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The half-space model of pseudo-hyperbolic space

M.Sc. Thesis

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Contents

Introduzione

L'obiettivo di questa tesi è quello di fornire una descrizione completa del modello del semispazio dello spazio pseudo-iperbolico, che è una generalizzazione del noto modello del semispazio di Poincaré per lo spazio iperbolico. Da questo lavoro è stato estratto l'articolo [ST21], che riassume i principali risultati dello studio.

Il modello del semispazio per lo spazio Anti-de Sitter $\mathbb{H}^{2,1}$ è stato introdotto da Danciger in [Dan11], ed è stato utilizzato ad esempio in [Tam21]. Il caso di $\mathbb{H}^{n,1}$, in dimensione arbitraria, è presente in [BS20]. È possibile definire lo spazio pseudo-sferico $\mathbb{S}^{p,q}$, che è anti-isometrico rispetto a $\mathbb{H}^{q,p}$. Un modello del semispazio si può definire in modo del tutto analogo per $\mathbb{S}^{p,q}$, assumendo $q \geq 1$. In particolare, il modello del semispazio per lo spazio de Sitter $\mathbb{S}^{p,1}$ è stato studiato in [Nom82]. Abbiamo deciso di concentrarci sul caso pseudo-iperbolico $\mathcal{H}^{p,q}$ per fissare le notazioni: basta cambiare segno alla pseudo-metrica per renderlo un modello di $\mathbb{S}^{q,p}$, per $p \geq 1$.

La trattazione si compone di due parti. La prima è un compendio di geometria pseudo-Riemanniana, dove abbiamo presentato i principali strumenti della teoria: la connessione di Levi-Civita (sezione 1.2), il tensore di curvatura di Riemann e la curvatura sezionale (sezione 1.3) e le geodetiche (sezione 1.4). Abbiamo cercato in questo capitolo di mettere in risalto le difficoltà tecniche che rendono non triviale l'estensione dei suddetti strumenti dall'ambiente Riemanniano a quello pseudo-Riemanniano.

Vengono presentati $\mathbb{R}^{p,q}$, ovvero lo spazio pseudo-Euclideo di segnatura (p,q), e $\mathbb{H}^{p,q}$, cioè lo spazio iperbolico della medesima segnatura, per mezzo di esempi utili a capire gli strumenti descritti. Per approfondimenti circa lo spazio iperbolico, si faccia riferimento a [BP92]; per $\mathbb{H}^{p,q}$ in segnatura e dimensione qualunque rimandiamo a [O'N, CTT19, DGK18].

Nella seconda parte, vengono utizzati gli strumenti di cui sopra per studiare la geometria del modello del semispazio, definito come il semispazio aperto $\{z > 0\}$ in $\mathbb{R}^{p+q}$, dotato della pseudo-metrica

$$\frac{dx_1^2 + \ldots + dx_{p-1}^2 - dy_1^2 - \ldots - dy_q^2 + dz^2}{z^2},$$

e denotato come $\mathcal{H}^{p,q}$. Se $q = 0$, ci troviamo alla presenza del classico modello del semispazio di $\mathbb{H}^n$. Per $q \geq 1$, il modello non è globalmente isometrico ad $\mathbb{H}^{p,q}$. Quest'ultimo infatti non è semplicemente connesso, e quindi non è neppure omeomorfo al semispazio. Nella sottosezione 2.1.2 forniremo un embedding isometrico di $\mathcal{H}^{p,q}$ in $\mathbb{H}^{p,q}$, la cui immagine è il complementare di un iperpiano totalmente geodetico degenere. Ne consegue che $\mathcal{H}^{p,q}$ *non* sia una varietà pseudo-Riemanniana geodeticamente completa.

Viene data una classificazione per i sottospazi totalmente geodetici di ogni dimensione (sezione 2.4) e successivamente una descrizione più dettagliata per il caso 1-dimensionale (sezione 2.5).

Nella sezione 2.6 viene presentato il bordo all'infinito $\partial_\infty \mathbb{H}^{p,q}$, visto dal punto di vista del modello del semispazio. In particolare il bordo $\partial \mathcal{H}^{p,q}$ in $\mathbb{R}^{p+q}$, che è una copia dello spazio pseudo-Euclideo $\mathbb{R}^{p-1,q}$, si embedda in maniera conforme in $\partial_\infty \mathbb{H}^{p,q}$; viene inoltre costruita la sua compattificazione topologica $\partial_\infty \mathcal{H}^{p,q}$, i cui punti sono limiti di ipersuperfici totalemente geodetiche degeneri. Questa descrizione è utile per dare un'ulteriore presentazione delle geodetiche di tipo spazio e luce, in funzione dei loro estremi.

Nella sezione 2.7 vengono presentate le orosfere del modello.

Infine, nella sezione 2.8, l'analisi delle geodetiche ci permette di dedurre il gruppo di isometrie $\mathrm{Isom}(\mathcal{H}^{p,q})$. Osserviamo che questo non corrisponde al gruppo di isometrie di $\mathbb{H}^{p,q}$, ma solo al sottogruppo che preserva l'iperpiano degenere che non compare nel modello. Ciononostante, nella sezione 2.8, siamo in grado di studiare l'azione di $\mathrm{Isom}(\mathbb{H}^{p,q})$ sul semispazio in funzione di $\mathrm{Isom}(\mathcal{H}^{p,q})$ e di trasformazioni che sono l'analogo delle inversioni in geometria iperbolica.

Ringraziamenti

Un ringraziamento particolare va ad Andrea Seppi, che ha dato un contributo fondamentale allo sviluppo di questa tesi e alla mia crescita come matematico.

Introduction

The aim of this thesis is to give a complete description of the half-space model of pseudo-hyperbolic space, which is a generalization of the well-known Poincaré half-space model of hyperbolic space. The main results of this study have been summarized in [ST21].

We remark that the half-space model for the Anti-de Sitter space $\mathbb{H}^{2,1}$ has been introduced by Danciger in [Dan11], and has been used for instance in [Tam21]; in any dimension, the half-space model of $\mathbb{H}^{n,1}$ also appears in [BS20]. Of course one analogously defines the pseudo-spherical space $\mathbb{S}^{p,q}$. The space obtained in this way is anti-isometric to $\mathbb{H}^{q,p}$. A half-space model of $\mathbb{S}^{p,q}$ is defined similarly, provided $q \geq 1$. As a particular case, the half-space model of the de Sitter space $\mathbb{S}^{p,1}$ has been studied in [Nom82]. We decided to focus on the case of $\mathbb{H}^{p,q}$ for the sake of definiteness: up to changing a sign to the pseudo-Riemannian metric, one recovers the half-space model for $\mathbb{S}^{q,p}$ if $p \geq 1$.

The discussion is divided in two parts: the first one is a summary of pseudo-Riemannian geometry, where we develop the main tools of the theory, namely Levi-Civita connection (Section 1.2), Riemann curvature tensor and sectional curvature (Section 1.3) and geodesics (Section 1.4). In this part, we have tried to emphasize the technical problem that make not trivial the extension of these tools from the Riemannian realm to the pseudo-Riemannian one.

In the first part we also describe $\mathbb{R}^{p,q}$, namely the pseudo-Euclidean space of signature (p, q), and $\mathbb{H}^{p,q}$, namely the pseudo-hyperbolic space of signature (p, q), as example to explain the tools introduced. For more details on the hyprbolic space, see for instance [BP92]; for $\mathbb{H}^{p,q}$ in arbitrary signature and dimension we recommend [O'N, CTT19, DGK18].

In the second part, the tools mentioned above are used to study the geometry of the half-space model. This is defined as the open half-space $\{z > 0\}$ in $\mathbb{R}^{p+q}$ endowed with the pseudo-Riemannian metric

$$\frac{dx_1^2 + \ldots + dx_{p-1}^2 - dy_1^2 - \ldots - dy_q^2 + dz^2}{z^2} ,$$

and denoted $\mathcal{H}^{p,q}$. When $q = 0$, this is the usual half-space model of $\mathbb{H}^n$. When $q \geq 1$, this space is not globally isometric to $\mathbb{H}^{p,q}$ (which is indeed not simply connected, hence not even homeomorphic to the half-space). In Subection 2.1.2 we show that $\mathcal{H}^{p,q}$ embeds isometrically into $\mathbb{H}^{p,q}$, with image the complement of a totally geodesic degenerate hyperplane. In other words, $\mathcal{H}^{p,q}$ is *not* a geodesically complete pseudo-Riemannian manifold.

We give a classification result for the totally geodesic subspaces of any dimension (Section 2.4) and then a more refined classification of the geodesics (Section 2.5).

In Section 2.6 we provide a description of the boundary at infinity $\partial_\infty \mathbb{H}^{p,q}$, seen from the half-space model. Of course the boundary $\partial\mathcal{H}^{p,q}$ in $\mathbb{R}^{p+q}$, which is a copy of the pseudo-Euclidean space $\mathbb{R}^{p-1,q}$, is conformally embedded in $\partial_\infty \mathbb{H}^{p,q}$; we describe topologically its compactification $\partial_\infty \mathcal{H}^{p,q}$ in terms of divergence of totally geodesic degenerate hypersurfaces. We use the description of the boundary to give a further description of spacelike and lightlike geodesic in terms of endpoints.

In Section 2.7 we describe the horospheres in the half-space model.

Finally, in Section 2.8, we compute the isometry group $\mathrm{Isom}(\mathcal{H}^{p,q})$, as a result of the analysis of geodesics. We remark that this does not correspond to the isometry group of $\mathbb{H}^{p,q}$, but only to a subgroup that preserves the complement of a degenerate hyperplane. Nevertheless, in Section 2.8, we study the action of $\mathrm{Isom}(\mathbb{H}^{p,q})$ on the half-space model in terms of $\mathrm{Isom}(\mathcal{H}^{p,q})$ and some transformations which are the analogue of inversions in hyperbolic geometry.

Requirements

The reader is supposed to have a basic knowledge of differential geometry. Nevertheless, except from the definition of (smooth) manifold, smooth map, tangent space and differential of a map, every tool that is used is defined along the text. We suggest to use [Laf96] as support.

A familiarity with Rimannian geometry is useful and allows to appreciate the technical difficulties of the pseudo-Riemannian case, but is not necessary. We based our study of it on [GHL04, DoC92]. Finally, the main reference for pseudo-Riemannian geometry is [O'N].

As for Riemannian geometry, a basic knowledge of hyperbolic and pseudo-hyperbolic geoemtry is not needed but recommended. Consider [BP92] to the Riemannian case and the first chapter of [RS19] for the Lorentzian case. The latter is highly recommended because is shorter and contains basically all the topics one will encounter in the following, namely pseudo-Riemannian geometry and pseudo-hyperbolic geometry.

Chapter 1

Pseudo-Riemannian Geometry

In this chapter we will study the basis of pseudo-Riemannian geometry. We will extend some foundamental tools of the Riemannian realm, such as covariant derivative (Section 1.2), curvature tensor (Section 1.3) and geodesics (Section 1.4), to the pseudo-Riemannian case.

The examples are chosen to introduce pseudo-Euclidean spaces and pseudo-hyperbolic spaces, in order to approach the second chapter with a sufficient knowledge of both spaces, on which is based the study of the half-space model.

1.1 Basic definitions

In the following, we will always suppose manifolds to be smooth.

Definition 1.1.0.1 (Pseudo-Riemannian metric). Let M be a smooth manifold. A collection $g = (g_m)_{m \in M}$ of bilinear symmetric 2-forms on $T_m M$ defines a *pseudo-metric* over M if

P1. g_m is non-degenerate $\forall m \in M$;

P2. the signature is constant, *i.e.* $(n_+(m), n_-(m)) = (n_+, n_-)$, $\forall m \in M$;

P3. the map $m \mapsto g_m$ is smooth.

A manifold endowed with a pseudo-metric is called *pseudo-Riemannian*.

Remark 1.1.0.2. (P2) is only a technical request, in fact, due to the symmetry of the form, it follows from (P1) if the manifold is connected (see Lemma 1.1.1.5).

Equivalently, g is a smooth and non-degenerate section of $S^2 T^* M$, namely the bundle of the symmetric 2-forms on M. In the language of tensors, g is a symmetric, non-degenerate $(0, 2)$-tensor field on M.

Let (U, x) be a local chart, we will write $g_{ij} = g(\partial_i, \partial_j)$, namely the coefficients of the matrix of g in the basis induced by the local chart.

Moreover, as g is non-degenerate, it admits inverse g^{-1}, which will be denoted $g^{ij} = g^{-1}(\partial_i, \partial_j)$ in local coordinates.

Example 1.1.0.3. A Riemannian metric is a pseudo-metric. Indeed, if g_m is positive-definite, (P1) and (P2) are automatically satisfied.

Definition 1.1.0.4 (Lorentzian manifold). A pseudo-Riemannian manifold (M, g) is called *Lorentzian* if the signature of the pseudo-metric is $(\dim M - 1, 1)$.

Example 1.1.0.5. (Pseudo-Euclidean space) The pseudo-Euclidean space $\mathbb{R}^{p,q}$ is obtained endowing $\mathbb{R}^{p+q} = \mathbb{R}^p_x \oplus \mathbb{R}^q_y$ with the pseudo-metric

$$\langle \cdot, \cdot \rangle_{p,q} := dx_1^2 + \ldots + dx_p^2 - dy_1^2 - \ldots - dy_q^2,$$

which clearly is non-degenerate and with signature (p, q).

Particularly, $\mathbb{R}^{p,0}$ is the Euclidean space $\mathbb{R}^p$, and $\mathbb{R}^{p,1}$ the $(p+1)$-dimensional Minkowski space, which is a Lorentzian manifold.

As the form over the tangent is not positive-definite, vectors can be divided in three *characters*, based on the value of $g(v, v)$, which is abusively called *squared norm* of v in analogy to the Riemannian case.

Definition 1.1.0.6 (Character). A vector $v \in T_mM$ is

- *timelike* if $g(v, v) < 0$;
- *spacelike* if $g(v, v) > 0$;
- *lightlike* (or *null*) if $g(v, v) = 0$.

The terminology comes from relativistic physics: consider for semplicity $\mathbb{R}^{1,1}$, the X-axis represents the space and the Y-axis the time. The physical meaning of a point (x, y) is to be at the place x at the time y, hence the origin is *here and now*.

Denote $(\partial s, \partial t) = (e_1, e_2)$ the basis of the tangent space. Let γ be a curve and (s, t) its tangent vector. The speed of γ, *i.e.* the ratio between space and time, is $|s|/|t|$. Up to scaling the space axis as $1\partial s = 299.792.458\,\mathrm{m}$ and time axis as $1\partial t = 1\,\mathrm{s}$, $|s|/|t| = 1$ when the speed is equal to the speed of light. Therefore, timelike vectors satisfy $|s|/|t| < 1$ while spacelike ones $|s|/|t| > 1$ (see Figure 1.1).

In literature, curves with either timelike or lightlike tangent vector are called them *causal*: they are the only path "physically possible" in our universe, since special relativity predicts that nothing can exceed the speed of light. In particular, the interior of a lightcone emanating from a point (x, y) represents the points that can be reached starting from (x, y) and moving less rapidly than the light.

The isometries preserve lightlike vectors, that is the speed of light does not depend on the observator, suggesting this is a good model for relativistic physic. In particular, the isometries of $\mathbb{R}^{1,1}$ are Lorentz transformations.

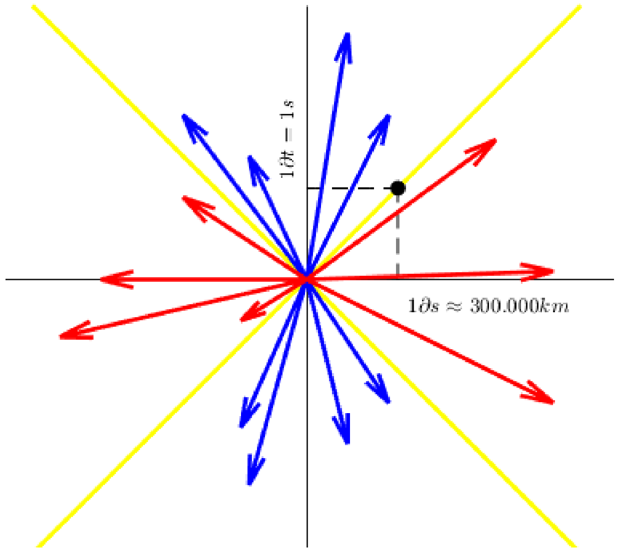

Figure 1.1: $\mathbb{R}^{1,1}$ with lightcone (yellow), some timelike vectors (blue) and spacelike ones (red).

1.1.1 Submanifolds

A submanifold of a Riemannian manifold inherits automatically the structure of Riemannian manifold, as the restriction of a positive-definite form is positive-definite, too. That is not true for a non-degenerate form.

Definition 1.1.1.1 (Pseudo-Riemannian submanifold)*.* Let (M, g) be a pseudo-Riemannian manifold. A smooth submanifold $N \subseteq M$ is a *pseudo-Riemannian submanifold* if the restriction of g on TN is still a pseudo-metric.

Example 1.1.1.2 (Degenerate planes)*.* In $\mathbb{R}^{2,1}$, consider the vectors $v = (1, 0, 0)$ and $w = (0, 1, 1)$.

$$\langle v, v\rangle_{2,1} = 1, \qquad \langle v, w\rangle_{2,1} = 0, \qquad \langle w, w\rangle_{2,1} = 0.$$

The restriction of the pseudo-metric induced on the smooth submanifold $\alpha = \mathrm{Span}(v, w)$, with respect to the basis $\{v, w\}$, is

$$\langle \cdot, \cdot\rangle_{2,1}|_\alpha = \begin{pmatrix} \langle v, v\rangle_{2,1} & \langle v, w\rangle_{2,1} \\ \langle w, v\rangle_{2,1} & \langle w, w\rangle_{2,1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

which is degenerate, that is $\langle \cdot, \cdot\rangle_{2,1}|_\alpha$ is not a pseudo-metric.

Example 1.1.1.3 (Sphere)*.* Consider the sphere $\mathbb{S}^2 = \{x_1^2 + x_2^2 + y_1^2 = 1\}$ in the 3-dimensional Minkowski space $\mathbb{R}^{2,1}$.

Let's study the restriction of the pseudo-metric using the local parameterization

$$\begin{aligned}\mathbb{R} \times (-\pi, \pi) &\to \mathbb{S}^2 \subseteq \mathbb{R}^{2,1}\\ (\phi, \theta) &\mapsto (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta).\end{aligned}$$

A basis of $T_m\mathbb{S}^2$ is $(\partial_\phi, \partial_\theta)$, where

$$\begin{cases}\partial_\phi = (-\sin\phi\sin\theta, \cos\phi\sin\theta, 0),\\ \partial_\theta = (\cos\phi\cos\theta, \sin\phi\cos\theta, -\sin\theta).\end{cases}$$

Then, the matrix of the restriction of the pseudo-metric is

$$\langle\cdot,\cdot\rangle_{2,1}|_{T_m\mathbb{S}^2} = \begin{pmatrix}\langle\partial_\phi, \partial_\phi\rangle_{2,1} & \langle\partial_\phi, \partial_\theta\rangle_{2,1}\\ \langle\partial_\theta, \partial_\phi\rangle_{2,1} & \langle\partial_\theta, \partial_\theta\rangle_{2,1}\end{pmatrix} = \begin{pmatrix}\sin^2\theta & 0\\ 0 & \cos^2\theta - \sin^2\theta\end{pmatrix}.$$

The matrix is degenerate if $\theta = \pm\frac{\pi}{4}$, namely when it intersects the lightcont, so it does not represent a pseudometric on the sphere. Moreover, the induced metric is Riemannian if $|\theta| < \pm\frac{\pi}{4}$ and Lorentzian if $|\theta| > \pm\frac{\pi}{4}$ (see Figure 1.2).

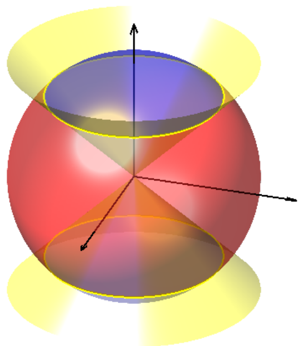

Figure 1.2: Restriction of $\langle\cdot,\cdot\rangle_{2,1}$ on $\mathbb{S}^2$.

These examples prove that not every submanifold of a pseudo-Riemannian manifold is a pseudo-Riemannian manifold itself. In the latter it can be noticed that the signature of the restricted form depends on the character of the vector $(T_m\mathbb{S}^2)^\perp$: when it is lightlike the form is degenerate, when it is timelike the form is Riemannian and when it is spacelike the signature is Lorentzian. This is not an isolated case, as we will show in the following results.

Lemma 1.1.1.4. *Let (M, g) be a pseudo-Riemannian manifold. Let $N \subseteq M$ be a smooth submanifold of codimension 1, $m \in N$, $v \in T_mN^\perp \setminus \{0\}$,* i.e. *a generator of the normal space to N at m. Denote (n_+, n_-) the signature of g. The signature of $g|_{T_mM}$ only depends on the character of v:*

- *if v timelike, $g|_{T_mN}$ has signature $(n_+, n_- - 1)$;*
- *if v spacelike, $g|_{T_mN}$ has signature $(n_+ - 1, n_-)$;*
- *if v lightlike, $g|_{T_mN}$ is degenerate.*

Proof. In the first cases v is not isotrophic whith respect to the form g_m, so $T_mM = v \oplus T_mN$, then

$$g = \begin{pmatrix} & & & 0 \\ & g|_{T_mN} & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & g(v,v) \end{pmatrix}.$$

g is non-degenerate by hypothesis, hence Sylvester's criterion provides us a way to compute the signature (n'_+, n'_-) of $g|_{T_mN}$:

$$(n_+, n_-) = \begin{cases} (n'_+, n'_- + 1) & \text{if } v \text{ timelike;} \\ (n'_+ + 1, n'_-) & \text{if } v \text{ spacelike.} \end{cases}$$

If v is lightlike, $v \in (T_mN^\perp)^\perp = T_mN$, hence $v \in T_mN \cap T_mN^\perp$. Then $g(v, w) = 0$, $\forall w \in T_mN$, namely $g|_{T_mN}$ is degenerate. $\square$

Lemma 1.1.1.5. *Let M be a connected manifold. If $A\colon M \to M_n(\mathbb{R})$ is a continuous function such that A_m is a non-degenerate, symmetric matrix, $\forall m \in M$, the signature of A_m is constant over M.*

In other words, in order to change signature, A_m has to be degenerate at least at one point.

Proof. Let $I_m = (i_+, i_-, i_0)$ the signature of A_m.

As A_m is symmetric, it is diagonalizable. Denote $\lambda_1, \ldots, \lambda_n$ its eigenvalues, counted with multiplicities, then

$$\begin{aligned} i_+ &= |\{\lambda_j > 0\}|; \\ i_- &= |\{\lambda_j < 0\}|; \\ i_0 &= |\{\lambda_j = 0\}| = n - i_+ - i_-. \end{aligned}$$

The eigenvalues depend continuously on m: indeed, they depend continuously on the coefficients of the characteristic polynomial, as they are its zeros, and the coefficients depend continuously on the matrix A_m, as they are found by computing the determinant of $A_m - t \cdot Id$.

Assume the signature changes, hence at least one eigenvalue has to change sign. This implies that it vanishes at one point $\bar{m}$, *i.e.* $A_{\bar{m}}$ is degenerate, which contradicts the hypothesis.

$\square$

Proposition 1.1.1.6. *Let (M, g) a pseudo-Riemannian manifold. For a smooth connected submanifold $N \subseteq M$, the following statements are equivalent:*

i. *N is a pseudo-Riemannian submanifold of M;*

ii. *$g|_{T_mN}$ is non-degenerate $\forall m \in N$;*

iii. *$T_mN \cap T_mN^{\perp} = \{0\}$, that is $T_mN \oplus T_mN^{\perp} = T_mM$, $\forall m \in N$;*

iv. *$g|_{T_mN^{\perp}}$ is non-degenerate $\forall m \in N$.*

Proof. *(i)* $\Rightarrow$ *(ii)* comes straight from the definition of pseudo-Riemannian submanifold, as a pseudo-metric is non-degenerate everywhere.

(i) $\Leftarrow$ *(ii)* is a direct corollary of Lemma 1.1.1.5: in local coordinates the restriction of the pseudo-metric can be seen as a smooth function $N \to M_k(\mathbb{R})$, $k = \dim N$. By hypothesis, $g|_{T_mN}$ is non-degenerate and N is connected, hence the lemma states that the signature is constant over N, *i.e.* N is a pseudo-Riemannian submanifold of M.

(ii) $\iff$ *(iii)* is a trivial exercise of linear algebra, as

$$T_mN \cap T_mN^{\perp} = \{w \in T_mN | g(w, v) = 0, \forall v \in T_mN\}.$$

(iii) $\iff$ *(iv)* is the same as above, due to the symmetry of the statements. $\square$

1.1.2 Isometries

We are interested in maps preserving the pseudo-Riemannian structure. They enable to study a pseudo-Riemannian manifold through another one, to induce pseudo-Riemannian structures on other manifolds and to understand symmetries of spaces.

Definition 1.1.2.1 (Pullback and push-forward). Let (M, g), (N, h) be two pseudo-Riemannian manifold, $\phi\colon M \to N$ a smooth diffeomorphism. ϕ induces two pseudo-Riemannian metric ϕ^*h on M and ϕ_*g N.

1. For $v, w \in T_mM$, we define the *pullback* of h as

$$(\phi^*h)_m := h_{\phi(m)}(d\phi_m v, d\phi_m w).$$

2. The *push-forward* of g is the pullback of the inverse map, namely $\phi_*g = (\phi^{-1})^*h$.

Definition 1.1.2.2 (Isometry). Let (M, g), (N, h) be two pseudo-Riemannian manifolds and $\phi\colon M \to N$ a diffeomorphism. ϕ is an *isometry* if $\phi^*h = g$ (or equivalenetly $\phi_*g = h$).

The isometries of M in itself form a group, with the composition as group operation, noted $\mathrm{Isom}(M)$ and called *isometry group of M*.

Remark 1.1.2.3. Let $(M, g), (N, h)$ be two pseudo-Riemannian manifolds, $\phi\colon M \to N$ a diffeomorphism. Consider a local chart (U, x) of M. Up to restrain U, we can assume it exists a local chart (V, y) of N containing $\phi(U)$.

Denote $G = (g_{ij})$ the matrix representing the pseudo-metric g in the basis induced by x, $H = (h_{ij})$ the analogue for h, and Φ the basis representing $d\phi_m$ in the basis induced by x in the domain and by y in the codomain. $\phi\colon U \to \phi(U)$ is an isometry if and only if

$$ {}^t\Phi_m H_{\phi(m)} \Phi_m = G_m, \qquad \forall m \in U. \tag{1.1}$$

As a consequence, $\phi\colon M \to N$ is an isometry if and only if it exists an atlas $\mathcal{A} = \{(U_i, x_i), i \in I\}$ of M such that (1.1) holds for all charts $(U_i, x_i) \in \mathcal{A}$.

Example 1.1.2.4 (Isometries of $\mathbb{R}^{p,q}$). Consider $\mathbb{R}^{p+q}$ in Cartesian coordinates, which is a global chart for the manifold. In this chart, the pseudo-metric can be written as

$$\langle v, w\rangle_{p,q} = {}^t v \begin{pmatrix} \mathrm{I}_p & \\ & -\mathrm{I}_q \end{pmatrix} w,$$

I_n being the identity matrix of dimension n.

The *pseudo-orthogonal* group of signature (p, q) is defined as

$$\mathrm{O}(p, q) := \left\{ A \in \mathrm{M}_{p+q}(\mathbb{R}), {}^t A \begin{pmatrix} \mathrm{I}_p & \\ & -\mathrm{I}_q \end{pmatrix} A = \begin{pmatrix} \mathrm{I}_p & \\ & -\mathrm{I}_q \end{pmatrix} \right\}.$$

By Remark 1.1.2.3, $\mathrm{O}(p, q) \subseteq \mathrm{Isom}(\mathbb{R}^{p,q})$, and all the affine maps having a pseudo-orthogonal matrix as linear part are isometries, too. Proposition 1.4.4.7 proves that they are the only isometries of $\mathbb{R}^{p,q}$.

Most of the properties we deal with are local, so we are interested also in maps preserving the pseudo-Riemannian structure only on portions of the spaces.

Definition 1.1.2.5 (Local isometry). Let (M, g), (N, h) be two pseudo-Riemannian manifolds, $U \subseteq M$ an open set, $\phi\colon U \to N$ a smooth function.

ϕ is a *local isometry* if $\forall m \in U$ it exists an open neighbourhood U_m of m such that $\phi|_{U_m}\colon U_m \to \phi(U_m)$ is an isometry.

We recall an important result about group action, whose proof can be found in [GHL04, Thm 49b, p.49].

Theorem 1.1.2.6. *Let M be a differentiable manifold, $G \leq \mathrm{Diff}(M)$ a group acting properly and freely, then $p\colon M \to M/G$ is a smooth covering map.*

Corollary 1.1.2.7. *Let (M, g) be a pseudo-Riemannian manifold, $G \leq \mathrm{Isom}(M)$ a group with the same properties as above, then $p\colon M \to M/G$ induces a structure of pseudo-Riemannian manifold over M/G, which is locally isomorphic to M.*

Proof. Let $U \subseteq M$ be a sheet of the covering space, then $p|_U \colon U \to p(U)$ is a diffeomorphism. Hence, it is well defined $(p|_U)_* g$, which is a pseudo-metric over U, and $p\colon (U, g) \to (p(U), (p|_U)_* g)$ is an isometry.

Let $V \subseteq M$ another sheet, such that $p(V) = p(U)$. By definition, up to restrain U and V, there is an isometry $\phi \in G$ such that $\phi(V) = U$. Remarking that $p \circ \phi = p$ and that the composition of isometries is an isometry, we have the following commutative diagram of isometries:

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[dr, "p"'] & U \arrow[d, "p"] \\ & p(V) = p(U) \end{tikzcd},$$

then $(p(V), (p|_V)_* g) = (p(U), (p|_U)_* g)$, that is $p_* g$ is well defined over M/G. $\square$

Example 1.1.2.8 (Pseudo-hyperbolic space). The pseudo-hyperboloid $\widetilde{\mathbb{H}}^{p,q}$ (see Figure 1.3) is the sphere of negative radius -1 in $\mathbb{R}^{p,q+1}$, with respect to the pseudo-metric, that is

$$\widetilde{\mathbb{H}}^{p,q} := \{v \in \mathbb{R}^{p,q+1}, \langle v, v\rangle_{p,q+1} = -1\}.$$

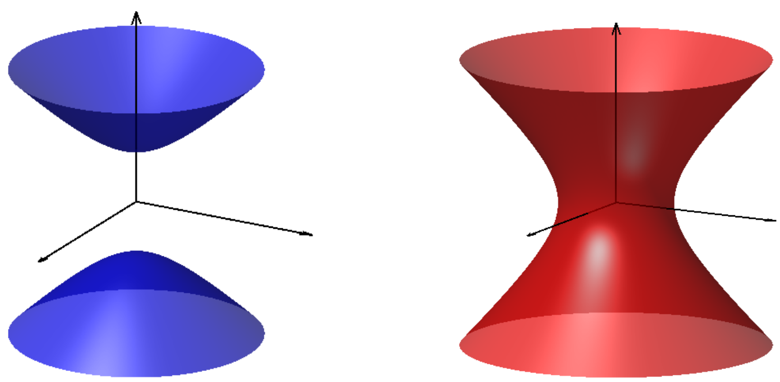

Figure 1.3: The Riemannian manifold $\widetilde{\mathbb{H}}^{2,0}$ (left) and the Lorentzian manifold $\widetilde{\mathbb{S}}^{1,1}$ (right). Both are pseudo-Riemannian submanifold of $\mathbb{R}^{2,1}$.

We recall that $T_v\widetilde{\mathbb{H}}^{p,q}$ is the kernel of $df(v)$, f being the submersion such that $f^{-1}(0) = \widetilde{\mathbb{H}}^{p,q}$, namely

$$f(v) = 1 + \langle v, v\rangle_{p,q+1} = 1 + x_1^2 + \ldots + x_p^2 - y_1^2 - \ldots - y_{q+1}^2.$$

Then $df(v) = 2(x_1, \ldots, x_p, -y_1, \ldots, -y_{q+1})$, and then $\ker(df(v)) = v^\perp$, where $v^\perp = \{w \in \mathbb{R}^{p,q+1}, \langle v, w\rangle_{p,q+1} = 0\}$. By definition, v is timelike, so $\widetilde{\mathbb{H}}^{p,q}$ is a pseudo-Riemannian submanifold of $\mathbb{R}^{p,q+1}$, with signature (p, q).

The pseudo-hyperbolic space is the projectivization of $\widetilde{\mathbb{H}}^{p,q}$, via the local isometry $\mathbb{P}\colon \mathbb{R}^{p,q+1} \to \mathbb{P}(\mathbb{R}^{p,q+1})$, namely

$$(\mathbb{H}^{p,q}, h) := \left(\mathbb{P}(\widetilde{\mathbb{H}}^{p,q}), \mathbb{P}_*\langle\cdot,\cdot\rangle_{p,q+1}|_{\widetilde{\mathbb{H}}^{p,q}}\right).$$

In other words, $\mathbb{H}^{p,q}$ is the quotient of $\widetilde{\mathbb{H}}^{p,q}$ by the action of $\{\pm \mathrm{Id}\}$. $\{\pm \mathrm{Id}\}$ respects the hypothesis of Corollary 1.1.2.7, hence $\mathbb{H}^{p,q}$ inherits a pseudo-Riemannian structure from $\widetilde{\mathbb{H}}^{p,q}$. Remarking that $\widetilde{\mathbb{H}}^{p,q}$ is defined by the equation $g(v,v) = -1$, which is bilinear, it is immediately checked that

$$\mathbb{H}^{p,q} = \mathbb{P}\left(\{v \in \mathbb{R}^{p,q+1}, \langle v,v\rangle_{p,q+1} < 0\}\right).$$

For the sake of completeness, we remark that $\widetilde{\mathbb{H}}^{p,q} \cong \mathbb{R}^p \times \mathbb{S}^q$ (see [O'N, Lemma 25, pp.110-111]), which is $(q-1)$-connected. In particular, this implies that the half-space model $\mathcal{H}^{p,q}$ (see Definition 2.1.1.1) is not a complete model for the pseudo-hyperbolic space for $q \neq 0$: indeed, $\pi_1(\mathbb{H}^{p,1}) = \mathbb{Z}$ and $\pi_1(\mathbb{H}^{p,q}) = \mathbb{Z}_2$ for $q > 1$, that is $\mathbb{H}^{p,q}$ is not simply connected for $q \neq 0$.

Example 1.1.2.9 (Pseudo-spheric space). The pseudo-sphere $\widetilde{\mathbb{S}}^{p,q} \subseteq \mathbb{R}^{p+1,q}$ positive counterpart of the pseudo-hyperboloid, that is

$$\widetilde{\mathbb{S}}^{p,q} := \{v \in \mathbb{R}^{p+1,q}, \langle v,v\rangle_{p+1,q} = 1\}.$$

Remarking that $\mathbb{R}^{p,q} = -\mathbb{R}^{q,p}$, that is an abusively way to write

$$(\mathbb{R}^{p,q}, \langle\cdot,\cdot\rangle_{p,q}) = (\mathbb{R}^{q,p}, -\langle\cdot,\cdot\rangle_{q,p}),$$

$\widetilde{\mathbb{S}}^{p,q} = -\widetilde{\mathbb{H}}^{q,p}$, so everything we wrote above extends to. The pseudo-spheric space is then $\mathbb{S}^{p,q} := -\mathbb{H}^{q,p}$ (see Figure 1.3).

Both spaces have been widely studied in Riemannian geometry and Lorentzian geometry. In the latter setting, they are called *de Sitter space*, noted $\mathrm{d}\mathbb{S}^{n+1} = \mathbb{S}^{n,1}$, and *Anti-de Sitter space*, noted $\mathbb{A}\mathrm{d}\mathbb{S}^{n+1} = \mathbb{H}^{n,1}$.

Pseudo-Riemannian submanifolds can inherits isometries by the environmental space, as explained in the following result.

Proposition 1.1.2.10. *Let (M,g) be a pseudo-Riemannian manifold and $N \subseteq M$ be a pseudo-Riemannian submanifold. If $\phi \in \mathrm{Isom}(M)$ is such that $\phi(N) = N$, then $\phi|_N \in \mathrm{Isom}(N)$.*

Proof. By hypothesis $\phi|_N$ is a smooth automorphism of N, then ϕ preserves TN, namely $\phi^*(TN) = TN$ (see Definition 1.1.2.1). The pseudometric on N is defined as $g|_{TN}$ and $\phi^*g = g$, hence $\phi^*(g|_{TN}) = (\phi^*g)|_{TN} = g|_{TN}$, *i.e.* $\phi|_N \in \mathrm{Isom}(N)$. ☐

Remark 1.1.2.11. Not all isometries of a submanifold can be obtained as restriction of isometries of the environmental manifold.

An easy counter-example is given by $M = \mathbb{R}^2 \setminus \{(0,0),(2,0),(0,2)\}$, endowed with the Euclidean metric: $\mathrm{Isom}(M) = \{\mathrm{Id}_M\}$, but $\mathbb{S}^1 \subseteq M$ has not trivial isometry group.

More generally, let $M = \mathbb{R}^n \setminus \{x_0, \dots, x_n\}$, x_i in general position, so that $\mathrm{Isom}(M) = \{\mathrm{Id}_M\}$. Consider a submanifold $N \subseteq \mathbb{R}^n$ whose isometry group is not trivial and $N \cap \{x_0, \dots, x_n\} \neq \emptyset$. Then N is a submanifold of M but its isometry group is not a subgroup of $\mathrm{Isom}(M)$.

Example 1.1.2.12 (Isometries of $\mathbb{H}^{p,q}$). We showed in Example 1.1.2.4 that $\mathrm{O}(p,q+1) \subseteq \mathrm{Isom}(\mathbb{R}^{p,q+1})$. By definition, it preserves $\widetilde{\mathbb{H}}^{p,q}$, hence

$$\mathrm{O}(p,q+1)|_{\widetilde{\mathbb{H}}^{p,q}} \subseteq \mathrm{Isom}(\widetilde{\mathbb{H}}^{p,q}).$$

The converse inclusion follows from Proposition 1.4.4.7. Indeed, take $v \in \widetilde{\mathbb{H}}^{p,q}$, that is $\langle v, v\rangle_{p,q+1} = -1$, and remark that $T_v\widetilde{\mathbb{H}}^{p,q} = v^\perp$, then if $\mathcal{B}_v$ is an orthonormal basis of $T_v\widetilde{\mathbb{H}}^{p,q}$ (with respect to the induced pseudo-metric), $\bar{\mathcal{B}}_v := \{v\} \cup \mathcal{B}_v$ is an orthonormal basis of $\mathbb{R}^{p,q+1}$ with respect to $\langle\cdot,\cdot\rangle_{p,q+1}$, and it contains v. $\mathrm{O}(p,q+1)$ is the space of orthogonal transformation of $\mathbb{R}^{p,q+1}$, namely it switches orthogonal basis, hence it exists a linear map in $\mathrm{O}(p,q+1)$ sending $\bar{\mathcal{B}}_v$ to $\bar{\mathcal{B}}_w$, and in particular v to w. We remark that the map depends on the basis chosen.

We claim that $\mathrm{Stab}_{\mathrm{O}(p,q+1)}(v) = \mathrm{Stab}_{\mathrm{Isom}(\widetilde{\mathbb{H}}^{p,q})}(v)$ (see Definition 2.1.3.8): it follows by transitivity that $\mathrm{O}(p,q+1) = \mathrm{Isom}(\widetilde{\mathbb{H}}^{p,q})$. To show the claim, consider the linear parameterization of $\mathbb{R}^{p,q+1}$ induced by the basis $\bar{\mathcal{B}}_v$. Noted g the metric of $\widetilde{\mathbb{H}}^{p,q}$, it is clear that the matrix associated to $\langle\cdot,\cdot\rangle_{p,q+1}$ in this setting is, up to permutation,

$$(\langle\cdot,\cdot\rangle_{p,q+1})_v = \begin{pmatrix} -1 & & \\ & \mathrm{I}_p & \\ & & -\mathrm{I}_q \end{pmatrix} = \begin{pmatrix} -1 & \\ & g_v \end{pmatrix}.$$

So if $\phi \in \mathrm{Stab}_{\mathrm{Isom}(\widetilde{\mathbb{H}}^{p,q})}(v)$, then $d\phi_v \in \mathrm{O}(p,q)$. Note A the matrix of $d\phi_v$ with respect to $\mathcal{B}_v$ and define the linear map

$$\Phi := \begin{pmatrix} -1 & \\ & A \end{pmatrix}.$$

Clearly, $\Phi \in \mathrm{O}(p,q+1)$. Hence, by Proposition 1.4.4.7, $\phi = \Phi$, which proves the claim. Then $\mathbb{P}\mathrm{O}(p,q+1) = \mathrm{O}(p,q+1)/\{\pm\mathrm{Id}\}$ is the isometry group of $\mathbb{H}^{p,q}$ and it acts transitively.

1.2 Levi-Civita connection

One powerful tool in Riemannian geometry is Levi-Civita connection, namely the only connection torsion-free and compatible with the metric. In fact, it allows to define curvature and geodesics, so we would gladly extend it.

1.2.1 Vector fields

Definition 1.2.1.1 (Vector field). A *vector field* on a smooth manifold M is a smooth map $X\colon M \to TM$ such that $X_m \in T_mM$, $\forall m \in M$. We denote $\Gamma(TM)$ the space of vector fields of M.

A vector field is then a smooth section of the bundle TM, *i.e* the canonical projection $TM \to M$, so $\Gamma(TM)$ is both a $\mathbb{R}$-vector space and $C^\infty(M)$-module, where sum and product are defined punctually.

We recall that a vector field $X \in \Gamma(TM)$ induces a derivative on $C^\infty(M, N)$: indeed, at a point m, it is the directional derivative along the vector X_m. For a smooth map $f\colon M \to N$, we will write $X(f)(m) = df_m X_m$.

Definition 1.2.1.2 (Push-forward). Let M, N be two smooth manifolds. A diffeomorphism $\phi\colon M \to N$ induces a bundle map $\phi_*\colon \Gamma(TM) \to \Gamma(TN)$, called *push-forward*, defined as

$$\begin{aligned} \phi_*\colon \Gamma(TM) &\to \Gamma(TN) \\ X = (X_m)_{m\in M} &\mapsto \phi_* X = \left(d\phi_{\phi^{-1}(n)} X_{\phi^{-1}(n)}\right)_{n\in N}. \end{aligned}$$

Remark 1.2.1.3. An equivalent definition it is the following: for $f \in C^\infty(N)$,

$$\phi_* X(f) = X(f \circ \phi) \circ \phi^{-1}. \tag{1.2}$$

In fact, by definition,

$$\begin{aligned} \left(X(f\circ\phi)\circ\phi^{-1}\right)_n &= d(f\circ\phi)_{\phi^{-1}(n)} X_{\phi^{-1}(n)} = \\ &= df_{\phi(\phi^{-1}(n))} d\phi_{\phi^{-1}(n)} X_{\phi^{-1}(n)} = \\ &= df_n d\phi_{\phi^{-1}(n)} X_{\phi^{-1}(n)} = \phi_* X(f)_n. \end{aligned}$$

Definition 1.2.1.4 (Pull-back). Let M, N be two smooth manifolds. A smooth map $\phi\colon M \to N$ induces a bundle map $\phi^*\colon \Gamma(TN) \to \Gamma(TM)$, called *pull-back*, defined as

$$\begin{aligned} \phi^*\colon \Gamma(TN) &\to \Gamma(TM) \\ Y = (Y_n)_{n\in N} &\mapsto \phi^* Y = \left(d\phi_m Y_{\phi(m)}\right)_{m\in M}. \end{aligned}$$

Definition 1.2.1.5 (Lie brackets). Let $X, Y \in \Gamma(TM)$, we define their *Lie bracket* as $[X, Y] := X(Y) - Y(X)$.

Proposition 1.2.1.6. *Lie brackets commutes with push-forward, namely is $\phi\colon M \to N$ is a diffemorphism, $\phi_*[X, Y] = [\phi_* X, \phi_* Y]$, $\forall X, Y \in \Gamma(TM)$,*

Proof. The thesis comes directly using the definition of push-forward as in (1.2): take $f \in C^\infty(N)$, $X, Y \in \Gamma(TM)$. Remarking that $\phi_* Y(f)$ belongs to $C^\infty(N)$, one has

$$\begin{aligned} \phi_* X(\phi_* Y(f)) &= X(\phi_* Y(f) \circ \phi) \circ \phi^{-1} = X\left(Y(f\circ\phi)\circ\phi^{-1}\circ\phi\right)\circ\phi^{-1} = \\ &= X(Y(f\circ\phi))\circ\phi^{-1} = X(Y)(f\circ\phi)\circ\phi^{-1} = \phi_*(X(Y)), \end{aligned}$$

which ends the proof. $\square$

1.2.2 Connections and Levi-Civita theorem

Definition 1.2.2.1 (Connection). A *(affine) connection* on a manifold M is a function

$$\begin{aligned}\nabla\colon \Gamma(TM)\times\Gamma(TM) &\to \Gamma(TM)\\ (X,Y) &\mapsto \nabla_X Y\end{aligned}$$

which satisfies the following properties:

1. $C^\infty(M)$-linear on the first argument, that is
$$\nabla_{(fX+Z)}Y = f\nabla_X Y + \nabla_Z Y, \qquad \begin{matrix}\forall f\in C^\infty(M)\\ \forall X,Y,Z\in\Gamma(TM)\end{matrix};$$

2. a derivative on the second argument, that is
$$\nabla_X(fY+Z) = f\nabla_X Y + X(f)Y + \nabla_X Z, \quad \begin{matrix}\forall f\in C^\infty(M)\\ \forall X,Y,Z\in\Gamma(TM)\end{matrix}.$$

This definition only depends on the differential structure on the manifold, so it is well defined in the pseudo-Riemannian domain, too.

Definition 1.2.2.2 (Torsion-free). A connection ∇ is *torsion-free* if

$$\nabla_X Y - \nabla_Y X = [X,Y],\ \forall X,Y\in\Gamma(TM).$$

Definition 1.2.2.3 (Compatible). A connection ∇ is *compatible with the metric* if

$$Z(g(X,Y)) = g(\nabla_Z X,Y) + g(X,\nabla_Z Y),\ \forall X,Y,Z\in\Gamma(TM).$$

Theorem 1.2.2.4 (Levi-Civita). *Let (M,g) be a pseudo-Riemannian manifold. It exists a unique connection torsion-free and compatible with the pseudo-metric, called Levi-Civita connection.*

Proof. First, we prove that it exists at most one torsion-free connection compatible with the metric over the same pseudo-Riemannian manifold.

Lemma 1.2.2.5. *Let (M,g) be a pseudo-Riemannian manifold, ∇ be a torsion-free connection compatible with the metric, then*

$$g(\nabla_X Y,Z) = \mathrm{Kos}(X,Y,Z), \qquad \forall X,Y,Z\in\Gamma(TM),$$

where $\mathrm{Kos}\colon\Gamma(TM)^3\to\mathbb{R}$ *is defined by* Koszul formula*, that is*

$$\begin{aligned}\mathrm{Kos}(X,Y,Z) :=& \frac{1}{2}\Big(X(g(Y,Z)) + Y(g(X,Z)) - Z(g(Y,X))\\ &- g([Y,X],Z) - g([X,Z],Y) - g([Y,Z],X)\Big).\end{aligned} \tag{1.3}$$

Proof. As ∇ is compatible with the metric, $\forall X, Y, Z \in \Gamma(TM)$ we can write

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{1.4}$$
$$Yg(X,Z) = g(\nabla_Y X, Z) + g(X, \nabla_Y Z) \tag{1.5}$$
$$Zg(Y,X) = g(\nabla_Z Y, X) + g(Y, \nabla_Z X) \tag{1.6}$$

Computing (1.4) + (1.5) − (1.6), since ∇ is torsion-free, we obtain

$$\begin{aligned}
&Xg(Y,Z) + Yg(X,Z) - Zg(Y,X) = \\
&= g(\nabla_X Y + \underbrace{\nabla_Y X}_{=\nabla_X Y + [Y,X]}, Z) + g(\underbrace{\nabla_X Z - \nabla_Z X}_{[X,Z]}, Y) + g(\underbrace{\nabla_Y Z - \nabla_Z Y}_{[Y,Z]}, X) = \\
&= 2g(\nabla_X Y, Z) + g([Y,X], Z) + g([X,Z], Y) + g([Y,Z], X).
\end{aligned}$$

Rearranging terms we obtain $g(\nabla_X Y, Z) = \mathrm{Kos}(X,Y,Z)$. □

The proof comes straight from Lemma 1.2.2.5: g being non-degenerate, that scripture defines uniquely the vector field $\nabla_X Y$, so ∇ is unique. □

Before starting with the second part of the proof, we report some result useful for the following.

Lemma 1.2.2.6. *The Lie bracket is a bilinear map* $\Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$.

Proof. Let X, Y, Z be vector fields on M, $a \in \mathbb{R}$,

$$\begin{aligned}
[aX + Z, Y] =& (aX + Z)(Y) - Y(aX + Z) = \\
=& aX(Y) + Z(Y) - aY(X) - Y(Z) = a[X,Y] + [Z,Y].
\end{aligned}$$

As Lie bracket is anti-symmetric, that ends the proof. □

Lemma 1.2.2.7. *Let* $f \in C^\infty(M)$, $X, Y \in \Gamma(TM)$, *then*

$$[fX, Y] = f[X,Y] - Y(f)X.$$

Proof. $[fX, Y] = fX(Y) - Y(fX) =$
$$= fX(Y) - fY(X) - Y(f)X = f[X,Y] - Y(f)X.$$
□

Lemma 1.2.2.8. *Let* $f \in C^\infty(M)$, $X, Y, Z \in \Gamma(TM)$, *then*

$$Yg(fX, Z) = Y(f)g(X,Z) + fYg(X,Z).$$

Proof. $Yg(fX,Z) = Y(fg(X,Z)) = Y(f)g(X,Z) + fYg(X,Z)$. □

Proposition 1.2.2.9. *Let M be a smooth manifold, let*

$$P\colon \Gamma(TM) \times \Gamma(TM)^k \to \Gamma(TM)^h$$

a smooth $\mathbb{R}$-linear map. If P is $C^\infty(M)$-linear on the first argument, then P only depends on the punctual value with respect to the first argument.

In other words, set $m \in M$, if $X, Z \in \Gamma(TM)$ are two vector fields such that $X_m = Z_m$, then $P(X,Y)_m = P(Z,Y)_m$, $\forall Y \in \Gamma(TM)^k$.

Remark 1.2.2.10. This is a specific case of a more general result, that can be found for instance [GHL04, Thm 1.114, p.40]. The original statement involves tensors, which have not been introducted in these notes. For this reason we present the theorem in this way.

Proof. It suffices to prove that $P(X,Y)|_m = 0$ for any vector field X such that $X_m = 0$.

Let X be such a vector field. In a local chart it can be written as

$$X = \sum_{i=1}^n X_i \partial_i.$$

Using $\mathbb{R}$-linearity and $\mathbb{C}^\infty(M)$-linearity on the first argument, one has

$$P(X,Y) = \sum_{i=1}^n P(X_i\partial_i, Y) = \sum_{i=1}^n X_i P(\partial_i, Y).$$

Since $X_m = 0$, $X_i(m) = 0$, $\forall i = 1, \dots, n$, hence

$$P(X,Y)_m = \sum_{i=1}^n X_i(m) P(\partial_i, Y)_m = 0,$$

. ☐

Remark 1.2.2.11. The statement of the proposition can be reformulated, saying that $\forall Y \in \Gamma(TM)^k$, P induces a collection of linear maps

$$\begin{aligned} P(\cdot, Y)_m \colon T_mM &\to T_mM \\ Z_m &\mapsto P(Z,Y)_m, \end{aligned}$$

smoothly depending on m.

Now we can return to the proof.

Proof. We need to find a connection satisfing the properties of symmetry and compatibility with the pseudo-metric. We proved that if the connection exists, it must satisfy Koszul formula (1.3), so we want to build a connection such that $g(\nabla_X Y, Z) = \mathrm{Kos}(X,Y,Z)$, and then prove that it defines a torsion-free connection compatible with the pseudo-metric.

It is clear that $\mathrm{Kos}\colon \Gamma(TM)^3 \to C^\infty(M)$ is a $\mathbb{R}$-multilinear function: indeed, every term of the sum is trivially $\mathbb{R}$-linear in every argument. Moreover, the maps $\mathrm{Kos}(\cdot, Y, Z)$ and $\mathrm{Kos}(X, Y, \cdot)$ are $C^\infty(M)$-linear, $\forall X, Y, Z \in \Gamma(TM)$.

Let's prove the former case, *i.e.* $\mathrm{Kos}(fX, Y, Z) = f\mathrm{Kos}(X, Y, Z)$.

$$\begin{aligned} 2\mathrm{Kos}(fX, Y, Z) = & fX(g(Y, Z)) + Y(g(fX, Z)) - Z(g(Y, fX)) \\ & - g([Y, fX], Z) - g([fX, Z], Y) - g([Y, Z], fX). \end{aligned}$$

Green terms are clearly $C^\infty(M)$-linear. The sum of red ones is, too:

$$\begin{aligned} & Y(g(fX, Z)) - g([Y, fX], Z) = \\ & = Y(f)g(X, Z) + fYg(X, Z) - g(Y(f)X, Z) - fg([Y, X], Z) = \\ & = fYg(X, Z) - fg([Y, X], Z). \end{aligned}$$

The same calculation holds for blue terms, hence $\mathrm{Kos}(\cdot, Y, Z)$ is $C^\infty(M)$-linear.

The latter case is proved alike, with a suitable choice of the couples, *i.e.*

$$\begin{aligned} 2\mathrm{Kos}(X, Y, fZ) = & X(g(Y, fZ)) + Y(g(X, fZ)) - fZ(g(Y, X)) \\ & - g([Y, X], fZ) - g([X, fZ], Y) - g([Y, fZ], X). \end{aligned}$$

Set $X, Y \in \Gamma(TM)$, since the map $Z \mapsto \mathrm{Kos}(X, Y, Z)$ is $C^\infty(M)$-linear, by Remark 1.2.2.11 it exists a smooth map $\Phi_{(X,Y)}\colon \Gamma(TM) \to C^\infty(M)$ such that

- $\Phi_{(X,Y)}|_{T_mM}\colon T_mM \to \mathbb{R}$ is $\mathbb{R}$-linear, $\forall m \in M$;
- $\Phi_{(X,Y)}(Z) = \mathrm{Kos}(X, Y, Z)$, $\forall Z \in T_mM$.

g being non-degenerate, $\forall m \in M$, it exists a unique vector $V_m^{(X,Y)} \in T_mM$ such that

$$\Phi_{(X,Y)}|_{T_mM}(Z_m) = g(V_m^{(X,Y)}, Z_m), \quad \forall Z \in T_mM.$$

$\Phi_{(X,Y)}$ smoothly depends on m, hence $V^{(X,Y)}$ is a vector field. We define then $\nabla_X Y := V^{(X,Y)}$.

Since g is non-degenerate, the fact that $X \mapsto \mathrm{Kos}(X, Y, Z) = g(\nabla_X Y, Z)$ is $C^\infty(M)$-linear $\forall Z \in \Gamma(TM)$ suggests that $X \mapsto \nabla_X Y$ is $C^\infty(M)$-linear, too. Indeed, let $f \in C^\infty(M)$, $X, Y, Z \in \Gamma(TM)$,

$$g(\nabla_{fX} Y, Z) = \mathrm{Kos}(fX, Y, Z) = f\mathrm{Kos}(X, Y, Z) = fg(\nabla_X Y, Z).$$

The same argument will be used repeatedly to prove properties of ∇ via Kos. Indeed, $Y \mapsto \mathrm{Kos}(X, Y, Z)$ is $\mathbb{R}$-linear, and so is $Y \mapsto \nabla_X Y$.

In order to prove this map to be a connection, one must check Liebniz rule on the second one, *i.e.* $\nabla_X(fY) = f\nabla_X Y + X(f)Y$. Once again, it suffices to show $\mathrm{Kos}(X, fY, Z) = f\mathrm{Kos}(X, Y, Z) + X(f)g(Y, Z)$.

$$\begin{aligned}2\mathrm{Kos}(X, fY, Z) =&X(g(fY, Z))+fY(g(X, Z))-Z(g(fY, X))\\&-g([fY, X], Z)-g([X, Z], fY)-g([fY, Z], X).\end{aligned}$$

As above, green terms are C^∞-linear. The blue ones are the same as above, up to switch X and Y, so their sum is C^∞-linear, too. On the other hand, the sum of red ones becomes

$$\begin{aligned}&Xg(fY, Z) - g([fY, X], Z) =\\&= X(f)g(Y, Z) + fXg(Y, Z) + g(X(f)Y, Z) - fg([Y, X], Z) =\\&= fXg(Y, Z) - fg([Y, X], Z) + 2g(X(f)Y, Z),\end{aligned}$$

which proves that ∇ is a connection.

The connection is torsion-free: indeed, using Koszul formula (1.3), one easily computes

$$g(\nabla_X Y, Z) - g(\nabla_Y X, Z) = -\frac{1}{2}g([Y, X], Z) + \frac{1}{2}g([X, Y], Z).$$

By definition $[Y, X] = -[X, Y]$, so $g(\nabla_X Y - \nabla_Y X, Z) = g([X, Y], Z)$.

Finally, a direct computation shows that ∇ is compatible with the pseudo-metric. □

Remark 1.2.2.12. A direct consequence of Koszul formula is that (M, g) and $(M, \lambda g)$, $\lambda \neq 0$, have the same Levi-Civita connection.

There is another interesting corollary due to Koszul formula, that is that Levi-Civita connection commutes with the push-forward.

Corollary 1.2.2.13. *Let (M, g), (N, h) be two pseudo-Riemannian manifold, ∇^M, ∇^N their Levi-Civita connections. If $\phi\colon M \to N$ is an isometry,*

$$\nabla^N_{\phi_* X}\phi_* Y = \phi_*\nabla^M_X Y, \quad \forall X, Y \in \Gamma(TM).$$

Proof. h is non-degenerate, hence it suffices to show that, $\forall \tilde{Z} \in \Gamma(TN)$,

$$h(\nabla^N_{\phi_* X}\phi_* Y, \tilde{Z}) = h(\phi_*\nabla^M_X Y, \tilde{Z}). \tag{1.7}$$

ϕ is a diffeomorphism, so we can write $\tilde{Z} = \phi_* Z$, $Z \in \Gamma(TM)$. Hence, the right term of (1.7) becomes

$$h(\phi_*\nabla^M_X Y, \tilde{Z}) = h(\phi_*\nabla^M_X Y, \phi_* Z) = g(\nabla^M_X Y, Z) = \mathrm{Kos}_M(X, Y, Z).$$

The left term is in fact $\mathrm{Kos}_N(\phi_* X, \phi_* Y, \phi_* Z)$. In Formula (1.3), up to switch the vector fields, there are two kind of terms:

- $\phi_* X(h(\phi_* Y, \phi_* Z)) = \phi_* Xg(Y, Z)$, as ϕ is an isometry;
- $h([\phi_* Y, \phi_* X], \phi_* Z) = h(\phi_*[Y, X], \phi_* Z) = g([Y, X], Z)$, using Proposition 1.2.1.6 and again the fact that ϕ is an isometry.

Hence $\mathrm{Kos}_N(\phi_* X, \phi_* Y, \phi_* Z) = \mathrm{Kos}_M(X, Y, Z)$, which ends the proof. □

1.2.3 Christoffel symbols

We need a way to write Levi-Civita connection explicitely, using Koszul formula on the local basis of TM induced by a local parameterization.

Definition 1.2.3.1 (Christoffel symbols). Let (U, x) be a local chart, $(\partial_1, \dots, \partial_n)$ the induced basis on TU. *Christoffel symbols* are the smooth functions $(\Gamma_{ij}^k)_{i,j,k=1}^n$ such that

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^n \Gamma_{ij}^k \partial_k. \tag{1.8}$$

Lemma 1.2.3.2. $\Gamma_{ji}^k = \Gamma_{ij}^k$ $k = 1, \dots, n$.

Proof. $\nabla_{\partial_j}\partial_i = \nabla_{\partial_i}\partial_j + [\partial_i, \partial_j]$ and, thanks to Schwarz theorem,

$$[\partial_i, \partial_j] = \frac{\partial^2}{\partial_i \partial_j} - \frac{\partial^2}{\partial_j \partial_i} = 0.$$

Hence $\nabla_{\partial_j}\partial_i = \nabla_{\partial_i}\partial_j$, and it follows that their coefficients are equal. □

Christoffel symbols permit to have an explicit formula for the connection.

Proposition 1.2.3.3. *Let (M, g) be a pseudo-Riemannian manifold, ∇ its Levi-Civita connection. Let (U, x) be a local chart, and take $X, Y \in \Gamma(TM)$.*

$$\nabla_X Y = X(Y) + \sum_{i,j,k=1}^n X_i Y_j \Gamma_{ij}^k \partial_k,$$

over U, where $X_i, Y_j \in C^\infty(U)$ are the coefficients of X, Y with respect to the basis induced by the chart, namely $X = \sum X_i \partial_i$, $Y = \sum Y_j \partial_j$.

Proof. It suffices to compute the connection using the local parameterization:

$$\begin{aligned}\nabla_X Y &= \sum_{i=1}^n X_i \nabla_{\partial_i} Y = \sum_{i=1}^n X_i \sum_{j=1}^n \nabla_{\partial_i}(Y_j \partial_j) = \\ &= \sum_{i=1}^n X_i \sum_{j=1}^n (Y_j \nabla_{\partial_i}\partial_j + \partial_i(Y_j)\partial_j) = \\ &= \sum_{i,j=1}^n X_i Y_j \underbrace{\nabla_{\partial_i}\partial_j}_{\sum_{k=1}^n \Gamma_{ij}^k \partial_k} + \underbrace{\sum_{i,j=1}^n X_i \partial_i(Y_j)\partial_j}_{=X(Y)}\end{aligned}$$

□

Proposition 1.2.3.4. *There is an explicit formula to compute Christoffel symbols, that is*

$$\Gamma_{ij}^{l} = \sum_{k=1}^{n} \frac{1}{2}\Big(\frac{\partial}{\partial_i} g_{jk} + \frac{\partial}{\partial_j} g_{ik} - \frac{\partial}{\partial_k} g_{ij}\Big) g^{lk}, \tag{1.9}$$

$(g^{ij}) = g^{-1}(\partial_i, \partial_j)$ being the inverse of the matrix which represents the pseudo-metric in the basis induced by the local parameterization.

Proof. An easy algebraic manipulation gives

$$\Gamma_{ij}^{l} = \sum_{h=1}^{n} \Gamma_{ij}^{h} \underbrace{\Big(\sum_{k=1}^{n} g_{hk} g^{kl}\Big)}_{\delta_{hl}} = \sum_{k=1}^{n} \Big(\sum_{h=1}^{n} \Gamma_{ij}^{h} g_{hk}\Big) g^{kl}. \tag{1.10}$$

Developing $g(\nabla_{\partial_i}\partial_j, \partial_k)$, one obtains

$$g(\nabla_{\partial_i}\partial_j, \partial_k) = \sum_{h=1}^{n} \Gamma_{ij}^{h} g(\partial_h, \partial_k) = \sum_{h=1}^{n} \Gamma_{ij}^{h} g_{hk},$$

which is the term in bracket in Formula (1.10).

Recalling that $[\partial_i, \partial_j] = 0$, Koszul formula (1.3), applied to $(\partial_i)_{i=1}^{n}$, ends the proof:

$$\Gamma_{ij}^{l} = \sum_{k=1}^{n} \Big(g(\nabla_{\partial_i}\partial_j, \partial_k)\Big) g^{kl} = \sum_{k=1}^{n} \frac{1}{2}\Big(\frac{\partial}{\partial_i} g_{jk} + \frac{\partial}{\partial_j} g_{ik} - \frac{\partial}{\partial_k} g_{ij}\Big) g^{lk}.$$

□

Example 1.2.3.5 (Levi-Civita connection of $\mathbb{R}^{p,q}$). Consider the parameterization given by Cartesian coordinates of $\mathbb{R}^{p+q}$. The pseudo-metric matrix induced is $g_{ij} \equiv \pm\delta_{ij}$, that is the functions $m \mapsto (g_{ij})_m$ are constant with respect to the parameterization. In other words, the derivative of these functions vanish identically, then substituting in Equation (1.9), one finds

$$\Gamma_{ij}^{l} = 0, \qquad\qquad \forall i, j, k = 1, \ldots, n.$$

It follows by Proposition 1.2.3.3 that $\nabla_X Y = X(Y)$, that is in $\mathbb{R}^{p,q}$ the Levi-Civita connection is the usual derivative.

One can compute the Levi-Civita connection of a submanifold intrinsecally, considering the submanifold as a manifold itself, or as induced from the enviromental one.

Lemma 1.2.3.6. *Let M be a smooth manifold and N a submanifold. Let $m \in M$, it exists an open set $U \subseteq M$ containing m such that $\forall X \in \Gamma(TN)$ it exists a vector field $\tilde{X} \in \Gamma(TU)$ which locally extends X, that is $\tilde{X}_l = X_l$, $\forall l \in N \cap U$.*

Proof. Assume $n = \dim N$ and k its codimension as submanifold, that is $\dim M = n+k$. By definition of submanifold, it exists an open neighborhood U of m and a diffeomorphism $\phi\colon U \to \mathbb{R}^{n+k}$ such that $\phi(N \cap U) \subseteq \mathbb{R}^n \times \{0\}$.

For $(x, 0) \in \mathbb{R}^n_x \times \{0\}$, denote $m' = \phi^{-1}(x, 0) \in N \cap U$. Let $X \in \Gamma(TN)$, hence

$$X_{m'} = X_{\phi^{-1}(x,0)} = \sum_{i=1}^{n} X_i(\phi^{-1}(x, 0))\partial_i.$$

For $(x, y) \in \mathbb{R}^n_x \times \mathbb{R}^k_y$, denote $m' = \phi^{-1}(x, y) \in U$. Define $\tilde{X} \in \Gamma(TN)$ as

$$\tilde{X}_{m'} = \tilde{X}_{\phi^{-1}(x,y)} := \sum_{i=1}^{n} X_i(\phi^{-1}(x, y))\partial_i.$$

Hence $\tilde{X}$ is a vector field in $\Gamma(TU)$ and $\tilde{X} = X$ on $N \cap U$. □

Proposition 1.2.3.7. *Let (M, g) be a pseudo-Riemannian manifold, D its Levi-Civita connection. Let $N \subseteq M$ be a pseudo-Riemannian submanifold and ∇ its Levi-Civita connection, then ∇ is the orthogonal projection of D over $\Gamma(TN)$.*

Proof. Levi-Civita connection can be computed locally, so set $m \in N$ and let U be an open neighborhood as in Lemma 1.2.3.6.

Taken $X, Y \in \Gamma(TN)$, denote $\tilde{X}, \tilde{Y} \in \Gamma(TU)$ their extensions on U. Remark that, over $N \cap U$, $g(X, Y) = g(\tilde{X}, \tilde{Y})$, $[X, Y] = [\tilde{X}, \tilde{Y}]$ and $X(f) = \tilde{X}(f)$, $\forall f \in C^\infty(N \cup U)$.

For $m' \in N \cap U$, $Z \in \Gamma(TN)$, comparing with (1.3), one finds

$$\mathrm{Kos}(\tilde{X}, \tilde{Y}, \tilde{Z})_{m'} = \mathrm{Kos}(X, Y, Z)_{m'}.$$

This implies, since $Z = \tilde{Z}$ on $N \cap U$,

$$g((D_{\tilde{X}}\tilde{Y})_{m'}, Z_{m'}) = g((\nabla_X Y)_{m'}, Z_{m'}),$$

namely $(D_{\tilde{X}}\tilde{Y} - \nabla_X Y)_{m'}$ is orthogonal to $Z_{m'}$.

$Z \in \Gamma(TN)$ is arbitrary, hence

$$(D_{\tilde{X}}\tilde{Y} - \nabla_X Y)_{m'} \in (T_{m'}N)^\perp, \quad \forall m' \in N \cap U.$$

Since N is a pseudo-Riemannian submanifold, $T_{m'}N \cap (T_{m'}N)^\perp = \{0\}$ (see Proposition 1.1.1.6 (iii)). $(\nabla_X Y)_{m'} \in T_{m'}N$ that is $\nabla_X Y = (D_{\tilde{X}}\tilde{Y})^\perp$ on $N \cap U$, which ends the proof. □

Example 1.2.3.8 (Levi-Civita connection of $\mathbb{H}^{p,q}$). First, we recall that the Levi-Civita connection is a local property and $\mathbb{H}^{p,q}$ is locally isometric to $\widetilde{\mathbb{H}}^{p,q}$. $\widetilde{\mathbb{H}}^{p,q}$ is a pseudo-Riemannian submanifold of $\mathbb{R}^{p,q+1}$, hence we need to describe the orthogonal projection of $D := \nabla^{\mathbb{R}^{p,q+1}}$ over $\Gamma(T\widetilde{\mathbb{H}}^{p,q})$.

$T_v\widetilde{\mathbb{H}}^{p,q} = v^\perp$, hence $T_v\mathbb{R}^{p,q+1} = v \oplus T_v\widetilde{\mathbb{H}}^{p,q}$. In other words, any vector $x \in T_v\mathbb{R}^{p,q+1} = \mathbb{R}^{p,q+1}$ can be written as $x = \lambda v + w$, $\lambda \in \mathbb{R}$, $w \in T_v\widetilde{\mathbb{H}}^{p,q}$, *i.e.* w is the orthogonal projection of x over $T_v\widetilde{\mathbb{H}}^{p,q}$. In particular,

$$\langle x, v\rangle_{p,q+1} = \lambda\langle v, v\rangle_{p,q+1} + \langle w, v\rangle_{p,q+1} = -\lambda,$$

hence $w = x + \langle x, v\rangle_{p,q+1} v$.

In Example 1.2.3.5, we showed $D_X Y = X(Y)$, so we conclude that

$$(\nabla_X Y)_v = X(Y)_v + \langle X(Y)_v, v\rangle_{p,q+1} v. \tag{1.11}$$

One can refine the formula above: let $I \in \Gamma(T\mathbb{R}^{p,q+1})$ be the vector field such that $I_v = v$, hence $X(I)_v = dI_v X_v = X_v$. D is compatible with the pseudo-metric $\langle\cdot,\cdot\rangle_{p,q+1}$ of $\mathbb{R}^{p,q+1}$, hence

$$X\langle Y, I\rangle_{p,q+1} = \langle X(Y), I\rangle_{p,q+1} + \langle Y, X(I)\rangle_{p,q+1}.$$

Recalling that $Y \in \Gamma(T_v\widetilde{\mathbb{H}}^{p,q}) = v^\perp$, $\langle Y, I\rangle_{p,q+1} \equiv 0$, and so is any of its derivative, *i.e.* $\langle X(Y), I\rangle_{p,q+1} = -\langle Y, X\rangle_{p,q+1}$. Substituting in Formula 1.11, one obtains

$$\nabla_X Y = X(Y) - \langle X, Y\rangle_{p,q+1} I. \tag{1.12}$$

The Levi-Civita connection of $\mathbb{H}^{p,q}$ is then obtained as the push-forward of the one of $\widetilde{\mathbb{H}}^{p,q}$ with respect to the quotient projection.

1.3 Curvature

Definition 1.3.0.1 (Curvature tensor). Let (M, g) a pseudo-Riemannian manifold, ∇ its the Levi-Civita connection. The *curvature tensor* R associates to each pair $X, Y \in \Gamma(TM)$ the operator $R(X, Y)$, defined by

$$R(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}.$$

We will write $R(X, Y)(Z) = R(X, Y)Z$.

Remark 1.3.0.2. The definition of the curvature tensor is up to sign: some authors define it as above, others, for example [DoC92], as

$$R(X, Y) = \nabla_Y\nabla_X - \nabla_X\nabla_Y + \nabla_{[X,Y]}.$$

Example 1.3.0.3 (Pseudo-Euclidean spaces). We saw in Example 1.2.3.5 that the classical derivative is the Levi-Civita connection of $\mathbb{R}^{p,q}$.

Let X, Y be two vector spaces on $\mathbb{R}^{p,q}$, then

$$\begin{aligned} R(X, Y) =& \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} = \\ =& X(Y) - Y(X) - (X(Y) - Y(X)) = 0. \end{aligned}$$

Another way to see it is remarking that the coefficents of the matrix of pseudo-metric with respect to the canonical basis are constant. Since the elements g_{ij} are derived in order to compute Christoffel symbols (see Formula (1.9)), they are null. We will prove in Proposition 1.3.1.10 that R can be obtained from Christoffel symbols, so R is null, too.

Example 1.3.0.4 (Pseudo-hyperbolic space). As in Example 1.2.3.8, we will compute the Riemann tensor of $\widetilde{\mathbb{H}}^{p,q}$ and then its push-forward will be the actual tensor of $\mathbb{H}^{(p,q)}$.

To compute it, we want to exploit the fact to be in a submanifold of $\mathbb{R}^{p,q+1}$. We recall that $\langle\cdot,\cdot\rangle_{p,q+1}$ is the exterior pseudo-metric, $D_X Y = X(Y)$ is the covariant derivative of $\mathbb{R}^{p,q+1}$ and $I \in \mathbb{R}^{p,q+1}$ is the vector field such that $I_v = v$. Then $\forall X, Y \in \Gamma(T\widetilde{\mathbb{H}}^{p,q}) \subseteq \Gamma(T\mathbb{R}^{p,q+1})$, $\langle X, I\rangle_{p,q+1} = 0$ and

$$\nabla_X Y = D_X Y - \langle X, Y\rangle_{p,q+1} I.$$

With these elements, we are ready to compute the Riemann tensor of $\widetilde{\mathbb{H}}^{p,q}$:

$$\begin{aligned}
\nabla_X \nabla_Y Z =& D_X(\nabla_Y Z) - \langle \nabla_Y Z, X\rangle_{p,q+1} I = \\
=& D_X \left[D_Y Z - \langle Z, Y\rangle_{p,q+1} I\right] - \langle D_Y Z - \langle Z, Y\rangle_{p,q+1} I, X\rangle_{p,q+1} I = \\
=& D_X D_Y Z - X\left(\langle Z, Y\rangle_{p,q+1}\right) I - \langle Z, Y\rangle_{p,q+1} X - \langle D_Y Z, X\rangle_{p,q+1} I + \\
& + \langle X, Z\rangle_{p,q+1} \underbrace{\langle I, X\rangle_{p,q+1}}_{=0} I = \\
=& D_X D_Y Z - \langle D_X Z, Y\rangle_{p,q+1} I - \langle Z, D_X Y\rangle_{p,q+1} I - \langle Z, Y\rangle_{p,q+1} X - \\
& - \langle D_Y Z, X\rangle_{p,q+1} I.
\end{aligned}$$

$-\nabla_Y \nabla_X Z$ is obtained switching X and Y in the previous formula and changing the sign, that is

$$\begin{aligned}
-\nabla_Y \nabla_X Z =& -D_Y D_X Z + \langle D_Y Z, X\rangle_{p,q+1} I + \langle Z, D_Y X\rangle_{p,q+1} I + \\
& + \langle Z, X\rangle_{p,q+1} Y + \langle D_X Z, Y\rangle_{p,q+1} I.
\end{aligned}$$

so the red terms cancel eachother, and the green ones become $-\langle [X,Y], Z\rangle_{p,q+1} I$. Finally,

$$-\nabla_{[X,Y]} Z = -D_{[X,Y]}(Z) + \langle Z, D_X Y\rangle_{p,q+1} I.$$

The green terms cancel, while the blue ones constitute the Riemann tensor of $\mathbb{R}^{p,q+1}$, which is 0, hence

$$R(X,Y)Z = \langle Z, X\rangle_{p,q+1} Y - \langle Z, Y\rangle_{p,q+1} X.$$

Proposition 1.3.0.5. *The properties of R hold in the pseudo-Riemannian realm:*

1. *The application $(X,Y) \mapsto R(X,Y)$ is bilinear;*

2. *the operator* $R(X,Y)\colon \Gamma(TM) \to \Gamma(TM)$ *is* $C^\infty(M)$*-linear* $\forall X, Y \in \Gamma(TM)$*;*

3. $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ *(first Bianchi identity);*

Proposition 1.3.0.6. *Denote* $R(X,Y,Z,W) := g(R(X,Y)Z,W)$*, the following properties hold in the pseudo-Riemannian realm:*

1. $R(\cdot,\cdot,Z,W)$ *is anti-symmetric* $\forall Z, W \in \Gamma(TM)$*;*

2. $R(X,Y,\cdot,\cdot)$ *is anti-symmetric* $\forall X, Y \in \Gamma(TM)$*;*

3. $R(X,Y,Z,W) = R(Z,W,X,Y)$*,* $\forall X, Y, Z, W$ *(symmetric on the couples).*

We won't give the proofs, as they don't differ from the classical case. The reference for this part are [DoC92] and [O'N].

1.3.1 Sectional curvature

The Riemann tensor is a heavy tool to handle, so we want to define another object that contains the same informations. In order to make a parallel with the Riemannian realm, we consider a Riemannian metric $\langle\cdot,\cdot\rangle$ and a pseudo-Riemannian metric $g(\cdot,\cdot)$ on M.

In the Riemannian domain, the tool containing all the informations is the sectional curvature, *i.e.*

$$K(m,\sigma) = \frac{\langle R(X,Y)Y, X\rangle}{|X \wedge Y|^2},$$

where X, Y is a basis of the 2-plane $\sigma \subseteq T_m M$ and

$$|X \wedge Y| = \sqrt{\langle X, X\rangle \langle Y, Y\rangle - \langle X, Y\rangle^2}.$$

If we try to substitute $\langle\cdot,\cdot\rangle$ with $g(\cdot,\cdot)$, there are two problems: the denominator can be a complex number or 0.

The first problem can be easily avoided using the quadric form

$$Q(X,Y) = g(X,X) + g(Y,Y) - g(X,Y)^2,$$

which is the determinant of the pseudo-metric induced on the plane σ, instead of $|X \wedge Y|^2$. Unfortunately, the form can be degenerate on σ, and there is no way to fix it, so we define the sectional curvature in the pseudo-Riemannian domain as it follows.

Definition 1.3.1.1 (Sectional curvature). Let $\sigma \subseteq T_m M$ a non-degenerate 2-plane and $\{X, Y\}$ a basis of σ.

$$K(m,\sigma) := \frac{g(R(X,Y)Y, X)}{Q(X,Y)}.$$

We need to prove that definition is well posed, that is it does not depend on the choice of the basis $\{X, Y\}$.

Lemma 1.3.1.2. *Let B a bilinear anti-symmetric form over a vector space, then $B(aX + bY, cX + dY) = (ad - bc)B(X, Y)$.*

Proof. $B(aX + bY, cX + dY) = aB(X, cX + dY) + bB(Y, cX + dY) =$

$$= \underbrace{acB(X, X)}_{=0} + adB(X, Y) + bcB(Y, X) + \underbrace{bdB(Y, Y)}_{=0} =$$

$$= (ad - bc)B(X, Y).$$

□

Corollary 1.3.1.3. *$K(p, \sigma)$ does not depend on the basis of σ chosen to compute it.*

Proof. Let $G_{X,Y}$ be the matrix of the pseudo-metric in the basis $\{X, Y\}$, so $Q(X, Y) = \det G_{X,Y}$.

Consider a change of basis

$$\begin{cases} W = aX + bY \\ Z = cX + dY \end{cases},$$

the matrix $G_{W,Z}$ of the pseudo-metric in the basis is

$$G_{W,Z} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} G_{X,Y} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T$$

Then $Q(W, Z) = \det G_{W,Z} = (ad - bc)^2 \det G_{X,Y} = (ad - bc)^2 Q(X, Y)$.

For all vector fields $U, V \in \Gamma(TM)$, $R(\cdot, \cdot, U, V)$ and $R(U, V, \cdot, \cdot)$ are bilinear and anti-symmetric forms. Hence, applying Lemma 1.3.1.2 twice, we obtain

$$\begin{aligned} &R(aX + bY, cX + dY, cX + dY, aX + bY) = \\ &\qquad = (ad - bc)R(X, Y, cX + dY, aX + bY) = \\ &\qquad = (ad - bc)^2 R(X, Y, Y, X). \end{aligned}$$

Computing the quotient concludes the proof. □

Example 1.3.1.4 (Pseudo-Euclidean space). $R(X, Y) = 0$ (see Example 1.3.0.3), so $K(m, \sigma) = 0$, $\forall m \in \mathbb{R}^{p,q}$, $\sigma \subseteq T_m\mathbb{R}^{p,q}$ non-degenerate plane.

Example 1.3.1.5 (Pseudo-hyperbolic space). A more interesting example is given by $\mathbb{H}^{p,q}$. We computed in Example 1.3.0.4 the Riemann tensor

$$R(X, Y)Z = \langle Z, X\rangle_{p,q+1} Y - \langle Z, Y\rangle_{p,q+1} X,$$

so computing $R(X,Y,Y,X)$ gives

$$\begin{aligned}R(X,Y,Y,X) &= \langle R(X,Y)Y, X\rangle_{p,q+1} = \\ &= \langle \langle Y,X\rangle_{p,q+1}Y - \langle Y,Y\rangle_{p,q+1}X, X\rangle_{p,q+1} = \\ &= \langle Y,X\rangle_{p,q+1}\langle Y,X\rangle_{p,q+1} - \langle Y,Y\rangle_{p,q+1}\langle X,X\rangle_{p,q+1} = \\ &= -Q(X,Y),\end{aligned}$$

that is $K(m,\sigma) = -1$, $\forall m \in \widetilde{\mathbb{H}}^{p,q}$, $\forall \sigma \subseteq T_m\widetilde{\mathbb{H}}^{p,q}$ non-degenerate plane, and so does for $\mathbb{H}^{p,q}$.

$\mathbb{H}^{p,q}$ is a space with constant negative sectional curvature. Spaces with this property are called *hyperbolic* in the Riemannian realm, *Anti-de Sitter* in the Minkowskian one and *pseudo-hyperbolic* in the most general context.

Proposition 1.3.1.6. *Let (M,g) be a pseudo-Riemannian manifold. $h = \lambda g$, is another pseudo-metric on M, $\lambda \in \mathbb{R}\setminus\{0\}$.*

1. *$R_h(X,Y)Z = R_g(X,Y)Z$, $\forall X,Y,Z \in \Gamma(TM)$;*
2. *$R_h(X,Y,Z,W) = \lambda R_g(X,Y,Z,W)$, $\forall X,Y,Z,W \in \Gamma(TM)$;*
3. *$K_h(m,\sigma) = \lambda^{-1}K_g(m,\sigma)$, $\forall m \in M$, $\forall \sigma$ non-degenerate 2-plane of T_mM.*

Proof. This result is a corollary of Remark 1.2.2.12. Indeed,

1. R only depends on ∇.
2. $R_h(X,Y,Z,W) = h(R_h(X,Y)Z,W) = \lambda g(R_g(X,Y)Z,W)$.
3. $K_h(m,\sigma) = \frac{R_h(X,Y,Y,X)}{Q_h(X,Y)} = \frac{\lambda R_g(X,Y,Y,X)}{\lambda^2 Q_g(X,Y)} = \frac{K_g(m,\sigma)}{\lambda}$.

□

Remark 1.3.1.7. If $\lambda > 0$ we are not changing signature, on the contrary if $\lambda < 0$ and $n_+ \neq n_-$, we are. Hence, when we focus on one specific geometry (*e.g.* Riemannian, Lorentzian, etc.), we are not allowed to scale the pseudo-metric by a negative value, except for the case $n_+ = n_-$.

A form $F\colon \Gamma(TM)^4 \to C^\infty(M)$ that respects the same symmetries showns in Proposition 1.3.0.6 is called *curvaturelike*. The following statement shows that there are not a lot of curvaturelike tensors.

Theorem 1.3.1.8. *Let E, F be a curvaturelike tensors on TM. If*

$$E(X,Y,Y,X) = F(X,Y,Y,X),$$

for every X,Y spanning a non-degenerate plane, then $E = F$.

The proof, which can be found in [O'N, pp. 77-80], is based on the fact that vectors couples spanning a non-degenerate plane form a dense subset of $TM \times TM$ and tensors are continuous functions.

Corollary 1.3.1.9. *The sectional curvature, together with the pseudo-metric, define univocally the curvature tensor.*

Proof. By definition, $K(p, \sigma)Q(X, Y) = R(X, Y, Y, X)$, $\forall X, Y$ spanning a non-degenerate plane. Hence, thanks to Theorem 1.3.1.8, R is the only tensor that satisfies that condition.

Once again, g being non-degenerate, $R(X, Y)Z$ is univocally defined by $R(X, Y, Z, \cdot)$, and so is the operator $Z \mapsto R(X, Y)Z$, that is the image of the curvature tensor. $\square$

From this result, we can collect all the informations of the curvature tensor by computing it on the basis $(\partial_1, \ldots, \partial_n)$, thanks to the linearity of the tensor.

Proposition 1.3.1.10 (Curvature tensor coefficients)**.** *Let R^l_{ijk} the l-th coefficient of $R(\partial_i, \partial_j)\partial_k$,* i.e. *$R(\partial_i, \partial_j)\partial_k = \sum R^l_{ijk}\partial_l$, then*

$$R^l_{ijk} = \frac{\partial}{\partial x_i}\Gamma^l_{jk} - \frac{\partial}{\partial x_j}\Gamma^l_{ik} + \sum_{m=1}^{n} \left(\Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}\right). \tag{1.13}$$

Proof. $[\partial_i, \partial_j] = 0$, so $R(\partial_i, \partial_j) = \nabla_{\partial_i}\nabla_{\partial_j} - \nabla_{\partial_j}\nabla_{\partial_i}$. $\nabla_{\partial_j}\partial_k = \sum_{l=1}^n \Gamma^l_{jk}\partial_l$ by definition, then

$$\begin{aligned}
\nabla_{\partial_i}\nabla_{\partial_j}\partial_k &= \sum_{l=1}^{n} \nabla_{\partial_i}\Gamma^l_{jk}\partial_l = \sum_{l=1}^{n} \left(\Gamma^l_{jk}\nabla_{\partial_i}\partial_l + \partial_i(\Gamma^l_{jk})\partial_l\right) = \\
&= \sum_{l=1}^{n} \Gamma^l_{jk}\Big(\sum_{m=1}^{n} \Gamma^m_{il}\partial_m\Big) + \sum_{l=1}^{n} \partial_i(\Gamma^l_{jk})\partial_l = \\
&= \sum_{l,m=1}^{n} \left(\Gamma^m_{jk}\Gamma^l_{im}\partial_l\right) + \sum_{l=1}^{n} \partial_i(\Gamma^l_{jk})\partial_l = \\
&= \sum_{l=1}^{n} \Big(\underbrace{\sum_{m=1}^{n} \Gamma^m_{jk}\Gamma^l_{im} + \partial_i(\Gamma^l_{jk})}_{\text{l-th coordinate of } \nabla_{\partial_i}\nabla_{\partial_j}\partial_k} \Big)\partial_l
\end{aligned}$$

We obtain the l-th coordinate of $\nabla_{\partial_i}\nabla_{\partial_j}\partial_k$ by switching the index i and j, and that concludes the proof. $\square$

Corollary 1.3.1.11. *Christoffel symbols completely determine the curvature tensor.*

Example 1.3.1.12 (Cylinder). $C = \{x^2 + y^2 = 1\} \subseteq \mathbb{R}^{2,1}$ is a Lorentzian submanifold. Indeed, consider the parameterization

$$\begin{aligned}(0, 2\pi) \times \mathbb{R} \colon &\to C \subseteq \mathbb{R}^{2,1} \\ (\phi, z) &\mapsto (\cos\phi, \sin\phi, z),\end{aligned}$$

the basis of the tangent space induced by the parametrization is

$$\begin{cases} \partial_\phi = (-\sin\phi, \cos\phi, 0) \\ \partial_z = (0, 0, 1) \end{cases}.$$

Hence the the restriction of $\langle \cdot, \cdot \rangle_{2,1}$ to C is

$$G = (g_{ij}) = \begin{pmatrix} \langle \partial_\phi, \partial_\phi \rangle_{2,1} & \langle \partial_\phi, \partial_z \rangle_{2,1} \\ \langle \partial_z, \partial_\phi \rangle_{2,1} & \langle \partial_z, \partial_z \rangle_{2,1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which is clearly Lorentzian. Moreover, it means that the parameterization is a loca isometry from $\mathbb{R}^{1,1}$ to the cylinder. The curvature is invariant by local isometry, hence the curvature is 0.

Example 1.3.1.13 (Hyperboloid of one sheet). $\widetilde{\mathbb{S}}^{1,1} = \{x^2 + y^2 - z^2 = 1\}$ is a Lorentzian submanifold of Minkowski 3-dimensional space (see Figure 1.3). We proved in Example 1.3.1.5 that it has constant curvature $K = 1$, namely it is a de Sitter space. Nevertheless, we give the explicit computation with Christoffel symbols in low dimension as an example.

A global parametrization of the surface is

$$\begin{aligned}\mathbb{R}^2 &\to \widetilde{\mathbb{S}}^{1,1}\mathbb{R}^{2,1} \\ (\phi, z) &\mapsto (\sqrt{1+z^2}\cos\phi, \sqrt{1+z^2}\sin\phi, z).\end{aligned}$$

In every point $p = p(\phi, z)$, the parametrization induces a basis on T_pH:

$$\begin{cases} \partial_\phi = (-\sqrt{1+z^2}\sin\phi, \sqrt{1+z^2}\cos\phi, 0) \\ \partial_z = (\frac{z}{\sqrt{1+z^2}}\cos\phi, \frac{z}{\sqrt{1+z^2}}\sin\phi, 1) \end{cases}.$$

The $G = (g_{ij})$ of the induced pseudo-metric in the basis $(\partial_\phi, \partial_z)$ is

$$G = \begin{pmatrix} g_{\phi\phi} & g_{\phi z} \\ g_{z\phi} & g_{zz} \end{pmatrix} = \begin{pmatrix} 1+z^2 & 0 \\ 0 & -\frac{1}{1+z^2} \end{pmatrix}.$$

It follows that its inverse $G^{-1} = (g^{ij})$ is

$$G^{-1} = \begin{pmatrix} g^{\phi\phi} & g^{\phi z} \\ g^{z\phi} & g^{zz} \end{pmatrix} = \begin{pmatrix} \frac{1}{1+z^2} & 0 \\ 0 & -(1+z^2) \end{pmatrix}.$$

Let's compute Christoffel symbols by means of Equation (1.9). As G^{-1} is diagonal, the formula becomes $\Gamma^k_{ij} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) g^{kk}$.

$$\Gamma^{\phi}_{\phi\phi} = 0, \qquad \Gamma^{\phi}_{\phi z} = \Gamma^{\phi}_{z\phi} = \frac{z}{1+z^2}, \qquad \Gamma^{\phi}_{zz} = 0,$$
$$\Gamma^{z}_{\phi\phi} = z(1+z^2), \qquad \Gamma^{z}_{\phi z} = \Gamma^{z}_{z\phi} = 0, \qquad \Gamma^{z}_{zz} = -\frac{z}{1+z^2}.$$

Since $\det G \equiv -1$, the sectional curvature is

$$K(\phi, z) = \frac{g(R(\partial_1, \partial_2)\partial_2, \partial_1)}{\det G} = -g(R(\partial_1, \partial_2)\partial_2, \partial_1) = g(R(\partial_1, \partial_2)\partial_1, \partial_2).$$

By means of Equation (1.13),

$$g(R(\partial_1, \partial_2)\partial_1, \partial_2) = R^1_{121}g_{12} + R^2_{121}g_{22}.$$

$g_{12} = 0$, hence we only need to compute $R^2_{121}g_{22}$.

$$\begin{aligned} R^2_{121} &= \underbrace{\frac{\partial}{\partial_1}\Gamma^2_{12}}_{=0} - \frac{\partial}{\partial_2}\Gamma^2_{11} + \Gamma^1_{21}\Gamma^2_{11} - \underbrace{\Gamma^1_{11}\Gamma^2_{21}}_{=0} + \underbrace{\Gamma^2_{21}\Gamma^2_{12}}_{=0} - \Gamma^2_{11}\Gamma^2_{22} + = \\ &= -\left((1+z^2) + 2z^2\right) + z^2 + z^2 = -(1+z^2) \end{aligned}$$

Then the sectional curvature is $K(\phi, z) = R^2_{121}g_{22} = 1$.

1.4 Geodesics

In Riemannian geometry, geodesics are curves locally minimizing distances. The distance is induced by the metric, and cannot be extended to the pseudo-Riemannian realm. Indeed, in the more general setting, curves can have negative or null length, so the notion of *minimize distances* is meaningless: any pair of points lying on the same lightlike curve has distance zero.

However, geodesics can be also defined as *straight lines* of the space, that is curves with no acceleration. This notion can be extended, involving only the Levi-Civita connection.

1.4.1 Curves

Let $c\colon I \to M$, $I \subseteq \mathbb{R}$, be a smooth connected curve on (M, g) pseudo-Riemannian manifold. We will call the curve *spacelike, timelike* or *lightlike* if its tangent vector is constant in character.

Definition 1.4.1.1 (Pseudo-length). The *pseudo-length* of a curve $c \in C^1(I, M)$ is defined as

$$L(c) := \left| \int_I \sqrt{|g\left(c'(t), c'(t)\right)|}dt \right|.$$

A Riemannian manifold M has a natural structure of metric space, induced by the metric. The distance from x to y is

$$d(x,y) = \inf\{L(c), c\colon [0,1] \to M, \text{ piecewise } C^1, c(0) = x, c(1) = y\}.$$

It is not possible to extend this definition on a pseudo-Riemannian manifold: the distance between points on the same lightlike curve is 0, that contradicts the axioms of distance.

1.4.2 Vector fields along curves

Definition 1.4.2.1 (Vector field along a curve). Let $c\colon I \to M$ be a smooth curve, a *vector field along* c is a smooth map $X\colon I \to TM$ such that $X_t \in T_{c(t)}M$, $\forall t \in I$.

Remark 1.4.2.2. Given a vector field $\tilde{X} \in \Gamma(TM)$ and a curve $c\colon I \to M$, the restriction of $\tilde{X}$ to c, *i.e.* $X_t := \tilde{X}_{c(t)}$, is a vector field along the curve c.

One could think that it is also possible to extend a vector field along a curve Y to a vector field $\tilde{Y}$ on M, that is finding $\tilde{Y} \in \Gamma(TM)$ such that $\tilde{Y}_{c(t)} = Y_t$. This is not true, and the following exemples will explain what kind of obstacles can occur.

Example 1.4.2.3 (Injectivity). If $c\colon I \to M$ is not injective, it exists $\bar{t} \neq \bar{s} \in I$ such that $c(\bar{t}) = c(\bar{s}) = m$, *i.e.* the curve self-intersects. Let X be a field along c such that $X_{\bar{t}} \neq X_{\bar{s}}$. Suppose it exists a vector field $\tilde{X} \in \Gamma(TM)$ that extends X, hence

$$\tilde{X}_{\bar{t}} = \tilde{X}_{\gamma(\bar{t})} = X_m = \tilde{X}_{\gamma(\bar{s})} = X_{\bar{s}},$$

which contradicts the hypothesis on X.

For an explicit example, take $X_t = c'(t)$, which is a vector field along $c(t)$ (see Figure 1.4).

In the previous example the vector field could not be extend due to punctual properties of the curve, suggesting that in a local chart X admits an extention. The next example shows that there are cases more pathological: the idea is to take a dense curve on M and a vector field which is continuous with respect to the topology of the curve but not to the manifold's one.

Example 1.4.2.4 (Density). Consider the torus $M = \mathbb{R}^2/\mathbb{Z}^2$ and the projection $p\colon \mathbb{R}^2 \to M$, which is a local diffeomorphism.

Consider the curve $\bar{c}\colon \mathbb{R} \to \mathbb{R}^2$ such that $\bar{c}(t) = (\pi t, t)$, which is a straight line with irrational slope. Hence the image of $c := p \circ \bar{c}$ is dense in M.

Define $\bar{X}_t = (\cos \pi t, \sin \pi t) \in \mathbb{R}^2 = T_{\bar{c}(t)}\mathbb{R}^2$, and $X := p_* \bar{X}$. $c(\mathbb{Z})$ is a dense subset of $p(\mathbb{R} \times \{0\})$, and $\bar{X}_t = ((-1)^t, 0)$ for $t \in \mathbb{Z}$. Since p is a local diffemorphism, $X_{2t} = -X_{2t+1}$, $\forall t \in \mathbb{Z}$, hence it cannot be extended to a smooth vector field (see Figure 1.5).

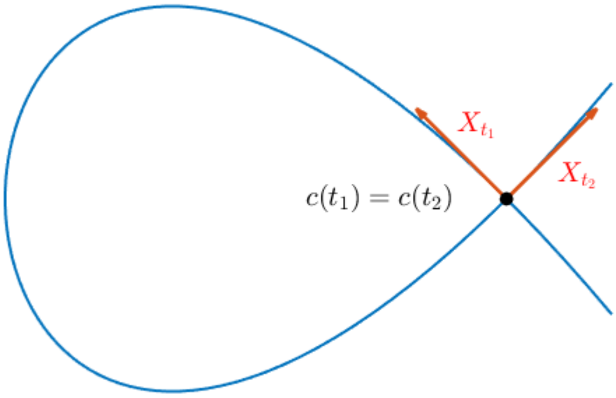

Figure 1.4: Vector field $X_t = c'(t)$ along a self-intersecting curve $c(t)$
.

Both examples suggest that, locally in I and in M, one can extend any vector field along a curve to a vector field on the manifold. The next example shows that we need additional assuptions on the derivative of the curve.

Example 1.4.2.5 (Regularity). Consider a point $m_0 \in M$ of a manifold. Let $c\colon I \to M$ be the curve $c(t) = m_0$. A field along c is a choice of infinite many vectors $(X_t)_{t\in I} \subseteq T_{m_0}M$, which clearly can not be extended even locally.

Nevertheless, it is possible to extend the Levi-Civita connection to vector fields defined along curves.

Theorem 1.4.2.6. *Let (M, g) be a pseudo-Riemannian manifold and $c\colon I \to M$ a smooth curve. It exists a unique operator ∇/dt from the space of vector fields along c to itself, such that $\forall t \in I$*

i. $\left(\frac{\nabla}{dt}(fX)\right)_t = f'(t)X_t + f(t)(\frac{\nabla}{dt}X)_t$, $\forall f\colon I \to \mathbb{R}$;

ii. if X can be (locally) extended to a vector field $\tilde{X} \in \Gamma(TM)$, then

$$\left(\frac{\nabla}{dt}X\right)_t = \left(\nabla_{c'(t)}\tilde{X}\right)_{c(t)}.$$

Proof. We will check first the unicity part of the statement. Suppose ∇/dt exists, set X a vector field along c. In a local chart, we can write

$$c(t) = (c_1(t), \dots, c_n(t)), \qquad c_i \in C^\infty(I, \mathbb{R});$$
$$X_t = \sum_{i=1}^n X_i(t)(\partial_i)_{c(t)}, \qquad X_i \in C^\infty(I, \mathbb{R}).$$

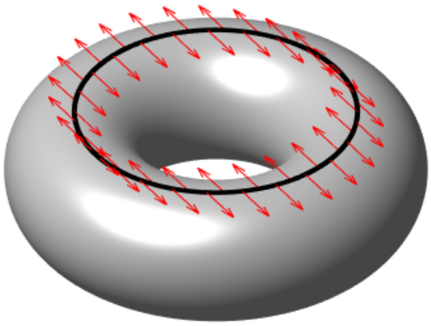

Figure 1.5: The vector field X along $c(t)$ restricted to the circle $p(\mathbb{R} \times \{0\})$ in $\mathbb{R}^2/\mathbb{Z}^2$ cannot be extended to a vector field along the circle, hence neither to a vector field on $\mathbb{R}^2/\mathbb{Z}^2$.

Using (i), one obtains

$$\frac{\nabla}{dt}X_t = \sum_{i=1}^{n} \frac{\nabla}{dt}\left(X_i(t)\partial_i\right) = \sum_{i=1}^{n} X_i'(t)(\partial_i)_{c(t)} + X_i(t)\frac{\nabla}{dt}(\partial_i)_{c(t)}. \tag{1.14}$$

Clearly $(\partial_i)_{c(t)}$ can be extended to a vector field on an open neighborhood of c. By the means of (ii) one computes

$$\begin{aligned}\frac{\nabla}{dt}(\partial_i)_{c(t)} = \nabla_{c'(t)}(\partial_i)_{c(t)} = \sum_{j=1}^{n} c_j'(t)\nabla_{(\partial_j)_{c(t)}}(\partial_i)_{c(t)} = \\ = \sum_{j,k=1}^{n} c_j'(t)\Gamma_{ij}^k(c(t))(\partial_k)_{c(t)}.\end{aligned}$$

Substituting in (1.14), we finds

$$\begin{aligned}\left(\frac{\nabla}{dt}X\right)_t &= \sum_{i,j,k=1}^{n} X_i'(t)(\partial_i)_{c(t)} + X_i(t)c_j'(t)\Gamma_{ij}^k(c(t))(\partial_k)_{c(t)} = \\ &= \sum_{i,j,k=1}^{n} \left[X_k'(t) + X_i(t)c_j'(t)\Gamma_{ij}^k(c(t))\right](\partial_k)_{c(t)}. \end{aligned} \tag{1.15}$$

which proves the unicity of such operator.

To prove that it exists, we define it as in (1.15) and show it satisfies the properties (i),(ii).

For (i), set $f \colon I \to \mathbb{R}$ and replace $X = fY$ in Equation (1.15):

$$\left(\frac{\nabla}{dt}(fY)\right)_t = \sum_{i,j,k=1}^{n} \left[(fY_k)'(t)(\partial_k)_{c(t)} + f(t)Y_k(t)c_j'(t)\Gamma_{ij}^k(c(t))\right](\partial_k)_{c(t)}.$$

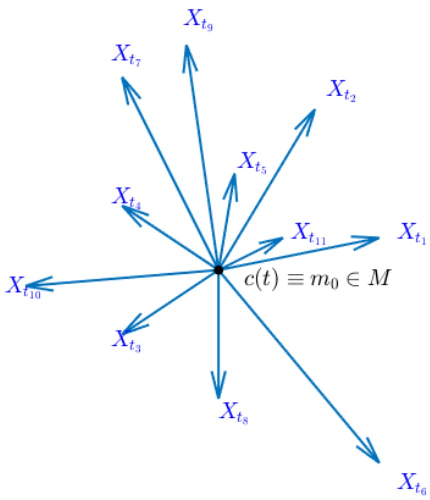

Figure 1.6: Vector field along a constant curve.

Since $(fY_k)' = f'Y_k + fY_k'$,

$$\left(\frac{\nabla}{dt}(fY)\right)_t = \sum_{i,j,k=1}^{n} \Big[f'(t)Y_k(t)(\partial_k)_{c(t)} + f(t)Y_k'(t)(\partial_k)_{c(t)} + \\ + f(t)Y_k(t)c_j'(t)\Gamma_{ij}^k(c(t))\Big](\partial_k)_{c(t)} = \\ = (f'Y)_t + f(t)\left(\frac{\nabla}{dt}Y\right)_t.$$

For (ii), let $t_0 \in I$, $I_{t_0} \subseteq I$ be an open neighborhood of t_0 and $\tilde{X} \in \Gamma(TM)$ such that $\tilde{X}_{c(t)} = X_t$, $\forall t \in I_{t_0}$. In a local chart, $\tilde{X} = \tilde{X}_i\partial_i$, and $\tilde{X}_i(c(t)) = X_i(t)$. Its derivative along $c'(t)$ is

$$c'(t)\left(\tilde{X}_i(c(s))\right) = \frac{d}{ds}\tilde{X}_i(c(s))|_{s=t} = X_i'(t).$$

Hence, replacing in (1.15) and using Proposition 1.2.3.3, one finds

$$\left(\frac{\nabla}{dt}X\right)_t = \sum_{i,j,k=1}^{n}\left[c'(t)\left(\tilde{X}_k(c(t))\right) + \tilde{X}_i(c(t))c_j'(t)\Gamma_{ij}^k(c(t))\right](\partial_k)_{c(t)} = \\ = c'(\tilde{X})_{c(t)} + \sum_{i,j,k=1}^{n}\tilde{X}_i(c(t))c_j'(t)\Gamma_{ij}^k(c(t))(\partial_k)_{c(t)} = \left(\nabla_{c'}\tilde{X}\right)_{c(t)}.$$

□

Lemma 1.4.2.7. *Let M be a smooth manifold, $c\colon I \to M$ a regular curve and $t_0 \in I$. It exists $\varepsilon > 0$ and $U \subseteq M$ open neighborhood of $c(t_0)$ such that*

$\forall X$ vector field along $c|_{(t_0-\varepsilon,t_0+\varepsilon)}$, it exists a vector field $\tilde{X} \in \Gamma(TU)$ which extends X.

Proof. It is a corollary of Proposition 1.2.3.7: indeed a regular curve is a parameterization for a smooth 1-submanifold of M. $\square$

Corollary 1.4.2.8. *If $c\colon I \to M$ is a regular or a constant curve, the operator ∇/dt coincide with the Levi-Civita operator restricted along c, so we will abusively write*

$$\frac{\nabla}{dt}X = \nabla_{c'}X.$$

Proof. ∇ can be computed locally, so Lemma 1.4.2.7 and Theorem 1.4.2.6 (ii) conclude the proof. $\square$

1.4.3 Geodesics

Definition 1.4.3.1 (Geodesic). A curve $\gamma\colon (a,b) \to M$ is a *geodesic* if

$$\nabla_{\gamma'(t)}\gamma'(t) = 0, \quad \forall t \in (a,b). \tag{1.16}$$

Remark 1.4.3.2. We wrote $\nabla_{\gamma'}\gamma'$ instead of $(\nabla/dt)(\gamma')$ because, as a corollary of Theorem 1.4.3.4, geodesic are either regular curves or constants.

It comes straight from the definition that a geodesic's tangent vectors have the same character, in fact

$$\frac{d}{dt}g\left(\gamma'(t),\gamma'(t)\right) = 2g\left(\nabla_{\gamma'(t)}\gamma'(t),\gamma'(t)\right) \equiv 0, \tag{1.17}$$

that means the norm of γ' is constant.

Locally, geodesics can be described as solution of ODEs, tied to Christoffel symbols.

Proposition 1.4.3.3. *Let (M,g) be a pseudo-Rieamnnian manifold and ∇ its Levi-Civita connection. Let $c\colon I \to M$ be a smooth curve, (U,x) a local chart, $c(t) = (c_1(t),\dots,c_n(t))$ in the local chart. c is a geodesic if and only if it satisfies the differential system*

$$c_k''(t) + \sum_{i,j=1}^{n} \Gamma_{ij}^k(c(t))c_i'(t)c_j'(t) = 0, \qquad \forall k = 1,\dots,n, \tag{1.18}$$

where Γ_{ij}^k are the Christoffel symbols with respect to the local basis $\{\partial_1,\dots,\partial_n\}$ induced by the local chart.

Proof. The proof is direct corollary of Theorem 1.4.2.6: in fact, substituting $X_t = c'(t)$ in (1.15), one obtains

$$\frac{\nabla}{dt}c'(t) = \sum_{k=1}^{n}\left[c_k''(t) + \sum_{i,j=1}^{n} c_i'(t)c_j'(t)\Gamma_{ij}^k(c(t))\right](\partial_k)_{c(t)}.$$

$(\nabla/dt)\gamma' = 0$ if and only if all its coefficents with respect to the basis vanish, which is exactly the statement. □

Theorem 1.4.3.4 (Existence and unicity of geodesics)**.** *Let (M, g) be a pseudo-Riemannian manifold, $m_0 \in M$, $v_0 \in T_{m_0}M$. There exist an open neighborhood $U \times V$ of $(m_0, v_0) \in TM$ and $\varepsilon > 0$ such that $\forall (m, v) \in U \times V$, it exists a unique geodesic $c_{(m,v)}\colon (-\varepsilon, \varepsilon) \to M$ such that $c_{(m,v)}(0) = m$, $c'_{(m,v)}(0) = v$.*

Moreover, the map $C\colon U \times V \times (-\varepsilon, \varepsilon) \to M$ such that $C(m, v, t) = c_{(m,v)}(t)$ is smooth.

Proof. Let A be an local chart of M containing m_0. The differential system (1.18) can be rearranged in a first order differential system

$$\begin{cases} x'_k = y_k & k = 1, \dots, n \\ y'_k = -\sum_{i,j=1}^{n} \Gamma_{ij}^k y_i y_j & k = 1, \dots, n \end{cases},$$

where $(x_1, \dots, x_n, y_1, \dots, y_n)$ are the coordinates of $(c(t), c'(t))$ in the local chart TA. The result is then a direct application of the fact that, given initial values (m, v), it exists a unique solution of an ODEs. The solutions of an ODE smoothly depend on the initial values, hence C is smooth. □

Corollary 1.4.3.5. *Let $\gamma_1\colon (a_1, b_1) \to M$, $\gamma_2\colon (a_2, b_2) \to M$ be two geodesics such that $\exists t_1 \in (a_1, b_1), t_2 \in (a_2, b_2)$ such that*

$$\begin{cases} \gamma_1(t_1) = \gamma_2(t_2), \\ \gamma'_1(t_1) = \gamma'_2(t_2). \end{cases}$$

It exists a geodesic $\gamma\colon (a_1 - t_1, b_1 - t_1) \cup (a_2 - t_2, b_2 - t_2) \to M$ such that

$$\gamma(t) = \begin{cases} \gamma_1(t + t_1) & \text{if } t \in (a_1 - t_1, b_1 - t_1) \\ \gamma_2(t + t_2) & \text{if } t \in (a_2 - t_2, b_2 - t_2) \end{cases}$$

Proof. The uniqueness of a local solution, proved in Theorem 1.4.3.4, implies that $\gamma_1 = \gamma_2$ over $(a_1 - t_1, b_1 - t_1) \cap (a_2 - t_2, b_2 - t_2)$, hence γ is well defined. By hypothesis, γ is a geodesic over three open sets which cover its domain, so it is a geodesic over all its domain. □

This result allows to glue geodesics coinciding over an open set, so it exists a maximal interval of definition for any geodesic, which bring to the following definition:

Definition 1.4.3.6. For $m \in M$ and $v \in T_mM$, *the* geodesic with initial point m and initial velocity v is the curve $\gamma_{(m,v)}\colon I_{(m,v)} \to M$ such that

i. $\gamma_{(m,v)}$ is a geodesic,

ii. $\gamma_{(m,v)}(0) = m$, $\gamma'_{(m,v)}(0) = v$,

iii. if a curve $c\colon J \to M$ satisfies the two previous points, $J \subseteq I_{(m,v)}$ and $c(t) = \gamma_{(m,v)}(t)$, $\forall t \in J$, that is c is a portion of $\gamma_{(m,v)}$.

Definition 1.4.3.7 (Completeness). A geodesic is *complete* if its maximal interval of definition is $\mathbb{R}$. It is complete *on one side* if its maximal interval of definition is an unlimited open subset of $\mathbb{R}$.

Definition 1.4.3.8 (Geodesically complete). A pseudo-Riemannian manifold is *geodesically complete* if every geodesic is complete.

Example 1.4.3.9 (Geodesics of $\mathbb{R}^{p,q}$). Consider a point $m \in \mathbb{R}^{p,q}$ and a vector $v \in T_m\mathbb{R}^{p,q} = \mathbb{R}^{p,q}$. The affine line $c(t) = m + tv$, defined on the whole real line, is the geodesic $\gamma_{(m,v)}$. Indeed, $c'(t) = v$, $\forall t \in \mathbb{R}$, that is the vector field $c'(t)$ is constant. Hence, recalling that Levi-Civita connection is the usual derivative in $\mathbb{R}^{p,q}$, we have

$$\nabla_{c'(t)} c'(t) = \nabla_v v = 0.$$

Theorem 1.4.3.4 concludes that affine lines are all and only geodesics of $\mathbb{R}^{p,q}$, hence the pseudo-Euclidean space is geodesically complete.

Incidentally, the example shows that, for $\lambda \in \mathbb{R}\{0\}$, $\gamma_{(m,v)}$ and $\gamma_{(m,\lambda v)}$ share the same image. This is in fact the statement of the following result.

Proposition 1.4.3.10. *Let $m \in M$, $v \in T_mM$ and $\lambda \in \mathbb{R}$, then*

$$\gamma_{(m,\lambda v)}(t) = \gamma_{(m,v)}(\lambda t).$$

Moreover, $I_{(m,\lambda v)} = I_{(m,v)}/\lambda$, where $(a,b)/\lambda = (a/\lambda, b/\lambda)$ for $\lambda \neq 0$ and $\mathbb{R}$, otherwise.

Proof. Let $c(t) := \gamma_{(m,v)}(\lambda t)$, Proposition 1.4.3.3 states that a geodesic satisfies a linear system of ODEs, so if $\gamma_{(m,v)}(t)$ is a solution, $c(t)$ is a solution, too. By definition,

$$\begin{aligned} c(0) &= \gamma_{(m,v)}(\lambda 0) = \gamma_{(m,v)}(0) = m, \\ c'(0) &= \big(\gamma_{(m,v)}(\lambda t)\big)'|_{t=0} = \lambda \gamma'_{(m,v)}(t)|_{t=0} = \lambda v, \end{aligned}$$

so $c(t)$ is a geodesic with initial values $(m, \lambda v)$.

$c(t)$ is defined on $I_{(m,v)}/\lambda$, hence $I_{(m,v)}/\lambda \subseteq I_{(m,\lambda v)}$. We need to check that c can not be extended, *i.e.* the interval is maximal.

If $\lambda = 0$, $I_{(m,v)}/\lambda = \mathbb{R}$, which cannot be extended. Otherwise, assume $I_{(m,v)}/\lambda \subsetneq I_{(m,\lambda v)}$. This implies that $\gamma_{(m,\lambda v)}(t/\lambda)$ is a geodesic with initial values (m, v) defined on $\lambda I_{(m,\lambda v)} \supsetneq I_{(m,v)}$, which contradicts the definition of $I_{(m,v)}$. $\square$

In general, it could be difficult to derive the geodesic equation from the differential system described in Proposition 1.4.3.3. The following result permits to check more easily if a smooth curve is a geodesic.

Proposition 1.4.3.11. *Let $I \subseteq \mathbb{R}$ be an open interval, $c\colon I \to M$ be a regular curve. c is an unparameterized geodesic $\iff$ it exists $f \in C^\infty(I, \mathbb{R})$ such that $\nabla_{c'(t)} c'(t) = f(t)c'(t)$, $\forall t \in I$.*

Typically, in Riemannian geometry, this statement is proved by reparameterizing c by its arc-length parameterization and checking that it makes c a geodesic. This approach fails in the pseudo-Riemannian case: the "arc-length parameterization" for lightlike geodesics is the constant one, since the length of the tangent vector is identically 0. It can still be used to prove the statement for non-degerate geodesics. However, the following approach permits to check directly both cases.

Proof. Take a reparameterization γ of c, that is $c(t) := \gamma(\phi(t))$, where $\phi\colon I \to J$ is a diffeomorphis between open intervals of $\mathbb{R}$, namely ϕ' never vanishes. Hence $\gamma'(t) = \phi'(t)c'(\phi(t))$ and

$$\begin{aligned}\nabla_{c'(t)} c'(t) &= \phi'(t)\nabla_{\gamma'(\phi(t))}\left(\phi'(t)\gamma'(\phi(t))\right) = \\ &= \phi'(t)\phi''(t)\gamma'(\phi(t)) + \phi'(t)^2\nabla_{\gamma'(\phi(t))}\gamma'(\phi(t)) = \\ &= \phi''(t)c'(t) + \phi'(t)^2\nabla_{\gamma'(\phi(t))}\gamma'(\phi(t)).\end{aligned}$$

If c is an unparameterized geodesic, let ϕ be a reparameterization that makes γ a geodesic. The formula above becomes

$$\nabla_{c'(t)} c'(t) = \phi''(t)c'(t) + \phi'(t)^2 \underbrace{\nabla_{\gamma'(\phi(t))}\gamma'(\phi(t))}_{=0}.$$

that is $\nabla_{c'(t)} c'(t) = f(t)c'(t)$, $f = \phi'' \in C^\infty(M)$.

Conversely, let $\nabla_{c'(t)} c'(t) = f(t)c'(t)$. From the formula above,

$$\phi'(t)^2\nabla_{\gamma'(\phi(t))}\gamma'(\phi(t)) = \left(f(t) - \phi''(t)\right)c'(t).$$

It follows that if a reparameterization ϕ solves the ODE $\phi''(t) = f(t)$, $\nabla_{\gamma'(s)}\gamma'(s) \equiv 0$, *i.e.* γ is a geodesic.

Set $t_0 \in I$ and impose the initial value $\phi'_0 > 0$. It exists a local solution such that ϕ' never vanishes. Hence we can build an open covering $\{I_x\}_{x\in I}$ of I such that for all $x \in I$ it exists a parameterization ϕ_x and a geodesic $\gamma_x\colon I_x \to M$ such that $c(t) = \gamma_x(\phi_x(t))$. We want to glue the pieces in a single geodesic defined on I.

We claim that given two geodesic $c(\phi_i(t))$ defined on I_i, $i = x, y$, such that $I_x \cap I_y \neq \emptyset$, we can find another parameterization ψ_y and a geodesic $\gamma(s)$ such that

$$\gamma(s) = \begin{cases} \gamma_x(s) & \text{if } s \in I_x, \\ \tilde{\gamma}_y(s) & \text{if } s \in I_y, \end{cases}$$

where $\tilde{\gamma}_y(s)$ is the geodesic such that $c(t) = \tilde{\gamma}_y(\psi_y(t))$. The claim clearly implies the statement.

To prove it, set x, y as above and take $t_0 \in I_x \cap I_y$. Up to substitute $\phi_i(t)$ with $\phi_i(t) - \phi_i(t_0)$, $i = x, y$, we can assume $\phi_x(t_0) = \phi_y(t_0) = 0$, then

$$\gamma_x(0) = \gamma_x(\phi_x(t_0)) = c(t_0) = \gamma_y(\phi_y(t_0)) = \gamma_y(0),$$

i.e. the two geodesics intersect in $s = 0$.

We study the tangent vectors of γ_x in 0 using their implicite definition:

$$c'(t_0) = \frac{d}{dt}\gamma_x(\phi_x(t))|_{t=t_0} = \phi'_x(t_0)\gamma_x(\phi_x(t))|_{t=t_0} = \phi'_x(t_0)\gamma'_x(0).$$

For the same reason $c'(t_0) = \phi'_y(t_0)\gamma'_y(0)$, and both ϕ'_x, ϕ'_y never vanishs, so

$$\gamma'_y(0) = \frac{\phi'_y(t_0)}{\phi'_x(t_0)}\gamma'_x(0) = \lambda\gamma'_x(0), \qquad \lambda \in \mathbb{R} \setminus \{0\}.$$

Set $\psi_y(t) := \lambda\phi_y(t)$, and $\tilde{\gamma}_y(s)$ such that $c(t) = \tilde{\gamma}_y(\psi_y(t))$. By Proposition 1.4.3.10, $\tilde{\gamma}_y(s)$ is a geodesic such that

$$\begin{aligned} \tilde{\gamma}_y(0) &= \gamma_y(0) = \gamma_x(0), \\ \tilde{\gamma}'_y(0) &= \frac{1}{\lambda}\gamma'_y(0) = \gamma'_x(0), \end{aligned}$$

hence, for Corollary 1.4.3.5, it exists a geodesic γ that satisfies the claim, which ends the proof. $\square$

Incidentally, the proposition allows to check completeness of non-degenerate geodesic from any of its parameterization.

Lemma 1.4.3.12. *Let $c\colon (a, b) \to M$, $a, b \in \mathbb{R} \cup \{\pm\infty\}$, an unparameterized non-degenerate geodesic,* i.e $\nabla_{c'}c' = fc'$. *Let γ be the geodesic such that $c(t) = \gamma(\phi(t))$. γ is complete if and only if*

$$L(c|(t_0, b)) = L(c|(a, t_0)) = +\infty, \quad \forall t_0 \in (a, b).$$

If only one among $L(c|(t_0, b))$ and $L(c|(a, t_0))$ is infinite, the geodesic is complete on one side.

Proof. Let ϕ be the parameterization such that $c(t) = \gamma(\phi(t))$ and $\gamma(s)$ is a geodesic. We proved in (1.17) that tangent vectors of a geodesic are constant in norm: let $\ell := |g(\gamma'(s), \gamma'(s))|$, then for $s_0, s_1 \in \phi^{-1}([a, b]) \subseteq \mathbb{R} \cup \{\pm\infty\}$,

$$L\left(\gamma|(s_1, s_2)\right) = \int_{s_1}^{s_2} \sqrt{|g(\gamma'(s), \gamma'(s))|}dt = \sqrt{\ell}\,|s_1 - s_0|.$$

For $s_2 = \phi^{-1}(b)$, $s_1 \neq \phi^{-1}(a)$, $L(c|(t_0, b)) = +\infty$ if and only if $\phi^{-1}(b) = +\infty$, and analogously for $s_1 = \phi^{-1}(a)$, hence we proved the statement for a *parameterized* geodesic.

To conclude the proof it suffices to notice that the length of a curve does not depend on the parameterization, but only on its support. Indeed, let $t_1, t_2 \in (a,b)$, $s_i := \phi(t_i)$, and remark that $c'(t) = \phi'(t)\gamma'(\phi(t))$ and $\operatorname{sgn}(\phi'(t))$ is constant.

$$\begin{aligned}
L(\gamma|(s_1,s_2)) &= \left|\int_{s_1}^{s_2} \sqrt{g(\gamma'(s),\gamma'(s))}dt\right| = \\
&= \left|\int_{t_1}^{t_2} \sqrt{g(\gamma'(\phi(t)),\gamma'(\phi(t)))}\,\phi'(t)dt\right| = \\
&= \left|\operatorname{sgn}(\phi')\int_{t_1}^{t_2} \sqrt{g(\gamma'(\phi(t)),\gamma'(\phi(t)))}\,|\phi'(t)|dt\right| \\
&= \left|\int_{t_1}^{t_2} \sqrt{\phi'(t)^2 g(\gamma'(\phi(t)),\gamma'(\phi(t)))}dt\right| = \\
&= \left|\int_{t_1}^{t_2} \sqrt{g(\phi'(t)\gamma'(\phi(t)),\phi'(t)\gamma'(\phi(t)))}dt\right| \\
&= \left|\int_{t_1}^{t_2} \sqrt{g(c'(t),c'(t))}dt\right| = L(c|(t_1,t_2)).
\end{aligned}$$

□

Physically speaking, geodesics are paths that an object follows if not subject to external forces. Their mathematical interest comes from the fact that they and their maximal intervals of definition are invariant by isometries.

Proposition 1.4.3.13. *Let (M,g) be a pseudo-Riemannian manifold and $\phi \in \operatorname{Isom}(M)$. Let $\gamma_{(m,v)}$ be a geodesic, then $\phi(\gamma_{(m,v)}) = \gamma_{(\phi(m),d\phi_m v)}$ and $I_{(\phi(m),d\phi_m v)} = I_{(m,v)}$.*

Proof. Consider the curve $c(t) = (\phi \circ \gamma_{(m,v)})(t)$, so $c(0) = \phi(m)$.

$$c'(t) = \frac{d}{dt}(\phi \circ \gamma_{(m,v)})(t) = d\phi_{\gamma_{(m,v)}(t)}\gamma'_{(m,v)}(t) = (\phi_*\gamma'_{(m,v)})_t,$$

then $c'(0) = d\phi_m v$, *i.e* c satisfies the initial condition of $\gamma_{(\phi(m),d\phi_m v)}$. Corollary 1.2.2.13 shows that c is a geodesic:

$$\nabla_{(\phi_*\gamma'_{(m,v)})_t}(\phi_*\gamma'_{(m,v)})_t = \phi_*\nabla_{\gamma'_{(m,v)}(t)}\gamma'_{(m,v)}(t) = \phi_* 0 = 0.$$

We still have to check that c is maximal: by definition, c is defined over $I_{(m,v)}$, that is $I_{(m,v)} \subseteq I_{(\phi(m),d\phi_m v)}$. Suppose $I_{(m,v)} \subsetneq I_{(\phi(m),d\phi_m v)}$, that is c can be extended. Then, $\phi^{-1}(\gamma_{(\phi(m),d\phi_m v)})$ is a geodesic whit initial values (m,v) and defined on an interval which strictly contains $I_{(m,v)}$, which is an absurd by definition of $I_{(m,v)}$. □

1.4.4 Exponential map

The exponential map connects a pseudo-Riemannian manifold (M, g) and its tangent space TM using the notion of geodesic. Roughly speaking, it projects a subset of tangent space T_mM onto a neighborhood of m in M.

In Theorem 1.4.3.4, we built a smooth map $C(m, v, t)$, which associate at any point $(m, v) \in TM$ a geodesic $t \mapsto C(m, v, t) = c_{(m,v)}(t)$. However, we want the domain of our map to be a subset of TM, so the naïve approach is to fix the time $t = 1$ and define the exponential map as $C(\cdot, \cdot, 1)$.

Remark 1.4.4.1. One could think that C is not well defined, since *a priori* $C = C_{U\times V\times(-\varepsilon,\varepsilon)}$, namely is a different function depending on the domain. However, Corollary 1.4.3.5 implies that it is not the case.

We remark that the geodesic $t \mapsto C(m, v, t)$ can be extended at most to $\gamma_{(m,v)}$ (see Corollary 1.4.3.5). In general $1 \notin I_{(m,v)}$ which means $C(\cdot, \cdot, 1)$ is not well defined on all TM.

Definition 1.4.4.2 (Exponential map). Let (M, g) be a pseudo-Riemannian manifold, the *exponential map* of M is defined as

$$\begin{aligned} \exp\colon \Omega &\to M \\ (m, v) &\mapsto \gamma_{(m,v)}(1), \end{aligned}$$

where $\Omega := \{(m, v) \in TM, 1 \in I_{(m,v)}\}$.

We denote $\exp_m\colon \Omega_m \to M$ the restriction of exp to T_mM, *i.e.* $\Omega_m := \Omega|_{T_mM}$.

Proposition 1.4.4.3. *Let (M, g) be a pseudo-Riemannian manifold, Ω_m is a connected open neighborhood of $0 \in T_mM$, $\forall m \in M$.*

Proof. $\gamma_{(m,0)} \equiv m$, that is $I_{(m,0)} = \mathbb{R}$, hence $0 \in \Omega_m$.

Set $(m, v_0) \in \Omega_m$, consider V, ε as in Theorem 1.4.3.4; $\gamma_{(m,v)}$ is defined at least over $(-\varepsilon, \varepsilon)$, $\forall v \in V|_{T_mM}$. Define $W := \{\lambda v, v \in V_m\}$, $\lambda := \varepsilon/2$. W is an open neighborhood of v_0 in T_mM, since the topology on $T_mM \cong \mathbb{R}^n$ is the Euclidean one. For $w \in W$, $\gamma_{(m,w)}$ is defined at least on $(-\varepsilon, \varepsilon)/\lambda = (-2, 2)$ (see Proposition 1.4.3.10), that is $W \subseteq \Omega_m$.

Finally, Ω_m is path-connected and so connected: indeed, let $v \in \Omega_m$, that is $1 \in I_{(m,v)}$. Since $I_{(m,\lambda v)} = I_{(m,v)}/\lambda$, $I_{(m,v)} \subseteq I_{(m,\lambda v)}$, $\forall\lambda \in [0, 1]$, that is the segment $[0, v] \subseteq \Omega_m$. □

Now that we have showed that the domain is open, we have a function between open subsets of smooth manifolds, then we can check its smoothness.

Proposition 1.4.4.4. *The exponential map is smooth.*

Proof. Set $(m_0, v_0) \in T_mM$, by definition $\exp(m, v) = C(m, v, 1)$ over $U\times W$, which is smooth by Theorem 1.4.3.4. □

Corollary 1.4.4.5. *Let* (M, g) *be a pseudo-Riemannian manifold,* $m \in M$*. It exists* U *open neighborhood of* $0 \in T_mM$ *such that* $\exp_m : U \to M$ *is a local diffeomorphism.*

Proof. We claim that $d(\exp_m)_0 = \mathrm{Id}_{T_mM}$, hence the differential of $\exp_m$ is invertible in $v = 0$, *i.e.* $\exp_m$ is a local diffeomorphism from an open neighborhood U of $0 \in T_mM$ to M.

To prove the claim, it suffice to remark that

$$\exp_m(tv) = \gamma_{(m,tv)}(1) = \gamma_{(m,v)}(t),$$

and that $\gamma'_{(m,v)}(0) = v$, hence

$$d(\exp_m)_0 v = \frac{d}{dt} \exp_m(tv)|_{t=0} = \frac{d}{dt}\gamma_{(m,v)}(t)|_{t=0} = v.$$

□

Corollary 1.4.4.6. *Let* (M, g) *be a pseudo-Riemannian manifold,* $m \in M$*,* $\phi \in \mathrm{Isom}(M)$*, then*

i. $\phi \circ \exp_m = \exp_{\phi(m)} \circ d\phi_m$*,*

ii. $\Omega_{\phi(m)} = \Omega_m$*.*

Proof. It comes straight from Proposition 1.4.3.13. □

Proposition 1.4.4.7. *Let* (M, g) *be a connected pseudo-Riemannian manifold,* $\phi, \psi \in \mathrm{Isom}(M)$*. If there exists* $\bar{m} \in M$ *such that* $\phi(\bar{m}) = \psi(\bar{m})$ *and* $d\phi_{\bar{m}} = d\psi_{\bar{m}}$*, then* $\phi = \psi$*.*

Proof. The set $A := \{m \in M | \phi(m) = \psi(m), d\phi_m = d\psi_m\}$ is not empty by hypothesis and closed, since ϕ, ψ and $d\phi, d\psi$ are continuous functions.

We claim that A is open. If this is the case, then $A = M$ because M is connected, that is $\phi = \psi$.

Choose $m \in A$, by Proposition 1.4.4.6

$$\phi \circ \exp_m = \exp_{\phi(m)} d\phi_m = \exp_{\psi(m)} d\psi_m = \psi \circ \exp_m .$$

By Proposition 1.4.4.5, $\exp_m$ is a local diffeomorphism on an open neighborhood U of $0 \in T_mM$, hence ϕ and ψ coincide over $\exp_m(U)$, and so do their differentials, that is $\exp_m(U) \subseteq A$. $\exp_m(U)$ is open, so the claim is proved. □

1.4.5 Totally geodesic submanifolds

Definition 1.4.5.1 (Totally geodesic submanifold). Let (M, g) be a pseudo-Riemannian manifold, a smooth submanifold $N \subseteq M$ is *totally geodesic* if all geodesics of N are geodesics of M, too.

Equivalentely, a smooth submanifold $N \subseteq M$ is *totally geodesic* if $\forall m \in N$, $\forall v \in T_m N$, it exists $\delta > 0$ such that the geodesic $\gamma_{(m,v)}$ of M is contained in N, $\forall t \in (-\delta, \delta)$.

When we talk about totally geodesic submanifold, we will assume they are maximal, *i.e* not properly included in any other totally geodesic submanifold of the same dimension.

Example 1.4.5.2. Geodesics are totally geodesic 1-submanifold. Particularly, lightlike geodesics are not pseudo-Riemannian submanifold, and we will call them *degenerate* totally geodesic submanifold.

Corollary 1.4.5.3. *Let (M, g) be a pseudo-Riemannian manifold, $\phi \in \mathrm{Isom}(M)$ and $N \subseteq M$ a totally geodesic submanifold. Then $\phi(N)$ is a totally geodesic submanifold. Moreover, N is maximal if and only if $\phi(N)$ is maximal.*

Proof. It comes straight from Proposition 1.4.3.13. □

Example 1.4.5.4 (Pseudo-Euclidean space). In $\mathbb{R}^{p,q}$ totally geodesic submanifold are affine subspaces.

Proposition 1.4.5.5. *Let (M, g) be a pseudo-Riemannian submanifold, $\phi \in \mathrm{Isom}(M)$, then $\mathrm{Fix}(\phi)$ is a totally geodesic submanifold.*

Proof. It is clear that fixed points of a diffeomorphism form a smooth submanifold: they are the zeros of the submersion $x(\phi(m)) - x(m)$, x being a local chart.

Note $F := \mathrm{Fix}(\phi)$, since $\phi|_F = \mathrm{Id}_F$, $d\phi|_{TF} = \mathrm{Id}_{TF}$. Let $m \in F$, $v \in T_m F$, $\gamma_{(m,v)}$ the geodesic with initial values (m, v). From Proposition 1.4.3.13,

$$\phi(\gamma_{(m,v)}(t)) = \gamma_{(\phi(m), d\phi_m v)}(t),$$

but $\phi(m) = m$ and $d\phi_m v = v$, hence

$$\phi(\gamma_{(m,v)}(t)) = \gamma_{(m,v)}(t),$$

that is $\gamma_{(m,v)}(t) \in \mathrm{Fix}(\phi) = F$, $\forall t$. □

Example 1.4.5.6 (Pseudo-hyperbolic space). As stated in Example 1.1.2.12, $\mathrm{O}(p, q+1)$ induces the isometry group of $\widetilde{\mathbb{H}}^{p,q}$, and it is an easy exercise to prove that is generated by reflections. This proves that any non-degenerate hyperplane is fixed by an isometry, and so is its intersection with $\widetilde{\mathbb{H}}^{p,q}$ is a totally geodesic submanifold. The same holds in higher codimension: by composing reflections, one can build an isometry fixing any non-degenerate vector subspace of $\mathbb{R}^{p,q+1}$.

A continuity argument or a direct computation using the differential system defined in Proposition 1.4.3.3 shows that degenerate totally geodesic subspace can be described in the same way, that is as intersection of $\widetilde{\mathbb{H}}^{p,q}$ and degenerate vector subspaces of $\mathbb{R}^{p,q+1}$.

Let V be a vector subspace of $\mathbb{R}^{p,q+1}$ with signature (n_+, n_-, n_0). Hence $\widetilde{S} = \widetilde{\mathbb{H}}^{p,q} \cap V$ is a totally geodesic submanifold of $\widetilde{\mathbb{H}}^{p,q}$ with signature $(n_+, n_- - 1, n_0)$. To see that, it suffices to remark that

$$T_m V = T_m \widetilde{S} \oplus (T_m \widetilde{\mathbb{H}}^{p,q})^{\perp},$$

is an orthogonal decomposition of $T_m V$, so the result follows from Sylvester's criterion. We add two remarks to this computation: if $n_- = 0$ the formula above makes no sense: that proves that the intersection is empty in such case (consider for example degenerates hyperplanes for $\widetilde{\mathbb{H}}^n$). Moreover, $\widetilde{S}$ is degenerate if and only if V is degenerate.

Recalling that $\mathbb{H}^{p,q} = \mathbb{P}(\widetilde{\mathbb{H}}^{p,q})$, totally geodesic subspaces of pseudo-hyperbolic space are the intersection of $\mathbb{P}\{\langle v, v\rangle_{p,q} < 0\}$ with projections of vector subspaces of $\mathbb{R}^{p,q+1}$. In particular, any geodesic of $\mathbb{H}^{p,q}$ is the projection via $\mathbb{P}$ of a vector 2-plane of $\mathbb{R}^{p,q+1}$, and it is lightlike if and only if the 2-plane is degenerate (to be precise, a 2-plane of signature $(0, 1, 1)$).

Totally geodesic submanifold are *flat* from the point of view of an inhabitant of the manifold, that is the intrinsical curvature tensor is the same as the extrinsical. In fact, this property characterizes non-degenerate totally geodesic submanifolds.

Proposition 1.4.5.7. *Let (M, g) be a pseudo-Riemannian manifold and $N \subseteq M$ be a pseudo-Riemannian submanifold. Denote D, ∇ their Levi-Civita connection, respectively. N is totally geodesic if and only if $D|_{\Gamma(TN)} = \nabla$.*

Proof. The implication $(\Leftarrow)$ is trivial: indeed, assume $\gamma \colon I \to N$ is a geodesic of N, namely it satisfies $\nabla_{\gamma'}\gamma' = 0$. Since $\gamma'(t) \in \Gamma(TN)$, $\forall t \in I$, one has

$$D_{\gamma'}\gamma' = (D|_{\Gamma(TN)})_{\gamma'}\gamma' = \nabla_{\gamma'}\gamma' = 0,$$

that is γ is a geodesic of M, too.

Conversely, we need to prove that $D_X Y \in \Gamma(TN)$, $\forall X, Y \in \Gamma(TN)$, which is equivalent to check $g(D_X Y, Z) = 0, \forall X, Y \in \Gamma(TN), \forall Z \in \Gamma(TN^{\perp})$.

We claim it suffice to prove the equation for $Y = X$: indeed, if it is the case, let $X, Y \in \Gamma(TN)$, $Z \in \Gamma(TN^{\perp})$,

$$\begin{aligned} 0 &= g(D_{X+Y}(X+Y), Z) = \\ &= \underbrace{g(D_X X, Z)}_{=0} + g(D_X Y, Z) + g(D_Y X, Z) + \underbrace{g(D_Y Y, Z)}_{=0} = \\ &= 2g(D_X Y, Z) + g([X, Y], Z). \end{aligned} \tag{1.19}$$

$[X,Y] \in \Gamma(TN)$, so it is orthogonal to Z, too, *i.e.* $g([X,Y],Z) = 0$. Equation (1.19) above becomes $g(D_X Y, Z) = 0$, which proves the claim.

$$g(D_X X, Z) = X(\underbrace{g(X,Z)}_{\equiv 0}) - g(X, D_X Z) = -g(X, D_X Z),$$

hence it suffices to prove $g(X, D_X Z) = 0$, $\forall X \in \Gamma(TN), Z \in \Gamma(TN^{\perp})$.

By the absurd, assume there exist $m \in N$, $X \in \Gamma(TN)$, $Z \in \Gamma(TN^{\perp})$ such that

$$g_m\left(X_m, (D_X Z)_m\right) \neq 0.$$

Since the equation is $C^{\infty}(M)$-linear with respect to X, one can substitute X with any vector field Y such that $Y_m = X_m$.

Choosing $Y := \gamma'_{(m, X_m)}$, which belongs to TN by hypothesis, one obtains

$$g(X, D_X Z) = g\left(\gamma', (D_{\gamma'} Z)\right) = \gamma'(\underbrace{g(\gamma', Z)}_{\equiv 0}) - g(\underbrace{D_{\gamma'}\gamma'}_{\equiv 0}, Z) = 0.$$

then it is 0 even at m, which is a contradiction. $\square$

Chapter 2

The half-space model

In this chapter, we will study the half space model. First we will introduce the model, justify the title *model of the pseudo-hyperbolic space*, and study some isometries.

After that we will study intrinsecally its geometry. The main results are the classification of totally geodesic subspace (Section 2.4) and the description of geodesics (Section 2.4). From the classification, it arises a way to describe the boundary of $\mathcal{H}^{p,q}$ and to extend it. The extended boundary $\partial_\infty \mathcal{H}^{p,q}$ will be proved to be homeomorphic to $\partial_\infty \mathbb{H}^{p,q}$ (Section 2.6). We will also described horospheres of the model in Section 2.7.

Finally, we will describe the isometries of $\mathcal{H}^{p,q}$. In Subection 2.8.1 we will present the actual isometry group, and in Section 2.8.2 the action of $\mathrm{Isom}(\mathbb{H}^{p,q})$ on $\mathcal{H}^{p,q}$ by local isometries, which are the analogue of inversions in the Riemannian case.

2.1 Introduction to the model

In this section we will introduce the half-space model $\mathcal{H}^{p,q}$ and justify the name of *model of the pseudo-hyperbolic space* by exhibiting an isometric embedding $\iota_{p,q}\colon \mathcal{H}^{p,q} \hookrightarrow \mathbb{H}^{p,q}$. Then we will provide a first attempt to discover the isometries of the model.

2.1.1 The half-space model

Definition 2.1.1.1 (Half-space). Let $p, q \in \mathbb{N}$, $p \geq 1$, the *half-space* of signature (p, q) is defined as

$$\mathcal{H}^{p,q} := \{(x, y, z) \in \mathbb{R}_x^{p-1} \oplus \mathbb{R}_y^q \oplus \mathbb{R}_z | z > 0\},$$

endowed with the pseudo-Riemannian metric

$$g_{p,q} = \frac{dx_1^2 + \ldots + dx_{p-1}^2 - dy_1^2 - \ldots - dy_q^2 + dz^2}{z^2}. \tag{2.1}$$

Example 2.1.1.2. When $q = 0$, one recovers Poincaré half-space, which is well known to be isomorphic to $\mathbb{H}^{p,0} = \mathbb{H}^p$. For references, see [BP92].

Example 2.1.1.3. For $q = 1$, $\mathcal{H}^{p,1}$ is the half-space model for the Anti-de Sitter space, namely $\mathbb{A}\mathrm{d}\mathbb{S}^{p+1} = \mathbb{H}^{p,1}$. This model is not complete and has been studied in [RS19].

Example 2.1.1.4. Finally, if $p = 1$ and $q \geq 1$, $\mathcal{H}^{1,q}$ is an anti-model for the de Sitter space $\mathrm{d}\mathbb{S}^{q+1}$, that is $(\mathcal{H}^{1,q}, -g_{p,q})$ is a model for the unitary pseudo-spheric space $\mathbb{S}^{q,1}$. This case has been studied with the name of *Lorentz-Poincaré half-space* in [Nom82].

2.1.2 An isometric embedding

The next proposition proves that $\mathcal{H}^{p,q}$ can be seen as an open subset of $\mathbb{H}^{p,q}$, so it inherits all its local properties, such as the sectional curvature, that is then constant $K = -1$. Nevertheless, we will check it directly in Section 2.3.

The image of $\iota_{p,q}$ is dense in $\mathbb{H}^{p,q}$. This result permits to extend $\mathcal{H}^{p,q}$ to a complete model of the pseudo-hyperbolic space, in a sense that we will explore in Subsection 2.6.4.

Remark 2.1.2.1 (Notation). By a small abuse of notation, from now on we will use $\|\cdot\|$ for the usual norm and $\langle\cdot,\cdot\rangle$ for the usual scalar product of $\mathbb{R}^{p-1}_x$ and $\mathbb{R}^q_y$.

Proposition 2.1.2.2. *There exists an isometric embedding*

$$\iota_{p,q} \colon \mathcal{H}^{p,q} \to \mathbb{H}^{p,q} \ .$$

If $q = 0$, $\iota_{p,q}$ is surjective. Otherwise, its image is the complement of a totally geodesic degenerate hyperplane in $\mathbb{H}^{p,q}$.

Proof. We will first define an embedding $\tilde{\iota}_{p,q} \colon \mathcal{H}^{p,q} \to \widetilde{\mathbb{H}}^{p,q} \subset \mathbb{R}^{p,q+1}$. Then define $\tilde{\iota}_{p,q}(x,y,z) = (X_1, \dots, X_{p+q+1})$ where:

$$\begin{aligned}
X_i &= \frac{x_i}{z} && i = 1, \dots, p-1,\\
X_p &= \frac{1 - \|x\|^2 + \|y\|^2 - z^2}{2z} && \\
X_{j+p} &= \frac{y_j}{z} && j = 1, \dots, q,\\
X_{p+q+1} &= \frac{1 + \|x\|^2 - \|y\|^2 + z^2}{2z}. &&
\end{aligned}$$

Since $(a-b)^2-(a+b)^2 = -4ab$, one checks immediately that $\forall(x,y,z) \in \mathcal{H}^{p,q}$

$$X_p^2 - X_{p+q+1}^2 = \left(\frac{1-\|x\|^2+\|y\|^2-z^2}{2z}\right)^2 - \left(\frac{1+\|x\|^2-\|y\|^2+z^2}{2z}\right)^2 =$$
$$= -\frac{\|x\|^2-\|y\|^2+z^2}{z^2} = -\left(\sum_{i=1}^{p-1} X_i^2 - \sum_{j=1}^{q} X_{j+p}^2\right) - 1,$$

hence $\langle \tilde{\iota}_{p,q}(x,y,z), \tilde{\iota}_{p,q}(x,y,z)\rangle_{p,q+1} = -1$, *i.e.* $\tilde{\iota}$ takes values in $\widetilde{\mathbb{H}}^{p,q}$.

To prove that $\tilde{\iota}_{p,q}$ is an isometry, one can easily compute the differential:

$$\begin{aligned}
d\tilde{\iota}_{p,q}\left(\frac{\partial}{\partial x_i}\right) =& \frac{1}{z}\frac{\partial}{\partial X_i} - \frac{x_i}{z}\frac{\partial}{\partial X_p} + \frac{x_i}{z}\frac{\partial}{\partial X_{p+q+1}},\\
d\tilde{\iota}_{p,q}\left(\frac{\partial}{\partial y_j}\right) =& \frac{y_j}{z}\frac{\partial}{\partial X_p} + \frac{1}{z}\frac{\partial}{\partial X_{j+p}} - \frac{y_j}{z}\frac{\partial}{\partial X_{p+q+1}},\\
d\tilde{\iota}_{p,q}\left(\frac{\partial}{\partial z}\right) =& -\sum_{i=1}^{p-1}\frac{x_i}{z^2}\frac{\partial}{\partial X_i} - \frac{1-\|x\|^2+\|y\|^2+z^2}{2z^2}\frac{\partial}{\partial X_p} -\\
& -\sum_{j=1}^{q}\frac{y_j}{z^2}\frac{\partial}{\partial X_{j+p}} - \frac{1+\|x\|^2-\|y\|^2-z^2}{2z^2}\frac{\partial}{\partial X_{p+q+1}}.
\end{aligned}$$

Above, we wrote the push-forward of the basis $\{\partial_{x_1},\dots,\partial_{x_{p-1}},\partial_{y_1},\dots,\partial_{y_q},\partial_z\}$ of $T\mathcal{H}^{p,q}$ with respect to the orthogonal basis $\{\partial_{X_1},\dots,\partial_{X_{p+q+1}}\}$ of $T\mathbb{R}^{p,q+1}$.

$$\begin{aligned}
\langle d\tilde{\iota}_{p,q}(\partial_{x_i}), d\tilde{\iota}_{p,q}(\partial_{x_j})\rangle_{p,q+1} =& \frac{1}{z^2}\underbrace{\langle\frac{\partial}{\partial X_i},\frac{\partial}{\partial X_j}\rangle_{p,q+1}}_{=\delta_{ij}}\\
+\frac{x_ix_j}{z^2}\underbrace{\langle\frac{\partial}{\partial X_p},\frac{\partial}{\partial X_p}\rangle_{p,q+1}}_{=1} &+ \frac{x_ix_j}{z^2}\underbrace{\langle\frac{\partial}{\partial X_{p+q+1}},\frac{\partial}{\partial X_{p+q+1}}\rangle_{p,q+1}}_{=-1} =\\
=\frac{\delta_{ij}}{z^2} = g_{p,q}(\partial_{x_i},\partial_{x_j}), &\qquad \forall i,j=1,\dots,p-1.
\end{aligned}$$

Similar calculations show

$$\begin{aligned}
&\langle d\tilde{\iota}_{p,q}(\partial_{x_i}), d\tilde{\iota}_{p,q}(\partial_{y_j})\rangle_{p,q+1} = 0 = g_{p,q}(\partial_{x_i},\partial_{y_j}), \qquad \begin{array}{l}\forall i=1,\dots,p-1,\\ \forall j=1,\dots,q,\end{array}\\
&\langle d\tilde{\iota}_{p,q}(\partial_{y_i}), d\tilde{\iota}_{p,q}(\partial_{y_j})\rangle_{p,q+1} = -\frac{\delta_{ij}}{z^2} = g_{p,q}(\partial_{y_i},\partial_{y_j}), \quad \forall i,j=1,\dots,q.
\end{aligned}$$

Now we compute it for ∂_z:

$$\begin{aligned}\langle d\tilde{\iota}_{p,q}(\partial_{x_i}), d\tilde{\iota}_{p,q}(\partial_z)\rangle_{p,q+1} &= -\frac{x_i}{z^3}\underbrace{\langle\frac{\partial}{\partial X_i},\frac{\partial}{\partial X_i}\rangle_{p,q+1}}_{=1}+\\ &+\frac{x_i(1-\|x\|^2+\|y\|^2+z^2)}{2z^3}\underbrace{\langle\frac{\partial}{\partial X_p},\frac{\partial}{\partial X_p}\rangle_{p,q+1}}_{=1}-\\ &-\frac{x_i(1+\|x\|^2-\|y\|^2-z^2)}{2z^3}\underbrace{\langle\frac{\partial}{\partial X_{p+q+1}},\frac{\partial}{\partial X_{p+q+1}}\rangle_{p,q+1}}_{=-1}=\\ &=-\frac{x_i}{z^3}+2\frac{x_i}{2z^3}=0=g_{p,q}(\partial_{x_i},\partial_z), \qquad \forall i=1,\dots,p-1,\end{aligned}$$

and analogously

$$\langle d\tilde{\iota}_{p,q}(\partial_{y_j}), d\tilde{\iota}_{p,q}(\partial_z)\rangle_{p,q+1}=0=g_{p,q}(\partial_{y_j},\partial_z), \qquad j=1,\dots,q.$$

Finally, one finds

$$\begin{aligned}&\langle d\tilde{\iota}_{p,q}(\partial_z), d\tilde{\iota}_{p,q}(\partial_z)\rangle_{p,q+1}=\\ &=\frac{\|x\|^2}{z^4}+\frac{(1-\|x\|^2+\|y\|^2+z^2)^2}{4z^4}-\frac{\|y\|^2}{z^4}-\frac{(1+\|x\|^2-\|y\|^2-z^2)^2}{4z^4}=\\ &=\frac{\|x\|^2}{z^4}-\frac{\|y\|^2}{z^4}-4\frac{\|x\|^2-\|y\|^2-z^2}{4z^4}=\frac{1}{z^2}=g_{p,q}(\partial_z,\partial_z).\end{aligned}$$

Hence we proved that $\tilde{\iota}_{p,q}^*\langle\cdot,\cdot\rangle_{p,q+1}=g_{p,q}(\cdot,\cdot)$, that is $\tilde{\iota}_{p,q}\colon \mathcal{H}^{p,q}\to\widetilde{\mathbb{H}}^{p,q}$ is a local isometry.

Let us now show that

$$\tilde{\iota}_{p,q}(\mathcal{H}^{p,q})=\widetilde{\mathbb{H}}^{p,q}\cap\{X_p+X_{p+q+1}>0\}. \tag{2.2}$$

The inclusion $\subseteq$ is trivial as $X_p+X_{p+q+1}=1/z>0$. For the other inclusion, given $(X_1,\dots,X_{p+q+1})$ such that $\langle X,X\rangle_{p,q+1}=-1$ and $X_p+X_{p+q+1}>0$, define

$$\begin{aligned}x_i&=\frac{X_i}{X_p+X_{p+q+1}} & i&=1,\dots,p-1,\\ y_j&=\frac{X_{j+p}}{X_p+X_{p+q+1}} & j&=1,\dots,q,\\ z&=\frac{1}{X_p+X_{p+q+1}}\ .\end{aligned}$$

Let $Y=(Y_1,\dots,Y_{p+q+1}):=\tilde{\iota}(x,y,z)$, we will check that $Y=X$:

$$\begin{aligned}Y_i&=\frac{x_i}{z}=\frac{X_i}{X_p+X_{p+q+1}}\Big/\frac{1}{X_p+X_{p+q+1}}=X_i & i&=1,\dots,p-1,\\ Y_{p+j}&=\frac{y_j}{z}=\frac{X_{p+j}}{X_p+X_{p+q+1}}\Big/\frac{1}{X_p+X_{p+q+1}}=X_{p+j} & j&=1,\dots,q.\end{aligned}$$

Finally, $\langle X, X\rangle_{p,q+1} = -1$, hence

$$\begin{aligned} X_p^2 - X_{p+q+1}^2 &= -\left(\sum_{i=1}^{p-1} X_i^2 - \sum_{j=1}^{q} X_{p+j}^2 + 1\right) = \\ &= -\left(\sum_{i=1}^{p-1} \frac{x_i^2}{z^2} - \sum_{j=1}^{q} \frac{y_j^2}{z^2} + 1\right) = \\ &- \frac{1}{z^2}(\|x\|^2 - \|y\|^2 + z^2), \end{aligned}$$

which leads to

$$\begin{aligned} Y_p =& \frac{1 - \|x\|^2 + \|y\|^2 - z^2}{2z} = \frac{1}{2}\left(\frac{1}{z} + z(X_p^2 - X_{p+q+1}^2)\right) = \\ =& \frac{1}{2}\left(X_p + X_{p+q+1} + \frac{X_p^2 - X_{p+q+1}^2}{X_p + X_{p+q+1}}\right) = X_p, \\ Y_{p+q+1} =& \frac{1 + \|x\|^2 - \|y\|^2 + z^2}{2z} = \frac{1}{2}\left(\frac{1}{z} - z(X_p^2 - X_{p+q+1}^2)\right) = \\ =& \frac{1}{2}\left(X_p + X_{p+q+1} - \frac{X_p^2 - X_{p+q+1}^2}{X_p + X_{p+q+1}}\right) = X_{p+q+1}. \end{aligned}$$

Incidentally, in the above argument we constructed an inverse of $\tilde{\iota}_{p,q}$ over its image, which implies that $\tilde{\iota}_{p,q}$ is injective, so an isometric embedding. It also follows from (2.2) that the restriction of $\mathbb{P}$ to the image of $\tilde{\iota}_{p,q}$ is injective, where $\mathbb{P}$ is the projection from $\widetilde{\mathbb{H}}^{p,q}$ to $\mathbb{H}^{p,q}$. Indeed, $\mathbb{P}$ is a 2-sheets covering, and $\mathbb{P}^{-1}([X]) = \{\pm X\}$, so the condition $X_p + X_{p+q+1} > 0$ is satisfied at most by one point of the preimage.

Hence, defining $\iota_{p,q} = \mathbb{P} \circ \tilde{\iota}_{p,q}$, $\iota_{p,q}$ is an isometric embedding whose image is the complement of $P \cap \mathbb{H}^{p,q}$, where P is the hyperplane defined by the condition $X_p + X_{p+q+1} = 0$.

Observe that for $q = 0$, the intersection $P \cap \mathbb{H}^{p,0}$ is empty: indeed X_{p+0+1} is the only negative contribute to the norm, hence $\langle X, X\rangle_{p,1} \geq 0$ if if $X_p^2 = X_{p+0+1}^2$. As a consequence, we recover that $\iota_{p,0}$ is a global isometry between the half-space model and the hyperboloid model of the hyperbolic space.

Otherwise, when $q \geq 1$, P is a totally geodesic hyperplane in $\mathbb{H}^{p,q}$ (see Example 1.4.5.6), which is degenerate because P is degenerate in $\mathbb{R}^{p,q+1}$, being the orthogonal complement of the line spanned by the isotropic vector $\partial_{X_p} - \partial_{X_{p+q+1}}$. $\square$

2.1.3 Symmetries

Here, we introduce the first isometries of $\mathcal{H}^{p,q}$.

Definition 2.1.3.1 (Group action). Let X be a topological space and G a group. An *action* of G on X is a homomorphism $G \to \mathrm{Homeo}(X)$.

Definition 2.1.3.2 (Faithful action). A group G acts *faithfully* on a space X if the homomorphism $G \to \mathrm{Isom}(X)$ is injective.

We remark that any slice $\{z = c\}$, $c \in \mathbb{R}^+$, is conformal to $\mathbb{R}^{p-1,q}$: indeed, the induced pseudo-metric is

$$g|_{\{z=c\}} = \frac{1}{c^2}(dx_1^2 + \ldots + dx_{p-1}^2 - dy_1^2 - \ldots - dy_q^2).$$

From this observation comes the following result:

Lemma 2.1.3.3. $\mathrm{Isom}(\mathbb{R}^{p-1,q})$ *acts faithfully on* $\mathcal{H}^{p,q}$ *by isometries of the form*

$$(x, y, z) \mapsto (A(x, y) + (x_0, y_0), z),$$

$A \in \mathrm{O}(p-1, q)$*,* $(x_0, y_0) \in \mathbb{R}_x^{p-1} \oplus \mathbb{R}_y^q = \mathbb{R}^{p-1,q}$*.*

Proof. Maps of that form preserve the slices $\{z = c\}$ and, as they are conformal to $\mathbb{R}^{p-1,q}$, are isometries on them. It is clear that the action is faithful: indeed, it is so on the slices by definition, and does not affect the last coordinate. $\square$

Proposition 2.1.3.4. $\mathbb{R}^+$ *acts faithfully by homotheties on* $\mathbb{R}^{p+q}$*, which are isometries of* $\mathcal{H}^{p,q}$*.*

Proof. Let $\lambda > 0$, define $\Lambda(x, y, z) = \lambda(x, y, z)$ the homothety with ratio λ. As $\lambda > 0$, λz is positive, too. Hence, it is well defined $\Lambda|_{\mathcal{H}^{p,q}} : \mathcal{H}^{p,q} \to \mathcal{H}^{p,q}$, which is clearly bijective. Moreover,

$$\begin{aligned}(\Lambda^* g_{p,q})_{\Lambda(x,y,z)} &= \frac{d(\lambda x_1)^2 + \ldots + d(\lambda x_{p-1})^2 - d(\lambda y_1)^2 - \ldots - d(\lambda y_q)^2 + d(\lambda z)^2}{(\lambda z)^2} = \\ &= (g_{p,q})_{(x,y,z)}, \qquad \forall (x, y, z) \in \mathcal{H}^{p,q},\end{aligned}$$

then Λ is an isometry of $\mathcal{H}^{p,q}$. $\square$

The representations of $\mathbb{R}^+$ and $\mathrm{Isom}(\mathbb{R}^{p-1,q})$ as subgroups of $\mathrm{Isom}(\mathcal{H}^{p,q})$ are injective, so we will abusively write $\mathbb{R}^+$ and $\mathrm{Isom}(\mathbb{R}^{p-1,q})$ for their isomorphic images in $\mathrm{Isom}(\mathcal{H}^{p,q})$. Now we are ready to give the following definition:

Definition 2.1.3.5. Let G is the set of maps defined as

$$(x, y, z) \mapsto \lambda \left(A(x, y) + (x_0, y_0), z\right),$$

$\lambda \in \mathbb{R}^+$, $A \in \mathrm{O}(p-1, q)$ and $(x_0, y_0) \in \mathbb{R}_x^{p-1} \oplus \mathbb{R}_y^q$.

Proposition 2.1.3.6. $G \leq \mathrm{Isom}(\mathcal{H}^{p,q})$.

Proof. $G = \mathbb{R}^+ \oplus \mathrm{Isom}(\mathbb{R}^{p-1,q})$, which is a subgroup $\mathrm{Isom}(\mathcal{H}^{p,q})$.

Indeed, every application in G can be obtained by composing two isometries of the two groups. Moreover,

$$\mathbb{R}^+ \cap \mathrm{Isom}(\mathbb{R}^{p-1,q}) = \mathrm{Id}_{\mathcal{H}^{p,q}},$$

since the only homothety preserving the slices $\{z = c\}$ is $\mathrm{Id}_{\mathcal{H}^{p,q}}$. □

Corollary 2.1.3.7. *G acts transitively on $\mathcal{H}^{p,q}$.*

Proof. One can bring a point to any height by the means of an homothety, and then change the other coordinates using an horizontal translation.

More precisely, given $(x, y, z), (v, w, t) \in \mathcal{H}^{p,q}$, we seek an isometry such that $(x, y, z) \mapsto (v, w, t)$.

Both z and t are positive, so $\lambda := t/z$ is positive, too. Hence

$$(x, y, z) \mapsto \lambda(x, y, z) = (\lambda x, \lambda y, t).$$

Then, taking $x_0 = v - \lambda x$, $y_0 = w - \lambda y$, one obtains

$$\begin{pmatrix} \lambda x \\ \lambda y \\ t \end{pmatrix} \longmapsto \begin{pmatrix} \lambda x + x_0 \\ \lambda y + y_0 \\ t \end{pmatrix} = \begin{pmatrix} \lambda x + (v - \lambda x) \\ \lambda y + (w - \lambda y) \\ t \end{pmatrix} = \begin{pmatrix} v \\ w \\ t \end{pmatrix}.$$

The map $(x, y, z) \mapsto \lambda(x + x_0, y + y_0, z)$ belongs to G, so it is the searched isometry. □

Definition 2.1.3.8 (Stabilizer). Let X be a set and G a group acting on X. For $x \in X$, the *stabilizer* subgroup of G with respect to x is

$$\mathrm{Stab}_G(x) := \{g \in G, g(x) = x\}.$$

Lemma 2.1.3.9. *The stabilizer of a point in G is isomorphic to* $\mathrm{O}(p-1, q)$.

Proof. Since G acts transitively, the proof does not depend on the point chosen. An isometry $g(x, y, z) = \lambda(A(x, y) + (x_0, y_0), z)$ of G sends $(0, 0, 1)$ to (x_0, y_0, λ), so $g \in \mathrm{Stab}_G(0, 0, 1)$ if and only if $\lambda = 1$, $x_0 = y_0 = 0$, then is uniquely defined by $A \in \mathrm{O}(p - 1, q)$. □

We will see in Theorem 2.8.1.1 that G is actually the full isometry group $\mathrm{Isom}(\mathcal{H}^{p,q})$ when $q \geq 1$. Since every local isometry between open neighbourhoods of $\mathbb{H}^{p,q}$ extends to a global isometry, the isometric embedding $\iota_{p,q}$ induces a group monomorphism from G to $\mathrm{Isom}(\mathbb{H}^{p,q})$, which is clearly not surjective because in $\mathrm{Isom}(\mathbb{H}^{p,q})$ there are isometries that do not preserve the totally geodesic hyperplane whose complement is the image of $\iota_{p,q}$. (Indeed if $n = p + q$, then G is a Lie group of dimension $(n^2 - n + 2)/2$, while $\mathrm{Isom}(\mathbb{H}^{p,q})$, which is isomorphic to a double quotient of $\mathrm{O}(p, q + 1)$, has dimension $n(n + 1)/2 = \dim G + n - 1$).

2.2 Levi-Civita connection of $\mathcal{H}^{p,q}$

From now on, when we work in charts, we will always refer to the one induced by the inclusion (as smooth manifold) of $\mathcal{H}^{p,q} \subseteq \mathbb{R}^{p+q}$, and we will note the basis induced on the tangent as $\{\partial_{x_1}, \dots, \partial_{x_{p-1}}, \dots, \partial_{y_1}, \dots, \partial_{y_q}, \partial_z\}$.

We start by calculating Christoffel symbols of $\mathcal{H}^{p,q}$.

Proposition 2.2.0.1. *The only non-vanishing Christoffel symbols of $\mathcal{H}^{p,q}$ are*

$$\begin{aligned}
&\Gamma^{x_i}_{x_i,z} = \Gamma^{x_i}_{z,x_i} = -1/z && i = 1, \dots, p-1,\\
&\Gamma^{y_j}_{y_j,z} = \Gamma^{y_j}_{z,y_j} = -1/z && j = 1, \dots, q,\\
&\Gamma^{z}_{x_i,x_i} = 1/z && i = 1, \dots, p-1,\\
&\Gamma^{z}_{y_j,y_j} = -1/z && j = 1, \dots, q,\\
&\Gamma^{z}_{z,z} = -1/z.
\end{aligned}$$

Proof. The basis is orthogonal with respect to the pseudo-metric, hence the matrix $(g_{ij})_{i,j=1}^n$ is diagonal. Equation (1.9) becomes

$$\Gamma^k_{ij} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) g^{kk}.$$

Moreover, it suggests that $\Gamma^k_{ij} = 0$ for $i \neq j \neq k$.

Let $\delta_{hl} := \mathrm{sgn}(g_{p,q}(\partial_h, \partial_l))$, that is $g_{hl} = \delta_{hl}/z^2$. Since g_{ij} only depends on z, if $z \notin \{i, j, k\}$, $\Gamma^k_{ij} = 0$.

The only non-vanishing candidates are then, for $i \neq z$:

1. $\Gamma^i_{iz} = \Gamma^i_{zi} = \frac{1}{2}(\partial_z g_{ii} + \partial_i g_{iz} - \partial_i g_{zi}) g^{ii} = \frac{1}{2}(-\delta_{ii}\frac{2}{z^3} + 0 - 0)\delta_{ii} z^2 = -\frac{1}{z}$;
2. $\Gamma^z_{ii} = \frac{1}{2}(\partial_i g_{iz} + \partial_i g_{iz} - \partial_z g_{ii}) g^{zz} = \frac{1}{2}(0 + 0 + \delta_{ii}\frac{2}{z^3}) z^2 = \delta_{ii}\frac{1}{z}$;
3. $\Gamma^z_{iz} = \Gamma^z_{zi} = \frac{1}{2}(\partial_z g_{iz} + \partial_i g_{zz} - \partial_z g_{zi}) g^{zz} = 0$;
4. $\Gamma^i_{zz} = \frac{1}{2}(\partial_z g_{zi} + \partial_z g_{zi} - \partial_i g_{zz}) g^{ii} = 0$;
5. $\Gamma^z_{zz} = \frac{1}{2}(\partial_z g_{zz} + \partial_z g_{zz} - \partial_z g_{zz}) g^{zz} = \frac{1}{2}(-\frac{2}{z^3}) z^2 = -\frac{1}{z}$.

□

Corollary 2.2.0.2. *The Levi-Civita connection computed on the basis is not* *zero in the following cases:*

$$\begin{aligned}
&\nabla_{\partial_{x_i}}\partial_z = \nabla_{\partial_z}\partial_{x_i} = -\frac{1}{z}\partial_{x_i} && i = 1, \ldots, p-1;\\
&\nabla_{\partial_{y_j}}\partial_z = \nabla_{\partial_z}\partial_{y_j} = -\frac{1}{z}\partial_{y_j} && j = 1, \ldots, q;\\
&\nabla_{\partial_{x_i}}\partial_{x_i} = \frac{1}{z}\partial_z = \frac{\delta_{x_i x_i}}{z}\partial_z && i = 1, \ldots, p-1;\\
&\nabla_{\partial_{y_j}}\partial_{y_j} = -\frac{1}{z}\partial_z = \frac{\delta_{y_j y_j}}{z}\partial_z && j = 1, \ldots, q;\\
&\nabla_{\partial_z}\partial_z = -\frac{1}{z}\partial_z
\end{aligned}$$

Proof. It suffices to substitute Christoffel symbols found in Proposition 2.2.0.1 in Equation (1.8). □

2.3 Curvature of $\mathcal{H}^{p,q}$

In this section we will compute the Riemann tensor and deduce from that the sectional curvature of $\mathcal{H}^{p,q}$.

Theorem 2.3.0.1. *Let R be the Riemann tensor of $\mathcal{H}^{p,q}$, $X, Y, Z \in \Gamma(T\mathcal{H}^{p,q})$, then*

$$R(X,Y)Z = g_{p,q}(X,Z)Y - g_{p,q}(Y,Z)X. \tag{2.3}$$

Before starting the proof, we recall that it follows directly that $\mathcal{H}^{p,q}$ is a pseudo-Riemannian manifold with constant sectional curvature $K = -1$. Indeed,

$$\begin{aligned}
R(X,Y,Y,X) =& g_{p,q}\left(R(X,Y)Y, X\right) =\\
=& g_{p,q}\left(g_{p,q}(X,Y)Y - g_{p,q}(Y,Y)X, X\right) =\\
=& g_{p,q}(X,Y)^2 - g_{p,q}(X,X)g_{p,q}(X,Y) = -Q(X,Y).
\end{aligned}$$

Lemma 2.3.0.2. *The Riemann tensor computed on the basis does not vanishes only in the following cases:*

$$\begin{aligned}
&R(\partial_a, \partial_b)\partial_b = -\frac{\delta_{bb}}{z^2}\partial_a, && a \neq b,\\
&R(\partial_b, \partial_a)\partial_b = \frac{\delta_{bb}}{z^2}\partial_a, && a \neq b.
\end{aligned}$$

Proof. Consider $R(\partial_a, \partial_b)\partial_c = \nabla_{\partial_a}\nabla_{\partial_b}\partial_c - \nabla_{\partial_b}\nabla_{\partial_a}\partial_c$, for Corollary 2.2.0.2, $\nabla_{\partial_b}\partial_c \neq 0$ if and only if $c = b$ or one among b, c is equal to z. We split the problem in three parts, namely $c = b \neq z$, $c = b = z$ and $c \neq b = z$. Moreover, in the proof we will always suppose $a \neq b$: otherwise for antisimmetry $R(\partial_a, \partial_b) = 0$.

If $c = b \neq z$,

$$\nabla_{\partial_b}\partial_b = \frac{\delta_{bb}}{z}\partial_z.$$

Suppose first $a \neq z$, so that δ_{bb}/z is a constant with respect to ∂_a, then

$$\nabla_{\partial_a}\nabla_{\partial_b}\partial_b = \frac{\delta_{bb}}{z}\nabla_{\partial_a}\partial_z = -\frac{\delta_{bb}}{z^2}\partial_a.$$

Since $a \neq b$, $\nabla_{\partial_a}\partial_b = 0$, hence $\nabla_{\partial_b}\nabla_{\partial_a}\partial_b = 0$. Then

$$R(\partial_a, \partial_b)\partial_b = -\frac{\delta_{bb}}{z^2}\partial_a - 0 = -\frac{\delta_{bb}}{z^2}\partial_a, \qquad \forall a, b, a \neq b \neq z,$$

Otherwise, if $a = z$, $b \neq z$, so

$$\nabla_{\partial_z}\nabla_{\partial_b}\partial_b = \frac{\delta_{bb}}{z}\nabla_{\partial_z}\partial_z - \frac{\delta_{bb}}{z^2}\partial_z = -\frac{2\delta_{bb}}{z^2}\partial_z,$$
$$\nabla_{\partial_b}\nabla_{\partial_z}\partial_b = -\frac{1}{z}\nabla_{\partial_b}\partial_b = -\frac{\delta_{bb}}{z^2}\partial_z,$$

so $R(\partial_z, \partial_b)\partial_b$ satisfies the statement.

Suppose now $c = b = z$, then

$$\nabla_{\partial_z}\partial_z = -\frac{1}{z}\partial_z.$$

$a \neq b$ means $a \neq z$, so $-1/z$ is a constant with respect to ∂_a, then

$$\nabla_{\partial_a}\nabla_{\partial_z}\partial_z = -\frac{1}{z}\nabla_{\partial_a}\partial_z = -\frac{\delta_{bb}}{z^2}\partial_a.$$

The equations of the second type follow by anti-symmetry of Riemann tensor. □

We are ready to prove Theorem 2.3.0.1:

Proof. Let $X, Y, Z \in \Gamma(T\mathcal{H}^{p,q})$. To lighten the notation, the indexes i, j, k of the sum will be supposed to lie in the set $I = \{x_1, \dots, x_{p-1}, y_1, \dots, y_q, z\}$, that is $X = \sum_{i \in I} X_i \partial_i$, and the same for Y, Z. The Riemann tensor is $C^\infty(\mathcal{H}^{p,q})$-linear, so

$$\begin{aligned} R(X, Y, Z) &= \sum_{i,j,k \in I} X_i Y_j Z_k R(\partial_i, \partial_j)\partial_k = \\ &= \sum_{i,j \in I, i \neq j} \left(X_j Y_i Z_j - X_i Y_j Z_j\right) R(\partial_j, \partial_i)\partial_j = \\ &= \sum_{i \in I}\left(\sum_{j \neq i} \frac{\delta_{jj}}{z^2} X_j Z_j\right) Y_i \partial_i - \sum_{i \in I}\left(\sum_{j \neq i} \frac{\delta_{jj}}{z^2} Y_j Z_j\right) X_i \partial_i. \end{aligned}$$

Remarking that

$$\sum_{j\neq i} \frac{\delta_{jj}}{z^2} X_j Z_j = g_{p,q}(X,Z) - \frac{\partial_{ii}}{z^2} X_i Z_i,$$

we find

$$\begin{aligned} R(X,Y,Z) &= g_{p,q}(X,Z)Y - \sum_i \frac{\partial_{ii}}{z^2} X_i Z_i Y_i \partial_i - g_{p,q}(Y,Z)X + \sum_i \frac{\partial_{ii}}{z^2} Y_i Z_i X_i \partial_i = \\ &= g_{p,q}(X,Z)Y - g_{p,q}(Y,Z)X. \end{aligned}$$

□

2.4 Totally geodesic subspaces of $\mathcal{H}^{p,q}$

In this section we will first derive the differential system characterizing geodesics, then describe totally geodesic hypersurfaces. After that, we will be able to find every totally geodesic subspace of $\mathcal{H}^{p,q}$. It will follow a more precise description of geodesic, namely the 1-dimensional case.

2.4.1 Geodesic ODEs system

In Proposition 1.4.3.3 we stated that geodesic must satisfy a system of ODEs involving Christoffel symbols, which we computed in Proposition 2.2.0.1. So the differential system is

$$\begin{cases} x'' - \frac{2}{z} x' z' = 0 \\ y'' - \frac{2}{z} y' z' = 0 \\ z'' + \frac{1}{z} \left(\|x'\|^2 - \|y'\|^2 - |z'|^2 \right) = 0 \end{cases} . \tag{2.4}$$

As a consequence of this expression of the geodesic equations, we show here that vertical affine subspaces of any dimension are totally geodesic.

Remark 2.4.1.1. We will describe accurately the boundary of $\mathcal{H}^{p,q}$ in Section 2.6, for now it suffice to remark that the topological boundary $\partial\mathcal{H}^{p,q}$ coincides with the hyperplane $\{z = 0\}$ which can be see as a copy of $\mathbb{R}^{p-1,q}$ via the inclusion of $\mathcal{H}^{p,q}$ in $\mathbb{R}^{p,q}$. When we want to emphasize that we are talking about the boundary, we will omitt the last coordinate, which is identically 0, while we will refer to $\partial\mathcal{H}^{p,q}$ as $\mathbb{R}^{p-1,q}$ when we want to emphasize the pseudo-metric or the vector structure.

Proposition 2.4.1.2. *Every submanifold of the form*

$$V_\ell := \left\{ (x,y,z) \in \mathbb{R}^{p-1} \oplus \mathbb{R}^q \oplus \mathbb{R} \,|\, (x,y) \in \ell, z > 0 \right\} ,$$

for ℓ an affine subspace of $\partial\mathcal{H}^{p,q}$, is totally geodesic.

Proof. Let us define ℓ as the set of solutions of a finite number of affine conditions of the form

$$\sum_{i=1}^{p-1} a_i x_i + \sum_{j=1}^{q} b_j y_j = c \ . \tag{2.5}$$

We remark that the same conditions define V_ℓ, since the parameter z is free.

We claim that if $\gamma(t) = (x(t), y(t), z(t))$ is a geodesic such that $\gamma'(0)$ is tangent to the subspace V_ℓ, namely

$$\sum_{i=1}^{p-1} a_i x_i'(0) + \sum_{j=1}^{q} b_j y_j'(0) = 0 \ , \tag{2.6}$$

then $\gamma(t)$ satisfies (2.5) for all times of definition. This clearly implies that $\gamma \subseteq V_\ell$ and then that V_ℓ is totally geodesic.

To show the claim, define the function

$$\chi(t) = \sum_{i=1}^{p-1} a_i x_i'(t) + \sum_{j=1}^{q} b_j y_j'(t) \ .$$

Taking a linear combination of the equations (2.4), χ satisfies the following ODE:

$$\chi'(t) = -2\frac{z'(t)}{z(t)}\chi(t).$$

By our hypothesis (2.6), $\chi(0) = 0$, hence $\chi \equiv 0$ solve the ODE. This proves our claim and concludes the proof. $\square$

2.4.2 Totally geodesic hypersurfaces

First we give the classification of totally geodesic hypersurfaces, namely the ones of codimension one. The general case will follow in Theorem 2.4.3.1.

Proposition 2.4.2.1. *The totally geodesic hypersurfaces of $\mathcal{H}^{p,q}$ are precisely:*

1. *the vertical hyperplanes $V_{\mathcal{L}}$, for $\mathcal{L}$ an affine hyperplane in $\partial\mathcal{H}^{p,q}$;*
2. *the quadric hypersurfaces of the form*

$$\|x - x_0\|^2 - \|y - y_0\|^2 + z^2 = c, \quad c \in \mathbb{R} \tag{Q}$$

for some $(x_0, y_0) \in \partial\mathcal{H}^{p,q}$. The hypersurfaces of the former type are degenerate if and only if $\mathcal{L}$ is degenerate in $\mathbb{R}^{p-1,q}$, and have signature $(n_+ + 1, n_-)$ where $(n_+ + 1, n_-)$ is the signature of $\mathcal{L}$. Those of the latter type are degenerate if and only if $c = 0$, and have signature $(p-1, q-1)$ if $c = 0$, $(p, q-1)$ if $c < 0$, and $(p-1, q)$ if $c > 0$.

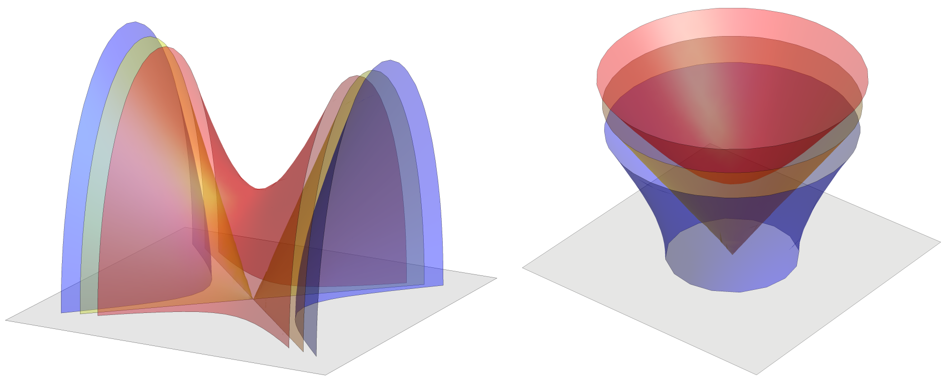

Figure 2.1: The totally geodesic quadric hypersurfaces in $\mathcal{H}^{2,1}$ (left) and $\mathcal{H}^{1,2}$ (right).

See also Figure 2.1 for some pictures in dimension 3.

Proof. It has been proved in Proposition 2.4.1.2 that vertical hyperplanes are totally geodesic. To prove that the quadric hypersurfaces as in the statement are totally geodesic, we will show that the intersection of the quadric hypersurface with any vertical 2-plane V_ℓ (for ℓ a line) is a geodesic of $\mathcal{H}^{p,q}$. This clearly implies that the hypersurface is totally geodesic, since any ambient geodesic that is tangent to the hypersurface at time zero remains in the hypersurface for all times.

To see this, set $(x_0, y_0) \in \partial\mathcal{H}^{p,q}$, $c \in \mathbb{R}$ as parameters for Equation (Q). Pick a 2-vertical vector space, *i.e.* set a line

$$\ell = \mathrm{Span}((u, v)) + (u_0, v_0) \subseteq \mathbb{R}^{p-1,q},$$

such that it intersects the hyperquadric in a curve. This curve can be parameterized as

$$\gamma(t) = (u_0 + tu, v_0 + tv, f(t)),$$

where the function f is determined by the quadric Equation (Q), namely

$$f(t) = \sqrt{c - \|x(t) - x_0\|^2 + \|y(t) - y_0\|^2}\ .$$

f is well-defined over an interval I and $f(t) > 0$, $\forall t \in I$.

Substituting in (2.4) one can compute

$$\nabla_{\gamma'}\gamma' = \left(-\frac{2f'}{f}u, -\frac{2f'}{f}v, f'' + \frac{\|u\|^2 - \|v\|^2 - |f'|^2}{f}\right). \tag{2.7}$$

Remarking that $f(t)^2 = c - \|tu + u_0 - x_0\|^2 + \|tv + v_0 - y_0\|^2$, differentiating with respect to t one obtains

$$2f(t)f'(t) = -2t\|u\|^2 - 2\langle u, u_0 - x_0\rangle + 2t\|v\|^2 + C,$$

$C = -2\langle u, u_0 - x_0\rangle + 2\langle v, v_0 - y_0\rangle$. Differentiating again one has

$$f'(t)^2 + f(t)f''(t) = -\|u\|^2 + \|v\|^2,$$

that is equivalent to

$$f''(t) = -\frac{\|u\|^2 - \|v\|^2 + f'(t)^2}{f(t)}.$$

Then the last term in (2.7) becomes $-(2f'/f)f'$, that means $\nabla_{\gamma'}\gamma' = -(2f'/f)\gamma'$, *i.e.* γ in an unparameterized geodesic by Proposition 1.4.3.11.

It only remains to show that these are all the totally geodesic hypersurfaces. For this purpose, we choose any vector (u, v, w) tangent to $\mathcal{H}^{p,q}$ at a point (x, y, z) and we show that there exists a totally geodesic hypersurface (which is necessarily unique) of the above two forms containing the point (x, y, z) and whose tangent space is orthogonal to (u, v, w) at (x, y, z). The key observation is that the orthogonality can be computed with respect to the flat metric

$$\overline{g}_{p,q} = dx_1^2 + \ldots + dx_{p-1}^2 - dy_1^2 - \ldots - dy_q^2 + dz^2 \ , \tag{2.8}$$

namely by seeing the half-space as a subset of a pseudo-Euclidean space of signature (p, q), since $\overline{g}_{p,q}$ is conformal to the metric $g_{p,q}$.

If $w = 0$, then clearly the hypersurface we are looking for is $V_{\mathcal{L}}$, for $\mathcal{L}$ the affine hyperplane of $\mathbb{R}^{p-1} \oplus \mathbb{R}^q$ containing the point (x, y) and whose underlying vector space is the orthogonal of (u, v).

So let us now assume $w \neq 0$. Let $(x_0, y_0, 0)$ be the point of intersection of the line through (x, y, z) having direction (u, v, w) with $\partial\mathcal{H}^{p,q}$. The quadric hypersurfaces of the form (Q) are precisely the sets of points at constant squared distance from $(x_0, y_0, 0)$ for the conformal pseudo-Euclidean metric $\overline{g}_{p,q}$. Hence there is one quadric hypersurface that contains the point (x, y, z), and it is orthogonal to the position vector $(x, y, z) - (x_0, y_0, 0)$, which is proportional to (u, v, w) by construction.

The statement about the signature is easily checked: we just proved that $(x, y, z) - (x_0, y_0, 0)$ is a generator of the normal space to the hypersurface in the point (x, y, z). The equation (2.1) states that

$$g_{p,q}\left((x, y, z) - (x_0, y_0, 0), (x, y, z) - (x_0, y_0, 0)\right) = z^2 c,$$

whose sign only depends on c. Then we can conclude by using Lemma 1.1.1.4.

□

Remark 2.4.2.2. If the metric is positive definite, namely $q = 0$, (Q) describe a half-sphere with center in $\partial\mathcal{H}^{p,0}$ for $c > 0$, a point for $c = 0$ and the empty set for $c < 0$, so we recover exactely that totally geodesic manifolds in the half-space model of hyperbolic space are vertical vector spaces and half-spheres with the center on the hyperplane $\{z = 0\}$.

2.4.3 The general classification

We can finally state the classification result for totally geodesic submanifolds.

Theorem 2.4.3.1. *The totally geodesic submanifolds of $\mathcal{H}^{p,q}$ are precisely:*

1. *the vertical subspaces,*
2. *the intersections of quadric hypersurfaces of the form* (Q) *with a vertical subspace.*

Proof. The submanifolds in the statement are totally geodesic: for the first item this follows from Proposition 2.4.1.2, while for the second item from Proposition 2.4.2.1 and the fact that the intersection of totally geodesic submanifolds is totally geodesic. To show that they exhaust the totally geodesic submanifolds, we remark that for a point (x, y, z) and a linear subspace W of $T_{(x,y,z)}\mathcal{H}^{p,q}$, it exists at most one maximal totally geodesic submanifolds of the same dimension as W, which is tangent to W at (x, y, z). We will show that that submanifold exists and belongs to the ones described above.

If W contains the vertical direction, then we can write $W = W_0 \oplus \partial_z$, for W_0 the orthogonal complement of ∂_z in W. Denoting by ℓ the affine subspace through the point (x, y) with underlying vector space W_0, V_ℓ is a totally geodesic subspace tangent to W at (x, y, z).

Now suppose that W does not contain ∂_z, and extend W to a subspace W_1 of codimension one which is still transverse to the vertical direction. By the proof of Proposition 2.4.2.1, there exists a quadric hypersurface Q which is tangent to W_1 at (x, y, z). Also, as in the first part of this proof, we find a vertical subspace V_ℓ which is tangent to $W \oplus \partial_z$ at (x, y, z). Then $Q \cap V_\ell$ is tangent to W at (x, y, z). This concludes the proof. $\square$

2.5 Geodesics of $\mathcal{H}^{p,q}$

The next step in our analysis is the study of the geodesics of $\mathcal{H}^{p,q}$. We will divide our analysis in three cases, namely lightlike, timelike and spacelike geodesics.

To simplify the statements of the following propositions, we refer to geodesics as unparametrized, *i.e.* our statements are actually about the image of the parametrized curves.

2.5.1 Lightlike geodesics

We start by lightlike geodesics, namely those for which the tangent vector is isotropic for the metric tensor (2.1).

Proposition 2.5.1.1. *Lightlike geodesics in $\mathcal{H}^{p,q}$ are precisely the straight lines spanned by a lightlike vector. These are incomplete as they escape from* *compact sets of $\mathcal{H}^{p,q} \cup \partial\mathcal{H}^{p,q}$, unless they are contained in a horizontal hyperplane $\{z = c\}$.*

Proof. Let (u, v, w) be a lightlike vector, that is, $\|u\|^2 - \|v\|^2 + |w|^2 = 0$. Up to changing the sign, we can assume $w \geq 0$. We claim that, if $w > 0$, then

$$\gamma(t) = (x_0, y_0, 0) + (1/t)(u, v, w)$$

is a parameterized geodesic; if instead $w = 0$, then

$$\gamma(t) = (x_0, y_0, z_0) + t(u, v, 0)$$

is a parameterized geodesic. This clearly implies the statement: these are all the lightlike geodesics because, up to choosing the parameter t suitably, one finds such a geodesic with tangent vector a multiple of (u, v, w) through any point of $\mathcal{H}^{p,q}$. Moreover, geodesics of the former type are defined on $(0, \infty)$, hence they are complete only when they approach $\partial\mathcal{H}^{p,q}$, while those of the latter type are defined on $\mathbb{R}$.

The claim is an easy computation from Equations (1.16). Indeed, since γ' is lightlike, we have $\|x'\|^2 - \|y'\|^2 = -|z'|^2$. Hence the equation becomes

$$(x'', y'', z'') = (2z'/z)(x', y', z') \ , \tag{2.9}$$

and one immediately checks that both expressions above for γ satisfy (2.9): in the first case we have

$$\gamma'(t) = -\frac{1}{t^2}(u, v, w); \qquad\qquad \gamma''(t) = \frac{2}{t^3}(u, v, w).$$

It follows that

$$\gamma''(t) - \frac{2z'(t)}{z(t)}\gamma'(t) = \left(\frac{2}{t^3} - 2\frac{w/t^2}{w/t}\frac{1}{t^2}\right)(u, v, w) = 0.$$

The case $w = 0$ is trivial: the parameterization is linear and $z(t)$ is constant, hence both γ'' and $2z'/z$ vanish identically. □

See Figure 2.2 to visualize the cone of lightlike geodesics emanating from a point in $\mathcal{H}^{2,1}$ and $\mathcal{H}^{1,2}$.

Remark 2.5.1.2. The fact that unparameterized lightlike geodesics are straight lines can also be proved by observing that $\mathcal{H}^{p,q}$ is conformal to the upper half-space in $\mathbb{R}^{p,q}$ endowed with the restriction of the pseudo-Euclidean metric, and applying the fact that two conformal pseudo-Riemannian metrics have the same unparameterized lightlike geodesics (see [GHL04, Proposition 2.131]).

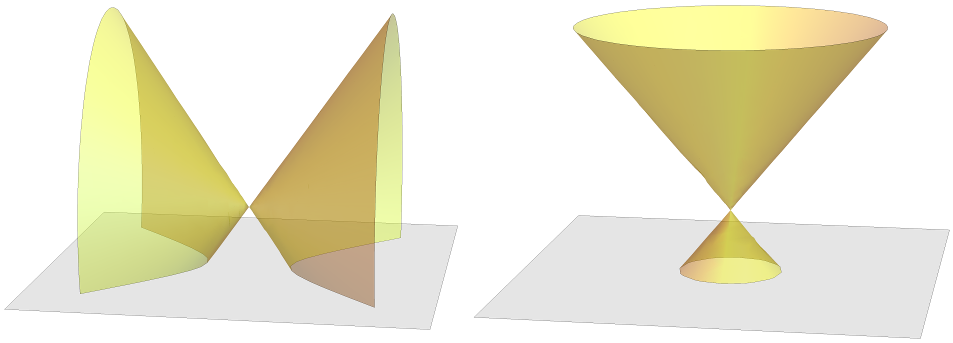

Figure 2.2: The lightcone from a point in $\mathcal{H}^{2,1}$ (left) and $\mathcal{H}^{1,2}$ (right).

2.5.2 A preliminary computation

As a consequence of the classification in Theorem 2.4.3.1, geodesics are either straight vertical lines or conics. We will give here a more precise classification in terms of the eccentricity, computed with respect to the Euclidean distances in $\mathcal{H}^{p,q} \subset \mathbb{R}^{p+q}$. We start by a general computation that we will apply repeatedly.

Lemma 2.5.2.1. *Geodesics of $\mathcal{H}^{p,q}$ are precisely:*

1. *vertical lines;*
2. *conics of equation*

$$\frac{\|u\|^2 - \|v\|^2}{\|u\|^2 + \|v\|^2} s^2 + z^2 + As = C\,, \qquad A, C \in \mathbb{R} \tag{Q'}$$

with respect to Euclidean coordinates (s, z) on a vertical 2-plane V_ℓ, where the underlying vector space of ℓ is spanned by (u, v).

Proof. By Theorem 2.4.3.1, geodesics are either vertical lines or obtained intersecting (Q) with the a 2-plane V_ℓ. Up to a horizontal translation, we can assume that the line ℓ contains the origin, hence it can be parameterized as

$$(x, y) = \frac{s}{\sqrt{\|u\|^2 + \|v\|^2}}(u, v)\ . \tag{2.10}$$

Substite the above formula in (Q):

$$\left\| \frac{s}{\sqrt{\|u\|^2 + \|v\|^2}} u - x_0 \right\|^2 - \left\| \frac{s}{\sqrt{\|u\|^2 + \|v\|^2}} v - x_0 \right\|^2 + z^2 = c$$
$$\iff \frac{\|u\|^2 - \|v\|^2}{\|u\|^2 + \|v\|^2} s^2 - 2\frac{(\langle u, x_0\rangle - \langle v, y_0\rangle)}{\sqrt{\|u\|^2 + \|v\|^2}} s + \|x_0\|^2 - \|y_0\|^2 = c,$$

which is Equation (Q') with the constants

$$A = -2\frac{u \cdot x_0 - v \cdot y_0}{\sqrt{\|u\|^2 + \|v\|^2}}, \qquad\qquad C = c - \|x_0\|^2 + \|y_0\|^2.$$

$\square$

Observe that if $\|u\|^2 - \|v\|^2 \neq 0$, then replacing the coordinate s by $s - s_0$, for a suitable choice of s_0, and relabeling the constant C, one then obtains the following equation:

$$\frac{\|u\|^2 - \|v\|^2}{\|u\|^2 + \|v\|^2} s^2 + z^2 = C\,, \qquad C \in \mathbb{R}\,. \tag{Q''}$$

In fact, the form of Equation (Q') is $Bs^2 + As - C = 0$; substituting s with $s - s_0$ we find

$$0 = B(s - s_0)^2 + A(s - s_0) - C = Bs^2 + (A - 2Bs_0)s + (Bs_0^2 - C).$$

The linear term vanishes when $s_0 = A/2B$, which is possible only if $B \neq 0$, that is $\|u\|^2 - \|v\|^2 \neq 0$.

Remark 2.5.2.2. Lemma 2.5.2.1 provides yet another method to obtain the straight lines as lightlike geodesics. Consider Equation (Q'') for $C = 0$, namely

$$\frac{\|u\|^2 - \|v\|^2}{\|u\|^2 + \|v\|^2} s^2 + z^2 = 0$$

which is indeed a double line through the origin if $\|u\|^2 - \|v\|^2 < 0$. An immediate computation shows that these lines are indeed lightlike: indeed

$$\begin{aligned} x(t) &= \frac{u}{\sqrt{\|u\|^2 + \|v\|^2}}(s - s_0), \\ y(t) &= \frac{v}{\sqrt{\|u\|^2 + \|v\|^2}}(s - s_0), \\ z(t) &= \sqrt{-\frac{\|u\|^2 - \|v\|^2}{\|u\|^2 + \|v\|^2}}\, s. \end{aligned}$$

Similarly, considering (Q') with $C > 0$, $A = 0$ and $\|u\|^2 - \|v\|^2 = 0$, we obtain $z^2 = C$ and then $z = \pm\sqrt{C}$. Since $\mathcal{H}^{p,q}$ only contains positive values of z, this is the equation a horizontal line with lightlike direction. A similar approach will be used in the next sections for the analysis of spacelike and timelike geodesics.

2.5.3 Timelike geodesics

We now move on to the study of timelike geodesics. Let us remark that if a vector (u, v, w) is tangent to a timelike curve at any point, then necessarily $\|u\| < \|v\|$ by the expression of the metric (2.1).

Proposition 2.5.3.1. *Timelike geodesics in $\mathcal{H}^{p,q}$ are exactly the branches of hyperbola with center on $\partial\mathcal{H}^{p,q}$, which do not meet $\partial\mathcal{H}^{p,q}$, of eccentricity*

$$e_T(u, v) = \sqrt{1 + \frac{\|u\|^2 + \|v\|^2}{\|v\|^2 - \|u\|^2}} \, ,$$

where (u, v, w) is a vector tangent to the geodesic at any point. These are incomplete on both sides.

See also Figure 2.3.

Proof. We will consider timelike geodesics as the intersections of quadric hypersurfaces with a vertical plane, as in Equation (Q'). In order to get a timelike geodesic, the vertical plane is necessarily of signature $(1, 1)$, hence $\|u\| < \|v\|$, in which case we can reduce to Equation (Q"). Indeed, the tangent space of a vertical 2-plane contains ∂_z, which is spacelike, so the signature can be either $(1, 1)$, $(2, 0)$ or degenerate of signature $(1, 0, 1)$. In the latter two cases the tangent space does not contain timelike vectors, and so the space cannot contain timelike curves.

As we observed in Remark 2.5.2.2, if $C = 0$ then Equation (Q") gives a pair of lightlike lines with the same endpoint on $\partial\mathcal{H}^{p,q}$. If $C < 0$, we obtain a pair of hyperbolas meeting $\partial\mathcal{H}^{p,q}$ orthogonally. Since the half-space metric is conformal to the pseudo-Euclidean metric on $\mathbb{R}^{p+q}$, these are spacelike (they tend to be vertical as they approach $\partial\mathcal{H}^{p,q}$).

We are left with the case of $C > 0$, which gives indeed a hyperbola that does not meet $\partial\mathcal{H}^{p,q}$. These are easily seen to be timelike, since the tangent vector at the minimum point of the z-coordinate along the hyperbola is proportional to $(u, v, 0)$, which is timelike by our initial assumption $\|u\| < \|v\|$. The eccentricity is $e_T(u, v)$.

It only remains to show that these are incomplete. First, observe that if H and H' are two such hyperbolas (considered as a subset of $\mathcal{H}^{p,q}$), then there is an element of G that mapping H to H'. Indeed, one can use a translation and a dilatation to map the minimum of the z-coordinate on H to that on H'. Composing with an isometry of the form $(x, y, z) \mapsto (A(x, y), z)$, one can then map the tangent vector to the tangent vector, and this concludes the claim.

To show incompleteness, it thus suffices to consider the hyperbola γ parameterized by $y_1(t) = \sinh(t)$, $z(t) = \cosh(t)$, and all other coordinates

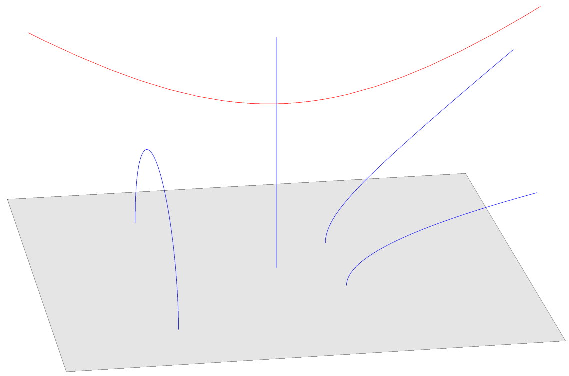

Figure 2.3: A timelike geodesic (in red) and the four types of spacelike geodesics (blue).

identically zero. Its length is

$$L(\gamma) = \left|\int_{-\infty}^{+\infty} \sqrt{|g_{p,q}(\gamma'(t),\gamma'(t))|}dt\right| = \left|\int_{-\infty}^{+\infty} \sqrt{\frac{-\cosh(t)^2+\sinh(t)^2}{\cosh(t)^2}}dt\right|$$
$$= \left|\int_{-\infty}^{+\infty} \frac{1}{\cosh(t)}dt\right| = \left[2\arctan(e^t)\right]_{-\infty}^{+\infty} = \pi\ .$$

Then all timelike geodesics are incomplete on both sides (see Lemma 1.4.3.12).

Remark 2.5.3.2. Timelike geodesic in $\mathbb{H}^{p,q}$ are periodic curves of length π. The fact that timelike geodesic of $\mathcal{H}^{p,q}$ have the same length prove that they are the complement (via $\iota_{p,q}$) of a set of Lebesgue measure null in $\mathbb{H}^{p,q}$. In fact we will see in Remark 2.6.6.3 that they are the complement of a single point.

□

2.5.4 Spacelike geodesics

Finally, we conclude by the analysis of spacelike geodesics. See again Figure 2.3.

Proposition 2.5.4.1. *Spacelike geodesics in $\mathcal{H}^{p,q}$ are exactly of one of the following types:*

1. *A vertical straight line;*
2. *A half-ellipse with foci on $\partial\mathcal{H}^{p,q}$, of eccentricity $e_S(u,v)$ for $\|u\| > \|v\|$;*
3. *A parabola with vertex and focus on $\partial\mathcal{H}^{p,q}$, for $\|u\| = \|v\|$;*

4. Half of a branch of hyperbola with foci on $\partial\mathcal{H}^{p,q}$, meeting $\partial\mathcal{H}^{p,q}$, of eccentricity $e_S(u,v)$, for $\|u\| < \|v\|$;

where

$$e_S(u,v) = \sqrt{1 + \frac{\|v\|^2 - \|u\|^2}{\|u\|^2 + \|v\|^2}}$$

and (u,v,w) is a vector tangent to the geodesic at any point. The first three types are complete, while the fourth type is incomplete as it escapes from compact sets of $\mathcal{H}^{p,q} \cup \partial\mathcal{H}^{p,q}$.

Before the proof, we observe that in particular all spacelike geodesics meet $\partial\mathcal{H}^{p,q}$ at right angles with respect to the conformal metric

$$\overline{g}_{p,q} = dx_1^2 + \ldots + dx_{p-1}^2 - dy_1^2 - \ldots - dy_q^2 + dz^2,$$

which extends over the horizontal hyperplane.

Proof. The first type follows from the first point of Lemma 2.5.2.1. The arc-length parameterization is $\gamma(t) = (x_0, y_0, z_0 e^{wt/z_0})$, which is defined for all times. Indeed, if $x(t)$ and $y(t)$ are constant, the last equation of (2.7) becomes

$$z''(t) = \frac{z'(t)^2}{z(t)},$$

whose solution is $z(t) = z_0 e^{wt/z_0}$, while the others become trivial.

Let us now consider the second point in Lemma 2.5.2.1, by distinguishing three cases according to the sign of $\|u\|^2 - \|v\|^2$.

If $\|u\| < \|v\|$, we have already seen in the proof of Proposition 2.5.3.1 that Equation (Q”) gives a spacelike geodesic if and only if $C < 0$. From the equation, in this case the geodesic is a branch of hyperbola that meets $\partial\mathcal{H}^{p,q}$ orthogonally, of eccentricity $e_S(u,v)$. To show that it has infinite length when approaching $\partial\mathcal{H}^{p,q}$ and finite length at the other end, we can assume that $(u,v) = (0, \partial_{y_1})$, up to isometry. Up to a translation, we can assume that the curve is parameterized by $y_1(t) = \cosh(t)$, $z(t) = \sinh(t)$, and all the other coordinates are identically zero. A direct computation shows that its length is

$$\begin{aligned} L(\gamma|[t_0,t_1]) &= \left| \int_{t_0}^{t_1} \sqrt{g(\gamma'(t),\gamma'(t))} dt \right| = \left| \int_{t_0}^{t_1} \frac{1}{\sinh(t)} dt \right| = \\ &= \left| [\log(\tanh(t/2))]_{t_0}^{t_1} \right| . \end{aligned}$$

Therefore it is complete as $t_0 \to 0^+$, and incomplete as $t_1 \to +\infty$.

If $\|u\| > \|v\|$, then (Q”) is the equation of an ellipse (for $C > 0$) that meets $\partial\mathcal{H}^{p,q}$ orthogonally, with eccentricity $e_S(u,v)$. We remark that it lies in a positive definite vertical 2-plane: the tangent space at the maximum of the ellipse is $(\tilde{u}, \tilde{v}, 0)$, which is both spacelike and orthogonal to ∂_z. So the plane containing γ is isometric to $\mathcal{H}^{2,0} \cong \mathbb{H}^2$, which is complete, hence γ is complete, too. Otherwise, one can compute the length as in the previous case, and find that it is infinite in both sides.

Finally, when $\|u\| = \|v\|$, (Q') becomes $z^2 + As = C$. We remark that necessarily $A \neq 0$, because otherwise we would obtain a lightlike line as observed in Remark 2.5.2.2. Namely, we obtained a parabola with vertex and focus both on $\partial\mathcal{H}^{p,q}$. To see that it is complete, observe that the last equation of (2.7) becomes the ODE $z'' = (z')^2/z$, whose solution $z(t) = z_0 e^{wt/z_0}$ is defined for all times. The parameterized geodesic is obtained by setting $s(t) = (1/A)(C - z^2)$; then $(x(t), y(t))$ is expressed from $s(t)$ as a function of t by Equation (2.10). Hence, the arc-length parameterization is

$$\begin{aligned}
x(t) &= \frac{u}{\sqrt{\|u\|^2 + \|v\|^2}} s(t) = \frac{u}{\sqrt{\|u\|^2 + \|v\|^2}} \frac{C - z(t)^2}{A}, \\
y(t) &= \frac{v}{\sqrt{\|u\|^2 + \|v\|^2}} s(t) = \frac{v}{\sqrt{\|u\|^2 + \|v\|^2}} \frac{C - z(t)^2}{A}, \\
z(t) &= z_0 e^{wt/z_0},
\end{aligned}$$

which is defined on $\mathbb{R}$, *i.e.* γ is complete. $\square$

As a consequence of this analysis of geodesics, we now have all the tools to prove that the group G (see Definition 2.1.3.5) is actually the full isometry group of the half-space model $\mathcal{H}^{p,q}$, if $q \neq 0$. However, we postpone the proof to Section 2.8 (see Theorem 2.8.1.1), where isometries are discussed in greater detail.

2.6 The boundary at infinity

In this section we will study the boundary at infinity of the pseudo-hyperbolic space $\mathbb{H}^{p,q}$ in the half-space model. We first show that the embedding $\iota_{p,q}$, introduced in Proposition 2.1.2.2, extends to a non-surjective embedding of $\partial\mathcal{H}^{p,q}$ into $\partial_\infty\mathbb{H}^{p,q}$; we then describe the missing points and the topology of the boundary in terms of lightlike cones and hyperplanes.

2.6.1 The conformal boundary

It is possible to endow the pseudo-hyperbolic space with a *conformal boundary*, which is a generalization of the visual boundary in Riemannian geometry, in order to study its asymptotic properties.

Since in the case of $\mathcal{H}^{p,q}$ this construction is trivial, we will give only the basic definitions and consider in the following the topological boundary. For details on the conformal boundary, we recommend [Fra02, Fra05, BS20, BCD+08].

Definition 2.6.1.1. (Conformal metric) Let (M, g) be a pseudo-Riemannian manifold. Another pseudo-metric h on M is *conformal* to g if it exists a smooth function $f \in C^\infty(M)$ such that $h = e^f g$, namely h is punctually a positive multiple of g.

Conformal metrics preserves angles and non parameterized lightlike geodesic, hence we can endow a manifold with different conformal metrics, in order to study these properties.

Definition 2.6.1.2. (Conformal class of metrics) Let (M, g) a pseudo-Riemannian manifold. The *conformal class* of g is the set of pseudo-metrics

$$[g] := \{e^f g, f \in C^\infty(M)\}.$$

We can endow a manifold with a conformal class of pseudo-metrics, so we are interested in maps preserving this structure.

Definition 2.6.1.3. (Conformal map) Let (M, g) and (N, h) be two pseudo-Riemannian manifolds, a (local) diffeomorphism $\phi\colon M \to N$ is a *conformal map* if it exists a smooth function $f \in C^\infty(M)$ such that $\phi^* h = e^f g$.

Equivalentely, a conformal map $\phi\colon (M, g) \to (N, h)$ is a (local) isometry between (M, g') and (N, h), where g' is a pseudo-metric conformal to g.

If we endowed two manifolds with conformal structures, namely $(M, [g])$ and $(N, [h])$, they are *conformally diffeomorphic* if it exists a diffeomorphism $\phi\colon M \to N$, which is a conformal map for some $g' \in [g]$ and $h' \in [h]$. We remark that if it is the case, ϕ is a conformal map $(M, g'') \to (N, h'')$, for all $g'' \in [g]$, $h'' \in [h]$.

Definition 2.6.1.4. (Conformal completion) A *conformal boundary completion* of a pseudo-Riemannian manifold (M, g) is a manifold $\overline{M}$ with boundary ∂M, endowed with a conformal structure $[\overline{g}]$ such that

i. the interior of $(\overline{M}, [\overline{g}])$ is conformally diffeomorphic to (M, g);

ii. it exists a metric $h \in [\overline{g}]$ and a smooth function $\rho \in C^\infty(\overline{M})$ such that

   a) $\rho^{-1}(\{0\}) = \partial M$,
   b) $d\rho \neq 0$ on ∂M,
   c) $h = \rho^2 g$ on the interior of $\overline{M}$, that is on M (up to scale ρ).

Remark 2.6.1.5. Again, if (c) holds for a single couple $h \in [\overline{g}]$, $\rho \in C^\infty(\overline{M})$, then for all $h' \in [\overline{g}]$, it exists $\rho' \in C^\infty(\overline{M})$ which satisfies the same requests.

Clearly the conformal completion of $(\mathcal{H}^{p,q}, g_{p,q})$ is

$$(\mathcal{H}^{p,q} \cup \{z = 0\}, [\overline{g}_{p,q}]),$$

where $\overline{g}_{p,q}$ is the flat metric of $\mathbb{R}^{p,q}$ defined as in (2.8). This implies that the conformal boundary is the same as the topological one. Indeed, $g_{p,q}$

is conformal to the Euclidean metric of $\mathbb{R}^{p,q}$ and one can easily check that $\rho(x,y,z) := z$ satisfies (c).

It is not trivial to show that even $\mathbb{H}^{p,q}$ admits a conformal boundary, and that it coincides with the topological boundary, namely the projectivization of the lightcone. For this construction in $\mathbb{H}^{n,1} = \mathbb{A}\mathrm{d}\mathbb{S}^{n+1}$, see for example [Fra05, BS20].

2.6.2 The extended embedding

Recall that Proposition 2.1.2.2 provided an isometric embedding $\iota_{p,q}$ of $\mathcal{H}^{p,q}$ into $\mathbb{H}^{p,q}$.

Proposition 2.6.2.1. *The isometric embedding $\iota_{p,q}\colon \mathcal{H}^{p,q} \hookrightarrow \mathbb{H}^{p,q}$ defined in Proposition 2.1.2.2 extends to an embedding $\partial\mathcal{H}^{p,q} \hookrightarrow \partial_\infty \mathbb{H}^{p,q}$.*

Proof. Consider the embedding $\iota_{p,q}$ as a map from $\mathcal{H}^{p,q}$ to the projective space $\mathbb{RP}^{p+q}$. From Proposition 2.1.2.2, it has the expression

$$\begin{aligned}\iota_{p,q}(x,y,z) &= \left[\frac{x}{z} : \frac{1-\|x\|^2+\|y\|^2-z^2}{2z} : \frac{y}{z} : \frac{1+\|x\|^2-\|y\|^2+z^2}{2z}\right] = \\ &= \left[x : \frac{1-\|x\|^2+\|y\|^2-z^2}{2} : y : \frac{1+\|x\|^2-\|y\|^2+z^2}{2}\right],\end{aligned}$$

hence it extends to $\partial\mathcal{H}^{p,q} = \{z=0\}$ by the above formula.

One can easily check that $\langle \iota_{p,q}(x,y,0), \iota_{p,q}(x,y,0)\rangle_{p,q+1} = 0$, *i.e.* $\iota_{p,q}(\partial\mathcal{H}^{p,q})$ is contained in $\partial_\infty\mathbb{H}^{p,q}$. Indeed

$$X_p^2 + X_{p+q+1}^2 = -\|x\|^2 + \|y\|^2 = -\sum_{i=1}^{p-1} X_i^2 + \sum_{j=1}^{q} X_{p+j}^2.$$

In particular $\iota_{p,q}(\mathcal{H}^{p,q}) \cap \iota_{p,q}(\partial\mathcal{H}^{p,q}) = \emptyset$. To show that $\iota_{p,q}$ is injective, it therefore suffices to show that it is injective when restricted to $\partial\mathcal{H}^{p,q}$, since we already showed in Proposition 2.1.2.2 the injectivity of $\iota_{p,q}$ on $\mathcal{H}^{p,q}$.

For this purpose, suppose there exist two points $(x,y,0), (t,w,0) \in \partial\mathcal{H}^{p,q}$ such that $\iota_{p,q}(x,y,0) = \iota_{p,q}(t,w,0)$, that is

$$\left[t : \frac{1-\|t\|^2+\|w\|^2}{2} : w : \frac{1+\|t\|^2-\|w\|^2}{2}\right] = \left[x : \frac{1-\|x\|^2+\|y\|^2}{2} : y : \frac{1+\|x\|^2-\|y\|^2}{2}\right].$$

It follows from the expression above that $(t,w) = \lambda(x,y)$ for some $\lambda \neq 0$ and

$$\begin{cases}\lambda\left(1-\|x\|^2+\|y\|^2\right) = 1-\|t\|^2+\|w\|^2 = 1-\lambda^2\|x\|^2+\lambda^2\|y\|^2 \\ \lambda\left(1+\|x\|^2-\|y\|^2\right) = 1+\|t\|^2-\|w\|^2 = 1+\lambda^2\|x\|^2-\lambda^2\|y\|^2\end{cases},$$

which can be rewritten equivalently as

$$\begin{cases}(\|x\|^2-\|y\|^2)\lambda(1-\lambda) = \lambda - 1 \\ (\|x\|^2-\|y\|^2)\lambda(1-\lambda) = 1-\lambda\end{cases},$$

whose only solution is $\lambda = 1$. Indeed, if $\lambda \neq 1$, we the system can be rearranged: dividing by $1-\lambda$ and summing the first eqation to the second, one obtains

$$\begin{cases} (\|x\|^2 - \|y\|^2)\lambda = -1 \\ (\|x\|^2 - \|y\|^2) = 0 \end{cases},$$

which is impossible. This concludes that $(t, w, 0) = (x, y, 0)$.

Moreover, using the same notation as in Proposition 2.1.2.2, clearly $X_p + X_{p+q+1} = 1$, *i.e.* $\iota_{p,q}(x, y, 0) \in \mathbb{P}\{X_p + X_{p+q+1} \neq 0\}$. In fact, one immediately checks that

$$\iota_{p,q}(\partial\mathcal{H}^{p,q}) = \partial_\infty \mathbb{H}^{p,q} \cap \mathbb{P}\{X_p + X_{p+q+1} \neq 0\} \subset \mathbb{RP}^{p+q} .$$

Indeed, given a null vector $(X_1, \dots, X_{p+q+1})$, up to rescaling we can assume $X_p + X_{p+q+1} = 1$; a direct computation shows that, for $x_i = X_i$ and $y_j = X_{j+p}$, $\iota(x, y, 0) = [X_1, \dots, X_{p+q+1}]$, since (X_p, X_{p+q+1}) is the unique couple solving

$$\begin{cases} X_p + X_{p+q+1} = 1 \\ X_p^2 - X_{p+q+1}^2 = -\sum_{i=1}^{p-1} X_i^2 + \sum_{j=1}^{q} X_{p+j}^2 \end{cases}.$$

$\square$

Corollary 2.6.2.2. *$\iota_{p,q}$ embeds conformally $\partial\mathcal{H}^{p,q}$ into $\partial_\infty\mathbb{H}^{p,q}$.*

Proof. Define the map

$$\begin{aligned} \bar{\iota}_{p,q} \colon \mathcal{H}^{p,q} \cup \partial\mathcal{H}^{p,q} &\to \mathbb{R}^{p,q+1} \\ (x, y, z) &\mapsto \left(x, \frac{1 - \|x\|^2 + \|y\|^2}{2}, y, \frac{1 + \|x\|^2 - \|y\|^2}{2} \right). \end{aligned}$$

$\bar{\iota}_{p,q}$ is a conformal map: one can easily compute it or remark that morally $\bar{\iota}_{p,q}(x, y, z) = z^2 \widetilde{\iota}_{p,q}(x, y, z)$. $\mathbb{P}$ is a local isometry, hence a conformal map. This implies that $\mathbb{P} \circ \bar{\iota}_{p,q}$ is a conformal map $\mathcal{H}^{p,q} \cup \partial\mathcal{H}^{p,q} \to \mathbb{H}^{p,q} \cup \partial_\infty\mathbb{H}^{p,q}$. One concludes the proof by remarking that $\mathbb{P} \circ \bar{\iota}_{p,q} = \mathbb{P} \circ \widetilde{\iota}_{p,q} = \iota_{p,q}$. $\square$

2.6.3 Hausdorff pseudo-metric

Our next goal is to describe the entire boundary $\partial_\infty\mathbb{H}^{p,q}$, seen in the half-space model. The starting observation is that $\partial_\infty\mathbb{H}^{p,q}$ is in bijection with the space of degenerate totally geodesic hyperplanes in $\mathbb{H}^{p,q}$. Indeed, to any $X \in \mathbb{R}^{p,q+1}$ such that $\langle X, X \rangle = 0$, one associates the intersection of the orthogonal subspace of X with $\mathbb{H}^{p,q}$, more precisely $(X^\perp \cap \widetilde{\mathbb{H}}^{p,q})/\{\pm \mathrm{Id}\}$, which is a totally geodesic hyperplane in $\mathbb{H}^{p,q}$ of degenerate type (see Example 1.4.5.6). We will simply denote it with $X^\perp$, by a small abuse of notation.

Example 2.6.3.1. The hyperplane "at infinity", which is the complement of the embedding $\iota_{p,q}$ in Proposition 2.1.2.2, is defined by the equation $X_p + \ X_{p+q+1} = 0$, hence it is the orthogonal of any nonzero vector proportional to $\partial_{X_p} - \partial_{X_{p+q+1}}$ in $\mathbb{R}^{p,q+1}$.

Clearly two hyperplanes $X^\perp$ and $Y^\perp$ coincide if and only if X and Y are proportional, and every degenerate totally geodesic hyperplane is obtained in this way.

To formalize the idea of *convergence of subset*, we introduct a topology on the power set of a metric space. We remark that $\mathcal{H}^{p,q}$ is not a metric space, nevertheless, we could use the Euclidean metric induced by the inclusion in $\mathbb{R}^{p+q}$.

Definition 2.6.3.2 (Hausdorff pseudo-metric). Let (X, d) be a metric space. Let $A \subseteq X$, $x \in X$, the distance between a point and a set is defined as

$$d(x, A) = \inf_{a \in A} d(x, a).$$

Then the *pseudo-distance* between two subset $A, B \subseteq X$ is

$$d_{\mathrm{H}}(A, B) := \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\}.$$

Lemma 2.6.3.3. *Let (X, d) be a metric space, $E_{\mathrm{H}} \subseteq \mathcal{P}(X)$ the set of closed bounded subspaces of X, then $(E_{\mathrm{H}}, d_{\mathrm{H}})$ is a metric space.*

Proof. The closed hypothesis is needed to distinguish points: otherwise if $x \in \bar{A} \setminus A$, then $d(x, A) = 0$, and so $d_{\mathrm{H}}(\{x\}, A)$. We ask the set to be bounded to assure that $d_{\mathrm{H}} < +\infty$.

It is clear that $d_{\mathrm{H}} \geq 0$, we show that different subset have positive distance. Let $A, B \in E_{\mathrm{H}}$, $A \neq B$. Up to switch names, we can assume $A \subseteq B^c$, that is an open subset of a metric space: then it exists $x \in A$, $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq B^c$, that is $d(x, B) > \varepsilon$. Hence

$$d_{\mathrm{H}}(A, B) \geq \sup_{a \in A} d(a, B) \geq d(x, B) \geq \varepsilon.$$

The symmetry is obvius by definition. We have to check triangular inequality. Set $A, B, C \in E_{\mathrm{H}}$; d is a distance, hence $d(a, b) \leq d(a, c) + d(c, b)$ for all $a \in A$, $b \in B$ and $c \in C$. It holds for all b in B, then

$$d(a, B) = \inf_{b \in B} d(a, b) \leq d(a, c) + \inf_{b \in B} d(c, b) = d(a, c) + d(c, B).$$

This is true for any $c \in C$, hence

$$d(a, B) \leq d(a, c) + \sup_{c \in C} d(c, B) \leq d(a, c) + d_{\mathrm{H}}(C, B).$$

Again, since a was arbitrary

$$\sup_{a\in A} d(a,B) \leq \sup_{a\in A} d(a,c) + d_{\mathrm{H}}(C,B) \leq d_{\mathrm{H}}(A,C) + d_{\mathrm{H}}(C,B).$$

One can prove the same inequality for $\sup_{b\in B} d(A,b)$, which ends the proof.
□

It is clear that the topology of $\partial_\infty \mathbb{H}^{p,q}$ is homeomorphic, under this correspondence, to the Hausdorff topology on closed subsets of $\partial_\infty \mathbb{H}^{p,q}$. Indeed, a sequence of vectors X_n converges projectively to X if and only if the orthogonal subspace of X_n converges to the orthogonal subspace of X. Our aim is to study the space of degenerate subspaces, which is a *visible* copy of the boundary $\partial_\infty \mathbb{H}^{p,q}$ in the half space model $\mathcal{H}^{p,q}$, but these space are not bounded with respect to the Euclidean metric. Nevertheless, we are interested only in topology, so we endow $\mathcal{H}^{p,q}$ with the metric of a bounded ball $B(0,r) \subseteq \mathbb{R}^{p+q}$, so that d_{H} induces a metric over it. We will consider the topology induced by this metric when we refer to the Hausdorff topology.

2.6.4 The full boundary in the half-space model

We are now ready to give the following definition.

Definition 2.6.4.1 (Extended boundary). We define

$$\partial_\infty \mathcal{H}^{p,q} = \{\text{degenerate totally geodesic hypersurfaces in } \mathcal{H}^{p,q}\} \cup \{\infty\} ,$$

where we endow the space of degenerate totally geodesic hypersurfaces with the Hausdorff topology described above, and $\partial_\infty \mathcal{H}^{p,q}$ with its one-point compactification, ∞ be the added point.

That definition permits to extend $\iota_{p,q}$ to an homeomorphism between the boundaries. The proof of that will be given through several lemmas, which explains the topology of the boundary. Recall from Proposition 2.4.2.1 that degenerate totally geodesic hypersurfaces in $\mathcal{H}^{p,q}$ are of the following two types:

- Vertical hyperplanes $V_{\mathcal{L}}$, where $\mathcal{L}$ is a degenerate affine hyperplane in $\mathbb{R}^{p-1,q}$;
- Quadric hypersurfaces $Q_{(x_0,y_0)}$ of equation

$$\|x - x_0\|^2 - \|y - y_0\|^2 + z^2 = 0 \tag{2.11}$$

We have already seen in Proposition 2.6.2.1 that $\iota_{p,q}$ maps $Q_{(x_0,y_0)}$ to $\iota_{p,q}(x_0, y_0, 0)$ and ∞ to the projective class of $\partial_{X_p} - \partial_{X_{p+q+1}}$ in $\mathbb{RP}^{p+q}$, now we are interested in its behaviour over degenerate hyperplanes.

Lemma 2.6.4.2. *Let $V_{\mathcal{L}}$ be degenerate vertical hyperplane defined by the equation $\langle x, u\rangle - \langle y, v\rangle = a$, for (u, v) lightlike vector of $\mathbb{R}^{p-1,q}$ and $a \in \mathbb{R}$. $\iota_{p,q}(V_{\mathcal{L}})$ is the projective class of $(u, -a, v, a)^{\perp}$.*

Proof. Let $X \in (u, -a, v, a)^{\perp} \cap \operatorname{Im} \widetilde{\iota}_{p,q}$, $X = (X_1, \dots, X_{p+q+1})$, then

$$\langle (X_1, \dots, X_{p-1}), u\rangle - aX_p - \langle (X_{p+1}, \dots, X_{p+q}), v\rangle - aX_{p+q+1} = 0.$$

$\operatorname{Im} \widetilde{\iota}_{p,q} = \{X_p + X_{p+1} > 0\}$, hence we can divide by $X_p + X_{p+1}$, obtaining

$$\langle \left(\frac{X_1}{X_p + X_{p+1}}, \dots, \frac{X_{p-1}}{X_p + X_{p+1}} \right), u\rangle - \langle \left(\frac{X_{p+1}}{X_p + X_{p+1}}, \dots, \frac{X_{p+q}}{X_p + X_{p+1}} \right), v\rangle = a.$$

We built the inverse of $\widetilde{\iota}_{p,q}$ (on its image) in Proposition 2.1.2.2, by defining

$$\begin{aligned} x_i &= \frac{X_i}{X_p + X_{p+q+1}}, & i &= 1, \dots, p-1 \\ y_j &= \frac{X_{j+p}}{X_p + X_{p+q+1}}, & j &= 1, \dots, q \\ z &= \frac{1}{X_p + X_{p+q+1}}. \end{aligned}$$

Hence the equation becomes $\langle x, u\rangle - \langle y, v\rangle = a$, which ends the proof. ☐

Proposition 2.6.4.3. *$\iota_{p,q} \colon \partial_\infty \mathcal{H}^{p,q} \to \partial_\infty \mathbb{H}^{p,q}$ is bijective.*

Proof. We proved in Proposition 2.6.2.1 that

$$\iota_{p,q} \colon \partial \mathcal{H}^{p,q} \to \partial_\infty \mathbb{H}^{p,q} \cap \{X_p + X_{p+q+1} \neq 0\}$$

is bijective and $\iota_{p,q}(\infty) = [0 : 1 : 0 : -1]$.

Lemma 2.6.4.2 states that $\iota_{p,q}$ is a bijection from the set of degenerate hyperplanes of $\mathcal{H}^{p,q}$, namely $\partial_\infty \mathcal{H}^{p,q} \setminus (\partial \mathcal{H}^{p,q} \cup \{\infty\})$, to the set of the projective classes of degenerate vectors of $\mathbb{R}^{p,q+1}$ such that $X_p + X_{p+q+1} = 0$, except for $[0 : 1 : 0 : -1]$, since $(u, v) \neq 0$, which ends the proof. ☐

We want now to study the topology of $\partial_\infty \mathcal{H}^{p,q}$ to prove that $\iota_{p,q}$ is continuous, therefore we are interested in convergence of sequences in $\partial_\infty \mathcal{H}^{p,q}$.

Remark 2.6.4.4. Consider a sequence $(A_n)_{n \in \mathbb{N}} \subseteq \partial_\infty \mathcal{H}^{p,q}$ such that A_n is described by the equation $f_n(x, y, z) = 0$. The sequence converges to $A_\infty = \{f_\infty(x, y, z) = 0\}$ if and only if $f_n \to f_\infty$ in $\mathrm{L}^\infty(\mathbb{R}^{p+q})$.

Moreover, we remark that any point in the boundary is represented by a degenerate maximal totally geodesic hypersurfaces of $\mathcal{H}^{p,q}$. Two maximal totally geodesic subspaces of the same dimension that coincide over an open set are equal. Since ours set are defined by smooth equations, this implies that $A_n \to A_\infty$ if and only is $f_n \to f_\infty$ in $\mathrm{L}^\infty(U)$, for U an open set. Particularly, we will take an open set having a compact neighborhood.

Let us describe more concretely the convergence of a sequence of elements in $\partial\mathcal{H}^{p,q}$, seen as a subset of $\partial_\infty\mathcal{H}^{p,q}$. The statements that we will give are *modulo* extraction of subsequences.

Lemma 2.6.4.5. *Consider a sequence (x_n, y_n) in $\partial\mathcal{H}^{p,q}$, we have the following possibilities:*

i. if (x_n, y_n) converges to (x_∞, y_∞), then $Q_{(x_n,y_n)}$ converges to $Q_{(x_\infty,y_\infty)}$, which still gives an element of $\partial\mathcal{H}^{p,q}$;

ii. if $\|x_n\| - \|y_n\|$ is bounded, $Q_{(x_n,y_n)}$ converges to the degenerate vertical hyperplane of equation $x \cdot u - y \cdot v = a$, where u, v and a are respectively the limits of $x_n/\|x_n\|$,$y_n/\|y_n\|$ and $\|x_n\| - \|y_n\|$;

iii. otherwise, (x_n, y_n) converges to the point $\infty \in \partial_\infty\mathcal{H}^{p,q}$.

Proof. For (i), it suffices to compute the limit with respect to Equation (2.11):

$$\|x - x_n\|^2 - \|y - y_n\|^2 + z^2 = 0 \longrightarrow \|x - x_\infty\|^2 - \|y - y_\infty\|^2 + z^2 = 0.$$

For (ii), assume (x_n, y_n) diverges in $\mathbb{R}^{p-1,q}$ and $\|x_n\| - \|y_n\|$ is bounded. Up to extracting a subsequence, $\|x_n\| - \|y_n\|$ has a finite limit $a \in \mathbb{R}$, so both $\|x_n\|, \|y_n\| \neq 0$ for n big enough. Hence, up to extract a second subsequence, $x_n/\|x_n\| \to u$ and $y_n/\|y_n\| \to v$, for $(u, v) \in \mathbb{S}^{p-2} \times \mathbb{S}^{q-1}$, namely $\|u\|^2 = \|v\|^2 = 1$, so (u, v) is lightlike. Remark that

$$\frac{\|x_n\|}{\|y_n\|} = \underbrace{\frac{\|x_n\| - \|y_n\|}{\|y_n\|}}_{\approx a/\|y_n\|} +1 \to 1.$$

By expanding Equation (2.11) and dividing it by $\|y_n\|$, one obtains

$$\frac{\|x\|^2 - \|y\|^2 + z^2}{\|y_n\|} - 2\langle x, \frac{x_n}{\|y_n\|}\rangle + 2\langle y, \frac{y_n}{\|y_n\|}\rangle + \frac{\|x_n\|^2 - \|y_n\|^2}{\|y_n\|} = 0.$$

Using Remark 2.6.4.4, we can restrain the function to a compact neighborhood of an open set, so that the variables x, y, z are bounded. Hence

$$\begin{array}{ccccccc}
\frac{\|x\|^2-\|y\|^2+z^2}{\|y_n\|} & - & 2\langle x, \frac{x_n}{\|y_n\|}\rangle + 2\langle y, \frac{y_n}{\|y_n\|}\rangle & + & \frac{\|x_n\|+\|y_n\|}{\|y_n\|}(\|x_n\| - \|y_n\|) & = & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & - & 2\langle x, u\rangle + 2\langle x, v\rangle & + & 2a & = & 0
\end{array}$$

namely $Q_{(x_n,y_n)}$ converges to $V_\mathcal{L}$, $\mathcal{L} \subseteq \mathbb{R}^{p-1,q}$ being an affine subspace whose underlying vector space is $(u, v)^\perp$, so it is a degenerate hyperplane of $\mathbb{R}^{p-1,q}$.

Finally, for (iii), we remark that (x_n, y_n) in $\partial\mathcal{H}^{p,q}$ converges to the point $\infty \in \partial_\infty\mathcal{H}^{p,q}$ if and only if $Q_{(x_n,y_n)}$ escapes from all compact sets of the half-space. Write (2.11) as

$$\|x\|^2 + z^2 - 2\langle x, x_n\rangle + \|x_n\|^2 - \|y_n\|^2 = \|y\|^2 - 2\langle y, y_n\rangle.$$

Assume $\|x_n\|/\|y_n\| \to 0$. Dividing by $\|y_n\|$ we obtain

$$\left|\frac{\|y\|^2}{\|y_n\|} - 2\langle y, \frac{y_n}{\|y_n\|}\rangle\right| \leq \frac{\|y\|^2}{\|y_n\|} + 2\frac{\|y\|\,\|y_n\|}{\|y_n\|} \leq \left(\|y\|^2 + 1\right)\left(\frac{1}{\|y_n\|} + 2\right).$$

Since $\|y_n\|$ diverges, $(2 + 1/\|y_n\|) \leq 3$ for n big enough. It follows that a point $(x, y, z) \in Q_{(x_n,y_n)}$ satisfies the following inequalities:

$$\begin{aligned} 3(\|y\|^2 + 1) &\geq \left|\frac{\|x\|^2 + z^2}{\|y_n\|} - 2\langle x, \frac{x_n}{\|y_n\|}\rangle + \frac{\|x_n\|^2 - \|y_n\|^2}{\|y_n\|}\right| \geq \\ &\geq \left|\frac{\|x_n\|^2 - \|y_n\|^2}{\|y_n\|}\right| - \left|\frac{\|x\|^2 + z^2}{\|y_n\|} - 2\langle x, \frac{x_n}{\|y_n\|}\rangle\right| \geq \\ &\geq \left|\frac{\|x_n\|^2 - \|y_n\|^2}{\|y_n\|}\right| - \frac{\|x\|^2 + z^2}{\|y_n\|} - 2\|x\|^2\frac{\|x_n\|^2}{\|y_n\|^2}. \end{aligned} \tag{2.12}$$

Let $K \subseteq \mathcal{H}^{p,q}$ be a compact subset of $\mathbb{R}^{p+q}$. The coordinates of its points are limited, namely it exists $C \in \mathbb{R}$ such that

$$\|x\|^2 + \|y\|^2 + z^2 \leq C, \qquad \forall (x, y, z) \in K. \tag{2.13}$$

Combining (2.12) with (2.13), a point $(x, y, z) \in Q_{(x_n,y_n)} \cap K$ satisfies

$$3C \geq \|y\|^2 \geq \left|\frac{\|x_n\|^2 - \|y_n\|^2}{\|y_n\|}\right| - \frac{C}{\|y_n\|} - \frac{\|x_n\|^2}{\|y_n\|^2}C - 3 =: F(x_n, y_n).$$

By hypothesis, $F(x_n, y_n) \to +\infty$ as $n \to \infty$, that is for n big enough $F(x_n, y_n) > 3C$, which means $Q_{(x_n,y_n)} \cap K = \emptyset$. K is arbitrary, hence $Q_{(x_n,y_n)}$ escapes from all compact sets, that is $(x_n, y_n) \to \infty$. □

Finally we can show that $\iota_{p,q}$ is an homeomorphism between the boundaries.

Proposition 2.6.4.6. *The embedding $\iota_{p,q}$ induces a homeomorphism between $\partial_\infty\mathcal{H}^{p,q}$ and $\partial_\infty\mathbb{H}^{p,q}$.*

Proof. It suffices to prove that $\iota_{p,q}$ is continuous, *i.e.* it commutes with limits.

Let $(x_n, y_n) \to V_{\mathcal{L}} = \{\langle x, u\rangle - \langle y, v\rangle = a\}$ in $\partial_\infty\mathcal{H}^{p,q}$. For Lemma 2.6.4.5,

$$\frac{x_n}{\|x_n\|} \to u, \qquad \frac{y_n}{\|y_n\|} \to v, \qquad \|x_n\| - \|y_n\| \to a.$$

Applying $\iota_{p,q}$ to the sequence, one has

$$\begin{aligned}\iota_{p,q}(x_n, y_n, 0) &= \left[x_n : \frac{1-\|x_n\|^2+\|y_n\|^2}{2} : y_n : \frac{1-\|x_n\|^2+\|y_n\|^2}{2}\right] = \\ &= \left[\frac{x_n}{\|y_n\|} : \frac{1-\|x_n\|^2+\|y_n\|^2}{2\|y_n\|} : \frac{y_n}{\|y_n\|} : \frac{1-\|x_n\|^2+\|y_n\|^2}{2\|y_n\|}\right].\end{aligned}$$

Recalling that $\|x_n\|/\|y_n\| \to 1$,

$$\frac{1-\|x_n\|^2+\|y_n\|^2}{2\|y_n\|} = \frac{1}{2\|y_n\|} - \underbrace{\frac{\|x_n\|+\|y_n\|}{2\|y_n\|}}_{=1}(\|x_n\|-\|y_n\|) \to -a.$$

Hence we proved

$$\lim_{n\to\infty}(\iota_{p,q}(x_n, y_n, 0)) = [u : -a : v : a] = \iota_{p,q}\left(\lim_{n\to\infty}(x_n, y_n, 0)\right).$$

We should check also that a sequence $(x_n, y_n) \to \infty$ commutes with $\iota_{p,q}$. This can be checked using the same computation as above, dividing by $\|y_n\|^2$ instead of $\|y_n\|$, if $\|x_n\|/\|y_n\| \to 0$, and by $\|x_n\|^2$ otherwise.

To show that a sequence $V_{\mathcal{L}_n} \to V_{\mathcal{L}_\infty}$ commutes with $\iota_{p,q}$ one can either compute it explicitely or take a family of sequences $(x_k^n, y_k^n)_{k\in\mathbb{N}}$, such that $(x_k^n, y_k^n) \to V_{\mathcal{L}_n}$ for $k \to \infty$, and use a diagonal argument to build a sequence

$$(x_k^{n_k}, y_k^{n_k}) \to V_{\mathcal{L}_\infty}.$$

One concludes by unicity of the limit. □

2.6.5 Examples

Let us now describe the topology of $\partial_\infty\mathcal{H}^{p,q}$ in two definite examples.

Example 2.6.5.1. Let us first consider $\mathcal{H}^{1,n}$, namely the half-space model of *minus* the de Sitter space. In this case $\partial\mathcal{H}^{1,n}$ is conformal to $\mathbb{R}^{0,n}$, hence is negative definite, therefore there are no degenerate affine hyperplanes in $\partial\mathcal{H}^{1,n}$. In other words, $\partial_\infty\mathcal{H}^{1,n}$ is the one-point compactification of $\partial\mathcal{H}^{1,n} \cong \mathbb{R}^n$, and therefore is homeomorphic to the sphere $\mathbb{S}^n$. This is not surprising indeed, as the $(n+1)$-dimensional de Sitter space shares the same boundary at infinity as the hyperbolic space $\mathbb{H}^{n+1}$ of the same dimension.

From the point of view of $\mathcal{H}^{1,n}$, this corresponds to the fact that a sequence of degenerate totally geodesic hypersurfaces Q_{y_n} defined by the equation $\|y - y_n\|^2 - z^2 = 0$, which is a cone over $(y_n, 0)$ (see Figure 2.1 on the right), escapes from compact sets in the half-space if the sequence y_n is diverging in $\mathbb{R}^n$.

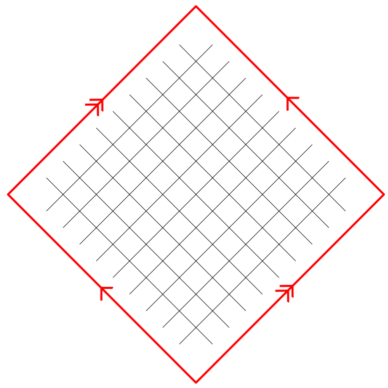

Figure 2.4: The compactification of $\partial\mathcal{H}^{2,1}$, which is a copy of $\mathbb{R}^{1,1}$ represented by the interior of the diamond, inside $\partial_\infty\mathcal{H}^{2,1}$. The lines at $\pm 45°$ represent the degenerate affine subspaces in $\mathbb{R}^{1,1}$, and each of them is compactified to a different point. The point ∞ then corresponds to the vertices of the diamond. The identifications of the sides clearly give the topology of a torus on $\partial_\infty\mathcal{H}^{2,1}$.

Example 2.6.5.2. Let us now consider a more interesting situation, namely the Anti-de Sitter half-space $\mathcal{H}^{n,1}$. In this case $\partial_\infty\mathcal{H}^{n,1}$ decomposes as the disjoint union of $\partial\mathcal{H}^{n,1}$, which is a copy of the n-dimensional Minkowski space $\mathbb{R}^{n-1,1}$, the singleton $\{\infty\}$, and the space of vertical hyperplanes $V_{\mathcal{L}}$. The latter is in bijection with the space of degenerate hyperplanes in $\mathbb{R}^{n-1,1}$, which is a trivial bundle $\mathbb{S}^{n-2} \times \mathbb{R}$, where the $\mathbb{S}^{n-2}$ factor determines the orthogonal direction (*i.e.* the projectivization of the cone $\{\|x\|^2 - y^2 = 0\}$ of null directions), and the $\mathbb{R}$ factor the intercept on the y axis. The complement $\partial_\infty\mathcal{H}^{n,1} \setminus \partial\mathcal{H}^{n,1}$ is therefore the one-point compactification of $\mathbb{S}^{n-2} \times \mathbb{R}$.

When $n = 2$, $\mathbb{S}^0 \times \mathbb{R}$ is the disjoint union of two lines, and its one-point compactification is homeomorphic to a wedge sum of two circles. Hence we directly recover the fact that $\partial_\infty\mathbb{H}^{2,1}$ is homeomorphic to a torus $\mathbb{S}^1 \times \mathbb{S}^1$. Indeed, from Remark 2.6.4.5 we see that $\mathbb{R}^{1,1}$ is compactified by adding a point to compactify every line of the form $y = x + a$ (this is the first copy of $\mathbb{R}$ in $\mathbb{S}^0 \times \mathbb{R}$), and a point for every line of the form $y = -x + a$ (the second copy of $\mathbb{R}$). By adding the point ∞, we then see that the obtained topology is that of a torus, see Figure 2.4. Compare also ([Dan11, Appendix]) for a more algebraic approach to this compactification.

2.6.6 Geodesics revisited

To conclude this section, we discuss again the geodesics in $\mathcal{H}^{p,q}$, now in terms of their endpoints in $\partial_\infty\mathcal{H}^{p,q}$. Indeed, in $\mathbb{H}^{p,q}$ the geodesics have the following topological behaviour:

- Spacelike geodesics converge to two different points in $\partial_\infty \mathcal{H}^{p,q}$ at the two ends.

- Lightlike geodesics converge to the same point in $\partial_\infty \mathcal{H}^{p,q}$ at the two ends.

- Timelike geodesics are closed, hence do not intersect $\partial_\infty \mathcal{H}^{p,q}$.

We will classify geodesics, distinguishing their type as usual, in relation with their endpoints.

Remark 2.6.6.1. Before stating the results, we give a preliminary observation that will be used repeatedly. It will be important to understand when a sequence of points $(x_n, y_n, z_n) \in \mathcal{H}^{p,q}$ converges to a point of $\partial_\infty \mathcal{H}^{p,q}$. In Section 2.6.4 we explained this for a sequence in $\partial \mathcal{H}^{p,q}$, *i.e.* for $z_n \equiv 0$ (see Lemma 2.6.4.5).

When the points are in the interior, we can apply a similar consideration, namely the fact that a sequence of points $X_n \in \mathbb{H}^{p,q}$ converges (projectively) to $X \in \partial_\infty \mathbb{H}^{p,q}$ if and only if the lightcone emanating from X_n converges to the totally geodesic degenerate hypersurface which corresponds to the orthogonal complement of X. Hence to check if a sequence $(x_n, y_n, z_n) \in \mathcal{H}^{p,q}$ converges to a point of $\partial_\infty \mathcal{H}^{p,q}$, which we recall is identified to the space of totally geodesic degenerate hypersurfaces, it suffices to check the convergence of the lightcones from $(x_n, y_n, z_n) \in \mathcal{H}^{p,q}$ (as in Figure 2.2).

In particular, it is clear that the topology on $\mathcal{H}^{p,q} \cup \partial \mathcal{H}^{p,q}$ coincides with that of the closed half-space, because if $(x_n, y_n, z_n) \to (x_0, y_0, 0)$, then the lightcones $\|x - x_n\|^2 - \|y - y_n\|^2 + |z - z_n|^2 = 0$ converge to the totally geodesic degenerate hypersurface $\|x - x_0\|^2 - \|y - y_0\|^2 + |z|^2 = 0$.

Let us first consider spacelike geodesics, beginning with the case where the two endpoints are both in $\partial \mathcal{H}^{p,q} \subset \partial_\infty \mathcal{H}^{p,q}$.

Proposition 2.6.6.2. *Let* $(x_0, y_0), (x_0', y_0') \in \partial \mathcal{H}^{p,q}$ *and define*

$$(u, v) := (x_0' - x_0, y_0' - y_0), \qquad (x_m, y_m) := ((x_0 + x_0')/2, (y_0 + y_0')/2).$$

Then:

- *If* $\|u\| > \|v\|$, *then the unique geodesic of* $\mathbb{H}^{p,q}$ *with endpoints* $\iota_{p,q}(x_0, y_0, 0)$ *and* $\iota_{p,q}(x_0', y_0', 0)$ *is contained in* $\mathcal{H}^{p,q}$, *and is the ellipse of eccentricity* $e_S(u, v)$ *with center* (x_m, y_m).

- *If* $\|u\| < \|v\|$, *then the unique geodesic of* $\mathbb{H}^{p,q}$ *with endpoints* $\iota_{p,q}(x_0, y_0)$ *and* $\iota_{p,q}(x_0', y_0')$ *is contained in* $\mathcal{H}^{p,q}$ *except for one point, and its intersection with* $\mathcal{H}^{p,q}$ *consists of the two upper half-branches of the hyperbola of eccentricity* $e_S(u, v)$ *with center* (x_m, y_m).

- *If* $\|u\| = \|v\|$, *there is no geodesic with endpoints* $\iota_{p,q}(x_0, y_0)$ *and* $\iota_{p,q}(x_0', y_0')$.

We remark that (u, v) is the vector joining the two points on the boundary, while (x_m, y_m) is the midpoint, and recall that the value $e_S(u, v)$ of the eccentricity appears in Proposition 2.5.4.1.

Proof. There is not much left to prove here. The first point follows from Proposition 2.5.4.1. For the third point, it is known that if two points in $\partial_\infty \mathbb{H}^{p,q}$ are connected by a lightlike segment in the boundary, then they are not connected by a spacelike geodesic; however the non-existence also follows from Proposition 2.5.4.1. For the second point, using again Proposition 2.5.4.1, the only thing left to prove is that the two half-branches of the same hyperbola are parts of the same spacelike geodesic in $\mathbb{H}^{p,q}$, and are separated by a single point. (We have showed that these branches are incomplete on the upper end, so they certainly converge to the interior of $\mathbb{H}^{p,q}$, since $\mathbb{H}^{p,q}$ is geodesically complete.)

To prove this statement, we can apply the isometry group of $\mathcal{H}^{p,q}$ and reduce to the curve parameterized by $y_1(t) = \pm\cosh(t)$, $z(t) = \sinh(t)$, and all the other coordinates identically zero (exactly as we did in the proof of Proposition 2.5.4.1). A direct computation shows that

$$\begin{aligned}\iota_{p,q}(\gamma(t)) &= \left[0 : 1 + \cosh(t)^2 - \sinh(t)^2 : \pm 2\cosh(t) : 1 - \cosh(t)^2 + \sinh(t)^2\right] = \\ &= [0 : 2 : \pm 2\cosh(t) : 0] = \left[0 : \frac{1}{\pm\cosh(t)} : 1 : 0\right].\end{aligned}$$

Clearly these points all lie on the same geodesic, because they are contained in a unique 2-plane in $\mathbb{R}^{p,q+1}$, and the limit as $t \to +\infty$ is $[0 : 0 : 1 : 0]$ regardless of the sign $\pm$ in front of $y_1(t)$. This concludes the proof. $\square$

Remark 2.6.6.3. A very similar computation shows that timelike geodesics of $\mathcal{H}^{p,q}$ are mapped to the complement of a point on a (closed) timelike geodesic of $\mathbb{H}^{p,q}$. Indeed, up to isometry, we can reduce to the branch of hyperbola $\gamma(t)$ given by $y_1(t) = \sinh(t)$, $z(t) = \cosh(t)$ and all the other coordinates identically zero. One can then show that the limit of the image under the embedding $\iota_{p,q}$ is the same point in $\mathbb{H}^{p,q}$ as $t \to \pm\infty$:

$$\begin{aligned}\iota_{p,q}(\gamma(t)) &= \left[0 : 1 + \sinh(t)^2 - \cosh(t)^2 : 2\sinh(t) : 1 - \sinh(t)^2 + \cosh(t)^2\right] = \\ &= [0 : 0 : 2\sinh(t) : 2] = \left[0 : 0 : 1 : \frac{1}{\sinh(t)}\right],\end{aligned}$$

which tends to $[0 : 0 : 1 : 0]$ for $t \to \pm\infty$.

The case where one point is on $\partial\mathcal{H}^{p,q}$ and the other is ∞ is very easy to deal with.

Proposition 2.6.6.4. *Let $(x_0, y_0) \in \partial\mathcal{H}^{p,q}$. The unique geodesic with endpoints (x_0, y_0) and ∞ is the vertical line over (x_0, y_0).*

Proof. Applying Remark 2.6.6.1, the endpoints of the vertical line over (x_0, y_0) are clearly (x_0, y_0) and ∞, for the lightcone over (x_0, y_0, z) converges to the totally geodesic hypersurface $\|x - x_0\|^2 - \|y - y_0\|^2 + |z|^2 = 0$ as $z \to 0$, and escapes from compact sets as $z \to +\infty$. $\square$

We are only left with the case where one point is on $\partial\mathcal{H}^{p,q}$, and the other is represented by a totally geodesic hypersurface $V_{\mathcal{L}}$. Indeed, after proving the next proposition, and comparing with Proposition 2.5.4.1, we see that *a posteriori* there are no geodesics in $\mathcal{H}^{p,q}$ connecting two points of the form $V_{\mathcal{L}}$ or ∞.

Proposition 2.6.6.5. *Let $(x_0, y_0) \in \partial\mathcal{H}^{p,q}$, $a \in \mathbb{R}$, $(u, v) \in \mathbb{R}^{p-1,q}$ such that $\|u\| = \|v\|$ and $\|u\|^2 + \|v\|^2 = 1$ and $\mathcal{L}^a_{(u,v)}$ the degenerate affine hyperplane in $\partial\mathcal{H}^{p,q}$ of equation $\langle (x - x_0), u\rangle - \langle (y - y_0)v\rangle = a$. Then the unique geodesic with endpoints (x_0, y_0) and $V_{\mathcal{L}^a_{(u,v)}}$ is the parabola*

$$x(t) = x_0 + \frac{t^2}{2a}u, \qquad y(t) = y_0 + \frac{t^2}{2a}v, \qquad z(t) = t\ . \tag{2.14}$$

Proof. Up to a horizontal translation, which does not affect the conclusion of the statement, we can assume $(x_0, y_0) = (0, 0)$. Set $\alpha = 1/4a$. The lightcones over $(x(t), y(t), z(t))$ satisfy the equation

$$\|x - 2\alpha t^2 u\|^2 - \|y - 2\alpha t^2 v\|^2 + |z - t|^2 = 0\ .$$

Dividing by t^2 and using $\|u\|^2 = \|v\|^2$,

$$\frac{\|x\|^2 - \|y\|^2}{t^2} + 2\alpha(\langle x, u\rangle - \langle y, v\rangle) + 4\alpha^2 t^2 \underbrace{(\|u\|^2 - \|v\|^2)}_{=0} + \frac{z^2 - 2zt + t^2}{t^2} = 0,$$

which converges as $t \to +\infty$ to the vertical hyperplane of equation

$$\langle x, u\rangle - \langle y, v\rangle = 1/4\alpha = a.$$

Clearly $(x(t), y(t), z(t))$ converges to $(x_0, y_0, 0) \in \partial\mathcal{H}^{p,q}$ as $t \to 0$. This concludes the proof. $\square$

Remark 2.6.6.6. One might wonder what is the geometric interpretation of the parameter a, which encodes the relation between the parabola and its endpoint at infinity, seen as a vertical hyperplane that does *not* contain the parabola itself. Let us describe the geometric intuition behind this relation. Given a parabola as in Equation (2.14), contained in a degenerate 2-plane V_ℓ, where ℓ is an affine line directed by (u, v), one can uniquely express this parabola as the intersection of V_ℓ and a totally geodesic degenerate hypersurface, which is a lightcone over a point $(\hat{x}, \hat{y}, 0)$. The vertical hyperplane to which the parabola is asymtoptic to is then the unique vertical hyperplane $V_{\mathcal{L}}$ such that the underlying vector space of $\mathcal{L}$ is the orthogonal of (u, v) and contains $(\hat{x}, \hat{y}, 0)$.

Let us now move on to lightlike geodesics.

Proposition 2.6.6.7. *Given $(x_0, y_0) \in \partial\mathcal{H}^{p,q}$, the lightlike geodesics of $\mathbb{H}^{p,q}$ with endpoint $\iota_{p,q}(x_0, y_0)$ are contained in $\mathcal{H}^{p,q}$ except for one point, and their intersection with $\mathcal{H}^{p,q}$ consists of two straight half-lines contained in the same vertical 2-plane.*

Proof. Up to a horizontal translation, it suffices to show that the half-lines $t \mapsto t(u, v, w)$ and $t \mapsto t(-u, -v, w)$, composed with the embedding $\iota_{p,q}$, converge to the same point in $\mathbb{H}^{p,q}$ at $t \to +\infty$, which can be checked similarly to Proposition 2.6.6.2 and Lemma 2.6.6.3. □

We now conclude our analysis by the only case left.

Proposition 2.6.6.8. *Given a degenerate affine hyperplane $\mathcal{L}$ in $\partial\mathcal{H}^{p,q}$, the lightlike geodesics of $\mathbb{H}^{p,q}$ with endpoint $V_{\mathcal{L}}$ are the horizontal straight lines contained in the vertical hyperplane $V_{\mathcal{L}}$ itself.*

Proof. By Remark 2.6.6.1, one has to check that the lightcones emanating from (x_0+tu, y_0+tv, z_0) converge to the vertical hyperplane through (x_0, y_0) whose underlying vector space is the orthogonal of (u, v). The computation is done exactly as in Remark 2.6.4.5. □

2.7 Horospheres

We now briefly turn the attention to the study of the horospheres, in the half-space model. Let us recall the definition of horosphere in $\mathbb{H}^{p,q}$.

Definition 2.7.0.1 (Horospheres). An *horosphere* in $\mathbb{H}^{p,q}$ is a smooth hypersurface S_a which is obtained as the projection in $\mathbb{H}^{p,q}$ of

$$\widetilde{S}_a = \{X \in \widetilde{\mathbb{H}}^{p,q}, \langle X, V\rangle_{p,q+1} = a\} \ , \tag{2.15}$$

for some null vector $V \in \mathbb{R}^{p,q+1}$ (*i.e.* $\langle V, V\rangle_{p,q+1} = 0$) and some constant $a \neq 0$. We say that the horosphere $\widetilde{S}_a$ has point at infinity $[V] \in \partial_\infty\mathbb{H}^{p,q}$.

Namely, a horosphere is the intersection between the projective class of an affine degenerate hyperplane of $\mathbb{R}^{p,q}$ and $\tilde{\mathbb{H}}^{p,q}$.

Observe that, when $V = \partial_{X_p} - \partial_{X_{p+q+1}}$, the corresponding horosphere S_a is precisely the image of $z = |a| \subset \mathcal{H}^{p,q}$ by the embedding $\iota_{p,q}$. We will prove this observation in the proof of Theorem 2.7.0.5 below.

For the sake of completeness, we provide a well-known characterization of horospheres in $\mathbb{H}^{p,q}$, that generalizes a classical description in hyperbolic space.

Lemma 2.7.0.2. *The horospheres S_a with point at infinity $[V]$ are precisely the smooth hypersurfaces orthogonal to all the spacelike geodesics having $[V]$ as an endpoint at infinity.*

Proof. To check the statement, since orthogonality can be computed locally, we will work in the double cover $\widetilde{\mathbb{H}}^{p,q}$. Let $\widetilde{S}_a = \{X \in \widetilde{\mathbb{H}}^{p,q}, \langle X, V\rangle_{p,q+1} = a\}$ and $X \in \widetilde{S}_a$. Then $\widetilde{S}_a = \widetilde{\mathbb{H}}^{p,q} \cap (V^\perp + X)$ and

$$T_X \widetilde{S}_a = T_X \widetilde{\mathbb{H}}^{p,q} \cap T_X(V^\perp + X) = X^\perp \cap V^\perp. \tag{2.16}$$

Now, every geodesic of $\widetilde{\mathbb{H}}^{p,q}$ is contained in the intersection of $\widetilde{\mathbb{H}}^{p,q}$ with a linear 2-dimensional subspace. In particular, the unique spacelike geodesic γ such that $\gamma(0) = X$ and having $[V]$ as an endpoint at infinity is contained in $\mathrm{Span}(X, V)$. So $\gamma'(0) \in \mathrm{Span}(X, V)$. Comparing with (2.16), we showed that $\gamma'(0)$ intersects $\widetilde{S}_a$ orthogonally.

Finally, observe that every spacelike geodesic in $\mathbb{H}^{p,q}$ with endpoint at infinity $[V]$ intersects S_a. Indeed, working again in the double cover, the preimages of spacelike geodesics of $\mathbb{H}^{p,q}$ are the intersection of $\widetilde{\mathbb{H}}^{p,q}$ with linear 2-dimensional subspaces. Given such a subspace containing the vector V, pick X such that $\langle X, V\rangle_{p,q+1} = a$. Then for every $\lambda \in \mathbb{R}$ we have $\langle X - \lambda V, V\rangle_{p,q+1} = a$. Choosing $\lambda = (\langle X, X\rangle + 1)/2a$, we obtain

$$\langle X - \lambda V, X - \lambda V\rangle_{p,q+1} = \langle X, X\rangle_{p,q+1} - 2\lambda a = -1,$$

hence $X - \lambda V \in \widetilde{\mathbb{H}}^{p,q}$, and therefore $X - \lambda V \in \widetilde{S}_a$. This concludes the proof. $\square$

Despite the term horospheres, which is borrowed from classical hyperbolic geometry, horospheres are not topologically spheres for $q \neq 0$. The boundary at infinity $\partial_\infty S_a$ of a horosphere S_a, namely its frontier in $\partial_\infty \mathbb{H}^{p,q}$, is precisely the lightcone in $\partial_\infty \mathbb{H}^{p,q}$ from $[V]$, hence $S_a \cup \partial_\infty S_a$ is homeomorphic to $\partial_\infty \mathbb{H}^{p,q}$.

Lemma 2.7.0.3. *$\partial_\infty S_a$ is the lightcone in $\partial_\infty \mathbb{H}^{p,q}$ from $[V]$, $[V]$ being the endpoint of S_a at infinity.*

Proof. Assume $[W] \in \partial_\infty S_a$ does not belong to the lightcone from $[V]$, *i.e.* $[V]$ and $[W]$ do not lie on a lightlike line. This implies that it exists a spacelike geodesic γ joining $[V]$ and $[W]$ (see Proposition 2.6.6.2). By Lemma 2.7.0.2, γ meets S_a transversally at a point $[X] \in \mathbb{H}^{p,q}$. γ is the projection of the 2-plane of $\mathbb{R}^{p,q+1}$ containing V, W, X. The three vectors are contained in $\{Y \in \mathbb{R}^{p,q+1}, \langle Y, V\rangle_{p,q+1} = a\}$ by construction, and none of them is colinear, hence the affine hyperplane contains the 2-plane $\mathrm{Span}(V, W, X)$, *i.e.* it is a vector hyperplane, which is an absurd.

Conversely, we recall that the lightlike geodesic of $\widetilde{\mathbb{H}}^{p,q}$ are the ligthlike lines $\mathbb{R}^{p,q+1}$ contained in the submanifold. We claim that for any $W \in \partial_\infty V^\perp$, the geodesic $\gamma(t) = X + tW$ is all contained in the affine hyperplane defined by the equation $\langle Y, V\rangle_{p,q+1} = a$, where $a = \langle X, V\rangle_{p,q+1}$. Indeed,

$$\langle \gamma(t), V\rangle_{p,q+1} = \langle X, V\rangle_{p,q+1} + t\langle W, V\rangle_{p,q+1} = a + 0, \quad \forall t \in \mathbb{R}.$$

Hence $[W] \in \mathbb{P}\{Y \in \mathbb{R}^{p,q+1}, \langle Y, V\rangle_{p,q+1} = a\} \cap \partial_\infty \mathbb{H}^{p,q}$, that is $[W] \in \partial_\infty S_a$. $\square$

Proposition 2.7.0.4. *$S_a \cup \partial_\infty S_a$ is homeomorphic to $\partial_\infty \mathbb{H}^{p,q}$.*

Proof. Consider the endpoint $[V] \in \partial_\infty \mathbb{H}^{p,q}$ of S_a at infinity. By Lemma 2.7.0.2, we can associate to any point $[X] \in S_a$ a point of the boundary, namely the other endpoint of the unique spacelike geodesic with endpoint in $[V]$ and intersecting S_a in $[X]$. This defines a injective continuous map $S_a \to \partial_\infty \mathbb{H}^{p,q}$. Its image is the complement of the cone from $[V]$, which coincides with $\partial_\infty S_a$ by Lemma 2.7.0.3. $\square$

In this section we will describe the horospheres in the half-space model $\mathcal{H}^{p,q}$.

Theorem 2.7.0.5. *The horospheres of $\mathcal{H}^{p,q}$ are, for a parameter $c > 0$:*

1. *horizontals hyperplanes $\{z = c\}$, if the point at infinity is ∞;*
2. *wedges of hyperplanes of the form*

$$z = c|\langle x, u\rangle - \langle y, v\rangle + d|$$

   if the point at infinity corresponds to the vertical hyperplane $V_{\mathcal{L}}$, for $\mathcal{L}$ the hyperplane of equation $\langle x, u\rangle - \langle y, v\rangle + d = 0$ (for $(u, v) \in \mathbb{R}^{p-1,q}$ a lightlike vector and $d \in \mathbb{R}$);

3. *piecewise quadric hypersurfaces of the form*

$$\|x - x_0\|^2 - \|y - y_0\|^2 + (z \pm c)^2 = c^2$$

   if the point at infinity is $(x_0, y_0, 0) \in \partial\mathcal{H}^{p,q}$.

See Figures 2.5 and 2.6.

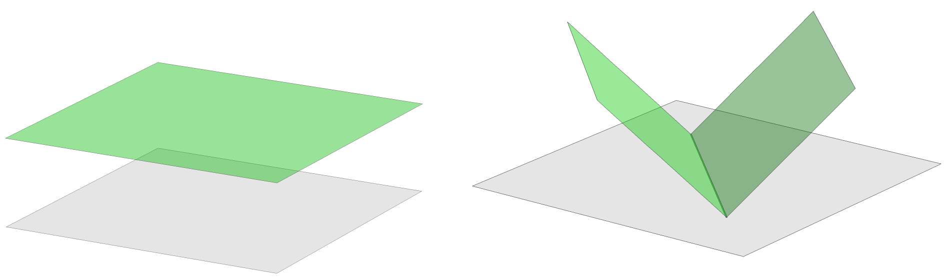

Figure 2.5: Horizontal horospheres, and wedges of hyperplanes.

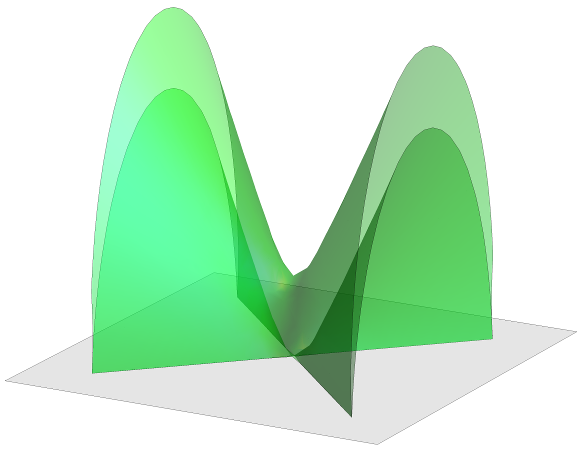

Figure 2.6: Horospheres in $\mathcal{H}^{2,1}$ corresponding to a point in $\partial\mathcal{H}^{2,1}$.

Proof. Recall that we introduced the embedding $\tilde{\iota}_{p,q}\colon \mathcal{H}^{p,q} \to \widetilde{\mathbb{H}}^{p,q}$ in the proof Proposition 2.1.2.2, that induces the embedding $\iota_{p,q}\colon \mathcal{H}^{p,q} \to \widetilde{\mathbb{H}}^{p,q}$ in the quotient. Also observe that every point in a horosphere S_a has two preimages X in $\widetilde{\mathbb{H}}^{p,q}$, which satisfy either $\langle X, V\rangle = a$ or $\langle X, V\rangle = -a$. Hence to determine the horospheres in $\mathcal{H}^{p,q}$ (or more precisely, the portion of horospheres contained in $\mathcal{H}^{p,q}$) it suffices to find the preimage of $|\langle X, V\rangle| = a$ under $\tilde{\iota}_{p,q}$, for $a > 0$.

In the first case, the point at infinity is ∞ and corresponds to $[V] = [0 : 1 : 0 : -1]$ in $\partial_\infty\mathbb{H}^{p,q}$. In Proposition 2.1.2.2 we showed that, via $\tilde{\iota}_{p,q}$, $X_p + X_{p+q+1} = 1/z$. This shows that the level sets $\{z = c\}$ are precisely the preimages of $\langle X, (0, 1, 0, -1)\rangle = a$, where $a = 1/c$.

Consider now the case where the point at infinity is $V_{\mathcal{L}}$. In Lemma 2.6.4.2 we computed the preimage of a degenerate vector space intersecting the boundary in the complement of $\operatorname{Im}\iota_{p,q}$. To do it in the affine case, it suffice to remark that the equation starts as $\langle X, V\rangle_{p,q+1} = a$ instead of $\langle X, V\rangle_{p,q+1} = 0$. Hence, taking $[V] = [u : d : v : -d]$, dividing by $X_p + X_{p+q+1} = 1/z$ we obtain

$$\langle x, u\rangle - \langle y, v\rangle + d = az.$$

As we stated above, the horosphere is the preimage of both $\langle X, V\rangle_{p,q+1} = \pm a$, so the equation above prove the statement for $c = 1/a$.

Finally, we consider the case where the point at infinity is $(x_0, y_0, 0) \in \partial\mathcal{H}^{p,q}$. Up to translation, we can assume $(x_0, y_0) = (0, 0)$. The corresponding point in $\partial_\infty\mathbb{H}^{p,q}$ is $[V]$, where $V_p = V_{p+q+1} = 1$ and the other coordinates of V are zero. Hence we need to determine the preimage of those X satisfying

$$|\langle X, (0, 1, 0, 1)\rangle| = |X_p - X_{p+q+1}| = a.$$

Observe that $X_p - X_{p+q+1} = -(\|x\|^2 - \|y\|^2 + z^2)/z$, hence we obtain the equation $|\|x\|^2 - \|y\|^2 + z^2| = az$. A simple manipulation gives the equivalent expression

$$\|x\|^2 - \|y\|^2 + \left(z \pm \frac{a}{2}\right)^2 = \frac{a^2}{4} \ ,$$

which is the desired formula, for $c = a/2$. $\square$

Remark 2.7.0.6. We conclude by remarking that the proof of Theorem 2.7.0.5 could have been done by checking directly that the hypersurfaces of the three types are orthogonal to all the spacelike geodesics in $\mathcal{H}^{p,q}$ which share an endpoint in $\partial_\infty \mathcal{H}^{p,q}$. This is evident for the horizontal horospheres $\{z = c\}$, which are orthogonal to all vertical geodesics, *i.e.* with endpoint ∞.

For the wedges of hyperplanes, one can show directly, using Proposition 2.6.6.5 and Remark 2.6.6.6, that the union of the hyperplanes

$$z = \pm(\langle x, u\rangle - \langle y, v\rangle)$$

is orthogonal to all the parabolas whose endpoint corresponds to the vertical hyperplane $V_{\mathcal{L}}$, $\mathcal{L} = \{\langle x, u\rangle - \langle y, v\rangle = 0\}$, namely those parabolas which are obtained as the intersection of a vertical 2-plane projecting to an affine line directed by (u, v), and a lightcone based on a point of $\mathcal{L}$.

Finally, for the horospheres of the third type, one could check that these are orthogonal to the geodesics with endpoint in $(x_0, y_0, 0)$ in the following way. Up to an isometry of $\mathcal{H}^{p,q}$, assume $(x_0, y_0) = (0, 0)$ and $c = 1$. Then one "sweeps" the hypersurface by curves of four types. The first case is that of a curve contained in a vertical 2-plane which is positive definite. Up to an isometry of the form $(x, y, z) \mapsto (A(x, y), z)$, which leaves the horosphere invariant, it suffices to consider the curve $x_1(t) = \sin(t)$, $z(t) = \cos(t) + 1$, and all the other coordinates identically zero. Then one shows that this curve is orthogonal to all the geodesics with endpoint $(0, 0)$ that it intersects, which are ellipses (circles, in this specific situation). This is exactly analogous to the half-space model of $\mathbb{H}^n$. Second, one consider curves in a vertical 2-plane which is indefinite. Again up to isometry, one reduces to two curves, defined by $y_1(t) = \sinh(t)$, and $z(t) = \cosh(t) + 1$ or $z(t) = \cosh(t) - 1$. These are orthogonal to all the spacelike geodesics which are hyperbolas and have $(0, 0)$ as an endpoint. In the former case, the curve intersects the branch containing $(0, 0)$; in the latter, the other branch. Finally, the horizontal planar curves contained in the horosphere are trivially ortogonal to all parabolas with endpoint $(0, 0)$, because they are lightlike and contained in a degenerate vertical 2-plane.

2.8 Isometries of $\mathcal{H}^{p,q}$

Let us conclude by describing the isometries of $\mathcal{H}^{p,q}$, and the action of those of $\mathbb{H}^{p,q}$, in the half-space model.

2.8.1 The isometry group $\mathrm{Isom}(\mathcal{H}^{p,q})$

We remarked in Subection 2.1.3 that the embedding $\iota_{p,q}$ induces a monomorphism $G \to \mathrm{Isom}(\mathbb{H}^{p,q})$, as a consequence of the fact that local isometries between open neighbourhoods of $\mathbb{H}^{p,q}$ uniquely extend to global isometries. From our study of geodesics in Section 2.5, we can deduce that the group G introduced in Subsection 2.1.3 (see Definition 2.1.3.5) is the full isometry group of the half-space model.

Theorem 2.8.1.1. *When $q \geq 1$, the group G coincides with the isometry group $\mathrm{Isom}(\mathcal{H}^{p,q})$. Moreover, G corresponds precisely to the isometries of $\mathrm{Isom}(\mathbb{H}^{p,q})$ that preserve the totally geodesic degenerate hyperplane $\mathbb{H}^{p,q} \setminus \iota_{p,q}(\mathcal{H}^{p,q})$.*

Proof. Since $G < \mathrm{Isom}(\mathcal{H}^{p,q})$ acts transitively by Proposition 2.1.3.6, it suffices to prove that $\mathrm{Stab}_G(0,0,1) = \mathrm{Stab}_{\mathrm{Isom}(\mathcal{H}^{p,q})}(0,0,1)$. Observe that $\mathrm{Stab}_G(0,0,1)$ is the subgroup of $\mathrm{Stab}_{\mathrm{Isom}(\mathcal{H}^{p,q})}(0,0,1)$ preserving oriented vertical lines, *i.e.* it consists of those isometries f such that $df_{(0,0,1)}(\partial_z) = \partial_z$. We claim that all isometries f in $\mathrm{Stab}_{\mathrm{Isom}(\mathcal{H}^{p,q})}(0,0,1)$ have this property.

By contradiction, assume $df_{(0,0,1)}(\partial_z) \neq \partial_z$. First, if $df_{(0,0,1)}(\partial_z) = -\partial_z$, then a lightlike geodesic starting at $(0,0,1)$ and parameterized in such a way that the z-coordinate is increasing along the geodesic (hence incomplete) would be sent to another lightlike geodesic parameterized in such a way that the z-coordinate is decreasing (hence complete) which is an absurd since isometries preserve completeness of geodesics. Otherwise,

$$df_{(0,0,1)}(\partial_z)^\perp \neq (\partial_z)^\perp.$$

The horizontal hyperplane $(\partial_z)^\perp \cong \mathbb{R}^{p-1,q}$ is generated by lightlike vectors, hence there exists a horizontal lightlike vector v such that $df_{(0,0,1)}(v)$ is not horizontal. This is an absurd as lightlike geodesics are complete (in both directions) if and only if the initial velocity is horizontal (Lemma 2.5.1.1), and again isometries preserve completeness.

The second part of the statement is clear, because every isometry of $\mathcal{H}^{p,q}$ extends to an isometry of $\mathbb{H}^{p,q}$ which preserves the image of $\iota_{p,q}$, hence also its complement. Conversely, every isometry of $\mathbb{H}^{p,q}$ that preserves the image of $\iota_{p,q}$ induces an isometry of $\mathcal{H}^{p,q}$, and therefore is in G. □

2.8.2 Inversions

In order to describe the action of the isometry group $\mathrm{Isom}(\mathbb{H}^{p,q})$ on the half-space model, we now introduce a new type of isometries, that are the analogous of inversions in hyperbolic geometry. Recall that, given a point $(x_0, y_0) \in \partial\mathcal{H}^{p,q}$, $Q_{(x_0,y_0)}$ denotes the totally geodesic hypersurface made of lightlike geodesics with endpoint (x_0, y_0), as in (2.11).

Proposition 2.8.2.1. *The involution* $\mathcal{J}\colon \mathcal{H}^{p,q}\setminus Q_{(0,0)} \to \mathcal{H}^{p,q}\setminus Q_{(0,0)}$ *defined by*

$$(x, y, z) \mapsto (\mu(x, y, z)x, \mu(x, y, z)y, |\mu(x, y, z)|z) \ ,$$

where $\mu(x, y, z) := (\|x\|^2 - \|y\|^2 + z^2)^{-1}$, *is an isometry which extends to a global isometry of* $\mathbb{H}^{p,q}$ *via* $\iota_{p,q}$.

We remark that if $q = 0$, $Q_{(0,0)} = \emptyset$ and $\mu > 0$, hence we recover the fact that $\mathcal{J}$ is a global isometry of the hyperbolic space.

Proof. Let $\phi \in \mathrm{Isom}(\mathbb{H}^{p,q})$ be the isometry induced by the reflection in the hyperplane $X_p = 0$. To prove the statement, we show that the following diagram commutes:

$$\begin{tikzcd} \mathcal{H}^{p,q} \setminus Q_{(0,0)} \arrow[r, "\iota_{p,q}"] \arrow[d, "\mathcal{J}"] & \mathbb{H}^{p,q} \arrow[d, "\phi"] \\ \mathcal{H}^{p,q} \setminus Q_{(0,0)} \arrow[r, "\iota_{p,q}"] & \mathbb{H}^{p,q} \end{tikzcd} .$$

We first remark that

$$\mu(\mathcal{J}(x, y, z)) = \mu(x, y, z)^{-2}(\|x\|^2 - \|y\|^2 + z^2)^{-1} = \mu(x, y, z)^{-1} \ , \qquad (2.17)$$

which also immediately implies that $\mathcal{J}$ is an involution. Observe that $\mathcal{J}$ is defined precisely on the complement of $\{\mu = 0\}$, which is $Q_{(0,0)}$. Suppose first $\mu > 0$. Denote $\tilde{\iota}_{p,q}(x, y, z) = (X_1, \dots, X_{p+q+1})$ (these are defined in the proof of Proposition 2.1.2.2) and $\tilde{\iota}_{p,q} \circ \mathcal{J}(x, y, z) = (Y_1, \dots, Y_{p+q+1})$. We have:

$$\begin{aligned} Y_i &= \frac{\mu x}{\mu z} = \frac{x}{z} = X_i && i = 1, \dots, p-1 \\ Y_p &= \frac{1-\mu}{2\mu z} = -\frac{1 - \|x\|^2 + \|y\|^2 - z^2}{2z} = -X_p \\ Y_{j+p} &= \frac{\mu y}{\mu z} = \frac{y}{z} = X_{j+p} && j = 1, \dots, q \\ Y_{p+q+1} &= \frac{1+\mu}{2\mu z} = \frac{1 + \|x\|^2 - \|y\|^2 + z^2}{2z} = X_{p+q+1} \end{aligned}$$

where in the second and fourth line we have used (2.17). This shows that $\tilde{\iota}_{p,q} \circ \mathcal{J} = \tilde{\phi} \circ \tilde{\iota}_{p,q}$, where $\tilde{\phi} \in \mathrm{O}(p, q+1)$ is the reflection fixing the hyperplane $X_p = 0$. One immediately checks that $\tilde{\iota}_{p,q} \circ \mathcal{J} = -\tilde{\phi} \circ \tilde{\iota}_{p,q}$ when $\mu < 0$. Since $\tilde{\phi}$ and $-\tilde{\phi}$ induce the same isometry on $\mathbb{H}^{p,q}$, the claim is proved.

Incidentally, this proves also that $\mathcal{J}$ is a local isometry with respect to the metric $g_{p,q}$: indeed $\phi \circ \iota_{p,q}(\mathcal{H}^{p,q} \setminus Q_{(0,0)}) \subseteq \mathrm{Im}\, \iota_{p,q}$, and $\iota_{p,q}$ is bijective over its image. Hence, $\mathcal{J} = \iota_{p,q}^{-1} \circ \phi \circ \iota_{p,q}$ over $\mathcal{H}^{p,q} \setminus Q_{(0,0)}$, namely $\mathcal{J}$ is a composition of isometries. $\square$

Remark 2.8.2.2. We saw that the involution $\mathcal{J}$ corresponds to a reflection in $\mathrm{Isom}(\mathbb{H}^{p,q})$. In fact its fixed point set is the totally geodesic hypersurface

$$\|x\|^2 - \|y\|^2 + z^2 = 1 \ .$$

The other inversions, fixing the general totally geodesic hypersurface of the form (Q) for $c > 0$, can be easily found conjugating $\mathcal{J}$ by elements of G.

In the following, we describe the action of $\mathcal{J}$ on totally geodesic hypersurfaces. We will denote $Q_{(x_0,y_0,c)}$ a quadric defined by Equation (Q) and $\partial Q_{(x_0,y_0,c)}$ its *boundary*, namely its intersection with the hyperplane $\{z = 0\}$. We remark that for an hyperplane $V_{\mathcal{L}}$, $\partial V_{\mathcal{L}} = \mathcal{L}$.

Recalling that a totally geodesic submanifold is defined by its local behaviour (see Remark 2.6.4.4), we will abusively talk about the whole totally geodesic hypersurface instead of its restriction to the domain of $\mathcal{J}$.

Proposition 2.8.2.3. *The action of $\mathcal{J}$ on the totally geodesic hypersurfaces of $\mathcal{H}^{p,q}$, except for $Q_{(0,0,0)}$ is the following:*

i. a quadric $Q_{(x_0,y_0,c)}$ such that $(0,0,0) \in \partial Q_{(x_0,y_0,c)}$ is sent to the vertical hyperplane $V_{\mathcal{L}}$ defined by the equation $\langle x, x_0\rangle - \langle y, y_0\rangle = 1/2$, which does not contain the origin, and conversely;

ii. a quadric $Q_{(x_0,y_0,c)}$ such that $(0,0,0) \notin \partial Q_{(x_0,y_0,c)}$ is sent to the quadric $Q_{(x_1,y_1,C)}$, where $x_1 = x_0/c_1$, $y_1 = y_0/c_1$ and $C = -(1 + \|x_0\|^2 - \|y_0\|^2)/c_1$, for $c_1 := (\|x_0\|^2 - \|y_0\|^2) - c$. The image also does not contain the origin;

iii. vertical hyperplanes $V_{\mathcal{L}}$ whose boundary contains the origin are (not punctually) preserved.

From a geometrical point of view, this is trivial: indeed vertical hyperplains are the only totally geodesic hypersurfaces containing ∞, hence a hypersurface can be send to a vertical hyperplain if and only if it contains the origin.

Proof. Since $\mathcal{J}$ is an involution, the image the quadric $Q_{(x_0,y_0,c)}$ satisfies

$$\|\mu x - x_0\|^2 - \|\mu y - y_0\|^2 + \mu^2 z^2 = c.$$

A simple algebraic manipulation, together with the fact that $\mu(x, y, z) = (\|x\|^2 - \|y\|^2 + z^2)^{-1}$ gives then

$$\mu \left(1 - 2(\langle x, x_0\rangle - \langle y, y_0\rangle)\right) = c - (\|x_0\|^2 - \|y_0\|^2).$$

Remarking that $c - (\|x_0\|^2 - \|y_0\|^2) = 0$ if and only if $(0,0,0) \in Q_{(x_0,y_0,c)}$, (i) is proved.

On the other hand, if $c_1 := c - (\|x_0\|^2 - \|y_0\|^2) - c \neq 0$, the equation can be rewritten as

$$1 - 2(\langle x, x_0 \rangle - \langle y, y_0 \rangle) = -c_1(\|x\|^2 - \|y\|^2 + z^2).$$

A simple algebraic manipulation shows that $\mathcal{J}(Q_{(x_0,y_0,c)}) = Q_{(x_1,y_1,C)}$, with x_1, y_1, C as in (ii).

Finally, if $\mathcal{L}$ contains the origin, then the equation that defines $V_{\mathcal{L}}$ is $\langle x, x_0 \rangle - \langle y, y_0 \rangle = 0$. As above, its image satisfies $\langle \mu x, x_0 \rangle - \langle \mu y, y_0 \rangle = 0$, *i.e.* $\mathcal{J}(V_{\mathcal{L}}) = V_{\mathcal{L}}$, which ends the proof. □

As a direct consequence of this result, we can describe the action of $\mathcal{J}$ on the boundary.

Corollary 2.8.2.4. *$\mathcal{J} \colon \partial_\infty \mathcal{H}^{p,q} \to \partial_\infty \mathcal{H}^{p,q}$ is an homeomorphism which*

i. *switches lightcones $Q_{(x_0,y_0)}$, such that $\|x_0\|^2 = \|y_0\|^2$, with vertical hyperplanes $V_{\mathcal{L}}$, $\mathcal{L} = \{\langle x, x_0 \rangle - \langle y, y_0 \rangle = 1/2\}$;*

ii. *switches lightcones $Q_{(x_0,y_0)}$, $\|x_0\|^2 \neq \|y_0\|^2$, with lightcones $Q_{(x_1,y_1)}$, where*
$$(x_1, y_1) = \frac{(x_0, y_0)}{\|x_0\|^2 - \|y_0\|^2};$$

iii. *preserves vertical hyperplanes $V_{\mathcal{L}}$, $\mathcal{L}$ vector hyperplane;*

iv. *switches the degenerate hyperplane $Q_{(0,0)}$ and ∞.*

Proof. The first points follow directly from the proposition remarking that $Q_{(x_0,y_0)} = Q_{(x_0,y_0,0)}$. The last one is trivial since $\iota_{p,q}(0,0) = [0 : 1 : 0 : 1]$ and $\iota_{p,q}(\infty) = [0 : -1 : 0 : 1]$, and we have just showed that $\mathcal{J}$ induces on $\mathbb{H}^{p,q}$ the reflection in the hyperplane $X_p = 0$. □

2.8.3 Action of $\mathrm{Isom}(\mathbb{H}^{p,q})$

We conclude by describing the action of the full isometry group $\mathrm{Isom}(\mathbb{H}^{p,q})$ on $\mathcal{H}^{p,q}$. Roughly speaking, the subgroup G and the inversion $\mathcal{J}$ (or more precisely, their extensions to $\mathbb{H}^{p,q}$) generate $\mathrm{Isom}(\mathbb{H}^{p,q})$.

Theorem 2.8.3.1. *Any isometry of $\mathbb{H}^{p,q}$ can be written in $\mathcal{H}^{p,q}$ as the composition of elements of G and $\mathcal{J}$.*

Proof. Since G corresponds precisely to the stabilizer of a point in $\partial_\infty \mathbb{H}^{p,q}$ by Theorem 2.8.1.1, it suffices to show that the elements of G, together with $\mathcal{J}$, induce a transitive action on $\partial_\infty \mathcal{H}^{p,q}$. Clearly G acts transitively on $\partial \mathcal{H}^{p,q}$, while $\mathcal{J}$ maps $(0,0) \in \partial \mathcal{H}^{p,q}$ to ∞ by Corollary 2.8.2.4.

Also, G acts transitively on the degenerate vertical hyperplanes of the form $V_{\mathcal{L}}$. Indeed it acts transitively on $\partial \mathcal{H}^{p,q} \cong \mathbb{R}^{p-1,q}$ and $V_{\mathcal{L}}$ is uniquly defined by $\mathcal{L} \subseteq \mathbb{R}^{p-1,q}$. Hence it remains to show that in the subgroup generated by G and $\mathcal{J}$, there is an element that maps some point in $\partial\mathcal{H}^{p,q}$ to some point in $\partial_\infty\mathcal{H}^{p,q}$ that corresponds to a vertical hyperplane $V_{\mathcal{L}}$. But this clear because by Proposition 2.8.2.1 $\mathcal{J}$ extends to an element $\phi \in \mathrm{Isom}(\mathbb{H}^{p,q})$, whose action on $\partial_\infty\mathbb{H}^{p,q}$ is a homeomorphism, hence it maps a neighbourhood of ∞ (which contains elements of the form $V_{\mathcal{L}}$) to a neighborhood of $(0,0)$ (which only contains points in $\partial\mathcal{H}^{p,q}$). $\square$

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