Alma Mater Studiorum · Università di Bologna

Scuola di Scienze
Corso di Laurea in Matematica

ADO'S THEOREM

Relatore:
Chiar.ma Prof.ssa
Nicoletta Cantarini

Presentata da:
Francesca Paganelli

Anno Accademico 2020/2021

To my grandparents

Introduction

This thesis is dedicated to the proof of Ado's Theorem for finite-dimensional Lie algebras over an algebraically closed field $\mathbb{K}$ of characteristic 0. In order to achieve this result we will describe some basic, fundamental properties of Lie algebras (see [1]). A Lie algebra is a $\mathbb{K}$-vector space endowed with a bilinear, antisymmetric product called *bracket*, which satisfies the Jacobi identity and is indicated by $[\cdot, \cdot]$.

In the first chapter we will give all the fundamental definitions necessary to study Lie algebras and we will present some classic examples. The notion of *representation* is probably one of the most important. A representation of a Lie algebra L is a Lie algebras homomorphism (that is a linear map preserving the bracket) of the form $\Phi : L \longrightarrow \mathfrak{gl}(V)$, where V is a $\mathbb{K}$-vector space and $\mathfrak{gl}(V)$ is the Lie algebra of endomorphisms of V. In addition, a representation is said to be *faithful* when it is injective. The definitions of Lie algebra and representation suffice to state Ado's Theorem:

Theorem 0.0.1 (Ado's Theorem)**.** *Let L be a finite-dimensional Lie algebra. Then there exists a finite-dimensional faithful representation of L.*

Observe that the theorem claims the existence of a *finite-dimensional* representation, namely the vector space V is finite-dimensional and so the same goes for $\mathfrak{gl}(V)$; furthermore the map is faithful and by the first Theorem of homomorphism (which, as one may expect, holds also for Lie algebras) this means that L is isomorphic to its image via the representation. In other words, Ado's theorem states that we can view any finite-dimensional Lie algebra L as a subalgebra (a subspace closed with respect to the bracket) of a Lie algebra consisting of endomorphisms. Of course, if V is an n-dimensional $\mathbb{K}$-vector space, then $\mathfrak{gl}(V)$ is isomorphic, as a Lie algebra, to $\mathfrak{gl}(n, \mathbb{K})$, so roughly speaking

L can be identified with a Lie algebra of matrices.
We will provide a proof of Ado's theorem following Terence Tao [2]. The proof develops gradually, from the easy case of *semisimple* algebras to the general case, going through the *nilpotent* and *solvable* cases. Instead of first listing all the preliminary stuff and next show the entire proof, we will divide the proof in different chapters, adding step by step the necessary ingredients to obtain the theorem in a little more complex context. So, in the second chapter we will analyze the *universal enveloping algebra* $\mathfrak{U}(L)$ of a Lie algebra L. We will use this algebraic structure heavily. Furthermore we will present a *weak version* of Ado's Theorem (namely we will claim that for any Lie algebra there exists a faithful representation on a space of endomorphisms) and we will prove the theorem in the case L is a semisimple or abelian algebra. The third and fourth chapters are dedicated to the proof of the theorem for nilpotent and solvable algebras. Treating these type of algebras will lead us to prove some other fundamental results in Lie algebra theory, like Engel's Theorem. Lastly, in the fifth chapter we will be able to prove Ado's theorem for any Lie algebra, but in order to do so we will need Levi decomposition. Thus, this thesis has been an opportunity to study the foundations of the important mathematical branch of Lie algebras, which is wide and sophisticated by itself, but is also fundamental in order to study other mathematical and physical theories.

We provide some information [3] about the history of Ado's Theorem. Igor Dmitrievich Ado (1910-1983) was born in Kazan (Russia) and he worked in that city during all his life. He studied at Kazan State University, at the faculty of mathematics and physics. He attended the PhD study in the same university, under the supervision of N.G. Chebotarev. In 1935, in his PhD thesis, he presented the proof of what we now call *Ado's Theorem.* Later, Ado himself and other mathematicians found different proofs and more complete statements. For example in 1948 K. Iwasawa proved the theorem for Lie algebras over a field of positive characteristic p. Ado worked as professor in Kazan State University and later in Kazan State Chemical Technological Institute. He is described as a wonderful teacher, loved and respected by colleagues and students.

Introduzione

Questa tesi è dedicata allo studio della dimostrazione del Teorema di Ado per algebre di Lie di dimensione finita su un campo $\mathbb{K}$ algebricamente chiuso e di caratteristica 0. Per ottenere questo risultato descriveremo alcune proprietà di base fondamentali delle algebre di Lie (si veda [1]). Un'algebra di Lie è un $\mathbb{K}$-spazio vettoriale dotato di un prodotto bilineare e antisimmetrico detto *bracket* che soddisfa l'identità di Jacobi e si denota con $[\cdot, \cdot]$.

Nel primo capitolo daremo tutte le definizioni fondamentali necessarie per lo studio delle algebre di Lie e presenteremo alcuni classici esempi. La nozione di *rappresentazione* è probabilmente una delle più importanti. Una rappresentazione di un'algebra di Lie è un omomorfismo di algebre di Lie (cioè una mappa lineare che preserva il bracket) della forma $\Phi : L \longrightarrow \mathfrak{gl}(V)$, dove V è un $\mathbb{K}$-spazio vettoriale e $\mathfrak{gl}(V)$ è l'algebra di Lie degli endomorfismi di V. Inoltre, una rappresentazione è detta *fedele* quando è iniettiva. Le definizioni di algebra di Lie e rappresentazione sono sufficienti per enunciare il Teorema di Ado:

Teorema 0.0.1 (Teorema di Ado)**.** *Sia L un'algebra di Lie di dimensione finita. Allora esiste una rappresentazione fedele e di dimensione finita di L.*

Osserviamo che il teorema afferma l'esistenza di una rappresentazione *finito dimensionale*, cioè lo spazio vettoriale V è di dimensione finita e quindi lo stesso vale per $\mathfrak{gl}(V)$; inoltre la mappa è fedele e per il primo Teorema di omomorfismo (che come ci si aspetta vale anche per algebre di Lie) questo significa che L è isomorfa alla sua immagine tramite la rappresentazione. In altre parole, il Teorema di Ado afferma che possiamo vedere ogni algebra di Lie di dimensione finita come una sottoalgebra (un sottospazio chiuso rispetto

al bracket) di un'algebra di Lie fatta di endomorfismi. Naturalmente, se V è un $\mathbb{K}$-spazio vettoriale V di dimensione n, $\mathfrak{gl}(V)$ è isomorfo, come algebra di Lie, a $\mathfrak{gl}(n, \mathbb{K})$ quindi sostanzialmente L può essere identificata con l'algebra di Lie delle matrici.
Forniremo una dimostrazione del Teorema di Ado seguendo Terence Tao [2]. La dimostrazione si sviluppa gradualmente, dal facile caso delle algebra *semisemplici* a quello generale, passando per il caso *nilpotente* e quello *risolubile*. Invece di elencare prima tutti i concetti preliminari e successivamente mostrare la dimostrazione tutta intera, divideremo la dimostrazione in diversi capitoli, aggiungendo passo dopo passo gli ingredienti necessari a dimostrare il teorema in casi via via più difficili. Così, nel secondo capitolo analizzeremo l'*algebra universale inviluppante* $\mathfrak{U}(L)$ di un'algebra di Lie L. Useremo pesantemente questa struttura. In aggiunta, presenteremo una *versione debole* del Teorema di Ado (cioè mostreremo che per ogni algebra di Lie esiste una rappresentazione fedele) e dimostreremo il teorema nei casi semisemplice e abeliano. Il terzo e il quarto capitolo sono dedicati alla dimostrazione del teorema per algebre nilpotenti e risolubili. Trattare questi tipi di algebre ci permetterà di mostrare altri risultati fondamentali della teoria delle algebre di Lie, come il Teorema di Engel. Infine, nel quinto capitolo saremo in grado di dimostrare il teorema per ogni algebra di Lie, ma per fare ciò avremo bisogno della decomposizione di Levi. Perciò, questa tesi offre la possibilità di studiare le basi della branca matematica delle algebre di Lie che è vasta, sofisticata e fondamentale per altre teorie matematiche e fisiche.

Diamo ora qualche informazione sulla storia del Teorema di Ado. Igor Dmitrievich Ado (1910-1983) nacque a Kazan (Russia) e lavorò in quella città per tutta la sua vita. Studiò all'Università Statale di Kazan, presso la facoltà di matematica e fisica. Svolse il dottorato nella medesima università, sotto la supervisione di N.G. Chebotarev. Nel 1935 presentò come tesi di dottorato quello che noi oggi chiamiamo *Teorema di Ado*. Successivamente, Ado e altri matematici trovarono altre dimostrazioni ed enunciati più completi per tale risultato. Per esempio, nel 1948 K. Iwasawa dimostrò il teorema per algebre di Lie su un campo di caratteristica p positiva. Ado fu professore all'Università Statale di Kazan e in seguito presso l'Istituto di Tecnologia Chimica di Kazan. Viene descritto come un professore fantastico, amato e rispettato da colleghi e alunni.

Contents

Chapter 1

Basic notions

In this chapter we introduce Lie algebras and some of their fundamental properties.

1.1 Lie algebras

Definition 1.1.1. *A* ***Lie algebra*** *L is a vector space over a field $\mathbb{K}$ equipped with a bilinear antisymmetric form $[\cdot,\cdot] : L \times L \longrightarrow L$ called* ***bracket*** *which satisfies the Jacobi identity*

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \quad \forall x,y,z \in L$$

Remark 1.1.1. *Because of antisymmetry, if char$\mathbb{K} \neq 2$, then $\forall x \in L$ $[x,x] = 0$.*

From now on, we assume that $\mathbb{K}$ has characteristic zero, all vector spaces will be $\mathbb{K}$-vector spaces.
In order to make the definition of Lie algebra clearer, we present some standard examples.

Example 1.1.1. *The vector space $\mathbb{R}^3$ with the operation of cross product is a Lie algebra.*

Example 1.1.2. *Let A be an associative algebra, i.e. a $\mathbb{K}$-vector space equipped with a bilinear, associative product $A \times A \longrightarrow A$ sending $(x,y) \mapsto xy$. Then A has a natural structure of Lie algebra given by $[x,y] := xy - yx$ (the bracket defined in this way is called* ***commutator****). Indeed, a simple computation shows that:*

- *bilinearity holds since the product in A is bilinear;*

- $[x, y] = xy - yx = -(yx - xy) = -[y, x] \; \forall x, y \in A$;

- $$\begin{aligned}[x,[y,z]]+[z,[x,y]]+[y,[z,x]] &= [x, yz - zy] + [z, xy - yx] + [y, zx - xz] \\ &= [x, yz] - [x, zy] + [z, xy] - [z, yx] + [y, zx] - [y, xz] \\ &= x(yz) - (yz)x - x(zy) + (zy)x + z(xy) - (xy)z \\ &\quad - z(yx) + (yx)z + y(zx) - (zx)y - y(xz) + (xz)y \\ &= 0\end{aligned}$$

  due to associativity.

Example 1.1.3. *Given a finite dimensional vector space V over a field $\mathbb{K}$, we denote by End(V) the set of endomorphisms of V. End(V) is a finite-dimensional vector space and also an associative algebra, with the product given by composition. Therefore, by Example 1.1.2 we can consider the associated Lie algebra structure. End(V) regarded as a Lie algebra with bracket $[x, y] = xy - yx$, ($xy = x \circ y$), is denoted by $\mathfrak{gl}(V)$ and it is called* ***general linear algebra****.*

Definition 1.1.2. *Given a Lie algebra L, a* ***subalgebra*** *K of L is a vector subspace which is closed under the Lie bracket.*

Definition 1.1.3. *A subspace I of a Lie algebra L is called an* ***ideal*** *of L if $[x, y] \in I, \forall x \in I, \forall y \in L$. The* ***centre*** *of L is the ideal $\boldsymbol{Z(L)} = \{z \in L \mid [x, z] = 0 \; \forall x \in L\}$. Given a subset $S \subset L$, the* ***normalizer*** *of S in L is the set $\boldsymbol{N_L(S)} = \{x \in L \,|\, [x, y] \in S, \; \forall y \in S\}$. A Lie algebra is* ***abelian*** *if $[x, y] = [y, x] \; \forall x, y \in L$. The* ***derived algebra*** *of L is the span of all brackets of elements in L and it is denoted by $\boldsymbol{[L, L]}$.*

Remark 1.1.2. *One can check that $[L, L]$ is an ideal of L and that $N_L(S)$ is a subalgebra. Also, L is abelian if and only if $Z(L) = L$.*

Definition 1.1.4. *Let L, H be Lie algebras. A linear map $\Phi : L \longrightarrow H$ is a* ***homomorphism*** *of Lie algebras if $\Phi([x, y]) = [\Phi(x), \Phi(y)] \; \forall \, x, y \; \in L$. An* ***isomorphism*** *of Lie algebras is an homomorphism such that it is invertible and the inverse function is still an homomorphism.*

Remark 1.1.3. *As we would expect, if* $\Phi : L \longrightarrow H$ *is a Lie algebra homomorphism, then* $Ker(\Phi)$ *is an ideal of* L *and* $Im(\Phi)$ *is a subalgebra of* H*. Furthermore, the first theorem of homomorphism holds, i.e.* $L/Ker(\Phi) \cong Im(\Phi)$*. As in other algebraic structures, given an ideal* $I \subset L$ *we can indeed define the* ***quotient algebra*** L/I*, where the bracket is well defined since* I *is an ideal.*

Example 1.1.4. *In Example 1.1.3, suppose* $dim(V) = n$*, fix a basis of* V*, then we can identify End(V) with the set* $M(n, \mathbb{K})$ *of* $n \times n$ *matrices with entries in* $\mathbb{K}$*, associating each linear transformation with its matrix. Again, defining* $[x, y] = xy - yx$*, where now* xy *is the standard matrix product, we give* $M(n, \mathbb{K})$ *the structure of a Lie algebra, which is denoted by* $\mathfrak{gl}(n, \mathbb{K})$*. Of course,* $\mathfrak{gl}(V)$ *and* $\mathfrak{gl}(n, \mathbb{K})$ *are isomorphic Lie algebras.*

Example 1.1.5. *Consider the following subset of* $\mathfrak{gl}(n, \mathbb{K})$

$$\mathfrak{sl}(n, \mathbb{K}) = \{a \in \mathfrak{gl}(n, \mathbb{K}) \,|\, tr(a) = 0\}.$$

Since $tr(a + b) = tr(a) + tr(b)$ *and* $tr(a \cdot b) = tr(b \cdot a)$*,* $\mathfrak{sl}(n, \mathbb{K})$ *is subalgebra of* $\mathfrak{gl}(n, \mathbb{K})$*.*

Example 1.1.6. *A* ***derivation*** *of a Lie algebra* L *is linear map* $\delta : L \longrightarrow L$ *which satisfies the Leibniz rule:* $\delta([a, b]) = [\delta(a), b] + [a, \delta(b)]$*. The set of derivations of* L *is denoted* $\boldsymbol{Der(L)}$ *and it is a subalgebra of* $\mathfrak{gl}(V)$*. Indeed,* $\forall\ \delta, \tilde{\delta} \in Der(L)$ *and* $a, b \in L$*, we have:*

$$\begin{aligned}
\delta\tilde{\delta}([a,b]) - \tilde{\delta}\delta([a,b]) &= \delta([\tilde{\delta}(a), b] + [a, \tilde{\delta}(b)]) - \tilde{\delta}([\delta(a), b] + [a, \delta(b)]) \\
&= \delta([\tilde{\delta}(a), b]) + \delta([a, \tilde{\delta}(b)]) - \tilde{\delta}([\delta(a), b]) - \tilde{\delta}([a, \delta(b)]) \\
&= [\delta\tilde{\delta}(a), b] + [\tilde{\delta}(a), \delta(b)] + [\delta(a), \tilde{\delta}(b)] + [a, \delta\tilde{\delta}(b)] - [\tilde{\delta}\delta(a), b] \\
&\quad - [\delta(a), \tilde{\delta}(b)] - [\tilde{\delta}(a), \delta(b)] - [a, \tilde{\delta}\delta(b)] \\
&= [\delta\tilde{\delta}(a), b] + [a, \delta\tilde{\delta}(b)] - ([\tilde{\delta}\delta(a), b] + [a, \tilde{\delta}\delta(b)]) \\
&= [[\delta, \tilde{\delta}](a), b] + [a, [\delta, \tilde{\delta}](b)].
\end{aligned}$$

Definition 1.1.5. *A Lie algebra is* ***simple*** *if it is not abelian and has no proper ideal.*

Definition 1.1.6. *Let L be Lie algebra and H, K two ideals of L. Assume $L = H \oplus K$ as a direct sum of vector spaces. We call it a direct sum of Lie algebras if $[H, K] = 0$.*

Example 1.1.7. *We claim that $\mathfrak{gl}(n, \mathbb{K}) = \mathfrak{sl}(n, \mathbb{K}) \oplus \mathbb{K}Id$. Indeed, we may consider the trace operator $tr : \mathfrak{gl}(n, \mathbb{K}) \longrightarrow \mathbb{K}$ between vector spaces and observe that $Ker(tr) = \mathfrak{sl}(n, \mathbb{K})$. So, by rank-nullity theorem, $dim(\mathfrak{sl}(n, \mathbb{K})) = n^2 - 1$. Therefore, we can search for a direct complement, which has to be one dimensional and with zero trace. Then of course we choose $\mathbb{K}Id$. Moreover, $[\mathfrak{sl}(n, \mathbb{K}), \mathbb{K}Id] = 0$, so $L = \mathfrak{sl}(n, \mathbb{K}) \oplus \mathbb{K}Id$ as Lie algebras.*
We have just proved that $dim(\mathfrak{sl}(n, \mathbb{K}))$ is $n^2 - 1$. We can also present a basis of $\mathfrak{sl}(n, \mathbb{K})$, i.e. the set consisting of the elementary matrices e_{ij} for $i \neq j$ together with the matrices of the form $e_{ii} - e_{i+1,i+1}$ for $i \leq 1 \leq n - 1$. Now we are showing that the derived algebra of $\mathfrak{gl}(n, \mathbb{K})$ is $\mathfrak{sl}(n, \mathbb{K})$ and so in particular $\mathfrak{sl}(n, \mathbb{K})$ is an ideal. Indeed, $\forall x, y \in \mathfrak{gl}(n, \mathbb{K})$, $tr([x, y]) = tr(xy) - tr(yx) = tr(xy) - tr(xy) = 0$, so $[\mathfrak{gl}(n, \mathbb{K}), \mathfrak{gl}(n, \mathbb{K})] \subset \mathfrak{sl}(n, \mathbb{K})$. On the other hand, any element of $\mathfrak{sl}(n, \mathbb{K})$ can be obtained as bracket of two elements of $\mathfrak{gl}(n, \mathbb{K})$. As a matter of fact, if $i \neq j$, $[e_{it}, e_{tj}] = e_{ij}$ and $[e_{i,i+1}, e_{i+1,i}] = e_{ii} - e_{i+1,i+1}$.

Example 1.1.8. *$\mathfrak{sl}(2, \mathbb{K})$ is a simple subalgebra of $\mathfrak{gl}(2, \mathbb{K})$. Fix the following basis of $\mathfrak{sl}(2, \mathbb{K})$:*

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

called the ***standard basis****, and observe that*

$$[x, y] = h \quad [h, x] = 2x \quad [h, y] = -2y. \tag{1.1}$$

Let $L = \mathfrak{sl}(2, \mathbb{K})$. Suppose $0 \neq I \subseteq L$ is an ideal and let $v = ax + by + ch$ be a nonzero element of I. If $b \neq 0$, $[x, v] \in I$ and $[x, v] = b[x, y] + c[x, h] = bh - 2cx \in I$. Applying again the bracket with x we get $[x, bh - 2cx] = b[x, h] = -2bx \in I$, so $x \in I$. By (1.1) it follows that also y and h belong to I, that is $I \equiv L$. If $a \neq 0$ we can repeat the same argument showing that $y \in I$. Finally, if both a and b are zero, then $h \in I$ and by (1.1) we get $I \equiv L$. We have shown that L has only trivial ideals and since it is not commutative, we can conclude that L is simple.

1.2 Representations of Lie algebras

Definition 1.2.1. *Given a vector space V and a Lie algebra L, a **representation** of L on V is a Lie algebras homomorphism $\Phi : L \longrightarrow \mathfrak{gl}(V)$. Sometimes, for convenience, $\forall\, x \in L$ we will write Φ_x, in order to emphasize the fact that it is an endomorphism of V.*

Definition 1.2.2. *An injective representation is called **faithful**.*

Example 1.2.1. *An example of representation of $\mathfrak{gl}(n, \mathbb{K})$ is the so-called "standard representation" on $\mathbb{K}^n$ given by the identity map.*

Example 1.2.2. *Let us now introduce the **adjoint representation**, which will be strongly needed in the sequel. If L is a Lie algebra, the adjoint representation of L is the following representation of L on itself:*

$$\begin{aligned} \boldsymbol{ad}:\ & L \longrightarrow \mathfrak{gl}(L) \\ & x \longmapsto ad_x \quad ad_x(y) = [x, y],\ y \in L. \end{aligned}$$

Let us check that this map is indeed a representation of L:

- *Let x be an element of L and consider $ad_x :\ L \longrightarrow L$. ad_x is a linear transformation of L due to linearity of the bracket on second entry. Hence, $ad(L) \subseteq \mathfrak{gl}(L)$.*

- *Now we show that ad is a homomorphism of Lie algebras. Linearity is again a consequence of the bilinearity of the bracket, so it remains to prove that $[ad_x, ad_y] = ad_{[x,y]}$.*

$$\begin{aligned} [ad_x, ad_y](z) &= ad_x(ad_y(z)) - ad_y(ad_x(z)) \\ &= ad_x([y, z]) - ad_y([x, z]) \\ &= [x, [y, z]] - [y, [x, z]] \\ &= [x, [y, z]] + [[x, z], y] \underbrace{=}_{\textit{using Jacobi id.}} [[x, y], z] = ad_{[x,y]} \end{aligned}$$

Furthermore, notice that $\forall x \in L$ ad_x is a derivation of L. Indeed, it satisfies the Leibniz rule:

$$\begin{aligned} ad_x([y,z]) &= [x,[y,z]] \\ &= -[y,[z,x]] - [z,[x,y]] \\ &= [y,[x,z]] + [[x,y],z] \\ &= [ad_x(y),z] + [y, ad_x(z)] \end{aligned}$$

(we used the Jacobi identity for the second step).

Remark 1.2.1. *The kernel of adjoint representation is the centre of the algebra:*

$$\begin{aligned} Ker(ad) &= \{x \in L \,|\, ad_x = 0\} \\ &= \{x \in L \,|\, ad_x(y) = 0 \, \forall y \in L\} \\ &= \{x \in L \,|\, [x,y] = 0 \, \forall y \in L\} = Z(L). \end{aligned}$$

Definition 1.2.3. *Let L be a Lie algebra. An **L-module** V is a vector space endowed with an operation*

$$\begin{aligned} . : \ L \times V &\longrightarrow V \\ (x,v) &\longmapsto x.v \end{aligned}$$

such that $\forall\, x,y \in L,\ \forall\, v,w \in V,\ \forall\, a,\, b \in \mathbb{K}$ the following properties are satisfied:

(m.1) $x.(av + bw) = a(x.v) + b(x.w)$

(m.2) $(ax + by).v = a(x.v) + b(y.v)$

(m.3) $[x,y].v = x.y.v - y.x.v.$

We will say in this case that L "acts" on V.

Remark 1.2.2. *Having an L-module V is equivalent to having a representation of L on V. Indeed, if we define* $\forall x \in L$, $\Phi(x)$ *as the action* $x.$, *thanks to (m.1)* $\Phi(x) \in \mathfrak{gl}(V)$ *and the map* $\Phi : L \longrightarrow \mathfrak{gl}(V),\ x \mapsto \Phi(x)$ *is an homomorphism thanks to (m.2) and (m.3). On the other hand, if* $\Phi : L \longrightarrow \mathfrak{gl}(V)$ *is a representation, we can give V the structure of L-module by setting* $x.v \coloneqq \Phi(x)(v)$.

Definition 1.2.4. *Let V be an L-module, then* $W \subseteq V$ *is an* ***L-submodule*** *of V if it is a vector subspace such that* $x.w \in W, \forall\, w \in W$.

Definition 1.2.5. *Let V and W be L-modules. A* ***homomorphism of L-modules*** *is a linear map* $f : V \longrightarrow W$ *such that* $f(x.v) = x.f(v), \forall\, x \in L, \forall v \in W$. *Then the kernel of f is a submodule of V and* $Im(f)$ *is a submodule of W.*

Definition 1.2.6. *An L-module V is called* ***irreducible*** *if it has no proper submodules. Equivalently, a representation of L on V is called irreducible if V is irreducible as an L-module with the action given by the representation map.*

Example 1.2.3. *The standard representations of* $\mathfrak{gl}(n, \mathbb{K})$ *and* $\mathfrak{sl}(n, \mathbb{K})$ *are irreducible. Indeed, suppose W is a nonzero* $\mathfrak{gl}(n, \mathbb{K})$*-submodule of* $\mathbb{K}^n$ *and let* $w \in W,\ w \neq 0$. *Then,* $\forall v \in \mathbb{K}^n \setminus W$ *there exists a linear map x which sends w to v, contradicting the hypothesis that W is a submodule. By Example 1.1.7 we know that* $\mathfrak{gl}(n, \mathbb{K}) = \mathfrak{sl}(n, \mathbb{K}) \oplus \mathbb{K}Id$. *Let W be a* $\mathfrak{sl}(n, \mathbb{K})$*-submodule of* $\mathbb{K}^n$, *then W is also a* $\mathbb{K}Id$*-submodule, since* $\mathbb{K}Id$ *stabilizes any subspace. Now, any element* $x \in \mathfrak{gl}(n, \mathbb{K})$ *can be decomposed as* $x = y + \lambda Id,\ y \in \mathfrak{sl}(n, \mathbb{K})$. *Both y and* λId *preserve W, so x preserves W too, but this contradicts the fact that the standard representation of* $\mathfrak{gl}(n, \mathbb{K})$ *is irreducible. Thus,* $\mathbb{K}^n$ *does not admit any* $\mathfrak{sl}(n, \mathbb{K})$*-submodules.*

Example 1.2.4. *The adjoint representation of L is irreducible if and only if L is simple. Indeed, by definition, the submodules of L are its ideals.*

1.3 Solvable and nilpotent algebras

In this section we present some preliminary notions about solvable and nilpotent algebras. More substantial results will be given in the following chapters.

Definition 1.3.1. *Let L be a Lie algebra. We define the **derived series** of L as the following sequence of ideals:*

$$\begin{aligned}
L^{(0)} &= L \\
L^{(1)} &= [L, L] \\
L^{(2)} &= [L^{(1)}, L^{(1)}] \\
&\vdots \\
L^{(i)} &= [L^{(i-1)}, L^{(i-1)}]
\end{aligned}$$

*and we say that L is **solvable** if $L^{(n)} = 0$ for some n.*

Example 1.3.1. *If L is abelian, then $[L, L] = 0$, so L is solvable.*

Example 1.3.2. *If L is simple, then it is not solvable. Indeed, $[L, L]$ is an ideal of L, so due to simplicity $L = [L, L]$ (recall that L simple implies L not abelian). Therefore, $L^{(1)} = L,\ L^{(2)} = [L^{(1)}, L^{(1)}] = L, \cdots L^{(i)} = [L^{(i-1)}, L^{(i-1)}] = L$.*

Example 1.3.3. *Consider the subalgebra of $\mathfrak{gl}(n, \mathbb{K})$ consisting of upper triangular matrices (a_{ij}) ($a_{ij} = 0$ if $i > j$). We denote it by $\mathfrak{t}(n, \mathbb{K})$ and we are going to show that this is a solvable algebra. We also introduce the subalgebra $\mathfrak{n}(n, \mathbb{K})$ of strictly upper triangular matrices (a_{ij}) ($a_{ij} = 0$ if $i \geq j$) and the subalgebra of diagonal matrices $\mathfrak{d}(n, \mathbb{K})$. A basis for $\mathfrak{t}(n, \mathbb{K})$ is $\{e_{ij}\}_{i \leq j}$, so its dimension is $n(n+1)/2$. We have $[e_{ij}, e_{kl}] = \delta_{jk}\, e_{il} - \delta_{il}\, e_{kj}$. Let $L = \mathfrak{t}(n, \mathbb{K})$. For $i < l$, $[e_{ii}, e_{il}] = e_{il} \in \mathfrak{n}(n, \mathbb{K})$, so $L^{(1)} = [L, L] \supset \mathfrak{n}(n, \mathbb{K})$.*
In addition, $\mathfrak{t}(n, \mathbb{K}) = \mathfrak{n}(n, \mathbb{K}) \oplus \mathfrak{d}(n, \mathbb{K})$, so $[L, L] \equiv \mathfrak{n}(n, \mathbb{K})$, since $\mathfrak{d}(n, \mathbb{K})$ is abelian (so its derived algebra is 0) and $[\mathfrak{n}(n, \mathbb{K}), \mathfrak{d}(n, \mathbb{K})] = 0$. This means $L^{(1)} = \mathfrak{n}(n, \mathbb{K})$.
Now we analyze $L^{(2)}$. We are dealing with strictly upper triangular matrices, so we come to the notion of "level" for e_{ij}, that is j-i. Suppose $i \neq l$, $i < j$, $k < l$, then $[e_{ij}, e_{kl}] = \delta_{jk} e_{il}$. The levels of e_{ij} and e_{kl} are both ≥ 1 and the level of their bracket (if it is not zero) is the sum of these levels, then it is ≥ 2. This means that any matrix in $L^{(2)}$ is spanned by matrices e_{ij} with $i < j$ and level ≥ 2. Going on like this we obtain that $L^{(i)}$ is spanned by elements e_{ij} with level $\geq 2^{i-1}$. Therefore, when $2^{i-1} > n - 1$, $L^{(i)} = 0$.

Proposition 1.3.1. *Let L be a Lie algebra.*

1. *If L is solvable, then its subalgebras and its homomorphic images are solvable.*

2. *If I is an ideal of L and both I and L/I are solvable, then L is solvable.*

3. *If I and J are solvable ideals, then $I + J$ is a solvable ideal.*

Proof. [1]

1. If K is a subalgebra of L, then $K^{(i)} \subset L^{(i)}$. Let $\phi : L \longrightarrow M$ a Lie algebras homomorphism, then by definition $\phi(L)^{(1)} = \phi(L^{(1)})$ and by induction, $\phi(L)^{(i)} = \phi(L^{(i)})$. Thus if $L^{(n)} = 0$, then $\phi(L)^{(n)} = 0$ too.

2. Consider the canonical projection $\pi : L \longrightarrow L/I$. By hypothesis, there exist two natural numbers m and n such that $I^{(m)} = 0$ and $\pi(L^{(n)}) = \pi(L)^{(n)} = 0$. Observe that $L^{(n)} \subset I$ and $(L^{(n)})^{(m)} = L^{(n+m)}$. Hence, $L^{(n+m)} \subset I^{(m)} = 0$.

3. One of homomorphism theorems state that given $I,\ J$ ideals of L, then $(I + J)/J$ is isomorphic to $I/(I \cap J)$. $I/(I \cap J)$ is solvable by (1), since it is an homomorphic image, then $(I + J)/J$ is solvable too and part (2) concludes.

□

Remark 1.3.1. *By Proposition 1.3.1 a Lie algebra L contains at least one maximal solvable ideal. In addition this ideal is unique. Indeed, let S be a maximal solvable ideal of L and let I be another solvable ideal. Then $I + S$ is still solvable and $I + S \supseteq S$, but S is maximal, so $S = I + S$, i.e. $I \subset S$. This means that any solvable ideal is contained in S.*

Definition 1.3.2. *The unique maximal solvable ideal is called the **radical** of L and is denoted by **Rad(L)**.*

Definition 1.3.3. *A Lie algebra L such that Rad(L)= 0 is called **semisimple**.*

Example 1.3.4. *If L is simple, then it is semisimple.*

Example 1.3.5. $\mathfrak{sl}(n,\mathbb{K})$ *is simple (hence it is semisimple).*

Proof. We know that $\mathfrak{gl}(n,\mathbb{K}) = \mathfrak{sl}(n,\mathbb{K}) \oplus \mathbb{K}Id$. Let I be an ideal of $\mathfrak{sl}(n,\mathbb{K})$ such that $I \neq 0, \mathbb{K}id$. Observe that any ideal of $\mathfrak{sl}(n,\mathbb{K})$ is an ideal of $\mathfrak{gl}(n,\mathbb{K})$ too. Indeed, $[I,\mathfrak{gl}(n,\mathbb{K})] = [I,\mathfrak{sl}(n,\mathbb{K})] + [I,\mathbb{K}Id] \subset I$, as $[I,\mathfrak{sl}(n,\mathbb{K})] \subset I$ by definition and $[I,\mathbb{K}] = 0$ due to commutativity of elements in $\mathbb{K}Id$. We will show that $\mathfrak{sl}(n,\mathbb{K}) \subset I$ and so we will conclude $\mathfrak{sl}(n,\mathbb{K}) = I$. To prove this, we need some premises.
Fact 1) Let V be a finite dimensional vector space over $\mathbb{K}$ and $A_1,\dots,A_k$ diagonalizable endomorphisms of V. Then $A_1,\dots,A_k$ are simultaneously diagonalizable if and only if they pairwise commute.
Fact 2) Under the same hypothesis as in *Fact 1*, assume that $A_1,\dots,A_k$ pairwise commute. Let W be an A_i-invariant subaspace of V for any $i = 1,\dots,k$.
Then, $W = \bigoplus(W \cap V_\lambda)$, where V_λ are the common eigenspaces (which are well defined by *Fact 1*).
We do not present the proofs of these facts as they are basic results of linear algebra.
Let us now consider the subalgebra $\mathfrak{h}$ of $\mathfrak{gl}(n,\mathbb{K})$ spanned by elements e_{ii}, $\forall i = 1,\dots,n$. The elements of $\mathfrak{h}$ are all ad-diagonalizable, i.e. the endomorphisms $ad_{e_{ii}}$ of $\mathfrak{gl}(n,\mathbb{K})$ are diagonalizable. Indeed, for all i we have $[e_{ii},e_{kl}] = \delta_{ik}e_{il} - \delta_{il}e_{ki}$, so for $i \neq j$ $[e_{ii},e_{ij}] = e_{ij}$, $[e_{ii},e_{ji}] = -e_{ji}$ and $[e_{ii},e_{jj}] = 0$. Therefore, if we set $\Phi_i = ad(e_{ii})$, we can affirm that $\mathfrak{h}$ together with $\{Span(e_{ij})\}_{i\neq j}$ are the eigenspaces of all Φ_i. Note that $\Phi_1,\dots,\Phi_n$ commute since $e_{11},\dots,e_{nn}$ commute, then $\Phi_1,\dots,\Phi_n$ are simultaneously diagonalizable. I is an ideal, so it is Φ_i-invariant for all i. Therefore, by *Fact 2*, $I = (I \cap \mathfrak{h}) \oplus \left(\bigoplus_{i\neq j}(I \cap Span(e_{ij}))\right)$. So, either I contains e_{ij} for some $i \neq j$ or $I \cap \mathfrak{h} \neq 0$, $\mathbb{K}id$. Take $x \in I \cap \mathfrak{h}$, $x = \sum a_i e_{ii}$ where $a_i \neq a_j$ for some $i \neq j$ (otherwise we may have $x \in \mathbb{K}id$). Now, $I \ni [x,e_{ij}] = \sum(a_i[e_{ii},e_{ij}]) = (a_i - a_j)e_{ij}$, thus in both cases $e_{ij} \in I$.
Moreover, for $k \neq i$, $[e_{ij},e_{jk}] = e_{ik}$, so $e_{ik} \in I$ and $I \ni [e_{ik},e_{ki}] = e_{ii} - e_{kk}$. In the end, for $r \neq s$ we obtain $I \ni [e_{rr} - e_{ss}, e_{rs}] = 2e_{rs}$. We have just shown that all the basis elements of $\mathfrak{sl}(n,\mathbb{K})$ are in I, so $\mathfrak{sl}(n,\mathbb{K}) \subset I$. Thus, $\mathfrak{sl}(n,\mathbb{K})$ has no proper ideal. □

Example 1.3.6. *$L/Rad(L)$ is semisimple.*

Definition 1.3.4. *Let L be a Lie algebra. We define the **lower central series** of L as the following sequence of ideals:*

$$\begin{aligned} L^0 &= L \\ L^1 &= [L, L] \\ L^2 &= [L, L^1] \\ &\vdots \\ L^i &= [L, L^{i-1}] \end{aligned}$$

*and we say that L is **nilpotent** if $L^n = 0$ for some n.*

Example 1.3.7. *If L is abelian then L is nilpotent.*

Remark 1.3.2. $L^{(i)} \subset L^i\ \forall i$*, so nilpotent algebras are solvable, but the converse is false. A counterexample is* $\mathfrak{t}(n, \mathbb{K})$*. In Example 1.3.3 we have shown that* $L^{(1)} = L^1 = \mathfrak{n}(n, \mathbb{K})$*, so* $L^2 = [\mathfrak{t}(n, \mathbb{K}), \mathfrak{n}(n, \mathbb{K})] = \mathfrak{n}(n, \mathbb{K})$*. Therefore,* $\forall i \geq 1,\ L^i = L^1 \neq 0$*.*

Proposition 1.3.2. *Let L be a Lie algebra.*

1. *If L is nilpotent, then its subalgebras and homomorphic images are nilpotent.*

2. *If $L/Z(L)$ is nilpotent, then L is nilpotent.*

3. *If L is nonzero and nilpotent, then $Z(L) \neq 0$.*

Proof. [1]

1. Same proof as in Proposition 1.3.1.

2. By hypothesis, there exists a natural number n such that $L^n \subset Z(L)$. Then, $L^{n+1} = [L, L^n] \subset [L, Z(L)] = 0$.

3. By definition, the last term of the central series is $L^n = [L, L^{n-1}] = 0$, thus $L^{n-1} \in Z(L)$.

$\square$

Chapter 2

A weak version of Ado's Theorem

As we explained in the introduction, the purpose of this thesis is to provide a step by step proof of Ado's Theorem. In this chapter, instead of setting all the preliminary notions that we need to prove the general theorem, we try to use what we already know about Lie algebras to give a proof of the theorem in the semisimple and abelian cases. Before doing this, we introduce the universal enveloping algebra $\mathfrak{U}(L)$ of a Lie algebra L, using which we can prove a "weak" version of Ado's theorem and that is going to be crucial in the following chapters.
In this chapter, we assume $\mathbb{K} = \mathbb{C}$, so all the vector spaces and algebras are $\mathbb{C}$-vector spaces and $\mathbb{C}$-algebras. Everything works in the same way on every algebraically closed field of characteristic zero. In addition, L will indicate always a Lie algebra.

2.1 Universal enveloping algebra

The aim of this section is to present the useful algebraic notion of universal enveloping algebra $\mathfrak{U}(L)$ of a Lie algebra L. We will explain the construction of $\mathfrak{U}(L)$, describe some of its properties and its first application in the proof of Ado's Theorem.
The idea behind the construction of $\mathfrak{U}(L)$ is to construct the "smallest" associative algebra containing L, i.e. to find a unital associative algebra together with a linear map $i : L \longrightarrow \mathfrak{U}(L)$ such that the bracket in L becomes a commutator in $\mathfrak{U}(L)$, namely $i([x, y]) = i(x)i(y) - i(y)i(x)$.

Definition 2.1.1. *Let V be a finite-dimensional vector space. We denote by V^n the cartesian product of V with itself n-times $V \times \cdots \times V$. The n-th tensor power of V is the only vector space $V^{\otimes n}$ (up to isomorphisms) together with a n-linear map $j : V^n \longrightarrow V^{\otimes n}$, $(v_1, \dots, v_n) \mapsto v_1 \otimes \dots \otimes v_n$ which satisfies the following universal property:*
if W is another vector space and $f : V^n \longrightarrow W$ is a n-linear map, then $\exists! \tilde{f} : V^{\otimes n} \longrightarrow W$ such that the following diagram is commutative

$$\begin{tikzcd} & V^{\otimes n} \arrow[d, "\tilde{f}"] \\ V^n \arrow[ur, "j"] \arrow[r, "f"'] & W \end{tikzcd}$$

We set $T^n(V) = V^{\otimes n}$ and, by convention, $T^0(V) = \mathbb{C}$.

Now we try to embed V in an associative algebra by "putting together" all these nth-tensor powers.

Definition 2.1.2. *The **tensor algebra** of a vector space V is $\mathfrak{T}(V) = \bigoplus_{n\geq 0} V^{\otimes n}$. We endow $\mathfrak{T}(V)$ with the structure of unital associative algebra defining a product in the natural way on the homogeneous vectors: if $v_1 \otimes \dots \otimes v_n \in V^{\otimes n}$ and $w_1 \otimes \dots \otimes w_m \in V^{\otimes m}$, then $(v_1 \otimes \dots \otimes v_n) \cdot (w_1 \otimes \dots \otimes w_m) = v_1 \otimes \dots \otimes v_n \otimes w_1 \otimes \dots \otimes w_m$. Observe that the product defined in such a way is a map $\cdot : V^{\otimes n} \times V^{\otimes m} \longrightarrow V^{\otimes n+m}$, namely $\mathfrak{T}(V)$ is a graded algebra. The unity is $1 \in \mathbb{C}$.*

Definition 2.1.3. *A **universal enveloping algebra** of a Lie algebra L is a pair $(\mathfrak{U}(L), i)$, where $\mathfrak{U}(L)$ is a unital, associative algebra and $i : L \longrightarrow \mathfrak{U}(L)$ is a Lie algebra homomorphism. In addition $(\mathfrak{U}(L), i)$ has to satisfy the following universal property:*
for any associative algebra A (where A is given the Lie algebra structure induced by the associative product), if $\varphi : L \longrightarrow A$ is a Lie algebras homomorphism, then $\exists!$ an associative algebra homomorphism $\tilde{\varphi} : \mathfrak{U}(L) \longrightarrow A$ such that the following diagram commutes

$$\begin{tikzcd} & \mathfrak{U}(L) \arrow[d, "\tilde{\varphi}"] \\ L \arrow[ur, "i"] \arrow[r, "\varphi"'] & A \end{tikzcd}$$

Theorem 2.1.1. *Let L be a Lie algebra. Then the universal enveloping algebra of L exists and it is unique.*

Proof. [4, pp. 1-2] (Uniqueness) Assume $(\mathfrak{U}(L), i)$ and $(\bar{\mathfrak{U}}(L), h)$ are both universal enveloping algebras of L, then we can combine their universal properties to obtain the result. Indeed, we have

$$\begin{tikzcd}
 & \mathfrak{U}(L) \arrow[d, "\exists! \tilde{h}"] \\
L \arrow[ur, "i"] \arrow[r, "h"] \arrow[dr, "i"'] & \bar{\mathfrak{U}}(L) \arrow[d, "\exists! \tilde{i}"] \\
 & \mathfrak{U}(L)
\end{tikzcd}$$

that is

$$h = \tilde{h} \circ i \quad \textit{and} \quad i = \tilde{i} \circ h \tag{2.1}$$

Putting together the equations in (2.1) we find that $i = \tilde{i} \circ \tilde{h} \circ i$ and $\tilde{i} \circ \tilde{h}$ is a map from $\mathfrak{U}(L)$ to $\mathfrak{U}(L)$. We can regard i as a map from L to an associative unital algebra, so due to the universal property there exists a unique map $f : \mathfrak{U}(L) \longrightarrow \mathfrak{U}(L)$ such that $i = f \circ i$. Both $\tilde{i} \circ \tilde{h}$ and $id_{\mathfrak{U}(L)}$ verify the condition, therefore $\tilde{i} \circ \tilde{h} = id_{\mathfrak{U}(L)}$ by uniqueness.
Now, on the other hand, $h = \tilde{h} \circ \tilde{i} \circ h$ and $\tilde{h} \circ \tilde{i}$ is a map from $\bar{\mathfrak{U}}(L)$ to itself, so repeating the preceding argument, we get $\tilde{h} \circ \tilde{i} = id_{\bar{\mathfrak{U}}(L)}$.
To summarize,

$$\begin{gathered}
\tilde{i} : \bar{\mathfrak{U}}(L) \longrightarrow \mathfrak{U}(L) \quad \textit{and} \quad \tilde{h} : \mathfrak{U}(L) \longrightarrow \bar{\mathfrak{U}}(L) \\
\tilde{i} \circ \tilde{h} = id_{\mathfrak{U}(L)} \quad \textit{and} \quad \tilde{h} \circ \tilde{i} = id_{\bar{\mathfrak{U}}(L)}
\end{gathered}$$

$\Longrightarrow$ $\bar{\mathfrak{U}}(L)$ *and* $\mathfrak{U}(L)$ *are isomorphic.*

(Existence) The notion of tensor algebra can be applied to a Lie algebra, as it is just a particular type of vector space. Let I be the two-sided ideal of $\mathfrak{T}(L)$ generated by elements of the form $[x, y] - x \otimes y - y \otimes x$, $\forall x, y \in L$. These generators are elements of $L \oplus (L \otimes L)$ and a general element of I is $x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes ([y_1, y_2] - y_1 \otimes y_2 - y_2 \otimes$

$y_1) \otimes z_1 \otimes \cdots \otimes z_m$ for some $x_1, \cdots, x_n, y_1, y_2, z_1, \cdots, z_m \in L$. Define $\mathfrak{U}(L) \coloneqq \mathfrak{T}(L)/I$, let $\pi : \mathfrak{T}(L) \longrightarrow \mathfrak{T}(L)/I$ be the natural projection and let $i \coloneqq \pi|_L$. Now we show that the pair $(\mathfrak{U}(L), i)$ is the universal enveloping algebra of L.
Let A be an associative unital algebra and $\varphi : L \longrightarrow A$ an homomorphism. We define $\tilde{\varphi} : \mathfrak{T}(L) \longrightarrow A$ on the homogeneous elements by taking $\tilde{\varphi}(x_1 \otimes \ldots \otimes x_n) = \varphi(x_1) \cdot \ldots \cdot \varphi(x_n)$ and extending linearly ($\cdot$ is the product in A). $\tilde{\varphi}$ just defined is a Lie algebra homomorphism. Now observe that for any $x, y \in L$, $\tilde{\varphi}([x, y] - x \otimes y - y \otimes x) = \varphi([x, y]) - \varphi(x) \cdot \varphi(y) - \varphi(y) \cdot \varphi(x) = 0$ since φ is a Lie algebra homomorphism. Thus, $\tilde{\varphi}$ descends to the quotient, namely $\tilde{\varphi} : \mathfrak{T}(L)/I \longrightarrow A$. Moreover, $\forall x \in L$, $(\tilde{\varphi} \circ i)(x) = \tilde{\varphi}([x]) = \varphi(x)$, so by linearity $\varphi = \tilde{\varphi} \circ i$. It only remains to check if $\tilde{\varphi}$ is unique. Assume $\psi : \mathfrak{U}(L) \longrightarrow A$ is another Lie algebra homomorphism such that $\varphi = \psi \circ i$, then $\forall x \in L$, $\tilde{\varphi}(i(x)) = \varphi(x) = \psi(i(x))$ and this is true also on 1. A set of generators of $\mathfrak{T}(L)$ is 1 and L, so 1 and $i(L)$ is a set of generators for $Im(i) = \mathfrak{U}(L)$ because the projection is surjective. Thus, $\tilde{\varphi}$ and ψ are homomorphisms that coincide on the generators of $\mathfrak{U}(L)$, therefore $\psi = \tilde{\varphi}$. $\square$

Remark 2.1.1. *From now on, when dealing with $\mathfrak{U}(L)$ the symbol $\otimes$ will be omitted, i.e. we shall indicate by $x_1 \cdots x_n$ the image in $\mathfrak{U}(L)$ of $x_1 \otimes \ldots \otimes x_n$. In $(\mathfrak{U}(L), i)$ defined in Theorem 2.1.1 the bracket between elements in L is a commutator as we have forced $[x, y] = xy - yx$ by cutting out the ideal I.*

Now we state the Poincaré-Birkhoff-Witt theorem, which describes the structure of $\mathfrak{U}(L)$.

Theorem 2.1.2 (Poincaré-Birkhoff-Witt (PBW))**.** *Let $\{x_i\}_{i \in I}$ be an ordered basis of a Lie algebra L. Then a basis of $\mathfrak{U}(L)$ is given by monomials of the form*

$$x_{i_1}^{m_1} \cdots x_{i_n}^{m_n} \tag{2.2}$$

with $i_1 < i_2 < \ldots < i_n$, $m_j \in \mathbb{Z}_+$ $\forall j \in \{1, \ldots, n\}$.

Example 2.1.1. *Consider $L = \mathfrak{sl}(2, \mathbb{C})$ and its standard basis $\{x, h, y\}$ introduced in Example 1.1.8. By PBW theorem, $\mathfrak{U}(L) = Span\{x^a h^b y^c \,|\, a, b, c \in \mathbb{Z}_+\}$. Moreover, taken for example the element $yxh \in \mathfrak{U}(L)$, we can rewrite it as a sum of monomials of the* *form* (2.2). *Indeed*

$$yxh = xyh + [y,x]h = xhy + x[y,h] - h^2 = xhy - 2xy - h^2.$$

Now we state Ado's Theorem and then we use the universal enveloping algebra to prove a weak version of it.

Theorem 2.1.3 (Ado's Theorem)**.** *Let L be a finite-dimensional Lie algebra. Then there exists a finite-dimensional, faithful representation $\Phi : L \longrightarrow \mathfrak{gl}(V)$ of L.*

Theorem 2.1.4 (*weak* version of Ado's Theorem)**.** *Let L be a Lie algebra. Then L is isomorphic to Lie algebra consisting of endomorphisms.*

Proof. L acts on its universal enveloping algebra $\mathfrak{U}(L)$ by left multiplication, namely we consider the map

$$\rho : L \longrightarrow \mathfrak{gl}(\mathfrak{U}(L)) \qquad \rho_x(y) = xy \quad \forall x \in L,\ \forall y \in \mathfrak{U}(L).$$

- ρ is a Lie algebra homomorphism because $\forall x,\ y \in L$ *and* $\forall z \in \mathfrak{U}(L)$, $\rho_{[x,y]}(z) = [x,y]z = xyz - yxz$ and similarly $[\rho_x, \rho_y](z) = \rho_x\rho_y(z) - \rho_y\rho_x(z) = xyz - yxz$.

- $\forall x \in L$, ρ_x is an endomorphism of $\mathfrak{U}(L)$. In fact, $\rho_x(v+w) = x(v+w) = xv+xw = \rho_x(v) + \rho_x(w)$.

Thus, ρ is a representation of L on its universal enveloping algebra.
Moreover, ρ is faithful since $\forall x \in L$ $\rho_x(1) = x$, so there are not two different elements of L which give the same endomorphism of $\mathfrak{U}(L)$. Then, using the theorem of homomorphism we conclude that $L \cong Im(\rho) \subseteq \mathfrak{gl}(\mathfrak{U}(L))$. □

Remark 2.1.2. *In Theorem 2.1.4 we have used the map of left multiplication and we could not use the right multiplication. Indeed, by a brief computation one sees that the map of right multiplication is not a Lie algebra homomorphism. In fact, using the same notation as in the proof of the theorem we get $\rho_{[x,y]}(z) = zxy - zyx \neq zyx - zxy = [\rho_x, \rho_y](z)$.*

Remark 2.1.3. *Even if L is a finite-dimensional Lie algebra, its universal enveloping algebra is always infinite dimensional, so Theorem 2.1.4 only tells us we can represent L on an infinite dimensional vector space. This is why we named this theorem a "weak" version of Ado's Theorem: to emphasize the difference between it and the real Ado's Theorem, which guarantees L is represented on a finite-dimensional space.*

We conclude this section showing the relation between the representations of a Lie algebra L and those of $\mathfrak{U}(L)$.

Remark 2.1.4. *Suppose $\varphi : L \longrightarrow \mathfrak{gl}(V)$ is a representation of L on V, then by the universal property $\exists!$ a homomorphism $\hat{\varphi} : \mathfrak{U}(L) \longrightarrow \mathfrak{U}(L)$ which extends φ. Conversely, given a homomorphism $\psi : \mathfrak{U}(L) \longrightarrow \mathfrak{gl}(V)$, we obtain a represenatation of L just by $\psi \circ\ i$.*

2.2 Semisimple and abelian cases

In this section we face the semisimple and abelian cases of the theorem. The proofs are elementary, but they are fundamental in order to understand what to do next.

Theorem 2.2.1 (Ado's Theorem for semisimple algebras)**.** *Let L be a finite-dimensional semisimple Lie algebra. Then there exists a faithful representation of L over a finite-dimensional vector space V. Equivalently, L is isomorphic to a subalgebra of $\mathfrak{gl}(V)$.*

Proof. The representation we are looking for is the adjoint representation

$$ad : L \longrightarrow \mathfrak{gl}(L)$$

and by Remark 1.2.1 we know that $Ker(ad) = Z(L)$. *Z(L)* is a solvable ideal since $[Z(L), Z(L)] = 0$, but L is semisimple, so the centre is null. Thus, $ad : L \longrightarrow \mathfrak{gl}(L)$ is a homomorphism with trivial kernel, and $L \cong Im(ad)$, i.e. L is isomorphic to a subalgebra of $\mathfrak{gl}(L)$. □

Remark 2.2.1. *The adjoint representation does not suffice to prove Ado's Theorem for a non-semisimple algebra, as it is not faithful on the centre. However, we can search for*

a complement of ad, in the following sense:
suppose $\rho : L \longrightarrow \mathfrak{gl}(V)$ *is a representation of* L *on a finite-dimensional vector space* V *which is faithful on* $Z(L)$*. Then we can define*

$$\begin{aligned} \Phi : L &\longrightarrow \mathfrak{gl}(L \oplus V) \\ x &\longmapsto \Phi(x) \end{aligned}$$

where $\Phi(x) : L \oplus V \longrightarrow L \oplus V$ *sends* $(l, v) \mapsto (ad_x(l), \rho_x(v))$*.* Φ *is still a representation of* L *on the finite-dimensional space* $L \oplus V$*.*
Moreover, Φ *is faithful on L. Indeed,* $\Phi(x) = 0$ *if and only if* $\forall\, l \in L, \forall\, v \in V, (ad_x(l), \rho_x(v)) = (0, 0)$*. Now,* $ad_x(l) = 0\ \forall l \in L \iff x \in Z(L)$ *and* $\rho_x(v) = 0\ \forall v \in V \iff x \in Ker(\rho)$*. But* ρ *is faithful on* $Z(L)$*, so* $Z(L) \cap Ker(\rho) = 0$*, which means* $\Phi(x) = 0 \iff x = 0$*.*

Due to Remark 2.2.1 we need to find a finite-dimensional representation of L which is faithful on *Z(L)* and this is easy when L is abelian because $L = Z(L)$.

Theorem 2.2.2 (Ado's Theorem for abelian algebras)**.** *Let L be a finite-dimensional, abelian Lie algebra. Then there exists a faithful representation of L over a finite-dimensional vector space.*

Proof. We claim that $\rho : L \longrightarrow \mathfrak{gl}(L \times \mathbb{C})$ defined as $\rho_x(y, t) = (tx, 0)$ is a faithful representation of L.

- ρ is clearly linear and it also satisfies $\rho([x, y]) = [\rho(x), \rho(y)]\ \forall x,\, y \in L$. Indeed:

  $$\rho([x, y]) = \rho(0) = 0 \text{ because } L \text{ is abelian} \tag{2.3}$$

  while

  $$\begin{aligned} \forall (z, t) \in L \times \mathbb{C},\ [\rho(x), \rho(y)](z, t) &= \rho_x(\rho_y(z, t)) - \rho_y(\rho_x(z, t)) \\ &= \rho_x(ty, 0) - \rho_y(tx, 0) \\ &= (0, 0) - (0, 0) \\ &= (0, 0) \end{aligned}$$

thus ρ is a representation of L.

- ρ is faithful because for $x \in L$

$$\begin{aligned}\rho_x = 0 &\iff \rho_x(y,t) = 0 \ \forall (y,t) \in L \times \mathbb{C} \\ &\iff (tx, 0) = (0,0) \ \forall \ t \in \mathbb{C} \\ &\iff x = 0\end{aligned}$$

$\square$

Remark 2.2.2. *If we consider only $Z(L)$ instead of the whole Lie algebra L, the map ρ in Theorem 2.2.2 is obviously a finite-dimensional representation of the centre. However, the direct sum ad $+\ \rho$ does not provide a faithful representation of L. In fact, ρ is a homomorphism only if L is abelian, as we noticed in 2.3.*

Chapter 3

The nilpotent case

In this chapter we present the proof of Ado's Theorem in the case of a nilpotent Lie algebra and then we show some important results about nilpotent Lie algebras.

3.1 Ado's Theorem in the nilpotent case

Theorem 3.1.1 (Ado's Theorem for nilpotent Lie algebras)**.** *Let* $\mathfrak{n}$ *be a finite-dimensional, nilpotent Lie algebra. Then there exists a finite-dimensional faithful representation of* $\mathfrak{n}$, $\Phi : \mathfrak{n} \longrightarrow \mathfrak{gl}(V)$. *Moreover, there exists* $k \in \mathbb{N}$ *such that* $\Phi(\mathfrak{n})^k = 0$*, namely* $\forall\, x_{i_1}, \dots, x_{i_k} \in \mathfrak{n}$ *we have* $\Phi(x_{i_1}) \cdots \Phi(x_{i_k}) = 0$*.*

Proof. We prove the theorem by induction on the dimension of $\mathfrak{n}$.
The result is trivially true for dimension zero, so suppose that $dim(\mathfrak{n}) > 0$ and that the theorem is valid for all nilpotent algebras with dimension lower than $dim(\mathfrak{n})$. We also assume $\mathfrak{n}$ is not abelian, otherwise we can use Theorem 2.2.2. Then $dim(Z(\mathfrak{n})) < dim(\mathfrak{n})$, so the quotient $\mathfrak{n}' \coloneqq \mathfrak{n}/Z(\mathfrak{n})$ has positive dimension and is still a nilpotent algebra. $\mathfrak{n}'$ is not abelian and by nilpotency, $[\mathfrak{n}', \mathfrak{n}'] \subsetneq \mathfrak{n}'$, so $\hat{\mathfrak{n}} \coloneqq \mathfrak{n}'/[\mathfrak{n}', \mathfrak{n}']$ has positive dimension and is nilpotent. Therefore, $\hat{\mathfrak{n}}$ has a one dimensional subspace, let's say $\mathbb{C}z$. We call V a complement of $\mathbb{C}z$, hence $dim(V) = dim(\hat{\mathfrak{n}}) - 1 = dim(\mathfrak{n}') - dim([\mathfrak{n}', \mathfrak{n}']) - 1$. Let $\pi : \mathfrak{n}' \longrightarrow \hat{\mathfrak{n}}$ be the natural projection. We claim that $\pi^{-1}(V)$ is an ideal of $\mathfrak{n}'$. Indeed, consider $\pi^{-1}(v)$ for some $v \in V$ a general element of $\pi^{-1}(V)$ and let x be a general element of $\mathfrak{n}'$: we wonder whether $[\pi^{-1}(v), x] \in \pi^{-1}(V)$. This is true if and only if

$\pi([\pi^{-1}(v), x]) \in V$. Note that $[\pi^{-1}(v), x] \in [\mathfrak{n}', \mathfrak{n}']$, so it is zero when passing to the quotient and then it is in V. With regard to the dimensions of these spaces,

$$dim(V) = dim(\mathfrak{n}') - dim([\mathfrak{n}', \mathfrak{n}']) - 1 \iff dim(V) + dim([\mathfrak{n}', \mathfrak{n}']) = dim(\mathfrak{n}') - 1$$

and $dim(V) + dim([\mathfrak{n}', \mathfrak{n}'])$ is exactly the dimension of $\pi^{-1}(V)$. Thus $\pi^{-1}(V)$ is a codimension 1 ideal of $\mathfrak{n}'$ and so it corresponds to a an ideal of $\mathfrak{n}$ containing $Z(\mathfrak{n})$. More precisely, $\pi^{-1}(V) = \mathfrak{a}/Z(\mathfrak{n})$ for some ideal $\mathfrak{a}$ of $\mathfrak{n}$. Finally,

$$dim(\mathfrak{n}') = dim(\mathfrak{a}/Z(\mathfrak{n})) + 1 \iff dim(\mathfrak{n}) - dim(Z(\mathfrak{n})) = dim(\mathfrak{a}) - dim(Z(\mathfrak{n})) + 1$$

that is $dim(\mathfrak{a}) = dim(\mathfrak{n}) - 1$, Therefore, $\mathfrak{a}$ is an ideal of codimension 1 in $\mathfrak{n}$ (then $\mathfrak{a}$ is nilpotent) and $\mathfrak{a} \supset Z(\mathfrak{n})$. Let $\mathfrak{h}$ be a complementary subspace of $\mathfrak{a}$ (note that $\mathfrak{h}$ is 1-dimensional, hence it is abelian), then we have the decomposition

$$\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{h}. \tag{3.1}$$

By Remark 2.2.1, we have to find a finite-dimensional representation ρ of $\mathfrak{n}$ which is faithful on $Z(\mathfrak{n})$ and then combine this with the adjoint action, so that $\Phi = \rho + ad$, (by the nilpotency of $\mathfrak{n}$, ad is nilpotent). First of all, by the inductive hypothesis, we know that there exists a finite-dimensional faithful representation of $\mathfrak{a}$, say

$$\rho_0 : \mathfrak{a} \longrightarrow \mathfrak{gl}(V_0)$$

with $\rho_0(\mathfrak{a})^{k_0} = 0$ for some $k_0 \in \mathbb{N}$. However, we will need ρ_0 only for inductive purpose; now we try to represent both $\mathfrak{a}$ and $\mathfrak{h}$ on $\mathfrak{U}(\mathfrak{a})$. As it is shown in Theorem 2.1.4, $\mathfrak{a}$ has a natural representation on its universal enveloping algebra, i.e. the left multiplication map. Recall $\mathfrak{a}$ is an ideal of $\mathfrak{n}$, hence $[\mathfrak{h}, \mathfrak{a}] \subset \mathfrak{a}$, then there is an adjoint action $ad : \mathfrak{h} \longrightarrow \mathfrak{gl}(\mathfrak{a})$. Now we extend this action to an action on $\mathfrak{U}(\mathfrak{a})$. Actually, for any $H \in \mathfrak{h}$ and for a general monomial in $\mathfrak{U}(\mathfrak{a})$ we define

$$[H, A_1 \cdots A_m] = \sum_{i=1}^{m} A_1 \cdots A_{i-1} [H, A_i] \cdots A_m. \tag{3.2}$$

and we extend it on $\mathfrak{U}(\mathfrak{a})$ by linearity. We have to prove that $ad : \mathfrak{h} \longrightarrow \mathfrak{gl}(\mathfrak{U}(\mathfrak{a}))$ with the bracket defined as is (3.2) is a representation. Assume H, $K \in \mathfrak{h}$ and $A_1 \cdots A_m \in \mathfrak{U}(\mathfrak{a})$, then:

- $$\begin{aligned} ad_{[H,K]}(A_1 \cdots A_m) &= \sum_{i=1}^{m} A_1 \cdots A_{i-1} \left[[H, K], A_i\right] \cdots A_m = \\ &= \sum_{i=1}^{m} A_1 \cdots A_{i-1} \left[H, [K, A_i]\right] \cdots A_m + \\ &\quad + \sum_{i=1}^{m} A_1 \cdots A_{i-1} \left[K, [A_i, H]\right] \cdots A_m \end{aligned}$$

- $$\begin{aligned} [ad_H, ad_K](A_1 \cdots A_m) =& ad_H(ad_K(A_1 \cdots A_m)) - ad_K(ad_H(A_1 \cdots A_m)) = \\ =& ad_H(\sum_{i=1}^{m} A_1 \cdots A_{i-1} \left[K, A_i\right] \cdots A_m) \\ & - ad_K(\sum_{i=1}^{m} A_1 \cdots A_{i-1} \left[H, A_i\right] \cdots A_m) = \\ =& \sum_{i=1}^{m} A_1 \cdots A_{i-1} \left[H, [K, A_i]\right] \cdots A_m \\ & - \sum_{i=1}^{m} A_1 \cdots A_{i-1} \left[K, [H, A_i]\right] \cdots A_m = \\ =& \sum_{i=1}^{m} A_1 \cdots A_{i-1} \left[H, [K, A_i]\right] \cdots A_m + \\ & + \sum_{i=1}^{m} A_1 \cdots A_{i-1} \left[K, [A_i, H]\right] \cdots A_m \end{aligned}$$

so $ad : \mathfrak{h} \longrightarrow \mathfrak{gl}(\mathfrak{U}(\mathfrak{a}))$ is a Lie algebra homomorphism, i.e. a representation of $\mathfrak{h}$ on $\mathfrak{U}(\mathfrak{a})$. Now we combine the adjoint action of $\mathfrak{h}$ with the left multiplication action of $\mathfrak{a}$ to get an action of the entire $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{h}$; namely we define

$$\hat{\rho} : \mathfrak{a} \oplus \mathfrak{h} \longrightarrow \mathfrak{gl}(\mathfrak{U}(\mathfrak{a})) \tag{3.3}$$

and set $\hat{\rho}_{A+H}(M) = AM + [H, M]$, $\forall A + H \in \mathfrak{a} \oplus \mathfrak{h}$ and $\forall M \in \mathfrak{U}(\mathfrak{a})$. Let us check that $\hat{\rho}$ is a genuine action, equivalently, that $\hat{\rho}_{[A+H,B+K]}(M) = [\hat{\rho}_{A+H}, \hat{\rho}_{B+K}](M)$ for all

A, $B \in \mathfrak{a}$ and H, $K \in \mathfrak{h}$. On one hand,

$$\begin{aligned}\hat{\rho}_{[A+H,B+K]}(M) =& \hat{\rho}([A,B]+[A,K]+[H,B]+[H,K])(M) = \quad \textit{(only [H,K] is in } \mathfrak{h}\textit{)}\\ =&[A,B]M+[A,K]M+[H,B]M+[[H,K],M].\end{aligned}$$

On the other hand

$$\begin{aligned}\left[\hat{\rho}_{A+H},\hat{\rho}_{B+K}\right](M) =& \hat{\rho}_{A+H}(\hat{\rho}_{B+K}(M)) - \hat{\rho}_{B+K}(\hat{\rho}_{A+H}(M)) =\\ =& \hat{\rho}_{A+H}(BM+[K,M]) - \hat{\rho}_{B+K}(AM+[H,M]) =\\ =& ABM + A[K,M] + [H,BM] + [H,[K,M]]\\ & - BAM - B[H,M] - [K,AM] - [K,[H,M]] =\end{aligned}$$

(we are dealing with $\mathfrak{U}(\mathfrak{a})$, so $ABM - BAM = [A,B]M$ and $[A,K]M = A[K,M] - [K,AM]$)

$$\begin{aligned}=&[A,B]M+[A,K]M+[H,B]M+[H,[K,M]]-[K,[H,M]] =\\ =&[A,B]M+[A,K]M+[H,B]M+[H,[K,M]]+[K,[M,H]] =\\ =&[A,B]M+[A,K]M+[H,B]M+[[H,K]M].\end{aligned}$$

Therefore, $\hat{\rho} : \mathfrak{a} \oplus \mathfrak{h} \longrightarrow \mathfrak{gl}(\mathfrak{U}(\mathfrak{a}))$ is a representation. We notice that $\hat{\rho}$ is faithful on $\mathfrak{a}$ since the left multiplication action of $\mathfrak{a}$ on $\mathfrak{U}(\mathfrak{a})$ was faithful. We want to modify $\hat{\rho}$ in order to obtain a finite-dimensional representation hence we look for an ideal I of $\mathfrak{U}(\mathfrak{a})$ such that $dim(\mathfrak{U}(\mathfrak{a})/I) < +\infty$. We consider the two-sided ideal $I = \langle(\mathfrak{a})^{k_0}\rangle$, generated by k_0-fold product of elements of $\mathfrak{a}$. Then we can consider the quotient $\mathfrak{U}(\mathfrak{a})/I$ and observe that it is finite-dimensional. In fact, $\mathfrak{a}$ is finite-dimensional and we can assume $a_1, \ldots, a_m$ is an ordered basis; then by PBW theorem we know that a basis of $\mathfrak{U}(\mathfrak{a})$ consists of monomials of the form $a_{i_1}^{m_1} \cdots a_{i_n}^{m_n}$. Passing to the quotient $\mathfrak{U}(\mathfrak{a})/I$ is generated only by those monomials whose degree $m_1 + \ldots + m_n < k_0$ and there are only finitely many such monomials.

Now we want to project $\hat{\rho}$ to a representation on $\mathfrak{U}(\mathfrak{a})/I$ and to do so we have to check if

$\forall A + H \in \mathfrak{a} \oplus \mathfrak{h}$, $\hat{\rho}_{A+H}$ descends to quotient. Consider $\mathfrak{U}(\mathfrak{a})$ as an $\mathfrak{a}$-module (action given by left multiplication), then I is an $\mathfrak{a}$-submodule and so it is stable under the action of $\mathfrak{a}$, i.e. $\mathfrak{a}.I \subseteq I$; similarly $[H, I] \subseteq I$ since I is an ideal, so I is also stable under the adjoint action of $\mathfrak{h}$. Therefore, $\hat{\rho}_{A+H}$ preserves the equivalence classes and this allows us to define

$$\rho : \mathfrak{a} \oplus \mathfrak{h} \longrightarrow \mathfrak{gl}\left(\frac{\mathfrak{U}(\mathfrak{a})}{I}\right). \tag{3.4}$$

It only remains to show that ρ is faithful on $\mathfrak{a}$ and that $\rho(\mathfrak{n})^k = 0$ for some k.
We have to prove the following implication:

$$\forall [z] \in \frac{\mathfrak{U}(\mathfrak{a})}{I},\ [x][z] = [xz] = [0] \qquad \Longrightarrow \qquad x = 0 \text{ (in } \mathfrak{a}\text{)}.$$

Suppose $[z] = [1]$, then $[x1] = 0 \iff [x] = [0]$, so it suffices to show $[x] = [0] \iff x = 0$. Equivalently, we shall prove that

$$\begin{aligned} \phi : \mathfrak{a} &\longrightarrow \frac{\mathfrak{U}(\mathfrak{a})}{I} \\ x &\longmapsto [x] \end{aligned}$$

is injective. We at last need $\rho_0 : \mathfrak{a} \longrightarrow \mathfrak{gl}(V_0)$ and we extend it to $\rho_0' : \mathfrak{U}(\mathfrak{a}) \longrightarrow \mathfrak{gl}(V_0)$ by defining the endomorphism $\rho_0'(a_{i_1} \cdots a_{i_m})$ as the composition $\rho_0(a_{i_1}) \circ \ldots \circ \rho_0(a_{i_m})$ (we define ρ_0' on a basis and then extend it by linearity). Since $\rho_0(\mathfrak{a})^{k_0} = 0$, we get that $\rho_0'(I) = 0$ and so ρ_0' descends to a representation of the quotient:

$$[\rho_0'] : \frac{\mathfrak{U}(\mathfrak{a})}{I} \longrightarrow \mathfrak{gl}(V_0)$$

Notice that the composition $[\rho_0'] \circ \phi$ gives exactly the representation ρ_0. More precisely, the following diagram is commutative:

$$\begin{tikzcd} \mathfrak{a} \arrow[r, "\phi"] \arrow[rr, bend right, "\rho_0"'] & \frac{\mathfrak{U}(\mathfrak{a})}{I} \arrow[r, "{[\rho_0']}"] & \mathfrak{gl}(V_0) \end{tikzcd}$$

Recall that by hypothesis ρ_0 is faithful on $\mathfrak{a}$, then necessarily ϕ is injective, thus ρ is

faithful on $\mathfrak{a}$.
Eventually we show that for some $k \in \mathbb{N}$ we get $\rho(\mathfrak{n})^k = 0$.
Precisely we need to check that for sufficient high k and $\forall A_1 \cdots A_m$ general monomial in $\mathfrak{U}(\mathfrak{a})$ it is true that $\rho_{x_1} \cdots \rho_{x_k}(A_1 \cdots A_m) = 0$, where by linearity we assume that x_i lies either in $\mathfrak{a}$ or in $\mathfrak{h}$. Now, if $x_i \in \mathfrak{a}$, $\rho_{x_i}(A_1 \cdots A_m) = x_1(A_1 \cdots A_m)$. That is, the degree of the monomial has increased, so in finitely many steps we get a monomial of degree k_0 which is 0 in $\mathfrak{U}(\mathfrak{a})/I$. On the other hand, if $x_i \in \mathfrak{h}$, its action gives a sum of monomials in which a term A_i has been replaced with $[H, A_i]$. Iterating this process and using the nilpotency of $\mathfrak{n}$ (which assures that sufficiently long iterated brackets vanish), we see that a repeated adjoint action of $\mathfrak{h}$ gives the null endomorphism. □

3.2 More about nilpotent Lie algebras

Now we present some results about nilpotent Lie algebras which will be useful in the following chapter. When dealing with endomorphisms or matrices, we will call an element f (***concretely***) ***nilpotent*** if $\exists\, m \in \mathbb{N}$ such that $f^m = 0$.

Definition 3.2.1. *Let L be a Lie algebra and $x \in L$. We call x* ***ad-nilpotent*** *if ad_x is a nilpotent endomorphism.*

Lemma 3.2.1. *Let V be a finite-dimensional vector space and $x \in \mathfrak{gl}(V)$. If x is a nilpotent endomorphism, then it is ad-nilpotent.*

Proof. [1, p. 12] We associate to x two endomorphisms, λ_x and ρ_x, which are the left and the right composition respectively, namely $\forall y \in \mathfrak{gl}(V)$

$$\lambda_x(y) = xy \qquad \rho_x(y) = yx.$$

Clearly these applications are both nilpotent and if $k \in \mathbb{N}$ is such that $x^k = 0$, then also $\lambda_x^k = 0$ and $\rho_x^k = 0$. Now, ρ_x and λ_x are elements of the ring of endomorphisms and moreover they commute, thus we can observe that

$$(\lambda_x - \rho_x)^{2k} = \sum_{t=0}^{2k} \binom{2k}{t} \lambda_x^t \, (-\rho_x)^{2k-t} = 0.$$

Therefore, $ad_x = \lambda_x - \rho_x$ is nilpotent, i.e. x is ad-nilpotent. $\square$

Theorem 3.2.1. *Let V be a finite-dimensional vector space and $L \subset \mathfrak{gl}(V)$ be a subalgebra consisting of nilpotent endomorphisms. Then*

$$\exists v \in V,\ v \neq 0 \text{ such that } L(v) = 0,$$

where by $L(v) = 0$ we mean that $\forall x \in L,\ x(v) = 0$.

Proof. [1, p. 12] We proceed by induction on the dimension of L. The base case ($dim(L) = 0$ or 1) is obvious.

Suppose $K \subset L$ is a proper subalgebra, then the elements of K are nilpotent and so by Lemma 3.2.1 they are also ad-nilpotent as endomorphisms of L. Moreover, $ad(K)$ is also a subalgebra of $\mathfrak{gl}(L/K)$ since $\forall x,\ z \in K,\ ad_x(z) = xz - zx \in K$, as K is a subalgebra. Thus, $ad(K)$ descends to the quotient. Observe that $0 < dim(L/K) < dim(L)$, so the dimension of $ad(K)$ as subalgebra of $\mathfrak{gl}(L/K)$ is lower than dimension of L. In addition, the elements of $ad(K)$ are nilpotent and so by inductive hypothesis there exists an element $x + K \in L/K$, $x \notin K$, such that $ad(K)(x + K) = 0$. Equivalently,

$$\forall y \in K,\ [y, x] \equiv 0 \quad mod\ K,$$

therefore $x \in N_L(K)$. This shows that $N_L(K)$ does not contain only K.

Now suppose K is a maximal proper subalgebra of L. Then, necessarily $N_L(K) = L$ and this means that K is an ideal of L. We claim that $dim(L/K) = 1$.

Indeed, let $\pi : L \longrightarrow L/K$ be the canonical projection and suppose $dim(L/K) > 1$. Then L/K contains a one-dimensional proper subalgebra, say $\mathbb{C}[z]$ for some $z \in L \setminus K$. $\pi^{-1}(\mathbb{C}[z]) = \mathbb{C}(z + K)$. Thus, $\pi^{-1}(\mathbb{C}[z])$ is a proper subalgebra of L which contains K and this is a contradiction by the maximality of K. Then K has codimension one in L and we can write $L = K + \mathbb{C}y$ for any $y \in L \setminus K$.

By the inductive hypothesis, $W := \{v \in V \mid x(v) = 0\ \forall x \in K\} \neq 0$. Note that W is stable under L. More precisely, if $t \in L,\ x \in K,\ v \in W$, then $x(t(v)) = t(x(v)) - [t, x](v) = 0$, because $t(x(v)) = 0$ by definition of W and $[t, x] \in K$ as K is an ideal. So, $t(v) \in W$. $\mathbb{C}y$ is nilpotent, then there exists a natural number k such that $y^k(v) \neq 0$, but $y^{k+1}(v) = 0$, for any $v \in V$. Besides, given $v \in W$, $w := y^k(v)$, is an element of W since W is stable under L and $y \in L$. So we have:

$$\begin{aligned} L(w) &= K(w) + \mathbb{C}y(w) \\ &= 0 + y(y^k(v)) \\ &= 0 + 0 = 0. \end{aligned}$$

Then w is the common eigenvector which satisfies the statement.

□

Corollary 3.2.1.1. *Let V be a finite-dimensional vector space (suppose $dim(V) = n$) and $L \subset \mathfrak{gl}(V)$ be a subalgebra consisting of nilpotent endomorphisms. Then there exists a basis of V relative to which the matrices of the endomorphisms of L are all in $\mathfrak{n}(n, \mathbb{C})$. In addition, $L^n = 0$, i.e. $x_1 \cdots x_n = 0$, $\forall x_i \in L$.*

Proof. [1, p. 13] We proceed by induction on the dimension of V. The cases $dim(V) = 0$ or 1 are trivial, so we assume $dim(V) \geq 2$ and that the theorem has been proved for lower dimensions. By Theorem 3.2.1, there exists a nonzero vector $v \in V$ such that $L(v) = 0$. Let V_1 be the subspace spanned by v and $W = V/V_1$. Observe that the elements in L descend to the quotient W as $L(v) = 0$ and they are still nilpotent as endomorphisms of W. We apply the inductive hypothesis on W (whose dimension is $n - 1$), finding a basis $\{v_1, \ldots, v_{n-1}\}$ of W relative to which the matrices of L "passed" to the quotient are strictly upper triangular. Then $\{v, v_1, \ldots, v_{n-1}\}$ is a basis of V relative to which the matrices of L are strictly upper triangular (in other words they are elements of $\mathfrak{n}(n, \mathbb{C})$). Then, since the matrices of L are $n \times n$ and nilpotent, the product of n matrices $x_1 \cdots x_n$ yields 0. □

Remark 3.2.1. *If L is nilpotent, then all its elements are ad-nilpotent. Indeed L is nilpotent if for some $n \in \mathbb{N}$ and $\forall x_i$, $y \in L$ one has $ad_{x_1} \cdots ad_{x_n}(y) = 0$ and in particular for any x, $(ad_x)^n = 0$.*

Engel's Theorem states that the converse is also true.

Theorem 3.2.2 (Engel's Theorem)**.** *Let L be a Lie algebra consisting of ad-nilpotent elements. Then L is nilpotent.*

Proof. [1, p. 13] By hypothesis, all the elements in L are ad-nilpotent, thus $ad(L) \subset \mathfrak{gl}(L)$ consists of nilpotent endomorphisms and so we can apply Theorem 3.2.1. Therefore there exists a nonzero element $x \in L$ such that $ad(L)(x) = 0$, i.e., $\forall y \in L,\ [y, x] = 0$. Equivalently, there exists $x \neq 0,\ x \in Z(L)$, so $Z(L) \neq 0$. Thus, $dim(L/Z(L)) < dim(L)$ and $L/Z(L)$ still consists of ad-nilpotent elements, then we can use an induction on the dimension of L (if it is 0 then it is obvious) and find that $L/Z(L)$ is nilpotent. Now, by Proposition 1.3.2 we get that L is nilpotent too. □

Example 3.2.1. *By Lemma 3.2.1 and Engel's Theorem* $\mathfrak{n}(n, \mathbb{C})$ *is a nilpotent Lie algebra.*

Remark 3.2.2. *By Lemma 3.2.1 and Engel's Theorem, a Lie algebra consisting of concretely nilpotent endomorphisms is nilpotent in the sense of Lie algebras too. The converse is not true. Actually the Lie algebra* $\mathfrak{d}(n, \mathbb{C})$ *of diagonal matrices is abelian, hence it is nilpotent; however diagonal matrices are not nilpotent, since the powers of a nonzero diagonal matrix are still nonzero (and diagonal).*

Definition 3.2.2. *The* ***nilradical*** *of a Lie algebra* L *is the maximal nilpotent ideal of* L *and we indicate it by* $Nil(L)$.

Proposition 3.2.1. *Let* V *be a finite-dimensional vector space and* $L \subset \mathfrak{gl}(V)$. *Let* $\mathfrak{a}$ *be a nilpotent ideal of* L. *Then* $\forall x_1, \ldots, x_m \in L$ *with at least one* $x_i \in \mathfrak{a}$, *one has*

$$tr(x_1 \cdots x_m) = 0.$$

Proof. [2] Let $V_i = \mathfrak{a}^i V$ be the vector space spanned by vectors of the form $a_1 \cdots a_i v$, for any $a_1, \ldots, a_i \in \mathfrak{a}$ and $v \in V$. Assume $dim(V) = n$, then $a_1 \cdots a_n = 0$ as these elements are nilpotent (recall Corollary 3.2.1.1). We then obtain the flag of vector subspaces $V = V_0 \supset V_1 \supset \ldots \supset V_n = \{0\}$.

Now, note that $\forall i,\ V_i$ is invariant under multiplication by an element of L, i.e. $x(V_i) = x\,\mathfrak{a}^i V \subseteq V_i, \forall x \in L$. Indeed, $x\mathfrak{a}^i V = \mathfrak{a}^i x V - [\mathfrak{a}^i, x]V$ and $\mathfrak{a}^i x V$ is contained in V_i by definition, whereas $[\mathfrak{a}^i, x] \in \mathfrak{a}^i$ because $\mathfrak{a}$ is an ideal. More explicitly, we can iterate this formula for endomorphisms: $[x, a_1 a_2] = a_1[x, a_2] + [x, a_1]a_2$, where the right-hand side lies in $\mathfrak{a}^2$. Lastly note that $\mathfrak{a}V_i \subset V_{i+1}$ and so the endomorphism $x_1 \cdots x_m$ with at least one $x_j \in \mathfrak{a}$ sends each V_i in V_{i+1}. It follows that $(x_1 \cdots x_m)^n(V) = 0$, i.e. $x_1 \cdots x_m$ is nilpotent and so it has zero trace. □

Proposition 3.2.2. *Let V be a finite-dimensional vector space and $L \subset \mathfrak{gl}(V)$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of L such that $\mathfrak{b} \subset [\mathfrak{a}, L]$. If $[\mathfrak{a}, \mathfrak{b}]$ is (concretely) nilpotent, then $\mathfrak{b}$ is (concretely) nilpotent.*

Proof. [2] Let B be an element of $\mathfrak{b}$. We have to show that B is nilpotent. We use the following sufficient condition:
if $tr(B^k) = 0$ $\forall k \geq 1$, then B is a nilpotent endomorphism.
We will give the proof of this fact after proving the Proposition.
B is an element of $\mathfrak{b}$, then by hypothesis, B is a linear combination of elements of the form $[A, X]$ where $A \in \mathfrak{a}$ and $X \in L$. So, proving $tr(B^k) = 0$ is equivalent to show that $tr([A, X]B^{k-1}) = 0$. Using the properties of the trace, we have $tr([A, X]B^{k-1}) = -tr(X[A, B^{k-1}])$. Notice that $[X, YZ] = Y[X, Z] + [X, Y]Z$, hence we can write $[A, B^{k-1}] = [A, BB^{k-2}] = B[A, B^{k-2}] + [A, B]B^{k-2}$ and going on like this we obtain

$$[A, B^{k-1}] = \sum_{i=1}^{k-1} B^{i-1}[A, B]B^{k-i-1}.$$

Now we calculate

$$\begin{aligned} tr(X[A, B^{k-1}]) &= tr(X \sum_{i=1}^{k-1} B^{i-1}[A, B]B^{k-i-1}) \\ &= tr(\sum_{i=1}^{k-1} XB^{i-1}[A, B]B^{k-i-1}) \\ &= \sum_{i=1}^{k-1} tr(XB^{i-1}[A, B]B^{k-i-1}). \end{aligned}$$

Observe that the term $[A, B]$ appears in each summand, and $[A, B] \in [\mathfrak{a}, \mathfrak{b}]$ which is a (concretely) nilpotent ideal by hypothesis.
Then, by Proposition 3.2.1, $\forall i \in \{1, \ldots, k-1\}$ we have $tr(XB^{i-1}[A, B]B^{k-i-1}) = 0$, therefore $tr(B^k) = tr([A, X]B^{k-1}) = -tr(X[A, B^{k-1}]) = 0$ for any $k \geq 1$, thus $B \in \mathfrak{b}$ is nilpotent.
Now we prove the sufficient condition for nilpotency. Recall that an endomorphism (equivalently a matrix) B is nilpotent if and only if all its eigenvalues are zero. Thus, we

want to show that if $\forall k \geq 1,\ tr(B^k) = 0$, then all the eigenvalues of B are zero. Suppose B is not nilpotent, let $\lambda_1, \dots, \lambda_t$ be its dinstict non-zero eigenvalues and $m_1, \dots, m_t$ their algebraic multiplicities. Observe that $\forall i$, λ_i^k is an eigenvalue of B^k, whereas the algebraic multiplicity of λ_i^k relative to B^k is still m_i.

We are working on $\mathbb{C}$, so every matrix is triangularizable and so for any k we have $0 = tr(B^k) = \sum_{i=1}^t m_i \lambda_i^k$. Therefore, we obtain the following linear system:

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_t \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_t^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^k & \lambda_2^k & \dots & \lambda_t^k \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.5}$$

and denote by A the matrix of coefficients. Note that $det(A) = \lambda_1 \cdots \lambda_t\, det(\tilde{A})$, where

$$\tilde{A} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_t \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \dots & \lambda_t^{k-1} \end{pmatrix}$$

is a Vandermonde matrix, thus $det(A) = \lambda_1 \cdots \lambda_t \prod_{1 \leq i < j \leq t} (\lambda_i - \lambda_j) \neq 0$, since $\lambda_1, \dots, \lambda_t$ are all distinct and non-zero. Therefore, system in (3.5) has only the trivial solution, i.e. $\forall 1 \leq i \leq t,\ m_i = 0$ and this means that all the eigenvalues λ_i are null and so we have found a contradiction. $\square$

Chapter 4

The solvable case

This chapter is dedicated to the proof of Ado's Theorem for solvable Lie algebras. We will use an argument similar to the one used for nilpotent algebras.

Theorem 4.0.1. *Let L be a finite-dimensional Lie algebra and let $\mathfrak{t}$ be a solvable ideal of L. Then $[L, L] \cap \mathfrak{t}$ is a nilpotent ideal.*

Proof. [2] We may take $\mathfrak{t}$ to be the radical of L, as it is the maximal solvable ideal. We divide the proof in two cases.
Case 1
We assume $L \subset \mathfrak{gl}(V)$ for a certain finite-dimensional vector space V. Recall the definition of the derived series of $\mathfrak{t}$ (Definition 1.3.1); now we claim that $\forall i, \mathfrak{t}^{(i)}$ is an ideal of L. We prove this claim by induction on i observing that the base case $\mathfrak{t}^{(0)} = \mathfrak{t}$ follows from the definition of the radical. So we suppose that $\mathfrak{t}^{(k-1)}$ is an ideal and we check that $\mathfrak{t}^{(k)} = [\mathfrak{t}^{(k-1)}, \mathfrak{t}^{(k-1)}]$ is an ideal too. Indeed, for any $x \in L$ we have

$$\begin{aligned} [x, \mathfrak{t}^{(k)}] &= [x, [\mathfrak{t}^{(k-1)}, \mathfrak{t}^{(k-1)}]] \\ &\subseteq [\mathfrak{t}^{(k-1)}, [\mathfrak{t}^{(k-1)}, x]] + [\mathfrak{t}^{(k-1)}, [x, \mathfrak{t}^{(k-1)}]] \in \mathfrak{t}^{(k)} \end{aligned}$$

as $[\mathfrak{t}^{(k-1)}, x] \subseteq \mathfrak{t}^{(k-1)}$ by inductive hypothesis.
Now we prove the statement of the theorem by downward induction. Indeed, $\mathfrak{t}$ is solvable, therefore, for sufficiently big, k we have $\mathfrak{t}^{(k)} = 0$ and so $[L, L] \cap \mathfrak{t}^{(k)} = 0$ is trivially nilpotent. Thus, we suppose that $[L, L] \cap \mathfrak{t}^{(j)}$ is nilpotent $\forall j > i$ and we are going to show

that $[L,L]\cap\mathfrak{t}^{(i)}$ is nilpotent too. Take $[\mathfrak{t}^{(i)},[\mathfrak{t}^{(i)},L]]$ and observe that it is contained both in $[L,L]$ and in $[\mathfrak{t}^{(i)},\mathfrak{t}^{(i)}]=\mathfrak{t}^{(i+1)}$. So, $[\mathfrak{t}^{(i)},[\mathfrak{t}^{(i)},L]]\subset[L,L]\cap\mathfrak{t}^{(i+1)}$ which is nilpotent by inductive hypothesis and so $[\mathfrak{t}^{(i)},[\mathfrak{t}^{(i)},L]]$ is nilpotent. Therefore, we can apply Proposition 3.2.2 with $\mathfrak{b}=[\mathfrak{t}^{(i)},L]$ and $\mathfrak{a}=\mathfrak{t}^{(i)}$ and conclude that $[\mathfrak{t}^{(i)},L]$ is nilpotent and moreover it is an ideal. Note that $[L,[L,L]\cap\mathfrak{t}^{(i)}]$ is nilpotent since it is in $[\mathfrak{t}^{(i)},L]$ and $[L,L]\cap\mathfrak{t}^{(i)}\subset[L,L]$. Consequently, we are able to apply again Proposition 3.2.2 with $\mathfrak{a}=L$ and $\mathfrak{b}=[L,L]\cap\mathfrak{t}^{(i)}$ and conclude $[L,L]\cap\mathfrak{t}^{(i)}$ is nilpotent.
Case 2
Now let L be any Lie algebra. Using the adjoint representation we can identify $L/Z(L)$ with a subalgebra of $\mathfrak{gl}(L)$ and so we can apply the previous case to it. Therefore, $([L,L]\cap\mathfrak{t})/Z(L)$ is nilpotent and by Proposition 1.3.2 the same goes for $[L,L]\cap\mathfrak{t}$. □

Corollary 4.0.1.1. *Let L be a finite-dimensional Lie algebra and let $\mathfrak{t}$ be a solvable ideal of L. Then for any derivation $\delta : L\longrightarrow L$ of L, $\delta(\mathfrak{t})$ is nilpotent.*

Proof. [2] We divide the proof in two cases. We first consider the inner derivations, i.e. the derivations of the form ad_x for $x\in L$.
Then $\delta(\mathfrak{t})=ad_x(\mathfrak{t})=[x,\mathfrak{t}]\subset[L,L]\cap\mathfrak{t}$ and we conclude using Theorem 4.0.1.
Now let δ be any derivation of L. The idea is to embed L in a larger algebra L' in such a way that δ can be regarded as an inner derivation of L', see [5, p. 6]. More precisely, we define the structure of ***semidirect product*** of L and $\mathbb{C}$ with respect to δ, which is indicated by $L\rtimes_\delta\mathbb{C}$. Here is how we construct this new space. We consider $L' := L\oplus\mathbb{C}$ as a direct sum of vector spaces and we define a bracket on L' as follows; for $x,y\in L$ and $a,b\in\mathbb{C}$:

$$[(x,a),(y,b)]=([x,y]+a\delta(y)-b\delta(x),0). \tag{4.1}$$

A standard check shows that $L'=L\rtimes_\delta\mathbb{C}$ with bracket (4.1) is a Lie algebra. Consider now the injection $\nu : L\longrightarrow L\rtimes_\delta\mathbb{C}$, $\nu(x)=(x,0)$, for $x\in L$. We note that $[\nu(x),\nu(y)]=[(x,0),(y,0)]=([x,y]+0\delta(y)-0\delta(x),0)=([x,y],0)=\nu([x,y])$, so ν is a Lie algebra homomorphism (linearity is obvious). Therefore we have an embedding of Lie algebras $L\hookrightarrow L'$ and we can identify $L\cong\{(x,0)|x\in L\}$. In particular, L is a subalgebra of $L\rtimes_\delta\mathbb{C}$. Now take $w=(0,1)\in L\rtimes_\delta\mathbb{C}$ and consider the inner derivation $ad_w : L'\longrightarrow L'$. For any

$x' = (x, 0) \in L$, we have $ad_w(x') = [(0,1),(x,0)] = ([0,x] + 1\delta(x) - 0\delta(0), 0) = (\delta(x), 0)$, i.e. δ is the restriction of the inner derivation ad_w to the subalgebra L. We make a few remarks before concluding. Firstly, $(\mathfrak{t}, 0)$ is a solvable ideal of $L \rtimes_\delta \mathbb{C}$. Indeed, by definition, there exists a natural number i such that $\mathfrak{t}^{(i)} = 0$. The derived series of $(\mathfrak{t}, 0)$ consists of terms $(\mathfrak{t}, 0)^{(k)} = (\mathfrak{t}^{(k)}, 0) = [(\mathfrak{t}^{(k-1)}, 0), (\mathfrak{t}^{k-1}, 0)]$. We note that $(\mathfrak{t}, 0)^{(i)} = [(\mathfrak{t}^{(i-1)}, 0), (\mathfrak{t}^{(i-1)}, 0)] = (0,0)$, so $(\mathfrak{t}, 0)$ is solvable. Similarly, if $\mathfrak{n}$ is a nilpotent ideal of L, then $\tilde{\mathfrak{n}} = (\mathfrak{n}, 0)$ is a nilpotent ideal of $L \rtimes_\delta \mathbb{C}$. Indeed, it is easy to see by induction that $\tilde{\mathfrak{n}}^k = (\mathfrak{n}^k, 0)$. Then, the lower central series of $\mathfrak{n}$ converges to 0 if and only if the lower central series of $\tilde{\mathfrak{n}}$ does the same. So, $(\delta(\mathfrak{t}), 0) = ad_w((\mathfrak{t}, 0))$ is nilpotent because $(\mathfrak{t}, 0)$ is solvable and ad_w is an inner derivation. Using what we have just observed about the relation between nilpotent ideals of L and those of L', we conclude that $\delta(\mathfrak{t})$ is nilpotent too. $\square$

Theorem 4.0.2 (Ado's Theorem for solvable Lie algebras)**.** *Let $\mathfrak{t}$ be a finite-dimensional, solvable Lie algebra.*
Then there exists a finite-dimensional, faithful representation Φ of $\mathfrak{t}$. Moreover, if $\mathfrak{n}$ is the nilradical of $\mathfrak{t}$, then $\Phi(\mathfrak{n})$ is nilpotent.

Proof. As in Theorem 3.1.1, we construct the map Φ combining the adjoint action of $\mathfrak{t}$ with a map ρ which is faithful on the centre. In addition we want both $ad(\mathfrak{n})$ and $\rho(\mathfrak{n})$ to be nilpotent. the map ad satisfies this property, since if $x \in \mathfrak{n}$ and $y \in \mathfrak{t}$ then $[x, y] \in \mathfrak{n}$, so applying ad_x sufficiently many times will yield 0.
We proceed by induction on the dimension of $\mathfrak{t}/\mathfrak{n}$. When $dim(\mathfrak{t}/\mathfrak{n}) = 0$, then $\mathfrak{t}$ is nilpotent and therefore Theorem 3.1.1 holds. So, we may suppose $dim(\mathfrak{t}/\mathfrak{n}) > 0$ and that the theorem has been proved for all solvable algebras $\tilde{\mathfrak{t}}$ such that $dim(\tilde{\mathfrak{t}}/\tilde{\mathfrak{n}}) < dim(\mathfrak{t}/\mathfrak{n})$. We are looking for a representation of $\mathfrak{t}$ which is faithful on the centre, but actually it suffices to construct a finite-dimensional representation of $\mathfrak{t}$ that is faithful on $\mathfrak{n}$, since $Z(\mathfrak{t}) \subset \mathfrak{n}$. Just as in Theorem 3.1.1, we decompose $\mathfrak{t}$ as

$$\mathfrak{t} = \mathfrak{a} \oplus \mathfrak{h}$$

where $\mathfrak{a}$ is a codimension 1 (hence solvable) ideal, $\mathfrak{a} \supset \mathfrak{n}$ and $\mathfrak{h}$ is a complementary subspace (which has dimension 1, so it is abelian). Notice that $Nil(\mathfrak{a}) = Nil(\mathfrak{t}) = \mathfrak{n}$.

By inductive hypothesis, there exists a finite-dimensional, faithful representation

$$\rho_0 : \mathfrak{a} \longrightarrow \mathfrak{gl}(V_0)$$

such that $\rho_0(\mathfrak{n})$ is nilpotent. In particular, by Corollary 3.2.1.1,

$$\exists k \in \mathbb{N} \text{ such that } \rho_0(\mathfrak{n})^k = 0. \tag{4.2}$$

We can extend ρ_0 to a representation of $\mathfrak{U}(\mathfrak{a})$ just as we did in the proof of Theorem 3.1.1, i.e. defining $\rho_0' : \mathfrak{U}(\mathfrak{a}) \longrightarrow \mathfrak{gl}(V_0)$, $\rho_0'(x_1 \cdots x_n) = \rho_0(x_1) \cdots \rho_0(x_n)$ for a general monomial $x_1 \cdots x_n \in \mathfrak{U}(\mathfrak{a})$.
Repeating the same argument used in the nilpotent case, we define a representation $\hat{\rho} : \mathfrak{a} \oplus \mathfrak{h} \longrightarrow \mathfrak{gl}(\mathfrak{U}(\mathfrak{a}))$ which is faithful on $\mathfrak{a}$ (recall that $\mathfrak{a}$ acts by left multiplication and $\mathfrak{h}$ by adjoint representation extended on $\mathfrak{U}(\mathfrak{a})$ by the Leibniz rule). Now we consider the two-sided ideal I of $\mathfrak{U}(\mathfrak{a})$ generated by $\mathfrak{n}$ together with $Ker(\rho_0')$. However, we are interested in I^k and we claim that $\mathfrak{U}(\mathfrak{a})/I^k$ is finite-dimensional. Indeed, for any element $A \in \mathfrak{a}$, $\rho_0(A) \in \mathfrak{gl}(V_0)$ and by Cayley-Hamilton Theorem there exists a monic polynomial p such that $\rho_0(p(A)) = p(\rho_0(A)) = 0$. Thus, $p(A) \in Ker(\rho_0) \subset I$, then $p(A)^k \in I^k$ and so $p(A)^k \equiv 0 \mod I^k$. This means that we can rewrite any sufficiently high power of A as a polynomial in A with lower degree, hence we can express each monomial of sufficiently high degree in $\mathfrak{U}(\mathfrak{a})$ in terms of monomials of lower degree modulo I^k. Therefore $\mathfrak{U}(\mathfrak{a})/I^k$ is generated by these monomials of bounded degree, since $\mathfrak{a}$ is finite-dimensional, there are only finitely many such monomials.
Now we want to project $\hat{\rho}$ to a finite-dimensional representation. In order to do so, first observe that I^k is stable under left multiplication by elements of $\mathfrak{a}$. Moreover, $\mathfrak{h}$ has an adjoint action on $\mathfrak{a}$ because $[\mathfrak{h}, \mathfrak{a}] \subset \mathfrak{a}$, and $\forall H \in \mathfrak{h}$, ad_H is a derivation of $\mathfrak{a}$, then by Corollary 4.0.1.1 $ad_H(\mathfrak{a}) \subset \mathfrak{n} \subset I$. Just as in (3.2) we extend the adjoint action of $\mathfrak{h}$ on the entire $\mathfrak{U}(\mathfrak{a})$ and we have $\mathfrak{h}.I \subset I$; in addition, $\mathfrak{h}.I^k \subset I^k$ thanks to Leibniz rule. Then, $\hat{\rho}$ descends to a map

$$\rho : \mathfrak{a} \oplus \mathfrak{h} \longrightarrow \mathfrak{gl}\left(\frac{\mathfrak{U}(\mathfrak{a})}{I^k}\right)$$

and we have to verify that it is faithful on $\mathfrak{a}$. Repeating the same argument as in

Theorem 3.1.1 we have to show that the map $\phi : \mathfrak{a} \longrightarrow \mathfrak{U}(\mathfrak{a})/I^k$ is injective. Observe that $\rho_0'(I^k) = 0$, since I^k is generated by $Ker(\rho_0')$ and $\mathfrak{n}$ and by (4.2) $\rho_0'(\mathfrak{n}) = 0$. Thus, ρ_0' passes to the quotient:

$$[\rho_0'] : \frac{\mathfrak{U}(\mathfrak{a})}{I^k} \longrightarrow \mathfrak{gl}(V_0).$$

We get the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{a} \arrow[r, "\phi"] \arrow[rr, bend right, "\rho_0"'] & \frac{\mathfrak{U}(\mathfrak{a})}{I^k} \arrow[r, "{[\rho_0']}"] & \mathfrak{gl}(V_0)
\end{tikzcd}$$

and by inductive hypothesis, ρ_0 is injective, hence the same goes for ϕ.
Finally, observe that for any $x \in \mathfrak{n}$, the action of x on $\mathfrak{U}(\mathfrak{a})/I^k$ is the left multiplication and due to the fact that $\mathfrak{n} \subset I$, $\rho_x(y) \in \mathfrak{n}$, $\forall y \in \mathfrak{U}(\mathfrak{a})/I^k$. So, $\rho_x^k(y) \in I^k$ and therefore is 0 in $\mathfrak{U}(\mathfrak{a})/I^k$. To summarize,

$$\rho(\mathfrak{n})^k \left(\frac{\mathfrak{U}(\mathfrak{a})}{I^k} \right) = 0,$$

i.e. $\rho(\mathfrak{n})$ is nilpotent. $\square$

Chapter 5

Levi decomposition and the general case

In this chapter we conclude the proof of Ado's Theorem, showing that the statement holds for any finite-dimensional Lie algebra. In order to obtain the result we will use the Levi decomposition, namely the decomposition of a Lie algebra L as $L = Rad(L) \oplus \mathfrak{h}$, where $\mathfrak{h}$ is a semisimple Lie algebra (necessarily). Levi decomposition enables us to reduce the proof to the solvable case.

Lemma 5.0.1. *Let L be a Lie algebra and $\mathfrak{a} \subset L$ be a solvable ideal. Then*

$$Rad\left(\frac{L}{\mathfrak{a}}\right) = \frac{Rad(L)}{\mathfrak{a}}.$$

Proof. $\subseteq$) $Rad(L/\mathfrak{a})$ is the maximal solvable ideal of $L/\mathfrak{a}$, then it has the form $\mathfrak{t}/\mathfrak{a}$, where $\mathfrak{t}$ is an ideal of L containing $\mathfrak{a}$. Both $\mathfrak{t}/\mathfrak{a}$ and $\mathfrak{a}$ are solvable, then by Proposition 1.3.1 $\mathfrak{t}$ is solvable too. Thus, $\mathfrak{t} \subseteq Rad(L)$ and so $Rad(L/\mathfrak{a}) = \mathfrak{t}/\mathfrak{a} \subseteq Rad(L)/\mathfrak{a}$.
$\supseteq$) $Rad(L)/\mathfrak{a}$ is a solvable ideal of $L/\mathfrak{a}$ and so it is contained in its maximal solvable ideal, i.e., $Rad(L)/\mathfrak{a} \subseteq Rad(L/\mathfrak{a})$. $\square$

Theorem 5.0.1 (Levi decomposition)**.** *Let L be a finite-dimensional Lie algebra. Then there exists a subalgebra $\mathfrak{h} \subset L$ called **Levi subalgebra**, which gives the vector space* *decomposition*

$$L = Rad(L) \oplus \mathfrak{h}$$

Proof. [6, pp. 499-500] Firstly, we make some useful reductions.

1. We may assume that $Rad(L)$ does not contain any proper nonzero ideal of L. Otherwise, let $\mathfrak{a} \neq 0$ be a solvable ideal of L (in particular $\mathfrak{a} \subset Rad(L)$) and consider the quotient algebra $L/\mathfrak{a}$. Then, by induction on the dimension of L, $L/\mathfrak{a}$ admits a Levi subalgebra, namely we can write $L/\mathfrak{a} = Rad(L/\mathfrak{a}) \oplus \mathfrak{s}/\mathfrak{a}$. By Lemma 5.0.1 we have $L/\mathfrak{a} = Rad(L)/\mathfrak{a} \oplus \mathfrak{s}/\mathfrak{a}$. $\mathfrak{s}/\mathfrak{a}$ is necessarily semisimple, so $\mathfrak{a} = Rad(\mathfrak{s})$. Using again the induction, $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{s}'$, with $\mathfrak{s}'$ semisimple. Consequently, $L = Rad(L) \oplus \mathfrak{s}'$, so $\mathfrak{s}'$ is the Levi subalgebra for L and the theorem is proved.

2. We may take $Rad(L)$ to be abelian. Otherwise, $Rad(L)^1 = [Rad(L), Rad(L)]$ would be a proper nonzero ideal of $Rad(L)$ and then we would conclude by reduction 1.

3. Note that $[L, Rad(L)]$ is an ideal of L contained in $Rad(L)$. Thus, by reduction 1 $[L, Rad(L)]$ is either 0 or L. Suppose $[L, Rad(L)] = 0$, then $Rad(L) = Z(L)$ hence $Rad(L) = Ker(ad)$, where ad is the adjoint representation of L. We get a representation of $L/Rad(L)$ on L and since $L/Rad(L)$ is semisimple, we can use the Weyl Theorem [1], stating that:
   every finite-dimensional representation of a semisimple Lie algebra is completely reducible.
   Thus, we can regard L as a completely reducible $L/Rad(L)$-module. This means that for any submodule of L there exists a direct summand. Therefore we have $L = Rad(L) \oplus \mathfrak{h}$, and $\mathfrak{h}$ is the Levi subalgebra. So, we may finally assume $[L, Rad(L)] = Rad(L)$. Notice that this reduction implies $Rad(L) \cap Z(L) = 0$. Otherwise, $Z(L)$ would be a nonzero ideal of $Rad(L)$ and by reduction 1 we would get $Z(L) = Rad(L)$ and this is a contradiction since $[L, Rad(L)] = Rad(L) \neq 0$.

Now we construct a representation of L on $\mathfrak{gl}(L)$ defining $\mu : L \longrightarrow \mathfrak{gl}(\mathfrak{gl}(L))$ as

$$\mu(x)(\xi) = [ad_x, \xi] = ad_x \circ \xi - \xi \circ ad_x \tag{5.1}$$

for $x \in L$ and $\xi \in \mathfrak{gl}(L)$. Equivalently, for any $y \in L$, $(x.\xi)(y) = [x, \xi(y)] - \xi([x, y])$. Consider the following subspaces of $\mathfrak{gl}(L)$:

$$
\begin{aligned}
A &= \{ad_x \,|\, x \in Rad(L)\} \\
B &= \{\xi \in \mathfrak{gl}(L) \,|\, \xi(L) \subset Rad(L),\ \xi(Rad(L)) = 0\} \\
C &= \{\xi \in \mathfrak{gl}(L) \,|\, \xi(L) \subset Rad(L),\ \xi|_{Rad(L)} \text{ is multiplication by a scalar}\}.
\end{aligned}
$$

Observe that $B \subset C$ by construction and $A \subset B$ because $Rad(L)$ is an ideal (so $ad_x(L) \subset Rad(L)$) and it is abelian by reduction 2 (so $ad_x(Rad(L)) = 0$ for any $x \in Rad(L)$).
Thus $A \subset B \subset C$. In addition, $A,\ B,\ C$ are all L-submodules of $\mathfrak{gl}(L)$. Indeed, for any $ad_x \in A$, $y \in L$, and $z \in \mathfrak{gl}(L)$ we have $y.\, ad_x(z) = [ad_y, ad_x](z) = ad_{[y,x]}(z) = ad_{-[x,y]}(z)$. As $[x, y] \in Rad(L)$ we have proved that $L.\, A \subset A$. By similar easy checks one can prove that also C and B are L-submodules.
We can set $\nu : C \longrightarrow \mathbb{C}$ as $\nu(\xi) = \lambda$ if ξ acts on $Rad(L)$ as the multiplication by λ. Notice that ν is a linear map between vector spaces, $Ker(\nu) = B$, so $C/B \cong \mathbb{C}$ as vector spaces. Moreover, $[\nu(\xi), \nu(\xi')] = [\lambda, \lambda'] = 0,\ \forall \xi, \xi' \in C$ and also $\nu_{[\xi,\xi']} = 0$, therefore $C/B \cong \mathbb{C}$ also as Lie algebras.
We also claim that $L.\, C \subset B$ and $Rad(L).C \subset A$. Indeed, take $\xi \in C$ and assume $\xi(y) = \lambda y, \forall y \in Rad(L)$. If $x \in L,\ y \in Rad(L)$, then

$$x.\xi(y) = [x, \lambda y] - \xi([x, y]) = \lambda[x, y] - \lambda[x, y] = 0 \tag{5.2}$$

so $x.\xi \in B$. If $x, y \in Rad(L)$, then

$$x.\xi(y) = [x, \xi(y)] - \lambda[x, y] = 0 + ad_{-\lambda x}(y), \tag{5.3}$$

so $x.\xi \in A$. Therefore, C/A and C/B are both $L/Rad(L)$-modules. Let us explain why C/A is an $L/Rad(L)$-module; almost the same argument can be used for C/B. We already know that C is an L-submodule, so we can consider the map $\mu : L \longrightarrow \mathfrak{gl}(C)$. We wonder whether for any $x \in L$, $\mu(x)$ can descend to an endomorphism of C/A. The answer is yes, as A is a L-submodule and so $L.A \subset A$. It only remains to verify that this map passes also to the quotient $L/Rad(L)$, but this is true as $Rad(L).C \subset A$ and so any element of $Rad(L)$ corresponds to the null endomorphism of C/A (indeed, such an element "sends" all C in A which is zero in the quotient).

Now, C/A is an $L/Rad(L)$-module, $L/Rad(L)$ is semisimple, hence by Weyl Theorem C/A splits as $C/A = D/A \oplus C/B$, where D/A is codimension-1 complement. C/B is 1-dimensional; let $[\varphi]$ be a generator of C/B in C/A. Up to normalization, we may assume $\varphi|_{Rad(L)} = id|_{Rad(L)}$. Moreover, $L/Rad(L)$ acts trivially on C/B since it is 1-dimensional, hence $L.\varphi \in A$.

We define $\mathfrak{h} = \{x \in L \mid x.\varphi = 0\}$ and observe that $\mathfrak{h}$ is a subalgebra of L. We are going to show that $\mathfrak{h}$ is the Levi subalgebra of L.

Firstly, $\mathfrak{h} \cap Rad(L) = 0$. Otherwise, let $x \neq 0$ be an element of the intersection. By construction, φ acts on $Rad(L)$ as the multiplication by 1, thus combining this with (5.3), we get $x.\varphi = ad_{-x}$. On the other hand, $x \in \mathfrak{h}$, so $x.\varphi = 0$, then $ad_{-x} = x.\varphi = 0$, that is $ad_x = 0$. This means $[L, x] = 0$, i.e. $x \in Z(L) \cap Rad(L)$ and this is a contradiction by reduction 3. Now we only have to show that $L = Rad(L) + \mathfrak{h}$. Take x in L. $L.\varphi \in A$, thus there exists $y \in Rad(L)$ such that $x.\varphi = ad_y$. Combining the actions of x and y and recalling $\varphi|_{Rad(L)} = id|_{Rad(L)}$, we get

$$\begin{aligned}
(x+y).\varphi &= x.\varphi + y.\varphi \\
&= ad_y + (ad_y \circ \varphi - \varphi \circ ad_y) \\
&= ad_y + (0 - ad_y) \\
&= 0,
\end{aligned}$$

thus $x + y \in \mathfrak{h}$. More precisely, there exists $z \in \mathfrak{h}$ such that $x + y = z$. To summarize, $\forall x \in L$ there exist $y' \in Rad(L)$, $y' = -y$ and $z \in \mathfrak{h}$ such that $x = y' + z$; in addition $Rad(L) \cap \mathfrak{h} = 0$. This means $L = Rad(L) \oplus \mathfrak{h}$. $\square$

Theorem 5.0.2 (Ado's Theorem)**.** *Let L be a finite-dimensional Lie algebra. Then there exists a finite-dimensional, faithful representation $\Phi : L \longrightarrow \mathfrak{gl}(V)$ of L. Furthermore, let $\mathfrak{n}$ be the nilradical of L. Then $\Phi(\mathfrak{n})$ is nilpotent.*

Proof. By Levi decomposition, we can split L in $L = Rad(L) \oplus \mathfrak{h}$, where $\mathfrak{h}$ is necessarily semisimple. Observe that $\mathfrak{n}$ is nilpotent and then solvable, so $\mathfrak{n} \subset Rad(L)$ and $Nil(Rad(L)) = \mathfrak{n}$. As we did in the nilpotent and solvable cases, we have to find a finite-dimensional representation ρ of L which is faithful on the centre (notice that the $Z(L) \subset \mathfrak{n}$) and sum it with the adjoint representation. We can repeat essentialy verbatim the proof of Ado's Theorem for solvable Lie algebras (Theorem 4.0.2). Indeed, in the solvable case there exists a finite-dimensional, faithful representation

$$\rho_0 : Rad(L) \longrightarrow \mathfrak{gl}(V_0)$$

such that $\rho_0(\mathfrak{n})^k = 0$ for some k. Furthermore, we can extend it on the universal enveloping algebra by

$$\rho_0' : \mathfrak{U}(Rad(L)) \longrightarrow \mathfrak{gl}(V_0).$$

We construct a representation of L on $\mathfrak{U}(Rad(L))$ in the usual way

$$\hat{\rho} : Rad(L) \oplus \mathfrak{h} \longrightarrow \mathfrak{gl}(\mathfrak{U}(Rad(L)))$$

where $Rad(L)$ acts on its universal enveloping algebra by left multiplication and $\mathfrak{h}$ by the "extended" adjoint representation. Then we consider the two sided ideal I of $\mathfrak{U}(Rad(L))$ generated by $Ker(\rho_0')$ together with $\mathfrak{n}$. Just as in Theorem 4.0.2 we can prove that $\mathfrak{U}(Rad(L))/I^k$ is finite-dimensional and that I^k is stable under the action of $Rad(L) \oplus \mathfrak{h}$. Therefore, we obtain

$$\rho : Rad(L) \oplus \mathfrak{h} \longrightarrow \frac{\mathfrak{gl}(\mathfrak{U}(Rad(L)))}{I^k}.$$

The proof of faithfulness and nilpotency of ρ is exactly the same as in the solvable case, so we do not repeat it now. $\square$

Appendix

We may compare the proof of Ado's Theorem presented in this thesis with an alternative proof, for example the one on [6]. This second proof is more direct, but needs a preliminary result which is quite complicated to prove. We only present the statement of this preliminary proposition and a useful definition; then we try to focus on the main steps of this different proof.

Definition 5.0.1. *Let ρ be a finite-dimensional representation of a Lie algebra L. If for every $x \in Nil(L)$ $\rho(x)$ is a nilpotent endomorphism, then ρ is called a **nilrepresentation**.*

Proposition 5.0.1. *Let L be a Lie algebra which is a sum $L = \mathfrak{a} \oplus \mathfrak{h}$ of a solvable ideal $\mathfrak{a}$ and a subalgebra $\mathfrak{h}$. Let σ be a finite-dimensional nilrepresentation of $\mathfrak{a}$. Then there exists a finite-dimensional representation ρ of L such that $\mathfrak{a} \cap Ker(\rho) \subset Ker(\sigma)$. If $Nil(L) = Nil(\mathfrak{a})$ or $Nil(L) = L$, then ρ may be taken to be a nilrepresentation.*

Here is a sketch of the proof Ado's Theorem following [6].

1. We construct a finite-dimensional, faithful representation of $Z(L)$. We may call this representation ρ_0.

2. We consider the following sequence of subalgebras of L:

$$Z(L) \subset L_1 \subset \ldots \subset L_k = Nil(L) \subset \ldots \subset L_m = Rad(L) \subset L_{m+1} = L,$$

   where each algebra is a solvable ideal of the next and $dim(L_{i+1}) = dim(L_i) + 1$ for $i \leq m$. Therefore for $i \leq m-1$ we have $L_{i+1} = L_i \oplus \mathbb{C}v_i$ for some $v_i \in L_{i+1} \setminus L_i$

and for $i = m$ we have $L = Rad(L) \oplus \mathfrak{h}$ by Levi decomposition.

3. We would like to use Proposition 5.0.1 inductively, as each subalgebra is a direct sum of a solvable ideal and a complement. We may verify that $\forall i$ the map $\rho_i : L_i \longrightarrow \mathfrak{gl}(V_i)$ is a nilrepresentation. ρ_0 satisfies the hypothesis; in addition, $\forall i \in \{1, \ldots, k\},\ Nil(L_i) = L_i$ and $\forall i \geq k\ Nil(L_i) = Nil(L)$, thus by Proposition 5.0.1, each ρ_i is a nilrepresentation.

4. We observe that $\forall i \in \{1, \ldots, m+1\},\ Ker(\rho_i) \cap Z(L) = 0$.

5. We define $\Phi \coloneqq \rho_{m+1} \oplus ad$, where $\rho_{m+1} : L \longrightarrow \mathfrak{gl}(V)$ is a nilrepresentation and $dim(V) < +\infty$. Φ is a finite-dimensional representation of L and it is faithful since $Ker(\Phi) = Ker(\rho_{m+1}) \cap Ker(ad) = Ker(\rho_{m+1}) \cap Z(L) = 0$ by (4).

Notice that this proof does not require directly the universal enveloping algebra; however, in order to prove Proposition 5.0.1 one needs it. We may also observe that both the proofs use the adjoint representation and the final representation is a sum of ad with another map. Finally, we notice that Levi decomposition is essential to prove Ado's Theorem.

Acknowledgements

Writing this thesis was the final act of a three year long journey as a student of mathematics, so it is also an opportunity to reflect on what I have learnt and experienced.

When I started studying math, I could not expect I would have loved it so much. I was not the kind of student naturally gifted for mathematics. I had never participated to Olympiads of mathematics, I had never participated in projects organized by universities to make young students love maths. I was just a good pupil with good marks. However, I have always perceived mathematics as the highest discipline a human being could study and I regarded mathematicians as genius and class of people I had nothing to do with because of their mind. Thus, probably I chose to study mathematics both to prove myself I was able to understand the most complicated theories and because solving high school's mathematical problems was enjoyable and relaxing (later I discovered maths' exercises are not just about calculating derivatives and drawing graphs). So, I enrolled in first year of university. I have to say the first semester was hard, but then little by little I learnt how to deal with new definitions, theorems ecc. In these three years I have improved my study method and the logic and abstract language of mathematics has become familiar to me. Moreover, the time of reasoning and understanding has become precious to me. When I am upset, or there is some unpleasant thought bothering me, I often take refugee in my homework or in some proof I have not yet understood, because maths absorbs completely my mind. I also have to thank mathematics for having taught me to accept (or at least to suffer) errors. Committing mistakes is normal when trying to understand something new; in addition a "good mistake" can help you comprehend in depth, as it shows which incoherent consequences a wrong argument may imply. More concretely, I have become used to mistakes also because university's exams are very demanding, so it is normal making a not excellent performances. Even though imperfection is an anguishing aspect of human existence, it is good to learn how to face it and realize we are not always the number one. Now I would like to thank the people who have helped me in these last months. Firstly, my supervisor, Professor Nicoletta Cantarini. You have always welcomed me in your office with kindness and you have been so generous answering my questions, suggesting me how to improve my work and mostly, making me feel at ease. I am very thankful for having addressed me in the study of Ado's Theorem, as it let me study many interesting and new things. I would also like to thank my university friends. We have supported each other with love, respect and sincere help. We have shared the same experiences and having you on my side was fundamental. In addition I would like to thank my parents, you are always there for me. Last, but not least I would like to thank Alessandro for always believing in me and pushing me to give my best. Any experience, when done with you, has a better taste.

Bibliography

[1] James E. Humphreys. *Introduction to Lie algebras and representation theory.* Springer-Verlag New York, 1972.

[2] T. Tao. *Ado's theorem.* `https://terrytao.wordpress.com/2011/05/10/ados-theorem/`.

[3] V. Gubarev D. Burde. *In memory of Igor Dmitrievich Ado.* `https://arxiv.org/pdf/1908.08361.pdf`.

[4] V. Kac. *Lecture 22 – The Universal Enveloping Algebra.* `https://math.mit.edu/classes/18.745/Notes/Lecture_22_Notes.pdf`.

[5] S. Asgarli. *Ado's theorem.* `https://personal.math.ubc.ca/~reichst/Ado's-Theorem.pdf`.

[6] W. Fulton and J. Harris. *Representation Theory: A First Course.* Springer New York, 1991.

[7] A. A. Swaminathan. *Lie algebras and Ado's theorem.* `https://scholar.princeton.edu/sites/default/files/ashvin/files/ado.pdf`.