_{m+1} \leq \Theta M_l \leq \Theta M_m$$

because $\{M_j\}_j$ is a decreasing sequence. As $0 < \Theta < 1$, the ratio test guarantees us that

$$\lim_{l \to \infty} M_l = 0$$

Since $u \geq 0$ and $\{0\} = \bigcap_{l \geq 0} S_l$, this makes us conclude that u must be continuous at 0 and $u(0) = 1 - \zeta_F(0) = 0$, hence 0 is a regular point. The proof is complete.

Further development

The Wiener's criterion has been studied in more general settings than the ones showed in this thesis; Garofalo and Lanconelli ([8]) proved it in the case of a parabolic operator with variable coefficients and showed, as a main consequence of it, that if L_1, L_2 are parabolic operators such that

$L_i u = \mathrm{div}(A_i(x,t)\nabla u) - \partial_t u$

A_i real symmetric matrix-valued function on $\mathbb{R}^{n+1}$ with C^∞ entries

$\exists 0 < \mu_i \leq 1 : \mu_i|\xi|^2 \leq \displaystyle\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \leq \mu_i^{-1}|\xi|^2, \forall \xi \in \mathbb{R}^n$

and Ω is a bounded open subset of $\mathbb{R}^{n+1}$ and $A_1(z_0) = A_2(z_0)$ for some $z_0 \in \partial\Omega$, then z_0 is L_1-regular if and only if it is L_2-regular.
Wiener's test for regularity has recently been studied for the p-Laplacian operator, defined by

$$\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u) \qquad 1 < p < \infty$$

Moreover, Fabes, Garofalo and Lanconelli ([6]) proved the same results in the case of parabolic operators with $C^{1,\alpha}$ coefficients. Lanconelli went on studying some sufficient and necessary conditions of regularity of boundary points also in the case of parabolic operators with discontinuous coefficients ([12]); he underlined the fact that in the parabolic case it is not possibile to find a *representative* operator, like in the elliptic case. In other words point can be regular for the operator $-a\Delta + \partial_t$ but not for $-b\Delta + \partial_t$ $(a > b > 0)$.
In 1994, Jan Maly and Tero Kilpeläinen ([10]) proved the criterion for a larger class of quasilinear partial differential operators that includes the p-Laplacian for any $p \geq 1$. The theorems is stated as follows: a finite boundary

point $x_0 \in \partial\Omega(\Omega \subset \mathbb{R}^n)$ is regular if and only if

$$\int_0^1 \Big(\frac{\mathrm{cap}_p(B(x_0,r)\cap\Omega^c, B(x_0,2r))}{\mathrm{cap}_p(B(x_0,r), B(x_0,2r))}\Big)^{\frac{1}{p-1}} \frac{dr}{r} = \infty$$

where cap_p is the p-capacity. Before them, Mazya found the proof of the necessity for every p and Lindqvist and Martio managed to prove both ways of the criterion in the case $p > n-1$.

The criterion can be stated also in the setting of homogeneous Carnot groups (see [2] for a reference) and takes the following form. Let $\mathbb{G} = (\mathbb{R}^n, \star, \delta_\lambda)$ be a homogeneous Carnot group, $\mathcal{L}$ a sub-Laplacian on $\mathbb{G}$ and Γ its fundamental solution with pole at 0. Let $y \in \mathbb{G}$ and $E \subset \mathbb{G}$. Pick $C > 1$ and define the anuli

$$A_j = \{x \in \mathbb{G} : C^j \le \Gamma(y^{-1} \star x) \le C^{j+1}\}$$

The following are equivalent:

(i) E is not $\mathcal{L}$-thin at y.

(ii) $\displaystyle\sum_{j=1}^{\infty} C^j \,\mathrm{cap}_{\mathcal{L}}(A_j \cap E) = \infty$.

where we denote by $\mathrm{cap}_{\mathcal{L}}(\cdot)$ the capacity with respect to Γ, defined in a similar way as for the Laplacian and the heat operator.

Bibliography

[1] D. H. Armitage, S. J. Gardiner, *Classical potential theory*, Springer, 2001.

[2] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, *Stratified Lie groups and potential theory for their sub-Laplacians*, Springer, 2007.

[3] E. DiBenedetto, *Partial differential equations*, Springer, 2010.

[4] L. C. Evans, R. F. Gariepy, *Wiener's criterion for the heat equation*, Arch. Rational Mech. and Analysis, **78**-4 (1982), 293-314.

[5] E. B. Fabes, N. Garofalo, *Mean value properties of solutions to parabolic equations with variable coefficients*, Journ. Math. Analysis and Appl., **121** (1987), 305-316.

[6] E. B. Fabes, N. Garofalo, E. Lanconelli, *Wiener's criterion for divergence form parabolic operators with C^1-Dini continuous coefficients*, Duke Math. J., **59** (1989), vol. 1, 191-232.

[7] H. Federer, *Geometric measure theory*, Springer-Verlag, 1969.

[8] N. Garofalo, E. Lanconelli, *Wiener's criterion for parabolic equations*, Trans. of the Am. Math. Soc., **308**-2 (1988), 811-836.

[9] L. L. Helms, *Introduction to potential theory*, Wiley Interscience, 1969.

[10] J. Maly, T. Kilpeläinen, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math., **172** (1994), 137-161.

[11] E. Lanconeli, *Sul problema di Dirichlet per l'equazione del calore*, Ann. Mat. Pura e Appl., IV, **97** (1973), 83-114.

[12] E. Lanconelli, *Sul problema di Dirichlet per equazioni paraboliche del secondo ordine a coefficienti discontinui*, Ann. Mat. Pura e Appl., IV, vol. CVI, 11-38.

[13] J. Moser, *On a pointwise estimate for parabolic differential equations*, Comm. Pure Appl. Math., **24** (1971), 727-740.

[14] B. Pini, *Sulle equazioni a derivate parziali, lineari del secondo ordine in due variabili, di tipo parabolico*, Ann. Mat. Pura Appl. (4), **32** (1951), 179-204.

[15] B. Pini, *Sulla soluzione generalizzata di Wiener per il primo problema di valori al contorno nel caso parabolico*, Rend. Sem. Mat. Univ. Padova, **23** (1954), 422-434.

[16] B. Pini, *Maggioranti e minoranti delle soluzioni delle equazioni paraboliche*, Ann. Mat. Pura e Appl. IV, **37** (1954), 422-434.

[17] S. Sternberg, *Lectures on differential geometry*, Prentice-Hall, Englewood Cliffs, N.J., 1964.

[18] N. A. Watson, *A theory of subtemperatures in several variables*, Proc. London Math. Soc., **26** (1973), 385-417.

[19] N. A. Watson, *Green functions, potentials and the Dirichlet problem for the heat equation*, Proc. London Math. Soc., **33** (1976), 251-297; Corrigendum, Proc. London Math. Soc., **37** (1978), 32-34.

[20] N. A. Watson, *Thermal capacity*, Proc. London Math. Soc., **37** (1978), 342-362.

[21] N. Wiener, *The Dirichlet problem*, J. Math. and Phys., **3** (1924), 127-146.

Ringraziamenti

Il giorno della laurea é allo stesso tempo un inizio ed una fine; é per il secondo suo aspetto che sento di dover rendere grazie a varie persone prima di chiudere questo capitolo della mia vita. La mia famiglia é stata una presenza costante nei miei anni di studi e li ha sostenuti economicamente, non senza sacrifici; sebbene a volte ci siano state divergenze nei punti di vista, non mi ha mai abbandonata. Questa tesi é dedicata ai miei genitori, per il duro lavoro che intraprendono ogni giorno per crescere me e Marco. Un enorme grazie spetta al mio compagno, Fabrizio, che non solo riesce a sopportarmi e supportarmi quotidianamente, ma ha anche immediatamente accettato di seguire me ed i miei sogni all'estero, lontano da tutto quello che conosce. Risulta impossibile non nominare il professor Giorgio Guerrini, la cui passione ed entusiasmo sono stati contagiosi per me fin dal primo istante; é riuscito a farmi innamorare di quello che faccio e so che questo é un raro privilegio. Come promesso, devo citare i passatelli di Maria Teresa, fonte delle idee migliori! Ringrazio in generale tutta la famiglia Bugané, che fin dai tempi del liceo mi ha trattata come una figlia. Sono perfettamente consapevole della fortuna che ho ad avere nella mia vita cosí tante persone che mi stimano, mi amano e mi sostengono, e non posso fare a meno di essere loro riconoscente per ció che mi regalano.