Alma Mater Studiorum · University of Bologna

School of Science
Department of Physics and Astronomy
Master Degree in Physics

Bouncing Universes

Analytic and effective dynamics in reduced-symmetry models of Loop Quantum Cosmology

Supervisor:
Prof. Roberto Balbinot

Co-supervisor:
Prof. Edward Wilson-Ewing

Submitted by:
Francesco Fazzini

Academic Year 2020/2021

Alla mia famiglia:
Giulia, Matteo, Carlo

Acknowledgements

I wish to thank Prof. Roberto Balbinot for two main reason: the first one is that during the Master program he showed me the beauty of classical General Relativity, the black hole analysis and the necessary techniques to quantize field theories on curved backgrounds. His passion and professionalism on these subjects have been an enormous source of inspiration for me and played an important role for the choice of a thesis project in the field of Gravity; then to have given to me the possibility of working on Quantum Gravity, supporting me and helping to face whatever obstacle I met during these months.
I also thank enormously prof. Edward Ewing-Wilson to have taught me the main features of this beautiful theory that is Loop Quantum Gravity and its cosmological models with passion, professionalism and constant support. I'm grateful for the extremely useful discussions we have had and for all his suggestions, that allowed me to make confidence with the main subtle points of the theory and be passionate about it; his knowledge and mastery of this subject have been of deep inspiration for me.
I thank also prof. Roberto Casadio and prof. Francisco Pedro, that in their extremely interesting course introduced me to classical and Quantum Cosmology, giving strong basis on which I could start to face the problem of Quantum Gravity in Cosmology with different approaches.
I finally thank each other professor of the Master to have contributed in a way or another to the knowledge, passion and rigour necessary for my future career in Theoretical Physics.
I thank my whole family, not only the closer one at which this work is dedicated, and all my friends that supported me and allowed me to conclude this amazing and important path.

Contents

Introduction

This work is based on Loop Quantum Gravity, one of the most promising and mathematically rigorous approaches to quantum gravity, that at the time being has collected a large series of theoretical successes like the derivation of the Bekenstein-Hawking entropy with the physical interpretation of the microstates of a non-rotating chargeless black hole, the derivation of the Hawking temperature and the resolution of the singularity problem that affects Einstein classical theory of gravity.
The work is divided in two main parts: in the first one we construct from the Hamiltonian formulation of the classical theory the full Loop Quantum Gravity theory. Following the Dirac quantization program for constrained system we construct initially the Kinematical Hilbert space of the theory, finding an explicit basis of states. Then we promote the classical constraints written in terms of Ashtekar-Barbero smeared variables to quantum operators and for each of them we look at the relative Hilbert space of the solutions, to arrive at the quantization of the Hamiltonian constraint and its solutions, the physical states of the theory. Along this path we construct the Area and Volume operators, that allow us to make considerations already at the Kinematical level about the resolution of the Singularity Problem in Loop Quantum Gravity.
In the second part of this work we study two models of Loop Quantum Cosmology, that despite its name is not properly the cosmological sector of the theory but is made of Minisuperspaces, so quantum models in which a certain symmetry is imposed already at the classical level. We'll focus initially on the simplest Minisuperspace, the flat FRW, where we impose homogeneity and isotropy at the classical level and assume a spatially flat solution. We loop-quantize this model and once we obtain the quantum Hamiltonian constraint we show some of its numerical solutions. We analyze qualitative features of such solutions, focusing in particular on their bouncing behaviour in the Planck regime, that in an extremely elegant and incredible way replaces the classical singularity that affect the classical theory. Then we introduce the effective approach to this model and we show explicit calculations for the solution of the effective equations. The effective

dynamics is an approximation of the analytic one, but for many initial states it reproduces extremely well the numerical solutions. We describe the physical and mathematical reason of this extraordinary agreement and we support such consideration with mathematical calculations. Then following the same path of the flat FRW Minisuperspace we study the second simplest Minisuperspace, the Bianchi I. Here we assume homogeneity but not isotropy: at the classical level we have only three space-like Killing vectors forming an abelian algebra. We quantize the system and we show qualitative features of the numerical solution, focusing in particular on the multi-bouncing behaviour of the wavefunction in the Planck regime. Then we look at its effective dynamics, that also here turns out to be extremely accurate for a large class of initial states. Thus also here we show explicit calculations that justify theoretically such strong agreement between the effective and the numerical results.

Chapter 1

Loop Quantum Gravity

As well as for the quantization of the other field theories like Electromagnetism, there are two approaches to quantize Einstein General Relativity within Loop Quantum Gravity: by a side the so called *Spin Foam approach*, that is based on the path integral formalism applied to Einstein General relativity written in terms of the so called *Ashtekar-Barbero variables* (that we'll present in this chapter), and the *Canonical approach* , that starting from the same classical setup follows instead the Dirac quantization program of constrained systems.
In this work we'll focus on the second one, analyzing it in detail and using it to derive solutions of the Cosmological sector of the theory.

1.1 Canonical approach

These are the steps we'll follow in this approach:

1. We introduce the ADM decomposition of the metric tensor, useful to construct the Canonical formulation of classical General Relativity.

2. We formulate classical General relativity in its hamiltonian version.

3. We introduce the tetrad formalism and we use it to write the classical theory in terms of the *Ashtekar-Barbero variables*, the real starting point of the *Loop quantization procedure*.

4. Following the Dirac prescription for the quantization of constrained systems we quantize the theory and we look at explicit but generic solutions for the constraints.

So let's start with the introduction of the powerful tool of the *ADM decomposition*.

1.1.1 ADM formalism

The ADM procedure starts by assuming that the *4-D* manifold $\mathcal{M}$ has the following topology:

$$M \simeq \mathbb{R} \times \Sigma$$

where Σ is a fixed *3-D* manifold of arbitrary topology and signature $(+++)$.
This physically means that instead of considering the space-time as a *4-D* manifold, we consider it as a *3-D* manifold evolving in time. In particular, these spatial hypersurfaces are assumed to be *Cauchy hypersurfaces*. Let's describe in detail this important concept.
Definition: given an hypersurface Σ we define the future (past) development of Σ and we call it $D^+(\Sigma)(D^-(\Sigma))$, as the set of the points of the manifold not belonging to Σ, living in the future (in the past) of the hypersurface such that, given a point $\in D^+(\Sigma)(D^-(\Sigma))$ each causal line which starts from the point, passes in the past (in the future) through Σ
Definition: we call Σ a Cauchy hypersurface if

$$D^+(\Sigma) \cup D^-(\Sigma) = \mathcal{M}$$

This means that a Cauchy hypersurface can be casually connected with the whole manifold.
Definition: A manifold that contains at least a Cauchy hypersurface is said to be *globally hyperbolic*.
Well, there is a theorem that states that we can always foliate a globally hyperbolic manifold with Cauchy hypersurfaces.
So if we consider $\mathcal{M}$ globally hyperbolic, the statement:

$$\mathcal{M} \simeq \mathbb{R} \times \Sigma$$

means we are choosing a particular foliation of $\mathcal{M}$ in Cauchy hypersurfaces, and this gives no restrictions to our globally hyperbolic manifold.
Remark: assuming that the space-time is globally hyperbolic means that there are no causally disconnected regions.
Let's come back now to the ADM procedure. Assuming a decomposition of our manifold in a real line $\mathbb{R}$ and space-like Cauchy hypersurfaces means that we associate to each sur-

face a value for the real parameter t belonging to $\mathbb{R}$. This foliation allows us to identify t as a time parameter, "decoupling" it from Σ.

Physically we don't have anymore a *4-D* manifold, but a *3-D* one that evolves accordingly with a parameter t.

Remark: The flow of t cannot be seen at this stage as the physical flow of time, since as we see in detail later it can be produced by time-like diffeomorphisms, that are unphysical infinitesimal transformations of coordinates. This is known in classical General Relativity as the *Problem of Time*, and we'll see how to solve it explicitally. We look for the moment at t as a generic parameter.

In order to fix ideas we choose a foliation using the adapted ADM coordinates $(t, \vec{x})$. For this purpose, we define the 4-vector X^μ_t, such that its first component points toward increasing values of t, while the other three are independent from t itself. Let's thus define the *normalized time-flow vector*:

$$t^\mu(x) \equiv \frac{\partial X^\mu_t}{\partial t} = (1, 0, 0, 0)$$

written in the basis $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}\}$. Well, the vector $t^\mu(x)$ in general is not normal to Σ_t.

We denote the normal to Σ with n^μ and we assume it is normalized: $n_\mu n^\mu = -1$. This means that n^μ and t^μ are both time-like, but in general not parallel.

So we can decompose t^μ into its normal and tangent part with respect to Σ:

$$t^\mu = N(x) \cdot n^\mu(x) + N^\mu(x)$$

We fix for simplicity: $n^\mu = (\frac{1}{N}, -\frac{N^a}{N})$, obtaining: $N^\mu = (0, N^a)$.

In fact since $t^\mu = (1, 0, 0, 0)$ in our coordinate system, then:

$$t^0 = 1 = \frac{N(x)}{N(x)} + N^0 = 1 + N^0 \iff N^0 = 0$$

N is called *Lapse function*, while N^a are called *Shift functions*.

If we write the norm of t^μ and the scalar product $N^\mu t_\mu$ in terms of these functions, we obtain the metric tensor (that allows to compute the scalar product) in terms of them:

$$g_{\mu\nu} t^\mu t^\nu = g_{00} = -N^2 + g_{ab} N^a N^b$$

While:

$$g_{\mu\nu} t^\mu N^\nu = g_{0b} N^b = g_{\mu\nu}(N n^\mu + N^\mu) N^\nu = g_{\mu\nu} N^\mu N^\nu = g_{ab} N^a N^b \iff g_{0b} = g_{ab} N^a \equiv N_b$$

Using these results, we can write:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(N^2 - N_a N^a)dt^2 + 2N_a dt dx^a + g_{ab}dx^a dx^b$$

where $a, b = 1, 2, 3$ are spatial indices contracted with the *3-metric* g_{ab}.
We use this form of the metric tensor within the hamiltonian formulation of the theory.
Remark: since t^μ is not orthogonal to Σ, in general g_{ab} is not its intrinsic metric.
The intrinsic metric is instead given by the spatial part of:

$$q_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$$

so that when we look at tensors defined on Σ_t, since they are orthogonal to n^μ can be equivalently contracted with $g_{\mu\nu}$ or $q_{\mu\nu}$.
The quantity $q^\mu{}_\nu = g^{\mu\rho}q_{\rho\nu}$ acts as projector on Σ_t, so acting on a generic tensor defined on the whole manifold projects it on the hypersurface Σ_t. Thus $q^\mu{}_\nu$ allows us to construct tensorial quantities on such hypersurface, starting from the ones defined on $\mathcal{M}$.
The intrinsic metric of the spatial slices allows us to define another important quantity, the *extrinsic curvature* of Σ_t:

$$K_{\mu\nu} = q^\rho{}_\mu q^\sigma{}_\nu n_{\sigma;\rho}$$

It can be proved that this tensor is symmetric and is linked to the Lie derivative of the intrinsic metric on the direction given by n^μ: $\mathcal{L}_{\vec{n}} = 2K_{\mu\nu}$.
For this reason we can use it to define at the Hamiltonian level the generalized conjugate momentum of q_{ab}, that we call π^{ab}.
The extrinsic curvature is present also in the relation between the Riemann tensor $\mathcal{R}^\mu{}_{\nu\rho\sigma}$ (for Σ_t) and the one for $\mathcal{M}$, the usual $R^\mu{}_{\nu\rho\sigma}$:

$$\mathcal{R}^\mu{}_{\nu\rho\sigma} = q^\mu{}_\alpha q^\beta{}_\nu q^\gamma{}_\rho q^\delta{}_\sigma R^\alpha{}_{\beta\gamma\delta} - K_{\nu\sigma}K^\mu{}_\rho - K_{\nu\rho}K^\mu{}_\sigma \qquad (\textit{Gauss-Codazzi equation})$$

1.1.2 Hamiltonian formulation of classical General Relativity

The Gauss-Codazzi equation and the ADM decomposition of the metric tensor allow us to rewrite the Einstein-Hilbert action in terms of $\mathcal{R}$ and K :

$$S = \int dt \int_{\Sigma_t} d^3x \sqrt{q} N [\mathcal{R} - K^2 + K_{\mu\nu}K^{\mu\nu}]$$

Where we set $16\pi G = 1$ for convenience (this choice will be changed for convention only in the cosmological sector of the theory). Even if not explicit, the previous action doesn't contain time derivatives of N, N^a. This means that:

$$\frac{\delta \mathcal{L}}{\delta \dot{N}} = \frac{\delta \mathcal{L}}{\delta \dot{N}^a} = 0 \tag{1.1}$$

N and N^a are not propagating degrees of freedom. In the Dirac nomenclature the relations 1.1 are called *Primary constraints* of the system, and their existence makes classical General Relativity a constrained system, that means that in its hamiltonian formulation has to me studied with the Dirac theory on constrained systems.
The conjugate conjugate momentum of q_{ab}, written in terms of the extrinsic curvature K:

$$\pi^{ab} = \frac{\delta \mathcal{L}}{\delta \dot{q}_{ab}} = \sqrt{q}(K^{ab} - Kq^{ab})$$

We can define the *Hamiltonian density* of the theory $\mathcal{H}$ by computing the Legendre transform of $\mathcal{L}$, obtaining

$$\mathcal{H} = N^a \mathcal{H}_a + N\mathcal{H}_0 \tag{1.2}$$

Where:

$$\mathcal{H}_a = -2\sqrt{q}\nabla_b \left(\frac{\pi^b{}_a}{\sqrt{q}} \right) ; \quad \mathcal{H}_0 = \frac{1}{\sqrt{q}} G_{abcd}\pi^{ab}\pi^{cd} - \sqrt{q}\mathcal{R} \tag{1.3}$$

and:

$$G_{abcd} = q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd}$$

is often called *supermetric*, or *DeWitt metric*. N, N^a are the shift and lapse functions of the ADM decomposition, and in this canonical formulation play the role of Lagrange multipliers for the constraints $\mathcal{H}_0$ and $\mathcal{H}_a$. In the Dirac nomenclature $\mathcal{H}_0$ and $\mathcal{H}_a$ are called secondary constraints, since they are obtained by imposing that the primary constraints 1.1 hold at each time, so for physical solutions:

$$\mathcal{H}_0 \approx 0, \qquad \mathcal{H}_a \approx 0 \tag{1.4}$$

The notation $\approx$ means that such relations hold on physical configurations (*weakly* in the Dirac nomenclature).
The relations 1.4 are not only secondary constraints for the system, but also *first class*: the Poisson algebra generated by $\mathcal{H}_0$ and $\mathcal{H}_a$ vanishes *on shell*, so on the physical solutions

$$\{\mathcal{H}_a, \mathcal{H}_0\} \approx 0, \quad \{\mathcal{H}_a, \mathcal{H}_b\} \approx 0, \quad \{\mathcal{H}_0, \mathcal{H}_0\} \approx 0 \tag{1.5}$$

As we'll see later these first class constraints generate gauge transformations, associated respectively with time-like and space-like diffeomorphisms.

In general any theory containing gauge invariance like General Relativity (the gauge invariance in this case comes from the diffeomorphism invariance of the theory) is in its hamiltonian formulation a constrained system that contains first class constraints, and such constraints turn out to be the generators of gauge transformations.

$\mathcal{H}_a \approx 0$ is usually called *Diffeomorphism constraint* (here for diffeo. we mean spatial diffeomorphisms), and $\mathcal{H}_0 \approx 0$ is called *Hamiltonian constraint*.

A system with an hamiltonian of kind 1.27, i.e. with the form (lagrange multipliers) $\times$ (first class constraints) is called *generally covariant system*, and it has very important features, we are about to explain. Usually for generic constrained systems the time evolution is not generated by some constraint, since the hamiltonian contains not only constraints but also other terms. In our case instead we can combine space-like and time-like diffeomorphisms to produce a gauge transformation of the metric tensor that reproduce exactly its infinitesimal variation under the time evolution generated by the Hamiltonian. This is a peculiar and central feature of the theory, and it brings some important consequences: by a side, solving the system given by the constraints means finding the solutions of the theory at each time. This holds both at the classical and quantum level, as we'll see. By the other side, this means that at this level we are not able to distinguish between a physical evolution and gauge transformations produced by diffeomorphisms, and this gives rise to the famous *problem of time* of General Relativity, and a possible way (not the unique one) to solve it is fixing the gauge for the solution, in such a way that the only possible time evolution is the physical one. We see how to do it in practise when we look at the cosmological solutions of the theory.

The same considerations can be performed on the system of the free relativistic particle: it can be seen as a gauge field theory, so at the hamiltonian level as a constrained system. Moreover, since as a field theory describes the evolution of four fields in 1-D it is usually studied as a toy model for the Einstein theory. We perform its analysis on *Appendix* B.

But let's come back to our Einstein-Hilbert action. In order to explore its Hamiltonian formulation let's look now at the symplectic structure.

Symplectic structure and gauge transformations

We saw in the previous paragraph that the physical phase space variables of our hamiltonian theory are q_{ab} and their conjugate momenta π^{ab}.
Let's construct their equal time canonical commutation relations:

$$\{\pi^{ab}(t,\vec{x}), q_{cd}(t,\vec{x'})\} = \delta^{a}{}_{(c}\delta^{b}{}_{d)}\delta(\vec{x}-\vec{x'}) \tag{1.6}$$

where with $(c\ d)$ we mean the symmetric part of the tensor, i.e.:

$$\delta^{a}{}_{(c}\delta^{b}{}_{d)} = \frac{1}{2}(\delta^{a}{}_{c}\delta^{b}{}_{d} + \delta^{a}{}_{d}\delta^{b}{}_{c})$$

and where the relations 1.6 are evaluated at equal t, so on the same Cauchy hypersurface Σ_t.
We recall that given two functions $A(x)$, $B(z)$, their Poisson brakets:

$$\{A(x), B(z)\} = \int d^3x \left[\frac{\delta A(y)}{\delta q_{ab}(x)}\frac{\delta B(z)}{\delta \pi^{ab}(x)} - \frac{\delta B(z)}{\delta q_{ab}(x)}\frac{\delta A(y)}{\delta \pi^{ab}(x)}\right] \tag{1.7}$$

Well, using 1.3 we can evaluate the equal-time Poisson brackets among the constraints. We obtain:

$$\begin{aligned}
&\{\mathcal{H}_a(x), \mathcal{H}_b(y)\} = \mathcal{H}_a(y)\partial_b[\delta(x-y)] - \mathcal{H}_b(x)\partial'_a[\delta(x-y)] \\
&\{\mathcal{H}_a(x), \mathcal{H}(y)\} = \mathcal{H}(x)\partial_a\delta(x-y) \\
&\{\mathcal{H}(x), \mathcal{H}(y)\} = \mathcal{H}^a(y)\partial_a\delta(x-y) - \mathcal{H}^a(y)\partial'_a\delta(x-y)
\end{aligned} \tag{1.8}$$

We notice as anticipated before that the Poisson brackets vanish on the constraint surface $(H^\mu \approx 0)$.
Let's also observe that these relations are not trivial on the constraint surface, since the weak equality holds only after the calculation of the Poisson brackets.
Let's look now at the gauge transformations of the theory. The Poisson brakets between the Diffeo. constraint and the variables of the phase space:

$$\begin{aligned}
\{H(\vec{N}), q_{ab}\} &= \mathcal{L}_{\vec{N}} q_{ab} \\
\{H(\vec{N}), \pi^{ab}\} &= \mathcal{L}_{\vec{N}} \pi^{ab}
\end{aligned}$$

where $H(\vec{N})$ is the smearing of the diffeomorphism constraint: $H(\vec{N}) = \int_\Sigma d^3x N^l \mathcal{H}_l$ and: $\mathcal{L}_{\vec{N}} q_{ab} = N_{a;b} + N_{b;a} = \delta_{-\vec{N}} q_{ab}$ is the Lie derivative of q_{ab} along $\vec{N}$, so an infinitesimal variation of q generated by the space-like diffeomorphism along the direction individuated by $\vec{N}$.
This means that $H(\vec{N})$ generates spatial diffeomorphisms on Σ, as anticipated before.
The situation is analogous for the Hamiltonian constraint:

$$\{H(N), q_{ab}\} = \mathcal{L}_{\vec{n}N} q_{ab}$$
$$\{H(N), \pi^{ab}\} = \mathcal{L}_{\vec{n}N} \pi^{ab} + \frac{1}{2} q^{ab} N \mathcal{H}_0 - 2N\sqrt{q} q^{c[a} q^{b]d} R_{cd}$$

where $H(N)$ is the smearing of the Hamiltonian constraint: $H(N) = \int_\Sigma d^3x N \mathcal{H}_0$.
The first Poisson bracket gives an infinitesimal variation of q_{ab} on the direction orthogonal to Σ (that in general is not the one individuated by t^μ) while the second term gives the same result for π^{ab} if we evaluate it on shell ($\mathcal{H}_0 \approx 0, R_{cd} \approx 0$ in vacuum).
So the Hamiltonian and the Diffeo. constraint generate the gauge transformations of the theory in this Hamiltonian formulation.
Let's perform now a brief analysis of the number of degrees of freedom of the theory.

Physical degrees of freedom of the classical theory

In the case of the linearized Einstein-Hilbert action the solution in vacuum has got only 2 physical degrees of freedom, that are the two polarizations of the gravitational wave. This counting of the degrees of freedom holds however not only for the linearized theory, but also for the generic theory in vacuum.
In the canonical formalism we can confirm this count with a general consideration.
To make it let's start with the classical phase space of a free particle : (q_i, p^j). It has got dimension 6, while as we know the physical degrees of freedom are only 3.
In the case of fied theories like GR, we expect that the number of d.o.f are infinite, since the field is defined on a space-time which is continuous, so on an infinite domain.
So let's see how "large" is this infinity in our canonical formulation.
The whole phase space: $(q_{ij}(x)\ \pi^{kl}(y))$ has $(6+6)\cdot\infty^3$ d.o.f., since q and π are symmetric tensors $(i, j = 1, 3)$ and each component is defined on the *3-D* manifold Σ.
But we remember that the physical configurations live on the constraint surface, and not in the whole space. So we have: $(12-4)\cdot\infty^3 = 8\cdot\infty^3$. This is not however the end of the count. As we said the first class constraints $\mathcal{H}^\mu$ generate gauge transformations on

the constraint surface, and this means that on the physical phase space we'll have gauge orbits that correspond to the same physical states.
Thus we have to gauge-fix the system to obtain the physical hypersurface, that technically means selecting a representant for each gauge orbit. This means reducing the number of d.o.f: $(8-4)\cdot\infty^3 = 4\cdot\infty^3$.
As in the classical case, in order to recover the dimension of the configuration space we have to divide this quantity by 2, obtaining the desired result.
Let's proceed now toward the next step of our program, formulating the Einstein theory in terms of Ashtekar-Barbero variables.

1.1.3 Classical background for LQG

The first tool we need to introduce in order to construct Loop Quantum Gravity is the Tetrad formalism. As we'll see in detail later, this formulation will allow us to write the theory in terms of an SU(2) connection and its conjugated variable, making gravity similar in this sense to other gauge theories like QED (U(1)), QCD (SU(3))..

Tetrad formalism

Definition: a tetrad is a quadruple of covariant tensors ${e^I}_\mu(x)dx^\mu$, with $I = 0,1,2,3$ with components that satisfy the following:

$$g_{\mu\nu}(x) = {e^I}_\mu(x){e^J}_\nu(x)\eta_{IJ} \tag{1.9}$$

Notation: we write conventionally latin indices: $I, J, K, ..$ to denote internal tetrad indices, while greek indices : $\mu, \nu, \rho, ..$ regarding space-time components written in a generic frame.
The dependence of e from x comes from the fact that the relation 1.9 depends on the space-time point on which is performed since the metric is a tensorial field that depends on the points on which is evaluated.
We could naively think that the previous relation is a sort of coordinate transformation, from a local inertial frame (in which the metric assumes the usual Minkowski form) to a generic one. So, let's focus on this point and see what is the difference between:

$$g_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\eta_{\rho\sigma}$$

and 1.9, since such difference is at the heart of tetrad formalism: when we use latin indices, we are forcing the tensor to be written (in those indices) in a local inertial frame, or in other words these indices are lowered and raised *always* by the Minkowski metric. So, if $\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}$ allows to pass from a generic frame to another, the second one describes the passage from a frame forced to be local inertial to a generic one. Thus we can see the tetrad indices as describing tensors written on frames forced to be locally inertial. As a consequence, the only transformations we can do on such tensors are Lorentz-Poincaré (since are the only isometries of the Minkowski metric):

$$\eta'_{IJ} = \Lambda^K{}_I \Lambda^M{}_J \eta_{KM}$$

The same for generic tensors written in the tetradic form: $A'^I = \Lambda^I{}_J A^J$.

Remark: when we define a tensor written in latin indices we don't mean in general a constant tensor field, and as well as all the other tensor fields it depends on x.

Notation: we consider $\tilde{e}^I$ as covariant basis vectors, while $\vec{e}_J$ as contravariant basis vectors. Tetrads provide the relation between basis vectors written in latin indices and greek ones:

$$\vec{e}_I = e^\mu{}_I(x)\frac{\partial}{\partial x^\mu}$$

While a rank-2 tensor with both latin indices has the following form:

$$\boldsymbol{\eta} = \eta_{IJ}\tilde{e}^I \otimes \tilde{e}^J$$

A rank-2 tensor with a latin index and a greek one:

$$\boldsymbol{\alpha} = \alpha_{I\mu}\tilde{e}^I \otimes dx^\mu$$

So we write a tensor with rank > 1 in such a way that different indices may mean different coordinate basis (differently from what we usually do in GR); in our case μ is written in a generic basis while I is written in a local inertial basis. Using the tetrads we can write the tensor $\boldsymbol{\alpha}$ in the usual basis:

$$\boldsymbol{\alpha} = \alpha_{I\mu}(e^I{}_\nu dx^\nu) \otimes dx^\mu = \alpha_{I\mu} e^I{}_\nu dx^\nu \otimes dx^\mu = \alpha_{\mu\nu} dx^\mu \otimes dx^\nu$$

Furthermore we have that basis vectors and their dual covectors can be contracted producing the usual Kronecker Delta:

$$\delta^I{}_J = \tilde{e}^I(\vec{e}_J) = e^I{}_\mu e^\nu{}_J dx^\mu(\partial_\nu) = e^I{}_\mu e^\mu{}_J$$

So far we introduced basis vectors ($\vec{e}_I$) and covectors ($\tilde{e}^J$) with latin indices. A generic vector in this basis: $\vec{\boldsymbol{V}} = V^I \vec{e}_I$. Since its components are labelled by a latin index, they clearly transform only under Lorentz-Poincaré. The covariant derivative of such tensor:

$$D_\mu V^I = \partial_\mu V^I(x) + \omega_\mu{}^I{}_J(x) V^J(x)$$

where $\omega_\mu{}^I{}_J(x)$ is called *Spin connection*, and it is a covector with values in the Lorentz algebra. We can define also derivatives of objects with both greek and latin indices

$$D_\mu e^I{}_\nu = \partial_\mu e^I{}_\nu + \omega_\mu{}^I{}_J(x) e^J{}_\nu - \Gamma^\rho{}_{\nu\mu} e^I{}_\rho$$

We notice that we have to use different connections for the covariant derivative of components defined in different spaces. We impose on ω the so called *Tetrad postulate*

$$D_\mu e^I{}_\nu = 0$$

Which implies: $\begin{cases} \partial_{(\mu} e^I{}_{\nu)} + \omega_{(\mu}{}^I{}_J(x) e^J{}_{\nu)} = \Gamma^\rho{}_{\nu\mu} e^I{}_\rho \\ \partial_{[\mu} e^I{}_{\nu]} + \omega_{[\mu}{}^I{}_J(x) e^J{}_{\nu]} = 0 \end{cases}$

With the squared brakets we mean the antisymmetric part of the μ, ν indices of the tensor : $\omega_{[\mu}{}^I{}_J(x) e^J{}_{\nu]} = \frac{1}{2}\big(\omega_\mu{}^I{}_J(x) e^J{}_\nu - \omega_\nu{}^I{}_J(x) e^J{}_\mu\big)$.
We want to prove now a relation that will be useful later

$$\omega^I{}_{\mu J} = e^I{}_\nu\big(\nabla_\mu e^\nu{}_J\big) \tag{1.10}$$

From the Tetrad postulate

$$\partial_\mu e^I{}_\nu - \Gamma^\rho{}_{\nu\mu} e^I{}_\rho = \nabla_\mu e^I{}_\nu = -\omega_\mu{}^I{}_J(x) e^J{}_\nu \quad \Rightarrow \quad e^\nu{}_K\big(\nabla_\mu e^I{}_\nu\big) = -\omega_\mu{}^I{}_J(x) e^J{}_\nu e^\nu{}_K$$

Remembering that $e^I{}_\mu e^\mu{}_J = \delta^I{}_J$, we have

$$e^\nu{}_K\big(\nabla_\mu e^I{}_\nu\big) = -\omega^I{}_{\mu K}$$

Using then the Leibnitz rule

$$\nabla_\mu\big(e^\nu{}_K e^I{}_\nu\big) - \big(\nabla_\mu e^\nu{}_K\big)e^I{}_\nu = -\omega^i{}_{\mu K} \quad \Rightarrow \quad \big(\nabla_\mu e^\nu{}_K\big)e^I{}_\nu = \omega^I{}_{\mu K}$$

Remark: ∇_μ and D_μ are different derivatives, and are linked by the following:

$$D_\mu e^I{}_\nu = \nabla_\mu e^I{}_\nu + \omega^I{}_{\mu J} e^J{}_\nu$$

With this formalism in hand we can construct the Einstein-Hilbert action in terms of tetrads. Once we do this, it will be sufficient a change of variables in order to reach the classical formulation we'll quantize to reach LQG. In order to do this, we introduce some important concept of tensorial calculus.

n-Forms

An n-form is a tensor field $\boldsymbol{\omega}$ of type $(0\ n)$, that is totally antisymmetric in its indices. For example $\omega_{\mu\nu} dx^\mu \otimes dx^\nu$ with $\omega_{\mu\nu} = \omega_{\nu\mu}$ is not a 2-form, while

$$\omega_{\mu\nu}(dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu)$$

Is a 2-form. We call

$$dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu \equiv dx^\mu \wedge dx^\nu$$

the wedge product between the basis 1-forms dx^μ and dx^ν. In general, the wedge product between two 1-form

$$\boldsymbol{\omega} \wedge \boldsymbol{v} = \omega_\mu dx^\mu \wedge v_\nu dx^\nu = \omega_\mu v_\nu (dx^\mu \wedge dx^\nu)$$

Exterior derivative

In general, the *Exterior derivative* of an n-form is an (n+1)-form of this kind: given

$$\boldsymbol{\omega} = \omega_{\mu_1 \ldots \mu_n} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n}$$

Then

$$d\boldsymbol{\omega} = \frac{\partial \omega_{\mu_1 \ldots \mu_n}}{\partial x^\nu} dx^\nu \wedge dx^{\mu_1} \wedge .. \wedge dx^{\mu_n}$$

is its exterior derivative. From the previous definition is clear that if $\boldsymbol{\omega}$ is an n-form defined on a manifold of dimension n, $\mathrm{d}\boldsymbol{\omega} = 0$.

We can now introduce the *Covariant exterior derivative* for the tetrads, and we call it d_ω to distinguish from the ordinary exterior derivative:

$$d_\omega \boldsymbol{e}^I = d\boldsymbol{e}^i + \boldsymbol{\omega}^I{}_J \wedge \boldsymbol{e}^J$$

Writing $\boldsymbol{e}$, $\boldsymbol{\omega}$ in the basis $\{dx^\mu\}$ we obtain

$$d_\omega e^I = d\big(e^I{}_\mu dx^\mu\big) + \omega^I{}_{\mu J} dx^\mu \wedge e^J{}_\nu dx^\nu = e^I{}_{\mu,J} dx^\nu \wedge dx^\mu + \omega^I{}_{\nu J} e^J{}_\mu dx^\nu \wedge dx^\mu = (e^I{}_{\mu,\nu} + \omega^I{}_{\nu J} e^J{}_\mu) dx^\nu \wedge dx^\mu$$

But we have that $dx^\nu \wedge dx^\mu$ is antisymmetric, so also the components of this 2-form have to be antisymmetric, thus

$$d_\omega e^I = (e^I{}_{[\mu,\nu]} + \omega^I{}_{[\nu J} e^J{}_{\mu]}) dx^\nu \wedge dx^\mu$$

We introduce now an object that represents the curvature tensor written in terms of the spin-connection ω

$$\boldsymbol{F}^{IJ} = d\boldsymbol{\omega}^{IJ} + \boldsymbol{\omega}^I{}_K \wedge \boldsymbol{\omega}^{KJ}$$

Well, it can be proved that such tensor is linked with the Riemann tensor

$$F^{IJ}{}_{\mu\nu}(\omega(e)) = e^{I\rho} e^{J\sigma} R_{\mu\nu\rho\sigma}(e) \qquad Curvature$$

F is fundamental to write the Einstein-Hilbert action in terms of tetrads, as well as the Riemann tensor is fundamental to write it in the usual way.
Before going on, we want to prove the following relation, useful in the next section: $g = det(g_{\mu\nu})) = -det(e)^2 = -e^2$
Recalling the Cayley formula for the determinant:

$$g = det(g_{\mu\nu}) = \frac{1}{4!} \epsilon^{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} g_{\mu\alpha} g_{\nu\beta} g_{\rho\gamma} g_{\sigma\delta}$$

And remembering that $g_{\mu\nu} = e^I{}_\mu e^J{}_\nu \eta_{IJ}$ we have

$$g = det(e^I{}_\mu e^J{}_\nu \eta_{IJ}) = \frac{1}{4!} \epsilon^{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} e^I{}_\mu e^J{}_\alpha \eta_{IJ} e^K{}_\nu e^L{}_\beta \eta_{KL} e^M{}_\rho e^N{}_\gamma \eta_{MN} e^O{}_\sigma e^P{}_\delta \eta_{OP}$$

Now, using the following

$$\epsilon^{\mu\nu\rho\sigma} e^I{}_\mu e^J{}_\nu e^K{}_\rho e^L{}_\sigma = e \epsilon^{IJKL}$$

We have

$$g = \frac{1}{4!}e^2\epsilon^{IKMO}\epsilon^{JLNP}\eta_{IJ}\eta_{KL}\eta_{MN}\eta_{OP} = \frac{1}{4!}e^2\epsilon^{IKMO}\epsilon_{IKMO} = -\frac{1}{4!}e^2\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma} = -e^2$$

Where we also used $\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma} = \frac{\sqrt{det((\eta))}}{sgn(det(\eta))}\epsilon^{IKMO}\epsilon_{IKMO}$

The Einstein-Hilbert action in terms of tetrads

From now on we'll work using the tetrad formalism, for a reason that will be clear later. The Einstein-Hilbert action as functional of tetrads

$$S_{EH}(e^I{}_\mu) = \frac{1}{2}\epsilon_{IJKL}\int e^I \wedge e^J \wedge F^{KL}(\omega(e)) \tag{1.11}$$

Let's prove it.

$$S_{EH}(g_{\mu\nu}(e)) = \int d^4x\sqrt{-g}R_{\mu\nu}g^{\mu\nu} = \int d^4x e e^\mu{}_I e^{\nu I} R_{\mu\rho\nu\sigma} e^\rho{}_J e^{\sigma J} = \int d^4x e e^\mu{}_I e^\rho{}_J F^{IJ}{}_{\mu\rho}(\omega(e))$$

Now we can prove with a bit of tensorial algebra that: $4 \cdot e e^\mu{}_I e^\rho{}_J = \epsilon_{IJKL}\epsilon^{\mu\rho\alpha\beta}e^K{}_\alpha e^L{}_\beta$. Substituting it into the action

$$S_{EH} = \frac{1}{2}\int d^4x \epsilon_{IJKL}\big(\frac{1}{2}\epsilon^{\mu\rho\alpha\beta}e^K{}_\alpha e^L{}_\beta F^{IJ}{}_{\mu\rho}(\omega(e))\big) = \frac{1}{2}\int \epsilon_{IJKL}e^K \wedge e^L \wedge F^{IJ}$$

In fact

$$e^K \wedge e^L \wedge F^{IJ} = \frac{1}{2}e^K{}_\mu e^L{}_\nu F^{IJ}{}_{\rho\sigma}dx^\mu \wedge dx^\nu \wedge (dx^\rho \wedge dx^\sigma) = \frac{1}{2}e^K{}_\alpha e^L{}_\beta F^{IJ}{}_{\mu\rho}\epsilon^{\mu\rho\alpha\beta}d^4x$$

Where we used the fact that F is antisymmetric in greek indices (it inherits this property from the Riemann tensor) and $\epsilon^{\mu\nu\rho\sigma}d^4x = dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$.
We notice that 1.11 is not only diffeomorphism invariant in the greek internal indices, but it is also Lorentz-Poincaré invariant in the latin indices. This larger symmetry group will play an important role in the formulation of Loop Quantum Gravity.
We can also consider ω as an extra variable of the action; in this case, if we vary 1.11 with respect of ω we obtain the relation 1.10 that we previously derived from the Tetrad postulate.
This formulation allows to add a term to the Lagrangian that contains all the features of 1.11, as the gauge invariance of greek indices and the invariance under local Lorentz

transformations of the latin ones

$$\delta_{IJKL} e^I \wedge e^J \wedge F^{KL}(\omega)$$

Where $\delta_{IJKL} \equiv \delta_{I[K}\delta_{L]J}$ is the so called *Generalized Kronecker Delta*, antisymmetric in the indices k, l as well as F.
The addition of this term is not relevant in vacuum GR since it leaves the equations of motion untouched, but will play an important role in the Loop quantization of the theory.
Adding this term in 1.11 we obtain

$$S_{Holst}(e,\omega) = \big(\frac{1}{2}\epsilon_{IJKL} + \frac{1}{\gamma}\delta_{IJKL}\big) \int e^I \wedge e^J \wedge F^{KL}(\omega) \qquad Holst \quad action$$

Where γ is called *Immirzi parameter*, and at this stage is a free parameter of the theory.
The variation of this action gives as anticipated

$$\begin{cases} \omega^{IJ}{}_{\mu\nu} = e^I{}_\nu \nabla_\mu e^{J\nu} \quad \rightarrow \quad variation \quad of \quad \omega \\ G_{\mu\nu}(e) = 0 \quad \rightarrow \quad variation \quad of \quad e \end{cases}$$

that are the field equations in vacuum, independent from γ.
Now we need to recover its Hamiltonian formulation. Once we have the hamiltonian formulation of the Holst action, we'll introduce the *Ashtekar-Barbero variables* and then proceed with the quantization.

Hamiltonian formulation with Ashtekar-Barbero variables

In order to recover the Hamiltonian formulation of the Holst action we proceed in the same way we did for the *Einstein-Hilbert* action.
We split our *4-D* space-time in Cauchy spatial hypersurfaces: $\mathcal{M} = \mathbb{R} \times \Sigma$, and choose a set of coordinates $(t, \vec{x})$ that fix the foliation. We introduce the lapse function N and the shift N^i, decomposing the metric in the ADM form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(N^2 - N_a N^a)dt^2 + 2N_a dt dx^a + g_{ab}dx^a dx^b$$

With $t^\mu = N(x)n^\mu + N^\mu$ the vector that individuates the t-flowing in the manifold.
Let's rewrite now t^μ (that is written in the coordinate system $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x^a}\}$) in a tetradic basis:

$$\tilde{t}^I = e^I{}_\mu t^\mu = N(x)e^I{}_\mu n^\mu(x) + e^I{}_\mu(x)N^\mu(x) = N(x)n^I + N^a(x)e^I{}_a(x)$$

where we recall that $n^\mu = \left(\frac{1}{N}, -\frac{N^a}{N}\right)$ and $N^\mu = \left(0, N^a\right)$. $\tilde{t}^I$ are the components of $\vec{t}$ in the tetradic basis $\{\vec{e_I}\}$.

In the Hamiltonian formulation tensors are written on the submanifolds Σ_t of $\mathcal{M}$, and this means that even if we can construct a tetradic basis on $\mathcal{M}$, let's say $\{\vec{e_I}\}$, we have to consider only its spatial part as basis for tensors. We call it $\{\vec{e_i}\}$, $i = 1, 2, 3$. On each Σ_t we have:

$$\delta_{ij} e^i{}_a e^j{}_b = g_{ab} \qquad i, j = 1, 2, 3 \tag{1.12}$$

δ_{ij} is the spatial (eucledian) part of η_{IJ}.

The basis $\{\vec{e}_i\}$ is called *triadic basis*, and is the spatial part of the tetradic basis $\{\vec{e}_I\} = \{\vec{e}_i, \vec{e}_0\}$.

The relation between the triadic basis and the tensorial one allows to define the *triads* $e^a{}_i(x)$:

$$\vec{e}_i = e^a{}_i(x) \frac{\partial}{\partial x^a} \tag{1.13}$$

As we did before, in order to reach the Hamiltonian formulation of the Holst action we need to define canonical conjugated variables from e, ω. However we won't work with the variables e, ω and their conjugated momenta, but we make a change of variables introducing the so called *Ashtekar-Barbero variables*. Before doing this to simplify the analysis, we introduce the so called *time gauge*:

$$e^I{}_\mu n^\mu = n^I = \delta^I{}_0$$

Which means that in the tetradic basis the vector n^μ is orthogonal to $\vec{e_i}$ $(i = 1, 2, 3)$. If now we solve the system:

$$\begin{cases} e^0{}_\mu n^\mu = 1 \\ e^0{}_\mu g^{\mu\nu} e^0{}_\nu = -1 \end{cases}$$

We obtain: $e^0{}_\mu = (N, 0, 0, 0)$ and consequently $\tilde{t}^I = (N, N^a e^I{}_a)$. Let's recall that $\tilde{t}^I$ are the components of the vector $\vec{t}$ (that determines the foliation) in the tetradic basis.

Now, in order to recover the Hamiltonian formulation of LQG at the classical level we introduce the so called *Ashtekar-Barbero* variables. We define:

$$E^a{}_i = e\, e^a{}_i = \frac{1}{2} \epsilon_{ijk} \epsilon^{abc} e^j{}_b e^k{}_c \qquad \textit{Densitized Triad} \tag{1.14}$$

$$A^i{}_a = \gamma\omega^{0i}{}_a + \frac{1}{2}\epsilon^i{}_{jk}\omega^{jk}{}_a \qquad \textit{Ashtekar-Barbero connection} \tag{1.15}$$

Where $w^{jk}{}_a$ is the spatial part of the *Spin connection*, contained in the covariant derivative of the tetrad:

$$D_\mu e^I{}_\nu = \partial_\mu e^I{}_\nu + w^I{}_{\mu J} e^J{}_\nu - \Gamma^\rho_{\nu\mu} e^I_\rho$$

while $e^a{}_i$ and: $e = \sqrt{|det(E)|}$. Well, these variables turn out to be canonically conjugated. Their canonical commutation relations:

$$\left\{ A^i{}_b(y), E^a{}_j(x) \right\} = \frac{1}{2}\gamma\delta^a{}_b\delta^i{}_j\delta(x-y) \tag{1.16}$$

$$\left\{ E^a{}_i(x), E^b{}_j(y) \right\} = 0 \tag{1.17}$$

$$\left\{ A^i{}_a(x), A^j{}_b(y) \right\} = 0 \tag{1.18}$$

Where γ is the *Immirzi parameter*. Let's recall that we assumed $16\pi G = 1$. If we restore such factor it appears in the numerator of the right-hand side of 1.16.
The hamiltonian formulation of the Holst action in these new variables:

$$S(A, E, N, N^a) = \frac{1}{\gamma}\int dt \int_\Sigma d^3x \left[\dot{A}^i_a E^a_i - A^i_0 \mathcal{G}_i - N\mathcal{H}_0 - N^a\mathcal{H}_a \right]$$

where

$$\begin{cases} \mathcal{G}_i = D_a E^a_i = \partial_a E^a_i + \epsilon_{jil} A^j_a E^{al} \\ \mathcal{H}_a = \frac{1}{\gamma} F^j_{ab} E^b_j - \frac{1+\gamma^2}{\gamma} K^i_a \mathcal{G}_i \\ \mathcal{H}_0 = \left[F^j_{ab} - (\gamma^2+1)\epsilon_{jmn} K^m_a K^n_b \right] \cdot \frac{\epsilon^{jkl} E^a_k E^b_l}{e} + \left(\frac{1+\gamma^2}{\gamma} \right) \mathcal{G}^i \partial_a \frac{E^a_i}{e} \end{cases}$$

with $K^i_a = \omega^{0i}_a$ the extrinsic curvature in the triadic form, and $F^k_{ab} = 2\partial_{[a} A^k_{b]} + \epsilon_{ij}{}^k A^i_a A^j_b$ the curvature tensor in terms of the Ashtekar connection.
We notice that the action has the same form of the usual canonical formulation of the E-H action, where here A takes the place of q, E of π. Let's make some further considerations. As usual N, N^a take the role of Lagrange multipliers, while $\mathcal{H}_0, \mathcal{H}_a$ are respectively the Hamiltonian and Diffeomorphism constraints.
We notice that here we have the extra-constraints $\mathcal{G}_i$, called *Gauss constraint*: this constraint generates gauge transformations for "triadic" indices, in particular transformations belonging to the $SU(2) \cong SO(3)$ group (spatial rotations), so a subgroup of the whole Lorentz-Poincaré.

The choice of the variables 2.102 and 1.15, that turned out to be crucial for the loop quantization, was made initially to write the theory as a Yang-Mills theory, so in terms of an $SU(2)$ connection and its conjugated variable, an $SU(2)$ vector. It can be proved in fact that A transforms like an $SU(2)$ connection, while E like an $SU(2)$ vector.
This however is not what we do in the Loop quantization of the theory. Instead of looking at the quantization of the phase space local variables (A, E), we construct non-local variables through their smearing along particular hypersurfaces. This by a side avoids having operators that are distributional, by the other complicates the quantization scheme (in particular the quantization of the Hamiltonian constraint). Despite this it is the fundamental step that produces a quantum theory different from the Wheeler-DeWitt one, and allows to solve its main issues. This is not true for example if we apply this scheme to classical Electromagnetism (see [15]).
We have to notice however a crucial difference with respect to other Yang-Mills theories: if there the constraint surface at the classical level is made only by the Gauss constraint $G_i \approx 0$, here we have in addition the Diffeo. and the Hamiltonian constraint, so a larger gauge group ([14]). This brings further complications that we'll face later.
Let's look now at the algebra generated by the Gauss constraint.
We define the smearing of the Gauss constraint:

$$G(\Lambda) = \int d^3x \mathcal{G}_i(x)\Lambda^i(x),$$

Where Λ^i is a generic field with a triadic index. Let's evaluate the Poisson brakets between G and E:

$$\begin{aligned}
\left\{ \int d^3x \Lambda^j(x)\mathcal{G}_j(x), E^a{}_i(x) \right\} &= \int d^3x \Lambda^j(x)\{\partial_b E^b{}_j + \epsilon_{mjn}A^m{}_b E^{bn}, E^a{}_i(y)\} = \\
&= \int d^3x \Lambda^j(x)\{\partial_b E^b_j(x), E^a{}_i(y)\} + \int d^3x \Lambda^j \epsilon_{mjn}\{A^m{}_b E^{bn}(x), E^a{}_i(y)\} = \\
&= \int d^3x \Lambda^j \epsilon_{mjn}\{A^m{}_b E^{bn}(x), E^a{}_i(y)\} = \\
&= \int d^3x \Lambda^j \epsilon_{mjn}\big(A^m{}_b\{E^{bn}(x), E^a{}_i(y)\} + \{A^m{}_b, E^a{}_i(y)\}E^{bn}(x)\big) = \\
&= \int d^3x \Lambda^j \epsilon_{mjn}\{A^m{}_b(x), E^a{}_i(y)\}E^{bn}(x) = \\
&= \gamma\Lambda^j(y)\epsilon_{ijn}E^{an}(y)
\end{aligned}$$

Instead:

$$\Big\{\int d^3x\Lambda^j(x)\mathcal{G}_j(x), A^i_a(y)\Big\} = \int d^3x\Lambda^j(x)\big\{\partial_b E^b_j + \epsilon_{mjn}A^m_b E^{bn}, A^i_a\big\} =$$
$$= \int d^3x\Lambda^j(x)\big(\big\{\partial_b E^b_j, A^i_a\big\} + \epsilon_{mjn}\big\{A^m_b E^{bn}, A^i_a\big\}\big)$$

Let's evaluate the two terms of the previous expression separately:

$$\int d^3x\Lambda^j(x)\big\{\partial_b E^b_j, A^i_a\big\} = \int d^3x\Lambda^j(x)\partial_b\big\{E^b_j, A^i_a\big\} = -\gamma\int d^3x\Lambda^j(x)\partial_b\delta^b{}_a\delta^i{}_j\delta(x-y) =$$
$$= \gamma(\partial_a\Lambda^i(y))$$

while:

$$\epsilon_{mjn}\int d^3x\Lambda^j(x)\big\{A^m_b E^{bn}, A^i_a\big\} = \epsilon_{mjn}\int d^3x\Lambda^j(x)A^m_b\big\{E^{bn}, A^i_a\big\} =$$
$$= -\gamma\epsilon_{mjn}\int d^3x\Lambda^j(x)A_b\delta^b{}_a\delta^{ni}\delta(x-y) =$$
$$= -\gamma\epsilon_{mjn}\delta^{ni}\Lambda^j(y)A^m{}_a(y) = -\gamma\epsilon_{mji}\Lambda^j(y)A^m{}_a(y)$$

Let's finally evaluate the Poisson brakets between $\mathcal{G}$ and itself:

$$\big\{G(\Lambda), \mathcal{G}_i(y)\big\} = \big\{\mathcal{G}(\Lambda), \partial_a E^a{}_i(y) + \epsilon_{jik}A^j{}_a E^{ak}(y)\big\}$$

We evaluate also in this case the two terms separately

$$\Big\{\int d^3xG_p(x)\Lambda^p(x), \partial_a E^a{}_i(y)\Big\} = \int d^3x\Lambda^p(x)\big\{\partial_b E^b{}_p + \epsilon_{jpl}A^j{}_a E^{al}, \partial_c E^c{}_i\big\} =$$
$$= \int d^3x\Lambda^p(x)\big\{\epsilon_{jpl}A^j{}_a E^{al}, \partial_c E^c{}_i\big\} = \gamma\delta^c{}_a\delta^j{}_i\int d^3x\Lambda^p(x)\epsilon_{jpl}\frac{\partial}{\partial y^c}(\delta(x-y))E^{al} =$$
$$= \gamma\int d^3x\Lambda^p(x)\epsilon_{ipl}\frac{\partial}{\partial y^c}(\delta(x-y))E^{cl} = -\gamma\int d^3x\Lambda^p(x)\epsilon_{ipl}\frac{\partial}{\partial x^c}(\delta(x-y))E^{cl} =$$
$$= \gamma\partial_c(\Lambda^j(y)\epsilon_{ijn}E^{cn})$$

While the second one

$$\epsilon_{jik}\big\{G(\Lambda), A^j{}_a E^{ak}(y)\big\} = \epsilon_{jik}\Big\{\int d^3xG_p(x)\Lambda^p(x), A^j{}_a E^{ak}(y)\Big\} =$$
$$= \epsilon_{jik}\int d^3x\Lambda^p(x)\Big\{G_p(x), A^j{}_a E^{ak}(y)\Big\} = \epsilon_{jik}\int d^3x\Lambda^p(x)\big\{\partial_b E^b{}_p + \epsilon_{jpl}A^j{}_a E^{al}, A^j{}_a E^{ak}(y)\big\}$$

That after a bit of calculations gives

$$\epsilon_{jik}\big\{G(\Lambda), A^j{}_a E^{ak}(y)\big\} = \gamma\epsilon_{jik}\epsilon^j_{ml}\Lambda^l A^m{}_a E^{ak} - \gamma\epsilon_{jik}\Lambda^l\epsilon^k_{ln}A^j{}_a E^{an} + \epsilon_{lik}(\partial_a\Lambda^l(y))E^{ak}(y)$$

Summing all together we have

$$\big\{G(\Lambda), \mathcal{G}_i(y)\big\} = \gamma\epsilon_{ilk}\Lambda^l(y)\mathcal{G}^k(y)$$

And finally, if we smear also the second term the Poisson brakets give

$$\big\{G(\Lambda_1), G(\Lambda_2)\big\} = \frac{\gamma}{2}G([\Lambda_1, \Lambda_2])$$

We notice that for $G \approx 0$ the Poisson braket vanishes. In a similar way can be shown that G Poisson-commutes with all the other constraints on shell. This means that it is also a first class constraint, and can be explicitally proved that generates gauge transformations. In this formulation of Einstein theory we have a phase space of dimension $18 \cdot \infty^3$ with the fundamental Poisson brackets:

$$\Big\{A^i_a(x), E^b_j(y)\Big\} = \delta^b_a\delta^i_j\gamma\delta^3(x-y)$$

We can recover the $12 \cdot \infty^3$ phase space from the $18 \cdot \infty^3$, firstly by imposing $G_i \approx 0$ and remaining with $15 \cdot \infty^3$ degrees of freedom, then by fixing the residual gauge freedom. If we do it we have no more gauge freedom for the symmetry $SU(2)$, and we remain only with the gauge freedom related with diffeomorphisms (generated by $\mathcal{H}_0$ and $\mathcal{H}_a$).

Smearing of the algebra

The next and last step we need to do at the classical level is the smearing of the variables A, E. We can smear $E^a{}_i$ on a surface S with normal n_a:

$$E_i(S) = \int_S n_a E^a{}_i d^2\sigma$$

The quantity $E_i(S)$ is the flux of $E^a{}_i$ across S. The reason why we considered a *2-D* surface of Σ to smear E comes from its definition: $E^a{}_i = \frac{1}{2}\epsilon_{ijk}\epsilon^{abc}e^j{}_b e^k{}_c$: even if $e's$ are contracted with the Levi-Civita tensor, we can see it as a combination of $\binom{0}{2}$ tensors of the form $e^j{}_b e^k{}_c$. By the other side $A^i{}_a = \gamma\omega^{0i}{}_a + \frac{1}{2}\epsilon^i{}_{jk}\omega^{jk}{}_a$, so is a $\binom{0}{1}$ tensor and it is natural to smear it along a *1-D* path of Σ. So let's consider a path γ and an its own

parametrization

$$x^a(s): \quad [0,1] \to \Sigma$$

Then, given a connection ${A^i}_a$ we can construct a generic element of the algebra of $SU(2)$: $A_a \equiv {A^i}_a \tau_i$, where τ_i are the SU(2) generators, and in the $j = \frac{1}{2}$ representation are: $\frac{i}{2}\cdot$ (Pauli matrices). Let's recall in fact that the upper index of A is an adjoint index of $SU(2)$. Finally we can integrate A_a along the path γ:

$${A^i}_a \quad \to \int_\gamma A \equiv \int_0^1 ds {A^i}_a(x(s)) \frac{dx^a}{ds} \tau_i$$

However this doesn't complete the smearing of the connection. For reasons that will be clear later we introduce the *holonomy* of A along γ:

$$\sum_{n=0}^{\infty} \int_0^t ds_1 \int_0^{s_1} ds_2 .. \int_0^{s_{n-1}} ds_n A(\gamma(s_1)) ... A(\gamma(s_n))$$

That is solution of the following Cauchy problem

$$\begin{cases} \frac{d}{dt} h_\gamma(t) - h_\gamma(t) A(\gamma(t)) = 0 \\ h_\gamma(0) = 1 \end{cases}$$

In fact, if we integrate this equation reiteratively:

$$\begin{aligned} h_\gamma(t) &= 1 + \int_0^t A(\gamma(s_1)) h_\gamma(s_1) ds_1 = 1 + \int_0^t ds_1 A(\gamma(s_1)) \left[1 + \int_0^{s_1} h_\gamma(s_2) A(\gamma(s_2) ds_2 \right] \\ &= 1 + \int_0^t A(\gamma(s_1)) ds_1 + \int_0^t A(\gamma(s_1)) ds_1 \int_0^{s_1} A(\gamma(s_2) ds_2 \left[1 + \int_0^{s_2} A(\gamma(s_3) h_\gamma(s_3) ds_3 \right] ... \\ &= \sum_{n=0}^{\infty} \int_0^t ds_1 \int_0^{s_1} ds_2 .. \int_0^{s_{n-1}} ds_n A(\gamma(s_1)) ... A(\gamma(s_n)) \end{aligned}$$

Let's fix n in such summation, and analyze the associated term:

$$\int_0^t ds_1 \int_0^{s_1} ds_2 .. \int_0^{s_{n-1}} ds_n A(\gamma(s_1)) ... A(\gamma(s_n)) \tag{1.19}$$

We firstly notice that the terms $A(\gamma(s_1))...A(\gamma(s_n))$ are in this form path ordered, since: $s_1 \geq s_2 ... \geq s_n$. This is relevant already at the classical level, since A contains elements of the algebra of $SU(2)$, and such algebra is not abelian. We also notice that we can rewrite 1.19 in a more useful way. For simplicity we fix $n = 2$, then we'll generalize the

consideration at arbitrary n. In this case 1.19 is:

$$\int_0^t ds_1 A(\gamma(s_1)) \int_0^{s_1} ds_2 A(\gamma(s_2)) \tag{1.20}$$

In 1.20 we see that the integration is made over a triangle in the (s_2, s_1) plane, since s_2 goes from 0 to s_1. If we double the area of integration we obtain an integral over a square of lenght t:

$$\int_0^t ds_1 A(\gamma(s_1)) \int_0^{s_1} ds_2 A(\gamma(s_2)) + \int_0^t ds_2 \int_0^{s_2} ds_1 A(\gamma(s_1)) A(\gamma(s_2)) =$$
$$= \int_0^t ds_1 A(\gamma(s_1)) \int_0^t ds_2 A(\gamma(s_2))$$

Where in the second integral of the first line we have to keep attention and preserve the order of the connections, since they don't commute. Now we apply to both members the path ordering operator

$$\mathcal{P}\Big[\int_0^t ds_1 A(\gamma(s_1)) \int_0^{s_1} ds_2 A(\gamma(s_2)) + \int_0^t ds_2 \int_0^{s_2} ds_1 A(\gamma(s_1)) A(\gamma(s_2)) \Big] =$$
$$= 2 \int_0^t ds_1 A(\gamma(s_1)) \int_0^{s_1} ds_2 A(\gamma(s_2)) = \int_0^t ds_1 \int_0^t ds_2 \mathcal{P}[A(\gamma(s_1)) A(\gamma(s_2))]$$

Thus

$$\int_0^t ds_1 A(\gamma(s_1)) \int_0^{s_1} ds_2 A(\gamma(s_2)) = \frac{1}{2} \int_0^t ds_1 \int_0^t ds_2 \mathcal{P}[A(\gamma(s_1)) A(\gamma(s_2))]$$

This result can be generalized for arbitrary n:

$$\int_0^t ds_1 \int_0^{s_1} ds_2 .. \int_0^{s_{n-1}} ds_n A(\gamma(s_1)) ... A(\gamma(s_n)) = \frac{1}{n!} \int_0^t ... \int_0^t \mathcal{P}[A(\gamma(s_1)) ... A(\gamma(s_n))] ds_1 ... ds_n$$

So:

$$h_\gamma(t) = \sum_{n=0}^{+\infty} \frac{1}{n!} \iiint_\square \mathcal{P}[A(\gamma(s_1)) ... A(\gamma(s_n))] ds_1 ... ds_n$$
$$\equiv \mathcal{P} \cdot exp\Big(\int_0^t A(\gamma(s)) ds \Big) = \mathcal{P} \cdot exp\Big(\int_\gamma A \Big)$$

where in our case:

$$A(\gamma(s)) = A_a^i(x) \frac{dx^a}{ds} \tau_i$$

$\mathcal{P}$ stays for the usual *path ordered product*. Let's look now at some important properties of the holonomy (also called *Path ordered exponential*), that are the reason why we smeared in this way the connection:

1. Given h_γ, h_δ the holonomies of two paths, the product $h_\gamma \cdot h_\delta$ is the holonomy of the composition of the paths:

$$h_\gamma \cdot h_\delta = h_{\gamma+\delta} = \mathcal{P}e^{\left(\int_{\gamma+\delta} A\right)}$$

2. Under a local gauge transformation $g \in SU(2)$ the holonomy transforms as:

$$h^g_\gamma = g_{s(\gamma)} h_\gamma g^{-1}_{t(\gamma)}$$

   Where s(γ) and t(γ) are respectively the initial and final points of γ, also called *source* and *target*. This property tells us that an $SU(2)$ transformation acts only on the initial and final point of γ.

3. Under the action of a generic spatial diffeomorphism the holonomy transforms as:

$$h_\gamma(\phi A) = h_{\phi\circ\gamma}(A)$$

4. The functional derivative of h with respect to the connection A gives:

$$\frac{\delta h_\gamma(A)}{\delta A^i{}_a(x)} = \begin{cases} \int ds \frac{1}{2}\dot{x}^a \delta^{(3)}(\gamma(s), x)\tau_i h_\gamma & if\ x\ is\ the\ source\ of\ \gamma \\ \int ds \frac{1}{2}\dot{x}^a \delta^{(3)}(\gamma(s), x) h_\gamma \tau_i & if\ x\ is\ the\ target\ of\ \gamma \\ \int ds \dot{x}^a \delta^{(3)}(\gamma(s), x) h_\gamma(0, s)\tau_i h_\gamma(s, 1) & if\ x\ is\ inside\ \gamma \end{cases} \tag{1.21}$$

Where we have to pay attention of the fact that $\tau_i h_\gamma \neq h_\gamma \tau_i$ since h_γ is an element of $SU(2)$ and the generators do not commute.
With the definition of the holonomy we completed the smearing of the variables A, E. The resulting smeared Algebra is called **Holonomy-flux Algebra**, and as we said earlier its introduction is the fundamental break from the old Wheeler-DeWitt theory. The quantization of such phase space is called *Loop quantization* since initially the holonomies have been thought as integrals along loops, not generic curves γ.

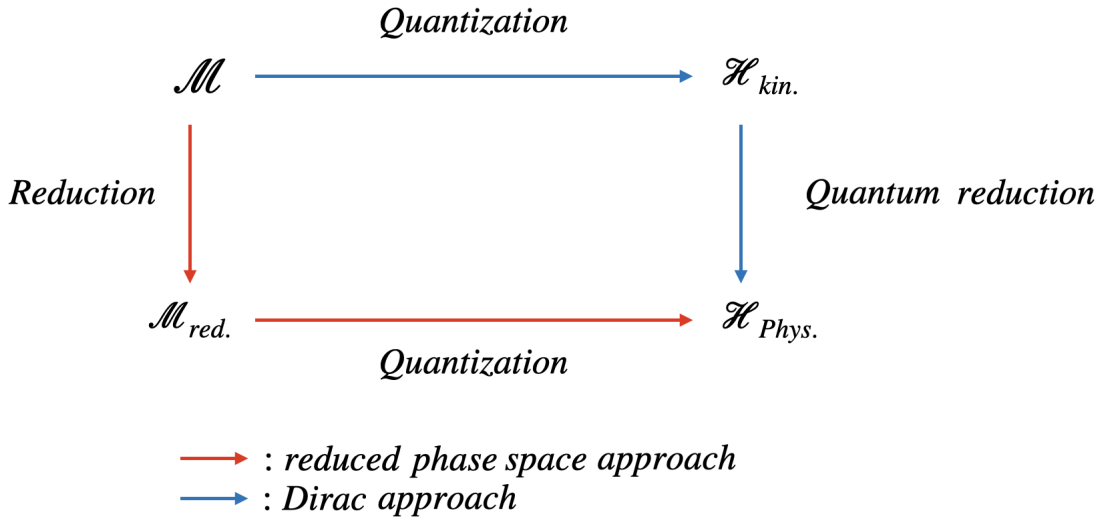
1.2 Quantization of the theory

Before applying the Dirac quantization program to our classical theory, let's make a brief recap about this procedure.

To quantize a constrained system there are two main approaches: the so called *reduced phase space approach* and the *Dirac approach*.

The first approach requires the following steps: constraining the system, constructing *Dirac observables*, which are quantities that commute with all the first class constraints: $\{O, C\} \approx 0$, and then quantize such observables making act them on a suitable Hilbert space. This procedure is in general quite involved, and we won't follow it. By the other side the Dirac approach starts from quantizing the theory without constraints, and then imposing constraints at the quantum level to the Kinematical Hilbert space.

Let's give a picture of these two different approaches:



Let's describe the Dirac program step by step:

1. Find a representation of the phase space variables of the theory as operators acting on the kinematical Hilbert space $\mathcal{H}_{kin}$, and promoting the Poisson brackets to commutators:
$$\{\,.\,,\,.\,\} \to -\frac{i}{\hbar}[\,.\,,\,.\,]$$

2. Promote the classical constraints of the system to self-adjoint operators acting on $\mathcal{H}_{kin}$.

3. Characterize the space of solutions of the quantum constraints by defining the corresponding inner product, that gives a notion of physical probability. This allows to construct the *Physical Hilbert space* of the theory, that is the quantum analog of the classical constraint surface.
   The states belonging to this space satisfy: $\widehat{C}_i \, |\psi_{phys}\rangle = 0$

4. Find a complete set of observables that commute with all the constraints:

$$\widehat{O} \;\; | \;\; [\widehat{O}, \widehat{C}_i] = 0$$

Well, this procedure is completely general and holds for any constrained system. So let's apply to our theory.
We start by defining the representation given by the connection A, so that a generic state of the Kinematical Hilbert space can be written as a wave functional depending on A:

$$\psi[A] = \langle A|\psi\rangle$$

Then we promote the phase space variables E, A to quantum operators; in such representation we have

$$\hat{A}^i_a \psi[A] = A^i_a \psi[A] \tag{1.22}$$

$$\hat{E}^a_i \psi[A] = -i\hbar\gamma \frac{\delta}{\delta A^i{}_a} \psi[A] \tag{1.23}$$

We notice that such operators, as well as $\widehat{q}_{ab}$ and $\widehat{\pi}^{cd}$ are distributional operators. Afterwards we promote the classical Poisson brakets to commutators:

$$[\hat{A}^i_a, \hat{E}^b_j] = i\hbar\gamma \delta^b{}_a \delta^i{}_j \delta^3(x-y) \tag{1.24}$$

The next step is construct a well-defined inner product for the Kinematical Hilbert space of such wave functionals. Here we meet the first difficulty, since the inner product requires the definition of a measure in the space of connections. In other gauge theories we have a fixed background metric to define the integration measure, while here the metric is a dynamical quantity, so we cannot follow the usual QFT procedure.
In order to reach this goal we need to come back to the classical theory and smear the canonical variables E and A. This by a side allows to work with operators at the quantum level that are not distributional, and by the other allow to write a well-defined

inner product for the Kinematical Hilbert space. To reach the goal we need to introduce the notion of *cylindrical functions*.

1.2.1 The Kinematical Hilbert space

A cylindrical function is a functional of a field that depends only on some subset of degrees of freedom of the field itself. The holonomies we introduced to smear the variable A:

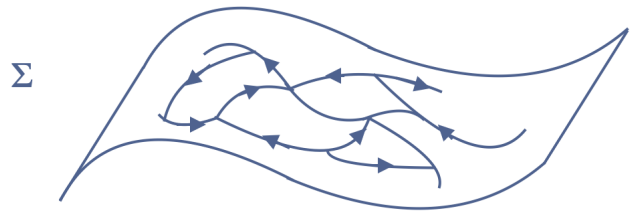
$$h_\gamma[A] = \mathcal{P}exp\left(\int_\gamma A\right)$$

are examples of cylindrical functions of A. We notice in fact that in the holonomy we don't consider all the degrees of freedom of the field A, but only the ones that regard the chosen path γ.

In order to construct more complicated cylindrical functions of the theory we introduce the definition of *graphs*.

Definition: we define a graph Γ as a collection of oriented paths $e \in \Sigma$.

We call such paths *links* of the graph, while the intersections between them *nodes*.



We can generalize now the previous example of cylindrical function:

a generic cylindrical function is a couple (Γ, f) of a graph Γ with L links and a smooth function

$$f : SU(2)^L \quad \longrightarrow \quad \mathbb{C}$$

with f given by a functional of the connection defined as

$$\langle A|\Gamma, f\rangle = \Psi_{(\Gamma,f)}[A] = f(h_{e_1}[A], ..., h_{e_L}[A]) \quad \in Cyl_\Gamma$$

In other words the state $|\Gamma, f\rangle$ written in the basis of the connections is a functional f depending on variables that are holonomies evaluated on the different paths that compose the graph Γ. In this generic case the cylindrical function captures only the degrees of freedom of A along the links of the graph; e_i, $i = 1, ..., L$ are the links of the graph Γ.

Remark: f takes values in $SU(2)^L$ since $h_{e_1}, ..., h_{e_L}$ are elements of $SU(2)$.
In this way we built an abstract space of functionals (Γ, f) that can be turned into an Hilbert space if we equip it with a scalar product.
The previous switch from the connection A to the holonomy:

$$h_\gamma = \mathcal{P} \cdot exp\Big(\int_\gamma A\Big)$$

is fundamental at this stage since the holonomy is an element of $SU(2)$ and the integration over $SU(2)$ is well defined [21]; moreover there is a unique gauge-invariant measure (normalized) dh, called *Haar measure* for such space.
So we can define the scalar product:

$$\langle \Psi_{(\Gamma,f)} | \Psi'_{(\Gamma,f')} \rangle = \int \prod_e dh_e \overline{f(h_{e_1}[A], ..., h_{e_L}[A])} \cdot f'(h_{e_1}[A]...h_{e_L}[A]) \tag{1.25}$$

where in principle f, f' are different functionals.
In this way, at fixed Γ, we send:

$$Cyl_\Gamma \to \mathcal{H}_\Gamma$$

Then we define the Hilbert space of all cylindrical functions for all graphs as the direct sum of Hilbert spaces on given graphs:

$$\mathcal{H}_{Kin} = \oplus \mathcal{H}_\Gamma$$

The scalar product on $\mathcal{H}_{Kin}$ follows from 1.25: if Ψ, Ψ' share the same Γ, we have 1.25, while if they regard different Γ, i.e.

$$\langle \Psi_{(\Gamma_1,f)} | \Psi'_{(\Gamma_2,f')} \rangle$$

we consider a graph: $\Gamma_3 \equiv \Gamma_1 \cup \Gamma_2$ and we extend f_1, and f_2 trivially on Γ_3 and define:

$$\langle \Psi_{(\Gamma_1,f_1)} | \Psi_{(\Gamma_2,f_2)} \rangle \equiv \langle \Psi_{(\Gamma_1 \cup \Gamma_2,f_1)} | \Psi_{(\Gamma_1 \cup \Gamma_2,f_2)} \rangle$$

Well, a key result due to Ashtekar and Lewandowski is that the Hilbert space

$$\mathcal{H}_{Kin} = \oplus \mathcal{H}_\Gamma$$

can be seen as an Hilbert space of gauge connections A:

$$\mathcal{H}_{Kin} = L_2[A, d\mu_{AL}]$$

where the measure $d\mu_{AL}$ is made over the space of connections, and not of holonomies. This means that we can see the scalar product 1.25 as

$$\langle \Psi_{(\Gamma_1,f_1)} | \Psi_{(\Gamma_2,f_2)} \rangle = \int d\mu_{AL} \overline{\Psi_{(\Gamma_1,f_1)}}[A] \Psi_{(\Gamma_2,f_2)}[A]$$

In this way we have an Hilbert space with a well-defined measure over the space of connections that doesn't require a background metric.
We completed the first step for quantization.
What we have to do now is promoting the holonomy-flux algebra to the quantum version. In order to do this is convenient to introduce an orthogonal basis on $\mathcal{H}_{Kin}$; this can be done using the Peter-Weyl theorem. Before doing this let's recall the notion of *Wigner matrices* of $SU(2)$: given an element $g \in SU(2)$ the Wigner matrix $D^j(g)$ represents the action of $g \in SU(2)$ on the $\{|j,n\rangle\}$ basis, which is the basis of common eigenstates of the operators $\hat{J}_z$ (generators of $SU(2)$), and $\hat{J}^2$ (the sum of squared generators), written in the $\langle j,m|$ representation. So:

$$D^j_{m,n}(g) \equiv \langle j,m| \, g \, |j,n\rangle$$

Well, the Peter-Weyl theorem states that given the Hilbert space:

$$\mathcal{L}_2(SU(2), d\mu_{Haar})$$

of functions of $SU(2)$, any of these functions (that in our case are the holonomies) can be written as a combination of D's:

$$f(g) = \sum_{j \in \frac{\mathbb{N}}{2}} \sum_{m=-j}^{j} \sum_{n=-j}^{j} f^j_{mn} D^{(j)}_{mn}(g)$$

with

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, ... \qquad m, n = -j, ..., j$$

where $D^{(j)}_{mn}$ are elements of the group written in the representation j of $SU(2)$.
This means that $\{D^j_{mn}\}$ is a complete basis for the space on which $f(g)$ lives : $\mathcal{L}_2(G, d\mu_{Haar})$.

$f^j_{mn} \in \mathbb{C}$ are the weights of the combination.
We can apply this consideration directly to $\mathcal{H}_\Gamma$. In particular, if

$$f(g) = \sum_{j \in \frac{\mathbb{N}}{2}} \sum_{m=-j}^{j} \sum_{n=-j}^{j} f^j_{mn} D^{(j)}_{mn}(g)$$

then

$$\langle A|\Gamma, f\rangle = \sum_{j_1}\sum_{j_2}\sum_{j_3}...\sum_{j_L}\sum_{m_1}\sum_{m_2}...\sum_{m_L}\sum_{n_1}...\sum_{n_L} \cdot D^{(j_1)}_{m_1 n_1} f^{j_1}_{m_1 n_1} ... D^{(j_L)}_{m_L n_L} f^{j_L}_{m_L n_L}$$

and the basis elements:

$$\langle A|\Gamma; j_l, m_l, n_l\rangle \equiv D^{(j_1)}_{m_1 n_1}(h_{e_1})...D^{(j_L)}_{m_L n_L}(h_{e_L})$$

Where $l = 1, 2, ...L$. This is a generic state of an orthonormal basis of our Hilbert space in the representation of A. Clearly L depends on the number of edges in Γ.
For the moment we are working with the $Haar$ measure, which means that our functionals are thought in the space of holonomies, not in the space of A; we'll see later how to write them in such space.
Using this basis for $\mathcal{H}_\Gamma$, we can construct easily quantum field operators associated with the phase space variables. In particular, if we consider:

$$\Psi_{(\Gamma,f)}[A] = D^{\frac{1}{2}}_{m,n}(h_e)$$

which means

$$f = f(h_e(A)), \quad f : SU(2) \to \mathbb{C}$$

so that Γ is made only of a link, and that we consider f as one of the basis elements $\{D^{(j)}_{mn}\}$ which span $\mathcal{L}_2(SU(2), d\mu_{Haar})$:

$$f = D^{\frac{1}{2}}(h_e) \equiv h_e$$

In this case if we promote the holonomy to an operator:

$$(\hat{h}_\gamma)_{mn}(h_e)_{pq} = (h_\gamma)_{mn}(h_e)_{pq} \tag{1.26}$$

with $m, n, p, q = \frac{1}{2}, -\frac{1}{2}$ and

$$(\hat{h}_\gamma)_{mn} = \left(Pexp\widehat{\left(\int_\gamma A \right)} \right)_{mn} \quad with \quad A = A^i_a(x)\tau_i$$

If instead we look at the action of $\hat{E}$ on the holonomy:

$$\hat{E}_i(S)h_\gamma[A] = -i\hbar\gamma \int_S d^2\sigma n_a \frac{\delta h_\gamma[A]}{\delta A^i_a(x(\sigma))} \tag{1.27}$$

where we used:

$$\hat{E}_i = \int_S d^2\sigma n_a \hat{E}^a_i \doteq \int_S d^2\sigma n_a \left(-i\hbar\gamma \frac{\delta}{\delta A^i_a} \right)$$

Let evaluate 1.27 explicitally. First of all we have to distinguish between different cases:

- $\gamma \cap S$=P, with P a point of the manifold Σ inner to γ.
- $\gamma \cap S$= $\emptyset$.
- $\gamma \cap S$=P, with P a point of the manifold Σ that is the source or the target of γ.

We consider here the first case, so we assume that such P exists and lies inside γ. Using the third relation of 1.21

$$\hat{E}_i h_\gamma[A] = -i\hbar\gamma \int_S d^2\sigma n_a \int_0^1 ds\ \dot{x}^a \delta^{(3)}(x^b(s) - y^b(\sigma)) h_{\gamma_1} \tau_i h_{\gamma_2}$$

Where γ is divided in γ_1 and γ_2 in P. We also used the following

$$\frac{\delta h_\gamma[A]}{\delta A^i{}_a(\vec{y}(\sigma_1,\sigma_2))} = \frac{\delta h_\gamma[A]}{\delta A^i{}_a(\vec{x}(s))} \frac{\delta A^i{}_a(\vec{x}(s))}{\delta A^i{}_a(\vec{y}(\sigma))} = \frac{\delta h_\gamma[A]}{\delta A^i{}_a(\vec{x}(s))} \delta^3(x^b(s) - y^b(\sigma))$$

We have

$$= -i\hbar\gamma \int_S d^2\sigma n_a \int_0^1 ds\ \dot{x}^a \delta^{(3)}(x^b(s) - y^b(\sigma)) h_\gamma(s=0, s=P) \tau_i h_\gamma(s=P, s=1) =$$
$$= -i\hbar\gamma\ h_\gamma(0,P)\tau_i h_\gamma(P,1) \int_S d^2\sigma n_a \int_0^1 ds\ \dot{x}^a \delta^{(3)}(x^b(s) - y^b(\sigma))$$

Let's so evaluate the integral:

$$\int_S d^2\sigma n_a \int_0^1 ds\ \dot{x}^a \delta^{(3)}(x^b(s) - y^b(\sigma)) = \int_S d^2\sigma \int_0^1 ds\ \epsilon_{abc} \frac{\partial y^b}{\partial \sigma_1} \frac{\partial y^c}{\partial \sigma_2} \frac{dx^a}{ds}\ \delta^{(3)}(x^d(s) - y^d(\sigma))$$

From the definition of the normal to a surface, and the scalar product between two vectors.

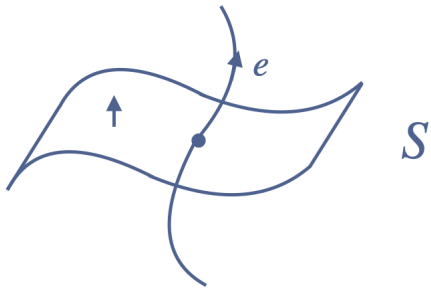
Now we have to distinguish between three different cases:

1. if the curve γ is tangent to the surface in P, then

$$\epsilon_{abc}\frac{\partial y^b}{\partial \sigma_1}\frac{\partial y^c}{\partial \sigma_2}\frac{dx^a}{ds}\bigg|_{\vec{x},\vec{y}=P} = 0$$

   Since in the tangent space of P the vector field $\frac{\partial x^a}{\partial s}$ will be a linear combination of $\frac{\partial y^b}{\partial \sigma_1}$ and $\frac{\partial y^c}{\partial \sigma_2}$. This gives a whole 0 result cause the Dirac delta.

2. If the curve γ intersects the surface in a non tangent way we have the only non trivial result: Now, assuming for clarity a coordinate system adapted to the surface S and



   to the curve γ, in such system we have: $x^d(s) = (s, 0, 0)$ and $y^d(\sigma) = (0, \sigma_1, \sigma_2)$ We have:

$$\begin{aligned}
&\int_S d^2\sigma \int_0^1 ds\ \epsilon_{abc}\frac{\partial y^b}{\partial \sigma_1}\frac{\partial y^c}{\partial \sigma_2}\frac{dx^a}{ds}\ \delta(s)\delta(\sigma_1)\delta(\sigma_2) = \\
&= \int_S d^2\sigma\ \epsilon_{abc}\frac{\partial y^b}{\partial \sigma_1}\frac{\partial y^c}{\partial \sigma_2}\frac{dx^a(s)}{ds}\bigg|_{s=0}\delta(\sigma_1)\delta(\sigma_2) = \\
&= \pm\int_S d^2\sigma\ \epsilon_{abc}u_1^a u_2^b u_3^c\ \delta(\sigma_1)\delta(\sigma_2) = \pm\int_S d^2\sigma\ \epsilon_{123}\ \delta(\sigma_1)\delta(\sigma_2) \\
&= \pm\int_S d^2\sigma\delta(\sigma_1)\delta(\sigma_2) = \pm 1
\end{aligned}$$

   Where we called

$$u_1^a = \begin{pmatrix}1\\0\\0\end{pmatrix}, u_2^b = \begin{pmatrix}0\\1\\0\end{pmatrix}, u_3^c = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

And the sign is fixed by the relative orientation between n_a and $\dot{x}^a$.
So in the end we have:

$$\hat{E}_i(S)h_\gamma = -i\gamma\hbar\ h_\gamma(0,s)\tau_i h_\gamma(s,1) = -i\gamma\hbar\ h_{\gamma_1}\tau_i h_{\gamma_2} \tag{1.28}$$

Where clearly s is the parameter of γ for which the curve meets the surface S.
With the previous result in hand, let's consider now the action of the scalar product of two fluxes acting inside γ:

$$\hat{E}_i(S)\hat{E}^i(S)h_\gamma(A) = -\hbar^2\gamma^2 h_{\gamma_1}(A)\tau^i\tau_i h_{\gamma_2}(A)$$

We notice that on the right-hand side we have the 2° Casimir of the $SU(2)$ group: $\tau^i\tau_i = C^2$. Moreover here we are dealing with the fundamental representation of the group, which means:

$$\tau_i = \langle \frac{1}{2}, m| \, \tau_i \, |\frac{1}{2}, n\rangle \equiv \frac{i}{2}\sigma_i$$

So for this representation: $C^2 = -\frac{3}{4}\mathbb{1}_{mn}$. This means that it commutes with all the elements of the group, so

$$\hat{E}_i(S)\hat{E}^i(S)h_\gamma(A) = -\hbar^2\gamma^2 C^2 h_\gamma(A) = \hbar^2\gamma^2\frac{3}{4}h_\gamma(A) \tag{1.29}$$

By the other side, if we have that the two fluxes act on an end point of γ, for example the target we have:

$$\hat{E}_i(S)\hat{E}_j(S)h_\gamma(A) = -\hbar^2\gamma^2 h_\gamma(A)\tau_i\tau_j$$

From this result we notice that two flux operators do not commute:

$$[\hat{E}_i(S), \hat{E}_j(S)]h_\gamma(A) = -\hbar^2\gamma^2 h_\gamma(A)[\tau_i, \tau_j] = -\hbar^2\gamma^2 h_\gamma(A)\epsilon_{ij}{}^k h_\gamma(A)\tau_k$$

The result 1.29 can be trivially extended to a generic basis element $D^j(h)$: in this case we don't have Pauli matrices, but higher dimensional unitary matrices, that we call J_i. The Casimir operator for such representation is : $C_j^2 = -j(j+1)\mathbb{1}_{2j+1}$, and:

$$\hat{E}_i(S)\hat{E}_j(S)D^j(h_\gamma(A)) = \hbar^2\gamma^2 j(j+1)D^j(h_\gamma(A)) \tag{1.30}$$

We can extend the action of $\hat{E}_i(S)\hat{E}_j(S)$ on a generic state of our Hilbert space $\mathcal{H}_{Kin}$.
Well, in this section we recovered a well-defined Hilbert space $\mathcal{H}_{Kin}$ for the theory, we defined a scalar product and a representation of the quantum version of the Algebra. Now we can proceed with the second step of the Dirac quantization program, i.e. promoting the classical constraint to quantum operators and find a basis of solutions for the relative quantum equations.

1.2.2 The Gauss constraint and the Spin Network States

In order to find the physical states of the theory belonging to $\mathcal{H}_{Phys}$ we need to impose all the constraints at the quantum level. We start with the Gauss constraint, the solutions of which are states invariant under a local $SU(2)$ transformation. They define the Hilbert space $\mathcal{H}^0_{Kin}$.
We recall that a property of the holonomies is:

$$h_\gamma \longrightarrow h'_\gamma = \hat{U}_G h_\gamma = g_{s(\gamma)} h_\gamma g^{-1}_{t(\gamma)} \tag{1.31}$$

Similarly, if we write h_γ in the representation j:

$$D^{(j)}(h_\gamma) \longrightarrow D^{(j)}(h'_\gamma) = D^{(j)}(g_{s(\gamma)} h_\gamma g^{-1}_{t(\gamma)}) = D^{(j)}(g_{s(\gamma)}) D^{(j)}(h_\gamma) D^{j}(g^{-1}_{t(\gamma)}) \tag{1.32}$$

Let's prove the last equality:

$$\begin{aligned}
&\left[D^{(j)}(g_{s(\gamma)} h_\gamma g^{-1}_{t(\gamma)})\right]_{mn} = \langle j, m| \left(g_{s(\gamma)} h_\gamma g^{-1}_{t(\gamma)}\right) |j, n\rangle = \\
&= \langle j, m| g_{s(\gamma)} \sum_{j'} \sum_{p} |j', p\rangle \langle j', p| h_\gamma \sum_{j''} \sum_{q} |j'', q\rangle \langle j'', q| g^{-1}_{t(\gamma)} |j, n\rangle
\end{aligned}$$

Now, the summation over j', j'' has to give respectively $\delta_{jj'}$, $\delta_{jj''}$; in fact:

$$g_s(\gamma) |j', p\rangle = \sum_k a_k |j', k\rangle$$

Because a generic element of $SU(2)$ acts as rotation and cannot change the total spin of the state, and: $\langle j, m|j', k\rangle = 0$ for $j' \neq j$.
So we have;

$$\begin{aligned}
&= \langle j, m| g_{s(\gamma)} \sum_{p} |j, p\rangle \langle j, p| h_\gamma \sum_{q} |j, q\rangle \langle j, q| g^{-1}_{t(\gamma)} |j, n\rangle = \\
&= \sum_{p} \sum_{q} \langle j, m| g_{s(\gamma)} |j, p\rangle \langle j, p| h_\gamma |j, q\rangle \langle j, q| g^{-1}_{t(\gamma)} |j, n\rangle = \\
&= \left[D^{(j)}(g_{s(\gamma)}) D^{(j)}(h_\gamma) D^{j}(g^{-1}_{t(\gamma)})\right]_{mn}
\end{aligned}$$

From 1.31 we see that a gauge $SU(2)$ transformation acts only on the source and the target of the link γ, $\forall\ \gamma \in \Gamma$. This means that it acts only on the nodes of the graph and that the gauge invariance of a state described by $f(h_1, .., h_L)$ is the gauge invariance of f at the nodes.

We have that $\psi \in \mathcal{H}^0_{Kin}$, so is solution of the Gauss constraint iff:

$$f_0(h_1, .., h_L) = f_0(g_{s_1} h_1 g_{t_1}^{-1}, ..., g_{s_L} h_L g_{t_L}^{-1})$$

This condition can be implemented through the *Group averaging* technique: given $f \in Cyl_\Gamma$, and Γ a graph with N nodes and L links, a gauge invariant state has the following form

$$f_0(h_1, ...h_L) = \int \prod_{n=1}^{N} dg_n \, f(g_{s_1} h_1 g_{t_1}^{-1}, ..., g_{s_L} h_L g_{t_L}^{-1}) \tag{1.33}$$

The previous statement can be easily proved:

$$f_0(\tilde{g}_{s_1} h_1 \tilde{g}_{s_1}^{-1}, ...\tilde{g}_{s_L} h_L \tilde{g}_{s_L}^{-1}) = \int \prod_{n} dg_n \, f(g_{s_1} \tilde{g}_{s_1} h_1 \tilde{g}_{t_1}^{-1} g_{t_1}^{-1}, ..., g_{s_L} \tilde{g}_{s_L} h_L \tilde{g}_{t_L}^{-1} g_{t_L}^{-1})$$

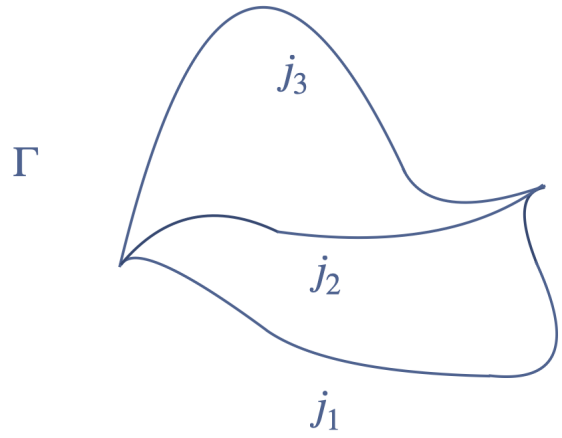
Let's call $\bar{g} = g\, \tilde{g}$, then $\bar{g}^{-1} = (g\, \tilde{g})^{-1} = \tilde{g}^{-1} g^{-1}$. We have:

$$f_0(\tilde{g}_{s_1} h_1 \tilde{g}_{s_1}^{-1}, ...\tilde{g}_{s_L} h_L \tilde{g}_{s_L}^{-1}) = \int \prod_{n} dg_n \, f(\bar{g}_{s_1} h_1 \bar{g}_{t_1}^{-1}, ..., \bar{g}_{s_L} h_L \bar{g}_{t_L}^{-1})$$

Now, since the integral is made over the whole group $SU(2)$, we have $\int dg = \int d\bar{g}$. This means

$$f_0(\tilde{g}_{s_1} h_1 \tilde{g}_{s_1}^{-1}, ...\tilde{g}_{s_L} h_L \tilde{g}_{s_L}^{-1}) = \int \prod_{n} d\bar{g}_n \, f(\bar{g}_{s_1} h_1 \bar{g}_{t_1}^{-1}, ..., \bar{g}_{s_L} h_L \bar{g}_{t_L}^{-1}) = f_0(h_1, ...h_L)$$

Let's consider an example, the *θ-graph*:



A generic cylindrical function for such graph:

$$\psi_{(\Gamma,f)}[A] = \sum_{j_l,m_l,n_l} f^{j_1,j_2,j_3}_{m_1,m_2,m_3,n_1,n_2,n_3} D^{j_1}_{m_1n_1}(h_{\gamma_1}) D^{j_2}_{m_2n_2}(h_{\gamma_2}) D^{j_3}_{m_3n_3}(h_{\gamma_3})$$

Well, in order to make it satisfying the Gauss constraint we have to impose gauge invariance. The gauge transformations don't act on the weights of the combination, that are scalars for such transformations, so:

$$\psi_{inv.} = \sum_{j_l,m_l,n_l} f^{j_1,j_2,j_3}_{m_1,m_2,m_3,n_1,n_2,n_3} \left[D^{j_1}_{m_1n_1}(h_{\gamma_1}) D^{j_2}_{m_2n_2}(h_{\gamma_2}) D^{j_3}_{m_3n_3}(h_{\gamma_3}) \right]_{inv.} \tag{1.34}$$

Using 1.33 we have

$$\left[D^{j_1}_{m_1n_1}(h_{\gamma_1}) D^{j_2}_{m_2n_2}(h_{\gamma_2}) D^{j_3}_{m_3n_3}(h_{\gamma_3}) \right]_{inv.} = \\ = \int dg_1 dg_2 D^{j_1}_{m_1n_1}(g_1 h_{\gamma_1} g_2^{-1}) D^{j_2}_{m_2n_2}(g_1 h_{\gamma_2} g_2^{-1}) D^{j_3}_{m_3n_3}(g_1 h_{\gamma_3} g_2^{-1})$$

where we have only two integrals since only two nodes. We can rewrite this expression

$$= P_{m_1m_2m_3\alpha_1\alpha_2\alpha_3} P_{\beta_1\beta_2\beta_3n_1n_2n_3} D^{j_1}_{\alpha_1\beta_1}(h_{\gamma_1}) D^{j_2}_{\alpha_2\beta_2}(h_{\gamma_2}) D^{j_3}_{\alpha_3\beta_3}(h_{\gamma_3}) \tag{1.35}$$

Where all the indices are summed over. We define

$$P_{m_1m_2m_3\alpha_1\alpha_2\alpha_3} \equiv \int dg_1 D^{(j_1)}_{m_1\alpha_1}(g_1) D^{(j_2)}_{m_2\alpha_2}(g_1) D^{(j_3)}_{m_3\alpha_3}(g_1)$$
$$P_{\beta_1\beta_2\beta_3n_1n_2n_3} \equiv \int dg_2 D^{(j_1)}_{\beta_1n_1}(g_2^{-1}) D^{(j_2)}_{\beta_2n_2}(g_2^{-1}) D^{(j_3)}_{\beta_3n_3}(g_2^{-1})$$

Where we notice that

$$\int dg_2 D^{(j_1)}_{\beta_1n_1}(g_2^{-1}) D^{(j_2)}_{\beta_2n_2}(g_2^{-1}) D^{(j_3)}_{\beta_3n_3}(g_2^{-1}) = \int dg_2 D^{(j_1)}_{\beta_1n_1}(g_2) D^{(j_2)}_{\beta_2n_2}(g_2) D^{(j_3)}_{\beta_3n_3}(g_2)$$

Since the integral is made over the whole group $SU(2)$.
P's are projectors from $\mathcal{H}_{Kin} \longrightarrow \mathcal{H}^0_{Kin}$.

Let's analyze in detail the equation 1.35:

$$P_{m_1m_2m_3\alpha_1\alpha_2\alpha_3} D^{j_1}_{\alpha_1\beta_1}(h_{\gamma_1}) D^{j_2}_{\alpha_2\beta_2}(h_{\gamma_2}) D^{j_3}_{\alpha_3\beta_3}(h_{\gamma_3}) =$$
$$= \sum_{\alpha_1,\alpha_2,\alpha_3} \int dg_1 \langle j_1, m_1 | g_1 | j_1, \alpha_1 \rangle \langle j_2, m_2 | g_1 | j_2, \alpha_2 \rangle \langle j_3, m_3 | g_1 | j_3\alpha_3 \rangle \langle j_1, \alpha_1 | h_{\gamma_1} | j_1, \beta_1 \rangle \cdot$$
$$\cdot \langle j_2, \alpha_2 | h_{\gamma_2} | j_2, \beta_2 \rangle \langle j_3, \alpha_3 | h_{\gamma_3} | j_3, \beta_3 \rangle =$$
$$= \int dg_1 \langle j_1, m_1 | g_1 \sum_{\alpha_1} | j_1, \alpha_1 \rangle \langle j_2, m_2 | g_1 \sum_{\alpha_2} | j_2, \alpha_2 \rangle \langle j_3, m_3 | g_1 \sum_{\alpha_3} | j_3\alpha_3 \rangle \langle j_1, \alpha_1 | h_{\gamma_1} | j_1, \beta_1 \rangle \cdot$$
$$\cdot \langle j_2, \alpha_2 | h_{\gamma_2} | j_2, \beta_2 \rangle \langle j_3, \alpha_3 | h_{\gamma_3} | j_3, \beta_3 \rangle =$$
$$\int dg_1 \langle j_1, m_1 | g_1 \sum_{\alpha_1} | j_1, \alpha_1 \rangle \langle j_1, \alpha_1 | h_{\gamma_1} | j_1, \beta_1 \rangle \langle j_2, m_2 | g_1 \sum_{\alpha_2} | j_2, \alpha_2 \rangle \langle j_2, \alpha_2 | h_{\gamma_2} | j_2, \beta_2 \rangle \cdot$$
$$\cdot \langle j_3, m_3 | g_1 \sum_{\alpha_3} | j_3\alpha_3 \rangle \langle j_3, \alpha_3 | h_{\gamma_3} | j_3, \beta_3 \rangle$$

Now, removing the three identities:

$$= \int dg_1 \langle j_1, m_1 | g_1(h_{\gamma_1} | j_1, \beta_1 \rangle) \langle j_2, m_2 | g_1(h_{\gamma_2} | j_2, \beta_2 \rangle) \langle j_3, m_3 | g_1(h_{\gamma_3} | j_3, \beta_3 \rangle)$$

Well, we can interpret the previous expression in the following way: the projector P acting on the state

$$h_{\gamma_1} | j_1, \beta_1 \rangle \otimes h_{\gamma_2} | j_2, \beta_2 \rangle \otimes h_{\gamma_3} | j_3, \beta_3 \rangle \tag{1.36}$$

that belongs to the space:

$$V = \bigotimes_{l=1}^{3} V^{(j_l)}$$

produces a state belonging to the subspace of V made of gauge invariant states, called the *Singlet space* V^0. Clearly a state of such space V^0 will be of kind 1.36, with the adding property of gauge invariance. The states belonging to V^0 are called *Intertwiners*. Well, if we consider a basis of intertwiners, let's say $\{|i_\alpha\rangle\}_{\alpha=1,..Dim(V^0)}$, appears clear that we can write P in the following way:

$$P = \sum_{\alpha=1}^{Dim(V^0)} |i_\alpha\rangle \langle i_\alpha|$$

In the case of the *θ-graph*, V is the tensor product of three spaces $V^{(j_l)}$ $(l = 1, 2, 3)$, and it can be proved that in this case we have only an intertwiner $|i\rangle$.

The intertwiner of the *θ-graph* is linked to the so called *Wigner's 3j-m symbols*:

$$\langle j_1, m_1; j_2, m_2; j_3, m_3 | i \rangle = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

That is a complex number at fixed j_i, m_i. Its explicit expression

$$\langle j_1, m_1; j_2, m_2; j_3, m_3 | i \rangle = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1 - j_2 - j_3}}{\sqrt{2j_3 + 1}} C^{j_3, -m_3}_{j_1 m_1 j_2 m_2} \tag{1.37}$$

Where C are the Clebsh-Gordon coefficients and are non vanishing if and only if $|j_2 - j_3| \leq j_1 \leq j_2 + j_3$. This means that if the spins of the graph don't satisfy this relation, there is no invariant state under the action of $SU(2)$.
Well, using this notation we can rewrite 1.35:

$$\left[D^{j_1}_{m_1 n_1}(h_{\gamma_1}) D^{j_2}_{m_2 n_2}(h_{\gamma_2}) D^{j_3}_{m_3 n_3}(h_{\gamma_3}) \right]_{inv.} =$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}^* \begin{pmatrix} j_1 & j_2 & j_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}^* D^{j_1}_{\alpha_1 \beta_1}(h_{\gamma_1}) D^{j_2}_{\alpha_2 \beta_2}(h_{\gamma_2}) D^{j_3}_{\alpha_3 \beta_3}(h_{\gamma_3})$$

Since

$$P_{m_1 m_2 m_3 \alpha_1 \alpha_2 \alpha_3} = \langle j_1, m_1; j_2, m_2; j_3, m_3 | i \rangle \langle i | j_1, \alpha_1; j_2, \alpha_2; j_3, \alpha_3 \rangle =$$

$$= \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}^*$$

Where * stays here for complex conjugation.

Let's rewrite the gauge invariant state in a more useful way:

$$\left[D^{j_1}_{m_1n_1}(h_{\gamma_1})D^{j_2}_{m_2n_2}(h_{\gamma_2})D^{j_3}_{m_3n_3}(h_{\gamma_3})\right]_{inv.} =$$

$$= \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}^* \sum_{\alpha_1,\alpha_2,\alpha_3} \sum_{\beta_1,\beta_2,\beta_3} D^{j_1}_{\alpha_1\beta_1}(h_{\gamma_1})D^{j_2}_{\alpha_2\beta_2}(h_{\gamma_2})D^{j_3}_{\alpha_3\beta_3}(h_{\gamma_3}) \langle i|\alpha_1,\alpha_2,\alpha_3\rangle \cdot$$

$$\cdot \langle \beta_1,\beta_2,\beta_3|i\rangle = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}^* i^* \left(\prod_{l=1}^{3} D^{j_l}(h_{\gamma_l})\right) i$$

Where in the last term the contractions of indices α_l and β_l are implicit. Finally:

$$\left[D^{j_1}_{m_1n_1}(h_{\gamma_1})D^{j_2}_{m_2n_2}(h_{\gamma_2})D^{j_3}_{m_3n_3}(h_{\gamma_3})\right]_{inv.} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}^* \left(\prod_{l=1}^{3} D^{j_l}(h_{\gamma_l})\right) \prod_n i_n$$

Where n as before is the number of nodes, that in our case are two, and i_n are the associated intertwiners.

Remark : if in this case we have two nodes and only an intertwiner for each of them, for a generic graph we can have more than 1 possible intertwiner for each node, and the state will depend also on what intertwiner we choose for each node. Thus i_n are quantum numbers for our state.

Let's write now the generic ψ_{inv} for such graph: recalling 1.39 we have

$$\begin{aligned} \psi_{inv.} &= \sum_{j_l,m_l,n_l} f^{j_1,j_2,j_3}_{m_1,m_2,m_3,n_1,n_2,n_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}^* \left(\prod_{l=1}^{3} D^{j_l}(h_{\gamma_l})\right) \prod_n i_n \\ &= \sum_{j_l} \tilde{f}^{j_1,j_2,j_3} \left(\prod_{l=1}^{3} D^{j_l}(h_{\gamma_l})\right) \prod_n i_n \end{aligned}$$

Where we defined

$$\tilde{f}^{j_1,j_2,j_3} \equiv \sum_{m_l,n_l} f^{j_1,j_2,j_3}_{m_1,m_2,m_3,n_1,n_2,n_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}^*$$

Well, we call

$$\psi_{inv.} = \sum_{j_l} \tilde{f}^{j_1,j_2,j_3} \Big(\prod_{l=1}^{3} D^{j_l}(h_{\gamma_l}) \Big) \prod_n i_n \tag{1.38}$$

a **Spin Network State**.

As we can see from its generic expression it is a gauge invariant state defined on a graph, with basis elements depending on j_l, i_n. This means in particular that:

$$\psi_{(\Gamma,j_l,i_n)} = \prod_{l=1}^{3} D^{j_l}(h_{\gamma_l}) \prod_n i_n \neq \prod_{l=1}^{3} D^{j_l}(h_{\gamma_l}) \prod_n \tilde{i}_n$$

If $\tilde{i}_n \neq i_n$ for some n. It is also clear that in the particular case of the *θ-graph* these basis element don't depend on i_n, since such graphs as we said previously admits only an intertwiner for each node.

What we found for the *θ-graph* can be extended to whatever kind of graph, with calculations that are naturally more involved for larger graphs. In particular for each graph we can construct the associated Hilbert space $\mathcal{H}^0_\Gamma$, and:

$$\mathcal{H}^0_{Kin} = \bigoplus_{\Gamma \subset \Sigma} \mathcal{H}^0_\Gamma$$

In order to conclude this section we want to prove that a Spin network state satisfies:

$$\hat{G}_i \, |\psi\rangle = 0$$

Let's prove it in the simple case of a node with only one link. The result can be immediately generalized to all the possible nodes.

We consider an infinitesimal cubic surface that contains the node, and we suppose that one of the sides of such surface intersects the link in an inner point infinitesimally close to the source of the link itself. We take a gauge invariant state ψ defined on a graph that contains such node. So

$$\hat{E}(S)\psi = \alpha^i \hat{E}_i \psi = \alpha^i \hat{E}_i \int dg_1 dg_2 D^{j_1}_{m_1 n_1}(g_1 h_\gamma g_2^{-1})$$

Where here we consider the only part of ψ that gives a non vainishing result, i.e. the matrix element of the holonomy of the link γ (the other links don't intersect the surface, so the action of $\hat{E}$ gives 0 acting on them). Using 1.28 in the limit in which $h_{\gamma_1} \to 1$, i.e.

the surface intersects the link arbritrarily near its source:

$$\hat{E}(S)\psi = -i\hbar\gamma \int dg_1 dg_2 D^{(j_1)}_{m_1 n_1}\big(g_1 \tau_i \alpha^i h_\gamma g_2^{-1}\big)$$

So:

$$\begin{aligned}\psi - \alpha^i \hat{E}_i \psi &= \int dg_1 dg_2 \big[D^{j_1}_{m_1 n_1}\big(i\hbar\gamma g_1 \tau_i \alpha^i h_\gamma g_2^{-1}\big) + D^{j_1}_{m_1 n_1}\big(g_1 h_\gamma g_2^{-1}\big)\big] = \\ &= \int dg_1 dg_2 D^{j_1}_{m_1 n_1}\big[g_1\big(1 + i\hbar\gamma\tau_i\alpha^i\big)h_\gamma g_2^{-1}\big] = \int dg_1 dg_2 D^{j_1}_{m_1 n_1}\big[\tilde{g}_1 h_\gamma g_2^{-1}\big] = \psi\end{aligned}$$

Thus: $\alpha^i \hat{E}_i \psi = 0$.
But since: $\hat{E}_i = \int d\sigma n_a \hat{E}^a_i$, and the only non-vanishing contribution to the integral is given by the infinitesimal part of the squared surface that intersects the link,

$$\hat{E}_i \psi = 0 \quad \Longleftrightarrow \quad \hat{E}^a{}_i \psi = 0$$

And since $\hat{G}_i = \hat{D}_a \hat{E}^a_i$, then:

$$\hat{G}_i \psi = 0$$

This result can be generalized to spin-network states with arbitrarily complicate graphs: $\widehat{E}_i$ vanishes at the nodes of the graph since the contributions from each link are summed up and the total result is zero.
This concludes our analysis of the gauge invariant Hilbert space of the theory.
At this stage of the construction of the theory, before proceeding with the quantization of the other constraints we can introduce two important quantum operators, constructed from their classical counterparts: the *Area operator* and the *Volume operator*.

1.2.3 The Area Operator

The simplest geometric operator we can build is the *Area Operator*.
At the classical level, given a surface $S \subset \Sigma_t$, the area of the surface can be given in terms of its normal and the densitized triad $E^a{}_i$:

$$A(S) = \int_S d\sigma_1 d\sigma_2 \sqrt{E^a{}_i E^{bi} n_a n_b} \tag{1.39}$$

Let's prove it. The area of S:

$$A(S) = \int_S d\sigma_1 d\sigma_2 \sqrt{-det\left(g_{ab}\frac{\partial x^a}{\partial \sigma^\alpha}\frac{\partial x^b}{\partial \sigma^\beta}\right)} \tag{1.40}$$

Where $g'_{\alpha\beta} = g_{ab}\frac{\partial x^a}{\partial \sigma^\alpha}\frac{\partial x^b}{\partial \sigma^\beta}$ with $\alpha, \beta = 1, 2$ is the induced metric on S. Thus:

$$\begin{aligned} -det\left(g_{ab}\frac{\partial x^a}{\partial \sigma^\alpha}\frac{\partial x^b}{\partial \sigma^\beta}\right) &= -g_{ab}g_{cd}[\partial_1 x^a \partial_1 x^b \partial_2 x^c \partial_2 x^d - \partial_1 x^a \partial_2 x^b \partial_1 x^c \partial_2 x^d] = \\ &= -g_{ab}g_{cd} 2\partial_1 x^a \partial_1 x^{[b} \partial_2 x^{c]} \partial_2 x^d = -2g_{a[b}g_{c]d}\partial_1 x^a \partial_1 x^b \partial_2 x^c \partial_2 x^d \end{aligned}$$

Now we use the following identities: $\begin{cases} g_{a[b}g_{c]d} = \frac{1}{2}\epsilon_{ace}\epsilon_{bdf} \cdot g \cdot g^{ef} \\ n_e = \epsilon_{eab}\frac{\partial x^a}{\partial \sigma^1}\frac{\partial x^b}{\partial \sigma^2} \end{cases}$

$$\begin{aligned} \implies \quad -det\left(g_{ab}\frac{\partial x^a}{\partial \sigma^\alpha}\frac{\partial x^b}{\partial \sigma^\beta}\right) &= -\epsilon_{ace}\epsilon_{bdf} g g^{ef} \partial_1 x^a \partial_1 x^b \partial_2 x^c \partial_2 x^d = -n_e n_f \cdot g g^{ef} = \\ &= n_e n_f e^2 e^e{}_i e^{if} \end{aligned}$$

obtained recalling that:

$$g = -e^2, \qquad g^{ef} = e^e{}_i e^{fi}$$

Plugging this result in 1.40, we obtain:

$$A(S) = \int_S d\sigma_1 d\sigma_2 \sqrt{e^2 e^e{}_i e^{fi} n_e n_f} = \int_S d\sigma_1 d\sigma_2 \sqrt{E^e{}_i E^{fi} n_e n_f} \tag{1.41}$$

Well, from 1.39 we notice that if we promote E to a quantum operator we should obtain automatically what we need. But let's proceed with caution: at the quantum level we know that E acts in our representation $\langle A|$ as a functional derivative. By the other side we saw how $E^i E_i$ acts on D if we consider only an intersecting link (1.30).
If instead of acting on an holonomy the operator $\hat{E}\hat{E}$ acts on a generic state:

$$f = D^{(j_1)}(h_{e_1}) D^{(j_2)}(h_{e_2}) ... D^{(j_L)}(h_{e_L})$$

where each link intersects the surface in some point (fig. 1.1), in order to give the right result we need a regularization. Thus we decompose S in a number N of 2-dimensional

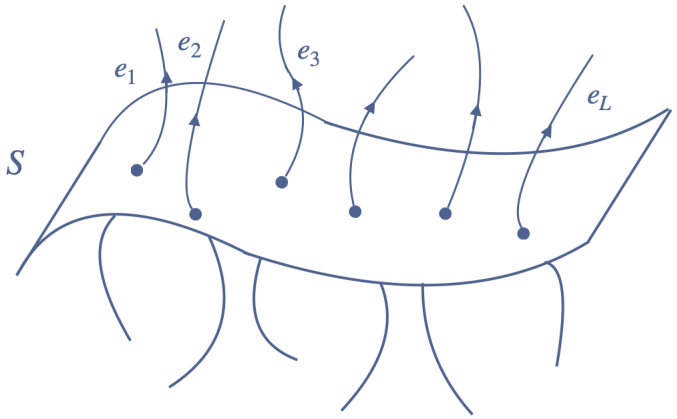


Figure 1.1

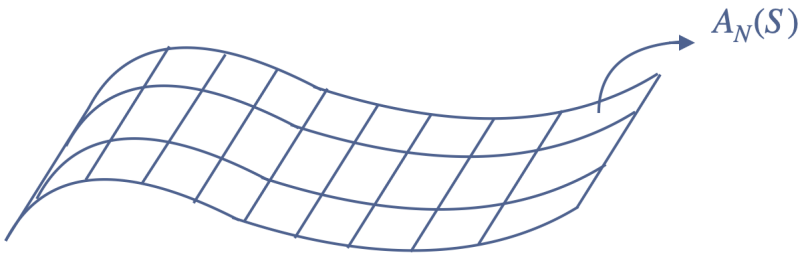


Figure 1.2

cells (fig. 1.2) and we can write:

$$A(S) = \lim_{N \to +\infty} A_N(S) \tag{1.42}$$

where

$$A_N(S) = \sum_{I=1}^{N} \sqrt{E_i(S_I) E^i(S_I)} \tag{1.43}$$

with $E_i(S_I) = \int_{S_I} n_a E_i^a d^2\sigma$, S_I infinitesimal. The proof of 1.42 is simple:
In the limit of infinitesimal cells S_I $(N \to \infty)$,

$$A_N(S) = \sum_{I=1}^{N} \sqrt{E_i(S_I) E^i(S_I)} \sim \sum_{I=1}^{N} \sqrt{E_i^a n_a^I S_I E^{bi} n_b^I S_I} = \sum_{I=1}^{N} S_I \sqrt{E_i^a n_a^I E^{bi} n_b^I} =$$

$$= \int_S d\sigma_1 d\sigma_2 \sqrt{E_i^a E^{bi} n_a n_b}$$

So we define the area operator as

$$\hat{A}(S) = \lim_{N\to\infty} \hat{A}_N(S)$$

where inside 1.43 we promote : $E^a_i \to \hat{E}^a{}_i$.
This operator acts on a generic Spin network state $\langle A|\Gamma; j_l, m_l, n_l\rangle$.
Remark : since $N \to \infty$, we have that each infinitesimal area S_I is punctured at most by one link. Once we reach N for which this condition is satisfied, a further refinement gives no consequences on the spectrum of the operator.
Let's consider for simplicity the action of such operator on $\langle A|\Gamma; j_1, m_1, n_1\rangle = D^{j_1}_{m_1 n_1}(h_{e_1})$, which means that we consider our graph Γ as composed by only one link. Well, since we have only one link intersecting the surface S, we expect that there will be only an infinitesimal element of the partition of S that is punctured by the link. This means that in the summation in 1.43 (once promoted to a sum of operators) there is only a term that acting on the state gives a non 0 contribution. According with 1.1 we have

$$\hat{A}(S)h_e[A] = \hbar\gamma\sqrt{j(j+1)}h_e[A] \tag{1.44}$$

The previous relation tells us that the spectrum of the Area operator of the quantum theory is *discrete*. In particular for a state with many links the eigenvalue of the Area depends on how many links of the graph that defines the state intersects the area, and on which representation of $SU(2)^M$ our state is written (M here is the number of links intersecting the surface). We also notice that the minimum of the spectrum is given by a state with only one intersecting link and the associated Wigner matrix with $j = \frac{1}{2}$; moreover its value is proportional to the square of the Planck lenght (we assumed $c = 16\pi G = 1$). In the case in which the graph intersects the surface in some node, there is a more general expression for the eigenvalues.
Let's construct now another fundamental geometric operator.

1.2.4 The Volume Operator

Following the same reasoning we used for the Area operator we can construct the *Volume Operator*. Given a region $R \subset \Sigma$, classically we define its volume as:

$$V(R) = \int_R d^3x\sqrt{h} = \int_R d^3x\sqrt{\left|\frac{1}{3!}\epsilon_{abc}\epsilon^{ijk}E^a{}_i E^b{}_j E^c{}_k\right|}$$

Where the last equality can be easily proved.

At the quantum level two different well-defined volume operators have been proposed, respectively by Rovelli-Smolin and Ashtekar-Lewandowski. In this work we introduce only the first one, since for our purposes is not relevant to describe both of them.

As well as we did for the area operator, given the volume R we replace the integral over it with the limit of a Riemannian sum of cubic cells C_I, so that $R = \cup_I C_I$. As before we'll have

$$\int_R d^3x \sim \sum_I Volume(C_I)$$

This partition allows us to write the volume of a cell in terms of fluxes. Let's consider for this purpose the following:

$$W_I \equiv \frac{1}{48}\int_{\partial C_I} d^2\sigma_1 \int_{\partial C_I} d^2\sigma_2 \int_{\partial C_I} d^2\sigma_3 \left|\epsilon_{ijk} E^a{}_i(\sigma_1) n_a(\sigma_1) E^b{}_j(\sigma_2) n_b(\sigma_2) E^c{}_k(\sigma_3) n_c(\sigma_3)\right|$$

In the limit we send the size of a cell to 0 ($\varepsilon \to 0$) and we shrink the cell to a point we obtain:

$$W_I = \frac{1}{48}\epsilon^{abc} n_a n_b n_c det\big(E^a{}_i(x)\big)\cdot\varepsilon^6 \simeq det(E^a_i(x))\varepsilon^6 \simeq Volume^2(C_I)$$

Where we used

$$\epsilon_{ijk}E^{ai}E^{bj}E^{ck} = det(E)\epsilon^{abc}$$

Hence we have:

$$V(R) = \lim_{\varepsilon\to 0}\sum_I \sqrt{W_I} \tag{1.45}$$

Now, let's divide each ∂C_Iin small surfaces S_I^α: $\partial C_I = \bigcup_\alpha S_I^\alpha$. We have:

$$\begin{aligned}
W_I &= \int_{\partial C_I} d^2\sigma_1 \int_{\partial C_I} d^2\sigma_2 \int_{\partial C_I} d^2\sigma_3 \left|\epsilon_{ijk} E^a{}_i(\sigma_1) n_a(\sigma_1) E^b{}_j(\sigma_2) n_b(\sigma_2) E^c{}_k(\sigma_3) n_c(\sigma_3)\right| \sim \\
&\sim \left| \int_{\partial C_I} d^2\sigma_1 \int_{\partial C_I} d^2\sigma_2 \int_{\partial C_I} d^2\sigma_3 \epsilon_{ijk} E^a{}_i(\sigma_1) n_a(\sigma_1) E^b{}_j(\sigma_2) n_b(\sigma_2) E^c{}_k(\sigma_3) n_c(\sigma_3)\right| \sim \\
&\sim \left| \sum_\alpha \int_{S_I^\alpha} d^2\sigma_1 \sum_\beta \int_{S_I^\beta} d^2\sigma_2 \sum_\gamma \int_{S_I^\gamma} d^2\sigma_3 \epsilon_{ijk} E^a{}_i(\sigma_1) n_a(\sigma_1) E^b{}_j(\sigma_2) n_b(\sigma_2) E^c{}_k(\sigma_3) n_c(\sigma_3)\right| \sim \\
&\sim \sum_\alpha \sum_\beta \sum_\gamma \left|\epsilon_{ijk} E_i(S_I^\alpha) E_j(S_I^\beta) E_k(S_I^\gamma)\right|
\end{aligned}$$

Substituting it in 1.45 we obtain:

$$V(R) = \lim_{\varepsilon \to 0} \sum_I \sqrt{\frac{1}{48} \sum_{\alpha,\beta,\gamma} \left| \epsilon_{ijk} E_i(S_I^\alpha) E_j(S_I^\beta) E_k(S_I^\gamma) \right|}$$

And turning the classical fluxes E_i into operators:

$$\hat{V}(R) = \lim_{\varepsilon \to 0} \sum_I \sqrt{\frac{1}{48} \sum_{\alpha,\beta,\gamma} \left| \epsilon_{ijk} \hat{E}_i(S_I^\alpha) \hat{E}_j(S_I^\beta) \hat{E}_k(S_I^\gamma) \right|} \tag{1.46}$$

this is the *Rovelli-Smolin Volume Operator*.

In this derivation we assumed $\begin{cases} \int |(...)| \sim |\int (...)| \\ \sum |(...)| \sim |\sum (...)| \end{cases}$

As well as for the Area operator there exists an optimal refinement such that a better refinement becomes unnecessary.

The optimal partition of the volume R in cells C_I is the following: the nodes of Γ can stay only in the interior of the cells, and not on their surface; a cell can contain at most one node, and if it doesn't contain any node it contains at most one link. Moreover the partition of the surfaces ∂C_I in cells S_I^α is made in such a way that links of Γ can intersect a cell S_I^α of the partition only in its interior, and each surface S_I^α is punctured at most by one link.

Remark: We notice that the presence of ϵ_{ijk} in 1.46 means that the three fluxes have to be different, otherwise their product gives a 0 contribution. By the other side S_I^α, S_I^β, S_I^γ can be different portions of the same face, or of different faces.

Well, let's study now the action of this operator on a Spin network state.

We look at the only two possible situations, illustrated in fig. 1.3. In the calculations we consider for simplicity only a cubic cell, and for a generic spin network state only the part associated with links and nodes that are inside such cell; the results can be easily extended to the whole states defined on the whole graph.

In the second case we have that the action of the volume operator on the state gives a 0 result. Let's prove it.

In this case we have a graph with only a link passing through the elementary volume C_I As we can see from the picture there are only two S_I's that give a non-0 contribution to the eigenvalue; let's call them for simplicity S_I^1, S_I^2.

When the operator 1.46 with I fixed acts on the holonomy, we notice that the only term

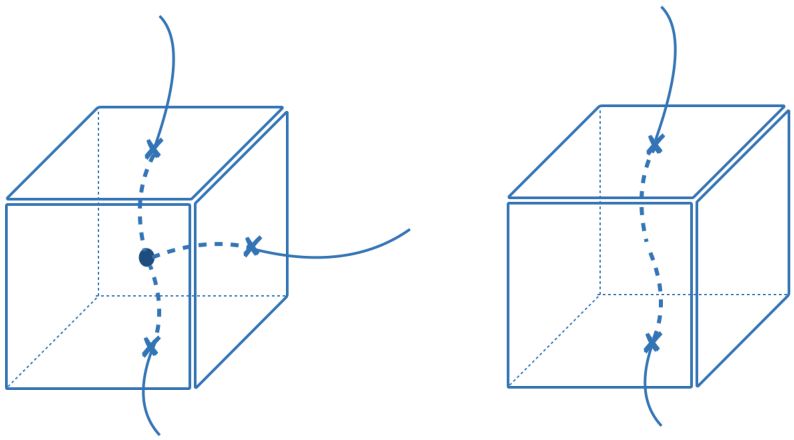


Figure 1.3

in the sum $\sum_{\alpha,\beta,\gamma}$ that gives a non trivial 0 result has two of the E's necessarily equal. This because we have only 2 surfaces of kind S_I^α punctured by the link. The presence of ϵ_{ijk} makes the result vanishing.

Let's look now at the second case. Here we have a node inside the elementary cell, and consequently three or more links. The volume operator associated with such cell:

$$\hat{U} = \lim_{\varepsilon \to 0} \sqrt{\frac{1}{48} \sum_{\alpha,\beta,\gamma} \left| \epsilon_{ijk} \hat{E}_i(S_I^\alpha) \hat{E}_j(S_I^\beta) \hat{E}_k(S_I^\gamma) \right|} \tag{1.47}$$

Let's restrict our attention on gauge invariant states and start with a three-valent node. For gauge invariant states as we found previously the Gauss constraint holds. But we remember that such constraint implies

$$\lim_{\varepsilon \to 0} \hat{E}_i(S) \left|\psi\right\rangle = 0$$

In the particular case of a three-valent node, like in the *θ-graph* , since only three S^α give a non 0 result we have:

$$\left[\hat{E}_i(S^1) + \hat{E}_i(S^2) + \hat{E}_i(S^3)\right] \left|\psi\right\rangle = 0$$

Which implies:

$$\hat{E}_i(S^1) \left|\psi\right\rangle = -\left[\hat{E}_i(S^2) + \hat{E}_i(S^3)\right] \left|\psi\right\rangle \tag{1.48}$$

By the other side the only non vanishing terms in the sum $\sum_{\alpha,\beta,\gamma}$ for such state are of kind

$$\pm\epsilon_{123}\hat{E}_1(S_I^1)\hat{E}_2(S_I^2)\hat{E}_3(S_I^3)\,|\psi\rangle$$

Using the relation 1.48 we have

$$\pm\epsilon_{123}\hat{E}_1(S_I^1)\hat{E}_2(S_I^2)\hat{E}_3(S_I^3)\,|\psi\rangle = \mp\epsilon_{123}\big[\hat{E}_1(S^2)+\hat{E}_1(S^3)\big]\hat{E}_2(S_I^2)\hat{E}_3(S_I^3)\,|\psi\rangle = 0$$

since ϵ_{ijk} is totally antisymmetric.
This means that the volume associated with a gauge invariant state on a three-valent node is zero.
Let's consider now the case of a four-valent node. Firstly we have to recall the fact that the intertwiners for such node is not unique but a genuine quantum number. This means that there is a degeneracy in the spectrum of the volume operator: gauge invariant states with different intertwiners have the same eigenvalue of V.
The Gauss constraint in this case implies

$$\big[\hat{E}_i(S^1)+\hat{E}_i(S^2)+\hat{E}_i(S^3)+\hat{E}_i(S^4)\big]\,|\psi\rangle = 0$$

Let's consider the action of the operator 1.47 in this case. We must ask to ourself how many non-zero contributions are present in the sum $\sum_{\alpha,\beta,\gamma}$. If we keep in account the relations that come from the Gauss constraint, we have 48 equal terms. This means:

$$\hat{U} = \lim_{\varepsilon\to 0}\sqrt{\big|\epsilon_{ijk}\hat{E}_i(S_I^1)\hat{E}_j(S_I^2)\hat{E}_k(S_I^3)\big|}$$

And:

$$\hat{U}\,|\psi\rangle = \sqrt{\hbar^3\gamma^3\big|\epsilon_{ijk}J_i^1J_j^2J_k^3\big|}\,|\psi\rangle$$

Where J_i are the generators of $SU(2)$. As usual their explicit form depends on the representation, and the representation depends on the spins that the state associates to the links: for example if $j=\frac{1}{2}$ we have that $J_i=\tau_i$, and so on.
This operator is well defined, with a discrete spectrum and minimal excitations proportional to $(\hbar G)^{\frac{3}{2}} = l_P^3$.
In order to complete the quantization of the theory, we need to look at the *dynamics*, studying the quantum version of the Diffeo. and Hamiltonian constraints, with related Hilbert spaces of solutions. Let's start with the Diffeo. constraint.

1.2.5 The Diffeomorphisms constraint

In this section we implement the diffeo. constraint $\mathcal{H}_a \approx 0$ in its quantum version and find a basis for the space $\mathcal{H}_{diff.}$, the space of Spin Network States that are also Diffeo-invariant states.
We start with the action of generic diffeomorphisms on a state ψ. Recalling the properties of the holonomy, the action of the operator $\widehat{\phi}$ on a Spin Network State:

$$\hat{\phi}\,\psi_\Gamma = \psi_{\phi\circ\Gamma}$$

where:

$$\hat{\phi}: \quad Cyl_\Gamma \longrightarrow Cyl_{\phi\circ\Gamma}$$

Let's list some important feature of such operator:

- It is a well defined unitary operator.
- $Cyl_\Gamma \perp Cyl_{\phi\circ\Gamma}$ independently on the particular diffeomorphism. This means that we cannot consider for such states infinitesimal diffeomorphisms: all the diffeomorphisms are finite for cylindrical functions. This comes from the fact that if we apply the diffeomorphism operator $\widehat{\phi}$ (that is assumed to be not the Identity operator) to a quantum state it modifies the associated graph, and the scalar product between two states with different graphs is always 0. In this sense $Cyl_\Gamma \perp Cyl_{\phi\circ\Gamma}$, and an infinitesimal diffeomorphism is finite from this point of view, since changes "drastically" the state. This as we see in a moment is not an obstacle for the construction of $\mathcal{H}_{diff.}$.

In order to construct $\mathcal{H}_{diff.}$ we can proceed with a group averaging as we did for the Gauss constraint, building in this way states invariant under finite diffeomorphisms. Before doing this, we need to identify the *symmetries* of the graph, i.e. diffeomorphisms that act trivially on them. We have two kind of such transformations:

1. The diffeomorphism that exchange links of the graph, without changing Γ.
   We call the subspace of these diffeomorphisms GS_Γ.
2. The ones that preserve the links, but shuffle the points inside them.
   We call the relative subspace $TDiff_\Gamma$. Well, these last ones have to be removed from $Diff_\Gamma$ because they are infinite and spoil the group averaging procedure.

There is however a residual issue for the group averaging procedure. The diffeomorphism group, as we know from classical General Relativity is not compact, differently from the group $SU(2)$; this means that a group-averaged state cannot belong to a subspace of $\mathcal{H}^0_{Kin}$; we encounter the same problem in Quantum Mechanics, when starting from a state $\psi \in \mathcal{L}_2(\mathbb{R}, dx)$ we require the translation invariance:

$$\psi \longrightarrow c, \quad c \in \mathbb{C}$$

and $\psi = c \notin \mathcal{L}_2$, since is not a normalized state. However we can recover the transation invariance if we define c as a linear functional:

$$c: \quad \psi \in \mathcal{L}_2[\mathbb{R}, dx] \longrightarrow \mathbb{C}$$

Of this kind:

$$c \int dx\psi(x) = c \int dx\psi(x)e^{ikx}\big|_{k=0} = c\tilde{\psi}(0)$$

In fact:

$$c \int dx\psi(x+a) = c \int dx'\psi(x') = c\tilde{\psi}(0)$$

We can do the same for our Hilbert space $\mathcal{H}^0_{Kin}$. For this purpose let's denote with $\mathcal{H}^{0*}_{Kin}$ the space of linear functionals acting on $\mathcal{H}^0_{Kin}$. Thus, $\eta \in \mathcal{H}^{0*}_{Kin}$ is a *Diffeomorphism-invariant functional*, if

$$\eta(\hat{\phi}\psi) = \eta(\psi) \quad , \forall\psi \in \mathcal{H}^0_{Kin} \tag{1.49}$$

In this way we don't restrict $\mathcal{H}^0_{Kin} \to \mathcal{H}^0_{Diff.}$, but $\mathcal{H}^{0*}_{Kin} \to \mathcal{H}^{0*}_{Diff.}$ so we restrict the space of functionals acting on $\mathcal{H}^0_{Kin}$, which is the dual of $\mathcal{H}^0_{Kin}$ itself.
Once we construct $\mathcal{H}^{0*}_{Diff.}$, we should be able to derive $\mathcal{H}^0_{Diff.}$.
We notice that the condition 1.49 is formally similar to the one associated with gauge invariance:

$$f_0(h_1, .., h_L) = f_0(g_{s_1}h_1g^{-1}_{t_1}, ..., g_{s_L}h_Lg^{-1}_{t_L})$$

With the important difference that $f_0 \in \mathcal{H}_{Kin}$, as well as f, while $\eta(\psi) \in \mathcal{H}^{0*}_{Diff.}$, not to $\mathcal{H}^0_{Diff.}$.
As well as we did before, we have to define a projector, called $P_{DIFF.}$ that allows to write diffeo. invariant functionals belonging to $\mathcal{H}^{0*}_{Diff.}$ out of generic functionals belonging to $\mathcal{H}^{0*}_{Kin.}$.
The result of this procedure are spin network states that instead of being defined on

particular graphs are defined on equivalence classes of graphs, in which each graph of a class can be transformed in whatever other graph of the same class through spatial diffeomophisms. These equivalence classes are called *Knots*. The name comes from the fact that a spatial diffeomorphism can change the form of the graph (fig. 1.4) but not

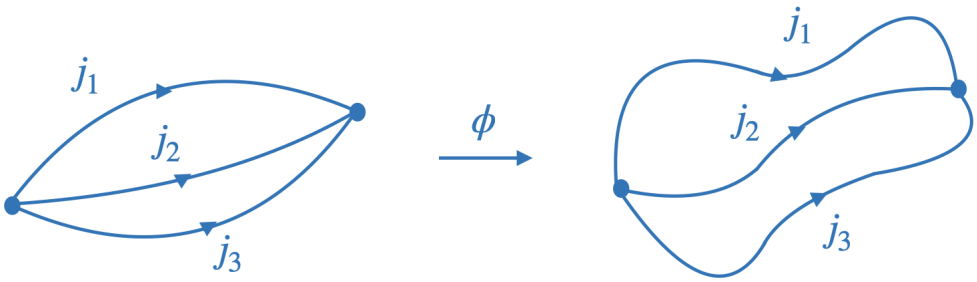


Figure 1.4

the number of its knots (fig. 1.5), so that each class is made of graphs with the same

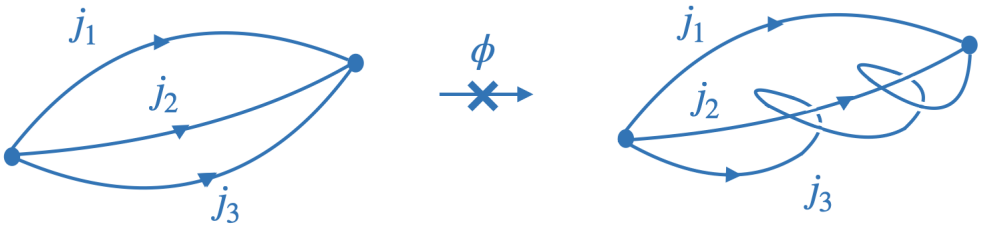


Figure 1.5

number of knots.

Thus the diffeomorphism-invariant Hilbert space of the theory is spanned by the so called *Knotted Spin Networks*.

We can now proceed with the quantization of the last constraint of our constrained system, the *Hamiltonian constraint*.

1.2.6 The Hamiltonian constraint

In order to reach the physical Hilbert space of the theory we have to impose the *Hamiltonian constraint* to our Knotted Spin Network States. The classical smeared constraint:

$$H_0(N) = \int d^3x N \left[F^j_{ab} - (\gamma^2+1)\epsilon^j{}_{mn} K^m_a K^n_b\right] \cdot \frac{\epsilon_j{}^{kl} E^a_k E^b_l}{e}$$

Where we don't consider the last term, since it contains G_i and acting on Spin Network states gives no contribution to the Hamiltonian constraint. Using the following relation:

$$\epsilon^{j}{}_{mn}\epsilon_{j}{}^{kl} = \delta^{k}{}_{m}\delta^{l}{}_{n} - \delta^{l}{}_{m}\delta^{k}{}_{n}$$

We can write:

$$\big[(\gamma^2+1)\epsilon_{jmn}K_a^m K_b^n\big]\cdot\frac{\epsilon_{jkl}E_k^a E_l^b}{e} = \big[(\gamma^2+1)K_a^m K_b^n\big]\cdot\frac{E_k^a E_l^b}{e}\cdot(\delta^{k}{}_{m}\delta^{l}{}_{n} - \delta^{l}{}_{m}\delta^{k}{}_{n}) =$$
$$= 2(\gamma^2+1)\cdot\frac{E_k^a E_l^b}{e}\cdot K^{k}{}_{[a}K^{l}{}_{b]}$$

So we have:

$$H_0(N) = \int d^3x N\big[F_{ab}^{j}\epsilon_{j}{}^{kl} - 2(\gamma^2+1)K_{[a}^{k}K_{b]}^{l}\big]\cdot\frac{E_k^a E_l^b}{e}$$
$$= H^E(N) - 2(1+\gamma^2)T(N)$$

where we introduced a short-hand notation for the first and second term in the summation. This object as we expect is non-linear in terms of E, A; this gives rise to some difficulty if we want to turn it into an operator. However we can write it in a suitable way for the quantization, due to Thiemann. Let's denote with $V = \int d^3x\sqrt{det(E)}$. After a bit of algebra we can rewrite $H^E(N)$, $T(N)$ in the following way:

$$H^E(N) = \int d^3x N\epsilon^{abc}\delta_{ij}F^{i}{}_{ab}\big\{A^{j}{}_{c}, V\big\}$$
$$T(N) = \int d^3x\frac{N}{\gamma^3}\epsilon^{abc}\epsilon_{ijk}\big\{A^{i}{}_{a}, \big\{H^E(1), V\big\}\big\}\cdot\big\{A^{j}{}_{b}, \big\{H^E(1), V\big\}\big\}\cdot\big\{A^{k}{}_{c}, V\big\}$$

In this way we mapped the non linearity of the constraint into Poisson brakets. Now we have to write this object in terms of holonomies and fluxes and promoting them to operators.

Let's do it for $H^E(N)$. For T the procedure is analogous but since the presence of more terms it turns out to be more complicate. We need to express the connection A and the curvature F in terms of the holonomies. For the connection this task is quite easy if we look at the general relation between them:

$$h_\gamma(t) = \sum_{n=0}^{+\infty}\int_0^t ... \int_0^t A(\gamma(s_1))...A(\gamma(s_n))ds_1...ds_n$$

For a path e_a of infinitesimal lenght ε along the coordinate x^a we have:

$$h_{e_a}[A] \simeq 1 + \varepsilon A^i{}_a \tau_i + O(\varepsilon^2)$$

Where we assumed:

$$\int_0^\varepsilon A^i{}_a \tau_i \frac{dx^a}{ds} ds = \int_0^\varepsilon A^i{}_a \tau_i dx^a \sim \varepsilon A^i{}_a \tau_i$$

and so:

$$\begin{aligned} h_{e_a}^{-1}\{h_{e_a}, V\} &= (1 - \varepsilon A^i{}_a \tau_i)\{1 + \varepsilon A^j{}_a \tau_j, V\} = (1 - \varepsilon A^i{}_a \tau_i)\{\varepsilon A^j{}_a \tau_j, V\} = \\ &= \varepsilon\{A_a, V\} + O(\varepsilon^2) \end{aligned} \tag{1.50}$$

For the curvature $F^i{}_{ab}$ we consider an infinitesimal triangular loop α_{ab} lying on the plane individuated by the coordinates x^a, x^b with two sides equal to ε. At the lowest order in ε the holonomy associated with such path:

$$h_{\alpha_{ab}} = 1 + \frac{1}{2}\varepsilon^2 F^i{}_{ab} \tau_i + O(\varepsilon^4)$$

So

$$h_{\alpha_{ab}}^{-1} = 1 - \frac{1}{2}\varepsilon^2 F^i{}_{ab} \tau_i + O(\varepsilon^4)$$

Thus

$$h_{\alpha_{ab}} - h_{\alpha_{ab}}^{-1} = \varepsilon^2 F^i{}_{ab} \tau_i + O(\varepsilon^4)$$

In this way we wrote all the objects that are in H^E in terms of holonomies.
As we did before for the Volume and the Area operator we need to regularize the integral: we decompose the volume Σ in cells, and transorm the integral in a summation:

$$H^E = \lim_{\varepsilon \to 0} \sum_I \varepsilon^3 N_I \epsilon^{abc} Tr\big(F_{ab}\{A_c, V(C_I)\}\big) \tag{1.51}$$

Where

$$Tr\big(F_{ab}\{A_c, V(C_I)\}\big) = F^i{}_{ab}\{A^j{}_c, V\} Tr(\tau_i \tau_j) = F^i{}_{ab}\{A^j{}_c, V\}\delta_{ij}$$

And where we substituted:

$$\int d^3x f(x) \longrightarrow \lim_{\varepsilon \to 0} \sum_I \varepsilon^3 f(I)$$

Writing 1.51 in terms of the holonomies:

$$H^E = \lim_{\varepsilon \to 0} \sum_I N_I \epsilon^{abc} Tr\big[\big(h_{\alpha_{ab}(I)} - h^{-1}_{\alpha_{ab}(I)}\big) h^{-1}_{e_c(I)} \big\{h_{e_c(I)}, V\big\}\big]$$

In principle the paths $\alpha_{ab}(I)$ and $e_c(I)$ have an arbitrary relative position. It is useful for this operator make a partition of the volume V in tetradic cells, and adapt these paths to such cells (fig. 1.6). Now we can promote this part of the Hamiltonian constraint to

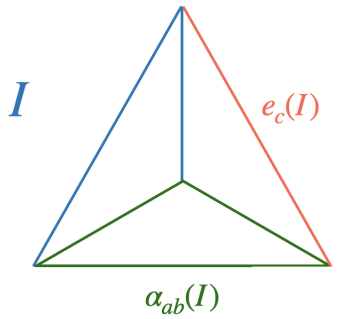


Figure 1.6

a quantum operator:

$$\hat{H}^E = \lim_{\varepsilon \to 0} \sum_I N_I \epsilon^{abc} Tr\big(\big(\hat{h}_{\alpha_{ab}(I)} - \hat{h}^{-1}_{\alpha_{ab}(I)}\big) \hat{h}^{-1}_{e_c(I)} \big[\hat{h}_{e_c(I)}, \hat{V}\big]\big)$$

It can be proved that this operator shares an important property with the Volume operator: it gives a non-zero result only if it acts on the nodes. This means that we can fix the partition of the volume in such a way that each node is contained in a tetrahedron C_I. In this way we can write the operator as a summation over the nodes of the graph on which it acts:

$$\hat{H}^E = \lim_{\varepsilon \to 0} \sum_{n \in \Gamma} N_n \epsilon^{abc} Tr\big(\big(\hat{h}_{\alpha_{ab}(I)} - \hat{h}^{-1}_{\alpha_{ab}(I)}\big) \hat{h}^{-1}_{e_c(I)} \big[\hat{h}_{e_c(I)}, \hat{V}\big]\big) \tag{1.52}$$

In the case of a three-valent node there is a natural choice for e_c and α_{ab}: we choose them in such a way that they are tangent to the links on the node. The contraction of a, b, c in 1.52 gives a sum over all the permutations of the three links. For an m-valent node (m > 4) we have the same expression, with $a, b, c = 1, 2, 3, 4$.

Let's look now at the action of such operator on a knotted spin network state: given a node n, the action of this operator on the node consists in the creation of 2 new nodes,

let's say n_a, n_b at a finite distance from n along the links a, b. The exact location of such new nodes is irrelevant, cause the diffeo. invariance of the state. It creates also a new link, let's say e_d of spin $\frac{1}{2}$ connecting the two nodes n_a and n_b (fig. 1.7). This new link is

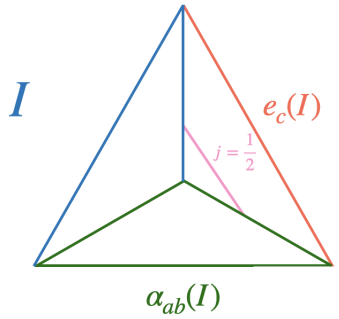


Figure 1.7

called *arc*. Finally it modifies the spin of the links that connect n with n_a and n_b of $\pm j$ with respect to the spins of e_a and e_b. Consequently if by a side changes the intertwiners of the node n, by the other the Clebsh-Gordon condition holds also for the modified node. This concludes the first chapter and our general treatment of Loop Quantum Gravity. For further details see [8]. The results we constructed are general and don't regard only some particular solution of the theory. With these tools in hand we can proceed to analyze some of the models of Loop Quantum Cosmology, that as we said in the introduction is not properly the Cosmological sector of the theory since is made of Minisuperspaces, in which we impose the symmetries of the solution already at the classical level.

We'll see in the next chapter that for the reduction of the degrees of freedom due to this symmetrization already at the classical level the huge computational difficulties that we meet in the general theory are extremely simplified.

We'll see different cosmological models, constructed on the assumption of different kind of symmetries at the classical level. We'll derive the solutions focusing on some of their features and in particular on a crucial property that join all of them: the *Bouncing behaviour* of the solutions, that is at the heart of this work.

Chapter 2

Loop Quantum Cosmology

Before moving on to the cosmological sector of LQG, we want to give a brief description of an important issue that affects classical Cosmology, and more in general Einstein theory of gravity: the *Singularity problem*. In classical Cosmology there exists a theorem, namely the *Hawking singularity theorem* which states:
A globally hyperbolic space-time $(\mathcal{M}, g)$ *in which there exists a Cauchy hypersurface* Σ *and a constant* $C_+ > 0$ $(C_- < 0)$*, such that the volume expansion* $\theta > C_+ (\theta < C_-)$*, everywhere on* Σ*, is past (future) geodesically incomplete if the Strong Energy Condition holds in the past (future) of* Σ.
This means in particular that the dynamics of a globally hyperbolic space-time with a source that satisfies the SEC and that expands (contracts) continually contains a singularity in the past (in the future). Generic ordinary matter (dust, photons,..) satisfies such condition, and this means that a model based on a source of this kind that describes an expanding or contracting Universe necessarily develops a singularity. Moreover the singularities predicted by such theorem are *physical singularities*, so that they cannot be eliminated by a change of coordinates. They are in general points of space-time in which the energy density diverges, and the scalar curvature becomes infinite, so clearly *unphysical*.
Singularities are features of many solutions of Einstein Equations not only in the cosmological sector, but also in the astrophysical one, like Black Holes, White Holes... Nowadays such odd predictions of Einstein theory are commonly considered as problems of the theory, not solutions, and they draw the limits of validity of the theory itself. The singularity problem of GR has been historically one of the reasons that brought physicists to look for a *Quantum theory of Gravity*. Physicists understood that only a Quantum theory of gravity could solve this problem, but why? We know from standard Quantum Mechanics

and from its application to Atomic physics that quantum effects of matter are already dominant at a lenght scale of $10^{-10}m$, the average atomic size. By the other side, the singularities predicted by Einstein theory have got 0 spatial dimension, which means technically that they are geometric points of space that contain a given finite quantity of energy. Now, even if the scale on which matter presents a quantum behaviour and the scale on which the space-time itself presents a quantum behaviour are not necessarily equal, and in fact ordinary Quantum Mechanics perfectly works in a classical spatial background (inside its limits of validity), we expect that there will be a lenght scale (or equivalently an energy scale) on which also the space-time (or the gravitational field) as well as all the other fields presents a quantum behaviour. Many physicists think that such behaviour should be dominant at the *Planck scale*: $l_P \sim 10^{-35}m$. Such scale is clearly much smaller with respect to the atomic scale, but "infinitely" much larger than the size of the singularities of Einstein theory. The previous is only one of the considerations that brought scientists to believe that the singularity problem could be solved only by a Quantum theory Gravity.

One of the beautiful and most important results of Loop Quantum Cosmology is the resolution of such classical problem in a *natural way*: in the cosmological sector the classical singularities are replaced by *Quantum Bounces* that result to be an intrinsic feature of the theory, as well as the classical singularities for Einstein theory. During the bounce the energy density of the solution reaches a minimum value and then, due to a *pure* quantum effect starts to expand going outside the Planck regime. As we'll see later, solutions with a finite small volume at the bounces reach the Planck volume (order of magnitude) and then start to expand. This amazing behaviour of the space-time regards at the same time Black Holes and all the cosmological solutions, even if at the time being the analysis on black holes is less developed in literature with respect to the cosmological one. With "natural" we mean that this problem is solved by the theory without requiring external ad hoc hypothesis, but its overcoming follows from the calculations.

We want to observe that such bouncing result couldn't absolutely be reached in a classical context by ordinary matter, cause a fundamental property of classical gravity: it is attractive. Technically such behaviour is forbidden for ordinary matter by the so called *Focusing theorem*. For this reason the bounce has to be interpreted as a genuine quantum effect and can arise only at the scale on which quantum effects of gravity start to be dominant, the Planck scale.

In this chapter we'll derive some solutions of Loop Quantum Cosmology, and we'll check

computationally that all of them share this feature at the Planck scale. Moreover we'll compare the results with the classical and Wheeler-DeWitt ones, showing by a side that only Loop Quantum Cosmology is able to solve the Singularity problem, by the other side that the results of the three theories are in perfect agreement outside the Planck scale, in the so called *Semiclassical region*.

Before moving on to the computations, we want here to point out the following important consideration: the ideal procedure to find symmetric solutions for the quantum theory would require to impose the symmetries we are interested in only after quantization, so to the quantum solutions. This procedure, even if the ideal one is very difficult to implement, and we won't follow it. Instead as commonly made in literature also for the Wheeler-DeWitt theory we'll construct *Minisuperspaces*: we impose the symmetries at the classical level, then we quantize the classical symmetric theory, and finally we find the solutions of the quantum constraints. As we said previously this way to proceed is not rigorous and the solution we'll find cannot considered rigorously exact. However this method allows to avoid all the computational complications of the full theory, and the solutions are expected to have many qualitative and quantitative features in common with the exact ones.

A very last observation: when we constructed in the full theory the Area and Volume operators, as well as the Hamiltonian constraint we introduced a regularization procedure, and at the very end we removed the regulator. We'll see that in these symmetrized models there is no regulator but a free parameter that has to be fixed taking hints from the full theory.

Let's proceed thus to the analysis of the various symmetric models: we start with the Friedmann-Robertson-Walker model, obtained by imposing homogeneity and isotropy to the classical solution, and we conclude with the Bianchi I Universe, that joins only spatial homogeneity.

2.1 FRW flat Universe

Classically the most generic form of an homogeneous and isotropic metric is the following:

$$ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \tag{2.1}$$

Where the only degree of freedom is a, usually called *scale factor*, that depends only on the variable t, accordingly with homogeneity. k is a parameter called *curvature constant*

and can have positive, negative or null value. Since it can be rescaled, its numerical value is not significant, and for this reason one distinguishes between three reference values: $k = 0, \pm 1$. The value of k determines the topology of the spatial hypersurfaces for such solution, and in this work we analyze only the simplest one, $k = 0$: this is the case of a spatially flat Universe.

The solution $a(t)$ depends on the kind of source we plug in Einstein equations, and if we consider a perfect fluid by its equation of state. In our analysis we consider a massless scalar field as the source of the expansion.

As we previously anticipated, if we assume that such field satisfies the SEC we obtain as classical solution an expanding (or contracting) Universe with a physical singularity in the past (future), and we cannot study it beyond such point in the past (future). For the expanding solution such point is often called theatrically *the beginning of Time and Space*.

Let's see what happens in Loop Quantum Cosmology.

In this section we follow mainly [9]. We start by writing the classical symmetrize model in terms of the Ashtekar-Barbero variables, then we proceed with the quantization.

2.1.1 Classical phase space and the Hamiltonian constraint

The classical phase space $(A^i{}_a, E^b{}_j)$ is drastically reduced, since the Gauss and Diffeo. constraints are trivially zero cause homogeneity and isotropy. The Hamiltonian constraint is the only that we have to impose. Moreover, since the spatial submanifolds Σ_t are flat, the spin connection vanishes:

$$\omega^{ij}{}_a = 0$$

This comes from 1.10 and the fact that $e^i{}_a$ depend only on t. Consequently we have a huge semplification of the classical Hamiltonian constraint:

$$C_{grav.}(N) = \int d^3x N \frac{-\epsilon_{ijk}}{16\pi G\gamma^2} \frac{F^i{}_{ab} E^{aj} E^{bk}}{e} \tag{2.2}$$

From this chapter on we restore the factor $16\pi G$.

Let's verify 2.2.We start by the following:

$$C_{grav.}(N) = \int d^3x \frac{N}{16\pi G} \left[F^k_{ab} \epsilon_k{}^{ij} - 2(\gamma^2 + 1) K^i_{[a} K^j_{b]}\right] \cdot \frac{E^a_i E^b_j}{e} \tag{2.3}$$

From the definition of the Ashtekar connection 1.15, we have in this model:

$$A^i{}_a = \gamma\omega^{0i}{}_a \equiv \gamma K^i{}_a \quad \Longrightarrow \quad 2K^i_{[a}K^j_{b]} = \frac{2}{\gamma^2}A^i_{[a}A^j_{b]} \tag{2.4}$$

By the other side:

$$F^k_{ab} = 2\partial_{[a}A^k_{b]} + \epsilon_{ij}{}^k A^i_a A^j_b$$

Thus:

$$\epsilon_{ij}{}^k\epsilon_k{}^{lm}A^i_a A^j_b = (\delta^l_i\delta^m_j - \delta^m_i\delta^l_j)A^i_a A^j_b = 2A^l{}_{[a}A^m{}_{b]} = \epsilon_k{}^{lm}F^k_{ab} - 2\epsilon_k{}^{lm}\partial_{[a}A^k_{b]}$$

Well, cause homogeneity the last term vanishes:

$$A^l_{[a}A^m_{b]} = \frac{1}{2}\epsilon_k{}^{lm}F^k_{ab}$$

Recalling 2.4:

$$2K^i_{[a}K^j_{b]} = \frac{1}{\gamma^2}\epsilon^{ij}_k F^k_{ab}$$

So we have:

$$\begin{aligned} C_{grav.}(N) &= \int_\Sigma d^3x N\frac{1}{16\pi G}\left[F^k_{ab}\epsilon_k{}^{ij} - \frac{1}{\gamma^2}(\gamma^2+1)F^k_{ab}\epsilon_k{}^{ij}\right]\cdot\frac{E^a_iE^b_j}{e} = \\ &= \int d^3x N\frac{-\epsilon_i{}^{jk}}{16\pi G\gamma^2}\frac{F^i{}_{ab}E^a_jE^b_k}{e} \end{aligned} \tag{2.5}$$

This integral in general diverges, since Σ is not compact and the fields are spatially homogeneous. To avoid this divergence we have to restrict our analysis to a finite cell $\mathcal{V}$; the homogeneity condition guarantees that the analysis on $\mathcal{V}$ can be extended on the whole Σ. If we require also isotropy $A^i{}_a$ and $E^b{}_j$ can be rewritten in this form

$$\begin{cases} A^i{}_a = c(t)V_0^{-\frac{1}{3}}\,{}^o e^i{}_a \\ E^a{}_i = p(t)V_0^{-\frac{2}{3}}\sqrt{{}^o q}\,{}^o e^a{}_i \end{cases} \tag{2.6}$$

Where $c(t)$ and $p(t)$ are functions that depend only on t. We describe the other quantities in a moment. Let's derive the first one.

We know from the full theory that:

$$A^i{}_a = \gamma \omega^{0i}{}_a$$

Where $\omega^{IJ}{}_\mu = e^I{}_\nu \nabla_\mu e^{J\nu}$. So

$$\omega^{0i}{}_a = e^0{}_\nu \nabla_a e^{i\nu} = e^0{}_\nu (\partial_a e^{i\nu} + \Gamma^\nu{}_{ab} e^{ib}) = e^0{}_\nu \Gamma^\nu{}_{ab} e^{ib}$$

Since homogeneity. Now, recalling that $e^0{}_\mu = \binom{N}{\vec{0}}$

$$\begin{aligned} e^0{}_\nu \Gamma^\nu{}_{ab} e^{ib} &= e^0{}_0 e^{ib} \frac{1}{2} g^{0\sigma} (g_{\sigma a,b} + g_{b\sigma,a} - g_{ab,\sigma}) = -N e^{ib} \frac{1}{2} g^{00} g_{ab,0} = \\ &= -\frac{N}{2} \frac{1}{a(t)^2} e^{ia} g_{aa,0} = N \frac{\dot{a}(t)}{a(t)} e^i{}_a \end{aligned}$$

So $A^i{}_a = \gamma N \frac{\dot{a}(t)}{a(t)} e^i{}_a$.

We introduce now the so called *fiducial metric* ${}^o q_{ab}$, defined by the following relation: $q_{ab} = {}^o q_{ab} a(t)^2$. This tensor doesn't contain any time evolution and allows to measure distancs in a flat space not evolving in time (so clearly not the physical one).

Recalling that:

$$\delta_{ij} e^i{}_a e^j{}_b = q_{ab}$$

We have

$$\delta_{ij} e^i{}_a e^j{}_b = {}^o q_{ab} a(t)^2 = \delta_{ij} {}^o e^i{}_a {}^o e^j{}_b a(t)^2$$

Thus

$$e^i{}_a = {}^o e^i{}_a a(t) \tag{2.7}$$

${}^o e^i{}_a$ is called *fiducial triad*.

These fiducial tensors are useful since allow to decouple the tensorial structure of the variables from their time dependence, and this can be done in virtue of the symmetries. So we have:

$$A^i{}_a = {}^o e^i{}_a \gamma N \dot{a}(t)$$

Finally to reach the first of 2.6 we simply call:

$$c(t) = \gamma N V_0^{\frac{1}{3}} \dot{a}(t) \tag{2.8}$$

where $V_0 = \int_{\mathcal{V}} d^3x\sqrt{q_0}$ is called *fiducial volume* of $\mathcal{V}$, so the volume measured with the fiducial metric ${}^o q_{ab}$. Clearly V_0 is not the physical volume of $\mathcal{V}$, since it doesn't contain any time evolution: it is a volume regarding a fictitious not evolving space-time.
Let's proceed with the second proof. Whe know that $E^a{}_i = e(t)e^a{}_i(t)$. But we also have

$$e^a{}_i = \frac{sgn(a)}{|a(t)|}{}^o e^a{}_i$$

And since

$$\epsilon^{abc} e^i{}_a e^j{}_b e^k{}_c = det(e^a{}_i)\epsilon^{ijk}$$

We have

$$det(e^a{}_i) = det({}^o e^a{}_i)|a|^3 sgn(a)$$

Thus

$$E^a{}_i = ee^a{}_i = |det(e^a{}_i)|e^a{}_i = sgn(a)^o e^o e^a{}_i a(t)^2 = \sqrt{{}^o q}\, p(t) V_0{}^{-\frac{2}{3}} {}^o e^a{}_i \tag{2.9}$$

Where we called

$$p(t) = sgn(a)a(t)^2 V_0{}^{\frac{2}{3}} \tag{2.10}$$

In this way we built a variable $p(t)$ with dimension $[L]^2$. We notice from 2.7 that $p(t)$ is positive if the physical triad and the fiducial one are parallel, negative if antiparallel. This means that its sign depends on our choice of the relative orientation of the two frames (the physical triadic frame and the fiducial one). For the moment we don't fix their relative orientation and the sign of p remains unfixed.
Let's see the relation between p and the physical volume V.

$$V = \int_{\mathcal{V}} d^3x\sqrt{q} = \int_{\mathcal{V}} d^3x\sqrt{{}^o q}a(t)^3 = V_0 a(t)^3$$

Thus:

$$V = |p|^{\frac{3}{2}} \tag{2.11}$$

Remark 1 : as we previously anticipated the tensorial structure of the Ashtekar-Babero variables is totally embedded in fiducial tensors. This means that written in this form we can study the evolution of these variables without looking at their tensorial structure.
Remark 2 : even if ${}^o q_{ab}$ describes a flat space not evolving in time, in general is not written in a Cartesian frame: ${}^o q_{ab} \neq \delta_{ab}$. However the choice of the fiducial frame is

arbitrary at this stage, so we can choose an orthonormal frame:

$$ {}^{o}q_{ab} = \delta_{ab} \quad \Longrightarrow \quad {}^{o}e^{a}{}_{i} = \delta^{a}{}_{i} $$

From now on we work with this assumption.
Since the decoupling due to homogeneity between the time dependence and the tensorial structure of the Ashtekar-Barbero variables we can consider directly $c(t)$ and $p(t)$ as variables of our classical phase space. They define a symplectic structure:

$$ \{c(t), p(t)\} = \frac{8}{3}\pi G\gamma \tag{2.12} $$

Let's prove it.
We start from 1.16. Using 2.6 we have

$$ \begin{aligned} \{A^{i}{}_{a}(x), E^{b}{}_{j}(y)\} &= \{cV_0{}^{-\frac{1}{3}}{}^{o}e^{i}{}_{a}, pV_0{}^{-\frac{2}{3}}\sqrt{{}^{o}q}\,{}^{o}e^{b}{}_{j}\} = \\ &= V_0^{-1}\delta^{i}{}_{a}\delta^{b}{}_{j}\sqrt{{}^{o}q}\,\{c,p\} = 8\pi G\delta^{b}{}_{a}\delta^{i}{}_{j}\gamma\delta^{3}(\vec{x}-\vec{y}) \end{aligned} $$

If now we integrate both members: $\int_{\mathcal{V}} d^3x$:

$$ \begin{aligned} &\int_{\mathcal{V}} d^3x\sqrt{{}^{o}q}V_0^{-1}\delta^{i}{}_{a}\delta^{b}{}_{j}\,\{c,p\} = 8\pi G\delta^{b}{}_{a}\delta^{i}{}_{j}\gamma\int_{\mathcal{V}} d^3x\delta^{3}(\vec{x}-\vec{y}) \\ &\Longleftrightarrow \quad \delta^{i}{}_{a}\delta^{b}{}_{j}\,\{c,p\} = 8\pi G\delta^{b}{}_{a}\delta^{i}{}_{j}\gamma \end{aligned} $$

Contracting now the indices $i-a$:

$$ 3\,\delta^{b}{}_{j}\,\{c,p\} = 8\pi G\delta^{b}{}_{j}\gamma \quad \Longleftrightarrow \quad \{c,p\} = \frac{8}{3}\pi G\gamma $$

Then we consider a massless scalar field ϕ as source of the expansion, with conjugate momentum P_ϕ. As we said previously it has to share the symmetry of the metric solution. The Poisson brakets for such field:

$$ \{\phi, P_\phi\} = 1 $$

Consequently the total Hamiltonian constraint contains a matter contribution; if we assume $N=1$ (synchronous gage), the Hamiltonian constraint of our constrained system:

$$ C(N=1) \equiv C = C_{grav.} + C_{matt.} = -\frac{6}{16\pi G\gamma^2}c^2\sqrt{|p|} + \frac{P_\phi^2}{2V} \approx 0 \tag{2.13} $$

Where as we previously noticed $V = |p|^{\frac{3}{2}}$ differently from V_0 is the physical volume of the cell $\mathcal{V}$ and depends on t. Let's verify the first term.
We start from

$$C_{grav.} = -\frac{1}{16\pi G\gamma^2}\int_{\mathcal{V}} d^3x\epsilon_{ijk}\frac{F^i{}_{ab}E^{aj}E^{bk}}{e} \tag{2.14}$$

And we assume 2.6. Firstly we notice that the choice ${}^oe^i{}_a = \delta^i{}_a$ (independent on $\vec{x}$ since homogeneity) means that A and E in such frame are diagonal matrices, and: $\sqrt{{}^oq} = 1$ since ${}^oq_{ab} = \delta_{ab}$. Given that

$$\begin{cases} det(E) = {}^oq^{\frac{3}{2}}p^3V_0^{-2} = p^3V_0^{-2} \\ F^i{}_{ab} = \epsilon^i{}_{jk}A^j{}_aA^k{}_b \end{cases} \tag{2.15}$$

and substituting them in 2.14

$$\begin{aligned} C_{grav.} &= -\frac{1}{16\pi G\gamma^2}\int_{\mathcal{V}} d^3x\epsilon_{ijk}\epsilon^i{}_{lm}\frac{A^l{}_aA^m{}_bE^{aj}E^{bk}}{e} = \\ &= -\frac{1}{16\pi G\gamma^2}\int_{\mathcal{V}} d^3x\epsilon_{ijk}\epsilon^i{}_{lm}c(t)^2p(t)^2{V_0}^{-2}\frac{{}^oe^l{}_a{}^oe^m{}_b({}^oe^{aj})({}^oe^{bk})}{e} = \\ &= -\frac{1}{16\pi G\gamma^2}c^2p^2V_0^{-2}\epsilon_{ijk}\epsilon^{ijk}\int_{\mathcal{V}} d^3x\frac{1}{e} = \\ &= -\frac{3}{8\pi G\gamma^2}c^2p^2V_0^{-2}\int_{\mathcal{V}} d^3x\frac{V_0}{p^{\frac{3}{2}}} = -\frac{3}{8\pi G\gamma^2}c^2\sqrt{|p|} \end{aligned}$$

This form for the Hamiltonian constraint even if elegant cannot be used as starting point for the quantization of the model. This because in order to construct a well-defined Hilbert space we have to smear the variables, and write such constraint in terms of holonomies and smeared triads, as well as in the full theory.
So let's look now at the smearing of the variables in this symmetrized model.

2.1.2 Holonomy-flux Algebra

In order to construct the classical holonomy-flux algebra we need to choose paths γ for the holonomies and surfaces S for the smearing of $E^a{}_i$. However, since the symmetry of the solution we don't need to consider all the possible edges and surfaces.
Let's see this in detail starting from the connection $A^i{}_a$. Since homogeneity and isotropy, we can smear the connection on a straight path contained in $\mathcal{V}$, obtaining a representant

of an equivalence class of states with generic graphs. In particular, given

$$h_\gamma = \mathcal{P}exp\bigg(\int_\gamma ds A^i{}_a \frac{dx^a}{ds} \tau_i \bigg)$$

we can consider a straight path, and set it on the direction individuated by a given x^b, that is a coordinate of the fiducial frame:

$$h_\gamma = \mathcal{P}exp\bigg(\int_\gamma dx^b A^i{}_a \frac{dx^a}{dx^b} \tau_i \bigg) = \mathcal{P}exp\bigg(\int_\gamma dx^b A^i{}_a \delta^a{}_b \tau_i \bigg) = \mathcal{P}exp\bigg(\int_\gamma dx^b A^i{}_b \tau_i \bigg)$$

with oriented fiducial lenght $l = \mu\, V_0^{\frac{1}{3}}$, $\mu \in (-\infty, +\infty)$.
Now, using:

$$A^i{}_b = c(t) V_0{}^{-\frac{1}{3}} \delta^i{}_b$$

We have

$$\begin{aligned} h_i^{(\mu)} &= \mathcal{P}exp\bigg(\int dx^b c(t) V_0^{-\frac{1}{3}} \delta^i{}_b \tau_i \bigg) = exp\bigg(\int dx^b c(t) V_0^{-\frac{1}{3}} \delta^i{}_b \tau_i \bigg) \\ &= exp\bigg(c(t) V_0^{-\frac{1}{3}} V_0^{\frac{1}{3}} \tau_i \mu \bigg) = exp\big(c(t) \mu \tau_i \big) \end{aligned}$$

Where in the first passage we removed the path ordering operator since cause homogeneity the integrand is independent on x^b, so on the point of the path on wich we evaluate it.
Remark : we notice that the subscript i in $h^{(\mu)}{}_i$ is not a triadic index, but a label that fixes the fiducial direction we choose for the holonomy. We can see this from the fact that the holonomy is not an element of the algebra of $SU(2)$, but is an element of $SU(2)$ itself so cannot have a non contracted triadic index.
As we see in a moment the Kinematical Hilbert space is independent on the $SU(2)$ representation we choose, so we can fix for simplicity $j = \frac{1}{2}$. Thus we can rewrite the holonomy in this way

$$h_i^{(\mu)} = exp\bigg(\frac{i c(t) \mu \sigma_i}{2} \bigg) = cos\bigg(\frac{c\mu}{2} \bigg) \mathbb{1} - i \sigma_i sin\bigg(\frac{c(t)\mu}{2} \bigg) = cos\bigg(\frac{c\mu}{2} \bigg) \mathbb{1} + 2\tau_i sin\bigg(\frac{c(t)\mu}{2} \bigg) \tag{2.16}$$

The homogeneity and isotropy ansatz brings to the fact that non-straight paths don't carry more informations than the one we considered here. The state associated with such link

$$D(h_i^{(\mu)})^{\frac{1}{2}}_{\alpha\beta} = exp\bigg(\frac{i\mu c(t) \sigma_i}{2} \bigg)_{\alpha\beta} \tag{2.17}$$

By the other side we can smear the densitized triad with the integral over a squared surface; so we can consider: $d^2\sigma = dx^d dx^e$, which means we adapt the square to two of the three axis of the fiducial coordinate system we use

$$E_i(S) = \int_S n_a E^a{}_i d^2\sigma = \int_S \epsilon_{abc} \frac{\partial x^b}{\partial \sigma^1} \frac{\partial x^c}{\partial \sigma^2} E^a{}_i d\sigma^1 d\sigma^2 = \int_S \epsilon_{ade} dx^d dx^e E^a{}_i$$

Where the indices d, e are not summed over. Using the explicit expression for E:

$$E_i(S) = p(t) V_0^{-\frac{2}{3}} \int_S \epsilon_{ade} dx^d dx^e \sqrt{{}^o q} \delta^a{}_i = p(t) V_0^{-\frac{2}{3}} A_{S_i}$$

So

$$E_i(S) f^i = p V_0^{-\frac{2}{3}} A_{S,f} \tag{2.18}$$

Where f^i is a generic test function. Let's observe that $\epsilon_{ibc} dx^b dx^c f^i$ describes the relative orientation between f^i and the surface and its sign depends on f once we fixed the surface. In particular if f lies on the surface the whole result is 0. 2.18 means that the flux is fundamentally described by $p(t)$.

What we need now is an algebra of variables to quantize. By a side we cannot use simply c, p since they are not smeared. By the other side is also unnecessary to use h, E since we don't need to look at the structure of a particular graph, since homogeneity and isotropy. From 2.18 we see that the time evolution of $E_i(S)$ is completely contained in $p(t)$, and the relation between E and p is linear. So we can think to use p in the place of E. Moreover if we look at 2.17, and in particular at the matrix elements of the holonomies we notice that they are linear combinations of exponentials of kind $exp\big(\frac{i\mu c(t)}{2}\big)$, independently on the j-representation we consider. This means that at the mathematical level the algebra generated by the holonomies is the same generated by $exp\big(\frac{i\mu c(t)}{2}\big)$. For this reason we can use it as a variable for the quantization of our Minisuperspace. The importance of taking $exp\big(\frac{i\mu c(t)}{2}\big)$ instead of $c(t)$ is that the first one captures the degrees of freedom of A only along a certain path with lenght μ, so is a non local variable.

Let's proceed with the construction of the Poisson brakets between such variables. We call $N_\mu(c) = exp\big(\frac{i\mu c(t)}{2}\big)$. Thus

$$\{N_\mu(c), p\} = \frac{8\pi G\gamma}{3} \frac{\partial e^{\frac{i\mu c}{2}}}{\partial c} = \frac{i4\pi G\gamma\mu}{3} N_\mu(c) \tag{2.19}$$

Well, since $N_\mu(c)$ belongs to the configuration space, the algebra is closed.

Mimicking the procedure we described in the first chapter, we can proceed now to quantize

the theory. We start by describing the Kinematical structure.

2.1.3 Kinematical Hilbert space

Following as usual the Dirac procedure, we quantize the system. Before proceeding with the calculations we want to notice an important fact: the system described by the variables N_μ and p is not a field theory, but a 1-particle theory in 1-D because the classical phase space is not ∞-dimensional (as in a field theory), but 2-dimensional. This is uniquely due to the symmetrization of the model at the classical level, that brought the system to be described by two variables $c(t)$, $p(t)$ instead of two fields A, E. Thus its quantization follows the one of a quantum mechanical system, producing a strong simplification.

To quantize the system we firstly need a basis for the Kinematical Hilbert space; it can be proved that the states $|\mu\rangle$, which represent the quantum version of $N_\mu(c)$, span a space of cylindrical functions that, embedded with the discrete scalar product $\langle\mu|\mu'\rangle = \delta_{\mu\mu'}$ is an Hilbert space. So we can use $\{|\mu\rangle\}$ as a basis for the Kinematical Hilbert space of our model, that we call $\mathcal{H}_{grav.}$.

This kind of quantization with a discrete inner product is called *polymeric quantization* and is inequivalent to the one we usually use for 1-D systems in Quantum mechanics [19]. It is however necessary to mimick the structure of the full theory that is discontinuous: in the full theory for example we cannot construct a variable that is canonically conjugated to the holonomy, since it should be (in the holonomy representation) a functional derivative with respect to an object that is not continuous (the holonomy itself), and here within this representation analogously we cannot construct a variable that is canonically conjugated with N_μ.

Well, promoting now p and N_μ to quantum operators, their action on our basis $|\mu\rangle$

$$\begin{cases} \widehat{N}_\mu |\mu'\rangle = |\mu + \mu'\rangle \\ \widehat{p}\,|\mu\rangle = p(\mu)\,|\mu\rangle \end{cases} \tag{2.20}$$

Where

$$p(\mu) = \frac{4\pi\gamma l_P^2}{3}\mu \tag{2.21}$$

Let's prove 2.21 We start from the Dirac rule

$$\big[\widehat{N'_\mu}, \widehat{p}\big] = i\hbar\widehat{\big\{N'_\mu, p\big\}} \tag{2.22}$$

Acting with both the members on $|\mu\rangle$ we have

$$\left[\widehat{N'_\mu}, \widehat{p}\right]|\mu\rangle = -\hbar\frac{4\pi G\gamma}{3}\mu'\widehat{N'_\mu}|\mu\rangle \quad \Longrightarrow \quad \left(\widehat{N'_\mu}\widehat{p} - \widehat{p}\widehat{N'_\mu}\right)|\mu\rangle = -\hbar\frac{4\pi G\gamma}{3}\mu'\widehat{N'_\mu}|\mu\rangle \quad \Longrightarrow$$
$$\widehat{N'_\mu}\widehat{p}|\mu\rangle = \widehat{p}\widehat{N'_\mu}|\mu\rangle - \hbar\frac{4\pi G\gamma}{3}\mu'\widehat{N'_\mu}|\mu\rangle$$

Using now the first relation of 2.20:

$$\widehat{p}|\mu+\mu'\rangle = \left(p(\mu) + \frac{l_P^2 4\pi\gamma\mu'}{3}\right)|\mu+\mu'\rangle \tag{2.23}$$

that tells that the states $|\mu\rangle$, that are eigenstates of the operator $\widehat{p}$ have eigenvalues $p(\mu) = \frac{l_P^2 4\pi\gamma\mu'}{3}$.

Remark 1 : in principle instead of considering $\hat{N}_\mu$ as operator we can consider directly the holonomy. This means

$$\widehat{h_i^{(\mu')}}|\mu\rangle = \frac{1}{2}(|\mu+\mu'\rangle + |\mu-\mu'\rangle)\mathbb{1} + \frac{1}{i}(|\mu+\mu'\rangle - |\mu-\mu'\rangle)\tau_i$$

That can be easily derived writing *cos* and *sin* in their exponential form. However as we can see in each matrix element h acts as $\hat{N}_{\mu'}$ or $\hat{N}_{-\mu'}$, so we can study directly the action of $\hat{N}_\mu$.

Remark 2 : looking at the second relation of 2.20, we notice that $|\mu\rangle$ is eigenstate of the operator $\hat{p}$, but recalling 2.11 we can construct immediately the volume operator $\hat{V}$, and obtain that $|\mu\rangle$ has to be eigenstate of $\hat{V}$ too. The action of the volume operator on the state $|\mu\rangle$:

$$\widehat{V}|\mu\rangle = \left(\frac{\gamma|\mu|4\pi}{3}\right)^{\frac{3}{2}} l_P^3|\mu\rangle = V(\mu)|\mu\rangle \tag{2.24}$$

This means that modulo a constant $|\mu|^{\frac{3}{2}}$ is the physical volume of the cell $\mathcal{V}$ when the Universe formally is in its quantum state $|\mu\rangle$.

Let's notice a big difference between the spectrum of the volume operator in the full theory, and the one here. The eigenvalues of $\hat{V}$ are a *fixed* (infinite) subset of the real line in the full theory. Here instead its eigenvalues are a infinite subset of the real line, but with values unfixed, since $\mu \in (-\infty, +\infty)$. In particular, it seems that the minimum of such spectrum is 0 since $|\mu|$ can assume arbitrary small values, which means in principle that in this model we cannot avoid the singularity. We'll see later that this is not the case.

Let's proceed now with the quantization of the massless scalar field.

For the matter field we use the usual Schrodinger-like representation with $\hat{\phi}$ acting by multiplication and $\hat{P}_\phi = -i\hbar\frac{\partial}{\partial\phi}$. Both act on states defined on the Hilbert space: $L^2(\mathbb{R}, d\phi)$. The whole Kinematical Hilbert space is

$$\mathcal{H}_{Kin} = \mathcal{H}_{grav} \otimes L^2(\mathbb{R}, d\phi)$$

Since as we previously said the classical Gauss and diffeo. constraints are identically 0 we can proceed directly with the quantization of the Hamiltonian constraint.

2.1.4 The classical Hamiltionian constraint and its quantization

In order to quantize the Hamiltonian constraint we follow the same procedure adopted in the full theory. We start expressing the curvature tensor F in terms of holonomies, that have a well defined quantum counterpart. If in the full theory we considered a small triangular loop to make the job, here we consider for simplicity a squared loop that depends parametrically on the lenght of the edge: $\mu V_0^{\frac{1}{3}}$ in the plane individuated by the coordinates x^i, x^j; we can write:

$$h^{(\mu)}_{\Box ij} = h^{(\mu)}_i h^{(\mu)}_j (h^{(\mu)}_i)^{-1} (h^{(\mu)}_j)^{-1}$$

Since the fundamental property of the holonomies: $h_{\alpha+\beta} = h_\alpha h_\beta$. We notice that such quantity is not trivially 1 since $[\tau_i, \tau_j] \neq 0$. This holonomy encolses a fiducial area: $A_\Box = \mu^2 V_0^{\frac{2}{3}}$. For the holonomies we use for the constraint we consider the $j = \frac{1}{2}$ representation, for a physical reason that we'll explain later. We only remark that such choice produces a form for the constraint different from other choices.
If we write the curvature tensor in terms of h, and we send $A_\Box = \mu^2 V_0^{\frac{2}{3}} \to 0$, we obtain

$$F^l{}_{ab} = \lim_{A_\Box \to 0} -2\frac{tr\{[h^{(\mu)}_{\Box_{ij}} - \mathbb{1}]\tau^l\}}{A_\Box} {}^o e^i{}_a {}^o e^j{}_b \tag{2.25}$$

Let's prove it. We start with the expression of holonomies for edges with oriented lenght $l = \mu V_0^{\frac{1}{3}}$:

$$
\begin{aligned}
h_{e_1} &= cos\left(\frac{\mu c}{2}\right)\mathbb{1} + 2\tau_i \cdot sin\left(\frac{\mu c}{2}\right) \\
h_{e_2} &= cos\left(\frac{\mu c}{2}\right)\mathbb{1} + 2\tau_j \cdot sin\left(\frac{\mu c}{2}\right) \\
h_{e_3} &= cos\left(\frac{\mu c}{2}\right)\mathbb{1} - 2\tau_i \cdot sin\left(\frac{\mu c}{2}\right) \\
h_{e_4} &= cos\left(\frac{\mu c}{2}\right)\mathbb{1} - 2\tau_j \cdot sin\left(\frac{\mu c}{2}\right)
\end{aligned}
$$

We need now to multiply them. Recalling the following relation

$$\tau_i\tau_j = \frac{1}{2}\epsilon_{ijk}\tau_k - \frac{1}{4}\delta_{ij}$$

And renaming for simplicity: $cos\left(\frac{\mu c}{2}\right) = C$ and $sin\left(\frac{\mu c}{2}\right) = S$, we have:

$$
\begin{aligned}
h_i^{(\mu)}h_j^{(\mu)} &= \mathbb{1}C^2 - \mathbb{1}\delta_{ij}S^2 + 2\epsilon_{ijk}\tau_k S^2 + 2\tau_j CS + 2\tau_i CS \quad \Longrightarrow \\
\Longrightarrow \quad h_i^{(\mu)}h_j^{(\mu)}h_i^{(\mu)-1} &= C^3\mathbb{1} + CS^2\mathbb{1} + 2\tau_j SC^2 + 4\epsilon_{ijk}\tau_k CS^2 \quad \Longrightarrow \\
\Longrightarrow \quad h_i^{(\mu)}h_j^{(\mu)}h_i^{(\mu)-1}h_j^{(\mu)-1} &= C^2(1+C^2)\mathbb{1} - 2CS^3(2\tau_i + \tau_j) + 4C^2S^2\epsilon_{ijk}\tau_k \qquad (2.26)
\end{aligned}
$$

Now, to recover 2.25 we have to assume μ small, since $A_\square \to 0$. So we can expand A.1 in powers of μ, stopping at $o(\mu^3)$. We obtain

$$h_{\square ij}^{(\mu)} \sim \mathbb{1} + \mu^2 c(t)^2 \epsilon_{ijk}\tau^k \qquad (2.27)$$

Now,

$$F^i{}_{ab} = \epsilon^i_{jk}A^j{}_a A^k{}_b = \epsilon^i_{jk}c(t)^2 V_0^{-\frac{2}{3}}\delta^j{}_a\delta^k{}_b = \epsilon^i_{ab}c^2 V_0^{-\frac{2}{3}} \qquad (2.28)$$

Substituting 2.27 in this:

$$F^k{}_{ij}\tau_k = \lim_{A_\square \to 0} \frac{\left(h_{\square ij}^{(\mu)} - \mathbb{1}\right)}{\mu^2 V_0^{\frac{2}{3}}}$$

Multiplying both members by ${}^o e^i{}_a {}^o e^j{}_b$

$${}^o e^i{}_a {}^o e^j{}_b F^k{}_{ij}\tau_k = F^k{}_{ab}\tau_k = \lim_{A_\square \to 0} \frac{(h_{\square ij}^{(\mu)} - \mathbb{1})}{A_\square} {}^o e^i{}_a {}^o e^j{}_b$$

Now we multiply both members by τ_l and we make the trace

$$F^k{}_{ab} tr(\tau_k \tau_l) = \lim_{A_\square \to 0} \frac{tr\{[h^{(\mu)}_{\square_{ij}} - \mathbb{1}]\tau_l\}}{A_\square} {}^o e^i{}_a {}^o e^j{}_b$$

And since $tr\left(\frac{\sigma_k \sigma_l}{4i^2}\right) = -\frac{\delta_{kl}}{2}$,

$$F^l{}_{ab} = \lim_{A_\square \to 0} -2\frac{tr\{[h^{(\mu)}_{\square_{ij}} - \mathbb{1}]\tau^l\}}{A_\square} {}^o e^i{}_a {}^o e^j{}_b \tag{2.29}$$

We notice that the direction individuated by the label l has to be orthogonal with respect to the ones individuated by i and j. This limit is classically well defined. However as we know from the full theory, the spectrum of the physical area is quantized, so at the quantum level we cannot send $A_\square \to 0$, since this would bring the value of the physical area to 0. For this reason, guided by a necessary reference to the full theory we have to assume that our area $A_\square$ has a minimal value.

But how do we choose it? In order to answer to this question we have to make a parallelism between LQG states and LQC states, since we need to use the the fact that the Area operator in LQG has got a minimum eigenvalue. In LQC the physical area of a side of the fiducial cell $\mathcal{V}$ is given simply by $|p|$. The LQC quantum state associated with such area is $|\mu(p)\rangle$. Let's ask now what is the associated LQG quantum state. The first ansatz for such state is that if the Area operator associated with the area of a side of the fiducial cell acts on it, it has to give a value equal to $|p|$. This means that some of the links of the graph of such state have to puncture such area. The second ansatz is that it has to be an homogeneous state on the area we are considering. In principle, in order to have an homogeneous graph on such area it should have an infinite number of links, in such a way that there is no empty space between them. This is however not possible, since the resulting eigenvalue of the Area operator would be infinite. So we have to choose the links that produce (each of them) the minimum possible value of the area, and we choose for this purpose links carrying $j = \frac{1}{2}$ spin. We choose for simplicity straight links for the graph. Now we have to understand how many links we need in our graph to reproduce the LQC state. In order to do this we remember that each link puncturing the area carries a contribution to the spectrum of the area of $4\pi\sqrt{3}\gamma l_p^2 = \Delta$. This means that we need N spins, with N such that

$$N 4\pi\sqrt{3}\gamma l_p^2 = |p| = N\Delta \tag{2.30}$$

Thus we can divide our area in N squared plaquettes, each of them with physical area $4\pi\sqrt{3}\gamma l_p^2$ and consider this one as the minimum value of the physical area (area evaluated with the physical metric) associated with the fiducial area $A_\square$ (evaluated with the fiducial metric). Calling the dimensionless fiducial area of such plaquettes $\bar{\mu}^2$, we have

$$A_{\square min} = \bar{\mu}^2 V_0^{\frac{2}{3}}$$

Now, since the fiducial area of a side of the volume $\mathcal{V}$ is $V_0^{\frac{2}{3}}$, we have

$$N\bar{\mu}^2 V_0^{\frac{2}{3}} = V_0^{\frac{2}{3}}$$

Thus $\bar{\mu}^2 = \frac{1}{N}$. Substituting this inside 2.30 we obtain

$$\bar{\mu}^2 = \frac{\Delta}{|p|} \tag{2.31}$$

This means that given $|p|$, we have to choose $\bar{\mu}$ such that $p(t) = \frac{\Delta}{\bar{\mu}^2}$. So we have:

$$F^k{}_{ab} = -2\frac{tr\{[h_{\square_{ij}}^{(\bar{\mu})} - \mathbb{1}]\tau^k\}}{\bar{\mu}^2 V_0^{\frac{2}{3}}} {}^o e^i{}_a {}^o e^j{}_b \tag{2.32}$$

The relation 2.31 is called *Improved dynamics* in LQC literature.
We can write 2.32 in a more useful way for its quantization. To do this we use 2.26, and we have

$$F^k{}_{ab} = -2\frac{tr\{[C^2(1+C^2)\mathbb{1} - \mathbb{1} - 2CS^3(2\tau_i + \tau_j) + 4C^2S^2\epsilon_{ijl}\tau_l]\tau^k\}}{\bar{\mu}^2 V_0^{\frac{2}{3}}} {}^o e^i{}_a {}^o e^j{}_b$$

Now recalling that ${}^o e^i{}_a = \delta^i{}_a$, and noticing that F has to have all the indices with different values (this can be seen easily from 2.28), from the property of the trace of the Pauli matrices we have:

$$\begin{aligned} F^k{}_{ab} =& -8C^2S^2\epsilon_{ijl}\frac{tr\{\tau_l\tau^k\}}{\bar{\mu}^2 V_0^{\frac{2}{3}}} {}^o e^i{}_a {}^o e^j{}_b = -8C^2S^2\epsilon_{abk}\frac{tr\{\tau_k\tau^k\}}{\bar{\mu}^2 V_0^{\frac{2}{3}}} = \\ =& \frac{4C^2S^2\epsilon^k_{ab}}{\bar{\mu}^2 V_0^{\frac{2}{3}}} = \epsilon^k_{ab}\frac{sin^2\big(2\bar{\mu}c\big)}{\bar{\mu}^2 V_0^{\frac{2}{3}}} \end{aligned} \tag{2.33}$$

Well, the curvature operator $\hat{F}$ is obtained by promoting 2.33 to a quantum operator. As usual τ inside F is a generator of $SU(2)$, and we fixed in the derivation the $j =$

$\frac{1}{2}$ representation for the holonomy along the squared plaquette, as well as for all the holonomies we consider in this model.

There is however a problem in the quantization of 2.33: ${F^l}_{ab}$ contains $h^{(\bar{\mu})}$, and this holonomy at the classical has the form $N^{(\bar{\mu})} = exp(\frac{i\bar{\mu}c(t)}{2}) = exp\big(\frac{ic(t)}{2}\sqrt{\frac{\Delta}{|p|}}\big)$. In order to promote F to an operator we need to promote $N^{(\bar{\mu})}$, but differently from $N^{(\mu)}$ it contains p, that is the value of a variable of the phase space. We can overcome such obstacle through a geometrical consideration. For the operator $\widehat{N}^{(\mu)}$, we have:

$$\widehat{exp\left(\frac{i\mu c(t)}{2}\right)}exp\left(\frac{i\mu' c(t)}{2}\right) = exp\left(\frac{ic(\mu+\mu')}{2}\right) = exp\left(\mu\frac{\partial}{\partial\mu'}\right)\exp\left(\frac{i\mu' c(t)}{2}\right)$$

Where μ is a constant. So we can think at $\widehat{exp\left(\frac{i\mu c(t)}{2}\right)}$ as a dragging for the state of the quantity μ along the direction individuated by the vector field $\frac{\partial}{\partial\mu'}$. In the same way we can think at $\widehat{exp\left(\frac{i\bar{\mu}c(t)}{2}\right)}$ as an operator that drags the state along $\frac{\partial}{\partial\mu'}$ of the quantity $\bar{\mu}(p)$. For a generic state:

$$\widehat{exp\left(\frac{i\bar{\mu}c(t)}{2}\right)}\tilde{\psi}(\mu) = exp\left(\bar{\mu}\frac{\partial}{\partial\mu}\right)\tilde{\psi}(\mu)$$

This means that this operator acts on the state as: $exp\left(\bar{\mu}(\mu)\frac{\partial}{\partial\mu}\right)$.

Let's call $\partial_v = \bar{\mu}\partial_\mu$. In this way the μ-dependence is absorbed in v. Clearly, if $\mu'\partial_\mu$ generates a shift of μ' along μ, ∂_v generates a shift of 1 along v.

We can fix v with the following relation:

$$v(p) = (2\pi\gamma l_P^2\sqrt{\Delta})^{-1} sgn(p)|p|^{\frac{3}{2}} \tag{2.34}$$

Where for p here we mean the eigenvalue of $\hat{p}$ on the state $|\mu\rangle$, i.e.:

$$p(\mu) = \frac{4\pi l_P^2\gamma}{3}\mu \tag{2.35}$$

Let's prove it. If we plug 2.35 inside 2.34:

$$v = \left(\frac{4\pi l_p^2\gamma}{3}\right)^{\frac{3}{2}}|\mu|^{\frac{3}{2}} sgn(\mu)(2\pi\gamma l_P^2\sqrt{\Delta})^{-1}$$

Now,

$$\frac{dv}{d\mu} = \frac{3}{2}\left(\frac{4\pi l_p^2\gamma}{3}\right)^{\frac{3}{2}}(2\pi\gamma l_P^2\sqrt{\Delta})^{-1}\sqrt{|\mu|} = \frac{1}{\sqrt{\Delta}}\sqrt{|\mu|}\left(\frac{4\pi l_P^2\gamma}{3}\right)^{\frac{1}{2}} = \sqrt{\frac{|p|}{\Delta}} = \frac{1}{\bar{\mu}(\mu)}$$

This means that: $\frac{d\mu}{dv} = \bar{\mu}(\mu)$, so:

$$\partial_v = \left(\frac{d\mu}{dv}\right)\partial_\mu = \bar{\mu}(\mu)\partial_\mu$$

The classical conjugated variable to v is $b = \frac{\bar{\mu}c}{2}$. They satisfy

$$\{b, v\} = 1 \tag{2.36}$$

Well, in this new basis $\{|v\rangle\}$, we have

$$\langle v|\,\widehat{N^{\bar{\mu}}}\,|v'\rangle = \langle v|v'+1\rangle$$

And this operator in such basis acts as $\widehat{N}^{\bar{\mu}} = e^{\frac{\partial}{\partial v}}$. From now on we'll work in the new basis $\{|v\rangle\}$. Let's see some features of such basis.
The action of $\widehat{p}$ on $|v\rangle$:

$$\widehat{p}\,|v\rangle = (2\pi\gamma l_P^2\sqrt{\Delta})^{\frac{2}{3}} sgn(v)|v|^{\frac{2}{3}}\,|v\rangle$$

The states $|v\rangle$ also form an orthonormal basis for $\mathcal{H}_{grav}$, so

$$\langle v_i|v_j\rangle = \delta_{ij}$$

Moreover the parameter v as well as p has a geometric interpretation: its absolute value is proportional to the physical volume of the cell $\mathcal{V}$; in fact:

$$\widehat{V} = \widehat{|p|^{\frac{3}{2}}} \quad , \quad \widehat{V}\,|v\rangle = 2\pi\gamma l_P^2\sqrt{\Delta}|v|\,|v\rangle \tag{2.37}$$

We can proceed with the quantization of F:

$$\widehat{F^k{}_{ab}} = \epsilon^k_{ab}\frac{\widehat{sin^2(2\bar{\mu}c)}}{\bar{\mu}^2 V_0^{\frac{2}{3}}} \tag{2.38}$$

Thus at this stage we quantized the curvature F and we found a basis of its eigenstates $|v\rangle$.
We can proceed with the quantization of the full Hamiltonian constraint, so the quanti-

zation of

$$\frac{E^a{}_j E^b{}_k}{\sqrt{|det(E)|}} = \frac{\sqrt{|p|}}{V_0^{\frac{1}{3}}}\delta^a{}_j\delta^b{}_k$$

But this is a simple task, since the quantization of p is straightforward. Then the gravitational part of the Hamiltonian constraint

$$\widehat{C}_{grav.}(N=1) = -\frac{6}{16\pi G\gamma^2\bar{\mu}^2}\widehat{\sqrt{|p|}sin^2(2\bar{\mu}c)} \tag{2.39}$$

For the moment we don't care about the operator ordering. We'll fix it later.
Let's deal now with the matter part of the hamiltonian constraint. It can be proved that the inverse of the volume can be written in the following way:

$$\left[\widehat{\frac{1}{\sqrt{|p|}}}\right] = \frac{3}{4\pi\gamma l_p^2\sqrt{\Delta}}\widehat{sgn(p)}\widehat{\sqrt{|p|}}\left(\widehat{N}_{-\bar{\mu}}\widehat{\sqrt{|p|}}\widehat{N}_{\bar{\mu}} - \widehat{N}_{\bar{\mu}}\widehat{\sqrt{|p|}}\widehat{N}_{-\bar{\mu}}\right) \tag{2.40}$$

Well, the matter Hamiltonian operator is given by

$$\widehat{C}_{matter} = -\frac{\hbar^2}{2}\widehat{|p|}^{-\frac{3}{2}}\partial_\phi^2 \tag{2.41}$$

Where the operator $\widehat{\frac{1}{V}}$ is written in abstract, while the conjugate momentum $\widehat{P_\phi}$ in the field representation. In this way we completed the construction of the quantum Hamiltonian constraint:

$$\widehat{C} = \widehat{C}_{grav} + \widehat{C}_{matter}$$

Now we need to require that $\widehat{C}$ is a self-adjoint operator, in order to generate time translations. Moreover we have to make a choice in the operator ordering. Several possibilities have been studied in literature and we make the choice called $sMMO$ (see [24] for more details). Its two main features are

- Decoupling of zero-volume states
- Decoupling of states with $|v<0\rangle$ from the ones $|v>0\rangle$.

This means that if the hamiltonian acts on states of kind $|v>0\rangle(|v<0\rangle)$ will give in general a combination of states with $v>0$ ($v<0$). The same holds for the zero-volume states, with the peculiarity that it annihilates them.
The Hamiltonian constraint with this prescription:

$$\widehat{C} = \left[\widehat{\frac{1}{V}}\right]^{\frac{1}{2}}\left(-\frac{6}{16\pi G\gamma^2}\widehat{\Omega}^2 + \frac{1}{2}\widehat{P}_\phi^2\right)\left[\widehat{\frac{1}{V}}\right]^{\frac{1}{2}} \tag{2.42}$$

Where $\widehat{\Omega}$ is defined as:

$$\widehat{\Omega} = \frac{1}{4i\sqrt{\Delta}}\widehat{|p|}^{\frac{3}{4}}\left[(\widehat{N}_{2\bar{\mu}} - \widehat{N}_{-2\bar{\mu}})\widehat{sgn(p)} + \widehat{sgn(p)}(\widehat{N}_{2\bar{\mu}} - \widehat{N}_{-2\bar{\mu}})\right]\widehat{|p|}^{\frac{3}{4}}$$

Now, since the decoupling of the states $|v\rangle$, we can study the different sectors of the Hilbert space separately, and in particular since one of the three sectors is made only by states with $|v = 0\rangle$, we can neglect it. We can do this if we assume that nowadays the Universe wavefunction has a contribution from the state $|v = 0\rangle$ that zero or very small. If so, since the time evolution generated by the Hamiltonian operator has to keep the $|v = 0\rangle$ part of the state untouched, it has to give at all times the same weight to the Universe wavefunction, that is zero or very small. Thus we can directly remove such state from the Kinematical Hilbert space, and consider the action of such quantum operators on the geometric sector of the Hilbert space $\mathcal{H}_{grav}$, defined as the Cauchy completion (with respect to the inner product $\langle v|v'\rangle = \delta_{vv'}$) of the following

$$Cyl_S = span\{|v\rangle\,; \quad v \in \mathbb{R}/\{0\}\}$$

With this prescription the *Big Bang singularity* is resolved already at the Kinematical level, since the quantum states equivalent to the classical singularity are already removed from the Kinematical Hilbert space (before looking at explicit solutions of 2.42).
Instead of working with 2.42 we can consider its densitized version

$$\widehat{\mathcal{C}} = \left[\frac{\widehat{1}}{V}\right]^{-\frac{1}{2}}\widehat{C}\left[\frac{\widehat{1}}{V}\right]^{-\frac{1}{2}} = -\frac{6}{16\pi G\gamma^2}\widehat{\Omega}^2 + \frac{1}{2}\widehat{P_\phi}^2$$

In this way both the operators $\widehat{\Omega}^2$ and $\widehat{P_\phi}^2$ commute with $\widehat{\mathcal{C}}$, since

$$\left[\widehat{\Omega}^2, \widehat{P_\phi}^2\right] = 0$$

The densitized constraint and the original one are equivalent, in the sense that their solutions are bijectively related.

2.1.5 Analysis of the Hamiltonian constraint operator

In order to find the physical solutions of the Minisuperspace we need to solve the following

$$\left[-\frac{3}{8\pi G\gamma^2}\widehat{\Omega}^2 + \frac{1}{2}\widehat{P_\phi}^2 \right] |\psi\rangle = 0 \tag{2.43}$$

Since it is an eigenvalue problem as usual we need to find a common basis of eigenstates of $\widehat{\Omega}^2$ and $\widehat{P_\phi}^2$. Let's look at these operators more in detail.
P_ϕ^2 is self-adjoint with a double degenerate continuum spectrum, and its eigenfunctions in the basis $|\phi\rangle$ are planewaves $e^{\pm i|\nu|\phi}$ with eigenvalues $(\hbar\nu)^2$. We can use such planewaves as basis for the eigenstates of P_ϕ^2. We notice that we can write a generic solution of 2.43 as

$$|\psi\rangle = |\Omega\rangle \otimes |\phi\rangle$$

With $|\Omega\rangle \otimes |\phi\rangle \in \mathcal{H}_{grav} \otimes L^2(\mathbb{R}, d\phi)$.
This because the operator is separable ($\widehat{\Omega}^2$ and $\widehat{P_\phi}^2$ commute). Let's look now at the eigenstates of $\widehat{\Omega}^2$.

Superselection sectors

The action of $\widehat{\Omega}^2$ on the states $|v\rangle$ of the kinematical sector $\mathcal{H}_{grav}$ is

$$\widehat{\Omega}^2 |v\rangle = -f_+(v)f_+(v+2)\,|v+4\rangle + \left[f_+^2(v) + f_-(v)^2\right]|v\rangle - f_-(v)f_-(v-2)\,|v-4\rangle \tag{2.44}$$

where

$$f_\pm(v) = \frac{\pi\gamma l_P^2}{2}\sqrt{|v\pm 2|}\sqrt{|v|}s_\pm(v)$$

with

$$s_\pm = sgn(v\pm 2) + sgn(v)$$

As we previously anticipated, $|v\rangle$ states are nor annihilated by $\widehat{\Omega}^2$ neither its eigenstates in general.
From 2.44 we notice that $\widehat{\Omega}^2$ is a difference operator of step 4. We also notice that

$$\begin{cases} f_-(v)f_-(v-2) = 0, & if\ \ v \in (0,4] \\ f_+(v)f_+(v+2) = 0, & if\ \ v \in [-4,0) \end{cases}$$

This means that if we have a state of kind: $|v = \varepsilon\rangle$, with $\varepsilon \in (0,4]$ the operator relates it with $|v = \varepsilon + 4\rangle$; it relates $|v = \varepsilon + 4\rangle$ with $|v = \varepsilon + 8\rangle$ and $|v = \varepsilon\rangle$, and so on. By the

other side, it relates $|v = -\varepsilon\rangle$ with $|v = -\varepsilon - 4\rangle$, $|v = -\varepsilon - 4\rangle$ with $|v = -\varepsilon - 8\rangle$ and $|v = -\varepsilon\rangle$... As we anticipated it creates two distinct sectors of the Hilbert space spanned by $|v\rangle$: the one containing states with positive v, and the one with negative v. Moreover it relates the $v's$ of the two sectors between themselves. More formally $\widehat{\Omega}^2$ relates $|v\rangle' s$ with support in a particular semilattice of step four of the form:

$$\mathcal{L}_\varepsilon^\pm = \{v = \pm(\varepsilon + 4n),\ \ n \in \mathbf{N}\} \quad ,\ \varepsilon \in (0, 4]$$

Then $\widehat{\Omega}^2$ is well defined in any of the Hilbert subspaces $\mathcal{H}_\varepsilon^\pm$ obtained as the completion of the respective domains

$$Cyl_\varepsilon^\pm = span\{|v\rangle\,,\ \ v \in \mathcal{L}_\varepsilon^\pm\}$$

With respect to the discrete inner product. So $\mathcal{H}_{grav}$ can be written as

$$\mathcal{H}_{grav} = \bigoplus_\varepsilon \left(\mathcal{H}_\varepsilon^+ \oplus \mathcal{H}_\varepsilon^-\right)$$

The action of the Hamiltonian constraint and that of the physical observables (as we'll see) preserve the spaces

$$\mathcal{H}_\varepsilon^\pm \otimes L^2(\mathbb{R}, d\phi)$$

that are called *Superselection sectors*. This means that instead of looking at the whole $\mathcal{H}_{grav} \otimes L^2(\mathbb{R}, d\phi)$ we can restrict our analysis to these spaces, for given $\varepsilon \in (0, 4]$.
The difference between our prescription $sMMO$ and other ones is that if here the sectors have support contained in semiaxis of the real line, the other ones in the whole real line. It can be proved that $\widehat{\Omega}^2$ is a self-adjoint and positive operator within $sMMO$ prescription.

Eigenfunctions of $\widehat{\Omega}^2$

As we said many times $|v\rangle$ is not in general eigenstate of $\widehat{\Omega}^2$. Let's denote with

$$|e_\lambda^\varepsilon\rangle = \sum_{v \in \mathcal{L}_\varepsilon^+} e_\lambda^\varepsilon(v)\,|v\rangle \tag{2.45}$$

the eigenstates of $\widehat{\Omega}^2$ corresponding to the eigenvalue $\lambda \in [0, +\infty)$.
It can be proved that for each λ there is a unique set of associated weights $e_\lambda^\varepsilon(v)$. In other words the spectrum of $\widehat{\Omega}^2$ is non-degenerate.

We can fix the complex norm of $|e^{\varepsilon}_{\lambda}\rangle$ by requiring that

$$\langle e^{\varepsilon}_{\lambda}|e^{\varepsilon}_{\lambda'}\rangle = \delta(\lambda - \lambda')$$

In this basis, the resolution of the identity reads

$$\mathbb{1} = \int_{\mathbb{R}^+} d\lambda \, |e^{\varepsilon}_{\lambda}\rangle \, \langle e^{\varepsilon}_{\lambda}|$$

That clearly holds for the Kinematical Hilbert space $\mathcal{H}^{+}_{\varepsilon}$.
The behaviour of such eigenfunctions $\langle v|e^{\varepsilon}_{\lambda}\rangle$, in the limit $v \to +\infty$ (so the classical limit) will allow us to understand the relation between the Loop-quantization of this symmetric model and its *Wheeler-DeWitt* quantization. Before looking at the explicit form of $e^{\varepsilon}_{\lambda}(v)$, we recall the Wheeler-DeWitt result.
In the Wheeler-DeWitt theory the analog of $\widehat{\Omega}^2$ is given by

$$\widehat{\Omega}^2_{WdW} = -\frac{\alpha^2}{4}\big[1 + 4v\partial_v + 4(v\partial_v)^2\big]$$

With α a prefactor. It is a well-defined operator on $L^2(\mathbb{R}, dv)$; it is self-adjoint and its spectrum is continuous with double degeneracy. Its eigenfunctions corresponding to $\lambda \in [0, +\infty)$, labelled with $\omega = \pm\sqrt{\lambda} \in \mathbb{R}$ (double degeneracy), are

$$e^{\omega}_{WdW}(v) = \frac{1}{\sqrt{2\pi\alpha|v|}} exp\bigg(- i\omega \frac{ln|v|}{\alpha}\bigg)$$

and provide an orthonormal basis for $L^2(\mathbb{R}, dv)$

$$\langle e^{\omega}_{WdW}|e^{\omega'}_{WdW}\rangle = \delta(\omega - \omega')$$

Let's compare such solutions with 2.45.
It can be proved that given an eigenfunction of $\widehat{\Omega}^2$:

$$\begin{pmatrix} e^{\varepsilon}_{\lambda}(\varepsilon) \\ e^{\varepsilon}_{\lambda}(\varepsilon + 4) \\ e^{\varepsilon}_{\lambda}(\varepsilon + 8) \\ . \\ . \\ . \end{pmatrix}$$

The components with large v behave in this way

$$e^{\varepsilon}_{\lambda} \quad \xrightarrow{v>>1} \quad r\{e^{i\phi_{\varepsilon}(\omega)}e^{\omega}_{WdW}(v) + e^{-i\phi_{\varepsilon}(\omega)}e^{-\omega}_{WdW}(v)\}$$

Where r is a normalization factor, and the phase $\phi_{\varepsilon}(\omega)$:

$$\phi_{\varepsilon}(\omega) = T(|\omega|) + c_{\varepsilon} + R_{\varepsilon}(|\omega|)$$

With c_{ε} a constant and $R_{\varepsilon}(|\omega|)$ such that

$$\lim_{\omega\to 0} R_{\varepsilon}(|\omega|) = 0$$

Thus we have that the components $\langle v|e^{\varepsilon}_{\lambda}\rangle$ are $\sim$ the solutions of WdW for large v. This means that the two theories are in strong agreement in the classical limit.

Physical Hilbert space

Now we can construct an explicit solution for the Hamiltonian quantum constraint 2.43. There are different approaches to solve this equation, and all of them give the same result:

$$\psi^{\varepsilon}(v,\phi) = \int_0^{+\infty} d\lambda e^{\varepsilon}_{\lambda}(v)\big[\tilde{\psi}_{+}(\lambda)e^{i\nu(\lambda)\phi} + \tilde{\psi}_{-}(\lambda)e^{-i\nu(\lambda)\phi}\big] = \psi^{\varepsilon}_{+}(v,\phi) + \psi^{\varepsilon}_{-}(v,\phi) \qquad (2.46)$$

where

$$\nu(\lambda) = \sqrt{\frac{3\lambda}{4\pi l_P^2 \hbar\gamma^2}} \qquad (2.47)$$

and where $\tilde{\psi}_{\pm}$ are fixed by the initial datum. We notice that both the geometric and scalar part of this solution are combinations of all the possible eigenfunctions respectively of $\widehat{\Omega}^2$ and $\widehat{P}^2_{\phi}$. Here a sort of superposition principle holds, and this is due to the fact that if $|\psi\rangle$, $|\psi'\rangle$ are solutions of the Hamiltonian constraint, the same holds for their sum. This solution is clearly written in the basis $\{|v\rangle \otimes |\phi\rangle\}$.
The relation 2.47 is fixed by requiring $\widehat{C}\,|\psi\rangle = 0$. The physical inner product

$$\langle\psi_1|\psi_2\rangle_{phys} = \int_0^{+\infty} d\lambda\big[\tilde{\psi}^{*}_{1+}(\lambda)\tilde{\psi}_{2+}(\lambda) + \tilde{\psi}^{*}_{1-}(\lambda)\tilde{\psi}_{2-}(\lambda)\big]$$

2.1.6 Problem of time and Physical observables

As we know from classical General Relativity in the Ashtekar-Barbero formulation physical solutions live on the constraint surface

$$\begin{cases} \mathcal{H}_0 \approx 0 \\ \mathcal{H}_a \approx 0 \\ \mathcal{G}_i \approx 0 \end{cases}$$

In particular the physical evolution is generated by the Hamiltonian, that is given by the sum of (constraints) $\cdot$ (lagrange multipliers), since the system is generally covariant. As we said many times the same object, the Hamiltonian, generates also gauge transformations in the time direction, producing what we technically call *gauge orbits* on the constraint surface in which physical solutions live. To the same gauge orbit belong different configurations of the field (classically speaking) that represent the same physical solution written in different coordinate systems (in the passive interpretation of the diffeomorphisms). This means that each physical evolution in the constraint surface can be mimicked by a gauge transformation in the same direction, and we call with "problem of time" this ambiguity. A way to solve this problem is through the gauge fixing of the solution after quantization, and we do it considering as Time the mean value of an operator on physical states. In the model we are studying a natural choice is fixing $\langle\hat{\phi}\rangle = t$: we consider the mean value of the scalar field operator $\hat{\phi}$ as a clock in our Universe, and only an its variation can be seen as a time variation. This gauge-fixing procedure is called *Time after Quantization*. Let's describe in this setup the time evolution of physical states.

First of all we consider separately positive and negative frequency states. Then, fixing the initial datum $\psi_\pm(v, \phi_0)$ we write its unitary evolution:

$$\psi^\varepsilon_\pm(v, \phi) = U_\pm(\phi - \phi_0)\psi^\varepsilon_\pm(v, \phi_0)$$

Where

$$U_\pm(\phi - \phi_0) = exp\Big[\pm i\sqrt{\frac{3}{4\pi l_P^2 \hbar\gamma^2}\widehat{\Omega}^2}(\phi - \phi_0)\Big]$$

Is the quantum time evolution operator if we fix time before quantization.

In this way we can define Dirac observables, that at the classical level we recall that are defined as quantities that Poisson-commute with all the first class constraints on the

constraint surface.
Let's start with v. Classically we know that $v(t)$ is not a constant of motion, so it cannot be a Dirac observable, since has non vanishing Poisson brakets with the Hamiltonian, the generator of time evolution. However, if we look at $v(\phi)\big|_{\phi=\phi_0}$ after the gauge fixing, it can be treated as a Dirac observable. This because at $\phi = \phi_0$ fixed, the quantity $v(\phi_0)$ doesn't evolve in time. This quantity classically measures the volume at time ϕ_0.
The quantum analog of the observable $v(\phi_0)$ is

$$\hat{v}\big|_{\phi_0}\psi^{\varepsilon}(v,\phi) = U_+(\phi-\phi_0)\big[v\psi^{\varepsilon}_+(v,\phi_0)\big] + U_-(\phi-\phi_0)\big[v\psi^{\varepsilon}_-(v,\phi_0)\big] \tag{2.48}$$

Let's analyze the previous equation. Given $\psi^{\varepsilon}(v,\phi)$ the action of this operator on such state is

1. Decomposing the solution in its positive and negative components.

2. Freezing them at the initial time $\phi = \phi_0$.

3. Multiplying $\psi^{\varepsilon}_{\pm}$ by v, that is physically the volume at time ϕ_0.

4. Evolving them through U.

We notice that $\widehat{v}\big|_{\phi}$ preserves not only the superselection sectors, but also the subspaces of positive and negative frequencies. Thus any of these subspaces (positive or negative frequency) provide an irreducible representation of he observable algebra, and the analysis can be restricted to the positive frequency sector.
The operator $\widehat{v}\big|_{\phi}$ allows to analyze the physical results during their evolution. We'll do it for semiclassical states. In the next section we look at the qualitative behaviour of the solution at the Planck scale, and compare it with the WdW behaviour.

2.1.7 Resolution of the singularity: the Quantum Bounce

In the classical theory, when the volume of the Universe vanishes ($a \to 0$), the energy density diverges, leading to a physical singularity. In the Wheeler-DeWitt theory, making the gauge fixing choice $\phi = t$, if we start with a solution that at $\phi = \phi_0$ is sharply peaked on a certain value $v = v^*$, and we look at the time evolution of such state, we obtain that

the wavefunction $\psi_{WdW}(v,\phi)$ is peaked on the classical trajectory

$$\phi = \sqrt{\frac{1}{12\pi G}} ln\left(\frac{|v|}{|v^*|}\right) + \phi_0 \tag{2.49}$$

for each time, and in particular in the Planck regime, until the physical singularity. In this sense the WdW theory doesn't resolve the singularity problem. The behaviour described by 2.49 is what we expect from a quantum solution only in the epoch for which the energy density of the Universe is much smaller than the Planck density (the semiclassical regime). In fact in such regime the wavefunction is peaked on the classical value, and for the Ehrenfest theorem we have $< \widehat{v} > \sim v_{cl}$ at each time. The behaviour described by 2.49 is correctly shared by the Loop-solution in the semiclassical regime, as we previously anticipated. What drastically changes in this Minisuperspace of Loop Quantum Cosmology is the behaviour of the solution *at the Planck scale*. The peak in the volume of our LQC solution starts to diverge from the classical value when we get closer the Planck regime (backward in time). In particular, the mean value of $\widehat{v}$ on a generic state (that we can assume for simplicity with positive frequency) when $\phi \to 0$ doesn't reach $v\big|_\phi = 0$ but stops at a certain v_{min} and then starts to enlarge.
This is the qualitative behaviour of the solutions in our Minisuperspace at the Planck scale, and is completely *independent* on the initial datum $\psi(v_0, \phi_0)$.
To fix ideas we can consider a physical profile for the initial datum given by a logarithmical normal distribution

$$\tilde{\psi}_+(\lambda) = \frac{1}{(2\pi)^{\frac{1}{4}}\sqrt{\sigma\lambda}} exp\left\{ - \frac{\left[ln\left(\frac{\lambda}{\lambda_0}\right)\right]^2}{4\sigma^2} \right\}$$

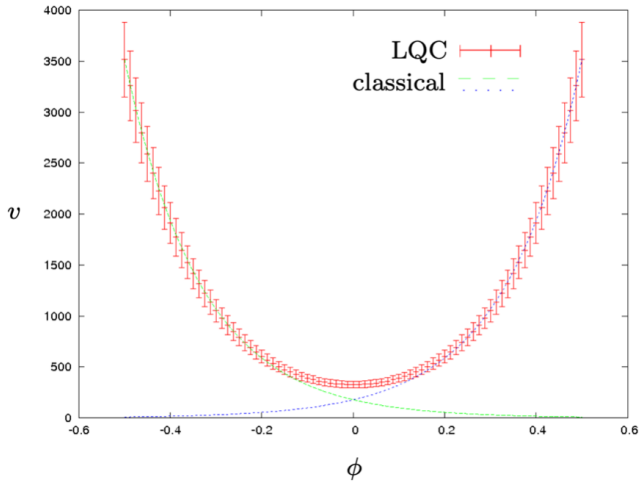
So the total wavefunction for positive frequency modes

$$\psi(v,\phi)\big|_{\phi_0} = \int_0^{+\infty} d\lambda \frac{1}{(2\pi)^{\frac{1}{4}}\sqrt{\sigma\lambda}} exp\left\{ - \frac{\left[ln\left(\frac{\lambda}{\lambda_0}\right)\right]^2}{4\sigma^2} \right\} \cdot e^{\varepsilon}_{\lambda}(v) e^{i\nu(\lambda)\phi}$$

where λ and σ are linked to $\langle\widehat{P_\phi}\rangle$ and $\Delta\widehat{P_\phi}$ by the relations

$$\langle\widehat{P_\phi}\rangle = \sqrt{12\pi G}\lambda_0 e^{-\frac{\sigma^2}{2}} \qquad \frac{\Delta\widehat{P_\phi}}{\langle\widehat{P_\phi}\rangle} = \sqrt{e^{\sigma^2} - 1}$$

For this kind of initial datum, we show the numerical plot made by J.Olmedo in fig. 2.1. It is shown the LQC solution compared with the classical one, for an expanding and contracting FRW solution [9]. We show in particular the expectation value $v = \langle\widehat{v}\rangle$ with respect to ϕ. As we can see from the plot the LQC solution reaches a minimum in



$$\varepsilon = 1$$

$$\langle \widehat{P_\phi} \rangle = 1000$$

$$\frac{\Delta \widehat{P_\phi}}{\langle \widehat{P_\phi} \rangle} = 0.1$$

Figure 2.1

volume that can be interpreted as a *bridge* from a contracting branch of the Universe to an expanding one.

We notice from the plot that the bounce occurs for $v \sim 400$, that means $V_{Bounce} >> l_P^3$. The fact that the wavefunctions doesn't reach the Planck volume doesn't mean that at each time it is outside the Planck regime. In fact if we plot the energy density ρ of the scalar field in terms of ϕ we obtain that at the bounce $\rho_{Bounce} \sim \rho_{crit}$, where $\rho_{crit} = \frac{3}{8\pi G\gamma^2\Delta}$. If one assumes $\gamma \approx 0.238$, usually used in LQC literature and that comes from the computation of non-rotating black holes isolated horizons, then $\rho_{crit} \sim 0.41\rho_P$, and it is coherent with what we expect for the energy density scale at which such phenomenon arises. Moreover it doesn't mean that the Universe doesn't reach the Planck volume at all. It means simply that if we start at the initial time with a wavefunction peaked on a large-volume state, it never reaches the Planck volume. To better understand this point we have to remember that we constructed our quantum theory starting from the classical formulation in terms of Ashtekar-Barbero variables, in which we fixed a certain value of the fiducial cell $\mathcal{V}$. The freedom in this choice as we previously said comes from the homogeneity of the model. Taking in the quantum theory an initial state with large V means choosing for such state a large fiducial cell $\mathcal{V}$. Thus the time evolution of such wavefunction doesn't represent the time evolution of the entire Universe. It represents instead the time evolution of a state that contains a certain number (large) of degrees of freedom of the gravitational field. If by the other side we start with a sharply-peaked initial state with small volume V, so we consider a small fiducial cell $\mathcal{V}$, the evolution of such state will be completely different (and more complicate) than the one shown in the plot, and it reaches at a certain point of the evolution the Planck volume with a density

that can be different from ρ_{crit} (usually a bit smaller) but that has ρ_{crit} as upper bound. There are plots that we don't show here that confirm this statement.

This means that in the deep Planck regime ($\rho_{Bounce} \sim \rho_{crit}$) the solution bounces, but the minimum volume it reaches depends on the portion of Universe we are considering. The apparent ambiguity of the different values V_{Bounce} comes from the fact that as we previously said in this model the volume of the Universe is at each time really infinite, also at the bounce, and what we can study is how finite portions of such Universe evolve and behave at the bounce.

As we can also see from the plot, near the bounce the expectation value approaches the classical one very rapidly, so that once we go away from the Planck region the semiclassical limit of the theory agrees with the classical $k = 0$ FRW as desired.

We conclude this section showing another plot [17] of the solution of such Minisuperspace (fig. 2.2) where on the z axis we have the modulus square of the wavefunction, while in the

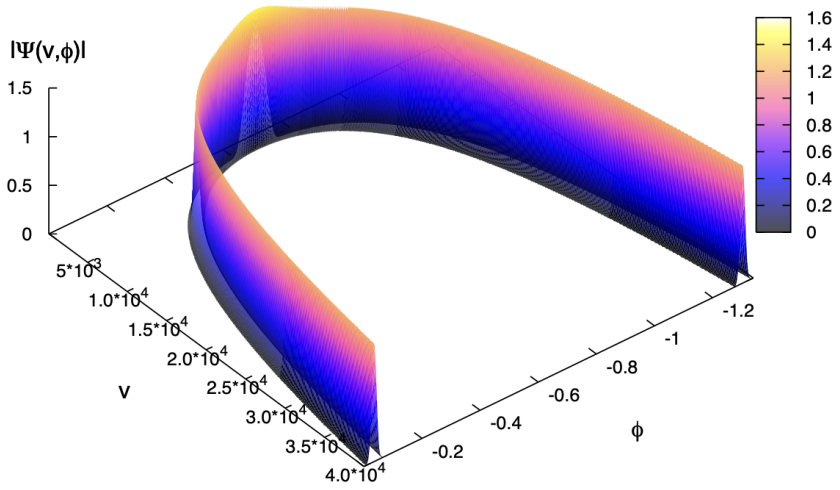


Figure 2.2

x and y axis respectively the eigenvalue of the volume and the time ϕ. Different colours describe the different values of $|\psi|^2$, and for clarity of visualization are represented only if $|\psi|^2 > 10^{-4}$. The plot is performed with the following data: $\varepsilon = 2$, $\langle \widehat{P}_\phi \rangle = 5 \cdot 10^3$ and $\frac{\Delta P_\phi}{\widehat{P}_\phi}$=0.025. After this qualitative description of such numerical solutions we proceed now introducing a powerful but approximate technique used in Loop Quantum Cosmology, and in particular in this Minisuperspace, that allows to describe well the dynamics of the solution for some particular initial states without solving analytically 2.43. Such technique brings to the so called *effective dynamics* of LQC.

2.1.8 Effective dynamics for flat FRW

The effective dynamics is a poweful tool that allows to study the solution of a model in LQC in an approximate but very accurate way, and is based on the analysis of the action of the quantum Hamiltonian constraint on large-volume states that are *at each time* sharply peaked on classical values of physical observables. We'll see in the next paragraph what does this assumption of sharply peaked states physically means. The equation of motion that arises from this approach is the classical Friedmann equation modified with quantum corrections, and are derived with the assumption that quantum fluctuations on such sharply peaked states are small. We start by considering the quantum Hamiltonian constraint we constructed in the previous paragraph

$$\widehat{C} = -\frac{6}{16\pi G\gamma^2\bar{\mu}^2}\widehat{\sqrt{|p|}sin^2(2\bar{\mu}c)} + \frac{1}{2}\frac{\widehat{1}}{V}\widehat{P_\phi}^2 \tag{2.50}$$

To reach the effective Hamiltonian constraint we evaluate such operator on states $|\psi_{SC}\rangle$ peaked at each time on some point of the phase space (b, v). If we assume that the volume V of such state is large enough at each time ($V >> l_P^3$, $v >> 1$), in such a way that the inverse volume corrections are negligible, we obtain:

$$\langle\psi_{SC}|\,\widehat{C}\,|\psi_{SC}\rangle = C_{eff.} \sim -\frac{6}{16\pi G\gamma^2}\sqrt{|p|}\frac{sin(\bar{\mu}c)^2}{\bar{\mu}^2} + |p|^{\frac{3}{2}}\rho \tag{2.51}$$

With

$$\rho = \frac{P_\phi^2}{2|p|^3} \tag{2.52}$$

the energy density of the massless scalar field ϕ.

Remark: the assumption $V >> l_P^3$ means that we are considering physical states governed by the effective Hamiltonian constraint 2.51 that at each time have a large volume. We notice that it is coherent with what we found in the plot of the numerical solution of the exact equations.

We notice also that to reach the effective Hamiltonian we don't need to care about the operator ordering in the quantum Hamiltonian constraint. This because we study its action on sharply-peaked states, and this makes the operator ordering we choose irrelevant. Well, firstly we notice that in the classical limit, so $\bar{\mu}c \to 0$

$$\langle\psi_{SC}|\,\widehat{C}\,|\psi_{SC}\rangle \quad \longrightarrow \quad -\frac{6}{16\pi G\gamma^2}\sqrt{|p|}c^2 + \frac{P_\phi^2}{2V} = C_{classic} \tag{2.53}$$

we recover the classical Hamiltonian constraint, as we expect.
In general the evolution of semiclassical states outside the semiclassical regime produce fluctuations that bring quantum corrections to 2.51. More we get off the semiclassical limit, larger such corrections are. This means that in principle

$$-\frac{6}{16\pi G\gamma^2}\sqrt{|p|}\frac{sin(\bar{\mu}c)^2}{\bar{\mu}^2}+|p|^{\frac{3}{2}}\rho=0 \tag{2.54}$$

doesn't hold at any times for physical states, and in particular approaching the Planck regime. Let's suppose for the moment however that these quantum fluctuations are negligible at each time. From 2.51 we can construct Hamilton equations of motion. The one for the densitized triad, described by p, gives the Hubble parameter

$$H=\frac{sin(2\bar{\mu}c)}{2\gamma V_0^{\frac{1}{3}}\bar{\mu}} \tag{2.55}$$

As we did for the Hamiltonian constraint, we can check the robustness of our procedure taking the classical limit: $\bar{\mu}\to 0$, then

$$H\longrightarrow\frac{\dot{a}}{a}$$

We want to write 2.54 in terms of 2.55. We start by writing

$$sin^2(\bar{\mu}c)=sin^4(\bar{\mu}c)+sin^2(\bar{\mu}c)cos^2(\bar{\mu}c)=sin^4(\bar{\mu}c)+\frac{sin^2(2\bar{\mu}c)}{4} \tag{2.56}$$

We write the first term using 2.54 in terms of ρ, and we plug 2.55 in the second one

$$sin^2(\bar{\mu}c)=\left(\frac{16\pi G|p|\rho\bar{\mu}^2\gamma^2}{6}\right)^2+H^2\gamma^2a^2\bar{\mu}^2V_0^{\frac{2}{3}} \tag{2.57}$$

Plugging the previous result inside 2.54 we obtain

$$\begin{aligned}-\frac{6}{\gamma^2}\left(\frac{16\pi G|p|\rho\bar{\mu}\gamma^2}{6}\right)^2-\frac{6H^2a^2}{V_0^{\frac{2}{3}}}+16\pi G|p|\rho=0\quad\Longrightarrow\\ \Longrightarrow H^2=\frac{8}{3}\pi G\rho\left(1-\frac{8\pi G|p|\rho\bar{\mu}^2\gamma^2}{3}\right)=\frac{8}{3}\pi G\rho\left(1-\frac{\rho}{\rho_{crit.}}\right)\end{aligned} \tag{2.58}$$

with

$$\rho_{crit.}=\frac{3}{8\pi G|p|\bar{\mu}^2\gamma^2}=\frac{3}{8\pi G\gamma^2\Delta}$$

is called *critical density*, and is constant.

Thus

$$H^2 = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_{crit.}}\right) \tag{2.59}$$

As we can see explicitally in 2.59 also in the effective dynamics we avoid the singularity: when the energy density of the scalar field reaches the critical value $\rho = \rho_{crit.}$($H = 0$), the Universe stops to contract, bounces and starts to expand. Thus the bouncing behaviour of the numerical simulation is confirmed by the effective dynamics.

The solution of 2.59 is a sharply peaked state on a trajectory that initially coincides with the classical one (contracting Universe with $k = 0$) going toward decreasing values of v; at a certain point it starts to diverge from the classical trajectory and instead of collapsing in a singularity it reaches a minimum in the value v of the phase space corresponding to a maximum in the energy density of the scalar field $\rho_{crit.}$, and then symmetrically starts to re-expand. In the expanding phase, at late times matches the classical solution of an expanding Universe ($k = 0$). Moreover the volume V_{bounce} reached by the solution as we'll show in the next paragraph is much larger than the Planck volume, coherently with the ansatz we assumed to build the effective Hamiltonian. The divergence from the classical behaviour is due to the presence of quantum corrections at the leading order in the effective Hamiltonian, in particular the dependence of the constraint from c through the holonomy. It can be shown that the effective solutions reproduce extremely well the numerical simulations for the same sharply-peaked initial states. We have to ask why this happens, since even if also the volume of the numerical solution doesn't reach the Planck volume, the energy density does and we should expect that quantum corrections to 2.51 arise in the Planck regime, so that the effective results should diverge from the numerical ones. The accuracy of the effective equations show that also the numerical (thus the exact) solution for such sharply peaked large-volume initial states is unaffected by quantum fluctuations, even if it reaches the Planck regime. In the next paragraph we'll give a mathematical proof and a physical interpretation of the previous statement.

2.1.9 Accuracy of the effective equations for the FRW Minisuperspace

In this section we show why the effective equations we built in the FRW Minisuperpace are accurate, and represent well the physical dynamics of sharply peaked large-volume quantum states also in the Planckian regime, where we expect that quantum corrections

become relevant. We'll prove that the explaination of this lies in the fact that for sharply peaked states of this model the Heisemberg uncertainty relations can be made arbitrarily small at *each time*. This means that quantum fluctuations for such states never become important, also in the Planckian regime, and the equation 2.59 that governs their motion doesn't acquire $\hbar$-corrections in such regime.
Let's start looking at fluctuations in the Wheeler-DeWitt theory. We can consider as classical phase space variables the following

$$\begin{cases} A^i_a = \tilde{c}(t)\delta^i_a = \gamma\dot{a}(t)\delta^i_a \\ E^a_i = \tilde{p}(t)\delta^a_i = a(t)^2\delta^a_i \end{cases} \tag{2.60}$$

where me made the same choices of the previous analysis for the fiducial triad and cotriad (${}^o e^a{}_i = \delta^a{}_i$) and for the lapse ($N = 1$) , while a different choice for the definition of c and p: we absorbed the powers of V_0 inside them. The relations with the variables c, p: $\tilde{c} = cV_0^{-\frac{1}{3}}$, $\tilde{p} = pV_0^{-\frac{2}{3}}$. We also assumed that $sgn(p) = 1$, that means e^i_a and ${}^o e^i_a$ parallel. With this choice the symplectic structure becomes

$$\{\tilde{c}(t), \tilde{p}(t)\} = \frac{8\pi G\gamma}{3V_0} \tag{2.61}$$

We notice here, even if we don't prove it explicitally that the dynamics as before is not affected by the choice of the fiducial cell, since with this choice appear a factor V_0 in the classical hamiltonian constraint 2.13, in such a way that the Poisson brakets between a variable and the constraint produce terms V_0-independent. Let's apply the Dirac quantization program and promote the relation 2.61 to its quantum version

$$\big[\,\widehat{\tilde{c}}, \widehat{\tilde{p}}\,\big] = i\frac{8\pi G\hbar\gamma}{3V_0} \tag{2.62}$$

Using now the Robertson inequality for two observables:

$$\Delta A\Delta B \geq \frac{1}{2}\big|\langle\big[\widehat{A}, \widehat{B}\big]\rangle\big| \tag{2.63}$$

on a generic state $|\psi\rangle$ we obtain

$$\Delta\tilde{c}\Delta\tilde{p} \geq \frac{4\pi G\hbar\gamma}{3V_0} \tag{2.64}$$

Where we notice the dependence at the denominator from V_0. This means that the choice of the fiducial cell $\mathcal{V}$, and its correspondent fiducial volume affects the quantum states. In fact if now we send $V_0 \to +\infty$, the product of the uncertanties has 0 as lower bound. In this case, we can have at all times states arbitrarily peaked both in the $\tilde{p}$ and $\tilde{c}$ variables. We can produce the same considerations to states in LQC, and this explains why the effective equations are so accurate: the uncertainties for peaked states don't spread necessarily during time evolution, and in particular near the bounce. Let's follow now the same argument in the LQC case. In order to do it is convenient to make a change of variables with respect to the usual ones

$$\begin{cases} \beta = \frac{\tilde{c}}{\sqrt{\tilde{p}}} \\ \tilde{V} = \tilde{p}^{\frac{3}{2}} \end{cases} \tag{2.65}$$

Where we notice here that $\tilde{V}$ is not the physical volume of the fiducial cell $\mathcal{V}$, since p doesn't contain V_0 anymore. We called it $\tilde{V}$ to distinguish it from the physical volume $V = V_0\tilde{V}$.
Their Poisson brakets

$$\{\beta, \tilde{V}\} = \frac{4\pi G\gamma}{V_0} \tag{2.66}$$

The proof of the previous relation is straightforward: we start from 2.61 and we have

$$\{\beta, \tilde{V}\} = \frac{8\pi G\gamma}{3V_0}\left\{\left(\frac{\partial}{\partial\tilde{c}}\frac{\tilde{c}}{\sqrt{\tilde{p}}}\right)\left(\frac{\partial}{\partial\tilde{p}}\tilde{p}^{\frac{3}{2}}\right)\right\} = \frac{4\pi G\gamma}{V_0} \tag{2.67}$$

In order to Loop-quantize this model we introduce as before the holonomy, so the smearing of the Ashtekar connection along a straight path: $exp(i\lambda\beta)$.
Here we consider directly an holonomy with minimum physical lenght $\sqrt{\Delta}$: $\lambda = V_0^{\frac{1}{3}}\bar{\mu}\sqrt{\tilde{p}} = \sqrt{\Delta}$. This choice implements authomatically the improved dynamics prescription when we use such holonomy for the Hamiltonian constraint.With these variables in hand we can quantize the classical phase space. Let's construct the commutator between them. Recalling the Dirac quantization rule

$$\big[\widehat{exp(i\beta\lambda)}, \hat{\tilde{V}}\big] = i\hbar\big\{\widehat{exp(i\beta\lambda)}, \tilde{V}\big\} \tag{2.68}$$

Now,

$$\big\{exp(i\beta\lambda), \tilde{V}\big\} = \frac{4\pi G\gamma}{V_0}exp(i\beta\lambda)i\lambda \tag{2.69}$$

Thus

$$\big[\widehat{exp(i\beta\lambda)}, \widehat{\tilde{V}}\big] = -\hbar\frac{4\pi G\lambda\gamma}{V_0}\widehat{e^{i\beta\lambda}} \tag{2.70}$$

As usual we can construct a basis $\{|\tilde{V}\rangle\}$ for the Kinematical Hilbert space, with $\langle\tilde{V}_1|\tilde{V}_2\rangle = \delta_{\tilde{V}_1\tilde{V}_2}$. The action of the operators $\widehat{\tilde{V}}$ and $\widehat{exp(il\beta)}$ on such basis

$$\widehat{\tilde{V}}\,|\tilde{V}\rangle = \tilde{V}\,|\tilde{V}\rangle \tag{2.71}$$

$$\widehat{exp(i\lambda\beta)}\,|\tilde{V}\rangle = |\tilde{V} + \frac{4\pi G\hbar\gamma\lambda}{V_0}\rangle \tag{2.72}$$

we can easily recover the second result using 2.70 and 2.71:

$$\begin{aligned}
&\big[\widehat{exp(i\beta\lambda)}, \widehat{\tilde{V}}\big]\,|\tilde{V}\rangle = -\hbar\frac{4\pi G\gamma}{V_0}\widehat{e^{i\beta\lambda}}\,|\tilde{V}\rangle \quad \Longrightarrow \\
\Longrightarrow \quad &\widehat{exp(i\beta\lambda)}\widehat{\tilde{V}}\,|\tilde{V}\rangle - \widehat{\tilde{V}}\widehat{exp(i\beta\lambda)}\,|\tilde{V}\rangle = -\hbar\frac{4\pi G\gamma}{V_0}\widehat{exp(i\beta\lambda)}\,|\tilde{V}\rangle \quad \Longrightarrow \\
\Longrightarrow \quad &\widehat{\tilde{V}}\widehat{exp(i\beta\lambda)}\,|\tilde{V}\rangle = \left(\tilde{V} + \frac{4l\pi\hbar G\gamma}{V_0}\right)\widehat{exp(i\beta\lambda)}\,|\tilde{V}\rangle
\end{aligned}$$

which means

$$\widehat{exp(i\lambda\beta)}\,|\tilde{V}\rangle = |\tilde{V} + \frac{4\pi G\hbar\gamma\lambda}{V_0}\rangle$$

In these variables the effective Hamiltonian constraint of the system, written in terms of holonomies of minimal physical lenght $\lambda = \sqrt{\Delta}$ is:

$$\langle\psi_{SC}|\,\widehat{C}\,|\psi_{SC}\rangle \sim -\frac{3V_0\tilde{V}}{8\pi G\gamma^2\lambda^2}sin^2(\lambda\beta) + V_0\tilde{V}\rho \approx 0 \tag{2.73}$$

with the usual Improved dynamics prescription implemented to define the curvature tensor F at the quantum level. As before we neglect the inverse volume corrections in the matter term, assuming large V. We notice that 2.73 is 2.51 written in different variables, and the same for its classical limit. Let's look at the matter term. In this section we generalize the source of the gravitational field to a scalar field ϕ considered as perfect fluid with a constant equation of state. The continuity equation of the perfect fluid in quantum average coincides with the classical one for this effective theory

$$\frac{d\rho}{dt} + \frac{1}{\tilde{V}}\frac{d\tilde{V}}{dt}(\rho + p) = 0 \tag{2.74}$$

And the equation of state

$$p = \omega\rho \tag{2.75}$$

with $-1 \leq \omega \leq 1$ constant. This means

$$\rho = \frac{\rho_0}{\tilde{V}^n} \tag{2.76}$$

with $0 \leq n \leq 2$. In fact, from the continuity equation

$$\frac{d\rho}{dt} + \frac{1}{a}\rho(1+\omega)\dot{a} = 0 \quad \Longrightarrow \quad \frac{d\rho}{\rho} = -\frac{3}{a}(1+\omega)da \tag{2.77}$$

And integrating both the members

$$\rho = \frac{\rho_0}{\tilde{V}^{1+\omega}} \tag{2.78}$$

If we call $n = 1 + \omega$ we recover 2.76. We notice that ρ and ρ_0 are independent on V_0. If now we evaluate the product between the uncertainties $\Delta\tilde{V}\Delta\big(\frac{sin(\lambda\beta)}{\lambda}\big)$ we have

$$\Delta\tilde{V}\Delta\left(\frac{sin(\lambda\beta)}{\lambda}\right) \geq \frac{2\pi G\hbar\gamma}{V_0}\big|\langle\widehat{cos(\lambda\beta)}\rangle\big| \tag{2.79}$$

In order to prove it we need to use 2.63, and in particular evaluate its right-hand side applied to this specific case:

$$\begin{aligned}
&\frac{1}{2}\left|\langle\left[\widehat{\tilde{V}}, \widehat{\frac{sin(\beta\lambda)}{\lambda}}\right]\rangle\right| = \frac{1}{2}\left|\langle\left[\widehat{\tilde{V}}, \widehat{\frac{exp(i\beta\lambda) - exp(-i\beta\lambda)}{2i\lambda}}\right]\rangle\right| = \\
=&\frac{1}{2}\left|\langle\left[\widehat{\tilde{V}}, \widehat{\frac{exp(i\beta\lambda)}{2i\lambda}}\right]\rangle - \langle\left[\widehat{\tilde{V}}, \widehat{\frac{exp(-i\beta\lambda)}{2i\lambda}}\right]\rangle\right| = \frac{2\pi G\hbar\gamma}{V_0}\left|\langle\widehat{cos(i\beta\lambda)}\rangle\right|
\end{aligned} \tag{2.80}$$

Where in the last passage we used 2.70. With this relation in hand we can evaluate, at fixed V_0 when the fluctuations of $\tilde{V}$ and $\frac{sin(\beta\lambda)}{\lambda}$ for the semiclassical state become important. By the other side from the Hamiltonian constraint we can derive what is the $\tilde{V}$ of the semiclassical state for which the bounce happens, and we can compare these two values. To do this we need the equation of motion for $\tilde{V}$. We can use the Heisemberg formula

$$\frac{d\widehat{\tilde{V}}}{dt} = \frac{i}{\hbar}[\widehat{C}, \widehat{\tilde{V}}] \tag{2.81}$$

with $\widehat{C}$ given by

$$\widehat{C} = -\frac{3\widehat{\tilde{V}}V_0}{8\pi G\gamma^2}\widehat{\frac{sin(\lambda\beta)}{\lambda}}\widehat{\frac{sin(\lambda\beta)}{\lambda}} + V_0\widehat{\tilde{V}}\widehat{\rho} \tag{2.82}$$

Thus we have

$$\begin{aligned}\frac{i}{\hbar}[\widehat{C},\widehat{\tilde{V}}] =&\frac{i}{\hbar}\bigg[-\frac{3V_0\widehat{\tilde{V}}}{8\pi G\gamma^2}\widehat{\frac{sin(\lambda\beta)}{\lambda}}\widehat{\frac{sin(\lambda\beta)}{\lambda}},\widehat{\tilde{V}}\bigg] = -\frac{3iV_0\widehat{\tilde{V}}}{8\pi G\gamma^2\hbar}\bigg[\widehat{\frac{sin(\lambda\beta)}{\lambda}}\widehat{\frac{sin(\lambda\beta)}{\lambda}},\widehat{\tilde{V}}\bigg]\\ =&-\frac{3iV_0\widehat{\tilde{V}}}{8\pi G\gamma^2\hbar}\bigg(\bigg[\widehat{\frac{sin(\lambda\beta)}{\lambda}},\widehat{\tilde{V}}\bigg]\widehat{\frac{sin(\beta\lambda)}{\lambda}}+\widehat{\frac{sin(\lambda\beta)}{\lambda}}\bigg[\widehat{\frac{sin(\beta\lambda)}{\lambda}},\widehat{\tilde{V}}\bigg]\bigg)\\ =&\frac{3\widehat{V}}{2\gamma}\bigg(2\widehat{cos(\lambda\beta)}\widehat{\frac{sin(\beta\lambda)}{\lambda}}\bigg)\end{aligned} \tag{2.83}$$

If we evaluate such relation as average on semiclassical states we obtain finally

$$\langle\frac{d\widehat{\tilde{V}}}{dt}\rangle = \frac{d\tilde{V}}{dt} = \frac{3\tilde{V}}{\gamma\lambda}cos(\lambda\beta)sin(\lambda\beta) \tag{2.84}$$

We notice that we can obtain the same result if we evaluate the classical Poisson brakets between the Hamiltonian constraint 2.73 and $\tilde{V}$, so the Hamilton equations for $\tilde{V}$, since we are considering averages on sharply-peaked states.
Let's solve 2.84. We firstly square it, and then we plug the Hamiltonian constraint 2.73. We have

$$\begin{aligned}\bigg(\frac{d\tilde{V}}{dt}\bigg)^2 =&\frac{9\tilde{V}^2}{\gamma^2\lambda^2}sin^2(\lambda\beta)cos^2(\lambda\beta) = \frac{9\tilde{V}^2}{\gamma^2\lambda^2}sin^2(\lambda\beta)\big[1-sin^2(\lambda\beta)\big] =\\ =&\frac{24\pi G\tilde{V}^2\rho_0}{\tilde{V}^{2n}}\bigg(\tilde{V}^n-\frac{\rho_0}{\rho_{crit}}\bigg)\end{aligned} \tag{2.85}$$

Where $\rho_{crit}\equiv\frac{3}{8\pi G\gamma^2\lambda^2}$. Now we square both the members and we integrate

$$\frac{d\tilde{V}\cdot\tilde{V}^{n-1}}{\sqrt{24\pi G\rho_0}\sqrt{\tilde{V}^n-\frac{\rho_0}{\rho_{crit}}}} = dt \quad\Longrightarrow\quad \frac{2}{n\sqrt{24\pi G\rho_0}}\bigg(\tilde{V}^n-\frac{\rho_0}{\rho_{crit}}\bigg)^{\frac{1}{2}} = t-t_0 \quad\Longrightarrow$$

$$\Longrightarrow\quad \bigg(\tilde{V}^n-\frac{\rho_0}{\rho_{crit}}\bigg)^{\frac{1}{2}} = n^2 6\pi G\rho_0(t-t_0)^2 \quad\Longrightarrow \tag{2.86}$$

$$\Longrightarrow\quad \tilde{V}(t) = \bigg(6\pi G\rho_0 n^2(t-t_0)^2+\frac{\rho_0}{\rho_{crit}}\bigg)^{\frac{1}{n}} \tag{2.87}$$

in the derivation we assumed $\frac{d\tilde{V}}{dt}\geq 0$, and that $\tilde{V}(t_0) = \big(\frac{\rho_0}{\rho_{crit}}\big)^{\frac{1}{n}}$. The minimum of this function is for $\tilde{V}(t) = \big(\frac{\rho_0}{\rho_{crit}}\big)^{\frac{1}{n}}$, that means $t = t_0$. Clearly we obtain the same result if we choose a different initial datum. Thus we have

$$\tilde{V}_{bounce} = \bigg(\frac{\rho_0}{\rho_{crit}}\bigg)^{\frac{1}{n}} \tag{2.88}$$

We notice that $\tilde{V}_{bounce}$ is independent on the fiducial volume V_0.
Now we have to determine for what value of $\tilde{V}$ the fluctuations become important, and compare it with the result we obtained for the bounce.
From the Hamiltonian constraint 2.73 we have

$$\frac{sin(\lambda\beta)}{\lambda} = \sqrt{\frac{\rho_0}{\rho_{crit}}}\frac{1}{\tilde{V}^{\frac{n}{2}}} \tag{2.89}$$

Now, using the following relation

$$\Delta[(f(x))] \sim \Delta(x)\left|\partial_x f(x)\right| \tag{2.90}$$

valid for small $\Delta(x)$, we have

$$\Delta\left(\frac{sin(\lambda\beta)}{\lambda}\right) = \frac{n}{2\lambda}\sqrt{\frac{\rho_0}{\rho_c}}\frac{\Delta\tilde{V}}{\tilde{V}^{\frac{n}{2}+1}} \tag{2.91}$$

Assuming now that the relation 2.79 is nearly saturated we obtain

$$\frac{\Delta\tilde{V}^2}{\tilde{V}^{\frac{n}{2}+1}} = \frac{4\pi G\hbar\gamma\lambda}{V_0 n}\sqrt{\frac{\rho_c}{\rho_0}}|cos(\lambda\beta)| \tag{2.92}$$

From the previous one we see that the quantum fluctuations in volume become important ($\tilde{V} \sim \Delta\tilde{V}$) for:

$$\tilde{V}_{qf} = \left[\frac{4\pi G\hbar\gamma\lambda}{V_0 n}\sqrt{\frac{\rho_c}{\rho_0}}|cos(\lambda\beta)|\right]^{\frac{2}{2-n}} \tag{2.93}$$

If we compare 2.88 with 2.93 we notice that for large V_0: $\tilde{V}_{bounce} >> \tilde{V}_{qf}$. This proves the correctness of the expansion 2.90 and the fact that the equation 2.84, valid only for semiclassical states, holds at each time, in particular near the bounce point. If we perform an analogous calculation for $sin(\lambda\beta)$ we obtain a similar result.
We notice from 2.93 that for $n = 2$ ($\omega = 1$) this relation is meaningless. However in such case we can conclude at the previous step that does not exist a value $\tilde{V}_{qf}$ for which quantum fluctuations become important, since for $n = 2$ we have that the ratio $\frac{\Delta\tilde{V}}{\tilde{V}}$ goes to 0 for $V_0 \to +\infty$.
By the other side we notice that for small V_0, or for states that are not sharply peaked, the effective equations fail. Summarizing, we proved that for the flat FRW Minisuperspace of LQC the effective equations work well not only in the semiclassical regime, but also in the deep Planck regime. This is due to the fact that the fluctuations of sharply peaked states can be taken arbitrarily small at each time, since they are parametrized with V_0,

that can be taken arbitrarily large. Exactly the same holds for the WdW Minisuperspace. The question that arises naturally from such conclusion is the following: does a quantum theory without quantum fluctuations of the observables make sense?

Physical interpretation of sharply-peaked states at each time

A quantum theory that predicts states sharply-peaked at each time sounds like a problematic theory, since the presence of quantum fluctuations usually allow to distinguish between the classical and the quantum behaviour of a system. In order to understand why this result is instead natural let's look at an example. Let's consider a system of N atoms with unit mass. Each of them can be described by the usual variables of the phase space x_i and p_i, satisfying at the quantum level the commutation relation $[\widehat{x}_i, \widehat{p}_j] = i\hbar\delta_{ij}$. Let's look now at the centre of mass of such system. We define as usual the position of the centre of mass and the momentum as

$$\widehat{X} = \frac{\sum_i \widehat{x}_i}{N} \tag{2.94}$$

$$\widehat{P} = \sum_i \widehat{p}_i \tag{2.95}$$

Their canonical commutation relations

$$[\widehat{X}, \widehat{P}] = \frac{i\hbar}{N} \tag{2.96}$$

and from the previous one we notice that even if we are looking at quantum non-commutating observables, for $N \to +\infty$ their commutator vanish. The same for the product of their relative uncertainties. So even if the position and momentum of the single atoms are subjected to quantum fluctuations, their average is blind to them. Moreover we are convinced that even if the centre of mass of such system behaves classically, the system continues to be quantum.

Exactly the same happens for our LQC minisuperspace. We recall in fact that we built our theory starting from a classical reduced phase space, in which we considered only the homogeneous and isotropic degrees of freedom of our fields A, E. We can see them as averages of the local variables, or better averages of the values of the fields made on different space points. So the fact that there are no fluctuations for our states at each time, doesn't imply that there are no quantum fluctuations for such fields at all. It simply means that the variables β and $\tilde{V}$ are blind to these at the quantum level, as well as the

centre of mass for the N-atom system. The theory with a fixed fiducial cell $\mathcal{V}$ describes the evolution of the modes of the field of size $\mathcal{V}$, and not all the other ones. Since we are interested in the dynamics of the large-scale structure of the Universe, we correctly take the $V_0 \to +\infty$ limit. If instead we are interested to what happens locally at each time, we have to choose V_0 small, in such a way that the quantum fluctuations are captured, and the evolution of the observables cannot be described by the effective equations.
For further details see [18] This concludes our analysis of such Minisuperspace of LQC. Let's proceed now with the Loop quantization of homogeneous (but not isotropic) cosmologies.

2.2 Bianchi-I Universe

In the previous section we loop-quantized homogeneous and isotropic cosmologies (FRW). Here we want to relax the symmetry assumptions and quantize a solution still homogeneous, but not isotropic. In this section we follow mainly [12]. At the classical level the homogeneous Universes are called *Bianchi*, and are not only three as in the FRW case ($k = 0, \pm 1$) but many more, labelled with latin numbers I, II,III,... Let's describe the difference between them. For generic homogeneous metrics we have three space-like Killing vectors that satisfy the following algebra:

$$\begin{cases} \big[X_1, X_2\big] = -aX_2 + n_3 X_3 \\ \big[X_2, X_3\big] = n_1 X_1 \\ \big[X_3, X_1\big] = aX_3 + n_2 X_2 \end{cases}$$

Where n_1, n_2, n_3, a are the structure constants of the algebra. The set of constants determines uniquely the Bianchi model. In this section we concentrate on the simplest Bianchi Universe, that is the Bianchi I: it has all the structure constants equal to 0, and this means that the algebra generated by the Killing vectors is abelian:

$$\begin{cases} \big[X_1, X_2\big] = 0 \\ \big[X_2, X_3\big] = 0 \\ \big[X_1, X_3\big] = 0 \end{cases}$$

In the classical ADM formulation, the line element of such model written in cartesian coordinates $(t, \vec{x})$:

$$ds^2 = -N^2 dt^2 + a_x(t)^2 dx^2 + a_y^2(t) dy^2 + a_z^2(t) dz^2 \tag{2.97}$$

Where N is the usual lapse function while $a_x(t), a_y(t), a_z(t)$ are called *directional scale factors* and are the dynamical variables of the theory. As we notice from 2.97 the metric is homogeneous, since the components don't depend on x, y, z but anisotropic, since we have different scale factors for different directions. We can recognize from 2.97 istantly the three Killing vectors of such metric: $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$.
We notice also that if we make a rescaling:

$$\begin{cases} x^i \longrightarrow x^i \alpha^i \\ a_i \longrightarrow a_i \alpha_i^{-1} \end{cases}$$

the line element is invariant. This rescale freedom means that at the classical level $a_i(t)$ are not observables: only the ratios $\frac{a_i(t)}{a_i(t')}$ can be measured. We are in the same situation of the FRW, where for the same reason only $\frac{a(t)}{a(t')}$ is an observable.
An important difference between FRW ($k = 0$) and this model is that the first one has a trivial dynamics in vacuum, while the second not. In fact, the Friedmann equations for a perfect fluid:

$$\begin{cases} \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} G\rho \\ \frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho + 3p) \end{cases}$$

and in absence of a source ($\rho = p = 0$):

$$\begin{cases} \left(\frac{\dot{a}}{a}\right)^2 = 0 \\ \frac{\ddot{a}}{a} = 0 \end{cases}$$

That means $\dot{a} = 0$, $\ddot{a} = 0$, with the trivial solution $a(t) = const.$, so the Minkowski space-time. By the other side the Bianchi I solution for an empty Universe (called also *Kasner solution*) is not Minkowski but describes a Universe with two expanding scale factors and the third contracting, or vice versa. So, even if we could in principle quantize this model in vacuum obtaining a non trivial Minisuperspace we'll do it in presence of a source.
A physical interesting property of this model is that for $t = const.$ describes a 3-$\mathcal{D}$ flat

space, in the sense that it has zero curvature, but expands anisotropically.
In order to Loop-quantize the system let's proceed with the classical formulation in terms of the Ashtekar-Barbero variables.

2.2.1 Classical phase space and the Hamiltonian constraint

As we can see from 2.97 the spatial part of the manifold is not compact, similarly to FRW with $k = 0, -1$, differently from the case $k = 1$. For this reason we have to restrict our analysis to a fiducial cell $\mathcal{V}$. The homogeneity of the space guarantees that the analysis made on $\mathcal{V}$ is representative of the whole space.
We choose $\mathcal{V}$ in a convenient way, so that its edges lie along the fixed coordinate axis x^i.
As in the FRW case is useful to fix a fiducial flat metric ${}^o q_{ab}$ with line element

$$ds_o^2 = {}^o q_{ab} dx^a dx^b$$

that doesn't evolve in time. As in the previous section we denote by ${}^o q$ the determinant of such metric, by L_i the lenghts of the three edges of $\mathcal{V}$ as measured with ${}^o q_{ab}$ and by $V_0 = L_1 L_2 L_3$ the fiducial volume of $\mathcal{V}$.
Finally as in FRW we intoduce fiducial triads $({}^o e^i{}_a)$ and co-triads $({}^o e^a{}_i)$ such that

$$\delta_{ij} = {}^o e^a{}_i \, {}^o e^b{}_j \, {}^o q_{ab}$$

Where in principle ${}^o q_{ab}$ is not written in a cartesian frame.
Let's fix the frame for our fiducial metric, and in particular ${}^o q_{ab} = \delta_{ab}$. In this way ${}^o e^a{}_i = \delta^a{}_i$, ${}^o e^i{}_a = \delta^i{}_a$. With this choice our fiducial line element:

$$ds_o^2 = \delta_{ab} dx^a dx^b$$

Independently on this choice instead we have a relation between the fiducial triad (co-triad) and the physical one:

$$\begin{cases} e^i{}_a = a_a(t) {}^o e^i{}_a \\ e^a{}_i = (a(t)^a)^{-1} {}^o e^a{}_i \end{cases}$$

where here two equal low (high) indices don't mean summation.
Remark : in the isotropic model we used $V_0^{\frac{1}{3}}$ in place of L_i; here we can use L_i since the associated physical lenghts are not equal because the loss of isotropy.

As in the previous case the homogeneity allows to write the variables E, A in a suitable way;

$$\begin{cases} A^i{}_a = c^i(t)(L^i)^{-1}{}^o e^i{}_a \\ E^a{}_i = p_i(t) L_i V_0^{-1} \sqrt{{}^o q}\, {}^o e^a{}_i \end{cases} \tag{2.98}$$

Well, if in the isotropic case:

$$p = sgn(a) a^2 V_0^{\frac{2}{3}}$$

Here:

$$\begin{cases} p_1 = sgn(a_1)|a_2 a_3| L_2 L_3 \\ p_2 = sgn(a_2)|a_1 a_3| L_1 L_3 \\ p_3 = sgn(a_3)|a_1 a_2| L_1 L_2 \end{cases} \tag{2.99}$$

where $sgn(a_i) = 1$ if ${}^o e^a{}_i$ and $e^a{}_i$ are parallel, -1 if antiparallel. We see later the explicit expression of c^i.

From 2.99 we can easily derive the relation between the physical volume and the $p's$

$$V = |a_1 a_2 a_3| L_1 L_2 L_3 = \sqrt{|p_1 p_2 p_3|} \tag{2.100}$$

Well, since the decoupling due to homogeneity between the time dependence and the tensorial structure in the Ashtekar-Barbero variables we can directly consider c^i and p_j as the variables of the classical phase space, and can be easily proved that satisfy the following

$$\{c^i, p_j\} = 8\pi G \gamma \delta^i{}_j \tag{2.101}$$

So the Bianchi-I phase space is 6-$\mathcal{D}$, differently from the 2-$\mathcal{D}$ FRW.

We notice from 2.99 that such phase space variables depend strongly on the choice of the elementary cell $\mathcal{V}$, but don't depend on the choice of the fiducial triads, co-triads and metric.

Let's now construct the constraints of the model at the classical level. As usual the constraint surface within the Ashtekar-Barbero formalism is the following:

$$\begin{cases} \mathcal{H}_\mu \approx 0 \\ \mathcal{G}_i \approx 0 \end{cases}$$

However as well as the isotropic case it can be easily proved that the Diffeo. and Gauss constraints are authomatically satisfied cause homogeneity. Thus we are left with the

Hamiltonian constraint:

$$C(N) = C_{grav.}(N) + C_{matt.}(N)$$

Where $C_{grav.}(N)$ is given by 2.3, while $C_{matt.}(N)$ is the hamiltonian constraint related to a massless scalar field ϕ.

Let's write down them explicitally, starting from the gravitational part. Firstly we notice that following the same reasoning of the previous chapter, we can pass from the expression 2.3 to 2.5, obtaining

$$\mathcal{C}_{grav.} = \frac{-\epsilon_{ijk}}{16\pi G\gamma^2} \frac{F^i{}_{ab} E^{aj} E^{bk}}{e} \tag{2.102}$$

Now, recalling 2.15, that holds since homogeneity, and using 2.98, we have:

$$\begin{aligned}\mathcal{C}_{grav.} &= -\frac{p_j L_j \sqrt{{}^o q}({}^o e^a{}_j) p_k L_k \sqrt{{}^o q}({}^o e^b{}_k)}{16\pi G\gamma^2 V_0^2 e} \epsilon_i^{jk} \epsilon^i{}_{lm} A^l{}_a A^m{}_b = \\ &= -\frac{p_j p_k L_j L_k {}^o q({}^o e^a{}_j)({}^o e^b{}_k)}{16\pi G\gamma^2 V_0^2 e} \epsilon_i^{jk} \epsilon^i{}_{lm} c^l (L^l)^{-1} c^m (L^m)^{-1} ({}^o e^l{}_a)({}^o e^m{}_b)\end{aligned}$$

Now, since we assumed ${}^o q_{ab} = \delta_{ab}$, that means ${}^o q = 1$, ${}^o e^a{}_j = \delta^a{}_j$, we have

$$\mathcal{C}_{grav.} = -\frac{p_l p_m}{V_0^2 16\pi G\gamma^2 e} c^l c^m \epsilon^{lm}{}_k \epsilon^k{}_{lm}$$

Now, recalling that

$$\epsilon^{ij}{}_k \epsilon^k{}_{lm} = \delta^i{}_l \delta^j{}_m - \delta^i{}_m \delta^j{}_l$$

We have, evaluating explicitally the summations over contracted indices:

$$\mathcal{C}_{grav.} = -\frac{1}{8\pi G\gamma^2 V_0^2 e}(p_1 p_2 c^1 c^2 + p_1 p_3 c^1 c^3 + p_2 p_3 c^2 c^3)$$

Recalling that: $e = \sqrt{|det(E)|} = \sqrt{|p_1 p_2 p_3|} V_0^{-1}$, and using 2.98 we finally have

$$\begin{aligned}\mathcal{C}_{grav.} &= -\frac{1}{8\pi G\gamma^2 V_0 \sqrt{|p_1 p_2 p_3|}}(p_1 p_2 c^1 c^2 + p_1 p_3 c^1 c^3 + p_2 p_3 c^2 c^3) \quad \Longrightarrow \\ \Longrightarrow \quad C_{grav.}(N) &= -\int d^3x N \frac{1}{V_0 8\pi G\gamma^2 \sqrt{|p_1 p_2 p_3|}}(p_1 p_2 c^1 c^2 + p_1 p_3 c^1 c^3 + p_2 p_3 c^2 c^3) = \\ &= -\frac{N}{8\pi G\gamma^2 \sqrt{|p_1 p_2 p_3|}}(p_1 p_2 c^1 c^2 + p_1 p_3 c^1 c^3 + p_2 p_3 c^2 c^3)\end{aligned} \tag{2.103}$$

Now, instead of taking $N = 1$ (synchronous gauge) as in the isotropic case, the simplest choice for N is

$$N = \sqrt{|p_1 p_2 p_3|} \tag{2.104}$$

called *harmonic gauge*. We write the associated time variable τ to distinguish it from the synchronous time t corresponding to the gauge fixing condition $N = 1$. Such variables are in the following relation: $N^2 d\tau^2 = dt^2$. In this way:

$$C_{grav.} = -\frac{1}{8\pi G\gamma^2}(p_1 p_2 c^1 c^2 + p_1 p_3 c^1 c^3 + p_2 p_3 c^2 c^3) \tag{2.105}$$

By the other side, for the matter constraint we have the same expression of the isotropic case:

$$C_{matt.} = \frac{1}{2}\frac{P_\phi^2}{V} \tag{2.106}$$

and the whole Hamiltonian constraint:

$$C = -\frac{1}{8\pi G\gamma^2}(p_1 p_2 c^1 c^2 + p_1 p_3 c^1 c^3 + p_2 p_3 c^2 c^3) + \frac{1}{2}P_\phi^2 \tag{2.107}$$

where for the Matter part we assorbed $\frac{1}{V}$ in the lapse function N as well as we did for $C_{grav.}$. The physical states of the classical theory lie on the constraint surface $C \approx 0$.
The classical solution $(c_i(t), p^j(t))$ is given by

$$\frac{dp_1}{d\tau} = \{p_1, C\} = -8\pi G\gamma\frac{\partial C}{\partial c^1} = \frac{p_1}{\gamma}(p_2 c^2 + p_3 c^3) \tag{2.108}$$

$$\frac{dc^1}{d\tau} = \{c^1, C\} = 8\pi G\gamma\frac{\partial C}{\partial p_1} = -\frac{c^1}{\gamma}(p_2 c^2 + p_3 c^3) \tag{2.109}$$

while the other time derivatives can be obtained by permutations.
Combining these equations with 2.99 we obtain

$$c^i = \gamma L^i V_0^{-1}(a_1 a_2 a_3)^{-1}\frac{da^i}{d\tau} \tag{2.110}$$

Now we can relate c^i with the Hubble parameters

$$H_i = \frac{da_i}{dt}\frac{1}{a_i}$$

written in terms of the synchronous time t. If we write 2.110 in terms of H_i, recalling that

$$\frac{d}{dt} = \frac{1}{\sqrt{|p_1 p_2 p_3|}}\frac{d}{d\tau}$$

using 2.100 we have:

$$c^i = \gamma L^i V_0^{-1}(a_1 a_2 a_3)^{-1}\sqrt{|p_1 p_2 p_3|}\frac{da^i}{dt} = \gamma L^i a^i H^i \tag{2.111}$$

Finally we want to find for this Bianchi I Universe the analogous of the first Friedmann equation. For this purpose we introduce a *mean scale factor*: $a \equiv (a_1 a_2 a_3)^{\frac{1}{3}}$ that encodes the physical volume 2.100 but ignores anisotropies. Then we can define the *mean Hubble parameter*:

$$\begin{aligned} H \equiv \frac{d}{dt} ln(a) = \frac{d}{dt} ln[(a_1 a_2 a_3)^{\frac{1}{3}}] = \frac{1}{a} \frac{d}{dt} (a_1 a_2 a_3)^{\frac{1}{3}} = \\ = \frac{1}{3a} (a_1 a_2 a_3)^{-\frac{2}{3}} (\dot{a}_1 a_2 a_3 + a_1 \dot{a}_2 a_3 + a_1 a_2 \dot{a}_3) = \\ = \frac{1}{3} \left(\frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3} \right) = \frac{1}{3} (H_1 + H_2 + H_3) \end{aligned} \tag{2.112}$$

Now, from the Hamiltonian constraint 2.107 that holds on-shell, calling $\rho_{matt.} = \frac{1}{2} \frac{P_\phi^2}{V^2}$, we have:

$$p_1 p_2 c^1 c^2 + p_1 p_3 c^1 c^3 + p_2 p_3 c^2 c^3 = 8\pi G \gamma^2 \rho_{matt.} V^2$$

using 2.99 and 2.111 :

$$\begin{aligned} & |a_1^2 a_2^2 a_3^2| \, L_1^2 L_2^2 L_3^2 \, (H_1 H_2 + H_2 H_3 + H_1 H_3) = 8\pi G \rho_{matt.} V^2 \quad \Longrightarrow \\ \Longrightarrow \quad & H_1 H_2 + H_1 H_3 + H_2 H_3 = 8\pi G \rho_{matt.} \end{aligned} \tag{2.113}$$

Then, combining 2.112 and 2.113 we obtain

$$H^2 = \frac{1}{9} (H_1^2 + H_2^2 + H_3^2 + 16\pi G \rho_{matt.})$$

that can be rewritten as

$$H^2 = \frac{8\pi G}{3} \rho_{matt.} + \frac{\Sigma^2}{a^6} \tag{2.114}$$

where

$$\Sigma^2 = \frac{a^6}{18} [(H_1 - H_2)^2 + (H_1 - H_3)^2 + (H_2 - H_3)^2] \tag{2.115}$$

is called *shear term*, and can be proved that classically is a constant of motion.
2.114 is the first generalized Friedman equation for the Bianchi-I Universe. We derived it within the Ashtekar-Barbero formulaton, but it can be equivalently derived from the canonical formulation of Einstein theory, by imposing the Hamiltonian constraint in the variables (q, π). We notice that if we assume in 2.114: $H_1 = H_2 = H_3$ we recover the first Friedmann equation ($k = 0$).
These considerations will be useful to explore some features of the quantum model.

Remark : as well as in the isotropic case we consider a massless scalar field ϕ not subjected to an external potential. This means that also here P_ϕ is a constant of motion, and ϕ grows linearly with τ. For this reason ϕ, or better $\langle\hat{\phi}\rangle$ is a good time parameter for our system, and we'll use it to solve the problem of time.
Let's conclude this paragraph focusing on the *reflection symmetries* of the theory, that will play an important role at the quantum level.
In the isotropic case we could restrict our attention on the sector $p > 0$, even if we kept the sign of p unfixed. This because by a side the classical Hamiltonian is invariant under:

$$\Pi(p) = -p \tag{2.116}$$

since it depends on $|p|$, and even if the quantum Hamiltonian depends explicitally on $\widehat{sgn(p)}$ (in the $sMMO$ quantization scheme) it can be proved that its total action on physical states doesn't depend on it. Moreover we recall that the transformation 2.116 physically means a change in the relative orientation between the axis of the triadic metric and the physical one. This can be easily seen from 2.9. Clearly the physical metric q_{ab} remains invariant under such transformation, which is global. In the Banchi I case we have three reflections that make the Hamiltonian constraint unchanged, described by three classical operators Π_i, each of them corresponding to the flip of only an axis of the triadic frame:

$$\Pi_1(p_1, p_2, p_3) = (-p_1, p_2, p_3) \tag{2.117}$$

2.2.2 Kinematical Hilbert space

Generalizing the procedure we followed for the isotropic case the elementary variables we consider here for the quantization of the model are the three momenta p_i and the holonomies $h_i^{(\mu_i)}$ evaluated along edges parallel to the three fiducial axis x_i, with lenghts $\mu_i L_i$ with respect to the fiducial metric ${}^o q_{ab}$. We recall that in the isotropic model $l = \mu V_0^{\frac{1}{3}}$, while here we take in account that $L_i \neq L_j$ for $i \neq j$. Thus

$$h_i^{(\mu_i)}(c_1, c_2, c_3) = \mathbb{1} \cdot cos(\mu_i c_i) + 2\tau_i sin(\mu_i c_i) \tag{2.118}$$

As well as the isotropic case the matrix elements of such holonomy are linear combinations of functions of kind $exp(i\mu_i c_i)$. For this reason we can consider as variables of the classical phase space directly $N_{\mu_i} = exp(i\mu_i c_i)$. Following instead the same consideration we did in the previous Minisuperspace for the triad, the other three variables we consider here

are p_j.

In the isotropic case we considered as basis $\{|\mu\rangle\}$ for $\mathcal{H}_{grav.}$ with the following features

$$\begin{aligned}
&\widehat{N_{\mu'}}\,|\mu\rangle = |\mu+\mu'\rangle \\
&\widehat{p}\,|\mu\rangle = p(\mu)\,|\mu\rangle\,, \quad with \quad p(\mu) = \frac{4\pi\gamma l_P^2\mu}{3} \\
&\langle\mu|\mu'\rangle = \delta_{\mu\mu'}
\end{aligned}$$

Here a basis for the Kinematical Hilbert space is $|\mu_1,\mu_2,\mu_3\rangle$. We label such states with the correspondent eigenvalue of $\widehat{p_i}$. A generic state in such basis:

$$|\psi\rangle = \sum_{p_1,p_2,p_3} \psi(p_1,p_2,p_3)\,|p_1,p_2,p_3\rangle \tag{2.119}$$

with

$$\langle p_1,p_2,p_3|p_1',p_2',p_3'\rangle = \delta_{p_1p_1'}\delta_{p_2p_2'}\delta_{p_3p_3'} \tag{2.120}$$

Where δ here are Kronecker deltas, that as in the isotropic case allow to mimick the structure of the full theory.

As we saw in the previous paragraph a change in the orientation of the axis of the tetrad frame with respect to the physical one produces a change in sign of the correspondent p_i. This classical property can be implemented at the quantum level assuming that ψ satisfies the following

$$\widehat{\Pi_1}\psi(p_1,p_2,p_3) = \psi(-p_1,p_2,p_3) \tag{2.121}$$

and since Π at the classical level doesn't affect the physics, at the quantum level that each operator has to commute with $\widehat{\Pi_i}$. Therefore we can restrict our Kinematical Hilbert space to the one spanned by states with a given eigenvalue of this operator. In particular we assume

$$\psi(p_1,p_2,p_3) = \psi(|p_1|,|p_2|,|p_3|) \tag{2.122}$$

so states that are eingenstates of $\widehat{\Pi_i}$ with eigenvalue $+1$.

Well, as in the isotropic case $|p_1,p_2,p_3\rangle$ are not only eigenstates of $\widehat{p_i}$, but also eigenstates of geometric operators like the area and the volume. In particular for the state $|p_1,p_2,p_3\rangle$ the action of the elementary operators is given by

$$\begin{cases}
\widehat{p_1}\,|p_1,p_2,p_3\rangle = p_1\,|p_1,p_2,p_3\rangle \\
\widehat{N_{\mu_1}}\,|p_1,p_2,p_3\rangle = \widehat{exp\big(i\mu_1 c_1\big)}\,|p_1,p_2,p_3\rangle = |p_1+8\pi\gamma l_P^2\mu_1,p_2,p_3\rangle
\end{cases} \tag{2.123}$$

and similarly for $\widehat{p_2}, \widehat{p_2}, \widehat{N}_{\mu_2}, \widehat{N}_{\mu_3}$. The only differences with respect to the isotropic case is the absence of the factor $\frac{1}{3}$, that comes from a different prefactor in the Poisson braket structure (2.101).
The whole Kinematical Hilbert space of the model is the tensor product

$$\mathcal{H}_{Kin} = \mathcal{H}^{grav.}_{Kin} \otimes \mathcal{H}^{matt.}_{Kin} \tag{2.124}$$

Where $\mathcal{H}^{matt.}_{Kin} = L^2(\mathbb{R}, d\phi)$. On this space $\widehat{\phi}$ acts by multiplication and $\widehat{P_\phi} = -i\hbar\frac{\partial}{\partial\phi}$.
Let's proceed now with the construction of the Hamiltonian operator.

2.2.3 The classical Hamiltonian constraint and its quantization

To construct a well defined quantum Hamiltonian operator we need to write the classical Hamiltonian constraint in terms of the holonomies and the fluxes.
Let's start with the curvature tensor F^i_{ab} contained in the Hamiltonian constraint. For such tensor we can use exactly the same expression we have in the isotropic case (2.29), that is useful to rewrite here

$$F^l{}_{ab} = \lim_{A_\square \to 0} -2\frac{tr\{[h^{(\mu)}_{\square_{ij}} - \mathbb{1}]\tau^l\}}{A_\square} {}^o e^i{}_a {}^o e^j{}_b \tag{2.125}$$

Where here $A_\square$ is the fiducial area of a rectangular plaquette. As we know from the full theory we cannot promote the connection A to a quantum operator, and since F is a combination of terms of kind AA, we cannot quantize the right hand side of the previous expression. We can see this recalling that the quantum area in LQG is quantized with a minimum non-zero value of the spectrum, and since we need to mimick the structure of the full theory we cannot send $A_\square$ to 0. We have to fix somehow the minimum area of such plaquettes. As in the isotropic case we can do it making a parallelism with LQG, using so the minimum value of the spectrum of the area operator. In the previous paragraph we found that a state of the Kinematical Hilbert space of kind $|p_1, p_2, p_3\rangle$ is eigenvalue of the volume and area operators, and in this quantum state the faces of the fiducial cell $\mathcal{V}$ have values $|p_1|, |p_2|, |p_3|$. What is the analogous of such state in LQG? As in the isotropic case, firstly we expect that the links of the graph associated with the LQG state have to puncture all the faces of the fiducial cell $\mathcal{V}$. Clearly such state cannot be perfectly homogeneous, since to be so it should have an infinite number of links. However we can achieve the best coarse grained geometry by assuming that such links carry spin $j = \frac{1}{2}$. In this way the separation area between them is the smallest possible. Finally we have to

understand how many links puncturing each surface such graph has to have. To simplify the analysis we consider one of the the surfaces of the cell $\mathcal{V}$, in particular one of the two orthogonal to the x^3 axis and we call it S_{12}. As we previously said the area of such surface for the state $|p_1, p_2, p_3\rangle$ is $|p_3|$. We also know that the area associated with each link of the spin-network state puncturing a surface is $\Delta = 4\pi\gamma\sqrt{3}l_P^2$. If we consider rectangular areas for each link, we need N links puncturing such surface such that

$$N\Delta = |p_3| \tag{2.126}$$

Let's call $\bar{\mu}_1$ and $\bar{\mu}_2$ the adimensional lenghts of the fiducial area associated with the physical area of such plaquettes. We have that the fiducial area of such plaquettes is: $\bar{\mu}_1\bar{\mu}_2 L_1 L_2$. By the other side, the fiducial area of the whole surface S_{12} is $L_1 L_2$. Thus we have

$$L_1 L_2 = N\bar{\mu}_1\bar{\mu}_2 L_1 L_2 \quad \Longleftrightarrow \quad N = \frac{1}{\bar{\mu}_1\bar{\mu}_2} \tag{2.127}$$

Substituting this in 2.126 we have

$$|p_3| = \frac{\Delta}{\bar{\mu}_1\bar{\mu}_2} \tag{2.128}$$

Differently from the isotropic case this equation is not sufficient to determine $\bar{\mu}_1$ and $\bar{\mu}_2$, but if we consider cyclic permutation of 2.128, so we consider the three orthogonal surfaces of $\mathcal{V}$ we obtain

$$\bar{\mu}_1 = \sqrt{\frac{|p_1|\Delta}{|p_2||p_3|}} \qquad \bar{\mu}_2 = \sqrt{\frac{|p_2|\Delta}{|p_1||p_3|}} \qquad \bar{\mu}_3 = \sqrt{\frac{|p_3|\Delta}{|p_1||p_2|}} \tag{2.129}$$

We notice that as in the isotropic case these $\bar{\mu}_i$ are not fixed, but depend on the state $|p_1, p_2, p_3\rangle$ on which the operator $\widehat{F}$ acts.
The resulting curvature tensor

$$F^l{}_{ab} = -2\frac{tr\{[h_{\Box_{ij}} - \mathbb{1}]\tau^l\}}{A_\Box} {}^o e^i{}_a {}^o e^j{}_b \tag{2.130}$$

with

$$h_{\Box_{ij}} = h_i^{(\bar{\mu}_i)} h_j^{(\bar{\mu}_j)} h_i^{(\bar{\mu}_i)-1} h_j^{(\bar{\mu}_j)-1}$$

and $A_\Box = L_i L_j \bar{\mu}_i \bar{\mu}_j$.
Now, let's write this tensor in a more useful form. Recalling 2.26 for the isotropic case,

here we have an analogous result with this form

$$h_{\square_{ij}} = f(C_iC_jS_iS_j)\mathbb{1} + g(C_iC_jS_iS_j)\tau_i + h(C_iC_jS_iS_j)\tau_j + 4C_iC_jS_iS_j\epsilon_{ijk}\tau_k \tag{2.131}$$

where f, g, h are linear functions of their argument, and $C_i = cos\big(\frac{\bar{\mu}_i c_i}{2}\big)$, $S_i = sin\big(\frac{\bar{\mu}_i c_i}{2}\big)$. Thus

$$h_{\square_{ij}} - \mathbb{1} = \tilde{f}(C_iC_jS_iS_j)\mathbb{1} + g(C_iC_jS_iS_j)\tau_i + h(C_iC_jS_iS_j)\tau_j + 4C_iC_jS_iS_j\epsilon_{ijk}\tau_k \tag{2.132}$$

If now we multiply this by τ^k and we make the trace, we notice that the first term vanishes, since $tr(\tau) = 0$. Moreover, recalling that

$$F^k{}_{ab} = \epsilon^k_{ij} A^i{}_a A^j{}_b = \epsilon^k_{ij} c^i c^j (L^i L^j)^{-1} {}^o e^i{}_a {}^o e^j{}_b$$

the indices i, j, k have to be all different. This means that when we multiply the second and third term in 2.132 by τ^k, and then we make the trace they vanish. The only non-zero contribution is given by the last term:

$$\begin{aligned}\widehat{F^k}_{ab} &= -2\frac{tr\{[h_{\square_{ij}} - \mathbb{1}]\tau^k\}}{A_\square} {}^o e^i{}_a {}^o e^j{}_b = -8\epsilon_{ij}{}^k C_iC_jS_iS_j \frac{tr\{\tau^k\tau^k\}}{A_\square} {}^o e^i{}_a {}^o e^j{}_b = \\ &= 4\epsilon_{ij}{}^k C_iC_jS_iS_j \frac{1}{A_\square} {}^o e^i{}_a {}^o e^j{}_b\end{aligned} \tag{2.133}$$

Now, to write it in a more convenient way we use the duplication formula for the sin: $sin(2x) = 2sin(x)cos(x)$:

$$F^k{}_{ab} = \epsilon_{ij}{}^k \left[\frac{sin(\bar{\mu}_i c_i)}{\bar{\mu}_i L_i} \frac{sin(\bar{\mu}_j c_j)}{\bar{\mu}_j L_j}\right] {}^o e^i{}_a {}^o e^j{}_b \tag{2.134}$$

where as before the indices i and j in the quantities in the squared brakets have to be thought as labelling factors of the components i, j of the fiducial triads. This is a very useful form for such tensor, since it can be immediately quantized:

$$\widehat{F^k}_{ab} = \epsilon_{ij}{}^k \left[\frac{\widehat{sin(\bar{\mu}_i c_i)}}{\bar{\mu}_i L_i} \frac{\widehat{sin(\bar{\mu}_j c_j)}}{\bar{\mu}_j L_j}\right] {}^o e^i{}_a {}^o e^j{}_b \tag{2.135}$$

We can proceed now with the quantization of the whole Hamiltonian constraint. Recalling 2.102, we have to quantize the term $\frac{E^{ai}E^{bk}}{\sqrt{|det(E)|}}$. We can quantize it in a direct way, since it contains only the variables p_i of our phase space, and we already promoted them to

quantum operators. Let's write it explicitally in terms of p_i:

$$\frac{E^a_j E^b_k}{\sqrt{|det(E)|}} = \frac{p_j p_k L_j L_k V_0^{-1} V_0^{-1} \sqrt{{}^o q}\sqrt{{}^o q}({}^o e^a_j)({}^o e^b_k)}{\sqrt{|p_1 p_2 p_3|} V_0^{-1}}$$

Thus the densitized hamiltonian constraint at the classical level:

$$\begin{aligned}\mathcal{C}_{grav.} &= \frac{-N\epsilon_i{}^{jk}}{16\pi G\gamma^2}\frac{F^i{}_{ab}E^a_j E^b_k}{eV_0^2} = \\ &= -\frac{N p_j p_k L_j L_k {}^o q({}^o e^a_j)({}^o e^b_k)}{16\pi G\gamma^2 V_0\sqrt{|p_1 p_2 p_3|}}\cdot \epsilon_i{}^{jk}\epsilon^i{}_{lm}\left[\frac{sin(\bar{\mu}_l c_l)}{\bar{\mu}_l L_l}\frac{sin(\bar{\mu}_m c_m)}{\bar{\mu}_m L_m}\right]({}^o e^l{}_a)({}^o e^m{}_b)\end{aligned}$$

Now we assumed ${}^o e^a{}_j = \delta^a{}_j$, thus:

$$\begin{aligned}\mathcal{C}_{grav.} &= -\frac{N p_l p_m L_l L_m}{16\pi G\gamma^2 V_0\sqrt{|p_1 p_2 p_3|}}\cdot \epsilon_i{}^{lm}\epsilon^i{}_{lm}\left[\frac{sin(\bar{\mu}_l c_l)}{\bar{\mu}_l L_l}\frac{sin(\bar{\mu}_m c_m)}{\bar{\mu}_m L_m}\right] \\ &= -\frac{N p_l p_m}{16\pi G\gamma^2 V_0\sqrt{|p_1 p_2 p_3|}}\cdot \epsilon_i{}^{lm}\epsilon^i{}_{lm}\left[\frac{sin(\bar{\mu}_l c_l)}{\bar{\mu}_l}\frac{sin(\bar{\mu}_m c_m)}{\bar{\mu}_m}\right]\end{aligned} \tag{2.136}$$

Now we integrate over d^3x and we obtain the Hamiltonian constraint

$$C_{grav.} = \int d^3x \mathcal{C}_{grav} = -\frac{N p_l p_m}{16\pi G\gamma^2\sqrt{|p_1 p_2 p_3|}}\cdot \epsilon_i{}^{lm}\epsilon^i{}_{lm}\left[\frac{sin(\bar{\mu}_l c_l)}{\bar{\mu}_l}\frac{sin(\bar{\mu}_m c_m)}{\bar{\mu}_m}\right] \tag{2.137}$$

As we did in the construction of the Hamiltonian constraint in terms of c^i and p_j (2.105) we make the gauge fixing $N = \sqrt{|p_1 p_2 p_3|}$ and we have

$$C_{grav.} = \int d^3x \mathcal{C}_{grav} = -\frac{p_l p_m}{16\pi G\gamma^2}\cdot \epsilon_i{}^{lm}\epsilon^i{}_{lm}\left[\frac{sin(\bar{\mu}_l c_l)}{\bar{\mu}_l}\frac{sin(\bar{\mu}_m c_m)}{\bar{\mu}_m}\right] \tag{2.138}$$

If now we expand $\epsilon_i{}^{lm}\epsilon^i{}_{lm}$, and we rewrite $\bar{\mu}_i$ in terms of p_j through 2.129 we obtain

$$\begin{aligned}C_{grav} = &-\frac{1}{8\pi G\gamma^2\Delta}[p_1 p_2|p_3|sin(\bar{\mu}_1 c_1)sin(\bar{\mu}_2 c_2) + p_2 p_3|p_1|sin(\bar{\mu}_2 c_2)sin(\bar{\mu}_3 c_3)+ \\ &p_3 p_1|p_2|sin(\bar{\mu}_3 c_3)sin(\bar{\mu}_1 c_1)]\end{aligned} \tag{2.139}$$

We can promote directly the previous expression to a quantum operator, obtaining the quantized gravitational Hamiltonian constraint of this Minisuperspace

$$\begin{aligned}\widehat{C}_{grav} = &-\frac{1}{8\pi G\gamma^2\Delta}\big[\widehat{p_1}\widehat{p_2}\widehat{|p_3|}\widehat{sin(\bar{\mu}_1 c_1)}\widehat{sin(\bar{\mu}_2 c_2)} + \widehat{p_2}\widehat{p_3}\widehat{|p_1|}\widehat{sin(\bar{\mu}_2 c_2)}\widehat{sin(\bar{\mu}_3 c_3)}+ \\ &\widehat{p_3}\widehat{p_1}\widehat{|p_2|}\widehat{sin(\bar{\mu}_3 c_3)}\widehat{sin(\bar{\mu}_1 c_1)}\big]\end{aligned} \tag{2.140}$$

For the moment we don't care about the operator ordering. In order to obtain the action of this operator on a state $|p_1, p_2, p_3\rangle$ is sufficient to look at the action of the operator $exp(i\bar{\mu}_i c_i)$, since the action of $\widehat{p_i}$ on such state is given by the first relation of 2.123. We observe that as in the isotropic case we cannot use for such purpose the second relation of 2.123, since here we have $\bar{\mu}$ that depends on p_i. If in the isotropic case we have

$$\widehat{exp\left(\frac{i\bar{\mu}c}{2}\right)}|p\rangle = exp\left(\bar{\mu}\frac{d}{d\mu}\right)|p\rangle = exp\left(\frac{4}{3}\pi\gamma\sqrt{\Delta}l_P^2\frac{1}{\sqrt{|p|}}\frac{d}{dp}\right)|p\rangle \tag{2.141}$$

In the anisotropic case we have analougsly

$$\widehat{exp(i\bar{\mu}_1 c_1)}\,|p_1, p_2, p_3\rangle = exp\left(8\pi\gamma\sqrt{\Delta}l_P^2\sqrt{\left|\frac{p_1}{p_2p_3}\right|}\frac{\partial}{\partial p_1}\right)|p_1, p_2, p_3\rangle \tag{2.142}$$

Differently from the isotropic case this expression is difficult to manage, since are present all the p_i. For this reason we cannot proceed as in the isotropic case introducing the differential operator $\partial_v = \bar{\mu}\partial_\mu$. In order to simplify it let's introduce new dimensionless variables

$$\lambda_i = \left(\frac{1}{4\pi|\gamma|l_P^2\sqrt{\Delta}}\right)^{\frac{1}{3}} sgn(p_i)\sqrt{|p_i|} \tag{2.143}$$

In such a way that $sgn(p_i) = sgn(\lambda_i)$.
Thus we can introduce a new orthonormal basis $\{|\lambda_1, \lambda_2, \lambda_3\rangle\}$ in $\mathcal{H}_{kin}^{grav}$. These states are again eigenstates of $\widehat{p_i}$:

$$\widehat{p_i}\,|\lambda_1, \lambda_2, \lambda_3\rangle = sgn(\lambda_i)\left(4\pi|\gamma|\sqrt{\Delta}l_P^2\right)^{\frac{2}{3}}\lambda_i^2\,|\lambda_1, \lambda_2, \lambda_3\rangle \tag{2.144}$$

The action of $\widehat{exp(i\bar{\mu}_1 c_1)}$ on these states

$$\widehat{exp(i\bar{\mu}_1 c_1)}\,|\lambda_1, \lambda_2, \lambda_3\rangle = exp\left(\frac{sgn(\lambda_1)}{\lambda_2\lambda_3}\frac{\partial}{\partial\lambda_1}\right)|\lambda_1, \lambda_2, \lambda_3\rangle \equiv \widehat{E_1}\,|\lambda_1, \lambda_2, \lambda_3\rangle \tag{2.145}$$

Where we introduce $\widehat{E_i}$ as a short-hand notation. In the previous expression we used the property of the Immirzi parameter $\gamma = |\gamma|sgn(p_1p_2p_3)$. Let's analyze the action of this operator. We notice that it contains the vector field $\frac{\partial}{\partial\lambda_1}$, and similarly as we did in the isotropic case we can conclude that it drags the wavefunction $\psi(\lambda_1, \lambda_2, \lambda_3)$ along the direction λ_1, of the quantity $\frac{sgn(\lambda_1)}{\lambda_2\lambda_3}$, where the factor $\frac{1}{\lambda_2\lambda_3}$ is constant along such direction,

since orthogonality. Thus its action on the state

$$\widehat{E_1}\,|\lambda_1,\lambda_2,\lambda_3\rangle = |\lambda_1 + \frac{sgn(\lambda_1)}{\lambda_2\lambda_3},\lambda_2,\lambda_3\rangle \tag{2.146}$$

In particular the action of this operator vanishes is not defined for states of kind $|\lambda_1,0,0\rangle$ since the wavefunction ψ doesn't have support in $\lambda_1=\infty$. This however is not a limitation since such states are annihilated by the volume operator before $\widehat{E}_i$ acts. With 2.143 in hand we can rewrite the gravitational part of the quantum hamiltonian constraint 2.140 in terms of λ_i

$$\widehat{C}_{grav} = \widehat{C}^{(1)}_{grav} + \widehat{C}^{(2)}_{grav} + \widehat{C}^{(3)}_{grav} \tag{2.147}$$

With

$$\begin{aligned}\widehat{C}^{(1)}_{grav} = &-\pi\hbar l_P^2 \widehat{\sqrt{|\lambda_1\lambda_2\lambda_3|}}\big[\widehat{sin(\bar{\mu}_2 c_2)}\widehat{sgn(\lambda_2)}|\widehat{\lambda_1\lambda_2\lambda_3}|\widehat{sgn(\lambda_3)}\widehat{sin(\bar{\mu}_3 c_3)}+\\ &+\widehat{sin(\bar{\mu}_3 c_3)}\widehat{sgn(\lambda_3)}|\widehat{\lambda_1\lambda_2\lambda_3}|\widehat{sgn(\lambda_2)}\widehat{sin(\bar{\mu}_2 c_2)}\big]\widehat{\sqrt{|\lambda_1\lambda_2\lambda_3|}}\end{aligned} \tag{2.148}$$

$\widehat{C}^{(2)}_{grav}$ and $\widehat{C}^{(3)}_{grav}$ are obtained from the previous expression through cyclic permutations. Let's observe that in 2.148 we make a precise choice in the operator ordering. As well as in the isotropic case, it can be proved that this operator is invariant under the action of $\widehat{\Pi_i}$ and self-adjoint. As we anticipated in the first paragraph of this section we assume that the eigenstates of this operator are symmetric under parity, so the eigenfunctions in these new variables

$$\psi(\lambda_1,\lambda_2,\lambda_3) = \psi(|\lambda_1|,|\lambda_2|,|\lambda_3|) \tag{2.149}$$

Let's conclude this paragraph with the construction of the matter Hamiltonian constraint. We have exactly the same operator of the isotropic case

$$\widehat{C}_{matt} = \frac{\widehat{P_\phi^2}}{2} = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial\phi^2} \tag{2.150}$$

In the $|\phi\rangle$ representation. Thus, the whole Hamiltonian constraint

$$\left(C_{grav} - \frac{\hbar^2}{2}\frac{\partial^2}{\partial\phi^2}\right)\psi(\lambda_1,\lambda_2,\lambda_3,\phi) = 0 \tag{2.151}$$

We finally notice that the choice 2.104 allowed us to avoid a term in the Hamiltonian of kind $\widehat{\frac{1}{V}}$, that is not only difficult to manage but has to be correctly ordered. If here we follow exactly the same procedure of the isotropic case, we end up with an Hamiltonian

constraint slightly different and more complicate, cause a different factor ordering. In the next paragraph we proceed with a further simplification of $\widehat{C}_{grav}$.

2.2.4 Simplification of $\widehat{C}_{grav}$

In the isotropic case we introduced $v(p)$ (2.34) such that

$$|v(p)| = (2\pi\gamma l_P^2\sqrt{\Delta})^{-1}|p|^{\frac{3}{2}} = (2\pi\gamma l_P^2\sqrt{\Delta})^{-1}V$$

Here if consider $|\lambda_1\lambda_2\lambda_3|$ we have

$$|\lambda_1\lambda_2\lambda_3| = \left(\frac{1}{4\pi|\gamma|l_P^2\sqrt{\Delta}}\right)\sqrt{|p_1p_2p_3|} = \left(\frac{1}{4\pi|\gamma|l_P^2\sqrt{\Delta}}\right)V \tag{2.152}$$

Thus if here we call

$$|v| = 2|\lambda_1\lambda_2\lambda_3| \tag{2.153}$$

we obtain an analogy between the two cases. As in the isotropic case, $|v|$ is directly linked with the volume of the elementary cell $\mathcal{V}$

$$\widehat{V}\psi(\lambda_1,\lambda_2,v) = 2\pi|\gamma|\sqrt{\Delta}|v|l_P^2\,\psi(\lambda_1,\lambda_2,v) \tag{2.154}$$

That is the analog of 2.37 for this anisotropic model. Differently from the isotropic case however it is not sufficient the variable v to completely determine the wavefunction, but we need other two, which we choose to be λ_1, λ_2. Well, let's proceed with the simplification of the Hamiltonian constraint. Firstly we notice that we can rewrite 2.148 as a linear combination of 24 terms of kind

$$\widehat{C}_{ij}^{\pm\pm} = \widehat{\sqrt{|v|}}\widehat{E}_i^{\pm}\widehat{sgn(\lambda_i)}\widehat{|v|}\widehat{sgn(\lambda_j)}\widehat{E}_j^{\pm}\widehat{\sqrt{|v|}} \qquad , i \neq j \tag{2.155}$$

obtained keeping in account 2.153 and writing $sin(\bar{\mu}_i c_i)$ as complex exponentials. Now we look at this constraint as acting on eigenfunctions of kind $\psi = \psi(\lambda_1,\lambda_2,v)$, and in particular on $\psi = \psi(|\lambda_1|,|\lambda_2|,|v|)$ (requiring symmetry under parity). This simplify enormously its expression, since we can use it to remove the factors $\widehat{sgn(\lambda_i)}$ from 2.155. In order to to this explicitally let's focus our attention on one of the 24 terms of kind

2.155:

$$\widehat{C}^{--}_{21}\psi(\lambda_1,\lambda_2,v) = \widehat{\sqrt{|v|}}\widehat{E}^{\pm}_2\widehat{sgn(\lambda_2)}\widehat{|v|}\widehat{sgn(\lambda_1)}\widehat{E}^{\pm}_1\sqrt{|v|}\,\psi(\lambda_1,\lambda_2,v) =$$
$$= \widehat{\sqrt{|v|}}\widehat{E}^{\pm}_2\widehat{sgn(\lambda_2)}\widehat{|v|}\widehat{sgn(\lambda_1)}\sqrt{|v-2sgn(\lambda_1)|}\,\psi\left(\lambda_1-\frac{2sgn(\lambda_1)\lambda_1}{v},\lambda_2,v-2sgn(\lambda_1)\right) =$$
$$= \widehat{\sqrt{|v|}}\widehat{E}^{\pm}_2\widehat{sgn(\lambda_2)}\widehat{|v|}sgn(\lambda_1)\sqrt{|v-2sgn(\lambda_1)|}\,\psi\left(\lambda_1-\frac{2sgn(\lambda_1)\lambda_1}{v},\lambda_2,v-2sgn(\lambda_1)\right) =$$
$$= \widehat{\sqrt{|v|}}\widehat{E}^{\pm}_2\widehat{sgn(\lambda_2)}|v|sgn(\lambda_1)\sqrt{|v-2sgn(\lambda_1)|}\,\psi\left(\lambda_1-\frac{2sgn(\lambda_1)\lambda_1}{v},\lambda_2,v-2sgn(\lambda_1)\right) =$$
$$= \widehat{\sqrt{|v|}}\widehat{E}^{\pm}_2sgn(\lambda_2)|v|sgn(\lambda_1)\sqrt{|v-2sgn(\lambda_1)|}\,\psi\left(\lambda_1-\frac{2sgn(\lambda_1)\lambda_1}{v},\lambda_2,v-2sgn(\lambda_1)\right) =$$
$$= \widehat{\sqrt{|v|}}sgn\left(\lambda_2-2\frac{sgn(\lambda_2)\lambda_2}{v}\right)|v-2sgn(\lambda_2)|sgn(\lambda_1)\sqrt{|v-2sgn(\lambda_1)-2sgn(\lambda_2)|}\times$$
$$\times\,\psi\left(\lambda_1-\frac{2sgn(\lambda_1)\lambda_1}{v-2sgn(\lambda_2)},\lambda_2-\frac{2sgn(\lambda_2)\lambda_2}{v},v-2sgn(\lambda_1)-2sgn(\lambda_2)\right) =$$
$$= \sqrt{|v|}sgn\left(\lambda_2-2\frac{sgn(\lambda_2)\lambda_2}{v}\right)|v-2sgn(\lambda_2)|sgn(\lambda_1)\sqrt{|v-2sgn(\lambda_1)-2sgn(\lambda_2)|}\times$$
$$\times\,\psi\left(\lambda_1-\frac{2sgn(\lambda_1)\lambda_1}{v-2sgn(\lambda_2)},\lambda_2-\frac{2sgn(\lambda_2)\lambda_2}{v},v-2sgn(\lambda_1)-2sgn(\lambda_2)\right) \tag{2.156}$$

If now we require symmetry under parity of the wavefuncton, we can restrict the action of the constraint on the positive octant ($\lambda_1 > 0, \lambda_2 > 0, v > 0$) of the wavefunction, obtaining

$$\left(\widehat{C}^{--}_{21}\psi(\lambda_1,\lambda_2,v)\right)\Big|_{+octant} = \left[\sqrt{|v|}(v-2)\sqrt{|v-4|}\,\right]\psi\left(\lambda_1\frac{v-4}{v-2},\lambda_2\frac{v-2}{v},v-4\right) \tag{2.157}$$

Remark: we observe that focusing on the positive octant here doesn't mean that the wavefunction has only positive arguments. At this stage we cannot exclude yet the negative arguments (as we did in the isotropic case fixing $\mathcal{L}^+_\varepsilon$ instead of $\mathcal{L}^-_\varepsilon$) since we didn't prove yet that the Hamiltonian constraint preserves such choice. We simply look at the positive arguments since the analysis of the negative ones gives exactly the same result, cause the symmetry assumption.

The action of $\widehat{C}^{--}_{21}$ is now much more clear: it multiplies the wavefunction by functions only of v, and the arguments of the wavefunction itself are modified in the following way: the volume is shifted by 4, while λ_1 and λ_2 are modified by a factor depending exclusively on the volume. Since all the other terms that compose the constraint have a similar form, its action on the state $\psi(\lambda_1,\lambda_2,v)$ depends mainly on the volume v of such state. This

feature will be crucial to analyze the comparison between such Minisuperspace and the isotropic one. Let's write down for completeness the action of the other therms $\widehat{C}_{21}^{\pm\pm}$ on such wavefunction

$$\begin{aligned}
\Big(\widehat{C}_{21}^{++}\psi(\lambda_1,\lambda_2,v)\Big)\Big|_{+octant} &= \big[\sqrt{v(v+4)}(v+2)\,\big]\psi\bigg(\lambda_1\frac{v+4}{v+2},\lambda_2\frac{v+2}{v},v+4\bigg)\\
\Big(\widehat{C}_{21}^{+-}\psi(\lambda_1,\lambda_2,v)\Big)\Big|_{+octant} &= \big[v(v+2)\,\big]\psi\bigg(\lambda_1\frac{v}{v+2},\lambda_2\frac{v+2}{v},v\bigg)\\
\Big(\widehat{C}_{21}^{-+}\psi(\lambda_1,\lambda_2,v)\Big)\Big|_{+octant} &= \big[v(v-2)\,\big]\psi\bigg(\lambda_1\frac{v}{v-2},\lambda_2\frac{v-2}{v},v\bigg)
\end{aligned}$$

The terms with i or j equal to 3 are also simpler.

We recall that the state $\psi(\lambda_1,\lambda_2,v)$ is eigenfunction of the volume operator, and we also know that the classical singularity corresponds to $a_i=0$, so $V=0$. Let's consider then a state $\psi(\lambda_1,\lambda_2,v)$ that at the initial time ϕ_0 is equal to zero for $v=0$ (has no support in $v=0$). We want see if it can reach the volume $v=0$, or better if ψ is different from zero for $v=0$ at late times. In order to do this let's decompose the Hilbert space $\mathcal{H}_{Kin}^{grav}$ in a singular and regular part: $\mathcal{H}_{Kin}^{grav}=\mathcal{H}_{sing}^{grav}\oplus\mathcal{H}_{reg}^{grav}$, where states belonging to the singular space have support only on $v=0$, while the ones belonging to the regular one have support on $\{v\in\mathbb{R},v\neq 0\}$. Now, all the operators in 2.155 have $\widehat{\sqrt{|v|}}$ on the right, so they annihilate each state belonging to $\mathcal{H}_{sing}^{grav}$. Therefore $\mathcal{H}_{sing}^{grav}$ is left invariant by time evolution. Moreover, since the prefactors in the action of the operators $\widehat{C}_{ij}^{\pm\pm}$, and in particular the fact that for each prefactor of kind $v\pm 2$ or $v\pm 4$, there is another operator with a prefactor in its action of kind $v\mp 2$ or $v\mp 4$, the Hamiltonian constraint preserves the structure of $\mathcal{H}_{reg}^{grav}$. This means that the two spaces are decoupled by the Hamiltonian constraint, and if one starts with a regular state, he ends up after any time evolution with a regular one. If we make the same well-motivated physical assumption that nowadays the Universe wavefunction has a contribution from the $|v=0\rangle$ state that is null or very small, this contribution must have been preserved during time evolution, until the Planck regime. Thus we can remove it by hand, and focusing on the regular sector of the Hilbert space. In this way we have that also in the Bianchi I case the singularity is resolved already at the Kinematical level. To analyze better the physical states of the model let's write the action of the full hamiltonian constraint

$$\begin{aligned}
\partial_\phi^2\psi(\lambda_1,\lambda_2,v,\phi) = \frac{\pi G}{2}\sqrt{v}\Big\{\big[(v+2)\sqrt{v+4}\,\big]\,\psi_4^+(\lambda_1,\lambda_2,v,\phi) - \big[(v+2)\sqrt{v}\,\big]\,\psi_0^+(\lambda_1,\lambda_2,v,\phi) -\\
-\big[(v-2)\sqrt{v}\,\big]\,\psi_0^-(\lambda_1,\lambda_2,v\phi) + \big[(v+2)\sqrt{|v-4|}\,\big]\,\psi_4^-(\lambda_1,\lambda_2,v,\phi)\Big\} \qquad (2.158)
\end{aligned}$$

where $\psi^{\pm}_{0,4}$ are defined in the following way

$$\begin{aligned}\psi^{\pm}_4(\lambda_1,\lambda_2,v,\phi) =&\psi\left(\frac{v\pm 4}{v\pm 2}\cdot\lambda_1,\frac{v\pm 2}{v}\cdot\lambda_2,v\pm 4,\phi\right)+\psi\left(\frac{v\pm 4}{v\pm 2}\cdot\lambda_1,\lambda_2,v\pm 4,\phi\right)+\\&+\psi\left(\frac{v\pm 2}{v}\cdot\lambda_1,\frac{v\pm 4}{v\pm 2}\cdot\lambda_2,v\pm 4,\phi\right)+\psi\left(\frac{v\pm 2}{v}\cdot\lambda_1,\lambda_2,v\pm 4,\phi\right)+\\&+\psi\left(\lambda_1,\frac{v\pm 2}{v}\cdot\lambda_2,v\pm 4,\phi\right)+\psi\left(\lambda_1,\frac{v\pm 4}{v\pm 2}\lambda_2,v\pm 4,\phi\right)\end{aligned}\tag{2.159}$$

and

$$\begin{aligned}\psi^{\pm}_0(\lambda_1,\lambda_2,v,\phi) =&\psi\left(\frac{v\pm 2}{v}\cdot\lambda_1,\frac{v}{v\pm 2}\cdot\lambda_2,v,\phi\right)+\psi\left(\frac{v\pm 2}{v}\cdot\lambda_1,\lambda_2,v,\phi\right)+\\&+\psi\left(\frac{v}{v\pm 2}\cdot\lambda_1,\frac{v\pm 2}{v}\cdot\lambda_2,v,\phi\right)+\psi\left(\frac{v}{v\pm 2}\cdot\lambda_1,\lambda_2,v,\phi\right)+\\&+\psi\left(\lambda_1,\frac{v}{v\pm 2}\cdot\lambda_2,v,\phi\right)+\psi\left(\lambda_1,\frac{v\pm 2}{v}\lambda_2,v,\phi\right)\end{aligned}\tag{2.160}$$

Let's analyze now the expression 2.158. Its right-hand side is a difference equation depending on v, in which the steps for v are uniform: as well as in the isotropic case the v-argument of the wavefunction is $v, v\pm 4$ as in the isotropic case (2.44).Thus the action of the Hamiltonian constraint produces *Superselection sectors*. As in the first model, we can construct a lattice $\mathcal{L}^{+}_{\varepsilon}$ with $\varepsilon\in(0,4]$ made of points

$$\mathcal{L}^{+}_{\varepsilon}=\{v=\varepsilon+4n,\quad n\in\mathbb{N}\}\tag{2.161}$$

We notice that differently from the isotropic case, since the assumption on the symmetry under parity of the wavefunctions we don't look at $\mathcal{L}^{-}_{\varepsilon}$. Well, the Hamiltonian constraint preserves the subspaces $\mathcal{H}^{\varepsilon}_{Kin}$ made of wavefunctions with argument $v\in\mathcal{L}^{+}_{\varepsilon}$, and as well as in the isotropic case we can look at solutions of the constraint in such subspace of the whole Kinematical Hilbert space. Even if we restrict our attention to a precise superselection sector, the explicit solution of 2.158 cannot be found analytically, but as in the isotropic case only numerically. We also notice that if we fix a superselection sector (or equivalently the v-argument of the wavefunctions), we cannot do the same for λ_1 and λ_2. This because if we start with a state with support on a lattice $\lambda=n\lambda_0$, the Hamiltonian operator modifies such support on the lattice $\lambda=\left(\frac{v\pm 2}{v}\right)n\lambda_0$, that is completely different from the initial one. Although in this model we have to deal with this further complication that produces difficulties to interpret and control the anisotropies of the Universe, we have that the evolution in volume is qualitatively similar to the one

of the isotropic model. For what concerns the anisotropies we can notice that the λ factors appear only in the arguments of the wavefunctions, and not on the prefactors. Moreover, they appear rescaled by factors of kind $\frac{v\pm a}{v\pm b}$, where $a, b = 0, 2, 4$. This means that for large volume (low density) they go to 1, and more precisely as $1 + O\big(\frac{\rho_{mat}}{\rho_P}\big)$. This means that in such semiclassical regime the quantum correctionsfor the anisotropies become smaller and smaller. As well as we did in the isotropic case, since the difficulty to find the analytic solution of the Hamiltonian constraint, in the next section we introduce briefly the effective equations for this anisotropic model, and through them we study the solution in a qualitative way.

2.2.5 Effective dynamics in Bianchi I

As we did in the isotropic case we can look at the effective dynamics of the Bianchi I quantization, in order to capture some qualitative feature of the exact solution. Following the same procedure of the isotropic case, in order to reach the effective equations of Bianchi I we can compute the mean value of the Hamiltonian constraint on semiclassical states

$$\langle \psi_{SC}| \, \widehat{C} \, |\psi_{SC}\rangle \tag{2.162}$$

with $\widehat{C_{grav}}$ given simply by 2.139 and $|\psi_{SC}\rangle$ peaked on given p_1, p_2, p_3. Thus we have

$$C^{eff.} = C^{eff.}_{grav} + C^{eff.}_{matt} = 0 \tag{2.163}$$

with

$$C^{eff.}_{grav} = -\frac{p_1 p_2 p_3}{8\pi G\gamma^2\Delta}\big[sin(\bar{\mu}_1 c_1)sin(\bar{\mu}_2 c_2) + sin(\bar{\mu}_2 c_2)sin(\bar{\mu}_3 c_3) + sin(\bar{\mu}_3 c_3)sin(\bar{\mu}_1 c_1)\big] \tag{2.164}$$

Looking at 2.164 we notice that since $sin(x)$ is at most equal to 1, we have

$$\frac{-3}{8\pi G\gamma^2\Delta} + \rho_{matt.} \leq 0 \quad \Longleftrightarrow \quad \rho_{matt.} \leq \frac{3}{8\pi G\gamma^2\Delta} = \rho_{crit.} \tag{2.165}$$

where the critical density $\rho_{crit.}$ is exactly the one found in the isotropic case ($\rho_{crit.} \sim 0.41\rho_P$). Thus we see already at this stage that also the effective evolution of this model avoids the classical singularity. As we did in the isotropic case, effective equations are derived by solving the Hamilton equations of motion for the variables p_i and c_i. If we

make calculations we obtain

$$\frac{dp_1}{d\tau} = \frac{p_1\sqrt{p_1p_2p_3}}{\sqrt{\Delta}\gamma}cos(\bar{\mu}_1c_1)\big[sin(\bar{\mu_2}c_2 + sin(\bar{\mu_3}c_3)\big] \tag{2.166}$$

$$\begin{aligned}\frac{dc_1}{d\tau} = &-\frac{p_2p_3}{\Delta\gamma}\big[sin(\bar{\mu_1}c_1)sin(\bar{\mu_2}c_2) + sin(\bar{\mu_1}c_1)sin(\bar{\mu_3}c_3) + sin(\bar{\mu_2}c_2)sin(\bar{\mu_3}c_3)+\\ &+ \frac{\bar{\mu_1}c_1}{2}cos(\bar{\mu_1}c_1)(sin(\bar{\mu_2}c_2) + sin(\bar{\mu_3}c_3)) - \frac{\bar{\mu_2}c_2}{2}cos(\bar{\mu_2}c_2)(sin(\bar{\mu_1}c_1) + sin(\bar{\mu_3}c_3))\\ &- \frac{\bar{\mu_3}c_3}{2}cos(\bar{\mu_3}c_3)(sin(\bar{\mu_1}c_1) + sin(\bar{\mu_2}c_2))\big]\end{aligned} \tag{2.167}$$

equatons for the other phase space variables are obtained through cyclic permutations. If we assume small $\bar{\mu}_ic_i$ we recover the classical relations 2.108 and 2.109. Moreover if we compute the effective shear term, so the quantum corrected 2.115 we have that it is no longer constant along physical solutions, and at late times it tends to a certain value, equal in the contracting and expanding phase.

Numerical simulations have been made to solve this set of equations, and it has been found that instead of having a single bounce, we have a bounce for each anisotropic direction. This can be physically understood since cause anisotropy the wavefunction doesn't reach the minimum spatial lenght at the same time ϕ in all the directions, so reaches more times the Planck regime (in energy density).

Such simulations also show that the bounces really happen for $\rho < \rho_{crit.}$. This is interpreted with the fact that what matters for the bounce is the whole energy density of the system, and in an anisotropic dynamics also gravitational waves play a role.

2.2.6 Accuracy of the effective equations for the Bianchi I Minisuperspace

In this section we follow the same considerations about the effective dynamics in the isotropic Minisuperspace to show why also in the anisotropic case we expect that the effective equations reproduce well the numerical simulations of the exact dynamcs. As in the isotropic case we evaluate the fluctuations of the observables and we show that they can be sent to 0 for observables that capture "large" degrees of freedom of the gravitational field.

Firslty we show the accuracy of the effective equations for the Wheeler-DeWitt theory:

we start with the classical phase space given by

$$\begin{cases} \tilde{c}_i = c_i L_i^{-1} \\ \tilde{p}_1 = p_1 (L_2 L_3)^{-1}, \quad \tilde{p}_2 = p_2 (L_1 L_3)^{-1}, \quad \tilde{p}_3 = p_3 (L_1 L_2)^{-1} \end{cases} \tag{2.168}$$

With this choice the symplectic structure of the classical theory:

$$\{\tilde{c}^i, \tilde{p}_j\} = \frac{8\pi G\gamma}{V_0} \delta^i{}_j \tag{2.169}$$

We can proceed directly with the quantization, obtaining

$$\left[\widehat{\tilde{c}}^i, \widehat{\tilde{p}}_j\right] = i\hbar \frac{8\pi G\gamma}{V_0} \delta^i{}_j \tag{2.170}$$

From the previous one, using the Robertson inequality:

$$\Delta\tilde{c}^i \Delta\tilde{p}_j \geq \frac{4\pi G\hbar\gamma}{V_0} \delta^i{}_j \tag{2.171}$$

where if we send $V_0 \to +\infty$ the product of the uncertainties goes to 0. This means that with the choice of a large fiducial cell $\mathcal{V}$ physical states can be sharply-peaked on some point of the phase space $(\tilde{c}^i, \tilde{p}_i)$ at each time, showing so semiclassicality. Let's proceed now with the Loop quantization of this model.
In this section, as in the previous one we restrict our attention on the positive octant ($p_i \geq 0$). We smear as before the variables $\tilde{c}_i$ through the holonomies with components given by combinations of functions of kind

$$exp(i\mu_i L_i \tilde{c}_i) \tag{2.172}$$

that also here we consider as three of the six variables of the classical phase space. To loop quantize the theory we firstly evaluate the classical Poisson brakets between the variables of the phase space. However for problems of operator ordering ambiguity we look at $\tilde{V}^2$ instead of $\tilde{V}$. The considerations we make are equivalent in the two cases.

$$\left\{exp(i\mu_i L_i \tilde{c}_i), \tilde{V}^2\right\} = \frac{8\pi G\gamma}{V_0} \left[\frac{\partial}{\partial \tilde{c}_i}\left(e^{i\mu_i L_i \tilde{c}_i}\right) \frac{\partial}{\partial \tilde{p}_i} \tilde{p}_1 \tilde{p}_2 \tilde{p}_3\right] = \frac{i4\pi G\gamma \mu_i L_i}{V_0} \frac{(\tilde{p}_1 \tilde{p}_2 \tilde{p}_3)}{\tilde{p}_i} e^{i\mu_i L_i \tilde{c}_i} \tag{2.173}$$

We notice that in the previous result there is no problem for the ordering since $\frac{\tilde{p}_1\tilde{p}_2\tilde{p}_3}{\tilde{p}_i}$ and $e^{i\mu_i L_i \tilde{c}_i}$ Poisson-commute. Then we apply the Dirac rule, obtaining

$$[\widehat{e^{i\mu_i L_i \tilde{c}_i}}, \widehat{\tilde{V}}^2] = -\frac{\hbar 4\pi G\gamma \mu_i L_i}{V_0} \frac{\widehat{\tilde{p}}_1\widehat{\tilde{p}}_2\widehat{\tilde{p}}_3}{\widehat{\tilde{p}}_i} \widehat{e^{i\mu_i L_i \tilde{c}_i}} \tag{2.174}$$

Similarly:

$$\{exp(i\mu_i \tilde{c}_i L_i), \tilde{p}_j\} = \frac{8\pi G\gamma}{V_0}\left[\frac{\partial}{\partial \tilde{c}_i}(e^{i\mu_i L_i \tilde{c}_i})\frac{\partial}{\partial \tilde{p}_i}\tilde{p}_j\right] = \frac{i8\pi G\gamma L_i \mu_i}{V_0} e^{i\mu_i L_i \tilde{c}_i}\delta_{ij} \tag{2.175}$$

And their quantization:

$$\left[\widehat{e^{i\mu_i L_i \tilde{c}_i}}, \widehat{\tilde{p}}_j\right] = -\frac{8\pi G\gamma\hbar\mu_i L_i}{V_0}\widehat{e^{i\mu_i \tilde{c}_i L_i}}\delta_{ij} \tag{2.176}$$

The other Poisson brakets (thus also the commutators) vanish.
With this in hand we can compute the product of the uncertainties:

$$\Delta\tilde{V}^2 \Delta\left(\frac{sin(\mu_i L_i \tilde{c}_i)}{\mu_i L_i}\right) \geq \frac{1}{2}\left|\langle\left[\widehat{\tilde{V}}^2, \widehat{\frac{sin(\mu_i L_i \tilde{c}_i)}{\mu_i L_i}}\right]\rangle\right| = \frac{2\pi G\hbar\gamma}{V_0}\left|\langle\frac{\widehat{\tilde{p}}_1\widehat{\tilde{p}}_2\widehat{\tilde{p}}_3}{\widehat{\tilde{p}}_i}\widehat{cos(i\mu_i L_i \tilde{c}_i)}\rangle\right| \tag{2.177}$$

$$\Delta\tilde{p}_j \Delta\left(\frac{sin(\mu_i L_i \tilde{c}_i)}{\mu_i L_i}\right) \geq \frac{1}{2}\left|\langle\left[\widehat{\tilde{p}}_j, \widehat{\frac{sin(\mu_i L_i \tilde{c}_i)}{\mu_i L_i}}\right]\rangle\right| = \frac{4\pi G\hbar\gamma}{V_0}|\langle\widehat{cos(i\mu_i L_i \tilde{c}_i)}\rangle|\delta_{ij} \tag{2.178}$$

Thus also in the loop quantization we notice that for states of the Kinematical Hilbert space for which $V_0 \to +\infty$ the product of the uncertainties goes to 0. This means that there exists a large class of initial states (the ones with large V_0) for which the exact dynamics is governed by the effective Hamiltonian. This explains why for such initial states the numerical simulations and the effective solutions are in strong agreement.
As in the isotropic case we can show the same concept from a different perspective: since numerical simulations of the effective equations show the existence of bounces, thus a minimum for the volume $\tilde{V}$, we can see for what value of $\tilde{V}$ (that we call $\tilde{V}_{qf}$) the quantum fluctuations become important, and compare such value with the minimum volume that the solution can assume ($\tilde{V}_{Bounce}$).
To make this calculation we assume as in the isotropic case as source a scalar field (perfect fluid) with a constant equation of state

$$p = \omega\rho \tag{2.179}$$

with $-1 \leq \omega \leq 1$. Since we have to deal with the quantum Hamiltonian constraint, as in the previous paragraph we need to consider the holonomies along paths of fiducial lenght $\bar{\mu}_i L_i$, where $\bar{\mu}_i$ are given by the Improved dynamics prescription 2.129:

$$\begin{aligned} exp(i\bar{\mu}_1 L_1 \tilde{c}_1) =& exp\left(i\sqrt{\frac{p_1\Delta}{p_2 p_3}} L_1 \tilde{c}_1 \right) = exp\left(i\sqrt{\frac{\tilde{p}_1 L_2 L_3 \Delta}{\tilde{p}_2 L_1 L_3 \tilde{p}_3 L_1 L_2}} L_1 \tilde{c}_1 \right) = \\ =& exp\left(i\sqrt{\frac{\tilde{p}_1\Delta}{\tilde{p}_2\tilde{p}_3}} \tilde{c}_1 \right) \end{aligned} \tag{2.180}$$

The other two are obtained with cyclic permutations of the indices.
As in the isotropic case here we use the following notation $\lambda = \sqrt{\Delta}$. The effective Hamiltonian of the system in such variables:

$$\begin{aligned} C^{eff.} = -\frac{V_0 \tilde{V}}{8\pi G\gamma^2\lambda^2} \big[& sin(\bar{\mu}_1\tilde{c}_1 L_1) sin(\bar{\mu}_2\tilde{c}_2 L_2) + sin(\bar{\mu}_2\tilde{c}_2 L_2) sin(\bar{\mu}_3\tilde{c}_3 L_3) + \\ & + sin(\bar{\mu}_1\tilde{c}_1 L_1) sin(\bar{\mu}_3\tilde{c}_3 L_3) \big] + (V_0\tilde{V})^2 \rho \end{aligned} \tag{2.181}$$

with

$$\rho = \frac{\rho_0}{\tilde{V}^n} \tag{2.182}$$

and $n = 1 + \omega \in [0, 2]$. It can be easily shown by the continuity equation for ρ that nor ρ neither ρ_0 depend on V_0.
Now, in order to find $\tilde{V}_{qf}$, so the volume at which quantum fluctuations become relevant we need to evaluate the following:

$$\Delta\tilde{V}\Delta\left(\frac{sin(\bar{\mu}_1\tilde{c}_1 L_1)}{\lambda} \right) \tag{2.183}$$

Using the Robertson inequality we have

$$\Delta\tilde{V}\Delta\left(\frac{sin(\bar{\mu}_1\tilde{c}_1 L_1)}{\lambda} \right) \geq \frac{2\pi G\gamma\hbar}{V_0} |\langle \widehat{cos(\bar{\mu}_1\tilde{c}_1 L_1)} \rangle| \tag{2.184}$$

In fact

$$\Delta\tilde{V}\Delta\left(\frac{sin(\bar{\mu}_1\tilde{c}_1L_1)}{\lambda}\right) \geq \frac{1}{2\lambda}\left|\langle\left[\widehat{\tilde{V}}, sin\left(\widehat{\sqrt{\frac{\tilde{p}_1\Delta}{\tilde{p}_2\tilde{p}_3}}\tilde{c}_1}\right)\right]\rangle\right| = \frac{1}{2\lambda}\left|\langle i\hbar\left\{\tilde{V}, sin\left(\sqrt{\frac{\tilde{p}_1\Delta}{\tilde{p}_2\tilde{p}_3}}\tilde{c}_1\right)\right\}\rangle\right| =$$
$$= \frac{4\pi\hbar G\gamma}{V_0\lambda}\left|\langle\left(\frac{\partial}{\partial\tilde{p}_1}\tilde{V}\right)\frac{\partial}{\partial\tilde{c}_1}sin\left(\sqrt{\frac{\tilde{p}_1\Delta}{\tilde{p}_2\tilde{p}_3}}\tilde{c}_1\right)\rangle\right| = \frac{2\pi\hbar G\gamma}{V_0}\left|\sqrt{\frac{\tilde{p}_2\tilde{p}_3}{\tilde{p}_1}}cos\left(\sqrt{\frac{\tilde{p}_1\Delta}{\tilde{p}_2\tilde{p}_3}}\tilde{c}_1\right)\sqrt{\frac{\tilde{p}_1}{\tilde{p}_2\tilde{p}_3}}\right| =$$
$$= \frac{2\pi\hbar G\gamma}{V_0}|\langle\widehat{cos(\bar{\mu}_1\tilde{c}_1}L_1)\rangle| \tag{2.185}$$

Then we evaluate

$$\Delta\tilde{V}\Delta\left(\frac{sin(\bar{\mu}_1\tilde{c}_1L_1)}{\lambda}\frac{sin(\bar{\mu}_2\tilde{c}_2L_2)}{\lambda}\right) \geq \frac{1}{2}\left|\langle\left[\widehat{\tilde{V}}, \frac{\widehat{sin(\bar{\mu}_1\tilde{c}_1}L_1)}{\lambda}\right]\frac{\widehat{sin(\bar{\mu}_2\tilde{c}_2}L_2)}{\lambda}+\right.$$
$$\left.+\frac{\widehat{sin(\bar{\mu}_1\tilde{c}_1}L_1)}{\lambda}\left[\widehat{\tilde{V}}, \frac{\widehat{sin(\bar{\mu}_2\tilde{c}_2}L_2)}{\lambda}\right]\rangle\right| = \frac{2\pi G\hbar\gamma}{V_0\lambda}\left|\langle\widehat{cos(\bar{\mu}_1\tilde{c}_1}L_1)\widehat{sin(\bar{\mu}_2\tilde{c}_2}L_2) + \widehat{sin(\bar{\mu}_1\tilde{c}_1}L_1)\widehat{cos(\bar{\mu}_2\tilde{c}_2}L_2)\rangle\right| \tag{2.186}$$

We want to reconstruct the uncertainty on the gravitational part of the effective Hamiltonian constraint. To do this we evaluate

$$\Delta\tilde{V}\Delta\left(\frac{sin(\bar{\mu}_1\tilde{c}_1L_1)}{\lambda}\frac{sin(\bar{\mu}_2\tilde{c}_2L_2)}{\lambda} + \frac{sin(\bar{\mu}_1\tilde{c}_1L_1)}{\lambda}\frac{sin(\bar{\mu}_3\tilde{c}_3L_3)}{\lambda} + \frac{sin(\bar{\mu}_2\tilde{c}_2L_2)}{\lambda}\frac{sin(\bar{\mu}_3\tilde{c}_3L_3)}{\lambda}\right) \geq$$
$$\geq \frac{2\pi G\hbar\gamma}{V_0\lambda}\left|\langle\widehat{cos(\bar{\mu}_1\tilde{c}_1}L_1)\widehat{sin(\bar{\mu}_2\tilde{c}_2}L_2) + \widehat{sin(\bar{\mu}_1\tilde{c}_1}L_1)\widehat{cos(\bar{\mu}_2\tilde{c}_2}L_2) + \widehat{cos(\bar{\mu}_1\tilde{c}_1}L_1)\widehat{sin(\bar{\mu}_3\tilde{c}_3}L_3)+\right.$$
$$\left.+\widehat{sin(\bar{\mu}_1\tilde{c}_1}L_1)\widehat{cos(\bar{\mu}_3\tilde{c}_3}L_3) + \widehat{cos(\bar{\mu}_2\tilde{c}_2}L_2)\widehat{sin(\bar{\mu}_2\tilde{c}_3}L_3) + \widehat{sin(\bar{\mu}_2\tilde{c}_2}L_2)\widehat{cos(\bar{\mu}_3\tilde{c}_3}L_3)\rangle\right| \tag{2.187}$$

Now, since we are assuming large V_0, we can consider the validity of the effective Hamiltonian constraint 2.181, and plug it in the left-hand side of the previous relation, obtaining:

$$\Delta\tilde{V}\Delta\left(\frac{8\pi G\gamma^2V_0\rho_0}{\tilde{V}^{n-1}}\right) \geq \frac{2\pi G\hbar\gamma}{V_0\lambda}\left|\langle\widehat{cos(\bar{\mu}_1\tilde{c}_1}L_1)\widehat{sin(\bar{\mu}_2\tilde{c}_2}L_2) + \widehat{sin(\bar{\mu}_1\tilde{c}_1}L_1)\widehat{cos(\bar{\mu}_2\tilde{c}_2}L_2)+\right.$$
$$+\widehat{cos(\bar{\mu}_1\tilde{c}_1}L_1)\widehat{sin(\bar{\mu}_3\tilde{c}_3}L_3) + \widehat{sin(\bar{\mu}_1\tilde{c}_1}L_1)\widehat{cos(\bar{\mu}_3\tilde{c}_3}L_3)+$$
$$\left.+\widehat{cos(\bar{\mu}_2\tilde{c}_2}L_2)\widehat{sin(\bar{\mu}_2\tilde{c}_3}L_3) + \widehat{sin(\bar{\mu}_2\tilde{c}_2}L_2)\widehat{cos(\bar{\mu}_3\tilde{c}_3}L_3)\rangle\right| \tag{2.188}$$

As in the isotropic case we can assume $\Delta\tilde{V}$ small, and write

$$\Delta\left(\frac{1}{\tilde{V}^{n-1}}\right) \sim \frac{n-1}{\tilde{V}^{n-2}}\Delta\tilde{V} \tag{2.189}$$

Substituting it in 2.188, and assuming that the relation is saturated we have

$$\frac{\Delta\tilde{V}^2}{\tilde{V}^{n-2}} = \frac{\hbar}{4\gamma\lambda^2\rho_0 V_0^2(n-1)}\Big|\langle\widehat{cos(\bar{\mu}_1\tilde{c}_1}L_1)\widehat{sin(\bar{\mu}_2\tilde{c}_2}L_2) + \widehat{sin(\bar{\mu}_1\tilde{c}_1}L_1)\widehat{cos(\bar{\mu}_2\tilde{c}_2}L_2)+$$
$$+ \widehat{cos(\bar{\mu}_1\tilde{c}_1}L_1)\widehat{sin(\bar{\mu}_3\tilde{c}_3}L_3) + \widehat{sin(\bar{\mu}_1\tilde{c}_1}L_1)\widehat{cos(\bar{\mu}_3\tilde{c}_3}L_3)+$$
$$+ \widehat{cos(\bar{\mu}_2\tilde{c}_2}L_2)\widehat{sin(\bar{\mu}_2\tilde{c}_3}L_3) + \widehat{sin(\bar{\mu}_2\tilde{c}_2}L_2)\widehat{cos(\bar{\mu}_3\tilde{c}_3}L_3)\rangle\Big| \tag{2.190}$$

The volume $\tilde{V}_{qf}$ for which the quantum fluctuations become important is given imposing $\tilde{V} \sim \Delta\tilde{V}$:

$$\tilde{V}_{qf} = \left(\frac{\hbar}{4\gamma\lambda\rho_0 V_0^2(n-1)}\right)^{\frac{1}{4-n}}\Big|\langle\widehat{cos(\bar{\mu}_1\tilde{c}_1}L_1)\widehat{sin(\bar{\mu}_2\tilde{c}_2}L_2) + \widehat{sin(\bar{\mu}_1\tilde{c}_1}L_1)\widehat{cos(\bar{\mu}_2\tilde{c}_2}L_2)+$$
$$+ \widehat{cos(\bar{\mu}_1\tilde{c}_1}L_1)\widehat{sin(\bar{\mu}_3\tilde{c}_3}L_3) + \widehat{sin(\bar{\mu}_1\tilde{c}_1}L_1)\widehat{cos(\bar{\mu}_3\tilde{c}_3}L_3)+$$
$$+ \widehat{cos(\bar{\mu}_2\tilde{c}_2}L_2)\widehat{sin(\bar{\mu}_2\tilde{c}_3}L_3) + \widehat{sin(\bar{\mu}_2\tilde{c}_2}L_2)\widehat{cos(\bar{\mu}_3\tilde{c}_3}L_3)\rangle\Big|^{\frac{1}{4-n}} \tag{2.191}$$

And we notice that independently from the precise value of n, for V_0 arbitrarily large $\tilde{V}_{qf}$ is arbitrarily small. Let's look now at $\tilde{V}_{Bounce}$. To find it we have to solve the following

$$\frac{d\tilde{V}}{d\tau} = \{\tilde{V}, C\} \tag{2.192}$$

Using 2.181 we have

$$\frac{\partial\tilde{V}}{\partial\tau} = -\{C, \tilde{V}\} = -\frac{8\pi G\gamma}{V_0}\left(\frac{\partial C}{\partial\tilde{c}_1}\frac{\partial\tilde{V}}{\partial\tilde{p}_1} + \frac{\partial C}{\partial\tilde{c}_2}\frac{\partial\tilde{V}}{\partial\tilde{p}_2} + \frac{\partial C}{\partial\tilde{c}_3}\frac{\partial\tilde{V}}{\partial\tilde{p}_3}\right) = \tag{2.193}$$
$$= \frac{\tilde{V}}{\gamma\lambda^2}\Big\{\big[cos(\bar{\mu}_1\tilde{c}_1L_1)sin(\bar{\mu}_2\tilde{c}_2L_2) + cos(\bar{\mu}_1\tilde{c}_1L_1)sin(\bar{\mu}_3\tilde{c}_3L_3)\big]\frac{\bar{\mu}_1L_1\tilde{V}}{2\tilde{p}_1}+$$
$$+ \big[sin(\bar{\mu}_1\tilde{c}_1L_1)cos(\bar{\mu}_2\tilde{c}_2L_2) + cos(\bar{\mu}_2\tilde{c}_2L_2)sin(\bar{\mu}_3\tilde{c}_3L_3)\big]\frac{\bar{\mu}_2L_2\tilde{V}}{2\tilde{p}_2}+$$
$$+ \big[sin(\bar{\mu}_2\tilde{c}_2L_2)cos(\bar{\mu}_3\tilde{c}_3L_3) + +sin(\bar{\mu}_1\tilde{c}_1L_1)cos(\bar{\mu}_3\tilde{c}_3L_3)\big]\frac{\bar{\mu}_3L_3\tilde{V}}{2\tilde{p}_3}\Big\} \tag{2.194}$$

Recalling 2.129, we can simplify it

$$\frac{\partial\tilde{V}}{\partial\tau} = \frac{\tilde{V}}{2\gamma\lambda}\big\{sin(\bar{\mu}_2\tilde{c}_2L_2 + \bar{\mu}_1\tilde{c}_1L_1) + sin(\bar{\mu}_3\tilde{c}_3L_3 + \bar{\mu}_1\tilde{c}_1L_1) + sin(\bar{\mu}_2\tilde{c}_2L_2 + \bar{\mu}_3\tilde{c}_3L_3)\big\} \tag{2.195}$$

We notice that if we restore isotropy $\bar{\mu}_1 = \bar{\mu}_2 = \bar{\mu}_3 \equiv \bar{\mu}$, $L_1 = L_2 = L_3 \equiv V_0^{\frac{1}{3}}$, and $\tilde{c}_1 = \tilde{c}_2 = \tilde{c}_3 \equiv \tilde{c}$ we recover exactly the isotropic equation of motion for $\tilde{V}$ (2.84).
Well, even if the solution of such equation is highly non-trivial, we notice that it doesn't contain V_0. This has to hold also for the solution. Moreover numerical simulations [26] confirm that the solution bounces in the deep Planck regime, thus even if we cannot find analytically $\tilde{V}_{bounce}$ we know that exists. So the solution is bounded from below, and such bound doesn't depend on V_0. Thus for V_0 sufficiently large, independently from the exact value of $\tilde{V}_{Bounce}$ the solutions never reach the value of the volume $\tilde{V}_{qf}$ for which quantum fluctuations become important (so the regime in which the effective equations loose their validity).
This argument proves the validity of the effective equations in Bianchi I for large V_0 initial states, and justifies formally the strong agreement between the numerical simulations of the effective dynamics and the analytic one.

Appendix A

Wheeler-DeWitt theory

To quantize General Relativity following the Wheeler-DeWitt approach we use the Dirac program for the quantization of constrained systems, starting from the classical hamiltonian formulation of Einstein gravity in terms of the canonical variables $q_{ab}(x)$ and $\pi^{cd}(y)$. Thus we promote the phase space variables $q_{ab}(x)$ and $\pi^{cd}(y)$ to quantum operators, and we promote the classical Poisson brakets to commutators:

$$[\widehat{q}_{ab}(x), \widehat{\pi}^{cd}(y)] = i\hbar\delta^{cd}_{(ab)}\delta^3(x-y)$$
$$[\widehat{q}_{ab}(x), \widehat{q}_{cd}(y)] = 0$$
$$[\widehat{\pi}^{ab}(x), \widehat{\pi}^{cd}(y)] = 0$$

where: $\delta^{cd}_{(ab)} \equiv \frac{1}{2}(\delta^c{}_a\delta^d{}_b + \delta^c{}_b\delta^d{}_a)$
. Here $\widehat{\pi}$ and $\widehat{q}$ are abstract operators written in this way. We can look at a representation of them in terms of the generalized coordinates:

$$\widehat{q}_{ab}(x) \doteq q_{ab}(x) \ , \quad \widehat{\pi}^{ab}(x) \doteq -i\hbar\frac{\delta}{\delta q_{ab}(x)}$$

that is the analogous of the coordinate representation for classical QM. These operators act on states ψ which are wavefunctional of the form: $\psi[q_{ab}(x)]$, written in the same representation. At this point we have to face the first serious issue of this approach: we should define a scalar product for the space of configurations of q, and this is not simple. But let's ignore for the moment this problem and proceed with the program. Supposing that we have been able to construct our Kinematical Hilbert space with a well defined measure, now we have to promote the classical constraints to quantum operators: $\widehat{H}^\mu\psi = 0$
We can do this in two steps:

$$\mathcal{H}_{kin} \xrightarrow{\widehat{\mathcal{H}}^a=0} \mathcal{H}_{diff.} \xrightarrow{\widehat{\mathcal{H}}^0=0} \mathcal{H}_{phys.}$$

So we pass from the initial Kinematical Hilbert space, to the physical Hilbert space, with an intermediate step given by the Hilbert space of Diffeo. invariant states: $\mathcal{H}_{diff.}$. For the constraint equations we will use their smeared version. Let proceed with order and solve the first equation:

$$\int_\Sigma d^3x N_a \widehat{H}^a \psi[q_{ab}] = 0$$

We recall that:

$$H^a = -2\sqrt{q}\nabla_b\left(\frac{\pi^{ab}}{\sqrt{q}}\right) \quad \Longrightarrow \quad \widehat{H}^a \doteq -2\sqrt{q}\nabla_c\left[\left(\frac{1}{\sqrt{q}}\right)\left(-i\hbar\frac{\delta}{\delta q_{ca}}\right)\right]$$

Thus, the wavefunctionals belonging to $\mathcal{H}_{diff.}$ have to satisfy

$$2i\hbar\int d^3x N_a\sqrt{q}\nabla_c\left[\frac{1}{\sqrt{q}}\left(\frac{\delta}{\delta q_{ca}}\right)\right]\psi[q_{ca}] = 0$$

Now, remembering that

$$\nabla_c(A^bB^c) = (\nabla_c A^b)B^c + (\nabla_c B^c)A^b$$

We have

$$2i\hbar\int d^3x\nabla_c\left(N_a\frac{1}{\sqrt{q}}\frac{\delta}{\delta q_{ca}}\psi[q_{ca}]\right)\sqrt{q} - 2i\hbar\int d^3x\nabla_c N_a\frac{\delta\psi}{\delta q_{ca}} =$$
$$-2i\hbar\int d^3x\nabla_b N_c\frac{\delta\psi}{\delta q_{bc}} = 0$$

where in the last line we neglect boundary terms using Gauss theorem. Now, since q_{bc} is symmetric under the exchange b, c we can write:

$$2i\hbar\int d^3x\nabla_{(b}N_{c)}\frac{\delta\psi}{\delta q_{bc}} = 0$$

And remembering that $2\nabla_{(b}N_{c)} = \delta_{\vec{N}}q_{bc}$

$$2i\hbar\int d^3x(\delta_{\vec{N}}q_{bc})\frac{\delta\psi}{\delta q_{bc}} = i\hbar\int d^3x\delta_{\vec{N}}\psi = 0$$

Which is solved by functionals that are invariant under spatial diffeomorphisms, i.e.

$$\psi[q_{bc} + 2\nabla_{(b}N_{c)}] = \psi[q_{bc}]$$

So we recovered a condition that allow us to construct $\mathcal{H}_{diff.}$. Here however arises the same problem we found previously: we are not able to define a scalar product for this Hilbert space, so we cannot proceed with the Dirac program. If however we omit also this issue and we impose the Hamiltonian constraint at the quantum level, we obtain

$$\widehat{H}(N)\psi(q_{ab}) = 0 \tag{A.1}$$

A.1 is called *Wheeler-DeWitt equation* in vacuum. Even if we can write it explicitally, in general we cannot find any generic property for the solution as we did in the previous case. If there are issues in this theory like the construction of the Hilbert spaces brought to the formulation of Loop Quantum gravity, there are other issues like the problem of time and the ambiguity of the operator ordering that this theory shares with LQG. We don't focus on them here but directly in the Loop context.

Appendix B

A toy model for Einstein Gravity: the free relativistic particle

Let's consider the following classical action

$$S = -m \int d\tau \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} \tag{B.1}$$

This is the action of a free relativistic particle in a fixed Minkowski space-time; m is the rest mass of the particle and τ the proper time.
The action B.1 is invariant under the reparametrization

$$\tau \longrightarrow \lambda = \lambda(\tau) \tag{B.2}$$

We can look at this theory as governing the dynamics of a relativistic particle, but also in a different way: instead of considering the variables x^μ as the space-time coordinates of the particle we can consider them as four different fields that live in a 1-D space parametrized with τ. In this perspective the Lorentz-Poincarèe transformations under which the theory is invariant have to be seen as internal symmetries of the fields x^μ, while B.2 is the only coordinate transformation that leaves the action unchanged.
If in General Relativity a generic diffeomorphism takes the form

$$x^\mu \longrightarrow x'^\mu = x^\mu + \varepsilon \xi^\mu(x), \quad \varepsilon << 1 \tag{B.3}$$

here a generic diffeomorphism, that is the infinitesimal version of B.2 has the form:

$$\tau \longrightarrow \tau' = \tau + \varepsilon f(\tau), \quad \varepsilon << 1 \tag{B.4}$$

Well, under this transformation the fields are subjected to an infinitesimal variation δx^μ:

$$\delta x^\mu \longrightarrow x'^\mu = x^\mu + \delta x'^\mu \tag{B.5}$$

that is a gauge transformation of the theory.
Since the theory is invariant under such gauge transformation of the fields, we can consider it as a constrained system, and as we expect from a theory that shows gauge invariance once we formulate it in the Hamiltonian form we expect a first class constraint that generate such gauge transformation. Let's construct explicitally such first class constraint in the Hamiltonian context.
From the action B.1 we can introduce the canonical momentum

$$p_\mu = \frac{\partial L}{\partial \dot{x}_\mu} = \frac{m\dot{x}^\mu}{\sqrt{-\dot{x}_\alpha \dot{x}^\alpha}} \tag{B.6}$$

From the previous relation and the lagrangian of this system we can construct directly the canonical Hamiltonian of the model

$$H_c = p_\mu \dot{x}^\mu - L = \frac{m\dot{x}^\mu \dot{x}_\mu}{\sqrt{-\dot{x}^\alpha \dot{x}_\alpha}} + m\sqrt{-\dot{x}_\mu \dot{x}^\mu} = 0 \tag{B.7}$$

The fact that the canonical Hamiltonian vanishes is a signal that such system is generally covariant, as well as Einstein theory of gravity.
We can easily recover from B.6 the only primary constraint of this system called Hamiltonian constraint:

$$H_0 = p_\mu p^\mu + m^2 = 0 \tag{B.8}$$

In fact, from B.6:

$$p_\mu p^\mu = \frac{m^2 \dot{x}^\mu \dot{x}_\mu}{-\dot{x}^\alpha \dot{x}_\alpha} = -m^2 \tag{B.9}$$

It is primary since we don't need to use the equations of motion to derive it. Moreover, since such constraint is unique is also a first class constraint (it Poisson commute with itself). It can be shown that the Hamiltonian constraint is the generator of the gauge transformations of the theory. Following the Dirac prescription for constrained systems we define the Hamiltonian of the system:

$$H = H_c + NH_0 = NH_0 = N(p_\mu p^\mu + m^2) \tag{B.10}$$

Where $N(\tau)$ is the Lagrange multiplier associated with the constraint.
Following the Dirac theory only physical solutions live on the hypersurface of the phase space

$$H = 0 \tag{B.11}$$

thus using the Dirac notation $H \approx 0$. But looking at B.10 this means that $H_0 \approx 0$, that is true only if

$$p^\mu \approx \frac{m\dot{x}^\mu}{\sqrt{-\dot{x}^\alpha \dot{x}_\alpha}} \tag{B.12}$$

This implies that for unphysical states p^μ cannot be evaluated using B.6.
We notice that the Hamiltonian has the form (Lagrange multiplier)$\times$(constraint), and this means that by definition the system is generally covariant. The action with the Dirac prescription:

$$S(x^\mu, N) = \int d\tau [p_\mu \dot{x}^\mu - N(p_\mu p^\mu + m^2)] \tag{B.13}$$

As we can see explicitally in B.10 H and H_0 generate the same transformations, and since H generates the time translations while H_0 the gauge transformations of the theory, this system as well as the Einstein theory is affected by the problem of time (the gauge orbits and the dynamical orbits on the constraint surface coincide).

Bibliography

1. Misner C.W., Thorne K.S. Wheeler J.A., *Gravitation.* Princeton University Press, 2017.

2. Schutz B.F. "A First Course in General Relativity". *Cambridge University Press*, 2009.

3. Schutz B.F. "Geometrical methods of mathematical physics". *Cambridge University Press*, 2009.

4. Rovelli C. "Quantum Gravity". *Cambridge University Press*, 2010.

5. Isham C.J., "Modern differential geometry for physicists". World Scientific Lecture Notes in Physics - Vol. 61

6. Martin-Dussaud P. "A primer of Group Theory for Loop Quantum Gravity and Spin-Foams". arXiv:1902.08439v3 [math-ph], 10.1007/s10714-019-2583-5

7. Rovelli C., Vidotto F., "Covariant Loop Quantum Gravity - an elementary introduction" to quantum gravity and spinfoam theory. Cambridge University Press 2015

8. Donà P., Speziale S., "Introductory lectures to Loop Quantum Gravity". arXiv:1007.0402 [gr-qc], (2013), unpublished.

9. Banerjee K., Calcagni G. Martìn-Benito M., "Introducton to Loop Quantum Cosmology". arXiv:1109.6801v2 [gr-qc], SIGMA 8 (2012), 016, 73 pages

10. Perez A., "Introduction to loop quantum gravity and spin foams". arXiv: 0409061v3 [gr-qc], (2004), Unpublished.

11. Ashtekar A., "Introduction to Loop Quantum Gravity and Cosmology" arXiv:1201.4598 [gr-qc], ," Lect. Notes Phys. 863 (2013) 31–56

12. Ashtekar A., Wilson-Ewing E. "Loop quantum cosmology of Bianchi I models". arXiv:0903.3397 [gr-qc], Phys.Rev.D79:083535,2009

13. Wilson-Ewing E. "The loop quantum cosmology bounce as a Kasner transition" arXiv:1711.10943 [gr-qc], Class.Quant.Grav. 35 (2018) 065005

14. Loll R. "Gauge Theory and Gravity in the Loop Formulation", Lect.Notes Phys. 434 (1994) 254-288.

15. Ashtekar A., Rovelli C. "A Loop Representation for the Quantum Maxwell Field" arXiv:hep-th/9202063, Class.Quant.Grav.9:1121-1150,1992

16. Ashtekar A., Bojowald M., Lewandowski J., "Mathematical structure of loop quantum cosmology" arXiv:gr-qc/0304074, 2003 International Press Adv. Theor. Math. Phys. 7 (2003) 233–268

17. Ashtekar A., Pawlowski T., Singh, P. "Quantum Nature of the Big Bang: Improved dynamics" arXiv:gr-qc/0607039, Phys.Rev.D74:084003,2006

18. Rovelli C., Wilson-Ewing E. "Why Are the Effective Equations of Loop Quantum Cosmology so Accurate?" arXiv:1310.8654 [gr-qc], Phys.Rev.D90:023538,2014

19. Corichi A., Vukasinac T., Zapata A. "Polymer Quantum Mechanics and its Continuum Limit". arXiv:0704.0007 [gr-qc], Phys.Rev.D76:044016,2007

20. Barca G., Giovannetti E., Montani G. "An Overview on the Nature of the Bounce in LQC and PQM" arXiv:2109.08645v1 [gr-qc], Universe 2021, 7, 327

21. Ashtekar A., Isham C.J. "Representations of the holonomy algebras of gravity and non-Abelian gauge theories" arXiv:hep-th/9202053, Class.Quant.Grav.9:1433-1468,1992

22. Ashtekar A., Pawlowski T., Singh P., Vandersloot K., "Loop quantum cosmology of k=1 FRW models" arXiv:gr-qc/0612104, Phys.Rev.D75:024035,2007

23. Gaul M., Rovelli C., "Loop Quantum Gravity and the Meaning of Diffeomorphism Invariance" arXiv:gr-qc/9910079, Lect.Notes Phys. 541 (2000) 277-324

24. Martin-Benito M., G. A. Mena Marugan, Olmedo J., "Further improvements in the Understanding of Isotropic Loop Quantum Cosmology" arXiv:0909.2829 [gr-qc], Phys.Rev.D80:104015,2009

25. Ashtekar A., Lewandowski J., "Background Independent Quantum Gravity: A Status Report" arXiv:gr-qc/0404018, Class.Quant.Grav.21:R53,2004

26. Diener P., Joe A., Megevand M., Singh P. "Numerical simulations of loop quantum Bianchi-I spacetimes" arXiv:1701.05824 [gr-qc], Class. Quantum Grav. 34 094004 (2017)