Alma Mater Studiorum · University of Bologna

School of Science

Department of Physics and Astronomy

Master Degree in Physics

GENERATING PRIMORDIAL BLACK HOLES IN NON-CANONICAL INFLATION

Supervisor:

Prof. Aleksandr Yu. Kamenshchik

Submitted by:

Lorenzo Copparoni

Co-supervisor:

Dott. Alessandro Tronconi

Academic Year 2020/2021

Abstract

Among the candidates for dark matter, Primordial Black Holes are extremely promising, as they do not require the introduction of new physics beyond the Standard Model of particle physics. These objects can arise from the collapse of overdense regions generated by the curvature perturbation generated during inflation. The probability distribution of these overdense regions is related to the power spectrum of the scalar perturbation which must be enhanced by compared to the perturbation at CMB scale in order for the collapse to generate enough Primordial Black Holes to account for the dark mater budget we observe today. In this thesis we discuss two possible amplification mechanism in non-canonical inflation, employing a decreasing sound speed. In the first case we consider a model which starts in a slow-roll regime at CMB scale. Later during inflation the field enters in a non-canonical regime and it approaches the k-inflation attractor. During this kinetically driven transient phase the sound speed and the slow roll parameter ϵ decrease exponentially and the power spectrum of the curvature perturbation gets amplified. We then consider a DBI model where we consider the presence of a gaussian spike in the warp factor, thus leading to the transition, from the canonical phase at CMB scales to a strongly non-canonical regime. In this latter phase, the sound speed of perturbation is much smaller than 1, and leads to an amplification of the scalar perturbation power spectrum.

Abstract

Fra i candidati per spiegare la materia oscura i buchi neri primordial sono fra i più promettenti, dato che non richiedono l'introduzione di nuova fisica al di fuori del Modello Standard della fisica delle particelle. Questi oggetti si formano dal collasso di regioni sovradense generate delle perturbazioni di curvatura generate durante l'inflazione, e la distribuzione di probabilità di queste sovradensità è collegata allo spettro di potenza delle perturbazioni scalari che deve essere amplificata rispetto alle scale della CMB affinché il collasso possa generare abbastanza buchi neri primordiali per spiegare la materia oscura che osserviamo oggi. In questa tesi andiamo a studiare due possibili meccanismi di amplificazione in modelli di inflazione non-canonica, sfruttando una decrescita della velocità del suono. Nel primo caso studiamo un modello che si trova in un regime di slow-roll sulla scala della CMB; a seguito durante l'inflazione il campo lascia l'attrattore di slow-roll e si avvicina all'attrattore di k-inflation. Durante questa fase transiente la velocità del suono ed il parametro di slow-roll ϵ calano esponenzialmente e lo spettro delle perturbazioni viene amplificato. Consideriamo poi un modello DBI dove studiamo uno spike nello warp factor, che porta alla transizione da un regime canonico nella scala della CMB ad un regime fortemente non canonico. Durante questa fase la velocità del suono delle perturbazioni è molto più piccola di 1 portando ad una amplificazione dello spettro di potenza delle perturbazioni scalari.

Contents

Introduction

Since its first run in 2015, LIGO has detected many gravitational waves genereated by the merging of two black holes in the $10 - 100M_{\odot}$ mass range. The astrophysical model of Supernova collapse does not allow black holes in this mass range, thereby, this discovery has spiked the interest in Primordial Black Holes (PBH) as candidates for Dark Matter. As of today there has not been a direct observation of Primordial Black Holes, rather the indirect observation leads to two interesting windows in the mass range:

- $10^{16} - 10^{24}$g in which the current observations allow all of Dark matter to be composed of Primordial Black Holes.

- $10 - 100M_{\odot}$ which is the LIGO mass range.

As postulated by Hawking and Carr, PBH can arise due to the collapse of an overdense region generated by primordial perturbations in the early universe. Thus, it is possible to relate it to the theory of inflation, which has been proven to be an extremely effective hypothesis to explain both the homogeneity of the universe on cosmological scale and the presence of structures on large scales such as Galaxies and Clusters. By using the Press-Schechter formalism it is possible to evaluate a threshold for collapse to occur. The abundance of PBH produced in a given mass range is then related to the power spectrum of curvature perturbation generated during inflation. A power spectrum 7 order of magnitude bigger than the COBE normalization is necessary for an efficient PBH formation, therefore to build an inflationary model which can replicate the data observed by PLANCK it is necessary to introduce a mechanism which amplifies the perturbation on small scales.

In this thesis we discuss a possible amplification mechanism for non canonical models of inflation exploiting a decreasing sound speed for perturbations since, naively one has the following expression for the power spectrum in a slow-roll regime $P^{\zeta} \simeq \frac{H^2}{8\pi^2 \epsilon c_s}$.

In the first model we employ a k-inflation model where the dynamics of the inflaton field during inflation is dominated by the kinetic terms and the model possesses an attractor trajectory in phase space where both ϵ and c_s vanish. We impose that the kinetical attractor trajectory in phase space is a solution of the equation of motion, thereby ensuring a k-inflation phase during which both the sound speed and the slow-roll parameter ϵ will decrease significantly. This kinetic phase though is not able to reproduce the observations at CMB scale, thus the introduction of an initial slow-roll phase is necessary. In particular, by choosing a lagrangian $\mathcal{L} = -X + L(\varphi)X^2 - V(\varphi)$, where $X = \frac{1}{2}\dot{\varphi}^2$ is the canonical kinetic term, with an exponential potential, one can recreate the expected perturbation at CMB scale and, during the kinetical regime, obtain $\epsilon \propto e^{-\gamma_s \Delta N}$, $c_s \propto e^{-\frac{\gamma_s}{2}\Delta N}$ thereby leading to an exponential growth of the power spectrum. Compared to other mechanisms for amplification, such as Ultra slow-roll, the amplitude of the modes does not undergo an amplification on superhorizon scales. Rather, due to the decreasing sound speed it freezes. In the second model we consider a DBI lagrangian and study the effect of a steep variation of the warp function through the introduction of a spike which causes a quick decrease of the sound speed for perturbations and of the slow-roll parameter ϵ. To ensure that inflation lasts for about 60 e-folds after the scales of the perturbation which are responsible for the CMB anisotropies cross the horizon, it is necessary to select an extremely narrow spike; this causes a violation of the slow-roll approximation, both during the growth of the warp factor and during its decrease. Interestingly during the latter phase one observes a quick growth of the sound speed which leads to an increase of the sound horizon for scalar perturbations. Such an increase should lead to the amplification of the perturbed modes on the superhorizon scale. This feature does not appear in our model, as due to the steep growth of the sound speed the sound horizon increases. Thus the modes which should have been amplified on superhorizon scale re-enter the horizon and their amplitude gets damped.

The thesis is organized as follows: in Chapter 1 we review the theory of inflation both in a canonical and non canonical case, explaining the shortcoming of the Hot Big Bang model and describing the theory of cosmological perturbation. In Chapter 2 we discuss Primordial Black Holes describing their formation mechanism, the current experimental constraints and reviewing some known model in canonical single field inflation. In Chapter 3 we describe how Primordial Black Holes may form as a consequence of non-canonical inflation discussing more in depth the k-inflation and DBI case.

Chapter 1

The Theory of Inflation

1.1 The FRW Universe

The theory of cosmology is based on two principles, the Copernical principle:

> *"We are not the preferred observer in the universe"*

which is strongly considered as a symbol for the antihomocentric nature of science. Beyond this philosophical meaning though, this principle alone does not have many practical uses. The second one is the so called Cosmological Principle:

> *"The universe is spatially homogeneous and isotropic."*

While the isotropic feature has been confirmed by the experimental observations, the homogeneity then comes naturally from the Copernical Principle. When introducing these principles, a short consideration is necessary: the existence of galaxies and larger structures violates the Cosmological principle. Rather it is necessary to introduce a so called *Cosmological Scale* beyond which it is possible to consider galaxies as components of an homogeneous fluid.

It is possible now to identify the structure of the universe by considering the internal symmetries fixed by the cosmological principle. These conditions impose the existence of three spatial translation killling vectors and the rotation killing vectors. We do not consider the existence of a time translation killing vector as it was shown that the universe is expanding. These conditions uniquely determine the Friedmann-Robertson-Walker-Lemaitre (or FRWL for short) metric:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a(t)^2 \left[\frac{\mathrm{d}r^2}{1 - Kr^2} + r^2 \mathrm{d}\Omega^2 \right] \tag{1.1}$$

where $a(t)$ is the scale factor of the universe and the factor K sets its geometry. Through a suthing rescale of the radius it is always possible to redifine K as $\pm 1, 0$. This marks three different cases:

- In the $K = 0$ case we have a flat universe and, if we consider the equal time slicing Σ_t this is a three dimensional euclidean space.
- The $K = -1$ case is for an open universe, the equal time slicing Σ_t is a three dimensional hyperboloid.
- The $K = +1$ case is for a closed universe, the equal time slicing Σ_t is a three dimensional sphere.

Coming back to the hypothesis of homogeneity and isotropy for our universe we can introduce a matter component. The previous conditions find an optimal candidate in a perfect fluid. It is easy to evaluate the energy momentum tensor for a perfect fluid in special relativity [44] and, by using the Einstein Equivalence principle, we obtain the expression for the energy momentum tensor on curved spacetime:

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu - p g_{\mu\nu} \tag{1.2}$$

where p, ρ are the pressure and the energy density of the fluid and u_μ is the 4-velocity of the fluid. In this context we suppose that the fluid is at rest. The first observation we have to make is that our energy momentum tensor satisfies the continuity equation $\nabla_\mu T^{\mu\nu} = 0$, which for an homogeneous and isotropic field translates into:

$$\dot{\rho} = -3H(\rho + p) \tag{1.3}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter. Now if we insert (1.2) into the Einstein equations we get the so called Friedmann equations:

$$H^2 - \frac{K}{a^2} = \frac{\rho}{3\mathrm{M_P}^2}, \qquad \frac{\ddot{a}}{a} = -\frac{\rho + 3p}{6\mathrm{M_P}^2}. \tag{1.4}$$

In the context of cosmology a perfect fluid having the equation of state $p = w\rho$, has an energy density which evolves as:

$$\rho \propto a^{-3(1+w)}. \tag{1.5}$$

The common fluids which are employed in cosmology are dust, radiation and vacuum energy.

- **Dust** is a fluid composed by non relativistic matter, with no other force acting beyond gravity so that $w = 0$. Inserting it into (1.5) we see that the energy density scales as a^{-3}, as one would expect for a set of particles in an expanding universe.
- **Radiation** is a fluid whose energy momentum tensor is traceless, hence we obtain $w = \frac{1}{3}$, and we can see that the energy density scales as $\rho \propto a^{-4}$.
- **Vacuum Energy** is a fluid with the energy density which does not change during the universe expansion with an equation of state $w = -1$. This latter case describes an empty universe with a cosmological constant Λ

It is possible to recast the Friedmann equations (1.4) by introducing a critical density $\rho_c = \frac{H^2}{3\mathrm{M_P}^2}$, and the density parameter $\Omega = \frac{\rho}{\rho_c}$:

$$\Omega - 1 = \frac{K}{a^2 H^2}. \tag{1.6}$$

The current observations show that our universe is highly compatible with $\Omega - 1 = 0$ therefore from now on we shall consider a flat universe.

Problems of the Hot Big Bang Model

The notion of the expansion of the universe brings into cosmology the problem of choosing a valid initial condition. Supposing that our universe is composed only by matter and radiation and, knowing that the energy density of the latter decreases faster, it is reasonable to assume that, at its very early stages the universe was filled with plasma with very high energy and positive pressure. In this situation though it is necessary to realize that, as we go back in time, we get close to the Big Bang singularity, at which point a quantum description of gravity is necessary. Hence someone could argue that the problem arising in the Hot Big Bang Model only exist because we do not have a Quantum Theory of Gravity whatever it is.

Horizon Problem

The Cosmic Microwave background gives us a picture of the early universe. It shows that at the time of recombination the primordial plasma was extreamely homogeneous and at thermal equilibrium. It is important now to make the following observation: if we apply a time slicing and consider the Universe at a fixed time t_0 a signal emitted at Big Bang is able to travel only inside the cosmological Horizon:

$$l_H(t_0) = a(t_0) \int_{t_P}^{t_0} \frac{\mathrm{d}t}{a(t)} = a(t_0) \int \frac{\mathrm{d}a}{a^2 H}. \tag{1.7}$$

But if we start from a radiation dominated universe we find that at recombination the cosmological horizon is too small, and we have about $10^4 - 10^5$ indipendent patches which are not casually connected. This of course is not compatible with the observations unless an excessive fine tuning is applyed to the initial conditions.

Flatness Problem

The current observation sets $|\Omega - 1| < 0.02$, and we can calculate this quantity at Planck time. Using equation (1.6) we can evaluate the ratio:

$$\frac{\Omega(t_0) - 1}{\Omega(t_\mathrm{P}) - 1} = \frac{(aH)^2|_{t_P}}{(aH)^2|_{t_0}}. \tag{1.8}$$

Inserting the values for the universe today, and supposing that at Planck time the Hubble parameter is of the order of the Planck mass, the above ratio turns out to be 10^{58}. The observed flatness would then require the initial conditions:

$$|\Omega(t_\mathrm{P}) - 1| \leq 10^{-60} \tag{1.9}$$

which is an unreasonable amount of fine tuning.

1.2 Homogeneous Inflation

Both the horizon and flatness problem arise because the comoving Hubble Radius $(aH)^{-1}$ monotonically increase. A simple solution to the fine tuning problem briefly illustrated above could be the introduction of an initial phase where, during the expansion of the universe, the comoving Hubble radius shrinks. Neither a matter dominated phase or a radiation phase are able to generate a decreasing comoving Hubble Horizon, and it does not arise naturally in hot big bang model. Rather we could add an initial phase before the radiation dominated phase where $\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} < 0$ which takes the name of inflation. Supposing that inflation starts at energies higher than the present ones, the causal physics before inflation established spatial homogeneity, thereby explaining the homogeneity observed in the CMB. Similarly it trivially solves the Flatness problem as, by looking at (1.6), a decreasing comoving Hubble Horizon automatically drives the evolution towards flatness.

To see how this inflationary phase could arise it is useful to look at the Friedmann equation. It is easy to see that the following 4 conditions are equivalent:

- Decreasing comoving Hubble Radius
- Accelerated expansion ($\ddot{a} > 0$)
- Slowly varying Hubble Parameter
- Negative pressure phase ($p < -\frac{1}{3}\rho$)

when one of them holds, automatically the system is in an inflationary phase. To translate this conditions into a possible model, we consider, for example, the slow variation of the Hubble parameter and we take the limit case $H = \text{const}$, corresponding to an exponential growth of the scale factor. During inflation both the scale factor, and the cosmological time t span many orders of magnitude[1], hence it is useful to introduce a new quantity to parametrize the universe expansion, the number of e-folds:

$$N = \log(a) = \int \mathrm{d}t\, H(t). \tag{1.10}$$

We can then apply some condition on the duration of inflation, by imposing that it lasts long enough to ensure the flatness of the universe. We compare $\Omega - 1$ at the onset of inflation t, where we expect $H \simeq \mathrm{M_P}$, and today:

$$\frac{|1-\Omega(t)|}{|1-\Omega(t_0)|} = \left(\frac{a_0 H_0}{a(t)H(t)}\right) \gg 1 \tag{1.11}$$

now we multiply and divide by aH evaluated at the end of inflation. As the Hubble factor is expected to be nearly constant during inflation we can set $\frac{H(t)}{H_{end}}$ to be 1 and, using (1.10) we can set $\frac{a_{end}}{a(t)} = e^{N_{TOT}}$. At the end of inflation the scale factor can be related to the the temperature of the universe as $a \simeq T^{-1}$:

$$N_{tot} > \log\left(a_{end}H_{end}\right) - \log\left(a_0 H_0\right) = \log\left(\frac{H_{end}}{T_{reh}}\right) - \log\left(\frac{H_0}{T_0}\right) \simeq 66 + \log\left(\frac{H_{end}}{T_{reh}}\right), \tag{1.12}$$

where T_{reh} is the reheating temperature according the current experimental data, and we therefore obtain: $-\log\left(\frac{H_0}{T_0}\right) \simeq 66$. This is a lower bound on the duration of inflation. A more precise calculation can be found in [27].

[1] Studies on the primordial fluctuations lead us to believe that inflation happened in the interval $10^{-43} - 10^{-36}$s

What is left is to identify possible ways to toggle the accelerated expansion of the universe. Historically the first satisfying model was proposed by Starobinsky [40, 41] and it consisted in an action with higher order terms in the Ricci scalar. Beyond the Starobinsky model other have been introduced. One of the most popular way to obtain an inflationary phase is to consider a universe whose matter component is completely described by a single scalar field (the inflation). The first models developed in the 80's (Old Inflation or Guth models) considered the consequences of a phase transition, where the system is in a false vacuum with a positive energy density. This false vacuum is unstable and it decays via a process of nucleation of bubbles to the true vacuum. As the universe expands the collision of the bubble wall create particles and their kinetic energy will turn into heat. These models though are not viable candidates to explain inflation as, imposing that the inflationary phase lasts long enough to explain the homogeneous nature of the universe, the bubble nucleation rate is too small to explain the thermalization of the universe. The Old Inflation model has been abandoned in favour of the New Inflation model employing a second order phase transition (which has been discarded too as it requires an excessive fine tuning) and the chaotic inflation model, which does not employ phase transitions and turned to be succesful. In this thesis we will consider the latter one.

Canonical Model of Inflation

We consider a generic scalar field φ in a FRW spacetime whose action takes the following form:

$$S = \int \mathrm{d}^4x\, a^3 \left[-\frac{R}{16\pi G_{\mathrm{N}}} + \frac{1}{2}\dot{\varphi}^2 - V(\varphi) \right] \tag{1.13}$$

where we suppose the dynamics of the scalar field significantly influences the evolution of the universe and that the field itself is homogeneous. We refer to these models as "*Canonical*" in the sense that the action of the scalar field has a canonical kinetic term. We can evaluate the energy momentum tensor $T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S_\varphi}{\delta g^{\mu\nu}}$ which, by employing the hydrodynamical formalism we expect it to be of the form (1.2), and identify the pressure and the energy density:

$$p = \frac{\dot{\varphi}^2}{2} - V(\varphi), \qquad \rho = \frac{\dot{\varphi}^2}{2} + V(\varphi). \tag{1.14}$$

Hence the equation describing the dynamics of the system are the Friedmann equation (1.4) and the Klein-Gordon (KG) Equation:

$$\ddot{\varphi} + 3H\dot{\varphi} + \partial_\varphi V = 0. \tag{1.15}$$

Can this model generate inflation? First of all we observe that:

$$\frac{\ddot{a}}{a} = H^2\left(1 + \frac{\dot{H}}{H^2}\right) \tag{1.16}$$

and, the request for an accelerated expansion, as mentioned before, is equivalent to a small variation of the Hubble parameter. It is useful to introduce the first slow-roll parameter:

$$\epsilon = -\frac{\mathrm{d}\log H}{\mathrm{d}N} = -\frac{\dot{H}}{H^2} = \frac{3}{2}\frac{\dot{\varphi}^2}{\frac{\dot{\varphi}^2}{2} + V(\varphi)} \tag{1.17}$$

and as long as $\epsilon < 1$ (see equation (1.16)) the accelerated expansion of the universe is ensured. Moreover for $\epsilon \to 0$ the Hubble parameter is a constant and we fall back to a de Sitter Universe and, in general, when ϵ is small the universe has a nearly exponential expansion. We notice that in a flat universe the slow-roll parameter ϵ will be non negative as long as the energy density is positive which, in a flat FRW universe, occurs. Since the Flatness problem requires more than 60 e-folds to be solved we need inflation lasting long enough. To ensure this we introduce the second slow-roll parameter ϵ_2 which measures how fast ϵ changes during the evolution of the universe:

$$\epsilon_2 = \frac{\mathrm{d}\log\epsilon}{\mathrm{d}N} = \frac{\dot{\epsilon}}{H\epsilon} \tag{1.18}$$

it should be noted though that the smallness of this latter parameter is actually not necessary to ensure an accelerated phase.

Evaluating the slow-roll parameter ϵ, we see that by imposing $\dot{\varphi}^2 \ll V(\varphi)$ it will take a small value and will ensure an inflationary evolution. Moreover the requirement that inflation lasts long enough to solve the flatness problem, turns into the conddition $\ddot{\varphi} \ll 3H\dot{\varphi}$ since, evaluating the second slow-roll parameter $\epsilon_2 = 2\epsilon + 2\frac{\ddot{\varphi}}{H\dot{\varphi}}$, we observe that it becomes small under this assumption. If we impose both of this conditions we can obtain an approximated form for the equation of motion which is usually referred to as the slow-roll approximation:

$$3H\dot{\varphi} \simeq -\partial_\varphi V, \qquad\qquad H^2 \simeq \frac{V}{3{\rm M_P}^2}. \tag{1.19}$$

It is useful now to introduce the potential slow-roll parameters:

$$\epsilon_{\rm SR} = \frac{{\rm M_P}^2}{2}\left(\frac{\partial_\varphi V}{V}\right)^2, \qquad\qquad \eta_{\rm SR} = {\rm M_P}^2\frac{\partial^2_\varphi V}{V} \tag{1.20}$$

which can be expressed in terms of ϵ, ϵ_2 by applying the approximation (1.19) obtaining $\epsilon \simeq \epsilon_{\rm SR}$ $\epsilon_2 \simeq 4\epsilon_{\rm SR} - 2\eta_{\rm SR}$. Thus the smallness of $\epsilon_{\rm SR}$, $\eta_{\rm SR}$ may lead to an inflationary evolution. It could be argued that those conditions are not actually sufficient to ensure an inflationary evolution, as they do not set any condition on the speed of the field. This does not pose an actual problem as we can show that the inflationary trajectory is an attractor [37]. The attractor nature of the inflationary trajectory also solves many possible doubts on the validity of the inflationary theory as it eliminates the necessity of an excessive fine tuning.

Some Models of Inflation

It is possible to introduce a very simple classification of potentials able to generate a slow-roll evolution according to their *duration* expressed by the inflaton field variation. We call $\varphi_{\rm CMB}$ the value of the inflaton field at the instant of horizon crossing for the modes of CMB and $\varphi_{\rm end}$ the value that the field has when the inflation ends. Comparing the variation of the scalar field with the Planck mass we can identify two possible cases:

- If during the evolution of the system the value of the field changes significantly compared to the Planck mass ($\frac{\varphi_{\rm end}-\varphi_{\rm CMB}}{\rm M_P} \gg 1$), we say we have a Large Field Model. Among the various potentials falling in this category the simplest is probably the monomial potential:

$$V = V_0\varphi^p \tag{1.21}$$

- The complete opposite case is the small field model in which $\frac{\varphi_{\rm end}-\varphi_{\rm CMB}}{\rm M_P} \ll 1$. These models can arise due to spontaneous symmetry breaking, a famous model which remains very popular in the phenomenological case is the Coleman-Weinberg potential:

$$V = V_0\left[\left(\frac{\varphi}{\mu}\right)^4\left(\log\left(\frac{\varphi}{\mu}\right) - \frac{1}{4}\right) + \frac{1}{4}\right] \tag{1.22}$$

which could arise due to the spontaneous symmetry breaking in grand unified theories [38]

1.3 Non-Canonical Inflation

The range of possible models able to generate an inflationary phase can be expanded by looking at lagrangians having non-canonical kinetic terms beyond the canonical one, such as those discussed in [3, 20]. A motivation to introduce such higher dimensional terms can be found either in string theory or in effective field theories for gravity. Here we shall introduce a generic non canonical lagrangian and see how that could start an inflationary phase.

A General Non-Canonical Model

We consider here a generic scalar field φ minimally coupled with gravity:

$$S = \int \mathrm{d}^4\mathbf{x}\,\sqrt{-g}\left[\frac{-R}{16\pi G_{\mathrm{N}}} + p(\varphi, \partial\varphi)\right], \tag{1.23}$$

where $p(\varphi, \partial\varphi)$ is the scalar field lagrangian density and we define:

$$X = \frac{1}{2} g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi. \tag{1.24}$$

Let us assume that p is a generic function of X. We can evaluate the energy momentum tensor for the matter field:

$$T_{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta\sqrt{g}p}{\delta g^{\mu\nu}} = \frac{\partial p}{\partial X}\partial_\mu\varphi\partial_\nu\varphi - p g_{\mu\nu}. \tag{1.25}$$

If $\partial_\mu\varphi$ is a time like vector it is possible to relate this energy momentum tensor to the one of a perfect fluid in General Relativity (1.2). So we can set the 4-velocity as: $u_\mu = \sigma\frac{\partial_\mu\varphi}{\sqrt{2X}}$ where σ is the sign of the speed of the field. We can observe that the pressure component of $T_{\mu\nu}$ coincides with the p functional in (1.23), while the energy density of the system is:

$$\rho = 2X\frac{\partial p}{\partial X} - p. \tag{1.26}$$

If we consider an homogenous and isotropic spacetime we need to impose that the field φ is homogeneous too, and the space dependence arise only for the perturbations over the homogeneous background. The equations describing the dynamics of the system are the

Friedmann equations (1.4) and the Klein Gordon equation:

$$\ddot{\varphi} + 3H\dot{\varphi}\frac{p_{,X}}{\rho_{,X}} + \frac{\partial_\varphi \rho}{\rho_{,X}} = 0. \tag{1.27}$$

An additional background parameter which plays an important role in non-canonical models for inflation is the sound speed for perturbations which, from the hydrodynamical theory, is defined as $c_s^2 = \frac{\delta p}{\delta \rho}$, and it is easy to verify that:

$$c_s^2 = \frac{\delta p}{\delta \rho} = \frac{\partial_X p}{\partial_X \rho} = \left(1 + 2X\frac{\partial_X^2 p}{\partial_X p}\right)^{-1} \tag{1.28}$$

when considering a canonical case this is always equal to one during the inflationary phase.

Now due to the more general form of the action (1.23) compared with the canonical one we have some regions in phase space which are not suitable for a field theory. As noted in [6] for any inflationary lagrangian we should require the null energy condition $\rho + p \geq 0$ which translates into $\partial_X p \geq 0$. Moreover the non-trivial nature of the sound speed requires further consideration, as we could have either super luminal pertutbation speed or non-physical cases for which $c_s^2 < 0$ so that the system is unstable. Remarkably those requirements translate into those following conditions

$$\partial_X p \geq 0, \qquad \partial_X^2 p > 0 \tag{1.29}$$

and requiring these conditions on the whole phase space is not possible, rather we can just require that the inflationary trajectory does not enters a forbidden region in phase space.

K-Inflation

In non-caonical models (1.23) it is possible to achieve an inflation evolution if we suppose the energy density is dominated by the potential or if $\partial_X p \simeq 0$. We focus on the latter case, which takes the name of kinetical driven inflation or k-inflation for short, showing that a purely kinetic lagrangian can generate an accelerated expansion of the universe. We consider thus a generic action where the dynamics is completely driven by the kinetic component of the field. As we neglect a potential term, if we impose that the pressure of the field vanish for $X = 0$, thus its expansion around this point shall be:

$$p(\varphi, X) = K(\varphi)X + L(\varphi)X^2 + \dots \tag{1.30}$$

To obtain an accelerated evolution of the universe we must impose that some of the coefficient $(K(\varphi),\ L(\varphi),\dots)$ in the expansion (1.30) takes a negative value.

We consider first a case where the coefficients in the matter lagrangian (1.30) are constants. Selecting for example a simple model:

$$p = KX + LX^2 \tag{1.31}$$

It possesses an attractor trajectory $\dot{\varphi}^2 = -\frac{K}{L}$. Imposing $K < 0$ and $L > 0$ we fix this attractor trajectory in an accessible region of phase space. We can evaluate the equation of motion of the system for the canonical momentum $\Pi = \dot{\varphi}(K + L\dot{\varphi}^2)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi + 3H\Pi = 0. \tag{1.32}$$

Solving (1.32) we see that $\Pi \propto e^{-3N}$, thus, setting the initial condition to be $\dot{\varphi} > -\frac{K}{L}$ the system approaches the attractor solution and both the sound speed and the first slow-roll parameter decrease exponentially. As noted in the original paper by Armendáriz-Picón et al. [3] this model is the equivalent of a de Sitter model with a constant energy density. Namely it does not offer a smooth transition to a Friedmann universe and no graceful exit. To avoid this problem one must impose a φ dependence in the coefficients of the lagrangian.

In general k-inflation is not a viable model to explain all of inflation as it leads to a small sound speed which is ruled out by the current observation of Non-Gaussianities in the CMB. Thus to build a viable non-canonical model we must introduce a potential term and suppose that at CMB scale the system was in a slow-roll regime.

Slow-Roll

In a similar fashion to the canonical case we should introduce the slow-roll parameters, we look back at the definition of the first slow roll parameter (1.17) and we can rewrite it in the more general form:

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{3}{2}\frac{\rho + p}{\rho} \tag{1.33}$$

and again we see that by imposing the null energy condition ϵ always has a positive value. Moreover we expect the inflation to last long enough to solve the Flatness problem, hence we should require that ϵ varies slowly during the inflationary evolution. Thus we evaluate the second slow-roll parameter:

$$\epsilon_2 = \frac{\mathrm{d}\log\epsilon}{\mathrm{d}N} = \frac{\ddot{\rho}}{H\dot{\rho}} - \frac{\dot{H}}{H^2} - \frac{\dot{\rho}}{H\rho} = 4\epsilon - \left(\epsilon - \frac{\dot{X}}{2HX}\right) - \left(\epsilon - \frac{\dot{\Pi}}{H\Pi}\right) \tag{1.34}$$

where $\Pi = \frac{\partial p}{\partial\dot{\varphi}}$ is the canonical momentum. For simplicity it is possible to define the parameters:

$$\eta_X = \epsilon - \frac{\dot{X}}{2HX} = \epsilon - \frac{1}{2}\frac{\mathrm{d}\log X}{\mathrm{d}N}, \qquad \eta_\Pi = \epsilon - \frac{\dot{\Pi}}{H\Pi} = \epsilon - \frac{\mathrm{d}\log\Pi}{\mathrm{d}N} \tag{1.35}$$

and we can notice that in canonical models $\eta_\Pi = \eta_X$. To ensure the small rate of change of the first slow-roll parameter during inflation, we should require $|\eta_X + \eta_\Pi| \ll 1$, for simplicity from now on we shall consider the two parameters (1.35) to be small separately. A similar request of slow variation during inflation should be made on the sound speed, hence the following slow-roll parameter for the sound speed is introduced

$$\kappa = \frac{\mathrm{d}\log c_s}{\mathrm{d}N} = \frac{\dot{c_s}}{Hc_s}. \tag{1.36}$$

Now the smallness of the slow-roll parameters can be obtained dinamically in various ways, for example it is easy to show that a purely kinetic model, shortly discussed before, can generate an inflation where those parameters are zero, but it is not able to generate a graceful exit.

The experimental data of PLANCK shows us that modes generating the CMB exited during a slow-roll phase, thus we need to translate the conditions imposed in the canonical case to a more general lagrangian and see whether this behaviour is possible.

First of all we need to restrict our study to a class of separable lagrangians: $p(\varphi, X) = q(\varphi, X) - V(\varphi)$ where $q(\varphi, X)$ cannot be further separated, and we suppose that the system is potential dominated so that $H^2 \simeq \frac{V}{3{\mathrm{M_P}}^2}$ which implies $w \simeq -1$. Thus we obtain a potential driven inflation in the non canonical case. To ensure a slow-roll evolution we also need to impose that ϵ is almost constant during inflation, this, as we can see in the equation (1.34), requires that the acceleration of the field and the time derivative of the conjugated momentum are negligible in their equation of motion so that:

$$\frac{\ddot{\varphi}}{H\dot{\varphi}}, \ll 1 \qquad \frac{\dot{\Pi}}{H\Pi} \ll 1. \tag{1.37}$$

Attractors

As illustrated before for the canonical case the inflationary trajectory is an attractor in phase space. To ensure that non-canonical inflation can generate inflation in a natural way, it is necessary to check that the inflationary solution has a similar property also in the non-canonical case.

There are various ways to show that an inflationary trajectory is an attractor in phase space, here we obtain this result by using the Hamilton-Jacobi formalism following the approach presented in [16]. We consider the hamiltonian as a function of the field parameter φ, and substitute the speed of the field in the action by employing the equations (1.4) and we finally obtain the field derivative of the Hubble parameter:

$$H'(\varphi) = \frac{\sqrt{2X}p_{,X}}{2{\mathrm{M_P}}^2} \tag{1.38}$$

where we have set $\dot{\varphi}$ and X to have an opposite sign without loss of generality. It is important to notice that we can not obtain an exact solution for $\dot{\varphi}$ as that would be model dependent. Substituting (1.38) into the Friedmann equation (1.4) we obtain the Hamilton-Jacobi equation for the general action:

$$3{\mathrm{M_P}}^2H^2 = \frac{4{\mathrm{M_P}}^2H'^2}{p_{,X}} - p. \tag{1.39}$$

Considering a general solution to (1.39) $H_0(\varphi)$, for the time being we do not need to impose it as an inflationary solution, and applying a small perturbation to this solution $H_0(1+\delta H)$, one can insert this perturbation into the Hamilton-Jacobi equation and evaluate the evolution of this perturbations expanding (1.39) to the first order:

$$6{\mathrm{M_P}}^2H_0^2\delta H = \frac{8{\mathrm{M_P}}^4H_0'(H_0'\delta H + H_0\delta H)}{p_{,X}} - \frac{4{\mathrm{M_P}}^4H_0'^2p_{,XX}}{p_{,X}^2}\delta X - p_{,X}\delta X. \tag{1.40}$$

We have to express δX and, to do so we can use the definition of the sound speed (1.28) and perturbing (1.38), we rewrite the Hamilton-Jacobi equation for the perturbations as:

$$\frac{\delta H'}{\delta H} = -\frac{3}{2{\mathrm{M_P}}^2}\frac{H_0}{H_0'}\left[\frac{c_s^2-2}{p_{,X}} - 4{\mathrm{M_P}}^2c_s^2H_0'^2\frac{p_{,XX}}{p_{,X}}\right]^{-1} - \frac{H_0'}{H_0}. \tag{1.41}$$

Using the definition of the sound speed and equation (1.38) it is possible to rewrite this equation in a much more simple form:

$$\frac{\delta H'}{\delta H} = \frac{3}{2{\rm M_P}^2}\frac{H_0}{H_0'}p_{,X} - \frac{H_0'}{H_0} \tag{1.42}$$

It is useful now to consider the definition of the e-folds parameter (1.10), we can use this definition and the Hamilton-Jacobi formalism ro obtain

$$\mathrm{d}N = -\frac{1}{2{\rm M_P}^2}\frac{H_0}{H_0'}p_{,X}\mathrm{d}\varphi. \tag{1.43}$$

Inserting this into equation (1.41) we can finally express the evolution of the perturbed Hubble parameter as a function of the e-folds number N as

$$\delta H = \exp\Big(- 3N + \frac{2{\rm M_P}^2}{p_{,X}}\frac{H_0'^2}{H_0^2}N\Big), \tag{1.44}$$

where we note that the second term in the exponential coincides with the first slow-roll parameter (1.17) multiplied by the number of e-folds. Hence if H_0 is an inflationary solution and we have $\epsilon \ll 1$, the perturbations evolve as e^{-3N}. Since the number of e-folds grows rapidly during inflation perturbations from non-canonical inflationary solutions are decreasing and become negligible. Interestingly this result coincides with the one obtained in the canonical case [37].

For the sake of completeness we consider a DBI model and obtain the same result. Starting from the DBI action:

$$S = \int \mathrm{d}^4 x a^3 \left[\frac{1}{h}(1 - \sqrt{1-2hX}) + V\right] \tag{1.45}$$

we apply the Hamilton-Jacobi formalsm and we evaluate:

$$2hX = \frac{4{\rm M_P}^4 hH'^2}{1+4{\rm M_P}^4 hH'^2} \qquad H^2 = \frac{1}{3{\rm M_P}^2}\left[\frac{1}{h}(\sqrt{1+4{\rm M_P}^4 hH'^2} - 1) + V\right]. \tag{1.46}$$

We then apply a perturbation around a solution to the Hamilton-Jacobi equation $H = H_0(\varphi)(1+\delta H)$ and we obtain the linearized perturbation equation:

$$\frac{\delta H'}{\delta H} = \frac{3}{2{\rm M_P}^2}\frac{H_0}{H_0'}\sqrt{1+4{\rm M_P}^4 hH_0'^2} - \frac{H_0'}{H_0}. \tag{1.47}$$

we again use the relationship (1.43), where in the DBI case $\partial_X p = \sqrt{1+4{\rm M_P}^4 hH_0'^2}$, and this equation coincides with the one obtained in the general case. Thus we finally obtain the expected solution

$$\delta H = \exp\big(N(\epsilon - 3)\big). \tag{1.48}$$

1.4 Perturbation

The model discussed so far describes an homogeneous picture of the universe in which the inflaton evolves generating an inflationary phase and, eventually, will reach a global minimum in its potential ending the accelerated expansion phase. This evolution can be evaluated in a classical context and, by itself, does not allow the generation of structures in the universe or even the anisotropies in the CMB. To actually obtain these features in a model of inflation, we need to consider the effects of the inflaton perturbations during its evolution in the form of quantum fluctuations. Hence we apply a semiclassical approximation by supposing a background classical field, which in this case is the homogeneous inflaton, and then quantize the perturbation over the background evolution, which here describe the space dependent part of the field.

In this procedure we must consider a quantization scheme for our perturbations, before going ahead with the description of cosmological perturbations we briefly review the quantization procedure on a de Sitter spacetime.

Quantization in de Sitter spacetime

We consider a scalar field ξ evolving on a de Sitter spacetime and we suppose that it is a spectator field, meaning that gravity does not get any back reaction from its evolution and just acts as an expanding background. For simplicity we take a massless field, thereby we have the well known action:

$$S = \frac{1}{2}\int \mathrm{d}t\mathrm{d}^3\mathbf{x}\, a^3\Big(\dot{\xi}^2 - \frac{1}{a^2}(\partial_i\xi)^2\Big). \tag{1.49}$$

It is useful, as we are working in a conformally flat spacetime, to introduce the conformal time $\tau = \int \mathrm{d}ta^{-1}$ which, in the de Sitter case, can easely shown to be $\tau = \frac{-1}{aH}$. We can also rescale the scalar field by introducing $\sigma = \frac{\xi}{a(\tau)}$, so that the action takes the following form:

$$S = \frac{1}{2}\int \mathrm{d}\tau\mathrm{d}^3\mathbf{x}\Big((\sigma')^2 - (\nabla\sigma)^2 - m_{\text{eff}}(\tau)\sigma^2\Big). \tag{1.50}$$

It is interesting to notice that through the introduction of the conformal time parameter and the rescaling of the scalar field we obtain an action similar to that of a Klein-Gordon field in a Minkowski spacetime with an effective mass $m_{\text{eff}} = -\frac{a''}{a}$. Indeed, all the information about the gravitational interaction are included in the definition of the effective mass.

As usual, we expand the field σ in Fourier modes, so we'll rewrite the action in Fourier space and we express the general solution as a linear combination of the mode functions $v_k(\tau)$, $v_k^*(\tau)$. As the energy depends only on the modulus of the momentum k the mode function will be a function only of the modulus of the momentum. We will see later that the perturbation equations too will depend only on the modulus of the momentum. Imposing the realness of the scalar field we set $\sigma^*_{\mathbf{k}} = \sigma_{-\mathbf{k}}$, thereby:

$$
\begin{aligned}
\sigma(\tau, \mathbf{x}) &= \int \mathrm{d}\tau \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2}} \Big[a^-_{\mathbf{k}} v^*_k(\tau) + a^+_{-\mathbf{k}} v_k(\tau) \Big] e^{\imath \mathbf{k}\mathbf{x}} \\
&= \int \mathrm{d}\tau \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2}} \Big[a^-_{\mathbf{k}} v^*_k(\tau) e^{\imath \mathbf{k}\mathbf{x}} + a^+_{\mathbf{k}} v_k(\tau) e^{-\imath \mathbf{k}\mathbf{x}} \Big]
\end{aligned}
\tag{1.51}
$$

and it follows that $a^+_{\mathbf{k}} = (a^-_{\mathbf{k}})^*$.

We can now apply the usual second quantization procedure by quantizing the original field σ substituting the coefficients $a^{\pm}_{\mathbf{k}}$ with the time independent operators $a_{\mathbf{k}}$, and its complex conjugate $a^\dagger_{\mathbf{k}}$, as the creation and annhilation operators, respectively. By then requiring the canonical equal time commutation relation between the field σ and its conjugated momentum $\pi = \frac{\delta S}{\delta \sigma'}$, we are led to the usual commutation relations for the creation and annhilation operators:

$$
[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta(\mathbf{k} - \mathbf{k}'), \qquad [a^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = [a_{\mathbf{k}}, a_{\mathbf{k}'}] = 0. \tag{1.52}
$$

What it is left to do is to evaluate the mode functions and, by inserting equation (1.51) into (1.50) we see that v_k must obey the equation:

$$
v''_k + (k^2 + {m_{\text{eff}}}^2(\tau)) v_k = 0 \tag{1.53}
$$

and, the commutation relations (1.52) lead to the normalization condition $v'_{\mathbf{k}} v^*_{\mathbf{k}} - v_{\mathbf{k}} v'^*_{\mathbf{k}} = 2i$, which is equivalent to requiring the linear indipendence of v_k and v^*_k.

To complete the quantization process we must choose the vacuum. In the Minkowski quantum theory the vacuum is defined as the minimum energy state. Here, though, the hamiltonian has a specific time dependence, thereby it does not have a time indipendent eigenstate. There are various prescription to define a suitable vacuum in a general spacetime. In an inflationary framework, the usual setup is to define the vacuum state as the eigenstate minimizing the instantaneous hamiltonian at a fixed time τ.

Thus we consider vacuum $|0_v\rangle$ chosen by taking a generic solution of the equation (1.53), such that $a_{\mathbf{k}}\,|0_v\rangle = 0$. We evaluate the mean value of the hamiltonian operator and find the function $v_k(\tau)$ minimizing it. We can evaluate the hamiltonian originating from the action (1.50), and, operating the mode expansion and the quantization procedure, we obtain:

$$H(\tau) = \frac{1}{4}\int \mathrm{d}^3\mathbf{k}\Big[a_{\mathbf{k}}a_{\mathbf{k}}F_{\mathbf{k}} * + a^\dagger_{\mathbf{k}}a^\dagger_{-\mathbf{k}}F_{\mathbf{k}} + (2a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + \delta^{(3)}(0))E_{\mathbf{k}}\Big] \tag{1.54}$$

where:

$$E_{\mathbf{k}} = |v'_{\mathbf{k}}|^2 + (k^2 + m_{\mathrm{eff}}{}^2)|v_{\mathbf{k}}|^2, \qquad\qquad F_{\mathbf{k}} = v'^2_{\mathbf{k}} + (k^2 + m_{\mathrm{eff}}{}^2)v^2_{\mathbf{k}}. \tag{1.55}$$

Evaluating the expectation value for a general vacuum state at a fixed time τ_0, we obtain the energy. The presence of the divergent term $\delta^{(3)}(0)$ could be worrying but that is just a volume normalization factor [31]. Evaluating the energy density:

$$\rho = \langle 0|H(\tau)|0\rangle = \frac{1}{4}\int \mathrm{d}^3\mathbf{k}(|v'_{\mathbf{k}}|^2 + (k^2 + m_{\mathrm{eff}}{}^2)|v_{\mathbf{k}}|^2) \tag{1.56}$$

for a fixed $\mathbf{k}$ the choice of the mode function is reduced to the choice of a suitable initial condition. Moreover if we set ourselves in the far past where $|m_{\mathrm{eff}}{}^2| \ll k^2$, and by imposing the consistency relationship for the modes and the minimization of $E_{\mathbf{k}}$, we see that the minimizing v_k is:

$$\lim_{\tau\to-\infty} v_k = \frac{1}{\sqrt{2k}}e^{ik\tau} \tag{1.57}$$

and this fixes the vacuum as the Bunch-Davies Vacuum [7], which is defined as the Minkowski vacuum in the early-time limit.

Now we can come back to the equation (1.53), and it is interesting to notice that it actually has an exact solution of the form:

$$v_k = A\frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) + B\frac{e^{ik\tau}}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right) \tag{1.58}$$

where A, B are integration constants. By imposing the Bunch-Davies vacuum and the quantization condition (1.52), we set $A = 0$, $B = 1$ obtaining the unique mode function:

$$v_k = \frac{e^{ik\tau}}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right). \tag{1.59}$$

We can then evaluate the 2 point function in Fourier space of the original field ξ:

$$\langle \xi_{\mathbf{k}} \xi_{\mathbf{k}'} \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \frac{|v_k|^2}{a^2} = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \frac{H^2}{2k^3} (1 + k^2 \tau^2). \tag{1.60}$$

Classical Perturbations

We are interested here in evaluating the first order perturbations of the inflation-gravity system, by solving the linearized Einstein equations:

$$\delta G_{\mu\nu} = 8\pi G_{\mathrm{N}} \delta T_{\mu\nu}. \tag{1.61}$$

To do this we will study the perturbations to the metric components and the matter components separately, and plug them into equation (1.61). It is possible to decompose both the metric and the stress energy tensor into independent scalar, vector and tensor modes having helicity $0, \pm 1, \pm 2$ respectively, and it can be shown that the different modes do not interact at the linear level and can therefore be studied separately.

The most general metric arising from this scalar, vector, tensor decomposition is:

$$\mathrm{d}s^2 = (1+2\phi)\mathrm{d}t^2 + 2a(B_{,i} + S_i)\mathrm{d}x^i \mathrm{d}t - a^2 \Big[((1+2\psi)\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij})\mathrm{d}x^i \mathrm{d}x^j \Big] \tag{1.62}$$

where the vector components S_i, F_i have null gradient, while the tensor perturbations h_{ij} are traceless and transverse. As expected the perturbations fields consist exactly of 10 degrees of freedom. From now on the vector perturbations will be ignored, since in single field inflation there are no non vanishing helicity 1 perturbations [21].

A generic spacetime in general relativity is invariant under diffeomorphism, so we can consider an infinitesimal transformation $\tilde{x}^\mu = x^\mu + \epsilon^\mu$, and evaluate the behaviour of the system under this transformation. We decompose the gauge transformation as $\epsilon^\mu = (\epsilon^0, \epsilon^i)$ with $\epsilon^i = \sigma \perp^i + \sigma^{,i}$ and $\sigma^i_{\perp\,,i} = 0$, so we insert this in (1.62) and see how this acts on the perturbations.

Under this gauge transformation the scalar transformation behave in this way:

$$\phi \to \tilde{\phi} = \phi - \frac{1}{a}(a\dot{\epsilon^0}), \qquad B \to \tilde{B} = B + \sigma' - \epsilon^0, \tag{1.63}$$

$$\psi \to \tilde{\psi} = \psi \frac{a'}{a} \epsilon^0, \qquad E \to \tilde{E} = E + \sigma. \tag{1.64}$$

It is natural to express the scalar perturbations in such a way to obtain the following gauge invariant quantities:[2]

$$\Phi = \phi - \frac{1}{a}\big[a(B-E')\big]', \qquad\qquad \Psi = \psi + \frac{a'}{a}(B-E') \tag{1.65}$$

which take the name of Bardeen potentials. These new fields do not change under infinitesimal coordinate transformations. Hence if they vanish in a given coordinate system they will vanish in any the other coordinate system.

The tensor perturbations h_{ij} do not change under gauge transformations, they describe gravitational waves in a gauge invariant way already.

We come back to the equation (1.61) and evaluate the first order perturbation of the Einstein and energy momentum tensor. In a similar fashion the perturbation components which are not gauge invariant must be rewritten in a gauge invariant way.

Scalar Perturbations

By expanding the gauge invariant version of the Einstein tensor, we obtain:

$$\Delta\Psi - 3\mathcal{H}(\Psi' + \mathcal{H}\Phi) = 4\pi G_{\mathrm{N}} a^2 \delta\bar{T}^0_0 \tag{1.66}$$

$$(\Psi' + \mathcal{H}\Phi)_{,i} = 4\pi G_{\mathrm{N}} a^2 \delta\bar{T}^0_i \tag{1.67}$$

$$\big[\Psi'' + \mathcal{H}(2\Psi + \Phi)' + (2\mathcal{H}' + \mathcal{H}^2)\Phi + \frac{1}{2}\Delta(\Phi - \Psi)\big]\delta_{ij} - \frac{1}{2}(\Phi - \Psi)_{,ij} = -4\pi a^2 G_{\mathrm{N}} \delta\bar{T}^i_j \tag{1.68}$$

where Φ, Ψ are the Bardeen potentials defined in (1.65), the barred terms $\delta\bar{T}^\mu_\nu$ are the gauge invariant component of the perturbed energy momentum tensor, while $\mathcal{H} = \frac{a'}{a}$ is the conformal Hubble parameter. Now we suppose that our matter field has a perfect fluid energy momentum tensor and the perturbed energy momentum tensor is

$$\delta\bar{T}^0_0 = \delta\bar{\rho}, \qquad \delta\bar{T}^0_i = \frac{1}{a}(\rho + p)\delta\bar{u}_i, \qquad \delta\bar{T}^i_j = -\delta\bar{p}\delta_{ij}, \tag{1.69}$$

where again the barred terms are to be interpreted as the gauge invariant components. By inserting this latter definition in the equation (1.68), we notice that there are no off-diagonal components, thereby $\Phi = \Psi$. It is important to notice that this relation has been evaluated in a gauge invariant way, and must hold regardless the gauge choice.

[2]This is not the only possible gauge invariant combination, is just the simplest one for the study we're carrying

From now on we shall consider the longitudinal gauge, by setting $B = E = 0$:

$$\mathrm{d}s^2 = a^2\left[(1+2\Phi)\mathrm{d}\tau^2 - (1-2\Phi)\mathrm{d}x^i\mathrm{d}x^j\delta_{ij}\right]. \tag{1.70}$$

Now we evaluate the δT_0^0 and δT_i^0 in this gauge as functions of the Bardeen potential Φ and $\frac{\delta\varphi}{\varphi_0}$:

$$a^2\delta T_0^0 = \frac{\rho+p}{c_s^2}\left[\partial_\tau\left(\frac{\delta\varphi}{\varphi_0'}\right) - \Phi\right] - 3\mathcal{H}(\rho+p)\frac{\delta\varphi}{\varphi_0'}, \qquad a^2\delta T_i^0 = (\rho+p)\left(\frac{\delta\varphi}{\varphi_0'}\right)_{,i} \tag{1.71}$$

where φ_0 is the unperturbed field. Considering the scalar perturbation, the discussion of the spatial component is not necessary, because when we insert the above equations into (1.66), (1.67) and (1.68) we can see that the ij equations are actually linear combinations of the 00 and $0i$ ones. The equations (1.66) and (1.67) can be rewritten in the following way:

$$\partial_\tau\left(\frac{\delta\varphi}{\varphi_0'}\right) = \left(1 + \frac{c_s^2\Delta}{4\pi G(\rho+p)}\right)\Phi, \qquad \partial_\tau(a\Phi) = 4\pi G a^2(\rho+p)\frac{\delta\varphi}{\varphi_0'}. \tag{1.72}$$

What is left to do is to rewrite these equations in a more useful way.

We'd expect to describe the scalar perturbation through a single perturbation field, to do that, we need to combine the Bardeen potential and the field perturbation. Thus, we introduce the fields ξ, and ζ where $\Phi a^2 = 4\pi G H\xi$ and $\frac{\delta\varphi}{\varphi'} = \frac{\zeta}{\mathcal{H}} - \frac{4\pi G}{a^2}\xi$. By using these relations and (1.72), it is possible to rewrite them in the much more compact form:

$$\xi' = \frac{a^4(\rho+p)}{\mathcal{H}^2}\zeta, \qquad \zeta' = \frac{c_s^2\mathcal{H}^2}{a^4(\rho+p)}\Delta\xi. \tag{1.73}$$

To normalize the amplitude of the quantum perturbation, it is necessary to obtain their action. The general action (1.49) can be expanded to the second order in the perturbation.

By applying the definitions of ξ, ζ it is possible to find their action (reproducing the equation of motion (1.73)):

$$S = \int \mathrm{d}\tau\mathrm{d}^3x\left[\xi\hat{O}(\Delta)\zeta' - \frac{\mathcal{H}^2c_s^2}{2a^4(\rho+p)}\xi\hat{O}(\Delta)\xi + \frac{a^4(\rho+p)}{2\mathcal{H}^2}\zeta\hat{O}(\Delta)\zeta\right] \tag{1.74}$$

where $\hat{O}$ is a time independent operator, which should be determined. As said before we would like to reduce the description to a single field,by evaluating the constraints. We observe that the ξ variable is not dynamical, so by using equation (1.73), we can substitute the ξ' with ζ. Thus, finally one obtains the action for the scalar perturbations:

$$S = \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3x\, z^2\left[(\partial_\tau\zeta)^2 + c_s^2\zeta\Delta\zeta\right] \tag{1.75}$$

where we have introduced the new variable $z = \frac{a\sqrt{2\epsilon}}{c_s}(\frac{\hat{O}(\Delta)}{\Delta})^{\frac{1}{2}}$. Here the Laplacian should be understood as a *c-number* representing the corresponding eigenvalue.

In order to quantize the perturbations we introduce a new variable $v = \zeta z$, also called Mukhanov-Sasaki variable. What is left to obtain is the function $\hat{O}$ which is fixed when we we compare the action (1.75) with the one obtained in the canonical theory of inflation [30] we are led to $\hat{O} = \Delta$. So we finally find the so called Mukhanov-Sasaki Action:

$$S = \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3x \left[v'^2 + c_s^2 v\Delta v + \frac{z''}{z}v^2\right] \tag{1.76}$$

which is the action of a scalar field with effective mass $-\frac{z''}{z}$.

We can apply here the quantization procedure described before. First of all we have to apply the Fourier mode decomposition, and the evolution of the modes will be described by the Mukhanov-Sasaki equation:

$$v_k'' + \left(c_s^2k^2 - \frac{z''}{z}\right)v_k = 0 \tag{1.77}$$

where as initial condition we'll choose the Bunch-Davies vacuum, which takes the following form due to the non trivial background sound speed:

$$\lim_{\tau\to-\infty} v_k = \frac{e^{-ic_sk\tau}}{\sqrt{2kc_s}}. \tag{1.78}$$

Obtaining an exact analytic solution to the equation (1.77) is not always possible and in general one should resort to numerical simulations. Still in various cases such as slow-roll or Ultra slow-roll it is possible to obtain an analytical approximate solution. We can write the effective mass as a function of the slow-roll parameters:

$$\frac{z''}{z} = (aH)^2\left[2 - \epsilon - 3\kappa - \epsilon_2\kappa + \frac{3}{2}\epsilon_2 + \frac{\epsilon\epsilon_2}{2} + \epsilon\kappa + \frac{\epsilon_2^2}{4} + \kappa^2 - \frac{\dot{\kappa}}{H} + \frac{\dot{\epsilon}_2}{2H}\right]. \tag{1.79}$$

and if the slow-roll parameters are small they can be treated as constant to the first order, $aH = \tau^{-1}$ and then (1.77) reduces to a Bessel equation with $\frac{z''}{z} \simeq \frac{\nu^2-\frac{1}{4}}{\tau^2}$. Imposing the

Bunch Davies condition (1.78) as initial conditions for the differential equation, one obtains:

$$v_k = \frac{\sqrt{-\pi\tau}}{2} H_\nu^{(2)}(-c_s k\tau). \tag{1.80}$$

During slow-roll inflation indeed the slow-roll parameters are small and almost constant, thus it is possible to neglect their derivatives in the effective mass term and take (1.80) as an approximated solution.

We now take a step back to relate the quantum scalar perturbations to an observable quantities. The Bardeen potential Φ, in the Longitudinal gauge coincides with the Newtonian Gravitational potential in General Relativity in the Newtonian limit, and during the evolution of the universe, the trajectory of the primordial photons will be influenced by its behaviour. This phenomena is called the Sachs-Wolfe Effect [36] and it relates the fluctuations of the temperature of CMB photons with the Bardeen potential; at large angular scales we have $\frac{\delta T}{T} \simeq \frac{\Phi}{3}$. Coming back to the definition of the Parameter ζ we see that by using the equations (1.73) it is:

$$\zeta = \frac{5\rho + 3p}{3(\rho + p)}\Phi + \frac{2\rho}{3(\rho + p)}\frac{\Phi'}{\mathcal{H}}. \tag{1.81}$$

Hence, we can relate the anisotropies of the CMB to the correlation functions of the perturbation ζ. A complete analysis would require the evaluation of all the n-point correlation function. Given the current theoretical development in field theory and experimental physics, this is unattainable, for the moment we consider only the 2-point correlation function. We shall discuss later the possible higher order correlation functions.

The first important consideration we should make regarding the evolution of the perturbations is the following. Taking the action for the scalar perturbations (1.75) and applying the Fourier decomposition, one finds the equation of motion for the comoving curvature perturbation:

$$\zeta_k'' + 2\frac{z'}{z}\zeta_k' + c_s^2 k^2 \zeta_k = 0. \tag{1.82}$$

We can immediately identify two different region in which the behaviour of the mode will be qualitatively different. In the far past the modes will start in a subhorizon scale where $\frac{c_s k}{aH} \gg 1$, then, as the universe expands, the comoving Hubble Horizon will decrease and the mode will eventually exit the horizon, reaching the superhorizon scale where $\frac{c_s k}{aH} \ll 1$. In this regime the term proportional to $c_s^2 k^2$ in (1.82) is negligible. Thus we can obtain an analytical solution to the modes equation on superhorizon scale:

$$\zeta_k = C_1 + C_2 \int \frac{\mathrm{d}a}{a} \frac{c_s^2 H}{2a^3 \varepsilon} \tag{1.83}$$

where C_1 and C_2 are integration constants. We can identify an important feature of the scalar perturbations on superhorizon scale, they are described by a constant mode C_1 and a mode which multiplies C_2. On superhorizon scale during slow-roll the modes freeze as the mode which multiplies C_2 quickly becomes negligible when compared to the constant modes. It should be noted however that, depending on the dynamics of the homogeneous background the decaying mode may turn into a growing one. This happens for $\frac{\mathrm{d}}{\mathrm{d}\log a}\frac{Hc_s^2}{\epsilon} > 3$ and the decaying mode will become a growing one the freezed solution becomes negligible. This happens for example in a Ultra slow-roll phase when $\epsilon \propto a^{-6}$ or in models with a quick growth of the sound speed. After inflation ends the comoving Hubble Horizon will start increasing and the modes will renter the horizon during the radiation or matter phase and they will leave significant effect on the structures in the universe.

To compare the theoretical models with the experimental observation we shall evaluate the 2-point correlation functions, the power spectrum:

$$P^{\zeta}(k) = \frac{k^3}{2\pi^2}|\zeta_k|^2 = \frac{k^3}{2\pi^2}\frac{|v_k|^2}{z^2}. \tag{1.84}$$

It is useful to apply a taylor expansion around a pivot scale k_* and express the power spectrum, close to the pivot scale, as a power law:

$$P^{\zeta}(k) = A_s \left(\frac{k}{k_*}\right)^{n_s - 1} \tag{1.85}$$

where we introduced the spectral index n_s describing the slope of the spectrum. We note that for $n_s = 1$ the power spectrum turns out to be scale invariant, for $n_s > 1$ we have a blue tilted power spectrum and for $n_s < 1$ a red tilted one.

We now study the behaviour of the perturbation in a slow-roll regime, first we remark that the freezing of the modes on superhorizon scale is expected as we assume both ϵ and c_s to behave as constants. Thereby, looking back at (1.83) we expect C_2 to be a decaying mode. Moreover, in the effective mass term in the Mukhanov-Sasaki equation (1.79), we can ignore all the ϵ_2 and κ terms. Thus we can suppose that the Mukhanov-Sasaki equation reduces to a Bessel equation having as a solution (1.80). Inserting this result into (1.84) and evaluating the limit to $\tau \to 0$ it is possible to obtain an analytical result:

$$P^\zeta = \frac{H^2}{8\pi^2 \epsilon c_s}|_{c_s k = aH} \tag{1.86}$$

where the term on the right side can be evaluated at horizon crossing for each mode, and this assumption holds only if the conditions for the freezing on superhorizon scale is satisfied. Similarly it is possible to evaluate the tilt of the power spectrum as a function of the slow-roll parameters, by inserting (1.86) into the definition of $n_s - 1$:

$$n_s - 1 = \frac{\mathrm{d}\log P^\zeta(k)}{\mathrm{d}\log k} \simeq -2\epsilon - \epsilon_2 - \kappa \tag{1.87}$$

where we have neglected non linear terms and we can notice that a slow-roll evolution is compatible with a flat or almost flat power spectrum. Moreover one could ask if $n_s - 1$ has a constant value, in which case (1.85) would describe the power spectrum at all scale, or it has a k dependence. The experimental data obtained by PLANCK[34] offer us some stringent constraints on the scalar perturbations at CMB scale $k_* = 0.05\mathrm{Mpc}^{-1}$:

$$A_s = 2 \cdot 10^{-9}, \qquad n_s = 0.965 \pm 0.0041, \qquad \frac{\mathrm{d}n_s}{\mathrm{d}\log k} = -0.003 \pm 0.0069. \tag{1.88}$$

Those data show the picture of an almost flat power spectrum at CMB scale, compatible with a negligible running.

Tensor Perturbations

Coming back to the general metric perturbations (1.62), we focus our attention on the tensor perturbations h_{ij}. As noted before, this tensor perturbations are already gauge invariant, so we directly move over to the linearized Einstein Equations.

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \Delta h_{ij} = 16\pi G_{\mathrm{N}} \delta T_{ij\,(T)} \tag{1.89}$$

where $\delta T_{ij\,(T)}$ is the transverse traceless component of the energy momentum tensor. Assuming that the universe is filled with a perfect fluid, we see that these components of the momentum energy tensor are equal to zero. Indeed at linear order, the primordial gravitational waves do not introduce perturbations in the perfect fluid. As said before, the tensor perturbations have 2 degrees of freedom. Knowing the theory for a spin-2 massless field we introduce the time indipendent polarization: $h_{ij} = h_+ e^+ + h_\times e^\times$ where the polarization tensor are, for example setting the propagation direction to be x^3:

$$e^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e^\times = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.90}$$

We can evaluate the evolution of the two polarizations separately. In a similar fashion to the scalar fields perturbations, we introduce the action reproducing this equation of motion. It is then useful now to introduce a canonically normalized variable $v^{(A)} = \frac{\mathrm{M_P}}{2} a h^{(A)}$ with $A = +, \times$, so that its action takes the following form:

$$S = \sum_A \frac{1}{2} \int \mathrm{d}^4 x \left[(v^{(A)\prime})^2 + \frac{a''}{a} v^{(A)\,2} - \left(\partial_i v^{(A)}\right)^2 \right] \tag{1.91}$$

and this actions coincides with (1.50) so we can proceed in the usual way by applying the mode decomposition. We can apply the second quantization setting the vacuum as the Bunch-Davies Vacuum (1.57), and evaluate the correlation function:

$$\langle h^{(A)}_{\mathbf{k}} h^{(B)}_{\mathbf{k}'} \rangle = \delta^{AB} \delta(\mathbf{k} - \mathbf{k}') |h^{(A)}_k|^2 \tag{1.92}$$

and the dimensionless power spectrum:

$$P_T = 2P^h(k) = 2\frac{k^3}{2\pi^2} |h^{(A)}_{\mathbf{k}}|^2 \simeq \frac{2}{\pi^2} \frac{H^2}{\mathrm{M_P}^2} |_{k=aH}. \tag{1.93}$$

We notice that in the non canonical context the horizon crossing of the scalar and tensor modes may not coincide.

As of today, there has been no direct and indirect detection of primordial gravitational waves. It is possible to obtain a constraint on the tensor to scalar ratio of the primordial perturbation by considering the temperature anisotropies of the CMB, and their polarization. The current observations obtained by PLANCK [34] constraint the tensor to scalar ration to be $r = \frac{P_T}{P^\zeta} < 0.07$. It is possible to express the tensor to scalar ratio as a function of the slow-roll parameters if the inflation occurs in a slow roll regime and it expression is given by

$$r = 16 \epsilon c_s. \tag{1.94}$$

1.5 Non Gaussian Features

All the calculations done up to now, allow us to derive a picture of the linear interactions happening in the primordial universe. To obtain a satisfying description, which could allow us to apply further constraints on the various inflationary models, it is necessary to evaluate the impact of the non linear interactions happening during inflation. To do that, it would be necessary to evaluate all the n-point correlation functions of the field ζ. Doing that in a theoretical framework is (with some exception) far beyond the current instruments developed in field theory.

In this section, we'll look at the constraints on the 3-point correlation functions and the role of non gaussianities. A complete description of the role of this non linear interactions is far beyond the scope of this thesis, rather we'll limit ourselves to a short introduction, and their role in non-canonical inflation. For a more in depth review we redirect to [5, 43].

Interaction Picture

To evaluate higher order correlation function, studying the first order perturbation dynamics is not enough and we should evaluate the third order perturbed action. To evaluate the correlation function of an interactive field in cosmology we have to use the in-in formalism.

We quickly review the interaction picture through a perturbative approach. Considering the hamiltonian $H[\varphi(\tau, \mathbf{x}), \Pi(\tau, \mathbf{x})]$ of an interacting single field theory, we need to separate the perturbed components from the background ones. In the context of cosmological perturbation, it is natural to select the background solution as the homogenous and isotropic part of the field. We apply the usual quantization procedure to the perturbed field and momentum and the hamiltonian can be expanded as:

$$H[\varphi, \Pi] = H[\varphi_0, \Pi_0] + \tilde{H}[\delta\varphi, \delta\Pi, \tau] + \int \mathrm{d}^3\mathbf{x} \frac{\delta H}{\delta\varphi}\bigg|_{\varphi_0,\Pi_0} \delta\varphi + \frac{\delta H}{\delta\Pi}\bigg|_{\varphi_0,\Pi_0} \delta\Pi \tag{1.95}$$

where the linear terms in $\delta\varphi$, $\delta\Pi$ vanish, as they coincide with the background equation of motion, and $\tilde{H}$ is the perturbation hamiltonian. If we restrict to first order perturbations, the perturbed hamiltonian can be expressed as the one of a free field ζ with a time dependent mass, as we did in the previous section. In studying this higher order interactions we do not expect to obtain an exact solution and we must apply perturbation theory. Hence we split the perturbed hamiltonian into the free part, containing the first order terms, H_0 and the interaction part, containing higher orders, H_I. We can introduce the time evolution operator for the complete theory and the free theory as the operators U, U_0, solving the following equations:

$$U'(\tau,\tau_0) = -i\tilde{H}U(\tau,\tau_0) \qquad\qquad U(\tau_0,\tau_0) = 1 \tag{1.96}$$

$$U_0'(\tau,\tau_0) - iH_0U_0(\tau,\tau_0) \qquad\qquad U_0(\tau_0,\tau_0) = 1 \tag{1.97}$$

using these operators we can define the time evolution in the interaction picture as:

$$\delta\varphi_{\mathrm{I}}' = i[H_0, \delta\varphi] \implies \delta\varphi(\tau,\mathbf{x})_{\mathrm{I}} = U_0^{-1}(\tau,\tau_0)\delta\varphi(\tau_0,\mathbf{x})U_0(\tau,\tau_0) \tag{1.98}$$

$$\delta\Pi_{\mathrm{I}}' = i[H_0, \delta\Pi] \implies \delta\Pi(\tau,\mathbf{x})_{\mathrm{I}} = U_0^{-1}(\tau,\tau_0)\delta\Pi(\tau_0,\mathbf{x})U_0(\tau,\tau_0). \tag{1.99}$$

The time evolution of course does not fix the initial conditions, and we do that by requiring that at a given fixed time τ_0 the interaction perturbations coincide with the ones obtained in the free case, thereby we can see that by introducing the operator:

$$F(\tau,\tau_0) = U_0^{-1}(\tau,\tau_0)U(\tau,\tau_0) = \hat{T}\exp\left(-i\int_{\tau_0}^{\tau} \mathrm{d}\bar{\tau} H_{\mathrm{I}}(\bar{\tau})\right) \tag{1.100}$$

where $\hat{T}$ is the time ordering operator.

We now take a step back and make the following consideration: in flat spacetime QFT, we usually consider the transition probability from a given initial non interacting state in the far past, to a non interacting state in the far future. Naturally a similar apporoach would not work in a cosmological context as:

1. We can only fix conditions on the far past,
2. We need to evaluate the expectation value of the observables at a given time.

We should fix the vacuum state in the far past as the Bunch-Davies one and use the operator (1.100) to evaluate the time evolution of the interacting vacuum state $|in(\tau)\rangle$. First of all, we expand the vacuum state in eigenstates of the free hamiltonian, then we consider the time evolution close to $\tau_0 = -\infty$:

$$|in(\tau)\rangle = U(\tau,\tau_1)\,|in(\tau_0)\rangle = |0\rangle\,\langle 0|in(\tau_0)\rangle + \sum_{n>0} e^{iE_n(\tau-\tau_0)}\,|n\rangle\,\langle n|in(\tau_0)\rangle\,. \tag{1.101}$$

We see that, by adding a small immaginary part to the time parameter $\tilde{\tau} = \tau(1-i\epsilon)$ and setting the system in the far past, all the excited components in (1.101) vanish and we can identify the interaction state at a given time as:

$$|in(\tau)\rangle = \hat{T}\exp\Big(-i\int_{-\infty(1-i\epsilon)}^{\tau} \mathrm{d}\bar{\tau} H_{\mathrm{I}}(\bar{\tau})\Big)|0\rangle\langle 0|in(\tau_0)\rangle\,. \tag{1.102}$$

We can evaluate the expectation value of a given operator $W(\tau)$ as:

$$\langle W(\tau)\rangle = \langle 0|F(\tau,-\infty(1-i\epsilon))^{\dagger}W(\tau)F(\tau,-\infty(1-i\epsilon)|0\rangle \tag{1.103}$$

To compute the expectation value we have to resort to a perturbative expansion of (1.103) in H_{I}:

$$\langle W(\tau)\rangle = \langle 0|W(\tau)|0\rangle + 2\,\mathrm{Im}\int_{-\infty(1-i\epsilon)}^{\tau} \mathrm{d}\tau'\,\langle 0|W(\tau')H_{\mathrm{I}}(\tau')|0\rangle + \ldots \tag{1.104}$$

Third Order perturbative Action

What is left is to evaluate are the three point function and the interaction hamiltonian. This is a rather tedious work, to evaluate them we have to come back to the action of the system and apply a perturbation:

$$S = S_0[\varphi(t), g_{\mu\nu}(t)] + S_2[\zeta^2] + S_3[\zeta^3] + \ldots \tag{1.105}$$

from the homogeneous action the background parameters are derived, while from S_2 one obtains the linear perturbations and, from higher orders, one obtains the interaction hamiltonian. In the current discussion the description up to cubic terms is sufficient. The interaction hamiltonian was studied in detail by Maldacena for a canonical single field model [28], while a more general field case has been tackled many times assuming the simplified case $c_s \simeq 1$ and for a generic sound speed by Chen et al. [12].

We consider the expectation value for the cubic function and make the following consideration: we know from Minkowski QFT that if a given scalar field has a gaussian distribution, then all the expectation values $\langle\zeta^{2n}\rangle$ can be expressed as products of $\langle\zeta^2\rangle$, while all the expectation values $\langle\zeta^{2n+1}\rangle$ vanish. Thus, one could restrict its attention to the non gaussian features in the CMB, to do that it comes natural to apply an expansion around the gaussian solution as:

$$\zeta(\tau,\mathbf{x}) = \zeta_g(\tau,\mathbf{x}) + \frac{3}{5}{f_{\mathrm{NL}}}^{\mathrm{local}}\Big(\zeta_g(\tau,\mathbf{x})^2 - \langle\zeta_g(\tau,\mathbf{x})^2\rangle\Big) + \ldots \tag{1.106}$$

and given the current experimental data we can restrict our discussion to the terms ζ_g^2 and the experimental constraints are set on the parameter $f_{\rm NL}$.

We move now to the momentum space by introducing the fourier transform of the three point function, the bispectrum:

$$\langle \zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)B_\zeta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3) \tag{1.107}$$

and by using the approximated form (1.106) and plugging it into (1.107) we derive:

$$B_\zeta(k_1,k_2,k_3) = \frac{6}{5}f_{\rm NL}\Big[|\zeta_{k_1}|^2|\zeta_{k_2}|^2 + |\zeta_{k_2}|^2|\zeta_{k_3}|^2 + |\zeta_{k_3}|^2|\zeta_{k_1}|^2\Big]. \tag{1.108}$$

The Dirac delta in (1.107) indicates that the modes of the bispectrum form a closed triangle. It is interesting to notice that different theoretical models will predict different maximal signals for different shapes, the most interesting cases one has are:

- Squeezed triangles ($k_1 \simeq k_2 \gg k_3$)
  This is the dominant non-gaussian effect for models with multiple light fields or curvaton scenario. Moreover for single field models of inflation it was proven by Creminelli and Zaldarriga [14]:

  $$\lim_{k_3\to 0}\langle \zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\rangle = (2\pi)\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)(1-n_s)|\zeta_{k_1}|^2|\zeta_{k_3}|^2 \tag{1.109}$$

  thereby for a scale invariant power spectrum non gaussian perturbation in the squeezed limit should vanish. As the experimental data suggest the power spectrum for CMB modes is almost flat and we expect that the non-gaussian perturbations in the squeezed limit have negligible contribution, their eventual detection would rule out the single field inflationary models.

- Equilateral ($k_1 = k_2 = k_3$)
  In models with higher derivative interaction such as non canonical single field inflation the signals are peaked for equilateral triangles with:

  $${f_{\rm NL}}^{\rm equil} = -\frac{35}{108}\left(\frac{1}{c_s^2}-1\right) + \frac{5}{81}\left(\frac{1}{c_s^2}-1-2A\right) \tag{1.110}$$

  where $A = \frac{X^2 p_{,XX}+\frac{2}{3}X^3 p_{,XXX}}{X p_{,X}+2X^2 p_{,XX}}$. It is easy to notice that for a sound speed close to one the non gaussianity is very small, and it increases quickly as c_s gets smaller.

The current observation brought up by PLANCK [33] have showed the following constraints to the non-gaussianity parameters evaluated at CMB scale:

$$f_{\mathrm{NL}}{}^{\mathrm{local}} = 0.5 \pm 5.0, \qquad\qquad f_{\mathrm{NL}}{}^{\mathrm{equil}} = -4 \pm 43. \tag{1.111}$$

The local non gaussianity parameter is compatible with the expected values from the Maldacena Theorem (1.109), and in general the small value of f_{NL} indicates that at least at CMB scale the perturbations are practically gaussian. Moreover in a non-canonical models of inflation the observed equilateral non gaussainity, poses a lower bound on the sound speed at CMB scales.

Chapter 2

Primordial Black Holes

2.1 Formation

The idea of black holes forming in the early stages of the universe by the collapse of overdense regions was first proposed in 1975 by Hawking and Carr [9]. Many possible formation mechanisms have been studied, in the context of this thesis we'll consider only the case of collapse from overdensities originated by the primordial perturbations formed during the inflationary phase.

Here we illustrate the procedure proposed by Carr in its original paper. We consider a flat background Friedmann universe (1.1) and we introduce some perturbations to the metric and the energy momentum tensor. To discuss the possible formation of primordial black holes it is useful to introduce the separate universe approach. We consider that each super-horizon sized overdense region of the universe evolves as a separate Friedmann-Robertson-Walker universe. Where energy density and pressure are locally homogeneous. Thereby we may consider these overdense regions as closed Friedmann universes with metric:

$$\mathrm{d}s^2 = \mathrm{d}t'^2 - R^2\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2\right] \tag{2.1}$$

where t' is the comoving time and R is the scale factor for the overdense universe. Then it is necessary to set the initial conditions of the perturbed region, and we choose it to be initially comoving with the background Friedmann universe hence imposing:

$$a(t_0) = R(t'_0), \qquad \left.\frac{\mathrm{d}a}{\mathrm{d}t}\right|_{t_0} = \left.\frac{\mathrm{d}R}{\mathrm{d}t'}\right|_{t'_0}. \tag{2.2}$$

We can now move to the time evolution of the overdense region, we fix this by using the Friedmann equation for a closed universe, it is useful to introduce the overdensity

$\delta_0 = \frac{\bar{\rho}-\rho}{\rho}$ evaluated at initial time t_0. The small overdense patch then will evolve up to a maximum extension before it will start collapsing; Inserting the initial condition (2.2) into the Friedmann equations we can obtain the maximum extension of the universe and the instant at which the collapse begins:

$$R_{\max} \simeq a_0 \delta_0^{\frac{1}{1+3\omega}}, \qquad t_{\max} \simeq t_0 \delta_0^{-3\frac{1+\omega}{2(1+3\omega)}} \tag{2.3}$$

where t_0 is the initial time for the homogeneous background universe.

Carr in its original paper studied the simplest possible overdense homogeneous model so that the condition for the formation of a primordial black hole is that the maximum extension of the overdense region is smaller than the particle horizon and bigger than its Jeans Length. By using the relationship (2.3) we can express this condition in a more useful way by employing the perturbation in the energy density. Supposing we are dealing with a perfect fluid with equation of state $p = \omega\rho$ we can approximate the Jeans length as $\sqrt{\omega}R_H$ where $R_H = H^{-1}$ is the particle horizon, and we can translate the condition for the black holes formation in a better way by using the perturbation density which leads to the condition:

$$\delta_c < \delta_0 < \delta_{\max} \simeq 1 \tag{2.4}$$

where δ_c is the threshold for PBH formation, δ_0 is the initial density perturbation evaluated at horizon crossing, and $\delta_{\max}$ is the value of the overdensity when the perturbed patch reaches its maximum extension.

The critical overdensity parameter will be dependent on the model chosen to describe the overdense patch, a first crude approximation is done by setting $\delta_c \simeq \omega$ as it was done in the original paper by Carr. Since then, many numerical and analytical calculation have been brought up. A more realistic analytical critical value was found by Harada et al. [22] where they considered a three zone model with the overdense region enveloped by an underdense region:

$${\delta_c}^{\mathrm{UH}} = \sin^2\left(\frac{\pi\sqrt{\omega}}{1+3\omega}\right) \tag{2.5}$$

where ${\delta^{\mathrm{UH}}}_c$ is the threshold evaluated in the uniform Hubble slice. This formula is in good agreement with numerical simulation of Black Hole collapse, it should be noted though that the numerical calculations have shown that the overdensity value on itself is not a sufficient parameter to understand wether a perturbation does collapse or not. Different effects arise due to the perturbation shape, and are expected. A simple example in the inflation case is the presence of non gaussian features in the perturbation spectrum, which can greately change the PBH abundance [17].

The overdensity perturbation parameter is not the only observable which could be used to understand whether a perturbation can generate a Black Hole. In 1999 Shibata and Sasaki obtained a threshold for the Bardeen Potential Φ [39] which can be quickly related to the amplitude of the comoving curvature perturbation. Such a formulation could actually lead to some inconsistencies when used to check whether a given perturbation in the early universe does generate a PBH, so it was found more useful to obtain some useful observables as functions of the comoving curvature perturbation. We consider a general perturbed universe in the ADM formalism, and we choose a comoving slicing. In such a way we can consider it as a flat Friedmann universe with spatial metric $g_{ij} = a(t)^2 e^{2\zeta_c(t,\mathbf{x})}\delta_{ij}$, thus we can obtain the various observables for the universe. We focus our attention to the overdensity of the perturbation, and we can obtain, at linear order:

$$\delta(t, \mathbf{x}) = \frac{2(1+\omega)}{5+3\omega} \frac{\Delta}{(aH)^2} \zeta_c(t, \mathbf{x}) \tag{2.6}$$

and to obtain a more useful form we can apply a Fourier transform and see:

$$\delta(t)_k = \frac{2(1+\omega)}{5+3\omega} \left(\frac{k}{aH}\right)^2 \zeta_{c\,k}(t). \tag{2.7}$$

Abundance of Primordial Black Holes

We can now relate the production of primordial black holes to the theory of perturbations generated during inflation. The perturbed modes generated in the early universe will re-enter the horizon during the radiation phase and, if the amplitude of the modes is large enough, it will generate overdense regions collapsing into primordial black holes. The mass of a PBH generated by the collapse of a mode is strictly related to the wavelength of the given mode. We obtain first the Horizon mass for a mode k as:

$$M_{\rm H} = \frac{4\pi}{3} \rho H^{-3}|_f \tag{2.8}$$

where the Hubble parameter is evaluated at the time of formation when $k = aH$. In a radiation dominated universe the wavelength and the Hubble parameter can be related as $H \propto k^2$ and in general it is useful to relate the Black hole mass to the horizon mass at matter radiation equivalence M_{eq} [26]:

$$M_{\rm PBH}(k) = \gamma M_{eq} \left(\frac{g_{\rm eq}}{g_{\rm BH}}\right)^{1/6} \left(\frac{k}{k_{eq}}\right)^{-2} \tag{2.9}$$

where γ is a numerical factor representing the efficiency of the collapse while $g_{\rm BH}$, $g_{\rm eq}$ are the relativistic degrees of freedom at formation and at matter radiation equivalence respectively. The first analytical calculations done by Carr [8] have set $\gamma \simeq 0.2$. Manipulating (2.9) we can extimate a relationship between the PBH mass and the number of e-folds between the horizon crossing of the modes generating the anisotropies on the CMB and the modes generating the overdensities collapsing into PBH

$$\Delta N = 18.4 - \frac{1}{12}\log\frac{g_{\rm BH}}{g_0} - \frac{1}{2}\log\frac{M}{M_\odot} + \log cs, BHc_{s,CMB} + \log\frac{H_{CMB}}{H_{BH}}. \tag{2.10}$$

Where g_0 are the relativistic degrees of freedom evaluated today, while H_{CMB}, $c_{s,CMB}$ are evluated at the horizon crossing for the modes generating the anisotropies of the CMB and H_{BH}, $c_{s,BH}$ are evaluated at the horizon crossing for the overdensities collapsing into PBH.

We now compare the amount of PBH generated from a given model of inflation with the exprimental data. We consider the fraction of total energy density in the primordial universe which is composed by PBH and hence we introduce:

$$\beta(M) = \frac{\rho_{\rm PBH}(M)}{\rho}\bigg|_f \tag{2.11}$$

which is evaluated at the time of formation. In the following calculations we'll suppose that the perturbations have a gaussian shape, eventual non gaussian effects may actually lead to different threshold on the formation of PBH from the collapse [17] and may significally increase and reduce their fraction in the early universe. Precise calculations in the non canonical model of inflation have not been brought up, thus for the moment this effect is ignored.

It is necessary to introduce the smoothed density contrast $\delta(x,R)$ evaluated by applying a convolution of the density contrast with a window function $W(x,R)$ where R is the comoving scale at formation. We can apply a Fourier transform and evaluate the variance as:

$$\langle\delta^2\rangle = \int_0^\infty \mathrm{d}\log k\, \tilde{W}(k,R)^2\frac{k^3}{2\pi^2}|\delta(k)|^2 = \int_0^\infty \mathrm{d}\log k\, \tilde{W}(k,R)^2\frac{4(1+\omega)^2}{(5+3\omega)^2}(kR)^4P^\zeta(k) \tag{2.12}$$

where P^ζ is the spectrum of the comoving perturbation. In order to relate the comoving density perturbation with the fraction of PBH (2.11) we could use either the Press-Schechter formalism or peaks theory, in this thesis we'll consider the former; hence supposing a gaussian distribution we obtain:

$$\beta = 2\int_{\delta_c}^{+\infty} \frac{d\delta}{\sqrt{2\pi\langle\delta^2\rangle}} \exp\left(-\frac{\delta^2}{2\langle\delta^2\rangle}\right) = \text{erfc}\left(\frac{\delta_c}{\sqrt{2\langle\delta^2\rangle}}\right) \tag{2.13}$$

where δ_c is the threshold value of the density perturbation for PBH formation. It is necessary now to apply a few considerations. First of all we suppose that the perturbation power spectrum is a power law (1.85). Moreover we guess that the choice of the window function $W(x, R)$ does influence the expected constraints on Black Holes formation [2]. We consider here a Gaussian window function $\frac{1}{\sqrt{(2\pi)^3R^3}}e^{-\frac{x^2}{2R^2}}$ and, inserting it into (2.12), we obtain:

$$\langle\delta^2\rangle = A_s\frac{2(1+\omega)^2}{(5+3\omega)^2(k_0R)^{n_s-1}}\Gamma\left(\frac{n_s+3}{2}\right). \tag{2.14}$$

We can now evaluate a threshold value for the power spectrum. We can also express the PBH abundance as [10]:

$$\beta_f(M) \simeq \frac{4}{\sqrt{\gamma}}10^{-9}\left(\frac{g_{*f}}{g_{*0}}\right)^{\frac{1}{4}}\sqrt{\frac{M}{M_\odot}}\frac{\rho_{\text{PBH}}(M)}{\rho_{\text{CDM}}} \tag{2.15}$$

where g_{*f} and g_{*0} are the relativistic degrees of freedom evaluated at formation and today. Setting the critical value of the density perturbation (2.5) for $\omega = \frac{1}{3}$ and supposing all of dark matter to be composed by PBH of mass $M = 10^{-15}M_\odot$ we extimate:

$$P^\zeta > 0.01 \tag{2.16}$$

for $n_s \simeq 1$ and, thus it is necessary an amplification of about 7 order of magnitude from the CMB scale.

2.2 PBH as dark matter

One of the main unsolved problems in modern astrophysics is the nature of dark matter. The current theoretical model on CMB and primordial nucleosynthesis clearly indicate that most of the matter in the universe is dark and non-baryonic [18] According to the current observation we have $\Omega_b = 0.04$ and $\Omega_M = 0.3$ where the first one is the density parameter for the baryonic (or ordinary) matter and the total matter in the universe. Many possible candidates have been brought up, it is possible to classificate them into hot dark Matter, relativistic matter at the present epoch, and cold dark matter, non relativisitic matter at the present epoch. One candidate for cold dark matter which has gained a lot of attention in the last years are primordial black holes. Moreover in 2015 LIGO detected the merging of two black holes of mass $36M_{\odot}$ $29M_{\odot}$, which are not allowed by the supernova collapse models for black holes and their formation mechanism is still unclear. It is thereby possible that the objects observed by LIGO are of primordial origin. Although there is no direct and conclusive evidence for the existence of primordial black holes, it is possible to put some severe constraints on their abundance, based on the current observation and theoretical expectation.

To discuss the constraints on masses beyond the lower bound we introduce a quantity which gives the ratio of black holes compared to the cold dark matter today. Rather than an extended mass function it is more useful to consider a monochromatic one:

$$f(M) = \frac{\rho_{\mathrm{PBH}}(M)}{\rho_{\mathrm{CDM}}} \tag{2.17}$$

following the discussion in [11] we can divide the constraints into 6 subgroups according to the physical observations.

Evaporation Constraints

It was theorized in the 70s by Hawking that black holes emit black body radiation at hawking temperature:

$$T_{\mathrm{H}} = \frac{\mathrm{M_P}}{Mk_B} \simeq \left(\frac{M_{\odot}}{M}\right)10^{-7}\mathrm{K} \tag{2.18}$$

where M is the mass of the black hole. Thus we can evaluate the time necessary for the complete evaporation of a black hole: $t \simeq M^3$. Considering the instant of formation during the radiation epoch it was shown that only the PBH with mass bigger than $5 \cdot 10^{14}$g can survive to the present epoch. Beyond this lower bound the Hawking radiation would significantly influence the γ-ray background if a significant amount of PBH were to fall in the $10^{15} - 10^{18}$g mass range.

Lensing Constraints

A significant amount of black holes in the universe would lead to lensing effects on the photons travelling through the universe; the observation of various sources allow significant constraint on the various mass ranges. The observation on Galactic sources for example offer constraints on the planetary mass black hole, while recent observation on the stars in the Magellanic Cloud show some light constraint on a wide mass range from planetary scale up to LIGO scale. Looking at the microlensig of quasar and the millilensing of compact radio sources also offer constraint on a wide scale of masses, in particular the first one shows the impossibility of all of dark matter to be composed by PBH in the mass range $10^{-3} < \frac{M}{M_\odot} < 60$, while the latter shows some extremely light constraint on the SuperMassive range.

Dynamical Constraints

The addition of MACHO population in stellar systems would lead to its expansion due to the increase in the kinetic energy of its stars. Populations with high mass to luminosity ratios are more sensitive to this effect, and in particular ultra faint dwarf galaxies would have been disrupted by the presence of massive primordial black holes, this fixes the bound on the abundance of black holes in the $10^4 M_\odot$ mass range to be $f \leq 10^{-3}$. At LIGO masses the dynamical constraints are not significative compared to the microlensing or gravitational waves constraints.

CMB constraints

Supposing a Gaussian distribution of the perturbations, primordial black holes are generated at the tail end of the perturbations, then they would introduce a μ deformation of the CMB power spectrum due to the Silk dumping of the density perturbation. These argument poses strong constraints over the mass range $10^3 < \frac{M}{M_\odot} < 10^{12}$. It should be noted that these assumptions rely on the gaussianity of the perturbations and the presence of non-gaussian features could mitigate these constraints. To avoid the constraints on the supermassive range though a huge non-gaussianity would be necessary, and this is not in accordance with the PLANCK observations. Still the supermassive black holes observed in the center of galaxies could be of primordial origin by supposing that the primordial black holes generated undergo an accretion after the μ era (10^6 s $< t < 10^9$ s).

Accretion Constraints

The accretion of background gas could lead to a large luminosity for Primordial Black Holes at early times, this luminosity will enhance the matter temperature of the background universe. Using FIRAS it was obtained $f < (\frac{M}{M_\odot})^{-2}$ in the mass range $1 < \frac{M}{M_\odot} < 10^3$ while WMAP imposes the constraints $f < (\frac{M}{30 M_\odot})$ in the $30 M_\odot < M < 10^4 M_\odot$ mass range. At solar masses these constraints should rule out $f \simeq 1$, although this results have turn out to be model dependent and incorrect due to a technical error, actually $f = 1$ is excluded only above $100 M_\odot$.

Gravitational Waves

Since its first run LIGO detected at least 40 merging of black holes in the mass range between $1-300M_\odot$, the rate of observation allows us to conclude that, if the compact object detected by LIGO are of primordial origin then the ratio of PBH in Cold dark matter must be smaller than 10^{-2} in this mass range. Moreover the presence of Black Holes as dark matter should lead to a gravitational wave background which, as of today, it has not been observed. This, imposes that the primordial black holes in this mass range can't provide all of the dark matter.

Window Functions

This set of observation allow us to sketch a series of interesting windows in the mass range for which PBH could compose a significant amount of dark matter, as one can notice in figure 2.1:

- $10^{16} - 10^{17}$g and $10^{20} - 10^{24}$g. In this mass range a monochromatic population of Primordial Black Holes can provide all of dark matter.
- $10 - 10^3 M_\odot$. Although this mass range can't provide all of dark matter the LIGO and VIRGO observations reveal a significant population of intermediate mass Black Holes.
- $10^{14} - 10^{18.5} M_\odot$. This window in the mass spectrum has long been neglected as such massive objects are nonsensically big, and cannot provide dark matter in the galactic halos. They may provide intergalactic dark matter but the possible constraint in this mass range have yet to be studied.

2.3 Ultra slow-roll Inflation

As stated before the generation of a significant amount of PBH requires that the power spectrum of the scalar perturbation is amplified up to $P_\zeta \simeq 10^{-2}$ (2.16)[1]. Comparing this threshold with the observed data from the cosmic microwave background we conclude that an amplification of at least seven order of magnitude is necessary. Such an amplification is impossible in single field inflationary models if the system is in a slow-roll regime [29], rather a violation of this regime can lead to a significant amplification to the power spectrum.

[1] different calculations may lead to a slightly smaller or bigger threshold

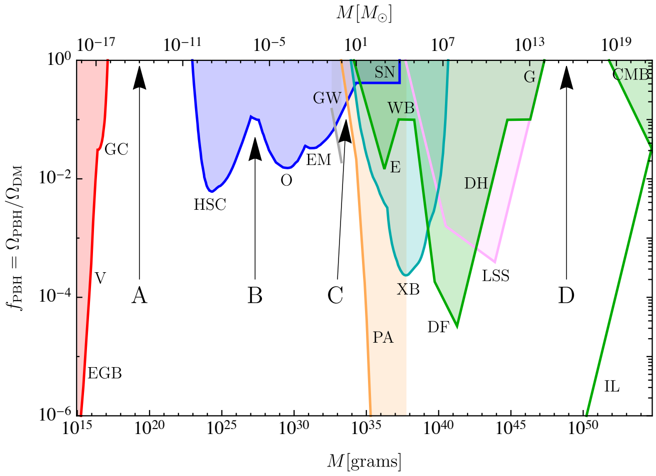

Figure 2.1: Constraints on $f(M)$ according to the current experimental observation, this graph has been taken from [11], here the interesting window functions are highlighted

Beyond the usual slow-roll inflation other evolutions leading to an accelerated expansion are possible in the canonical inflation case. An interesting mechanism which is worth investigating can be obtained by supposing that instead of a slowly changing potential we have a constant one. This mechanism is called Ultra slow-roll. The Klein-Gordon equation (1.15) then becomes:

$$\ddot{\varphi} + 3H\dot{\varphi} = 0 \tag{2.19}$$

so the acceleration of the field is compensated by the friction term. Interestingly if we solve this equation we obtain $\dot{\phi} \propto e^{-3N}$ where N is the number of e-folds, and inserting this solution in the energy density we see that the Hubble parameter quickly reaches a constant value $H^2 \simeq \frac{V_0}{3\mathrm{M_P}}$ and so an inflation phase is ensured. Inserting the equation (2.19) in the definition of ϵ_2 we get (in a canonical case):

$$\epsilon_2 = 2\epsilon + 2\frac{\ddot{\phi}}{H\dot{\phi}} = 2\epsilon - 6 \simeq -6 \tag{2.20}$$

and then, during an inflation phase with $\epsilon \ll 1$ it is reasonable to approximate $\epsilon_2 \simeq -6$, so that the slow-roll approximation is violated and ϵ quickly decreases.

This model of course is not a viable possibility to describe the universe and the inflationary phase as it does not replicate the observed CMB spectrum and it does not offer graceful exit from the inflation phase. A more realistic possibility would be to build an inflation model which, after an initial slow-roll phase, enters an almost constant potential phase in which the system behaves in a similar way to a constant potential and eventually it comes back to a slow-roll approximation. We can describe the evolution of the system in the following way: if the factor $|V'|$ will reduce drastically, when the system is in the slow-roll limit, this will correspond to a steep decrease of $|\dot{\phi}|$. As the speed of the field decreases the acceleration term becomes significant and it locks to the friction term so that the system is described by equation (2.19). Looking at the case of a constant potential we see that the speed of the field evolves as a^{-3}, which takes the name of free fall case. Comparing this to the expected value for the speed of the field in the slow-roll approximation $\dot{\phi}_{\rm SR} = \frac{-V'}{3H}$ we see that if the expected value for slow-roll decreases faster than in the free fall case the system moves away from the slow-roll attractor and enters in a ultra slow roll phase. It is easy to notice that, as the system moves into the ultra slow-roll phase, ϵ_V becomes smaller than the slow-roll parameter ϵ, thus the number of e-folds necessary to fly over the flat patch of the potential is actually far smaller then expected when evaluating it using the slow roll approximation [15].

Now we consider the behaviour of the scalar perturbations during the ultra slow-roll phase. It is important to notice that the analytical result (1.86) will not hold. Looking back at the perturbation equation (1.82) we see that the friction term:

$$\frac{z'}{z} = \left(1 + \frac{\epsilon_2}{2} - \kappa\right)\frac{1}{aH} \tag{2.21}$$

has a negative value during the ultra slow-roll phase as $\epsilon_2 \simeq -6$. Thus when evaluating the analytical solution for the modes equation on superhorizon scale (1.83) the decreasing mode will turn into a growing one and the perturbation will grow on superhorizon scale. Due to this feature the analytical evaluation for the power spectrum (1.86) does not hold and numerical evaluation of the Mukhanov Equation is required.

In the last years some approaches have been brought up to construct an inflation toy model with an ultra slow-roll phase. Probably the most successful class of models have been those with a potential with an inflection point. These models can arise in the Standard Model or Minimal Supersymmetric Standard Model theories [23] as one can obtain an inflationary evolution with a simple renormalizable potential such as:

$$V = \frac{m^2}{2}\varphi^2 - \frac{g}{3}\varphi^3 + \frac{\lambda}{4}\varphi^4. \tag{2.22}$$

Setting $g = 2m\sqrt{\lambda}$ the potential (2.22) has an inflection point in $\varphi_c = \frac{m}{\sqrt{\lambda}}$. Close to the inflection point ϕ_c the system can undergo a slow-roll inflation as ϵ_V, $\eta_V \ll 1$. When building those models one supposes that the inflation starts after the inflection point exploiting the flatness of the potential in this region without actually having to go through the inflection point. Obviously such a system undergoes a slow-roll evolution and it can not generate an amplification of the power spectrum. As noted by García-Bellido and Ruiz Morales these models can actually be repurposed to have an USR phase and generate a significant amount of black holes [19]. If for example the CMB modes exit significantly before the inflection point, in stark opposition to the typical inflection point models, the system will eventually reach the inflection point later. Here the speed of the field in the slow-roll approximation will decrease faster than the in the free fall case and the system will abandon the slow-roll attractor. To avoid an eternal inflation, which would be triggered if the system reaches a point where both $V' = V'' = 0$, one should resort to a near-inflection point. In their analysis they proposed a toy model with polynomial potential:

$$V = \frac{\lambda v^4}{12} \frac{x^2(6 - 4ax + 3x^2)}{(1 + bx^2)^2} \tag{2.23}$$

where $x = \varphi/v$, and the potential can be obtained multiplying a renormalizable potential such as (2.22) by a factor $(1 + c\varphi^2)^{-2}$. The behaviour of the system will be strongly dependent on the two parameter a, b while v, λ are essenetially an energy scale. A short calculation reveals that in order to have an inflection point $V''(x_0) = 0$ for a real x_0 one must impose a relationship between the b and a as:

$$b_c(a) = 1 - \frac{a^2}{3}\left(1 - \left(\frac{9}{2a^2} - 1\right)^{2/3}\right) \tag{2.24}$$

we then introduce a resonance parameter to avoid an eternal inflation: $b = b_c(a) + b_r$ so that we obtain a near inflection point, obviously the smaller the resonance parameter is the longer the expected duration of the USR phase is. The model they presented was not actually able to generate PBH as the maximum value obtained in the power spectrum was smaller than the expected threshold (2.16) and only generated an amplification up to 10^{-4}.

More recently some models have been found which both show connections to known phenomenological models of inflation and describe an USR phase which amplifies the power spectrum enough to generate a significant amount of black holes. One model was developed by Özsoy et al. [32] by employing axion inflation; here the perturbative axion shift symmetry is spontaneously broken and a large field inflation model with either monomial or cosine potential arise. Supposing that the non-perturbative effects breaking the symmetry are large enough to superimpose the oscillation on the potential.

$$V = V_0 + \frac{1}{2}m^2\varphi^2 + \Lambda_1^4 \frac{\varphi}{f} \cos\left(\frac{\varphi}{f}\right) + \Lambda_2^4 \sin\left(\frac{\varphi}{f}\right) \tag{2.25}$$

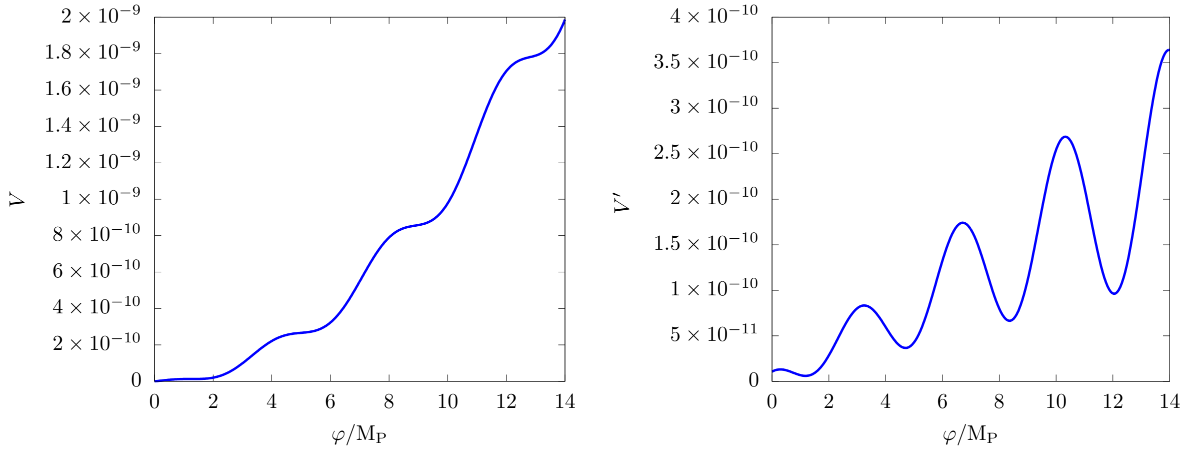

Figure 2.2: The potential and its first order derivative in the axion model developed by Özsoy et al. using the coefficients of Case 1 shown in the table 2.1

	Λ_1^4	Λ_2^4	$\frac{\mathrm{M_P}}{f}$
Case 1	$2.202\frac{\mathrm{M_P}^2}{m^2}$	$0.64\frac{\mathrm{M_P}^2}{m^2}$	1.6
Case 2	$2.601\frac{\mathrm{M_P}^2}{m^2}$	$0.462\frac{\mathrm{M_P}^2}{m^2}$	1.7

Table 2.1: Coefficients for the potential (2.25) used in [32]

and depending on the amplitude of the perturbations new local maximum or minumum appear in the thoery. If $\frac{\Lambda_i}{m^2 f^d} > 1$ then we would expect the presence of some critical point beyond the global minimum at $\varphi = 0$, rather if one sets $\frac{\Lambda_i}{f} < 1$ but not negligible then some near inflection point could come up. In their calculations they introduced two cases: where the mass m is fixed by imposing the COBE normalization to the power spectrum at CMB scale. In particular in case 2 the bumpy feature of the potential temporarily halts the inflation before the Ultra slow-roll phase.

Other phenomenological models were recently developed such as the one proposed by Cicoli et al. [13] in the context of Fibre Inflation, a model built on type IIB compactification or the one developed by Ballesteros et al. [4] in which a polynomial potential with a non minimal coupling between an inflaton and gravity was considered. All the models quickly discussed here produce primordial black holes in the asteroid mass range where they can describe all dark matter. It should be noted that, as of today, all the models using Ultra slow-roll to generate PBH show a red tilted power spectrum not compatible with the PLANCK data.

Chapter 3

Possible PBH formation mechanism in Non-Canonical Inflation

3.1 Kinetical Amplification

As remarked in the first chapter a non-canonical theory of inflation may possess, beyond the slow-roll attractor, a kinetical attractor defined by $\partial_X p = 0$. Looking back at the definition of the sound speed (1.28) and of the first slow-roll parameter ϵ (1.33) we see that both of them vanish on the attractor. We will show that while the system approaches the k-inflation attractor the perturbation modes will freeze on the superhorizon scale and it is possible to recover an analytical solution for the power spectrum which actually coincides with the slow-roll one (1.86). Thereby the kinetically driven inflation can trigger an amplification of the power spectrum which, as we will show, is able to produce a significant amount of Primordial Black Holes [25]. As discussed in Chapter 1, k-inflation on CMB scale is ruled out by the observational constraints of the non-gaussian features in the temperature anisotropies imposed by PLANCK. To alleviate this problem we can introduce a potential term so that the inflation is divided in two phases: in the first phase the energy density will be dominated by the potential and the system will be in a slow-roll regime. During this phase the modes on the CMB scales will exit the horizon. In the second phase the system evolves towards the k-inflation attractor with $c_s \ll 1$. This second phase occurs later during inflation so that the modes exiting the horizon are unconstrained by the CMB observations. We will study the two phases separately in the following non-canonical model:

$$p = KX + LX^2 - V, \tag{3.1}$$

and we solve the equations governing the homogeneous dynamics:

$$\ddot{\varphi} + 3H\dot{\varphi}\frac{K + L\dot{\varphi}^2}{K + 3L\dot{\varphi}^2} + \frac{\partial_\varphi K\frac{\dot{\varphi}^2}{2} + \frac{3}{4}\partial_\varphi L\dot{\varphi}^2 + \partial_\varphi V}{K + 3L\dot{\varphi}^2} = 0 \tag{3.2}$$

$$H^2 = \frac{1}{3{\mathrm{M_P}}^2}\left[K\frac{\dot{\varphi}^2}{2} + 3L\frac{\dot{\varphi}^4}{4} + V\right]. \tag{3.3}$$

Slow-roll evolution

We apply the slow-roll condition to the system first and we suppose that the system energy density is dominated by the potential. We then to ensure the smalleness of the slow roll parameter ϵ and then impose the smallness of the second slow-roll parameter through the smallness of the parameters η_X and η_Π. This requires that the acceleration of the field is negligible with respect to the friction term in the Klein-Gordon equation and that the time derivative of the momentum is negligible when compared to the friction term $H\Pi$. We evaluate the conjugated momentum and we see that it takes the following form $\Pi = \dot{\varphi}(L\dot{\varphi}^2 + K)$. Thus we obtain:

$$\frac{\dot{\Pi}}{H\Pi} = \frac{\ddot{\varphi}}{H\dot{\varphi}} + \frac{\partial_t(L\dot{\varphi}^2 + K)}{H(L\dot{\varphi}^2 + K)} \tag{3.4}$$

and, by requiring it to be small we obtain the following conditions:

$$\frac{\ddot{\varphi}}{H\dot{\varphi}} \ll 1, \qquad \frac{\partial_t(L\dot{\varphi}^2 + K)}{H(L\dot{\varphi}^2 + K)} \ll 1. \tag{3.5}$$

We insert the condition (3.5) in the Klein-Gordon equation and we impose that the dynamic of the system is dominated by the potential term. Thus we obtain the slow-roll equation:

$$\dot{\varphi}(L\dot{\varphi}^2 + K) \simeq -\frac{\partial_\varphi V}{3H}. \tag{3.6}$$

We apply the latter result to the Friedmann equations, and we can obtain the first slow-roll parameter:

$$\epsilon \simeq \frac{\epsilon_V}{L\dot{\varphi}^2 + K}. \tag{3.7}$$

Moreover we can evaluate the time derivative of the slow-roll equation (3.7), and we can express the variation of the momentum as a function of the slow roll parameters:

$$\frac{\dot{\Pi}}{H\Pi} \simeq -\frac{\epsilon_V - \eta_V}{L\dot{\varphi}^2 + K}. \tag{3.8}$$

The factor $K + L\dot{\varphi}^2$ can either amplify or reduce the slow-roll parameters with respect to the canonical framework. We can notice that if the factor $K + L\dot{\varphi}^2$ approaches 0 the analytical results will deviate from the numerically evaluated slow-roll parameters. In this regime it is still possible to obtain an accelerated expansion of the universe as we can notice that the field is approaching the k-inflation attractor.

Approaching the k-inflation attractor

We now study the transient phase in which the system approaches the k-inflation attractor. First we solve the equation $\partial_X p = 0$, and we obtain the attractor trajectory in phase space:

$$X = -\frac{K}{2L}. \tag{3.9}$$

We want this trajectory to be in an accessible region of phase space, thus we must impose that K and L have opposite sign. Selecting a negative L would lead to a negative kinetic term for high momentum, thus we will impose K to be negative and L positive. Moreover we can, through a field redefinition, fix either K or L to have a constant value and rewrite the lagrangian in a simpler form. Here we fix $K = -1$:

$$\varphi^{\text{new}} = \int \mathrm{d}\varphi^{\text{old}} \sqrt{|K(\varphi^{\text{old}})|}. \tag{3.10}$$

If we impose that the attractor solution $X = \frac{1}{2L}$ solves the Klein-Gordon equation, the potential must satisfy the following condition

$$\partial_\varphi \Big(- V - \frac{1}{4L} \Big) = 0. \tag{3.11}$$

This condition fixes the potential in the action.

Now we study how the field approaches the attractor solution during the transient phase and we study the Klein-Gordon equation for $X = \frac{1}{L} + D(\varphi)$ and we evaluate it at linear order in D:

$$\partial_\varphi D + \sigma \frac{3H_0\sqrt{L}}{\mathrm{M_P}} D = 0 \tag{3.12}$$

where H_0 is the Hubble parameter evaluated on the attractor and $\sigma = \operatorname{sign}\dot{\varphi}$. We solve equation (3.12) obtaining:

$$D = D_0 \exp\Big(- \frac{3\sigma}{\mathrm{M_P}} \int \mathrm{d}\bar{\varphi} H_0 \sqrt{L} \Big) \tag{3.13}$$

which describes the transient approach of the system to the k-inflation attractor. Let us note that H_0 is the Hubble parameter evaluate on the attractor. Close to the attractor we have $\dot{\varphi} \simeq \frac{\sigma}{\sqrt{L}}$, thus the integral term in (3.13) can be solved by applying a change in integration variable: $\mathrm{d}t = \mathrm{d}\bar{\varphi}\sqrt{L}$. Thus we can solve this integral obtaining $D = D_0 e^{-3N}$ and we notice that this coincides with the result obtained in Chapter 1 using the Hamilton-Jacobi formalism.

We now insert this solution into the definition of the slow-roll parameter ϵ and the sound speed, to study the behaviour of the system as it approaches the k-inflation attractor. We apply a Taylor expansion around the attractor solution and we obtain, at linear order in D:

$$\epsilon = \frac{D}{H_0^2}, \qquad\qquad c_s^2 = DL. \tag{3.14}$$

Because $D \propto e^{-3N}$, we notice that as the system approaches the attractor solution both the sound speed and the slow-roll parameter tend to 0. We then evaluate the parameters ϵ_2 and κ, obtaining:

$$\epsilon_2 = -3 + 2\sigma \mathrm{M_P} \partial_\varphi \log(L) \qquad\qquad \kappa = -\frac{3}{2} + \sigma \mathrm{M_P} \partial_\varphi \log(L). \tag{3.15}$$

We notice that if we impose $\partial_\varphi L = 0$ both ϵ_2 and κ are constants and both the sound speed and the slow-roll parameter ϵ decrease exponentially.

First we focus our attention to a model where L has no dependence on the field φ. Looking back at (3.11) we see that this choice fixes the potential to be constant, thus such a model can't undergo a slow-roll phase and is not able to replicate the perturbations at CMB scale. We can mitigate this problem by introducing an additional potential term in the action which will dominate the dynamic of the model at CMB scale. Later, during inflation, in the kinetically driven phase, the potential will not dominate the energy density and it will behave as a perturbation term. We come back to the equation (3.11) and we add a perturbation term f:

$$\partial_\varphi \Big(V + \frac{1}{4L} \Big) = f \tag{3.16}$$

imposing L to be a constant we notice that the perturbation term f coincides with the field derivative of the potential V. We study again the dynamics of the system as it approaches the k-inflation attractor with the addition of a perturbation in the potential. We evaluate the Hubble parameter on the attractor as: $H_0 = \frac{1}{4L}$ and we insert $X = \frac{1}{2L} + D(\varphi)$ in the Klein-Gordon equation obtaining, after linearizing

$$\partial_\varphi D + \frac{\sigma\sqrt{3}}{2\mathrm{M_P}} D + \frac{f}{2} = 0. \tag{3.17}$$

We solve this equation and we find:

$$D(\varphi) = \exp\left(-\frac{\sigma\sqrt{3}}{2\mathrm{M_P}}\varphi\right)\left[D_0 - \int \mathrm{d}\bar{\varphi}\exp\left(\frac{\sigma\sqrt{3}}{2\mathrm{M_P}}\varphi\right)\frac{f}{2}\right]. \tag{3.18}$$

The term multiplying the constant D_0 will quickly vanish and the approach to the k-inflation attractor will be described by the term in f.

We can again evaluate the sound speed and the slow-roll parameter ϵ at linear order obtaining:

$$\epsilon = 12DL, \qquad\qquad c_s^2 = DL \tag{3.19}$$

and the slow-roll parameters ϵ_2 and κ

$$\epsilon_2 = 2\kappa = -3 - \sigma 2\sqrt{3}\frac{f}{D}. \tag{3.20}$$

We now select an exponential potential $V = V_0 e^{\alpha\frac{\varphi}{M_P}}$, and we insert it into (3.17). It is easy to see that:

$$D = D_0 e^{\frac{\alpha\varphi}{\mathrm{M}_P}} \tag{3.21}$$

solves equation (3.17), where

$$D_0 = \frac{-\alpha}{2\left(\alpha + \frac{\sigma\sqrt{3}}{2}\right)}. \tag{3.22}$$

Requiring D_0 to be positive we see that α must be smaller than $\frac{\sigma\sqrt{3}}{2}$.

We can then evaluate the value of the field as a function of the number of e-folds. During the kinetically driven phase the speed of the field and the Hubble parameter are approximately constant, thus we can write:

$$\dot{\varphi}^2 = H_0^2\left(\frac{\mathrm{d}\varphi}{\mathrm{d}N}\right)^2 = \frac{1}{L}. \tag{3.23}$$

As $H_0^2 = \frac{1}{4L}$ we can obtain: $\varphi(N) = \sigma 2\sqrt{3}N - N_0$ where N_0 is the number of e-folds at the onset of the kinetically driven phase. We insert this result into (3.21) and we see that, as the system approaches the k-inflation attractor, D and by extension ϵ and c_s decrease as exponentially in N:

$$D = D_0 e^{-\gamma_s \Delta N} \tag{3.24}$$

where

$$\gamma_s = -\sigma 2\sqrt{3}\alpha. \tag{3.25}$$

Moreover looking back at (3.20) we find $\epsilon_2 = -\gamma_s$, $\kappa = -\frac{\gamma_s}{2}$.

We now consider the behaviour of the perturbations during the transient phase, first of all we insert the slow-roll parameters in the perturbation equation (1.82). We see that the friction coefficient $\frac{z'}{z}$ is always positive regardless of the value of γ_s, thus the perturbation modes ζ_k will freeze on superhorizon scale.

We then insert the slow-roll parameters into the effective mass in the Mukhanov-Sasaki equation (1.79). We notice that all the terms in ϵ_2 and κ cancel each other and the mass term becomes:

$$\frac{z''}{z} = (aH)^2\Big[2 - \epsilon - \epsilon\gamma_s\Big] \simeq 2(aH)^2 \tag{3.26}$$

as ϵ decreases exponentially the last equality holds. Then the Mukhanov-Sasaki equation will reduce to a Bessel equation, and we can obtain an analytical value of the power spectrum, which coincides with the analytical result obtained in the slow-roll case (1.86). Thus as the system approaches the k-inflation attractor the power spectrum (1.86) will grow as $e^{\frac{3}{2}\gamma_s N}$. It is thereby easy to do a rough estimate of the number of e-folds necessary to obtain an amplification up to the threshold for PBH formation (2.16), setting an amplification by 7 order of magnitude we obtain: $\Delta N = \frac{14}{3\gamma_s}\log 10$.

One could notice that the behaviour of the field and the slow-roll parameters is similar to the ultra slow-roll model for single field canonical inflation. In both cases the speed of the field decrease exponentially during the transient phase and the power spectrum undergoes an amplfication. It should be noted that in a model employing a kinetically driven inflation the power spectrum can be evaluated analytically while the ultra slow-roll case require numerical investigation. Moreover as we will show in the following section, it can replicate the observables at CMB scale and generate a significant amount of Primordial Black Holes while the ultra slow-roll models present a red tilted power spectrum.

α	c_{s*}	r	$\frac{\mathrm{d}n_s}{\mathrm{d}\log k}$	ΔN	$\frac{M}{M_\odot}$
$5{\cdot}10^{-2}$	$9{\cdot}10^{-2}$	$4{\cdot}10^{-2}$	$-5{\cdot}10^{-3}$	75	$1{\cdot}10^{-55}$
0.12	0.12	0.15	$4.3 \cdot 10^{-2}$	30	$1{\cdot}10^{-16}$
0.15	0.13	0.20	$6.85 \cdot 10^{-2}$	24	$1{\cdot}10^{-10}$

Table 3.1: Estimate for models with different α with L and V selected to obtain the COBE normalization and $n_s = 0.965$ at CMB scale.

A Simple K-Inflation Model

We now consider some toy models with an exponential potential where we will set L and V_0 to have $P^\zeta = 2.4 \cdot 10^{-9}$ and $n_s = 0.965$ at CMB scale. The choice of α (see above) will set the expected number of e-folds necessary to reach the threshold value of the power spectrum for PBH formation. A flatter potential will require more e-folds to undergo a significant amplification. We evaluate the behaviour of the system for different values of the parameter α and we present various observables in Table 3.1. One notices that for small α ($\alpha = 0.05$ in Table 3.1) we obtain both a running tilt parameter and ratio between the scalar and tensor perturbation in agreement with the PLANCK observation, but Black Holes with an excessively small mass. Choosing for higher value of α ($\alpha = 0.15$) we expect the amplification during k-inflation to be faster, thus we can obtain Black Holes in the asteroid mass range or even higher mass ranges. In those latter cases however the ratio between scalar and tensor perturbation is too big and not compatible with the PLANCK data.

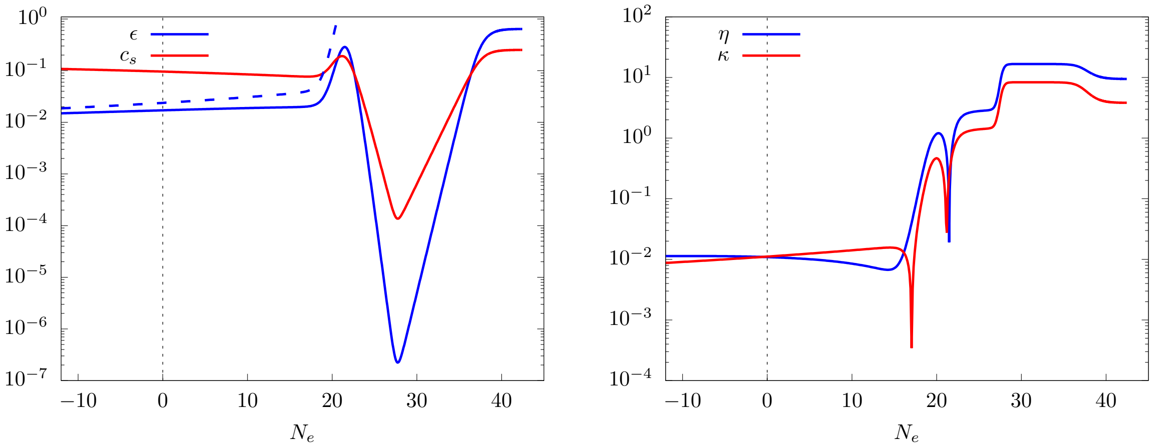

Figure 3.1: The horizon crossing for CMB modes is set at $N = 0$,on the right we plot the first slow-roll parameter and the sound speed for perturbation in the second model, the analytical slow roll approximation is plotted as a dotted line. On the right hand side ϵ_2 and κ are plotted.

Thereby a more realistic model can be introduced by setting a small modification to the potential selecting:

$$V = V_0 e^{\frac{\alpha\varphi}{15\mathrm{M_P}}}\left[1+\tanh\left(\frac{\alpha\varphi}{\mathrm{M_P}}\right)\right] \tag{3.27}$$

so that for $\varphi > 0$ we have a very flat potential which is more suitable to replicate the system at CMB scale. As $\varphi < 0$ the potential becomes steeper thus leading to a very quick amplification. To stop the amplification we apply a small variation to the coefficients in the lagrangian (3.1) which is negligible during the k-inflation phase and here we have chosen to set $L = L_0\left(1+\delta e^{-\frac{\alpha\varphi}{2\mathrm{M_P}}}\right)$. As L grows the system will move away from the attractor trajectory. On figure 3.2 we plot the power spectrum for the two different model where we've set for the simple exponential model

$$L_0 = 1.15\cdot 10^8 {\mathrm{M_P}}^{-4}, \qquad \alpha = 0.12, \qquad V_0 = 10^{-8}{\mathrm{M_P}}^4, \qquad \delta = 2\cdot 10^{-9} \tag{3.28}$$

while in the latter case we set

$$L_0 = 1.81\cdot 10^9 {\mathrm{M_P}}^{-4}, \qquad \alpha = 0.45, \qquad V_0 = 1.4\cdot 10^{-10}{\mathrm{M_P}}^4, \qquad \delta = 10^{-11}, \tag{3.29}$$

obtaining the observables:

$$c_{s*} = 0.09 \qquad r = 0.03 \qquad \frac{\mathrm{d}n_s}{\mathrm{d}\log k} = -0.0001 \qquad M_{\mathrm{PBH}} = 10^{-13}M_\odot \tag{3.30}$$

It should be noted that the following model does not possess a mechanism for the graceful exit, as the k-inflationary phase does not allow the speed of the field to be smaller than $\sqrt{\frac{1}{L}}$ and to make inflation end we need to introduce a mechanism which makes K change sign, so that the system abandons the kinetical attractor. Moreover we should introduce a modification to the potential and allow the existence of a minimum, so that the system re-approaches the canonical slow-roll attractor and inflation eventually ends.

3.2 Brane Inflation

One of the most succesful model of non-canonical inflation arises in string theory, from the model of Brane Inflation. In this class of models inflation is driven by the motion of a D3-brane in a warped throat of a stabilized compact space [24]. To preserve the four dimensional Lorentz (or de Sitter) inrvariance the D3-brane fills the four dimensional spacetime and is pointlike in extra dimensions. It is possible to rearrange this model into a four dimensional field theory in a FRW spacetime by introducing an inflaton scalar field

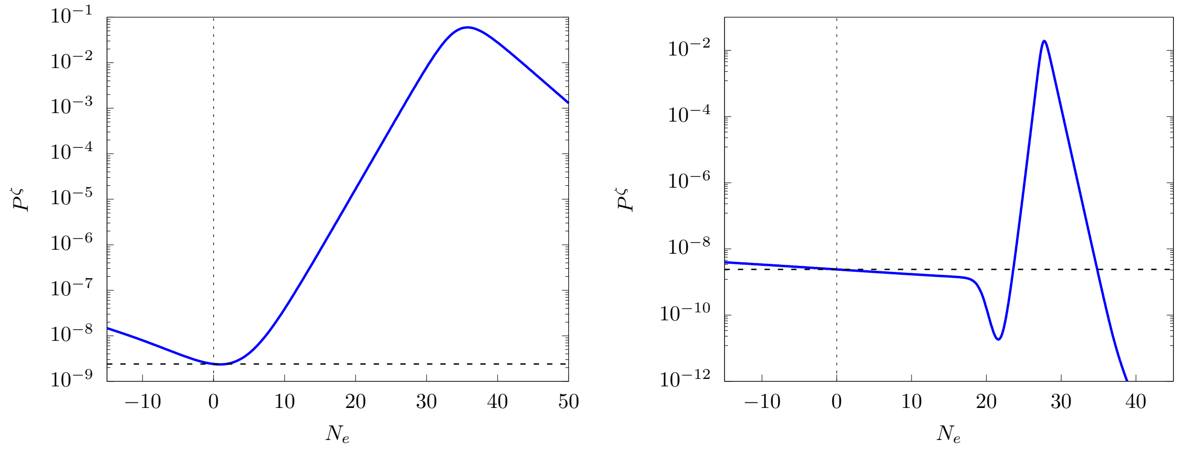

Figure 3.2: The power spectrum of the comoving curvature perturbation, the horizon crossing for the CMB modes is set as $N = 0$, on the left we plot the first model setting α=0.12 and on the right the second model is considered.

parametrizing the position of the brane. We introduce the Dirac Born Infeld (DBI) action:

$$S_{\mathrm{DBI}} = -\int \mathrm{d}^4\mathbf{x}\sqrt{-g}\frac{1}{h}\sqrt{1 - h\partial_\mu\varphi\partial^\mu\varphi} \tag{3.31}$$

where $h(\varphi)$ is the rescaled warp factor describing the geometry of the throat. If we consider the couplings of the D-Brane with the other background fields we can introduce a potential term in the DBI action [1]. We write the potential as: $V - h^{-1}$ and, adding it to the DBI action (3.31) we obtain:

$$S = \int \mathrm{d}t\mathrm{d}^3\mathbf{x}a^3\left[\frac{1}{h}\left(1 - \sqrt{1 - h\dot{\varphi}^2}\right) - V(\varphi)\right] \tag{3.32}$$

where we have considered a homogeneous scalar field on a FRW background. We can evaluate the Klein-Gordon equation:

$$\ddot{\varphi} + 3H\dot{\varphi}c_s^2 + \partial_\varphi V c_s^3 + \frac{\partial_\varphi h}{h}\dot{\varphi}^2\frac{1-c_s}{c_s}\frac{1-2c_s}{1+c_s} = 0 \tag{3.33}$$

where c_s is the sound speed for perturbations:

$$c_s^2 = 1 - h(\varphi)\dot{\varphi}^2 = \gamma^{-1}. \tag{3.34}$$

we notice that the sound speed for perturbation is the inverse of the Lorenz factor γ for the Brane. The Friedmann equation is:

$$H^2 = \frac{1}{3\mathrm{M_P}^2}\left[\frac{1}{h}(\frac{1}{\sqrt{1-h\dot{\varphi}^2}} - 1) + V\right]. \tag{3.35}$$

The DBI action employs an interesting dynamics as, if we suppose $h\dot{\varphi}^2 \ll 1$ we recover the canonical action (1.13) and the Klein-Gordon equation (3.33) coincides with the canonical one. In this regime the sound speed (3.34) is very close to 1. Instead if $h\dot{\varphi}^2 \simeq 1$ the sound speed is far smaller than 1, we will show that a DBI model can generate inflation also in this phase. We see that if $h\dot{\varphi}^2 = 1$ the sound speed is equal to 0, remarkably this is a solution to the Klein-Gordon equation (3.33), but is a singular trajectory for the Friedmann equation (3.35). If we consider instead $h\dot{\varphi}^2 > 1$ the action is not real. Thus we must restrict the phase space to regions where $\dot{\varphi}^2 < h^{-1}$ and we observe that the Dirac-Born Infield action possesses a speed limit. Interestingly one notices that in this region of phase space both the null energy condition and the stability of the perturbation are satisfied.

We can now evaluate the slow-roll parameter ϵ for a DBI lagrangian:

$$\epsilon = \frac{3}{2}\frac{\rho + p}{\rho} = 3\frac{\gamma\frac{\dot{\varphi}^2}{2}}{\frac{\gamma^2}{\gamma+1}\dot{\varphi}^2 + V} \tag{3.36}$$

as expected, when $\gamma \simeq 1$ it coincides with the canonical case. If $\gamma \gg 1$ (3.36) does not coincide with the slow-roll parameter in a canonical case. We notice though that also in the non-canonical regime an accelerated evolution requires the energy density to be dominated by the potential.

3.3 Attractor in Brane Inflation

As discussed in section 1.3 in a non-canonical model the inflationary trajectory is an attractor, here we'll obtain the attractor trajectory for the DBI model. We should notice though that the strongly non canonical evolution of the system does not coincide with the k-inflation described by Mukhanov [3] as, looking at the DBI lagrangian we see that a trajectory in phase space where $\partial_X p = 0$ does not exist .

To evaluate the attractor trajectory we can introduce the parameter $\chi = h^{\frac{1}{2}}|\dot{\varphi}|$, so that we can rewrite the sound speed for perturbations as $\sqrt{1-\chi^2}$. Now we rewrite the DBI action (3.32) as a function of φ and χ, although χ is not the canonical momentum associated to φ it can still be used to evaluate the attractor solution of the theory [42]. We obtain the linear equations of motion:

$$\chi = h^{\frac{1}{2}}|\dot{\varphi}| \tag{3.37}$$

$$\frac{\mathrm{d}\chi}{\mathrm{d}t} = (1-\chi^2)\left[h^{\frac{1}{2}}\sqrt{1-\chi^2}\partial_\varphi V - 3H\chi - \frac{\partial_\varphi h}{h^{3/2}}\left(1-\sqrt{1-\chi^2}\right)\right] \tag{3.38}$$

$$H^2 = \frac{1}{3{\mathrm{M_P}}^2}\left[h^{-1}\left(\frac{1}{\sqrt{1-\chi^2}}-1\right)+V(\varphi)\right] \tag{3.39}$$

one of them has the trivial solution $\chi = 1$ which coincides with the singular trajectory in phase space. To obtain the attractor solution it is useful now to introduce two non-canonicalness parameters:

$$A = \frac{\partial_\varphi V}{3H}h^{\frac{1}{2}} \qquad\qquad \Delta = \frac{\partial_\varphi h}{3Hh^{3/2}} \tag{3.40}$$

and dividing the equation (3.39) by $3H$ on both sides and rewriting it as:

$$\frac{\mathrm{d}\chi}{\mathrm{d}t} = (1-\chi^2)\left[A\sqrt{1-\chi^2}-\chi-\Delta\left(1-\sqrt{1-\chi^2}\right)\right] \tag{3.41}$$

we can evaluate the attractor solution as:

$$\chi_{\mathrm{att}} = \frac{-\Delta+(A+\Delta)\sqrt{1+2A\Delta+A^2}}{1+(A+\Delta)^2}. \tag{3.42}$$

We see that this solution is the attractor trajectory of the model by inserting in (3.39) $\chi = \chi_{\mathrm{att}} + \delta\chi$ and expanding at linear order in $\delta\chi$:

$$\dot{\delta\chi} + 3H\lambda\delta\chi \simeq 0 \tag{3.43}$$

where $\lambda > 0$. By solving equation (3.43) we see that regardless of the sign of $\delta\chi$ the system will be driven towards χ_{att} [42] thus confirming the attractor nature of the trajectory (3.42). We expect to apply this results to an inflationary phase therefore we need the energy density to be potential dominated, and we can express the non-canonicalness parameters (3.40) as functions of the field:

$$A = \mathrm{M_P}\sqrt{\frac{2}{3}\epsilon_V Vh}, \qquad\qquad \Delta = \mathrm{M_P}\frac{\partial_\varphi h}{\sqrt{3Vh^3}}. \tag{3.44}$$

Now we evaluate the sound speed and the slow-roll parameters on the attractor solution. During the accelerated expansion of the universe we expect that Δ is negligible. We make this assumption because if Δ becomes big both η_X, and η_Π become significantly bigger than 1 [16], leading to a fast variation of the first slow-roll parameter ϵ and of the sound speed which is incompatible with the observation at CMB scale.

Thus we evaluate the attractor solution (3.42) for $\Delta = 0$:

$$
{\chi_{\rm att}}^2 = \frac{A^2}{1+A^2} \tag{3.45}
$$

and, inserting this result into (3.34) we evaluate:

$$
c_s^2 = \frac{1}{1+A^2}. \tag{3.46}
$$

In a similar fashion we evaluate the first slow-roll parameter ϵ and we obtain:

$$
\epsilon = \frac{3}{2}\frac{\chi^2}{\sqrt{1-\chi^2}((1-\chi^2)^{-\frac{1}{2}}-1+Vh)} = \begin{cases} \frac{3}{2}\frac{A^2}{Vh} = \epsilon_V, & \text{for } A \ll 1 \\ \frac{3}{2}\frac{A}{Vh} = \frac{\epsilon_V}{A}, & \text{for } A \gg 1 \end{cases} \tag{3.47}
$$

We see that a sound speed close to 1 requires $A \ll 1$, and in this case the model is in a canonical regime, while for $A \gg 1$ the sound speed is small, we call this latter phase strongly non-canonical or DBI regime. Therefore we observe that when $A \gg 1$ the system is in a strongly non-canonical regime with a small sound speed while for $A \ll 1$ it behaves as a canonical model. Then we evaluate the coefficients η_X and η_Π (1.35) on the attractor in the canonical and non-canonical limit:

$$
\eta_X = \epsilon - \frac{\ddot{\varphi}}{H\dot{\varphi}} = \epsilon + \frac{\eta_V - \epsilon_V}{(1+A^2)^{3/2})} = \begin{cases} \eta_V & \text{for } A \ll 1 \\ \frac{\epsilon_V}{A} & \text{for } A \gg 1 \end{cases} \tag{3.48}
$$

$$
\eta_\Pi = \epsilon - \frac{\dot{\Pi}}{H\Pi} = \frac{\eta_V}{(1+A^2)^{1/2}} = \begin{cases} \eta_V & \text{for } A \ll 1 \\ \frac{\eta_V}{A} & \text{for } A \gg 1 \end{cases} \tag{3.49}
$$

where Π is the canonical momentum. Lastly we evaluate κ, and we notice that $\Pi = \frac{\dot{\varphi}}{c_s}$, hence we have:

$$
\kappa = \frac{\ddot{\varphi}}{H\dot{\varphi}} - \frac{\dot{\Pi}}{H\Pi} = \eta_\Pi - \eta_X \tag{3.50}
$$

which we can estimate both in the canonical and non canonical limit to be

$$
\kappa = \eta_\Pi - \eta_X = \begin{cases} 0 & \text{for } A \ll 1 \\ \frac{\eta_V - \epsilon_V}{A} & \text{for } A \gg 1 \end{cases} \tag{3.51}
$$

and as expected, in the canonical limit, the sound speed will be constant and equal to 1, thus κ will vanish while in the non canonical limit it will undergo small variations.

3.4 PBH production in Brane Inflation

To generate a significant amount of primordial black holes from our theory we require an amplification of the power spectrum with respect to its value at CMB scales, to about 10^{-2}, which is the commonly assumed threshold for the power spectrum excluding possible non gaussian effects. The known phenomenological models for DBI inflation are not able to produce Primoridial Black Holes. It is possible though to introduce some modifications to the known phenomenological cases and study some toy model which could account for some possible mechanism for the production of PBH in a Brane inflation. To do that we could consider some possible modification of the warp factor as, by looking at (3.34) and (3.40) we see that it generates significant variations to the sound speed and the dynamics of the system. One model for PBH formation was proposed by Özsoy et al. [32], where by introducing a step in the warp factor one expects a steep growth of the sound speed. This leads to a negative friction term in the perturbation equation (1.82) an and thus we should observe an amplification of the curvature perturbations on super horizon scale.

Introducing a Spike in the Warp Factor

In a canonical model during a slow-roll evolution the power spectrum can be evaluated analytically using the equation (1.86). In order for this approximation to hold we need the effective mass in the Mukhanov-Sasaki equation to behave as $\frac{\nu^2-\frac{1}{4}}{\tau^2}$ so that we can analytically solve it. Interestingly during a DBI inflation phase, with $A \gg 1$, the slow-roll parameters are small, in a similar fashion to the slow-roll approximation in a canonical model of inflation. Thus we see that the power spectrum can be evaluated analytically using (1.86).

By inserting the analytical values of ϵ (3.47) and the sound speed (3.46) into (1.86) we can obtain a simple expression for the power spectrum in a canonical slow-roll phase and in a DBI regime:

$$P^{\zeta} = A^2 \frac{H^2}{8\pi^2 \epsilon_V} \text{ for } A \gg 1. \tag{3.52}$$

We can obtain an amplification by imposing that the non-canonical parameter undergoes a quick growth, which can be achieved by introducing a spike in the warp factor which moves the system in a strongly non canonical regime.

An important consideration should be done regarding the width of the spike as, estimating analytically the duration of inflation we find:

$$N = \int \mathrm{d}tH = \int \mathrm{d}\varphi \frac{1}{\sqrt{2\epsilon c_s}} \simeq \begin{cases} \int \mathrm{d}\varphi \frac{1}{\sqrt{2\epsilon_V}} & \text{for } A \ll 1 \\ \int \mathrm{d}\varphi \sqrt{\frac{A^2}{2\epsilon_V}} & \text{for } A \gg 1 \end{cases} \tag{3.53}$$

As we could expect we see that, in a non canonical phase, the number of e-folds will grow faster compared to the canonical case. In particular a change of several order of magnitude of h leads to a similar increase of the number of e-folds necessary to traverse the spike. To make the system compatible with the expected duration of the inflation (about 60 e-folds) and obtain an amplification of several order of magnitude, a very slim spike is required. Taking for example a gaussian spike in the warp factor $h = h_0(\varphi)\Big[1 + \delta \exp\big(-\frac{(\varphi-\varphi_0)^2}{2\sigma^2 \mathrm{M_P}^2}\big)\Big]$, where $h_0(\varphi)$ is a given background warp factor, we can evaluate the non canonicalness parameter Δ:

$$\Delta = \frac{\Delta_0}{\sqrt{1+\delta e^{-\frac{(\varphi-\varphi_0)^2}{2\sigma^2 \mathrm{M_P}^2}}}} - \delta \frac{\frac{\varphi-\varphi_0}{\sigma^2 \mathrm{M_P}^2} e^{-\frac{(\varphi-\varphi_0)^2}{2\sigma^2 \mathrm{M_P}^2}}}{\sqrt{3h_0 V\left(1+\delta e^{-\frac{(\varphi-\varphi_0)^2}{2\sigma^2 \mathrm{M_P}^2}}\right)^3}}, \tag{3.54}$$

where Δ_0 is the delta factor linked to the original warp factor. We select a spike sufficiently high to generate a significant amplification of the power spectrum, and sufficiently narrow to make the amplified phase lasts a short number of e-folds. This latter requirement leads to a non negligible Δ during the evolution. As the field gets closer to the spike the warp factor grows until $|\Delta| \gg 1$ and, in this conditions the slow-roll approximation breaks down. If we evaluate η_X, η_Π [16] at first order expansion in Δ we see that neither of them are small. Moreover looking at equation (3.54) we can compare the Δ factor to A, and across the spike we can suppose $A_0 = \sqrt{\frac{2}{3}\epsilon_V V h_0}$ to be almost constant and we obtain:

$$\frac{\Delta}{A} \simeq -\frac{\delta}{A_0 \sigma} y \frac{e^{-\frac{y^2}{2}}}{\left(1+\delta e^{-\frac{y^2}{2}}\right)^2} \tag{3.55}$$

where $y^2 = \frac{\varphi-\varphi_0}{\sigma^2 \mathrm{M_P}^2}$. Thus if the spike is sufficiently narrow and we have: $\Delta \gg -A$ close to the minimum of Δ. If we insert this assumption into the attractor solution (3.42) we see that the term inside the square root becomes negative and the attractor trajectory disappears. As the system reaches the top of the spike Δ becomes small again, thus the field re-enters the attractor solution and its dynamics can be studied analytically. As the system exits the spike Δ becomes large again and the η_X, η_Π terms are not small. In this case though the Δ factor is positive and, looking back at (3.42) we see that the attractor solution is well defined. Moreover we observe a quick growth of the sound speed as we can see that in this regime $\kappa > 1$ and this implies that the horizon will start growing during this phase. This can be shown easily by evaluating the scale of the modes crossing the horizon as $k_s = \frac{aH}{c_s}$, and calculating the derivative of $\log k_s$ by the number of e-folds:

$$\frac{\mathrm{d}\log k_s}{\mathrm{d}N} = 1 - \epsilon - \kappa \tag{3.56}$$

we see that the horizon grows when $\kappa > 1 - \epsilon$. To avoid confusion with the comoving Hubble horizon $(aH)^{-1}$, which decreases during inflation regardless of the sound speed variation, from now on we will refer to the horizon for the scalar perturbations as the *sound horizon*. Due to the growing horizon for scalar perturbations the modes close to the top of the spike will re-enter the horizon and, when they eventually re-exit their amplitude shall be significantly damped.

Constant warp factor model

We now study the follwing two models with a *gaussian spike in a constant warp factor* selecting an exponential and quadratic potential

$$V = \frac{m^2}{2}\varphi^2, \qquad V = V_0 e^{\alpha \frac{\varphi}{\mathrm{M_P}}}, \qquad h = h_0\Big[1 + \delta \exp\Big(- \frac{(\varphi - \varphi_0)^2}{2\sigma^2 \mathrm{M_P}^2}\Big)\Big] \tag{3.57}$$

Since we fixed the potential and the warp factor we can evaluate the number of e-folds necessary to get across the spike. Close to φ_0 the system is in a strong DBI regime and we can approximate the warp factor as $h \simeq h_0 \delta \exp\big(- \frac{-(\varphi-\varphi_0)^2}{2\sigma^2 \mathrm{M_P}}\big)$ and, inserting it into (3.53) we can obtain analytically the number of e-folds necessary to cross the spike.

For the exponential potential we obtain:

$$\Delta N_{\text{spike}} = 2\sigma\sqrt{\frac{\pi V_0 h_0 \delta}{3}} \exp\Big(- 2\frac{\alpha\varphi_0}{\mathrm{M_P}} + 2\alpha^2\sigma^2\Big) \tag{3.58}$$

while for a quadratic potential:

$$\Delta N_{\text{spike}} = 2\sqrt{\frac{2}{3}\pi\delta h_0 V_0 \sigma\varphi_0} \tag{3.59}$$

where we used the fact that the contribution at the tail of the gaussian is negligible to evaluate the integral in the whole interval of φ.

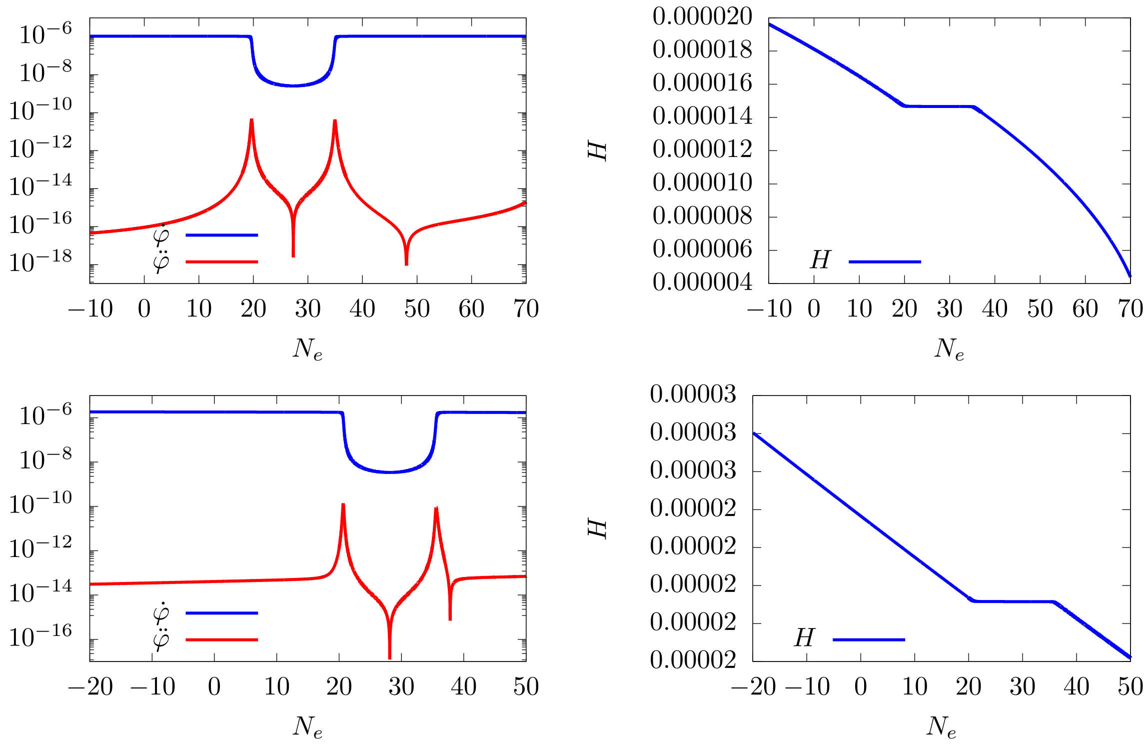

Figure 3.3: Comparison between the speed of the field and its acceleration on the left hand side and the Hubble parameter on the right side. On the top we plot the top we plot the quadratic potential case and on the bottom the exponential potential case

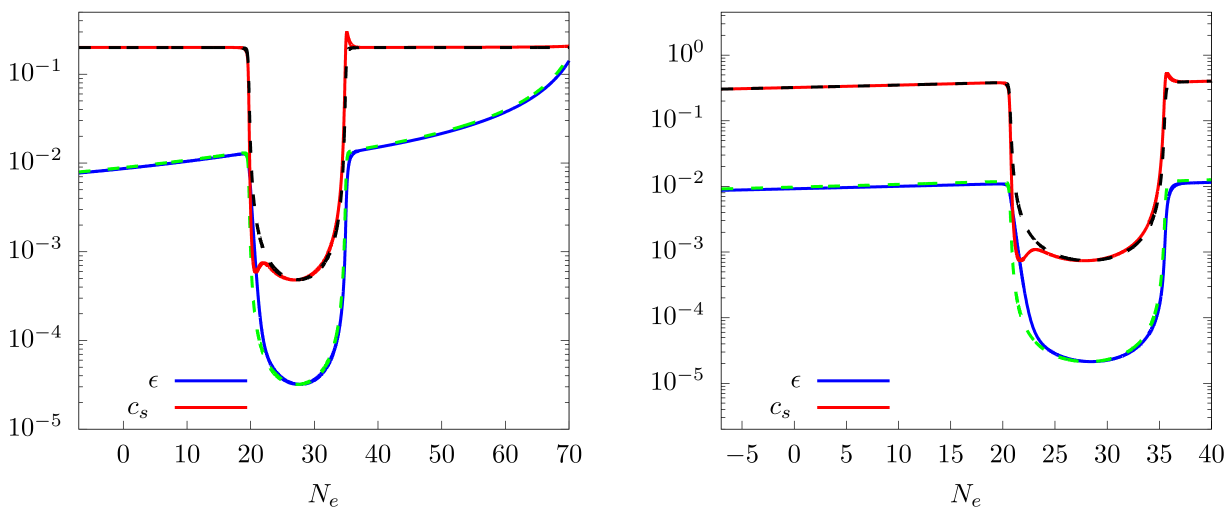

Figure 3.4: First slow-roll parameter and sound speed for perturbation compared with their analytical value (green and black dotted line respectively). On the right hand side we have the quadratic potential model while on the left the exponential potential. We can notice that at the beginning of the spike when the system enters a transient phase, the analytical value of the sound speed does not coincide with the numerical evaluation. Moreover we can see that with the exponential potential the analytical value of ϵ deviates from the numerical evaluation outside of the spike as there A is of the order of unity and we can not completely resort to the DBI regime approximation.

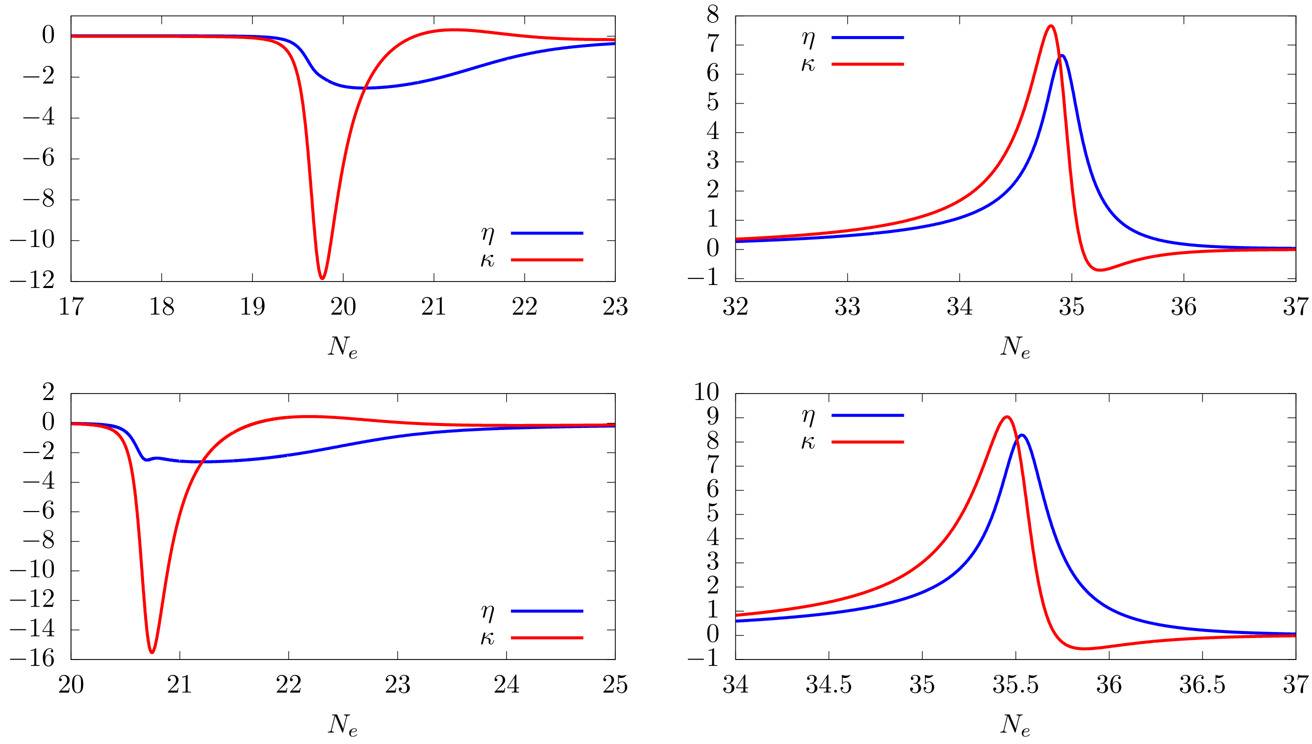

Figure 3.5: ϵ_2 and κ in the quadratic model (top) and exponential model (bottom) , on the left hand side they are plotted during the first initial spike and the exit from the spike. We notice that both of them are bigger than unity, interestingly during the exit from the spike κ is positive and we expect a significant amount of modes to re-enter.

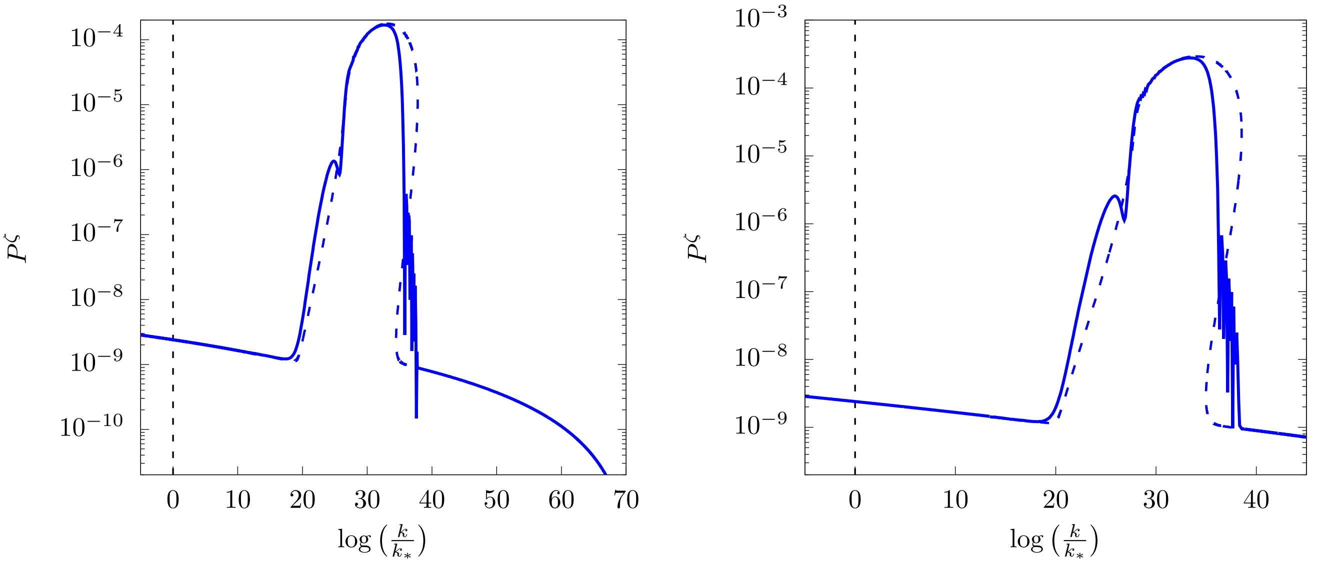

Figure 3.6: The power spectrum evaluated both numerically and using analytical approximation, plotted as a function of the momentum k setting the momentum of the modes exiting at CMB scale as k_*, on the right side we plot the quadratic model while on the left the exponential model

Imposing the COBE normalization and the almost flatness of the power spectrum at CMB scale setting $n_s = 0.965$ we select:

$$V_0 = 10^{-9} \qquad h_0 = 2.71 \cdot 10^{11} \qquad \alpha = 0.24 \tag{3.60}$$

for the exponential potential and:

$$V_0 = 4.264 \cdot 10^{-11} \qquad h_{\text{CMB}} = 8.44 \cdot 10^{13} \tag{3.61}$$

for the quadratic potential. We suppose that the spike is far enough from the horizon crossing of the CMB modes not to influence the dynamic of the system, thus we can evaluate the observables regardless of its height and narrowness. We obtain:

$$\frac{\mathrm{d}n_s}{\mathrm{d}\log k} = -0.0003 \qquad c_s = 0.32 \qquad r = 0.0047 \tag{3.62}$$

for the exponential potential and:

$$\frac{\mathrm{d}n_s}{\mathrm{d}\log k} = -0.0003 \qquad c_s = 0.20 \qquad r = 0.03 \tag{3.63}$$

for the quadratic potential. The height of the spike and its narrowness will set the maximum value of the power spectrum and its duration, we choose the position of the spike to be about 20 e-folds after the CMB horizon crossing, thus selecting $\varphi_0 = 0.4$ for the exponential potential and $\varphi_0 = 5.5$ for the quadratic potential. To reach the threshold for

σ	$\log_{10}(\delta)$	$\Delta N_{\text{analytical}}$	ΔN	$P_{\max}$
$2.51 \cdot 10^{-3}$	4.00	8.78	8.83	$9.25 \cdot 10^{-6}$
$1.58 \cdot 10^{-3}$	4.50	9.85	9.87	$2.92 \cdot 10^{-5}$
$1{\cdot}10^{-3}$	5.00	11.05	11.06	$9.25 \cdot 10^{-5}$
$6.31 \cdot 10^{-4}$	5.50	12.40	12.39	$2.92 \cdot 10^{-4}$
$3.98 \cdot 10^{-4}$	6.00	13.91	13.91	$9.25 \cdot 10^{-4}$

Table 3.2: Here we compare the duration of the spike evaluated numerically and using the equation (3.58) for an exponential model varying the height and narrowness of the spike

PBH formation we need an amplification of about 7 order of magnitude. We were not able to solve numerically the dynamics for the parameters leading to an amplification. However it has been possible to study the system with shorter spikes and observe that the analytical approximation coincide with the exact numerical results. It is thereby reasonable to extrapolate these numerical results to different choices of the parameters without numerically solving the Mukhanov equation.

σ	$\log_{10}(\delta)$	$\Delta N_{\text{analytical}}$	ΔN	P_{max}
$2.51 \cdot 10^{-3}$	4.00	11.99	11.99	$9.88 \cdot 10^{-6}$
$1.58 \cdot 10^{-3}$	4.50	13.46	13.45	$3.13 \cdot 10^{-5}$
$6.31 \cdot 10^{-4}$	5.25	12.70	12.70	$1.76 \cdot 10^{-4}$
$3.98 \cdot 10^{-4}$	5.75	14.25	14.25	$5.56 \cdot 10^{-4}$
$1.58 \cdot 10^{-4}$	6.25	10.09	10.09	$1.76 \cdot 10^{-3}$

Table 3.3: Here we compare the duration of the spike evaluated numerically and using the equation (3.59) for the quadratic model varying the height and narrowness of the spike

In the figures we have plotted as examples the power spectrum and the slow-roll parameters evaluated setting $\sigma = 6.31 \cdot 10^{-4}$ and $\delta = 10^{5.5}$ for the exponential potential, and $\sigma = 6.31 \cdot 10^{-4}$ and $\delta = 10^{5.25}$ for the quadratic potential. [1] Using equation (2.10) we can evaluate the mass of the PBH produced by the modes that exit the horizon while the field is on top of the spike. We find for the exponential potential $M = 10^{-14} M_{\odot}$ and for quadratic potential $M = 10^{-13} M_{\odot}$.

Modes freezing on super horizon scales

During the exit from the spike one should expect $\kappa > 1$ and, as mentioned by Romano [35] and, such a background evolution should lead to a growth of the modes on super horizon scale, adding thereby a second amplification mechanism. In this model we can expect this feature to be still present as, by evaluating the coefficient of amplification (2.21) we observe a negative value of the friction term in the perturbation equation. Looking at the numerically evaluated power spectrum in figure 3.6 though, we notice that, at the top of the spike, the numerical estimate and the slow-roll analytical approximation of P^{ζ} do coincide. Moreover, evaluating the time evolution of the different modes we observes that the modes that exit the horizon when the field is on top of the spike do freeze on superhorizon scale. Modes on smaller scales exit the horizon and they initially freeze, during this initial frozen phase the system enters in a phase where $\frac{z'}{z}$ is negative and thereby we should expect an amplification on super horizon scale. Importantly though, since during the evolution the $(c_s k)^2$ term in the Mukhanov-Sasaki equation grows and becomes non-negligible, we can't completely ignore the sound speed component in the perturbation equation and, thus the superhorizon approximation does not hold. A significant amount of the modes will actually re-enter the sound horizon, so that their amplitude get significantly damped as one can see

[1] One could notice that the duration of the spike in the two cases plotted is slightly larger than expected using the analytical result. The choice of a purely gaussian spike (3.57) turns out to be excessively demanding to the computation and we had to resort to a slight modification in the numerical simulations setting $h = h_0 \delta \exp\left(-\frac{y_0^2}{2\sigma^2}\frac{(x-x_0)^2}{y_0^2+(x-x_0)^2}\right)$ with $y_0 = \sqrt{2\log(\delta)}\sigma$. This model deviates slightly from the gaussian case and the number of e-folds necessary to cross the spike is slightly by a few e-folds

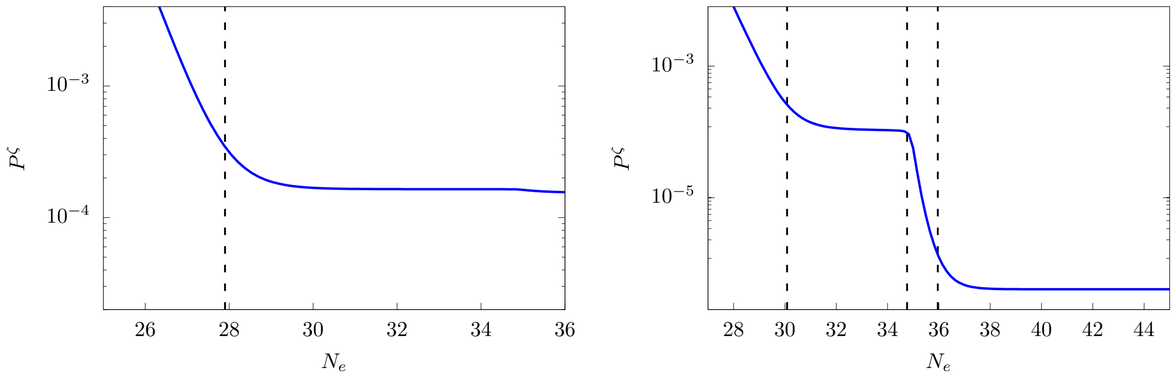

Figure 3.7: Time evolution of the modes exiting at $N = 30$ and $N = 34.9$ in the quadratic potential model, in both cases the exit from the horizon is highlighted by a black dotted line, in the figure on the right also the re-entry of the mode is highlighted. We notice that both the modes freeze on superhorizon scale, the first one does not amplify during the exit phase while the second mode re-enters and its amplitude gets significantly damped

in figure 3.7.

We note that the amplification of the modes on superhorizon scales, in single field models they require a negative friction term on a superhorizon scale which could arise due to a significant growth of the sound speed, or a fast decrease of the slow-roll parameter ϵ. Relying only on the sound speed variation though we'd require $\kappa > 1$ which, in turn, imposes that the sound horizon is not monotonically decreasing and, thereby a significant number of modes which should have undergone an amplification on super horizon scale actually re-enter the horizon. Deeper studies should be brought up to obtain an actual threshold for the maximum κ and the necessary duration of this phase which allows the superhorizon amplification both in the case where the negative value of the friction term is solely due to κ or cases where $\epsilon_2 < 0$ and $\kappa > 0$ which can arise by introducing a near inflection point in the potential during a strongly non canonical regime. In any case this effects are subdominant and, as shown in the figures still the analytical approximations employed are quite accurate and lead to a quite affordable prediction on the amplification of the power spectrum.

Conclusions

In this thesis we studied some possibel mechanism to generate primordial Black Holes in a non-canonical model of inflation.

In the first chapter we offered an introduction to standard cosmology, describing the shortcomings of the Hot Big Bang Model. We then studied the model of inflation as a viable solution to these problems, focusing our attention on single field models. Here we studied the background dynamics of the inflaton field both in a canonical and a non-canonical context, focusing in particular on the latter case. We've shown that is is possible to achieve an accelerated phase both described by a slow-roll evolution where the energy density is dominated by the potential, and through a kinetically dominated energy density. Subsequently we considered the scalar perturbation, studying their generation and evolution through the Mukhanov-Sasaki equation. Using the quantization formalism in a quasi de Sitter spacetime we introduced the power spectrum P^{ζ} and the spectral tilt n_s. The current observational constraints by PLANCK offer stringent bounds on these observables at CMB scales ($k_{\text{CMB}} \simeq 0.05\text{Mpc}^{-1}$). Lastly we quickly reviewed the non-gaussian features of the perturbations discussing how the perturbations may become significantly non-gaussian for a small small value of the sound speed c_s.

In the second chapter we discuss Primordial Black Holes as viable candidates for Dark Matter. We reviewed their formation mechanism due to the gravitational collapse and their abundance calculated by assuming a gaussian probability distribution of primordial density fluctuations, the collapse occurs when overdensities re-enter the horizon in a radiation dominated phase. We then shown that Primordial Black Hole dark matter requires an amplification of the power spectrum up to 10^{-2} on PBH scales. Lastly we discussed the current experimental constraints on PBH abundance and the Ultra slow-roll mechanism in single field canonical inflation.

In the third chapter we studied some possible amplification mechanisms in non-canonical inflation. First we studied a polynomial non canonical model $p = -X + LX^2 + V$, we selected a model which starts in a slow-roll regime at CMB scale. Later during inflation it will abandon the slow-roll trajectory and approach the k-inflation attractor. During this transient phase the sound speed and the slow roll parameter ϵ decrease exponentially, and we have shown that during this transient phase the power spectrum can be evaluated analytically and gets enhanced. We also considered a DBI toy model where we introduced a spike in the warp factor. We considered a gaussian spike and showed that it determines a transition for the system from a canonical evolution to a strongly non canonical one. During the transient phase approaching the strongly non-canonical phase both the sound speed and the slow-roll parameter ϵ decrease quickly. This leads to a steep growth of the power spectrum. It was also observed that after this phase the sound speed undergoes a steep growth, which should lead, in principle, to an amplification of the modes on super-horizon scale which was not observed. However showed that the steep growth of the sound speed causes the sound horizon to grow, thus the modes which should have undergone an amplification on superhorizon scale re-enter the horizon and their amplitude is significantly damped.

It should be noted that the first toy model does not include a mechanism describing a graceful exit. Moreover that significant non-gaussian features could be generated during the enhancement of the power spectrum and this may alter the estimates regarding the amount of amplification necessary to produce a certain abundance of Primordial Black Holes. Still the study of non-gaussianities is very complicated and goes beyond the scope of this thesis.

Bibliography

[1] M. Alishahiha, E. Silverstein, and D. Tong, “DBI in the Sky”, Physical Review D - Particles, Fields, Gravitation and Cosmology **70**, 15 (2004).

[2] K. Ando, K. Inomata, and M. Kawasaki, “Primordial black holes and uncertainties in the choice of the window function”, Physical Review D **97**, `10.1103/PhysRevD.97.103528` (2018).

[3] C. Armendáriz-Picón, T. Damour, and V. Mukhanov, “K-Inflation”, Physics Letters, Section B: Nuclear, Elementary Particle and High-Energy Physics **458**, 209–218 (1999).

[4] G. Ballesteros, J. Rey, M. Taoso, and A. Urbano, “Primordial black holes as dark matter and gravitational waves from single-field polynomial inflation”, Journal of Cosmology and Astroparticle Physics **2020**, 1–28 (2020).

[5] N. Bartolo, E. Komatsu, S. Matarrese, and A. Riotto, “Non-Gaussianity from Inflation: Theory and Observations”, Physics Reports **402**, 103–266 (2004).

[6] R. Bean, D. J. Chung, and G. Geshnizjani, “Reconstructing a general inflationary action”, Physical Review D - Particles, Fields, Gravitation and Cosmology **78**, `10.1103/PhysRevD.78.023517` (2008).

[7] T. S. Bunch and P. Davies, “Quantum field theory in de Sitter space: renormalization by point-splitting”, Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences **360**, 117–134 (1978).

[8] B. J. Carr, “The primordial black hole mass spectrum”, The Astrophysical Journal **201**, 1 (1975).

[9] B. J. Carr and S. W. Hawking, “Black Holes in the Early Universe”, Monthly Notices of the Royal Astronomical Society **168**, 399–415 (1974).

[10] B. J. Carr, K. Kohri, Y. Sendouda, and J. Yokoyama, “New cosmological constraints on primordial black holes”, Physical Review D - Particles, Fields, Gravitation and Cosmology **81**, 10.1103/PhysRevD.81.104019 (2009).

[11] B. Carr and F. Kuhnel, “Primordial black holes as dark matter: Recent developments”, Annual Review of Nuclear and Particle Science **70**, 355–394 (2020).

[12] X. Chen, M.-x. Huang, S. Kachru, and G. Shiu, “Observational Signatures and Non-Gaussianities of General Single Field Inflation”, Journal of Cosmology and Astroparticle Physics, 10.1088/1475-7516/2007/01/002 (2006).

[13] M. Cicoli, V. A. Diaz, and F. G. Pedro, “Primordial black holes from string inflation”, en, Journal of Cosmology and Astroparticle Physics **2018**, 034–034 (2018).

[14] P. Creminelli and M. Zaldarriaga, “Single field consistency relation for the 3-point function”, Journal of Cosmology and Astroparticle Physics, 101–107 (2004).

[15] K. Dimopoulos, “Ultra slow-roll inflation demystified”, Physics Letters, Section B: Nuclear, Elementary Particle and High-Energy Physics **775**, 262–265 (2017).

[16] P. Franche, R. Gwyn, B. Underwood, and A. Wissanji, “Attractive Lagrangians for noncanonical inflation”, Physical Review D - Particles, Fields, Gravitation and Cosmology **81**, 1–39 (2010).

[17] G. Franciolini, A. Kehagias, S. Matarrese, and A. Riotto, “Primordial black holes from inflation and non-Gaussianity”, en, Journal of Cosmology and Astroparticle Physics **2018**, 016–016 (2018).

[18] K. Freese, “Review of Observational Evidence for Dark Matter in the Universe and in upcoming searches for Dark Stars”, 10.1051/eas/0936016 (2008).

[19] J. García-Bellido and E. Ruiz Morales, “Primordial black holes from single field models of inflation”, en, Physics of the Dark Universe **18**, 47–54 (2017).

[20] J. Garriga and V. F. Mukhanov, “Perturbations in k-inflation”, Physics Letters, Section B: Nuclear, Elementary Particle and High-Energy Physics **458**, 219–225 (1999).

[21] D. S. Gorbunov and V. A. Rubakov, *Introduction to the Theory of the Early Universe* (World Scientific Publishing Company, Singapore ; Hackensack, N.J, Feb. 2011), p. 489.

[22] T. Harada, C.-M. Yoo, and K. Kohri, “Threshold of primordial black hole formation”, Physical Review D **88**, 084051 (2013).

[23] S. Hotchkiss, A. Mazumdar, and S. Nadathur, “Inflection point inflation: WMAP constraints and a solution to the fine-tuning problem”, Journal of Cosmology and Astroparticle Physics **2011**, 10.1088/1475-7516/2011/06/002 (2011).

[24] S. Kachru, R. Kallosh, A. Linde, J. Maldacena, L. McAllister, and S. P. Trivedi, “Towards Inflation in String Theory”, 10.1088/1475-7516/2003/10/013 (2003).

[25] A. Y. Kamenshchik, A. Tronconi, T. Vardanyan, and G. Venturi, “Non-canonical inflation and primordial black holes production”, en, Physics Letters B **791**, 201–205 (2019).

[26] M. Kawasaki, A. Kusenko, Y. Tada, and T. T. Yanagida, “PBH Dark Matter in Supergravity Inflation Models”, 10.1103/PhysRevD.94.083523 (2016).

[27] A. R. Liddle and S. M. Leach, “How long before the end of inflation were observable perturbations produced?”, Physical Review D - Particles, Fields, Gravitation and Cosmology **68**, 10.1103/PhysRevD.68.103503 (2003).

[28] J. Maldacena, “Non-gaussian features of primordial fluctuations in single field inflationary models”, en, Journal of High Energy Physics **2003**, 013–013 (2003).

[29] H. Motohashi and W. Hu, “Primordial black holes and slow-roll violation”, Physical Review D **96**, 063503 (2017).

[30] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, *Theory of cosmological perturbations*, June 1992.

[31] V. Mukhanov and S. Winitzki, *Introduction to Quantum Effects in Gravity* (Cambridge University Press, Cambridge, 2007), p. 273.

[32] O. Özsoy, S. Parameswaran, G. Tasinato, and I. Zavala, “Mechanisms for primordial black hole production in string theory”, Journal of Cosmology and Astroparticle Physics **2018**, 10.1088/1475-7516/2018/07/005 (2018).

[33] Planck Collaboration et al., “Planck 2015 results. XVII. Constraints on primordial non-Gaussianity”, Astronomy and Astrophysics **594**, 32 (2015).

[34] Planck Collaboration et al., “Planck 2018 results. X. Constraints on inflation”, Astronomy and Astrophysics **641**, 61 (2018).

[35] A. E. Romano, “Sound speed induced production of primordial black holes”, en, arXiv:2006.07321 [astro-ph] (2020).

[36] R. K. Sachs and A. M. Wolfe, “Perturbations of a Cosmological Model and Angular Variations of the Microwave Background”, The Astrophysical Journal **147**, 73 (1967).

[37] D. S. Salopek and J. R. Bond, “Nonlinear evolution of long-wavelength metric fluctuations in inflationary models”, Physical Review D **42**, 3936–3962 (1990).

[38] Q. Shafi and V. N. Senoguz, “Coleman-Weinberg Potential In Good Agreement With WMAP”, `10.1103/PhysRevD.73.127301` (2006).

[39] M. Shibata and M. Sasaki, “Black hole formation in the Friedmann universe: Formulation and computation in numerical relativity”, Physical Review D **60**, 084002 (1999).

[40] A. A. Starobinskii, Starobinskii, and A. A., “Spectrum of relict gravitational radiation and the early state of the universe”, ZhPmR **30**, 719–723 (1979).

[41] A. A. Starobinsky, “A new type of isotropic cosmological models without singularity”, Physics Letters B **91**, 99–102 (1980).

[42] B. Underwood, “Brane Inflation is Attractive”, Physical Review D - Particles, Fields, Gravitation and Cosmology **78**, `10.1103/PhysRevD.78.023509` (2008).

[43] Y. Wang, “Inflation, Cosmic Perturbations and Non-Gaussianities”, Communications in Theoretical Physics **62**, 109–166 (2014).

[44] S. Weinberg, *Gravitatation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley India Pvt. Limited, 2008).