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SYSTEM OF IMPRIMITIVITY AND MACKEY'S THEOREM

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Introduction

The aim of this thesis is to study and classify the representations of the *Poincaré group* $\mathcal{P}$, whose elements are all the isometries of the Minkowski Spacetime. These representations are important in particle physics since they can be considered the mathematical equivalent of the elementary particles of the Standard Model, hence it is possible to identify and classify all such particles through the analysis of the Poincaré representations. In order to achieve this we introduce the fundamentals of the *system of imprimitivity.* This concept was developed by George Mackey in an effort to study in general terms the theory of induced representation of locally compact groups and it has been widely applied in physics. In particular, through the *Mackey's Theorem*, those results are able to give us information on the representations of the semidirect product of two groups $G = A \rtimes H$ starting from the representations of the groups A and H themselves.
We start this dissertation with an introduction about groups and their actions, as it is necessary to correctly define a group representation and a semidirect product of groups, followed by a brief introduction of short sequences and their role in the description of those groups. In the second chapter we define smooth manifolds and tangent spaces, with smooth functions on manifolds and their differentials. These concepts are necessary to build Lie groups and Lie algebras: they are essential to the study of the representations of the Poincaré group not only because it is a Lie group itself, but because the matrix groups with whom we work are Lie groups, and their representations can be easier derived through the study of their Lie algebra. Then the third chapter is dedicated to the computations of the representations of the Lie group $SL(2,\mathbb{C})$ and its Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, which is key

to obtain all the necessary representations of the Restricted Lorentz group $SO(1,3)^0$, one of the main constituents of the Poincaré group. In the fourth and final chapter we describe first the Poincaré group and then we state Mackey's Theorem, giving an introduction of characters and their action. We conclude with a brief explanation of the theorem, its connection with the previous topics and how to obtain the Poincaré group representations.

Chapter 1

Preliminaries on Groups

In this chapter we introduce the preliminary definitions about groups, their morphisms and their action on sets, giving examples relevant to the topics treated in the following chapters. Specifically, we are going to define the semidirect product of groups, briefly talking about its properties and how these are connected to the concept of exact sequences of groups.

1.1 Groups and Morphisms

First we give the definition of group, which needs the basic idea of a binary operation on a set. After an essential introduction of subgroups, normality and direct product of groups (this, in particular, is necessary for future comparisons) we shall go to the description of the structure-preserving functions, or homomorphisms, and their properties.

Definition 1.1.1. Given a set S, we call *binary operation on* S a function $\mu : S \times S \longrightarrow S$ and we denote it with $\mu(a, b) = a * b$.
A *group* is a pair $(G, *)$, where G is a nonempty set and $*$ is a binary operation on it such that:

1. $*$ is associative: $(a * b) * c = a * (b * c)$ for every $a, b, c \in G$;

2. There exists $e \in G$ *identity element* such that $e * a = a * e = a$ for every $a \in G$;

3. For every $a \in G$ there exists an *inverse element* a^{-1} such that $a * a^{-1} = a^{-1} * a = e$.

A group $(G, *)$ is said to be *abelian* if $a * b = b * a$ for every $a, b \in G$.

Definition 1.1.2. Given a group $(G, *)$ and a nonempty subset $S \subset G$ closed under $*$, if $(S, *)$ is a group we say that S is a *subgroup of* G. This is denoted by $S < G$.

Observation 1.1.3. Every group $(G, *)$ has a *trivial subgroup* $\{e\}$, containing only the identity element $e \in G$. We denote this subgroup $\{0\}$ or simply 0 if the additive notation $+$ is used for the binary operation.

We now give the notion of normal subgroup, which is especially important when talking about homomorphisms and quotients.

Definition 1.1.4. Let $(G, *)$ be a group with $N < G$ a subgroup. We say that N is a *normal subgroup of* G, written $N \lhd G$, if $a * n * a^{-1} \in N$ for every $a \in G$ and all $n \in N$. Every subgroup of an abelian group is normal.

Given two or more groups, it is possible to construct new ones through the direct product. It consists of the cartesian product of the underlying sets with a component-wise binary operation.

Definition 1.1.5. Given a finite collection of n groups $(G_i, *_i)_{i=1,\dots,n}$, it is possible to define a new group $G = \bigotimes_{i=1,\dots,n} G_i$ whose set is $G = \prod_{i=1,\dots,n} G_i$ and the operation $*$ is defined component-wise on the $*_i$:

$$(a_1, ..., a_n) * (b_1, ..., b_n) = (a_1 *_1 b_1, ..., a_n *_n b_n).$$

This is a group, since it inherits the associativity of the operation $*$ from the original operations $*_i$. The identity element e of G is $(e_1, ..., e_n)$ and the inverse of an element $(a_1, ..., a_n) \in G$ is $(a_1^{-1}, ..., a_n^{-1})$.
G is abelian if and only if all the $(G_i, *_i)_{i=1,\dots,n}$ are abelian.

Now we are interested in functions that preserve the group structure:

Definition 1.1.6. Let $(G, *)$ and $(H, \star)$ be groups. A function $f : G \longrightarrow H$ is said to be a *group homomorphism* if $f(a * b) = f(a) \star f(b)$ for every $a, b \in G$.
In particular, if f is a bijective function, it is called an *isomorphism*, while groups G, H are said to be *isomorphic* and we write $G \cong H$. Moreover, if $G = H$ the homomorphism f is called *endomorphism*, while a bijective endomorphism is called *automorphism*.
We define the image and kernel of f as follows:

$$Im(f) = \{h \in H \mid h = f(g) \text{ for some } g \in G\}$$

$$Ker(f) = \{g \in H \mid f(g) = e_H\}$$

Notation. For the sake of brevity we sometime refer to a homomorphism f only as a *morphism*. We also denote respectively with $End(G)$ and $Aut(G)$ the endomorphisms and automorphisms of G. Notice that $Aut(G)$ is a group.

Observation 1.1.7. A simple consequence of the above definition is that for a morphism of groups $f : (G, *) \longrightarrow (H, \star)$ we have $f(e_G) = e_H$ and $f(a^{-1}) = (f(a))^{-1}$. From this follows easily that $Im(f)$ is a subgroup of H and $Ker(f)$ is a normal subgroup of G. For a detailed proof of these claims, see [6] chapter 1, section 2.

The following example introduce groups that will be used further in this thesis for the definition of group representation and the analysis of the Poincaré group.

Notation. For brevity, we use $\mathbb{K}$ from now on for $\mathbb{R}$ or $\mathbb{C}$.

Example 1.1.8. One of the most important groups is the *general linear group*. Given a $\mathbb{K}$-vector space V of $dim = n$, the group $Aut(V)$ of linear maps from V to V is isomorphic to the group of all $n \times n$ invertible matrices with coefficient in the field $\mathbb{K}$, together with matrix multiplication. Such group is denoted $GL(n, \mathbb{K})$.
The subset of all such matrices with determinant 1 is also a group, called the *special linear group*, and we denote it with $SL(n, \mathbb{K})$. It is a normal subgroup of $GL(n, \mathbb{K})$.

Suppose now that such vector space V is equipped with a positive defined symmetric (for $\mathbb{K} = \mathbb{R}$) or hermitian (for $\mathbb{K} = \mathbb{C}$) bilinear form $\langle \cdot\, , \cdot \rangle$. Let $U \in Aut(V)$ be a linear map preserving $\langle \cdot\, , \cdot \rangle$:

$$\langle Ux, Uy \rangle = \langle x, y \rangle \text{ for every } x, y \in V.$$

U is said to be *unitary*. Once we fix an orthonormal basis we call the matrix associated with U also unitary. The set of all unitary matrices, denoted $O(n)$ for $\mathbb{K} = \mathbb{R}$ or $U(n)$ for $\mathbb{K} = \mathbb{C}$, is a subgroup of $GL(n, \mathbb{K})$. In particular, $O(n)$ is called *orthogonal group*. The subset of all unitary matrices with determinant 1 is called *special orthogonal* or *special unitary* and denoted $SO(n)$ or $SU(n)$ depending on the field $\mathbb{K}$. It is a subgroup of $SL(n, \mathbb{K})$ and a normal subgroup of its respective orthogonal or unitary matrix group. All these groups are obviously non abelian since matrix multiplication (or function composition) is not commutative.
For the properties of unitary matrices here used and for an accurate insight of the relation between groups, vector spaces and matrices, see chapters 5,7 and 8 of [7].

1.2 Group Action

In order to understand the concept of group representation and, later in this chapter, of semidirect product, we need to introduce the important notion of group action. First of all, we give the basic definition of action of a group on a set. Starting from this, we derive an equivalent concept of group action that can be better generalized for our purposes.

Definition 1.2.1. An *action of a group* $(G, *)$ *on a set* S is a function $G \times S \longrightarrow S$, denoted $(g, x) \longmapsto gx$ for a generic $g \in G, x \in S$, such that:

- $(e, x) \mapsto ex = x$ for every $x \in S$;
- $(g_1, (g_2, x)) \mapsto g_1(g_2x) = (g_1 * g_2)x$ for every $g_1, g_2 \in G, x \in S$.

The action is said to be *transitive* if, for each pair $x, y \in S$, there exists a $g \in G$ such that $y = gx$, while it is said to be *faithful* if, for each $g \in G\,, g \neq e\,$, there exists an $x \in S$ such that $gx \neq x$.
We call S a *G-set* and we say that a map $f : S \longrightarrow Z$ between G-sets S and Z is a *morphism of G-sets* if it preserves the action of G, that is:

$$f(gs) = gf(s) \quad \text{for every } g \in G, s \in S.$$

Observation 1.2.2. The concept of action in definition 1.2.1 can be expressed equivalently as a function $\sigma : G \longrightarrow Aut(S)$. Such σ is a homomorphism, and also the converse is true: given a homomorphism σ as before, it identifies a unique action $G \times S \longrightarrow S$.
In fact, suppose that we have the action

$$G \times S \longrightarrow\ S\ ,\ (g, x) \longmapsto gx.$$

Define $\sigma\ :\ G\ \longrightarrow\ Aut(S)\ ,\ g\ \longmapsto\ \sigma(g)\,$, as $\sigma(g)(x)\ =\ gx\,$. We immediately verify it is a group homomorphism:

$$\sigma(g_1 * g_2)(x) = (g_1 * g_2)x = g_1(g_2 x) = \sigma(g_1)(\sigma(g_2)(x)) = (\sigma(g_1) \circ \sigma(g_2))(x)$$

because of definition of group action. Hence, σ is a homomorphism.
Suppose now that we have a homomorphism

$$\sigma : G \longrightarrow Aut(S)\ , g \longmapsto \sigma(g).$$

Define $G \times S \longrightarrow\ S\ ,\ (g, x)\ \mapsto \sigma(g)(x)\,$. It is a group action since, following observation 1.1.7, we have:

$$(e, x) \longmapsto \sigma(e)(x) = Id(x) = x\,.$$

Moreover

$$(g_1(g_2, x)) \longmapsto \sigma(g_1)(\sigma(g_2)(x)) = \sigma(g_1 * g_2)(x) = (g_1 * g_2)(x)\,,$$

because σ is a homomorphism. Hence, σ induces a group action.

Definition 1.2.3. We define any morphism $\sigma : G \longrightarrow Aut(S)$ a *representation of* G into S. From our previous observation we have that an action of G on S is equivalent to a representation of G in S.

Observation 1.2.4. It is clear that an action is faithful if and only if $Ker(\sigma) = \{e\}$, that is, the kernel of its representation is trivial. Indeed, if the action is faithful, there could not be any $g \neq e \in G$ such that $\sigma(g) = Id_S$, otherwise

$$(g, x) \mapsto gx = \sigma(g)(x) = Id_S(x) = x \quad \text{for every } x \in S.$$

Differently, with a non-faithful action, there would be an element $g \in G$ such that $gx = x$ for every $x \in S$. Thus $\sigma(g) = Id_S$ and the kernel of the representation is not trivial.

Example 1.2.5. Consider the group $(\mathbb{R}^n, +)$ acting on itself through the homomorphism (additive notation) $\tau : \mathbb{R}^n \longrightarrow Aut(\mathbb{R}^n)\,, \;\; \tau(a)(x) = x + a$. Such τ is clearly an homomorphism, while every function $\tau(a)$ is a *translation* of $\mathbb{R}^n$, which is a bijection (but not an automorphism) of the group.

Example 1.2.6. Given a group $(G, *)$, it acts on itself by left multiplication: $G \times G \longrightarrow G\ ,\ (g, h) \mapsto g * h$. Moreover, given a subgroup H of G, the left multiplication induces an action of G on the sets of left cosets of H, or G/H: $G \times G/H \longrightarrow G/H\ , (g, xH) = (g * x)H$. It is easy to see that both these maps are actions. Therefore, both G and G/H are G-sets.
For a complete introduction of cosets and quotient groups, see chapter 1, section 4-5 of [4].

As a consequence of the previous result, we can extend the group action definition to a generic algebraic structure as a homomorphism between the acting group and the group of automorphisms of the structure, that is, the group of structure-preserving bijections. Despite that, for the aim of this dissertation we are restricting to the case of group acting on other groups. Note that the preceding definition of group acting on a set can fit perfectly this generalization, since there is no binary operation naturally defined on a set, hence the automorphisms of the structure are simply bijections.

Observation 1.2.7. Given two groups $(H, \star), (G, *)$ and a representation of H into G, $\sigma : H \longrightarrow Aut(G)$, we have that H *is acting on* G (through σ) with the action $(h, g) \longmapsto \sigma(h)(g)$.

Example 1.2.8. Let H be a subgroup of $(G, *)$. We define the action of H on G by *conjugation* $C : H \times G \longrightarrow G$, $(h, g) \mapsto h * g * h^{-1}$, or equivalently by the representation $\sigma : H \longrightarrow Aut(G)$, where $\sigma(h)(g) = h * g * h^{-1}$. It is easy to see that such σ is a homomorphism, thus proving that also the action is well defined. In fact:

$$\sigma(h_1 * h_2)(g) = (h_1 * h_2) * g * (h_1 * h_2)^{-1} = (h_1 * h_2) * g * (h_2^{-1} * h_1^{-1}) =$$
$$= \sigma(h_1)(\sigma(h_2)(g)) \qquad \text{for every } g \in G, h_1, h_2 \in H.$$

Moreover, $\sigma(h)$ is an automorphism of G:

$$\sigma(h)(g_1 * g_2) = h * g_1 * g_2 * h^{-1} = h * g * h^{-1} * h * g_2 * h^{-1} =$$
$$= \sigma(h)(g_1) * \sigma(h)(g_2).$$

We say that $h * g * h^{-1}$ is the *conjugate* of g.

Observation 1.2.9. We can define, for H normal subgroup of G, another important action: define $\sigma : G \longrightarrow Aut(H)$, $\sigma(g)(h) = g * h * g^{-1}$. Clearly this action is well defined since H is normal in G.

Going back to the basic definition of group action on a set, we want to show a general result necessary for the introduction of two core concepts for the the last chapter, orbit and stabilizer.

Proposition 1.2.10. *Let G be a group acting on a set S through $G \times S \longrightarrow S$, $(g, x) \mapsto gx$. Then:*

- *The relation on S defined by*

$$x_1 \sim x_2 \Leftrightarrow \exists\, g \in G \text{ such that } gx_1 = x_2$$

  is an equivalence relation.

- *For each $x \in S$, $G_x = \{g \in G \mid gx = x\}$ is a subgroup of G.*

Proof. For the first part, let us show that $\sim$ is an equivalence relation: Reflexive property: consider $g = e_G \in G$, then $e_G x_1 = x_1 \Rightarrow x_1 \sim x_1$.
Symmetric property: let $x_1 \sim x_2$ through g. Consider then g^{-1} :

$$g^{-1}x_2 = g^{-1}(gx_1) = (g^{-1} * g)x_1 = e_G x_1 = x_1 \Rightarrow x_2 \sim x_1 \,.$$

Transitive property: let $x_1 \sim x_2$ and $x_2 \sim x_3$ through g and h, respectively. Then we have:

$$(h * g)x_1 = h(gx_1) = h(x_2) = x_3 \Rightarrow x_1 \sim x_3 \,.$$

For the second part, we will show that G_x is closed for $*$: if $g, h \in G$, then

$$(g * h)x = g(hx) = gx = x \Rightarrow (g * h) \in G_x.$$

Additionally, $e_G \in G_x$ and for every $g \in G$, $g^{-1} \in G$. Indeed, $e_G x = x$ and

$$g^{-1}x = g^{-1}(gx) = (g^{-1} * g)x = e_G x = x.$$

Thus, G_x is a subgroup of G. □

Definition 1.2.11. Consider a group $(G, *)$ acting on a set S, according to definition 1.2.1. The equivalence classes of the previous theorem are called the *orbits of* G on S, while the subgroup G_x is called the *isotropy group of* x or the *stabilizer of* x. Namely $Ox = \{ z \in S \mid \exists g \in G \;\; z = gx \}$ is the orbit of x and $G_x = \{ g \in G \mid gx = g \}$ is its stabilizer.

Observation 1.2.12. An action $G \times S \longrightarrow S$ is transitive if and only if there is only one orbit in S. Indeed, if the action is transitive, for any given $x \in S$ we have $Ox = S$, as $Ox \subset S$ and for every $z \in S$ there exists a $g \in G$ such that $z = gx$. Hence $z \in Ox$ and $S \subset Ox$. Every orbit is S, or equivalently there is only one orbit. On the contrary, if there is only one orbit, for every pair $x, z \in S$ there exists $g \in G$ such that $z = gx$, thus the action is transitive.

Observation 1.2.13. Let $(G, *)$ be a group acting transitively on a set S. Then for every $x \in S$ the stabilizers G_x are conjugate, that is, for every $x_1, x_2 \in S$ there exists a $g \in G$ such that $G_{x_2} = g * G_{x_1} * g^{-1}$. Indeed, if the action is transitive there exists a $g \in G$ such that $g^{-1}x_2 = x_1$, so for every $h \in G_{x_1}$ we have

$$(g * h * g^{-1})x_2 = (g * h)(g^{-1}x_2) = (g * h)(x_1) = g(hx_1) = gx_1 = x_2.$$

Hence $g * h * g^{-1} \in G_{x_2}$. This procedure can be simmetrically repeated for G_{x_2}, obtaining that for every $h' \in G_{x_2}$ there exists a $g' \in G$ such that $h'' = g'h'g'^{-1} \in G_{x_1}$, which means that $h' = g'^{-1}h''g'$. Thus every element of G_{x_2} is the conjugate of an element of G_{x_1}.
In particular it is clear that, for a generic non-transitive action, the stabilizers of the elements of the same orbit are conjugate.

The following proposition shows a fundamental relation between orbits and stabilizers.

Proposition 1.2.14. *Let $(G, *)$ be a group acting on a set S, $(g, s) \mapsto gs$, Os the orbit of $s \in S$. Then there exists a bijective morphism of G-sets between G/G_s and Os, or equivalently $G/G_s \cong Os$.*

Proof. The action of G on G/G_s is the left multiplication, as seen in example 1.2.6, while Os, being a subset of S, inherits the action of G on S.
Define $f : G/G_s \longrightarrow Os$, $f(gG_s) = gs$. Such function is well defined: considering $h \in gG_s$, we have $hG_s = gG_s$ and $h = g * x$ for a given $x \in G_s$. So we have

$$f(hG_s) = f((g * x)G_s) = (g * x)s = g(xs) = gs = f(gG_s),$$

by definition of stabilizer and group action. f is clearly surjective, since for every $z \in Os$, $z = f(zG_s)$. Now we show that f is injective: consider $g, h \in G$ such that $f(gG_s) = f(hG_s)$. Then we have $gs = hs$, thus g and h have the same action, which means $gG_s = hG_s$. Hence, f is a bijection. Last, we show that f is a morphism of G-sets:

$$f(g(hG_s)) = f((g * h)G_s) = (g * h)s = g(hs) = gf(hG_s).$$

$\square$

1.3 Semidirect Product

In this section we exhibit a new way of building groups through the semidirect product of existing groups, a generalization, involving group action, of the direct product previously defined in 1.1.5. Such operation is the key to define the Euclidean group and the Poincaré group, that will be the main focus of this thesis.

Observation 1.3.1. Given a representation $\sigma : H \longrightarrow Aut(G)$ between groups $(G, *)$ and $(H, \star)$, we can describe a new operation on G for every $h \in H \, : \, (g_1, g_2) \longmapsto g_1 * \sigma(h)(g_2)\,.$

Proposition 1.3.2. *Given two groups $(G, *), (H, \star)$ and a representation $\sigma : H \longrightarrow Aut(G)$, consider the set $G \times H$ with the operation induced by σ as in the preceding observation : $(g_1, h_1)(g_2, h_2) = (g_1 * \sigma(h_1)(g_2), h_1 \star h_2)$. Then $G \times H$ with this operation is a group with identity element (e_G, e_H) and inverse $(g, h) \mapsto (\sigma(h)^{-1}(g^{-1}), h^{-1}) = (\sigma(h^{-1})(g^{-1}), h^{-1})$.*

Proof. Let us verify that such structure is effectively a group. Associativity:

$$(g_1, h_1)[(g_2, h_2)(g_3, h_3)] = (g_1, h_1)(g_2 * \sigma(h_2)(g_3), h_2 \star h_3) =$$

$$= (g_1 * \sigma(h_1)(g_2 * \sigma(h_2)(g_3)), h_1 \star (h_2 \star h_3)) =$$

$$= (g_1 * (\sigma(h_1)(g_2) * \sigma(h_1)(\sigma(h_2)(g_3))), (h_1 \star h_2) \star h_3)\,.$$

Since $\star$ is associative and $\sigma(h_1)$ is a homomorphism. Now, $*$ is associative and σ is a homomorphism, so we have:

$$(g_1, h_1)[(g_2, h_2)(g_3, h_3)] = ((g_1 * \sigma(h_1)(g_2)) * \sigma(h_1 \star h_2)(g_3), (h_1 \star h_2) \star h_3) =$$

$$= (g_1 * \sigma(h_1)(g_2), h_1 \star h_2)(g_3, h_3) = [(g_1, h_1)(g_2, h_2)](g_3, h_3),$$

which proves associativity.
Identity element:

$$(e_G, e_H)(g, h) = (e_G * \sigma(e_H)(g), e_H \star h) = (e_G * g, h) = (g, h)$$

following observation 1.1.7.

Inverse:

$$(g,h)(\sigma(h)^{-1}(g^{-1}),h^{-1}) = (g * \sigma(h)(\sigma(h)^{-1}(g^{-1})), h \star h^{-1}) =$$

$$= (g * g^{-1}, e_H) = (e_G, e_H).$$

$\square$

Definition 1.3.3. We define $G \times H$ together with the operation introduced above the *semidirect product of* H *and* G and we denote it $G \rtimes H$, or $G \rtimes_\sigma H$ if the action is not obvious from the context.

Observation 1.3.4. The direct product of $(G,*)$ and $(H,\star)$ (see definition 1.1.5) can be thought of as a semidirect product between H and G where the representation defining the binary operation is the trivial homomorphism $f : H \longrightarrow Aut(G)$, $f(h) = Id_G = e_{Aut(G)}$ for every $h \in H$. So we have: $(g_1,h_1)(g_2,h_2) = (g_1 * f(h_1)(g_2), h_1 * h_2) = (g_1 * g_2, h_1 * h_2) \Rightarrow G \rtimes H = G \times H$.

Example 1.3.5. Let us define the Euclidean group $E(n)$ as the group of all isometries, or the affine transformations that preserve the metric, of the Euclidean space $\mathbb{R}^n$, that is the vector space $\mathbb{R}^n$ equipped with the Euclidean distance. Such group is the result of the semidirect product between $T_{\mathbb{R}^n}$, the group of translations of $\mathbb{R}^n$, and the orthogonal group $O(n)$, considering the representation:

$$\sigma : O(n) \longrightarrow Aut(T_{\mathbb{R}^n}) \ , \ \sigma(U)(\tau_a) = U\tau_a,$$

where

$$U\tau_a(x) = \tau_{Ua}(x) = x + Ua \quad \text{with } \ x, a \in \mathbb{R}^n \ , \ U \in O(n).$$

Hence, the resulting binary operation is:

$$(\tau_b, O_2)(\tau_a, O_1) = (\tau_b(O_2\tau_a), O_2O_1).$$

Considering that $(\tau_t, O)(x) = Ox + t$, we have:

$$(\tau_b(O_2\tau_a), O_2O_1)(x) = O_2O_1x + O_2a + b.$$

Which is exactly what we expect from the composition of the two isometries (τ_b, O_2) and (τ_a, O_1) applied on a vector $x \in \mathbb{R}^n$.

Observation 1.3.6. Consider a subgroup H of $(G, *)$, with the action being given by the conjugation, as seen in example 1.2.8. In particular, when H is a normal subgroup of G both $G \rtimes H$ and $H \rtimes G$ are well defined, taking as action the conjugation. Furthermore, the subgroup (H, e_G) is normal both in $G \rtimes H$ and in $H \rtimes G$. Since the proof is virtually the same for the two products, we prove only the first assertion.
We want to show that for every $(x, e_G) \in (H, e_G)$ and $(g, h) \in G \rtimes H$ we have $(g, h)(x, e_G)(g, h)^{-1} \in (H, e_g)$.
Since $(g, h)^{-1} = (\sigma(h^{-1})(g^{-1}), h^{-1})$, we have by semidirect product definition:

$$\begin{aligned}(g,h)(x,e_G)(g,h)^{-1} &= (g,h)(x*\sigma(e_G)(\sigma(h^{-1})(g^{-1}), e_G*h^{-1}) = \\ &= (g,h)(x*\sigma(h^{-1})(g^{-1}), h^{-1}) = (g*\sigma(h)(x*\sigma(h^{-1})(g^{-1})), h*h^{-1}) = \\ &= (g*h*x*h^{-1}*g^{-1}*h*h^{-1}, e_G) = (\underbrace{g*h*x*h^{-1}*g^{-1}}_{\in H}, e_G).\end{aligned}$$

Which implies that $(h, g)(H, e_G)(h, g)^{-1} \subset (H, e_G)$ and, by definition 1.1.4, $(H, e_G) \lhd G \rtimes H$.

1.4 Exact Sequences

Now, for a more comprehesive point of view on the semidirect product, we will briefly talk about exact sequences of groups. This is a powerful tool for a broad investigation of group extensions, but we limit ourselves to a narrow spectrum of examples and observations necessary for the understanding of the core relations between the groups and their semidirect product.

Definition 1.4.1. An *exact sequence of groups* is a sequence, finite or infinite, of groups $\{G_i\}_{i\in I}$ and morphisms $\{\phi_i\}_{i\in I}$ such that $Ker(\phi_{i+1}) = Im(\phi_i)$ for every $i \in I$.

$$G_0 \xrightarrow{\phi_1} G_1 \xrightarrow{\phi_2} G_2 \xrightarrow{\phi_3} \dots \xrightarrow{\phi_n} G_n \xrightarrow{\phi_{n+1}} \dots$$

In particular, we call *exact short sequence of groups* a sequence

$$\{e\} \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow \{e\}$$

where $\{e\}$ is the trivial group. The first and the last function can only be, following observation 1.1.7, trivial homomorphisms $f_1(e) = e_A$ and $f_2(c) = e$.

An exact sequence is said to be *split* if there exists a right-inverse for β, which is a homomorphism $\gamma : C \longrightarrow B$ such that $\beta \circ \gamma = Id_C$.

Observation 1.4.2. From the condition on the morphisms of an exact sequence, we have that $Im(\beta) = Ker(f_2) = C$, so β must be surjective. At the same time, $Ker(\alpha) = Im(f_1) = e_A$, which means that α must be injective. Furthermore, by the *First Theorem of Homomorphism*, β induces an isomorphism between C and $B/Im(\alpha)$ and every time we have a quotient of a group G by a normal subgroup H we can write the exact sequence:

$$\{e\} \longrightarrow H \xrightarrow{\imath} G \xrightarrow{\pi} G/H \longrightarrow \{e\}$$

With $\imath$ being the inclusion $H \subset G$ and $\pi : G \longrightarrow G/H$ the projection into the quotient.

Example 1.4.3. Consider the following short sequence:

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{f} G \xrightarrow{g} \mathbb{Z}_2 \longrightarrow 0$$

- For $G = \mathbb{Z}_4$, the (unique) homomorphisms are $f(a) = 2a$ and $g(0) = g(2) = 0$, $g(1) = g(3) = 1$. This sequence is not split, since the only possible homomorphism from $\mathbb{Z}_2$ to $\mathbb{Z}_4$ is f, for what we asserted in observation 1.1.7. Indeed, $g \circ f \neq Id_{\mathbb{Z}_2}$, as $g(f(1)) = 0$.

- For $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, the homomorphism are (not unique!) $f(0) = (0, 0)$, $f(1) = (1, 0)$ and $g((0, 0) = g((1, 0)) = 0$, $g((0, 1)) = g((1, 1)) = 1$. This sequence is split, considering as right-inverse for g the homomorphism $h(0) = (0, 0)$, $h(1) = (0, 1)$.

1.5 Semidirect Product and Exact Sequences

In this section we see a characterization of semidirect products via exact sequences.

Let $(C, *)$ be a group and $A \lhd C$. Following example 1.3.6, we can define $A \rtimes C$ using the conjugation $\sigma : C \longrightarrow Aut(A)$. Consider the short sequence:

$$\{e\} \longrightarrow A \hookrightarrow A \rtimes C \xrightarrow{\pi} C \longrightarrow \{e\}$$

Where the first non trivial morphism is the embedding $\imath : A \longrightarrow A \rtimes C$, $\imath(a) = (a, e_C)$, and the second one, from $A \rtimes C$ to C, is the projection $\pi(a, c) = c$. This is an exact sequence since: $Im(f_1) = e_A = Ker(\imath)$, $Im(\imath) = (A, e_C) = Ker(\pi)$ and $Im(\pi) = C = Ker(f_2)$. Both $\imath$ and π are homomorphisms, in fact:

$$\begin{aligned}\imath(a_1)\imath(a_2) &= (a_1, e_C)(a_2, e_C) = (a_1 * \sigma(e_C)(a_2), e_C * e_C) = (a_1 * a_2, e_C) = \\ &= \imath(a_1 * a_2).\end{aligned}$$

and

$$\begin{aligned}\pi((a_1, c_1)(a_2, c_2)) &= \pi((a_1 * \sigma(c_1)(a_2), c_1 * c_2)) = c_1 * c_2 = \\ &= \pi((a_1, c_1)) * \pi((a_2, c_2)).\end{aligned}$$

This sequence is split: the homomorphism $\phi : C \longrightarrow A \rtimes C$, $\phi(c) = (e_C, c)$ is a right inverse for π : $\pi \circ \phi = Id_C$. The proof of ϕ homomorphism is essentially the same as the one given for $\imath$.

Suppose now that the short sequence in definition 1.4.1 is split:

$$\{e\} \longrightarrow A \xrightarrow{\alpha} B \underset{\gamma}{\overset{\beta}{\rightleftarrows}} C \longrightarrow \{e\}$$

with $\beta \circ \gamma = Id_C$. We want to show that every $b \in (B, \star)$ can be written uniquely as $b = a \star c$, with $a \in Im(\alpha) = Ker(\beta)$ and $c \in Im(\gamma)$:

$$b = b \star [\gamma(\beta(b))]^{-1} \star \gamma(\beta(b)) = [b \star \gamma(\beta(b^{-1}))] \star \underbrace{\gamma(\beta(b))}_{\in Im(\gamma)},$$

while $b \star \gamma(\beta(b^{-1})) \in Ker(\beta)$ since

$$\beta(b \star \gamma(\beta(b^{-1}))) = \beta(b) * \beta(\gamma(\beta(b^{-1}))) = \beta(b) * \beta(b^{-1}) = e_C$$

for the properties of morphisms.
Suppose now that $b = x_1 \star y_1 = x_2 \star y_2$ with $x_1, x_2 \in Ker(\beta))$ and $y_1, y_2 \in Im(\gamma)$. Applying β both sides we have:

$$\beta(x_1) * \beta(y_1) = \beta(x_2) * \beta(y_2) \;\Rightarrow\; \beta(y_1) = \beta(y_2).$$

There exists $z_1, z_2 \in C$ such that $\gamma(z_1) = y_1$ and $\gamma(z_2) = y_2$, so:

$$\beta(\gamma(z_1)) = \beta(\gamma(z_2)) \;\Rightarrow\; z_1 = z_2 \;\Rightarrow\; y_1 = y_2 \;\text{ and }\; x_1 = x_2.$$

Hence, the factorization is unique.
Consider the group $Ker(\beta) \rtimes Im(\gamma)$, with the binary operation induced by the conjugation, as in 1.3.6: $(a_1, c_1)(a_2, c_2) = (a_1 \star c_1 \star a_2 \star c_1^{-1}, c_1 \star c_2)$. Such operation is well defined since $Ker(\beta) \lhd B$.
Define

$$\phi : B \longrightarrow Ker(\beta) \rtimes Im(\gamma)\;,\; \phi(b) = (a, c),$$

where $a \in Ker(\beta)$ and $c \in Im(\gamma)$ such that $b = a \star c$.
For what we showed before, ϕ is a bijection. But it is also a homomorphism: given $b_1, b_2 \in B$ with factorization $a_1 \star c_1$ and $a_2 \star c_2$ respectively, we have that

$$\phi(b_1 \star b_2) = \phi((a_1 \star c_1) \star (a_2 \star c_2)) = \phi(\underbrace{a_1 \star c_1 \star a_2 \star c_1^{-1}}_{\in Ker(\beta)} \star \underbrace{c_1 \star c_2}_{\in Im(\gamma)}) =$$

$$= (a_1 \star c_1 \star a_2 \star c_1^{-1}, c_1 \star c_2) = (a_1, c_1)(a_2, c_2) = \phi(b_1)\phi(b_2).$$

Hence, B is isomorphic to $Ker(\beta) \rtimes Im(\gamma)$.

Now we want to point out that $Im(\alpha) \cong A$ since α is an injection, so $Im(\alpha) = Ker(\beta) \cong A$, and also that $Im(\gamma) \cong C$, because γ is an injection. Finally, we proved that $B \cong A \rtimes C$.

We have thus proved the following important result concerning exact sequences and semidirect products:

Theorem 1.5.1. *Let A be a normal subgroup in a group $(C, *)$, then the group $(B, \star)$ is isomorphic to the semidirect product $A \rtimes C$ if and only if the sequence*

$$\{e\} \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow \{e\}$$

splits.

For abelian groups, such as the one seen in example 1.4.3, the semidirect product becomes a direct product, or equivalently a direct sum. Moreover, recalling what we asserted in observation 1.4.2, being split implies that $C \cong A \rtimes C/Im(\alpha)$.
One last important example, concerning the above results, is about the structure of the orthogonal group $O(n)$. As in example 1.1.8, we take for granted the basic knowledge in linear algebra. See [7] for more details.

Example 1.5.2. Consider the sequence:

$$\{e\} \longrightarrow SO(n) \hookrightarrow O(n) \xrightarrow{\pi} O(n)/SO(n) \longrightarrow \{e\}$$

It is a particular case of the sequence seen in observation 1.4.2, where the first non trivial homomorphism of the sequence is the inclusion $\imath$, and the projection to the quotient π is given by the determinant function $det : GL(n, \mathbb{R}) \longrightarrow \mathbb{R}\backslash\{0\}$. Orthogonal matrices have unitary determinant, which means, for $O(n) \subset GL(n, \mathbb{R})$, that it can only be ± 1. Since the determinant function det is a group homomorphism, and $SO(n)$ is its kernel, it induces an isomorphism between $O(n)/SO(n)$ and $(\{\pm 1\}, \cdot)$. Hence, we obtained that $O(n)/SO(n) \cong C_2$, where C_2 is the cyclic group of order 2. Given a reflection r through an hyperplane in $\mathbb{R}^n$, such group can be expressed in a more relevant way as $(\{Id, R\}, *)$, where Id is the identity matrix, R is the matrix associated with r and $*$ is the matrix multiplication. Since a reflection is an orthogonal transformation, and in particular an inverse isometry and an involution, it is clear that $(\{Id, R\}, *) \cong (\{\pm 1\}, \cdot) \cong C_2$. Such sequence is exact, because $Im(\imath) = Ker(\pi) = SO(n)$. Moreover, it is also split: we can define the right-inverse for π as

$$\phi : \{\pm 1\} \longrightarrow O(n) \; , \text{ with } \; \phi(1) = Id \; , \; \phi(-1) = R$$

which can be seen as the inclusion $\{Id, R\} \subset O(n)$. Using now the previous and general conclusions on split exact sequences, we obtain that $SO(n) \rtimes \{Id, R\} \cong O(n)$, with the representation $\sigma : \{Id, R\} \longrightarrow Aut(SO(n))$ being the conjugation.

Chapter 2

Smooth Manifolds and Lie Groups

In this second chapter we focus on the preliminary knowledge necessary for the study of the main object of this thesis, the Poincaré group and its representations. Here we are going to introduce Lie groups, without going into a detailed description of their properties, as it would bring us too far. In order to give a precise definition of Lie groups we need to introduce smooth manifolds, therefore we start with a short section about manifolds and smooth functions defined on them. For an extensive discussion on this topic, we refer the reader to [11], chapters 5 and 6.
We take for granted the topological concepts, such as homeomorphism, neighborhood, Hausdorff space, topological base and second-countable set. For such concepts, see [11], appendix A.

2.1 Manifolds and C^∞ Atlases

In this section we define first what a topological manifold is, and then, through the concept of a C^∞ atlas, we put a differentiable structure on the manifold, obtaining a smooth manifold. Such construction relies on the following basic geometric idea: a manifold is a topological space that locally resembles $\mathbb{R}^n$ for a given n, with various degrees of regolarity on the functions involved depending on the kind of manifolds we are interested

in (for topological ones we just need continuity, while smooth manifolds requires C^∞ functions).
A formalization of this concepts can be expressed as follows.

Definition 2.1.1. A topological space M is *locally euclidean of dimension* n if every point p in M has a neighborhood U and a homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$. A *topological manifold* is a Hausdorff, second countable and locally euclidean space. We call n the dimension of M.
The pair (U, ϕ) is called *chart*, while U is called a *coordinate neighborhood* and ϕ a *coordinate map*.

The purpose of a chart is to create a local coordinate system on the manifold. Since the coordinate neighborhoods on a manifold are not disjoint, we require a certain compatibility of the coordinate maps acting on the same domain.

Definition 2.1.2. Suppose (U, ϕ) and (V, ψ) are two charts on a topological manifold such that $U \cap V \neq \varnothing$. These two charts are C^∞*-compatible*, or simply *compatible*, if the two following maps, called *transition functions*,
$\phi \circ \psi^{-1} \,:\, \psi\,(U \cap V) \longrightarrow \phi\,(U \cap V) \;\; ; \;\; \psi \circ \phi^{-1} \,:\, \phi\,(U \cap V) \longrightarrow \psi\,(U \cap V)$
are C^∞.
Two coordinate functions whose domain does not intersect are trivially compatible charts.

We are ready to define the notion of differentiable structure as the collection of compatible charts of a manifold.

Definition 2.1.3. An *atlas* on a locally euclidean space M is a collection $\mathscr{U} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ of pairwise C^∞-compatible charts that covers M, or explicitly, such that $M = \bigcup_{\alpha \in A} U_\alpha$. An atlas $\mathscr{U}$ is said to be *maximal* if it is not contained in any larger atlas, that is, if $\mathscr{M}$ is another atlas and $\mathscr{U} \subset \mathscr{M}$, then $\mathscr{U} = \mathscr{M}$.
A *smooth* or C^∞ *manifold* is a topological manifold together with a maximal atlas.

However, to prove that a topological manifold is a smooth manifold it is not necessary to exhibit a maximal atlas, since the existence of a generic one is sufficient. Indeed, in the following proposition we prove that given an atlas it is possibile to obtain a unique maximal atlas containing our starting one. Hence, a topological manifold with an atlas automatically possesses a maximal atlas and so it is, by definition, a smooth manifold.

Proposition 2.1.4. *Given an atlas $\mathscr{U} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ on a topological manifold, it is contained in a unique maximal atlas $\mathscr{M}$.*

Proof. Consider all the charts $(V_i, \psi_i)_{i \in I}$ compatible with every chart in $\mathscr{U}$. It can be shown (proposition 5.8 in [11]) that all charts $(V_i, \psi_i)_{i \in I}$ are compatible with each other. Hence, if we adjoin the charts $(V_i, \psi_i)_{i \in I}$ to the atlas $\mathscr{U}$ we obtain a larger atlas $\mathscr{M}$. Such new atlas is maximal, because any other chart compatible with every chart in $\mathscr{M}$ is in particular compatibile with those of $\mathscr{U}$, so it is already in $\mathscr{M}$.
This maximal atlas $\mathscr{M}$ is unique: if there was another maximal atlas $\mathscr{M}'$ containing $\mathscr{U}$, all the charts in $\mathscr{M}'$ would be compatible with those in $\mathscr{U}$. Hence $\mathscr{M}' \subset \mathscr{M}$. Since they are both maximal we would have $\mathscr{M}' = \mathscr{M}$. □

Observation 2.1.5. Notice that an open set in a smooth manifold is a smooth manifold: in fact an open subset of a topological space, with the induced topology, inherits the properties of being Hausdorff and second countable from the ambient space. Moreover, if $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ is an atlas for our smooth n-manifold M and $V \subset M$ is the open subset considered, then $\{(U_\alpha \cap V, \phi_\alpha|_{U_\alpha \cap V})\}_{\alpha \in A}$ is an atlas for V: it clearly covers V and the maps $\phi_\alpha|_{U_\alpha \cap V} : U_\alpha \cap V \longrightarrow \mathbb{R}^n$ are a restriction of the coordinate maps ϕ_α, hence they are compatible charts for V.

Observation 2.1.6. We know that if we have two topological spaces that are both Hausdorff and second countable, their cartesian product retains those properties, provided that it is equipped with the product topology. So we conclude that, given a smooth m-manifold M and a smooth n-manifold N with atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ respectively, their product $M \times N$ is a smooth $(m+n)$-manifold with the atlas

$$\{(U_\alpha \times V_\beta\,,\ \phi_\alpha \times \psi_\beta\ :\ U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n)\},$$

where $(\phi_\alpha \times \psi_\beta)(p,q) = (\phi_\alpha(p), \psi_\beta(q))$.
Such collection of charts is effectively an atlas since they are pairwise compatible.

$$\text{Consider : } \phi_{\alpha_1} \times \psi_{\beta_1} \ : U_{\alpha_1} \times V_{\beta_1} \to \mathbb{R}^m \times \mathbb{R}^n \ \text{ and}$$
$$\phi_{\alpha_2} \times \psi_{\beta_2} \ : U_{\alpha_2} \times V_{\beta_2} \to \mathbb{R}^m \times \mathbb{R}^n.$$

Define $W := (U_{\alpha_1} \times V_{\beta_1}) \cap (U_{\alpha_2} \times V_{\beta_2}) = (U_{\alpha_1} \cap U_{\alpha_2}) \times (V_{\beta_1} \cap V_{\beta_2})$.
The two transition functions

$$(\phi_{\alpha_2} \times \psi_{\beta_2}) \circ (\phi_{\alpha_1} \times \psi_{\beta_1})^{-1} \ : (\phi_{\alpha_1} \times \psi_{\beta_1})(W) \longrightarrow (\phi_{\alpha_2} \times \psi_{\beta_2})(W)$$
$$(\phi_{\alpha_1} \times \psi_{\beta_1}) \circ (\phi_{\alpha_2} \times \psi_{\beta_2})^{-1} \ : (\phi_{\alpha_2} \times \psi_{\beta_2})(W) \longrightarrow (\phi_{\alpha_1} \times \psi_{\beta_1})(W)$$

are C^∞, since:

$$(\phi_{\alpha_2} \times \psi_{\beta_2}) \circ (\phi_{\alpha_1} \times \psi_{\beta_1})^{-1}(x,y) = (\underbrace{(\phi_{\alpha_2} \circ \phi_{\alpha_1}^{-1})}_{\in C^\infty \ Hp.}(x), \underbrace{(\psi_{\beta_2} \circ \psi_{\beta_1}^{-1})}_{\in C^\infty \ Hp.}(y)),$$

and $\mathbb{R}^m \times \mathbb{R}^n$ is equipped with the product topology, which means that the smoothness of a function derives from the smoothness of its component. The same holds for the second transition function.

Example 2.1.7. The vector space $\mathbb{R}^n$ is a smooth manifold with the atlas made up of the trivial chart $(\mathbb{R}^n, Id_{\mathbb{R}^n})$ alone. By the observation 2.1.5, every open subset of $\mathbb{R}^n$ is a manifold with the induced atlas.

Example 2.1.8. The general linear group $GL(n, \mathbb{R})$ is an n^2-manifold : the vector space of all $m \times n$ matrices, $\mathbb{R}^{m\times n}$, is isomorphic to $\mathbb{R}^{mn}$, so we can put the euclidean topology of $\mathbb{R}^{n^2}$ on $\mathbb{R}^{n\times n}$. By definition of $GL(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid det(A) \neq 0\}$ and the continuity of the determinant function $det : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$, we obtain that $GL(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$, because it is the preimage of the open set $\mathbb{R} \setminus \{0\}$ through a continous function. Following observation 2.1.5, we conclude that $GL(n,\mathbb{R})$ is an n^2-manifold, as an open subset of $\mathbb{R}^{n^2}$.
Similar conclusions hold for the complex case: the group $GL(n,\mathbb{C})$ is the preimage of the open set $\mathbb{C} \setminus \{0\}$ through the continuous function det :

$\mathbb{C}^{n\times n} \longrightarrow \mathbb{C}$. As such, it is an open subset of C^{n^2} and thus a $2n^2$-manifold.

We now want to introduce the concept of smooth function between smooth manifolds.

Definition 2.1.9. Let M be a smooth m-manifold and N a smooth n-manifold. A map $f : M \longrightarrow N$ is C^∞, or *smooth, at a point* $p \in M$ if there are charts (V, ψ) in N and (U, ϕ) in M such that $p \in U$, $\phi(p) \in V$ and the composition $\psi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \longrightarrow \mathbb{R}^n$ is C^∞ at $\phi(p)$. The function f is said to be C^∞, or *smooth, on* M if it is smooth at every point of M.

Observation 2.1.10. Let M and N be two open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. As in example 2.1.7, we can consider these sets as smooth manifolds with the trivial chart only. Following the previous definition, a function $f : M \longrightarrow N$ is a smooth function at $p \in \mathbb{R}^m$ if $Id \circ f \circ Id^{-1} = f$ is C^∞ at p. This means that the regularity of f ensures the smoothness of f as a function between manifolds. Such conclusion will come in handy working with matrix groups.

2.2 Tangent Space

The tangent space naturally arises during the study of smooth manifolds. It is the vector space of derivations on germs of functions and allows us to define other important objects such as vector fields and the differential of a smooth map.
First we define the germs of functions:

Definition 2.2.1. Let M be a manifold and p a point in M. Consider the set of pairs (f, U) where $f \in C^\infty(U)$ and U is an open set in M such that $p \in U$. We introduce the following equivalence relation: given two pairs (f, U) and (g, V), we say that (f, U) is equivalent to (g, V) if it exists an open neighborhood $W \subset U \cap V$ of p such that $f|_W = g|_W$.
We call *germ of* f *at* p the equivalence class of (f, U), while $C_p^\infty(M)$ is the set of all such equivalence classes.

Definition 2.2.2. Given a manifold M and a point $p \in M$, a *tangent vector at p in M* is a derivation on germs of functions at p, that is, a linear map $D : C_p^\infty(M) \longrightarrow \mathbb{R}$ satisfying the Leibniz rule:

$$D(fg) = (Df)g(p) + f(p)D(g) \quad \text{for every } f, g \in C_p^\infty(M).$$

The *tangent space of M in p*, written T_pM, is the set of all derivations at p. We call *vector field* on an open subset U of M a function X that assigns to each point $p \in U$ a tangent vector X_p in T_pM.

For every point p in a manifold M, the tangent space T_pM is a real vector space of the same dimension as the manifold. Let $p \in M$ and consider a coordinate neighborhood U of p with coordinate maps $\{x^i\}_{i=1,\dots n}$. A basis for the tangent space T_pM is $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}_{i=1,\dots n}$ (see [11], chapter 8, section 4). Thus, a vector field X on a n-manifold can be expressed as a linear combination of functions $\{a^i\}_{i=1,\dots n}$ defined on an open subset U of the manifold:

$$X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i} \quad \text{and} \quad X_p = \sum_{i=1}^{n} a^i(p) \frac{\partial}{\partial x^i}\Big|_p.$$

In particular, we say that a vector field X is *smooth* if all the coefficient functions $\{a^i\}_{i=1,\dots n}$ are smooth. Notice that this property is independent of the local coordinates chosen (Proposition 14.2 in [11]).
Hence, a smooth vector field can be considered as a derivation $X : C^\infty(U) \longrightarrow C^\infty(U)$: given a smooth map on an open subset U of a manifold, $f \in C^\infty(U)$, the function

$$X(f) = \sum_{i=1}^{n} a^i \frac{\partial f}{\partial x^i}$$

is pointwise defined by

$$(Xf)(p) = X_p(f) = \sum_{i=1}^{n} a^i(p) \frac{\partial f}{\partial x^i}(p)$$

and it is smooth.
For a comprehensive introduction about vector fields, derivations and tangent spaces, with all the necessary proofs, see [11], chapter 2 and 8.

2.3 Differential of a Function

Now we go to the definition of the differential of a smooth function.

Definition 2.3.1. Let N, M be two manifolds and $F : N \longrightarrow M$ a smooth function between them. Let $p \in N$. The *differential of* F *at* p*,* dF_p, is a linear map between the tangent spaces T_pN and $T_{F(p)}M$ such that if $X_p \in T_pN$ is a tangent vector at p, then $dF_p(X_p) \in T_{F(p)}M$ is a tangent vector at $F(p)$ defined by:

$$(dF(X_p))f = X_p(f \circ F) \quad \text{for } f \in C^\infty_{F(p)}(M).$$

Such map is clearly linear since for any tangent vectors X_p, Y_p at p and $f \in C^\infty_{F(p)}(M)$:

$$\begin{aligned}(dF_p(X_p + Y_p))f &= (X_p + Y_p)(f \circ F) = X_p(f \circ F) + Y_p(f \circ F) = \\ &= (dF_p(X_p))f + (dF_p(Y_p))f,\end{aligned}$$

by definition of tangent space as a real vector space.

One common technique that we will frequently employ from now on is the computation of differentials using curves on the manifold. The idea behind this procedure is clarified by the following proposition.

Proposition 2.3.2. *Let N, M be two manifolds and $F : N \longrightarrow M$ a smooth function between them. Consider a point $p \in N$, a tangent vector $X_p \in T_pN$ and c a smooth curve starting at p with velocity X_p, that is, a smooth function between manifolds $c : (a, b) \longrightarrow N$ such that $0 \in (a, b)$, $c(0) = p$ and $c'(0) = X_p$. Then we have:*

$$dF_p(X_p) = \left.\frac{d}{dt}\right|_{t=0} (F \circ c)(t)$$

Which means that $dF_p(X_p)$, the differential of F in p evaluated on X_p, is the velocity vector of the curve $F \circ c$ at $F(p)$.

Proof. Since $c'(0) = X_p$ and $c(0) = p$, we have that

$$dF_p(X_p) = dF_p(c'(0)) = (dF_p \circ dc_0)\left(\left.\frac{d}{dt}\right|_{t=0}\right) =$$

(for the chain rule, see Theorem 8.5 in [11])

$$= d(F \circ c)_0 \left(\frac{d}{dt}\Big|_{t=0} \right) =$$
$$= \frac{d}{dt}\Big|_{t=0} (F \circ c)(t),$$

by definition of the differential. $\square$

2.4 Lie Groups

We introduce now Lie groups, since the main object we are interested in, the Poincaré group, is a semidirect product of Lie groups. This concept combines both the ideas of group and smooth manifold, requiring a compatibility between these two structures: the binary operation and the inverse function of the group have to be smooth functions.

Definition 2.4.1. A *Lie group* is a smooth manifold G with a group structure such that the binary operation $\mu : G \times G \longrightarrow G$, $\mu(x, y) = x * y$ and the inverse operation $\iota : G \longrightarrow G$, $\iota(x) = x^{-1}$ are C^∞.

Observation 2.4.2. Note that, in this definition, the binary operation is required to be a smooth function between manifolds $G \times G$, built through the cartesian product as seen in 2.1.6, and G.
Notice that the *left multiplication* (also called *left translation*) map $l_g : G \longrightarrow G$, $l_g(h) = g * h$ is a smooth map for every $g \in G$, since the composition of this function with the coordinate charts can be seen as a restriction of the composition of the group operation μ with suitable charts, which is smooth by definition.

Example 2.4.3. The identity-connected component G^0 of a Lie group G is itself a Lie group. Let $g_1, g_2 \in G^0$ and $\gamma_i : [0,1] \longrightarrow G^0$, $t \longmapsto \gamma_i(t)$ two continuous curves connecting e with g_i , $i = 1, 2$. We have $g_1 * g_2 = \mu(g_1, g_2) \in G^0$ since μ is a smooth function and

$$\gamma_{g_2} = \mu(\gamma_1, g_2) : [0,1] \longrightarrow G^0 \ , \gamma_{g_2}(t) = \gamma_1(t) * g_2$$

is a continuous curve connecting g_2 with $g_1 * g_2$. Hence the curve

$$\gamma(t) : [0,1] \longrightarrow G^0 \quad , \gamma(t) = \begin{cases} \gamma_1(2t) & t \in [0, \frac{1}{2}] \\ \gamma_{g_2}(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

connects e with $g_1 * g_2$.
Moreover, $\iota(g_1) = g_1^{-1} \in G^0$ because ι is smooth and

$$\gamma = \iota(\gamma_1) : [0,1] \longrightarrow G^0 \ , \ \gamma(t) = \gamma_1(t)^{-1}$$

is a continuous curve connecting e with g_1^{-1}. So we obtained that G^0 is a subgroup of G.
Moreover, being G a real manifold, it is locally homeomorphic to $\mathbb{R}^n$ for a given n. As such, G is locally path-connected, since path-connectedness is a topological invariant. A connected component in a locally path-connected space is open, hence G^0 is a smooth manifold of the same dimension as G, being an open set of the smooth manifold G. We conclude that G^0 is a Lie group since it is a smooth manifold and a group whose binary operation and inverse inherit the smoothness from μ and ι.

Example 2.4.4. We already saw in example 2.1.8 that the general linear group $GL(n,\mathbb{R})$ is a n^2-manifold, since it is an open subset of $\mathbb{R}^{n^2}$. Now we show that it is also a Lie group. In the first chapter we introduced $GL(n,\mathbb{R})$ as a group with matrix multiplication, which can be expressed in global coordinates of $\mathbb{R}^{n^2}$ as follows.
Given two matrices $A, B \in GL(n,\mathbb{R})$, $A = (a)_{ij}$, $B = (b)_{ij}$ and the matrix multiplication $\mu : GL(n,\mathbb{R}) \times GL(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$, $\mu(A,B) = AB$ we have that $(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. Hence, every component of μ is a polynomial in the coordinates of A and B. Recalling observation 2.1.10, we can conclude that μ is C^∞.
For the inverse function: given $A \in GL(n,\mathbb{R})$ and $\iota : GL(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$, $\iota(A) = A^{-1}$, we have $(A^{-1})_{ij} = \dfrac{1}{det(A)} \cdot (-1)^{i+j}(A_{ij})$, where A_{ij} is the determinant of the submatrix of A obtained removing the jth-row and the ith-column. Every component of $\iota(A)$ is a rational function in the coordi-

nates of A with a non-zero denominator, since $det(A) \neq 0$. This confirms that ι is C^∞ and effectively proves that $GL(n,\mathbb{R})$ is a Lie group of dimension n^2.

Observation 2.4.5. Notice that results obtained for a real matrix group can be applied to its complex counterpart through the realification: given a complex number $z = a + ib$, we can write it as the square real matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in M(2,\mathbb{R})$. This means that every complex matrix in $M(n,\mathbb{C})$ can be written as a real matrix with doubled dimensions in $M(2n,\mathbb{R})$. Hence, a completely analogous argument as the one given in the previous example shows that $GL(n,\mathbb{C})$ is a real Lie group of dimension $2n^2$, which means it has a differentiable atlas of transition functions between open sets in $\mathbb{R}^{2n^2}$.

All the matrix subgroups introduced in example 1.1.8 are Lie groups. We will not give a proof of the following result, so we refer to [12], chapter 2, section 1.

Theorem 2.4.6. *Let G be a subgroup of $GL(n,\mathbb{R})$ (or $GL(n,\mathbb{C})$) defined through algebraic equations. Then G is a closed real (or complex) Lie group.*

Example 2.4.7. We already know that $SL(n,\mathbb{K})$ is a subgroup of $GL(n,\mathbb{K})$. Moreover, being $det : GL(n,\mathbb{K}) \longrightarrow \mathbb{K}$ a polynomial function in the matrix entries, $SL(n,\mathbb{K})$ is a subgroup $GL(n,\mathbb{K})$ defined through an algebraic equation. Hence, given the previous theorem, we conclude that $SL(n,\mathbb{K})$ is a Lie Group. We will discuss later about its dimension, as we need to use some concepts that will be introduced in the following sections. See example 2.6.7 for the conclusion.

Example 2.4.8. The orthogonal group $O(n)$ is a Lie group, since its defining conditions are algebraic. In fact, considering the euclidean topology on $\mathbb{R}^n$ given by the standard inner product $\langle v, w \rangle = \sum_{i=1}^{n} v_i w_i$ and a matrix $A \in O(n)$, the condition $A^T A = Id$ means that the columns $(a_1, a_2, \cdots, a_n) = A$ satisfy the polynomial equations $\sum_{k=1}^{n} a_{ik} a_{kj} = \langle a_i, a_j \rangle = \delta_{ij}$.

Furthermore, following the previous example, we can say that $SO(n)$ is a Lie group, since it is a subgroup of the Lie group $O(n)$ defined by the algebraic condition that the determinant is 1.

Observation 2.4.9. Analogous results can be obtained for the unitary group $U(n)$ and its subgroup $SU(n)$, applying first the realification seen in observation 2.4.5 and then concluding that the defining condition $A^\dagger A = Id$ is now algebraic. Hence, they are both Lie groups.

2.5 Lie Algebras

As we will see in the next chapter, Lie algebras are fundamental for the computation of the representations of their associated Lie groups.
We give first the general definition of Lie algebra with its bracket operation, then we prove that this structure is effectively associated with the tangent space of a Lie group G and its Lie bracket.

Definition 2.5.1. Given a field $\mathbb{K}$, a Lie algebra $\mathfrak{A}$ is a vector space over $\mathbb{K}$ equipped with an antisymmetric bilinear map, called the *bracket*, from $\mathfrak{A} \times \mathfrak{A}$ into $\mathfrak{A}$

$$(X, Y) \longmapsto [X, Y] \quad \text{with } X, Y \in \mathfrak{A}$$

satisfying the *Jacobi identity*, that is:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad \text{for every } X, Y, Z \in \mathfrak{A}$$

The definition of a Lie algebra associated with a Lie group relies on a particular class of vector fields, the left invariant ones. After their definition we will be able to geometrically identify the Lie algebra of a Lie group with its tangent space at the identity.

Observation 2.5.2. Given a Lie group G, we already saw in example 2.4.2 that the left multiplication l_g is a smooth function. So it can be differentiated:

$$dl_g : T_hG \longrightarrow T_{l_g(h)}G = T_{g*h}G \ , \ X_h \longmapsto dl_g(X_h)$$

In particular, considering $G \subset GL(n, \mathbb{K})$, we observe that l_g is a linear function in the vector space $M(n, \mathbb{K})$:

$$l_g(h+k) = g(h+k) = gh + gk = l_g(h) + l_g(k) \quad \text{for every } h, k \in G$$

so the differential dl_g is l_g itself, and the above expression becomes $X_h \longmapsto dl_g(X_h) = gX_h$, with $g, X_h \in M(n, \mathbb{K})$

Definition 2.5.3. We say that a vector field X on a Lie group G is *left invariant* if $dl_g(X_h) = X_{g*h}$, for every $g, h \in G$.
In particular, for $G \subset GL(n, \mathbb{K})$, this means that $dl_g(X_h) = gX_h = X_{gh}$

Observation 2.5.4. Let G be a Lie group and X a left invariant vector field on it. Considering the identity element e of G and the tangent vector $X(e) = X_e$, we have:

$$dl_g : T_eG \longrightarrow T_gG \ , \ X_e \longmapsto dl_g(X_e) = X_{g*e} = X_g$$

This means that a left invariant vector field X is completely determined by its value at e.

Now we have all the preliminary concepts to define the structure that we will prove to be the Lie algebra associated with a Lie group.

Definition 2.5.5. Given a Lie group G, we define the Lie algebra $Lie(G) = \mathfrak{g}$ associated with G as the set of all the left invariant vector fields:

$$\mathfrak{g} = \{X : g \longmapsto X_g = dl_g(X_e)\}$$

Given two vector fields $X, Y \in \mathfrak{g}$, we define the *Lie bracket* $[X, Y]$ as:

$$[X, Y](f) = (XY - YX)(f) = X(Y(f)) - Y(X(f)) \text{ with } f \in C^\infty(G)$$

One can easily see that this is a vector field. Additionally, it can be shown ([11], chapter 14, section 6) that $Z = [X, Y]$ is a left invariant vector field, hence $Z \in \mathfrak{g}$ and the Lie bracket is effectively defined on $\mathfrak{g}$.

Now we are going to show that $\mathfrak{g}$ is isomorphic, as a vector space, to the tangent space in the identity element e of the Lie group G, and then we will prove that $\mathfrak{g}$, with the Lie bracket, is effectively a Lie algebra, as we defined it in 2.5.1.

As we have seen in observation 2.5.4, a left invariant vector field is determined by its value at e. This means that every vector field X can be represented by a vector $X_e \in T_eG$. Moreover, for every vector X_e in T_eG we have a (left invariant) vector field X applying dl_g to X_e for every $g \in G$. Hence, considering that the set of all vector fields is a vector space, we conclude that the set of all the left invariant vector fields is isomorphic to the tangent space of G in e. We can now extend the previous definition:

$$\mathfrak{g} = \{X : g \longmapsto X_g = dl_g(X_e)\} \cong T_eG$$

See [12], chapter 2, section 3, for more details. In particular, it is necessary to prove that such left invariant vector field is smooth.

We have already seen in definition 2.5.5 that, given a Lie group G, the vector space $\mathfrak{g}$ is equipped with a product map, the Lie bracket. Now, considering that a derivation is a linear map and the set of all vector fields is a vector space, we conclude that the Lie bracket is an anticommutative bilinear product on $\mathfrak{g}$:

$$[Y, X](f) = Y(X(f)) - X(Y(f)) = -[X, Y](f)$$

and

$$\begin{aligned}[rX + sZ, Y](f) &= (rX + sZ)(Y(f)) - Y((rX + sZ)(f)) = \\ &= rX(Y(f)) + sZ(Y(f)) - Y(rX(f)) - Y(sZ(f)) = \\ &= r[X, Y](f) + s[Z, Y](f).\end{aligned}$$

For every X, Y, Z vector fields on G and $r, s \in \mathbb{K}$. Simmetrically for the second argument.
Moreover, the Lie bracket satisfies the Jacobi identity:

$$[X, [Y, Z]](f) + [Y, [Z, X]](f) + [Z, [X, Y]](f) =$$

$$= [X, (YZ - ZY)](f) + [Y, (ZX - XZ)](f) + [Z, (XY - YX)](f) =$$

$$\begin{aligned} &= (XYZ - XZY - YZX + ZYX)(f) + \\ &+ (YZX - YXZ - ZXY + XZY)(f) + \\ &+ (ZXY - ZYX - XYZ + YXZ)(f). \end{aligned}$$

Using the linearity of the derivation it becomes easy to see that all those terms are mutually cancelled.
Hence, for every Lie group G, the set of all left invariant vector fields (or tangent space in the identity) $Lie(G) = \mathfrak{g}$ with the Lie bracket is effectively a Lie algebra.

Example 2.5.6. Consider the group $G = SO(2) \subset GL(2, \mathbb{R})$. For what we have seen in example 2.4.8, it is a Lie group. We already know that

$$SO(2) \cong \left\{ \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}, \theta \in [0, 2\pi] \right\} \cong S^1.$$

As such, $Lie(SO(2)) = \mathfrak{so}(2) \cong \mathbb{R}$, since the tangent space of S^1 in $e = 1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = Id_{\mathbb{R}^2}$ is geometrically a line, or 1-dimensional real vector space.
Left invariant vector fields on $SO(2)$ are determined applying rotations, that are left translations in $SO(2)$, to a tangent vector in e, which can be represented as a real number.

Observation 2.5.7. Let G be a subgroup of $GL(n, \mathbb{R})$ with the identification $\mathfrak{g} = Lie(G) \cong T_e G$:

$$X_e = \sum_{i,j=1}^{n} a_{ij} \frac{\partial}{\partial x_{ij}}\Big|_e \longleftrightarrow (a_{ij})_{i,j=1,\dots n} = A,$$

the Lie bracket on $\mathfrak{g}$ behaves as a matrix product.
Indeed, given $X = \sum_{i,j=1}^{n} v_{ij} \frac{\partial}{\partial x_{ij}}$ and $Y = \sum_{i,j=1}^{n} w_{ij} \frac{\partial}{\partial x_{ij}}$ left invariant vector fields such that $X_e \leftrightarrow A = (a_{ij}) = (v_{ij}(e))$ and $Y_e \leftrightarrow B = (b_{ij}) = (w_{ij}(e))$,

we have

$$\begin{aligned}[X,Y]_e x_{ij} &= X_e(Y(x_{ij})) - Y_e(X(x_{ij})) \\ &= X_e(\sum_{l,m=1}^{n} w_{lm}\frac{\partial}{\partial x_{lm}} x_{ij}) - Y_e(\sum_{l,m=1}^{n} v_{lm}\frac{\partial}{\partial x_{lm}} x_{ij}) = \\ &= X(w_{ij}) - Y(v_{ij}).\end{aligned}$$

Since X, Y are left invariant vector fields, they are determined in every point $p = (x_{ij}(p)) \in G$ by $dl_p X_e = pX_e = X_p$, which means

$$X_p = \sum_{i,j=1}^{n} (pA)_{ij} \frac{\partial}{\partial x_{ij}}\Big|_p \Longrightarrow v_{ij} = \sum_{k=1}^{n} x_{ik} a_{kj}$$

and the same for Y. So it follows that

$$\begin{aligned}[X,Y]_e x_{ij} &= \sum_{l,m=1}^{n} a_{lm}\frac{\partial}{\partial x_{lm}}(\sum_{k=1}^{n} x_{ik} b_{kj}) - \sum_{l,m=1}^{n} b_{lm}\frac{\partial}{\partial x_{lm}}(\sum_{k=1}^{n} x_{ik} a_{kj}) = \\ &= \sum_{l,m,k=1}^{n} a_{lm} b_{kj}\delta_{il}\delta_{km} - \sum_{l,m,k=1}^{n} b_{lm} a_{kj}\delta_{il}\delta_{km} = \\ &= \sum_{k=1}^{n} a_{ik} b_{kj} - \sum_{k=1}^{n} b_{ik} a_{kj} = (AB - BA)_{ij},\end{aligned}$$

which gives exactly what we were looking for:

$$[X,Y]_e = AB - BA.$$

Example 2.5.8. Consider $\mathfrak{gl}(2,\mathbb{R}) = Lie(GL(2,\mathbb{R}))$. Following example 2.1.8, we conclude that $\mathfrak{gl}(2,\mathbb{R}) \cong \mathbb{R}^4$, since the Lie group $GL(2,\mathbb{R})$ is an open set in $\mathbb{R}^4$. We want to study the behaviour of the Lie bracket applied to the following matrices:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ ,\ X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ ,\ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Note that H, X, Y are matrices in $\mathfrak{gl}(2,\mathbb{R})$, not in $GL(2,\mathbb{R})$. Indeed, $det(X) = det(Y) = 0$.

$$\begin{aligned}[X,Y] = XY - YX &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = H\end{aligned}$$

$$\begin{aligned}[H,X] = HX - XH &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 2X\end{aligned}$$

$$\begin{aligned}[H,Y] = HY - YH &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = -2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = -2Y\end{aligned}$$

We will see that this results are extremely important for the structure of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ and its representations.

2.6 Matrix Exponential

In this section we briefly introduce the matrix exponential, a fundamental tool in Lie Theory. We are not going to use intensively this useful tool in this thesis, so our exposition of the topic will be quite simple and focused on our needs. For more details, see [12], chapter 2, section 10.

Definition 2.6.1. The matrix exponential is a function of square matrices

$$exp : M(n,\mathbb{K}) \longrightarrow M(n,\mathbb{K})\ ,\ exp(A) = e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

Observation 2.6.2. The matrix exponential is defined through the power series $\displaystyle\sum_{k=0}^{\infty} \frac{A^k}{k!}$, which is a generalization of the exponential map on real or complex numbers. As such, it has the same property of convergence: we know that in a Banach space, such as $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$ with the euclidean norm, the absolute convergence of a series implies the convergence. So we have to study the convergence of the series $\sum_{k=0}^{\infty} \left\| \frac{A^k}{k!} \right\|$, which is bounded by

$$\sqrt{n} + \|X\| + \frac{1}{2!}\|A\|^2 + \frac{1}{3!}\|A\|^3 + \cdots = (\sqrt{n} - 1) + e^{\|A\|},$$

using the property $\|AB\| \leq \|A\|\|B\| \Rightarrow \|A^n\| \leq \|A\|^n$ for every $A \in \mathbb{R}^{n\times n}$. Since $(\sqrt{n} - 1) + e^{\|A\|}$ is finite for every $A \in \mathbb{R}^{n\times n}$, the matrix exponential is absolutely convergent for every $A \in \mathbb{R}^{n\times n}$. For more details, see [11], chapter 15, section 3.

Observation 2.6.3. Notice that for a diagonal matrix the following property holds:

$$A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} \longrightarrow A^k = \begin{pmatrix} a_1^k & & & \\ & a_2^k & & \\ & & \ddots & \\ & & & a_n^k \end{pmatrix}$$

So it is immediate, by definition of matrix exponential, to conclude that e^{tA} assumes the simple expression

$$e^{tA} = \begin{pmatrix} e^{ta_1} & & & \\ & e^{ta_2} & & \\ & & \ddots & \\ & & & e^{ta_n} \end{pmatrix}.$$

The same is true for triangular matrices. For example, if A is an upper triangular matrix:

$$A = \begin{pmatrix} a_1 & \cdots & * & * \\ & a_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & a_n \end{pmatrix} \longrightarrow A^k = \begin{pmatrix} a_1^k & \cdots & \star & \star \\ & a_2^k & \cdots & \star \\ & & \ddots & \vdots \\ & & & a_n^k \end{pmatrix}.$$

So we have

$$e^{tA} = \begin{pmatrix} e^{ta_1} & \cdots & \bullet & \bullet \\ & e^{ta_2} & \cdots & \bullet \\ & & \ddots & \vdots \\ & & & e^{ta_n} \end{pmatrix}.$$

This result will be useful for the next proposition.

Proposition 2.6.4. *For every $A \in \mathbb{K}^{n\times n}$, $det(e^{tA}) = e^{tr(A)}$.*

Proof. If A is upper triangular as in the previous observation, we have

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} a_1^k & \cdots & \star & \star \\ & a_2^k & \cdots & \star \\ & & \ddots & \vdots \\ & & & a_n^k \end{pmatrix} = \begin{pmatrix} e^{ta_1} & \cdots & \bullet & \bullet \\ & e^{ta_2} & \cdots & \bullet \\ & & \ddots & \vdots \\ & & & e^{ta_n} \end{pmatrix}.$$

Which implies that

$$det(e^A) = \prod_{i=0}^{n} e^{a_i} = e^{\sum_{i=0}^{n} a_i} = e^{tr(A)}.$$

By Jordan's Theorem, for a matrix A there exists an invertible matrix $M \in GL(n, \mathbb{C})$ such that

$$MAM^{-1} = \begin{pmatrix} \lambda_1 & \cdots & * & * \\ & \lambda_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & \lambda_n \end{pmatrix},$$

where $\{\lambda_1, \cdots, \lambda_n\}$ is the (complex) spectrum of A. So we have

$$\begin{aligned} e^{MAM^{-1}} &= I + MAM^{-1} + \frac{1}{2!}MA^2M^{-1} + \frac{1}{3!}MA^3M^{-1} + \cdots = \\ &= M(I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots)M^{-1} = Me^AM^{-1}. \end{aligned}$$

Considering that MAM^{-1} is upper triangular, we obtain

$$det(e^A) = det(Me^AM^{-1}) = det(e^{MAM^{-1}}) = e^{tr(MAM^{-1})} = e^{tr(A)},$$

since the determinant and the trace of a matrix are both invariant by similarity. □

This means that the image of the matrix exponential is in $GL(n, \mathbb{K})$, and not simply in $M(n, \mathbb{K})$: since $det(e^{tA}) = e^{tr(A)}$, for the properties of the complex exponential, the right-hand side is always non-zero. Hence the matrix e^A is always invertible. In particular, for a Lie group $G \subset GL(n, \mathbb{K})$ and its Lie algebra $\mathfrak{g} \subset M(n, \mathbb{K})$, the matrix exponential sends matrices $A \in \mathfrak{g}$ into matrices $e^A \in G$ (for an exhaustive derivation of this result, see [12], chapter 2, section 10).

Proposition 2.6.5. *For every $A \in M(n, \mathbb{K})$ we have* $\frac{d}{dt} e^{tA} = Ae^{tA}$

Proof. Since every entry in the matrix e^{tA} is a function in t defined through a convergent power series, we can compute the derivative of each component function with a term by term differentiation of the power series:

$$\begin{aligned} \frac{d}{dt} e^{tA} &= \frac{d}{dt}\left(I + tA + t^2\frac{A^2}{2!} + t^3\frac{A^3}{3!} + \cdots\right) \\ &= A + 2t\frac{A^2}{2!} + 3t^2\frac{A^3}{3!} + \cdots = A\left(I + tA + t^2\frac{A^2}{2!} + \cdots\right) \\ &= Ae^{tA} \end{aligned}$$

□

Observation 2.6.6. The matrix exponential is particularly useful to write curves in $GL(n, \mathbb{K})$ with a given initial point and velocity. For example, consider Id as the starting point and $A \in M(n, \mathbb{K})$ as the initial velocity. Now look at the curve $c(t) = e^{tA} : \mathbb{R} \longrightarrow GL(n, \mathbb{K})$. We have

$$c(0) = e^{0A} = e^0 = I \quad \text{and} \quad c'(0) = \frac{d}{dt}\Big|_{t=0} e^{tA} = Ae^0 = A$$

by the previous proposition. The curve e^{tA} has also a group structure and realizes a group homomorphism between $\mathbb{R}$ and $GL(n, \mathbb{R})$:

$$(t_1 + t_2) \longmapsto e^{(t_1+t_2)A} = e^{t_1 A} e^{t_2 A} \quad \text{for every } t_1, t_2 \in \mathbb{R}$$

since $t_1 A$ and $t_2 A$ commute ([12], Lemma 2.10.2). For this reason e^{tA}, $t \in \mathbb{R}$, is called a *one parameter subgroup.*

Example 2.6.7. There are different ways to determine the Lie algebra associated to a Lie group. Since we are working with matrix Lie groups, the easiest strategy makes use of curves, similarly to what we have seen in proposition

2.3.2 for the differential of a smooth map.
We will give an example of this procedure for the group $SL(2,\mathbb{K})$. Let $X \in M(2,\mathbb{K})$. Define the smooth curve γ in $M(2,\mathbb{K})$ as

$$\gamma : (-\epsilon,\epsilon) \subset \mathbb{R} \longrightarrow M(2,\mathbb{K}) \ , \ \gamma(t) = e^{tX} \ , \ \epsilon \in \mathbb{R}.$$

Such curve is effectively in $GL(2,\mathbb{K})$ for the properties of the matrix exponential, but we are interested in $SL(2,\mathbb{K})$ and its Lie algebra $\mathfrak{sl}(2,\mathbb{K})$. Notice that

$$X \in \mathfrak{sl}(2,\mathbb{K}) \Longleftrightarrow det(e^{tX}) = det(\gamma(t)) = 1 \quad t \in (-\epsilon,\epsilon).$$

This implication relies on the fact that whenever $det(\gamma(t)) = 1$ we are moving in $SL(2,\mathbb{K})$, hence the tangent vector $\frac{d}{dt}\gamma(t) = \gamma'(t)$ is in the tangent space of $SL(2,\mathbb{K})$. In particular, this is true for $t = 0$ and thus $\gamma'(0) = X \in T_{Id}SL(2,\mathbb{K}) = \mathfrak{sl}(2,\mathbb{K})$.
For the properties of the matrix exponential, we have

$$\begin{aligned} det(e^{tX}) = e^{tr(tX)} = 1 \quad t \in (-\epsilon,\epsilon) \\ \Longleftrightarrow tr(X) = 0. \end{aligned}$$

We conclude that $\mathfrak{sl}(2,\mathbb{K})$ is the vector space of 2×2 matrices with null trace. Hence:

$$\begin{aligned} \mathfrak{sl}(2,\mathbb{K}) &= \left\{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix} \ , \ x_{ij} \in \mathbb{K} \right\} = \\ &= span_{\mathbb{K}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}. \end{aligned}$$

For the Lie structure, see example 2.5.8.
This also confirms that $dim(SL(2,\mathbb{K})) = 3$ as a Lie group, and in general $dim(SL(n,\mathbb{K})) = n^2 - 1$, since the approach here presented can be repeated for an arbitrary n.

Chapter 3

Representations of $SL(2, \mathbb{C})$

In this chapter we study the representations of the Lie group $SL(2, \mathbb{C})$, necessary to study the representations of the Poincaré group $\mathcal{P}$. Note that the Poincaré group is a real group, so we consider $SL(2, \mathbb{C})$ as a real group, even if the matrices have complex entries. The choice to work with complex matrices relies on the greater ease of diagonalization.
In this chapter $\mathbb{K}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$.

3.1 Preliminaries on Representation Theory

Before we talk about the classification of all possibile representations of $SL(2, \mathbb{C})$, we define a Lie group and a Lie algebra representation. Moreover, we show some important connections in Lie theory between Lie groups and their algebras when dealing with representations.

Definition 3.1.1. Given a Lie group G and a finite dimensional vector space V over a field $\mathbb{K}$, a *linear representation of* G is a smooth homomorphism $\sigma : G \longrightarrow Aut(V)$.
Given a linear representation σ, a subspace W of V is said to be *invariant* if $\sigma(g)w \in W$ for every $g \in G$ and $w \in W$. A representation is *irreducible* if the only invariant subspaces are trivial.
In particular, if V is a complex Hilbert space, we say that σ is *unitary* if $\sigma(g)$ is a unitary operator for every $g \in G$.
Moreover, if G is a matrix group, we say that a representation is *standard*

if $\sigma \equiv Id$.

Definition 3.1.2. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and V a finite dimensional vector space over a field $\mathbb{K}$. A *representation of* $\mathfrak{g}$ is an algebra homomorphism $\rho : \mathfrak{g} \longrightarrow End(V)$, that is, a linear function preserving the Lie bracket: $\rho : [X,Y] \longmapsto [\rho(X), \rho(Y)]$, for every $X, Y \in \mathfrak{g}$.
Similarly to definition 3.1.1, we define an *irreducible* and a *standard representation* for a Lie algebra.

So, a Lie group representation is a group representation, as we have seen in 1.2.7, which is smooth.

Observation 3.1.3. Let G be a Lie group and V a n-dimensional $\mathbb{K}$-vector space, with a representation $\sigma : G \longrightarrow Aut(V)$. We know that σ is a smooth map by definition, hence it is differentiable. Suppose that the identity element of G is e, the differential $d\sigma$ evaluated in e is a function between $\mathfrak{g} = T_eG$ and $T_{Id}Aut(V)$. If we fix a basis for V, we obtain that $Aut(V) \cong GL(n,\mathbb{K})$ and $T_{Id}Aut(V) = \mathfrak{gl}(n,\mathbb{K}) \cong M(n,\mathbb{K})$, since $dim(GL(n,\mathbb{K})) = n^2$. The matrix exponential makes the following diagram commutative:

$$\begin{tikzcd} G \arrow[r, "\sigma"] & Aut(V) \cong GL(n,\mathbb{K}) \\ \mathfrak{g} = T_eG \arrow[u, "exp"] \arrow[r, "(d\sigma)_e"] & T_{Id}Aut(V) \cong M_n \arrow[u, "exp"'] \end{tikzcd}$$

This result is found in [12], chapter 2, section 10.
Notice that, by Theorem 2.7.3 in [12], $\rho = (d\sigma)_e$ is an algebra homomorphism between $\mathfrak{g}$ and $M(n,\mathbb{K})$. Indeed, not only ρ is a linear function, but it preserves the bracket: given $A, B \in \mathfrak{g}$, $\rho([A,B]) = [\rho(A), \rho(B)]$. Hence, for every morphism (or representation) of G in $GL(n,\mathbb{K})$ we have a morphism (or representation) of $\mathfrak{g}$ in $M(n,\mathbb{K})$.

$$\begin{tikzcd} \sigma : G \to GL(n,\mathbb{K}) \arrow[r, dashed, "diff"] & (d\sigma)_e = \rho : \mathfrak{g} \to M(n,\mathbb{K}) \arrow[l, dashed, bend right, "(\ddagger)"'] \\ \sigma(g_1g_2) = \sigma(g_1)\sigma(g_2) \arrow[r, dashed] & \rho([A,B]) = [\rho(A), \rho(B)] \end{tikzcd}$$

Now we are interested in ($\ddagger$), that is, we want to know under which condition it is possibile to obtain a representation of G starting from a representation of $\mathfrak{g}$. Such question is of utter importance, since it is commonly easier to work with representations of $\mathfrak{g}$, as they are linear maps between vector spaces, than with representations of G, which are smooth functions between manifolds. From Proposition 2.7.5 in [12] we obtain that such correspondence exists if G is simply connected:

Proposition 3.1.4. *Let G be a simply connected Lie group and $\mathfrak{g}$ its Lie algebra. Then for every morphism of algebras $\rho : \mathfrak{g} \longrightarrow M(n,\mathbb{K})$ there exists a unique smooth group homomorphism $\sigma : G \longrightarrow GL(n,\mathbb{K})$ such that $(d\sigma)_e = \rho$.*

Since $SL(2,\mathbb{C})$ is simply connected (see [2], chapter 13, section 3), we conclude that its representations can be obtained studying and classifying the representations of its Lie algebra. In order to achieve this, we classify the representations of $\mathfrak{sl}(2,\mathbb{C})$.

3.2 Representations of $\mathfrak{sl}(2,\mathbb{C})$

We begin our inquiry with a standard representation of $SL(2,\mathbb{C})$, which will be fundamental for our subsequent generalization.

$$\mu : SL(2,\mathbb{C}) \times \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \ , \ \mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}. \tag{3.1}$$

It can be expressed equivalently by the morphism $\sigma : SL(2,\mathbb{C}) \longmapsto GL(2,\mathbb{C})$, $\sigma(A) = A$ for every $A \in SL(2,\mathbb{C})$. For a clearer notation we will write $A \cdot v = \sigma(A)v$ instead of $\mu(A,v)$, with $A \in SL(2,\mathbb{C})$ and $v \in \mathbb{C}^2$.
We are now interested in the Lie algebra represention $\rho = d\sigma_e : \mathfrak{sl}(2,\mathbb{C}) \longrightarrow M(2,\mathbb{C})$ induced by σ, that is, the action of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathbb{C}^2$. Following observation 2.6.7, we recall that

$$\mathfrak{sl}(2,\mathbb{C}) = span_{\mathbb{C}}\left\{H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \ X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}.$$

Hence, considering that ρ is a linear function, it is sufficient to study its behaviour on the basis $\{H, X, Y\}$.

To achieve this we will use curves, in a way similar to the above mentioned observation 2.6.7. We already showed in observation 2.6.6 that it is possibile to build a curve with given initial point and velocity using the matrix exponential. So, considering H first, the curve we use is

$$\gamma(t) = e^{tH} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in SL(2,\mathbb{C}) \text{ with } t \in (-\epsilon, \epsilon) \text{ and } \epsilon \in \mathbb{R}^+.$$

Since $\gamma(0) = I$ and $\gamma'(0) = H$, computing $\frac{d}{dt}\Big|_{t=0}(\gamma(t) \cdot v)$ will give as result $H \cdot v = \rho(H)v$.

$$\frac{d}{dt}\Big|_{t=0}\left[\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}\right] = \frac{d}{dt}\Big|_{t=0}\begin{pmatrix} e^t x \\ e^{-t} y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}.$$

We conclude that $\begin{pmatrix} x \\ 0 \end{pmatrix}$ is an eigenvector of H with eigenvalue 1, while $\begin{pmatrix} 0 \\ y \end{pmatrix}$ is an eigenvector with eigenvalue -1.
We repeat this procedure with X and Y, using respectively $\gamma(t) = I + tX$ and $\gamma(t) = I + tY$ as curves.

$$\frac{d}{dt}\Big|_{t=0}\left[\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}\right] = \frac{d}{dt}\Big|_{t=0}\begin{pmatrix} x + ty \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}. \tag{3.2}$$

$$\frac{d}{dt}\Big|_{t=0}\left[\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}\right] = \frac{d}{dt}\Big|_{t=0}\begin{pmatrix} x \\ tx + y \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix}. \tag{3.3}$$

Note that we could omit the $\cdot$ in the previous equations, as the action described by μ is exactly the vector-matrix product. Indeed, considering that $\sigma = Id$, we could conclude that $\rho = d\sigma_e = Id$ and $\rho(H) = H$, which is exactly what we obtained through the above computations: $H \cdot v = \rho(H)v = Hv$. The same is true for X and Y.
Hence, the representation of $\mathfrak{sl}(2,\mathbb{C})$ on $M(2,\mathbb{C})$ is $\mathfrak{sl}(2,\mathbb{C})$ itself. We will return later on this representation.

The group action μ is fundamental for our generalization on $\mathbb{C}^n$. For example, let us consider $\mathbb{C}^3$, seen as $span_{\mathbb{C}}\{\, x^2\,,\, xy\,,\, y^2\,\}$, that is the space of all second degree homogeneous polynomials in x, y. The action of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ is based on the matrix-vector product $A^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$, resulting in: $x \longmapsto dx - by$ and $y \longmapsto -cx + ay$. Here x, y are intended as complex numbers, but starting from this we obtain our action on the polynomials:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x^2 = (dx - by)^2 = d^2x^2 - 2dbxy + b^2y^2;$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot xy = (dx - by)(-cx + ay) = -cdx^2 + (ad + bc)xy - bay^2;$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot y^2 = (-cx + ay)^2 = c^2x^2 - 2caxy + a^2y^2;$$

by linearity it is extendend to all $\mathbb{C}^3$. This approach can be further generalized to $\mathbb{C}^{n+1} = span_{\mathbb{C}}\{\, x^n\,,\, x^{n-1}y\,,\, \cdots\,,\, y^n\,\}$.

Observation 3.2.1. The procedure used here to extend the action of $SL(2,\mathbb{C})$ on $\mathbb{C}^{n+1}$ is the application of a more general method: let G be a group acting on a $\mathbb{K}$ vector space V. This induces an action of G on the functions $f : V \longrightarrow \mathbb{K}$

$$(g, f)(v) = f(g^{-1}v) \quad \text{for every } \; g \in G, v \in V.$$

This action is well defined, since:

$$(e, f)(v) = f(e^{-1}v) = f(v)$$

and

$$\begin{aligned}(g_1(g_2, f))(v) = (g_2, f)(g_1^{-1}v) = f(g_2^{-1}g_1^{-1}v) = \\ = f((g_1g_2)^{-1}v) = (g_1g_2, f)(v).\end{aligned}$$

So we have a representation σ_n, or equivalently an action μ_n, of $SL(2,\mathbb{C})$ for every $n \geq 1$, where the first and basic action μ introduced in 3.1 can now be considered μ_1 and its representation σ_1. We will often omit this subscript to keep a lighter notation, as it is commonly obvious from the context.

As we did before with the representation in 3.1, we are now interested in the action of the algebra $\mathfrak{sl}(2,\mathbb{C})$ on $\mathbb{C}^n$. We exhibit some examples on this to better understand the classification result, which we will enunciate later. Starting from $\mathbb{C}^3$ and the action of H:

$$\frac{d}{dt}\Big|_{t=0}\left[\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}\cdot x^2\right] = \frac{d}{dt}\Big|_{t=0}(e^{-t}x)^2 = \frac{d}{dt}\Big|_{t=0} e^{-2t}x^2 = -2x^2;$$

$$\frac{d}{dt}\Big|_{t=0}\left[\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}\cdot xy\right] = \frac{d}{dt}\Big|_{t=0}(e^{-t}x)(e^{t}y) = \frac{d}{dt}\Big|_{t=0} xy = 0;$$

$$\frac{d}{dt}\Big|_{t=0}\left[\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}\cdot y^2\right] = \frac{d}{dt}\Big|_{t=0}(e^{t}y)^2 = \frac{d}{dt}\Big|_{t=0} e^{2t}y^2 = 2y^2.$$

We conclude that the basis vectors x^2 , xy , y^2 are eigenvector of H with eigenvalue -2 , 0 , 2 respectively.

Using the same curves of equations 3.2 and 3.3, for X and Y we have:

$$\frac{d}{dt}\Big|_{t=0}\left[\begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}\cdot x^2\right] = \frac{d}{dt}\Big|_{t=0}(x-ty)^2 = \frac{d}{dt}\Big|_{t=0} x^2 - 2txy + t^2y^2 = -2xy;$$

$$\frac{d}{dt}\Big|_{t=0}\left[\begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}\cdot xy\right] = \frac{d}{dt}\Big|_{t=0}(x-ty)y = \frac{d}{dt}\Big|_{t=0} xy - ty^2 = -y^2;$$

$$\frac{d}{dt}\Big|_{t=0}\left[\begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}\cdot y^2\right] = \frac{d}{dt}\Big|_{t=0} y^2 = 0;$$

$$\frac{d}{dt}\Big|_{t=0}\left[\begin{pmatrix} 1 & 0 \\ t & 1\end{pmatrix}\cdot x^2\right] = \frac{d}{dt}\Big|_{t=0} x^2 = 0;$$

and so on for the action of Y on the basis vectors xy and y^2.
We have thus described the action of H in terms of its eigenvectors, which are a basis for $\mathbb{C}^3$, and the action of X, Y through the relations between

these vectors. We can summarise these passages in a scheme:

$$\begin{tikzcd}
\text{Eigenvector} & \text{Eigenvalue} \\
\mathbf{0} & \\
\mathbf{y^2} \arrow[u, bend left, "X"] \arrow[loop right, "H"] \arrow[d, bend left, "Y"] & 2 \\
\mathbf{xy} \arrow[u, bend left, "X"] \arrow[loop right, "H"] \arrow[d, bend left, "Y"] & 0 \\
\mathbf{x^2} \arrow[u, bend left, "X"] \arrow[loop right, "H"] \arrow[d, bend left, "Y"] & -2 \\
\mathbf{0} &
\end{tikzcd}$$

All these results can be extended to $\mathbb{C}^{n+1} = span_{\mathbb{C}}\{\, x^n \,,\, x^{n-1}y \,,\, \cdots \,,\, y^n \,\}$ for any n. The computations above can be carefully replicated for H, X, Y with the same curves as before, but on the now $n+1$ basis vectors. So we have:

$$\frac{d}{dt}\Big|_{t=0}\left[\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}\cdot x^n\right] = \frac{d}{dt}\Big|_{t=0}(e^{-t}x)^n = \frac{d}{dt}\Big|_{t=0}e^{-nt}x^n = -nx^n;$$

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\left[\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}\cdot x^{n-1}y\right] &= \frac{d}{dt}\Big|_{t=0}(e^{-t}x)^{n-1}(e^t y) = \\ &= \frac{d}{dt}\Big|_{t=0}(e^{-(n-1)t}x^{n-1})(e^t y) = -(n-2)x^{n-1}y;\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\left[\begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}\cdot x^n\right] &= \frac{d}{dt}\Big|_{t=0}(x-ty)^n = \\ &= \frac{d}{dt}\Big|_{t=0}\left[\sum_{k=0}^{n}\begin{pmatrix} n \\ k\end{pmatrix} x^k(-ty)^{n-k}\right] = -nx^{n-1}y.\end{aligned}$$

Analogous results can be found with with the other basis vectors. So it can be shown that the action of H, X, Y on $\mathbb{C}^{n+1}$ is a consistent extension of the previous case with $\mathbb{C}^3$, obtaining the following scheme based on the eigenvectors of H:

Eigenvector	Eigenvalue
$\mathbf{0}$	
$\mathbf{y^n}$	n
$\mathbf{y^{n-1}x}$	$n-2$
$\mathbf{y^{n-2}x^2}$	$n-4$
$\vdots$	$\vdots$
$\mathbf{x^n}$	$-n$
$\mathbf{0}$	

Note that for even n there will be a sequence of $2\,;\,0\,;\,-2$ in the eigenvalues, while for odd n the sequence will be $1\,;\,-1$. In both these situation, the smallest eigenvalue is $-n$. The full derivation of these representations, called *Ladder Representations*, which can also be proven to be irreducible, is found in [2], chapter 4, section 2.

3.3 Classification Theorem

Now that we have introduced the Ladder Representations, we are ready to state the classification theorem for $\mathfrak{sl}(2,\mathbb{C})$ and show the connections between this general result and the previous representations. For the proof, see [12], chapter 4, Theorem 4.2.2 .

Theorem 3.3.1. *For every $m \in \mathbb{Z}^+$ there exists a unique irreducible complex representation of $\mathfrak{sl}(2,\mathbb{C})$ with dimension $m+1$ given as above and every irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$ with dimension $m+1$ is one of those.*

Example 3.3.2. Suppose that we want to study the representation for $m = 1$: the space is $V_1 = \mathbb{C}^{1+1} = \mathbb{C}^2$. With the function ρ having $M(2,\mathbb{C})$ as codomain and the eigenvalues of $\rho(H)$ being $\{1; -1\}$, it is clear that this is a standard representation, the same as the one seen in 3.1. Thus $\rho \equiv Id$ and $v_0 = e_1, v_1 = e_2$, while $\rho(X)e_2 = Xe_2 = e_1$ and $\rho(Y)e_1 = Ye_1 = e_2$.

Example 3.3.3. Another important representation is the *Adjoint*, where $\mathfrak{sl}(2,\mathbb{C})$ acts on itself through the bracket:

$$\begin{aligned} \rho(H):\ & X \to [H,X] = 2X \\ & Y \to [H,Y] = -2Y \\ & H \to [H,H] = 0 \end{aligned} \qquad\qquad \begin{aligned} \rho(X):\ & X \to [X,X] = 0 \\ & Y \to [X,Y] = H \\ & H \to [X,H] = -2X \end{aligned}$$

$$\begin{aligned} \rho(Y):\ & X \to [Y,X] = -H \\ & Y \to [Y,Y] = 0 \\ & H \to [Y,H] = 2Y \end{aligned}$$

Which is exactly the representations with $m = 2$, or dimension 3, that is $\mathfrak{sl}(2,\mathbb{C})$ acting on $\mathbb{C}^3$. Indeed, we know that both $\mathbb{C}^3$ and $\mathfrak{sl}(2,\mathbb{C})$ are complex vector spaces of dimension 3, hence they are isomorphic once we fix a basis. In this situation, the most situable basis for $\mathfrak{sl}(2,\mathbb{C})$ is $\{X; H; Y\}$, as it is made of eigenvectors. For example, with this basis, $\rho(H) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

Chapter 4

Poincaré Group and Mackey's Theorem

In this chapter we discuss the "Mackey Machine", a method to classify the representations of a group which is a semidirect product. This technique is very important for its physics application. In the first section we introduce the Poincaré group, describing first its structure and then showing the Penrose realisation. This is the most convenient expression of the group for our analysis of its representations, which is treated in the second part of this chapter. We then enunciate the Mackey's theorem, which is the key result for the system of imprimitivity, that is, the idea of deriving the representation of a semidirect product of groups through the representation of the groups themselves.

4.1 The Poincaré Group

Physically, the Poincaré group is the group of all Minkowksi spacetime isometries, or affine transformations preserving the Minkowski metric. As such, it is a proper subgroup of the affine group (see example 1.3.5). Let us briefly recall what the structure of the affine group in $\mathbb{R}^n$ is, and then we will see how Minkowski spacetime comes into this framework.

Consider V a real n-vector space. We call *affine group*, denoted $Aff(V)$, the group of all the affine transformations of V, which are all the linear invertible transformations of V together with the translations $T_v : V \longrightarrow V$, $T_v(w) = w + v$ for every $v, w \in V$. This group can be also written as

$$Aff(V) = \{(v, h) \ , \ v \in V \ , \ h \in GL(V)\}.$$

It is clear from our introduction on groups, in particular from example 1.3.5, that such group is described by a semidirect product, with $GL(V)$ acting on the group of translations, which is clearly isomorphic to the space V itself, through the standard action of $GL(V)$ on V. Hence,

$$Aff(V) = V \rtimes GL(V) = \{(v, h) \ , \ v \in V \ , \ h \in GL(V)\},$$

with

$$(v_1, h_1)(v_2, h_2) = (v_1 + h_1 v_2, h_1 h_2).$$

In particular, if we fix a basis for V, we can write the affine group in terms of matrices:

$$Aff(\mathbb{R}^n) = \mathbb{R}^n \rtimes GL(n, \mathbb{R}) = \{(v, M) \ , \ v \in \mathbb{R}^n \ , \ M \in GL(n, \mathbb{R})\}.$$

with the same group operation as before, which is now computed through matrix products. We can view $Aff(\mathbb{R}^n)$ as a subset of $GL(n+1, \mathbb{R})$, through the identification

$$Aff(\mathbb{R}^n) \ni (v, M) \longleftrightarrow \begin{pmatrix} M & u \\ \mathbf{0} & 1 \end{pmatrix} \in GL(n+1, \mathbb{R}).$$

where $\mathbf{0}$ is the zero row-vector of $\mathbb{R}^n$. Such matrix is in $GL(n+1, \mathbb{R})$ since its determinant is $det(M)$, which is non-zero by definition. A easy computation show that this structure, with the matrix product, behaves exactly as the semidirect product written before.

Definition 4.1.1. The *Minkowski spacetime* is a 4-dimensional real vector space equipped with the *Minkowski inner product*, a nondegenerate symmetric bilinear form with metric signature $+, -, -, -$. Once we fix a suitable basis we can identify the Minkowski spacetime with $\mathbb{R}^4$ with the scalar product:

$$\langle\, u\,,\, v\,\rangle = u_0v_0 - u_1v_1 - u_2v_2 - u_3v_3 \quad u, v \in \mathbb{R}^4.$$

Given a point $x = (x_0, x_1, x_2, x_3)$ in the Minkowski spacetime, we will usually refer to the coordinate x_0 as the *time coordinate* and to (x_1, x_2, x_3) as the *spatial coordinates.*

The *Lorentz group*, denoted $O(1,3)$, is the group of all linear isometries of the Minkowski spacetime. Hence, the elements of this group, called *Lorentz transformations*, are the automorphisms of $\mathbb{R}^4$ that preserve the Minkowski inner product.

The *Poincaré group* is $\mathcal{P} = T_{\mathbb{R}^4} \rtimes O(1,3)$, that is the group of all the affine trasformations of $\mathbb{R}^4$ preserving the Minkowski inner product, or isometries of the Minkowski spacetime. Since $T_{\mathbb{R}^4} \cong \mathbb{R}^4$, we can write the group directly as $\mathcal{P} = \mathbb{R}^4 \rtimes O(1,3)$.

Observation 4.1.2. It is possibile to give a description of $O(1,3)$ in terms of matrices. It is a subgroup of $GL(4,\mathbb{R})$ whose elements preserve the Minkowski inner product:

$$\langle\, gu\,,\, gv\,\rangle = \langle\, u\,,\, v\,\rangle \quad u, v \in \mathbb{R}^4\,,\, g \in O(1,3).$$

Hence,

$$O(1,3) = \{g \in GL(4,\mathbb{R}) \mid g^T I_{1,3} g = I_{1,3}\},$$

with $I_{1,3} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$.

In particular, we have

$$SO(1,3)^0 = \{g \in GL(4,\mathbb{R}) \mid g^T I_{1,3} g = I_{1,3}\,,\ det(g) = 1\}^0.$$

Observation 4.1.3. $O(1,3)$ is the isotropy subgroup of $0 \in \mathbb{R}^4$ of the Poincaré group, since every linear transformation fixes the origin. Following examples 1.1.8 and 2.4.8, it easy to see that $O(1,3)$ is a Lie group. Computing $Lie(O(,1,3)) = \mathfrak{o}(1,3)$ we obtain a vector space of $dim = 6$, hence the Lorentz group is a six-dimensional Lie group. Intuitively, this is reasonable since the Lorentz transformations are spatial rotations (three degrees of freedom) and Lorentz boosts, which is a coordinate transformation between systems in relative motion and can be viewed as an hyperbolic rotation in a 3-dimensional space: given two frames of reference with coordinates respectively (x_0, x_1, x_2, x_3) and (x'_0, x'_1, x'_2, x'_3) in relative motion along the x-axis with velocity v, the equations for the Lorentz transformation (boost) between the two frames of reference are

$$x'_0 = \gamma(x_0 - \frac{v}{c}x_1) \ ; \ x'_1 = \gamma(x_1 - \frac{v}{c}x_0) \ ; \ x'_2 = x_2 \ ; \ x'_3 = x_3$$

where c is the speed of light in a vacuum and $\gamma = \dfrac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} \in [1, +\infty)$.
Such transformation can be rewritten as

$$x'_0 = -x_1 sinh(\phi) + x_0 cosh(\phi) \ ; \ x'_1 = x_1 cosh(\phi) - x_0 sinh(\phi)$$

$$x'_2 = x_2 \ ; \ x'_3 = x_3$$

with $cosh(\phi) = \gamma$ and $sinh(\phi) = \gamma\frac{v}{c}$, which means that $\phi \in \mathbb{R}$.
For a generic Lorentz boost, we have three possibile spatial direction for the relative velocity v, hence for this kind of transformation we have another three degrees of freedom. In particular, with the previous basis, this transformation can be expressed with the matrix

$$\begin{pmatrix} cosh(\phi) & -sinh(\phi) & 0 & 0 \\ -sinh(\phi) & cosh(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which has determinant $cosh^2(\phi) - sinh^2(\phi) = 1$, so it is a proper Lorentz transformation.
The connected component of the identity is the *Restricted Lorentz group* $SO(1,3)^0$, which is itself a Lie group (See example 2.4.3). Moreover, being an open set in $O(1,3)$, it still has $dim = 6$. Physically it consists of *proper* and *orthochronous* Lorentz transformations, which are linear isometries with $det = 1$ preserving both space and time orientation.

For a detailed derivation of these results, see [5], chapter 3, section 1-2.

For the analysis of the Poincaré group representations it is necessary to restrict the Lorentz group $O(1,3)$ to the Restricted Lorentz group $SO(1,3)^0$, as this group is much easier to study and the conclusions are still valid for the entire group. Hence, from now on we take as the Poincaré group $\mathcal{P} = \mathbb{R}^4 \rtimes SO(1,3)^0$.

4.2 The Penrose Realisation of the Poincaré Group

We introduce now the Penrose realisation of the Poincaré group: we realise $\mathcal{P}$ as $\mathbb{R}^4 \rtimes SL(2,\mathbb{C})$, where $SL(2,\mathbb{C})$ is viewed as a real Lie group. Indeed, it is possibile to show that there exists a local isomorphism $SO(1,3)^0 \cong SL(2,\mathbb{C})$. We are not going to prove this (see [9], chapter 2, section 11), but we will show why it is reasonable in terms of group action.
In order to show the action of $\mathcal{P}$ on $\mathbb{R}^4$ we need to consider this vector space as the space of the 2×2 hermitian matrices through the identification:

$$x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \longleftrightarrow \begin{pmatrix} x_0 - x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 + x_1 \end{pmatrix} = M.$$

It is easy to see that $M^\dagger = M$ and the above matrix entries generate every 2×2 hermitian matrix. Moreover, we have:

$$det \begin{pmatrix} x_0 - x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 + x_1 \end{pmatrix} = x_0^2 - x_1^2 - x_2^2 - x_3^2,$$

which is exactly the norm of the corresponding vector $(x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ in the Minkowski metric. The action of $L \in SL(2,\mathbb{C})$ on a 2×2 hermitian matrix is defined as

$$M = \begin{pmatrix} x_0 - x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 + x_1 \end{pmatrix} \longmapsto L^\dagger M L^{-1}.$$

This action, which is not linear, preserves the Minkowski metric. In fact, by Binet's Theorem

$$det(LXL^\dagger) = det(L^\dagger) det(X) det(L^{-1}) = det(X).$$

The complete action of $\mathcal{P}$ on $\mathbb{R}^4$ is thus:

$$(u, L) : M \longmapsto L^\dagger M L^{-1} + u.$$

So it is reasonable to use $SL(2, \mathbb{C})$ with this action instead of $SO(1, 3)^0$, because they both preserve the metric.

Observation 4.2.1. We can view the Penrose realisation considering the Minkowski spacetime as the subset of 2×2 hermitian matrix of the Grassmanian $Gr(2, \mathbb{C}^4)$ and $\mathcal{P} = \{(u, L) \ , \ u \in \mathbb{R}^4 \ , \ L \in SL(2, \mathbb{C})\}$ as a subgroup of the conformal group $SL(4, \mathbb{C})$. There is indeed a bijection between $\mathcal{P}$ and the subgroup $\left\{ \begin{pmatrix} L & \mathbf{0} \\ uL & L^\dagger \end{pmatrix} \right\}$ of $SL(4, \mathbb{C})$. Notice that we are effectively in $SL(4, \mathbb{C})$ since every block is a 2×2 complex matrix and the determinant is $det(L)det(L^\dagger) = 1$. Hence, given M a 2×2 coordinate matrix for the space of hermitian matrices, the Poincaré group acts as:

$$\begin{pmatrix} L & \mathbf{0} \\ uL & L^\dagger \end{pmatrix} \begin{pmatrix} 1 \ 0 \\ 0 \ 1 \\ M \end{pmatrix} = \begin{pmatrix} L \\ uL + L^\dagger M \end{pmatrix} = \begin{pmatrix} 1 \ 0 \\ 0 \ 1 \\ u + L^\dagger M L^{-1} \end{pmatrix},$$

where we applied $L^{-1} \in SL(2, \mathbb{C})$ in the last equivalence, since it left unchanged the subspace expressed by the matrix and put it in an explicit form. As we see, the action of $(u, L) \in \mathcal{P}$ on M is the same as the one introduced above, now expressed though a matrix product.

4.3 Characters

We introduce now the concept of character, which is a one-dimensional representation.

Definition 4.3.1. A complex *character* of a group A is a group homomorphism $\chi : A \longrightarrow \mathbb{C}^\times$, where $\mathbb{C}^\times$ is the multiplicative group of $\mathbb{C}$.
The set of all the characters of a group A is denoted $\widehat{A}$. We define in $\widehat{A}$ a pointwise multiplication:

$$(\chi_1 \cdot \chi_2)(a) = \chi_1(a)\chi_2(a) \quad \text{with } a \in A, \chi_1, \chi_2 \in \widehat{A}.$$

$\widehat{A}$ is an abelian group.

Example 4.3.2. Consider $A = \mathbb{R}^4$. For every $p = (p_0, p_1, p_2, p_3) \in \mathbb{R}^4$ we have a character

$$\chi_p : \mathbb{R}^4 \longrightarrow \mathbb{C}^\times \ , \ \chi_p(x) = e^{i\langle\, x\, ,\, p\,\rangle} = e^{i(x_0p_0 - x_1p_1 - x_2p_2 - x_3p_3)}. \qquad (4.1)$$

It is an homomorphism since $x + y \longmapsto e^{i\langle\, x+y\, ,\, p\,\rangle} = \chi_p(x)\chi_p(y)$.
We refer to p as the *momentum*.

Observation 4.3.3. It is not difficult to see that all the characters of $\mathbb{R}^4$ are of the form 4.1 (see theorem 8.19 in [1] for the complete and detailed proof): the only (smooth) morphisms from $\mathbb{R}$ to $\mathbb{C}^\times$ are of the form $t \longmapsto e^{i\lambda t}$, $\lambda \in \mathbb{R}$. Now, since every $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ can be written as $v = \sum_{i=1}^{n} v_i e_i$ and our character $\chi : \mathbb{R}^n \longrightarrow \mathbb{C}^\times$ is a homomorphism, we have

$$\chi(v) = \chi(\sum_{i=1}^{n} v_i e_i) = \prod_{i=1}^{n} \chi(v_i e_i) = \prod_{i=1}^{n} e^{iv_i\lambda_i} = e^{i\langle\, v\, ,\, p\,\rangle}$$

for some $\lambda_i \in \mathbb{R}$, $i = 1, \dots, n$ and $p = (\lambda_1, \dots, \lambda_n)$.
We have thus obtained that there is a group isomorphism between $\widehat{\mathbb{R}^4}$ and $\mathbb{R}^4$:

$$\widehat{\mathbb{R}^4} = \{\chi_p : \mathbb{R}^4 \to \mathbb{C}^\times \ , \ \chi_p(x) = e^{i\langle\, x\, ,\, p\,\rangle}\} \cong \mathbb{R}^4 \ , \ \chi_p \longmapsto p$$

Notice that this isomorphism depends on the inner product chosen for χ.

Since we are interested in $G = A \rtimes H$, there is an action $\mu : H \times A \longrightarrow A$ which defines the group operation in the semidirect product. This action of H on A induces a group action $\widehat{\mu} : H \times \widehat{A} \longrightarrow \widehat{A}$ on the characters of A. We have:

$$(\widehat{\mu}(h, \chi))(x) = \chi(h^{-1}x).$$

For what we said in observation 3.2.1, this action is well defined.

Observation 4.3.4. For $H = SO(1,3)^0$, the action $\widehat{\mu}$ induces an action on the momenta:

$$\widehat{\mu}(h, \chi_p)(x) = \chi_p(h^{-1}x) = e^{i\langle\, h^{-1}x\, ,\, p\,\rangle},$$

so, by definition of $h \in SO(1,3)^0$,

$$\langle\, h^{-1}x\,,\, p\,\rangle = x^T(h^{-T}I_{1,3}h^{-1})hp = x^T I_{1,3}hp,$$

which means that $\widehat{\mu}(h, \chi_p) = \chi_{hp}$. Hence, the action $\widehat{\mu}$ of $SO(1,3)^0$ on the characters $\widehat{\mathbb{R}^4}$ can be viewed as an action on the momenta, or $\mathbb{R}^4$. Notice that this action is formally the same as the one expressed by μ. However, μ acts on the points of the Minkowski spacetime, while $\widehat{\mu}$ acts on the space of the momenta.

4.4 Mackey's Theorem

We have now all the premiliminary knowledge necessary to formulate the Mackey's Theorem. We shall not go in details on the concepts of unitary representations when V is ∞-dimensional. For a complete proof of this result, its derivation and a full discussion we refer to [8], chapter 3, section 8.

Theorem 4.4.1. *Let $G = A \rtimes H$ be a group. There exists a bijection between the unitary irreducible representations of G and the pairs $(O_\chi, \sigma_{H_\chi})$, where O_χ is the orbit of $\chi \in \widehat{A}$ through the action of H and σ_{H_χ} is a representation of H_χ, the stabilizer of χ for the action of H.*

Our main interest for the Mackey's Theorem is in the Poincaré group $\mathcal{P}$, so our analysis of the pairs $(O_\chi, \sigma_{H_\chi})$ will always consider $A = \mathbb{R}^4$, $\chi = \chi_p$ and $H = SO(1,3)^0$.
Following observation 4.3.4, we have that the orbits O_{χ_p} are subsets of $\mathbb{R}^4$, precisely:

$$O_{\chi_p} = \{p' = hp \text{ , for every } h \in SO(1,3)^0\}.$$

In order to determine the explicit structure of the orbits, we prove the following classification result:

Proposition 4.4.2. *The orbits for the Restricted Lorentz group $SO(1,3)^0$ are the hyper-surfaces $X_m^\pm = \{x_0^2 - x_1^2 - x_2^2 - x_3^2 = \pm m^2, x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \text{ , } m \in \mathbb{R} \geq 0\}$, with X_m^+ associated to $+m^2, x_0 > 0$; X_m^- associated with*

$+m^2, x_0 < 0$ and X_m with $-m^2$. In particular, they are of the following types:

- *Q_1, for $+m^2$ and $x_0 > 0$: upper branch of a two-sheet hyperboloid.*
- *Q_2, for $+m^2$ and $x_0 < 0$: lower branch of a two-sheet hyperboloid.*
- *Q_3, for $m = 0$ and $x_0 > 0$: upper branch of a cone.*
- *Q_4, for $m = 0$ and $x_0 < 0$: lower branch of a cone.*
- *Q_5, for $-m^2$: a one-sheet hyperboloid.*
- *Q_6, for $x_0 = x_1 = x_2 = x_3 = 0$: the origin of coordinates.*

Sketch of the proof. It is clear that every transformation $h \in SO(1,3)^0$ maps points of Q_i in Q_i, $i = 1, ..., 6$, as it is a transformation that preserves the Minkowski metric, so it leaves the norm unchanged.
Consider now the intersection of the surfaces Q_i, $i = 1, ..., 6$, with the half-plane generated by x_0 and $x_3 > 0$. We obtain six types of curves on this half-plane:

- The upper and lower half-branch ($x_3 > 0$) of the hyperbola $x_0^2 - x_3^2 = +m^2$, for $x_0 \gtrless 0$.
- Half upper asyntote $x_0 = x_3$ for $x_0 > 0$ and half lower asyntote $x_0 = -x_3$ for $x_0 < 0$.
- The right-hand branch of an hyperbola $x_0^2 - x_3^2 = -m^2$.
- The origin of coordinates $x_0 = x_3 = 0$.

Every one of these curves is transitive for an hyperbolic rotation in the plane (x_0, x_3). We have seen in observation 4.1.3 that such transformations are proper, and it is clear that a transformation along a single axis is orthochronous, hence it is $SO(1,3)^0$.
Given now two points q_1 and q_2 on a surface Q_i, it is possibile to carry them on the half-plane (x_0, x_3) , $x_3 > 0$ through a spatial rotation which does not alter the first coordinate x_0. Such rotations are transformations in $SO(1,3)^0$, since they preserve both time and space orientation. The two

point obtained thus lie on the same curve and we can map one on the other through an hyperbolic rotation. Since the transformations used to map q_1 to q_2 are all in $SO(1,3)^0$, we conclude that the above surface is transitive for the Restricted Lorentz group, and thus is an orbit. □

Observation 4.4.3. Starting from the Restricted Lorentz group $SO(1,3)^0$, we have to add spatial and temporal reflection to the transformations to obtain the whole Lorentz group $O(1,3)$ (see [5], chapter 3, section 2). A spatial reflection preserves the orbits seen in the previous proposition, but a temporal reflection interchanges the two branches of the hyperboloid and the cone, hence the orbits for $O(1,3)$ are of only four types:

- Q'_1, for $+m^2$: a two-sheet hyperboloid.
- Q'_2, for $m = 0$: a cone.
- Q'_3, for $-m^2$: a one-sheet hyperboloid.
- Q'_4, for $x_0 = x_1 = x_2 = x_3 = 0$: the origin of coordinates.

Now we the study of the representations of H_{χ_p}, the stabilizer of χ_p, also called the *little group* at p. Let χ_p be a character with $p = (p_0, p_1, p_2, p_3)$ and $p_0^2 - p_1^2 - p_2^2 - p_3^2 = +m^2$. We know that the action of $H = SO(1,3)^0$ on χ_p can be transferred on the element $p \in \mathbb{R}^4$. Therefore it is sufficient to study the stabilizer of a generic element of the orbit of p since, following observation 1.2.13, the stabilizers of the elements in the same orbit are conjugate. It is not difficult to see that for conjugate subgroups the representations are the same, since the groups are isomorphic.
Consider the point $q = (m, 0, 0, 0)$ in the orbit of p. Notice that for $(-m, 0, 0, 0)$ we would have a different orbit, since it would change the sign of p_0. Every element of $h \in H_{\chi_q}$ shall satisfy the condition $hq = q$, which in terms of matrices is

$$\left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & h' & \\ 0 & & & \end{array}\right)\begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

with $h' \in SO(3)$. It is clear that the first row has to be $(1,0,0,0)$ or such equality would not hold. This means that the first column has the same entries, since $h \in SO(1,3)^0$ implies that $h^{-1} = h^T$. So we reduced our problem to the determination of the representations of $SO(3)$. Since the Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are isomorphic (see [3], chapter 10, section 6), they have the same representations. Moreover, since $SU(2) \subset SL(2,\mathbb{C})$ is a compact group and the complexification $\mathfrak{su}(2) \otimes \mathbb{C}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ (see [3] chapter 7, section 6) the representations of $SU(2)$ are the same of $SL(2,\mathbb{C})$ (See [12], chapter 4, section 11). Hence, following theorem 3.3.1, we have all the finite-dimensional irreducible unitary representation of $SL(2,\mathbb{C})$ and thus we obtain those of $\mathfrak{so}(3)$.
However, $SO(3)$ is not simply connected ([2], chapter 1, section 3), so we cannot apply proposition 3.1.4: there is no bijection between the representations of the Lie algebra $\mathfrak{so}(3)$ and the representations of the Lie group $SO(3)$. Still, all the representations of the group $SO(3)$ can be found between the representations of $\mathfrak{sl}(2,\mathbb{C})$, precisely, they are the representations corresponding to odd $j \in \mathbb{Z}^+$ (see [10], chapter 2, section 5).
Unfortunately this result is not physically relevant, since it would not identify all the elementary particles that have been experimentally found. For a comprehensive mathematical modelization of the elementary particles is thus necessary to consider the group $SL(2,\mathbb{C})$ instead of $SO(1,3)^0$, as we have seen in the Penrose realisation of the Poincaré group. The group $SL(2,\mathbb{C})$ is the universal (2-fold) cover of $SO(1,3)^0$ and the local isomorphism ϕ between these two groups can be restricted to their respective subgroups $SU(2)$ and $SO(3)$ ([3], chapter 10, "Further Results") :

$$\begin{tikzcd} SU(2) \arrow[r, "\phi|_{SU(2)}", "2:1"'] \arrow[d, hook] & SO(3) \arrow[d, hook] \\ SL(2,\mathbb{C}) \arrow[r, "\phi", "2:1"'] & SO(1,3)^0 \end{tikzcd}$$

This means that we can consider all the representations of $SU(2)$, which are those of $SL(2,\mathbb{C})$, and obtain a representation of the little group at q for every $j \in \mathbb{Z}^+$. Notice that in Physics it is preferred to classify these representations with the half-integers $\dfrac{j}{2}$, $j \in \mathbb{Z}^+$.

This conclusion also holds for the little group of the orbits X_m^-, where the starting point is $q = (-m, 0, 0, 0)$.

We have thus obtained a class of representation of the Poincaré group, identified by the pair (m, j) $m > 0$, where m is given by the orbit and identifies the mass of the particle, and j is given by the representation and $\frac{j}{2}$ is the spin of the particle. Those particles are the massive particles of the Standard Model, such as the fermions (quarks and leptons) and the Z, W, and Higgs bosons. The representations given by $(-, m, j)$ $m > 0$ are those obtained starting from an orbit X_m^-. There is an antiunitary isomorphism (see [13], chapter 1, section 5) between the representations given by $(-, m, j)$ and those of (m, j), so the first one can be interpreted as the antiparticles with opposite charge.

Analogous conclusions can be derived for the the other types of orbit shown in proposition 4.4.2, obtaining the following results ([9], chapter 3, section 2):

Orbit	Point	Little Group
X_m^+	$(m, 0, 0, 0)$	$SU(2)$
X_m^-	$(-m, 0, 0, 0)$	$SU(2)$
X_0^+	$(1, 0, 0, 1)$	$\mathbb{R}^2 \rtimes U(1)$
X_0^-	$(-1, 0, 0, -1)$	$\mathbb{R}^2 \rtimes U(1)$
$(0, 0, 0, 0)$	$(0, 0, 0, 0)$	$SL(2, \mathbb{C})$
$X_m\,,\,-m^2$	$(0, im, 0, 0)$	$SL(2, \mathbb{R})$

We conclude that the orbits in $X_0^{\pm}$ identify particles with zero mass travelling at the speed of light, which are photons and, hypothetically, gravitons. The orbit $(0, 0, 0, 0)$ has different representations, but they are all considered unphysical except for the trivial one-dimensional representation of the little group, which can be considered as a model of the vacuum ([13], chapter 1, section 5). The last ones, X_m, $-m^2$, should be particles with imaginary mass im, which are experimentally considered unphysical.

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