Alma Mater Studiorum · University of Bologna

School of Science
Department of Physics and Astronomy
Master Degree in Physics

Anomalies of a Weyl fermion in a gauge background

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Academic Year 2020/2021

Abstract

In this thesis we study the anomalies of a Weyl fermion in an abelian gauge background. It is well-know that this model has a chiral anomaly which breaks the gauge invariance and prevents a consistent quantization of the corresponding gauge theory. In addition, there is a trace anomaly, whose precise structure is the main focus of the present study.

We describe a derivation of these anomalies in terms of Feynman diagrams computation, using dimensional regularization and the Breitenlohner-Maison scheme for treating the chiral matrix γ^5.

The issue discussed here is analogous to the one present in the context of a Weyl fermion coupled to a gravitational background. In that case, there has been recently a debate about the presence or absence of a contribution of a parity-odd term in the trace anomaly. The coupling of the Weyl fermion to an abelian gauge field provides a simpler setting for discussing the possible presence or absence of a parity-odd term in the trace anomaly. Our final result indicates that parity-odd terms do not arise.

Abstract

In questa tesi studiamo le anomalie di un fermione di Weyl in un background di gauge abeliano. È ben noto che questo modello ha un'anomalia chirale che rompe l'invarianza di gauge e impedisce una consistente quantizzazione della corrispondente teoria di gauge. Inoltre, è presente un'anomalia di traccia, la cui precisa struttura è l'obiettivo principale del presente studio.

Presentiamo una derivazione di queste anomalie in termini del calcolo di diagrammi di Feynman, usando la regolarizzazione dimensionale e lo schema di Breitenlohner-Maison per trattare la matrice chirale γ^5.

L'argomento qui discusso è analogo a quello di un fermione di Weyl accoppiato ad un background gravitazionale. In questo caso è presente in letteratura un dibattito sulla presenza o assenza di un contributo di un termine di parità dispari nell'anomalia di traccia. L'accoppiamento di un fermione di Weyl ad un campo di gauge abeliano costituisce uno scenario più semplice per discutere la possibile presenza o assenza di un termine di parità dispari nell'anomalia di traccia. Il nostro risultato finale mostra che termini di parità dispari non compaiono.

Contents

Introduction

Symmetries and their corresponding conservation laws play an important role in describing the fundamental forces of nature. Actions are constructed in order to be invariant under transformations (symmetries) which are supposed to be dictated by the nature. However, it might turn out that a certain conservation law, valid in the classical theory, is violated in the quantum formulation. Then we speak of an *anomaly*.

Since symmetries are extremely important to build a consistent quantum field theory, anomalies are important as well. One can distinguish two types of anomalies, anomalies in global symmetries, *global anomalies*, and anomalies in local (gauge) symmetries, *gauge anomalies*. The latter prevent to consistently define a quantum theory, because they afflict symmetries which are necessary to renormalise the theory, unlike global anomalies which do not affect the renormalizability.

Also, the two most important classes of anomalies are named *chiral anomalies* and *trace anomalies*. Chiral anomalies appear in fermionic systems in which the axial current related to a chiral symmetry is not conserved, which often happens when the quantum theory is defined to preserve vector currents (whenever they exist). We may also mention here that the so-called gravitational anomalies are anomalies in the conservation of the stress tensor, and are closely related to the chiral anomalies in that they can only appear genuinely in chiral theories in 2, 6 and 10 dimensions (more generally in $2+4k$ dimensions with k an integer). In 4 dimensions they are absent, and if they appear it is only because of the use of non-invariant regulators, in which case they can be removed by adding local counterterms to the effective action. Trace anomalies are instead related to the breaking of the scale invariance of a scale invariant classical theory. The stress tensor ceases to have a vanishing (integrated) trace, which otherwise would be a consequence of scale invariance. Scale invariance can often be extended to an invariance under the conformal group, which implies that the unintegrated trace of the stress tensor vanishes classically (and hence the often used name of conformal anomalies). When a conformal theory is coupled to background gravity, the extended theory shows also a Weyl symmetry, i.e. symmetry under a Weyl transformation that rescales the background metric by an arbitrary (nowhere vanishing) function. In this context Weyl symmetry also implies the vanishing of the trace of the stress tensor (and hence the name of Weyl anomalies).

In this thesis, we study the anomalies of a Weyl fermion in an abelian gauge background. It is well-know that this model has an anomaly in the conservation of the gauge current which prevents the full gauge theory to be quantized consistently. In addition, this model has an anomaly in the trace of the stress tensor which has been computed recently in [1] with particular attention to the chiral properties of the model.

One of the reason to study this model is that it is supposed to be analogous to the one in which a Weyl fermion is coupled to a gravitational background. In particular, in that case there is a dispute in the literature concerning the presence or absence of a parity-odd term in the trace anomaly. This term has been found in [2], see also [3, 4, 5]. However, there are indications and results in support of the fact that such a term cannot be present in the trace anomaly, as shown in [6], [7], [8], [9] as well as [10] where the model has been recast in terms of Majorana fermions.

Here, we consider a Weyl fermion coupled to a $U(1)$ gauge background and derive its full set of anomalies. In particular, we compute the trace anomaly in order to determine whether or not it has a parity-odd term using a different regularization scheme than the one used in [1], where the trace anomaly has been computed using the Pauli-Villars regularization. Such a parity-odd term, if present, should have the form of a Chern-Pontryagin density $F\tilde{F}$ [11], [12].

We propose a derivation by means of a Feynman diagrams computation using dimensional regularization. In addition, because of the presence of the chiral matrix γ^5 there is the need of a prescription in order to treat it in dimensional regularization in a consistent manner. We adopt the Breitenlohner-Maison scheme [13] which is the same one used in [9]. Furthermore, this scheme breaks local Lorentz covariance and the conservation of the stress tensor is not guaranteed a the quantum level and has to be inspected, unlike in [1]. We find that the trace anomaly does not contain any parity-odd contributions.

The thesis is organized as follows. In chapter 1 we give a historical introduction of anomalies in quantum field theory with a focus on chiral theories. In chapter 2 we briefly summarise the main features of dimensional regularization and present the Breitenlohner-Maison prescription for γ^5. The model we are interested in is described in chapter 3 where the classical properties of stress tensor are studied and the Feynman rules are derived. The latter are needed for the calculations carried out in chapter 4 where we derive the anomalies of our model. In particular, we find as main result the absence of parity-odd contributions to the trace anomaly. In chapter 5 we introduce Majorana fermions, recast the model in the Majorana basis and compute again the trace anomaly. The last chapter 6 is dedicated to our conclusions. The final appendices are reserved to conventions, useful relations, formulae and integrals, and more details of the calculations presented in the main part of the thesis.

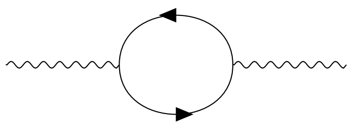
Chapter 1

Introduction to Anomalies

In this chapter we give an introduction to anomalies [14]. The first section is dedicated to a historical introduction based on [15] and [16], where more details and references can be found. In the second section we introduce anomalies in chiral theories. The latter are the main topic of the thesis and this section provides a background of knowledge about them.

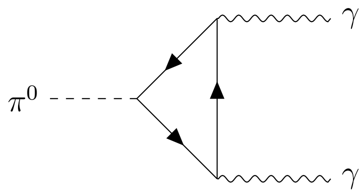
1.1 A brief history of anomalies

Physicists started computing radiative corrections to processes in Quantum Electrodynamics (QED) in 1930's and very soon they realized that the theory contained divergences and other inconsistency. Even the simplest one-loop diagram, that is the photon self-energy due to an electron loop, presented these difficulties.



Gauge invariance requires that this diagram be transversal and vanish on-shell because the photon should remain massless, but Tomonaga and collaborators found a divergent and not gauge invariant result. This divergence could be identified with a photon mass which could not be removed by renormalization because there is no photon mass as required by gauge invariance.

Two of Tomonaga's collaborators, Fukuda and Miyamoto, studied the next simplest diagram, namely the triangle diagram which was supposed to describe the decay $\pi \to \gamma\gamma$ mediated by fermions.



They found inconsistent results and concluded that the inconsistency arose from the mathematical difficulty of dealing with the Pauli-Jordan distributions and they did not know how to resolve the ambiguity.

Steinberger and Tomonaga, with Fukuda, Miyamoto and Miyazima applied the Pauli-Villars regularization to the triangle diagram. This regularization scheme seemed to to maintain gauge invariance and Lorentz covariance and led to a finite result for the triangle diagram. However, the result seemed to depend on how the calculations were performed and as a consequence there still was an ambiguity about the lifetime of the neutral pion. They suggested that such an ambiguity could be solved only by some experiment designed to detect the $\pi^0 \to \gamma\gamma$ decay. In other words, this inconsistency is nothing but the chiral anomaly.

Schwinger introduced a new regularization scheme (point splitting) which preserved gauge invariance at all intermediate stages, and, in 1951, he used it to compute the photon selfenergy and the triangle diagram. He found finite and gauge invariant result for the photon selfenergy and an apparent anomaly-free result for the triangle diagram.

Two important works appeared in 1969 signed by Bell and Jackiw and Adler. Bell and Jackiw studied the amplitude for the $\pi \to \gamma\gamma$ decay and they noted that it could be parametrized as

$$M^{\mu\nu}(p,q) = \epsilon^{\mu\nu\rho\sigma} p_\rho q_\sigma M(k^2) \tag{1.1}$$

where p and q are the on-shell photon momenta, and $k = p+q$ was the pion momentum. The amplitude was gauge invariant and symmetric. It satisfied the Ward identities $p_\mu M^{\mu\nu}(p,q) = 0$, $q_\nu M^{\mu\nu}(p,q) = 0$ and $M^{\mu\nu}(p,q) = M^{\nu\mu}(q,p)$. Bell and Jackiw focused on the computation of $M(0)$ which was already calculated by Steinberger, who found a non zero result, and by Veltman and Sutherland, who found a vanishing result using an off-mass-shell pion field equal to the divergence af the axial current (PCAC, the partially conserved axial current). Bell and Jackiw considered both the off-shell and the on-shell cases in the linear sigma model and found the result $M(0) = 0$. It seemed that there was no anomaly in their work, although in the appendix they commented that when dealing with linearly divergent integral a shift of variable produces a boundary term. So, they noted the hallmark of an anomaly.

Adler just studied the AVV triangle diagram in QED and proved the uniqueness of the triangle diagrams by imposing vector gauge invariance and discussed a possible connection with the $\pi^0 \to \gamma\gamma$ decay. The triangle diagram is the lowest and unique non zero diagram which contributes to the calculation of the anomaly. Indeed, it was shown by Adler and Bardeen that the ABJ anomaly is exact at one loop, which means that higher order loop diagrams do not contribute at all to the structure of the anomaly.

In 1971 't Hooft demonstrated the renormalizability of non abelian pure gauge theories and it was realized that anomalies would spoil renormalizability and unitarity. Thus, one has to make sure that anomalies in the gauge transformations of the effective action with chiral fermions, quarks and leptons, would cancel. In the Standard Model, which is a quantum field theory with gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$, one has to check that the currents associates to this group of symmetry are not anomalous. Only the $U(1)_Y$ hypercharge gauge group is potentially anomalous, but it was found that anomalies cancel and the Standard Model is free from chiral anomalies. A more exhaustive discussion about cancellation of anomalies in the Standard Model can be found in [17].

Anomalies can also arise if one couples fermions to gravity instead of electromagnetism. This was realised first by Kimura, later by Delbourgo and Salam, and then by Eguchi. They studied the triangle diagrams with non chiral fermions running in the loop, the axial current at one vertex and two other vertices given by $h^{\mu\nu}T_{\mu\nu}$, where $T^{\mu\nu}$ is the fermionic stress tensor and the coupling is obtained expanding the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and keeping only quadratic terms in $h_{\mu\nu}$. They found an anomaly of the form $\epsilon^{\mu\nu\rho\sigma} R^{\alpha\beta}_{\mu\nu} R_{\rho\sigma\alpha\beta}$, which is the gravitational contribution to the chiral anomaly.

This in turn leads to a related problem: if chiral fermions are coupled to a gravitational background, are there anomalies in the conservation of the stress tensor which are the counterpart of the anomalies in the gauge invariance of chiral gauge theories? The non conservation of the stress tensor is related to presence of local Lorentz anomaly. However, it was found that there is no potential problems for the Standard Model, because gravitational contributions to the chiral anomaly cancel, while local Lorentz anomalies can only occur in $4k + 2$ dimensions, with k an integer, and thus yield no potential problems for the Standard Model.

In addition to anomalies in chiral theories, there are also trace anomalies which occur when (rigid or local) scale invariance of the classical action is broken at the quantum level. This kind of anomaly associated to the breaking of local (Weyl) scale invariance was first observed by Capper and Duff in 1973. They tried to apply dimensional regularization to calculate corrections to the graviton propagator due to closed loops of massless vector and spinor fields. They computed the quantity

$$\Pi_{\mu\nu\rho\sigma}(p) = \int d^n x e^{ipx} \langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle |_{g_{\mu\nu}=\eta_{\mu\nu}} \tag{1.2}$$

where n is the spacetime dimension and $T_{\mu\nu}$ the stress tensor of the massless particles. They wanted to verify that this new regularization scheme correctly preserved the Ward identity $p^\mu \Pi_{\mu\nu\rho\sigma}(p) = 0$. They found the result

$$\Pi_{\mu\nu\rho\sigma} = \frac{1}{\varepsilon} \Pi_{\mu\nu\rho\sigma}(pole) + \Pi_{\mu\nu\rho\sigma}(finite) \tag{1.3}$$

where ε is the deviation from the physical spacetime dimension and the above result was obtain by expanding around $\varepsilon = 0$. They verified that both the pole part and the finite part obeyed the Ward identity for general covariance and that the infinity could then be removed by a generally covariant counterterm. So, there was no diffeomorphism anomaly. Since the systems under study were invariant under Weyl transformations of the metric, together with appropriate rescaling of the matter fields, the classical stress tensor was traceless and the self energy should also satisfy the identity $\Pi^{\mu}{}_{\mu\rho\sigma} = 0$. They verify the tracelessness of the pole part but found a non zero result for the finite part. This implied that conformal invariance was not preserved in the quantum theory and there was an anomaly.

The most general expression for the trace anomaly for spinors in four dimensions was found to be

$$\langle T^{\mu}{}_{\mu}\rangle = aR^2 + bR_{\mu\nu}R^{\mu\nu} + cR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + d\Box R + eF^{a}{}_{\mu\nu}F^{a\mu\nu} \tag{1.4}$$

The term $\Box R$ can be removed by a local counterterm $\Delta\mathcal{L} \sim R^2$, but the other terms are genuine anomalies. The coefficients are not all independent, but they combine to give

$$\langle T^{\mu}{}_{\mu}\rangle = \alpha\left(C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{2}{3}\Box R\right) + \beta E_4 + eF^{a}{}_{\mu\nu}F^{a\mu\nu} \tag{1.5}$$

where the last term is due to the possible presence of an external gauge field in addition to the gravitational field, $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$ is the square of the Weyl tensor, $E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ yields the Euler invariant and the constants α, β, γ also parametrize the one loop divergences due to matter loops with external gravity.

A rederivation of the conformal anomaly for a Majorana fermion (without external gauge fields) was given by Godazgar and Nicolai [10]. The computation is carried out in terms of Feynman diagrams and by using dimensional regularization and the result proves that the trace anomaly is half that of a Dirac fermion.

1.2 Anomalies in chiral theories

In [2], [4], [5] Bonora and collaborators computed the trace anomaly of a Weyl fermion coupled to a gravitational background and found that it contained a parity-odd term with imaginary coefficient

$$\langle T^{\mu}{}_{\mu}\rangle = \frac{1}{360(4\pi)^2}\left(9C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} - \frac{11}{2}E_4 - i\frac{15}{2}P\right) \tag{1.6}$$

where P is the Pontryagin density

$$P = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\alpha\beta}R_{\rho\sigma}{}^{\alpha\beta}. \tag{1.7}$$

This result has been derived in dimensional regularization, both using perturbative computation around Minkowski space time using Feynman diagrams [2], and introducing a metric-axial-tensor gravity approach through which the same result is derived using Dirac fermions, [4], [5], similar to the axial vector potential used in gauge theories.

The presence of such a term in the trace anomaly was conjectured by Nakayama [11]. It can not be excluded a priori because it satisfies the Wess-Zumino consistency condition [18], according to which the anomalous Ward identities must satisfy integrability relations which follow from the structure of the gauge group and are non trivial in the case of non abelian gauge groups. The consistency condition allows to determine the form of the anomaly in a model-invariant way, up to an overall constant. The values of these constants differentiate one model from the other and need to be obtained by explicit calculations.

The CP odd Chern-Pontryagin term obtained by Bonora and collaborators has an imaginary coefficient which is a problem for the unitarity of the model. Indeed, this would imply that the hamiltonian is complex and unitary is broken.

The same model has been studied in [6], [7], [8], and the trace anomaly has been computed using strictly four-dimensional regularization methods such as Pauli-Villars regularization, employed by Bastianelli and Martelli [6] and Bastianelli and Broccoli [7], and Hadamard subtraction [8]. They found no Chern-Pontryagin density contribution and that the trace anomaly was half that of a Dirac fermion.

In a recent work [9], the trace anomaly for a Weyl fermion has been computed using dimensional regularization and the Breitenlohner-Maison scheme for treating the chiral matrix γ^5 [13]. The authors computed the full quantum expectation value of the stress tensor and evaluated the trace and the conservation. The expectation value of the stress tensor does not contain any parity-odd contribution and therefore no Chern-Pontryagin density term appears in the trace anomaly. The result derived therein agrees with [6], [7], [10], [8]

$$\langle T^\mu{}_\mu \rangle = \frac{1}{360(4\pi)^2}\left(9C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} - \frac{11}{2}E_4\right). \tag{1.8}$$

The regularization scheme used in [9] breaks the Lorentz covariance and anomalies in the conservation of the stress tensor (Einstein anomalies, also called gravitational anomalies) may appear. However, the conservation of the stress tensor has been checked and no anomalies of this kind has been found. This does not happen when working with Pauli-Villars regularization which preserves Lorentz symmetry, while in [8] a vanishing divergence of the stress tensor is imposed and allows to cancel parity-odd terms appearing at intermediate steps in the calculation of the trace anomaly.

An apparently analogous and perhaps simpler model to study is that of a Weyl fermion coupled to an external gauge field instead of a gravitational background. The trace anomaly for this model has been computed by Bastianelli and Broccoli in the case of an external abelian gauge background [1] and for a non abelian gauge background

[19]. The regularization method employed is the Pauli-Villars and the result confirms the absence of a Chern-Pontryagin term. The stress tensor does not suffer any anomaly in the conservation because the regularization scheme employed preserves the local Lorentz covariance.

Recently the same model of [1] has been studied in [12] by Bonora. Therein the trace anomaly is derived in terms of Feynman diagrams computation and dimensional regularization, and contains a contribution of the parity-odd term $F\tilde{F}$.

This model possesses another anomaly in the conservation of the gauge current, called chiral anomaly which prevents it to be consistently quantized. We conclude this chapter by reporting the expressions for the chiral and trace anomalies derived in [1] because our aim is to test them by using a different regularization method, that is, dimensional regularization and Breitenlohner-Maison prescription for γ^5

$$\partial_a \langle J^a \rangle = \frac{1}{96\pi^2} \epsilon^{abcd} F_{ab} F_{cd} \tag{1.9}$$

$$\langle T^a{}_a \rangle = -\frac{1}{48\pi^2} F^{ab} F_{ab}. \tag{1.10}$$

Chapter 2

Dimensional Regularization

2.1 Generalities on Dimensional Regularization

Dimensional Regularization (DR) is the regularization scheme in which the regularization parameter is the dimension of the spacetime [20] [21] [22]. DR allows to regulate divergent integrals arising from loop calculations in perturbation theory in four dimensions.

The definition of any object involved in the loop integration is extended from the four-dimensional spacetime to the n-dimensional one and any amplitude or Feynman diagram one wishes to compute has to be considered as an analytic function of the spacetime dimension.

Calculations are carried out in n dimensions and integrals are evaluated after Wick-rotating the integration variable to Euclidean space where they are well defined[1]. After that, one takes the limit $n \to 4$ to restore the ordinary spacetime dimension and obtain the desired result in four dimensions. In this limit singularities may appear and DR allows to separate the finite part from the divergent part of the final result which is said to be regularized. These singularities can be removed if DR is supported by a subtraction scheme in which one adds n-dimensional counterterms to the action whose contribution is to cancel these singular terms once the limit $n \to 4$ is taken, so as to obtain a finite result in four dimensions. A theory is said to be renormalizable if all infinities appearing in perturbation calculations are removable by a finite number of counterterms.

More details and examples of application of DR can be found in many textbooks, for instance [23], [17], [24]. In this thesis we will apply dimensional regularization in order to regulate Feynman diagrams to compute anomalies.

[1]See appendix D.

2.2 The Breitenlohner-Maison prescription

One of the main problem when dealing with DR concerns the extension of purely four-dimensional object to n dimensions, such as the chiral gamma matrix[2] γ^5 or the Levi-Civita symbol ϵ_{abcd}.

In four dimensions one has

$$\gamma^5 = \frac{i}{4!}\epsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d = -i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{2.1}$$

where the Levi-Civita symbol ϵ_{abcd} is normalised as $\epsilon_{0123} = -1$ and $\epsilon^{0123} = 1$. It follows that

$$\left\{\gamma^5, \gamma^a\right\} = 0 \tag{2.2}$$

$$(\gamma^5)^2 = \mathbb{1} \tag{2.3}$$

and one can compute the trace of γ^5 with an even number, greater than or equal to four, of Dirac gamma matrices. For instance, the trace of γ^5 with four gamma matrices is given by

$$Tr\left\{\gamma^5\gamma^a\gamma^b\gamma^c\gamma^d\right\} = 4i\epsilon^{abcd} \tag{2.4}$$

where $a, b, c, d = 0, 1, 2, 3$.

In n dimensions there are n gamma matrices which satisfy the Clifford algebra

$$\left\{\gamma^a, \gamma^b\right\} = 2\eta^{ab}, \quad a, b = 0, 1, ...n \tag{2.5}$$

and the simplest thing one can do is to extend (2.2) to n dimensions, but, unfortunately, a fully anticommuting γ^5 leads to inconsistency for parity-odd calculations, that is calculations in which there are traces involving an odd number of γ^5 matrices [13] [25] [26] [27]. Indeed, using an anticommuting γ^5 one can easily derive the identity

$$(n-4)Tr\left(\gamma^5\gamma^a\gamma^b\gamma^c\gamma^d\right) = 0 \tag{2.6}$$

which admits the two solutions $n = 4$ and $Tr\left(\gamma^5\gamma^a\gamma^b\gamma^c\gamma^d\right) = 0$. We reject the latter solution, otherwise the parity-odd calculation would vanish identically. However, the other solution $n = 4$ prevents us from considering n to be a variable.

In [22] 't Hooft and Veltman proposed a generalization of γ^5 in n dimensions which was such that γ^5 anticommuted with the first four gamma matrices and commuted with the remaining $n - 4$, and derived the ABJ anomaly within this scheme by computing the AVV triangle diagram. This proposal was further developed by Breitenlohner and

[2] Sometimes the chiral matrix in n dimensions is denoted by γ_* to avoid confusion with γ^a when $a = 5$. We keep using the ordinary notation since there is no situation that could create confusion.

Maison [13], who proved its consistency at all orders in perturbation theory. That is why this scheme is called Breitenlohner-Maison prescription[3].

According to this scheme, the n-dimensional Minkowski spacetime splits in the product of a four-dimensional and a $(n-4)$-dimensional subspace. Any n-dimensional object, such as metric tensor, gamma matrices, momenta, ecc., decomposes into a four-dimensional part (denoted by an overbar) and a $(n-4)$-dimensional part (denoted by a hat), for instance

$$\eta^{ab} = \bar{\eta}^{ab} + \hat{\eta}^{ab}, \quad \gamma^a = \bar{\gamma}^a + \hat{\gamma}^a, \quad p^a = \bar{p}^a + \hat{p}^a, \quad ... \tag{2.7}$$

Contractions between indices belonging to different subspaces vanish[4]. The chiral matrix γ^5 is defined as in four dimensions by (2.1)

$$\gamma^5 = \frac{i}{4!}\epsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d = -i\bar{\gamma}^0\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^3 \tag{2.8}$$

where ϵ_{abcd} is purely four-dimensional, and anticommutes with gamma matrices in the four-dimensional subspace, while it commutes with those belonging to the $(n-4)$-dimensional one

$$\left\{\gamma^5, \bar{\gamma}^a\right\} = 0 \quad \text{for } a = 0, 1, 2, 3 \tag{2.9a}$$

$$\left[\gamma^5, \hat{\gamma}^a\right] = 0 \quad \text{for } a \geq 4. \tag{2.9b}$$

Moreover, this prescription is such to preserve the square (2.3) and the ciclicity of the trace.

This scheme breaks n-dimensional Lorentz covariance for chiral objects and, as a consequence, spurious non covariant terms may appear in the calculations, which however are expected to be removable with finite non covariant counterterms.

The Breitenlohner-Maison prescription is not the unique one for dealing with γ^5 in DR and a comparison between different proposals can be found in [25]. However, among these, only the Breitenlohner-Maison scheme has been shown to give mathematically consistent results at arbitrary loop orders [13] [28].

We conclude this chapter with an important remark. A prescription for γ^5 in n dimensions is necessary to overcome inconsistencies in parity-odd calculations, as explained above. However, whenever parity-even calculations are concerned, in which traces containing an even number of γ^5 matrices appear, no such inconsistencies arise and one can safely extend (2.2) to n dimensions and use the square (2.3) to completely eliminate γ^5 from the traces [25] [26] [27].

[3]Sometimes it is also called Breitenlohner-Maison-'t Hooft-Veltman prescription.

[4] Useful relations are collected in appendix B.

Chapter 3

The Weyl Fermion

3.1 The model

The model we are interested in is that of a massless Weyl fermion coupled to a background abelian gauge field described by the lagrangian

$$\mathcal{L} = -\bar{\lambda}\gamma^a \left(\partial_a - iA_a\right)\lambda = -\bar{\lambda}\gamma^a D_a(A)\lambda = -\bar{\lambda}\not{D}(A)\lambda. \tag{3.1}$$

$A_a(x)$ is the background $U(1)$ gauge field and λ is a left handed Weyl spinor defined by $\lambda = P_L\lambda$ and $\bar{\lambda} = \bar{\lambda}P_R$, where

$$P_L = \frac{1}{2}\left(\mathbb{1} + \gamma^5\right) \qquad P_R = \frac{1}{2}\left(\mathbb{1} - \gamma^5\right) \tag{3.2}$$

are the left and right chiral projectors, respectively. They are idempotent

$$P_L^2 = P_L \quad \text{and} \quad P_R^2 = P_R \tag{3.3}$$

and orthogonal

$$P_L P_R = 0. \tag{3.4}$$

The latter property forbids to write a Dirac mass term for Weyl fermions in the lagrangian. Under a parity transformation left handed Weyl fermions are mapped into right handed ones and vice versa since $\gamma^5 \to -\gamma^5$.

Symmetries of the classical action are gauge and conformal (Weyl) transformations. Under a gauge transformation fields transform as

$$\begin{cases} \lambda(x) \to \lambda(x)' = e^{i\alpha(x)}\lambda \\ \bar{\lambda}(x) \to \bar{\lambda}'(x) = e^{-i\alpha(x)}\bar{\lambda} \\ A_a(x) \to A_a(x)' = A_a(x) + \partial_a\alpha(x) \end{cases} \tag{3.5}$$

and the corresponding Noether's current

$$J^a = i\bar{\lambda}\gamma^a\lambda \tag{3.6}$$

is conserved on-shell $\partial_a J^a = 0$.

The invariance under conformal (or Weyl) transformations implies that the classical stress energy tensor is traceless. The stress energy tensor is the Noether's current associated to the invariance of the action under spacetime translations.

A simple way to obtain it is to couple the theory to gravity by introducing the *vielbein* $e_\mu{}^a$ and the related spin connection $\omega_\mu{}^{ab}$ [1]. The coupling to gravity is described by the curved-space lagrangian

$$\mathcal{L} = -e\bar{\lambda}\gamma^\mu\nabla_\mu\lambda \tag{3.7}$$

where $\gamma^\mu = e^\mu{}_a\gamma^a$ are the gamma matrices with curved indices ($e^\mu{}_a$ is the inverse of the vielbein), e is the determinant of the vielbein and ∇_μ is the covariant derivative containing both the gauge field and spin connection

$$\nabla_\mu = \partial_\mu - iA_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^a\gamma^b. \tag{3.8}$$

The local Weyl symmetry in n dimensions is given by

$$\begin{cases} \lambda(x) \to \lambda'(x) = e^{\frac{1-n}{2}\sigma(x)}\lambda(x) \\ \bar{\lambda}(x) \to \bar{\lambda}'(x) = e^{\frac{1-n}{2}\sigma(x)}\bar{\lambda}(x) \\ A_a(x) \to A'_a(x) = A_a(x) \\ e_\mu{}^a(x) \to e'_\mu{}^a(x) = e^{\sigma(x)}e_\mu{}^a(x) \end{cases} \tag{3.9}$$

where $\sigma(x)$ is an arbitrary function. In curved space the action acquires two new symmetries being invariant under general coordinate and local Lorentz transformations.

The stress tensor is defined as the functional derivative of the action with respect to the vielbein

$$T^\mu{}_a = \frac{1}{e}\frac{\delta S}{\delta e_\mu{}^a}. \tag{3.10}$$

The symmetries of the action fix the properties of the classical stress tensor which turns out to be covariantly conserved, symmetric and traceless on-shell

$$\nabla_\mu T^\mu{}_a = 0, \quad T^{ab} = T^{ba}, \quad T^a{}_a = 0. \tag{3.11}$$

The explicit expression of the stress energy tensor in flat spacetime in terms of the covariant gauge derivative is

$$T_{ab} = \frac{1}{4}\bar{\lambda}\left(\gamma_a\overleftrightarrow{D}_b + \gamma_b\overleftrightarrow{D}_a\right)\lambda - \frac{1}{2}\bar{\lambda}\eta_{ab}\overleftrightarrow{\not{D}}\lambda \tag{3.12}$$

where $\overleftrightarrow{D}_a = D_a - \overleftarrow{D}_a$ and $\overleftarrow{D}_a = \overleftarrow{\partial}_a + iA_a$. The last term vanishes on-shell.

[1] More details can be found in appendix C or chapter 12 of [29].

3.2 Properties of the Stress Tensor

In this section the classical properties of the stress tensor (3.11) are derived.

The infinitesimal transformations associated to general coordinate invariance (diffeomorphism with infinitesimal local parameter ξ^μ), local Lorentz invariance (with infinitesimal local parameter ω_{ab}) and Weyl invariance (with infinitesimal local parameter σ) take the form

$$\begin{cases} \delta e_\mu{}^a & = \xi^\nu \partial_\nu e_\mu{}^a + (\partial_\mu \xi^\nu)\, e_\nu{}^a + \omega^a{}_b e_\mu{}^b + \sigma e_\mu{}^a \\ \delta A_\mu & = \xi^\nu \partial_\nu A_\mu + (\partial_\mu \xi^\nu)\, A_\nu \\ \delta\lambda & = \xi^\nu \partial_\mu \lambda + \frac{1}{4}\omega_{\mu ab}\gamma^a\gamma^b\lambda + \frac{1-n}{2}\sigma\lambda \end{cases} \tag{3.13}$$

Under Weyl symmetry the gauge field does not transform and the invariance of the action implies the tracelessness of the stress tensor

$$\begin{aligned} \delta_\sigma S &= \int d^n x \left(\frac{\delta S}{\delta e_\mu{}^a(x)} \delta_\sigma e_\mu{}^a(x) + \frac{\delta S}{\delta\lambda(x)}\delta_\sigma\lambda(x) + \delta_\sigma\bar{\lambda}(x)\frac{\delta S}{\delta\bar{\lambda}(x)} \right) \\ &= \int d^n x e T^\mu_{\ a}(x)\delta_\sigma e_\mu{}^a(x) = \int d^4 x e T^a_{\ a}(x)\sigma(x) = 0 \end{aligned} \tag{3.14}$$

where the last two terms in the first line vanish because of the equations of motion of the Weyl spinors.

Under a local Lorentz symmetry one has

$$\begin{aligned} \delta_\omega S &= \int d^n x \left(\frac{\delta S}{\delta e_\mu{}^a(x)} \delta_\omega e_\mu{}^a(x) + \frac{\delta S}{\delta\lambda(x)}\delta_\omega\lambda(x) + \delta_\omega\bar{\lambda}(x)\frac{\delta S}{\delta\bar{\lambda}(x)} \right) \\ &= \int d^n x e T^\mu_{\ a}(x)\delta_\omega e_\mu{}^a(x) = \int d^4 x e T^\mu_{\ a}(x)\omega^a{}_b e_\mu{}^b \\ &= \int d^n x e T^{ba}\omega_{ab} = 0 \end{aligned} \tag{3.15}$$

where the vanishing of last two terms in the first line is due to the fermionic equations of motion, and the antisymmetry of the local parameters $\omega_{ab}(x)$ constrains the stress tensor to be symmetric.

Finally, the conservation of the stress tensor comes from the diffeomorphism invariance of the action

$$
\begin{aligned}
\delta_\xi S &= \int d^n x \left(\frac{\delta S}{\delta e_\mu{}^a(x)} \delta_\xi e_\mu{}^a(x) + \frac{\delta S}{\delta A_\mu(x)} \delta_\xi A_\mu + \frac{\delta S}{\delta \lambda(x)} \delta_\xi \lambda(x) + \delta_\xi \bar{\lambda}(x) \frac{\delta S}{\delta \bar{\lambda}(x)} \right) \\
&= \int d^n x e \left(T^\mu_{\ a} \mathcal{L}_\xi e_\mu{}^a(x) + J^\mu(x) \mathcal{L}_\xi A_\mu \right) \\
&= \int d^n x e \left(T^\mu_{\ a} \nabla_\mu \xi^a + J^\mu \xi^\nu F_{\nu\mu} \right) \\
&= -\int d^n x e \xi^a \left(\nabla_\mu T^\mu_{\ a} - J^b F_{ab} \right) = 0
\end{aligned}
\tag{3.16}
$$

where, once again, the fermionic equations of motion has been used in the first line. In the second line the variations under diffeomorphism of the vielbein and gauge field are replaced by Lie derivatives. In the third line, due to symmetry of the stress tensor, a spin connection term, $\omega_\mu{}^a{}_b \xi^b$, is added to the vielbein's Lie derivative to reconstruct the covariant derivative, while, in the second term, the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ appears after applying a gauge transformation to the Lie derivative of the abelian gauge field. In the last line we have integrated by parts and get the conservation equation

$$
\nabla_\mu T^\mu_{\ a} = J^b F_{ab}. \tag{3.17}
$$

This equation contains a contribution from the gauge field on the right hand side and in order to cancel it one has to make the gauge field dynamical, that is, add the kinetic term $-\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$ which contributes with a term like $F_{ab} \partial_c F^{cb}$ in the last line of (3.16) and cancels the second term once the equation of motion get used. In this way the conservation law appearing in (3.11) is obtained. Note, however, that keeping the gauge field only as a background, one finds that the correct equation describing the conservation properties of the stress tensor of the Weyl fermion only is the one in (3.17).

A more extensive discussion about the construction and properties of the stress tensor can be found in [30].

3.3 Feynman rules

We now derive Feynman rules in n dimensions needed to compute Feynman diagrams in order to find the anomalies. The choice of working in n dimensions from the beginning is dictated by the requirement to correctly employ dimensional regularization. Indeed, we want to find a set of Feynman rules which are equivalent in four and in n dimensions in order to avoid inconsistency when moving from four to n dimensions.

Let us start from the action in curved n-dimensional spacetime

$$
S = -\int d^n x e \bar{\lambda} P_R \gamma^\mu \nabla_\mu P_L \lambda \tag{3.18}
$$

where we have explicitly displayed the projectors, and we recall the definition of the covariant derivative

$$\nabla_\mu = \partial_\mu - iA_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^a\gamma^b. \tag{3.19}$$

In order to derive Feynman rules we consider fluctuations of the metric around Minkowski spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{3.20}$$

which imply the following expansions at first order

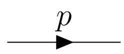
$$\begin{aligned}
g^{\mu\nu} &= \eta^{\mu\nu} - h^{\mu\nu} \\
e_\mu{}^a &= \delta^a_\mu + \frac{1}{2}h_\mu{}^a \\
e^\mu{}_a &= \delta^\mu_a - \frac{1}{2}h^\mu{}_a \\
e &= 1 + \frac{1}{2}h \\
e^{-1} &= 1 - \frac{1}{2}h
\end{aligned} \tag{3.21}$$

with $h = h^\mu{}_\mu$, and insert them in the action (3.18)

$$\begin{aligned}
S &= -\frac{1}{2}\int d^nxe\bar{\lambda}P_R\gamma^\mu\overleftrightarrow{D}_\mu P_L\lambda && (3.22\text{a})\\
&= -\frac{1}{2}\int d^nxe\bar{\lambda}P_R\gamma^\mu\overleftrightarrow{\partial}_\mu P_L\lambda + i\int d^nxe\bar{\lambda}P_R\gamma^\mu A_\mu P_L\lambda && (3.22\text{b})\\
&= -\frac{1}{2}\int d^nx\left(1+\frac{1}{2}h\right)\bar{\lambda}P_R\gamma^a\left(\delta^\mu{}_a - \frac{1}{2}h^\mu{}_a\right)\overleftrightarrow{\partial}_\mu P_L\lambda && (3.22\text{c})\\
&\quad + i\int d^nx\left(1+\frac{1}{2}h\right)\bar{\lambda}P_R\gamma^a\left(\delta^\mu{}_a - \frac{1}{2}h^\mu{}_a\right)A_\mu P_L\lambda && (3.22\text{d})\\
&= -\frac{1}{2}\int d^nx\bar{\lambda}P_R\gamma^a\overleftrightarrow{\partial}_a P_L\lambda + i\int d^nx\bar{\lambda}P_R\gamma^a A_a P_L\lambda && (3.22\text{e})\\
&\quad + \frac{1}{4}\int d^nxh^{ab}\bar{\lambda}P_R\gamma_b\overleftrightarrow{\partial}_a P_L\lambda - \frac{1}{4}\int d^nxh\bar{\lambda}P_R\gamma^a\overleftrightarrow{\partial}_a P_L\lambda && (3.22\text{f})\\
&\quad - \frac{i}{2}\int d^nxh^{ab}\bar{\lambda}P_R\gamma_b A_a P_L\lambda + \frac{i}{2}\int d^nxh^{bc}\eta_{bc}\bar{\lambda}P_R\gamma^a A_a P_L\lambda && (3.22\text{g})
\end{aligned}$$

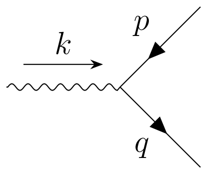
From the above terms we can derive Feynman rules in momentum space. We denote fermions by straight lines, gauge fields by wavy lines and metric fluctuations, h_{ab}, by curled lines.

The first term of (3.22e) is not invertible because of the projectors. To remedy this problem and determine the propagator, we add a free right handed Weyl fermion to the action[2]. Therefore, the fermionic propagator is



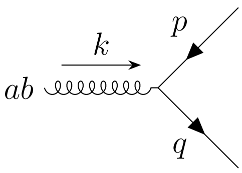
$$= -i\frac{-i\not{p}}{p^2 - i\epsilon} \tag{3.23}$$

From the second term of (3.22e) we read the fermion-gauge field vertex



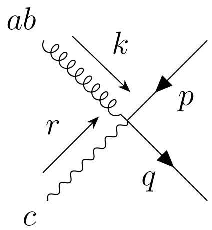
$$= \quad -P_R\gamma^a P_L \tag{3.24}$$

From line (3.22f) the two-fermion-gravity vertex is



$$= \quad -\frac{1}{8}P_R\left[(p+q)_a\gamma_b + (p+q)_b\gamma_a - 2\eta_{ab}(\not{p}+\not{q})\right]P_L \tag{3.25}$$

and from (3.22g) the two-fermion-gravity-gauge vertex



$$= \frac{1}{4}P_R\left(\gamma^a\eta^{bc} + \gamma^b\eta^{ac} - 2\eta^{ab}\gamma^c\right)P_L \tag{3.26}$$

In addition, we recall that any diagram has to be multiplied by -1 and a trace over spinor indices has to be taken for every fermionic loop.

Even though the propagator is that one of a Dirac fermion, the information about chirality is stored in the interaction terms and in the corresponding vertices, in which every fermion possesses its own projector. This is perhaps the most straightforward way to define chiral fermions in $n \neq 4$ dimensions because the following three terms

$$\bar{\lambda}\gamma^a P_L\lambda, \qquad \bar{\lambda}P_R\gamma^a\lambda, \qquad \bar{\lambda}P_R\gamma^a P_L\lambda, \tag{3.27}$$

which are equivalent in four dimensions, are inequivalent in $n \neq 4$ dimensions since $\gamma^a P_L \neq P_R\gamma^a$.

[2]This is the same route followed in [2], [3], [9] and [31].

Chapter 4

Anomalies

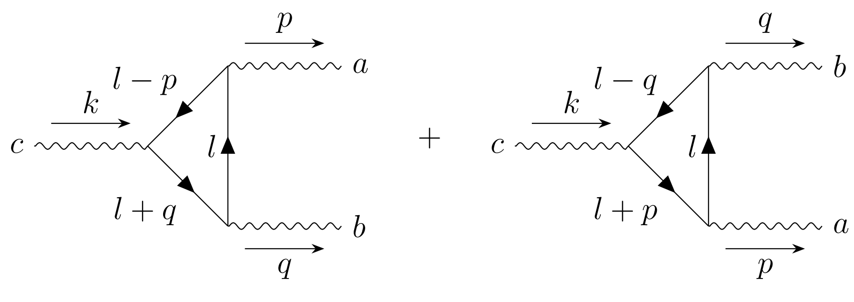
In this chapter we derive anomalies by computing Feynman diagrams. All diagrams we will encounter diverge in four dimensions and we regularize them by means of dimensional regularization and employing the Breitenlohner-Maison scheme for the chiral gamma matrix γ^5.

4.1 Chiral anomaly

In order to derive the chiral anomaly we have to compute the expectation value $\langle J^a(x)\rangle$ which is given at second order in the background gauge field by

$$\begin{aligned}\langle J^c(x)\rangle = -\frac{1}{2}\int d^4y\int d^4z\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\times\\ \times\, e^{ikx}e^{-ipy}e^{-iqz}\delta(k-p-q)\mathcal{M}^{cab}(p,q)A_a(y)A_b(z).\end{aligned}\tag{4.1}$$

External momenta k, p and q can be taken in four dimensions and the Dirac delta ensures the momentum conservation. The matrix element $\mathcal{M}^{cab}(p,q)$ is the Fourier transform of the three-point function $\langle J^c(x)J^a(y)J^b(z)\rangle$ and is obtained by computing the following triangle diagrams



(4.2)

where the second one is obtained from the first one by replacing $(p,a)\leftrightarrow(q,b)$, p and q are two outgoing momenta and momentum conservation requires $k=p+q$. By making use of Feynman rules we have

$$\begin{aligned}\mathcal{M}^{cab} = i \int \frac{d^n l}{(2\pi)^n} & \frac{Tr\left\{P_R\gamma^c P_L(\not{l}-\not{p})P_R\gamma^a P_L \not{l} P_R \gamma^b P_L(\not{l}+\not{q})\right\}}{l^2(l-p)^2(l+q)^2} \\ & + (p,a) \leftrightarrow (q,b).\end{aligned} \tag{4.3}$$

where $l = \bar{l} + \hat{l} = \bar{l} + s$, $\frac{d^n l}{(2\pi)^n} = \frac{d^4 l}{(2\pi)^4}\frac{d^{n-4}s}{(2\pi)^{n-4}}$. Taking the divergence of (4.1) is equivalent to contract the above expression by ik_c

$$\begin{aligned}ik_c\mathcal{M}^{cab} = & -\int \frac{d^n l}{(2\pi)^n}\frac{Tr\left\{P_R \not{k} P_L(\not{l}-\not{p})P_R\gamma^a P_L \not{l} P_R \gamma^b P_L(\not{l}+\not{q})\right\}}{l^2(l-p)^2(l+q)^2} \\ & + (p,a) \leftrightarrow (q,b).\end{aligned} \tag{4.4}$$

Using properties (B.9) we can write

$$\begin{aligned}ik_c\mathcal{M}^{cab} = & -\int \frac{d^n l}{(2\pi)^n}\frac{Tr\left\{\not{k} P_L(\not{l}-\not{p})\bar{\gamma}^a P_L \not{l} \bar{\gamma}^b P_L(\not{l}+\not{q})\right\}}{l^2(l-p)^2(l+q)^2} \\ & + (p,a) \leftrightarrow (q,b)\end{aligned} \tag{4.5}$$

where we avoid to put a bar over k because external momenta remains four-dimensional.

By explicitly displaying the projectors, there are eight terms in the numerator. Most of them cancel with those of the diagram with $(p,a) \leftrightarrow (q,b)$[1] and only four of them survive that are

$$\begin{aligned}& -\frac{1}{8}\int \frac{d^n l}{(2\pi)^n} Tr \frac{\not{k}\gamma^5(\not{l}-\not{p})\bar{\gamma}^a \not{l} \bar{\gamma}^b(\not{l}+\not{q})}{l^2(l-p)^2(l+q)^2} \\ & -\frac{1}{8}\int \frac{d^n l}{(2\pi)^n} Tr \frac{\not{k}\gamma^5(\not{l}-\not{p})\bar{\gamma}^a \gamma^5 \not{l} \bar{\gamma}^b \gamma^5(\not{l}+\not{q})}{l^2(l-p)^2(l+q)^2} \\ & -\frac{1}{8}\int \frac{d^n l}{(2\pi)^n} Tr \frac{\not{k}\gamma^5(\not{l}-\not{q})\bar{\gamma}^b \not{l} \bar{\gamma}^a(\not{l}+\not{p})}{l^2(l-q)^2(l+p)^2} \\ & -\frac{1}{8}\int \frac{d^n l}{(2\pi)^n} Tr \frac{\not{k}\gamma^5(\not{l}-\not{q})\bar{\gamma}^b \gamma^5 \not{l} \bar{\gamma}^a \gamma^5(\not{l}+\not{p})}{l^2(l-q)^2(l+p)^2}\end{aligned} \tag{4.6}$$

If we use the identity

$$\not{k}\gamma^5 = -\gamma^5\not{k} = -\gamma^5(\not{p}+\not{q}) = \gamma^5(\not{l}-\not{p}) + (\not{l}+\not{q})\gamma^5 - 2\gamma^5\not{s} \tag{4.7}$$

in the first two integrals and

$$\not{k}\gamma^5 = -\gamma^5\not{k} = -\gamma^5(\not{p}+\not{q}) = \gamma^5(\not{l}-\not{q}) + (\not{l}+\not{p})\gamma^5 - 2\gamma^5\not{s} \tag{4.8}$$

[1]See appendix E.1.

in the last two, then the contributions due to the first two terms of these identities cancel out after shifting the loop variable $l \to l - p$ and $l \to l + q$, using the cyclicity of the trace and the anticommutator $\{\bar{\gamma}^a, \gamma^5\} = 0$ in the integrals of the crossed diagram.

Thus, the anomaly comes from

$$
\begin{aligned}
&\frac{1}{4}\int \frac{d^n l}{(2\pi)^n} Tr \frac{\gamma^5 \not{s}(\not{\bar{l}} + \not{s} - \not{p})\bar{\gamma}^a \left(\not{\bar{l}} + \not{s}\right) \bar{\gamma}^b(\not{\bar{l}} + \not{s} + \not{q})}{l^2(l-p)^2(l+q)^2} \\
&+ \frac{1}{4}\int \frac{d^n l}{(2\pi)^n} Tr \frac{\gamma^5 \not{s}(\not{\bar{l}} + \not{s} - \not{p})\bar{\gamma}^a \left(\not{\bar{l}} - \not{s}\right) \bar{\gamma}^b(\not{\bar{l}} + \not{s} + \not{q})}{l^2(l-p)^2(l+q)^2}
\end{aligned}
\tag{4.9}
$$

where we have multiplied by two since terms of the crossed diagram gives an equal contribution. In the second line properties (2.9) have been used to bring the two γ^5 matrices together and this leads to replace $\bar{l} + \not{s}$ by $\bar{l} - \not{s}$.

Making use of the Feynman parametric formula[2], the denominator can be rewritten in symmetric form as

$$
2\int_0^1 dx \int_0^{1-x} dy \left[s^2 + \bar{r}^2 + f\right]^{-3} \tag{4.10}
$$

where $\bar{r} = \bar{l} + xq - yp$, $f = f(x, y, p, q) = xq^2 + yp^2 - (xq - yp)^2$, and the integration variable can be shifted $\bar{l} \to \bar{r}$. Since the denominator is symmetric in s, only terms with an even number of s can contribute, otherwise the integral would vanish by symmetry. However, a term like $Tr\left(\gamma^5 \not{s}\not{s}\bar{\gamma}^a \not{s}\bar{\gamma}^b \not{s}\right) = s^4 Tr\left(\gamma^5 \bar{\gamma}^a \bar{\gamma}^b\right) = 0$, and only terms proportional to s^2 give a non zero result. They are

$$
\begin{aligned}
&Tr\left\{\gamma^5 \not{s}\not{s}\bar{\gamma}^a \not{\bar{l}}\bar{\gamma}^b(\not{\bar{l}} + \not{q}) + \gamma^5 \not{s}(\not{\bar{l}} - \not{p})\bar{\gamma}^a \not{s}\bar{\gamma}^b(\not{\bar{l}} + \not{q}) + \gamma^5 \not{s}(\not{\bar{l}} - \not{p})\bar{\gamma}^a \not{\bar{l}}\bar{\gamma}^b \not{s}\right\} \\
&+ Tr\left\{\gamma^5 \not{s}\not{s}\bar{\gamma}^a \not{\bar{l}}\bar{\gamma}^b(\not{\bar{l}} + \not{q}) - \gamma^5 \not{s}(\not{\bar{l}} - \not{p})\bar{\gamma}^a \not{s}\bar{\gamma}^b(\not{\bar{l}} + \not{q}) + \gamma^5 \not{s}(\not{\bar{l}} - \not{p})\bar{\gamma}^a \not{\bar{l}}\bar{\gamma}^b \not{s}\right\}
\end{aligned}
\tag{4.11}
$$

since $\left\{\bar{\gamma}^a, \hat{\gamma}^b\right\} = 0$ and $\not{s}\not{s} = s^2$, these terms contain only traces of γ^5 with four-dimensional gamma matrices given by (2.4)

$$
\begin{aligned}
&4is^2\epsilon^{abcd}\left\{-\bar{l}_c(\bar{l} + q)_d + (\bar{l} - p)_c(\bar{l} + q)_d - (\bar{l} - p)_c\bar{l}_d\right\} \\
&+ 4is^2\epsilon^{abcd}\left\{-\bar{l}_c(\bar{l} + q)_d - (\bar{l} - p)_c(\bar{l} + q)_d - (\bar{l} - p)_c\bar{l}_d\right\} \\
&= -8is^2\epsilon^{abcd}p_c q_d\,(x + y)
\end{aligned}
\tag{4.12}
$$

where we have replaced $\bar{l} = \bar{r} - xq + yp$, and neglected linear terms in $\bar{r}$, which vanish by integration, and terms which vanish by symmetry because of the antysimmetric Levi-Civita symbol.

The integral

$$
\int \frac{d^n r}{(2\pi)^n} \frac{s^2}{(r^2 + f)^3} \tag{4.13}
$$

[2] Equation (D.1b) with $A = (l + q)^2, B = (l - p)^2, C = l^2$.

where $r = \bar{r} + s$, is easily evaluated thanks to (D.14) and gives the finite result $-\frac{i}{32\pi^2}$. The integration over x and y yields $\frac{2}{3}$ and adding the contribution of the crossed diagram we get the result in the momentum space

$$ik_c \mathcal{M}^{cab} = -\frac{8}{3}\frac{1}{32\pi^2}\epsilon^{abcd} p_c q_d. \tag{4.14}$$

In order to move to position space, this result has to be insert in the divergence of (4.1) and yields

$$\begin{aligned}
\partial_c \langle J^c(x) \rangle &= \frac{4}{96\pi^2}\epsilon^{abcd} \int d^4y \int d^4z \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \times \\
&\qquad \times e^{ikx} e^{-ipy} e^{-iqz} \delta(k-p-q) p_c q_d A_a(y) A_b(z) \\
&= \frac{4}{96\pi^2}\epsilon^{abcd} \int d^4y \int d^4z \, (i\partial^y_c)\, (i\partial^z_d)\, A_a(y) A_b(z) \times \\
&\quad \times \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} e^{ikx} e^{-ipy} e^{-iqz} \delta(k-p-q) \\
&= -\frac{4}{96\pi^2}\epsilon^{abcd} \int d^4y \int d^4z \, (\partial^y_c A_a(y))\, (\partial^z_d A_b(z))\, \delta(x-y)\delta(x-z) \\
&= \frac{4}{96\pi^2}\epsilon^{abcd} \, (\partial_a A_b)\, (\partial_c A_d) \\
&= \frac{1}{96\pi^2}\epsilon^{abcd} F_{ab} F_{cd}
\end{aligned} \tag{4.15}$$

where in the last line the abelian field strength $F_{ab} = \partial_a A_b - \partial_b A_a$ is introduced.

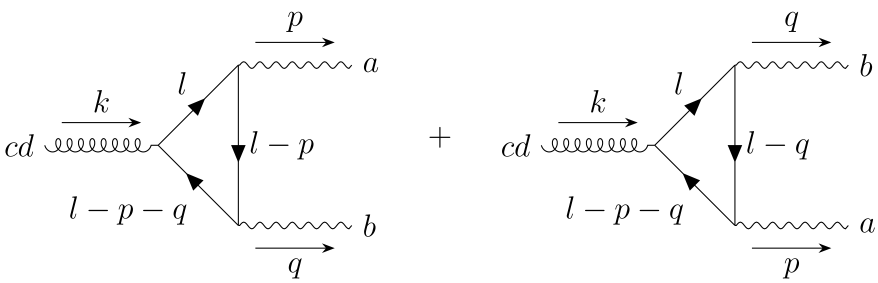
This result reproduces the standard chiral anomaly of a single Weyl fermion, also reproduced in [1] by a Pauli-Villars regularization.

4.2 Stress Tensor

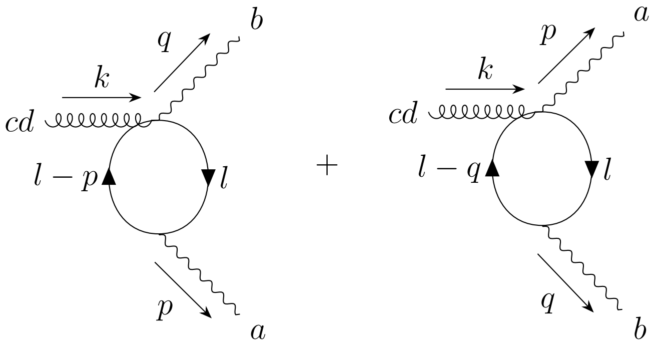
In order to study the anomalies related to the stress tensor, one has to first consider the quantum expectation value of the stress tensor which at second order in the background gauge field is given by

$$\begin{aligned}
\langle T^{cd}(x) \rangle = \int d^4y \int d^4z \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \times \\
\times e^{ikx} e^{-ipy} e^{-iqz} \delta(k-p-q) \mathcal{T}^{cdab}(p,q) A_a(y) A_b(z)
\end{aligned} \tag{4.16}$$

where the matrix element $\mathcal{T}^{cdab} = \mathcal{T}^{cdab}_{(1)} + \mathcal{T}^{cdab}_{(2)}$ receive contributions from the following Feynman diagrams



(4.17)



(4.18)

whit the ones on the right obtained form those on the left by exchanging the external gauge fields, i.e. by replacing $(p,a) \leftrightarrow (q,b)$, p and q are two outgoing momenta and momentum conservation requires $k = p + q$. By making use of Feynman rules we can write

$$\mathcal{T}^{cdab}_{(1)} = \frac{i}{8} \int \frac{d^n l}{(2\pi)^n} \frac{\mathcal{N}^{cdab}_{(1)}}{l^2 (l-p)^2 (l-p-q)^2} + (p,a) \leftrightarrow (q,b) \tag{4.19}$$

where the numerator is

$$\begin{aligned} \mathcal{N}^{cdab}_{(1)} = Tr\Big(& P_R \left[(2l-p-q)^c \gamma^d + (2l-p-q)^d \gamma^c - 2\eta^{cd} (2\not{l} - \not{p} - \not{q}) \right] P_L \times \\ & \times \not{l} P_R \gamma^a P_L (\not{l} - \not{p}) P_R \gamma^b P_L (\not{l} - \not{p} - \not{q}) \Big) \end{aligned} \tag{4.20}$$

and

$$\mathcal{T}^{cdab}_{(2)} = -\frac{i}{4} \int \frac{d^n l}{(2\pi)^n} \frac{\mathcal{N}^{cdab}_{(2)}}{l^2 (l-p)^2} + (p,a) \leftrightarrow (q,b) \tag{4.21}$$

with

$$\mathcal{N}^{cdab}_{(2)} = Tr\Big(P_R \left(\gamma^c \eta^{bd} + \gamma^d \eta^{bc} - 2\eta^{cd} \gamma^b \right) P_L \not{l} P_R \gamma^a P_L (\not{l} - \not{p}) \Big) \tag{4.22}$$

and the n-dimensional integrated momentum $l = \bar{l} + \hat{l} = \bar{l} + s$ and $\frac{d^n l}{(2\pi)^n} = \frac{d^4 l}{(2\pi)^4} \frac{d^{n-4} s}{(2\pi)^{n-4}}$.

The properties of γ^5 in the Breitenlohner-Maison prescription allow us to simplify these expressions. Thanks to the identities (B.9)

$$P_R\gamma^a P_L = \bar{\gamma}^a P_L = P_R\bar{\gamma}^a, \qquad P_L\gamma^a P_R = \bar{\gamma}^a P_R = P_L\bar{\gamma}^a \tag{4.23}$$

the two numerators can be rewritten as

$$\begin{aligned}
\mathcal{N}^{cdab}_{(1)} &= Tr\Big(P_R\Big[(2l-p-q)^c\bar{\gamma}^d + (2l-p-q)^d\bar{\gamma}^c - 2\eta^{cd}(2\bar{\not{l}} - \not{p} - \not{q})\Big]\times \\
&\quad \times \bar{\not{l}}\bar{\gamma}^a(\bar{\not{l}} - \not{p})\bar{\gamma}^b(\bar{\not{l}} - \not{p} - \not{q})\Big) \\
&= \frac{1}{2}Tr\Big(\Big[(2l-p-q)^c\bar{\gamma}^d + (2l-p-q)^d\bar{\gamma}^c - 2\eta^{cd}(2\bar{\not{l}} - \not{p} - \not{q})\Big]\times \\
&\quad \times \bar{\not{l}}\bar{\gamma}^a(\bar{\not{l}} - \not{p})\bar{\gamma}^b(\bar{\not{l}} - \not{p} - \not{q})\Big) \\
&\quad - \frac{1}{2}Tr\Big(\gamma^5\Big[(2l-p-q)^c\bar{\gamma}^d + (2l-p-q)^d\bar{\gamma}^c - 2\eta^{cd}(2\bar{\not{l}} - \not{p} - \not{q})\Big]\times \\
&\quad \times \bar{\not{l}}\bar{\gamma}^a(\bar{\not{l}} - \not{p})\bar{\gamma}^b(\bar{\not{l}} - \not{p} - \not{q})\Big)
\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}
\mathcal{N}^{cdab}_{(2)} =& Tr\Big(P_R\left(\bar{\gamma}^c\eta^{bd} + \bar{\gamma}^d\eta^{bc} - 2\eta^{cd}\bar{\gamma}^b\right)\bar{\not{l}}\bar{\gamma}^a(\bar{\not{l}} - \not{p})\Big) \\
&= \frac{1}{2}Tr\Big(\left(\bar{\gamma}^c\eta^{bd} + \bar{\gamma}^d\eta^{bc} - 2\eta^{cd}\bar{\gamma}^b\right)\bar{\not{l}}\bar{\gamma}^a(\bar{\not{l}} - \not{p})\Big) \\
&\quad - \frac{1}{2}Tr\Big(\gamma^5\left(\bar{\gamma}^c\eta^{bd} + \bar{\gamma}^d\eta^{bc} - 2\eta^{cd}\bar{\gamma}^b\right)\bar{\not{l}}\bar{\gamma}^a(\bar{\not{l}} - \not{p})\Big).
\end{aligned} \tag{4.25}$$

where we do not put a bar over p and q because external momenta remains four-dimensional in the Breitenlohner–Maison scheme.

One can shows that traces containing γ^5 which would produce parity-odd contributions vanish. This can be done by using the invariance of the trace under transposition, i.e. the property of the trace of a matrix to be equal to the trace of the transposed matrix, using properties of Dirac gamma matrices, shifting the momentum variable and using the anticommutator $\{\gamma^5, \bar{\gamma}^a\} = 0$. Details of the computation are shown in appendix E.2.

As explained in section 2.2, in the parity-even sector one can use an anticommuting γ^5, this implies that the two numerators become

$$\begin{aligned}
\mathcal{N}^{cdab}_{(1)} =& \frac{1}{2}Tr\Big(\left[(2l-p-q)^c\gamma^d + (2l-p-q)^d\gamma^c - 2\eta^{cd}(2\not{l} - \not{p} - \not{q})\right]\times \\
&\times \not{l}\gamma^a(\not{l} - \not{p})\gamma^b(\not{l} - \not{p} - \not{q})\Big)
\end{aligned} \tag{4.26}$$

and

$$\mathcal{N}^{cdab}_{(2)} = \frac{1}{2}\left(\gamma^c\eta^{bd} + \gamma^d\eta^{bc} - 2\eta^{cd}\gamma^b\right)\not{l}\gamma^a(\not{l} - \not{p}) \tag{4.27}$$

which differ form the previous expressions by the fact that the gamma matrices are now taken in n dimensions[3].

Let us now look at the terms proportional to η^{cd} in both integrals. We can use the following identity in the first integral

$$\frac{(\not{l}-\not{p}-\not{q})(2\not{l}-\not{p}-\not{q})}{l^2(l-p)^2(l-p-q)^2}=\frac{2}{l^2(l-p)^2}+\frac{(\not{l}-\not{p}-\not{q})(\not{p}+\not{q})}{l^2(l-p)^2(l-p-q)^2}. \tag{4.28}$$

From the first term of the above identity we get an expression which cancels the integral of the second diagram. Moreover, the second term gives a zero result, since it is nothing but the integral that one has to evaluate to compute the Ward identity for the conservation of the $U(1)$ gauge current of QED which is conserved both at classical and quantum level. In addition, this would prove that the second term of the stress tensor (3.12), which vanishes on-shell, does not even contribute in the quantum case.

At this point we are left with

$$\begin{aligned}\mathcal{T}^{cdab}=&\frac{i}{16}\int\frac{d^nl}{(2\pi)^n}Tr\frac{\left[(2l-p-q)^c\gamma^d+(2l-p-q)^d\gamma^c\right]\not{l}\gamma^a(\not{l}-\not{p})\gamma^b(\not{l}-\not{p}-\not{q})}{l^2(l-p)^2(l-p-q)^2}\\&-\frac{i}{8}\int\frac{d^nl}{(2\pi)^n}Tr\frac{\left(\gamma^c\eta^{bd}+\gamma^d\eta^{bc}\right)\not{l}\gamma^a(\not{l}-\not{p})}{l^2(l-p)^2}\\&+(p,a)\leftrightarrow(q,b).\end{aligned} \tag{4.29}$$

4.2.1 Trace anomaly

In order to determine the trace anomaly we contract (4.16) with the four-dimensional metric tensor $\bar{\eta}_{cd}$

$$\begin{aligned}\langle T^a{}_a\rangle=\bar{\eta}_{cd}\langle T^{cd}(x)\rangle=&\int d^4y\int d^4z\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\times\\&\times e^{ikx}e^{-ipy}e^{-iqz}\delta(k-p-q)\mathcal{T}^{ab}(p,q)A_a(y)A_b(z)\end{aligned} \tag{4.30}$$

where

$$\begin{aligned}\mathcal{T}^{ab}\equiv\bar{\eta}_{cd}\mathcal{T}^{cdab}=&\frac{i}{8}\int\frac{d^nl}{(2\pi)^n}Tr\frac{(2\bar{\not{l}}-\not{p}-\not{q})\not{l}\gamma^a(\not{l}-\not{p})\gamma^b(\not{l}-\not{p}-\not{q})}{l^2(l-p)^2(l-p-q)^2}\\&-\frac{i}{4}\int\frac{d^nl}{(2\pi)^n}Tr\frac{\gamma^b\not{l}\gamma^a(\not{l}-\not{p})}{l^2(l-p)^2}\\&+(p,a)\leftrightarrow(q,b).\end{aligned} \tag{4.31}$$

[3] This is the same strategy adopted in [9].

In the second term we should have written $\bar{\gamma}^b$ instead of γ^b, however, we can treat γ^a and γ^b as purely four-dimensional since they are contracted with external gauge fields which remains four-dimensional and we can omit the overbar. Writing $\bar{\not{l}} = \not{l} - \not{s}$ and using again the identity (4.28) the above expression can be further simplified to

$$\mathcal{T}^{ab} = -\frac{i}{4} \int \frac{d^n l}{(2\pi)^n} Tr \frac{\not{s}\bar{\not{l}}\gamma^a(\not{l} - \not{p})\gamma^b(\not{l} - \not{p} - \not{q})}{l^2(l-p)^2(l-p-q)^2} + (p,a) \leftrightarrow (q,b). \tag{4.32}$$

Using the Feynman parametric formula[4] to rewrite the denominator in symmetric form, the above integral becomes

$$-\frac{i}{2}\int_0^1 dx \int_0^{1-y} dy \int \frac{d^n l}{(2\pi)^n} \frac{N^{ab}}{(\bar{l}^2 + s^2 + f)^3} \tag{4.33}$$

where we have shifted the integration variable $\bar{l} \to \bar{l} + p(x+y) + qx$, the numerator is

$$\begin{aligned} N^{ab} = Tr\Big[&\not{s}(\bar{\not{l}} + \not{s} + \not{p}(x+y) + \not{q}x)\gamma^a(\bar{\not{l}} + \not{s} + \not{p}(x+y-1) + \not{q}x)\times \\ &\times \gamma^b(\bar{\not{l}} + \not{s} + \not{p}(x+y-1) + \not{q}(x-1))\Big] \end{aligned} \tag{4.34}$$

and $f = f(x,p,q) = p^2[x(1-x) + y(1-y) - 2xy] + x(1-x)q^2 + 2p\cdot qx(1-x-y)$. By symmetry only terms proportional to even powers of s give a non zero contribution. There are three terms proportional to s^2 and one proportional to s^4, they are

$$\begin{aligned} &s^2 Tr\Big((\bar{\not{l}} + \not{p}(x+y) + \not{q}x)\gamma^a\gamma^b(\bar{\not{l}} + \not{p}(x+y-1) + \not{q}(x-1)\Big) \\ &+ s^2 Tr\Big(\gamma^a(\bar{\not{l}} + \not{p}(x+y-1) + \not{q}x)\gamma^b(\bar{\not{l}} + \not{p}(x+y-1) + \not{q}(x-1))\Big) \\ &+ s^2 Tr\Big((\bar{\not{l}} + \not{p}(x+y) + \not{q}x)\gamma^a(\bar{\not{l}} + \not{p}(x+y-1) + \not{q}x)\gamma^b\Big) \\ &- s^4 Tr\Big(\gamma^a\gamma^b\Big). \end{aligned} \tag{4.35}$$

After working the traces out by using (B.10) and (B.11), neglecting terms proportional to $\bar{l}$ because they vanish by symmetric integration, replacing $\bar{l}^a\bar{l}^b \to \frac{1}{4}\eta^{ab}\bar{l}^2$ and evaluating the loop integrals using (D.14) and (D.17), we obtain

$$-\frac{1}{16\pi^2}\int_0^1 dx \int_0^{1-y} dy \; N^{ab} \tag{4.36}$$

where

$$\begin{aligned} N^{ab} = &p^a p^b 2(2x^2 + 2y^2 + 4xy - 3x - 3y + 1) + p^a q^b(4x^2 + 4xy - 4x - 2y + 1) + \\ &+ q^a p^b(4x^2 + 4xy - 4x + 1) + q^a q^b(4x^2 - 2x) + \\ &- \eta^{ab}\Big[p^2(2x^2 + 4xy - 3x + 2y^2 - 3y + 1) + q^2(2x^2 - x) \\ &\quad + p\cdot q(4x^2 + 4xy - 4x + 1)\Big]. \end{aligned} \tag{4.37}$$

[4] (D.1b) with $A = (l-p-q)^2$, $B = (l-p)^2$, $C = l^2$.

Integrating[5] over x and y and adding the contribution of the cross diagram we finally obtain

$$\mathcal{T}^{ab} = -\frac{1}{24\pi^2}\Big(q^a p^b - \eta^{ab} p\cdot q\Big). \tag{4.38}$$

This result has to be inserted in (4.30) and yields[6]

$$\langle T^a_{\ a}\rangle = -\frac{1}{48\pi^2} F^{ab}F_{ab}. \tag{4.39}$$

Even though this result reproduces exactly the one obtained in [1] we cannot yet assert with certainty that it is exact without first checking the conservation of the stress tensor. That because the Breitenlohner-Maison scheme breaks Lorentz covariance for chiral objects and the conservation of the stress tensor is not guaranteed at quantum level. As a consequence, it may happen that in order to preserve the conservation one needs to introduce counterterms which in turn may modify the expression of the trace anomaly.

4.2.2 Stress tensor conservation

In order to verify the conservation of the stress tensor we have to take the divergence of (4.16)

$$\begin{aligned}\partial_c\langle T^{cd}(x)\rangle = \int d^4y \int d^4z \int \frac{d^4k}{(2\pi)^4}\int \frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\times \\ \times\, e^{ikx}e^{-ipy}e^{-iqz}\delta(k-p-q) ik_c\mathcal{T}^{cdab}(p,q)A_a(y)A_b(z)\end{aligned} \tag{4.40}$$

where $k_c = p_c + q_c$ because of the Dirac delta function and

$$\begin{aligned} ik_c\mathcal{T}^{cdab} &= -\frac{1}{16}\int\frac{d^nl}{(2\pi)^n} Tr\frac{\left[k_c(2l-k)^c\gamma^d + (2l-k)^d\not{k}\right]\not{l}\gamma^a(\not{l}-\not{p})\gamma^b(\not{l}-\not{k})}{l^2(l-p)^2(l-k)^2} \\ &+ \frac{1}{8}\int\frac{d^nl}{(2\pi)^n}Tr\frac{\left(\not{k}\eta^{bd}+\gamma^d k^b\right)\not{l}\gamma^a(\not{l}-\not{p})}{l^2(l-p)^2} \\ &+ (p,a)\leftrightarrow(q,b).\end{aligned} \tag{4.41}$$

The following two identities can be used to simplify the calculation

$$k_c(2l-k)^c = k\cdot(2l-k) = 2l\cdot k - k^2 = l^2-(l-k)^2 \tag{4.42}$$

$$\not{k} = \not{l} - (\not{l}-\not{k}) \tag{4.43}$$

[5] The only non zero integral is $\int_0^1 dx \int_0^{1-y} dy(4x^2+4xy-4x+1) = \frac{1}{3}$.

[6] See appendix E.3.

and one has

$$ik_c \mathcal{T}^{cdab} = -\frac{1}{8}\int \frac{d^n l}{(2\pi)^n} Tr\Big(\gamma^d \not{l}\gamma^a(\not{l}-\not{p})\gamma^b(\not{l}-\not{k})\Big)\left(\frac{1}{(l-p)^2(l-k)^2} - \frac{1}{l^2(l-p)^2}\right) \tag{4.44a}$$

$$-\frac{1}{8}\int \frac{d^n l}{(2\pi)^n}(2l-k)^d\left(Tr\frac{\gamma^a(\not{l}-\not{p})\gamma^b(\not{l}-\not{k})}{(l-p)^2(l-k)^2} - Tr\frac{\not{l}\gamma^a(\not{l}-\not{p})\gamma^b}{l^2(l-p)^2}\right) \tag{4.44b}$$

$$+\frac{1}{8}\int \frac{d^n l}{(2\pi)^n} k^b Tr\frac{\gamma^d \not{l}\gamma^a(\not{l}-\not{p})}{l^2(l-p)^2} \tag{4.44c}$$

$$+\frac{1}{8}\int \frac{d^n l}{(2\pi)^n}\eta^{bd} Tr\frac{\not{q}\not{l}\gamma^a(\not{l}-\not{p})}{l^2(l-p)^2} \tag{4.44d}$$

$$+\frac{1}{8}\int \frac{d^n l}{(2\pi)^n} k^a Tr\frac{\gamma^d \not{l}\gamma^b(\not{l}-\not{q})}{l^2(l-q)^2} \tag{4.44e}$$

$$+\frac{1}{8}\int \frac{d^n l}{(2\pi)^n}\eta^{ad} Tr\frac{\not{p}\not{l}\gamma^b(\not{l}-\not{q})}{l^2(l-q)^2} \tag{4.44f}$$

where the first two integrals have been multiplied by two since those of the cross diagram are equal after shifting the integration momentum and using the invariance of the trace under transposition, while in the last terms we have neglected tadpole integrals which vanish. The calculation is quite long and we refer to appendix E.4 for details. Here we give the final result in momentum space

$$\begin{aligned} ik_c \mathcal{T}^{cdab} = {} & \frac{i}{24\pi^2}\Big(q^d\left(p^a p^b - \eta^{ab}p^2\right) + \eta^{bd}\left(q^a p^2 - p^a p\cdot q\right)\Big)\times \\ & \times\left(\frac{2}{4-n} + \frac{5}{3} - \log p^2 - \gamma + \log 4\pi + O(4-n)\right) + \\ & + \frac{i}{24\pi^2}\Big(p^d\left(q^a q^b - \eta^{ab}q^2\right) + \eta^{ad}\left(p^b q^2 - q^b p\cdot q\right)\Big)\times \\ & \times\left(\frac{2}{4-n} + \frac{5}{3} - \log q^2 - \gamma + \log 4\pi + O(4-n)\right). \end{aligned} \tag{4.45}$$

This expression is semi-local. Having found a non zero result does not mean that there is an anomaly in the conservation of the stress tensor and there is need of counterterms to restore the conservation. Indeed, we recall that the classical conservation equation of the stress tensor has a vanishing right hand side only after making the gauge field dynamical and using its equations of motion, otherwise it is given by (3.17) which in flat space reduces to

$$\partial_a T^{ab} = F^{ab} J_b. \tag{4.46}$$

One can show that the semi-local terms appearing in the above equations are related to the right hand side of this equation.

To do so we can re-express the classical conservation equation as an equation of generating functionals in the abelian background gauge field[7]

$$\partial_a \langle T^{ab}\rangle_A = F^{ab}\langle J_b\rangle_A \tag{4.47}$$

where formally

$$\langle T^{ab}\rangle_A = \int D\lambda D\bar{\lambda}\; T^{ab}\; e^{iS} = \langle T^{ab} e^{i\int d^4x J^a A_a}\rangle \tag{4.48a}$$

$$\langle J_b\rangle_A = \int D\lambda D\bar{\lambda}\; J_b\; e^{iS} = \langle J_b\; e^{i\int d^4x J^a A_a}\rangle \tag{4.48b}$$

and J^a is the quantum operator of the gauge current (3.6) which couples to the classical background abelian gauge field. Expanding equation (4.47) one obtain

$$\langle T^{ab}\rangle_A = \langle T^{ab}\rangle + i\langle T^{ab}(J\cdot A)\rangle + \frac{i^2}{2}\langle T^{ab}(J\cdot A)(J\cdot A)\rangle + ... \tag{4.49}$$

from the left hand side, and

$$F^{ab}\langle J_b\rangle_A = F^{ab}\langle J_b\rangle + iF^{ab}\langle J_b(J\cdot A)\rangle + ... \tag{4.50}$$

from the right hand side, where $(J\cdot A) = \int d^4x J^a A_a$. By taking twice the functional derivative with respect to the gauge field, setting $A = 0$ and moving to momentum space, one derives the Ward identity corresponding to the conservation equation (4.47) which is

$$ik_c\mathcal{T}^{cdab} = \left(\delta^a_c p^d - \eta^{da}p_c\right)\Pi^{bc}(q) + \left(\delta^b_c q^d - \eta^{db}q_c\right)\Pi^{ac}(p) \tag{4.51}$$

where

$$\Pi^{ab}(p) = \int \frac{d^n l}{(2\pi)^n} Tr \frac{P_R\gamma^a P_L \not{l} P_R\gamma^b P_L(\not{l}-\not{p})}{l^2(l-p)^2}. \tag{4.52}$$

The quantity on the left hand side $ik_c\mathcal{T}^{cdab}$ has been already computed and is given by (4.41), while the two integrals appearing on the right hand side reduce easily to one half the integral of the photon self-energy and yields

$$\begin{aligned}
&\frac{i}{24\pi^2}\Big(q^d\left(p^a p^b - \eta^{ab}p^2\right) + \eta^{bd}\left(q^a p^2 - p^a p\cdot q\right)\Big)\times \\
&\qquad\times\left(\frac{2}{4-n} + \frac{5}{3} - \log p^2 - \gamma + \log 4\pi + O(4-n)\right) + \\
&+\frac{i}{24\pi^2}\Big(p^d\left(q^a q^b - \eta^{ab}q^2\right) + \eta^{ad}\left(p^b q^2 - q^b p\cdot q\right)\Big)\times \\
&\qquad\times\left(\frac{2}{4-n} + \frac{5}{3} - \log q^2 - \gamma + \log 4\pi + O(4-n)\right).
\end{aligned} \tag{4.53}$$

[7] In [32] and [33] the authors used this strategy in order to determine the structure of the Ward identity for the conservation of the stress tensor of a Dirac fermion. We do the same thing but for a Weyl fermion and compare the result with the one obtained from the explicit calculation of the left hand side of the conservation equation (4.46).

By comparing this result with (4.45) we can assert that the left and right hand sides of the conservation equation (4.47) are equal and the conservation of the stress tensor is not broken by the regularization scheme we have used. Therefore, being the conservation of the stress tensor preserved even at the quantum level we can conclude that (4.39) gives the correct result for the trace anomaly.

Chapter 5

The Majorana Fermion

It may be useful to cast the previous model of a Weyl fermion using a Majorana fermion, and check again the correctness of our previous results. The approach with Majorana fermions has been used recently in curved space in [10].

5.1 Majorana fermions

A Majorana spinor χ is a spinor which satisfy the reality constraint

$$\chi = \chi_c \tag{5.1}$$

where χ_c is the charged conjugated spinor which is defined by

$$\chi_c = C^{-1}\bar{\chi}^T. \tag{5.2}$$

Further details on conventions can be found in appendix A. Thus, a Majorana spinor satisfies the relations

$$\chi = C^{-1}\bar{\chi}^T, \qquad \bar{\chi} = -\chi^T C. \tag{5.3}$$

The condition (5.1) implies that a Majorana particle is indistinguishable from its own antiparticle.

Using for simplicity the chiral representations of the gamma matrices the Majorana condition (5.1) can be solved in terms of a two-dimensional left handed Weyl fermion l as

$$\chi = \begin{pmatrix} l \\ -i\sigma^2 l^* \end{pmatrix}. \tag{5.4}$$

Note that the two-dimensional Weyl fermion l appears in the four-dimensional Weyl spinor λ and its charge conjugated λ_c as

$$\lambda = \begin{pmatrix} l \\ 0 \end{pmatrix}, \qquad \lambda_c = \begin{pmatrix} 0 \\ -i\sigma^2 l^* \end{pmatrix}. \tag{5.5}$$

Alternatively, one can use a right handed Weyl fermion to solve the Majorana constraint. We refer to appendix A for details about that.

5.2 Action and symmetries

The action for a free massless Majorana fermion is

$$S = -\frac{1}{2}\int d^4x \bar{\chi}\gamma^a\partial_a\chi. \tag{5.6}$$

It is conventionally normalized to be half the action of a Dirac fermion. Because of the condition (5.1) the action is not invariant under a $U(1)$ transformation. Indeed, the Majorana condition is inconsistent with a $U(1)$ vector symmetry, as the transformation

$$\chi \to e^{i\alpha}\chi \tag{5.7}$$

implies

$$\bar{\chi} \to e^{-i\alpha}\bar{\chi} \qquad \text{i.e.} \qquad \chi_c \to e^{-i\alpha}\chi_c \tag{5.8}$$

which violate the constraint (5.1). Related to this fact, one may recall that the vector current vanishes for a Majorana spinor, $\bar{\chi}\gamma^a\chi = 0$. This is seen by manipulating the current as follows

$$\begin{aligned}\bar{\chi}\gamma^a\chi &= -\chi^T C\gamma^a\chi = -(\chi^T C\gamma^a\chi)^T = \chi^T\gamma^{aT}C^T\chi = -\chi^T\gamma^{aT}C\chi \\ &= -\chi^T CC^{-1}\gamma^{aT}C\chi = \chi^T C\gamma^a\chi = -\bar{\chi}\gamma^a\chi\end{aligned} \tag{5.9}$$

where in the third equality we have used the Grassmann property of the spinor, and then used the antisymmetry of the charge conjugation matrix C. This implies $\bar{\chi}\gamma^a\chi = 0$.

However, the action is invariant under an axial $U(1)$ transformation so that a Majorana fermion can be coupled to an axial gauge field. The reason is that since a Majorana fermion is a four-component spinor containing a Weyl fermion and its charge conjugate, in order to leave the action invariant the conjugate must transform with a phase having an opposite sign with respect to the one acting on the Weyl fermion. The needed opposite sign in the phase comes from the γ^5 matrix appearing in the exponential of the axial $U(1)$ transformation. To check this statement directly, one may use the chiral representation of the Dirac matrices A, so that the the Majorana spinor can be written as in (5.4), realize that γ^5 is diagonal and of the form in (A.15), and recognize that

$$\chi \to e^{i\alpha\gamma^5}\chi \tag{5.10}$$

is indeed consistent with the Majoarana condition.

The coupling is described by the action

$$S = -\frac{1}{2}\int d^4x \bar{\chi}\gamma^a D_a\chi \tag{5.11}$$

where D_a is the covariant gauge derivative

$$D_a = \partial_a - i\gamma^5 A_a. \tag{5.12}$$

Under an axial $U(1)$ transformation, fields transform as

$$\begin{cases} \chi(x) \to \chi'(x) = e^{i\gamma^5 \alpha(x)} \chi(x) \\ A_a(x) \to A'_a(x) = A_a(x) + \partial_a \alpha(x) \end{cases} \tag{5.13}$$

and the action is invariant. The related Noether's current is

$$J_5^a = \frac{i}{2} \bar{\chi} \gamma^5 \gamma^a \chi. \tag{5.14}$$

and is conserved on-shell, i.e. $\partial_a J_5^a = 0$.

A mass term can also be added, but it violates the chiral symmetry. Indeed, a standard Dirac mass term of the form

$$\mathcal{L}_m = -\frac{m}{2} \bar{\chi} \chi \tag{5.15}$$

can be added to the integral (5.11) defining the action. Because of the Majorana condition (5.1) it takes the equivalent form

$$\mathcal{L}_m = \frac{m}{2} \chi^T C \chi \tag{5.16}$$

which is known as the Majorana mass. Evidently, it breaks the gauge symmetry, and the axial $U(1)$ current in (5.14) now satisfies the equation

$$\partial_a J_5^a = -im \bar{\chi} \gamma^5 \chi. \tag{5.17}$$

A Majorana fermion can be coupled to gravity in the same way as we did for the Weyl fermion, i.e. by introducing the vielbein and the spin connection, so that the action in curved spacetime is

$$S = -\frac{1}{2} \int d^4 x e \bar{\chi} \gamma^\mu \nabla_\mu \chi \tag{5.18}$$

where e is the determinant of the vielbein, γ^μ are the curved-space gamma matrices and ∇_μ is the covariant derivative

$$\nabla_\mu = D_\mu + \frac{1}{4} \omega_{\mu ab} \gamma^a \gamma^b. \tag{5.19}$$

The stress tensor is defined as the functional derivative of the action with respect to the vielbein and is covariantly conserved, symmetric and traceless on-shell as consequence of the invariance of the action under general coordinate, local Lorentz and conformal transformations, respectively. The most general form of these transformations is defined by (3.13), with the only exception that λ is now replaced by χ.

The explicit expression of the stress tensor in flat space is given by

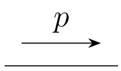
$$T_{ab} = \frac{1}{8}\bar{\chi}\left(\gamma_a \overleftrightarrow{D}_b + \gamma_b \overleftrightarrow{D}_a\right) - \frac{1}{4}\bar{\chi}\eta_{ab}\overleftrightarrow{\not{D}}\chi \tag{5.20}$$

with the last term which vanishes on-shell.

5.3 Feynman rules

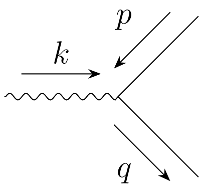
In order to find the Feynman rules in momentum space we follow the same strategy employed in the case of the Weyl fermion. We replace the expansion (3.21) into the action (5.18). In the following we collect the relevant Feynman rules. We denote Majorana fermions by straight lines[1], gauge fields by wavy lines and metric fluctuations by curled lines. The propagator and the interaction vertices are the same as for a Dirac fermion

- propagator



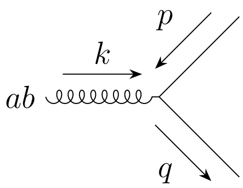
$$= -\frac{\not{p}}{p^2 - i\epsilon} \tag{5.21}$$

- fermion-gauge field vertex



$$= -\gamma^a \gamma^5 \tag{5.22}$$

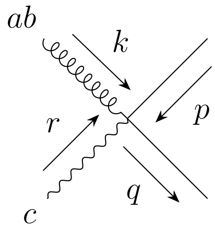
- two-fermion-gravity vertex



$$= -\frac{1}{8}\left[(p+q)_a \gamma_b + (p+q)_b \gamma_a - 2\eta_{ab}(\not{p} + \not{q})\right] \tag{5.23}$$

[1]Fermionic lines do not have any arrow because a Majorana particles is indistinguishable from its own antiparticle and the arrow over a line only denotes the momentum direction. In the case of Dirac fermions, the lines denoting propagators have an arrow whose direction allows to distinguish a Dirac particle from its own antiparticle, since the arrows have opposite directions and follow the flow of some charge which has to be conserved at the vertex. On the contrary, Majorana fermions do not carry any charge and the direction of the arrow is arbitrary.

- two-fermion-gravity-gauge vertex



$$= \frac{1}{4}\left(\gamma^a\eta^{bc} + \gamma^b\eta^{ac} - 2\eta^{ab}\gamma^c\right)\gamma^5 \tag{5.24}$$

The overall coefficient $\frac{1}{2}$ of the action does not appear in the interaction vertices because it is compensated by a factor 2 due to the possible ways to attach two identical lines to the vertex. This is a consequence of the reality constraint (5.1) which implies that Majorana fermions behave like real bosonic fields in the determination of the symmetry factor of any diagram. This leads to a new Feynman rule for what concerns loops in which Majorana fermions run. Each diagram containing a Majorana fermion loop gets multiplied by a factor $\frac{1}{2}$ due to the permutation symmetry between internal lines [34], [35], [36], [37]. The origin of this factor can be thought as due to the possible ways in which near vertices can be connected by means of identical lines in order to close the loop. In this way, all coefficients $\frac{1}{2}$ in vertices except one are compensated by a factor 2 due to the possible ways in which two near vertices can be connected by identical lines and there is just one way to attach the last vertex to the first one and close the loop. Therefore, a factor $\frac{1}{2}$ survives and multiplies the loop diagram[2].

In addition, we recall that whenever a diagram contains a fermionic loop a trace over spinor indices has to be taken and the diagram has to be multiplied by -1.

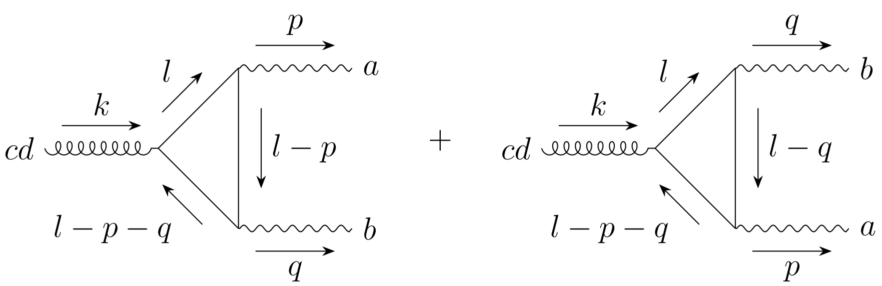
5.4 Anomalies

In this section we derive the anomalies related to the stress tensor. Let us consider the quantum expectation value of the stress tensor which at second order in the background gauge field is given by (4.16)

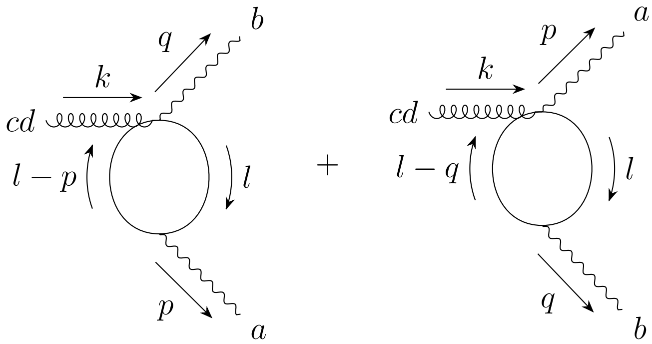
$$\langle T^{cd}(x)\rangle = \int d^4y \int d^4z \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} e^{ip(x-y)} e^{iq(x-z)} \mathcal{T}^{cdab}(p,q) A_a(y) A_b(z). \tag{5.25}$$

[2] An alternative way of understanding this factor is to recall that a gaussian path integral on complex Grassmann variables produces a determinant, while a gaussian path integral on real Grassmann variables produces a pfaffian, the square root of a determinant. More on this later on.

The matrix element $\mathcal{T}^{cdab} = \mathcal{T}^{cdab}_{(1)} + \mathcal{T}^{cdab}_{(2)}$ receive contributions from the following Feynman diagrams



(5.26)



(5.27)

whit the ones on the right obtained form those on the left by exchanging the external gauge fields, that is by replacing $(p,a) \leftrightarrow (q,b)$, p and q are two outgoing momenta and momentum conservation requires $k = p+q$. Thanks to Feynman rules for Majorana fermions we can write

$$\mathcal{T}^{cdab}_{(1)} = \frac{i}{16}\int \frac{d^n l}{(2\pi)^2}\frac{\mathcal{N}^{cdab}_{(1)}}{l^2(l-p)^2(l-p-q)^2} + (p,a) \leftrightarrow (q,b) \tag{5.28}$$

where

$$\mathcal{N}^{cdab}_{(1)} = Tr\Big([(2l-p-q)^c\gamma^d + (2l-p-q)^d\gamma^c - 2\eta^{cd}(2\not{l}-\not{p}-\not{q})]\not{l}\gamma^a\gamma^5(\not{l}-\not{p})\gamma^b\gamma^5(\not{l}-\not{p}-\not{q})\Big) \tag{5.29}$$

and

$$\mathcal{T}^{cdab}_{(2)} = -\frac{i}{8}\int \frac{d^n l}{(2\pi)^2}\frac{\mathcal{N}^{cdab}_{(2)}}{l^2(l-p)^2} + (p,a) \leftrightarrow (q,b) \tag{5.30}$$

with

$$\mathcal{N}^{cdab}_{(2)} = Tr\Big([\gamma^c\eta^{bd} + \gamma^d\eta^{bc} - 2\eta^{cd}\gamma^b]\gamma^5\not{l}\gamma^a\gamma^5(\not{l}-\not{p})\Big). \tag{5.31}$$

Since there are two γ^5 matrices the final result is parity-even and we can safely use an anticommunting γ^5. Therefore, moving γ^5 and using $(\gamma^5)^2 = 1$ we can eliminate γ^5 from the traces. Moreover, terms proportional to η^{cd} can be eliminated using the identity (4.28) in the first integral. Hence, we are left with

$$\begin{aligned}
\mathcal{T}^{cdab} =& \frac{i}{16}\int \frac{d^n l}{(2\pi)^n} Tr \frac{\left[(2l-p-q)^c\gamma^d + (2l-p-q)^d\gamma^c\right] \not{l}\gamma^a(\not{l}-\not{p})\gamma^b(\not{l}-\not{p}-\not{q})}{l^2(l-p)^2(l-p-q)^2} \\
&- \frac{i}{8}\int \frac{d^n l}{(2\pi)^n} Tr \frac{\left(\gamma^c\eta^{bd} + \gamma^d\eta^{bc}\right) \not{l}\gamma^a(\not{l}-\not{p})}{l^2(l-p)^2} \\
&+ (p,a) \leftrightarrow (q,b).
\end{aligned} \tag{5.32}$$

This is the same expression we have found for a Weyl fermion (4.29) and the computation of the anomalies is carried out in the same way and leads to the same results. For this reason, we refer to chapter 4 for details of the calculations and just display the final results. In particular, the trace anomaly is the same as that of a Weyl fermion and given by

$$\langle T^a_{\ a}\rangle = -\frac{1}{48\pi^2}F^{ab}F_{ab}. \tag{5.33}$$

The trace anomaly turns out to be half that of a Dirac fermion and does not contain any parity-odd contribution as already observed in [10] where the Majorana fermion is coupled to a gravitational background.

More generally, the anomalies for a Majorana fermion are half those of a Dirac fermion. This can be seen also at the level of path integrals as briefly described below. The path integral for Dirac fermions is

$$\int D\psi D\bar{\psi} e^{i\int d^4x \mathcal{L}} \tag{5.34}$$

with lagrangian $\mathcal{L} = \bar{\psi}(K+V)\psi$, where K is the kinetic term and V is the interaction to treat perturbatively. Formally the above path integral yields the determinant of the operator $K+V$

$$\mathrm{Det}(K+V) = \mathrm{Det}K \ \mathrm{Det}(1+K^{-1}V) = \mathrm{Det}K \exp\left\{Tr\log(1+K^{-1}V)\right\} \tag{5.35}$$

where K^{-1} is the inverse of the kinetic term, i.e. the propagator. By Taylor expanding one obtains a perturbative expansion in terms of the interaction and derive all possible Feynman diagrams. In this way one can extract the Feynman rule for fermion loops, i.e. the rule according to which for every fermionic loop a trace over spinor indices as to be taken and the diagram gets multiplied by -1.

On the contrary, the path integral for Majorana fermions does not give the determinant of the operator $K+V$, but, instead, its square root. This is a consequence of the Majorana constraint (5.1) which implies that the integration measure of the path integral is just $D\chi$. Thus, one has

$$(\mathrm{Det}(K+V))^{\frac{1}{2}} = (\mathrm{Det}K)^{\frac{1}{2}}(\mathrm{Det}(1+K^{-1}V))^{\frac{1}{2}} = (\mathrm{Det}K)^{\frac{1}{2}} \exp\left\{\frac{1}{2}Tr\log(1+K^{-1}V)\right\} \tag{5.36}$$

and Taylor expanding one obtains an expansions in terms of Feynman diagrams and derive Feynman rules. The coefficient $\frac{1}{2}$ appearing in the exponential will multiply every diagram arising from the perturbative expansion, which are the same as for Dirac fermions, and gives rise to the Feynman rule according to which each diagram containing a Majorana fermion loop gets multiplied by $\frac{1}{2}$. As already said, in the framework of Feynman diagrams this factor can be understood as a symmetry factor due to the possible equivalent ways in which the vertices of the loop can be connected by identical lines.

Chapter 6

Conclusions

In this thesis we have studied the anomalies of a Weyl fermion in an abelian gauge background. We have presented a derivation in terms of a Feynman diagrams computation, using dimensional regularization and the Breitenlohner-Maison scheme for treating the chiral matrix γ^5 in a consistent manner.

The aim of this work was to test the results obtained in [1], where the same model has been studied but a different regularization scheme was employed.

We have derived the chiral anomaly (4.15) reproducing the result known in literature. After that, we have computed the trace anomaly. To do so, we have first considered the quantum expectation value of the stress tensor (4.16), used the properties of γ^5 to simplify the computation of the related Feynman diagrams and shown that parity-odd terms do not contribute to the calculation as well as the second term of the classical stress tensor (3.12) which vanishes on-shell. Since the Breitenlohner-Maison scheme breaks the local Lorentz invariance we had to check whether or not there was an anomaly associated to the conservation of the stress tensor. Indeed, the presence of this kind of anomaly implies that in order to preserve the conservation one has to introduce counterterms which in turn may modify the expression of the trace anomaly. We have found that the stress tensor is conserved even at the quantum level and that the trace anomaly is given by (4.39).

Then, we have recast the model in terms of Majorana fermions, computed the trace anomaly and verified that it is still given by (4.39).

Our final result (4.39), namely

$$\langle T^a{}_a \rangle = -\frac{1}{48\pi^2} F^{ab} F_{ab} \tag{6.1}$$

proves that the trace anomaly does not contain the Chern-Pontryagin density term $F\tilde{F}$, agrees with the one derived in [1] and is half that of a Dirac fermion.

Appendix A

Conventions

We use a mostly plus Minkowski metric η_{ab}. The Dirac gamma matrices satisfy the Clifford's algebra

$$\left\{\gamma^a, \gamma^b\right\} = 2\eta^{ab}. \tag{A.1}$$

We recall that γ^0 is antihermitian and γ^i, $i = 1, 2, 3$, is hermitian

$$(\gamma^0)^\dagger = -\gamma^0 \quad \text{and} \quad (\gamma^i)^\dagger = \gamma^i \tag{A.2}$$

and that all gamma matrices are traceless.

The hermitian and traceless chiral matrix γ^5 is defined as

$$\gamma^5 = \frac{i}{4!}\epsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d = -i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{A.3}$$

where the Levi-Civita symbol ϵ_{abcd} is normalised as $\epsilon_{0123} = -1$ and $\epsilon^{0123} = 1$, so that the trace

$$Tr\left(\gamma^5\gamma^a\gamma^b\gamma^c\gamma^d\right) = 4i\epsilon^{abcd}. \tag{A.4}$$

It is used to define chiral projectors

$$P_L = \frac{\mathbb{1} + \gamma^5}{2} \quad \text{and} \quad P_R = \frac{\mathbb{1} - \gamma^5}{2} \tag{A.5}$$

which split a Dirac spinor ψ into its left and right Weyl components

$$\psi = \lambda + \rho, \qquad \lambda = P_L\psi, \qquad \rho = P_R\psi. \tag{A.6}$$

They are idempotent

$$P_L^2 = P_L \quad \text{and} \quad P_R^2 = P_R \tag{A.7}$$

and orthogonal

$$P_LP_R = P_RP_L = 0. \tag{A.8}$$

The conjugate Dirac spinor $\bar{\psi}$ is defined using $\beta = i\gamma^0$ by

$$\bar{\psi} = \psi^\dagger \beta. \tag{A.9}$$

The charge conjugation matrix C is defined by

$$C\gamma^a C^{-1} = -\gamma^{aT} \tag{A.10}$$

and is used to define the charge conjugation of the Dirac spinor

$$\psi_c = C^{-1}\bar{\psi}^T \tag{A.11}$$

for which the roles of particles and antiparticles get interchanged. It also satisfies

$$C\gamma^5 C^{-1} = (\gamma^5)^T. \tag{A.12}$$

Note that a chiral spinor λ has its charge conjugated field λ_c of opposite chirality. A Majorana spinor χ is a spinor that equals its charge conjugated spinor

$$\chi = \chi_c. \tag{A.13}$$

This condition is called Majorana constraint and is incompatible with the chiral constraint, and Majorana-Weyl spinors do not exist in four dimensions.

In order ot check the above formulae we find convenient to use the chiral representation of gamma matrices. In terms of 2×2 blocks they are given by

$$\gamma^0 = -i \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \gamma^i = -i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{A.14}$$

where σ^i are the Pauli matrices, and

$$\gamma^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{A.15}$$

The chiral representation makes evident that the Lorentz generators in the spinor representation $\Sigma^{ab} = \frac{1}{4}\left[\gamma^a, \gamma^b\right] = \frac{1}{2}\gamma^{ab}$ take a block diagonal form

$$\Sigma^{0i} = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \qquad \Sigma^{ij} = \frac{i}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix} \tag{A.16}$$

and do not mix the chiral components of a Dirac spinor.

In the chiral representation the charge conjugation matrix C is given by

$$C = \gamma^2\beta = -i\begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} \tag{A.17}$$

and satisfies

$$C = -C^T = -C^{-1} = -C^\dagger = C^* \tag{A.18}$$

where some of these relations depend on the representation.

The two-dimensional Weyl-spinors appear inside a four-dimensional Dirac spinor as

$$\psi = \begin{pmatrix} l \\ r \end{pmatrix}, \quad \lambda = \begin{pmatrix} l \\ 0 \end{pmatrix}, \quad \rho = \begin{pmatrix} 0 \\ r \end{pmatrix} \tag{A.19}$$

where l and r indicate the two-dimensional independent spinors of opposite chirality.

In the chiral representation the Majorana constraint (A.13) takes the form

$$\chi = \chi_c \quad \rightarrow \quad \begin{pmatrix} l \\ r \end{pmatrix} = \begin{pmatrix} i\sigma^2 r^* \\ -i\sigma^2 l^* \end{pmatrix} \tag{A.20}$$

which shows that the two-dimensional spinors l and r cannot be independent. The Majorana condition can be solved in terms of a single two-dimensional left handed spinor l as

$$\chi = \begin{pmatrix} l \\ -i\sigma^2 l^* \end{pmatrix} \tag{A.21}$$

which appear also in the four-dimensional chiral spinors λ and λ_c

$$\lambda = \begin{pmatrix} l \\ 0 \end{pmatrix}, \qquad \lambda_c = \begin{pmatrix} 0 \\ -i\sigma^2 l^* \end{pmatrix}. \tag{A.22}$$

In four dimensional spinor notation one can write $\chi = \lambda + \lambda_c$, as well as $\lambda = P_L\chi$ and $\lambda_c = P_R\chi$. Alternatively, the Majorana condition can be solved in terms of a single two-dimensional right handed spinor r as

$$\chi = \begin{pmatrix} i\sigma^2 r^* \\ r \end{pmatrix}. \tag{A.23}$$

which contains the four-dimensional chiral spinors ρ and ρ_c defined as

$$\rho = \begin{pmatrix} 0 \\ r \end{pmatrix}, \qquad \rho_c = \begin{pmatrix} i\sigma^2 r^* \\ 0 \end{pmatrix} \tag{A.24}$$

so that $\chi = \rho + \rho_c$. This solution is the same as the previous one since one may identify $\lambda = \rho_c$.

The explicit dictionary between Weyl and Majorana spinors shows clearly that the field theory of a Weyl spinor is equivalent to that of a Majorana spinor, being their actions fixed uniquely by Lorentz symmetry, and thus bound to be identical.

Appendix B

Useful relations and formulae

In this appendix we collect useful relations and formulae.

According to the Breitenlohner-Maison scheme the n-dimensional Minkoski spacetime splits in the product of a four-dimensional subspace and a $(n-4)$-dimensional one. Any n-dimensional object decomposes into a four-dimensional part (denoted by an overbar) and an $(n-4)$-dimensional part (denoted by a hat). The metric tensor is defined as

$$\eta^{ab} = \bar{\eta}^{ab} + \hat{\eta}^{ab} \tag{B.1}$$

with

$$\eta_{ab}\eta^{ab} = n, \qquad \bar{\eta}_{ab}\bar{\eta}^{ab} = 4, \qquad \hat{\eta}_{ab}\hat{\eta}^{ab} = n-4, \qquad \bar{\eta}_{ab}\hat{\eta}^{ab} = 0. \tag{B.2}$$

The last of these relations shows that contractions between indices belonging to different subspaces vanish.

Any vector decomposes as

$$k^a = \bar{k}^a + \hat{k}^a \tag{B.3}$$

and metric tensors act as projectors onto different subspaces

$$\begin{aligned} &k^a = \eta^{ab}k_b, \qquad k_a = \eta_{ab}k^b, \qquad \bar{k}_a = \bar{\eta}_{ab}k^b, \qquad \hat{k}_a = \hat{\eta}_{ab}k^b, \qquad k^2 = \bar{k}^2 + \hat{k}^2, \\ &k^2 = k^a k_a = \eta^{ab}k_a k_b = \eta_{ab}k^a k^b, \qquad \bar{k}^2 = \bar{k}_a\bar{k}^a = \bar{\eta}^{ab}k_a k_b = \bar{\eta}_{ab}k^a k^b, \\ &\hat{k}^2 = \hat{k}_a\hat{k}^a = \hat{\eta}^{ab}k_a k_b = \hat{\eta}_{ab}k^a k^b, \qquad \bar{\eta}_{ab}\hat{k}^b = 0, \qquad \hat{\eta}_{ab}\bar{k}^b = 0. \end{aligned} \tag{B.4}$$

Dirac gamma matrices decompose as

$$\gamma^a = \bar{\gamma}^a + \hat{\gamma}^a \tag{B.5}$$

and satisfy

$$\begin{aligned} &\left\{\gamma^a, \gamma^b\right\} = 2\eta^{ab}, \qquad \gamma^a\gamma_a = n, \qquad Tr\gamma^a = 0, \\ &\left\{\gamma^a, \bar{\gamma}^b\right\} = \left\{\bar{\gamma}^a, \bar{\gamma}^b\right\} = 2\bar{\eta}^{ab}, \qquad \gamma^a\bar{\gamma}_a = \bar{\gamma}^a\bar{\gamma}_a = 4, \qquad Tr\bar{\gamma}^a = 0, \\ &\left\{\gamma^a, \hat{\gamma}^b\right\} = \left\{\hat{\gamma}^a, \hat{\gamma}^b\right\} = 2\hat{\eta}^{ab}, \qquad \gamma^a\hat{\gamma}_a = \hat{\gamma}^a\hat{\gamma}_a = n-4, \qquad Tr\hat{\gamma}^a = 0, \\ &\left\{\bar{\gamma}^a, \hat{\gamma}^b\right\} = 0, \qquad \bar{\gamma}^a\hat{\gamma}_a = 0. \end{aligned} \tag{B.6}$$

γ^5 is defined as in four dimensions (2.1) and anticommutes with gamma matrices of the four-dimensional subspace and commutes with those in the $(n-4)$-dimensional subspace

$$\{\gamma^5, \bar{\gamma}^a\} = 0, \qquad [\gamma^5, \hat{\gamma}^a] = 0 \tag{B.7}$$

which implies

$$\{\gamma^5, \gamma^a\} = \{\gamma^5, \hat{\gamma}^a\} = 2\gamma^5\hat{\gamma}^a, \qquad [\gamma^5, \gamma^a] = [\gamma^5, \bar{\gamma}^a] = 2\gamma^5\bar{\gamma}^a. \tag{B.8}$$

From the definition of γ^5 (2.1), its square (2.3) and projectors (3.2) one can also derive the following identities

$$P_R\gamma^a P_L = \bar{\gamma}^a P_L = P_R\bar{\gamma}^a, \qquad P_L\gamma^a P_R = \bar{\gamma}^a P_R = P_L\bar{\gamma}^a. \tag{B.9}$$

We conclude this appendix by writing the explicit expression of traces involving two, four and six gamma matrices in n dimensions

$$Tr\Big(\gamma^a\gamma^b\Big) = 2^{\frac{n}{2}}\eta^{ab} \tag{B.10}$$

$$Tr\Big(\gamma^a\gamma^b\gamma^c\gamma^d\Big) = 2^{\frac{n}{2}}\left(\eta^{ab}\eta^{cd} - \eta^{ac}\eta^{bd} + \eta^{ad}\eta^{bc}\right) \tag{B.11}$$

$$\begin{aligned}
&Tr\Big(\gamma^d\gamma^e\gamma^a\gamma^f\gamma^b\gamma^g\Big) = \\
&\quad = 2^{\frac{n}{2}}\Big(\eta^{de}\eta^{af}\eta^{bg} - \eta^{de}\eta^{ab}\eta^{fg} + \eta^{de}\eta^{ag}\eta^{fb} + \\
&\qquad - \eta^{da}\eta^{ef}\eta^{bg} + \eta^{da}\eta^{eb}\eta^{fg} - \eta^{da}\eta^{eg}\eta^{fb} + \\
&\qquad + \eta^{ea}\eta^{df}\eta^{bg} - \eta^{ea}\eta^{db}\eta^{fg} + \eta^{ea}\eta^{dg}\eta^{fb} + \\
&\qquad - \eta^{df}\eta^{eb}\eta^{ag} + \eta^{df}\eta^{ab}\eta^{eg} - \eta^{ef}\eta^{ab}\eta^{dg} + \\
&\qquad + \eta^{ef}\eta^{db}\eta^{ag} - \eta^{af}\eta^{db}\eta^{eg} + \eta^{af}\eta^{eb}\eta^{dg}\Big).
\end{aligned} \tag{B.12}$$

Appendix C

Vielbein and Spin Connection

Let $\mathcal{M}$ be a manifold with metric tensor $g_{\mu\nu}$, the vielbein is defined by

$$g_{\mu\nu} = e_\mu^{\ a} e_\nu^{\ b} \eta_{ab} \tag{C.1}$$

The vectors $e_\mu^{\ a}$ form a basis[1] in the tangent space of the manifold $\mathcal{M}$ at the point x. The Lorentz index a of the vielbein is raised and lowered with the Lorentz metric η_{ab}, while the spacetime index μ is raised and lowered with the metric tensor $g_{\mu\nu}$.

The vielbein is defined up to a local Lorentz transformation

$$e_\mu^{\ a} \to e_\mu^{'\ a}(x) = \Lambda^a_{\ b}(x) e_\mu^{\ b} \tag{C.2}$$

In order to achieve local Lorentz invariance one needs to introduce a gauge field $\omega_\mu^{\ ab}(x)$of the Lorentz group usually called *spin connection*.

The covariant derivative of a vector field V^μ is defined in general relativity as

$$D_\lambda V^\mu = \partial_\lambda V^\mu + \Gamma^\mu_{\ \lambda\nu} V^\nu \tag{C.3}$$

with $\Gamma^\mu_{\ \lambda\nu}$ being the Christoffel symbols given by

$$\Gamma^\mu_{\ \lambda\nu} = \frac{1}{2} g^{\mu\rho} \left(\partial_\lambda g_{\rho\nu} + \partial_\nu g_{\lambda\rho} - \partial_\rho g_{\lambda\nu} \right) \tag{C.4}$$

in terms of the metric tensor. Once the vielbein is introduces one can choose to work with $V^a = V^\mu e_\mu^{\ a}$ with covariant derivative

$$\nabla_\mu V^a = \partial_\mu V^a + \omega_\mu^{\ a}{}_b V^b. \tag{C.5}$$

The two definitions of covariant derivative must be equivalent because we do not want to modify standard content of general relativity, meaning that

$$\nabla_\mu V^a = e_\nu^{\ a} D_\mu V^\nu. \tag{C.6}$$

[1]This basis is usually called vielbein (from the German in which it means "many legs"), in four dimensions, it is also called *vierbein* or *tetrad*.

By expanding this equation one has

$$
\begin{aligned}
\nabla_\mu V^a &= \nabla_\mu \left(e_\nu{}^a V^\nu\right) = \partial_\mu \left(e_\nu{}^a V^\nu\right) + \omega_\mu{}^a{}_b e_\nu{}^b V^\nu \\
&= \left(\partial_\mu e_\nu{}^a\right) V^\nu + e_\nu{}^a \partial_\mu V^\nu + \omega_\mu{}^a{}_b e_\nu{}^b V^\nu \\
&= e_\nu{}^a \left(\partial_\mu V^\nu + \Gamma^\nu{}_{\mu\lambda} V^\lambda\right)
\end{aligned}
\tag{C.7}
$$

where the last line comes from the right hand side of (C.6). By comparing the two expressions one has

$$
\Gamma^\nu{}_{\mu\lambda} = e^\nu{}_a \left(\partial_\mu e_\lambda{}^a + \omega_\mu{}^a{}_b e_\lambda{}^b\right) \tag{C.8}
$$

which can be rewritten as

$$
\partial_\mu e_\nu{}^a - \Gamma^\lambda_{\mu\nu} e_\lambda{}^a + \omega_\mu{}^a{}_b e_\nu{}^b = \tilde{\nabla}_\mu e_\nu{}^a = 0 \tag{C.9}
$$

where $\tilde{\nabla}_\mu$ is the covariant derivative acting on an object with both Lorentz and space-time indices like $e_\nu{}^a$. This result tells that the vielbein is covariantly conserved and is known as the *vielbein postulate*.

The explicit expression of the spin connection in terms of the vielbein is

$$
\begin{aligned}
\omega_\mu{}^{ab} =& \frac{1}{2} e^{\nu a} \left(\partial_\mu e_\nu{}^b - \partial_\nu e_\mu{}^b\right) - \frac{1}{2} e^{\nu b} \left(\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a\right) \\
& - \frac{1}{2} e^{\rho a} e^{\sigma b} \left(\partial_\rho e_{\sigma c} - \partial_\sigma e_{\rho c}\right) e_\mu{}^c
\end{aligned}
\tag{C.10}
$$

which is antisymmetric in the indices a and b, and transforms under a local Lorentz transformation as

$$
\omega_\mu{}^a{}_b \to \omega_\mu^{\prime}{}^a{}_b = \Lambda^a{}_c \Lambda_b{}^d \omega_\mu{}^c{}_d - \Lambda_b{}^c \partial_\mu \Lambda^a{}_c \tag{C.11}
$$

that is the standard transformation of a gauge field.

Thanks to the spin connection is not difficult to couple spinors to general relativity. The covariant derivative of a field $\psi(x)$ in the spinor representation of the Lorentz group is

$$
\nabla_\mu \psi = \partial_\mu \psi + \frac{1}{2} \omega_\mu{}^{ab} \Sigma_{ab} \psi \tag{C.12}
$$

where Σ_{ab} are the generators of the Lorentz group in the spinor representation. Under a local Lorentz transformation the covariant derivative transforms homogeneously, $\nabla_\mu \psi \to \Lambda(x) \nabla_\mu \psi$, since the spinor field transforms as $\psi \to \Lambda(x)\psi$ and the spin connection in the standard way.

The variation of the spin connection induced by a variation of the vielbein $\delta e_{\mu a}$ is

$$
\delta\omega_{\mu ab} = \frac{1}{2} \left(\nabla_\mu \delta e_{ab} - \nabla_a \delta e_{\mu b} - \nabla_a \delta e_{b\mu}\right) - (a \leftrightarrow b) \tag{C.13}
$$

This formula can be specialized for a Weyl transformation $\delta e_{\mu a} = \sigma e^a_\mu$ in flat space to

$$
\delta\omega_{\mu ab} = e_{\mu a} \partial_b \sigma - e_{\mu b} \partial_a \sigma. \tag{C.14}
$$

Appendix D

Loop integrals and dimensional regularization

In order to combine propagator denominators in loop integrals, we have used Feynman parametric formulae

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2} \tag{D.1a}$$

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^{1-x} dy \frac{1}{[xA + yB + (1-x-y)C]^3} \tag{D.1b}$$

Once this is done, the denominator becomes a quadratic function of the integration variable l. Then, one completes the square and shifts the integration variable to absorb liner terms in l. After that, the denominator takes the form $(l^2 + f)^m$, where $m = 2, 3$ and f is a function of the Feynman parameters and external momenta. In the numerator, terms with an odd number of powers of l vanish by symmetric integration and symmetry allows to replace

$$l^a l^b \to \frac{1}{n}\eta^{ab} l^2 \tag{D.2a}$$

$$l^a l^b l^c l^d \to \frac{1}{n(n+2)} l^4 \left(\eta^{ab}\eta^{cd} + \eta^{ac}\eta^{bd} + \eta^{ad}\eta^{bc}\right) \tag{D.2b}$$

where n is the spacetime dimension. Integrals are most conveniently evaluated after Wick-rotating the integration variable to Euclidean space, that is by replacing $l^0 \to il^0$.

One can use the following table of n-dimensional integrals in Minkowski space

$$\int \frac{d^n l}{(2\pi)^n} \frac{1}{(l^2 + f)^m} = \frac{i}{(4\pi)^{\frac{n}{2}}} \frac{\Gamma\left(m - \frac{n}{2}\right)}{\Gamma(m)} \left(\frac{1}{f}\right)^{m - \frac{n}{2}} \tag{D.3}$$

$$\int \frac{d^n l}{(2\pi)^n} \frac{l^2}{(l^2+f)^m} = \frac{i}{(4\pi)^2} \frac{n}{2} \frac{\Gamma(m-\frac{n}{2}-1)}{\Gamma(m)} \left(\frac{1}{f}\right)^{m-\frac{n}{2}-1} \tag{D.4}$$

$$\int \frac{d^n l}{(2\pi)^n} \frac{l^4}{(l^2+f)^m} = \frac{i}{(4\pi)^{\frac{n}{2}}} \frac{n(n+2)}{4} \frac{\Gamma(m-\frac{n}{2}-2)}{\Gamma(m)} \left(\frac{1}{f}\right)^{m-\frac{n}{2}-2} \tag{D.5}$$

where the overall factor i comes from the Wick rotation of the integration variable. We need also the following expansion

$$\frac{\Gamma(2-\frac{n}{2})}{f^{2-\frac{n}{2}}} \overset{n\to 4}{=} \frac{2}{4-n} - \log f - \gamma + \log 4\pi + O(4-n). \tag{D.6}$$

where γ is the Euler-Mascheroni constant.

Thanks to this table we can easily evaluate integrals appearing in the calculations. In the following we collect relevant integrals employed during the calculations derived from the above table

$$\begin{aligned}
\left(\frac{2}{n}-1\right) \int \frac{d^n l}{(2\pi)^n} \frac{l^2}{(l^2+f)^2} &= \left(\frac{2}{n}-1\right) \frac{i}{(4\pi)^{\frac{n}{2}}} \frac{n}{2} \frac{\Gamma(1-\frac{n}{2})}{\Gamma(2)} \left(\frac{1}{f}\right)^{1-\frac{n}{2}} = \\
&= \frac{i}{(4\pi)^{\frac{n}{2}}} \left(1-\frac{n}{2}\right) \frac{\Gamma(1-\frac{n}{2})}{f^{1-\frac{n}{2}}} = \\
&= \frac{i}{(4\pi)^{\frac{n}{2}}} \frac{\Gamma(2-\frac{n}{2})}{f^{2-\frac{n}{2}}} f = \\
&\overset{n\to 4}{=} \frac{i}{16\pi^2} f \left(\frac{2}{4-n} - \log f - \gamma + \log 4\pi + O(4-n)\right)
\end{aligned} \tag{D.7}$$

$$\begin{aligned}
\int \frac{d^n l}{(2\pi)^n} \frac{1}{(l^2+f)^2} &= \frac{i}{(4\pi)^2} \frac{\Gamma(2-\frac{n}{2})}{\Gamma(2)} \left(\frac{1}{f}\right)^{2-\frac{n}{2}} \\
&\overset{n\to 4}{=} \frac{i}{16\pi^2} \left(\frac{2}{4-n} - \log f - \gamma + \log 4\pi + O(4-n)\right)
\end{aligned} \tag{D.8}$$

$$\begin{aligned}
\int \frac{d^n l}{(2\pi)^n} \frac{l^2}{(l^2+f)^3} &= \frac{i}{(4\pi)^2} \frac{n}{2} \frac{\Gamma(2-\frac{n}{2})}{\Gamma(3)} \left(\frac{1}{f}\right)^{2-\frac{n}{2}} \\
&\overset{n\to 4}{=} \frac{i}{16\pi^2} \left(\frac{2}{4-n} - \log f - \gamma + \log 4\pi + O(4-n)\right)
\end{aligned} \tag{D.9}$$

Defining the loop momentum $l = \bar{l} + s$ we can evaluate the integrals

$$\int \frac{d^4 \bar{l}}{(2\pi)^4} \int \frac{d^{n-4} s}{(2\pi)^{n-4}} \frac{s^2}{(\bar{l}^2+s^2+f)^3} \tag{D.10}$$

and

$$\int \frac{d^4\bar{l}}{(2\pi)^4} \int \frac{d^{n-4}s}{(2\pi)^{n-4}} \frac{s^4}{(\bar{l}^2+s^2+f)^3}. \tag{D.11}$$

Let us first perform integration over s and define $t = \bar{l}^2 + f$

$$\int \frac{d^{n-4}s}{(2\pi)^{n-4}} \frac{s^2}{(s^2+t)^3} = \frac{n-4}{2(4\pi)^{\frac{n-4}{2}}} \frac{\Gamma(4-\frac{n}{2})}{\Gamma(3)} \left(\frac{1}{t}\right)^{4-\frac{n}{2}} \tag{D.12}$$

then, integrate over $\bar{l}$

$$\int \frac{d^4\bar{l}}{(2\pi)^4} \frac{1}{(\bar{l}^2+f)^{4-\frac{n}{2}}} = \frac{i}{(4\pi)^2} \frac{\Gamma(2-\frac{n}{2})}{\Gamma(4-\frac{n}{2})} \left(\frac{1}{f}\right)^{2-\frac{n}{2}}. \tag{D.13}$$

Putting everything together and using the expansion (D.6) we get the finite result

$$\int \frac{d^4\bar{l}}{(2\pi)^4} \int \frac{d^{n-4}s}{(2\pi)^{n-4}} \frac{s^2}{(\bar{l}^2+s^2+f)^3} = -\frac{i}{32\pi^2} \tag{D.14}$$

Following similar steps one has

$$\int \frac{d^{n-4}s}{(2\pi)^{n-4}} \frac{s^4}{(s^2+t)^3} = \frac{(n-4)(n-2)}{(4\pi)^{\frac{n-4}{2}}4} \frac{\Gamma(3-\frac{n}{2})}{\Gamma(3)} \left(\frac{1}{t}\right)^{3-\frac{n}{2}} \tag{D.15}$$

from the integration over s, and

$$\int \frac{d^4\bar{l}}{(2\pi)^4} \frac{1}{(\bar{l}^2+f)^{3-\frac{n}{2}}} = \frac{i}{(4\pi)^2} \frac{\Gamma(1-\frac{n}{2})}{\Gamma(3-\frac{n}{2})} \left(\frac{1}{f}\right)^{1-\frac{n}{2}} \tag{D.16}$$

from integrating over $\bar{l}$, and putting everything together we obtain the finite result

$$\int \frac{d^4\bar{l}}{(2\pi)^4} \int \frac{d^{n-4}s}{(2\pi)^{n-4}} \frac{s^4}{(\bar{l}^2+s^2+f)^3} = \frac{i}{32\pi^2} f. \tag{D.17}$$

We see that the factor $(n-4)$ arising from the integration over s kills the singularity in the expansion (D.6) and cancel all other terms, once the limit $n \to 4$ is taken, leading to a finite result.

Appendix E

Details of calculations

In this appendix we report more details about the calculations carried out in the main part of the thesis which have been omitted in order to make the reading less heavy.

E.1 Chiral anomaly

Let us recall the integrals (4.5)

$$
-\int \frac{d^n l}{(2\pi)^n} \frac{Tr\left\{\not{k} P_L(\not{l}-\not{p})\bar{\gamma}^a P_L \not{l} \bar{\gamma}^b P_L(\not{l}+\not{q})\right\}}{l^2(l-p)^2(l+q)^2} \tag{E.1a}
$$

$$
-\int \frac{d^n l}{(2\pi)^n} \frac{Tr\left\{\not{k} P_L(\not{l}-\not{q})\bar{\gamma}^b P_L \not{l} \bar{\gamma}^a P_L(\not{l}+\not{p})\right\}}{l^2(l-q)^2(l+p)^2} \tag{E.1b}
$$

where the second line is the contribution of the diagram with the two external gauge fields exchanged. By making explicit the projectors, each numerator gives the trace of eight terms multiplied by $\frac{1}{8}$, which are

$$(\not{p}+\not{q})(\not{l}-\not{p})\bar{\gamma}^a \not{l} \bar{\gamma}^b (\not{l}+\not{q})+ \tag{E.2a}$$

$$(\not{p}+\not{q})(\not{l}-\not{p})\bar{\gamma}^a \not{l} \bar{\gamma}^b \gamma^5 (\not{l}+\not{q})+ \tag{E.2b}$$

$$(\not{p}+\not{q})(\not{l}-\not{p})\bar{\gamma}^a \gamma^5 \not{l} \bar{\gamma}^b (\not{l}+\not{q})+ \tag{E.2c}$$

$$(\not{p}+\not{q})(\not{l}-\not{p})\bar{\gamma}^a \gamma^5 \not{l} \bar{\gamma}^b \gamma^5 (\not{l}+\not{q})+ \tag{E.2d}$$

$$(\not{p}+\not{q})\gamma^5(\not{l}-\not{p})\bar{\gamma}^a \not{l} \bar{\gamma}^b (\not{l}+\not{q})+ \tag{E.2e}$$

$$(\not{p}+\not{q})\gamma^5(\not{l}-\not{p})\gamma^a \not{l} \gamma^b \gamma^5 (\not{l}+\not{q})+ \tag{E.2f}$$

$$(\not{p}+\not{q})\gamma^5(\not{l}-\not{p})\bar{\gamma}^a \gamma^5 \not{l} \bar{\gamma}^b (\not{l}+\not{q})+ \tag{E.2g}$$

$$(\not{p}+\not{q})\gamma^5(\not{l}-\not{p})\bar{\gamma}^a \gamma^5 \not{l} \bar{\gamma}^b \gamma^5 (\not{l}+\not{q}) \tag{E.2h}$$

from the numerator of (E.1a), and

$$(\not{p}+\not{q})(\not{l}-\not{q})\bar{\gamma}^b\not{l}\bar{\gamma}^a(\not{l}+\not{p})+ \tag{E.3a}$$
$$(\not{p}+\not{q})(\not{l}-\not{q})\bar{\gamma}^b\not{l}\bar{\gamma}^a\gamma^5(\not{l}+\not{p})+ \tag{E.3b}$$
$$(\not{p}+\not{q})(\not{l}-\not{q})\bar{\gamma}^b\gamma^5\not{l}\bar{\gamma}^a(\not{l}+\not{p})+ \tag{E.3c}$$
$$(\not{p}+\not{q})(\not{l}-\not{q})\bar{\gamma}^b\gamma^5\not{l}\bar{\gamma}^a\gamma^5(\not{l}+\not{p})+ \tag{E.3d}$$
$$(\not{p}+\not{q})\gamma^5(\not{l}-\not{q})\bar{\gamma}^b\not{l}\bar{\gamma}^a(\not{l}+\not{p})+ \tag{E.3e}$$
$$(\not{p}+\not{q})\gamma^5(\not{l}-\not{q})\bar{\gamma}^b\not{l}\bar{\gamma}^a\gamma^5(\not{l}+\not{p})+ \tag{E.3f}$$
$$(\not{p}+\not{q})\gamma^5(\not{l}-\not{q})\bar{\gamma}^b\gamma^5\not{l}\bar{\gamma}^a(\not{l}+\not{p})+ \tag{E.3g}$$
$$(\not{p}+\not{q})\gamma^5(\not{l}-\not{q})\bar{\gamma}^b\gamma^5\not{l}\bar{\gamma}^a\gamma^5(\not{l}+\not{p}) \tag{E.3h}$$

from the numerator of (E.1b).

We will use the following identities to make simplifications

$$\not{p}+\not{q}=\not{l}+\not{q}-(\not{l}-\not{p}) \tag{E.4a}$$
$$(\not{p}+\not{q})\gamma^5=(\not{l}+\not{q})\gamma^5+\gamma^5(\not{l}-\not{p})-2\not{s}\gamma^5 \tag{E.4b}$$

in (E.2), and

$$\not{p}+\not{q}=\not{l}+\not{p}-(\not{l}-\not{q}) \tag{E.5a}$$
$$(\not{p}+\not{q})\gamma^5=(\not{l}+\not{p})\gamma^5+\gamma^5(\not{l}-\not{q})-2\not{s}\gamma^5 \tag{E.5b}$$

in (E.3).

Every term in which there is an even number of γ^5 is parity-even. Such terms are (E.2d), (E.2f), (E.2g), (E.3d), (E.3f), (E.3g). In this case, it is possible to use an anti-commuting γ^5 together with $(\gamma^5)^2=1$ to complete remove γ^5 from these terms. After doing so, they become equal to (E.2a) and (E.3a) and, up to an overall coefficient, we have the integrals

$$\begin{aligned}&\int\frac{d^nl}{(2\pi)^n}Tr\frac{(\not{p}+\not{q})(\not{l}-\not{p})\bar{\gamma}^a\not{l}\bar{\gamma}^b(\not{l}+\not{q})}{l^2(l-p)^2(l+q)^2}+\\&+\int\frac{d^nl}{(2\pi)^n}Tr\frac{(\not{p}+\not{q})(\not{l}-\not{q})\bar{\gamma}^b\not{l}\bar{\gamma}^a(\not{l}+\not{p})}{l^2(l-q)^2(l+p)^2}\end{aligned} \tag{E.6}$$

If we use (E.4a) in the first integral and (E.5a) in the second integral, then we obtain the following four terms up to an overall constant

$$\begin{aligned}&\int\frac{d^nl}{(2\pi)^n}Tr\frac{(\not{l}-\not{p})\bar{\gamma}^a\not{l}\bar{\gamma}^b}{l^2(l-p)^2}-\int\frac{d^nl}{(2\pi)^n}Tr\frac{\bar{\gamma}^a\not{l}\bar{\gamma}^b(\not{l}+\not{q})}{l^2(l+q)^2}+\\&-\int\frac{d^nl}{(2\pi)^n}Tr\frac{(\not{l}+\not{p})\bar{\gamma}^b\not{l}\bar{\gamma}^a}{l^2(l+p)^2}+\int\frac{d^nl}{(2\pi)^n}Tr\frac{\bar{\gamma}^b\not{l}\bar{\gamma}^a(\not{l}-\not{q})}{l^2(l-q)^2}\end{aligned} \tag{E.7}$$

Making the change of variable $l \to -l$ and using the ciclicity of the trace in integrals of the second line, all these terms cancel pairwise. This was expected because this is nothing but the way in which one can prove the Ward identity for the $U(1)$ gauge current of QED, which is conserved both at classical and quantum level. This shows that parity-even terms do not contribute to the chiral anomaly.

Let us now consider parity-odd terms (E.2c) and (E.3b), and use (E.4a) and (E.5a) to obtain the following four terms

$$\begin{aligned} &\int \frac{d^n l}{(2\pi)^n} Tr \frac{(\not{l} - \not{p})\bar{\gamma}^a \gamma^5 \not{l} \bar{\gamma}^b}{l^2 (l-p)^2} - \int \frac{d^n l}{(2\pi)^n} Tr \frac{\bar{\gamma}^a \gamma^5 \not{l} \bar{\gamma}^b (\not{l} + \not{q})}{l^2 (l+q)^2} + \\ &- \int \frac{d^n l}{(2\pi)^n} Tr \frac{(\not{l} + \not{p})\bar{\gamma}^b \not{l} \bar{\gamma}^a \gamma^5}{l^2 (l+p)^2} + \int \frac{d^n l}{(2\pi)^n} Tr \frac{\bar{\gamma}^b \not{l} \bar{\gamma}^a \gamma^5 (\not{l} - \not{q})}{l^2 (l-q)^2} \end{aligned} \tag{E.8}$$

which differ from the previous ones for the presence of γ^5. Following the same steps as before and using $\{\bar{\gamma}^a, \gamma^5\} = 0$, terms in the second line cancel those in the first line. By a similar reasoning one can show that (E.2b) and (E.3c), and (E.2d) and (E.3d) cancel pairwise.

E.2 Cancellation of parity-odd terms

In this appendix we show how parity-odd terms in the quantum expectation value of the stress tensor (4.16) cancel.

Let us consider the odd part of the the cross term of (4.19)

$$-\frac{i}{16} \int \frac{d^n l}{(2\pi)^n} \frac{\mathcal{N}^{cdba}_{(1)odd}}{l^2 (l-q)^2 (l-p-q)^2} \tag{E.9}$$

where the numerator is

$$\begin{aligned} \mathcal{N}^{cdba}_{(1)odd} &= Tr\Big(\gamma^5 \Big[(2l-p-q)^c \bar{\gamma}^d + (2l-p-q)^d \bar{\gamma}^c - 2\eta^{cd}(2\bar{\not{l}} - \not{p} - \not{q})\Big] \times \\ &\qquad \times \bar{\not{l}} \bar{\gamma}^b (\bar{\not{l}} - \not{q}) \bar{\gamma}^a (\bar{\not{l}} - \not{p} - \not{q})\Big) = \\ &= Tr\Big(\gamma^5 \Big[(2l-p-q)^c \bar{\gamma}^d + (2l-p-q)^d \bar{\gamma}^c - 2\eta^{cd}(2\bar{\not{l}} - \not{p} - \not{q})\Big] \times \\ &\qquad \times \bar{\not{l}} \bar{\gamma}^b (\bar{\not{l}} - \not{q}) \bar{\gamma}^a (\bar{\not{l}} - \not{p} - \not{q})\Big)^T = \\ &= Tr\Big((\bar{\not{l}} - \not{p} - \not{q})^T (\bar{\gamma}^a)^T (\bar{\not{l}} - \not{q})^T (\bar{\gamma}^b)^T \bar{\not{l}}^T \times \\ &\times \Big[(2l-p-q)^c \bar{\gamma}^d + (2l-p-q)^d \bar{\gamma}^c - 2\eta^{cd}(2\bar{\not{l}} - \not{p} - \not{q})\Big]^T (\gamma^5)^T\Big). \end{aligned} \tag{E.10}$$

After using

$$C\bar{\gamma}^a C^{-1} = -(\bar{\gamma}^a)^T \quad \text{and} \quad C\gamma^5 C^{-1} = (\gamma^5)^T \tag{E.11}$$

where C is the charge conjugation matrix whose definition and properties are collected in appendix A and C^{-1} is its inverse. They disappear from the trace because $C^{-1}C = \mathbb{1}$ and the above expression reduces to

$$\begin{aligned} \mathcal{N}^{cdba}_{(1)odd} &= Tr\Big((\bar{\not{l}} - \not{p} - \not{q})\bar{\gamma}^a(\bar{\not{l}} - \not{q})\bar{\gamma}^b\bar{\not{l}}\times \\ &\times \Big[(2l-p-q)^c\bar{\gamma}^d + (2l-p-q)^d\bar{\gamma}^c - 2\eta^{cd}(2\bar{\not{l}} - \not{p} - \not{q})\Big]\gamma^5\Big) = \\ &= Tr\Big(\Big[(2l-p-q)^c\bar{\gamma}^d + (2l-p-q)^d\bar{\gamma}^c - 2\eta^{cd}(2\bar{\not{l}} - \not{p} - \not{q})\Big]\gamma^5\times \\ &\qquad \times (\bar{\not{l}} - \not{p} - \not{q})\bar{\gamma}^a(\bar{\not{l}} - \not{q})\bar{\gamma}^b\bar{\not{l}}\Big) \end{aligned} \tag{E.12}$$

where the ciclicity of the trace has been used in the second line. if we now shift the integration variable $l \to -l + p + q$, we obtain

$$\begin{aligned} \mathcal{N}^{cdba}_{(1)odd} &= Tr\Big(\Big[(2l-p-q)^c\bar{\gamma}^d + (2l-p-q)^d\bar{\gamma}^c - 2\eta^{cd}(2\bar{\not{l}} - \not{p} - \not{q})\Big]\gamma^5\times \\ &\qquad \times \bar{\not{l}}\bar{\gamma}^a(\bar{\not{l}} - \not{p})\bar{\gamma}^b(\bar{\not{l}} - \not{p} - \not{q})\Big) \end{aligned} \tag{E.13}$$

and the denominator of the integral (E.9) becomes $l^2(l-p)^2(l-p-q)^2$. Using the anticommutator $\{\bar{\gamma}^a, \gamma^5\} = 0$ we get an overall minus sign

$$\begin{aligned} \mathcal{N}^{cdba}_{(1)odd} &= -Tr\Big(\gamma^5\Big[(2l-p-q)^c\bar{\gamma}^d + (2l-p-q)^d\bar{\gamma}^c - 2\eta^{cd}(2\bar{\not{l}} - \not{p} - \not{q})\Big]\times \\ &\qquad \times \bar{\not{l}}\bar{\gamma}^a(\bar{\not{l}} - \not{p})\bar{\gamma}^b(\bar{\not{l}} - \not{p} - \not{q})\Big) \end{aligned} \tag{E.14}$$

and now integral (E.9) cancels the odd part of the first term of (4.19). By a similar calculation one can show that odd terms in the integral (4.21) cancel. In this way one can prove that there is no parity-odd contribution in the quantum expectation value of the stress tensor and, in turn, in the trace anomaly.

E.3 Trace anomaly

In order to derive (4.39) we insert (4.38) in (4.30) and interpret p and q as derivatives acting on the gauge fields

$$
\begin{aligned}
\langle T^a_{\ a}\rangle = \bar{\eta}_{cd}\langle T^{cd}(x)\rangle = \int d^4y \int d^4z \int \frac{d^4k}{(2\pi)^4}\int \frac{d^4p}{(2\pi)^4}\int \frac{d^4q}{(2\pi)^4}\times \\
\times\, e^{ikx}e^{-ipy}e^{-iqz}\delta(k-p-q)\mathcal{T}^{ab}(p,q)A_a(y)A_b(z) \\
= -\frac{1}{24\pi^2}\int d^4y \int d^4z \int \frac{d^4k}{(2\pi)^4}\int \frac{d^4p}{(2\pi)^4}\int \frac{d^4q}{(2\pi)^4}\times \\
\times\, e^{ikx}e^{-ipy}e^{-iqz}\delta(k-p-q)\Big(q^a p^b - \eta^{ab} p\cdot q\Big)A_a(y)A_b(z) \\
= -\frac{1}{24\pi^2}\int d^4y \int d^4z \Big[\eta^{ab}\partial^y_a\partial^a_z - \partial^a_z\partial^b_y\Big]A_a(y)A_b(z)\times \\
\times \int \frac{d^4k}{(2\pi)^4}\int \frac{d^4p}{(2\pi)^4}\int \frac{d^4q}{(2\pi)^4} e^{ikx}e^{-ipy}e^{-iqz}\delta(k-p-q) \\
= -\frac{1}{24\pi^2}\int d^4y \int d^4z \Big[\,(\partial_a A_b(y))\left(\partial^a A^b(z)\right) - (\partial_a A_b(\,z))\left(\partial^b A^a(y)\right)\Big]\delta(x-y)\delta(x-z) \\
= -\frac{1}{24\pi^2}\Big[\,(\partial_a A_b(x))\left(\partial^a A^b(x)\right) - (\partial_a A_b(\,x))\left(\partial^b A^a(x)\right)\Big] \\
= -\frac{1}{48\pi^2}F^{ab}F_{ab}.
\end{aligned}
\tag{E.15}
$$

E.4 Conservation of the stress tensor

Let us compute

$$ik_c\mathcal{T}^{cdab} = -\frac{1}{8}\int\frac{d^nl}{(2\pi)^n}Tr\Big(\gamma^d \not{l}\gamma^a(\not{l}-\not{p})\gamma^b(\not{l}-\not{k})\Big)\times \tag{E.16a}$$
$$\times\left(\frac{1}{(l-p)^2(l-k)^2}-\frac{1}{l^2(l-p)^2}\right)$$
$$-\frac{1}{8}\int\frac{d^nl}{(2\pi)^n}\left(2l-k\right)^d\left(Tr\frac{\gamma^a(\not{l}-\not{p})\gamma^b(\not{l}-\not{k})}{(l-p)^2(l-k)^2}-Tr\frac{\not{l}\gamma^a(\not{l}-\not{p})\gamma^b}{l^2(l-p)^2}\right) \tag{E.16b}$$
$$+\frac{1}{8}\int\frac{d^nl}{(2\pi)^n}k^bTr\frac{\gamma^d\not{l}\gamma^a(\not{l}-\not{p})}{l^2(l-p)^2} \tag{E.16c}$$
$$+\frac{1}{8}\int\frac{d^nl}{(2\pi)^n}\eta^{bd}Tr\frac{\not{q}\not{l}\gamma^a(\not{l}-\not{p})}{l^2(l-p)^2} \tag{E.16d}$$
$$+\frac{1}{8}\int\frac{d^nl}{(2\pi)^n}k^aTr\frac{\gamma^d\not{l}\gamma^b(\not{l}-\not{q})}{l^2(l-q)^2} \tag{E.16e}$$
$$+\frac{1}{8}\int\frac{d^nl}{(2\pi)^n}\eta^{ad}Tr\frac{\not{p}\not{l}\gamma^b(\not{l}-\not{q})}{l^2(l-q)^2} \tag{E.16f}$$

(E.16c) and (E.16e) have the same structure of the integral appearing in the computation of the vacuum polarization (or photon self-energy) diagram[1] and yield

$$\frac{i}{16\pi^2}(p+q)^b\left(p^ap^d-\eta^{ad}p^2\right)\int_0^1 dxx(x-1)\left(\frac{2}{4-n}-\log f-\gamma+\log 4\pi+O(4-n)\right)+ \tag{E.17a}$$
$$+\frac{i}{16\pi^2}(p+q)^a\left(q^bq^d-\eta^{bd}q^2\right)\int_0^1 dxx(x-1)\left(\frac{2}{4-n}-\log g-\gamma+\log 4\pi+O(4-n)\right) \tag{E.17b}$$

where $f=f(x,p)=p^2x(1-x)$, $g=g(q,x)=q^2x(1-x)$ and γ is the Euler-Mascheroni constant.

Let us consider (E.16d). Using Feynman parametric formula (D.1a) it becomes

$$\frac{1}{8}\int_0^1 dx\int\frac{d^nl}{(2\pi)^n}\eta^{bd}Tr\frac{\not{q}(\not{l}+\not{p}x)\gamma^a(\not{l}+\not{p}(x-1))}{(l^2+f)^2} \tag{E.18}$$

where $f=f(x,p)=p^2x(1-x)$ and we have shifted the integration variable $l\to l+px$.

[1] See [17], [23] and [24].

Then, we evaluate the trace

$$
\begin{aligned}
&Tr\Big(\gamma^e\gamma^f\gamma^a\gamma^g\Big)q_e(l+px)_f(l+p(x-1))_g\\
&Tr\Big(\gamma^e\gamma^f\gamma^a\gamma^g\Big)\Big(q_e l_f l_g+q_e p_f p_g x(x-1)\Big)
\end{aligned}
\tag{E.19}
$$

where in the second line we have neglected term with one l because they vanish by symmetric integration. After replacing $l_f l_g \to \frac{1}{n}\eta_{fg}l^2$, using (B.11) and (D.7) and (D.8) we obtain

$$
-\frac{i}{32\pi^2}\eta^{bd}\Big(2q^a p^2-2p\cdot qp^a\Big)\int_0^1 dxx(x-1)\left(\frac{2}{4-n}-\log f-\gamma+\log 4\pi+O(4-n)\right). \tag{E.20}
$$

By an analogous reasoning we can compute (E.16f) which gives

$$
-\frac{i}{32\pi^2}\eta^{ad}\Big(2p^b q^2-2p\cdot qq^b\Big)\int_0^1 dxx(x-1)\left(\frac{2}{4-n}-\log g-\gamma+\log 4\pi+O(4-n)\right). \tag{E.21}
$$

where $g=g(q,x)=q^2x(1-x)$.

Let us now consider the first integral of (E.16b), after using the Feynman parametric formula and shifting the integration variable $l\to l+p+qx$, this becomes

$$
-\frac{1}{8}\int_0^1 dx\int\frac{d^nl}{(2\pi)^n}\Big(2l+p+q(2x-1)\Big)^d Tr\frac{\gamma^a(\not{l}+\not{q}x)\gamma^b(\not{l}+\not{q}(x-1))}{(l^2+g)^2} \tag{E.22}
$$

where $g=g(q,x)=q^2x(1-x)$. Term proportional to $(2x-1)$ vanish by integration over x. The integral proportional to p^d has the same structure of that one of the photon self-energy and yields

$$
-\frac{i}{16\pi^2}p^d\left(q^aq^b-\eta^{ab}q^2\right)\int_0^1 dxx(x-1)\left(\frac{2}{4-n}-\log g-\gamma+\log 4\pi+O(4-n)\right). \tag{E.23}
$$

Now, we evaluate

$$
-\frac{1}{4}\int_0^1 dx\int\frac{d^nl}{(2\pi)^n}l^d Tr\frac{\gamma^a(\not{l}+\not{q}x)\gamma^b(\not{l}+\not{q}(x-1))}{(l^2+g)^2} \tag{E.24}
$$

Let us compute the trace using (B.11)

$$
\begin{aligned}
&Tr\Big(\gamma^a\gamma^c\gamma^b\gamma^e\Big)\left(l+qx\right)_c\left(l+q(x-1)\right)_e=\\
&=4\Big(\left(l+qx\right)^a\left(l+q(x-1)\right)^b+\left(l+qx\right)^b\left(l+q(x-1)\right)^a+\\
&\qquad-\eta^{ab}\left(l+qx\right)\cdot\left(l+q(x-1)\right)\Big).
\end{aligned}
\tag{E.25}
$$

Since the integral is non zero only if an even power of l appears in the numerator, we can keep terms of this trace with one l, and obtain

$$-\int_0^1 dx(2x-1)\int\frac{d^n l}{(2\pi)^n}\frac{l^d l^a q^b + l^d l^b q^a - \eta^{ab} l^d l\cdot q}{(l^2+g)^2} = 0 \tag{E.26}$$

which vanishes by integration over x. Let us now focus on the second integral of (E.16b) which can be rewritten as

$$\frac{1}{8}\int_0^1 dx\int\frac{d^n l}{(2\pi)^n}\left(2l+p(2x-1)-q\right)^d Tr\frac{(\not{l}+\not{p}x)\gamma^a(\not{l}+\not{p}(x-1))\gamma^b}{(l^2+f)^2} \tag{E.27}$$

with $f=f(x,p)=p^2x(1-x)$. By following a similar reasoning as before, the unique non zero term is

$$-\frac{i}{16\pi^2}q^d\left(p^a p^b-\eta^{ab}p^2\right)\int_0^1 dx x(x-1)\left(\frac{2}{4-n}-\log f-\gamma+\log 4\pi+O(4-n)\right). \tag{E.28}$$

Let us now evaluate (E.16a) and start from the first term

$$-\frac{1}{8}\int\frac{d^n l}{(2\pi)^n}Tr\frac{\left(\gamma^d\not{l}\gamma^a(\not{l}-\not{p})\gamma^b(\not{l}-\not{k})\right)}{(l-p)^2(l-k)^2} \tag{E.29}$$

Introducing Feynman parameter and shifting $l\to l+p+qx$, this becomes

$$-\frac{1}{8}\int_0^1 dx\int\frac{d^n l}{(2\pi)^n}Tr\frac{\gamma^d(\not{l}+\not{p}+\not{q}x)\gamma^a(\not{l}+\not{q}x)\gamma^b(\not{l}+\not{q}(x-1))}{(l^2+g)^2} \tag{E.30}$$

where $g=g(q,x)=q^2x(1-x)$. In evaluating the trace we neglect terms containing an odd number of l because they vanish by symmetric integration. Thus, one has

$$\begin{aligned}&Tr\left(\gamma^d\gamma^e\gamma^a\gamma^f\gamma^b\gamma^g\right)\Big(l_e l_f q_g(x-1)+l_e l_g q_f x+\\&+l_f l_g(p+qx)_e+(p+qx)_e q_f x q_g(x-1)\Big)\end{aligned} \tag{E.31}$$

By symmetric integration we can replace $l_a l_b\to\frac{1}{n}\eta_{ab}l^2$, use (B.12) and compute

$$\begin{aligned}&Tr\left(\gamma^d\gamma^e\gamma^a\gamma^f\gamma^b\gamma^g\right)l_e l_f q_g(x-1)=\\&=\frac{1}{n}l^2 Tr\left(\gamma^d\gamma^e\gamma^a\gamma^f\gamma^b\gamma^g\right)\eta_{ef}q_g(x-1)=\\&=\frac{1}{n}l^2 2^{\frac{n}{2}}\Big[\eta^{ab}(2-n)q^d(x-1)+\\&\quad+\eta^{db}(n-2)q^a(x-1)+\eta^{da}(2-n)q^b(x-1)\Big]=\\&=\left(\frac{2}{n}-1\right)2^{\frac{n}{2}}l^2\Big[\eta^{ab}q^d(x-1)-\eta^{db}q^a(x-1)+\eta^{da}q^b(x-1)\Big]\end{aligned} \tag{E.32}$$

$$
\begin{aligned}
&Tr\Big(\gamma^d\gamma^e\gamma^a\gamma^f\gamma^b\gamma^g\Big)l_f l_g(p+qx)_e=\frac{1}{n}l^2Tr\Big(\gamma^d\gamma^e\gamma^a\gamma^f\gamma^b\gamma^g\Big)\eta_{fg}(p+qx)_e=\\
&=\left(\frac{2}{n}-1\right)2^{\frac{n}{2}}l^2\Big[\eta^{ab}(p+qx)^d-\eta^{da}(p+qx)^b+\eta^{db}(p+qx)^a\Big]
\end{aligned}
\tag{E.33}
$$

$$
\begin{aligned}
&Tr\Big(\gamma^d\gamma^e\gamma^a\gamma^f\gamma^b\gamma^g\Big)l_e l_g q_f x=\frac{1}{n}l^2Tr\Big(\gamma^d\gamma^e\gamma^a\gamma^f\gamma^b\gamma^g\Big)\eta_{eg}q_f x=\\
&=\left(\frac{2}{n}-1\right)2^{\frac{n}{2}}l^2\Big[\eta^{db}q^a x+\eta^{da}q^b x-\eta^{ab}q^d x\Big]
\end{aligned}
\tag{E.34}
$$

Putting everything together

$$
\left(\frac{2}{n}-1\right)2^{\frac{n}{2}}l^2\Big[\eta^{ab}\left(q^d(x-1)+p^d\right)+\eta^{da}\left(-p^b+q^b(x-1)\right)+\eta^{db}\left(q^a(x+1)+p^a\right)\Big]
\tag{E.35}
$$

Integrating over l using (D.7), one obtains

$$
\begin{aligned}
&\frac{i}{32\pi^2}\int_0^1 dx\Big(\eta^{ab}\left(q^d(x-1)+p^d\right)+\eta^{da}\left(-p^b+q^b(x-1)\right)+\\
&+\eta^{db}\left(q^a(x+1)+p^a\right)\Big)g\left(\frac{2}{4-n}-\log g-\gamma+\log 4\pi+O(4-n)\right)
\end{aligned}
\tag{E.36}
$$

From the term with no l we obtain

$$
\begin{aligned}
&Tr\Big(\gamma^d\gamma^e\gamma^a\gamma^f\gamma^b\gamma^g\Big)=(p+qx)_e\,q_f q_g x(x-1)=\\
&=2^{\frac{n}{2}}x(x-1)\Big(\,(p+qx)^d\left(2q^aq^b-\eta^{ab}q^2\right)+\\
&\quad+(p+qx)^a\left(2q^dq^b-\eta^{db}q^2\right)+\eta^{ad}\left(q^2(p+qx)^b-2q^b(p+qx)\cdot q\right)\Big)
\end{aligned}
\tag{E.37}
$$

and after integrating over l using (D.8)

$$
\begin{aligned}
&-\frac{i}{32\pi^2}\int_0^1 dx x(x-1)\times\\
&\times\Big(\,(p+qx)^d\left(2q^aq^b-\eta^{ab}q^2\right)+(p+qx)^a\left(2q^dq^b-\eta^{db}q^2\right)+\\
&+\eta^{ad}\left(q^2(p+qx)^b-2q^b(p+qx)\cdot q\right)\Big)\times\\
&\left(\frac{2}{4-n}-\log g-\gamma+\log 4\pi+O(4-n)\right)
\end{aligned}
\tag{E.38}
$$

By adding (E.17b), (E.21), (E.23), (E.36) and (E.38) we obtain

$$
\begin{aligned}
& -\frac{i}{32\pi^2}\int_0^1 dx x(x-1)\times \\
& \times\Big(4p^d\left(q^a q^b-\eta^{ab}q^2\right)+4\eta^{ad}\left(q^2p^b-q^bp\cdot q\right)+ \\
& \quad (2x-1)\left(-\eta^{ad}q^bq^2-\eta^{ab}q^dq^2+2q^aq^bq^d+\eta^{bd}q^aq^2\right)\Big)\times \\
& \qquad \left(\frac{2}{4-n}-\log g-\gamma+\log 4\pi+O(4-n)\right)
\end{aligned}
\tag{E.39}
$$

and after integrating over x

$$
\begin{aligned}
& \frac{i}{24\pi^2}\Big(p^d\left(q^aq^b-\eta^{ab}q^2\right)+\eta^{ad}\left(q^2p^b-q^bp\cdot q\right)\Big)\times \\
& \qquad \times\left(\frac{2}{4-n}+\frac{5}{3}-\log q^2-\gamma+\log 4\pi+O(4-n)\right).
\end{aligned}
\tag{E.40}
$$

Let us now consider the second term of (E.16a)

$$
\frac{1}{8}\int\frac{d^nl}{(2\pi)^n}Tr\frac{\gamma^d \not{l}\gamma^a(\not{l}-\not{p})\gamma^b(\not{l}-\not{k})}{l^2(l-p)^2}
\tag{E.41}
$$

Making use of Feynman parametric formula for rewriting the denominator and shifting the integration variable $l\to l+px$, this becomes

$$
\frac{1}{8}\int_0^1 dx\int\frac{d^nl}{(2\pi)^n}Tr\frac{\gamma^d(\not{l}+\not{p}x)\gamma^a(\not{l}+\not{p}(x-1))\gamma^b(\not{l}-\not{q}+\not{p}(x-1))}{(l^2+f)^2}
\tag{E.42}
$$

As before, we keep only terms containing an even number of l, and they are

$$
\begin{aligned}
& Tr\Big(\gamma^d\gamma^e\gamma^a\gamma^f\gamma^b\gamma^g\Big)\Big(l_el_f(-q+p(x-1))_g+l_el_gp_f(x-1)+ \\
& \qquad +l_fl_gp_ex+p_expf(x-1)(-q+p(x-1))_g\Big).
\end{aligned}
\tag{E.43}
$$

Then, we compute

$$
\begin{aligned}
& Tr\Big(\gamma^d\gamma^e\gamma^a\gamma^f\gamma^b\gamma^g\Big)l_el_f(-q+p(x-1))_g= \\
& =\Big(\frac{2}{n}-1\Big)2^{\frac{n}{2}}l^2\Big[\eta^{ab}(p(x-1)-q)^d-\eta^{db}(p(x-1)-q)^a+\eta^{da}(p(x-1)-q)^b\Big]
\end{aligned}
\tag{E.44}
$$

$$
\begin{aligned}
& Tr\Big(\gamma^d\gamma^e\gamma^a\gamma^f\gamma^b\gamma^g\Big)l_el_gp_f(x-1)= \\
& =\Big(\frac{2}{n}-1\Big)2^{\frac{n}{2}}l^2\Big[\eta^{db}p^a(x-1)+\eta^{da}p^b(x-1)-\eta^{ab}p^d(x-1)\Big]
\end{aligned}
\tag{E.45}
$$

$$
\begin{aligned}
&Tr\Big(\gamma^d\gamma^e\gamma^a\gamma^f\gamma^b\gamma^g\Big)l_f l_g p_e x = \\
&= \Big(\frac{2}{n}-1\Big)2^{\frac{n}{2}}l^2\Big[\eta^{ab}p^d x - \eta^{da}p^b x + \eta^{db}p^a x\Big]
\end{aligned}
\tag{E.46}
$$

After putting everything together and integrating over l, one has

$$
\begin{aligned}
&-\frac{i}{32\pi^2}\int_0^1 dx\Big(\eta^{ab}\left(p^d x - q^d\right) + \eta^{da}\left(p^b(x-2) - q^b\right) + \\
&+\eta^{db}\left(p^a x + q^a\right)\Big) f\left(\frac{2}{4-n} - \log f - \gamma + \log 4\pi + O(4-n)\right).
\end{aligned}
\tag{E.47}
$$

From the term with no l one obtains

$$
\begin{aligned}
&Tr\Big(\gamma^d\gamma^e\gamma^a\gamma^f\gamma^b\gamma^g\Big)p_e x p_f (x-1)(-q+p(x-1))_g = \\
&= 2^{\frac{n}{2}}x(x-1)\Big(\,(-q+p(x-1))^b\left(2p^a p^d - \eta^{ad}p^2\right) + \\
&+(-q+p(x-1))^d\left(2p^a p^b - \eta^{ab}p^2\right) - \eta^{bd}p^a p^2(x-1) + \eta^{bd}\left(2p^a p\cdot q - q^a p^2\right)\Big)
\end{aligned}
\tag{E.48}
$$

and integrating over l

$$
\begin{aligned}
&\frac{i}{32\pi^2}\int_0^1 dx x(x-1)\times \\
&\quad\times\Big(\,(-q+p(x-1))^b\left(2p^a p^d - \eta^{ad}p^2\right) + (-q+p(x-1))^d\left(2p^a p^b - \eta^{ab}p^2\right) + \\
&\qquad - \eta^{bd}p^a p^2(x-1) + \eta^{bd}\left(2p^a p\cdot q - q^a p^2\right)\Big)\times \\
&\qquad\quad\times\left(\frac{2}{4-n} - \log f - \gamma + \log 4\pi + O(4-n)\right).
\end{aligned}
\tag{E.49}
$$

Adding (E.17a), (E.20), (E.28), (E.47) and (E.49), we obtain

$$
\begin{aligned}
&\frac{i}{32\pi^2}\int_0^1 dx x(x-1)\times \\
&\times\Big(-4q^d\left(p^a p^b - \eta^{ab}p^2\right) - 4\eta^{bd}\left(q^a p^2 - p^a p\cdot q\right) + \\
&\quad + (2x-1)\left(2p^a p^b p^d - \eta^{ab}p^2 p^d - \eta^{bd}p^a p^2 - p^b\eta^{ad}p^2\right)\Big)\times \\
&\qquad\times\left(\frac{2}{4-n} - \log f - \gamma + \log 4\pi + O(4-n)\right)
\end{aligned}
\tag{E.50}
$$

and integrating over x

$$\begin{aligned}&\frac{i}{24\pi^2}\Big(q^d\left(p^a p^b-\eta^{ab}p^2\right)+\eta^{bd}\left(q^a p^2-p^a p\cdot q\right)\Big)\times\\&\qquad\times\left(\frac{2}{4-n}+\frac{5}{3}-\log p^2-\gamma+\log 4\pi+O(4-n)\right).\end{aligned}\tag{E.51}$$

The final result in momentum space is given by (E.51) and (E.40) which lead to (4.45).

Bibliography

[1] F. Bastianelli and M. Broccoli, *On the trace anomaly of a Weyl fermion in a gauge background*, Eur. Phys. J. C **79** (2019), arXiv:1808.03489v2 [hep-th].

[2] L. Bonora, S. Giaccari and B. Lima de Souza, *Trace anomalies in chiral teories revisited*, JHEP **07** (2014) 117, arXiv:1403.2606v3 [hep-th].

[3] L. Bonora, A. D. Pereira and B. Lima de Souza, *Regularization of the energy-momentum tensor correlators correlators and parity-odd terms*, JHEP **06** (2015) 024, arXiv:1503.03326v2 [hep-th].

[4] L. Bonora, M. Cvitan, P. Dominis Prester, A. Duarte Pereira, S. Giaccari and T. Štemberga, *Axial gravity, massless fermions and trace anomalies*, Eur. Phys. J. C **77** (2017) 511, arXiv:1703.10473v3 [hep-th].

[5] L. Bonora, M. Cvitan, P. Dominis Prester, S. Giaccari, M. Paulišić and T. Štemberga, *Axial gravity: a non-pertirbative approach to split anomalies*, Eur. Phys. J. C **78** (2018) 8, 652, arXiv:1807.01249 [hep-th].

[6] F. Bastianelli and R. Martelli, *On the trace anomaly of a Weyl fermion*, JHEP **11** (2016) 178, arXiv:1610.02304 [hep-th].

[7] F. Bastianelli and M. Broccoli, *Axial gravity and anomalies of fermions*, Eur. Phys. J. C **80** (2020) 276, arXiv:1911.02271v1 [hep-th].

[8] M. B. Fröb and J. Zahn, *Trace anomaly for chiral fermions via Hadamard subtraction*, JHEP **10** (2019) 223, arXiv:1904.10982 [hep-th].

[9] S. Abdallah, S. A. Franchino-Viñas and M. B. Fröb, *Trace anomlay for Weyl fermions using the Breitenlohner-Maison scheme for γ_**, JHEP **03** (2021) 271, arXiv:2101.11382 [hep-th].

[10] H. Godazgar and H. Nicolai, *A rederivation of the conformal anomaly for spin-$\frac{1}{2}$*, Class. Quant. Grav. **35** (2018) 10, 105013, arXiv:1801.01728v1 [hep-th].

[11] Y. Nakayama, *CP-violating CFT and trace anomaly*, Nucl. Phys. B **859** (2012) 288-298, arXiv:1201.3428 [hep-th].

[12] L. Bonora, *Perturbative and non-perturbative trace anomalies*, arXiv:2107.07918 [hep-th].

[13] P. Breitenlohner and D. Maison, *Dimensional Renormalization and the Action Principle*, Commun. Math. Phys. **52**, 11-38 (1977).

[14] R. A. Bertlmann, *Anomalies in Quantum Field Theory*, Clarendon Press, Oxford (1996).

[15] F.Bastianelli and P. van Nieuwenhuizen, *Path Integrals and Anomalies in Curved Space*, Cambridge University Press (2006).

[16] M. J. Duff, *Twenty Years of the Weyl Anomaly*, arXiv:hep-th/9308075v1.

[17] M. D. Schwartz, *Quantum Field Theory and the Standard Model*, Cambridge University Press (2014).

[18] J. Wess and B. Zumino, *Consequences of Anomalous Ward Identities*, Phys. Lett. **37B** (1971), 95-97.

[19] F. Bastianelli and M. Broccoli, *Weyl fermions in a non-abelian gauge background and trace anomalies*, JHEP **10** (2019) 241, arXiv:1908.03750v1 [hep-th].

[20] J. F. Ashmore, *A Method of Gauge-Invariant Regularization*, Nuovo Cimento Lett. **4** (1972) 289.

[21] C. G. Bollini and J. J. Giambiagi, *Dimensional Renormalization: The number of Dimensions as a Regulating Parameter*, Nuovo Cimento **12B** (1972) 20.

[22] G. 't Hooft and M. Veltman, *Regularization and Renormalization of Gauge Fields*, Nucl. Phys. **B44** (1972) 189.

[23] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*, Persues Books (1995).

[24] M. Srednicki, *Quantum Field Theory*, Cambridge University Press (2007).

[25] P. A. Baikov and V. A. Il'in, *Status of γ^5 in dimensional regularization*, Theor. Math. Phys. **88** (1991) 789-809.

[26] G. Bonneau, *Consistency in dimensional regularization with γ_5*, Phys. Lett. **96B** (1980), 147.

[27] S. Gottlieb and J. T. Donohue, *Axial-vector current and dimensional regularization*, Phys. Rev. **D20** (1979), 3378.

[28] G. Rufa, *Mathematical foundations of the dimensional method*, Annalen Phys. **47** (1990), 6-26.

[29] M. G. Green, J. H. Schwarz and E. Witten, *Superstring Theory, Volume 2: Loop Amplitudes, Anomalies and Phenomenology*, Cambridge University Press (2012).

[30] M. Forger and H. Romer, *Currents and the energy momentum tensor in classical field theory: a fresh look at an old problem*, Annals Phys. **309** (2004) 306, arXiv:hep-th/0307199.

[31] H. Bélusca-Maïto, A. Ilakovac, M. Mađor-Božinović and D. Stöckinger, *Dimensional Regularization and Breitenlonher-Maison/'t Hooft-Veltman Scheme for γ_5 applied to Chiral YM Theories: Full One-Loop Counterterm and RGE Structure*, JHEP **08** (2020) 024, arXiv:2004.14398 [hep-ph].

[32] M. Giannotti and E. Mottola, *The Trace Anomaly and Massless Scalar Degrees of Freedom in Gravity*, Phys. Rev. D **79** (2009) 045014, arXiv:0812.0351 [hep-th].

[33] R. Armillis, C. Coriano and L. Delle Rose, *Conformal Anomalies and the Gravitational Effective Action: The TJJ Correlator for a Dirac Fermion*, Phys. Rev. D **81** (2010) 085001, arXiv:0910.3381 [hep-ph].

[34] S. K. Jones and C. H. Llewellyn Smith, *Leptoproduction of Supersymmetric Particles*, Nucl. Phys. B **217** (1983) 145-171.

[35] H.E. Haber and G. L. Kane, *The search for supersymmetry: Probing physics beyond the standard model, Appendix D*, Phys. Rept. 117 (1985) 75.

[36] J. Gluza and M. Zraek, *Feynman rules for Majorana-neutrino interactions*, Phys. Rev D **45** (1992) 1693.

[37] E. Akhmedov, *Majorana neutrinos and other Majorana particles: Theory and experiment*, arXiv:1412.3320 [hep-ph].