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FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI
Corso di Laurea Magistrale in Matematica

A HOMOLOGICAL DEFINITION OF THE HOMFLY POLYNOMIAL

Tesi di Laurea in Topologia Algebrica

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Introduzione

La teoria dei nodi è la branca della topologia che studia il comportamento di un cerchio, o di un'unione disgiunta di cerchi, chiamata link, nello spazio $\mathbb{R}^3$. Per i risultati ottenuti nello studio di varietà di dimensione 3 e 4, la teoria dei nodi è spesso considerata parte della topologia di dimensione bassa. Problemi riguardanti i nodi emergono anche nella teoria della singolarità e nella geometria simplettica. Notevoli sono anche le applicazioni ad altre discipline scientifiche, in particolar modo la fisica e la biologia.

La nozione di equivalenza tra nodi è l'isotopia, che corrisponde a considerare un nodo come un filo unidimensionale, flessibile ed elastico. Se però dimostrare che due nodi sono equivalenti corrisponde a trovare un'isotopia tra i due, per provare la non equivalenza è necessario dimostrare che una tale isotopia non possa esistere. E' quindi necessario studiare delle caratteristiche geometriche invarianti per isotopia. I primi invarianti trovati erano però troppo difficili da calcolare.

Per questo motivo si studiano non tanto caratteristiche geometrico topologiche dei nodi, ma piuttosto si cerca di utilizzare gli strumenti della topologia algebrica. Uno dei primi invarianti è stato il gruppo fondamentale del nodo, definito come il gruppo fondamentale del complementare del nodo, visto in $\mathbb{R}^3$ o, equivalentemente, in S^3. Il primo invariante efficientemente calcolabile è però il polinomio di Alexander, sviluppato contemporaneamente negli anni venti da Alexander e Reidemeister. Lo sviluppo della teoria si concentrò sullo studio delle proprietà del polinomio di Alexander, dandone diverse interpretazioni (tra cui il calcolo di Fox), nonché sui legami con la teoria della torsione di Reidemeister. Sempre tramite la teoria di Reidemeister si ha un'interpretazione degli zeri del polinomio nell'ambito della teoria della rappresentazione del gruppo fondamentale dei nodi.

Nel 1981 L. Kauffman caratterizza il polinomio di Alexander puramente in termine combinatori. Considerando un nodo L, si focalizza su un incrocio e chiama K_+ il nodo in cui l'incrocio è 'positivo' e K_- il nodo con incrocio 'negativo'. Ottiene così la seguente relazione, detta relazione di Alexander-Conway,

- $\Delta_O(t) = 1$
- $\Delta_{K+}(t) - \Delta_{K-}(t) = (t^{1/2} + t^{-1/2})\Delta_K(t)$

dove $\Delta_K(t)$ è il polinomio di Alexander di un dato nodo K. Questa relazione permette di calcolare in maniera induttiva il polinomio, dal momento che qualsiasi nodo può essere trasformato nel nodo banale con un numero finito di cambi nel diagramma.

Nel 1984 Vaughan Jones, durante una ricerca sulla teoria delle algebre di Von Neumann, scopre un nuovo polinomio per nodi orientati, il polinomio di Jones. Questo polinomio è univocamente determinato dalle due condizioni

1. $V_O(t) = 1$
2. $tVK+(t) - t^{-1}V_{K-}(t) = (t^{1/2} + t^{-1/2})V_K(t)$

vedi [15].

La costruzione attraverso le relazioni 1 e 2 ha ispirato alcuni ricercatori sulla possibilità di una possibile generalizzazione ad un polinomio in due variabili. Nello stesso periodo, diversi autori hanno scoperto un nuovo polinomio, o, meglio, differenti versioni isomorfe delle stesso polinomio, [12]. Il nuovo invariante, il polinomio HOMFLY, prese il suo nome dalle iniziali dei suoi 6 scopritori: J. Hoste, A. Ocneanu, K. Millett, P. Freyd, W.B.R. Lickorish, D. Yetter. Al momento, sono conosciuti tre polinomi associati ai nodi:

l'Alexander-Conway, quello di Jones e il polinomio HOMFLY, l'ultimo dei quali contenente come casi particolari i primi due.

Nel 2001 Bigelow ha presentato una interpretazione geometrica del polinomio di Jones, [2]. L'obiettivo di questa tesi è studiare una definizione omologica del polinomio HOMFLY, sempre scoperta da Bigelow e pubblicata in [4]. Questa è definita non sui nodi, ma sulle trecce. Il gruppo di trecce fu introdotto da Emil Artin negli anni '20 e presto emersero connessioni con la teoria dei nodi. E' infatti possibile ottenere link dalle trecce in una maniera standard, con un'operazione chiamata chiusura. Per un teorema di Alexander, è possibile ottenere ogni link come chiusura di un'opportuna treccia. Vedremo quindi una definizione del polinomio HOMFLY in termini di chiusura di trecce, più esattamente con una differente chiusura, chiamata 'plat closure', definita su trecce con un numero pari di corde. Questa costruzione è dovuta a Birman, [5].

Vediamo ora brevemente gli argomenti trattati nei vari capitoli.

Nel primo capitolo sono presentati i risultati classici della teoria dei nodi. Il primo paragrafo è dedicato a illustrare le definizioni fondamentali della teoria dei nodi e alla costruzione del primo invariante algebrico, il gruppo del nodo. Nel secondo paragrafo definiamo il polinomio di Alexander in quattro maniere diverse, attraverso spazi di ricoprimento, superfici di Seifert, calcolo di Fox e relazioni del tipo 1 e 2, dette relazioni 'skein'. Nel terzo paragrafo è introdotto il polinomio di Jones.

Il secondo capitolo è completamente dedicato alle trecce. Nel primo paragrafo introduciamo differenti definizioni equivalenti del gruppo di trecce. Nel paragrafo successivo sono studiate alcune delle sue proprietà. Il terzo paragrafo è dedicato a investigare l'equivalenza tra trecce e link. Nell'ultimo paragrafo sono presentati alcuni risultati relativi alla rappresentazione del gruppo lineare, per arrivare a citare il teorema di linearità del gruppo di trecce.

Nel terzo capitolo ci occupiamo del risultato principale della tesi, la costruzione di una definizione omologica del polinomio HOMFLY. Nel primo paragrafo è definito il polinomio. Nel secondo paragrafo viene presentata la costruzione geometrica utilizzata nei paragrafi successivi. Nei due successivi, è definito un invariante sulle trecce, insieme a una spiegazione del suo significato topologico. Nell'ultimo paragrafo dimostriamo il teorema fondamentale, l'equivalenza dell'invariante definito sulle trecce e del polinomio HOMFLY.

Introduction

Knot theory is the branch of topology which studies the behaviour of a circle, or of a disjoint union of circles, a link, in the space $\mathbb{R}^3$. Due to the results concerning topological manifolds of dimension 3 and 4, knot theory is often considered as part of low dimensional topology. Problems about knots also emerge in singularity theory and in simplectic geometry. Remarkable are also the applications in other scientific disciplines, as physics and biology.

The notion of knot equivalence is known as isotopy, which consists in considering a knot as a unidimensional thread, flexible and elastic. While proving two knots' equivalence is proving the existence of a particular isotopy, proving that two knots are not equivalent consists in showing the impossibility of such an isotopy. It is thus necessart to study geometric features which are invariant for isotopy. The first invariants found were very difficult to compute.

This is the reason we do not study the geometrical-topological knot behaviour, but rather we try to use algebraic topology tools. One of the first invariants was the knot fundamental group, defined as the fundamental group of the knot complementary, seen in $\mathbb{R}^3$ or, equivalently, in S^3. The first invariant effectively computable is the Alexander polynomial, developed at the same time in the '20 by Alexander and Reidemeister. Further developments of the theory focused on the study of these polynomial proprieties, giving different interpretations of it (like Fox calculus), as well as on the bonds to Reidemeister torsion theory. Thanks moreover to Reidemeister's theory we have an interpretation of the zeros of the polynomial in the theory of representation of the knot fundamental group.

In 1981 L. Kauffman defines Alexander's polynomial in a purely combinatorial way. Taking a knot K, we focus on a crossing and we call K_+ the knot where this crossing is 'positive' and K_- the knot where this crossing is 'negative'. We obtain then the following relation, the Alexander-Conway relation,

- $\Delta_O(t) = 1$
- $\Delta_{K+}(t) - \Delta_{K-}(t) = (t^{1/2} + t^{-1/2})\Delta_K(t)$

where $\Delta_K(t)$ is the Alexander's polynomial of a given know K. This relation allows to compute in an inductive way the polynomial, since every knot can be transformed in the trivial knot with a finite number of transformations of the crossing type in the diagram.

In 1984 Vaughan Jones, during a research on Von Neumann algebras, found a new oriented knot polynomial, the Jones polynomial. This polynomial is uniquely determined by the two conditions

1. $V_O(t) = 1$
2. $tVK{+}(t) - t^{-1}V_{K-}(t) = (t^{1/2} + t^{-1/2})V_K(t)$

see [15].

The construction via relations 1 and 2 pointed towards the possibility of generalisation in two indeterminate. At the same time, different authors discovered a new polynomial, or rather isomorphic versions of the same polynomial, [12]. This new invariant, the HOMFLY polynomial, took its name from the initials of the 6 discoverers: J. Hoste, A. Ocneanu, K. Millett, P. Freyd, W.B.R. Lickorish, D. Yetter. At present, there are three main known knot polynomials: the Alexander-Conway, the Jones and the HOMFLY polynomials, the last one containing as particular cases the first two.

In 2001 Bigelow presented a geometric interpretation of the Jones polynomial, see [2]. In this thesis we studied a homological definition of the HOMFLY polynomial, discovered as well by Bigelow, see [4]. This construction was not defined on the basis of knots, but braids. These were introduced by Emil Artin in the 1920s. It is possible to pass in a standard way from braids to links with an operation called closure. Through a theorem by Alexander, it is possible to take each link as a closure of an opportune braid. There is a definition of the HOMFLY polynomial in terms of closures of braids, more exactly as plat closures of braids with an even number of strings. This construction is due to Birman.

Now we briefly summarise the contents of this thesis.

In the first chapter we give the classic results of knot theory. The first section is dedicated to illustrate the fundamental definitions of knot theory and of the first algebraic invariant, the knot group. In the second section we define the Alexander polynomial in four different ways, via covering spaces, via Seifert surfaces, via Fox calculus and via skein relations. In the third section we introduce the Jones polynomial.

The second chapter is dedicated to braids. In the first section we introduce different equivalent definitions of the braid group. In the following section, we study some of its proprieties. The third section is dedicated to investigate the equivalence between braids and links. In the last section we see some linear representations of braid groups, stating the linearity of braid groups.

In the third chapter we prove the main statement of this work, the building of a homological definition of the Homfly polynomial. In the first section we define the polynomial. The fundamental geometric constructions are presented in the second section. In the following two sections, we define an invariant on braids and we find a topological meaning for this invariant. In the last section we prove the fundamental theorem, the equivalence of the invariant on braids and the HOMFLY polynomial.

Contents

Chapter 1

Knots

1.1 Knots and fundamental group

1.1.1 Definitions

Definition 1. A knot K is a smooth sub-variety in S^3 diffeomorphic to the circle S^1. If S^1 is oriented, the diffeomorphism preserves the orientation and we say that the knot is oriented.

Example 1. *The trivial knot:*

$$\{(x, y, z) \in S^3 | \ x^2 + y^2 = 1, \ z = 0\}$$

Example 2. *Knots lying on the surface of an unknotted torus in S^3 are called torus knots. Each knot torus can be specified by two coprime integers p and q, where p is the number of windings along the longitude and q is the number of windings along the meridian of the torus T^2.*

Let us see an example of parametrization. The curve $\gamma_{p,q} : S^1 \to \mathbb{R}^3$ is given by

$$\theta \mapsto (\frac{cos(p\theta)}{1 - sin(q\theta)}, \frac{sin(p\theta)}{1 - sin(q\theta)}, \frac{cos(q\theta)}{1 - sin(q\theta)}).$$

Definition 2. A n-component link L is a smooth subvariety in S^3 diffeomorphic to a disjoint union on n circles S^1.

Definition 3. An isotopy between two knots K and K' in S^3 is a map

$$h : [0,1] \times S^3 \to S^3$$

$$(t, x) \mapsto h_t(x)$$

such that:

- $h_0 = Id_{S^3}$
- $\forall t$ h_t is a diffeomorphism
- $K' = h_1(K)$

Two knots K and K' are isotopic if and only if there is an isotopy h between them.

Remark 1. The relation 'K is isotopic to K'' is an equivalence relation.

Definition 4. The knot obtained from K by inverting its orientation is called the inverted knot and it is denoted by $-K$.

The mirrored knot of K is obtained by a reflection of K in a plane and it is denoted by K^*.

A knot K is called invertible is $K = -K$ up to isotopy and amphicheiral if $K = K^*$ up to isotopy.

It is possible to define the notion of knot and isotopy also in the piecewise linear category, rather than in the smooth category. It turns out that in three dimensions, the smooth category and the piecewise category are the same, so there is no substantial difference between the two definitions.

Definition 5. A knot diagram is a regular projection, i.e. it is such that multiple points are double points with different tangent vectors, of a circle S^1 in the oriented plane $\mathbb{R}^2$, with an over-under information for every double point.

Remark 2. The set of double points is a discrete set.

Every diagram can be associated to a isotopy class of knots in S^3. The diagram is immersed in $\mathbb{R}^2 \times \{0\}$ and to every crossing just push the under point in a smooth way so that it lies in the plane $z = -\epsilon$, $\epsilon > 0$.

Proposition 1.1.1. *The set of regular projections is open and dense in the space of all projections.*

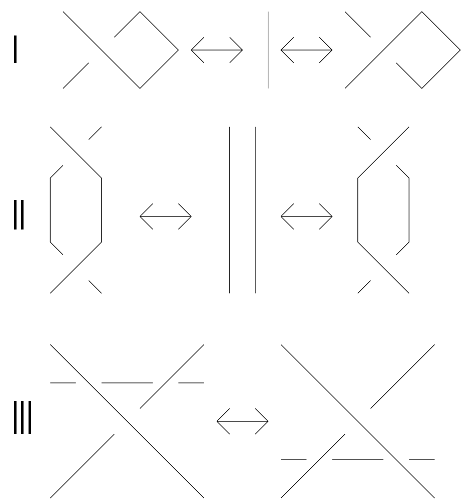
A proof of this proposition in PL situation can be find in [Bur85].

It follows from the proposition:

Theorem 1.1.2. *Every knot in S^3 is isotopic to a knot with regular projection in $\mathbb{R}^2$. Every knot can be defined as a diagram, up to isotopy.*

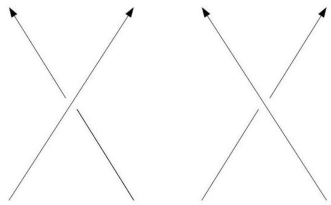
Now we want to see how the condition of isotopy equivalence can be expressed in terms of diagrams.

Theorem 1.1.3. *Two knot are isotopic if and only if their diagrams are connected by a finite sequence of Reidemeister moves or their inverses:*



A proof can be find, in the PL case, in [8].

Let L be an oriented link with two connected components, $L = (K_1, K_2)$. We can assign $+1$ to the crossing such as the one on the left and -1 to the crossings such as the one on the right, obtaining a function $\epsilon(L) = \sum_c \epsilon(c)$, where c are all the crossing between D_1 and D_2.



Proposition 1.1.4. *$\epsilon(L)$ is even and $\frac{1}{2}\epsilon(L)$ is invariant under Reidemeister moves.*

Proof. The first Reidemeister move does not change the linking number.

In the second Reidemeister move there is always one positive and one negative crossing, so the move does not change the linking number.

The third Reidemeister move does not change the signs of the crossings.

□

Definition 6. The function $lk(K_1, K_2) = \frac{1}{2}\epsilon(L)$ is called the linking number of the two connected components K_1, K_2 of L.

1.1.2 Knot groups

Remark 3. The fundamental group of a knot K is $\pi_1(K) = \mathbb{Z}$.

So the fundamental group does not give any useful information for comparing knots. The same is for homology and cohomology theory. Instead, it is really useful to study the complement of the knot $S^3 - K$, which is a strong invariant. Actually, we do not study the space $S^3 - K$, but the space $M_K = S^3 - V(K)$, where $V(K)$ is a closed tubular neighbourhood of the knot K. They have the same homology, because they are homotopically equivalent, but M_K is a compact 3-manifold with boundary.

Theorem 1.1.5. • $H_i(M_K) = \begin{cases} \mathbb{Z}, & i = 0,\ 1 \\ 0, & i \geq 2 \end{cases}$

- *There are two simple curves m and l in $T^2 = M_K \cap V$ such that:*
    1. *$m \cap l = \{P\}$*
    2. *$m \sim 0,\ \ l \sim K$ in $V(K)$*
    3. *$l \sim 0$ in M_K*
    4. *$lk(m, K) = 1$, $lk(l, K) = 0$ in S^3*

m and l are determined up to isotopy in T^2 by these proprieties and are called meridian and longitude of the knot K.

Proof. The first part is a direct consequence of the Mayer-Vietoris sequence. $S^3 \simeq M_K \cup V$, $M_K \cap V \simeq T^2$. Being each one or a 2 or a 3 dimensional CW complex, it is possible to apply the Mayer-Vietoris sequence.

$$0 \to H_*(T^2) \to H_*(M_K) \oplus H_*(V) \to H_*(S^3) \to 0$$

From the homological proprieties of S^3, T^2 and $V \simeq S^1$ we obtain:

$$0 \to H_3(M_K) \to \mathbb{Z} \to \mathbb{Z} \to H_2(M_K) \to 0$$

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to H_1(M_K) \oplus \mathbb{Z} \to 0$$

$$0 \to \mathbb{Z} \to H_0(M_K) \oplus \mathbb{Z} \to \mathbb{Z} \to 0$$

From the last two we have immediately that $H_0(M_K) \simeq \mathbb{Z}$, $H_1(M_K) \simeq \mathbb{Z}$.

Now, T^2 is the boundary of a compact orientable 3-manifold. The immersion $T^2 \hookrightarrow C$ maps the group $H_2(T^2) \mapsto 0$, so that $H_2(M_K) = 0$, $H_3(M_K) = 0$.

Now we take the isomorphism

$$\mathbb{Z} \oplus \mathbb{Z} \simeq H_1(T^2)$$

in the Mayer-Vietoris sequence. The generators of $H_1(V) \simeq \mathbb{Z}$ and $H_1(M_K) \simeq \mathbb{Z}$ are determined up to the inverse. We can choose l, a representative of the homology class of K, as a generator of $H_1(V)$ with the condition that l be homologous to 0 in $H_1(M_K)$. So l is unique up to isotopy in T^2. Equivalently, a generator of $H_1(M_K)$ can be represented by a curve m in T^2 that is homologous to 0 in V. With this choices of l and m we obtain a system of generators of $H_1(T^2) \simeq \mathbb{Z} \oplus \mathbb{Z}$. We can also assume that m is simple and intersects l in one point, see [23].

The linking number proprieties follow from the construction. $\square$

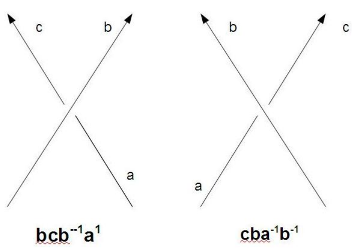
The most important invariant of a knot (and of a link) is the so called knot group (link group).

Definition 7. The group $\pi_1(M_L)$, where L is a link, is called the link group.

Remark 4. Generally, we take as basepoint the point $(0, i)$, viewing S^3 as a subspace of $\mathbb{C}^2$.

Remark 5. The knot group is independent of the choice of the orientation, but the orientation defines uniquely the generators.

Theorem 1.1.6. *Let K be a knot (L be a link). Let D be an associated diagram, formed by n arcs and m crossings. The Wirtinger presentation of the knot group is $P =< x_1, \ldots, x_n \mid r_1, \ldots, r_{n-1} >$, where x_i are meridians passing for the basepoint associated to every arc and r_j are the relations associated to every crossing such that:*



The theorem follows from a particular case of the Van Kampen theorem.

Theorem 1.1.7. *Let (X, x_0) be a pointed topological space, union of two open sets U_1 and U_2 such that $U_1 \cap U_2$ is arc-connected and $x_0 \in U_1 \cap U_2$. Let us also suppose that U_2 is simply connected. Let i and j be the applications:*

$$i : U_1 \cap U_2 \to U_1$$

$$j : U_1 \to X$$

Then the application

$$j_{\#} : \pi_1(U_1, x_0) \to \pi_1(X, x_0)$$

induces an isomorphism

$$\frac{\pi_1(U_1, x_0)}{i_{\#}(\pi_1(U_1 \cap U_2, x_0))^N} \simeq \pi_1(X, x_0)$$

Proof. A proof can we found in [14]. □

Now we can prove the main theorem on the Wirtinger presentation.

Proof. We can take the knot K as being immersed in $\mathbb{R}^3$. Let X_K be $X_K = \mathbb{R}^3 - K$. We take two sets $U_1, U_2 \subset X_K$ such that:

$$U_1 = (\{z \geq -\epsilon\} \cup \{\infty\}) \cap X_K$$

$$U_2 = (\{z \leq -\epsilon\} \cup \{\infty\}) \cap X_K$$

where, up to homeomorphism, we can take the basepoint x_0 to be ∞. Again up to homeomorphism, we can choose ∞ such that $\infty \notin K$.

We can not apply directly theorem 1.1.7 because the intersection of the two sets is closed. But in this situation it is not a big problem, because $3-$manifolds can be always seen as CW-complexes and theorem 1.1.7 is true also with closed sets.

Now we have a sequence

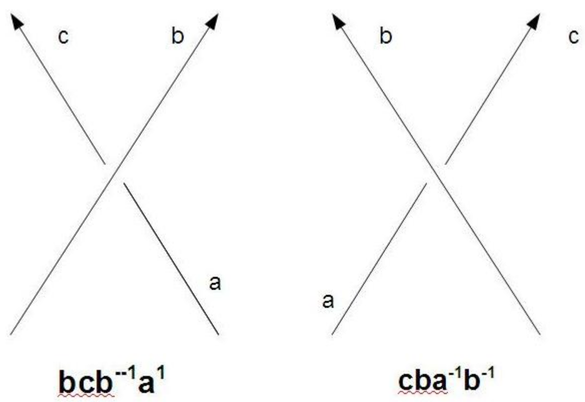
$$U_1 \cap U_2 \xrightarrow{j} U_1 \xrightarrow{i} X_K$$

Applying the Van Kampen theorem, $i_\#$ induces an isomorphism

$$\frac{\pi_1(U_1,\infty)}{j_\#(\pi_1(U_1\cap U_2))^N}\xrightarrow{\sim}\pi_1(X_K,\infty)$$

Now we can take U_1 as a 3-dimensional disk, and so U_1 becomes homeomorphic to a 3-dimensional disk without some arcs. By recursion, we can apply many times the Van Kampen theorem. The group $\pi_1(U_1,\infty)$ is a free group generated by the meridian associated to the arc. Instead $U_1 \cap U_2$ is a sphere minus $n-1$ arcs. It remains to study $j_\#$.

For every crossing (positive or negative) there is a relation:



This is well defined up to conjugacy classes because we are taking the normal set. □

1.1.3 Knots classification

Definition 8. The peripheral system of a knot K is a triple $(\pi_1(M_K), l, m)$, where $\pi_1(M_K)$ is the knot group, l and m are the homology classes of a longitude and a meridian.

Remark 6. m and l commute, $l\cdot m = m\cdot l$.

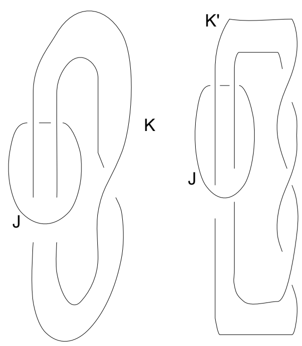
The pair (l, m) is uniquely determined up to a common conjugacy element in $\pi_1(M_K)$.

Theorem 1.1.8. *(Waldhausen)*

Two knots K_1 and K_2 in S^3 with peripheral systems $(\pi_1(M_{K_1}), l_1, m_1)$, $(\pi_1(M_{K_2}), l_2, m_2)$ are equal if and only if there is an isomorphism $\varphi : \pi_1(M_{K_1}) \to \pi_1(M_{K_2})$ such that $\varphi(l_1) = l_2$ and $\varphi(m_1) = m_2$.

Proof. A proof can be found in [8]. □

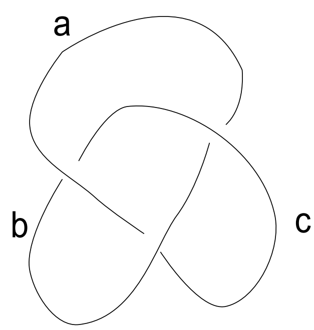
Example 3. *In theorem 1.1.8 the hypothesis K knot is essential. For links, it is possible to find inequivalent links with homeomorphic complements. Take the two links*



We can notice that K' is a trefoil knot, while K is the trivial knot. These two knots are inequivalent, so the links must be inequivalent too. We now show that their complements are homeomorphic. First, $S^3 - J$ is homeomorphic to $S^1 \times intD_2$. Applying the homeomorphism $h(z, w) = (z, zw)$, after identification of S^1 and D_2 with sets of complex numbers, we have $h(K) = K'$. So $S^3 - (J \cup K)$ and $S^3 - (J \cup K')$ are homeomorphic.

1.1.4 Some examples

We want to compute the fundamental group of the trefoil knot.



We obtain the presentation:

$$\Gamma_K =< a, b, c \mid ac = cb, \;\; cb = ba >=$$

$$=< a, b, c \mid c = bab^{-1}, \;\; ac = cb >=< a, b | \; aba = bab >$$

Setting $x = ab$, $y = aba$, we have the presentation (that is not a Wirtinger presentation), $P =< x, y \mid x^3 = y^2 >$.

We want now to show how the fundamental group allows to discriminate different knots in a large class of knots, the torus knots.

Definition 9. Let S^3 be such that $S^3 = W \cup W'$, where W is an unknotted solid torus and $T = W \cap W'$ is a torus, with orientation given by W. The couple (W, W') is called a Heegard splitting of genus 1 of the oriented 3-sphere S^3.

Proposition 1.1.9. *$H_1(T) = \mu\mathbb{Z} \oplus \nu\mathbb{Z}$, where μ is the meridian and ν is the longitude of T, such that they intersect in the basepoint P with intersection number* 1.

Any closed curve on T is homotopic to a curve $\mu^a \cdot v^b$, a, $b \in \mathbb{Z}$, with a and b relatively prime.

Proof. A proof can be found in every book of algebraic topology, for example [7] or [14]. $\square$

Definition 10. Let (W, W') be a Heegard splitting of genus 1 of S^3. If K is a simple closed curve on F with intersection numbers p, q with ν and μ, $|p|$, $|q| \geq 2$, then K is called the torus knot $K(p, q)$.

Remark 7.

$$K(-p, -q) = -K(p, q)$$

$$K(p, -q) = K^*(p, q)$$

$$K(p, q) = K(-p, -q) = K(q, p)$$

Proposition 1.1.10. *A presentation of the group $\Gamma(p, q)$ of the torus knot $K(p, q)$ is*

$$\Gamma(p, q) =< u, v \mid u^p v^{-q} >$$

where u and v represents μ and ν. It has the following proprieties:

1. *The centre is $< u^a > \simeq \mathbb{Z}$.*

2. *$m = u^r v^s$, $l = u^p m^{-pq}$, $ps+qr = 1$, m and l are meridian and longitude of $K(p, q)$ for a suitable basepoint P.*

3. *$K(p, q)$ and $K(p', q')$ have isomorphic groups if and only if $|p| = |p'|$ and $|q| = |q'|$ or $|p| = |q'|$ and $|q| = |p'|$.*

Proof. It follows from standard results in Heegard splitting, see [8]. □

Theorem 1.1.11. *$K(p, q) = K(p', q')$ if and only if (p', q') is equal to: (p, q), (q, p), $(-p, -q)$ or $(-q, -p)$. Also, torus knots are invertible, but not amphicheiral.*

Proof. Sufficiency has been proven in the previous proposition. Suppose now $K(p, q) = K(p', q')$. The centre $Z(\Gamma)$ is a characteristic subgroup, so $\Gamma(p, q)/Z$ is a knot invariant. Also, the integers $|p|$, $|q|$ are invariants of $\mathbb{Z}_{|p|} * \mathbb{Z}_{|q|}$ and they are the orders of maximal finite subgroups of $\mathbb{Z}_{|p|} * \mathbb{Z}_q$, which are not conjugate. This proves the first part of the statement.

Let now p, q be such that $p,\ q > 0$. There is an isomorphism φ : $\Gamma(p,q) \to \Gamma(p',q')$ such that it maps the peripheral system $(\Gamma(p,q),m,l)$ in $(\Gamma(p',q'),m',l')$:

$$m' = \varphi(u^r v^s) = u'^{r'} v'^{s'}$$

$$l' = \varphi(u^p (c^r v^s)^{-pq}) = u'^p (u'^{r'} v'^{d'})^{-pq}$$

where $ps + qr = ps' - qr' = 1$, so that $s' = s + jq$ and $r' = r + jp$, for $j \in \mathbb{Z}$.

The isomorphism φ maps the centre $Z(\Gamma(p,q))$ in $Z(\Gamma(p',q'))$, so that $\varphi(u^p) = (u'^p)^\epsilon$, $\epsilon = \pm 1$. We have

$$u'^p (u'^{r'} v'^{s'})^{pq} = \varphi(u^p (u^r v^s)^{-pq}) = \varphi(u^p)\varphi(u^r v^s)^{-pq} = (u'^p)^\epsilon (u'^{r'} v'^{s'})^{-pq}$$

and so $(u'^p)^{1-\epsilon} = (u'^{r'} v'^{s'})^{-2pq}$, that is impossible. In fact, the homomorphism $\Gamma(p',q') \to \Gamma(p',q')/Z(\Gamma(p',q'))$ maps the terms on the left to 1, while the term on the right represents a non-trivial element.

□

1.2 The Alexander-Conway polynomial

1.2.1 The Alexander module

Definition 11. The infinite cyclic cover of a knot K is the regular cover $\tilde{X}_K$ of the complement of the knot K, associated to the morphism:

$$h : \pi_1(X_k, \infty) \to H_1(X_K, \mathbb{Z})$$

Let us give a constructive definition of the space. Let $\Omega(X_K, \infty, x)$ be the space of all paths from ∞ to the point x in X_K. By the theory of covering spaces, $\tilde{X}_K$ is defined as

$$\{(x,\gamma), x \in X_K, \gamma \in \Omega(X_K, \infty, x)\}/\sim$$

where $\sim$ is the relation:

$$(x,\gamma) \sim (x,\gamma') \Leftrightarrow x = x',\ h[\gamma'\gamma^{-1}] = 0$$

The group $\pi_1(X_K, \infty)$ acts on $\tilde{X}_K$ via the application h, that factorises by the quotient of $H_1(X_K, \mathbb{Z}) \simeq \mathbb{Z}$. Let t be the generator of $H_1(X_K)$. t acts by a transformation $t : \tilde{X}_K \to X_K$. So the transformations group of the covering space is $< t >$, which acts on $C_*(X_K)$ by translation. More, it acts on $H_*(\tilde{X}_K)$ and this action extends to the group ring $\mathbb{Z}[< t >] = \mathbb{Z}[t^{\pm}]$.

Definition 12. The $\mathbb{Z}[t^{\pm}]$-module $H_1(\tilde{X}_K, \mathbb{Z})$ is called the Alexander module of the knot K.

Remark 8. $H_1(\tilde{X}_K; \mathbb{Z})$ is independent of the orientation of K, but the action of $\mathbb{Z}[t^{\pm}]$ is not. Changing the orientation exchanges t and t^{-1}.

Definition 13. Let K be a knot and D an associated diagram. Let F be the set of the connected bounded components of $\mathbb{R}^2 - D$. For every $X \in F$ we can define

$$\gamma_X : [0,1] \to \hat{\mathbb{R}}^3$$

where $\hat{\mathbb{R}}^3 = \mathbb{R}^3 \cup \infty$, such that

$$\gamma_X(t) = (x_X, y_X, \tan(\frac{\pi}{2}(2t-1)))$$

The curve γ_X is called a Seifert generator.

Proposition 1.2.1. *The laces γ_X generate $\pi_1(X_K)$.*

Proof. As in the proof of the Wirtinger presentation of the knot group, it is sufficient to apply the Van Kampen theorem to appropriate subsets.

□

Remark 9. If K' is a knot, $K' \in S^3$, then $[K'] \in H_1(M_K)$. We denote $I(K') = lk(K', K)$. If $lk(K', K) = 0$, $[K'] = 0 \in H_1(M_K)$.

γ_X can be seen as a knot in M_K, so by remark 9 $\gamma_X \in H_1(M_K)$.

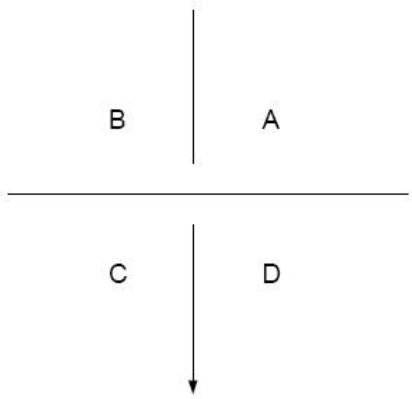
Remark 10. The action of the lace γ_X is

$$[\gamma_X].\tilde{\infty} = t^{I(\gamma_X)}.\tilde{\infty}$$

Remark 11. For every face X, $\tilde{\gamma}_X$ is a path from $\tilde{\infty}$ to $t^{I(\gamma_X)}.\tilde{\infty}$.

We can now prove the main theorem about the presentation of the Alexander module.

Theorem 1.2.2. *Let K be a knot and Z a face of a diagram D such that $I(\gamma_Z) = \pm 1$. A presentation of the Alexander module $H_1(\tilde{M}_K, \mathbb{Z})$ as $\mathbb{Z}[t^{\pm}]$-module is given by taking as generators the laces γ_X, $X \neq Z$, with relations given on every crossing*



by $A - tB + tC - D = 0$. We can identify the generators γ_X as the faces X, $X \neq Z$.

Proof. Let $\tilde{\gamma}_X$ be a lift of γ_X, starting by the basepoint $\tilde{\infty}$. By definition:

$$\partial\tilde{\gamma}_X = \gamma_X.\tilde{\infty} - \tilde{\infty} = t^{I(\gamma_X)}.\tilde{\infty} - \tilde{\infty}$$

We can define c_X to be

$$c_X = \tilde{\gamma}_X - \frac{t^{I(\gamma_X)} - 1}{t^{I(\gamma_Z)} - 1}\tilde{\gamma}_Z$$

So,

$$\partial c_X = \partial\tilde{\gamma}_X - \partial\frac{t^{I(\gamma_X)} - 1}{t^{I(\gamma_Z)} - 1}\tilde{\gamma}_Z = 0$$

and it is well defined $[c_X] = [X]$.

We want to apply the Mayer Vietoris sequence to the sets

$$V_1 = M_K - \bigcup_{c\ crossings} (x_c, y_c) \times [-\epsilon, 0]$$

$$V_2 = \bigcup_{c\ crossings} D_r((x_c, y_c), \frac{\epsilon}{2})$$

We have $V_1 \cup V_2 \simeq M_K$.

Let $\tilde{V}_1$ and $\tilde{V}_2$ be the images of the two sets in $\tilde{M}_K$. If we take $X_K^1 = \bigcup_{X \in F} \gamma_X \hookrightarrow V_1$, we obtain a homotopy equivalence $H_1(\tilde{X}_K^1) \xrightarrow{\sim} H_1(\tilde{V}_1)$. As a $\mathbb{Z}[t^{\pm}]$-module the cellular complex is free with basis ∞ at degree 0, $\tilde{\gamma}_X$ at degree 1. We also have a definition for the boundary operator, so we can compute the homology. We obtain that $H_1(\tilde{V}_1)$ is free on $\mathbb{Z}[t^{\pm}]$ with basis $\{c_X\}$, $X \neq Z$. Now we want to study $\tilde{V}_2$. We have that $V_2 \simeq D_2$, so $\tilde{V}_2 \simeq V_2 \times \mathbb{Z}$. Then $H_*(\tilde{V}_2) = 0$ for every $* \geq 1$.

We also have that $\tilde{V}_1 \cap \tilde{V}_2 \simeq \partial V_2 \times \mathbb{Z}$, so that we have, for every crossing c, $S^1 \times \mathbb{Z}$. Let l_c be

$$l_c = S_r^1((x_c, y_c), -\frac{\epsilon}{2}) \times \mathbb{Z}$$

Then the lace $l_c \in \tilde{V}_1$ is homologous to

$$\gamma_A \widetilde{\overline{\gamma}_B \gamma_C} \overline{\gamma}_D$$

We have the following proprieties:

1. $\widetilde{uv} = \tilde{u}(u\tilde{v})$

2. $\widetilde{u\overline{u}} = \tilde{u}(u\tilde{\overline{u}})$

3. $\overline{\tilde{u}} = u\tilde{\overline{u}}$

We can also write $\tilde{\overline{u}} = t^{-I(u)}\overline{\tilde{u}}$

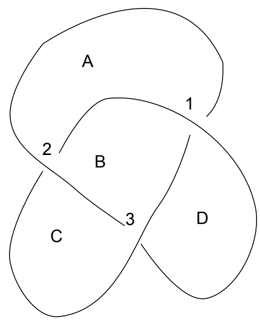
$$\gamma_A \widetilde{\overline{\gamma}_B \gamma_C} \overline{\gamma}_D = \tilde{\gamma}_A t^{I(\gamma_A)-I(\gamma_B)} \tilde{\gamma}_B t^{I(\gamma_A)-I(\gamma_B)} \tilde{\overline{\gamma}}_C t^{I(\gamma_A)-I(\gamma_B)-I(\gamma_C)+I(\gamma_D)} \tilde{\gamma}_D$$

Applying $I(\gamma_A) = I(\gamma_B) + 1$, $I(\gamma_C) = I(\gamma_D) - 1$ and writing in additive notation we obtain

$$\tilde{\gamma}_A - t\tilde{\gamma}_B + t\tilde{\gamma}_C - \tilde{\gamma}_D$$

□

Example 1. Let us calculate the Alexander module for the trefoil knot.



$I(\gamma_D) = \pm 1$, depending on the orientation, so we can take $D = Z$. Now we have for each crossing:

1. $-A + B = 0$
2. $-tA + B - C = 0$
3. $B - tC = 0$

We obtain then a presentation:

$$\begin{pmatrix} 1 & -t & 0 \\ -1 & 1 & 1 \\ 0 & -1 & -t \end{pmatrix}$$

Definition 14. The Alexander polynomial $\Delta_K(t)$ is defined as the determinant of the presentation matrix, modulo $\pm t^{\pm k}$.

Remark 12. Usually, it is written $P \doteq Q$ for an equality modulo $\pm t^{\pm k}$ of two polynomials P and Q in $\mathbb{Z}[t^{\pm 1}]$.

Example 2. Using the matrix presentation computed above, we find that the Alexander polynomial for the trefoil know is

$$\Delta_K(t) = t^2 - t + 1 \doteq 1 - (t + t^{-1})$$

1.2.2 Seifert surfaces

Proposition 1.2.3. *A simple closed curve $K \subset S^3$ is the boundary of an orientable surface Σ, embedded in S^3. Σ is called a Seifert surface.*

Proof. A proof can be found in [8].

□

Definition 15. Suppose that M is a surface and that there is a solid cylinder $D^2 \times [0,1]$ such that $(D^2 \times [0,1]) \cap F = \{0,1\} \times D^2$, respecting the orientation. Let M' be such that $M' = ((M - \{0,1\}) \times D^2) \cup [0,1] \times \partial D^2$. M' is said to be obtained from M by surgery along the arc $[0,1] \times 0$.

Two surfaces M and N are said to be tube equivalent if one is obtained from the other by surgery along the arc $[0,1] \times 0$.

Theorem 1.2.4. *Suppose that Σ and Σ' are two Seifert surfaces for an oriented link L in S^3. Then there is a sequence of Seifert surfaces $\{\Sigma_1, \Sigma_2, \ldots, \Sigma_n\}$ with $\Sigma_1 = \Sigma$ and $\Sigma_n = \Sigma'$, such that, for every i, Σ_i and Σ_{i-1} are tube-equivalents.*

Proof. A proof can be found in [17]. □

We will now use the Seifert surfaces to give a different interpretation of the Alexander polynomial. The Seifert surfaces are connected compact and oriented surfaces, then they are completely classified by their genus g.

Proposition 1.2.5. *Let K be an oriented knot, Σ a Seifert Surface and K' a knot transversal to S. Then*

$$lk(K, K') = I(\Sigma \cap K')$$

where $I(\Sigma, K')$ is the algebraic intersection.

Proof. We write the exact sequence:

$$H_2(S^3) \to H_2(S^3, V_K) \to H_1(V_K) \to H_1(S^3)$$

For the homological proprieties of S^3, $H_1(S^3) = H_2(S^3) = 0$, so there is an isomorphism between the two middle terms. For homotopy equivalence $H_1(V_K) \simeq H_1(K)$. We have now:

$$H_2(S^3, V_K) \simeq H_2(S^3 - \overset{\circ}{V}_K, \partial(S^3 - \overset{\circ}{V}_K)) \simeq H^1(S^3 - V_K)$$

where the first equivalence is given by excision and the second by Poincaré duality.

$$H^1(S^3 - V_K) \simeq H_1(S^3 - V(K))^* \simeq \mathbb{Z}.[m_K]^*$$

Now, for $[K'] = lk(K, K').[M_K] \in H_1(S^3 - V_K)$, we have

$$lk(K, K') =< [m_K]^*, [K'] >= D[m_K]^*.[K']$$

Also, $[\Sigma].[m_K] = 1$, then $[\Sigma] = D[m_K]$.

□

Proposition 1.2.6. *There is a duality*

$$H_1(S^3 - \Sigma) \overset{\sim}{\to} H^1(\Sigma) \simeq H_1(\Sigma)^*.$$

This is called Alexander duality.

Proof.

$$H_1(S^3 - \Sigma) \simeq H_1(S^3 - \overset{\circ}{V}_\Sigma) \simeq H^2(S^3 - \overset{\circ}{V}_\Sigma, \partial(S^3 - V_\Sigma)) \simeq H^2(S^3, V_\Sigma)$$

where the second equality is given by Poincaré duality and the last by excision. If we write the long exact sequence of (S^3, V_Σ) we obtain:

$$0 \to H^1(V_\Sigma) \to H^2(S^3, V_\Sigma) \to 0$$

and for homotopy equivalence $H^1(V_\Sigma) \simeq H^1(\Sigma)$. □

Proposition 1.2.7. *The bilinear non-singular form*

$$lk : H_1(S^3 - \Sigma) \otimes H_1(\Sigma) \to \mathbb{Z}$$

is well defined.

Proof. We have a bilinear form:

$$\varphi : H_1(S^3 - \overset{\circ}{V}_\Sigma) \otimes H_2(S^3 - V_\Sigma, \partial(S^3 - V_\Sigma)) \to \mathbb{Z}$$

As we have already seen, $H_1(S^3 - \overset{\circ}{V}_\Sigma) \simeq H_1(S^3 - \Sigma)$ and $H_2(S^3 - V_\Sigma, \partial(S^3 - V_\Sigma)) \simeq H_1(\Sigma)$. It is sufficient then to prove that this bilinear form is the linking. Let K' be a knot in $S^3 - V_\Sigma$ and K'' be a knot in Σ. By construction:

$$\varphi([K] \otimes [K']) = I(K', \Sigma'') = lk(K', K'')$$

□

Let L be an oriented link and Σ a Seifert surface associated to L. We have that $\Sigma \times [-1, 1] \subset S^3 - V_L$, with $\Sigma \times \{0\} = \Sigma$. Let $i^\pm$ be the applications $i^\pm : \Sigma \to S^3 - \Sigma$ given by $i^\pm(x) = x \times \pm 1$. If c is an oriented simple closed curve in Σ, $c^\pm = i^\pm_*(c)$.

Definition 16. We can associate to the Seifert surface Σ of an oriented link L the Seifert form

$$\alpha : H_1(\Sigma) \otimes H_1(\Sigma) \to \mathbb{Z}$$

defined by $\alpha(x, y) = lk((i^-)_* x, y)$.

Remark 13. By sliding with respect to second coordinate of $\Sigma \times [-1, 1]$ we obtain:

$$lk(a^-, b) = lk(a, b^+)$$

Definition 17. Let $\{f_i\}$ be a basis of $H_1(\Sigma)$. We define the Seifert matrix A associated to α the matrix whose component A_{ij} is

$$A_{ij} = \alpha(f_i, f_j) = lk(f_i^-, f_j) = lk(f_i, f_j^+)$$

Proposition 1.2.8. *If $\{e_i\} \in H_1(S^3 - \Sigma)$ is a β-dual basis of $\{f_i\}$, then $f_i^- = \sum_j A_{ij} e_j$ and $f_j^+ = \sum_i A_{ij} e_i$.*

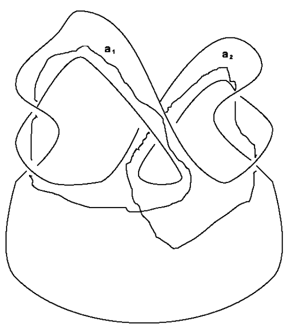
It is possible to compute a presentation for the Alexander module by the Seifert matrix.

Theorem 1.2.9. *Let S be a Seifert matrix for a knot K. Then $tS - S^T$ is a presentation matrix for the Alexander module $H_1(\tilde{X}_K)$.*

Proof. A proof can be find in [17] □

Corollary 1.2.10. *The Alexander polynomial can be computed as $\Delta_K(t) \doteq \det(tS - S^T)$.*

Example 4. *Let us calculate the Seifert matrix S for the trefoil knot. We can draw the trefoil know in a way such that it would be easier to see the Seifert surface.*



It is possible to see that

$lk(a_1, a_1^+) = -1$

$lk(a_1, a_2^+) = 1$

$lk(a_2, a_1^+) = 0$

$lk(a_2, a_2^+) = 1$

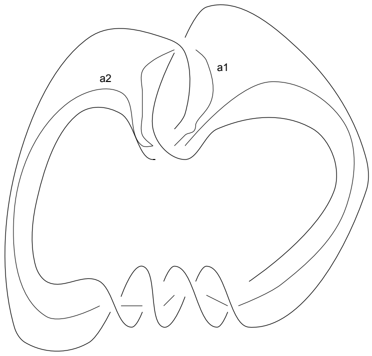
We obtain the matrix

$$S = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

We can compute the Alexander polynomial of the trefoil knot as

$$\Delta_K(t) = \det(tS - S^T) = -t^2 - 1 + t \doteq 1 - (t + t^{-1})$$

Example 5. *We want to compute the Alexander polynomial for a more complicated case, the twisted knots K_n defined as*



where the lower part has $2n-1$ crossings.

For the Seifert surface Σ we can take the surface having as boundary the knot, with generators for $H_1(\Sigma)$ represented by a_1 and a_2. Computing the Seifert matrix, we find:

$$A = \begin{pmatrix} 1 & 0 \\ -1 & n \end{pmatrix}$$

It follows that

$$tA - A^T = \begin{pmatrix} t-1 & 1 \\ -t & n(t-1) \end{pmatrix}$$

whose determinant is $n(t-1)^2 + t$. So the Alexander polynomial is

$$\Delta_{K_n} \doteq n(t^2 - 2t + 1) + t$$

.

We can notice that the trefoil knot is K_1 and recover the above calculation.

Using this definition it is very easy to prove some results about the Alexander polynomial.

Theorem 1.2.11. *For any oriented link* L*,* $\Delta_L(t) \doteq \Delta_L(t^{-1})$*. For any oriented knot* K*,* $\Delta_K(1) = \pm 1$*.*

Proof. The first statement is just an application of the standard theorems on linear algebra.

$$\Delta_L(t^{-1}) = \det(t^{-1}A - A^T) = t^{-n}\det(A - tA^t) \doteq \det(tA^t - A) = \Delta_L(t)$$

Now, let A be the Seifert matrix for K and Σ be the associated Seifert surface. If Σ has genus g, $H_1(\Sigma)$ is $2g$-dimensional and a basis is given by $\{f_i\}$. By theorem 1.2.9, $\Delta_K(1) = \pm\det(A - A^T)$, where $(A - A^T)_{ij} = lk(f_i^-, f_j) - lk(f_i^+, f_j)$, which is the algebraic number of intersections of f_i and f_j on Σ. We can see $A - A^T$ as made by g blocks

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

in the diagonal and 0 elsewhere. So, computing the determinant, we have proven the statement.

□

Corollary 1.2.12. *For any knot* K*,* $\Delta_K(t) \doteq a_0 + a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) + \ldots + a_n(t^n + t^{-n})$*, where* a_i *are integers and* a_0 *is odd.*

Proof. If the degree of the Alexander polynomial $\Delta_K(t)$ is odd, there is a contradiction with theorem 1.2.11. By construction of the Seifert matrix, all the a_i must be integers. If we calculate $\Delta_K(1) = a_0 + 2a_1 + \ldots + 2a_n = \pm 1$ we have that $a_0 \pm 1 = 2(a_0 + \ldots + a_n)$, so it has to be odd.

□

Remark 14. Usually, the signs for the coefficients a_i are chosen such that $\Delta_K(1) = 1$.

Corollary 1.2.13. *Let L be an oriented link. Let $\overline{L}$ be the reflection of L and rL its reverse. Then $\Delta_L(t) = \Delta_{\overline{L}}(t) = \Delta_{rL}(t)$.*

Proof. If S is the Seifert matrix for L, then $-A$ is the Seifert matrix for $\overline{L}$ and A^T is the Seifert matrix for rL.

□

1.2.3 Fox differential calculus

Given a group G, we want to define the group ring $\mathbb{Z}[G]$. Let $\nu : G \to \mathbb{Z}$ be such that $\nu(g) = 0$ except for a finite number of elements of G. The homomorphism ν has the following proprieties:

- $(\nu_1 + \nu_2)(g) = \nu_1(g) + \nu_2(g)$,
- $(\nu_1\nu_2)(g) = \sum_{h\in G}(\nu_1(g)\nu_2(h^{-1}g))$.

Let us define the 'dual space' G^*, consisting of elements g^* such that

$$g^*(h) = \begin{cases} 1 & \textit{if g=h} \\ 0 & \textit{otherwise} \end{cases}$$

So, if $g_1, \ldots, g_k \in G$ and $n_i = \nu(g_i)$ we can set

$$\nu = n_1 g_1^* + \ldots + n_k g_k^*$$

In the following we will denote $\mathbb{Z}[G]$ as $\mathbb{Z}G$.

Remark 15. $\mathbb{Z}G$ is a commutative ring if and only if G is an abelian group.

Remark 16. If $\Phi : G \to A$, A abelian ring, then there exists an extension $\overline{\Phi} : ZG \to A$, $\overline{\Phi}$ ring homomorphism.

Remark 17. If $\Phi : G \to G'$ is a group homomorphism, then Φ admits an extension $\overline{\Phi} : \mathbb{Z}G \to \mathbb{Z}G$, $\overline{\Phi}$ a ring homomorphism.

Let $\theta : G \to \mathbb{Z}$ be the trivial morphism, $\theta(g) = 1 \; \forall g \in G$. This morphism can be extended to $aug : \mathbb{Z}G \to \mathbb{Z}$, $aug(\sum_{i=1}^k n_i g_i) = \sum_{i=1}^k n_i$.

Definition 18. A derivation D is an application $D : \mathbb{Z}G \to \mathbb{Z}G$ such that

1. $D(\nu_1 + \nu_2) = D(\nu_1) + D(\nu_2)$,

2. $D(\nu_1\nu_2) = (D\nu_1)aug(\nu_2) + \nu_1 D(\nu_2)$.

Remark 18. If g_1, $g_2 \in G$, $D(g_1g_2) = D(g_1) + g_1D(g_2)$.

Proposition 1.2.14. *If D is a derivation then*

- $D(\sum_i n_i g_i) = \sum_i n_i D(g_i)$,
- $D(n) = 0$, $\forall n \in \mathbb{Z}$,
- $\forall g \in G$, $Dg^{-1} = -g^{-1}Dg$,

Proof. The first statement is a trivial consequence of linearity.

$D(1) = D(1 \cdot 1) = D(1) + 1 \cdot D(1) = D(1) + D(1)$, then $D(1) = 0$. By linearity, $D(n) = 0$ $\forall n \in \mathbb{Z}$.

$0 = D(1) = D(g^{-1}g) = D(g^{-1}) + g^{-1}D(g)$, then $D(g^{-1}) = -g^{-1}D(g)$. □

Proposition 1.2.15. $\forall n > 0$ $Dg^n = \sum_{i=0}^{n-1} g^i$ *and* $Dg^{-n} = -\sum_{i=-n}^{-1} g^i$.

Proof. It is easily seen by induction on n. □

Theorem 1.2.16. *Let F be the free group generated by n elements,*

$$F =< x_1, \dots, x_n > .$$

For every generator x_i of F there is an unique derivation $D_i = \frac{\partial}{\partial x_i}$ in $\mathbb{Z}F$ such that

$$\frac{\partial x_j}{\partial x_i} = \delta_{ij}$$

Proof. The proof, made by Fox, is included in his works [9], [10] and [11]. □

Corollary 1.2.17. *Let $h_1(x), \dots, h_n(x)$ be such that $h_1(x), \dots, h_n(x) \in \mathbb{Z}[F]$. It exists an unique derivation D in $\mathbb{Z}F$ such that $Dx_j = h_j(x)$. For all $f(x) \in \mathbb{Z}F$, $Df(x) = \sum \frac{\partial f}{\partial x_j} h_j(x)$.*

Theorem 1.2.18. *$\beta : f(x) \mapsto f(x) - f(1)$ is a derivation.*

$$f(x) - f(1) = \sum_1 \frac{\partial f}{\partial x_i}(x_i - 1)$$

Theorem 1.2.18 is called the fundamental theorem of Fox calculus.

Let G be a group and P be a finite representation, $P =< x_1, \ldots, x_n | \, r_1, \ldots, r_k >$. Let $F_n =< x_1, \ldots, x_n >$ be the free group generated by $x_1, \ldots, x_n$, H the normal subgroup generated by $r_1, \ldots, r_k$. Then $F_n/H \simeq G$, $g \mapsto \bar{g}$. This function can be extended to $\mathbb{Z}F_n \to \mathbb{Z}G$, $a \mapsto \bar{a}$.

Definition 19. The Fox matrix associated to the group G with presentation P is the $k \times n$ matrix with coefficients in $\mathbb{Z}G$

$$F(G, P) = (\overline{\frac{\partial r_i}{\partial j}}), \ 1 \leq i \leq k, \ 1 \leq j \leq n$$

We now show that the Fox matrix is well defined, i.e. it is independent of the choice of the presentation P. To do that, we will use the so-called Tietze presentation.

Definition 20. Let A and A' be two matrices with coefficients in the same ring R. A and A' are T-equivalent if it is possible to obtain one from the other by a sequence of following operations or of their inverses:

1. permutations of rows and columns

2. adding to a row (respectively column) a linear combination of other rows (respectively columns)

3. changing the matrix A by the matrix

$$\begin{pmatrix} 1 & * \\ 0 & A \end{pmatrix}$$

Theorem 1.2.19. *Let $P =< x_1, \ldots, x_n | \, r_1, \ldots, r_k >$ be a presentation of the group G. If P' is another presentation, then it is possible to obtain P' from P by a finite number of Tietze transformations and their inverses:*

1. *adding a new relation r such that $r \in H$,*

2. *adding a new generator x and a relation xw, where w is a word in $x_1, \ldots, x_n$.*

Proof. A proof can be found in [18].

□

Theorem 1.2.20. *Let P and P' be two finite representations of the same group G. Then the Fox matrices $F(G, P)$ and $F(G, Q)$ are T-equivalent.*

Proof. A permutation of rows and columns corresponds to a permutation of generators and relations.

Let now $P =< x_1 \ldots, x_n | \; r_1, \ldots, r_k >$ be a presentation of G. We add a relation $s \in H$. So,

$$s = \prod_{i=1}^{m} u_i r_i^{\epsilon_i} u_i^{-1}$$

where u_i is a word in x_i, $\epsilon_i = \pm 1$, $r_i \in H$. We can split it in three different cases and get the propriety for linearity.

Case $s = r_1 r_2$. In $\mathbb{Z}F_n$, $\frac{\partial s}{\partial x_i} = \frac{\partial r_1}{\partial x_i} + r_1 \frac{\partial r_2}{\partial x_i}$. So in $\mathbb{Z}G$ we get $\overline{\frac{\partial s}{\partial x_i}} = \overline{\frac{\partial r_1}{\partial x_i}} + \overline{\frac{\partial r_2}{\partial x_i}}$, that is adding a row.

Case $s = uru^{-1}$, $r \in H$, u word in $x_1, \ldots, x_n$.

$$\frac{\partial s}{\partial x_i} = \frac{\partial u}{\partial x_i} + u(\frac{\partial r}{\partial x_i} - ru^{-1}\frac{\partial u}{\partial x_i}) = u\frac{\partial r}{\partial x_i} + (1 - uru^{-1})\frac{\partial u}{\partial x_i}$$

So $\frac{\partial s}{\partial x_i} = u\frac{\partial r}{\partial x_i} + (1 - s)\frac{\partial u}{\partial x_i}$. In $\mathbb{Z}G$ we have

$$\overline{\frac{\partial s}{\partial x_i}} = u\overline{\frac{\partial r}{\partial x_i}}$$

which is multiplying a row on the left.

Case $s = r^{-1}$, $r \in H$.

$$\frac{\partial s}{\partial x_i} = -r^{-1}\frac{\partial r}{\partial x_i}$$

So $\overline{\frac{\partial s}{\partial x_i}} = -\overline{\frac{\partial r}{\partial x_i}}$, multiplying a row for -1.

Now, if $P' =< x_1, \ldots, x_n, x| \ xw, r_1, \ldots, r_k >$, we get the matrix

$$F(G, P') = \begin{pmatrix} 1 & \partial w / \partial x_i \\ 0 & F(G, P) \end{pmatrix}$$

$\square$

Definition 21. Let R be a commutative ring, $\alpha : \mathbb{Z}G \to R$ a homomorphism,

$$F(G, P, \alpha) = \alpha(\overline{\frac{\partial r_i}{\partial x_j}})$$

We call $E(G, P, \alpha)$ the ideal generated by the minors of order $n-1$.

Lemma 1.2.21. $E(G, P, \alpha)$ *is independent of the presentation* P.

Proof. If P' is another presentation of the group G, then $F(G, P, \alpha)$ and $F(G, Q, \alpha)$ are T-equivalent.

Permuting rows and columns and adding linear combination of rows (columns) to another row (column) leaves unchanged the minors. We have just to check the case F is changed in $F' = \begin{pmatrix} 1 & * \\ 0 & F \end{pmatrix}$, where $*$ is a $n-1$ array. F has n columns, while F' has $n+1$-columns.

Every minor of F of order $n-1$ lies in a minor of F' of order n, so that $E \subset E'$.

Every minor of order n of F' is a linear combination of minors of F of order $n-1$, so $E' \subset E$.

$\square$

1.2.4 Fox differential calculus for knots groups

Let us now focus on knots groups. Let $K \subset S^3$ be a knot, $\Gamma_K = \pi_1(S^3 - K)$ the knot group. We take $P =< x_1, \ldots, x_n| \ r_1, \ldots, r_{n-1} >$ as the Wirtinger presentation of Γ_K. Let $\alpha : \Gamma_K \to \mathbb{Z}$ be such that $\alpha(x_i) = t$, $\forall i, \ 1 \leq i \leq n$. We have $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$, the ring of Laurent polynomials. We can extend α to $\overline{\alpha} : \mathbb{Z}[\Gamma_K] \to \mathbb{Z}[t, t^{-1}]$, a ring homomorphism. In the following we will call, with a little abuse of notation, $\overline{\alpha} = \alpha$.

Definition 22. The Alexander ideal of a knot K is $E(K) = E(\Gamma_K, \alpha)$.

Theorem 1.2.22. *E(K) is a principal ideal.*

Proof. The fundamental theorem of Fox calculus implies that if

$$x \in F_n =< x_1, \ldots, x_n >,$$

then

$$x - 1 = \sum_{j=1}^{n} \frac{\partial x}{\partial x_j}(x_j - 1).$$

Also,

$$r_i - 1 = \sum_{j=1}^{n} \frac{\partial r_i}{\partial x_j}(x_j - 1), \forall\ i = 1, \ldots, n-1.$$

So, $\sum_{j=1}^{n} \overline{\frac{\partial r_i}{\partial x_j}}(\bar{x}_j - 1) = \bar{r}_i - 1 = 0$ and $\sum_{j=1}^{n} \alpha(\frac{\partial r_i}{\partial x_j}) = 0$ in $\mathbb{Z}[t, t^{-1}]$.

We have then that if the matrix is $F = (a_{ij})$, $\sum_{j=1}^{n} a_{ij} = 0$, $\forall\ i : 1 \leq i \leq n-1$. If $c_1, \ldots, c_n$ are the columns of $F(\Gamma_K, F, \alpha) = (a_{ij})$, then $c_1 + \ldots + c_n = 0$.

By definition, the Alexander ideal is generated by $\det(A_1), \ldots, \det(A_n)$. We have

$$\det(A_1) = \det(c_2, \ldots, c_n) = \det(c_2 + \ldots + c_n, c_3, \ldots, c_n) =$$

$$\det(-c_1, c_3, \ldots, c_n) = -\det(A_1).$$

So, with similar considerations, we get $\det(A_i) = \det(\pm A_j)$ and $E(K)$ is then generated by only one of the $\det(A_i)$.

$\square$

Theorem 1.2.23. *A generator of the ideal $E(K)$ is the Alexander polynomial $\Delta_K(t)$.*

Proof. A proof can be found in [8] or [17]. $\square$

Remark 19. All the considerations above can be done if we took the Wirtinger presentation. If we have another presentation, it is not true in general that the ideal $E(K)$ is generated by a minor A_i.

Example 6. *We want to compute the Alexander polynomial for the trefoil knot using the Fox calculus.*

$$P =< x_1, x_2, x_3 |\ x_3x_2x_3^{-1}x_1^{-1},\ x_1x_3x_1^{-1}x_2^{-1} >$$

is the Wirtinger presentation of the trefoil knot. So we have

$$\frac{\partial r_1}{\partial x_1} = \frac{\partial x_3x_2x_3^{-1}x_1^{-1}}{x_1} = \frac{\partial x_3x_2x_3^{-1}}{\partial x_1} + x_3x_2x_3^{-1}\frac{\partial x_1^{-1}}{\partial x_1} = -x_3x_2x_3^{-1}x_1^{-1}$$

$$\frac{\partial r_1}{\partial x_2} = \frac{\partial x_3x_2x_3^{-1}x_1^{-1}}{x_2} = \frac{\partial x_3}{\partial x_2} + x_3\frac{x_2x_3^{-1}}{\partial x_2} = -x_3 + x_3x_2\frac{\partial x_3^{-1}x_1^{-1}}{\partial x_2} = x_3$$

$$\frac{\partial r_1}{\partial x_3} = \frac{\partial x_3x_2x_3^{-1}x_1^{-1}}{\partial x_3} = \frac{\partial x_3}{\partial x_3} + x_3\frac{\partial x_2x_3^{-1}x_1^{-1}}{\partial x_3} =$$

$$1 + x_3x_2\frac{\partial x_3^{-1}x_1^{-1}}{\partial x_3} = 1 - x_3x_2x_3^{-1}$$

In the same way we get

$$\frac{\partial r_2}{\partial x_1} = 1 - x_1x_3x_1^{-1},\ \frac{\partial r_2}{\partial x_2} = -x_1x_3x_1^{-1}x_2^{-1},\ \frac{\partial r_1}{\partial x_3} = x_1$$

So, seeing now it in Γ_K, we get

$$\overline{\frac{\partial r_1}{\partial x_1}} = -1,\ \overline{\frac{\partial r_1}{\partial x_2}} = \overline{x}_3,\ \overline{\frac{\partial r_1}{\partial x_1}}1 - \overline{x}_1$$

$$\overline{\frac{\partial r_2}{\partial x_1}} = 1 - \overline{x}_2,\ \overline{\frac{\partial r_2}{\partial x_2}} = -1,\ \overline{\frac{\partial r_2}{\partial x_3}} = 1 - \overline{x}_1$$

and applying α we get the matrix

$$F(\Gamma_K, P, \alpha) = \begin{pmatrix} -1 & t & 1-t \\ 1-t & -1 & t \end{pmatrix}$$

So, $\det(A_1) = t^2 + 1 - t$*,* $\det(A_2) = 1 + t^2 - t$ *and* $\det(A_3) = 1 - t + t^2$*, so that* $\Delta_K(t) \doteq \det(A_i)$*,* $i = 1, 2, 3$*.*

Let us now see what happens if we take a group representation that is not Wirtinger's. We take as presentation

$$Q =< a, b|\ a^2b^{-3} > .$$

We had calculated it by passing by the presentation $< x, y|xyx = yxy >$ *and taking* $xyx = a$, $xy = b$. *So,* $\alpha(x) = t^3$, $\alpha(y) = t^2$. *We have*

$$\frac{\partial r}{\partial a} = (1 + a),\ \frac{\partial r}{\partial b} = a^2(-b^{-1} - b^{-2} - b^{-3})$$

So we get

$$F(\Gamma_K, Q, \alpha) = (1 + t^3 \ \ - 1 - t^2 - t^4).$$

By factorisation

$$1 + t^3 = (1 + t)(1 - t + t^2)$$

$$1 + t^2 + t^4 = (1 + t + t^2)(1 - t + t^2)$$

and $E(K) = \Delta_K(t) = 1 - t + t^2$.

Example 7. *We want now to compute the Alexander polynomial for a generic torus knot* $K(p, q)$. *We have seen that a Wirtinger presentation is*

$$Q =< x, y|\ x^p = y^q > .$$

As above, $\alpha(x) = t^q$, $\alpha(y) = t^p$.

$$F(\Gamma_{K(p,q)}, Q, \alpha) = (\frac{1 - t^{pq}}{1 - t^q} \quad - \frac{t^{pq}t^{-p}(1 - t^{-pq})}{1 - t^{-p}})$$

The highest common factor of $(1 - t^{pq})/(1 - t^q)$ *and* $(1 - t^{pq})/(1 - t^p)$ *is*

$$\frac{(1 - t)(1 - t^{pq})}{(1 - t^p)(1 - t^q)} = \Delta_{K(p,q)}.$$

We can see that if we take $p = 3$, $q = 2$, *that is the trefoil knot, we get*

$$\frac{(1 - t)(1 - t^6)}{(1 - t^3)(1 - t^2)} = \frac{(1 - t)(t^4 + t^2 + 1)(1 - t^2)}{(1 - t)(t^2 + t + 1)(1 - t^2)} =$$

$$\frac{(1 - t)(t^2 - t + 1)(t^2 + t + 1)(1 - t^2)}{(1 - t)(t^2 + t + 1)(1 - t^2)} = t^2 - t + 1$$

the same as in the previous computation.

1.2.5 The Alexander-Conway polynomial

We will now show how it is possible to normalise the Alexander polynomial such that it has no more the ambiguity concerning multiplication by units of $\mathbb{Z}[t^{\pm}]$. We start by some considerations about Seifert matrices.

Definition 23. Let A be a square matrix over $\mathbb{Z}$. A square matrix B such that

$$B = \begin{pmatrix} A & \lambda & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

or

$$B = \begin{pmatrix} A & 0 & 0 \\ \eta^T & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

where λ is a column and η^T a row, is called an elementary enlargement of A, and respectively A is called an elementary reduction of B. Two matrices A and B are said to be S-equivalent if they are related by a sequence of elementary enlargements, elementary reductions and unimodular congruences, i.e. there is P, $\det(P) = \pm 1$, $B = P^T AP$.

Theorem 1.2.24. *If A and B are two Seifert matrices for an oriented link L, then A and B are S-equivalent.*

Proof. Suppose that A is a $n \times n$ matrix corresponding to a Seifert surface Σ and to a choice of a basis of $H_1(\Sigma, \mathbb{Z})$. If the basis of $H_1(\Sigma, \mathbb{Z})$ is changed, the matrix A is changed by a unimodular congruence. Now suppose we change the Seifert surface. We can take another surface Σ', tube equivalent to Σ. Let $\{f_i\}_{1 \leq i \leq n}$ be a basis for $H_1(\Sigma, \mathbb{Z})$.We can then choose a basis for $H_1(\Sigma', \mathbb{Z})$ as the homology class of the curves $\{f_i\}$ and of two curves f_{n+1}, f_{n+2}, such that f_{n+1} goes over the solid cylinder and f_{n+2} around the middle of it.

By definition, f_{n+2} can be chosen such that it is disjoint from all curves $\{f_i\}$, $1 \leq i \leq n$, so that $lk(f_{n+2}^{\pm}, f_i) = 0$, $\forall i \neq n-1$. By definition, and with a nice choice of orientations, either $lk(f_{n+1}^{+}, f_{n+2}) = 0$ and $lk(f_{n+1}^{-}, f_{n+2}) = 1$, or $lk(f^+_{n+1}, f_{n+2}) = 1$ and $lk(f^-_{n+1}, f_{n+2}) = 0$. So in the first case we have a Seifert matrix of the form

$$\begin{pmatrix} A & \lambda & 0 \\ a & b & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

that is congruent to

$$\begin{pmatrix} A & \lambda & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

In the second case, we have

$$\begin{pmatrix} A & 0 & 0 \\ \eta^T & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

So matrices relatives to different Seifert surfaces are S-equivalent.

□

Definition 24. Let S be the matrix associated to a Seifert surface Σ of a link L. The Alexander-Conway polynomial is $\Delta_L(t) \in \mathbb{Z}[t^{\pm 1/2}]$ defined as

$$\Delta_L(t) = \det(t^{1/2}S - t^{-1/2}S^T).$$

Remark 20. Up to unit of $\mathbb{Z}[t^{\pm 1/2}]$, $\Delta_L(t)$ is the Alexander polynomial of L, that is why we are keeping the same notation.

Theorem 1.2.25. *The Conway-normalised Alexander polynomial is a well-defined invariant of the oriented link L.*

Proof. We have, for P unimodular congruence,

$$\det(t^{1/2}P^TSP - t^{-1/2}P^TS^TP) = (\det(P))^2 \det(t^{1/2}S - t^{-1/2}S).$$

Now, let S' such that

$$S' = \begin{pmatrix} S & \lambda & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

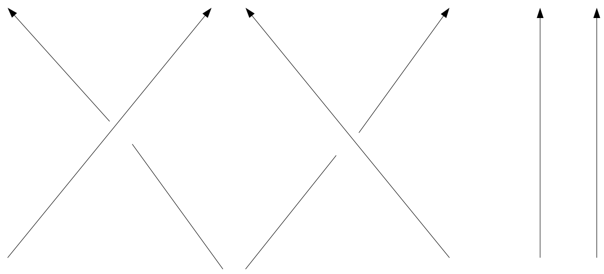
So,

$$(t^{1/2}S' - t^{-1/2}S'^T) = \begin{pmatrix} t^{1/2}S - t^{1/2}S^t & t^{1/2}\lambda & 0 \\ -t^{-1/2}\lambda^T & 0 & t^{1/2} \\ 0 & t^{-1/2} & 0 \end{pmatrix}$$

which has the same determinant as $t^{1/2}S - t^{-1/2}S^T$. In the same way, the other type of elementary enlargement of S has no effect on the determinant. □

Theorem 1.2.26. *For oriented links L, the Alexander-Conway polynomial $\Delta_L(t) \in \mathbb{Z}[t^{\pm 1/2}]$ is such that $\Delta_{unknot}(t) = 1$.*

Moreover, we take three different links that are the same except in a ball where they are like



From left to right, we call them L_+, L_- and L_0 and we have

$$\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{-1/2} + t^{1/2})\Delta_{L_0}(t).$$

Proof. Let Σ_0, Σ_+ and Σ_- be Seifert surfaces for L_0, L_+ and L_-. L_+ and L_- can be constructed by adding a short twisted strips to Σ_0. Let $\{f_2, f_3, \dots, f_n\}$ be oriented closed curves forming a basis for $H_1(\Sigma_0, \mathbb{Z})$. For $H_1(\Sigma_\pm, \mathbb{Z})$ we can take $\{f_1, f_2, \dots, f_n\}$, where $f_2, \dots, f_n$ are the same curves forming the basis of $H_1(\Sigma_0, \mathbb{Z})$ and f_1 is a curve going once along the twisted strips. Let S_0 be the Seifert matrix for L_0. The Seifert matrix for L_- is then

$$S_- = \begin{pmatrix} N & \lambda \\ \eta & S_0 \end{pmatrix}$$

whereas the Seifert matrix for L_+ is

$$S_+ = \begin{pmatrix} N-1 & \lambda \\ \eta & S_0 \end{pmatrix}$$

Now, computing $\det(t^{1/2}S_+ - t^{-1/2}S_+^T) - \det(t^{1/2}S_- - t^{1/2}S_-^T)$ gives exactly $\det(t^{1/2}S_0 - t^{-1/2}S_0)$.

□

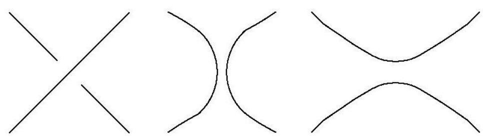
1.3 The Jones polynomial

1.3.1 Definition by Kaufmann brackets

Definition 25. The Kauffman bracket is a function from unoriented link diagrams to Laurent polynomials with integer coefficients in an indeterminate A, $<>: D \to \mathbb{Z}[A^{\pm}]$, such that

1. $< unknot >= 1$,
2. $< D \amalg unknot >= (-A^{-2} - A^{2}) < D >$,
3. $< C_+ >= A < C_1 > + A^{-1} < C_2 >$.

where C_+, C_1 and C_2 are such that



The third propriety means that the three link diagrams are the same, except near a point where they differ in the way indicated in the picture.

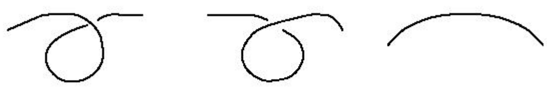
Remark 21. The bracket polynomial of a diagram with n crossings can be calculated by expressing it as a sum of 2^n diagrams with no crossings, then applying proprieties 1 and 2.

We want now to see what happens changing the diagram D by a Reidemeister move.

Lemma 1.3.1. *If a diagram D is changed by a Reideimesteir move of first type, its bracket changes in the following way:*

$$< R_1^+ >= -A^3 < R_0 >, \quad < R_1^- >= -A^{-3} < R_0 >$$

where R_1^+, R_1^- and R_0 are such that, from left to right,



Proof. We have

$$< R_1^+ >= A < R_0 \amalg unknot > +A^{-1} < R_0 >=$$

$$= (A(-A^{-2} - A^2 + A^{-1})) < R_0 >= -A^3 < R_0 >$$

and

$$< R_1^- >= A < R_0 > +A^{-1} < R_0 \amalg unknot >=$$

$$= (A + A^{-1}(-A^{-2} - A^2)) < R_0 >= -A^3 < R_0 > .$$

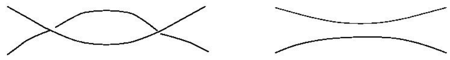
□

So it is clear that the Kauffman bracket is not an invariant for link diagrams, but it is possible to renormalise it to have an invariant.

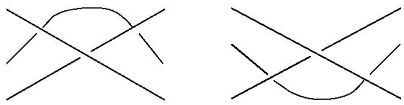
Lemma 1.3.2.

$$< R_2^+ >=< R_2 >, \quad < R_3^+ >=< R_3 >$$

where R_2^+ and R_2 are, from left to right

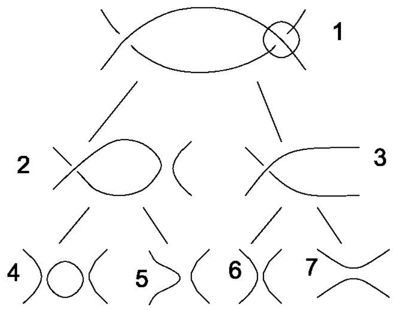


and R_3^+ and R_3^- are, from left to right



So $< D >$ is invariant under Reidemeister moves of second and third type.

Proof. It follows from the proprieties of the diagram and from lemma 1.3.1. For the second Reidemeister move we make the following simplifications:



Now, by iteration, we get:

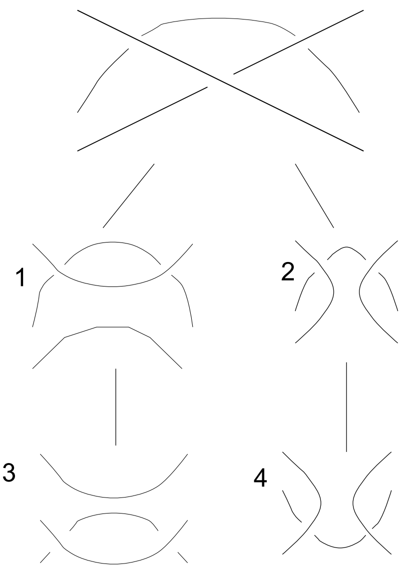
$$< D_1 >= A^{-1} < D_2 > +A < D_3 >=$$

$$A^{-1}(A < D_4 > +A^{-1} < D_5 >) + A(A < D_6 > +A^{-1} < D^{-7} >).$$

By definition of the bracket and noticing that $< D_5 >=< D_6 >$ we get

$$< D_5 > (-A^{-2} - A^2 + A^{-2} + A^2)+ < D_7 >=< D_7 >$$

For the third Reidemeister move we get:



We pass from 1 to 3 and from 2 to 4 by a Reidemeister move of second type. Then

$$< R_3^+ >= A < D_1 > +A^{-1} < D_2 >=$$

$$= A < D_3 > +A^{-1} < D_4 >=< R_3^- >$$

□

Definition 26. We call the sum of the signs of all the crossings of a diagram D of an oriented link the writhe $w(D)$ of the diagram.

Remark 22. The writhe $w(d)$ is not a link invariant. It is invariant for the second and the third Reidemeister move, but not for the first.

Theorem 1.3.3. *Let D be a diagram of an oriented link L. Then*

$$(-A)^{-3w(D)} < D >$$

is an invariant of the oriented link L.

Proof. By lemma 1.3.2 and remark 22 this expression is invariant for second and third Reidemeister move. By the computation done for the first Reidemeister move we have that is also unchanged for first Reidemeister move. □

Definition 27. Given a diagram D of a link L, we define the Jones polynomial $V(L)$ as the Laurent polynomial in $t^{1/2}$ with integer coefficients, defined by

$$V(L) = ((-A)^{-3w(D)} < D >)_{t^{1/2}=A^{-2}}.$$

For theorem 1.3.3, $V(L)$ is well defined.

1.3.2 Definition by skein relations

As for the Alexander-Conway polynomial, we now show that it is possible to define it using skein relations.

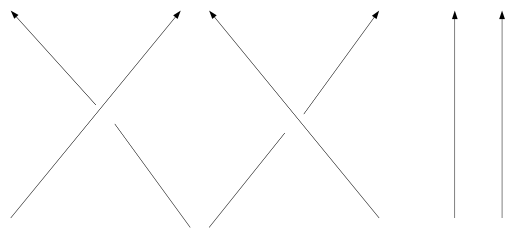
Proposition 1.3.4. *The Jones polynomial is a function*

$$V : \{\textit{oriented links in } S^3\} \to \mathbb{Z}[t^{\pm 1/2}]$$

such that $V(unknot) = 1$ *and that*

$$t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_0) = 0$$

where L_+*,* L_- *and* L_0 *mean that the diagrams are the same, except in a neighbourhood of a point where is as in the following picture, reading from left to right:*



Proof. Let us take into consideration the diagrams $C_1 = L_0$, C_2 of the definition of the Kauffman bracket. We have

$$< L_+ >= A < C_1 > +A^{-1} < C_2 >, \quad < L_- >= A^{-1} < C_1 > +A < C_2 >$$

so that $A < L_+ > -A^{-1} < L_- >= (A^2 - A^{-2}) < L_0 >$.

By definition, $w(L_+) - 1 = w(L_0)) = w(L_-) + 1$ and so

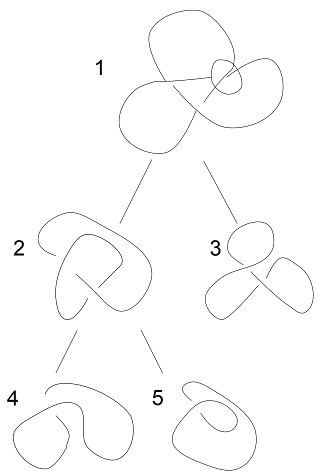
$$-A^4 V(L_+) + A^{-4} V(L_-) = (A^2 - A^{-2}) V(L_0).$$

Up to the substitution $t^{1/2} = A^{-2}$ we have the thesis.

□

Example 8. *We want now to calculate the Jones polynomial for the right and left trefoil.*

We take the diagram



So we get that

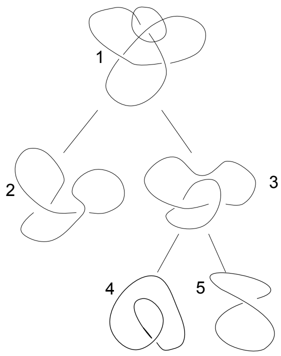
$$< D_1 >= A < D_2 > +A^{-1} < D_3 >=$$

$$= A(A^{-1} < D_4 > +A < D_5 >) + A^{-1} < D_3 >= -A^{-3} - A^5 + A^{-7}.$$

All the crossings are positive, so that $w(D) = 3$. *Then the Jones polynomial is*

$$V(3_1^+) = ((-A)^{-9}(A^{-7} - A^{-3} + A^{-5}))_{t^{1/2}=A^{-2}} = -t^4 + t^3 + t.$$

Now we do the same computation for the left trefoil.



Then

$$< D_1 >= A < D_2 > +A^{-1}D_3 =$$

$$= A < D_2 > +A^{-1}(A^{-1} < D_4 + A < D_5 >) = A^7 - A^{-5} - A^3$$

In this case $w(D_1) = -3$ *and the Jones polynomial is*

$$V(3_1^-) = ((-A)^9(A^7 - A^{-5} - A^3))_{t^{1/2}=A^{-2}} = -t^{-4} + t^{-3} + t^{-1}$$

We have then shown that the right trefoil and the left trefoil are not isotopic, so that the trefoil knot is not amphicheiral.

Chapter 2

Braid groups

2.1 Some equivalent definitions

2.1.1 Configuration spaces

Definition 28. Let $\Delta \subset \mathbb{C}^n$, $n \in \mathbb{N}$ be

$$\Delta = \cup_{i,j=1}^{n} \{z_i = z_j\}, \quad i \neq j.$$

Let C and $Conf_n$ be such that $C = \mathbb{C}^n - \Delta$ and $Conf_n(\mathbb{C}) = (\mathbb{C}^n - \Delta)/S_n$, where S_n is the group of permutation on n elements.

Remark 23. The space C, associated to the map $C \to Conf_n(\mathbb{C})$, is a covering space of $Conf_n(\mathbb{C})$ and the associated group is S_n.

Remark 24. Up to homeomorphism, it is possible to consider just points in the interior of $D^2 \subset \mathbb{C}$.

Let $*$ be the point on the real line, $* = \{\frac{2k-1-n}{n},\ 1 \leq k \leq n\}$.

Definition 29. We call n-pure braid group the group

$$P_n = \pi_1(C, *).$$

We call n-braid group the group

$$B_n = \pi_1(Conf_n(\mathbb{C}), *).$$

2.1.2 Diagrams

Let I be the closed interval $[0,1] \subset \mathbb{R}$. A topological interval is a topological space homeomorphic to I.

Definition 30. A geometric braid on n strings, $n \geq 1$, is a set $b \subset \mathbb{C} \times I$ formed by n disjoints topological intervals, the strings of b, such that the projection $\mathbb{C} \times I \to I$ maps each string homeomorphically onto I and

$$b \cap (\mathbb{C} \times \{0\}) = * \times \{0\}$$

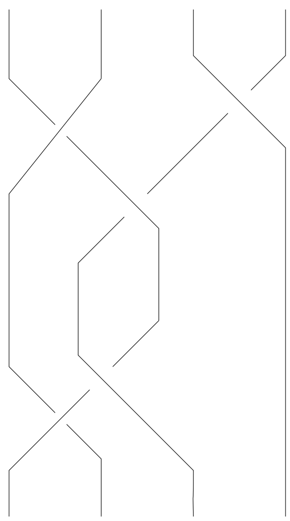
$$b \cap (\mathbb{C} \times \{1\}) = * \times \{1\}$$

where $*$ is the set of points defined above. We can label the n points by $P = (p_1, \ldots, p_n) = \{1, \ldots, n\}$.

Remark 25. Every string of b meets each $\mathbb{C} \times \{t\}$, $t \in I$, in exactly one point and connects a point in $P \times \{0\}$ to a point in $P \times \{1\}$.

We can associate to each string b a permutation $\pi \in S_n$ such that $b(P_i, 1) = \pi(i)$.

An example of geometric braid is:



The underlying permutations is (3 4).

We want now to define a class of isotopy for braids, as it is for knots.

Definition 31. Let b and b' be two geometric braids. They are said to be isotopic if there is a continuous map $F : b \times I \to \mathbb{C} \times I$ such that

- for each $s \in I$ $F_s : b \to \mathbb{C} \times I$ is an embedding, whose image is a geometric braid on n strings,
- $F_0 = id_b : b \to b$,
- $F_1(b) = b'$.

Remark 26. If b and b' are isotopic, then the underlying permutations π_b and $\pi_{b'}$ must be the same.

Proposition 2.1.1. *The relation of isotopy is an equivalence relation. We call the equivalence classes braids on n strings.*

We want now to introduce a canonical operation between two braids. We will see that the set of geometric braids with this operation is actually a group.

Definition 32. Let b_1 and b_2 be two n-strands geometric braids. We define $b = b_1 b_2$ as the set of points $\{z, t\} \in \mathbb{C} \times I$ such that $\{z, 2t\} \in b_1$, for $0 \leq t \leq 1/2$ and $\{z, 2t - 1\} \in b_2$ for $1/2 \leq t \leq 1$. b is still a geometric braid with n strands.

Remark 27. If b_1 is isotopic to b'_1 and b_2 is isotopic to b'_2, then $b_1 b_2$ is isotopic to $b'_1 b'_2$.

As we have done for knots, we would like to represent geometric braids on the plane. In the following we will identify $\mathbb{C}$ with $\mathbb{R}^2$.

Definition 33. A braid diagram on n strands is a set $D \subset \mathbb{R} \times I$, union of n topological intervals, such that:

- the projection $R \times I \to I$ maps each strand homeomorphically onto I,

- every point of $* \times \{0,1\}$ is the endpoint of a unique strand,
- every point of $\mathbb{R} \times I$ belongs to at most two strands. At each crossing these strands meet transversely, with one undergoing and the other overgoing.

Remark 28. By compactness of the strands, the number of crossing of a diagram D is finite.

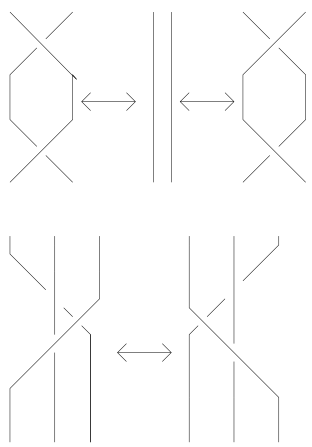
Proposition 2.1.2. *Every geometric braid can be represented by a braid diagram.*

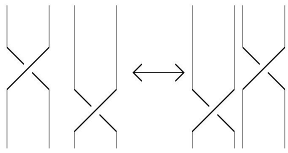
Every braid diagram represents a geometric braid, up to isotopy.

Definition 34. Two braid diagrams D and D' on n strands are said to be isotopic if there is a map $F : D \times I \to \mathbb{R} \times I$ such that

- $F(D \times \{s\})$ is a braid diagram on n strands $\forall s \in I$,
- $D_0 = D$,
- $D_1 = D'$.

Theorem 2.1.3. *Two braid diagrams D and D' define the same geometric braid if and only if it is possible to get one from the other by a sequence of diagram moves and their inverses as following:*



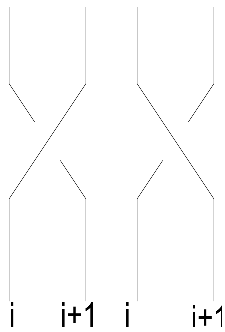


Proof. A proof can be found in [16].

□

2.1.3 Presentation

Let σ_i and σ_i^{-1} be the braids:



with all the other strands being straights.

Theorem 2.1.4. *The braid diagrams are generated by σ_1, σ_2, ..., σ_{n-1}.*

The group B_n admits a presentation:

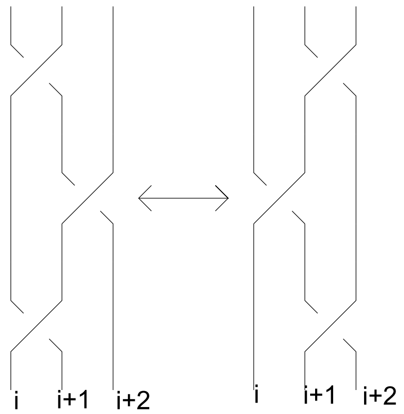
$$P =< \sigma_1, \ldots, \sigma_{n-1} | \ (1), (2) >$$

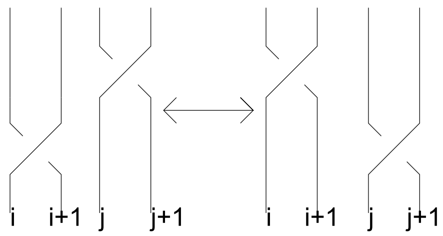
where (1) *and* (2) *are the relations:*

1. $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, $\forall i : \ 1 \leq i \leq n-2$

2. $\sigma_i\sigma_j = \sigma_j\sigma_i$, $\forall i, j : \ |i-j| \geq 2$

Conditions 1 and 2 can be represented by the following diagrams:





Proof. It is trivial that every diagram can be constructed by a sequence of elements σ_i and σ_i^{-1}.

The three diagram movements of theorem 2.3.1 are covered by relations (1) and (2) and by $\sigma_i \sigma_i^{-1} = 1$. $\square$

Corollary 2.1.5. *The group B_2 is the infinite cyclic group generated by the element σ_1.*

Remark 29. $\forall i : \ 1 \leq i \leq n-1$ we have:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1}, \ \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i = \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1}, \ \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i^{-1} = \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1},$$

$$\sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} = \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}, \ \sigma_i^{-1} \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1}.$$

$\forall i, j : \| i - j| \geq 2$

$$\sigma_i^{-1} \sigma_j = \sigma_j \sigma_i^{-1}, \ \sigma_i \sigma_j^{-1} = \sigma_j^{-1} \sigma_i, \ \sigma_i^{-1} \sigma_j^{-1} = \sigma_j^{-1} \sigma_i^{-1}.$$

Proposition 2.1.6. *If $s_1, \ldots, s_{n-1}$ are elements of a group G satisfying the braid relations, then there is a unique group homomorphism $f : B_n \to G$ such that $s_i = f(\sigma_i)$ $\forall i : 1 \leq i \leq n-1$.*

Proof. Let F_n be such that $F_n =< \sigma_1, \ldots, \sigma_{n-1} >$. There is a unique group homomorphism $\tilde{f} : F_n \to G$ such that $\tilde{f}(\sigma_i) = s_i$. This homomorphism induces a group homomorphism $f : B_n \to G$ if and only if $\tilde{f}(r) = \tilde{f}(r')$ for all braid relations $r = r'$.

For the first braid relation:

$$\tilde{f}(\sigma_i \sigma_{i+1} \sigma_i) = s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} = \tilde{f}(\sigma_{i+1} \sigma_i \sigma_{i+1}).$$

For the second braid relation:

$$\tilde{f}(\sigma_i \sigma_j) = s_i s_j = s_j s_i = \tilde{f}(\sigma_j \sigma_i).$$

□

Let S_n be the group of permutations on n elements. Let $\tau_i = (i\ i+1)$ be a transposition. S_n is generated by the elements τ_i, $i = 1, \ldots, n-1$, such that $\tau_i^2 = 2$. Also, $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$ $\forall i : \ 1 \leq i \leq n-1$ and $\tau_i \tau_j = \tau_j \tau_i$ $|i-j| \geq 2$.

Theorem 2.1.7. *S_n admits a presentation P such that*

$$P =< \tau_1, \ldots, \tau_{n-1} |\ \tau_i^2 = 1\ \forall i,\ (1),\ (2) >$$

where (1) *and* (2) *are the relations of theorem 2.1.4.*

By proposition 2.1.6, there exists a unique group epimorphism $\pi : B_n \to S_n$ such that $\tau_i = \pi(\sigma_i)$, $\forall i : \ 1 \leq i \leq n-1$.

Proposition 2.1.8. *The group B_n, $n \geq 3$, is nonabelian.*

Proof. S_n, $n \geq 3$, is a nonabelian group and $\pi : B_n \to S_n$ is an epimorphism. So also B_n is non abelian. □

Proposition 2.1.9. *The map $i : B_n \to B_{n+1}$, $i(\sigma_j) = (\sigma_j)\ \forall\ j = 1, \ldots, n-1$ is an injective homomorphism.*

We can see this map on geometric braids as the map sending every braid $\beta \in B_n$ in itself, with an additional straight strand on the far right. We have then a sequence of inclusions $B_1 \subset B_2 \subset B_2 \subset \ldots$.

Composing i with the projection $\pi : B_{n+1} \to S_{n+1}$ it is the same than composing $\pi : B_n \to S_n$ with the inclusion $S_n \hookrightarrow S_{n+1}$. So there is a commutative diagram

$$\begin{array}{ccc} B_n & \longrightarrow & S_n \\ \downarrow & & \downarrow \\ B_{n+1} & \longrightarrow & S_{n+1} \end{array}$$

Example 9. *By definition, $B_3 =< \sigma_1, \sigma_2 | \ \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 >$.*

This is exactly the knot group of the trefoil knot, as we have computed in the first chapter. We can represent it also as $< a, b | \ a^2 = b^3 >$. We can notice that, with that presentation, the element a^2 lies in the center of B_3.

Proposition 2.1.10. *The group B_n admits a presentation with two generators.*

Proof. Let α and β be such that $\alpha = \sigma_1$, $\beta = \sigma_1\sigma_2, \ldots, \sigma_{n-1}$,

For $1 \leq i \leq n-2$ we have $\beta\sigma_i = \sigma_{i+1}\beta$.

$$(\sigma_1\sigma_2 \ldots \sigma_{n-1})\sigma_i = \sigma_1\sigma_2 \ldots \sigma_{i-1}\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+2} \ldots \sigma_{n-1} =$$

$$\sigma_1\sigma_2 \ldots \sigma_{i-1}\sigma_{i+1}\sigma_i\sigma_{i+1} \ldots \sigma_{n-1} = \sigma_{i+1}(\sigma_1\sigma_2 \ldots \sigma_{n-1}).$$

So it is easy to see that $\sigma_i = \beta^{i-1}\alpha\beta^{1-i}$ and that α, β generate all B_n.

□

2.1.4 Mapping class groups

In the following P_n represents a set of n points on the real line of $\mathbb{C}$ (or equivalently on the line $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$).

Definition 35. Let $Diff(D_2, P_n, S^1)$ be the set of diffeomorphisms $f : D_2 \to D_2$ such that $f(P_n) = P_n$ and $f|_{S^1} = Id_{S^1}$, also called self diffeomorphisms. Let $Diff_0(D_2, P_n, S^1)$ be the set of the self diffeomorphisms $f : D_2 \to D_2$

such that every f is isotopic to Id_{D_2}, i.e. there is a continuous family $f_t(x)$ of self diffeomorphisms such that $f_0(x) = f(x)$ and $f_1(x) = x$.

Proposition 2.1.11. *$Diff_0(D_2, P_n, S^1)$ is a normal subgroup of $Diff(D_2, P_n, S^1)$.*

Proof. It is easy to see that $Diff_0(D_2, P_n, S^1)$ is a subgroup.

If $f \in Diff_0(D_2, P_n, S^1)$, $g \in Diff(D_2, P_n, S^1)$ then $g^{-1}fg \simeq g^{-1}Idg \simeq gg^{-1} \simeq Id_{D_2}$. □

Definition 36. Let D_2^n be D_2 with a choice of a set of n points P_n. We define $M(D_2^n)$ as

$$M(D_2^n) = \frac{Diff(D_2, P_n, S^1)}{Diff_0(D_2, P_n, S^1)}.$$

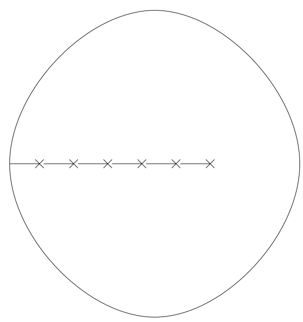
Theorem 2.1.12. *$M(D_2^0) = M(D_2)$ is trivial.*

Proof. We use the Alexander trick. Let f be such that $f \in Diff(D_2, S^1)$. Let us take

$$h(s, re^{i\theta}) = \begin{cases} f(\frac{r}{1-s}e^{i\theta}), & r \leq 1-s \\ re^{i\theta}, & r \geq 1-s \end{cases}$$

□

Corollary 2.1.13. *Let $f \in Diff(D_2, P_n)$ be such that $f(a_i) = a_i$, where a_i are segments linking every point i to the point $i+1$ and 1 to S^1, i.e.*



Then f is isotopic to the identity.

Proof. If f is the identity on the segments, then it is isotopic to the identity in a small open set U such that $a_1, \dots, a_n \subset U$. $(D_2 - P_n) - U$ is diffeomorphic to D_2, so we can apply theorem 2.1.12, $f \simeq Id$ in $(D_2 - P_n) - U$. Then $f \simeq Id$ in $(D_2 - P_n) - U$.

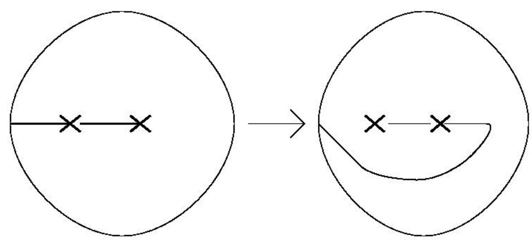
□

We want now to find a group of generators for $M(D_2^n)$. We take $n = 2$, so that $P_2 = \{(-1/2, 0), (1/2, 0)\}$.

Let $t : D_2 \to D_2$ be

$$t : re^{i2\pi\theta} \mapsto re^{i2\pi(\theta+1-r)}.$$

The map t is a diffeomorphism such that $t((1/2, 0)) = (-1/2, 0)$, $t((-1/2, 0)) = (1/2, 0)$. We can represent it as:



Now we extend this function to every n. Let p_k and p_{k+1} be two points and U an open set diffeomorphic to D_2 and such that $p_i \notin U$ for $i \neq k, k+1$. We can define t_k as acting as t on U and being the identity on $D_2 - U$.

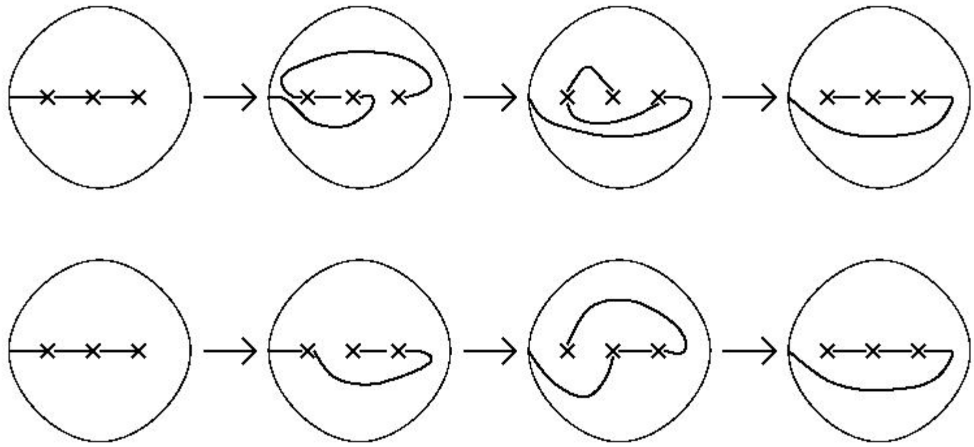
Proposition 2.1.14. *The functions t_k, $1 \leq k \leq n-1$, satisfy the braid relations*

- $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$, $\forall i : \ 1 \leq i \leq n-2$,
- $t_i t_j = t_j t_i$, $|i - j| \geq 2$.

Proof. The second relation follows from the construction. We just have to check the first one.

So $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$.

□



Theorem 2.1.15. *Let $\psi : B_n \to M(D_2^n)$ be defined by*

$$\psi(\sigma_k) = t_k.$$

Then ψ is an isomorphism.

Proof. We show that ψ is an isomorphism by defining an inverse.

Let f be in $Diff(D_2, P_n, S^1)$. f is isotopic to the identity in $Diff(D_2, S^1)$. Let $h : [0,1] \times [0,1] \to D_2$ be $(s, z) \mapsto h_s(z)$, $h_0 = Id$, $h_1 = f$.

$\sigma_f : s \mapsto h_s(P_n)$ is then a braid and $t_k \mapsto \sigma_k$. σ_f is well-defined because $Diff(D_2, S^1)$ is contractible. It is sufficient to see that σ is an homomorphism, $\sigma_{gf} = \sigma_g \sigma_f$.

Let h_s be an isotopy from Id to g and h'_s be an isotopy from Id to f. Then $h_s \circ f$ is an isotopy from f to $g \circ f$ and $h'_s(h_s \circ f)$ is an isotopy from Id to $g \circ f$. So this is a morphism.

□

2.2 Proprieties

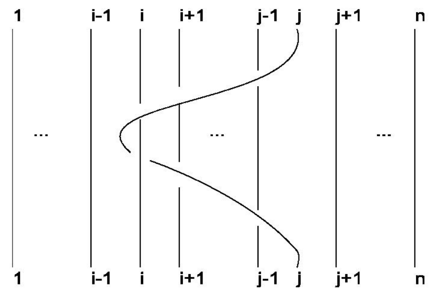
2.2.1 Pure braid group

Let us focus on the projection $\pi : B_n \to S_n$. The kernel of π is a group, called the pure braid group P_n. Its elements are called pure braids on n strands. A geometric braid on n strands represents an element in P_n if and only if every strand starting by $(i, 0, 0)$ ends in $(i, 0, 1)$, $\forall i : \ 1 \leq i \leq n$.

We want to find a set of generators for this group. The group P_n is generated by elements $A_{i,j}$, $1 \leq i < j \leq n$ such that

$$A_{i,j} = \sigma_{j-1}\sigma_{j-2} \dots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \dots \sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$$

represented by the geometric braid



P_n is a normal subgroup of B_n. Moreover, the braids $A_{i,j}$ are all conjugate each other in B_n.

Proposition 2.2.1. *Let $\alpha_{i,j}$ be $\alpha = \sigma_{j-1}\sigma_{j-2} \dots \sigma_i$. Then for $1 \leq i < j < k \leq n$*

$$\alpha_{j,k}A_{i,j}\alpha_{j,k}^{-1} = A_{i,k}, \quad \sigma_i A_{i,j}\sigma_i^{-1} = A_{i+1,j}.$$

Proof.

$$\alpha_{j,k}A_{i,j}\alpha_{j,k}^{-1} = \sigma_{k-1}\sigma_{k-2} \dots \sigma_j\sigma_{j-1} \dots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \dots \sigma_{j-1}^{-1}\sigma_j^{-1} \dots \sigma_k^{-1} = A_{i,k}.$$

$$\sigma_i\sigma_{j-1} \dots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \dots \sigma_{j-1}^{-1}\sigma_i^{-1} = \sigma_{j-1} \dots \sigma_{i+2}\sigma_i\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+2}^{-1} \dots \sigma j - 1^{-1} =$$

$$= \sigma_{j-1} \dots \sigma_{i+2}\sigma_{i+1}\sigma_i\sigma_{i+1}\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}\sigma_{i+2}^{-1} \dots \sigma_{j-1}^{-1} = A_{i+1,j}.$$

$\square$

Remark 30. The inclusion $i : B_n \to B_{n+1}$ maps P_n to P_{n+1}, an homomorphism $i|_{P_n} : P_n \to P_{n+1}$. As in B_n, we can see this application geometrically as the addiction of a vertical strand on the far right. Moreover, $P_n \to P_{n+1}$ is injective.

We want now to define a forgetting homomorphism $f_n : P_n \to P_{n-1}$. Let b be a pure braid. By definition, the i-th strand connects $(i, 0, 0)$ to $(i, 0, 1)$. By convention, we eliminate the n-string.

Remark 31. If b is isotopic to b', then $f_n(b)$ is isotopic to $f_n(b')$.

By remark, the function f_n is well defined from P_n to P_{n-1} and it is a group homomorphism.

Proposition 2.2.2. *If $i : P_{n-1} \to P_n$ is the natural inclusion and $f_n : P_n \to P_{n-1}$ is the forgetting homomorphism, then $f_n \circ i = id_{P_{n-1}}$.*

The proposition easily follows from the geometric construction.

Corollary 2.2.3. *$i : P_{n-1} \to P_n$ is into and $f_n : P_n \to P_{n-1}$ is onto.*

Let U_n be $\ker(f_n : P_n \to P_{n-1})$. There is an exact sequence

$$0 \to U \to P_n \to P_{n-1} \to 0$$

There is a section $i : P_{n-1} \to P_n$, so the sequence splits and

$$P_n = U_n \rtimes P_{n-1}.$$

Then every braid $\beta \in B_n$ can be written as $\beta = i(\beta')\beta_n$, where $\beta' \in P_{n-1}$ and $\beta_n \in U$.

In particular, $\ker(P_n \to P_{n-1}) = \pi_1(\mathbb{C} - \{z_1, \dots, z_{n-1}\}) = F_{n-1}$, the free group on $n-1$ generator.

Iterating this construction, we obtain that $P_n = F_{n-1} \rtimes F_{n-2} \rtimes \ldots \rtimes F_2 \rtimes F_1$, so that

$$\beta = \beta_2 \beta_3 \ldots \beta_n,$$

$\beta_i \in U_i \subset P_i$.

Proposition 2.2.4. *The group P_n admits a normal filtration*

$$1 = U_n^{(0)} \subset U_n^{(1)} \subset \ldots \subset U_n^{(n-1)} = P_n$$

such that $U_n^{(i)}/U_n^{(i-1)}$ is a free group of rank $n-i$ for all i.

Proof. Take $U_n^{(0)} = 1$ and

$$U_n^{(i)} = \ker(f_{n-i+1} \ldots f_{n-1} f_n : P_n \to P_{n-1})$$

for all i, $1 \leq i \leq n-1$. Then

$$U_n^{(i)}/U_n^{(i-1)} \simeq \ker(f_{n-i+1} : P_{n-i+1} \to P_{n-i}) = U_{n-i+1}.$$

□

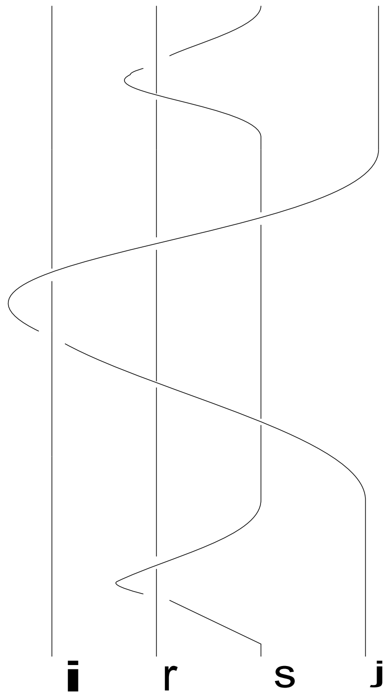
Corollary 2.2.5. *The group P_n is torsion free.*

Proof. The group P_n can be decomposed as semidirect product of free groups, which are torsion free. □

Theorem 2.2.6. *The group P_n admits a presentation with $\frac{n(n-1)}{2}$ generators $\{A_{i,j}\}$, $1 \leq i < j \leq n$ and relations*

$$A_{r,s}^{-1} A_{i,j} A_{r,s} = \begin{cases} A_{i,j} & if \;\; s < i. \\ A_{i,j} & if \;\; i < r < s < j. \\ A_{r,j} A_{i,j} A_{r,j}^{-1} & if \;\; s = i. \\ A_{r,j} A_{s,j} A_{i,j} A_{s,j}^{-1} A_{r,j}^{-1} & if \;\; i = r < s < j. \\ A_{r,j} A_{s,j} A_{r,j}^{-1} A_{s,j}^{-1} A_{i,j} A_{s,j} A_{r,j} A_{s,j}^{-1} A_{r,j}^{-1} & if \;\; r < i < s < j. \end{cases}$$

Proof. We have seen that $\{A_{i,j}\}$ generate P_n. The relations can be proven by drawing the diagrams. For example, for the second relation:



It is easy to see that this is actually $A_{i,j}$. All the other relations are proven in the same way, see [13].

□

Corollary 2.2.7. $P_n/[P_n, P_n] \simeq \mathbb{Z}^{n(n-1)/2}$.

Proof. It is sufficient to show that the elements $A_{i,j}$ are linearly indipendent, i.e. it is sufficient to construct a group homomorphism $l_{i,j} : P_n \to \mathbb{Z}$ such that $l_{i,j}(A_{r,s}) = 1$ for $(r, s) = (i, j)$, $l_{i,j}(A_{r,s}) = 0$ otherwise.

Let β be a braid and D an associated braid diagram. We can orient the strands of this diagram from the level $t = 0$ to the level $t = 1$. So we have two functions $l^+_{i,j}(D)$ and $l^-_{i,j}(D)$, such that $l^+_{i,j}$ is the number of crossing of D where the ith strands goes over the jth strand from left to right and $l^-_{i,j}$ is the number of crossing of D where the ith strands goes over the jth strand from right to left. Let $l_{i,j}$ be $l_{i,j} = l^+_{i,j} - l^-_{i,j}$.

The function $l_{i,j}$ is invariant under Reidemeister moves on D, so it is well defined, $l_{i,j} : P_n \to \mathbb{Z}$. It is straightforward to see that $l_{i,j}(A_{r,s}) = 1$ if

$(r, s) = (i, j)$, 0 otherwise.

□

Definition 37. A group G is residually finite if for each $g \in G - \{1\}$ there is a homomorphism f from G to a finite group, $f(g) \neq 1$.

Proposition 2.2.8. *Free groups are residually finite and a semidirect product of two finitely generated residually finite groups is residually finite.*

Proof. A proof of the first fact can be found in [18], while a proof of the second can be found in [19].

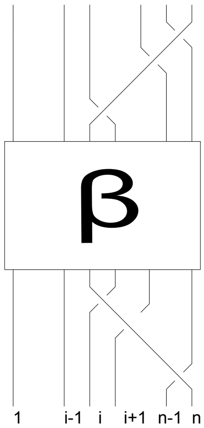
□

Corollary 2.2.9. *B_n is residually finite and so are all its subgroups.*

Proof. We have seen that P_n is a semidirect product of free groups, so it is residually finite. Moreover, every extension of a residually finite group by a finite group is still residually finite, because the intersection of a finite family of subgroups of finite index is a subgroup of finite index. Now, B_n is an extension of P_n by S_n, which is a finite group, so that B_n is residually finite too. It is easy to see that every subgroup of a residually finite group is residually finite. □

Corollary 2.2.10. *$f_n^i : P_n \to P_{n-1}$ is defined as the map forgetting the ith string, for $i = 1, 2 \ldots, n$. The kernel of f_n^i is a free group of rank $n - 1$ with free generators $A_{i,1}, \ldots, A_{i-1,i}, A_{i,i+1}, \ldots, A_{i,n}$.*

Proof. Let $\alpha_{i,n}$ be $\alpha = \sigma_{n-1}\sigma_{n-2} \ldots \sigma_i$ and let β be a pure braid in P_n. We take into consideration the braid $\alpha_{i,n}\beta\alpha_{i,n}^{-1}$



Applying f_n, the usual forgetting homomorphism, we find

$$f_n(\alpha_{i,n}\beta\alpha_{i,n}^{-1}) = 1_{n-1}f_n^i(\beta)1_{n-1} = f_n^i(\beta).$$

So we have that $f_n^i(\beta) = f_n(\alpha_{i,n}\beta\alpha_{i,n}^{-1})$ and then

$$\ker(f_n^i) = \alpha_{i,n}^{-1}\ker(f_n)\alpha_{i,n} = \alpha_{i,n}^{-1}U_n\alpha_{i,n}.$$

Now, by the proprieties of P_n, we get that conjugating by $\alpha_{i,n}^{-1}$ transforms the set $\{A_{j,n}\}_{j=1,2,\dots,n-1}$ into the set $\{A_{i,1}\dots, A_{i-1,i}, A_{i,i+1}\dots, A_{i,n}\}$.

□

2.2.2 The center of B_n

Definition 38. The center of a group G is the set

$$Z(G) = \{h \in G |\ hg = gh,\ \forall g \in G\}.$$

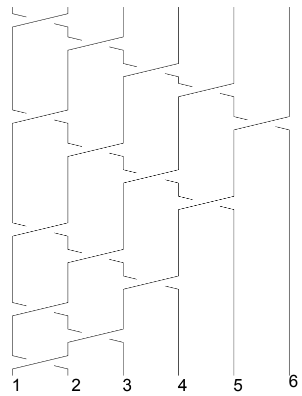
Remark 32. $Z(G)$ is a subgroup of G.

Definition 39. Let $\Delta_n \in B_n$ be the braid

$$\Delta_n = (\sigma_1\sigma_2\dots\sigma_{n-1})(\sigma_1\sigma_2\dots\sigma_{n-2})\dots(\sigma_1\sigma_2)\sigma_1.$$

Δ_n is called the Garside element.

Example 10. *For $n = 6$ the Garside element is*



Lemma 2.2.11. $\sigma_i \Delta_n = \Delta_n \sigma_{n-i}, \quad \forall i = 1, \ldots, n-1,$

Proof. We will see it by induction.
If $n = 2$ it is trivial. If $n = 3$, we have

$$\sigma_1(\sigma_1\sigma_2)\sigma_1 = \sigma_1\sigma_2\sigma_1\sigma_2, \quad \sigma_2(\sigma_1\sigma_2)\sigma_1 = \sigma_1\sigma_2\sigma_1\sigma_1$$

Take $i = 2, \ldots, n-1$. Then

$$\sigma_i(\sigma_1\sigma_2 \ldots, \sigma_{n-1})(\sigma_1\sigma_2 \ldots, \sigma_{n-2}) \ldots (\sigma_1\sigma_2\sigma_1) =$$

$$= \sigma_1\sigma_2 \ldots \sigma_i\sigma_{i-1}\sigma_i\sigma_{i+1} \ldots, \sigma_{n-1}(\sigma_1\sigma_2 \ldots, \sigma_{n-2}) \ldots (\sigma_1\sigma_2\sigma_1) =$$

$$= (\sigma_1\sigma_2 \ldots \sigma_{i-1}\sigma_i\sigma_{i+1} \ldots \sigma_{n-1})\sigma_{i-1}\Delta_{n-1} = \Delta_n\sigma_{n-i}$$

Take now the case $i = 1$.

$$\sigma_1\sigma_1\sigma_2 \ldots \sigma_{n-1}\sigma_1\sigma_2 \ldots \sigma_{n-2}\Delta_{n-2} = \sigma_1\sigma_1\sigma_2\sigma_1\sigma_3 \ldots \sigma_{n-1}\sigma_2\sigma_3 \ldots \sigma_{n-2}\Delta_{n-2} =$$

$$= \sigma_1\sigma_2\sigma_2\sigma_3 \ldots \sigma_{n-1}\sigma_2\sigma_3 \ldots \sigma_{n-2}\Delta_{n-2} = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2 \ldots \sigma_{n-1}\sigma_3 \ldots \sigma_{n-2}\Delta_{n-2} =$$

$$= \sigma_1\sigma_2\sigma_3 \ldots \sigma_{n-1}\sigma_1\sigma_2 \ldots \sigma_{n-2}\sigma_{n-1}\Delta_{n-2} = \Delta_n\sigma_n.$$

□

Corollary 2.2.12. *Let θ_n be $\theta_n = \Delta_n^2$. Then $\sigma_i\theta_n = \theta_n\sigma_i$. Moreover, $\theta_n = i(\theta_{n-1})\gamma_n$, where*

$$\gamma_n = A_{1,n}A_{2,n}\dots A_{n-1,n}$$

and $i : P_{n-1} \to P_n$ is the natural inclusion.

Theorem 2.2.13. *$Z(B_n) = Z(P_n)$ is the infinite cyclic group generated by θ_n.*

Proof. We have seen that $\theta_n \in Z(B_n)$. We should prove that every element of $Z(B_n)$ is a power of θ_n. We start by focusing on P_2.

For $n = 2$ it is obvious, because P_2 is the infinite cyclic group generated by $\sigma_1^2 = \theta_2$.

Let now β be in $Z(P_n)$, $n \geq 3$. We can decompose β as $\beta = i(\beta')\beta_n$, $\beta' = f_n(\beta) \in P_{n-1}$ and $\beta \in U_n$. We can notice that the braid γ_n commutes with any elements of $i(P_{n-1})$. Moreover, β commutes with γ, because we have supposed $\beta \in Z(P_n)$. So γ commutes with $\beta_n = i(\beta')^{-1}\beta$ and the group G generated by γ and β_n is an abelian group, $G \subset U_n$. U_n is a free group, so all its subgroups are free groups too. This implies that G is an infinite cyclic group.

Let $l_{i,j} : P_n \to \mathbb{Z}$ be the homomorphism defined in Corollary 2.2.7. $l_{1,n}(\gamma) = 1$, so γ is a generator of G and $\exists k :\ \beta_n = \gamma^k$. By the induction assumption there is m such that $\beta' = f_n(\beta) = (\theta_{n-1}^m)$. Then we have $l_{i,n} = k\ \forall i = 1, 2, \dots, n-1$, so it is independent from i. Since $\beta \in Z(P_n)$, also $\sigma_{n-1}\beta\sigma_{n-1}^{-1} \in Z(P_n)$ and $l_{i,n}(\sigma_{n-1}\beta\sigma_{n-1}^{-1})$ is independent from i. By definition we obtain

$$l_{1,n}(\sigma_{n-1}\beta\sigma_{n-1}^{-1}) = l_{1,n-1}(\beta) = m.$$

$$l_{n-1,n}(\sigma_{n-1}\beta\sigma_{n-1}) = l_{n-1,n}(\beta) = k.$$

and $m = k$ and $\beta = i(\theta_{n-1}\gamma)^k = \theta_n^k$.

For $n \geq 3$ the center of B_n projects to the trivial subgroup of S_n, because $Z(S_n) = \{1\}$. So $Z(B_n) \subset Z(P_n)$. Since $\theta_n \in Z(B_n)$, $Z(B_n) = (\theta_n)$.

□

Corollary 2.2.14. *B_m and B_n are isomorphic if and only if $m = n$.*

Proof. The image of $Z(B_n)$ in $B_n/[B_n, B_n]$ is a subgroup of $\mathbb{Z}$ of index $n(n-1)$. If B_n is isomorphic to B_m then $m(m-1) = n(n-1)$, so that $m = n$. $\square$

2.2.3 Homotopy groups

Theorem 2.2.15. *Let (S^n, s_0) and (X, x_o) be two topological pointed spaces. Let $\pi_n(X, x_0)$ be the set of homotopy classes of maps $f : S^n \to X$ such that $f(s_0) = x_0$. Then $\pi_x(X, x_0)$ is a group, called the n-homotopy group of (X, x_0).*

Proof. A proof can be found in every book of algebraic topology, see [14] or [7]. $\square$

Proposition 2.2.16. *$\pi_n(X, x_0)$ is a commutative group for $n \geq 2$.*

Definition 40. A map $p : E \to X$ is locally trivial if and only if $\forall x \in X$ there is a neighbourhood $V = V_x$ and a homeomorphism $p^{-1}(V) \xrightarrow{\sim} p^{-1}(x) \times V$ such that there is a commutative diagram

$$\begin{array}{ccc} p^{-1}(V) & \xrightarrow{\sim} & p^{-1}(x) \times V \\ \downarrow & & \downarrow \\ V & \to & V \end{array}$$

Theorem 2.2.17. *Let $p : E \to X$, X arc connected, be a locally trivial fibration. Let e_0 be in E, $p(e_0) = b_0$, $F = p^{-1}(b_0)$ and $f_0 \in F$. There is a long exact sequence in homotopy:*

$$\ldots \to \pi_n(F, f_0) \to \pi_n(E, e_0) \to \pi_n(X, x_0) \to \pi_{n-1}(F, f_0) \to \ldots \to \pi_1(F, f_0)$$

$$\to \pi_1(E, e_0) \to \pi_1(X, x_0) \to \pi_0(F, f_0) \to \pi_0(E, e_0) \to \pi_0(X, x_0) \to 0.$$

A proof can be found in [14].

Corollary 2.2.18. *If p is a covering map, then $\pi_n(F) = 0$ for $n \geq 2$ and $\pi_n(E) \simeq \pi_n(X)$ for $n \geq 2$*

Corollary 2.2.18 follows from theorem 2.2.17 by setting $\pi_n(F) = 0$ in the long exact sequence.

Let us now focus on the configuration spaces. There is a map

$$p : \mathbb{C}^n - \Delta_n \to \mathbb{C}^{n-1} - \Delta_{n-1}$$

such that $p((z_1, \ldots, z_n)) = (z_1, \ldots, z_{n-1})$, so that

$$p^{-1}((q_1, \ldots, q_n)) = \mathbb{C} - Q_{n-1}$$

where $Q_{n-1} = \{q_1, \ldots, q_{n-1}\}$. The map p is locally trivial.

Theorem 2.2.19. *$\pi_k(Conf_n(\mathbb{C})) = 0$ for $k \geq 2$, $n \geq 1$.*

Proof. We apply theorem 2.217 and corollary 2.2.18. $F = \mathbb{C} - Q_{n-1} \sim \vee_{i=1}^{n-1} S^i$, so $\pi_k(F) = 0$ for $k \geq 2$.

We prove it by induction of n. For $n = 1$, $Conf_1(\mathbb{C}) = \mathbb{C}$ and so we have the claim.

For $n > 1$ we take the sequence:

$$\ldots \to \pi_k(F) \to \pi_k(\mathbb{C}^n - \Delta_n) \to \pi_k(C_{n-1} - \Delta_{n-1}) \to \ldots$$

By induction, $\pi_k(C_{n-1} - \Delta_{n-1}) = 0$ for $k \geq 2$ and we have seen that $\pi_k(C_{n-1} - \Delta_{n-1}) = 0$, so that $\pi_k(Conf_n(\mathbb{C})) = 0$ for $k \geq 2$.

□

2.3 Representation of links by braids

2.3.1 Construction of links by braids

Definition 41. Let L be a link in the solid torus $T = D_2 \times S^1$. L is called a closed n-braid if L meets each 2-disk $D_2 \times \{z\}$, $z \in S^1$, transversely in n points.

Remark 33. The projection $V \to S^1$ restricted to L gives a n-fold covering $L \to S^1$.

We provide L with the canonical orientation obtained by lifting the counterclockwise orientation of S^1.

Definition 42. Two closed braids in V are isotopic if they are isotopic as oriented links.

Remark 34. A link in V does not need to be isotopic to a closed braid in V, as for example a link lying inside a small 3-ball in V.

Such links are called closed n-braid because, given a braid β on n strands, there is a standard way to obtain from it a closed n-braid in the solid torus V. A solid torus can be seen as the cylinder $D_2 \times I$ with identifications $(x, 0) \sim (x, 1)$. So, given a braid β, take the geometric braid $b \subset D_2 \times I$. Then $\hat{b}$, image of b in the projection $D_2 \times I \to V$, is a closed n-braid in V.

Proposition 2.3.1. *Let β be a n-braid and b a geometric braid associated to β. The isotopy class of $\hat{b}$ depends only on β.*

Proof. Let b and b' be two geometric braids in $D_2 \times I$. Then b is isotopic to b', i.e. there is an isotopy of $D \times I$ constant on the boundary and transforming b in b'. This isotopy induces an isotopy between $\hat{b}$ and $\hat{b'}$ in V, thus the isotopy class of $\hat{b}$ does not depend on the choice of the geometric braid.

□

Theorem 2.3.2. *For any $n \geq 1$ and any β and $\beta' \in B_n$ with geometric braids b and b', the closed braids $\hat{b}$ and $\hat{b'}$ are isotopic in the solid torus if and only if β and β' are conjugate in B_n.*

Proof. A proof can be found in [16]. □

Definition 43. Let β be a braid diagram. The closure $\hat{\beta}$ of β is the link diagram obtained by joining the bottom endpoints with the top endpoints by n standard arcs. We call the oriented link which diagram is $\hat{\beta}$ as $\hat{\beta}$ too.

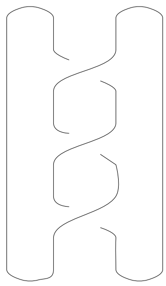
Proposition 2.3.3. *Two closed braid diagrams D and D' in $S^1 \times I$ represent isotopic closed braids in the solid torus $S^1 \times I \times I$ if and only if D can be transformed in D' by a finite sequence of isotopies and Reidemeister moves.*

Proof. A proof can be found in [16].

□

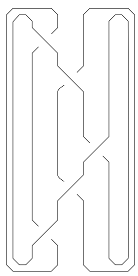
Example 11. *If 1_n is the trivial braid on n strands, its closure is the union on n disjoints trivial links.*

Example 12. *Let β be $\beta = \sigma_1^2$. Then $\hat{\beta}$ is the knot:*



By computing the fundamental group, it is possible to see that $\hat{\beta}$ is actually isotopic to the trefoil knot.

Example 13. *Let β be $\beta = \sigma_1^{-1}\sigma_2^{-1}\dots\sigma_{n-2}^{-1}\sigma_{n-1}\sigma_{n-2}\dots\sigma_2\sigma_1$ be a braid on n strands. Its closure $\hat{\beta}$ is isotopic to the disjoint union of $n-1$ trivial knots. For example, for $n = 4$*



2.3.2 Alexander's theorem

We give now an equivalent definition of closed braid in $\mathbb{R}^3$. Let $l = \{(0,0)\} \times \mathbb{R} \subset \mathbb{R}^3$ be the coordinate axis meeting the plane $\mathbb{R}^2 \times \{0\}$ at the origin $O = (0, 0, 0)$. We can choose a positive direction of rotation about l as the counterclockwise direction about O in the plane $\mathbb{R}^2 \times \{0\}$.

Definition 44. An oriented geometric link $L \subset \mathbb{R}^3 - l$ is a closed n-braid if the vector from O to any point $X \in L$ rotates in the positive direction about l when X moves along L in the direction determined by the orientation of L.

Lemma 2.3.4. *The two definitions are equivalent.*

Proof. Let D_2 be a disk lying in an open half-plane bounded by l in $\mathbb{R}^3$ and having its center in $\mathbb{R}^2 \times \{0\}$. Now, rotating D around l we obtain a solid torus $V = D_2 \times S^1$. The assumption then follows by considering that, if D is big enough, a given link $L \subset \mathbb{R}^3 - l$ lies in V.

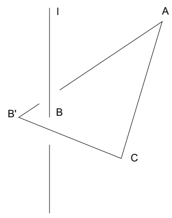
□

Theorem 2.3.5. *Any oriented link in* $\mathbb{R}^3$ *is isotopic to a closed braid.*

Proof. Any link in $\mathbb{R}^3$ is isotopic to a polygonal link, so it is sufficient to prove the theorem for such links. By moving the vertices of L we can assume that $L \subset \mathbb{R}^3 - l$, where $l = \{(0, 0)\} \times \mathbb{R}$, and that the edges of L do not lie in planes containing the axis l. Let AC be an edge of L, labeling the vertices such that L is oriented from A to C. The edge AC is said to be positive (resp. negative) if the vector from the origin $0 \in l$ to a point $X \in AC$ rotates in the positive (resp. negative) direction about l when X moves from A to C. By the assumption that AC does not lie in a plane containing l, AC is necessarily positive or negative. The edge AC is now said to be accessible if there is a point $B \in l$ such that the triangle ABC meets L only along AC.

If all edges of L are positive, L is a closed braid.

Let AC be a negative edge of L. It is possible to replace AC with a sequence of positive edges. Assume AC is accessible. Then there exists $B \in l$ such that the triangle ABC meets L only along AC. In the plane containing ABC we take a bigger triangle $AB'C$ containing B in its interior, meeting l only at B and meeting L only along AC.



We can replace then the edge AC with the two positive edges AB' and $B'C$, a Δ-move $\Delta(AB'C)$. Then the resulting polygonal link is isotopic to L and has one negative edge fewer than L.

Assume now that the edge AC is not accessible. Let B be in L and let P be in AC such that the segment PB meets L only at P. Now it is possible to thicken this segment inside the triangle ABC to obtain a triangle P^-BP^+ meeting L along its side P^-P^+. Then P^-P^+ is an accessible subsegment of AC. Since AC is compact it is possible to split it into a finite number of consecutive accessible subsegments. Then we apply to each of them the Δ-move as above choosing distinct points $B \in l$ and choosing B' such that it does not lie in the other edges of L. Since AC does not lie in a plane containing l, the triangles determining the Δ -moves meet only at the common vertices of the consecutive subsegments of AC. Then they replace $AC \subset L$ with a finite sequence of positive edges, beginning at A and ending at C and the resulting polygonal link is isotopic to L. Applying this procedure to all negative edges of L we obtain a closed braid isotopic to L.

□

In his proof, Alexander modifies the diagram of an oriented link to obtain a closed braid, by changing the geometry of the picture. Applying his method, it is frequent to obtain a diagram with many more crossings than the initial one. This is why in application is usually used an other algorithm, which gives a upper bound to the number of crossing added. The algorithm can be found in [25].

2.3.3 Markov theorem

We have a method for building a link diagram as a closed braid. Now we want to investigate the other problem: how to describe all braids with isotopic closures in $\mathbb{R}^3$.

We have seen that if two braids β and β' are conjugate then they have isotopic closures, but the converse is not true.

Example 14. *Let us take σ_1, $\sigma_1^{-1} \in B_2$. They are not conjugate each to the other, but their closures are both isotopic to the trivial knot.*

Example 15. *Let $i : B_n \to B_{n+1}$ be the natural embedding of B_n in B_{n+1}. Let β be a braid in B_n. Then the braids $\sigma_n i(\beta)$ and $\sigma_n^{-1} i(\beta)$ have isotopic closures. Also, their closures are isotopic to $\hat{\beta}$.*

Definition 45. Two braids β and β' are said to be M-equivalent if they can be related by a finite sequence of moves M1 and M2, and their inverses, given by

1. if β, $\gamma \in B_n$, $\beta \mapsto \gamma\beta\gamma^{-1}$;
2. if $\beta \in B_n$, $i : B_n \to B_{n+1}$ is the canonical embedding, $\beta \mapsto \sigma_n^{\pm 1} i(\beta)$.

The moves M1 and M2 are called Markov moves.

Remark 35. The relation of M-equivalence is an equivalence relation.

Remark 36. Two braids can be M-equivalent also if they have a different number of strands.

Example 16. *Let us show that the braids σ_1 and $\sigma_1^{-1} \in B_2$ are equivalent.*

$$\sigma_1 \sim \sigma_2^{-1}\sigma_1 \sim (\sigma_1\sigma_2)^{-1}(\sigma_2^{-1}\sigma_1)(\sigma_1\sigma_2) =$$

$$= \sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^2\sigma_2 = \sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2 =$$

$$= \sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_2 = \sigma_2\sigma_1^{-1} \sim \sigma_1^{-1}.$$

Theorem 2.3.6. *Two braids have isotopic closures in* $\mathbb{R}^3$ *if and only if they are M-equivalent.*

Proof. A proof can be found in [16].

□

Corollary 2.3.7. *Let* Λ *be the set of all isotopy classes of nonempty oriented links in* $\mathbb{R}^3$*. The mapping* $\coprod_{n\geq 1} B_n \to \Lambda$ *assigning to a braid the isotopy class of its closure induces a bijection from the quotient set* $\frac{(\coprod_{n\geq 1} B_n)}{\sim} \xrightarrow{\sim} \Lambda$.

From corollary 2.3.7, it is possible to define functions acting on braids and such that they are well defined on their closures, i.e. on links.

Definition 46. A Markov function with values in a set E is a sequence of functions $\{f_n : B_n \to E\}_{n\geq 1}$, satisfying the following conditions:

1. $\forall\, n \geq 1$ and $\forall\, \alpha,\ \beta\ \in\ B_n$,

$$f_n(\beta) = f_n(\alpha^{-1}\beta\alpha)$$

2. $\forall\, n \geq 1$ and all $\beta\ \in\ B_n$,

$$f_n(\beta) = f_{n+1}(\sigma_n i(\beta)),\quad f_n(\beta) = f_{n+1}(\sigma_n^{-1}(\beta)).$$

Proposition 2.3.8. *Any Markov function* $\{f_n : B_n \to E\}_{n\geq 1}$ *determines an E-valued isotopy invariant* $\hat{f}$ *of oriented links in* $\mathbb{R}^3$.

Proof. Let L be an oriented link in $\mathbb{R}^3$. For Alexander theorem, it is possible to take a braid $\beta \in B_n$ such that L is isotopic to $\hat{\beta}$. Let $\hat{f}(L)$ be $\hat{f}(L) = f_n(\beta)$. Let β' be another braid whose closure is isotopic to L. β and β' are M-equivalent, so there is a finite sequence of Markov moves starting from β' and ending in β. By definition the Markov functions are invariant on M-moves, so the function $\hat{f}$ is well defined. Let now L' be an oriented link isotopic to L. Let $\beta \in B_n$ be a braid whose closure is isotopic to L. Then the closure of β is also isotopic to L' and $\hat{f}(L) = f_n(\beta) = \hat{f}(L')$.

□

2.4 Representations of braid groups

2.4.1 The Burau representation

Definition 47. Let us take $n \geq 2$. For $i = 1, \ldots, n-1$ we take the following $n \times n$ matrix over the ring $\Lambda = \mathbb{Z}[t^{\pm}]$:

$$U_i = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix}.$$

where I_m denotes the unit $m \times m$ matrix. If $i = 1$ the first matrix disappears, and so does the second if $i = n-1$.

Remark 37. For $t = 1$ the matrices U_i become the permutation matrices.

Proposition 2.4.1. *The matrix U_i is invertible with inverse given by*

$$U_i^{-1} = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & t^{-1} & 1-t^{-1} & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix}.$$

Proposition 2.4.2. *The matrices U_i, $i = 0, \ldots, n-1$ satisfy the following proprieties:*

1. $U_iU_j = U_jU_i$, $\forall i, j = 0, \ldots, n-1 : |i-j| \geq 2$,

2. $U_iU_{i+1}U_i = U_{i+1}U_iU_{i+1}$, $\forall i = 0, \ldots, n-2$.

Proof. These two propositions are proven by direct computation. $\square$

It is possible to define a group homomorphism $\psi_n : B_n \to GL_n(\Lambda)$, $n \geq 2$, by $\psi_n(\sigma_i) = U_i$. By proposition 2.4.1 and proposition 2.4.2 this is well defined.

Definition 48. The map $\psi_n : B_n \to GL_n(\Lambda)$ such that $\psi_n(\sigma_i) = U_i$ is called the Burau representation of B_n.

Remark 38. The Burau representations $\{\psi_n\}_{n\geq 1}$ are compatible with the natural inclusions $i : B_n \to B_{n+1}$ for any $\beta \in B_n$, i.e.

$$\psi_{n+1}(i(\beta)) = \begin{pmatrix} \psi_n(\beta) & 0 \\ 0 & 1 \end{pmatrix}.$$

Definition 49. A group homomorphism $\varphi : G \to G'$ is said to be faithful is its kernel is trivial.

Theorem 2.4.3. *The Burau representation $\{\psi_n\}$ is faithful for $n = 2, 3$, unfaithful for $n \geq 5$.*

Proof. $\ker(\psi_n) \subset \ker(\psi_{n+1})$. Then it is sufficient to prove the unfaithfulness for $n = 5$. This can be made by showing a non-trivial element in B_n such that his image in $GL_n(\Lambda)$ is 0. Set

$$\gamma = \sigma_4\sigma_3^{-1}\sigma_2^{-1}\sigma_1^2\sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-2}\sigma_1^{-1}\sigma_4^{-5}\sigma_2\sigma_3\sigma_4^3\sigma_2\sigma_1^2\sigma_2\sigma_3^{-1}$$

Then the commutator $\rho = [\gamma\sigma_4\sigma^{-1}, \sigma_4\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_3\sigma_4]$ is a non trivial element in $\ker(\psi_5) \subset B_5$.

Let us see the case $n = 2$, $B_2 \simeq \mathbb{Z}$. Then it is sufficient to see that $U^n \neq I_2$ $\forall n \in \mathbb{Z}_0$. We have $(1, -1)U = (-t, t) = -t(1, -1)$. Then $(1, -1)U^k = t^k(1, -1)$ for all $k \in \mathbb{Z}$. Then $U^k \neq U^h$ for $k \neq h$ and $< U >= \mathbb{Z}$.

The proof for the case $n = 3$ can be found in [16].

□

Remark 39. The vector $(1, 1, \ldots, 1)$ is an eigenvector for every matrix U_i.

By remark 39, it is possible to decompose the representation matrix in a direct sum of a 1-dimensional representation and of a $n-1$-dimensional representation. Let P be the matrix

$$P = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 0 & 1 & \ldots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 \end{pmatrix}$$

Then $P^{-1}U_iP$ is the matrix $\begin{pmatrix} \tilde{U}_i & 0 \\ 0 & 1 \end{pmatrix}$, where $\tilde{U}_i$ is the matrix, for $1 < i < n-1$

$$\tilde{U}_i = \begin{pmatrix} I_{i-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-i-2} \end{pmatrix}$$

For $i = 1$ and $i = n-1$ the matrix is

$$\tilde{U}_1 = \begin{pmatrix} -t & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix} \tilde{U}_i = \begin{pmatrix} I_{n-3} & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & -t \end{pmatrix}$$

The group homomorphism $\tilde{\psi}_n : B_n \to GL_{n-1}(\Lambda)$ such that $\psi_n(\sigma_i) = \tilde{U}_i$ is called the reduced Burau representation.

2.4.2 A homological description of the Burau representation

We want now to understand the geometric meaning of the Burau representation. To do that, we shall remember that braid groups can be defined as mapping class groups on punctured disks, as it has been done in 2.1.4.

Remark 40. $H_1(D_2 - \{x\}) = \mathbb{Z}$, generated by the class of loops encircling x in counterclockwise direction. Each loop represents k times the generator, where k is the winding number of the loop around x.

Remark 41. $H_n(D_2 - P_n) = \mathbb{Z}^n$, where P_n are n distinct points. The generators are the classes of small loops encircling only one point in P_n in the counterclockwise direction.

Definition 50. Let γ be a loop in $D_2 - P_n$. We define the total winding number of γ as the sum of its winding numbers around $x_1, \ldots, x_n$. The total winding number defines a homomorphism $H_1(D_n) \to \mathbb{Z}$.

Let $\tilde{D}_n \to D_n$ be the regular covering corresponding to the homomorphism $H_1(D_n) \to \mathbb{Z}$ such that $(k_1, \ldots, k_n) \mapsto k_1 + \ldots + k_n$. Then the group of covering transformations of $\tilde{D}_n$ is $\mathbb{Z}$, a multiplicative group with generator t. Thus the group $H_1(\tilde{D}_n)$ acquires the structures of a Λ-module.

Let $d \in \partial D$ be a basepoint for D_n. Any diffeomorphism $d : D_n \to D_n$ lifts uniquely to a diffeomorphism $\tilde{h} : \tilde{D}_n \to \tilde{D}_n$ which fixes the fiber over d pointwise. Thus there is an induced homomorphism $B_n \to Aut(H_1(\tilde{D}_n))$ defined by $h \to \tilde{h}_*$, where $\tilde{h}_*$ is a Λ-linear automorphism of $H_1(\tilde{D}_n)$. We will see that this map is equivalent to the reduced Burau representation.

Definition 51. Let α, β be two embedded arcs in D_n with endpoints in the set P_n, $n \geq 4$. Let $\tilde{\alpha}$, $\tilde{\beta}$ be lifts of α and β to $\tilde{D}_n$. Then

$$< \alpha, \beta >= \sum_{k \in \mathbb{Z}} (t^k \tilde{\alpha} \cdot \tilde{\beta}) t^k \in \Lambda$$

where $(t^k \tilde{\alpha} \cdot \tilde{\beta})$ is the algebraic intersection of the arcs $t^k \tilde{\alpha}$ and $\tilde{\beta}$ in $\tilde{D}_n$. This intersection is also called Blanchfield intersection.

Remark 42. The sum is finite and it is defined up to multiplication by a power of t depending on the choices of the lifts $\tilde{\alpha}$ and $\tilde{\beta}$.

Now, to compute $< \alpha, \beta >$ it is possible to deform α with respect to β and compute the geometric intersection. We call ϵ_p the intersection sign of the two loops at an intersection point $p \in \alpha \cap \beta$. We determine also a number $k_p \in \mathbb{Z}$ by the following. Let us take two points p, $q \in \alpha \cap \beta$. Then $k_p - k_q$ is the total winding number of the loop going from p to q along α and then from q to p along β. Then

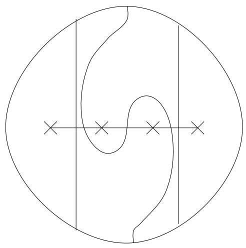
$$< \alpha, \beta >= \sum_{p \in \alpha \cap \beta} \epsilon_p t^{k_p}$$

For $1 \leq j \leq n$ let a_j be the oriented segment joining q_j and q_{j+1}, where q_i are points in P_n. Let us link a_j to the point $(-1, 0) \in D_2$ that we take as basepoint. For all j let b_j be a vertical segment, oriented downward, passing for the middle of a_j. Let $\tilde{a}_j$ and $\tilde{b}_j$ be their lifts.

Remark 43. The lifts $\tilde{b}_j$ represent a basis for $H_1(\tilde{D}_2^n, \partial D_2)$.

The lifts $\tilde{a}_j$ represent a basis for $Hom(H_1(\tilde{D}_2^n, \partial D_2))$.

We have seen in 2.1.4 that we can take as generators of the braid group B_n the half twists t_k. Let us see how a half twist acts on the generators b_j and a_j. We find, focusing on a neighborhood of 4 punctures points



So, by using the definition of the Blanchfield intersection, we obtain the matrix

$$\begin{pmatrix} 1 & t & 0 \\ 0 & -t & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

that is exactly the matrix of the reduced Burau representation.

2.4.3 Krammer-Bigelow-Lawrence representation

We introduce now the Krammer-Bigelow-Lawrence representation, a faithful linear representation of B_n.

Definition 52. Let D_2^n be $D_2^n = D_n - P_n$, where $P_n = \{q_1, \dots, q_n\}$. Let C_n be $C_n = (D_2^n \times D_2^n - \Delta)/\sim$, where $\Delta = \{(x, y) \in D_2^n \times D_2^n | \ x = y\}$ and $\sim$ is the equivalence relation $(x, y) \sim (y, x)$ for any distinct x, $y \in D_2^n$.

A closed curve $\alpha : [0, 1] \to C_n$ can be written in the form $\alpha(s) = (\alpha_1(s), \alpha_2(s))$, $s \in [0, 1]$ and α_1, α_2 are arcs in D_2^n such that $\{\alpha_1(0), \alpha_2(0)\} =$

$\{\alpha_1(1), \alpha_2(1)\}$. Thus the arcs α_1 and α_2 are either loops or can be composed one with the other.

Remark 44. The curves $\alpha = (\alpha_1, \alpha_2)$ form a closed oriented one-manifold γ mapped to D_2^n.

Let $a(\alpha) \in \mathbb{Z}$ be the total winding number of γ around the points $\{q_1, \dots, q_n\}$. For the construction of the space, $\alpha_1(s) \neq \alpha_2(s)$ $\forall s \in [0,1]$. Then it is well defined the map

$$s \mapsto \frac{\alpha_1(s) - \alpha_2(s)}{|\alpha_1(s) - \alpha_2(s)}$$

Composing this map with the projection $S^1 \to \mathbb{R}P^1$ we obtain a loop in $\mathbb{R}P^1$. We call $b(\alpha)$ the corresponding element in $H_1(\mathbb{R}P^1)$. So we have a map $\psi : H_1(C) \to \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ such that $\alpha \mapsto q^{a(\alpha)} t^{b\alpha}$.

There is a regular covering $\tilde{C} \to C$ associated to the kernel of ψ, so that the generators q, t acts on $\tilde{C}$ as commuting covering transformations. Now any diffeomorphism h of D_2^n induces a diffeomorphism $C \to C$ by $h(\{x, y\}) = \{h_x, h_y\}$. We denote this diffeomorphism with h too. The diffeomorphism h lifts to a map $\tilde{h} : \tilde{C} \to \tilde{C}$. Thus there is a representation $B_n \to Aut(H_2(\tilde{C}))$ such that $[h] \in B_n$ maps to the automorphism $\tilde{h}_*$ of $H_2(\tilde{C})$.

Theorem 2.4.4. *The representation $B_n \to Aut(H_2(\tilde{C}))$ is faithful for all $n \geq 1$.*

Proof. A proof can be found in [16] or in [26]. □

The proof consists in a homological construction, as it has been done for the Burau representation. This construction is far more complicated, but it allows to prove that braid groups are linear, a statement really difficult to prove only with algebraic methods.

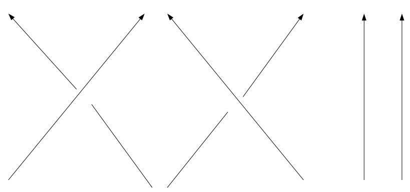
Chapter 3

The HOMFLY polynomial

3.1 Construction of the polynomial

3.1.1 Definition

Let L_+, L_- and L_0 be three links such that they are the same, except in the neighbourhood of a point where they are as shown in the following diagram, reading from left to right.



We want to define a polynomial P_L such that

$$xP_{L_+}(x,y,z) + yP_{L_-}(x,y,z) + zP_{L_0}(x,y,z) = 0.$$

Remark 45. From this polynomial it is possible to recover the Alexander-Conway and the Jones polynomial.

$$\Delta_L(t) = P_L(1, -1, t^{1/2} - t^{-1/2})$$

$$V_L(t) = P_L(t, -t^{-1}, t^{1/2} - t^{-1/2})$$

Moreover, being an homogeneous polynomial in 3 variables, it is possible to take it as a non-homogeneous polynomial in two variables. In literature usually it is taken the polynomial $P_L(l,m) = P_L(l, l^{-1}, m)$, so the relation becomes:

$$lP_{L_{+1}}(l,m) + l^{-1}P_{L_{-1}} + mP_{L_0} = 0$$

Theorem 3.1.1. *There is a unique function*

$$P : \{\textit{Oriented links in } S^3\} \to Z[l^{\pm}, m^{\pm}]$$

well defined up to isotopy such that $P(unknot) = 1$ *and*

$$lP_{L_+}(l,m) + l^{-1}P_{L_-}(l,m) + mP_{L_0} = 0$$

We will see a proof in the next section. Other proofs of theorem 3.1.1 can be found in [12] or in [17].

Definition 53. Given an integer $N > 1$, we call invariant of type A_N the polynomial satisfying:

- $P_N(unknot) = 1$
- $q^{(N+1)/2}P_N(L_{+1}) - q^{-(N+1)/2}P_N(L_{-1}) = (q^{1/2} - q^{-1/2})P_N(L_0)$

In the following chapters we will build a homological definition for this invariant.

There is a theorem claiming that it is possible to reconstruct the HOMFLY polynomial by the values of the invariant of type A_N, see [21] or [24] for a more complete explanation. This is why we can say that we are able to construct a general homological definition for the HOMFLY polynomial.

3.1.2 Construction by Hecke algebra

Proposition 3.1.2. *The symmetric group S_n admits a presentation*

$$S_n =< \tau_1, \dots, \tau_{n-1} |\ 1,\ 2,\ 3 >$$

where $\tau_i = (i\ i+1)$ and

1. $\tau_i^2 = 1, \forall\ i\ : 1 \leq i \leq n-1,$
2. $\tau_i \tau_j = \tau_j \tau_i, \forall\ i,\ j\ : 1 \leq i < j-1 \leq n-2,$
3. $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \forall\ i :\ 1 \leq i \leq n-2.$

The group operation in S_n is written from left to right, so that for example $(1\ 2)(2\ 3) = (1\ 3\ 2)$

Proof. A proof can be found in every book of algebra, for example [1]. □

Remark 46. It is possible to identify S_{n-1} with the subgroup of permutations in S_n leaving n fixed.

Definition 54. Given a permutation $\pi \in S_n$ such that $\pi(n) = j$, we define

$$b_\pi(\tau_i) = (j\ j+1)(j+1\ j+2) \dots (n-1\ n) \cdot b_{\pi'}(\tau_i)$$

where $\pi' \in S_{n-1}$.

Proposition 3.1.3. *Let W_n be $W_n = \{b_\pi(\tau_i) | \pi \in S_n\}$. The elements of W_n satisfy the Schreier condition:*

- *if $b_\pi(\tau_i) = w(\tau_i)\tau_k$ then $w(\tau_i) = b_{\pi \cdot \tau_k^{-1}}(\tau_i)$,*
- *b_{id} is the empty word.*

Also, $b_\pi(\tau_i)$ are of minimal length and τ_{n-1} occurs at most once in each $b_\pi(\tau_i) \in W_n$.

Proof. A proof can be found in [8]. □

Definition 55. $\hat{S}_n =< \hat{\tau}_1, \hat{\tau}_2, \ldots, \hat{\tau}_{n-1} |\ 1,\ 2 >$ where

1. $\hat{\tau}_i\hat{\tau}_j = \hat{\tau}_j\hat{\tau}_i$, $\forall\ i,\ j\ :\ 1 \leq i < j-1 \leq n-2$,

2. $\hat{\tau}_i\hat{\tau}_{i+1}\hat{\tau}_i = \hat{\tau}_{i+1}\hat{\tau}_i\hat{\tau}_{i+1}$, $\forall\ i:\ 1 \leq i \leq n-2$

and such that inverses are not allowed. Then $\hat{S}_n$ is a semigroup.

There is a canonical homomorphism $k : \hat{S}_n \to S_n$ such that $k(\hat{\tau}_i) = \tau_i$. Let $\hat{b}_\pi$ and $\hat{W}_n$ be respectively $\hat{b}_\pi = b_\pi(\hat{\tau}_i)$ and $\hat{W}_n = \{\hat{b}_\pi \mid \pi \in S_n\}$.

We should see now what happens to the products $\hat{b}_\pi\hat{\tau}_k = \hat{b}_\rho$. There are two cases:

1. the class of $\hat{b}_\pi\hat{\tau}_k$ contains a representative $\hat{b}_\rho \in W_n$

2. the class of $\hat{b}_\pi\hat{\tau}_k$ does not contain a representative $\hat{b}_\rho \in W_n$

Remark 47. The first case occurs when the strings crossing at τ_k do not cross in b_π, $\rho = \pi\tau_k$. In the second case they do and $\hat{b}_\pi\hat{\tau}_k = \hat{b}_\rho\hat{\tau}_k^2$, so that

$$\hat{b}_\pi \cdot \hat{\tau}_k = \begin{cases} \hat{b}_\rho, & \rho = \pi\tau_k \\ \hat{b}_\rho\hat{\tau}_k^2, & \rho\tau_k = \pi \end{cases}$$

Let M_n be a free module of rank $n!$ over a unitary commutative ring R using the $n!$ words of W_n. We take the generator c_i instead of $\hat{\tau}_i$, $1 \leq i \leq n-1$ and we take $w(c_i) = w'_{c_i}$ if and only if $\hat{w}(\hat{\tau}_i) = \hat{w}(\hat{\tau}_i)$. Let M_n be the free R-module with basis $W_n(c_i) = \{b_\pi(c_i) | \pi \in S_n\}$, so that $c_j = b_{\tau_j}(c_i) \in W_n(c_i)$. Introducing an associative product in M_n we get an R-algebra $H_n(z)$ of rank $n!$.

Definition 56. Let C_k^2 be $C_k^2 = zc_k + 1$, $1 \leq k \leq n-1$ for some fixed element $z \in R$. Then

$$b_\pi(c_i) \cdot c_k = \begin{cases} b_{\pi\tau_k}(c_i), & & \textit{first case} \\ zb_\pi(c_i) + b_\rho(c_i), & \rho\tau_k = \pi & \textit{second case} \end{cases}$$

By iteration we can define a product for the elements of the basis $W_n(c_i)$ and so a product on M_n.

Lemma 3.1.4. *The product defined above is associative on* $W_n(c_i)$*.*

Proof. A proof can be found in [8]. □

Definition 57. The module M_n, seen as an R-algebra of rank $n!$ is a Hecke algebra. It is denoted by $H_n(z)$.

Proposition 3.1.5. *Let* R *be a commutative unitary ring,* $z \in R$*. The algebra generated by elements* $\{c_i,\ 1 \leq i \leq n-1 |\ 1,\ 2,\ 3\}$ *with*

1. $c_i c_{i+1} c_i = c_{i+1} c_i c_{i+1}$, $\forall i\ :\ 1 \leq i \leq n-2$,

2. $c_i c_j = c_j c_i$, $\forall i,\ j\ :\ 1 \leq i < j-1 \leq n-2$,

3. $c_i^2 = z c_i + 1$, $\forall i\ :\ 1 \leq i \leq n-1$

is isomorphic to the Hecke algebra $H_n(z)$*.*

Corollary 3.1.6. $c_j^{-1} = c_j - z$*.*

Proof. $(c_j - z)c_j = c_j^2 - zc_j = zc_j + 1 - zc_j = 1$. □

Let $R = \mathbb{Z}[z^{\pm}, v^{\pm}]$ be the 2-variable ring of Laurent polynomials. We denote by $H_n(z, v) = H_n$ the Hecke algebra with respect to R.

Proposition 3.1.7. *Given a braid group* B_n*,* $\rho_v : B_n \to H_n$ *defined by* $\rho_v(\sigma_j) = v c_j$ *defines a group representation.*

Proof. A proof can be found in [8]. □

Remark 48. There are natural inclusions $H_{n-1} \hookrightarrow H_n$, $W_{n-1}(c_i) \hookrightarrow W_n(c_i)$.

Let H be $H = \cup_{n=1}^{\infty} H_n$, $W(c_i) = \cup_{n=1}^{\infty} W_n(c_i)$, $H_1 = R$. By remark 48, H and $W(c_i)$ are well defined.

Definition 58. Let α_π in $\mathbb{Z}[z^{\pm}, v^{\pm}]$, $a, b \in H_n$, $a' \in H_{n-1}$. A function

$$tr : H_n \to \mathbb{Z}[z^{\pm}, v^{\pm}, T]$$

is called a trace on H if, $\forall n \in \mathbb{N}$ it has the following proprieties:

- $tr(\sum_{\pi \in S_n} \alpha_\pi b_\pi) = \sum_{\pi \in S_n} \alpha_\pi tr(b_\pi)$,
- $tr(ab) = tr(ba)$,
- $tr(1) = 1$
- $tr(a' c_{n-1}) = T \cdot tr(a')$

Lemma 3.1.8. *There exists only one trace on H.*

Also, $\forall a \in H_n$, $tr(ac_n^{-1}) = tr(ac_n) - z \cdot tr(a) = (T - z) \cdot tr(a)$.

Proof. A proof can be found in [8].

□

We will now use the language of Hecke algebras to prove the existence of the HOMFLY polynomial. In the following we will not be using this convention, but the one used in 3.1.1.

Proposition 3.1.9. *Let H_n be the Hecke algebra over $\mathbb{Z}[z^\pm, v^\pm, T]$.*

Let $\rho_v : B_b \to H_n$ be such that $r_v(\sigma_i) = vc_i$ is a representation

Let P_{ξ_n} be $P_{\xi_n} = k_n \cdot tr(\rho_v(\xi_n))$, $\xi_n \in B_n$, for some $k_n \in \mathbb{Z}[z^\pm, v^\pm, T]$. By the proprieties of the trace, $P_{\xi_n} \in \mathbb{Z}[z^\pm, v^\pm, T]$ is invariant under conjugation of ξ_n in B_n. So P_{ξ_n} can be seen as a polynomial $P_{\hat{\xi}_n}$ assigned to the closed braid ξ_n.

To turn effectively P_{ξ_n} to an invariant of the link represented by ξ_n we have to see that it is invariant under Markov moves, see [20]. Assume

$$k_n \cdot tr(\rho_v(\xi_n)) = j_{n+1} tr(\rho_v(\xi_n))$$

As $tr(\rho_v(\xi_n \sigma_n)) = v \cdot tr(\rho_v(\xi_n \cdot c_n)) = v \cdot T \cdot tr(\rho_v(\xi_n))$, we have $k_n = k_{n+1} \cdot v \cdot T$. Moreover, we have

$$k_{n+1} tr(\rho_v(\xi_n \sigma_n^{-1})) = k_{n+1} v^{-1} tr(\rho_v(\xi_n) c_n^{-1}) = k_{n+1} v^{-1} (T - z) \cdot tr(\rho_v(\xi_n))$$

and so $k_n = k_{n+1} \cdot v^{-1}(T - z)$. We have then $v^{-1}(T - v) = vT$, so that

$$T = \frac{zv^{-1}}{v^{-1} - v}.$$

We can now compute inductively k_n as

$$k_{n+1} = k_n \cdot \frac{1}{v \cdot T} = k_n \cdot z^{-1}(v^{-1} - v)$$

with the condition $k_1 = 1$. So

$$k_n = \frac{(v^{-1} - v)^{n-1}}{z^{n-1}}.$$

Theorem 3.1.10. *The Laurent polynomial*

$$P_{\xi_n}(z, v) = \frac{(v^{-1} - v)^{n-1}}{z^{n-1}} \cdot tr(\rho(\xi_n))$$

is an invariant of the oriented link K represented by $\hat{\xi}_n$. This polynomial is the HOMFLY polynomial of the oriented link K, where $\xi_n \in B_n$ is a braid.

Proof. The proof follows from the considerations above and from the fact the $\rho_{\xi_n} = 1$ if ξ_n is the trivial braid.

□

Corollary 3.1.11. *The trivial braid with n strings represents the trivial link with n components. Its HOMFLY polynomial is $\frac{(v^{-1}-v)^{n-1}}{z^{n-1}}$.*

Proposition 3.1.12. *Let L_+, L_- and L_0 be as in* 1.1*. Then there is a skein relation*

$$v^{-1}P_{L_+} - vP_{L_-} = zP_{L_0}.$$

A proof can be found in [8].

Remark 49. Using the skein relations, it is possible to compute the HOMFLY polynomial in an algorithmic way. Choosing a crossing, we set, according to the sign

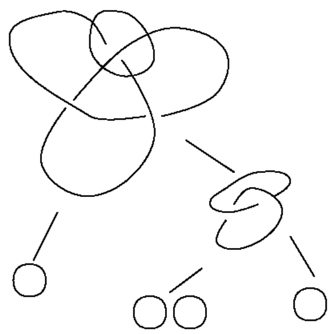
$$P = v^2 P_{L_-} + vzP_{L_0}$$

$$P = v^{-2} P_{L_+} - v^{-1}zP_{L_0}$$

where we take two new diagrams with crossing changed according to the figures. So we get an iterating algorithm, where, by changing crossings, we simplify the knot. This algorithm is very difficult to compute directly for knots with a large number of crossings, because it has an exponential time complexity.

We end this section by computing the HOMFLY polynomial for the right and left trefoil knot.

Example 17. *We start by the right handed trefoil. The circles indicate where we are applying the skein relation.*



So we have

$$P = v^2 P_{L_{-1}} + zvP_0 = v^2 + zv(v^2 \frac{v^{-1} - v}{z} + zv) =$$
$$= v^2 + v^2 - v^4 + z^2v^2 = -v^4 + 2v^2 + z^2v^2$$

For the left handed trefoil, we have, by the same considerations:

$$P = -v^{-4} + 2v^{-2} + z^2v^{-2}.$$

3.2 The configuration space

3.2.1 Construction of the space

We start now by defining the configuration space C, which is the base for all the geometric constructions in this chapter. Let q be a transcendental complex number with unit norm 1, i.e. such that $q^k \neq q^h$ for every $q \neq k$, $q, k \in \mathbb{Z}$.

We want to deal with oriented braids, so we have to define a proper subset. Let $\mathbf{p} = (c_1, \ldots, c_k)$ be a k-tuple of elements of $\{0, N+1\}$. Let $p_1, \ldots,$ p_k be points in the unit disk in the complex plane, called punctured points. Up to diffeomorphism, we can choose the puncture points as lying on the real line. To each point p_i we can associate the number c_i, called the colour of the puncture point. The disk D with coloured puncture points is denoted as $D_{\mathbf{p}}$. Now, braids preserving the colours of the puncture points form a subgroup of the braid group B_k, called the mixed braid group $B_{\mathbf{p}}$.

Now, suppose $\mathbf{m} = (c'_1, \dots, c'_m)$, $c'_i \in \{1, 2, \dots, N\}$. We define a space $\tilde{C}$ as the set of m-tuples $(x_1, x_2, \dots, x_m)$, $x_i \in D$, such that:

- if $1 \leq i < j \leq m$ and $|c'_i - c'_j| \leq 1$, $\Rightarrow\ x_i \neq x_j$
- if $1 \leq i \leq m$, $1 \leq j \leq k$ and $|c'_i - c_j| = 1$, $\Rightarrow\ x_i \neq p_j$

We can consider the subgroup of permutations $W \subset S_m$ such that $c'_i = c'_{w(i)}$ for all $i = 1, \dots, m$. W induces an action on $\tilde{C}$. We call $C_{\mathbf{m}}(D_{\mathbf{p}})$ the quotient of $\tilde{C}$ by this action.

The m-tuple $(x_1, \dots, x_m)$ can be seen as a configuration of m points in D, which we call mobile points. Every mobile point has colour assigned by the array $\mathbf{m}$. The first condition implies that two mobile points can coincide if and only if their colours differ by at least two. The second condition implies the same for a mobile point and a punctured point.

We have defined the configuration space, so the next step will be studying its fundamental group. It is natural to try to represent its elements by using braids. We can take the set $\mathbf{p} + \mathbf{m}$ as

$$\mathbf{p} + \mathbf{m} = (c_1, \dots, c_k, c'_1, \dots, c'_m)$$

So it is well defined the braid group $B_{\mathbf{p}+\mathbf{m}}$. Let G be the subgroup of those mixed braids whose first k strands are straight. Now, if a pair of strands is such that the colours differ by at least two, we can change the over/under information of the crossing. Then the group $\pi_1(C)$ is the group obtained by quotienting G by this relations.

We have defined the configuration space in a very general fashion, but in the following we will focus on a particular case. We take $\mathbf{p}$ as the $2n$-tuple

$$\mathbf{p} = (0, N + 1, 0, N + 1, \dots, 0, N + 1)$$

In this way, the braid β we want to study is just an element in $B_{\mathbf{p}}$, where we take the strands whose colour is 0 as oriented downwards and the strands whose colour is $N+1$ as oriented upwards. Now, every knot or link can be reconstructed from the plat closure of such a braid β, where the plat closure $\hat{\beta}$ is the knot or link obtained by joining adjacent pairs of nodes at the top and at the bottom of β. A proof of this statement can be found in [5].

Let $m = Nn$ and $\mathbf{m}$ be the m-tuple

$$\mathbf{m} = (1, 2, \ldots, N, 1, 2, \ldots, N, \ldots, 1, 2, \ldots, N)$$

We denote the configuration space $C_{\mathbf{m}}(D_{\mathbf{p}})$ as C. Now, we take an homomorphism $\rho_{\mathbf{m}} : \pi_1(C) \to \{\pm q^k | k \in \mathbb{Z}\}$. We associate at every positive crossing of a representation of $g \in \pi_1(C)$ as a braid group

- $-q^{-1}$ if the two strands have the same colour
- $q^{1/2}$ if the two strands have colours differing by one
- 1 otherwise

Let $\rho_m(g)$ be the product of all the terms associated to the crossings of a braid diagram. If we want it to be invariant by the second Reidemeister move, it is straightforward to define a negative crossing as the reciprocal of the analogous positive crossing. Also, the third condition implies that this is well defined. For the definition of the space, the number of crossing whose strand colours differ by 1 is even, so the exponent of q is effectively an integer.

Now, we want to define a homomorphism $\rho_{\mathbf{p}} : B_{\mathbf{p}} \to \{\pm q^{k/2} | k \in \mathbb{Z}\}$. As earlier, we associate to every positive crossing of a braid representation of g

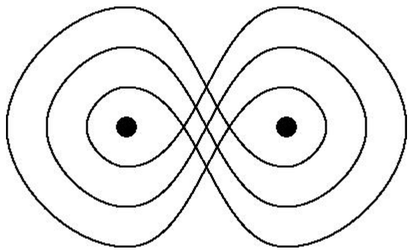
- $q^{N/2}$ if the two strands have the same colour
- $q^{-(N+1)/2}$ otherwise

As before, to every negative crossing we associate the reciprocal term associated to the analogous positive crossing and we define $\rho_{\mathbf{p}}(g)$ as the product of all the terms associated to the crossings.

3.2.2 A torus and a ball

We want now to define two geometric spaces, a torus and a ball, in the configuration space C. We will use later these two spaces to define the invariant $Q(\beta)$. In particular, we want to define an immersed m-dimensional torus and an embedded m-dimensional ball.

As usual, we can take $S^1 = \{|z| = 1, z \in \mathbb{C}\}$ and $T = S^1 \times S^1 \times \ldots \times S^1$. We also define $A = S^1 \cap \{z \in \mathbb{C}| \Im(z) \geq 0\}$ and $B = S^1 \cap \{z \in \mathbb{C}| \Im(z) \leq 0\}$. Let $\gamma_1, \ldots \gamma_N$ be such that



where the point on the left is p_1 and the point on the right is p_2. Assume that γ_i is parametrised so that $\gamma_i|_A$ is a loop around p_1 and $\gamma_i|_B$ is a loop around p_2. We assume that the γ_i are parametrised so that $\gamma_i(1) \neq p_1, p_2$ is on the real line. We label these curves so that $p_1 < \gamma_1(1) < \gamma_2(1) < \ldots < \gamma_N(1) < p_2$.

Remark 50. $\gamma_i(A)$ is a concentric loop around p_1 and $\gamma_i(B)$ is a concentric loop around p_2.

3.2.3 Construction in the case N=2

We suppose $N = 2$. We start by constructing the immersion $\Phi : T \to C$.

Proposition 3.2.1. *There is an immersion $\Phi_1 : B \times A \to C$ such that*

$$\Phi_1|_{\partial(B \times A)} = (\gamma_1 \times \gamma_2)|_{\partial(B \times A)}$$

Φ_1 *can also be chosen such that, for every $x_1,\ x_2 \in Im(\Phi_1)$*

- $x_1 \in D_2(\gamma_1(B))$*, the closed disk bounded by* $\gamma_1(B)$*,*

- *$x_2 \in D_2(\gamma_2(A))$, the closed disk bounded by $\gamma_2(A)$,*
- *given (x_1, x_2), $\exists\ i$ such that $x_i \in \gamma_1(B) \cap \gamma_2(A)$.*

Proof. If $(\gamma_1 \times \gamma_2)|_{\partial(B\times A)}$ is nullhomotopic in C, it is possible to extend Φ_1 to a function on $B \times A$, which implies the statement.

$$(\gamma_1 \times \gamma_2)|_{\partial(B\times A)} = (\gamma_1 \times \gamma_2)|_{(\partial B\times A)\cup(B\times\partial A)} =$$

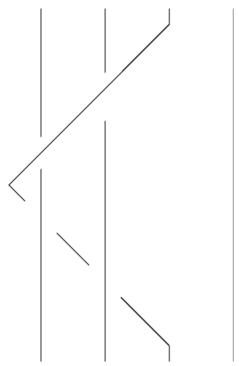
$$= (\gamma_1(1), \gamma_2(s))(\gamma_1(s), \gamma_2(1)) - (\gamma_1(s), \gamma_2(1))(\gamma_1(1), \gamma_2(s))$$

Let $\alpha : A \to C$, $\beta : B \to C$ be given by

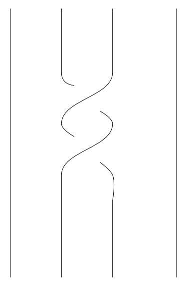
$$\alpha(s) = (\gamma_1(1), \gamma_2(s))$$

$$\beta(s) = (\gamma_1(s), \gamma_2(1))$$

As a braid, α can be represented as



where the strands on the far left and on the far right represent p_1 and p_2, the punctured points. The strands are also coloured, from left to right, as 0, 1, 2, 3. Because we are dealing with coloured braids, we can change the lower point of intersection in such a way that the third strand passes under the first strand. By applying the second Reidemeister move, we obtain α' such that



In the same way, β is homotopic to $\beta' = (\alpha')^{-1}$. Then $\alpha\beta - \beta\alpha = 0$ and $(\gamma_1 \times \gamma_2)|_{\partial(B\times A)}$ is nullhomotopic.
Now, let C' and C'' be such that

$$C' = \{(x_1, x_2) \in C|\ x_1,\ x_2 \textit{ satisfy the three conditions}\}$$

and

$$C'' = \{(x_1, x_2) \in C|\ x_1, x_2\ \in D_2(\gamma_1(B)) \cap D_2(\gamma_2(A))\}.$$

It is possible to find an homotopy between α and α' lying completely in C' and we can also assume that α' lies in C''. With the same considerations for β, we have that $\alpha'\beta' - \beta'\alpha'$ is null homotopic as a loop in C''.

$\square$

Now, it is straightforward to prove the following proposition.

Proposition 3.2.2. *$\Phi : T \to C$ defined as*

$$\Phi(s_1, s_2) = \begin{cases} \Phi_1(s_1, s_2) & \forall (s_1, s_2) \in B \times A \\ (\gamma_1(s_1), \gamma_2(s_2)) & \textit{otherwise} \end{cases}$$

is an immersion.

3.2.4 Construction for general values of n

We want to define $\Phi : T \to C$ for general values of N. Let $\Phi_1, \ldots, \Phi_N$ be functions

$$\Phi_i : B \times A \to D \times D$$

such that

$$\Phi_i|_{\partial(B\times A)} = (\gamma_i \times \gamma_{i+1})|_{\partial(B\times A)}$$

and such that for all $(x_i, x_{i+1}) \in Im(\Phi_i)$

- $x_i \neq x_{i+1}$
- $x_i \in D_2(\gamma_i(B))$
- $x_{i+1} \in D_2(\gamma_{i+1}(A))$
- at least one between x_i and x_{i+1} lies in $D_2(\gamma_i(B)) \cap D_2(\gamma_{i+1}(A))$

Now suppose $(s_1, \dots, s_N) \in T$. We let x_i, $i = 1, \dots, N$ be such that

- if s_{i-1}, $s_i \in A$ then $x_i = \gamma_i(s_i)$
- if $s_i \in A$ and $s_{i-1} \in B$ then x_i is the second coordinate of $\Phi_{i-1}(s_{i-1}, s_i)$
- if s_i, $s_{i+1} \in B$ then $x_i = \gamma_i(s_i)$
- if $s_i \in B$ and $s_{i+1} \in A$ then x_i is the first coordinate of $\Phi_i(s_i, s_{i+1})$

By convention, $s_0 \in A - B$ and $s_N \in B - A$. $\Phi(s_1, \dots, s_n) = (x_1, \dots, x_n)$.

Proposition 3.2.3. $\Phi : T \to C$, *defined as* $\Phi(s_1, \dots, s_n) = (x_1, \dots, x_n)$, *is well defined.*

Proof. If at least two of the conditions apply in the definition of x_i, then they all give $x_i = \gamma_i(s_i)$.

Since either $x_1 = \gamma_1(s_1)$ or x_1 is the first coordinate of $\Phi(s_1, s_2)$, $x_1 \neq p_1$ and similarly $x_N \neq p_2$.

We now check that $x_i \neq x_{i+1}$, $\forall i = 1, \dots, N-1$.

We can suppose $s_i \in A$. Then for the first two conditions, x_i should lie in $D_2(\gamma_1(A))$ and x_{i+1} should not, thus $x_i \neq x_{i+1}$. The same if $x_{i+1} \in B$. If $s_i \in B$ and $s_{i+1} \in A$, $\Phi(s_i, s_{i+1}) = (x_i, x_{i+1})$, so $x_i \neq x_{i+1}$.

□

Proposition 3.2.4.

$$\rho_{\mathbf{m}} \circ \Phi_{\#}(\pi_1(T)) = \{1\}$$

Proof. The group $\pi(T) = \mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z}$ is generated by $g_i : S^1 \to T$, $i = 1, \ldots, N$, such that

$$g_i(s) = (\gamma_1(1), \ldots, \gamma_{i-1}(1), \gamma_i(s), \gamma_{i+1}(1), \ldots, \gamma_N(1))$$

The loop g_i can be represented as a mixed braid with $N+2$ strands, with every braid straight except the strand with colour i, describing a figure of eight. We have two positive crossings involving a pair of strands with colours i and $i-1$ and two negative crossings involving a pair of strands with colours i and $i+1$. By definition of $\rho_{\mathbf{m}}$, $\rho_{\mathbf{m}}(g_i) = 1$. Every $g \in \pi_1(T)$ is combination of elements g_i, so $\rho_{\mathbf{m}}$ being a homomorphism, $\rho_{\mathbf{m}}(g) = 1$. $\square$

Let X be $X = D_2(\gamma_1(B)) \cap D_2(\gamma_N(A))$, $C_X = \{(x_1, \ldots, x_N) \in C | \, \exists i, x_i \in X\}$ and $T_X = \{(s_1, \ldots, s_N \in T | \, \exists i, s_i \in B \text{ and } s_{i+1} \in A)\}$. So $\Phi(T_X) \subset C_X$ and $\Phi(T - T_X)$ is a disjoint union of $N+1$ embedded N-balls. If X is small, it is possible to ignore $\Phi(T_X)$ and just consider $\Phi(T - T_X)$.

We have defined T as an oriented N-dimensional sub-manifold of C. We want now to see that it is a pointed space, with a canonical choice for the basepoint t. We start by N points $t_1, \ldots, t_N$ in the disk such that

- $t_i \in \gamma_i(B)$
- $\Im(t_i) < 0$
- $\gamma_N(1) < \Re(t_1) < \Re(t_2) < \ldots < \Re(t_N) < p_2$

Then $t = (t_1, \ldots, t_N)$ is the canonical basepoint of T.

We want now to define a basepoint for C. For every $i = 1, \ldots, N$, $\tau_i : I \to D$ is a vertical edge, $\tau_i(0)$ lying in the lower half of ∂D and $\tau_i(1) = t_i$. $x = \tau(0)$ is the canonical basepoint for C and τ is a path between the two basepoints

We now define an embedded ball in C. We start by the open N- ball $S = \{(s_1, \ldots, s_N) \in \mathbb{R}^N | \ 0 < s_1 < \ldots < s_N < 1\}$. We parametrise the edge from p_1 to p_2 by $\gamma : I \to D$. We can define an embedding $\psi : S \to C$ by

$$\psi(s_1, \ldots, s_N) = (\gamma(s_1), \ldots, \gamma(s_N))$$

We want now to define a canonical basepoint for S. Let $\eta_i : I \to D$ be a vertical edge from x_i to a point in γ. $\eta : I \to C$ is the map

$$\eta(s) = (\eta_1(s), \ldots, \eta_N(s))$$

such that $\eta(0) = x$. We take the basepoint of S as $s = \eta(1)$, and η becomes a path from the basepoint x of C to the basepoint s of S.

We have defined S and T in the case $C_1 = C_{1,\ldots,N}(D_{0,N+1})$. We can define an embedding $\coprod_n C_1 \to C$ by gluing together n copies of $D_{0,N+1}$. By taking n copies of the immersed torus and of the embedded ball, we obtain the general construction.

3.3 The invariant on braids

3.3.1 Definition

Now we define the invariant for a braid β by using the two spaces T and S.

Remark 51. β is a diffeomorphism from D to itself, $\beta(\{p_1, \ldots, p_{2n}\}) = \{p_1, \ldots, p_{2n}\}$ and is such that it preserves the colours of the punctured points. β induces an application from C to C, that we will also call β.

By construction, C is a $2m$-dimensional manifold, while S and T are two immersed m-manifolds. We can assume, up to isotopy, that they intersect transversely at a finite number of points $y_1, \ldots, y_k$. To each intersection point y_i we associate the sign at the intersection, ϵ_{y_i}. We consider the path

$\xi_y = (\beta \circ \tau)(\gamma_1)(\gamma_2)(\overline{\eta})$, where γ_1 is a path in $\beta(T)$, $\gamma_1(0) = \beta(t)$, $\gamma_1(1) = y$ and γ_2 is a path in S, $\gamma_2(0) = y$, $\gamma_2(1) = s$. So we have:

$$< S, \beta(T) >= \sum_{y \in S \cap \beta(T)} \epsilon_y \rho_{\mathbf{m}}(\xi_y)$$

It is very hard to compute directly this intersection. It is also necessary to show that this definition is well defined, independent from the choices of paths. This is why we are going to search an interpretation of this intersection using algebraic topology.

3.4 A homological definition

Let L be the flat complex line bundle over C with monodromy given by $\rho_{\mathbf{m}}$. If we take $(\mathbb{C}, \tau_d)$, where τ_d is the discrete topology, we can think of L as a covering space of C, with every fibre given by $\mathbb{C}$.

We denote by $H_m(C; L)$ the m-dimensional homology of C with local coefficients. We denote by $H_m^{lf}(C; L)$ the m-dimensional locally finite homology of C with local coefficients. In the following, we will write $H_m(C)$ for $H_m(C; L)$. See the appendix A for the two definitions.

Theorem 3.4.1. *$H^m(C)$ and $H_m^{lf}(C, \partial C)$ are isomorphic.*

Proof. A proof can be found in [22]. □

Theorem 3.4.2. *$H^m(C)$ and $Hom(H_m(C), \mathbb{C})$ are conjugate-isomorphic.*

Proof. The theorem is an easy consequence of the universal coefficients theorem. A prove can be found in [14]. □

Corollary 3.4.3. *$H_m^{lf}(C, \partial C)$ and $Hom(H_m(C), \mathbb{C})$ are conjugate isomorphic. Also, there is a sesquilinear pairing:*

$$< \cdot, \cdot >: H_m^{lf}(C, \partial C) \times H_m(C) \to \mathbb{C}$$

Remark 52. β induces an automorphism $\beta_\# : \pi_1(C) \to \pi_1(C)$ such that $\rho_{\mathbf{m}} \circ \beta_\# = \rho_{\mathbf{m}}$.

By remark 52, β lifts to an action on L, so there are induced actions on $H_m(C)$, $H_m(C, \partial C)$ and $H_m^{lf}(C)$.

We can identify the fibre over a point x with $\mathbb{C}$. We have two lifts $\tilde{T}$ and $\tilde{S}$ of the torus T and the ball S, leading to the following proposition.

Proposition 3.4.4. *$< S, \beta(T) >$ is the sesquilinear pairing of $S \in H_m^{lf}(C, \partial C)$ and $\beta(T) \in H_m(C)$. It has the following proprieties:*

- $< S, \beta(T) >= \sum_{y \in S \cap \beta(T)} \epsilon_y \rho_{\mathbf{m}}(\xi_y)$
- *it is invariant under the action of $B_{\mathbf{p}}$*
- *if $v_1,\ v_2 \in H_m(C)$, $v_1',\ v_2' \in H_m^{lf}(C, \partial C)$ are their images, then* $< v_1', v_2 >= (-1)^m \overline{< v_2', v_1' >}$

Proof. The existence of the pairing is given by previous considerations. The proprieties are proven in every book of algebraic topology, for example [14] or [7].

□

Definition 59.

$$Q(\beta) = \frac{\rho_{\mathbf{p}}(\beta)}{[N+1]q^{m/2}} < S, \beta(T) >$$

where $[N+1]$, the quantum integer corresponding to $N+1$, is

$$[N+1] = \frac{q^{(N+1)/2} - q^{-(N+1)/2}}{q^{1/2} - q^{-1/2}}$$

3.4.1 Barcodes

Definition 60. Let C_R be $C_R = \{(x_1, \dots, x_n) \in C |\ \Re(x_i) = x_i,\ \forall i = 1, \dots N\}$.

Lemma 3.4.5. *The map $H_m^{lf}(C_R) \to H_m^{lf}(C)$ induced by inclusion is an isomorphism.*

Proof. A proof can be found in [3].

□

Definition 61. A code sequence is a permutation of the sequence $\mathbf{p}+\mathbf{m}$ containing $\mathbf{p}$ as a subsequence.

Let S' be a connected component of C_R and let y be $y=(y_1,\dots,y_m)\in S'$, $y_i \neq y_j\ \forall i,j$, $y_i \neq p_j\ \forall i,j$. We call $c=(c_1,\dots,c_{m+n})$ a sequence of colours of mobile points and puncture points, reading from right to left on the real line. By definition, c is a code, representing S'.

We want now to see some conditions about the code sequence. If there is $i \in \{1,\dots N\}$ such that $|c_i - c_{i+1}| \geq 2$, then it is possible to exchange c_i and c_{i+1} without altering the connected component of C_R. This follows from the construction of C and the conditions about the movements of the mobile points.

Definition 62. The code sequences are equivalent if they are related by exchanges between c_i and c_{i+1}, where at least one among c_i and c_{i+1} lies in $\{1,\dots,N\}$ and $|c_{i+1}-c_i| \geq 2$.

If a code sequence c is equivalent to a code sequence whose first or last entry lies in $\{1,\dots N\}$, then c is said to be trivial.

Remark 53. The equivalence classes of code sequences are in bijective correspondence with the connected components of C_R.

Proposition 3.4.6. *Let c be a sequence and S' the associated connected component of C_R.*

If c is trivial, S' is homeomorphic to the upper half space in $\mathbb{R}^m$.

If c is not trivial, S' is homeomorphic to an open m-ball.

Proof. If c is trivial, then S' contains a point $(y_1,\dots,y_m)$ such that y_1 or y_m lies in ∂D.

If c is not trivial, every point in S' is a configuration of points between p_1 and p_{2n}.

□

Proposition 3.4.7. *Let $\mathbb{R}^m_+$ be the upper half space in $\mathbb{R}^m$. Then*

$H^{lf}_m(\mathbb{R}^m_+)=0$,

$H^{lf}_m(\mathbb{R}^m)=\mathbb{C}$.

Proof. A proof can be found in [14] or in [6].

□

3.4.2 Basis for the homology

Let us choose a non zero element in $H_m^{lf}(S')$ associated to a given non-trivial code sequence c. By lemma 3.4.5 and proposition 3.4.6 this is a basis for $H_m^{lf}(C)$, for which we should choose an orientation and a lift to L.

Let $< \cdot, \cdot >$ be the non-degenerate sesquilinear pairing

$$< \cdot, \cdot >' \colon H_m^{lf}(C) \times H_m(C, \partial C) \to \mathbb{C}$$

Then we can define a basis for $H_m(C, \partial C)$ by dualising the basis for H_m^{lf} with respect to the pairing. We will now give a geometric construction of the homology classes.

Let D_2 be a disk in the complex plane and let $E_1, \ldots,\ E_m$ be properly embedded vertical edges in D_2, $E_i \cap E_j = \emptyset$, $\forall i \neq j$ and $p_i \notin E_j$ $\forall i, j$. Now, the product of these edges is an embedded closed m-ball in C. We will call this ball Z. We can take the code sequence $c = (c_1, \ldots, c_{m+n})$, given by the sequence of colours of the punctured points or of vertical edges.

It is possible to lift every Z to L, representing an element in $H_m(C, \partial C)$.

Definition 63. Z denotes both the embedded m-ball and an element in $H_m(C, \partial C)$. Z will be called the barcode corresponding to the code sequence c.

Remark 54. Two equivalent code sequences give rise to the same barcode in $H_m(C, \partial C)$, up to choices of lifts to L.

Proposition 3.4.8. *If c is a trivial code, then any barcode corresponding to c is zero.*

A basis for $H_m(C, \partial C)$ is given by non-zero barcodes corresponding to each non-trivial code sequence c.

Proof. Let S' be a component of C_R and Z be a barcode. If S' and Z correspond to the same non-trivial code sequence c, then $S' \cap Z = \{point\}$, so, up to choices of orientations and lifts, $< S', Z >= 1$. Instead, if S' and Z correspond to different code sequences, then $S \cap Z = \emptyset$ and $< S', Z >= 0$. Thus the basis is well defined as the dual of the basis of $H_m^{lf}(C)$.

□

3.4.3 The image of T

In this paragraph we want to compute T in $H_m^{lf}(C)$ and in $H_m(C, \partial C)$. To start, we suppose $n = 1$.

Remark 55. If $n = 1$, the only non trivial sequence is $(0, 1, \ldots, N + 1)$. We will call simply as Z its barcode, product of the edges $1, \ldots, N$ in order and we choose a lift to L such that it contains the point 1 in the fibre over the basepoint x.

Proposition 3.4.9. *The image of $T \in H_m^{lf}(C)$ is $(q-1)^m S$.*

Proof. Having assumed $n = 1$, by remark 55 there is only one non-trivialcode sequence, so $H_m^{lf}(C) = \mathbb{C}$ and $T = \lambda S$.

We have defined T by figures of eight $\gamma_1, \ldots, \gamma_N$ and Z by edges $E_1, \ldots, E_N$. By construction, γ_i intersects E_i at two points, y_i^+ and y_i^-. So, T and Z intersect at the 2^N points

$$(y_1^{\pm}, \ldots, y_N^{\pm})$$

Every such point contributes a monomial $\pm q^k$ to $< T, Z >$.

Let t be $t = (y_1^-, \ldots, y_N^-)$ such that the orientation of the intersection at y_i^- is positive. Thus t contributes 1 to $< T, Z >'$.

Let us compute by induction the contributions of the other points. Let y, y' be points differing at the mobile point of colour i. We can assume y taking the value y_i^- and y' taking the value y_i^+. We define a loop η, $\eta(0) = \eta(1) = y'$, such that it follows a path in T from y to y' and then back to y' along a path in Z. A braid representing it is such that all the strands are straight, with the exception of the strand of colour i, which makes a positive full twist around the strands of colour $i+1, \ldots, N+1$. By construction, $\rho_{\mathbf{m}}(\eta) = q$. So if y contributes $\pm q^k$ to $< T, Z >'$, then y' contributes $\mp q^{k+1}$, with the sign given by the opposite orientation.

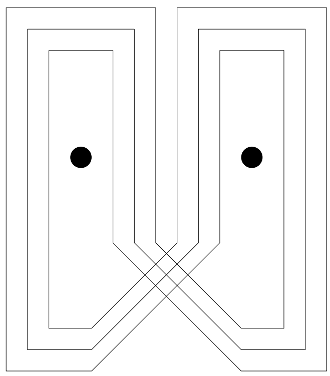
So all the contributions give

$$< T, Z >' = \sum_{k=0}^{N} (-1)^k \left(\begin{array}{c} N \\ k \end{array} \right) q^k = (1-q)^N$$

□

Proposition 3.4.10. *The image of $T \in H_m(C, \partial C)$ is $(1 + q + \ldots + q^N)Z$.*

Proof. As above, let $\gamma_1, \ldots, \gamma_N$ be the figures of eight used to define T. It is possible to isotope T so that the set X is below $[p_1, p_2]$, obtaining for example if $N = 3$:



Let $\{a_i, b_i\}$ be such that $\{a_i, b_i\} = \gamma_i \cap [p_1, p_2]$, $\Re(a_i) < \Re(b_i)$. So there are $2N$ ordered points $a_i, \ldots, a_N, b_1, \ldots, b_N$. For $i = 0, \ldots, N$ let

$$y_i = (a_1, \ldots, a_i, b_{i+1}, \ldots, b_N)$$

Each one of these $N+1$ points is a point of intersection between S and T, such that each contributes a monomial $\pm q^k$ to $< S, T >$. With this choice of orientations, the signs of the intersections of S and T at y_i are positive for every i. Also, y_N contributes 1 to $< S, T >$. As in the previous proof, we assume that η is a loop, $\eta(0) = \eta(1) = y$, following a path in T from y_i to y_{i-1}, and then a patch in S from y_{i-1} to y_i. Again, if we see this loop as a braid, all the strands are straight, except the strand of colour i, acting as in the last proof. So again $\rho_{\mathbf{m}}(\eta) = q$. So, if y_i contributes q^k, y_i contributes q^{k-1}. Then

$$< S, T >= 1 + q + \ldots + q^N$$

and comparing with $< S, Z >= 1$ we obtain the claim. □

Proposition 3.4.11. *The image of $T \in H_m^{lf}(C)$ is $(q-1)^m S$.*

Proof. By construction of T, the general case is a trivial consequence of the case $n = 1$.

□

3.4.4 Partial bar codes

We had defined S as the product of N-balls $S_1, \ldots, S_N$. Being Z the non-trivial barcode for $n = 1$, let Z_i be

$$Z_i = S_1 \times \ldots \times S_{i-1} \times Z \times S_{i+1} \times \ldots \times S_N$$

with basepoint s lying in Z_i. So the path η determines a lift of Z_i to L, giving an element in $H_m^{lf}(C, \partial C)$.

Proposition 3.4.12. *$(q-1)^N S = (1 + q + \ldots + q^N) Z_i \in H_m^{lf}(C, \partial C)$.*

Proof. For the case $n = 1$, by previous computation we have

$$T = (1 + q + \ldots + q^N) Z_1, \; T = (q-1)^N S$$

.

We assume now $n > 1$. We start by stretching T and using excision to obtain a disjoint union of bar codes, where one is Z_i and the others are trivial code sequences. Such a barcode is $0 \in H^{lf}_m(C, \partial C)$, since one of the vertical edges can be slid to the boundary of the disk, ending the construction for $n = 1$.

We want now to do something similar for the case $n > 1$. Let $\mathbf{m}'$ be $\mathbf{m}' = (1, 2, \ldots, N, j)$, $j = 1, \ldots, N$. Let C' be the configuration space

$$C' = C_{\mathbf{m}'}(D_{(0,N+1)})$$

Let S' be the product of the N-ball in $C_{(1,\ldots,N)}(D_{(0,N+1)})$ and a circle of colour j around $[p_1, p_2]$. S' is a $(N+1)$-dimensional sub-manifold of C'. For homotopy equivalence, $\pi_1(S') = \mathbb{Z}$, generated by g. The loop g can be represented by a braid with straight strands, with the exception of the strand of colour i making a positive full twist around the other strands, so that $\rho_{\mathbf{m}}(g) = 1$. So it is possible to lift S' to L, representing an element in $H^{lf}_m(C)$ also called S'.

We want now to show that $S' = 0 \in H^{lf}_m(C)$. Assuming $N = 1$, S' is simply the product of an edge γ between the two puncture points and a circle δ around γ. We call Z the barcode corresponding to the only non-trivial sequence (0,1,1,2). Let E and E' be the two vertical edges of colour 1 between the two punctured points, E being to the right of E. So we can see Z as a closed 2-ball, being the product of E and E'.

We have $E \cap (\gamma \cup \delta) = \{a_1, a_2, a_3\}$, $E' \cap (\gamma \cup \delta) = \{b_1, b_2, b_3\}$. We then have 4 points of intersection between S' and Z,

- $y_1 = (a_2, b_1)$
- $y_2 = (a_2, b_3)$
- $y_3 = (a_1, b_2)$
- $y_4 = (a_3, b_2)$

We are supposed to have chosen y_1 such that its contribution is 1. Now, for $i = 2, 3, , 4$, ξ_i is a loop following a path in Z from y_1 to y_i and a path in S' from y_i to y_1. So we obtain

- $\rho_{\mathbf{m}}(\xi_2) = (q^{1/2})^2 = q$
- $\rho_{\mathbf{m}}(\xi_3) = (-q^{-1})^{-1} = -q$
- $\rho_{\mathbf{m}}(\xi_4) = (-q^{-1})(q^{1/2})^2 = -1$

$\square$

We have assumed that the sign of the intersection at y_1, ϵ_1, is equal to 1. It is easy to see that the signs of intersection are the same for couples (a_i, b_j) and (a_i, b_k), (a_j, b_i) and (a_k, b_i). Also, the intersections at a_1 and a_3 have opposite signs. Then

$$< S', Z >' = 1 - q - (-q) + (-1) = 0$$

If $N > 1$, the only non-trivial code sequence is

$$(0, 1, \ldots, j-1, j, j, j+1, \ldots, N, N+1)$$

with corresponding barcode Z, product of vertical edges $E_1 \ldots, E_{j-1}, E_j$, $E'_j, E_{j+1}, \ldots, E_N$, such that E_k has colour k, E'_j has colour j. Let y_k be the point of intersection between E_k and $[p_1, p_2]$. Given a point of intersection between S' and Z, it must include the mobile points y_k, $\forall k \neq j$, which remain the same throughout the proof. The computation is the same as in the case $N = 1$.

3.5 The equivalence of the two invariants

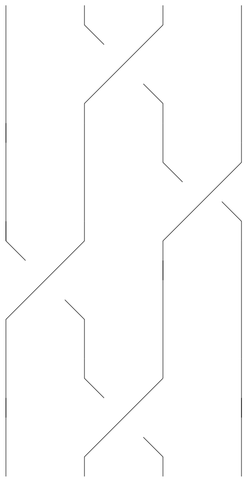
3.5.1 Invariance under movements

We want now to show that the invariant for braids we have defined, $q(\beta)$, is such that $q(\beta) = p(\hat{\beta})$. As proven in [5], we have to see that $q(\beta)$ is invariant under certain moves to give $q(\beta) = q'(\hat{\beta})$. Then, it is sufficient to see if q' respects the skein relations and if it is such that $q'(unknot) = 1$.

Let σ'_i be $\sigma'_i = \sigma_{2i}\sigma_{2i+1}\sigma_{2i-1}\sigma_{2i}$, $i = 1, \ldots, b-1$.

Lemma 3.5.1. *$\sigma_1^2\beta$, $\sigma_i\beta$, $\beta\sigma_1^2$ and $\beta\sigma_i$ respects the orientation of braids.*

Proof. As a braid, σ_i' is



So adding β at left or at right is compatible with the orientations, i.e. with the colours The same for σ_1^2.

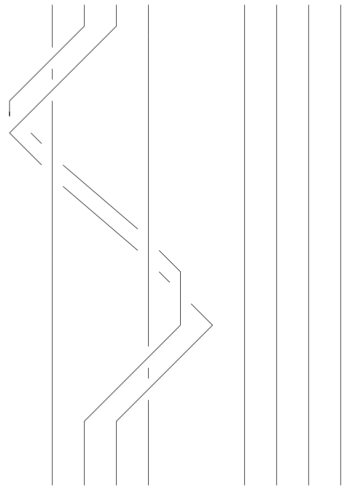
□

Proposition 3.5.2. $Q(\sigma_1^2\beta) = Q(\sigma_i\beta = \beta\sigma_1^2) = Q(\beta\sigma_i) = Q(\beta)$.

Proof. By construction of q_1^2 and $\rho_{\mathbf{p}}$

$$\rho_{\mathbf{p}}(\sigma_1^2\beta) = q^{-(N+1)}\rho_{\mathbf{p}}(\beta)$$

It is sufficient to show that $\sigma_1^2 S = q^{N+1}S$, by proprieties of the sesquilinear pairing. We can assume that, as a function, σ_1^2 acts as the identity on S, so it is sufficient to restrict it on the fibre over s. Let ξ be $\xi = (\sigma_2\eta)(\overline{\eta})$. This path is represented by a braid in which strands of colour $1, \dots, N$ make a positive full twist around two with colours 0 and $N+1$. For example, for $n = 2$ and $N = 2$



So it is easy to see that

$$\rho_{\mathbf{m}}(\xi) = (q^{1/2})^{2N+2} = q^{N+1}$$

so that $\sigma_1^2(S) = q^{N+1}S$.

Now we want to compute $Q(\sigma_i'(\beta))$. We get

$$\rho_{\mathbf{p}}(\sigma_i'\beta) = q^{-1}\rho_{\mathbf{p}}(\beta)$$

As in the previous case, it is sufficient to show $\sigma_i'S = qS$. We can again assume that the function σ_i' acts as the identity on $S \subset C$ and we let ξ_i be $\xi_i = (\sigma_i'\eta)\overline{\eta}$. This is a braid in which two collections of N parallel strands $1, \ldots, N$ form a large figure of X, enclosing two strands of colours 0 and $N+1$. Then

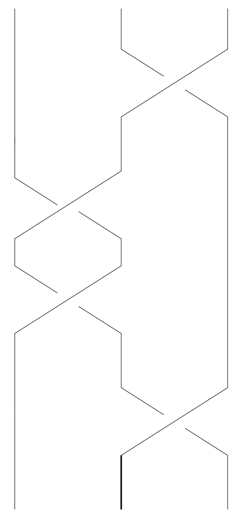
$$\rho_{\mathbf{m}}(\xi_i) = (q^{-1})^N (q^{1/2})^{2N+2} = q$$

and $\sigma_i(S) = qS$.

To show the two remaining cases, it is sufficient to notice that in this case we have to study the action on T and to show that $\sigma_1^2 T = q^{N+1}T$ and $\sigma_i'T = qT$. The proof is the same as in the two previous cases.

□

In the following, we denote by σ_{2112} the braid $\sigma_2\sigma_1^2\sigma$, that is



Proposition 3.5.3.

$$Q(\sigma_{2112}\beta) = Q(\beta\sigma_{2112}) = Q(\beta)$$

Proof.

$$\rho_{\mathbf{p}}(\sigma_{2112}\beta) = q^{-1}\rho_{\mathbf{p}}(\beta)$$

As in the previous proposition, it is sufficient to show that in $H_m^{lf}(C, \partial C)$ we have

$$\sigma_{2112}S = qS$$

By construction of the spaces Z_i, it is sufficient to show that $\sigma_{2112}Z_2 = qZ_2$. As before, we can choose the function σ_{2112} such that it acts as the identity on $Z_2 \subset C$. So it is sufficient to focus on the fibre over s. Let ξ be $\xi = (\sigma_{2112}\eta) = \overline{\eta}$. ξ is represented by a braid in which strands of colour $1, \ldots, N$ wind in parallel around a strand of colour 0, so that $\rho_{\mathbf{m}}(\xi) = q$, which implies the claim.

By the proprieties of the sesquilinear pairing, proving $Q(\beta\sigma_{2112}) = Q(\beta)$ is equivalent to proving $Q(\beta^{-q}) = \overline{Q(\beta)}$. We have the identities

- $q^{m/2} = q^m \overline{q^{m/2}}$
- $[N+1] = \overline{[N+1]}$
- $\rho_{\mathbf{p}}(\beta^{-1}) = \overline{\rho_{\mathbf{p}}(\beta)}$

Then it is sufficient to show

$$< S, \beta^{-1}(T) >= q^m \overline{< S, \beta(T) >}$$

that is equivalent to

$$< \beta(T), T >= (-1)^m \overline{< T, \beta T >},$$

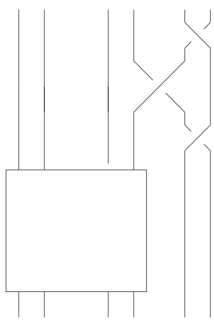
the symmetry property of the pairing

$\square$

3.5.2 The Markov Birman stabilization

Definition 64. Let $\mathbf{p}' = (0, N+1, 0, N+1, \ldots, 0, N+1)$ be a $2n$-tuple and $i : B_{\mathbf{p}} \to B_{\mathbf{p}'}$ the inclusion map, i.e. the map sending the braid to itself and adding at the right two straight brands of colour 0 and $N+1$. The Markov-Birman stabilization of β is the braid

$$\beta' = (\sigma_{2n+1}^{-1}\sigma_{2n}\sigma_{2n+1})i(\beta)$$



Proposition 3.5.4. *Let β be a braid and β' its Markov-Birman stabilization. Then $Q(\beta') = Q(\beta)$.*

Proof. Let m' be the $m+N$-tuple $(1,\dots,N,1,\dots,N,\dots,1,\dots,N)$, $D' = D_{p'}$, $C' = C_{m'}(D')$ and S' and T' the embedded m'-ball and the immersed m'-torus in C'. Then

$$< S', \beta'(T') >=< S, \beta(T) >$$

by the identities

- $\rho_{\mathbf{p}}(\beta') = q^{N/2}\rho_{\mathbf{p}}(\beta)$
- $q^{m'/2} = q^{N/2}q^{m/2}$

Let now $Z_n \subset C$ and $Z'_N \subset C'$ be defined by replacing the second to rightmost N-ball of S' by a barcode. So we have

$$< Z'_n, \beta'(T') >=< Z_n, \beta(T) >$$

which is equivalent to

$$< \sigma(Z'_n), i(\beta)(T') >=< Z_n, \beta(T) >$$

with $\sigma = \sigma_{2n+1}^{-1}\sigma_{2n}^{-1}\sigma_{2n+1}$. It is sufficient to compute this intersection.

Let D_3 be a disk with three punctured points. We imagine D_3 as the set of points in D' to the right of a vertical line between p_{2n-1} and p_{2n}. So we can identify the three punctured points with p_{2n}, p_{2n+1} and p_{2n+2}. We can define an embedding

$$C_{\mathbf{m}}(D' - D_3) \times C_{(1,\dots,N)}(D_3) \to C'$$

Assuming that Z_n lies in $C_{\mathbf{m}}(D' - D_3)$ and taking S_{n+1} as the N-ball in $C_{(1,\dots,N)}(D_3)$, we get

$$Z'_n = Z_n \times S_{n+1}$$

Then

$$\sigma(Z'_n) = Z_n \times \sigma(S_{n+1})$$

Now, let D_2 be a disk with two punctured points. As above, we can imagine it as the set of points in D' to the right of a vertical line between p_{2n} and p_{2n+1}, so that we can identify the two punctured points with p_{2n+1} and p_{2n+2}. There is an embedding

$$C_{\mathbf{m}}(D' - D_2) \times C_{(1,\ldots,N)}(D_2) \to C'$$

Assuming that T lies in $C_{\mathbf{m}}(D'-D_2)$ and letting T_{N+1} be the N-dimensional torus in $C_{(1,\ldots,N)}(D_2)$, we get

$$T' = T \times T_{n+1}$$

so that

$$i(\beta)(T') = \beta(T) \times T_{n+1}$$

By the above computation, we obtain

$$< Z \times \sigma(S_{n+1}), \beta(T) \times T_{n+1} >=< Z, \beta(T) >$$

Any point of intersection between $Z \times \sigma(S_{n+1})$ and $\beta(T) \times T_{n+1}$ lies in the intersection of the two product spaces

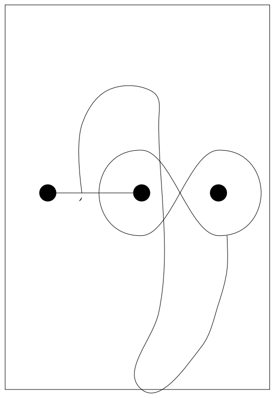
$$C_{\mathbf{m}}(D' - D_3) \times C_{(1,\ldots,N)}(D_2)$$

So it is sufficient to show that

$$< \sigma(S_{n+1}), T_{n+1} >= 1$$

By restriction, we can see this intersection pairing as being between submanifolds of $C_{(1,\ldots,N)}(D_3)$. We should now compute directly this pairing.

There is one point of intersection y between $\sigma(S_{n+1})$ and T_{n+1}. We can assume the sign of this intersection to be positive. Now, both $\sigma(S_{n+1})$ and T_{n+1} have associated paths from a configuration of points on ∂D_3 to y, paths that are homotopic relative to endpoints, which completes the proof. For example, in the case $N = 1$ we have



□

3.5.3 Equivalence with the HOMFLY polynomial

Definition 65. Let K_n be the subgroup of B_{2n} generated by

$$\{\sigma_1,\ \sigma_2\sigma_1^2\sigma_2,\ \sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}\sigma_{2i},\ 1 \leq i \leq n-1\}.$$

Theorem 3.5.5. *Let β_1 and β_2 be two braids, L_1 and L_2 the associated plat closures. L_1 and L_2 are isotopic if and only if, after adding a suitable number of trivial loops to each component of L_1 and L_2, we obtain two braids β_1' and β_2' such that they are in the some double coset of B_2 modulo the subgroup K_{2n}.*

A proof can be found in [5]

So, in the two above paragraphs we have proved the following theorem

Theorem 3.5.6. *There exists an invariant of knots P' such that $P'(\hat{\beta}) = Q(\beta)$, where $\hat{\beta}$ is the plat closure of β.*

We have now all the necessary tools to prove the equivalence between the invariant P' and the HOMFLY polynomial.

Theorem 3.5.7.

$$P'(\hat{\beta}) = P(\hat{\beta})$$

Proof. It has been proven in [12] that there exists an unique polynomial P such that

- P respects the skein relation,
- $P(unknot) = 1$.

Then it is sufficient to show that P' also respects the two conditions.

We begin by proving that P' respects the skein relation. Let β_+ and β_- be

$$\beta_+ = \sigma_2^{-1}\sigma_1\sigma_2\beta,\ \beta_- = \sigma_2^{-1}\sigma_1^{-1}\sigma_2\beta.$$

Up to isotopy, it is sufficient then to show

$$q^{(N+1)/2}Q(\beta_-) - q^{-(N+1)/2}Q(\beta_+) = (q^{1/2} - q^{-1/2})Q(\beta)$$

By definition

- $\rho_{\mathbf{p}}(\beta_+) = q^{N/2}\rho_{\mathbf{p}}(\beta)$
- $\rho_{\mathbf{p}}(\beta_-) = q^{-N/2}\rho_{\mathbf{p}}(\beta)$

So we should show

$$q^{1/2} < S, \beta_-(T) > -q^{-1/2} < S, \beta_+(T) >= (q^{1/2} - q^{-1/2}) < S, \beta(T) >$$

We can take, instead of S, the subspace Z_2, so that we can restrict to show

$$q^{1/2} < Z_2, \beta_-(T) > -q^{-1/2} < Z_2, \beta_+(T) >= (q^{1/2} - q^{-1/2}) < Z_2, \beta(T) >$$

that is equivalent to

$$< \sigma_2^{-1}(\sigma_1 - 1)(1 + q\sigma_1^{-1})\sigma_2(Z_2), \beta(T) >= 0$$

We will not compute the intersection pairing, but we will show that in $H_m^{lf}(C, \partial C)$ we have

$$\sigma_2^{-1}(\sigma_1 - 1)(1 + q\sigma_1^{-1})\sigma_2(Z_2) = 0$$

Let D_3 be a three times punctured disk, which can be identified with the set of points in D to the left of a vertical line between p_3 and p_4, such that the three punctured points can be identified with p_1, p_2 and p_3. Let $C_1 = C_{(1\ldots,N)}(D_3)$ and $\mathbf{m_2}$ be the $m-N$-tuple $\mathbf{m_2} = (1,\ldots,N,1,\ldots,N,\ldots,1,\ldots,N)$ and $C_2 = C_{\mathbf{m_2}}(D - D_3)$. We consider the embedding

$$C_1 \times C_2 \to C.$$

Taking S_1 as the N-ball in C_1 and Z' as an $(N-m)$-manifold in C_2 we have

$$Z_2 = S_1 \times Z'.$$

σ_1 and σ_2 are such that they act as the identity on $D - D_3$. Thus it is sufficient to show that in $H_N^{lf}(C_1)$

$$\sigma_2^{-1}(\sigma_1 - 1)(1 + q\sigma_1^{-1}\sigma_2)(S_1) = 0$$

Let now D_3' be $D_3' = \sigma D_3$, a three times punctured disk with colours, from left to right, 0, 0 and $N+1$. Let C_1' be $C_1' = C_{(1,\ldots,N)}(D_3')$ and S_1' be $S_1' = \sigma_2 S_1$. Then we have

$$(\sigma_1 - 1)(1 + q\sigma_1^{-1})(S_1') = 0$$

in $H_N^{lf}(C_1')$.

It is easy to see that in this setting there are only two non-trivial sequences, $(0,1,2,\ldots,N+1,0)$ and $(0,N+1,N,\ldots,1,0)$. Let Z be the barcode corresponding to the non-trivial code sequence $(0,1,2,\ldots,N+1,0)$. So S_1' and Z do not intersect and $< S_1', Z >' = 0$.

Instead, $\sigma_1(S_1')$ and Z intersect at a single point, and so do $\sigma_1^{-1}(S_1')$ and Z. The two intersection points are such that their signs are the same. We assume that the two points of intersection coincide on y. Now, each of $\sigma_1(S_1')$ and $\sigma_1^{-1}(S_1')$ are associated to a path from y to x, paths that differ by the direction the points of colours $1,\ldots,N$ pass around the middle puncture point, so that

$$< \sigma_1^{-1}(S_1'), Z' >' = q < \sigma_1(S_1'), Z > .$$

By computation,

$$< (\sigma_1 - 1)(1 + q\sigma_1^{-1})(S_1'), Z >' = 0.$$

Let Z be the barcode corresponding to $(0, N+1, N, \ldots, 1, 0)$ and let us assume that σ_1 acts as the identity on Z. So we have

$$< S_1', Z >' = < \sigma_1^{-1}(S_1'), Z >'$$

and by computation

$$< (\sigma_1 - 1)(1 + q\sigma_1^{-1})(S_1'), Z >' = 0.$$

So we have proven that P' respects the skein relation. We now see that $P'(unknot) = 1$. Suppose $n = 1$ and β is the identity braid. We have proven that

$$< S, T > = 1 + q + \ldots + q^N$$

and so

$$Q(\beta) = \frac{1}{[N+1]q^{N/2}}(1 + q + \ldots + q^N) = 1$$

So we have proven that $P'(\hat{\beta}) = Q(\beta) = 1$, which ends the proof. □

Appendix A

Local Coefficients via Bundles of Groups

Definition 66. Let E and X be two topological spaces and let p be a map $p : E \to X$ such that each point of X admits a neighborhood U for which there is a homeomorphism $\varphi_U : p^{-1}(U) \to U \times G$ taking each $p^{-1}(x)$ to $\{x\} \times G$ by a group isomorphism. The map p is called a bundle of groups, the subsets $p^{-1}(x)$ are called the fibers of p and E is said to be a bundle of groups with fiber G.

Remark 56. If we take (G, τ_D), where τ_D is the discrete topology, p is a covering space.

Let $\sigma_i : \Delta_n \to X$ be a singular n-simplex in X and let $n_i : \Delta_n \to E$ be a lifting of σ_i. We can then take finite sums $\sum_{i=1}^{m} n_i\sigma_i$. Given two lifts n_i and m_i of the same lift σ_i, we can define their sum as $(n_i+m_i)(s) = n_i(s)+m_i(s)$, which is still a lift of σ_i. So we have defined an abelian group $C_n(X; E)$. We want it to be a chain complex.

We can define a boundary homomorphism $\partial : C_n(X; E) \to C_{n-1}(X; E)$ by

$$\partial(\sum_i n_i\sigma_i) = \sum_{i,j} (-1)^j n_i|_{v_0,\ldots,\hat{v}_j,\ldots,v_n} \sigma_i|_{v_0,\ldots,\hat{v}_j,\ldots,v_n}$$

We can notice that the boundary is linear and it acts on σ_i as the usual boundary for singular homology. Then it is obvious that $\partial^2 = 0$ and $(C_*(X;E), \partial)$ is a chain complex, whose homology groups are denoted as $H_*(X;E)$.

Let $f : X \to X'$ be a map. We want to find conditions for which it induces a map in homology with local coefficients. Let $p' : E' \to X'$ be a bundle of groups. We define E as

$$E = \{(x, e') \in X \times E' |\ f(x) = p'(e')\}$$

and $p : E \to X$ as $p(x, e') = x$. Let $\tilde{f} : E \to E'$ be $\tilde{f}(x, e') = e'$. By definition, the fiber $p^{-1}(x)$ is the set $\{(x, e') \in X \times E |\ f(x) = p'(e')\}$, so that $\tilde{f}$ is a bijection between $p^{-1}(x)$ and $p'^{-1}(f(x))$ and $p^{-1}(x)$ has a group structure.

By definition of bundle of groups, $\varphi' : (p')^{-1}(U') \to U' \times G$ is an isomorphism. Let U be $U = f^{-1}(U')$ and let φ be $\varphi : p^{-1}(U) \to U \times G$, defined by $\varphi(x, e') = (x, \varphi'_2(e'))$. So φ admits as inverse and then it is an isomorphism on each fiber. The bundle of groups p is called the pullback of p', or the induced bundle.

Let us now define the cohomology groups. Let Φ be a function from the set of singular n-simpleces to E, assigning to each $\sigma \in \Delta_n$ a lift $\Phi(\sigma) \in E$. Let $C^n(X;E)$ be the group of such functions. The definition of the coboundary map $\delta : C^n(X;E) \to C^{n+1}(X;E)$ is exactly as in singular cohomology, so that $\delta^2 = 0$ and $(C^n(X;E), \delta)$ is a cochain complex. Let $H^n(X;E)$ be the cohomology of such a complex.

We now define the homology groups $H_n^{lf}(X;G)$. Let $\sigma : \Delta_n \to X$ be a singular simplex. We take the set C_n^{lf} of locally finite chains, which are formal sums $\sum_\sigma g_\sigma \sigma$, $g_\sigma \in G$, such that every $x \in X$ admits a neighborhood U such that U meets the images of a finite number of σ with $g_\sigma \neq 0$. The boundary operator is obviously well defined, because if σ satisfies the condition, so does its boundary. The homology $H_*^{lf}(X;G)$ is the homology of this chain complex.

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