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Nilpotent orbits in semisiple Lie algebras

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Introduction

This thesis is dedicated to the introductory study of the so-called nilpotent orbits in a semisimple complex Lie algebra $\mathfrak{g}$, i.e., the orbits of nilpotent elements under the adjoint action of the adjoint group G_{ad} with Lie algebra $\mathfrak{g}$. These orbits have an extremely rich structure and lie at the interface of Lie theory, algebraic geometry, symplectic geometry, and geometric representation theory. The interest in these objects has been long-standing, ranging from Kostant's foundational work in the 1950s and 1960s to Kronheimer's realization of nilpotent orbits as moduli spaces. At the same time, nilpotent orbits are often studied for the sake of understanding closely associated varieties.

In the case of a linear Lie algebra, i.e., a subalgebra of $\mathfrak{gl}(\mathfrak{n}, \mathbb{C})$, it is clear what a nilpotent element is. For an abstract Lie algebra $\mathfrak{g}$ a nilpotent element x is an element such that $ad(x)$ is nilpotent in $\mathfrak{gl}(\mathfrak{g})$. Of course, in the case of a linear Lie algebra the two definitions coincide.

The Jacobson and Morozov Theorem relates the orbit of a nilpotent element X in a semisimple complex Lie algebra $\mathfrak{g}$ with a triple $\{H, X, Y\}$ that generates a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. There is a parabolic subalgebra associated to this triple that permits to attach a weight to each node of the Dynkin diagram of $\mathfrak{g}$. The resulting diagram is called a weighted Dynkin diagram associated with the nilpotent orbit of X. This is a complete invariant of the orbit (see Theorem 5.9) that one can use in order to show that there are only finitely many nilpotent orbits in $\mathfrak{g}$.

The classical Dynkin-Kostant classification of nilpotent orbits is given. First, one constructs a one-to-one correspondence between nilpotent orbits and conjugacy classes of standard triples in $\mathfrak{g}$. The proof that this correspondence is surjective depends on the Jacobson-Morozov Theorem and a theorem of Kostant proves it is injective. Second, one shows that conjugacy classes of standard triples in $\mathfrak{g}$ are in one-to-one correspondence with certain distinguished semisimple orbits; this uses a second conjugacy theorem, this time due to Mal'cev.

In a sense, all these classification results are a demonstration of the magnificent effectiveness of the representation theory of $\mathfrak{sl}(2, \mathbb{C})$.

The thesis is organized as follows: the first three chapters contain some preliminary material on Lie algebras (Chapter 1), on Lie groups (Chapter 3) and on the representation theory of $\mathfrak{sl}(2, \mathbb{C})$ (Chapter 2). Chapter 4 and 5 are the heart of the thesis. Namely, Jacobson-Morozov, Kostant and Mal'cev Theorems are proved in Chapter 4 and Chapter 5 is dedicated to the construction of weighted Dynkin diagrams. As an example the conjugacy classes of nilpotent elements in $\mathfrak{sl}(n, \mathbb{C})$ are described in detail and a formula for their dimension is given (see Theorem 5.17). In this case, as well as in the case of all classical Lie algebras, the description of the orbits can be done in terms of partitions and tableaux.

Introduzione

Questa tesi è un'introduzione allo studio delle cosiddette orbite nilpotenti di un'algebra di Lie complessa e semisemplice $\mathfrak{g}$, cioè le orbite di elementi nilpotenti rispetto all'azione del gruppo aggiunto G_{ad} con algebra di Lie $\mathfrak{g}$. Queste orbite hanno una struttura estremamente ricca e sono alla base della teoria di Lie, della geometria algebrica, della geometria simplettica e della teoria geometrica delle rappresentazioni. L'interesse verso questi oggetti è di lunga data, partendo dal lavoro fondamentale di Kostant negli anni '50 e '60 fino alla realizzazione delle orbite nilpotenti come spazio di moduli dovuta a Kronheimer. Allo stesso tempo, le orbite nilpotenti sono studiate spesso per comprendere le varietà ad esse associate.

Nel caso di un'algebra di Lie lineare, cioè di una sottoalgebra di $\mathfrak{gl}(\mathfrak{n}, \mathbb{C})$, è chiaro cosa sia un elemento nilpotente. Per un'algebra di Lie $\mathfrak{g}$ astratta, un elemento nilpotente x è un elemento tale che $ad(x)$ sia nilpotente in $\mathfrak{gl}(\mathfrak{g})$. Naturalmente, nel caso di un'algebra di Lie lineare le due definizioni coincidono.

Il teorema di Jacobson e Morozov mette in relazione l'orbita di un elemento nilpotente X di un'algebra di Lie complessa e semisemplice $\mathfrak{g}$ con una tripla $\{H, X, Y\}$ che genera una sottoalgebra di $\mathfrak{g}$ isomorfa a $\mathfrak{sl}(\mathfrak{n}, \mathbb{C})$. A questa tripla si può associare una sottoalgebra parabolica che permette di attribuire un peso ad ogni nodo del diagramma di Dynkin di $\mathfrak{g}$. Il diagramma che si ottiene è detto diagramma di Dynkin pesato associato all'orbita nilpotente di X. Questo è un invariante completo dell'orbita (vedi Teorema 5.9) che si può usare per dimostrare che le orbite nilpotenti di $\mathfrak{g}$ sono finite.

Si ottiene dunque la classificazione di Dynkin-Kostant delle orbite nilpotenti. Dapprima, si costruisce una corrispondenza biiettiva tra le orbite nilpotenti e le classi di coniugio delle triple standard di $\mathfrak{g}$. Il teorema di Jacobson-Morozov prova la suriettività di questa corrispondenza e un teorema di Kostant ne prova l'iniettivitá. Si prosegue dimostrando che le classi di coniugio delle triple standard di $\mathfrak{g}$ sono in corrispondenza biunivoca con certe orbite semisemplici, dette distinte; per ottenere questo risultato si usa un secondo teorema di coniugio, questa volta dovuto a Mal'cev.

In un certo senso, tutti questi risultati sono una dimostrazione della stupefacente efficacia della teoria delle rappresentazioni di $\mathfrak{sl}(2, \mathbb{C})$

La tesi è organizzata come segue: i primi tre capitoli contengono del materiale preliminare sulle algebre di Lie (Capitolo 1), sui gruppi di Lie (Capitolo 3) e sulla teoria delle rappresentazioni di $\mathfrak{sl}(2, \mathbb{C})$. Il Capitolo 4 ed il Capitolo 5 sono il cuore della tesi. Precisamente, i teoremi di Jacobson-Morozov, di Kostant e di Mal'cev sono dimostrati nel Capitolo 4 ed il Capitolo 5 è dedicato alla costruzione dei diagrammi di Dynkin pesati. Come esempio, vengono trattate dettagliatamente le classi di coniugio di elementi nilpotenti di $\mathfrak{sl}(\mathfrak{n}, \mathbb{C})$ e viene dimostrata una formula per la loro dimensione (vedi Teorema 5.17). In questo caso, come per tutte le algebre di Lie classiche, la descrizione delle orbite può essere fatta in termini di partizioni e tableaux.

Chapter 1

Lie algebras

In this section we review some basic concepts on Lie algebras.

Definition 1.1. (Lie algebra)
A vector space $\mathfrak{g}$ over a field $\mathbb{K}$ with a bilinear operation $[,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, called bracket or commutator, is said to be a Lie algebra if the following properties are satisfied:

1. $[x, x] = 0 \ \ \forall x \in \mathfrak{g}$;

2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \ \ \forall x, y, z \in \mathfrak{g}$ (Jacobi identity).

If the commutator is trivial, i.e., $[x, y] = 0 \ \ \forall x, y \in \mathfrak{g}$, $\mathfrak{g}$ is said commutative.

Remark 1.2. By condition 1. we have that $[x, y] = -[y, x]$, i.e., the product is anticommutative. (If char $\mathbb{K} \neq 2$, anticommutativity is equivalent to 1. indeed it is sufficient to notice that $[x + y, x + y] = 0$ for every $x, y \in \mathfrak{g}$).

Remark 1.3. Using Remark 1.2, condition 2. can be rewritten as a derivation, i.e.:

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]] \quad (\textit{Leibniz rule})$$

Remark 1.4. Lie algebras naturally arise from associative algebras endowed with a new operation. Namely given an associative algebra A, we can define a bracket on A as follows: $[,] : A \times A \longrightarrow A$, $(x, y) \mapsto [x, y] := xy - yx$. This new operation is obviously bilinear and satisfies condition 1.; besides an easy computation shows that the Jacobi identity also holds. For example, $\mathfrak{gl}(\mathfrak{n}, \mathbb{C})$ is the Lie algebra of $n \times n$ matrices with the bracket induced by the standard matrices product. Equivalently, given a vector space V, we denote by $\mathfrak{gl}(V)$ the Lie algebra of endomorphisms of V with the bracket induced by the composition of endomorphisms.

Definition 1.5. (Lie subalgebra)
A Lie subalgebra of a Lie algebra $\mathfrak{g}$ is a vector subspace W of $\mathfrak{g}$ such that $[x, y] \in W$ for every $x, y \in W$.

Definition 1.6. (Centralizer of an element)
Let $\mathfrak{g}$ be a Lie algebra and let $x \in \mathfrak{g}$. The centralizer of x in $\mathfrak{g}$ is $\mathfrak{g}^x := \left\{ y \in \mathfrak{g} \;\middle|\; [x, y] = 0 \right\}$.

Remark 1.7. Let $\mathfrak{g}$ be a Lie algebra and let $x \in \mathfrak{g}$, then $\mathfrak{g}^x$ is a subalgebra of $\mathfrak{g}$. Indeed it is a vector subspace of $\mathfrak{g}$ by the bilinearity of the bracket and, by the Leibniz rule, we have that for $y, z \in \mathfrak{g}^x$:

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]] = 0.$$

Definition 1.8. (Ideal)
A vector subspace I of a Lie algebra $\mathfrak{g}$ is an ideal of $\mathfrak{g}$ if $[x, y] \in I \quad \forall x \in I, \ \forall y \in \mathfrak{g}$.

Remark 1.9. If I, J are two ideals of a Lie algebra $\mathfrak{g}$, then $I + J, [I, J]$ and $I \cap J$ are also ideals of $\mathfrak{g}$. In particular, $[I, J] \subseteq I \cap J$ and if the sum $I + J$ is direct, $[I, J] \subseteq I \cap J = 0$, i.e., I and J commute.

Example 1. (Examples of ideals)
Let $\mathfrak{g}$ be a Lie algebra.

- 0 and $\mathfrak{g}$ are always ideals (said trivial).
- An example of a commutative ideal is the center of a Lie algebra $\mathfrak{g}$, that is

$$Z(\mathfrak{g}) = \{ z \in \mathfrak{g} | \, [x, z] = 0 \quad \forall x \in \mathfrak{g} \}.$$

  $\mathfrak{g}$ is commutative if and only if $Z(\mathfrak{g}) = \mathfrak{g}$.

- The derived algebra of $\mathfrak{g}$, namely $[\mathfrak{g}, \mathfrak{g}]$, consists of the linear combination of commutators of elements in $\mathfrak{g}$. $\mathfrak{g}$ is commutative if and only if $[\mathfrak{g}, \mathfrak{g}] = 0$.

Example 2. If $\mathfrak{g}$ is a Lie algebra and $x \in \mathfrak{g}, x \neq 0$, then $Span\{x\}$ is a commutative Lie subalgebra.
Let now $\mathfrak{g} = \mathfrak{gl}(\mathfrak{n}, \mathbb{C})$; the subspace $\mathfrak{sl}(\mathfrak{n}, \mathbb{C})$ of $n \times n$ matrices with trace equal to 0 is a Lie subalgebra of $\mathfrak{g}$, indeed:

$$tr([x, y]) = tr(xy) - tr(yx) = 0 \qquad \text{for every } x, y \text{ in } \mathfrak{g}.$$

This shows in fact that $[\mathfrak{gl}(\mathfrak{n}, \mathbb{C}), \mathfrak{gl}(\mathfrak{n}, \mathbb{C})] \subseteq \mathfrak{sl}(\mathfrak{n}, \mathbb{C})$. Moreover, using the bracket rule $[e_{i,j}, e_{h,l}] = \delta_{jh} e_{il} - \delta_{li} e_{hj}$, one can show that the equality holds. Thus we have the following decomposition in direct sum of ideals:

$$\mathfrak{gl}(\mathfrak{n}, \mathbb{C}) = \mathfrak{sl}(\mathfrak{n}, \mathbb{C}) \oplus \mathbb{C}I = [\mathfrak{gl}(\mathfrak{n}, \mathbb{C}), \mathfrak{gl}(\mathfrak{n}, \mathbb{C})] \oplus Z(\mathfrak{gl}(\mathfrak{n}, \mathbb{C})).$$

Definition 1.10. (Simple Lie algebras)
A Lie algebra is called simple if it is not commutative and it does not contain non-trivial ideals.

Remark 1.11. If $\mathfrak{g}$ is simple, then $Z(\mathfrak{g}) = 0$ and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Indeed $\mathfrak{g}$ is non-commutative hence $Z(\mathfrak{g}) \neq \mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}] \neq 0$. Moreover, since the only ideals of $\mathfrak{g}$ are trivial our claim follows.

Example 3. We shall prove that the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{K})$ is simple if char $\mathbb{K} \neq 2$. Let us set

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\{e, h, f\}$ is a basis for $\mathfrak{sl}(2, \mathbb{K})$ satisfying the following commutation rules:

$$[e, f] = h, \; [h, e] = 2e, \; [h, f] = -2f.$$

In particular, $\mathfrak{g}$ is not commutative. We will call $\{e, h, f\}$ the standard basis of $\mathfrak{sl}(2, \mathbb{K})$. Let us now show that $\mathfrak{g}$ is simple.
Let I be an ideal of $\mathfrak{g}$, $I \neq 0$. We want to show that $I = \mathfrak{g}$. If t is a non-zero element of $\mathfrak{sl}(2, \mathbb{K})$, then we can write it in the form $t = ae + bh + cf$ with $a, b, c \in \mathbb{K}$, $(a, b, c) \neq (0, 0, 0)$. We have

$$[t, e] = 2be - ch, \; so \; [[t, e], e] = -2ce. \text{ Similarly } [[t, f], f] = -2af.$$

Therefore if $a \neq 0$ (or $c \neq 0$), then $f \in I$ (or $e \in I$) and this implies $[e, f] = h \in I$ hence $e, f \in I$, obtaining $I = \mathfrak{g}$. If, instead, $a = c = 0$, then h belongs to I and so do the commutators of h with e and f; it follows again $I = \mathfrak{g}$. We conclude that $\mathfrak{sl}(2, \mathbb{K})$ is simple.

Let us now define a particular class of Lie algebras which are the main object of our study.

Definition 1.12. (Semisimple Lie algebra)
A Lie algebra $\mathfrak{g}$ is said to be semisimple if it is the direct sum of simple ideals $\mathfrak{g} = \oplus_i \mathfrak{g}_i$.

Remark 1.13. Definition 1.12 is one of the possible characterizations of a semisimple Lie algebra. It is worth recalling that it is possible to define a semisimple Lie algebra as a Lie algebra containing no solvable ideals (or, equivalently, containing no non-zero commutative ideals). If $\mathfrak{g}$ is a semisimple Lie algebra, then $Z(\mathfrak{g})$ is a commutative ideal, so it is zero. Moreover,

$$[\mathfrak{g}, \mathfrak{g}] = \left[\oplus_{i \in \{1,\dots,n\}} \mathfrak{g}_i, \oplus_{i \in \{1,\dots,n\}} \mathfrak{g}_i\right] = \oplus_{i \in \{1,\dots,n\}} [\mathfrak{g}_i, \mathfrak{g}_i] = \oplus_{i \in \{1,\dots,n\}} \mathfrak{g}_i = \mathfrak{g},$$

where the second equality follows from Remark 1.9 and the third equality from Remark 1.11. Thus for a semisimple Lie algebra $\mathfrak{g}$ we have that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and $Z(\mathfrak{g}) = 0$.

Definition 1.14. (Lie algebra homomorphism)
Let $\mathfrak{g}, \mathfrak{g}'$ be two Lie algebras. A linear map $\Phi : \mathfrak{g} \mapsto \mathfrak{g}'$ is called a Lie algebra homomorphism if $\Phi([x, y]) = [\Phi(x), \Phi(y)]$ for every $x, y \in \mathfrak{g}$. A Lie algebra homomorphism Φ is called isomorphism if it is bijective.

Remark 1.15. It is easy to check that if $\Phi : \mathfrak{g} \mapsto \mathfrak{g}'$ is a Lie algebra homomorphism, then $Ker(\Phi)$ is an ideal of $\mathfrak{g}$ and $\Phi(\mathfrak{g})$ is a subalgebra of $\mathfrak{g}'$.

Definition 1.16. (Representation)
Let $\mathfrak{g}$ be a Lie algebra and V a vector space, both over the field $\mathbb{K}$. A representation of $\mathfrak{g}$ on V is a Lie algebra homomorphism $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$.
A representation is said to be finite dimensional if $dim\ V < \infty$.
A subspace W of V is said stable under the representation ρ if $\rho(\mathfrak{g})(W) \subseteq W$. ρ is said irreducible if V contains no proper stable subspaces.

Remark 1.17. If $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$ on V and W is a stable subspace, then ρ induces a representation on W defined by $\rho_W(x) = (\rho(x))_{|W}$.

Definition 1.18. (Completely reducible representation)
Let $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ on V. ρ is said completely reducible if there exist $W_1, \ldots, W_n$ stable subspaces of V such that $V = W_1 \oplus \ldots \oplus W_n$ and ρ_{W_i} is irreducible for every $i \in \{1, \ldots, n\}$.

Example 4. (Adjoint representation)
For $x \in \mathfrak{g}$, let us consider the map $ad(x) : \mathfrak{g} \longrightarrow \mathfrak{g},\ y \mapsto [x, y]$. This is a linear map which is also a derivation of $\mathfrak{g}$ by the Jacobi identity. Now we can consider $ad : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}),\ x \mapsto ad(x)$. This is a Lie algebra homomorphism, indeed it is a linear map by the bilinearity of the bracket and

$$ad([x,y])(z) = [[x,y],z] = [x,[y,z]] - [y,[x,z]] = ad(x)([y,z]) - ad(y)([x,z]) =$$

$$= ad(x)\,ad(y)(z) - ad(y)\,ad(x)(z) = [ad(x), ad(y)](z).$$

Therefore ad defines a representation of a Lie algebra on itself, called the adjoint representation. An element lies in $Ker(ad)$ if and only if it commutes with every x in $\mathfrak{g}$, i.e., $Ker(ad) = Z(\mathfrak{g})$. By definition of ad, a stable subspace of $\mathfrak{g}$ is an ideal of $\mathfrak{g}$, thus a Lie algebra $\mathfrak{g}$ is simple if and only if $dim\ \mathfrak{g} > 1$ and its adjoint representation is irreducible.

Remark 1.19. We recall that if $\mathbb{K}$ is an algebraically closed field and $char\mathbb{K} = 0$, another equivalent and extremely useful characterization of a semisimple Lie algebra is the non-degeneracy of the killing form, which is the bilinear map on $\mathfrak{g}$ defined by $k(x,y) = \operatorname{tr}(ad(x) \cdot ad(y))$. Using the properties of the trace one can see that the killing form is associative, i.e., $k([x,y],z) = k(x,[y,z])$ indeed:

$$k([x,y],z) = \operatorname{tr}(ad(x) \cdot ad(y) \cdot ad(z)) - \operatorname{tr}(ad(y) \cdot ad(x) \cdot ad(z)) =$$

$$= \operatorname{tr}(ad(x) \cdot ad(y) \cdot ad(z)) - \operatorname{tr}(ad(x) \cdot ad(z) \cdot ad(y)) = k(x,[y,z]).$$

Definition 1.20. (Reductive Lie algebra)
A Lie algebra $\mathfrak{g}$ is said to be reductive if $\mathfrak{g} = Z(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$, with $[\mathfrak{g}, \mathfrak{g}]$ a semisimple ideal of $\mathfrak{g}$.

Example 5. We have seen in Example 2 that we can decompose $\mathfrak{gl}(\mathfrak{n},\mathbb{C})$ as a direct sum of ideals:

$$\mathfrak{gl}(\mathfrak{n},\mathbb{C}) = \mathfrak{sl}(\mathfrak{n},\mathbb{C}) \oplus \mathbb{C}I = [\mathfrak{gl}(\mathfrak{n},\mathbb{C}), \mathfrak{gl}(\mathfrak{n},\mathbb{C})] \oplus Z(\mathfrak{gl}(\mathfrak{n},\mathbb{C})).$$

On can show that $\mathfrak{sl}(\mathfrak{n},\mathbb{C})$ is simple, thus $\mathfrak{gl}(\mathfrak{n},\mathbb{C})$ is reductive.

Definition 1.21. (Cartan subalgebra)
Let $\mathfrak{g}$ be a reductive Lie algebra. A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal abelian subalgebra consisting of ad-semisimple elements.

Remark 1.22. Let us consider the pair $(\mathfrak{g},\mathfrak{h})$, where $\mathfrak{g}$ is a reductive Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Then $ad(\mathfrak{h})$ consists of semisimple endomorphisms of $\mathfrak{g}$ that commute with each other and are therefore simultaneously diagonalizable. For $\alpha \in \mathfrak{h}^*$, we define $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | ad(H)(X) = [H,X] = \alpha(H)X \quad \forall H \in \mathfrak{h}\}$. We notice that $\mathfrak{g}_0$ is the centralizer of $\mathfrak{h}$ and one can demonstrate that $C_\mathfrak{g}(\mathfrak{h}) = \mathfrak{h}$ (see [3], Chapter 8). Thus we have that

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha\right), \quad \text{where } \Phi = \{\alpha \in \mathfrak{h}^* \setminus \{0\} | \mathfrak{g}_\alpha \neq 0\}. \tag{1.1}$$

Decomposition (1.1) is called Cartan decomposition, and Φ is called the root system of $\mathfrak{g}$. Notice that if X is an ad-semisimple element, then there exists a Cartan subalgebra containing X. Indeed $Span\{X\}$ is an abelian subalgebra consisting of semisimple elements and containing X; if it is maximal we have finished, if not we consider a bigger subalgebra consisting of ad-semisimple elements and we argue in the same way. Since $dim\ \mathfrak{g} < \infty$, this process ends and we obtain a Cartan subalgebra containing X.

1.1 Semisimple Lie algebras

The aim of this section is to recall the general facts about complex semisimple Lie algebras, in particular to explore the Cartan decomposition (1.1). Proofs are omitted and can be found in [3], Chapters 8,9,14 and 18. From now on we assume that the base field is $\mathbb{K} = \mathbb{C}$.
The restriction of the killing form k to the Cartan subalgebra $\mathfrak{h}$ is nondegenerate, thus

we can identify $\mathfrak{h}$ with $\mathfrak{h}^*$. For every $\alpha \in \mathfrak{h}^*$ there exists a unique element $t_\alpha \in \mathfrak{h}$ such that $k(t_\alpha, h) = \alpha(h)$ for every $h \in \mathfrak{h}$.
The following properties are satisfied:

1. Φ spans $\mathfrak{h}^*$;
2. if $\alpha \in \Phi$, then $-\alpha \in \Phi$;
3. let $\alpha \in \Phi,\ x_\alpha \in \mathfrak{g}_\alpha,\ y_\alpha \in \mathfrak{g}_{-\alpha}$, then $[x, y] = k(x, y)\, t_\alpha$;
4. if $\alpha \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one dimensional, with basis t_α;
5. $\alpha(t_\alpha) = k(t_\alpha, t_\alpha) \neq 0$, for $\alpha \in \Phi$;
6. if $\alpha \in \Phi$ and x_α is a non-zero element of $\mathfrak{g}_\alpha$, then there exists $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $x_\alpha, y_\alpha, h_\alpha = [x_\alpha, y_\alpha]$ span a three dimensional subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ via $x_\alpha \mapsto e,\ y_\alpha \mapsto f, h_\alpha \mapsto h$;
7. $h_\alpha = \frac{2t_\alpha}{k(t_\alpha, t_\alpha)}$; $h_{-\alpha} = -h_\alpha$.

Remark 1.23. By property 6. above, we have that $\mathfrak{sl}(2, \mathbb{C})$ is the only three dimensional semisimple algebra, up to isomorphism. Notice that a Lie algebra of dimension one is commutative and a non commutative two dimensional Lie algebra $\mathfrak{g}$ has a proper commutative ideal spanned by $[x, y]$, where $\{x, y\}$ is a basis of $\mathfrak{g}$.

The following orthogonality and integrality properties hold:

a) if $\alpha \in \Phi$, then $\pm\alpha$ are the only scalar multiples of α that lie in Φ and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one dimensional with basis t_α;

b) if $\alpha \in \Phi$, then $dim\ \mathfrak{g}_\alpha = 1$;

c) if $\alpha, \beta \in \Phi$, then $\beta(h_\alpha) \in \mathbb{Z}$ and $\beta - \beta(h_\alpha)\, \alpha \in \Phi$;

d) if $\alpha, \beta \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. Moreover if $\alpha + \beta \in \Phi$ the equality holds;

e) $\mathfrak{g}$ is generated, as a Lie algebra, by the root spaces $\mathfrak{g}_\alpha$;

f) Let $\{\alpha_1, \ldots, \alpha_n\} \subseteq \Phi$ be a basis of $\mathfrak{h}^*$ and $\beta \in \Phi$. Then

$$\beta = \sum_{i=1}^{n} c_i \alpha_i, \quad c_i \in \mathbb{Q}.$$

One can define an "abstract root system" and check that the root system of a semisimple Lie algebra is an abstract root system. First we recall that an Euclidean space E is a finite dimensional vector space over $\mathbb{R}$ with a positive definite symmetric bilinear form $(\cdot, \cdot)$. A reflection of E is a linear map that fixes an hyperplane and sends any vector orthogonal to that hyperplane to its opposite. A non-zero vector α determines the reflection σ_α with respect to the hyperplane orthogonal to α:

$$\sigma_\alpha (\beta) = \beta - < \beta, \alpha > \alpha, \quad < \beta, \alpha > := \frac{2 (\beta, \alpha)}{(\alpha, \alpha)}.$$

Reflections are isometries of the space E.

Definition 1.24. (Abstract root system)
A subset Φ of a Euclidean space E is called abstract root system if the followings are satisfied:

(R1) Φ is finite, spans E and does not contain 0.

(R2) If $\alpha \in \Phi$, the only scalar multiples of α in Φ are $\pm\alpha$.

(R3) If $\alpha \in \Phi$, the reflection σ_α leaves Φ invariant.

(R4) If $\alpha, \beta \in \Phi$, then $< \beta, \alpha > \in \mathbb{Z}$.

We call rank of Φ the dimension of E. We say that two root systems are isomorphic if there exists an isomorphism $\phi : E \longrightarrow E'$ of the corresponding euclidean spaces such that $\phi(\Phi) = \Phi'$ and $< \phi(\alpha), \phi(\beta) > = < \alpha, \beta >$ for every $\alpha, \beta \in \Phi$.

The Weyl group of Φ is the subgroup of $GL(E)$ generated by the reflections σ_α for $\alpha \in \Phi$. Since reflections are isometries and leave Φ invariant, we can identify the Weyl group with a subgroup of the symmetric group on $|\Phi|$ elements.

Definition 1.25. (Base)
Let E be an euclidean space and $\Phi \subset E$ an abstract root system. A subset Δ of Φ is called a base of Φ if:

B1) Δ is a basis of E;

B2) each root $\beta \in \Phi$ can be written as $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$ with c_α integer coefficients, all nonnegative or all nonpositive.

The elements in Δ are called simple roots.

The fact that bases exist (see [3], Chapter 10) allows a decomposition $\Phi = \Phi^+ \cup \Phi^-$ in positive and negative roots: a root is said to be positive (negative) if its coefficient in B2) are all nonnegative (nonpositive). The proof of the existence of a base shows $< \alpha, \beta > \leq 0$ for every pair of simple roots $\alpha \neq \beta$.

Definition 1.26. (Irreducible root system)
An abstract root system is said irreducible if it cannot be partitioned in the union of two proper subsets such that each root of the first set is orthogonal to each root of the second set.

One immediately checks that the notion of irreducible root system is equivalent to the irreducibility of a base Δ, where the definition of an irreducible base is essentially the same of the one for abstract root systems.
To a base $\Delta = \{\alpha_1, \dots, \alpha_n\}$ of Φ, we can associate an $n \times n$ matrix C with integral entries $C_{i,j} =< \alpha_i, \alpha_j >$. It is immediate to check that a root system is irreducible if and only if C is a diagonal block matrix. Moreover $C_{i,i} = \frac{2(\alpha_i, \alpha_i)}{(\alpha_i, \alpha_i)} = 2$ and $C_{i,j} \leq 0$ if $i \neq j$ due to what we noticed after the definition of a base.
We can associate a graph with a Cartan matrix: the Coxeter graph of Φ is a graph of n vertices with the i-th joined with the j-th by $< \alpha_i, \alpha_j >< \alpha_j, \alpha_i >$ vertices. From the Coxeter graph we can construct another graph, called the Dynkyn diagram of Φ, adding an arrow pointing the shortest root (i.e., the vertex associated with the shortest simple root) when there are multiple edges between two vertices. This graph is connected if and only if Δ is irreducible, i.e., if and only if Φ is irreducible. The classification of root systems is therefore reduced to that of irreducible root systems.

A complete list of irreducible root systems consists of four infinite families A_n, B_n, C_n, D_n and five special graphs E_6, E_7, E_8, F_4, G_2. We saw that a root system hence, a Dynkin diagram, is associated to a semisimple Lie algebra. The root system of a semisimple Lie algebra is irreducible if and only if the Lie algebra is simple. Dynkin diagrams completely determine semisimple Lie algebras.

Chapter 2

On the representations of $\mathfrak{sl}(2, \mathbb{C})$

We now give a complete description of the finite dimensional representations of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ that will be strongly used in Chapter 3. We first study the irreducible representations. In the whole chapter, all the representations are finite dimensional.

Definition 2.1. ($\mathfrak{g}$-module)
Let $\mathfrak{g}$ be a Lie algebra. A vector space V endowed with an operation

$$. : \mathfrak{g} \times \mathfrak{g} \longrightarrow V$$

$$(x, v) \mapsto x.v$$

is called a $\mathfrak{g}$-module if the following conditions are satisfied:
for every $x, y \in \mathfrak{g}$, for every $v, w \in V$ for every $a, b \in \mathbb{C}$,

1. $(ax + by).v = a(x.v) + b(y.v)$

2. $x.(av + bw) = a(x.v) + b(x.w)$

3. $[x, y]\,.v = x.y.v - y.x.v$

Definition 2.2. (Submodule, irreducible module)
Let V be a $\mathfrak{g}$-module. A subspace W of V is called a $\mathfrak{g}$-submodule if $x.w \in W$ for every $x \in \mathfrak{g}$, for every $w \in W$. V is said irreducible if its only submodules are 0 and V.

Definition 2.3. (Homomorphism of $\mathfrak{g}$-modules)
Let V,W be $\mathfrak{g}$-modules. A linear map $f : V \longrightarrow W$ is a homomorphism of $\mathfrak{g}$-modules if $x.f(v) = f(x.v)$ for every x in $\mathfrak{g}$, for every v in V.

Remark 2.4. In Definition 2.3 we are using the same notation for, in general, different actions of $\mathfrak{g}$ on V and W. When $V = W$ we mean that the action is the same.

Lemma 2.5. *If $f : V \longrightarrow W$ is a homomorphism of $\mathfrak{g}$-modules, then $Kerf$ is a $\mathfrak{g}$-submodule of V and Imf is a $\mathfrak{g}$-submodule of W.*

Proof. Since f is a linear map, we only have to prove that $Kerf$ and Imf are closed under the multiplication by elements in $\mathfrak{g}$. Let $x \in \mathfrak{g}$.
If v is an element of $Kerf$, then $f(x.v) = x.f(v) = x.0 = 0$. If w is an element of Imf, then there exists $v \in V$ such that $f(v) = w$, thus $x.w = x.f(v) = f(x.v)$. □

Lemma 2.6. *Let $f : V \longrightarrow V$ be a homomorphism of $\mathfrak{g}$-modules and λ an eigenvalue. Then the eigenspace V_λ is a submodule of V.*

Proof. Let $v \in V_\lambda$ and $x \in \mathfrak{g}$, then $f(x.v) = x.f(v) = x.(\lambda v) = \lambda(x.v)$. □

Remark 2.7. A representation of $\mathfrak{g}$ is equivalent to a $\mathfrak{g}$-module, indeed:

- if ρ is a representation of $\mathfrak{g}$ on V, then V is a $\mathfrak{g}$-module with $x.v := \rho(x)(v)$;
- if V is a $\mathfrak{g}$-module, then ρ, defined by $\rho(x)(v) = x.v$, is a representation of $\mathfrak{g}$ on V.

Irreducible modules correspond to irreducible representations; completely reducible representations correspond to direct sum of submodules. A homomorphism of $\mathfrak{g}$-modules correspond to a map that commutes with all the elements in the image of the representation.

2.1 Irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$

We will study the representations of $\mathfrak{sl}(2,\mathbb{C})$ using the language of $\mathfrak{sl}(2,\mathbb{C})$-modules. Let $\{e, h, f\}$ be the standard basis of $\mathfrak{sl}(2,\mathbb{C})$.

Let V be an $\mathfrak{sl}(2,\mathbb{C})$-module with $0 < dim\ V < \infty$. Since $\mathbb{C}$ is algebraically closed there exists $v \neq 0$ such that $h.v = \lambda v$, i.e., v is an eigenvector of h of eigenvalue λ. Let us denote by V_λ the eigenspace of h of eigenvalue λ.

Lemma 2.8. *If $v \in V_\lambda$, then $e^r.v \in V_{\lambda+2r}$ and $f^r.v \in V_{\lambda-2r}$.*

Proof. We use induction on r:
if $r = 1$,

$$h.e.v = [h,e].v + e.h.v = 2e.v + e.(\lambda v) = (2+\lambda)\, e.v$$

$$h.f.v = [h,f].v + f.h.v = -2f.v + f.(\lambda v) = (-2+\lambda)\, f.v$$

Suppose now that the statement is true for $r = n$, then

$$h.e^{n+1}.v = [h,e].e^n.v + e.h.(e^n.v) = 2e^{n+1}.v + e.((\lambda+2n)\, e^n.v) = (2(n+1)+\lambda)\, e^{n+1}.v$$

$$h.f^{n+1}.v = [h,f].f^n.v + f.h.(f^n.v) = -2f^{n+1}.v + f.((\lambda-2n)\, f^n.v) = (-2(n+1)+\lambda)\, f^{n+1}.v$$

□

Corollary 2.9. *If v is an eigenvector of eigenvalue λ, then there exists $r \in \mathbb{N}$ such that $e^r.v \neq 0$ and $e^{r+1}.v = 0$.*

Proof. By Lemma 2.8, $(e^n.v)_{n\in\mathbb{N}}$ is a sequence of zeros and eigenvectors relative to different eigenvalues. Since V is finite dimensional, there must be a minimum $r \in \mathbb{N}$ such that $e^r.v \neq 0$ and $e^k.v = 0$ for every $k > r$. □

Let $v_0 = e^r.v$ and let $\lambda' = \lambda + 2r$, where v is an eigenvector of eigenvalue λ and r is as in Corollary 2.9. Then v_0 is called maximal weight vector. By Lemma 2.8 and the same argument as in Corollary 2.9, there exists $k \in \mathbb{N}$ such that $f^k.v_0 \neq 0$ and $f^{k+1}.v_0 = 0$. Let us define $v_{i+1} = \frac{1}{i+1} f.v_i$, for $i = 0, \ldots, k-1$ and $W = Span\{v_0, v_1, \ldots, v_k\}$.

Lemma 2.10. *W is a submodule of V and $dim\ W = k+1$.*

Proof. Every vector $f^i.w$ is an eigenvector of h of eigenvalue $\lambda' - 2i$, thus W is closed under the action of h and the vectors $w, f.w, \ldots, f^k.w$ are linearly independent, since they are eigenvectors of different eigenvalues. Besides, W is closed under the action of f, since $f.v_i = (i+1)\, v_{i+1}$. It remains to prove that W is closed under the action of e.

This follows from the fact that $e.v_0 = 0$ and from the relation $e.v_i = (\lambda' - i + 1)\, v_{i-1}$ for $i \geq 1$, which we prove by induction on i.

If $i = 1$, $e.v_1 = e.f.v_0 = [e, f]\,.v_0 + f.e.v_0 = h.v_0 + f.0 = \lambda' v_0$. Now suppose that the formula is true for $i = n - 1$, then

$$n\,(e.v_n) = e.\,(nv_n) = e.f.v_{n-1} = [e, f]\,.v_{n-1} + f.\,(e.v_{n-1}) = h.v_{n-1} + f.\left((\lambda' - n + 1 + 1)\, f^{n-2}\right) =$$
$$= (\lambda' - 2n + 2)\, v_{n-1} + (\lambda' - n + 2)\,(n - 1)\, v_{n-1} = -nv_{n-1} + (\lambda' - n + 2)\, v_{n-1} +$$
$$+ n\,(\lambda' - n + 2)\, v_{n-1} - (\lambda' - n + 2)\, v_{n-1} = n\,(\lambda' - n + 1)\, v_{n-1},$$

thus, $e.v_n = (\lambda' - n + 1)\, v_{n-1}$. □

Theorem 2.11. *($\mathfrak{sl}(2,\mathbb{C})$ irreducible modules)*
For every λ in $\mathbb{N}$ there exists a unique (up to isomorphism) $\mathfrak{sl}(2,\mathbb{C})$ irreducible module $V(\lambda)$ of dimension $\lambda + 1$ and it has a basis $\{v_0, v_1, \ldots, v_\lambda\}$ of eigenvectors for h that satisfies the following conditions:

$$h.v_i = (\lambda - 2i)\, v_i, \;\; f.v_i = (i + 1)\, v_{i+1}, \;\; e.v_i = (\lambda - i + 1)\, v_{i-1}.$$

Proof. Using the notation above, W is a non-zero submodule of V hence if V is irreducible, then $W = V$ and $dim\ V = k + 1$.
We have that $0 = e.0 = e.v_{k+1} = (\lambda' + 1 - k - 1)\, v_k$ and $v_k \neq 0$, thus it must be $\lambda' = k$.
The uniqueness of $V(\lambda)$ up to isomorphism follows from the previous lemmas. In order to prove the existence, we consider a space V of dimension $\lambda + 1$ and a basis $\{v_0, v_1, \ldots, v_\lambda\}$.
We then define ρ as the linear map $\rho : \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathfrak{gl}(V)$ defined by

$$\rho(e) = \begin{pmatrix} 0 & \lambda & 0 & \ldots & 0 \\ 0 & 0 & \lambda - 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \; \rho(f) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda & 0 \end{pmatrix},$$

$$\rho(h) = \begin{pmatrix} \lambda & 0 & 0 & \ldots & 0 \\ 0 & \lambda - 2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -\lambda + 2 & 0 \\ 0 & 0 & \ldots & 0 & -\lambda \end{pmatrix}.$$

Then we have that $[\rho(e), \rho(f)] = \rho(h)$, $[\rho(h), \rho(e)] = 2\rho(e)$, $[\rho(h), \rho(f)] = -2\rho(f)$, thus ρ is an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ with highest weight vector v_0 of weight λ. □

We will call $V = V(\lambda)$ a highest weight module of weight λ.

Remark 2.12. We have an explicit decomposition of $V(\lambda)$ in h-eigenspaces: every eigenvalue is integer and differs by 2 from the previous and (or) by the following, the relative eigenspace is one-dimensional and there is a maximal eigenvalue $\lambda = dim\ V - 1$. V decomposes as follows:

$$V = V_\lambda \oplus V_{\lambda-2} \oplus \ldots \oplus V_{-\lambda+2} \oplus V_{-\lambda}.$$

2.2 Weyl's Theorem

The study of the irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ is sufficient to classify all its finite dimensional representations. This comes from a more general result:

Theorem 2.13. *(Weyl)*
Let $\mathfrak{g}$ be a semisimple Lie algebra and $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ a representation of $\mathfrak{g}$ on V, with $dim\ V < \infty$. Then ρ is completely reducible.

In order to prove this theorem, we will need some preliminary results.

Lemma 2.14. *(Schur)*
Let $f : V \longrightarrow V$ be a homomorphism of $\mathfrak{g}$-modules. If V is irreducible, then there exists $\lambda \in \mathbb{C}$ such that $f = \lambda Id$.

Proof. Since $\mathbb{C}$ is algebraically closed, f has an eigenvalue $\lambda \in \mathbb{C}$. V is irreducible and we showed that V_λ is a submodule of V, thus $V = V_\lambda$, i.e, $f = \lambda Id$. □

As in the case of the killing form, one can show that, given a representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, the bilinear form defined by $\beta(x, y) = trace(\rho(x) \cdot \rho(y))$ is a symmetric associative bilinear form on $\mathfrak{g}$. Moreover if $\mathfrak{g}$ is semisimple, β is non-degenerate. One obtains the killing form when $\rho = ad$.

Remark 2.15. If β is a bilinear non-degenerate form and $\{x_1,\dots,x_n\}$ is a basis of $\mathfrak{g}$, there exists a unique dual basis, that is a basis $\{y_1,\dots,y_n\}$ such that $\beta(x_i,y_j)=\delta_{i,j}$ where δ is the Dirac delta. Indeed $\beta(\cdot,y)$ is an isomorphism between $\mathfrak{g}$ and its dual by the non-degenerancy of the form, thus there is a unique y_j in $\mathfrak{g}$ such that $\beta(\cdot,y_j)=\delta_{i,j}$.

Definition 2.16. (Casimir element)
In the notation above, the Casimir element of ρ is the endomorphism of V defined as

$$C_\rho=\sum_{i=1}^{n}\rho(x_i)\cdot\rho(y_i).$$

Remark 2.17. We have:

$$trace\,(C_\rho)=\sum_{i=1}^{n}trace\,(\rho(x_i)\cdot\rho(y_i))=\sum_{i=1}^{n}\beta(x_i,y_i)=n=dim\ \mathfrak{g}.$$

Remark 2.18. We will need the following formula: let $a,b,c\in End\,(V)$, then

$$[a,b\cdot c]=a\cdot b\cdot c-b\cdot c\cdot a=a\cdot b\cdot c-b\cdot a\cdot c+b\cdot a\cdot c-b\cdot c\cdot a=[a,b]\cdot c+b\cdot[a,c]\,.$$

Lemma 2.19. *Let $\rho:\mathfrak{g}\longrightarrow\mathfrak{gl}(V)$ be a representation $\mathfrak{g}$ and let $\{x_1,\dots,x_n\}$ be a basis of $\mathfrak{g}$. The Casimir element of ρ commutes with all the elements of $\rho(\mathfrak{g})$. In particular, if ρ is irreducible, C_ρ is a scalar, precisely $C_\rho=\frac{dim\ \mathfrak{g}}{dim\ V}Id$. In this case, the Casimir element does not depend on the choice of a basis.*

Proof. Let $\{y_1,\dots,y_n\}$ be the dual basis of $\{x_1,\dots,x_n\}$ and let x be an element of $\mathfrak{g}$. We want to prove that $[\rho(x),C_\rho]=0$. We have:

$$[\rho(x),C_\rho]=\sum_{i=1}^{n}([\rho(x),\rho(x_i)\cdot\rho(y_i)])=\sum_{i=1}^{n}([\rho(x),\rho(x_i)]\cdot\rho(y_i)+\rho(x_i)\cdot[\rho(x),\rho(y_i)])=$$

$$=\sum_{i=1}^{n}(\rho([x,x_i])\cdot\rho(y_i)+\rho(x_i)\cdot\rho([x,y_i])).$$

Note that for every $i\in\{1,\dots,n\}$ there exist unique coefficients $a_{ij},b_{ij},\ j\in 1,\dots,n$, such that

$$[x,x_i]=\sum_{j=1}^{n}a_{ij}x_j,\quad [x,y_i]=\sum_{j=1}^{n}b_{ij}y_j.$$

Moreover $a_{ik} = -b_{ki}$, indeed by the associativity of the form we have:

$$a_{ik} = \sum_{j=1}^{n} a_{ij}\beta\left(x_j, y_k\right) = \beta\left(\left[x, x_i\right], y_k\right) = -\beta\left(x_i, \left[x, y_k\right]\right) = -\sum_{j=1}^{n} b_{kj}\beta\left(x_i, y_j\right) = -b_{ki}.$$

Thus we have that:

$$\left[\rho\left(x\right), C_\rho\right] = \sum_{i=1}^{n}\left(\rho\left(\left[x, x_i\right]\right)\cdot\rho\left(y_i\right) + \rho\left(x_i\right)\cdot\rho\left(\left[x, y_i\right]\right)\right) =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\rho\left(x_j\right)\cdot\rho\left(y_i\right) + \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\rho\left(x_i\right)\cdot\rho\left(y_j\right) = 0.$$

If ρ is irreducible, $C_\rho = \lambda Id$, $\lambda \in \mathbb{C}$, by Schur's lemma. The associated matrix is a $dim\ V \times dim\ V$ matrix with trace $\lambda \cdot dim\ V$. We noticed before that C_ρ has trace equal to $dim\ \mathfrak{g}$, thus $\lambda = \frac{dim\ \mathfrak{g}}{dim\ V}$. □

Lemma 2.20. *Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ on V. Then $\rho\left(\mathfrak{g}\right) \subseteq \mathfrak{sl}(dim\ V, \mathbb{C})$. In particular, if $dim\ V = 1$, $\mathfrak{g}$ acts trivially on V.*

Proof. In the previous chapter we have seen that if $\mathfrak{g}$ is a semisimple Lie algebra, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Thus, for every x in $\mathfrak{g}$ there exist $y_i, z_i \in \mathfrak{g}$ such that $x = \sum_i \left[y_i, z_i\right]$. Since ρ is a representation, $\rho\left(x\right) = \rho\left(\sum_i \left[y_i, z_i\right]\right) = \sum_i \left[\rho\left(y_i\right), \rho\left(z_i\right)\right]$ and $trace\left(\rho\left(x\right)\right) = \sum_i \left(trace\left(\rho\left(y_i\right)\cdot\rho\left(z_i\right)\right) - trace\left(\rho\left(z_i\right)\cdot\rho\left(y_i\right)\right)\right) = 0$.
The last statement follows from the fact that if $dim\ V = 1$, $\mathfrak{sl}(dim\ V, \mathbb{C}) = \mathfrak{sl}(1, \mathbb{C}) = 0$. □

Remark 2.21. If V and U are $\mathfrak{g}$-modules, we can define on $Hom\left(V, U\right)$ a structure of $\mathfrak{g}$-module as follows:

$$\left(x.f\right)\left(v\right) = x.f\left(v\right) - f\left(x.v\right) \qquad \text{for } x \in \mathfrak{g},\ v \in V.$$

We now have all the tools to prove Weyl's Theorem.

Proof of Weyl's Theorem. Without loss of generality we can assume that ρ is injective (if it is not, we can consider $\mathfrak{g}/_{Ker\rho}$ that acts in the same way). The proof is organized in three steps.

- First step: suppose that there exists an irreducible submodule W of V of codimension 1.

Since the Casimir element $C_\rho : V \longrightarrow V$ commutes with every $\rho(x)$, it is a homomorphism of $\mathfrak{g}$-modules, thus $KerC_\rho$ is a submodule of V. We want to show that $V = W \oplus KerC_\rho$.
${}^{V}\!/_{W}$ has dimension 1, thus $\mathfrak{g}$ acts trivially on it by Lemma 2.20, i.e., $\rho(x)$ maps V in W for every $x \in \mathfrak{g}$. Since C_ρ is a sum of compositions of such endomorphisms, it maps V in W. Thus $KerC_\rho \neq 0$ and the trace of an element in $\rho(\mathfrak{g})$ is the same as that of its restriction to W. In particular the Casimir element of $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is the same as that of $\rho_W : \mathfrak{g} \longrightarrow \mathfrak{gl}(W)$. Since W is irreducible, the latter is a non-zero scalar multiple of the identity, thus $W \cap KerC_\rho = 0$ and $V = W \oplus KerC_\rho$.

- Second step: suppose that there exists a sumbmodule W of V of codimension 1.

We argue by induction on $n = dim\ V$. If $n = 1$ it is trivial. Let $dim\ V > 1$. If W is irreducible, we are in the previous case. If W is not irreducible, there exists a submodule $W' \neq 0$ of W (which is also a submodule of V). Thus ${}^{W}\!/_{W'}$ is a submodule of ${}^{V}\!/_{W'}$ of codimension 1 and, by the inductive hypothesis, there exists a submodule ${}^{\tilde{W}}\!/_{W'}$ of ${}^{V}\!/_{W}$ such that ${}^{V}\!/_{W'} = {}^{W}\!/_{W'} \oplus {}^{\tilde{W}}\!/_{W'}$. Since ${}^{\tilde{W}}\!/_{W'}$ has dimension 1, W' is a submodule of $\tilde{W}$ of codimension 1 and, again by induction, there exists a submodule X of $\tilde{W}$ such that $\tilde{W} = W' \oplus X$ and $dimX = 1$. Thus ${}^{V}\!/_{W'} \cong {}^{W}\!/_{W'} \oplus X$ and $V = W \oplus X$.

- Third step: general case.

By induction on $dim\ V$. If $dim\ V = 1$ it is obvious. Let $dim\ V > 1$.
If V is irreducible, there is nothing to prove. Suppose that V is not irreducible and let U be a non-zero submodule of V.
Consider $S = \left\{ f \in Hom(V, U) \ \middle|\ f_{|_U} = \lambda Id,\ \lambda \in \mathbb{C} \right\}$ and $T = \left\{ f \in Hom(V, U) \ \middle|\ f_{|_U} = 0 \right\}$.
These are two subspaces of $Hom(V, U)$ and $S = T \oplus Span\{h\}$ as vector spaces, where h is an element of S such that $h_{|_U} \neq 0$, thus T has codimension 1 in S. Let $g \in V$, $g_{|_U} = \lambda Id$ and let $x \in \mathfrak{g}$, then we have:

$$(x.g)(u) = x.g(u) - g(x.u) = x.(\lambda u) - \lambda x.u = 0, \qquad \text{for every } u \in U.$$

Thus $\mathfrak{g}.(S) \subseteq T$ and S,T are submodules of $Hom(V,U)$. By the second step, there exists f in $Hom(V,U)$ such that $S = T \oplus Span\{f\}$. Eventually normalizing f, we can assume that $f_{|_U} = id$. By Lemma 2.20 $\mathfrak{g}$ acts trivially on $Span\{f\}$, so f is a homomorphism of $\mathfrak{g}$-modules. Indeed, for every v in V and for every x in $\mathfrak{g}$, $0 = x.f(v) - f(x.v)$. It follows that $Kerf$ is a submodule of V. Moreover $Kerf \cap Imf = Kerf \cap U = 0$ since f acts as the identity on U. By the rank-nullity Theorem we have that $V = U \oplus Kerf$ and we can conclude applying the inductive hypothesis on both summands. □

Theorem 2.22. *($\mathfrak{sl}(2,\mathbb{C})$-modules)*
Let V be a finite-dimensional $\mathfrak{sl}(2,\mathbb{C})$-module. Then V decomposes into the direct sum of highest weight modules $V(\lambda)$, for some λ in $\mathbb{N}$. In particular all the h-eigenvalues are integer and V is the sum of $m_g(0) + m_g(1)$ irreducible submodules, where $m_g(\lambda)$ is the geometric multiplicity of the eigenvalue λ.

Proof. The first part is just Weyl's Theorem applied to $\mathfrak{sl}(2,\mathbb{C})$-modules. The second part follows from Theorem 2.11: every irreducible $\mathfrak{sl}(2,\mathbb{C})$-module is isomorphic to $V(\lambda)$ for some natural λ; $V(\lambda)$ has eigenvalues of the same parity of λ and symmetric with respect to zero, thus one and only one between 0 and 1 occurs as eigenvalue in $V(\lambda)$. □

Chapter 3

Lie groups

A Lie algebra is associated with a Lie group. We refer to [4], [5] and [6] for the results in this chapter.

Definition 3.1. (Lie group)
A Lie group G is a topological group with the structure of a smooth manifold such that multiplication and inversion are smooth. An analytic group is a connected Lie group.

Example 6. Let U be an open subset of $\mathbb{R}^n$. A smooth vector field on U is any operator X on smooth functions on U of the form $X = \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i}$ with $a_i(x) \in C^\infty(U)$. The real vector space $\mathfrak{g}$ of all smooth vector fields on U is a Lie algebra with the standard bracket $[X, Y] = XY - YX$. This example generalizes to any smooth manifold M.

If $p \in M$ and X is a vector field on M, we denote by $T_p(M)$ the tangent space of M at p and by X_p the value of X at p. If $\Phi : M \longrightarrow N$ is a smooth map between smooth manifolds, we write $d\Phi_p : T_p(M) \longrightarrow T_{\Phi(p)}(N)$ for the differential of Φ at p.

Let G be a Lie group and $x \in G$. The map $L_x : G \longrightarrow G$ defined by $y \mapsto xy$ is called left translation by x. If $f \in C^\infty(M; \mathbb{R})$, we set $f_x = f \circ L_x$.

Definition 3.2. (Left-invariant vector field)
A vector field X on G is said left-invariant if, for any x and y in G, $(dL_{yx^{-1}})(X_x) = X_y$.

Remark 3.3. If we consider the vector field X as an operator on smooth real-valued functions, Definition 3.2 says that X commutes with left translations, indeed for $f \in$

$C^{\infty}(M;\mathbb{R})$ we have:

$$(dL_{yx^{-1}})(X_x) = X_y \Leftrightarrow X_x(f \circ L_{yx^{-1}}) = X_y(f) \Leftrightarrow X_x(f_{yx^{-1}}) = X_y(f) \Leftrightarrow$$

$$\Leftrightarrow X(f_g) = (X(f))_g \quad \forall g \in G.$$

Remark 3.4. The bracket of two left-invariant vector fields is a left-invariant vector field. Indeed, if $f \in C^{\infty}(M;\mathbb{R})$, $g \in G$ and X, Y are left-invariant vector fields on G, we have:

$$([X,Y])_g(f) = (X(Y(f)))_g - (Y(X(f)))_g = X\Big((Y(f))_g\Big) - Y\Big((X(f))_g\Big) =$$

$$= XY(f_g) - YX(f_g) = [X,Y](f_g).$$

Theorem 3.5. *(Lie algebra of a Lie group)*
Let G be a Lie group. The map

$$\psi : \{\textit{left invariant vector fields on } G\} \longrightarrow T_1(G)$$

$$X \mapsto X_1$$

is a real vector space isomorphism. In particular, $\mathfrak{g} = T_1(G)$ becomes a Lie algebra with the bracket induced by ψ and it is called the Lie algebra of the Lie group G.

Proof. ψ is clearly linear, thus it is sufficient to show that it is invertible. The map defined by $v \mapsto X$, where $v \in T_1(G)$ and $Xf(x) := v(L_{x^{-1}} \cdot f)$ (with $L_{x^{-1}} \cdot f(y) = f(xy)$) is the inverse of ψ. □

Definition 3.6. An analytic subgroup H of a Lie group G is a subgroup of G with the structure of analytic group such that the inclusion mapping is smooth and everywhere regular.

If $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H respectively, then the differential of the inclusion at 1 carries $\mathfrak{h}$ to a Lie subalgebra of $\mathfrak{g}$ and it is one-to-one, thus we can identify $\mathfrak{h}$ with its image. The correspondence between analytic subgroups of G and Lie subalgebras of $\mathfrak{g}$ is bijective.
Let $\Phi : G \longrightarrow H$ be a smooth homomorphism between Lie groups and let $d\Phi_x : \mathfrak{g} \longrightarrow \mathfrak{h}$

be the differential at $x \in \mathfrak{g}$. Then the following property holds: if X is a left-invariant vector field on G and Y is the left invariant vector field on H such that $Y_1 = (d\Phi)_1 (X_1)$, we have

$$(d\Phi)_x (X_x) = Y_{\Phi(x)} \qquad \text{for all } x \in G. \tag{3.1}$$

Lemma 3.7. *$d\Phi_1$ is a Lie algebra homomorphism.*

Proof.

$$(d\Phi_1)([X_1, X_1']) = (d\Phi_1)([X, X']_1) = [Y, Y']_1 = [Y_1, Y_1'] = [d\Phi_1 (X_1), d\Phi_1 (X_1')] .$$

□

Remark 3.8. If G is connected, Φ is uniquely identified by $d\Phi$.

Let G be an analytic group and let $\tilde{G}$ be its universal covering, with covering map e. Let $\tilde{1}$ be an element in $e^{-1}(1)$, then there exists a unique multiplication on $\tilde{G}$ that makes $\tilde{G}$ an analytical group in such a way that e is a group homomorphism and $\tilde{1}$ is the identity in $\tilde{G}$. e is a smooth homomorphism and the Lie algebras of G and $\tilde{G}$ are isomorphic via de_1. $\tilde{G}$ is called the simply connected covering group of G.
Moreover if G and H are analytic groups, with G simply connected, and $\phi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a Lie algebra homomorphism between their Lie algebras, then there exists a smooth homomorphism $\Phi : G \longrightarrow H$ such that $d\Phi_1 = \phi$.

We want now to construct a map from the Lie algebra $\mathfrak{g}$ of G to G. In order to do this, we need the one dimensional additive group $\mathbb{R}$; its Lie algebra τ is commutative and it is generated by $\left(\frac{d}{dt}\right)_0$. Let now G be an analytic group with Lie algebra $\mathfrak{g}$ and X an element in $\mathfrak{g}$. We define a Lie algebra homomorphism by requiring that $\left(\frac{d}{dt}\right)_0$ maps to X. Since $\mathbb{R}$ is simply connected, there exists a smooth homomorphism $exp_X : \mathbb{R} \longrightarrow G$, $t \mapsto exp(tX)$ that lifts up the Lie algebra homomorphism to the Lie groups.
Set $c(t) = exp(tX)$ and let $\left(\frac{d}{dt}\right)$ and $\tilde{X}$ be the left-invariant vector fields on $\mathbb{R}$ and G, respectively, that extend $\left(\frac{d}{dt}\right)_0$ and X. Since exp is a group homomorphism, we have $c(0) = 1$. By Equation (3.1), we have that $(dc)_{\bar{t}} \left(\frac{d}{dt}\right)_{\bar{t}} = \tilde{X}_{c(\bar{t})}$. Let us now compute the

left side on a function $f \in C^{\infty}(G;\mathbb{R})$:

$$(dc)_{\bar{t}}\left(\frac{d}{dt}\right)_{\bar{t}} f = \frac{d}{dt}\left(f \circ c(t)\right)_{|t=\bar{t}} = \frac{d}{dt}\left(f\left(exp\left(tX\right)\right)\right)_{|t=\bar{t}},$$

thus we have:

$$\tilde{X} f\left(exp\left(tX\right)\right) = \frac{d}{dt}\left(f\left(exp\left(tX\right)\right)\right). \tag{3.2}$$

The flux given by $(t, X) \mapsto exp(tX)$ is smooth, thus the map $exp : \mathfrak{g} \longrightarrow G$, $X \mapsto exp_X$ is smooth. Moreover it is locally invertible about the origin. This is the exponential map for G. The exponential map satisfies the following property, that will turn out to be helpful in the next section: if $\Phi : G \longrightarrow H$ is a Lie group homomorphism, then

$$exp_H \circ d\Phi_1 = \Phi \circ exp_G. \tag{3.3}$$

3.1 Adjoint representation of a Lie group on its Lie algebra

A Lie group G naturally acts on its Lie algebra $\mathfrak{g}$ via the so-called "adjoint action". This section is dedicated to the study of this action.

Theorem 3.9. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If X is in $\mathfrak{g}$ and $\tilde{X}$ denotes the corresponding left-invariant vector field, and if $f \in C^{\infty}(G;\mathbb{R})$, then*

$$\left(\tilde{X}^n f\right)\left(g \; exp\left(tX\right)\right) = \frac{d^n}{dt^n}\left(f\left(g \; exp\left(tX\right)\right)\right) \qquad \text{for } g \text{ in } G.$$

Proof. For $g = 1$ it is sufficient to iterate the formula in Equation (3.2). The general case follows from the left-invariance of $\tilde{X}$, after replacing f with f_g. □

Corollary 3.10. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If X is in $\mathfrak{g}$ and $\tilde{X}$ denotes the corresponding left-invariant vector field, and if $f \in C^{\infty}(G;\mathbb{R})$, then*

$$\tilde{X} f(g) = \frac{d}{dt}\left(f\left(g \; exp\left(tX\right)\right)\right)_{|t=0}$$

Proof. Recalling that $exp_X(0) = 1$, it is sufficient to apply Theorem 3.9 with $n = 1$ and $t = 0$. □

The adjoint representation of a Lie group on its lie algebra is defined as follows. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Fix an element g in G and consider the smooth isomorphism $\Phi^g : G \longrightarrow G$, $\Phi^g(x) = gxg^{-1}$. The corresponding isomorphism $d\Phi_1^g : \mathfrak{g} \longrightarrow \mathfrak{g}$ is denoted by $Ad(g)$. By Equation (3.3) we have that:

$$exp\,(Ad\,(g)\,X) = g\,(expX)\,g^{-1}. \tag{3.4}$$

Relation (3.4) and the fact that exp has a smooth inverse in a neighborhood of the identity in G, imply that the map

$$g \mapsto Ad(g)$$

is smooth from a neighborhood of 1 in G into $GL\,(\mathfrak{g})$. Since $\Phi^{g_1} \circ \Phi^{g_2} = \Phi^{g_1 g_2}$, using the chain rule for differentials, we obtain $Ad\,(g_1) \circ Ad\,(g_2) = Ad\,(g_1 g_2)$, thus the smoothness is valid everywhere on G. Therefore we proved the following result:

Theorem 3.11. *If G is a Lie group and $\mathfrak{g}$ is its Lie algebra, then Ad is a smooth homomorphism from G into $GL\,(\mathfrak{g})$.*

Definition 3.12. (Adjoint representation)
Let G be a Lie group with Lie algebra $\mathfrak{g}$. Ad is called the adjoint representation of G on $\mathfrak{g}$.

Definition 3.13. (Complex Lie group)
A complex Lie group is a Lie group G possessing a complex analytic structure such that multiplication and inversion are holomorphic.

Remark 3.14. For such a group the complex structure induces a multiplication-by-i mapping in the Lie algebra of $\mathfrak{g} = T_1(G)$ such that $\mathfrak{g}$ becomes a Lie algebra over $\mathbb{C}$. Every left-invariant vector field has holomorphic coefficients and exp is a holomorphic mapping.

To a complex Lie algebra $\mathfrak{g}$ we can associate a connected complex Lie group, called the adjoint group G_{ad}.

Definition 3.15. The group of automorphisms of $\mathfrak{g}$ generated by the elements $exp(ad(x))$, with $x \in \mathfrak{g}$ is called the adjoint group G_{ad} of $\mathfrak{g} : G_{ad} := Ad\,(G)$.

Remark 3.16. If G is connected then the kernel of the adjoint representation of G on $\mathfrak{g}$ is the center $Z(G)$ of G. More generally the kernel of the adjoint map is the centralizer of the identity component G^0 of G, hence

$$G_{ad} \cong {}^{G}\!\big/_{Z^{G^0}}.$$

Remark 3.17. G_{ad} is the connected subgroup of $GL(\mathfrak{g})$ with Lie algebra $ad(\mathfrak{g})$.

The theory of covering groups tells us that, given a Lie algebra $\mathfrak{g}$, there exists a simply connected complex Lie group G_{sc} with Lie algebra $\mathfrak{g}$, and every other connected group G with Lie algebra $\mathfrak{g}$ is a quotient of G_{sc} by a finite central subgroup. In particular, there are finitely many such G; G_{ad} is the smallest group with Lie algebra $\mathfrak{g}$, while G_{sc} is the largest.

We shall now compute the differential of Ad, that will turn out to be ad.

Lemma 3.18. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If X and Y are in $\mathfrak{g}$, then*

a) $exp(tX)\,exp(tY) = exp\left\{t(X+Y) + \frac{1}{2}t^2[X,Y] + O(t^3)\right\}$, *as* $t \to 0$*;*

b) $exp(tX)\,exp(tY)\,(exp(tX))^{-1} = exp\{tY + t^2[X,Y] + O(t^3)\}$, *as* $t \to 0$.

Proof. See [5], Section 1.10. □

In order to compute the differential of Ad, we need to develop a theory on linear Lie groups, that is itself interesting and makes more explicit the description of the Lie algebra of a linear Lie group.

3.2 Linear Lie groups

Definition 3.19. (Closed linear group)
A closed linear group is a closed subgroup G of nonsingular real or complex matrices.

Definition 3.20. (Linear Lie algebra)
Let G be a closed linear group. The linear Lie algebra of G is

$$\mathfrak{g} = \left\{c'(0) \;\middle|\; c : \mathbb{R} \xrightarrow{C^\infty} G, \quad c(0) = 1\right\}.$$

The following lemma justifies Definition 3.20.

Lemma 3.21. *Let G be a closed linear group of $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$. The linear Lie algebra $\mathfrak{g}$ of G is a Lie algebra.*

Proof. We begin by showing that $\mathfrak{g}$ is a vector subspace of $M_n(\mathbb{R})$ (possibly the whole $M_n(\mathbb{R})$). Let $\lambda \in \mathbb{R}$, $v, w \in \mathfrak{g}$ and let c, d be curves in G as in Definition 3.20 such that $c'(0) = v$ and $d'(0) = w$. By the linearity of the derivative, we have that $\lambda v = \lambda c'(0) = (\lambda c)'(0) \in \mathfrak{g}$. By the chain rule, $v + w = c'(0) + d'(0) = c'(0)\, d(0) + c(0)\, d'(0) = (cd)'(0) \in \mathfrak{g}$.
Let us now show that $[v, w] = vw - wv \in \mathfrak{g}$. For every $g \in G$, $Ad_{lin}(g)\, w := gwg^{-1} \in \mathfrak{g}$, since it is the derivative in zero of the curve $gd(t)\, g^{-1}$; in particular $\mathfrak{g}$ contains $Ad_{lin}(c(t))\, w$ for every $t \in \mathbb{R}$. Since $\mathfrak{g}$ is a vector subspace, it is closed. Thus $\mathfrak{g}$ contains $\frac{Ad_{lin}(c(t+h))w - Ad_{lin}(c(t))w}{h}$ and the limit for $h \to 0$, i.e., $\frac{d}{dt}(Ad_{lin}(c(t))\, w) \in \mathfrak{g}$. An easy calculation shows that $\frac{d}{dt}\left(c(t)^{-1}\right) = -c(t)^{-1} c'(t)\, c(t)^{-1}$ and using Leibniz rule for derivative, we obtain:

$$\frac{d}{dt}(Ad_{lin}(c(t))\, w) \in \mathfrak{g} = c'(t)\, wc(t)^{-1} - c(t)\, w(t)^{-1} c'(t)\, c(t)^{-1}.$$

For $t = 0$, we have $[v, w] = \frac{d}{dt}(Ad_{lin}(c(t))\, w)\big|_{t=0} \in \mathfrak{g}$. □

The exponential map for matrices is the map defined by

$$e^A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

Using a matrix norm, one can show that this series always converge. With essentially the same proof for the complex exponential function, one can show that $e^{A+B} = e^A e^B$ if A and B commute; since $e^0 = 1$ we have that e^A is always invertible and its inverse is e^{-A}. The matrix exponential function is enough to describe the linear Lie algebra $\mathfrak{g}$. This is the content of the following theorem:

Theorem 3.22. *If G is a closed linear group and $\mathfrak{g}$ is its linear Lie algebra, then the matrix exponential function carries $\mathfrak{g}$ into G. Consequently*

$$\mathfrak{g} = \left\{ A \in \mathfrak{gl}(\mathfrak{n}, \mathbb{C}) \;\middle|\; e^{tA} \textit{ is in } G \textit{ for all } t \in \mathbb{R} \right\}.$$

This theorem can be shown directly, but it will be an obvious consequence of the next results, namely, we will show that the linear Lie algebra of a linear group G is isomorphic to the Lie algebra of G and that the exponential map described in the previous section coincide with the matrix exponential function under this identification.

Corollary 3.23. *The linear Lie algebra of $GL(n,\mathbb{C})$ is $\mathfrak{gl}(\mathfrak{n},\mathbb{C})$.*

Proof. We have already pointed out that e^A is invertible for every $A \in \mathfrak{gl}(\mathfrak{n},\mathbb{C})$, thus $e^{tX} \in GL(n,\mathbb{C})$ for every $t \in \mathbb{R}$ and $X \in \mathfrak{gl}(\mathfrak{n},\mathbb{C})$. We conclude by Theorem 3.22. □

Remark 3.24. Theorem 3.22, together with the Inverse Function Theorem, can be used to prove the following. A closed linear group G with its relative topology has a natural structure of Lie group such that:

- the real and imaginary part of each entry function are smooth;
- every smooth function from a smooth manifold M to $GL(n,\mathbb{C})$ that takes values in G is smooth as function from M to G.

Moreover *dim* $\mathfrak{g}$ = *dim* G. (See [5], Section 10.1).

Theorem 3.25. *Let G be a closed linear group of $n \times n$ matrices, denote by $\mathfrak{g}_1$ the Lie algebra of G and by $\mathfrak{g}_2$ its linear Lie algebra. Then the map $\mu : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ given by*

$$\mu\left(X\right)_{ij} = X_1\left(Re\ e_{ij}\right) + iX_1\left(Im\ e_{ij}\right) \qquad \text{with } e_{ij}\left(A\right) = A_{ij}$$

is a Lie algebra isomorphism.

Proof. In order to avoid a heavy notation, we extend the definition of $X \in \mathfrak{g}_1$ to complex-valued function as $Xf := X\left(Re\ f\right) + iX\left(Im\ f\right)$. The Leibniz rule for differentiation is still valid.
First, we prove that μ is a Lie algebra homomorphism. We have:

$$e_{ij} \circ L_x\left(y\right) = e_{ij}\left(xy\right) = \sum_{k=1}^{n} e_{ik}\left(x\right) e_{kj}\left(y\right).$$

Applying $X \in \mathfrak{g}_1$ we obtain:

$$Xe_{ij}\left(x\right) = X_1\left(e_{ij} \circ L_x\right) = X_1\left(\sum_{k=1}^{n} e_{ik}\left(x\right) e_{kj}\right) = \sum_{k=1}^{n} e_{ik}\left(x\right) X_1 e_{kj} = \sum_{k=1}^{n} e_{ik}\left(x\right) \mu\left(X\right)_{kj}.$$

If also $Y \in \mathfrak{g}_1$, then

$$YXe_{ij}(x) = Y_1((Xe_{ij}) \circ L_x) = Y_1\left(\sum_{k,l=1}^{n} e_{il}(x)\, e_{lk}(y)\, \mu(X)_{kj}\right) = \sum_{k,l=1}^{n} e_{il}(x)\, \mu(Y)_{lk}\, \mu(X)_{kj}.$$

For $x = 1$, $e_{il}(x) = \delta_{il}$ where δ_{il} is the Kronecker delta, thus

$$YXe_{ij}(1) = \sum_{k=1}^{n} \mu(Y)_{ik}\, \mu(X)_{kj} = (\mu(Y)\, \mu(X))_{ij}.$$

Reversing the roles of X and Y we finally have:

$$\mu([X,Y])_{ij} = ([X,Y]e_{ij})(1) = XYe_{ij}(1) - YXe_{ij}(1) = (\mu(X)\, \mu(Y))_{ij} - (\mu(Y)\, \mu(X))_{ij} =$$

$$= [\mu(X), \mu(Y)]_{ij},$$

i.e., μ is a Lie algebra homomorphism.

Our next goal is to show that $\mathfrak{g}_2 \subseteq Im\mu$. Let $A \in \mathfrak{g}_2$ and choose a curve $c(t)$ in G with $c'(0) = A$. Put

$$Xf(x) = \frac{d}{dt}(f(xc(t)))_{|t=0}.$$

X is a left-invariant vector field, indeed:

$$X(f_g)(x) = \frac{d}{dt}(f(gxc(t)))_{|t=0} = (Xf)_g(x).$$

Moreover

$$\mu(X)_{ij} = X_1 e_{ij} = Xe_{ij}(1) = \frac{d}{dt}(e_{ij}(c(t)))_{|t=0} = \frac{d}{dt}\left(c(t)_{ij}\right)_{|t=0} = c'(0)_{ij} = A_{ij},$$

thus $\mathfrak{g}_2 \subseteq Im\mu$. This allow us to complete the proof, indeed by Remark 3.24 we have:

$$dim\ \mathfrak{g}_1 = dim\ G = dim\ \mathfrak{g}_2 \leq dim\,(Im\mu) \leq dim\,(Dom\mu) = dim\ \mathfrak{g}_1,$$

and equality must hold throughout. μ is therefore an isomorphism. □

Remark 3.26. The proof shows what μ^{-1} maps a matrix $A \in \mathfrak{g}_2$ to the left-invariant vector field X defined by $Xf(x) = \frac{d}{dt}(f(x\ c(t)))_{|t=0}$.

We shall now go deeper in the correspondence between the Lie algebra of a Lie group and its linear Lie algebra, in order to explicit how to compute differentials at the level of linear Lie algebras.

Lemma 3.27. *Let G be a closed linear group and let $c(t)$ be a smooth curve in G with $c(0) = 1$. If μ is the isomorphism of Theorem 3.25, then we have*

$$\mu\left(dc_0\left(\frac{d}{dt}\right)\right) = c'(0).$$

Proof. By definition of μ we have:

$$\mu\left(dc_0\left(\frac{d}{dt}\right)\right)_{ij} = dc_0\left(\frac{d}{dt}\right)(e_{ij}) = \frac{d}{dt}\left(c(t)_{ij}\right)_{|t=0} = c'(0)_{ij}.$$

□

Theorem 3.28. *Let $\Phi : G \longrightarrow H$ be a smooth homomorphism between closed linear group and let μ_G and μ_H be the corresponding Lie algebra isomorphisms of Theorem 3.25. Let X be in the Lie algebra of G and put $Y = (d\Phi)_1(X)$. If $c(t)$ is a smooth curve in G with $c(0) = 1$ such that $\mu_G(X) = c'(0)$, then $\mu_H(Y) = \frac{d}{dt}\Phi(c(t))_{|t=0}$.*

Proof. By lemma 3.27, $X = \mu_G^{-1}(c'(0)) = \mu_G^{-1}\left(\mu\left(dc_0\left(\frac{d}{dt}\right)\right)\right) = dc_0\left(\frac{d}{dt}\right)$. Thus we have:

$$\mu_H(Y) = \mu_H((d\Phi)_1(X)) = \mu_H\left((d\Phi)_1(dc_0)\left(\frac{d}{dt}\right)\right) = \mu_H\left(d(\Phi\circ c)_0\left(\frac{d}{dt}\right)\right),$$

and applying Lemma 3.27 again, we can identify the right side with $\frac{d}{dt}\Phi(c(t))_{|t=0}$. □

We are now ready to prove that the exponential map is the matrix exponential function under the identification of the Lie algebra of a Lie group with its linear Lie algebra.

Theorem 3.29. *Let G be a closed linear group of $n \times n$ matrices and let $\mathfrak{g}$ its linear Lie algebra. If the exponential map is regarded as carrying $\mathfrak{g}$ to G, then it is given by the exponential matrix function.*

Proof. We first consider the case $G = GL(n, \mathbb{C})$. Let X be in the Lie algebra of G and $\mu(X)$ the correspondent element in $\mathfrak{g}$; let $\tilde{X}$ the left-invariant vector field with X. By Equation (3.2), with $f = e_{ij}$, we have

$$\frac{d}{dt}\left(exp(tX)_{ij}\right) = \tilde{X}e_{ij}(exp(tX)) = \sum_{k=1}^{n} e_{ik}(exp(tX))\,\mu(X)_{kj},$$

i.e., the smooth curve $c(t) = (exp(tX))$ satisfy the differential equation

$$c'(t) = c(t)\,\mu(X), \qquad \text{with } c(0) = 1$$

that has unique solution $c(t) = e^{t\mu(X)}$.
For the general case of a closed Lie group G it is sufficient to apply Theorem 3.28 to the inclusion map $i : G \longrightarrow GL(n, \mathbb{C})$. □

Remark 3.30. Let G be a closed linear group, then we can consider X and g in (3.4) as matrices and we can think of *exp* as the usual exponential of matrices. If we replace X with tX, differentiate and set $t = 0$, we see that $Ad(g)(X)$ is given by the element gXg^{-1} in $\mathfrak{g}$.

Corollary 3.31. *Let G be an analytic group and $\Phi : G \longrightarrow GL(n, \mathbb{C})$ a smooth homomorphism. If we identify the Lie algebra of $GL(n, \mathbb{C})$ with $\mathfrak{gl}(\mathfrak{n}, \mathbb{C})$, then $\Phi \circ exp_g$ can be computed as $e^{d\Phi}$.*

Proof. It is just the formula in Equation (3.3) with $exp_{GL(n,\mathbb{C})}$ identified with the matrix exponential function. □

We are now ready to prove that the differential of *Ad* is *ad*.

Theorem 3.32. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. The differential of $Ad : G \longrightarrow GL(\mathfrak{g})$ is $ad : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$, where the Lie algebra of $GL(\mathfrak{g})$ has been identified with the Lie algebra $\mathfrak{gl}(\mathfrak{g})$.*

Proof. Let $L : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ be the differential of Ad. By Lemma 3.18 and Equation (3.4), for $X, Y \in \mathfrak{g}$ we have:

$$Ad(exp(tX))\,tY = tY + t^2[X,Y] + O\left(t^3\right) \qquad \text{as } t \to 0.$$

Dividing by t we obtain $Ad(exp(tX))\,Y = Y + t[X,Y] + O(t^2)$ as $t \to 0$; differentiating and putting $t = 0$ we get

$$L(X)\,Y = [X,Y] = ad(X)(Y).$$

Therefore $L = ad$. □

Remark 3.33. In the special case of $\Phi = Ad$ in Corollary 3.31, we have that $Ad\,(expX) = e^{ad(X)}$.

We end this section by showing that the linear Lie algebra of $SL(n,\mathbb{C})$ is $\mathfrak{sl}(\mathfrak{n},\mathbb{C})$. We will need an easy result about the matrix exponential function.

Lemma 3.34. *Let $A \in \mathfrak{gl}(\mathfrak{n},\mathbb{C})$, then $det\left(e^A\right) = e^{trace(A)}$.*

Proof. If T is upper triangular, then e^T is upper triangular with $e^T_{ii} = e^{T_{ii}}$ where in the left side e is the matrix exponential function, while in the left side it is the complex exponential; then we have:

$$det\left(e^T\right) = \prod_{k=1}^{n} e^{T_{ii}} = e^{\sum_{i=1}^{n} e^{T_{ii}}} = e^{trace(T)}.$$

Let now analyze the general case. If $A \in \mathfrak{gl}(\mathfrak{n},\mathbb{C})$, then there exists $X \in GL(n,\mathbb{C})$ such that $A = XTX^{-1}$ with T upper triangular; thus

$$det\left(e^A\right) = det\left(e^{XTX^{-1}}\right) = det\left(Xe^TX^{-1}\right) = det\left(e^T\right) = e^{trace(T)} = e^{trace\left(XTX^{-1}\right)} = e^{trace(A)}.$$

□

Theorem 3.35. *The linear Lie algebra of $SL(n,\mathbb{C})$ is $\mathfrak{sl}(\mathfrak{n},\mathbb{C})$.*

Proof. By Theorem 3.22, the linear Lie algebra of $SL(n,\mathbb{C})$ consists of matrices X such that $det\left(e^{tX}\right) = 1$ for every $t \in \mathbb{R}$; by Lemma 3.34, this is possible if and only if $trace\,(X) = 0$, thus if and only if $X \in \mathfrak{sl}(\mathfrak{n},\mathbb{C})$. □

Remark 3.36. The adjoint group of $SL(n,\mathbb{C})$ is $SL(n,\mathbb{C})_{ad} = PSL(n,\mathbb{C})$, where $PSL(n,\mathbb{C}) := {SL(n,\mathbb{C})}/{Z}$ with Z the center of $SL(n,\mathbb{C})$.

Chapter 4

Nilpotent orbits

The main goal of this thesis is the classification of the nilpotent orbits of a complex semisimple Lie algebra under the adjoint action.

Definition 4.1. (Nilpotent and semisimple elements)
Let $\mathfrak{g}$ be a semisimple Lie algebra and let $x \in \mathfrak{g}$. We say that x is a nilpotent (semisimple) element of $\mathfrak{g}$ if $ad\,(x)$ is a nilpotent (semisimple) endomorphism of $\mathfrak{g}$. We might refer to such x as an ad-nilpotent (ad-semisimple) element of $\mathfrak{g}$.

Let $Aut\,(\mathfrak{g}) = \left\{\Phi \in GL\,(\mathfrak{g}) \;\middle|\; [\Phi\,(X)\,, \Phi\,(Y)] = \Phi\,([X, Y])\,,\ \forall X, Y \in \mathfrak{g}\right\}$ be the automorphisms group of $\mathfrak{g}$. For every Φ in $Aut\,(\mathfrak{g})$ and X in $\mathfrak{g}$, we have that:
$ad_{\Phi(X)}\,(Y) = [\Phi\,(X)\,, Y] = [\Phi\,(X)\,, \Phi\,(\Phi^{-1})Y))] = \Phi\,([X, \Phi^{-1}\,(Y)]) = \Phi \cdot ad_X \cdot \Phi^{-1}\,(Y)$ for every Y in $\mathfrak{g}$, i.e.,

$$ad_{\Phi(X)} = \Phi \cdot ad_X \cdot \Phi^{-1} \tag{4.1}$$

From Equation (4.1), we have that an element $X \in \mathfrak{g}$ is nilpotent (semisimple) if and only if $\Phi\,(X)$ is nilpotent (semisimple) for every Φ in $Aut\,(\mathfrak{g})$. Since $G_{ad} \subseteq Aut(\mathfrak{g})$, $X \in \mathfrak{g}$ is nilpotent (semisimple) if and only if every element in its orbit is nilpotent (semisimple).

Remark 4.2. We noticed in Chapter 1 that for a semisimple Lie algebra $\mathfrak{g}$ the adjoint map is injective. The image of ad lies in

$$Der\,(\mathfrak{g}) = \{\delta \in End\,(\mathfrak{g})\,|\delta\,([x, y]) = [\delta\,(x)\,, y] + [x, \delta\,(y)] \text{ for every } x, y \in \mathfrak{g}\}\,.$$

An easy calculation shows that the bracket of two derivations is a derivation and that $Der\,(\mathfrak{g})$ is a vector subspace of $End\,(\mathfrak{g})$, thus $Der\,(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{gl}\,(\mathfrak{g})$. By

Jordan Theorem we can decompose a derivation into the sum of a semisimple and a nilpotent part that commute; moreover both parts are derivations, indeed: let $\delta = \sigma + \tau$ be a derivation, σ its semisimple part and τ its nilpotent part. If x and y are generalized eigenvectors of δ of eigenvalues λ and μ respectively, $[x, y]$ is either a generalized eigenvector of eigenvalue $\lambda + \mu$ or it is zero, since the following formula holds:

$$(\delta - (\lambda + \mu) Id)^n ([x, y]) = \sum_{i=0}^{n} \binom{n}{i} [(\delta - \lambda Id)^i (x), (\delta - \mu Id)^{n-i} (y)],$$

giving zero for a large integer n. Thus we have:

$$\sigma ([x, y]) = (\lambda + \mu) [x, y] = [\lambda x, y] + [x, \mu y] = [\sigma (x), y] + [x, \sigma (y)],$$

i.e., σ is a derivation. Then $\tau = \delta - \sigma$ is a derivation too. Moreover, if $\mathfrak{g}$ is semisimple then $Der(\mathfrak{g}) = ad(\mathfrak{g})$. (see [3], Section 5.3).
This remark leads us to the notion of the abstract Jordan decomposition: let H be an element of a semisimple Lie algebra $\mathfrak{g}$, then there exist unique $H_s, H_n \in \mathfrak{g}$ such that $H = H_s + H_n$ and $ad(H) = ad(H_s) + ad(H_n)$ is the Jordan decomposition of $ad(H)$. We call H_s, H_n, respectively, the semisimple and the nilpotent part of H and $H = H_s + H_n$ the abstract Jordan decomposition of H.

When both the "classical" and the "abstract" Jordan decomposition are defined, they coincide. In order to prove this one needs to show that the classical semisimple and nilpotent parts of an element still lie in $\mathfrak{g}$ (a proof can be found in [3], Section 6.4) and that these are semisimple and nilpotent as elements of $\mathfrak{g}$ (see Theorem 4.3); by the uniqueness of both decompositions, they must coincide.

For the special case of $\mathfrak{sl}(\mathfrak{n}, \mathbb{C})$ we can classify nilpotent conjugacy classes using Jordan classification Theorem. In order to analyze this case, we recall the following powerful theorem.

Theorem 4.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra, $x \in \mathfrak{g}$. Then the following claims are equivalent:*

1. *x is a nilpotent (semisimple) element of $\mathfrak{g}$ (i.e., ad-nilpotent);*

2. *for every finite dimensional representation* $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, $\rho(x)$ *is a nilpotent (semisimple) element of* $End(V)$*;*

3. *if* $\mathfrak{g}$ *is a Lie subalgebra of* $\mathfrak{gl}(\mathfrak{n}, \mathbb{C})$, x *is a nilpotent (semisimple) endomorphism of* $\mathbb{C}^n$.

Proof. 1. $\Rightarrow$ 2. Let x be a nilpotent element of $\mathfrak{g}$, i.e., x is ad-nilpotent. Then $\rho(x)$ is an ad-nilpotent element of $\mathfrak{gl}(V)$ because $ad^n(\rho(x)) = \rho(ad^n(x))$.
If now x is a semisimple element of $\mathfrak{g}$ and $\{y_1, ..., y_n\}$ is a basis of $\mathfrak{g}$ that diagonalizes $ad(x)$, then $\{\rho(y_1), ..., \rho(y_n)\}$ diagonalizes $ad(\rho(x))$ in $Im\rho$ since $[\rho(x), \rho(y_i)] = \rho([x, y_i]) = \lambda_i \rho(y_i)$.
The claim follows since the abstract Jordan decomposition and the classical Jordan decomposition coincide (when both are defined).
(2. $\Rightarrow$ 1. It is sufficient to apply 2. to the case of the adjoint representation.)
2. $\Rightarrow$ 3. It is sufficient to apply 2. to the case of the natural representation.
3. $\Rightarrow$ 1. We first analyze the case of x being a nilpotent endomorphism of $\mathbb{C}^n$. We define two maps $\phi, \psi : End(\mathbb{C}^n) \longrightarrow End(\mathbb{C}^n)$, $\phi(y) = xy$, $\psi(y) = yx$ for every $y \in \mathbb{C}^n$. These two maps are nilpotent commuting endomorphism of $End(\mathbb{C}^n)$, thus their difference, which is exactly $ad(x)$, is nilpotent.
Let now x be a semisimple endomorphism of $\mathbb{C}^n$. Fix a basis of $\mathbb{C}^n$ such that x has matrix $d = diag(a_1, ...a_n)$. The elementary matrices e_{ij} form a basis of $End(\mathbb{C}^n)$ that diagonalizes $ad(x)$, since $[d, e_{ij}] = (a_i - a_j) e_{ij}$. Thus $ad(x)$ is semisimple. □

4.1 Conjugacy classes in $\mathfrak{sl}(\mathfrak{n}, \mathbb{C})$

Let us consider the space $M_n(\mathbb{C})$ of $n \times n$ matrices over $\mathbb{C}$. The group $GL(n, \mathbb{C})$ acts on it by conjugation and its orbits are the conjugacy classes of matrices. The scalar matrices act trivially hence we have a representation of the quotient group $PGL(n, \mathbb{C}) := GL(n, \mathbb{C})/_Z$ where Z is the center of $GL(n, \mathbb{C})$. Since $\mathbb{C}$ is algebraically closed $PGL(n, \mathbb{C}) \simeq PSL(n, \mathbb{C})$. We shall denote by O_X the conjugacy class of $X \in \mathfrak{sl}(\mathfrak{n}, \mathbb{C})$ under the action of $PSL(n, \mathbb{C})$, i.e., its orbit under the adjoint action (see Remark 3.36).

Let us now examine the case $n = 2$. We know that the characteristic polynomial of a matrix A in $M_2(\mathbb{C})$ is of the form $(t-\lambda_1)(t-\lambda_2)$, so that two possible situations may occur:

- $\lambda_1 \neq \lambda_2$: i.e., A is diagonalizable and $0 = tr(A) = \lambda_1 + \lambda_2$ hence A is conjugate to $X(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$;
- $\lambda_1 = \lambda_2 = \lambda$: in this case we have $0 = tr(A) = 2\lambda$ which implies $\lambda = 0$. Looking at the minimal polynomial we conclude that A is conjugate either to $Y(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ or to $Y(1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

With the notation above, and defining $\Lambda_s = \{\mathbb{C}^* | \{\lambda \sim -\lambda\}\}$,

$$\mathfrak{sl}(2,\mathbb{C}) = \bigcup_{\lambda \in \Lambda_s} O_{X(\lambda)} \cup O_{Y(0)} \cup O_{Y(1)}.$$

In this elementary case we can recognize the general structure of the semisimple and nilpotent orbits in a semisimple Lie algebra, namely for the conjugacy classes of $\mathfrak{sl}(2,\mathbb{C})$ we have that:

1. there exist infinitely many semisimple orbits;
2. there exists only a finite number of nilpotent orbits;
3. an orbit is both semisimple and nilpotent if and only if it is zero.

Let us analyze the case of $\mathfrak{sl}(\mathfrak{n},\mathbb{C})$, with a deeper inspection of the case $n = 3$.

Definition 4.4. (Soft partition)
A soft partition of $n \in N$ is a tuple of natural integers $[d_1, d_2, \ldots, d_n]$ such that $d_1 + d_2 + \ldots + d_n = n$ and there exists an integer k with $1 \leq k \leq n$ such that $d_1 \geq d_2 \geq \ldots \geq d_k > 0$ and $d_{k+1} = \ldots = d_n = 0$.

Example 7. The soft partitions of 4 are:

$$[4,0,0,0]\,,\ [3,1,0,0]\,,\ [2,2,0,0]\,,\ [2,1,1,0]\,,\ [1,1,1,1]\,.$$

Let us denote by J_i the $i \times i$ Jordan block

$$J_i = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \in M_i\left(\mathbb{C}\right)$$

We can associate a soft partition $[d_1, d_2, \dots, d_n]$ with a $n \times n$ matrix in normal Jordan form with k blocks $J_{d_1}, \dots, J_{d_k}$,

$$X_{[d_1,d_2,\dots,d_n]} = \begin{pmatrix} J_{d_1} & 0 & 0 & \dots & 0 \\ 0 & J_{d_2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & J_{d_k} \end{pmatrix}.$$

This is of course a nilpotent matrix in $M_n\left(\mathbb{C}\right)$.

For example, for $n = 3$ we have:

$$X_{[3,0,0]} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ X_{[2,1,0]} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ X_{[1,1,1]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By the normal Jordan form theorem and since an element of a Lie subalgebra of $\mathfrak{gl}(V)$ is nilpotent if and only if it is nilpotent in $End(V)$, we have proved the existence of a "partition classification" for nilpotent elements of $\mathfrak{sl}(\mathfrak{n},\mathbb{C})$.
Let us compute the dimension of the three nilpotent orbits of $\mathfrak{sl}(3,\mathbb{C})$. In order to do this it is sufficient to calculate the dimension of the centralizer of a representative of each orbit and then use the formula $dimO_X = dim\ \mathfrak{g} - dim\ \mathfrak{g}^X$. Obviously, the centralizer of $X_{[1,1,1]}$ is $\mathfrak{g}$, thus $dimO_{X_{[1,1,1]}} = 0$. If we consider the matrix $X_{[2,1,0]}$, an easy calculation shows that

$$\mathfrak{sl}(3,\mathbb{C})^{X_{[2,1,0]}} = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11} & 0 \\ 0 & a_{32} & -2a_{11} \end{pmatrix} : \ a_{ij} \in \mathbb{C} \right\}$$

which has dimension 4, thus $dimO_{X_{[2,1,0]}} = 8 - 4 = 4$. Finally,

$$\mathfrak{sl}(3,\mathbb{C})^{X_{[3,0,0]}} = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{12} \\ 0 & 0 & 0 \end{pmatrix} : \ a_{ij} \in \mathbb{C} \right\}$$

which has dimension 2, thus $dimO_{X_{[3,0,0]}} = 8 - 2 = 6$.

4.2 Jacobson-Morozov Theorem

The purpose of this paragraph is to prove a fundamental theorem which is the first step in our walk toward a classification of nilpotent orbits. More precisely, we are going to prove that with every non-zero nilpotent element in a Lie algebra $\mathfrak{g}$ we can associate a so called "standard triple".

Definition 4.5. (Standard triple)
Let $\mathfrak{g}$ be a Lie algebra. A standard triple in $\mathfrak{g}$ is a triple of elements $\{H, X, Y\} \subset \mathfrak{g}$ satisfying the following bracket relations:

$$[X, Y] = H, \ [H, X] = 2X, \ [H, Y] = -2Y.$$

H,X,Y are called the semisimple, nilpositive and nilnegative element of the triple, respectively.

Remark 4.6. One can immediately check that a standard triple spans a Lie subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.
The nilpositive and nilnegative elements are nilpotent as elements of $Span\{H, X, Y\}$, thus by Theorem 4.3 they are nilpotent as elements of $\mathfrak{g}$. The same holds for the semisimple element of the standard triple.

G_{ad}-invariance of standard triples and a first isomorphism Let $\mathfrak{g}$ be a Lie algebra. We want to prove that there exists a bijection between $\mathrm{Hom}^{\times}(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{g})$ and the set of standard triples in $\mathfrak{g}$. Define

$$\Gamma : \mathrm{Hom}^{\times}(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{g}) \longrightarrow \{\text{standard triples in } \mathfrak{g}\}$$

$$\Phi \mapsto \{\Phi(h), \Phi(e), \Phi(f)\}.$$

We can construct the inverse map of Γ in the following way: to a standard triple $\left\{\tilde{H}, \tilde{E}, \tilde{F}\right\}$ in $\mathfrak{g}$ we associate the Lie algebra homomorphism Φ defined as the homomorphism that sends the semisimple element of the triple to h, the nilnegative to f and the nilpositive to e. Φ is an isomorphism between the standard triple and $\mathfrak{sl}(2,\mathbb{C})$ and its inverse lies in $\mathrm{Hom}^{\times}(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{g})$. The map defined by $\Gamma^{-1}\left(\left\{\tilde{H}, \tilde{E}, \tilde{F}\right\}\right) := \Phi^{-1}$ is the inverse of Γ.
G_{ad} acts on $\mathfrak{g}$ via automorphisms, so we define its action on standard triples as

$$x.\{H, X, Y\} := \{x.H, x.X, x.Y\}.$$

We then define the action on $\mathrm{Hom}^{\times}(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{g})$ as $(X \cdot \Phi)(v) = X \cdot \Phi(v)$. If these sets are G_{ad}-invariant, the action is well defined and $\Gamma(X \cdot \Phi) = X \cdot \Gamma(\Phi)$. Our bijection is a G_{ad}-sets isomorphism (this works because we used the bijection to transfer the action from $\{$standard triples in $\mathfrak{g}\}$ to $\mathrm{Hom}^{\times}(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{g})$). Thus, it is sufficient to prove that the set of standard triples in $\mathfrak{g}$ is G_{ad}-invariant.
The action is via automorphisms, so $x \cdot \tilde{H}, x \cdot \tilde{X}, x \cdot \tilde{Y} \neq 0$ $\forall x \in G_{ad}$ and $x \cdot [\alpha, \beta] = [x \cdot \alpha, x \cdot \beta]$. It follows that $\left\{x \cdot \tilde{H}, x \cdot \tilde{E}, x \cdot \tilde{F}\right\}$ is still a standard triple, indeed $\forall x \in G_{ad}$, we have:

$$\left[x \cdot \tilde{H}, x \cdot \tilde{X}\right] = x \cdot \left[\tilde{H}, \tilde{X}\right] = x \cdot \left(2\tilde{X}\right) = 2\left(x \cdot \tilde{X}\right),$$

$$\left[x \cdot \tilde{X}, x \cdot \tilde{Y}\right] = x \cdot \left[\tilde{X}, \tilde{Y}\right] = x \cdot \tilde{H},$$

$$\left[x \cdot \tilde{H}, x \cdot \tilde{Y}\right] = x \cdot \left[\tilde{H}, \tilde{Y}\right] = x \cdot \left(-2\tilde{Y}\right) = -2\left(x \cdot \tilde{Y}\right).$$

We will need the following results:

Lemma 4.7. *Let $\mathfrak{g}$ be a semisimple Lie algebra and let k its killing form. Then $\left(\mathfrak{g}^X\right)^{\perp} = [\mathfrak{g}, X]$. Moreover if $X \in \mathfrak{g}$ is nilpotent, then $X \in \left(\mathfrak{g}^X\right)^{\perp}$.*

Proof. Let $W \in \mathfrak{g}^X$, then $k([\mathfrak{g}, X], W) = k(\mathfrak{g}, [X, W]) = 0$ by the associativity of the killing form. We have proved that $[\mathfrak{g}, X] \subseteq \left(\mathfrak{g}^X\right)^{\perp}$. Since $\mathfrak{g}$ is semisimple, k is non-degenerate thus $dim\ \mathfrak{g} = dim\ \mathfrak{g}^X + dim\left(\mathfrak{g}^X\right)^{\perp}$ and, by the rank-nullity theorem applied

to $ad\,(X)$, $dim\ \mathfrak{g} = dim\ \mathfrak{g}^X + dim\,[\mathfrak{g}, X]$. So we can conclude that $dim\,[\mathfrak{g}, X] = dim\left(\mathfrak{g}^X\right)^{\perp}$ that implies $[\mathfrak{g}, X] = \left(\mathfrak{g}^X\right)^{\perp}$.
Let now X be nilpotent. Since $ad\,(X)$ and $ad\,(Z)$ commute for every Z in $\mathfrak{g}^X$, $ad\,(X)\,ad\,(Z)$ is nilpotent. Thus for every $Z \in \mathfrak{g}^X$, $k\left(X, \mathfrak{g}^X\right) = trace\,(ad\,(X) \cdot ad\,(Z)) = 0$, i.e., $X \in \left(\mathfrak{g}^X\right)^{\perp}$. □

Lemma 4.8. *Let* $\mathfrak{g}$ *be a complex reductive Lie algebra and* H *a semisimple element in* $\mathfrak{g}$*. Then* $\mathfrak{g}^H$ *is reductive and there exists a Cartan subalgebra* $\mathfrak{h}$ *containing* H*. If* Φ *denotes the root system for the pair* $(\mathfrak{g}, \mathfrak{h})$*, then*

$$\mathfrak{g}^H = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_H} \mathfrak{g}_\alpha, \qquad \textit{where } \Phi_H = \{\alpha \in \Phi | \alpha\,(H) = 0\}\,.$$

Proof. We have already observed that there exists a Cartan subalgebra containing H. A complete proof can be found in [1], Chapter 2. □

Lemma 4.9. *Let* $\mathfrak{g}$ *be a complex Lie algebra and let* H *be a semisimple element of* $\mathfrak{g}$*. If* X *is an eigenvector of* $ad\,(H)$*, then* $\mathfrak{g}^X$ *is* $ad\,(H)$*-stable.*

Proof. Let X be an eigenvector of $ad\,(H)$ of eigeinvalue λ and let W be an element of $\mathfrak{g}^X$. We want to show that $ad\,(H)\,(W) = [H, W]$ lies in $\mathfrak{g}^X$. We have that

$$[X, [H, W]] = -\left[[H, X]\,, W\right] + [H, [X, W]] = -\lambda\,[X, W] + 0 = 0.$$

□

We are now ready to prove

Theorem 4.10. *(Jacobson-Morozov)*
Let $\mathfrak{g}$ *be a complex semisimple Lie algebra and* X *a non-zero nilpotent element of* $\mathfrak{g}$*. Then there exists a standard triple in* $\mathfrak{g}$ *whose nilpositive element is* X*.*

Proof. We will proceed by induction on $dim\ \mathfrak{g} \geq 3$.
If $dim\ \mathfrak{g} = 3$ then $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ (see Remark 1.23) and we can identify $\mathfrak{g}$ and $\mathfrak{sl}(2, \mathbb{C})$. X is an ad-nilpotent element and, by Theorem 4.3, it is a nilpotent endomorphism of $\mathbb{C}^2$. By the Jordan normal form theorem, there exists $A \in GL\,(2, \mathbb{C})$

such that $X = AeA^{-1}$. Let $Y = AfA^{-1}$, $H = AhA^{-1}$, then $\{H, X, Y\}$ is a standard triple with nilpositive element X, indeed:

$$[X, Y] = AeA^{-1}AfA^{-1} - AfA^{-1}AeA^{-1} = AefA^{-1} - AfeA^{-1} = A[e,f]A^{-1} = AhA^{-1} = H,$$

$$[H, X] = A[h,e]A^{-1} = 2AeA^{-1} = 2X, \quad [H, Y] = A[h,f]A^{-1} = -2AfA^{-1} = -2Y.$$

Let now be $dim\ \mathfrak{g} > 3$. If X lies in any proper semisimple subalgebra of $\mathfrak{g}$ we can conclude by the inductive hypothesis. Suppose that X is not contained in any such subalgebra. We first look for a candidate for the semisimple element of the standard triple. By Lemma 4.7 we have that $X \in \left(\mathfrak{g}^X\right)^\perp = [\mathfrak{g}, X]$, then there exists an element $\overline{H}$ such that $\left[\overline{H}, X\right] = 2X$. In order to construct a standard triple, we want $\overline{H}$ to be semisimple in $\mathfrak{g}$ by Remark 4.6. From Jordan Theorem we can decompose $ad\left(\overline{H}\right) = ad\left(\overline{H}\right)_s + ad\left(\overline{H}\right)_n$, the sum of its semisimple and nilpotent parts. By Remark 4.2 there exist $\overline{H}_s, \overline{H}_n \in \mathfrak{g}$ that act, respectively, semisimply and nilpotently on $\mathfrak{g}$ and $\overline{H} = \overline{H}_s + \overline{H}_n$. $\overline{H}$ acts semisimply on X, so $2X = \left[\overline{H}, X\right] = \left[\overline{H_s}, X\right] + \left[\overline{H_n}, X\right]$ and it must be $\left[\overline{H_s}, X\right] = 2X$, $\left[\overline{H_n}, X\right] = 0$ (it is sufficient to take a basis with respect to which the matrix of $ad\left(\overline{H}\right)$ is in Jordan normal form and to look how its nilpotent and semisimple parts act on an eigeinvector). We then set $H = \overline{H}_s$. Thus, we have proved that there exists a semisimple element in $\mathfrak{g}$ such that $[H, X] = 2X$. It remains to prove that there is an element Y in $\mathfrak{g}$ that makes $\{H, X, Y\}$ a standard triple. We can decompose $\mathfrak{g}$ in eigenspaces of $ad(H)$:

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{C}} \mathfrak{g}_\lambda, \ \ \mathfrak{g}_\lambda = \{W \in \mathfrak{g} |\, [H, W] = \lambda W\}.$$

This is actually a finite sum since $\mathfrak{g}$ is finite dimensional. H lies in $\mathfrak{g}_0$, while $X \in \mathfrak{g}_2$. Moreover, if $W \in \mathfrak{g}_\lambda$ we have

$$ad(H)([X, W]) = [H, [X, W]] = [[H, X], W] + [X, [H, W]] = (2 + \lambda)[X, W],$$

i.e., $[X, \mathfrak{g}_\lambda] \subseteq \mathfrak{g}_{\lambda+2}$. In order to conclude our proof, it is sufficient to show that $H \in [X, \mathfrak{g}]$. Indeed if this is the case and $\hat{Y}$ is an element such that $H = \left[X, \hat{Y}\right]$, we can write $Y = \sum_{\lambda \in \mathbb{C}} Y_\lambda$ where $Y_\lambda \in \mathfrak{g}_\lambda$. Since the sum of different eigenspaces is direct and $H \in \mathfrak{g}_0$,

$$H = \left[X, \hat{Y}\right] = \sum_{\lambda \in \mathbb{C}} [X, Y_\lambda] \Rightarrow H = [X, Y_{-2}].$$

Therefore, if we let $Y = Y_{-2}$ we have that $\{H, X, Y\}$ is a standard triple with nilpositive element X.
Now we prove that H lies in $[X, \mathfrak{g}] = \left(\mathfrak{g}^X\right)^{\perp}$ by contradiction. Suppose that $H \notin \left(\mathfrak{g}^X\right)^{\perp}$, then there exists an element Z in $\mathfrak{g}^X$ such that $k(H, Z) \neq 0$. By Lemma 4.9, $\mathfrak{g}^X$ is $ad(H)$-invariant and we can consider the $ad(H)$-eigenspaces decomposition

$$\mathfrak{g}^X = \bigoplus_{\lambda\in\mathbb{C}} \mathfrak{g}^X_\lambda, \ \mathfrak{g}^X_\lambda = \mathfrak{g}_\lambda \cap \mathfrak{g}^X.$$

We note that $\mathfrak{g}^X_0$ is the centralizer of H in $\mathfrak{g}^X$, so we can write $\mathfrak{g}^X$ as

$$\mathfrak{g}^X = \left(\mathfrak{g}^X\right)^H \oplus \bigoplus_{\lambda\in\mathbb{C}\setminus\{0\}} \mathfrak{g}^X_\lambda.$$

If $Z \neq 0$ and $Z \in \mathfrak{g}^X_\lambda$ with $\lambda \neq 0$, then we have

$$0 = k([H, H], Z]) = k(H, [H, Z]]) = \lambda k(H, Z)$$

therefore $H \in \left(\mathfrak{g}^X_\lambda\right)^{\perp}$. Thus there must be a non-zero element $Z \in \left(\mathfrak{g}^X\right)^H$ with the property $k(H, Z) \neq 0$ (if not, $k\left(H, \mathfrak{g}^X\right) = 0$ by bilinearity, that would contradict our hypothesis).
By Lemma 4.7, Z cannot be nilpotent, indeed $Z \in \left(\mathfrak{g}^X\right)^H$ if and only if $H \in \left(\mathfrak{g}^X\right)^Z$ and Z being nilpotent would imply $Z \in \left(\left(\mathfrak{g}^X\right)^Z\right)^{\perp}$; then it would be $k(H, Z) = 0$. Thus the semisimple part of Z is non-zero. We want to show that this situation leads to an absurd, in particular that X lies in a proper semisimple subalgebra of $\mathfrak{g}$.
Let us notice that:

a) $H \in \mathfrak{g}^{Z_s}$: H acts semisimply on Z and this is equivalent to the fact that Z acts semisimply on H, then $0 = [H, Z] = -[Z_s, H]$.

b) $X \in \mathfrak{g}^{Z_s}$: this is obvious since $Z \in \mathfrak{g}^X$, thus it acts semisimply on X.

c) $\mathfrak{g}^{Z_s}$ is a proper subalgebra of $\mathfrak{g}$: if $\mathfrak{g}^{Z_s} = \mathfrak{g}$ then $Z_s = 0$, because it would be in the center of $\mathfrak{g}$, which is trivial.

These three fact imply that $X = \frac{1}{2}[H, X] \in \left[\mathfrak{g}^{Z_s}, \mathfrak{g}^{Z_s}\right]$, that is a proper subalgebra of $\mathfrak{g}$. By Lemma 4.8 this subalgebra is also semisimple and this is in contrast with our initial hypothesis. □

Jacobson-Morozov Theorem will provide us the surjectivity of a map from the set of standard triples to the set of nilpotent orbits of $\mathfrak{g}$. In the next section we will explore the injectivity of the same map.

4.3 Kostant and Mal'cev's theorems

In this section we want to prove that two standard triples with the same nilpositive element or the same semisimple element are conjugate under the action of G_{ad}. The representation theory of $\mathfrak{sl}(2,\mathbb{C})$ (see Chapter 2) will be used.
By Weyl's Theorem every finite-dimensional representation of a semisimple Lie algebra is completely reducible. We can apply this result to the adjoint representation of a standard triple $\{H, X, Y\}$ (which can be identified with $\mathfrak{sl}(2,\mathbb{C})$). H acts semisimply, thus we can decompose $\mathfrak{g}$ in $ad\,(H)$-eigenspaces as we did in the proof of Jacobson-Morozov's theorem. In Lemma 4.9 we proved the stability of $\mathfrak{g}^X$ under the action of H. Due to the representation theory of $\mathfrak{sl}(2,\mathbb{C})$, in every irreducible representation of the standard triple the eigenvalues of $ad\,(H)$ are integers; moreover, $\mathfrak{g}^X$ is the sum of highest weight spaces with non-negative weight, since an eigenvector W of H such that $X.W = [X, W] = 0$ is a maximal weight vector. Then we can refine the decomposition obtained in Jacobson-Morozov's theorem as follows:

$$\mathfrak{g}^X = \bigoplus_{i\geq 0} \mathfrak{g}_i^X,\ \mathfrak{g}_i^X = \mathfrak{g}_i \cap \mathfrak{g}^X. \tag{4.2}$$

This allows us to prove that a standard triple is uniquely determined by its nilpositive and semisimple elements.

Lemma 4.11. *Let $\mathfrak{g}$ be a semisimple algebra and $\{H, X, Y\}$, $\{H, X, Y'\}$ two standard triples. Then $Y = Y'$.*

Proof. Notice that $Y - Y' \in \mathfrak{g}_{-2}$ since both Y and Y' do. Moreover $Y - Y' \in \mathfrak{g}^X$, in fact $[X, Y - Y'] = H - H = 0$. From decomposition (4.2) we have that $\mathfrak{g}^X \cap \mathfrak{g}_{-2} = \{0\}$. It follows that $Y = Y'$. □

We want to prove that two standard triples with the same nilpositive element are conjugate under G_{ad}; in order to obtain such a result we don't need the whole G_{ad}, but it is enough to consider a subgroup of G_{ad}, that we will denote by U^X. U^X is defined as the centralizer of X in the connected component of the subgroup of G_{ad} with Lie algebra $u = \oplus_{i>0}\mathfrak{g}_i$. We now introduce the Lie algebra u^X associated with U^X.

Lemma 4.12. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\{H, X, Y\}$ a standard triple in $\mathfrak{g}$. Then $u^X = \mathfrak{g}^X \cap [\mathfrak{g}, X]$ is an $ad\,(H)$-invariant nilpotent ideal of $\mathfrak{g}^X$. More precisely, $u^X = \bigoplus_{i>0} \mathfrak{g}_i^X$.*

Proof. u^X is $ad\,(H)$-invariant since both $\mathfrak{g}^X$ (see Lemma 4.9) and $[\mathfrak{g}, X]$ are $ad\,(H)$-invariant, indeed for $Z \in [\mathfrak{g}, X]$, there exists $W \in \mathfrak{g}$ such that $Z = [W, X]$ and we have

$$[H, Z] = [H, [W, X]] = [[H, W], X] + [W, [H, X]] = [[H, W] + 2W, X] \in [\mathfrak{g}, X].$$

Let now $T \in \mathfrak{g}^X$ and $W \in u^X$; we have that:

$$[X, [W, T]] = [[X, W], T] + [W, [X, T]] = 0 + 0 = 0, \quad \text{i.e, } [W, T] \in \mathfrak{g}^X$$

and there exists $Z \in \mathfrak{g}$ such that $W = [Z, X]$, so that

$$[W, T] = [[Z, X], T] = [Z, [X, T]] - [X, [Z, T]] = -[X, [Z, T]] \in [\mathfrak{g}, X].$$

Thus $[W, T] \in u^X$, i.e., u^X is an ideal of $\mathfrak{g}^X$. Moreover, it must be $u^X \subseteq \bigoplus_{i>0} \mathfrak{g}_i^X$, indeed if $T \in \mathfrak{g}_0^X$, it must be a highest weight vector of weight zero; thus there is not an element S in $\mathfrak{g}_{-2}^X$ such that $[S, X] = T$. Since $[\mathfrak{g}, X] = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_{j+2}^X$, its intersection with $\mathfrak{g}^X$ must be contained in $\bigoplus_{i>0} \mathfrak{g}_i^X$. The representation theory of $\mathfrak{sl}(2, \mathbb{C})$ tell us that the equality holds. u^X is nilpotent since $\left[\mathfrak{g}_i^X, \mathfrak{g}_j^X\right] \subseteq \mathfrak{g}_{i+j}^X$ and $\mathfrak{g}$ is finite dimensional. □

As a general fact, we can rebuild the group action from the Lie algebra using the exponential map *Exp*: for $Z, H \in \mathfrak{g}$,

$$Exp\ Z \cdot H = \sum_{i=0}^{\infty} \frac{(ad\,(Z))^i\,(H)}{i!}.$$

We notice that if Z is nilpotent, the sum makes sense because it is actually finite. Moreover if Z lies in u^X, then every summand but H lie in u^X. This means that $U^X \cdot H \subseteq u^X + H$, where U^X is the connected Lie subgroup of G_{ad}^X with Lie algebra u^X (since the last is nilpotent, *Exp* is a diffeomorphism).

Lemma 4.13. *Let $\mathfrak{g}$ be a semisimple Lie algebra and H a semisimple element. For every $V \in u^X$ there exists a unique $Z \in u^X$ such that $Exp\ Z \cdot H = V + H$.*
Equivalently, for every $V \in u^X$ there exists a unique $x \in U^X$ such that $x \cdot H = V + H$.

Proof. Since $\mathfrak{g}$ is finite dimensional, there exists an integer m such that $u^X = \bigoplus_{1 \leq i \leq m} \mathfrak{g}_i^X$. We construct the element Z in the statement inductively.
Let V_1 be the component of V in $\mathfrak{g}_1^X$ and $Z_1 = -V_1$. We have that $[Z_1, H] = -Z_1 = V_1$. So we have that

$$Exp\ Z_1 \cdot H - (H+V) = \sum_{i=0}^{\infty} \frac{(ad\,(Z_1))^i\,(H)}{i!} - (H+V) = H + V_1 - H - V =$$
$$= V_1 - V \in \bigoplus\nolimits_{2 \leq i \leq m} \mathfrak{g}_i^X.$$

Suppose now that there exists an element Z_j such that

$$Z_j \in \bigoplus_{1 \leq i \leq j} u_i^X,\ Exp\ Z_j \cdot H - (H+V) \in \bigoplus_{j+1 \leq i \leq m} \mathfrak{g}_i^X.$$

We define Z'_{j+1} as the component of $Exp\ Z_j \cdot H - (H+V) \in \bigoplus_{j+1 \leq i \leq m} \mathfrak{g}_i^X$ that lies in $\mathfrak{g}_{j+1}$ and $Z_{j+1} = Z_j + \frac{Z'_{j+1}}{j+1}$. Obviously $Z_{j+1} \in \bigoplus_{1 \leq i \leq j+1} u_i^X$. We also have

$$Exp\ Z_{j+1} \cdot H = Exp\ Z_j \cdot H + \frac{1}{j+1}\left[Z'_{j+1}, H\right] + \ldots = Exp\ Z_j \cdot H - \frac{j+1}{j+1} Z'_{j+1} + \ldots$$

where the dots refer to terms that lie in eingenspaces with weight higher than $j+1$, so $Exp\ Z_{j+1} \cdot H - (H+V) \in \bigoplus_{j+2 \leq i \leq m} \mathfrak{g}_i^X$. This concludes our induction and, for $Z = Z_m$, $Exp\ Z \cdot H = V + H$.
In order to prove the uniqueness of the element Z, we can notice that the projections on the $\mathfrak{g}_i^X$'s are uniquely determined. The existence of the element Z is equivalent to the existence of the element $x \in U^X$ in the statement since Exp is a diffeomorphism. □

We are now ready to prove the following crucial result.

Theorem 4.14. *(Kostant)*
Let $\mathfrak{g}$ be a semisimple algebra and $\{H, X, Y\}$, $\{H', X, Y'\}$ two standard triples in $\mathfrak{g}$ with the same nilpositive element. Then there exists $x \in U^X$ such that $x \cdot H = H'$, $x \cdot X = X$, $x \cdot Y = Y'$.

Proof. We have that $[H' - H, X] = 2X - 2X = 0$, i.e. $H' - H \in \mathfrak{g}^X$. Moreover H and H' obviously lie in $[\mathfrak{g}, X]$, thus $H' - H \in u^X$. We can apply Lemma 4.13 with $V = H - H'$, so there exists $x \in U^X$ such that $x \cdot H = H + H' - H = H'$. Moreover $x \cdot X = X$ since $x = Exp\ Z$ for an opportune Z in u^X, so the only non-zero term in the sum will be X. The invariance of standard triples under the action of G_{ad} implies that $\{H', X, x \cdot Y\}$ is a standard triple. By Lemma 4.11, $x \cdot Y = Y'$. □

Remark 4.15. We are now ready to define the following bijection: let $\mathfrak{g}$ be a semisimple Lie algebra and let us consider the following map

$$\Omega : \{G_{ad} - \text{conjugacy classes of standard triples in } \mathfrak{g}\} \longrightarrow \{\text{nilpotent orbits in } \mathfrak{g}\}$$

$$[\{H, X, Y\}] \mapsto O_X.$$

Jacobson-Morozov Theorem gives us the surjectivity of this map, while the injectivity follows from Kostant Theorem.

We can make a further step building a bijection between these sets and a certain subset of semisimple orbits, the set of distinguished semisimple orbits. We already know that there is no chance to have a bijection between nilpotent orbits and semisimple orbits, since already in the case of $\mathfrak{sl}(2, \mathbb{C})$ we noticed that the former is a finite set, while the latter is infinite.

There is a natural way to produce such a map, namely

$$\Upsilon : \{G_{ad} - \text{conjugacy classes of standard triples in } \mathfrak{g}\} \longrightarrow \{\text{semisimple orbits in } \mathfrak{g}\}$$

$$[\{H, X, Y\}] \mapsto O_H$$

We define $S_{dist} = \text{Image}\Upsilon$. We will prove that this map is injective and this will be the content of Mal'cev's Theorem.

We want to prove that two standard triple with the same semisimple element are conjugate under G_{ad}. As it happened for Kostant Theorem, it is enough to consider a subgroup of G_{ad}; in particular, we consider the centralizer of H in G_{ad}, i.e., $G_{ad}^H := \{x \in G_{ad} \mid x \cdot H = H\}$.

Theorem 4.16. *(Mal'cev)*
Let $\mathfrak{g}$ be a semisimple Lie algebra and $\{H, X, Y\}$, $\{H, X', Y'\}$ two standard triples with the same semisimple element. Then there exists $x \in \left(G_{ad}^H\right)^\circ$ such that $\{x \cdot H, x \cdot X, x \cdot Y\} = \{H, X', Y'\}$, where $\left(G_{ad}^H\right)^\circ$ is the connected component of the identity of G_{ad}^H.

Proof. We can decompose $\mathfrak{g}$ into $ad\,(H)$-eigenspaces, $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ as we did before. We define

$$P = \left\{Z \in \mathfrak{g}_2 | \mathfrak{g}^Z \cap \mathfrak{g}_{-2} = 0\right\}.$$

For a nilpositive element E of a standard triple we already know that $\mathfrak{g}^E$ is the sum of eigenspaces relative to non-negative eigenvalues; this pose X, X' in P. Moreover $\mathfrak{g}_2$ is invariant under the action of $\left(G_{ad}^H\right)^\circ$ since $\mathfrak{g}_0$ is the tangent space of G_{ad}^H and $[\mathfrak{g}_0, \mathfrak{g}_2] \subseteq \mathfrak{g}_2$ (and the same holds for $\mathfrak{g}_{-2}$).
Let now x be an element of $\left(G_{ad}^H\right)^\circ$,

$$Z \in P \Leftrightarrow \mathfrak{g}^Z \cap \mathfrak{g}_{-2} = 0 \Leftrightarrow x \cdot \left(\mathfrak{g}^Z \cap \mathfrak{g}_{-2}\right) = 0 \Leftrightarrow \mathfrak{g}^{x \cdot Z} \cap x \cdot \mathfrak{g}_{-2} = 0 \Leftrightarrow \mathfrak{g}^{x \cdot Z} \cap \mathfrak{g}_{-2} = 0 \Leftrightarrow x \cdot Z \in P$$

where the third and the fourth equivalences come, respectively, from the fact that $x \cdot [a, b] = [x \cdot a, x \cdot b]$ and the invariance of $\mathfrak{g}_{-2}$.
The theorem will be proved once we show that $\left(G_{ad}^H\right)^\circ$ acts transitively on P, for then there exists an element x in $\left(G_{ad}^H\right)^\circ$ that sends X to X' and fixes H and we can conclude that the two triples are conjugate by the stability of the set of standard triples under the action of G_{ad} and Lemma 4.3. We will use a bit of topology to show that $\left(G_{ad}^H\right)^\circ$ has only one orbit on P.

- P is path connected (and, in particular, connected):

we want to establish a correspondence between P and a Zariski open set of $\mathfrak{g}_2$. Indeed, this is sufficient to prove that it is an Euclidean open path connected set (it is open since the Euclidean topology is finer then the Zariski, while it is path connected because the line $\lambda A + (1-\lambda) B$ linking two elements A, B is contained in P except for a finite number of $\lambda \in \mathbb{C}$ and $\mathbb{C}$ minus a finite number of elements is path connected).
Let us now exhibit such a correspondence. We set:

$$T : \mathfrak{g}_2 \longrightarrow Hom\,(\mathfrak{g}_{-2}, \mathfrak{g}_0)$$

$$Z \mapsto ad(Z).$$

Notice that $T(Z)$ is represented by a $dim\ \mathfrak{g}_0 \times dim\ \mathfrak{g}_{-2}$ matrix, whose entries depend linearly on Z (so this is, in fact, an homeomorphism). Moreover, $KerT(Z) = \mathfrak{g}_{-2} \cap \mathfrak{g}_0$ and $ImT(Z) = [Z, \mathfrak{g}_{-2}]$, therefore

$$Z \in P \Leftrightarrow \mathfrak{g}^Z \cap \mathfrak{g}_{-2} = 0 \Leftrightarrow KerT(Z) = 0 \Leftrightarrow dim\,[Z, \mathfrak{g}_{-2}] = dim\ \mathfrak{g}_{-2} = dim\ \mathfrak{g}_2$$

where the last equality comes from the $\mathfrak{sl}(2, \mathbb{C})$ representation theory. This proves that $Z \in P$ if and only if $T(Z)$ has full rank. The complement of full rank matrices are matrices whose columns satisfy a system of linear equations, so it is a Zariski closed set. Thus P is homeomorphic to a Zariski open set.

- Each $\left(G_{ad}^H\right)^\circ$-orbit is open and closed (in the Euclidean topology):

it is sufficient to prove that such orbits are open, as their complement is open since it is a disjoint union of orbits. In order to do this, we prove that the tangent space of an orbit is the whole space.
By the associativity of the killing form,

$$0 = k\left(\left[\mathfrak{g}^Z \cap \mathfrak{g}_0, Z\right], \mathfrak{g}_{-2}\right) = k\left(\mathfrak{g}^Z \cap \mathfrak{g}_0, [Z, \mathfrak{g}_{-2}]\right)$$

and since $[Z, \mathfrak{g}_{-2}] \subseteq \mathfrak{g}_0$, we have that $[Z, \mathfrak{g}_{-2}] \subseteq \left(\mathfrak{g}^Z \cap \mathfrak{g}_0\right)^\perp \cap \mathfrak{g}_0$. By the proof of Lemma 4.8, the restriction of the killing form to $\mathfrak{g}_0 = \mathfrak{g}^H$ is non-degenerate and this implies that

$$dim\ \mathfrak{g}_0 = dim\left(\left(\mathfrak{g}^Z \cap \mathfrak{g}_0\right)^\perp \cap \mathfrak{g}_0\right) + dim\left(\mathfrak{g}^Z \cap \mathfrak{g}_0\right) \geq dim\,[Z, \mathfrak{g}_{-2}] + dim\left(\mathfrak{g}^Z \cap \mathfrak{g}_0\right),$$

which can be rewritten as

$$dim\left(\mathfrak{g}^Z \cap \mathfrak{g}_0\right) \leq dim\ \mathfrak{g}_0 - di\,[Z, \mathfrak{g}_{-2}].$$

We already proved that $Z \in P \Leftrightarrow dim\,[Z, \mathfrak{g}_{-2}] = dim\ \mathfrak{g}_{-2}$, so that for such an element the inequality becomes

$$dim\left(\mathfrak{g}^Z \cap \mathfrak{g}_0\right) \leq dim\ \mathfrak{g}_0 - dim\ \mathfrak{g}_{-2} = dim\ \mathfrak{g}_0 - dim\ \mathfrak{g}_2.$$

Now we can conclude that $\left(G_{ad}^{H}\right)^{\circ}$-orbits have the same dimension of $\mathfrak{g}_2$, indeed:

$$dim\left(\left(G_{ad}^{H}\right)^{\circ}\cdot Z\right)=dim\,[Z,\mathfrak{g}_0]=dim\ \mathfrak{g}_0-dim\left(\mathfrak{g}^{Z}\cap\mathfrak{g}_0\right)\geq dim\ \mathfrak{g}_0-dim\ \mathfrak{g}_0+dim\ \mathfrak{g}_2=$$

$$=dim\ \mathfrak{g}_2,$$

where the second equality is the rank-nullity Theorem applied to the map $\alpha:\mathfrak{g}_0\longrightarrow\mathfrak{g}_2,\ \alpha(X)=[Z,X]$.

It follows that $\left(G_{ad}^{H}\right)^{\circ}$ acts transitively on P. $\square$

Chapter 5

Weighted Dynkin diagrams

5.1 Kostant Theorem

Our ultimate goal is to establish a correspondence between the set of distinguished semisimple orbits and the so called weighted Dynkin diagrams. As a consequence, we will prove that there are only finitely many nilpotent orbits.
Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Consider a base Δ of the root system Φ of $\mathfrak{g}$ and the relative set Φ^+ of positive root. Let $\mathfrak{n} = \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$ and $\bar{\mathfrak{n}} = \sum_{\alpha\in\Phi^-} \mathfrak{g}_\alpha$. An element $Z \in \mathfrak{g}$ is said to be $\Delta -$ *dominant* if $\alpha(Z)$ is real and nonnegative for every α in Δ.

Definition 5.1. (Borel subalgebra)
$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is called a Borel subalgebra of $\mathfrak{g}$. The opposite Borel subalgebra is $\bar{\mathfrak{b}} = \mathfrak{h} \oplus \bar{\mathfrak{n}}$.

Definition 5.2. (Fundamental domain)
We call fundamental domain the set

$$D_\Delta = \left\{ x \in \mathfrak{h} \;\middle|\; Re(\alpha(x)) > 0 \textit{ or } Re(\alpha(x)) = 0 \textit{ and } Im(\alpha(x)) \geq 0,\ \forall \alpha \in \Delta \right\}.$$

Remark 5.3. In [1], Section 2.2, it is shown that every semisimple orbit can be parametrized by a fundamental domain and we can always conjugate an element $H \in \mathfrak{h}$ so that it lies in D_Δ.

A proof of the following theorem can be found in [3], Chapter 16.

Theorem 5.4. *If $\mathfrak{g}$ is reductive and $\mathfrak{h}_1$, $\mathfrak{h}_2$ are Cartan subalgebras, then there exists $x \in G_{ad}$ such that $x \cdot \mathfrak{h}_1 = \mathfrak{h}_2$, i.e., all Cartan subalgebras are conjugate.*

Let O be a nilpotent orbit and X be a representative of O. By Jacobson-Morozov Theorem, X embeds into a standard triple $\{H, X, Y\}$. Since H is semisimple, by Remark 1.22, there exists a Cartan subalgebra $\mathfrak{h}_X$ containing H. Theorem 5.4 and Remark 5.3 imply that we can assume without loss of generality that H lies in $\mathfrak{h}$ and in D_Δ.

Lemma 5.5. $\alpha(H) \in \mathbb{N}$ *for every* $\alpha \in \Delta$. *In particular,* H *is* $\Delta -$ *dominant.*

Proof. H lies in D_Δ and we know from Chapter 2 that $\mathfrak{g}$ decomposes into the direct sum of $ad(H)$-eigenspaces with integral eigenvalues; thus for every simple root α we have that $\alpha(H)$ must be a nonnegative integer, i.e., $\alpha(H) \in \mathbb{N}$. □

Remark 5.6. We point out that Y lies in $\bar{\mathfrak{n}}$. Indeed by Lemma 5.5 we have that every eigenvector of $ad(H)$ that lies in $\mathfrak{b}$ must have a positive eigenvalue, since positive roots are nonnegative sums of simple roots. Moreover $Y \in \mathfrak{g}_{-2}$, thus $Y \in \bar{\mathfrak{n}}$.

Lemma 5.7. $\alpha(H) \in \{0, 1, 2\}$ *for every* $\alpha \in \Delta$.

Proof. We first show that $[X_\alpha, Y] \in \bar{\mathfrak{b}}$ for every $\alpha \in \Delta$. Since Y lies in $\bar{\mathfrak{n}}$,

$$Y = \sum_{\beta \in \Phi^-} c_\beta X_\beta, \quad c_\beta \in \mathbb{C}.$$

Moreover $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ and any negative root is a non-positive sum of simple roots, thus:

$$[X_\alpha, Y] \in \bigoplus_{\beta \in \Phi^-} \mathfrak{g}_{\alpha+\beta} \ \text{ where } \beta \text{ can be written as } \beta = \sum_{\gamma \in \Delta} c_\gamma \gamma, \ c_\gamma \leq 0.$$

Thus we have three possibilities:

1. $\beta + \alpha = 0$: $\beta = -\alpha$ and $\mathfrak{g}_{\alpha+\beta} \subseteq \mathfrak{h}$;

2. $\beta + \alpha \in \Phi$: it must be a negative root, since there is at least one negative coefficient when writing $\beta + \alpha$ as a sum of simple roots;

3. $\beta + \alpha \notin \Phi$, i.e., $\mathfrak{g}_{\alpha+\beta} = 0$.

In any of these possibilities, $\mathfrak{g}_{\alpha+\beta} \subseteq \bar{\mathfrak{b}}$ thus $[X_\alpha, Y] \in \bar{\mathfrak{b}}$.
If $X_\alpha \in \mathfrak{g}_\alpha$ centralizes Y, we can argue as we did in order to obtain decomposition (4.2) and conclude that

$$\mathfrak{g}^Y = \bigoplus_{i \leq 0} \mathfrak{g}_i.$$

This implies that $\alpha(H) \in \mathbb{Z}_{\leq 0} \cap \mathbb{N} = \{0\}$. Suppose now that $[X_\alpha, Y] \neq 0$; we have just observed that it must lie in $\bar{\mathfrak{b}}$ and by Lemma 2.8 we have that $[X_\alpha, Y]$ is an eingenvector of ad_H of eigenvalue $\alpha(H) - 2$, thus it must be $\alpha(H) - 2 \in -\mathbb{N}$, i.e., $\alpha(H) \in \{0, 1, 2\}$. □

Definition 5.8. (Weighted Dynkin diagram)
The weighted Dynkin diagram of O_X is the Dynkin diagram of $\mathfrak{g}$ where the node corresponding to the simple root α is labeled with $\alpha(H)$. We denote such a diagram by $\Delta(O_X)$.
By convention, the zero nilpotent orbit is represented by a Dynkin diagram with every node is labeled by 0, even though we don't consider $\{0, 0, 0\}$ as a standard triple.

Theorem 5.9. *(Kostant)*
There are only finitely many, and in fact at most $3^{rank\ \mathfrak{g}}$, nilpotent orbits in $\mathfrak{g}$. The weighted Dynkin diagram is a complete invariant, i.e., $\Delta(O_X) = \Delta(O_{X'})$ if and only if $O_X = O_{X'}$.

Proof. Every node has at most three possible labels, thus the number of nilpotent orbits in $\mathfrak{g}$ is less than or equal to $3^{rank\ \mathfrak{g}}$.
Let $\{H, X, Y\}$, $\{H', X', Y'\}$ be two standard triples. Then $\Delta(O_X) = \Delta(O_{X'})$ if and only if $\alpha(H) = \alpha(H')$ for every $\alpha \in \Delta$. Since Δ is a basis of $\mathfrak{g}^*$, these values completely determine H and H', thus $H = H'$. By Theorem 4.16, the two standard triples are conjugate, i.e., $O_X = O_{X'}$. □

5.2 The case of $\mathfrak{sl}(\mathfrak{n}, \mathbb{C})$

The Dynkin diagram of $\mathfrak{sl}(\mathfrak{n}, \mathbb{C})$ is the diagram of Type A_n:

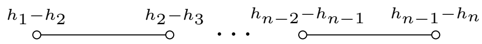
We have seen that the nilpotent orbits of $\mathfrak{sl}(\mathfrak{n},\mathbb{C})$ are parametrized by the soft partitions of n in this way:

$$[d_1, d_2, \dots, d_n] \longleftrightarrow X_{[d_1,d_2,\dots,d_n]} = \begin{pmatrix} J_{d_1} & 0 & 0 & \dots & 0 \\ 0 & J_{d_2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & J_{d_k} \end{pmatrix}$$

where k is the largest integer such that d_k is non-zero.
The standard triple $\left\{X_{[d_1,d_2,\dots,d_n]}, H_{[d_1,d_2,\dots,d_n]}, Y_{[d_1,d_2,\dots,d_n]}\right\}$ with nilpotent element $X_{[d_1,d_2,\dots,d_n]}$ consists of diagonal blocks matrices

$$H_{[d_1,d_2,\dots,d_n]} = \begin{pmatrix} D_{d_1} & 0 & 0 & \dots & 0 \\ 0 & D_{d_2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & D_{d_k} \end{pmatrix}, \quad Y_{[d_1,d_2,\dots,d_n]} = \begin{pmatrix} Y_{d_1} & 0 & 0 & \dots & 0 \\ 0 & Y_{d_2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & Y_{d_k} \end{pmatrix}$$

where blocks D_{d_i}, Y_{d_i} are as in the proof of Theorem 2.11 with $\lambda = d_i - 1$. It is possible to choose the set of positive roots in such a way that the corresponding Borel subalgebra is the subalgebra of upper triangular matrices of trace 0; this choice corresponds to the base $\Delta = \{\alpha_1, \dots, \alpha_{n-1}\}$ where $\alpha_i(e_{k,j})$ is defined by the equation $[e_{i,i} - e_{i+1,i+1}, e_{k,j}] = \alpha_i(e_{k,j})\, e_{k,j}$. It is immediately verified that if $h = diag\,(h_1, \dots, h_n)\,,\ h_i \in \mathbb{Z}$, then $[h, e_{ij}] = (h_i - h_j)\, e_{ij}$ and h is $\Delta - dominant$ if and only if $h_1 \geq h_2 \geq \dots \geq h_n$. In order to compute the weighted Dynkin diagram of $O_{[d_1,d_2,\dots,d_n]}$ it is sufficient to conjugate $H_{[d_1,d_2,\dots,d_n]}$ with a permutation matrix so that it becomes $\Delta - dominant$. Then we have that $\Delta\left(O_{[d_1,d_2,\dots,d_n]}\right)$ is



In the two following tables we write all the possibilities for $n = 3$ and $n = 4$. These examples show that the number of nilpotent orbits is in fact strictly lower than 3^n.

Nilpotent orbits in $\mathfrak{sl}(3, \mathbb{C})$			
Orbit	$H_{[d_1,\ldots,d_n]}$	$\overline{H}_{[d_1,\ldots,d_n]}$	$\Delta\left(O_{[d_1,\ldots,d_n]}\right)$
$O_{[3,0,0]}$	$diag\,(2,0,-2)$	$diag\,(2,0,-2)$	2 — 2
$O_{[2,1,0]}$	$diag\,(1,-1,0)$	$diag\,(1,0,-1)$	1 — 1
$O_{[1,1,1]}$	$diag\,(0,0,0)$	$diag\,(0,0,0)$	0 — 0

Nilpotent orbits in $\mathfrak{sl}(4, \mathbb{C})$			
Orbit	$H_{[d_1,\ldots,d_n]}$	$\overline{H}_{[d_1,\ldots,d_n]}$	$\Delta\left(O_{[d_1,\ldots,d_n]}\right)$
$O_{[4,0,0,0]}$	$diag\,(3,1,-1,-3)$	$diag\,(3,1,-1,-3)$	2 — 2 — 2
$O_{[3,1,0,0]}$	$diag\,(2,0,-2,-0)$	$diag\,(2,0,0,-2)$	2 — 0 — 2
$O_{[2,2,0,0]}$	$diag\,(1,-1,1,-1)$	$diag\,(1,1,-1,-1)$	0 — 2 — 0
$O_{[2,1,1,0]}$	$diag\,(1,-1,0,0)$	$diag\,(1,0,0,-1)$	1 — 0 — 1
$O_{[1,1,1,1]}$	$diag\,(0,0,0,0)$	$diag\,(0,0,0,0)$	0 — 0 — 0

In what follows we will use the weighted Dynkin diagrams of nilpotent orbits in $\mathfrak{sl}(\mathfrak{n}, \mathbb{C})$ to compute the dimension of the orbits. Our goal is to express this dimension in terms of the Young diagram associated with the partition of n (that corresponds to a Jordan block matrix). We shall use the weighted Dynkin diagram and the representation theory of $\mathfrak{sl}(2, \mathbb{C})$.
Let $\{H, X, Y\}$ be a standard triple in $\mathfrak{g}$ and $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ the decomposition of $\mathfrak{g}$ in $ad\,(H)$-eigenspaces. We recall that $dim O_X = dim\ \mathfrak{g} - dim\ \mathfrak{g}^X$.

Lemma 5.10. *$dim\ \mathfrak{g}^X = dim\ \mathfrak{g}_0 + dim\ \mathfrak{g}_1$.*

Proof. This result follows from the representation theory of $\mathfrak{sl}(2, \mathbb{C})$ applied to the adjoint action on $\mathfrak{g}$ of the subalgebra spanned by the standard triple. An element that commutes with X is a highest weight vector and every irreducible submodule contains only one such vector (up to scalar multiples); this implies that the dimension of the centralizer of X is the number of irreducible submodules of $\mathfrak{g}$ and, by Theorem 2.22, there are exactly $dim\ \mathfrak{g}_1 + dim\ \mathfrak{g}_0$ irreducible submodules. □

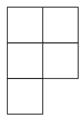
Let now $\mathfrak{g} = \mathfrak{sl}(\mathfrak{n}, \mathbb{C})$. Lemma 5.10 suggests how to calculate the dimension of the centralizer of X by looking at the weighted Dynkin diagram of the orbit O_X. Since $\mathfrak{h}$ is commutative and $H \in \mathfrak{h}$, we have that $\mathfrak{h} \subseteq \mathfrak{g}_0$. The dimension of $\mathfrak{g}_0$ is thus the sum of the dimension of $\mathfrak{h}$, i.e., the number of nodes of the diagram, and the number of roots $\alpha \in \Phi$ such that $\alpha\,(H) = 0$. Obviously if a root is zero on H so is its negative, thus we can restrict our attention to the positive roots that are zero on H and then double this number. With respect to Δ, every positive root is of the form $\alpha_i + \alpha_{i+1} + \dots \alpha_{i+k},\ i \in \{1, \dots, n-1\},\ k \leq n - 1 - i$. In order to see how many positive roots are zero on H, it is thus sufficient to sum up consecutive labels of the nodes and see how many of these are zero. For example, if we consider the weighted Dynkin diagram associated with the soft partition $[2, 2, 1, 0, 0]$ of 5:

the number of nodes is four and the sum of consecutive labels is 0 if and only if it is the sum of just one label, the first or the fourth. Thus the dimension of $\mathfrak{g}_0$ is $4 + 2 \cdot 2 = 8$.
Now we compute the dimension of $\mathfrak{g}_1$. We notice that only positive roots are nonnegative

on H (since simple roots are nonnegative on H), thus if $\alpha(H) = 1$, it must be $\alpha \in \Phi^+$; thus it is sufficient to sum consecutive labels of the nodes and see how many of these sums are equal to one. In the previous example, there are four possibilities, namely: α_2, α_3, $\alpha_1 + \alpha_2$, $\alpha_3 + \alpha_4$.
We can now compute the dimension of the nilpotent orbit $O_{X_{[2,2,1,0,0]}}$:

$$dim O_{X_{[2,2,1,0,0]}} = dim\ \mathfrak{sl}(5, \mathbb{C}) - dim\ \mathfrak{g}_0 - dim\ \mathfrak{g}_1 = 24 - 8 - 4 = 12.$$

Remark 5.11. In order to compute the dimension of the centralizer of a Jordan matrix in $\mathfrak{sl}(\mathfrak{n}, \mathbb{C})$, we can reduce this calculation to the dimension of the centralizer of the same matrix in $\mathfrak{gl}(\mathfrak{n}, \mathbb{C})$, indeed these dimensions differ by one: scalar matrices commute with every element and the condition for a matrix to have zero trace is linear, hence $\mathfrak{gl}(\mathfrak{n}, \mathbb{C})^X = \mathfrak{sl}(\mathfrak{n}, \mathbb{C})^X \oplus \mathbb{C}I$. For example, the dimension of the centralizer of $X_{[2,2,1,0,0]}$ in $\mathfrak{gl}(\mathfrak{n}, \mathbb{C})$ is 13. The Young diagram associated with the soft partition $[2, 2, 1, 0, 0]$ is the following:



Notice that the sum of the squares of the columns' length is exactly 9+4=13. We will show that this is not just a case (see Theorem 3.24 below).

We fix the following notation: $[d_1, \ldots, d_n]$ is a soft partition of n, k is the largest integer for which $d_k > 0$; $O_{X_{[d_1,\ldots,d_n]}}$ is a nilpotent orbit in $\mathfrak{sl}(\mathfrak{n}, \mathbb{C})$ associated with the soft partition $[d_1, \ldots, d_n]$; $H_{[d_1,\ldots,d_n]}$ is the semisimple element of the standard triple with nilpositive element $X_{[d_1,\ldots,d_n]}$.

Lemma 5.12. *Let $1 \leq i < j \leq k$. If d_i, d_j have same parity, there are at least d_j labels equal to zero in the weighted Dynkin diagram of the orbit. In particular, there are at least $2d_j$ roots that are 0 on $H_{[d_1,\ldots,d_n]}$.*

Proof. In the diagonal of the matrix $H_{[d_1,\ldots,d_n]}$ we have two strings $\{d_i - 1, \ldots, -d_i + 1\}$, $\{d_j - 1, \ldots, -d_j + 1\}$. Since $i < j$ and d_i, d_j have the same parity, $d_i \geq d_j$ and the second string is included in the first (for example, if we consider the soft partition $[4, 2, 0, 0, 0, 0]$ of 6, we have two non-zero parts with the same parity, namely $d_1 = 4$ and $d_2 = 2$, that are associated with the strings $\{3, 1, -1, -3\}$ and $\{1, -1\}$; we have that $H_{[4,2,0,0,0,0]} = diag\,(3, 1, -1, -3, 1, -1)$ and the correspondent $\Delta -$ *dominant* matrix is $\bar{H}_{[4,2,0,0,0,0]} = diag\,(3, 1, 1, -1, -1, -3)$); thus we have d_j labels that are zero (2, in the example). In particular there are d_j simple roots that are zero on $H_{[d_1,\ldots,d_n]}$ and their negatives are zero on $H_{[d_1,\ldots,d_n]}$ too. □

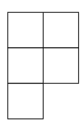
Lemma 5.13. *Let $1 \leq i < j \leq k$. If d_i, d_j have different parities, there are at least $2d_j$ labels equal to one in the weighted Dynkin diagram of the orbit. In particular, there are at least $2d_j$ simple roots that are 1 on $H_{[d_1,\ldots,d_n]}$.*

Proof. As in the previous lemma, in the diagonal of the matrix we have two blocks strings $\{d_i - 1, \ldots, -d_i + 1\}$, $\{d_j - 1, \ldots, -d_j + 1\}$. $d_i > d_j$ since $i < j$ and they have different parities; thus there are at least two labels that are equal to one for each $d_j - k$ contained in the second string, indeed $d_j - k + 1$ and $d_j - k - 1$ are contained in the first string (for example, if we consider the soft partition $[3, 2, 0, 0, 0]$ of 5 we have two non-zero parts with different parity, namely 3 and 2, that are associated with the strings $\{2, 0, -2\}$ and $\{1, -1\}$). □

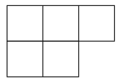
By Lemmas 5.12 and 5.13 we have that, for different reasons, every d_j (with $j \neq 1$) increases the dimension of $\mathfrak{g}^X$ of $2d_j$. What we have not considered yet, is the chance to have consecutive zeros in the labeling and to have a zero label near a one label. In these cases it is possible to sum labels and still obtain a zero or a one.

Definition 5.14. (Dual soft partition)
Let $[d_1, \ldots, d_n]$ be a soft partition of n and Y the associated Young diagram. The dual soft partition is the soft partition of n associated with the transposed Young diagram of Y.

Example 8. Let us consider the soft partition $[2, 2, 1, 0, 0]$ of 5, i.e., the partition associated with is associated with the Young diagram



Then the transposed Young diagram is



that is associated with the (dual) soft partition $[3,2,0,0,0]$.

Definition 5.15. Let $j \in \{1,\ldots,n\}$. We define

$$r_j = max\left\{i \in \{1,\ldots,n\} \;\middle|\; d_i \geq j\right\} \text{ if } j \leq d_1; \;\; r_j = 0 \text{ if } j > d_1.$$

Example 9. Let us consider the soft partition $[2,2,1,0,0]$ of 5. We have that $d_1 = 2$, thus $r_3 = r_4 = r_5 = 0$; moreover $r_1 = 3$ and $r_2 = 2$.

Remark 5.16. Let r_j be as in Definition 5.15. Then the r_j-th row of the Young diagram associated with the partition is the lowest row whose length is equal to or greater than j, thus it is the length of the j-th column of the Young diagram. If there are no rows of length j, then $r_j = r_{j+1}$; more generally, the nonnegative integer $r_j - r_{j+1}$ counts the number of rows of length j in the Young diagram.
Notice that

$$\sum_{j=1}^{n} r_j = n \tag{5.1}$$

since $[r_1,\ldots,r_n]$ is the dual partition of $[d_1,\ldots,d_n]$.

Theorem 5.17. *The dimension of the centralizer of $X_{[d_1,\ldots,d_n]}$ in $\mathfrak{gl}(\mathfrak{n},\mathbb{C})$ is*

$$dim\ \mathfrak{gl}(\mathfrak{n},\mathbb{C})^{X_{[d_1,\ldots,d_n]}} = \sum_{j=1}^{n} {r_j}^2.$$

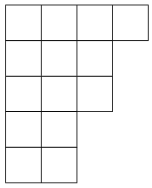
It follows that $dim O_{X_{[d_1,\ldots,d_n]}} = n^2 - \sum_{j=1}^{n} {r_j}^2$.

Proof. We use the previous results to calculate the dimension of the centralizer of $X_{[d_1,\ldots,d_n]}$ in $\mathfrak{sl}(\mathfrak{n},\mathbb{C})$.
Let us consider the Young diagram associated with the orbit $O_{X_{[d_1,\ldots,d_n]}}$. Then there are r_{j+1} rows longer than j $\left(= d_{r_j}\right)$ and each of them gives a contribution of $2j$ to the dimension of the centralizer. As we noticed before, we have $r_j - r_{j+1}$ parts of length j; once we choose a row of length j, we are forced to sum all the (zero) labels before

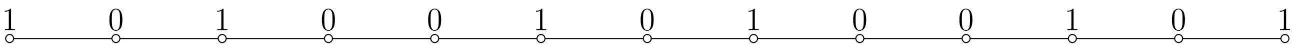
the zero or one label. Thus once we choose a row of length j the rows above it give a contribution of

$$2j \cdot r_{j+1} \cdot (r_j - r_{j+1})$$

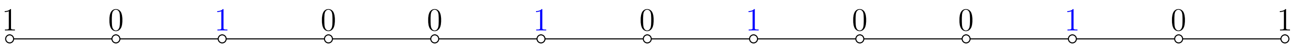
to the dimension of the centralizer of $X_{[d_1,\ldots,d_n]}$. Let us clarify this argument through an example. Let us consider the orbit associated with the Young diagram



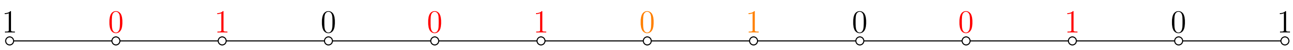
and fix $j = 2$; then $r_j = 5$ and $r_{j+1} = 3$. The associated $\Delta - \textit{dominant}$ matrix is $\textit{diag}\,(3, 2, 2, 1, 1, 1, 0, 0, -1, -1, -1, -2, -2, -3)$ (it obviously makes no difference, but to better understand the example we agree on the following ordering: if there are one or more equal integerrs m, the first comes from the first row of length $m+1$ and so on if there are other rows of length $m+1$, then we put the integers m due to longer rows). The associated Dynkin diagram is the following:



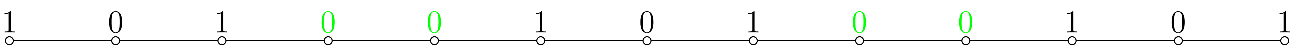
Consider the first row of length j, i.e., the fourth row. There are four labels equal to one due to the fact that 3 and 2 have different parities, that we evidence in blue:



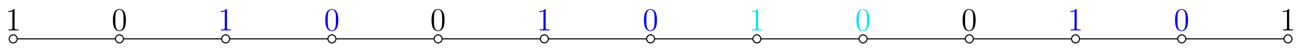
There is another row of length three and, due to this fact, for each of the previous one labels there is a zero label before it, thus the effect of this row of length three on the considered row of length two is unique, for each label one; we put this in evidence with red and orange:



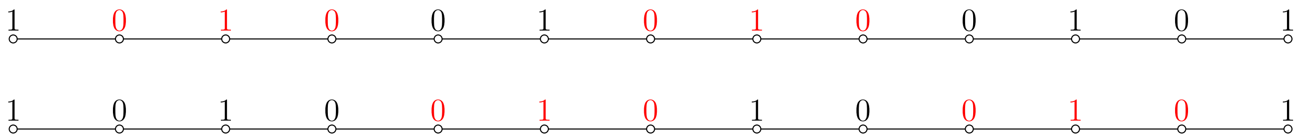
It remains to consider the row of length 4, that affects the dimension of an orbit by two strings of zeros (thus it increases of 4 the dimension of the centralizer), that we evidence in green:



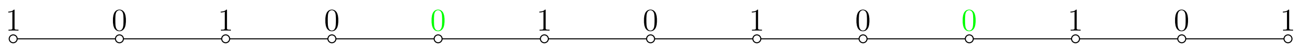
The way the second row of length two gives a contribution of $2j \cdot r_{j+1} = 12$ (equal to the contribution of the previous case) is similar to the case of the second part of length three. We now give the details. We evidence in blue the contribution that comes from the first part of length three:



We now put in evidence the effect due to the second part of length 3 (we need two different diagrams to avoid "intersections":



Finally, we consider the row of length 4, that produces two zeros in the Dynkin diagram (and thus affects by 4 the dimension of the centralizer):



We argue in the same way for every j (if it is not the maximum in the soft partition), and this shows that, once a row of length j has been fixed, then each of the r_{j+1} rows of higher length gives a contribution of $2j$, thus we have a total contribution $2j \cdot r_{j+1}$ for each row of length j, as stated before the example. If j is the maximum integer in the soft partition, there is no effect on higher parts (because there are not) and the formula $2j \cdot r_{j+1} \cdot (r_j - r_{j+1})$ still holds since $r_{j+1} = 0$.

Now we need to calculate how much the number of rows of length j influences the dimension of the centralizer. We recall that every pair of rows of length j gives a contribution of $2j$. Choosing two rows of length j is equivalent to choosing a string of consecutive labels that we need to sum (we will choose the first and the last row of length j and consider the string in between), thus there are $\binom{r_j - r_{j+1}}{2}$ possible choices. The contributions are

$$\binom{r_j - r_{j+1}}{2} \cdot 2j = (r_j - r_{j+1})\,(r_j - r_{j+1} - 1)\,.$$

The sum of the two contributions is $2j \cdot r_{j+1} \cdot (r_j - r_{j+1}) + (r_j - r_{j+1})(r_j - r_{j+1} - 1) = j(r_j - r_{j+1})(r_{j+1} + r_j - 1)$. In order to obtain the dimension of the centralizer of X in $\mathfrak{gl}(\mathfrak{n}, \mathbb{C})$, what is left to do is to sum over $j \in \{1, \ldots, n\}$ and add the number of nodes $(n-1)$ plus one:

$$dim\ \mathfrak{gl}(\mathfrak{n}, \mathbb{C})^{X_{[d_1,\ldots,d_n]}} = dim\ \mathfrak{sl}(\mathfrak{n}, \mathbb{C})^{X_{[d_1,\ldots,d_n]}} + 1 = n + \sum_{j=1}^{n-1} j(r_j - r_{j+1})(r_j - r_{j+1} - 1) =$$

$$= \sum_{j=1}^{n} r_j + \sum_{j=1}^{n-1} j\left(\left(r_j^2 - r_{j+1}^2\right) + (-r_j + r_{j+1})\right) = \sum_{j=1}^{n} r_j + \sum_{j=1}^{n} r_j^2 - \sum_{j=1}^{n} r_j = \sum_{j=1}^{n} r_j^2$$

where in the second to last equality we used the fact that for $j = k$ and $j = k+1$ there are common terms. The last assertion is an immediate consequence of the formula we just proved:

$$dim O_{X_{[d_1,\ldots,d_n]}} = dim\, \mathfrak{sl}(\mathfrak{n}, \mathbb{C}) - dim\, \mathfrak{sl}(\mathfrak{n}, \mathbb{C})^{X_{[d_1,\ldots,d_n]}} = dim\, \mathfrak{sl}(\mathfrak{n}, \mathbb{C}) + 1 - dim\, \mathfrak{sl}(\mathfrak{n}, \mathbb{C})^{X_{[d_1,\ldots,d_n]}} - 1 =$$

$$= dim\ \mathfrak{gl}(\mathfrak{n}, \mathbb{C}) - dim\ \mathfrak{gl}(\mathfrak{n}, \mathbb{C})^{X_{[d_1,\ldots,d_n]}} = n^2 - \sum_{j=1}^{n} r_j^2.$$

□

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