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Reconstruction of schemes via the tensor triangulated category of perfect complexes

M.Sc. Thesis

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Introduction

This thesis aims to cover the knowledge needed in order to present notions and results, introduced for the first time by Paul Balmer, which gave rise to the field of tensor triangular geometry, with a particular focus on the applications to algebraic geometry. The definition of spectrum of a tensor triangulated category is central, and the general theory that can be developed around this tool has, as a guiding example, the case of the category of perfect complexes of sheaves of modules on a scheme X, endowed with the derived tensor product. In this case, the spectrum coincides with X, and therefore if the triangulated categories of perfect complexes on two schemes X and Y are tensor equivalent, then the schemes are isomorphic. It's however remarkable that tensor triangulated categories, and therefore applications of tensor triangular geometry, arise in many areas of mathematics, such as stable homotopy theory and modular representation theory.

Questions as to what extent and under which conditions a category of sheaves on a scheme determines the scheme go back in the years. In the end of its 1962 thesis *Des Catégories Abéliennes* ([Gab62]), Pierre Gabriel introduced the notion of spectrum of the category of sheaves of modules, aimed to reconstruct a scheme. In 1980s, Alexander L. Rosenberg defined the spectrum of any abelian category, and proved that a quasi-compact and quasi-separated scheme can be reconstructed from the abelian category of quasi-coherent sheaves of modules. This spectrum is constructed as the set of equivalence classes of certain subcategories, called topologizing, together with a natural topology and a structure sheaf defined in terms of centers (endotransformation ring of the identity functor) of some abelian categories. It follows that two schemes having equivalent categories of quasi-coherent sheaves of modules are isomorphic.

A slightly different question concerns the derived category of the abelian category of (coherent, quasi-coherent) sheaves of modules. Bondal and Orlov proved in 1997 (see [BO97]) that for a smooth irreducible projective variety X with ample canonical bundle, the bounded derived category $\mathrm{D}^b(\mathbf{Coh}(\mathcal{O}_X))$ characterizes X among all other smooth varieties. This requires a lot of assumptions, and in fact, derived categories of abelian categories of sheaves comes with a quite large loss of information about the scheme, with respect to the abelian categories. This is in fact one of the reasons why derived categories are introduced, in order to look closely to cohomological properties. However, in 2004 Paul Balmer, probably inspired by the Robert W. Thomason's classification of certain thick subcategories of the category of perfect complexes, observed that the latter, a triangulated subcategory of the derived category of sheaves of modules on X, is suitable in order to recover X, using the derived tensor product structure. It's then defined in [Bal04] the spectrum of any triangulated category endowed with a symmetric monoidal structure as the set of those thick subcategories which behave like prime ideals under the tensor product. The topology will be an analogous of the Zariski topology, and eventually the structure sheaf will be defined in terms of endomorphism rings of the unit for the tensor structure.

Chapter 1

Categories and sheaves for geometry

Through this chapter we are going to recall some basic notions and facts from category theory and sheaf theory, highlighting how they arise from topological and geometric examples. Let's start recalling the fundamental result about presheaves on a locally small category.

1.1 The Yoneda Lemma

Definition 1.1.1. Let $\mathbf{C}$ be a locally small category. The category of *presheaves of sets* $\hat{\mathbf{C}}$ is the category whose objects are functors $\mathbf{C}^{op} \to \mathbf{Set}$, and morphisms are all the the natural transformations between these functors. Namely, $\hat{\mathbf{C}} = \mathbf{Set}^{\mathbf{C}^{op}}$.

In order to study geometric spaces, are more useful sets with an algebraic structure rather than just sets. Therefore, can analogously be defined the category of presheaves of abelian groups, or commutative unital rings, by replacing **Set** with respectively **Ab** or **ComRing**.

Example 1.1.2. The functor $\mathcal{P} : \mathbf{Set}^{op} \to \mathbf{Set}$ mapping a set X to its power set $\mathcal{P}(X)$, and a function $f : X \to Y$ to the inverse image mapping $\mathcal{P}(f) : V \mapsto f^{-1}(V)$ is a presheaf. Observe that $\mathcal{P}$ arise in the form of $\mathrm{Hom}(-, \Omega)$, for a set Ω. More precisely, if we take $\Omega = \{0, 1\}$, there is a natural isomorphism $\mathcal{P} \cong \mathrm{Hom}(-, \Omega)$ given object-wise by taking an element $A \in \mathcal{P}(X)$ to the characteristic function $\chi_A : X \to \{0, 1\}$.

Example 1.1.3. Let $U : \mathbf{ComRing} \to \mathbf{Ab}$, or equivalently $\mathbf{AffSch}^{op} \to \mathbf{Ab}$, where **AffSch** is the category of affine schemes (see Remark 1.4.5), be the functor associating to a ring R its group of units $R^{\times}$. This is a presheaf over the category of affine schemes into the category of abelian groups, mapping a morphism of affine schemes $f : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$, equivalently a morphism of rings $g : B \to A$, to $U(f) : B^\times \to A^\times$ defined as a restriction of g. This is certainly a group morphism because g is a ring morphism.

This functor also arise as functor of the form $\mathrm{Hom}(-, Y)$, precisely for the affine scheme $Y = \mathrm{Spec}(\mathbb{Z}[X, X^{-1}])$. There's in fact an isomorphism

$$\begin{aligned} U(R) &\xrightarrow{\psi} \mathrm{Hom}_{\mathbf{AffSch}}(\mathrm{Spec}R, \mathrm{Spec}(\mathbb{Z}[X, X^{-1}])) \\ u &\longmapsto \psi(u) \end{aligned}$$

where $\psi(u)$, seen as a morphism in $\mathrm{Hom}_{\mathbf{ComRing}}(\mathbb{Z}[X, X^{-1}], R)$, evaluates polynomials $p(X, X^{-1}) \mapsto p(u, u^{-1})$. Conversely, for any ring morphism $\mathbb{Z}[X, X^{-1}] \to R$, the image of X need to be a unit with inverse the image of X^{-1}. Quite obviously, these operations are inverses to each other.

We saw two examples of presheaves on a category $\mathbf{C}$ arising in the same way from an object in $\mathbf{C}$. On the other hand, any object C in a locally small category $\mathbf{C}$ gives rise to a presheaf.

Definition 1.1.4. The *Yoneda functor* $y : \mathbf{C} \to \hat{\mathbf{C}}$ is the functor defined, on an object C, to be the presheaf

$$y_C = \mathrm{Hom}_{\mathbf{C}}(-, C).$$

This is a presheaf since it maps an object A of $\mathbf{C}$ to the set $\mathrm{Hom}(A, C)$, and a morphism $g : A \to A'$ to the function

$$\begin{aligned} \mathrm{Hom}(g, C) : \mathrm{Hom}(A', C) &\longrightarrow \mathrm{Hom}(A, C) \\ \alpha &\longmapsto \alpha g \end{aligned}$$

On a morphism $f : C \to C'$, the functor y is defined object-wise to be the precomposition by f

$$\begin{aligned} y(f)_A : \mathrm{Hom}(A, C) &\longrightarrow \mathrm{Hom}(A, C') \\ \alpha &\longmapsto f\alpha \end{aligned}$$

The main foundational result describing the category of presheaves, which is a starting point for the whole category theory, is the following theorem, and is a result about morphisms from a representable presheaf.

Theorem 1.1.5 (Yoneda Lemma)**.** *For a presheaf F on a category $\mathbf{C}$ and an object C in $\mathbf{C}$, there is a bijection*

$$f_{C,F} : \mathrm{Hom}_{\hat{\mathbf{C}}}(y_C, F) \longrightarrow F(C)$$

Moreover, this bijection is natural in both F and C in the following sense: if $\eta : F \to F'$ and $g : C' \to C$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}(y_C, F) & \xrightarrow{f_{C,F}} & F(C) \\ {\scriptstyle \mathrm{Hom}(y(g),\eta)}\downarrow & & \downarrow{\scriptstyle \eta_{C'}F(g)} \\ \mathrm{Hom}(y_{C'}, F') & \xrightarrow[f_{C',F'}]{} & F'(C') \end{array}$$

*commutes in **Set**.*

Proof. Note, first of all, that $\mathrm{Hom}(y(g), \eta)$ is the function mapping $\nu \mapsto \eta\nu y(g)$.

The bijection is given on one hand by evaluation at the identity: the map $f_{C,F}$ sends $\eta : y_C \to F$ to $\eta_C(1_C) \in F(C)$. Conversely, an element $x \in F(C)$ determines a natural transformation that at the object C' is $\eta^x_{C'} : y_C(C') = \mathrm{Hom}(C', C) \to F(C')$, given by mapping $f : C' \to C$ to the evaluation at x of $F(f) : F(C) \to F(C')$. Let's prove that this defines a bijection. The element $\eta_C(1_C)$ uniquely determines $\eta : y_C \to F$, since it holds, for C' in $\mathbf{C}$ and $f : C' \to C$, that $\eta_{C'}(f) = F(f)(\eta_C(1_C))$, by the naturality of η at C

$$\begin{array}{ccc} y_C(C) & \xrightarrow{\eta_C} & F(C) \\ {\scriptstyle y_C(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ y_C(C') & \xrightarrow[\eta_{C'}]{} & F(C') \end{array}$$

So, the first map is injective. To associate to $x \in F(C)$ the natural transformation $\eta^x : y_C \to F$ also is an injective operation, and this can be checked at 1_C. More precisely, if $\eta_1 = \eta_2$ are such that for every C' and every $f : C' \to C$ one has $\eta_{1C'} : f \mapsto F(f)(x_1)$, and $\eta_{2C'} : f \mapsto F(f)(x_2)$, then in particular this holds for $C' = C$ and $f = 1_C$, showing that if $\eta_1 = \eta_2$, then $x_1 = x_2$.

Moreover, let $g : C' \to C$ in $\mathbf{C}$ and $\eta : F \to F'$ in $\hat{\mathbf{C}}$. The map $\mathrm{Hom}(y(g), \eta)$ sends the natural transformation $\nu : y_C \to F$ to the natural transformation $\lambda : y_{C'} \to F'$ defined, for any C'' and $h : C'' \to C'$, by

$$\lambda_{C''}(h) = \eta_{C''}(\nu_{C''}(y(g)_{C''}(h))),$$

that is $\eta_{C''}(\nu_{C''}(gh))$. Observe then that

$$f_{C',F'}(\lambda) = \lambda_{C'}(1_{C'}) = \eta_{C'}(\nu_{C'}(g)) =$$
$$= \eta_{C'}(F(g)(\nu_C(1_C))) = (\eta_{C'}F(g))(f_{C,F}\nu).$$

This proves naturality of the bijection. □

Corollary 1.1.6. *The functor $y : \mathbf{C} \to \hat{\mathbf{C}}$ is fully faithful.*

Proof. for C, C' in $\mathbf{C}$ it holds, applying Yoneda to the functor $F = y_{C'}$,

$$Hom_{\mathbf{C}}(C, C') = y_{C'}(C) \cong Hom_{\hat{\mathbf{C}}}(y_C, y_{C'})$$

□

Corollary 1.1.6 says that y is an embedding, and hence that we can look at $\mathbf{C}$ as sitting in its category of presheaves $\hat{\mathbf{C}}$. Objects in $\hat{\mathbf{C}}$ of the form y_C for some C in $\mathbf{C}$ are called *representable* functors. As we are going to remark, the category $\hat{\mathbf{C}}$ is in general much larger that $\mathbf{C}$, nevertheless, representable functors contains all the information needed to compute any presheaf. More precisely, the following theorem holds true.

Theorem 1.1.7. *Any presheaf is colimit of a diagram of representable presheaves.*

Proof. The construction is explicit and canonical. Let P be a presheaf and consider the category $\int_{\mathbf{C}} P$ whose objects are pairs $(C, \mu : y_C \to P)$ and morphisms $(C, \mu) \to (C', \mu')$ are the arrows $f : C \to C'$ in $\mathbf{C}$ making the triangle

$$\begin{tikzcd} y_C \arrow[rr, "y(f)"] \arrow[dr, "\mu"'] & & y_{C'} \arrow[dl, "\mu'"] \\ & P & \end{tikzcd}$$

to commute. Observe that there is a projection functor $\pi_P : \int_{\mathbf{C}} P \to \mathbf{C}$ mapping $(C, \mu) \mapsto C$. The composition of this functor with y defines a diagram in $\hat{\mathbf{C}}$

$$\int_{\mathbf{C}} P \xrightarrow{\pi_P} \mathbf{C} \xrightarrow{y} \hat{\mathbf{C}}$$

with a cocone having vertex P and cocone morphisms

$$\rho_{(C,\mu)} = \mu : y\pi_P(C, \mu) = y_C \longrightarrow P$$

In order to prove this cocone to be limiting, suppose Q, with morphisms $\lambda_{(C,\mu)} : y_C \to Q$, to be another cocone over the same diagram. Let's define a morphism $\nu : P \to Q$ such that $\nu\rho_{(C,\mu)} = \lambda_{(C,\mu)}$. For any C and $\xi \in P(C)$, consider the morphism $\mu : y_C \to P$ corresponding to ξ under the Yoneda Lemma, and set $\nu_C(\xi) = (\lambda_{(C,\mu)})_C(\mathrm{id}_C)$.

More explicitly, $\mu : y_C \to P$ is the morphism defined on the component D to be $\mu_D : (g : D \to C) \mapsto Pg(\xi)$.

Therefore, a morphism $f : C' \to C$ defines a morphism $(C', \mu') \to (C, \mu)$ where $\mu' : y_{C'} \to P$ is the morphism corresponding to $Pf(\xi) \in PC'$, so that for any object D and morphism $g : D \to C'$ one has

$$\mu_D y f_D(g) = \mu_D(fg) = P(fg)(\xi) = Pg(Pf(\xi)) = \mu'_D(g),$$

namely, the triangle

$$\begin{tikzcd} y_{C'} \arrow[rr, "yf"] \arrow[dr, "\mu'"'] & & y_C \arrow[dl, "\mu"] \\ & P & \end{tikzcd}$$

commutes. Then, observe that also the triangle

$$\begin{tikzcd} y_{C'} \arrow[rr, "yf"] \arrow[dr, "\lambda_{(C',\mu')}"'] & & y_C \arrow[dl, "\lambda_{(C,\mu)}"] \\ & Q & \end{tikzcd}$$

is commutative because λ defines a cocone. Now we need to prove the commutativity of each of the triangles of the form

$$\begin{tikzcd} P \arrow[r, "\nu"] & Q \\ y_C \arrow[u, "\mu"] \arrow[ur, "\lambda_{(C,\mu)}"'] & \end{tikzcd}$$

Let $f : C' \to C$ be a morphism and suppose μ to correspond under the Yoneda Lemma to an element which we call ξ. Therefore, consider the morphism $\mu' : y_{C'} \to P$ constructed above corresponding to $Pf(\xi)$. We can compute

$$\nu_{C'}(\mu_{C'}(f)) = \nu_{C'}(\mu_{C'}(yf_{C'}(\mathrm{id}_{C'}))) = \nu_{C'}(\mu'_{C'}(\mathrm{id}_{C'})) = \nu_{C'}(P(\mathrm{id}_{C'})Pf(\xi)) = \nu_{C'}(Pf(\xi)) = (\lambda_{(C',\mu')})_{C'}(\mathrm{id}_{C'}) = (\lambda_{(C,\mu)})_{C'}(yf(\mathrm{id}_{C'})) = (\lambda_{(C,\mu)})_{C'}(f).$$

This proves the commutativity of the desired triangles. Uniqueness is easily given because if ν and ν' are such that $\nu\mu = \lambda_{(C,\mu)} = \nu'\mu$ for every $\mu : y_C \to P$, then for any C and $\xi \in P(C)$ one can consider the morphism $\mu : y_C \to P$ corresponding to ξ, and observe that

$$\nu_C(\xi) = \nu_C(\mu_C(\mathrm{id}_C)) = (\lambda_{(C,\mu)})_C(\mathrm{id}_C) = \nu'_C(\mu_C(\mathrm{id}_C)) = \nu'_C(\xi)$$

□

Remark 1.1.8. From the fact that **Set**, **Ab**, **ComRing** are complete and cocomplete, we can deduce that the category of presheaves on a category **C** with values in sets, abelian groups or commutative rings is both complete and cocomplete. Limits and colimits of presheaves are computed "pointwise", namely if

$$D : \mathbf{I} \to \hat{\mathbf{C}}$$

is a diagram, consider the class of diagrams indexed by Ob(**C**) from **I** into **Set**, **Ab** or **ComRing** given by $i \mapsto D(i)(C)$, and compute the colimit $X(C)$ for this diagram. This defines a presheaf X which is colimit for D. The same holds true for limits.

Remark 1.1.9. In order to taste how many non-representable presheaves there could be, one can consider a category which is not (co)complete, and compute in $\hat{\mathbf{C}}$ any (co)limit which doesn't exist in **C**. Since y is full and faithful, it reflects (co)limits, hence the computed presheaf cannot be representable.

Not only the category of presheaves is complete and cocomplete, but also the Yoneda embedding $y : \mathbf{C} \to \hat{\mathbf{C}}$ is the "free colimit completion" of $\mathbf{C}$ in the following sense.

Theorem 1.1.10. *Let $F : \boldsymbol{C} \to \boldsymbol{D}$ be a functor into a cocomplete category $\boldsymbol{D}$. Then, there exists a unique, up to isomorphism, functor $\tilde{F} : \hat{\boldsymbol{C}} \to \boldsymbol{D}$ that preserves colimits and makes the following diagram to commute*

$$\begin{tikzcd} \boldsymbol{C} \arrow[r, hook, "y"] \arrow[dr, "F"'] & \hat{\boldsymbol{C}} \arrow[d, dashed, "\tilde{F}"] \\ & \boldsymbol{D} \end{tikzcd}$$

Proof's idea. In the notations of Theorem 1.1.7, $\tilde{F}(P)$ is defined to be the colimit of the diagram

$$\int_{\mathbf{C}} P \xrightarrow{\pi_P} \mathbf{C} \xrightarrow{F} \mathbf{D}$$

□

Remark 1.1.11. The functor $\tilde{F}$ is called *left Kan extension* of F. From Theorem 1.1.10 it follows that for every functor $G : \mathbf{C} \to \mathbf{D}$, its composition with the Yoneda embedding $y_{\mathbf{D}}$ of $\mathbf{D}$, induces a functor $\widetilde{y_{\mathbf{D}}G} : \hat{\mathbf{C}} \to \hat{\mathbf{D}}$. This construction happens to define a functor from the category $\mathbf{Cat}$ to its full subcategory of cocomplete categories

$$\hat{(-)} : \mathbf{Cat} \longrightarrow \mathbf{CocompCat}$$

which is left adjoint to the forgetful functor $U : \mathbf{CocompCat} \to \mathbf{Cat}$.

1.2 Sheaves on topological spaces

The theory of sheaves has a very general flavor which allows to talk about them over any (locally small) category. However, for our purposes aimed to the study of schemes, it will be sufficient to consider sheaves on a topological space, namely we restrict ourselves to the case where the category considered is the partially ordered set ΩX of the open sets of a fixed topological space X, where the arrows are the inclusion morphisms.

Definition 1.2.1. Let X be a topological space. A *sheaf on* X is a presheaf F in $\hat{\Omega X}$ such that for any $U \subseteq X$, any open cover $\{U_i\}_{i \in I}$ of U and any set $\{x_i \in F(U_i) | i \in I\}$ such that $\forall i, j \in I$

$$x_i|_{U_i \cap U_j} = x_j|_{U_i \cap U_j}$$

there exists a unique $x \in F(U)$ such that $x|_{U_i} = x_i$ for all $i \in I$.

The full subcategory of $\hat{\Omega X}$ whose objects are sheaves is denoted by $\mathbf{Sh}(X)$. The category $\hat{\Omega X}$ is usually denoted by $\mathbf{Psh}(X)$. Whenever $i : V \subseteq$

U is a morphism in ΩX and P is a (pre)sheaf on X, the induced morphism $P(i) : P(U) \to P(V)$ is called *restriction*, and $P(i)(s)$ is denoted by $s|_V$.

One of the main reason why we care about sheaves is the notion of *stalk*, and in particular Proposition 1.2.11 below. The following constructions will be done in the setting of sheaves of sets, and then extended to the case of sets with an algebraic structure.

Definition 1.2.2. Denote by $\Omega_x X \subseteq \Omega X$ the partially ordered set of the open neighborhoods of x. The *stalk* at a point $x \in X$ of a presheaf P on X (i.e. a presheaf on ΩX) is the colimit of the diagram

$$\Omega_x X^{op} \longrightarrow \mathbf{Set}$$

induced by P, that associate to any $U \ni x$ the set $P(U)$, and acting the same as P on morphisms $V \subseteq U$.

Such colimit is denoted by P_x, and the images under the colimit morphisms of a section $s \in P(U)$ (which certainly does not depend on U) is denoted by s_x, and is called *germ* of s at x.

The stalk of a sheaf is a particular case of a general kind of colimit.

Definition 1.2.3. A non-empty category $\mathbf{I}$ is said to be *filtered* if

(a) for any two objects i, i' there exists an object k with morphism $i \to k$ and $i' \to k$,

(b) for any pair of morphisms $u, v : i \to j$ there is a morphism $w : j \to k$ such that $wu = wv$.

Remark 1.2.4. The opposite category of open neighborhood of a fixed $x \in X$ is filtered. Condition (a) is provided by intersection, while condition (b) is trivial because by definition there are no distinct parallel arrows in a partially ordered set.

Definition 1.2.5. The colimit of a diagram $\mathbf{I} \to \mathbf{C}$ is said to be *filtered* if the category $\mathbf{I}$ is filtered.

Remark 1.2.6. For general diagrams in a complete and cocomplete category $\mathbf{C}$, limits does not commutes with colimits, in the sense that for a diagram

$$D : \mathbf{I} \times \mathbf{J} \longrightarrow \mathbf{C}$$

one can consider limits or colimits over the diagrams with a fixed index, and then take the other one letting vary the previous fixed index. More precisely, we have for a fixed $i \in \mathrm{Ob}(\mathbf{I})$, a functor $D(i, -) : \mathbf{J} \to \mathbf{C}$, of which we can consider the limit, with morphisms

$$\mathrm{Lim}_j D(i,j) \xrightarrow{\eta_j} D(i,j)$$

This defines a diagram $\mathbf{I} \to \mathbf{Set}$ mapping $i \mapsto \mathrm{Lim}_j D(i,j)$, with morphisms, for $f : i \to i'$,

$$\mathrm{Lim}_j D(i,j) \longrightarrow \mathrm{Lim}_j D(i',j)$$

induced by $D(f,j)\eta_j : \mathrm{Lim}_j D(i,j) \to D(i',j)$. We can take the colimit of this diagram, with morphisms

$$\mathrm{Lim}_j D(i,j) \xrightarrow{\lambda_i} \mathrm{Colim}_i \mathrm{Lim}_j D(i,j)$$

Analogously, we can consider the colimit

$$D(i,j) \xrightarrow{\nu_i} \mathrm{Colim}_i D(i,j)$$

and, as before, the limit of the induced diagram $\mathrm{Lim}_j \mathrm{Colim}_i D(i,j)$.

Observe then that there is a canonical morphism which we may call *switching* morphism

$$\mathrm{Colim}_i \mathrm{Lim}_j D(i,j) \longrightarrow \mathrm{Lim}_j \mathrm{Colim}_i D(i,j) \tag{1.1}$$

induced by the family of morphisms

$$\mathrm{Lim}_j D(i,j) \longrightarrow \mathrm{Lim}_j \mathrm{Colim}_i D(i,j)$$

which is in turn induced by the morphisms

$$\nu_i \eta_j : \mathrm{Lim}_j D(i,j) \longrightarrow \mathrm{Colim}_i D(i,j)$$

In general, the switching morphism is not an isomorphism

Example 1.2.7. Consider the categories $\mathbf{I} = (\mathbb{N}, \leq)$ as a partially ordered set, and $\mathbf{J} = \mathbb{N}$ as a discrete category. Define a functor

$$\mathbf{I} \times \mathbf{J} \longrightarrow \mathbf{Set}$$

by $A_{i,j} = A(i,j) = \{0, ..., i\}$, with inclusion morphisms for every $i \leq i'$ arrow in $\mathbf{I}$.

Observe that $\mathrm{Colim}_i A_{i,j} = \bigcup\limits_{i \in \mathbb{N}} \{0, ..., i\} = \mathbb{N}$, with inclusions $\{0, ..., i\} \hookrightarrow \mathbb{N}$ as cocone morphisms, hence

$$\mathrm{Lim}_j \mathrm{Colim}_i A_{i,j} = \prod_{j \in \mathbb{N}} \mathbb{N} = \mathbb{N}^{\mathbb{N}}.$$

On the other hand, $\mathrm{Lim}_j A_{i,j} = \prod\limits_{j \in \mathbb{N}} \{0, ..., i\}$, and the colimit of the induced diagram is

$$\begin{tikzcd}
\mathrm{Colim}_i(\prod_{\mathbb{N}}\{0, ..., i\}) & & & \\
\prod_{\mathbb{N}}\{0\} \arrow[u] \arrow[r] & \prod_{\mathbb{N}}\{0,1\} \arrow[ul] \arrow[r] & \prod_{\mathbb{N}}\{0,1,2\} \arrow[ull] \arrow[r] & \cdots
\end{tikzcd}$$

where the base morphisms are the inclusions induced by $\prod_{\mathbb{N}}\{0, ..., i\} \to \{0, ..., i\} \hookrightarrow \{0, ..., i+1\}$. Therefore, the coproduct is again the union of these objects, namely

$$\mathrm{Colim}_i\mathrm{Lim}_j A_{i,j} = \bigcup_{i\in\mathbb{N}}\{0, ..., i\}^{\mathbb{N}}.$$

The natural morphism

$$\bigcup_{i\in\mathbb{N}}\{0, ..., i\}^{\mathbb{N}} \longrightarrow \mathbb{N}^{\mathbb{N}}$$

maps a function $f : \mathbb{N} \to \{0, ..., i\}$ to the composition with the inclusion

$$\mathbb{N} \xrightarrow{f} \{0, ..., i\} \longrightarrow \mathbb{N}$$

and cannot be surjective because it cannot have unbounded functions in its codomain.

Example 1.2.8. An even easier example of limits not commuting with colimits is the case of binary products and coproducts: If $\mathbf{I} = \mathbf{J} = \{0, 1\}$ are discrete categories with two elements, a diagram in **Set** consists of four sets $A_{0,0}$, $A_{0,1}$, $A_{1,0}$ and $A_{1,1}$. Certainly we can choose this sets in a way that the morphism

$$(A_{1,0} \times A_{1,1}) \sqcup (A_{0,0} \times A_{0,1}) \longrightarrow (A_{0,0} \sqcup A_{1,0}) \times (A_{0,1} \sqcup A_{1,1})$$

is not an isomorphism, because if for example all the sets are the one point set, cardinality fails to be the same because $(1\times1)+(1\times1) \neq (1+1)\times(1+1)$.

The two previous examples shows the failure of a desirable result about commutativity of limits with colimits. In the first one the category **I** is filtered but **J** is infinite, while in the second one the category **J** is finite, but **I** is not filtered. One could then suspect then, that filtered colimits commutes with finite limits, and in fact this is the case for **Set**.

Theorem 1.2.9. *Let $D : \boldsymbol{I} \times \boldsymbol{J} \to \boldsymbol{Set}$ be a diagram where $\boldsymbol{I}$ is filtered and $\boldsymbol{J}$ finite. Then, for any diagram $D : \boldsymbol{I} \times \boldsymbol{J} \to \boldsymbol{Set}$, the morphism*

$$\mathrm{Colim}_i\mathrm{Lim}_j D(i, j) \longrightarrow \mathrm{Lim}_j\mathrm{Colim}_i D(i, j)$$

defined in (1.1) is an isomorphism.

Proof. See [ML97], IX.1.2.1. □

Remark 1.2.10. Taking stalks defines a functor $\mathrm{St}_x : \mathbf{Sh}(X) \to \mathbf{Set}$. On a morphism of sheaves $\phi : F \to G$, this functor is defined to be the colimit map induced by the set of morphisms $\lambda_U\phi_U$, where λ defines the cocone structure of G_x

$$\begin{tikzcd} F_x \arrow[r, dashed, "\phi_x"] & G_x \\ F(U) \arrow[u, "\eta_U"] \arrow[r, "\phi_U"] & G(U) \arrow[u, "\lambda_U"'] \end{tikzcd}$$

Proposition 1.2.11. *A morphism $\phi : F \to G$ in the category of sheaves is a monomorphism (resp. epimorphism) in $\boldsymbol{Sh}(X)$ if an only if for every $x \in X$ the morphism $\phi_x : F_x \to G_x$ is a injective (resp. surjective) map.*

Proof. See [MLM91] II.6.6. □

One way to prove the result above is to make use of the following functor.

Definition 1.2.12. Let X be a topological space and $x \in X$. The *skyscraper sheaf* over x is the functor

$$\mathrm{Sk}_x : \mathbf{Set} \longrightarrow \mathbf{Sh}(X)$$

mapping a set A to the sheaf defined for $U \in \Omega X$ by

$$\mathrm{Sk}_x(A)(U) = \begin{cases} A & \text{if } x \in U \\ \{0\} & \text{if } x \notin U \end{cases}$$

with the obvious restriction morphisms. The identity on A if $x \in V \subseteq U$, and the only existing morphism into the terminal object in the other cases.

Lemma 1.2.13. *Let X be a topological space and $x \in X$. The functor Sk_x is right adjoint to the stalk functor St_x.*

Proof. Let's prove the bijection, for a set A and a sheaf F,

$$\mathrm{Hom}(F_x, A) \cong \mathrm{Hom}(F, \mathrm{Sk}_x(A)).$$

Given a function $f : F_x \to A$ set the morphism $h : F \to \mathrm{Sk}_x(A)$ on a component U to be

$$h_U : F(U) \to \mathrm{Sk}_x(A)(U)$$

defined as the unique possible function $F(U) \to \{0\}$ if $x \notin U$, and as the function $s \mapsto f(s_x)$ if $x \in U$.

Conversely, if $h : F \to \mathrm{Sk}_x(A)$ is a morphism of sheaves, one can define a morphism $f : F_x \to A$ as the colimit map induced by a family of morphisms $\{F(U) \to A\}_{U \ni x}$ commuting with restrictions. This is easily done mapping $s \mapsto h_U(s)$.

These mappings are clearly inverses to each other: if $f : F_x \to A$ is associated to a morphism $h : F \to \mathrm{Sk}_x(A)$, then the morphism f' associated to this h is the unique morphism such that $f'(s_x) = h_U(x)$, but this is f by definition of h. On the other hand, if $h : F \to \mathrm{Sk}_x(A)$ and $f : F_x \to A$ is the morphism associated to it, then the morphism h' associated to f is, on a section U over which is non-trivial, $h'_U : s \mapsto f(s_x)$. By definition of the colimit morphism f, it is $f(s_x) = h_U(s)$, therefore $h = h'$. □

Remark 1.2.14. By Theorem 1.2.9, the functor St_x preserves finite limits. The previous lemma says that it also preserve, as a left adjoint, any colimit.

The following construction provides a canonical procedure to determine a sheaf from a given presheaf. Such a construction will satisfy a universal property, and will be used when operations defined on sheaves will not provide another sheaf, but just a presheaf. The main case of this happening is taking colimits.

Definition 1.2.15. The *étale space* of a presheaf P on a topological space X is the space

$$\Lambda_P = \coprod_{x \in X} P_x$$

with topology generated by those sets of the form $\{(x, s_x) | x \in U\}$, with U ranging over the topology of X and $s \in P(U)$.

Definition 1.2.16. A *bundle* on a topological space X is an object in the category $\mathbf{Top}/X$ whose objects are morphisms of topological spaces

$$p : E \longrightarrow X$$

and morphisms are commutative triangles

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Remark 1.2.17. Observe that the étale space of a presheaf P on X defines a bundle over X with the canonical projection

$$\Lambda_P \longrightarrow X$$

given by $(x, s_x) \mapsto x$, which is continuous by the very definition of the topology on Λ_P.

Moreover, Λ defines in fact a functor $\mathbf{Psh}(X) \to \mathbf{Top}/X$, because to any morphism of presheaves $\eta : P \to Q$ correspond a morphism

$$\Lambda_P = \coprod_{x \in X} P_x \longrightarrow \coprod_{x \in X} Q_x = \Lambda_Q$$

induced by the morphisms $P_x \to \coprod_{x \in X} Q_x$, each of which is induced in turn by the family of morphisms $\eta_U : P(U) \to Q(U)$, for any $U \ni x$, composed with the composition of the two cocone morphisms

$$Q(U) \longrightarrow Q_x \longrightarrow \coprod_{x \in X} Q_x$$

The resulting morphism $\Lambda_P \to \Lambda_Q$ certainly maps each P_x to Q_x, hence it defines a morphism of bundles over X.

Remark 1.2.18. Any bundle determines a sheaf as its sheaf of sections. More in detail, if $f : E \to X$ is a bundle, one can consider the sheaf $\Gamma_f : \Omega X^{op} \to \mathbf{Set}$ defined on an open subset $i : U \hookrightarrow X$ to be the set

$$\{s : U \to E | fs = i\}.$$

This is a clearly a sheaf, and Γ is a functor $\mathbf{Top}/X \to \mathbf{Sh}(X)$, because a morphism of bundles

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "f_1"'] & & E_2 \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

defines a morphism

$$\Gamma_{f_1} \longrightarrow \Gamma_{f_2}$$

defined on an open $U \subseteq X$ to maps sections as $s \mapsto fs$, which is a section on E_2 because $f_2(fs) = f_1 s$ is the inclusion of U in X.

Proposition 1.2.19. *There is an adjunction* $\Gamma \vdash \Lambda$ *between the section and the étale bundle functors*

$$\boldsymbol{Top}/X \underset{\Lambda}{\overset{\Gamma}{\rightleftarrows}} \boldsymbol{Sh}(X)$$

Proof. See [MLM91] II.6.2. □

Remark 1.2.20. As a consequence, the functor Γ preserves limits, while Λ preserves colimits. Moreover, by Theorem 1.2.9 we have that Λ also preserves finite limits. In fact, Λ_P can be described, as a set, as the colimit over the filtered diagram

$$\Omega_x X^{op} \longrightarrow \mathbf{Set}$$
$$U \longmapsto \coprod_{x \in X} P(U)$$

which is in fact $\underset{U \ni x}{\mathrm{Colim}}(\coprod_{x \in X} P(U)) \cong \coprod_{x \in X} \underset{U \ni x}{\mathrm{Colim}} P(U) = \Lambda_P$.

Not only the section functor defines sheaves, the next result says that any sheaf arises as a sheaf of sections, precisely on its étale space.

Definition 1.2.21. Let P be a presheaf on a topological space X and Λ_P its étale space. The sheaf of sections $\Gamma\Lambda_P$ is called *sheafification* of P.

Theorem 1.2.22. *Let P be a presheaf on a topological space X. The natural transformation*

$$\eta : P \longrightarrow \Gamma\Lambda_P,$$

defined on each open U to be $\eta_U(s)(x) = (x, s_x)$, is an isomorphism whenever P is a sheaf.

Proof. See [MLM91] II.5.1. □

The sheafification (morphism) is universal for morphisms into a sheaf.

Theorem 1.2.23. *Let P be a presheaf on a space X and $\eta : P \to \Gamma\Lambda_P$ its sheafification morphism. For any sheaf F and morphism $\sigma : P \to F$ there exists a unique morphism of sheaves $\bar{\sigma} : \Gamma\Lambda_P \to F$ such that the diagram*

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[dr, "\sigma"'] & \Gamma\Lambda_P \arrow[d, dashed, "\bar{\sigma}"] \\ & F \end{tikzcd}$$

commutes in $\boldsymbol{Psh}(X)$.

Proof. See [MLM91] II.5.2. □

More in general, the sheafification functor is a particular case of a more general one arising whenever $f : X \to Y$ is a morphism of topological spaces.

Definition 1.2.24. Let $f : X \to Y$ be a morphism of topological spaces.

If F is a sheaf on X, the *direct image presheaf* of F is the presheaf f_*F on Y defined by

$$V \mapsto F(f^{-1}(V))$$

with restriction morphisms induced by F.

If G a sheaf on Y, the *inverse image presheaf* of G is the presheaf $f^{-1}G$ on X defined by

$$f^{-1}(G)(U) = \operatorname*{Colim}_{V \supseteq f(U)} G(V)$$

where the colimit is taken on the diagram induced by G from the partially ordered set of the open containing $f(U)$. Restriction morphisms are here colimit morphisms, induced by themselves because whenever $U' \subseteq U$, any cocone map $G(V) \to \operatorname*{Colim}_{V \supseteq f(U)} G(V)$ is also a cocone morphism for $\operatorname*{Colim}_{V \supseteq f(U')} G(V)$, since $V \supseteq f(U) \supseteq f(U')$.

Remark 1.2.25. For a morphism of topological spaces $f : X \to Y$ and sheaves F on X and G on Y, the presheaf f_*F is actually a sheaf, while $f^{-1}G$ isn't in general. From now on, unless otherwise stated, by $f^{-1}G$ we mean its sheafification $\Gamma\Lambda_{f^{-1}G}$.

These constructions define functors $f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ and $f^{-1} : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$.

Theorem 1.2.26. *Let $f : X \to Y$ be a morphism of topological spaces. The functor f^{-1} is left adjoint to f_*.*

Proof. The argument goes on proving that there is an adjunction in the category of presheaves

$$\mathrm{Hom}_{\mathbf{Psh}(X)}(f^{-1}G, F) \cong \mathrm{Hom}_{\mathbf{Psh}(Y)}(G, f_*F)$$

if we look at $f^{-1}G$ before the sheafification. Then, from the universal property of the sheafification in Theorem 1.2.23, the left hand side is isomorphic to $\mathrm{Hom}_{\mathbf{Sh}(X)}(f^{-1}G, F)$, while the right hand side is $\mathrm{Hom}_{\mathbf{Sh}(Y)}(G, f_*F)$, because sheaves are defined as a full subcategory of presheaves.

Hence, let's prove the adjunction at the level of presheaves. Let $\eta \in \mathrm{Hom}_{\mathbf{Psh}(Y)}(G, f_*F)$, and consider, for a fixed $U \subseteq X$, the cocone morphisms $\theta_W : G(W) \to f^{-1}G(U)$ for every open $W \supseteq f(U)$. The family of morphisms

$$G(W) \xrightarrow{\eta_W} F(f^{-1}(W)) \xrightarrow{|_U} F(U)$$

defines another cocone structure over the same diagram $\{G_W\}_{W \supseteq f(U)}$, inducing a colimit morphism which we call $(\phi\eta)_U : f^{-1}G(U) \to F(U)$

$$\begin{tikzcd}
G(W) \arrow[r, "\eta_W"] \arrow[d, "\theta_W"'] & F(f^{-1}(W)) \arrow[d, "|_U"] \\
f^{-1}G(U) \arrow[r, dashed, "(\phi\eta)_U"'] & F(U)
\end{tikzcd}$$

This defines a natural transformation $\phi\eta : f^{-1}G \to F$, and hence a morphism

$$\phi : \mathrm{Hom}_{\mathbf{Psh}(Y)}(G, f_*F) \longrightarrow \mathrm{Hom}_{\mathbf{Psh}(X)}(f^{-1}, F)$$

Let's find an inverse of this morphism. Observe that for any open $W \subseteq Y$

$$f_*f^{-1}G(W) = f^{-1}G(f^{-1}(W)) = \operatorname*{Colim}_{V \supseteq f(f^{-1}(W))} G(W) = G(W),$$

therefore the functor f_* induces on morphisms a mapping

$$\mathrm{Hom}_{\mathbf{Psh}(X)}(f^{-1}G, F) \xrightarrow{\psi} \mathrm{Hom}_{\mathbf{Psh}(Y)}(f_*f^{-1}G, f_*F) \cong \mathrm{Hom}_{\mathbf{Psh}(Y)}(G, f_*F)$$

The mappings ϕ and ψ are inverses. The definition of $\phi\eta$ on the component $f^{-1}(W)$ gives

$$(\phi\eta)_{f^{-1}(W)} = \eta_W,$$

but $(\phi\eta)_{f^{-1}(W)}$ is, by definition of f_* on morphism, $(\psi(\phi\eta))_W$. This gives $\psi(\phi(\eta)) = \eta$. On the other hand, suppose $\xi : f^{-1}G \to F$ and use again the definition of the morphism $(\phi\psi\xi)_{f^{-1}W}$, for any open $W \subseteq Y$, which is the unique such that the diagram

$$\begin{tikzcd}
G(W) \cong f_*f^{-1}G(W) \arrow[d, "\theta_W"'] \arrow[dr, "(\psi\xi)_W"] & \\
f^{-1}G(f^{-1}(W)) \arrow[r, dashed] & F(f^{-1}(W))
\end{tikzcd}$$

commutes, where θ_W is, since $f(f^{-1}(W)) = W$, the cocone morphism

$$G(W) \longrightarrow \operatorname*{Colim}_{V \supseteq W} G(V) = G(W),$$

namely the identity. Therefore it suffices. by uniqueness, to prove that $\xi_{f^{-1}(W)}$ is such that it makes commute the above triangle. That is straightforward from the definition of ψ, because

$$(\psi\xi)_W = (f_*\xi)_W = \xi_{f^{-1}(W)} = \theta_W \xi_{f^{-1}(W)}.$$

$\square$

Remark 1.2.27. In the form of one of the isomorphisms stated above, precisely the one

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(f^{-1}G, F) \cong \mathrm{Hom}_{\mathbf{Psh}(Y)}(G, f_*F),$$

the adjoint functors $f^{-1} \dashv f_*$ reduce, in the case $f = \mathrm{id}_X : X \to X$, to be f_* the forgetful functor $U : \mathbf{Sh}(X) \to \mathbf{Psh}(X)$, while f^{-1} the sheafification. Therefore we see that there is an adjunction

$$\Gamma\Lambda \dashv U.$$

Remark 1.2.28. In particular, by the previous Remark we get that the sheafification functor $\Gamma\Lambda$ preserves all colimits, while the forgetful functor U preserves all limits.

In other words, we can say that the limit for a diagram $F : \mathbf{I} \to \mathbf{Sh}(X)$ in the category of sheaves on a space X, seen as a presheaf, is

$$U(\mathrm{Lim}_i F(i)) = \mathrm{Lim}_i UF(i),$$

which means that the limit in $\mathbf{Sh}(X)$ can be computed as the limit in $\mathbf{Psh}(X)$ obtained applying the forgetful functor.

The fact that the sheafification functor preserves colimits implies that if we consider a diagram $\mathbf{I} \to \mathbf{Sh}(X)$ and the composed diagram $UF : \mathbf{I} \to \mathbf{Psh}(X)$, the sheafification of the colimit for this diagram is the colimit of the sheafification of the $UF(i)$'s, which are sheaves yet. Therefore the colimit of sheaves is computed by the sheafification of the colimit of presheaves obtained applying the forgetful functor.

1.3 Sheaves with algebraic structure

In the beginning we have introduced presheaves with values on some categories different from **Set**, but the results and the construction for sheaves in the previous section was entirely focused on the case of sets. In this section we are going to argue that the main results proved about sheaves hold similarly for the categories which we will be interested in, such as sheaves of rings and abelian groups, and can be described starting from the category of sheaves of sets.

I has to be said that to consider algebraic structure on the category of sheaves is going to brutally change many categorical properties of the category of sheaves of sets. Nevertheless, from the fact that both **Ab** and **ComRing** admits forgetful functors with nice algebraic properties, we can specialize most of the arguments above.

Lemma 1.3.1. *The forgetful functors $U : \textbf{Ab} \to \textbf{Set}$ and $U' : \textbf{ComRing} \to \textbf{Set}$ have left adjoint functors.*

Proof. Let $F : \mathbf{Set} \to \mathbf{Ab}$ be the functor associating to any set A the free abelian group on A, that is the abelian group presented by the set A together with only the commutators $a^{-1}b^{-1}ab$ for $a, b \in A$ as relations. A function $g : A \to B$ defines a group morphism in the obvious way, mapping $a_1 \cdots a_n \mapsto g(a_1) \cdots g(a_n)$. Given a group morphism $h : FA \to G$, consider the function $\bar{h} : A \to UG$ defined by $a \mapsto h(a)$, applied to the one element word. Conversely, to any set function $f : A \to UG$ we can associate the group morphism

$$\bar{f} : FA \longrightarrow G$$

mapping a word $a_1 \ldots a_n$ to the product element $f(a_1) \cdots f(a_n)$. These mappings are natural in A and G, and clearly inverses to each other, so that $U \vdash F$.

In order to prove the same for **ComRing**, we consider the functor F' mapping a set A to the polynomial ring $F'(A) = \mathbb{Z}[A]$ with integer coefficients and variables from A. This naturally defines a functor since any function $g : A \to B$ defines a ring morphism $F'(g) : p(a_1, ..., a_n) \mapsto p(g(a_1), ..., g(a_n))$. Similarly as above, a ring morphism $h : \mathbb{Z}[A] \to R$ defines a function $\bar{h}$ on A as seen applied to the one-variable polynomial, and a function $f : A \to U'R$ defines a ring homomorphism

$$\bar{f} : \mathbb{Z}[A] \longrightarrow R$$

mapping $p(a_1, ..., a_n)$ to the evaluation in R of $p(f(a_1), ..., f(a_n))$. Again, it's immediate to see that these mappings are natural in A and R and inverses to each other. $\square$

Remark 1.3.2. In particular, Lemma 1.3.1 says that limits in **Ab** and **ComRing** are computed as in set. In particular, will sounds familiar that the product of abelian groups is the obvious group structure given on the cartesian product of the underlying sets, and the same is true for commutative rings.

Observe, moreover, that any abelian groups or rings morphism which is a bijection of the underlying sets is an isomorphism. That means forgetful functors U and U' reflects isomorphisms.

Remark 1.3.3. A category $\mathbf{C}$ whose objects are defined as sets and morphisms are functions (with additional structure), admits in general a forgetful functor to $U : \mathbf{C} \to \mathbf{Set}$. However, it's not guaranteed for U to reflect isomorphism. Any continuous bijection of topological spaces which is not an isomorphism provides a counterexample. The forgetful functor $\mathbf{Top} \to \mathbf{Set}$ is however faithful.

There's also a forgetful functor, from the category $\mathbf{Sch}$ of schemes to the category of sets, which isn't even faithful. In fact, any morphism

$$f : \operatorname{Spec} \mathbb{C} \longrightarrow \operatorname{Spec} \mathbb{C}$$

is defined by a continuous morphism between the underlined topological spaces, which are just one point sets 1, and a ring morphism $\mathcal{O}_{\operatorname{Spec}\mathbb{C}} \to f_* \mathcal{O}_{\operatorname{Spec}\mathbb{C}}$, namely a ring morphism $\mathbb{C} \to \mathbb{C}$. As well known, there are $2^{2^{\aleph_0}}$ of such morphisms, but only one map of sets $1 \to 1$.

What about colimits? The example of coproducts of two abelian groups A and B, which we know to be the direct sum $A \oplus B$, says that the forgetful functor doesn't preserve colimits. In fact, the underlying set of $A \oplus B$ is emphatically not the coproduct of sets $A \sqcup B$.

Recall, moreover, that the coproduct of commutative rings A and B is given by the tensor product as $\mathbb{Z}$-module $A \otimes B$, with cocone morphisms $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$. Again, the underlying set isn't in general the disjoint union.

However, may be that some kind of colimits do commute with these forgetful functors. This is in fact the case for filtered colimits. Let's see how it works for the case of abelian groups.

Remark 1.3.4. Recall that the colimit of a diagram of sets $D : \mathbf{I} \to \mathbf{Set}$ can be explicitly described as

$$\coprod_{i \in \operatorname{Ob}(\mathbf{I})} D(i) /_{\sim}$$

where $\sim$ is the smallest equivalence relation containing a pair $((i, d_i), (j, d_i))$ whenever there exists $\phi : i \to j$ such that $D(\phi)(d_i) = d_j$. More explicitly, two pairs (i, d_i), (j, d_j) are equivalent if and only if there exists a chain of morphisms in $\mathbf{I}$

$$\begin{tikzcd}
 & i_1 \arrow[dl] \arrow[dr] & & i_3 \arrow[dl] & & i_{2n-1} \arrow[dr] & \\
i = i_0 & & i_2 & & \cdots & & i_{2n} = j
\end{tikzcd}$$

with objects $d_{i_j} \in D(i_j)$ mapping each other through the morphisms induced by D.

If the small category I is filtered, there is an easier description. Let's claim that if $\mathbf{I}$ is filtered, $(i, d_i) \sim (j, d_j)$ if and only if

$$\exists k \exists \phi_i : i \to k, \phi_j : j \to k \text{ s.t. } D(\phi_i)(d_i) = D(\phi_j)(d_j).$$

One one hand it's clear that if this is the case, then $(i, d_i) \sim (k, D(\phi_i)(d_i)) = (k, D(\phi_j)(d_j)) \sim (i, d_j)$. Conversely, if there is a chain of morphisms as described above, one can iteratively apply the two conditions defining a filtered category, finding k_1' with morphisms $i_0 \to k_1' \leftarrow i_2$, and then consider an object k_1 with morphism $k_1' \to k_1$ such that the resulting pair of morphisms $i_1 \to k_1$ coincide. The next step consists of taking an object k_2 and morphisms $k_1 \to k_2 \leftarrow i_4$ such that the composites $i_3 \to k_2$ are the same. This process eventually gives $k = k_n$ as desired.

The stalk of sheaf of sets F, for example, can be explicitly described as

$$F_x = \coprod_{U \ni x} F(U) /\!\sim$$

where $(U, s) \sim (V, t)$, with $s \in F(U)$ and $t \in F(V)$, if and only if there exists $W \subseteq U \cap V$ such that $s|_W = t|_W$. Both the categorical and this more concrete description of a stalk are important.

Here it comes the case of abelian groups

Proposition 1.3.5. *The filtered colimit of a diagram $D : \boldsymbol{I} \to \boldsymbol{Ab}$ is computed as*

$$\mathrm{Colim}_i D(i) = \bigoplus_{i \in \mathrm{Ob}(\boldsymbol{I})} D(i) /\!\sim$$

where $\sim$ is the equivalence relation generated by (i.e. the smallest equivalence relation compatible with the group structure containing) the pairs $(d_i, F(\phi)(d_i))$ whenever $\phi : i \to j$ is a morphism in $\boldsymbol{I}$.

Proof. Cocone morphism for the diagram are the natural inclusions

$$\begin{aligned} D(i) &\longrightarrow \bigoplus_{i \in \mathrm{Ob}(\mathbf{I})} D(i) /\!\sim \\ d_i &\longmapsto [(d_i)] \end{aligned}$$

where (d_i) denotes the sequence where only the i-th element is not the neutral element. The imposed equivalence relation says exactly that these morphisms define a cocone. If Q is an abelian group with morphisms $\eta_i : D(i) \to Q$ defining a cocone structure, the colimit map

$$\bigoplus_{i \in \mathrm{Ob}(\mathbf{I})} D(i) /\!\sim \longrightarrow Q$$

is defined by $(d_i)_{i \in I} = \sum_{i \in \mathrm{Ob}(\mathbf{I})} (d_i) \mapsto \sum_{i \in \mathrm{Ob}(\mathbf{I})} \eta_i(d_i)$. This is certainly the unique morphisms of abelian groups mapping $(d_i) \mapsto \eta_i(d_i)$. $\square$

Proposition 1.3.6. *Let $D : \boldsymbol{I} \to \boldsymbol{Ab}$ be a diagram from a filtered category to the category of abelian groups, and consider the diagram $UD : \boldsymbol{I} \to \boldsymbol{Set}$. The canonical morphism*

$$\mathrm{Colim}_i UD(i) \longrightarrow U(\mathrm{Colim}_i D(i))$$

is an isomorphism of sets.

Proof. It has to be proven that the morphism of sets

$$\psi : \coprod_{i \in \mathrm{Ob}(\mathbf{I})} UD(i) \,/\!\sim \longrightarrow U\Big(\bigoplus_{i \in \mathrm{Ob}(\mathbf{I})} D(i) \,/\!\sim\Big)$$
$$[d_i] \longmapsto [(d_i)]$$

is a bijection. If $[d_i]$ and $[d_j]$ are classes of element $d_i \in UD(i) \subseteq \coprod UD(i)$ and $d_j \in UD(j) \subseteq \coprod UD(j)$ such that $[(d_i)] = [(d_j)]$, we can find a finite chain of morphisms

$$\begin{tikzcd}
 & i_1 \arrow[dl] \arrow[dr] & & i_3 \arrow[dl] & & i_{2n-1} \arrow[dr] & \\
i = i_0 & & i_2 & & \cdots & & i_{2n} = j
\end{tikzcd}$$

such that $d_i \in D(i)$ comes from an element in $d_{i_1} \in D(i_1)$, which is mapped to an element $d_{i_2} \in D(i_2)$, and so on up to reach $d_j \in D(j)$. Since $\mathbf{I}$ is filtered, we can iterate the same process described above of finding an object k_1' with morphisms $i \to k_1' \leftarrow i_2$, and an object k_1 with a morphism $k_1' \to k_1$ such that the two resulting morphisms $i_1 \to k_1$ coincide. This process ends providing two morphisms $\phi_i : i \to k_n$ and $\phi_j : j \to k_n$ such that the induced morphisms $D(\phi_i)$ and $D(\phi_j)$ map respectively d_i and d_j to the same element. Therefore, we get that $[d_i] = [d_j]$ in the domain of ψ. Hence, that the morphism ψ is injective.

In order to prove surjectivity, consider an element

$$[(d_i)_{i \in I}] \in \bigoplus_{i \in \mathrm{Ob}(\mathbf{I})} D(i)/\sim$$

where all but finitely many are the neutral element of the group, say $(d_i) = (d_{i_1}, \dots, d_{i_n})$. Since $\mathbf{I}$ is filtered, we can find an object k with morphisms

$$\phi_j : i_j \to k$$

for every $j \in \{1, \dots, n\}$. Therefore, we can consider the finite sum

$$\sum_{j=1}^{n} D(\phi_j)(d_{i_j}) \in D(k).$$

This element is a one element sequence, so it is in the range of ψ, and it's in the same equivalence class of

$$(d_i)_{i\in I} = (d_{i_1}, \ldots, d_{i_n}) = \sum_{j=1}^{n} d_{i_j}$$

because the equivalence relation on the codomain of ψ is compatible with the group structure. $\square$

It follows the analogous result to Theorem 1.2.9.

Corollary 1.3.7. *Let $D : \boldsymbol{I} \times \boldsymbol{J} \to \boldsymbol{Ab}$ be a diagram where $\boldsymbol{I}$ is filtered and $\boldsymbol{J}$ is finite. Then the canonical morphism*

$$\mathrm{Colim}_i \mathrm{Lim}_j D(i,j) \longrightarrow \mathrm{Lim}_j \mathrm{Colim}_i D(i,j)$$

is an isomorphism.

Proof. Since the result holds true for the functor $UD : \mathbf{I} \to \mathbf{Set}$, and U preserves (all) limits and filtered colimits, it holds

$$\begin{aligned} U(\mathrm{Colim}_i \mathrm{Lim}_j D(i,j)) &\cong \mathrm{Colim}_i \mathrm{Lim}_j UD(i,j) \\ \cong \mathrm{Lim}_j \mathrm{Colim}_i UD(i,j) &\cong U(\mathrm{Lim}_j \mathrm{Colim}_i D(i,j)). \end{aligned}$$

The result follows since U also reflects isomorphisms. $\square$

Definition 1.3.8. A *sheaf of abelian groups* on a topological space X is a functor $F : \Omega X^{op} \to \mathbf{Ab}$ such that the composition $UF : \Omega X^{op} \to \mathbf{Set}$ of F with the forgetful functor U is a sheaf of sets. Morphisms between sheaves of abelian groups are all natural transformations. This defines the category $\mathbf{Ab}(\mathbf{Sh}(X))$.

Analogously, a sheaf of commutative rings on X is a functor $F : \Omega X^{op} \to \mathbf{ComRing}$ such that $U'F$ is a sheaf of sets, where $U' : \mathbf{ComRing} \to \mathbf{Set}$ is the forgetful functor.

Definition 1.3.9. A *bundle of abelian groups* is a surjective bundle $p : E \to X$ in $\mathbf{Top}/X$ together with a structure of abelian group on each fiber $(p^{-1}(x), +)$ in such a way that abelian group operation

$$m : Y \times_X Y \longrightarrow Y$$

mapping (y, y') with $x = p(y) = p(y')$ to $y + y' \in p^{-1}(x)$, and

$$i : Y \to Y$$

mapping $y \mapsto y^{-1} \in p^{-1}(p(y))$, are continuous morphisms.

Both these constructions are a special case of the general notion of *inner abelian group*.

Definition 1.3.10. Let $\mathbf{C}$ be a locally small category. An object G in $\mathbf{C}$ is an *abelian group object* if for every C in $\mathbf{C}$ there is an abelian group structure on the set $\mathrm{Hom}(C, G)$ in such a way that y_G defines a functor $\mathbf{C}^{op} \to \mathbf{Ab}$. A morphism of group objects G, G' is a morphism $f : G \to G'$ in $\mathbf{C}$ such that for every C in $\mathbf{C}$ the morphism $y(f)(C) : \mathrm{Hom}(C, G) \to \mathrm{Hom}(C, G')$ is a group morphism.

This defines a category denoted by $\mathbf{Ab}(\mathbf{C})$.

Remark 1.3.11. To give a natural structure of abelian group to the set $\mathrm{Hom}_{\mathbf{Sh}(X)}(y_C, F)$ for any C, is the same as to give a natural structure of an abelian group to $F(C)$, by Yoneda Lemma. Therefore, sheaves of abelian groups are the abelian group object in the category $\mathbf{Sh}(X)$. It's straightforward to prove as well that the category of bundles of abelian groups is the category $\mathbf{Ab}(\mathbf{Top}/X)$.

Remark 1.3.12. Observe that the definition of abelian group object in a category $\mathbf{C}$ with finite products, and hence with a terminal object 1, gives the following morphisms:

$$u : 1 \longrightarrow G,$$

the neutral element in $\mathrm{Hom}(1, G)$, called *unit*,

$$m : G \times G \longrightarrow G$$

defined by the sum in $\mathrm{Hom}(G \times G, G)$ of the two projections $p, q : G \times G \to G$, called *multiplication*, and eventually the *inverse* morphism

$$i : G \longrightarrow G$$

defined to be the group inverse of $\mathrm{id}_G \in \mathrm{Hom}(G, G)$.

Proposition 1.3.13. *Let $F : \boldsymbol{C} \to \boldsymbol{D}$ be a functor between categories with finite products, and suppose F to preserve them. Then F maps abelian group objects of $\boldsymbol{C}$ to abelian group objects of $\boldsymbol{D}$.*

Proof. Let G be a group object in $\mathbf{C}$ and D any object in D, let's define a group structure for

$$\mathrm{Hom}(D, FG).$$

The neutral element of this group is defined to be the morphism

$$D \longrightarrow 1_{\mathbf{D}} \cong F(1_{\mathbf{C}}) \xrightarrow{F(u)} FG$$

where $D \to 1_{\mathbf{D}}$ is the unique morphism into the terminal object, while u is the unit morphism of the group object G.

Given two morphisms $f, g \in \mathrm{Hom}(D, FG)$, they induce a morphism $(f, g) : D \to FG \times FG \cong F(G \times G)$. The morphism $f + g \in \mathrm{Hom}(D, FG)$ is then defined to be the composition

$$D \xrightarrow{(f,g)} F(G \times G) \xrightarrow{F(m)} F(G)$$

where $m : G \times G \to G$ is the multiplication morphism.

Eventually, if $f \in \mathrm{Hom}(D, FG)$, we analogously define $-f$ as the composition

$$D \xrightarrow{f} FG \xrightarrow{F(i)} FG$$

where i is the inverse morphism.

The verifications that this actually defines an abelian group structure on $\mathrm{Hom}(D, FG)$ is straightforward. $\square$

Remark 1.3.14. It follows that both Γ and Λ, which preserves respectively all and finite limits, and hence products, define functors

$$\mathbf{Ab}(\mathbf{Top}(X)) \underset{\Lambda}{\overset{\Gamma}{\rightleftarrows}} \mathbf{Ab}(\mathbf{Psh}(X))$$

which still give rise to an adjunction, whose unit $\eta_P : P \to \Gamma\Lambda_P$ is said to be *sheafification* of the presheaf of abelian groups P.

From now on, when no confusion arises, we denote the category of sheaves of abelian groups on X just by $\mathbf{Sh}(X)$.

The notions of direct and inverse images can be transposed in the context of sheaves of abelian groups, too. If $f : X \to Y$ is a morphism of topological spaces and F, G are sheaves of abelian groups respectively on X and Y, the direct image f_*F and the inverse image $f^{-1}G$ are defined in the same way as sheaves of abelian groups, and give rise to the adjunction $f^{-1} \dashv f_*$

$$\mathbf{Sh}(X) \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} \mathbf{Sh}(Y)$$

between categories of sheaves of abelian groups.

1.4 Sheaves of modules

For a commutative unital ring R there's a category $\mathbf{Mod}(R)$ of modules over it, which generalizes notions such as ideals, quotients, and provides a nice (abelian) category where to study geometrical properties of $\mathrm{Spec}(R)$.

Analogously, to endow with a sheaf of rings $\mathcal{O}_X$ a topological space X turns it into a so called *ringed space*, and there is a category $\mathbf{Mod}(\mathcal{O}_X)$ which is useful in order to study $(X, \mathcal{O}_X)$, especially when the space $(X, \mathcal{O}_X)$ is a scheme.

Definition 1.4.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module F is a sheaf of abelian groups on X such that for any open $U \subseteq X$ the abelian group $F(U)$ is endowed with the structure of an $\mathcal{O}_X(U)$-module in a way compatible with the restrictions, namely it's required that $(st)|_V = s|_V t|_V$ whenever $V \subseteq U$, and $s \in \mathcal{O}_X(U)$ acts on $t \in F(U)$.

A morphism of $\mathcal{O}_X$-modules is a morphism of sheaves $\eta : F \to G$ such that for any $U \subseteq X$ the morphism η_U is a morphism of $\mathcal{O}_X(U)$-modules. This defines the category $\mathbf{Mod}(\mathcal{O}_X)$.

$\mathcal{O}_X$-modules are also referred to as *sheaves of modules* when the sheaf of rings $\mathcal{O}_X$ is understood. The notion of sheaf of modules allows to import constructions from commutative algebra. One of the main one is the tensor product.

Definition 1.4.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Given $\mathcal{O}_X$-modules F and G consider the presheaf

$$U \longmapsto F(U) \otimes_{\mathcal{O}_X(U)} G(U)$$

and define $F \otimes_{\mathcal{O}_X} G$ to be its sheafification.

Remark 1.4.3. From the fact that the tensor product is commutative and associative over the category of R-modules for a commutative ring R, the same follows for

$$- \otimes_{\mathcal{O}_X} - : \mathbf{Mod}(\mathcal{O}_X) \times \mathbf{Mod}(\mathcal{O}_X) \longrightarrow \mathbf{Mod}(\mathcal{O}_X)$$

More precisely, there are isomorphisms

$$\alpha_{A,B,C} : (A \otimes_{\mathcal{O}_X} B) \otimes_{\mathcal{O}_X} C \longrightarrow A \otimes_{\mathcal{O}_X} (B \otimes_{\mathcal{O}_X} C)$$

natural in A, B and C, and

$$\gamma_{A,B} : A \otimes_{\mathcal{O}_X} B \longrightarrow B \otimes_{\mathcal{O}_X} A$$

natural in A and B. In other words, α and γ defines natural isomorphisms.

In the following, we will focus on schemes, abandoning the full generality of ringed spaces.

Definition 1.4.4. A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that each stalk of $\mathcal{O}_X$ at a point $x \in X$, denoted by $\mathcal{O}_{X,x}$, is a local ring.

Since each ring $\mathcal{O}_{X,x}$ is local, it has a unique maximal ideal which we denote by m_x, and we indicate by $k(x)$ the *residue field* $\mathcal{O}_{X,x}/m_x$.

A morphism of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, i.e. a pair of morphisms $(f, f^{\#})$ with $f : X \to Y$ in **Top** and $f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$ a morphism of sheaves of rings, such that for every $x \in X$ the morphism at the stalk $f(x)$

$$f^{\#}_{f(x)} : \mathcal{O}_{Y,f(x)} \longrightarrow \underset{f^{-1}(U) \ni x}{\mathrm{Colim}} \, \mathcal{O}_X(f^{-1}(U)) = \mathcal{O}_{X,x}$$

is a local ring morphism, meaning that the image of the maximal ideal is contained in the maximal ideal.

Remark 1.4.5. A motivation for locally ringed spaces comes from geometrical observations. One of the first examples of $\mathcal{O}_X$-module that one encounter is the sheaf of continuous real valued functions $\mathcal{C}_X$ on a topological space X, associating to any open subset $U \subseteq X$ the ring of continuous functions $U \to \mathbb{R}$. The stalk at x of this sheaf is the ring of germs, namely equivalence classes of functions $[h]$ where $h \sim h'$ if they agree on a neighborhood of x. This ring is local, with unique maximal ideal $m_x = \{[h] \in \mathcal{C}_{X,x} | h(x) = 0\}$. If we consider the morphism of locally ringed spaces $(f, f^{\#}) : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$, where

$$f^{\#}_U : \mathcal{C}_Y(U) \longrightarrow \mathcal{C}_X(f^{-1}(U))$$

maps a function h to hf, we get that on stalks at $f(x)$ this morphism become

$$f^{\#} : \mathcal{C}_{Y,f(x)} \longrightarrow \mathcal{C}_{X,x}$$

mapping $[h] \mapsto [hf]$, and if $h(f(x)) \neq 0$, then certainly $hf(x) \neq 0$, too. So this is a local ring morphism.

From a more general perspective, after observing that a natural sheaf of rings $\tilde{R}$ on $\mathrm{Spec}(R)$ turns it into a locally ringed space, we can argue that locally ringed spaces are suitable in order to define Spec as a full and faithful functor from $\mathbf{ComRing}^{op}$ into the category of locally ringed spaces. This point of view leads to define **AffSch** as the essential image of the functor Spec, over which this functor will be full, faithful and essentially surjective, i.e. an equivalence $\mathbf{ComRing}^{op} \simeq \mathbf{AffSch}$.

Remark 1.4.6. For an R-module M the sheaf $\tilde{M}$ is the sheaf of modules on the topological space $\mathrm{Spec}(R)$ defined on basic opens $D(f) = \{p \in \mathrm{Spec}(R) | f \notin p\}$ to be the localization $\tilde{M}(D(f)) := M_f = M \otimes_R R_f$, with restriction morphisms for $D(f) \subseteq D(g) \subseteq \mathrm{Spec}(R)$, given by the universal properties of localization and tensor product.

More precisely, if $D(f) \subseteq D(g)$, if follows that g divides some power f^n of f, say $mg = f^n$, hence the natural morphism $R \to R_f$ maps g to the invertible element $\frac{g}{1} = (\frac{m}{f^n})^{-1}$. Therefore it's induced a morphism $R_g \to R_f$ by the universal property of the localization, and we get two morphisms $M \to M_f = M \otimes_R R_f$, the inclusion, and $R_g \to M \otimes_R R_f$, composing $R_g \to R_f$ with the other inclusion. This gives a morphism $M_g \to M_f$ induced by the pullback property defining the tensor product in

$$
\begin{tikzcd}
R \arrow[r] \arrow[d] & M \arrow[d] \arrow[ddr] & \\
R_g \arrow[r] \arrow[drr] & M \otimes_R R_g \arrow[dr, dashed] & \\
& & M \otimes_R R_f
\end{tikzcd}
$$

The particular case $\tilde{R}$ clearly defines a sheaf of rings whose stalks are local rings.

Definition 1.4.7. An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to the spectrum of a ring $(\mathrm{Spec}(R), \tilde{R})$.

More in general, a *scheme* is a locally ringed spaces $(X, \mathcal{O}_X)$ which is locally isomorphic to an affine scheme, i.e. such that for every $x \in X$ there exists an open neighborhood U of x such that $(U, \mathcal{O}_X|_U)$ is isomorphic to an affine scheme.

Definition 1.4.8. Let $(X, \mathcal{O}_X)$ be a scheme. An $\mathcal{O}_X$-module F is said to be *quasi-coherent* if for every $x \in X$ there is an affine open neighborhood $U \cong \mathrm{Spec}(R)$ of x such that $F|_U$ is isomorphic to the sheaf $\tilde{M}$ in $\mathrm{Spec}(R)$ associated to an R-module M.

Remark 1.4.9. There's a full subcategory $\mathbf{QCoh}(\mathcal{O}_X) \subseteq \mathbf{Mod}(\mathcal{O}_X)$ whose objects are quasi-coherent $\mathcal{O}_X$-modules. Moreover, if R is a commutative ring, there's an equivalence of categories $\mathbf{QCoh}(\tilde{R}) \simeq \mathbf{Mod}(R)$.

Here's an important example of how some $\mathcal{O}_X$-modules on a scheme X arise.

Example 1.4.10. Recall that a *closed subscheme* of a scheme $(X, \mathcal{O}_X)$ is given by a scheme $(Z, \mathcal{O}_Z)$ together with a locally ringed space morphism $(i, i^\#) : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ such that the morphism $i : Z \to X$ is a closed immersion, and $i^\# : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is an epimorphism of sheaves.

There's a correspondence

$$\{\text{closed subschemes of } (X, \mathcal{O}_X)\} \leftrightarrow \{\text{quasi-coherent } J \subseteq \mathcal{O}_X \text{ in } \mathbf{Mod}(\mathcal{O}_X)\}$$

This correspondence is given on one hand by taking, if $J \subseteq \mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_X$-module, the support Z of the quotient sheaf $\mathcal{O}_X/\mathcal{J}$, with the induced structure sheaf. Conversely, a close subscheme $(i, i^\#) : (Z, \mathcal{O}_Z) \subseteq (X, \mathcal{O}_X)$ defines a $\mathcal{O}_X$-submodule of $\mathcal{O}_X$ by taking the kernel of $i^\# : \mathcal{O}_X \to i_*\mathcal{O}_Z$.

Definition 1.4.11. Let $(X, \mathcal{O}_X)$ be a scheme. An $\mathcal{O}_X$-module F is said to be *free* if it's isomorphic to a direct sum $\bigoplus\limits_{i \in I} \mathcal{O}_X$ for a set I, i.e. $\mathcal{O}_X$-module associating to each open U the direct sum of modules $\bigoplus_{i \in I} \mathcal{O}_X(U)$, with the obvious action by $\mathcal{O}_X(U)$. A free sheaf is said to be of *rank* $n \in \mathbb{N}$ if the set I can be chosen to be of cardinality n.

The $\mathcal{O}_X$-module F is said to be *locally free* (of *rank* n) if it's locally isomorphic to a free module (of rank n). More precisely, if for every $x \in X$ there exists an open neighborhood U of x and a set I (with $|I| = n$) such that

$$F|_U \cong \bigoplus_{i \in I} \mathcal{O}_U,$$

where by $\mathcal{O}_U$ we indicate the restriction $\mathcal{O}_X|_U$.

Remark 1.4.12. Certainly, a locally free sheaf F is quasi-coherent. For any $x \in X$ consider an affine open $U \cong \mathrm{Spec}(R)$ containing x, over which $F|_U \cong \bigoplus_{i \in I} \mathcal{O}_U$, then the module $M = \bigoplus_I R$ shows $F|_U \cong \tilde{M}$.

The category of $\mathcal{O}_X$ modules can be seen as a generalization of the concept of vector bundle. Those $\mathcal{O}_X$-modules specializing to the latter are in fact the locally free $\mathcal{O}_X$-modules. Let's now briefly sketch how this correspondence works, introducing the basic notions.

Definition 1.4.13. Let $(X, \mathcal{O}_X)$ be a scheme. An $\mathcal{O}_X$-algebra is an $\mathcal{O}_X$-module $\mathcal{A}$ which is also a sheaf of rings. These objects form a category $\mathbf{Alg}(\mathcal{O}_X)$ whose morphisms are the morphisms of $\mathcal{O}_X$-modules which are also morphisms of sheaves of rings.

Example 1.4.14. Given an $\mathcal{O}_X$-module E, we can construct its *symmetric sheaf algebra* $\mathrm{Sym}(E)$ as the sheafification of the presheaf $U \mapsto \mathrm{Sym}(E(U))$. The symmetric algebra $\mathrm{Sym}(M)$ for an R-module M is the quotient of the tensor algebra $T(M) = \bigoplus_{n \geq 0} T^n(M)$, with $T^n(M) = R \otimes_R M \otimes_R \cdots \otimes_R M$ (with $n+1$ factors), by the two-sided ideal generated by the elements of the form $x \otimes y - y \otimes x \in T^2(M)$. The same way, the sheaf $\mathrm{Sym}(E)$ can be described as analogous quotient of the *tensor sheaf algebra* $T(E) = \bigoplus_{n \geq 0} T^n(E)$, with $T^n(E) = \mathcal{O}_X \otimes_{\mathcal{O}_X} E \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} E$.

The symmetric sheaf algebra certainly is an $\mathcal{O}_X$-algebra, and it satisfies the same universal property of the symmetric algebra. More precisely, $\mathrm{Sym}(E)$ comes with a morphism $E \to \mathrm{Sym}(E)$ such that for any morphisms of $\mathcal{O}_X$-modules $h : E \to F$ into an $\mathcal{O}_X$-algebra F there exists a unique morphism $\bar{h} : \mathrm{Sym}(E) \to F$ making

$$\begin{tikzcd} E \arrow[r] \arrow[dr, "h"'] & \mathrm{Sym}(E) \arrow[d, dashed, "\bar{h}"] \\ & F \end{tikzcd}$$

to commute.

Remark 1.4.15. The notion of spectrum of a commutative ring can be generalized by considering the spectrum of an $\mathcal{O}_X$-algebra. Fix a scheme X and an $\mathcal{O}_X$-algebra $\mathcal{A}$, and consider the family of affine scheme

$$\{\mathrm{Spec}(\mathcal{A}(U_i))\}_i$$

indexed by the set of affine open subsets $\{U_i \cong \mathrm{Spec}(R_i)\}$ of the scheme X. These affine schemes can be glued together by morphisms induced by those morphisms defining X as gluing of its affine subschemes, in order to define a scheme which we call $\underline{\mathrm{Spec}}_X(\mathcal{A})$. Such a scheme comes together with a natural morphism

$$f : \underline{\mathrm{Spec}}_X(\mathcal{A}) \to X.$$

such that $f^{-1}(U) \cong \mathrm{Spec}(\mathcal{A}(U))$.

Recall by A.0.1 that the functor Spec can be defined to be the right adjoint to the global section functor Γ. It follows that for a fixed ring A the scheme $\mathrm{Spec}(B)$ with a morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is characterized by the adjunction isomorphism

$$\mathrm{Hom}_{\mathbf{Alg}(A)}(B, \Gamma(Y)) \cong \mathrm{Hom}_{\mathbf{Sch}/\,\mathrm{Spec}(A)}(Y, \mathrm{Spec}(B))$$

describing the morphisms into $\mathrm{Spec}(B)$ as those morphisms of A-algebras from B to $\Gamma(Y, \mathcal{O}_Y)$.

Analogously, $\underline{\mathrm{Spec}}_X(\mathcal{A})$ can be characterized by the adjunction isomorphism, for any scheme Y over X, i.e. with a morphism $\pi : Y \to X$,

$$\mathrm{Hom}_{\mathbf{Alg}(\mathcal{O}_X)}(\mathcal{A}, \pi_*\mathcal{O}_X) \cong \mathrm{Hom}_{\mathbf{Sch}/X}(Y, \underline{\mathrm{Spec}}_X(\mathcal{A})).$$

Definition 1.4.16. Let X be a scheme. Set, for any open $U \subseteq X$, the scheme

$$\mathbb{A}^n_U := \mathrm{Spec}\,\mathbb{Z}[x_1, \ldots, x_n] \times_{\mathrm{Spec}\,\mathbb{Z}} U$$

A *geometric vector bundle* of rank n is a morphism of schemes $f : Y \to X$ with an open cover $X = \bigcup_{i \in I} U_i$ and isomorphisms in the category $\mathbf{Sch}/U_i$

$$\psi_i : f^{-1}(U_i) \longrightarrow \mathbb{A}^n_{U_i}$$

such that for every affine open $\mathrm{Spec}(A) \cong U \subseteq U_i \cap U_j$ the morphism

$$\psi_i \psi_j^{-1}|_U : \mathbb{A}^n_U = \mathrm{Spec}\,A[x_1, \ldots, x_n] \to \mathbb{A}^n_U = \mathrm{Spec}\,A[x_1, \ldots, x_n]$$

is $\mathrm{Spec}(\phi)$ for a automorphism of A-algebras $\phi : A[x_1, \ldots, x_n] \to A[x_1, \ldots, x_n]$.

There's a category of geometric vector bundles whose objects are geometric vector bundles over X, and morphisms $f \to f'$ are commutative triangles

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, "f"'] & & Y' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

in the category of schemes.

Definition 1.4.17. For any $\mathcal{O}_X$-module E consider its symmetric $\mathcal{O}_X$-algebra $\mathrm{Sym}(E)$, and set

$$\mathbb{V}(E) = \underline{\mathrm{Spec}}_X(\mathrm{Sym}(E)),$$

and call the natural morphism $f : \mathbb{V}(E) \to X$ the *vector bundle associated to E*.

Remark 1.4.18. In the case of E locally free of rank n over X, the scheme morphism $\mathbb{V}(E) \to X$ defined above gives in fact a geometric vector bundle. Let U be an open such that $E|_U$ is free, hence $E(U)$ has a basis $\{x_1, \dots, x_n\}$. Consider the identification $\mathrm{Sym}(E(U)) \cong \mathcal{O}(U)[x_1, \dots, x_n]$ to which we can apply Spec in order to get an isomorphism of schemes

$$\psi : \mathrm{Spec}(\mathrm{Sym}(E(U))) = f^{-1}(U) \longrightarrow \mathrm{Spec}(\mathcal{O}(U)[x_1, \dots, x_n]) = \mathbb{A}^n_U$$

These isomorphisms provide the structure of geometric vector bundle.

Conversely, if we consider a geometric vector bundle $f : Y \to X$, we can take the sheaf of sections Γ_f defined on $U \subseteq X$ to be the set of morphisms $s : U \to Y$ such that $fs = i : U \subseteq X$. This module has a natural structure of locally free $\mathcal{O}_X$-module. It's know that it suffices to define the structure of module on a basis. So, suppose $U \cong \mathrm{Spec}(A)$ to be affine, and moreover assume it to be trivializing for the vector bundle, i.e. such that $f^{-1}(U) \cong \mathrm{Spec}\, A[x_1, \dots, x_n]$. Therefore, a section $s \in \Gamma_f(U)$ is a morphism

$$s : U \longrightarrow \mathrm{Spec}\, A[x_1, \dots, x_n],$$

coming from a morphism of A-modules

$$\theta : A[x_1, ..., x_n] \longrightarrow A$$

This shows a correspondence between n-tuples $(\theta(x_1), \dots, \theta(x_n))$ of elements in A (i.e. elements of the A-module $\bigoplus_n A$) and sections $s \in \Gamma_f(U)$. Therefore, the module structure of Γ_f is just the multiplication of an n-tuples in $\bigoplus_n A$ by an element $r \in \mathcal{O}_X(U) = A$. It's clear that restricted to each open in the chosen basis for the topology of X, the module Γ_f is free, hence Γ_f is locally free.

Now, if E is a locally free $\mathcal{O}_X$-module and $f : \mathbb{V}(E) \to X$ is its associated vector bundle, then there is an isomorphism of locally free sheaves

$$\Gamma_f \cong E^{\vee}.$$

If $U \subseteq X$ is open and $E^\vee = \mathcal{H}om(E, \mathcal{O}_X)$ is the dual sheaf of E, i.e. the sheaf $U \mapsto \operatorname{Hom}(E|_U, \mathcal{O}_U)$, an element $t \in E^\vee(U)$ is a morphism $s : E|_U \to \mathcal{O}_U$ of $\mathcal{O}_U$-modules. From the universal property of the symmetric algebra, this determines a morphism of $\mathcal{O}_U$-algebras

$$\operatorname{Sym}(E|_U) \longrightarrow \mathcal{O}_U$$

which, applying the functor $\underline{\operatorname{Spec}}_X$, gives rise to a morphism of schemes, which we still call s,

$$s : U = \underline{\operatorname{Spec}}_X(\mathcal{O}_U) \longrightarrow \mathbb{V}(E|_U) = f^{-1}(U)$$

and which happens to be a section in $\Gamma_f(U)$.

Therefore, this leads us to conclude that the mappings

$$\begin{aligned} E &\mapsto \mathbb{V}(E) \\ (\Gamma_f)^\vee &\leftarrow\!\shortmid f \end{aligned}$$

define a one-to-one correspondence between geometric vector bundles of rank n over a scheme X and locally free $\mathcal{O}_X$-modules of rank n.

Now we would like to specialize the adjunction between the operation of taking, for a morphism $f : X \to Y$, direct image f_*F and inverse image $f^{-1}G$ of sheaves F on X and G on Y respectively, to the case of a morphism of schemes $X \to Y$ and sheaves of modules F and G. The main adversity is that the sheaf $f^{-1}G$ may not carry the structure of an $\mathcal{O}_X$-module. However, it has a natural structure of $f^{-1}\mathcal{O}_Y$-module.

Remark 1.4.19. Let $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes, and suppose F to be an $\mathcal{O}_X$-module and G an $\mathcal{O}_Y$-module.

On each open subset $V \subseteq Y$ the morphism

$$\begin{aligned} \mathcal{O}_Y(V) \times f_*F(V) &\longrightarrow f_*F(V) \\ (s, t) &\longmapsto f_V^\#(s)t \end{aligned}$$

defines a $\mathcal{O}_Y$-module structure on f_*F. Therefore, there is a well defined functor

$$f_* : \mathbf{Mod}(\mathcal{O}_X) \longrightarrow \mathbf{Mod}(\mathcal{O}_Y)$$

The action, for any open $U \subseteq X$,

$$\begin{aligned} f^{-1}\mathcal{O}_Y(U) \times f^{-1}G(U) &\longrightarrow f^{-1}G(U) \\ ([(V, s_V \in \mathcal{O}_Y(V))], [(V, t_V \in G(V))]) &\longmapsto [(V, s_V t_V)] \end{aligned}$$

defines an $f^{-1}\mathcal{O}_Y$-module structure on $f^{-1}G$.

Moreover, the structure sheaf $\mathcal{O}_X$ also carries a natural structure of $f^{-1}\mathcal{O}_Y$-module, given by the action, for any $U \subseteq X$,

$$\begin{aligned} f^{-1}\mathcal{O}_Y(U) \times \mathcal{O}_X(U) &\longrightarrow \mathcal{O}_X(U) \\ ([(V, t_V \in \mathcal{O}_Y(V))], s) &\longmapsto f_V^{\#}(t_V)|_U s \end{aligned}$$

where this mapping is in fact well defined because it doesn't really depend on V. If $(V, t_V) \sim (W, t_W)$ one can take H with $f(U) \subseteq H \subseteq V \cap W$ such that $t_V|_H = t_W|_H$ and consider, since $U \subseteq f^{-1}(H)$

$$f_V^{\#}(t_V)|_U = f_V^{\#}(t_V|_H)|_U = f_H^{\#}(t_W|_H)|_U = f_W^{\#}(t_W)|_U$$

Definition 1.4.20. If $f : X \to Y$ is a morphism of schemes and G is an $\mathcal{O}_Y$-module, the *inverse image sheaf of modules* of G is defined to be the $\mathcal{O}_X$-module

$$f^*G = f^{-1}G \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

This certainly defines a functor

$$f^* : \mathbf{Mod}(\mathcal{O}_Y) \longrightarrow \mathbf{Mod}(\mathcal{O}_X)$$

It's important to highlight that this construction is not just "bug fixing" of the functor f^{-1} in the case of modules. The functor f^* is in fact left adjoint to the functor f_* between categories of sheaves of modules. This property is known to characterize f^* up to unique isomorphism.

Theorem 1.4.21. *Let $f : X \to Y$ be a morphism of schemes. There is an adjunction $f^* \dashv f_*$ between the functors*

$$\boldsymbol{Mod}(\mathcal{O}_X) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \boldsymbol{Mod}(\mathcal{O}_Y)$$

Proof. From what argued in Remark 1.4.19, if F is an $\mathcal{O}_X$-module and G an $\mathcal{O}_Y$-module, the adjunction isomorphism of Theorem 1.2.26, induces an isomorphism between the subgroups of morphisms of sheaves of modules

$$\mathrm{Hom}_{\mathbf{Mod}(f^{-1}\mathcal{O}_Y)}(f^{-1}G, F) \cong \mathrm{Hom}_{\mathbf{Mod}(\mathcal{O}_Y)}(G, f_*F) \tag{1.2}$$

Now, this reduces to a general fact in abstract algebra ensuring an adjunction between the so called *restriction of scalars* and *extension of scalars*. More precisely, if $h : A \to B$ is a ring homomorphism there are functors

$$\rho_f : \mathbf{Mod}(B) \longrightarrow \mathbf{Mod}(A)$$

mapping a module M over B to itself with the structure of A-module induced by h with action $am = h(a)m$ for $a \in A$ and $m \in M$. This functor, which is somehow forgetful, admits a left adjoint construction of a B-module starting from a A-module N, which is the tensor product functor with the A-module B

$$- \otimes_A B : \mathbf{Mod}(A) \longrightarrow \mathbf{Mod}(B)$$

providing a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Mod}(B)}(N \otimes_A B, M) \cong \mathrm{Hom}_{\mathbf{Mod}(A)}(N, M)$$

This argument clearly generalizes to sheaves of rings and modules. In conclusion, the morphism of sheaves of rings $f^\flat : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, which is the transposed of $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$, induces a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Mod}(\mathcal{O}_X)}(f^{-1}G \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X, F) \cong \mathrm{Hom}_{\mathbf{Mod}(f^{-1}\mathcal{O}_Y)}(f^{-1}G, F)$$

giving, together with (1.2), the desired adjunction. □

Chapter 2

Triangulated categories

Triangulated categories naturally arise inverting some classes of morphisms of an abelian category. Intuitively, to invert morphisms is a method for losing some information and focus on the properties which are of our interest. The triangulated structure is then a more flexible environment aimed to deal with the analogous of properties of an abelian category that we lose after the operation of inverting some morphisms.

2.1 Abelian categories

Let's recall definition and first properties of an *abelian category.* The reason why here we care about them is that examples of abelian categories are the categories mainly used to study geometric spaces, such as the categories of modules and the categories of sheaves of modules.

Definition 2.1.1. A category $\mathbf{A}$ is called *pre-additive* if for every two objects A and B in $\mathbf{A}$ the set $\mathrm{Hom}(A, B)$ is endowed with an abelian group structure in such a way that whenever C is another object in $\mathbf{A}$, the composition

$$\circ : \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \longrightarrow \mathrm{Hom}(A, C)$$

is bilinear with respect to the group structure. That is $f \circ (g+h) = f \circ g + f \circ h$ as well as $(f + g) \circ h = f \circ h + g \circ h$.

Example 2.1.2. Rings are precisely pre-additive categories with exactly one object. A category with just one object is in fact precisely a monoid under composition, while the abelian group structure of the morphisms, together with the distributive laws, concludes the definition of a ring.

Definition 2.1.3. A pre-additive category $\mathbf{A}$ is called *additive* if it admits finite products

Definition 2.1.4. A functor $F : \mathbf{A} \to \mathbf{B}$ between additive categories is said to be *additive* if the map induced between groups

$$\mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(FA, FB)$$

is a group homomorphism.

This notion defines a subcategory of **Cat** whose objects are additive categories and morphisms between them are additive functors.

Remark 2.1.5. Having finite products for an additive category $\mathbf{C}$ implies of course to have a terminal object 1. Let's show that 1 is also initial object. In order to prove that, consider any object A in $\mathbf{C}$ and any $f \in \mathrm{Hom}(1, A)$, which, being a group, is not empty. Then it holds $f = f \circ \mathrm{id}_1 = f \circ e$, where e is the unique element, by terminality, of $\mathrm{Hom}(1, 1)$. In particular $e = e - e$, and since composition is bilinear, the last term is just $f \circ (e - e) = e'$, the neutral element of $\mathrm{Hom}(1, A)$. This concludes the proof of the claim, since shows that there is a unique map $1 \to A$.

More in general, the additive structure provides the following.

Proposition 2.1.6. *Let $\mathbf{C}$ be an additive category and $\{A_i\}_{i \in I}$ be a finite set of objects in $\mathbf{C}$, then the coproduct $\coprod_{i \in I} A_i$ exists and is naturally isomorphic to the product $\prod_{j \in I} A_j$.*

Proof. The proof is by induction, with base step provided by Remark 2.1.5. The inductive step just consists of proving the binary case. Consider two objects A and B and their binary product $A \times B$, with projection maps $p : A \times B \to A$ and $q : A \times B \to B$.

$$\begin{tikzcd} & A \times B \arrow[dl, "p"'] \arrow[dr, "q"] & \\ A & & B \end{tikzcd}$$

Let the morphism $i : A \to A \times B$ be induced as a product map by $\mathrm{id} : A \to A$ and $0 : A \to B$, as well as $j : B \to A \times B$ induced by $0 : B \to A$ and $\mathrm{id} : B \to B$. That is

$$\begin{tikzcd} & A \arrow[d, dashed, "i"] \arrow[ddl, "\mathrm{id}"'] \arrow[ddr, "0"] & \\ & A \times B \arrow[dl, "p"] \arrow[dr, "q"'] & \\ A & & B \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[d, dashed, "j"] \arrow[ddl, "0"'] \arrow[ddr, "\mathrm{id}"] & \\ & A \times B \arrow[dl, "p"] \arrow[dr, "q"'] & \\ A & & B \end{tikzcd}$$

Now, let's prove that i and j are coproduct morphisms. In order to do so, consider any object C with morphisms $f : A \to C$ and $g : B \to C$, and define explicitly the desired morphism as $h = fp + gq : A \times B \to C$. The commutativity in

$$\begin{tikzcd}
 & C & \\
 & A \times B \arrow[u, "h"] & \\
A \arrow[uur, "f"] \arrow[ur, "i"'] & & B \arrow[uul, "g"'] \arrow[ul, "j"]
\end{tikzcd}$$

is given by the calculation

$$\begin{aligned}
hi &= (fp + gq)i = f\underbrace{(pi)}_{=\mathrm{id}_A} + g\underbrace{(qi)}_{=0} = f \\
hj &= (fp + gq)j = f\underbrace{(pj)}_{=0} + g\underbrace{(qj)}_{=\mathrm{id}_B} = g.
\end{aligned}$$

This gives existence. In order to prove uniqueness, observe that $ip + jq$ is a morphism $A \times B \to A \times B$ such that $p(ip + jq) = pip + pjq = \mathrm{id}_A p + 0q = p$, as well as $q(ip + jq) = qip + qjq = q$. Another map making this service is the identity of $A \times B$, hence by uniqueness in the property of product $ip + jq = \mathrm{id}_{A\times B}$. Now, if another map h' is such that $h'i = f$ and $h'j = g$, then

$$h - h' = (h - h')\mathrm{id}_{A\times B} = (h - h')(ip + jq) = (fp + gq) - (fp + gq) = 0,$$

that is $h' = h$. $\square$

Corollary 2.1.7. *Let $F : \boldsymbol{A} \to \boldsymbol{B}$ be an additive functor between additive categories with finite products. Then F preserves finite products, i.e. for every X and Y in $\boldsymbol{A}$ there is a canonical isomorphism $F(X \oplus Y) \cong F(X) \oplus F(Y)$.*

Proof. The coproduct $X \oplus Y$ comes with canonical inclusions

$$\begin{tikzcd}
 & X \oplus Y & \\
X \arrow[ur, "i"] & & Y \arrow[ul, "j"']
\end{tikzcd}$$

and, thanks to Proposition 2.1.6, with canonical projections

$$\begin{tikzcd}
 & X \oplus Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X & & Y
\end{tikzcd}$$

It suffices to prove that $F(X \oplus Y)$ satisfies the definition of $F(X) \oplus F(Y)$. The cocone structure is provided by

$$\begin{tikzcd} & F(X \oplus Y) \arrow[dl, "Fp"'] \arrow[dr, "Fq"] & \\ F(X) & & F(Y) \end{tikzcd}$$

and whenever we consider two morphisms $h : Q \to F(X)$ and $k : Q \to F(Y)$, the morphism

$$F(i)h + F(j)k : Q \longrightarrow F(X \oplus Y)$$

is such that

$$F(p)(F(i)h + F(j)k) = \underbrace{F(pi)}_{=\mathrm{id}} h + \underbrace{F(pj)}_{=0} k = h,$$

and similarly $F(q)(F(i)h + F(j)k) = k$.

Uniqueness is given because whenever $f, g : Q \to F(X \oplus Y)$ are morphisms such that, for example, $F(p)f = F(p)g$, then $F(i)F(p)f = F(i)F(p)g$, that is $f = g$. $\square$

Example 2.1.8. The category **Ab** of abelian groups is additive, and, more in general, is additive the category $\mathbf{Mod}(R)$ of modules over any commutative unital ring R.

Example 2.1.9. The category of commutative unital rings **ComRing** is not an additive category. One way to prove that is to observe that initial and terminal objects, which are respectively $\mathbb{Z}$ and the zero ring $\{0\}$, does not coincide.

Definition 2.1.10. Let **C** be a category with initial and terminal objects respectively 0 and 1, and let $f : A \to B$ be a morphism in **C**. Define its *kernel*, if it exists, as the pullback $\mathrm{Ker}(f)$ in the diagram

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r] \arrow[d] & 0 \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Dually, define its *cokernel*, if it exists, as the pushout $\mathrm{Coker}(f)$ in the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ 1 \arrow[r] & \mathrm{Coker}(f) \end{tikzcd}$$

Usually, when no confusion arises, we talk about kernel and cokernel to mean either the objects or the maps $\mathrm{Ker}(f) \to A$ and $B \to \mathrm{Coker}(f)$ respectively.

Definition 2.1.11. For a morphism $f : A \to B$ with cokernel $\mathrm{Coker}(f)$, the *image* of f is defined as the kernel, if it exists, of the map $B \to \mathrm{Coker}(f)$. Dually, the *coimage* is, if it exists, the cokernel of the kernel map.

Observe that since the pullback of any mono is mono, and pushout of any epi is epi, a kernel is a subobject of the domain. On the other hand, we can define the quotient object for any mono $i : A \hookrightarrow B$ as $B/A = \mathrm{Coker}(i)$. This, when $A = \mathrm{Im}(f)$, fit the usual definition of cokernel of a linear map as codomain modulo image.

Definition 2.1.12. An additive category $\mathbf{A}$ is called *abelian* if every morphism $f : A \to B$ admits kernel and cokernel, and the natural map

$$\mathrm{Coim}(f) \longrightarrow \mathrm{Im}(f)$$

is an isomorphism.

Remark 2.1.13. The natural map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is given by the following observations: one has pullback and pushout squares defining kernel and cokernel (recall that 0=1 in any abelian category), that now exists by assumption. The definition of $\mathrm{Ker}(f)$ gives that the composition $\mathrm{Ker}(f) \xrightarrow{i} A \xrightarrow{f} B$ is zero, since the square commute passing through the terminal object. Thus there exists unique a map $\mathrm{Coker}(i) \to B$ making the following commute

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "i"] \arrow[d] & A \arrow[d] \arrow[dr, "f"] & \\
0 \arrow[r] & \mathrm{Coker}(i) \arrow[r, dashed] & B
\end{tikzcd}$$

Now, $A \xrightarrow{f} B \xrightarrow{\pi} \mathrm{Coker}(f)$ is zero by definition of the cokernel, thus by commutativity of the previous diagram, is zero the the factorization $A \to \mathrm{Coker}(i) \to B \to \mathrm{Coker}(f)$. Observe that the map $A \to \mathrm{Coker}(i)$ is the pushout of the epimorphism $\mathrm{Ker}(f) \to 0$, hence is epimorphism, from what we deduce that the composition $\mathrm{Coker}(i) \to B \to \mathrm{Coker}(f)$ is zero, and hence defines a cone over the following pullback diagram:

$$\begin{tikzcd}
\mathrm{Coker}(i) \arrow[r, dashed] \arrow[dr] & \mathrm{Ker}(\pi) \arrow[r] \arrow[d] & 0 \arrow[d] \\
& B \arrow[r, "\pi"] & \mathrm{Coker}(f)
\end{tikzcd}$$

So, in any abelian category, every morphism $f : A \to B$ factors as

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "i"] & A \arrow[rr, "f"] \arrow[dr] & & B \arrow[r, "\pi"] & \mathrm{Coker}(f) \\
& & \mathrm{Coker}(i) \arrow[r, "\cong"] & \mathrm{Ker}(\pi) \arrow[u] &
\end{tikzcd}$$

Proposition 2.1.14. *In any abelian category* $\boldsymbol{C}$*, for a morphism* $f : A \to B$*, it holds*

(*i*) *f is monomorphism if and only if* $\mathrm{Ker}(f) = 0$*. And dually, f is epimorphism if and only if* $\mathrm{Coker}(f) = 0$

(*ii*) *f is isomorphism if and only if f is monomorphism and epimorphism.*

Proof. (i): If f is mono, then the fact that $\mathrm{Ker}(f) \to A \to B$ is zero, implies by definition that $\mathrm{Ker}(f) \to A$ is zero, hence 0 works as kernel. Conversely, if $\mathrm{Ker}(f) = 0$, any pair of morphism $C \underset{g}{\overset{f}{\rightrightarrows}} A \to B$ such that compositions coincide, gives a composition $C \overset{f-g}{\to} A \to B$ which is zero, and pullback property gives that $f - g = 0$ because factors through $\mathrm{Ker}(f) = 0$. The dual property is proved by reversing arrows.

(ii): On one hand it's certainly true in any category that isomorphisms are both mono and epi. Conversely, suppose, by (i), to have $\mathrm{Ker}(f) = \mathrm{Coker}(f) = 0$. The decomposition of f defining abelian categories is given by

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[rr] \arrow[dr] & & Y \arrow[r] & 0 \\
& & \mathrm{Coker}(0 \to X) \cong \mathrm{Ker}(Y \to 0) \arrow[ur] & &
\end{tikzcd}$$

but $\mathrm{Coker}(0 \to X) \cong X$ and $\mathrm{Ker}(Y \to 0) \cong Y$. $\square$

Remark 2.1.15. If $\mathbf{A}$ is an abelian category, every monomorphism $f : A \to B$ is isomorphic to the kernel map of the cokernel $\pi : B \to \mathrm{Coker}(f)$ as objects in the category of objects over B. In particular $A \cong \mathrm{Ker}(B \to \mathrm{Coker}(f))$ (which is by definition $\mathrm{Im}(f)$). This follows straightforward form the observation that $f : A \to B$ being mono gives $\mathrm{Ker}(f) = 0$, hence $A \cong \mathrm{Coker}(0 \to A) = \mathrm{Coker}(i)$, and the isomorphism $\mathrm{Coker}(i) \cong \mathrm{Ker}(\pi)$ through which f factors in the diagram of Remark 2.1.13.

Dually, every epimorphism is a cokernel.

Lemma 2.1.16. *Let* $\boldsymbol{A}$ *be an abelian category and*

$$\begin{tikzcd}
A \arrow[r, "k"] \arrow[d, "h"'] & B \arrow[d, "g"] \\
C \arrow[r, "f"] & D
\end{tikzcd}$$

a pullback square. Then, $\mathrm{Ker}(g) \cong \mathrm{Ker}(h)$*.*

Proof. Consider $\mathrm{Ker}(g)$ and the morphisms $\mathrm{Ker}(g) \hookrightarrow B$ and $\mathrm{Ker}(g) \xrightarrow{0} C$ inducing

$$\begin{tikzcd}
\mathrm{Ker}(g) \arrow[dr, dashed] \arrow[drr, hook, bend left] \arrow[ddr, "0"'] & & \\
& A \arrow[r, "k"] \arrow[d, "h"'] & B \arrow[d, "g"] \\
& C \arrow[r, "f"] & D
\end{tikzcd}$$

By commutativity of the pullback diagram, the morphism $\mathrm{Ker}(g) \to A$ induces a morphism $\phi : \mathrm{Ker}(g) \to \mathrm{Ker}(h)$. Now, consider the object $\mathrm{Ker}(h) \subseteq A$ and observe that by commutativity of the pullback square, k induces a morphism $\mathrm{Ker}(h) \to \mathrm{Ker}(g)$, which is inverse to ϕ. ☐

Proposition 2.1.17. *Let* $\boldsymbol{A}$ *be an abelian category and*

$$\begin{tikzcd}
A \arrow[r, "k"] \arrow[d, "h"'] & B \arrow[d, "g"] \\
C \arrow[r, "f"] & D
\end{tikzcd}$$

a commutative diagram. Then, such a diagram is a pullback if and only

$$0 \longrightarrow A \xrightarrow{{}^t(k,h)} B \oplus C \xrightarrow{(g,-f)} D$$

is an exact sequence.

Dually, the square is a pushout if an only if the sequence

$$A \xrightarrow{{}^t(k,h)} B \oplus C \xrightarrow{(g,-f)} D \longrightarrow 0$$

is exact.

Proof. Certainly, commutativity of the diagram implies that $\mathrm{Im}^t(k,h) \subseteq \mathrm{Ker}(g,-f)$ and vice versa. Now, suppose the sequence to be exact, and consider two morphisms

$$A \xrightarrow{{}^t(k,h)} B \oplus C \xrightarrow{(g,-f)} D \longrightarrow 0$$

$k' : A' \to B$ and $h' : A' \to C$ such that $gk' = fh'$. Then we have $\mathrm{Im}^t(k',h') \subseteq \mathrm{Ker}(g,-f) = \mathrm{Im}^t(k,h)$. Therefore, since ${}^t(k,h) : A \to B \oplus C$ is monomorphism, is induced a unique morphism $P' \to P$ proving the square to be pullback. On the other hand, if the square is a pullback, we need to prove that $\mathrm{Ker}^t(k,h) = 0$ as well as $\mathrm{Ker}(g,-f) \subseteq \mathrm{Im}^t(k,h)$. The inclusion defining the subobject $\mathrm{Ker}^t(k,h) \subseteq A$ is the unique morphism induced by the restrictions of k and h to $\mathrm{Ker}^t(k,h)$. However, we can observe that the zero morphism works in making the diagram

$$
\begin{tikzcd}
\mathrm{Ker}^t(k,h) \arrow[drr, hook, "0"] \arrow[drrr, "k|_{\mathrm{Ker}^t(k,h)}"] \arrow[ddrr, "h|_{\mathrm{Ker}^t(k,h)}"'] & & & \\
& & A \arrow[r, "k"] \arrow[d, "h"'] & B \arrow[d, "g"] \\
& & C \arrow[r, "f"] & D
\end{tikzcd}
$$

to commute. It follows that $\mathrm{Ker}^t(k,h) = 0$, and hence that $\begin{pmatrix} k \\ h \end{pmatrix}$ is mono. Now, consider the object $\mathrm{Ker}(g, -f)$ with its inclusion morphism $\mathrm{Ker}(g,-f) \to B \oplus C$, which we can compose with the two projections in order to obtain two morphisms

$$C \xleftarrow{q} \mathrm{Ker}(g,-f) \xrightarrow{p} B$$

such that $fp = gq$. This induces a unique pullback morphism $u : \mathrm{Ker}(g,-f) \to A$ such that $\begin{pmatrix} k \\ h \end{pmatrix} u = \begin{pmatrix} p \\ q \end{pmatrix}$. Therefore, the image of $\begin{pmatrix} k \\ h \end{pmatrix}$ contains the domain of u, which is $\mathrm{Ker}(f,-g)$, which is what we needed in order to prove exactness.

The dual statement is analogous. $\square$

Remark 2.1.18. It follows from Proposition 2.1.17 that a necessary and sufficient condition for a square diagram to be both a pullback and a pushout is the exactness of the sequence

$$0 \longrightarrow A \xrightarrow{{}^t(k,h)} B \oplus C \xrightarrow{(g,-f)} D \longrightarrow 0$$

In particular, not only the pullback of a monomorphism is a monomorphism as in any category with pullbacks, but also the pushout of a monomorphism is a monomorphism. In fact, for m mono in a pushout square

$$
\begin{tikzcd}
A \arrow[r, "n"] \arrow[d, "m"'] & B \arrow[d, "g"] \\
C \arrow[r, "f"] & D
\end{tikzcd}
$$

we find an exact sequence

$$A \xrightarrow{{}^t(n,m)} B \oplus C \xrightarrow{(g,-f)} D \longrightarrow 0$$

but since m is mono, this extend to a short exact sequence

$$0 \longrightarrow A \xrightarrow{{}^t(n,m)} B \oplus C \xrightarrow{(g,-f)} D \longrightarrow 0$$

Therefore, the square is also a pullback. Now, from Lemma 2.1.16, we get that $0 = \mathrm{Ker}(m) \cong \mathrm{Ker} g$, hence g is a monomorphism.

Dually, the pullback of an epimorphism is epi in any abelian category.

Definition 2.1.19. An additive functor $F : \mathbf{A} \to \mathbf{B}$ between abelian categories is said to be *exact* if any short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is such that

$$F0 = 0 \longrightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \longrightarrow F0 = 0$$

is again a (short) exact sequence.

The use of the decomposition for proving (ii) in Proposition 2.1.14 is not just convenient, it's necessary. In fact, there are additive (non-abelian) categories in which a morphism that is mono and epi need not to be an isomorphism:

Example 2.1.20. Consider the full subcategory of $\mathbf{Ab}$ whose objects are torsion-free abelian groups, and any non-zero, morphism $\phi : \mathbb{Z} \to \mathbb{Z}$, say $\phi(k) = nk$ for some fixed $n \in \mathbb{Z} \setminus \{0\}$. Then ϕ is certainly a monomorphism, since whenever G is an abelian group with group morphisms $G \underset{g}{\overset{f}{\rightrightarrows}} \mathbb{Z}$ such that compositions with ϕ are the same, that is $nf(k) = ng(k) \in \mathbb{Z}$, and since $n \neq 0$ we get $f(k) = g(k)$ for any $k \in G$. The morphism ϕ, however, is also an epimorphism: consider in fact a torsion free abelian group G and morphisms $\mathbb{Z} \underset{g}{\overset{f}{\rightrightarrows}} G$ such that precompositions with ϕ are equal, that is for all $k \in \mathbb{Z}$ it holds $f(nk) = g(nk)$. That means $nf(k) = ng(k) \in G$, and hence $n(f(k) - g(k)) = 0$. Since G is torsion free, that implies $f(k) = g(k)$ for every $k \in \mathbb{Z}$.

As any full subcategory of $\mathbf{Ab}$ closed by taking finite products, the category of torsion-free abelian groups is additive. However, it's not abelian, since we can take ϕ to be non-isomorphism, but epi and mono.

Some important abelian categories are the following.

Example 2.1.21. The category $\mathbf{Ab}$ of abelian groups is abelian. It admits finite products, the additive structure is clearly given by $(f+g)(a) = f(a) + g(a)$ for $f, g \in \operatorname{Hom}(A, B)$. Composition is bilinear. Kernel and cokernel of a morphism $f : A \to B$ certainly exists, and the well known First Isomorphism Theorem $A/\operatorname{Ker}(f) \cong \operatorname{Im}(f)$ is another way of stating the isomorphisms

$$\operatorname{Coker}(\operatorname{Ker}(f) \longrightarrow A) \cong \operatorname{Ker}(B \longrightarrow \operatorname{Coker}(f)).$$

More in general, the category $\mathbf{Mod}(R)$ of R-modules, as well as its full subcategory of finitely generated R-modules, is abelian for any commutative unital ring R. Just as for abelian groups, the additive structure is obvious as well as the existence of kernel and cokernel of any morphism, and again the First Isomorphism Theorem for modules concludes.

Example 2.1.22. The category $\mathbf{Psh}(X)$ of presheaves of abelian groups on a topological space X is abelian. Recall from Remark 1.1.8 in the context of sheaves of abelian groups, that we can compute kernels and cokernels pointwise, just as any limit and colimit. Moreover, the isomorphisms of presheaves are just the open-wise isomorphisms, therefore the proof is an immediate consequence of the case of abelian groups.

Example 2.1.23. The category $\mathbf{Sh}(X)$ of sheaves of abelian groups on a topological space X is abelian. Remember that for a morphism $\phi : F \to G$ in $\mathbf{Sh}(X)$, the sheaf $\mathrm{Ker}(\phi)$ is computed as in presheaves, just as any limit. That is, the presheaf $U \mapsto \mathrm{Ker}(\phi_U)$ is actually a sheaf. However, to deal with cokernels, and in general with colimits, requires the sheafification of the corresponding presheaf. That is, the sheaf $\mathrm{Coker}(\phi)$ is the sheafification of the presheaf associating $U \mapsto \mathrm{Coker}(\phi_U)$.

Consider then a morphism $\phi : F \to G$, and let's prove that the morphism of sheaves $\mathrm{Coker}(\mathrm{Ker}(\phi) \xrightarrow{i} F) \to \mathrm{Ker}(G \xrightarrow{\pi} \mathrm{Coker}(\phi))$ is an isomorphism. This can be done by checking on stalks.

Since colimits commutes with colimits, the stalk $\mathrm{Coker}(i)_x$ is the same as $\mathrm{Coker}(i_x)$, where $i_x : \mathrm{Ker}(\phi)_x \to F_x$. Kernels also, being finite limits, commutes with stalks which are filtered colimits. Therefore, since $\mathbf{Ab}$ is abelian

$$\mathrm{Coker}(i_x) \cong \mathrm{Ker}(G_x \to \mathrm{Coker}(\phi_x)) \cong \mathrm{Ker}(G_x \to \mathrm{Coker}(\phi)_x) = \mathrm{Ker}(\pi_x)$$

which is isomorphic to $\mathrm{Ker}(\pi)_x$ as desired.

Example 2.1.24. On a ringed space $(X, \mathcal{O}_X)$ the category $\mathbf{Mod}(\mathcal{O}_X)$ of sheaves of modules is abelian. The same argument as above works, and the result follows since $\mathbf{Mod}(R)$ is abelian.

Example 2.1.25. It's a well established result that on any scheme X, both the categories of quasi-coherent sheaves $\mathbf{QCoh}(X)$ and its full subcategory $\mathbf{Coh}(X)$ of coherent sheaves are abelian categories.

Here there are some important non-examples.

Example 2.1.26. For a fixed scheme X the category of vector bundles over X, i.e. the category of locally free $\mathcal{O}_X$-modules, is not abelian. More precisely, if we look at the vector bundles as a full subcategory of $\mathbf{Mod}(\mathcal{O}_X)$, then kernel and the cokernel of morphisms need not to be vector bundles.

An easy example happens on the affine line $\mathrm{Spec}(k[x])$ for, let's say, $k = \mathbb{C}$. Consider the vector bundles $\mathcal{O}_X$, the structure sheaf, associated to the module $k[x]$, and the vector bundle F associated to the free module $xk[x]$. This is a subsheaf of the structure sheaf, in the sense that it comes with a monomorphism

$$m : F \longrightarrow \mathcal{O}_X$$

which is of course given because on any open set $D(f) = \{p \in \mathrm{Spec}(k[x]), f \notin p\}$, the module $F(D(f)) = (xk[x])_f$ is a submodule of $(k[x])_f = \mathcal{O}_X(D(f))$. The cokernel of this morphism can be computed by looking at its stalks. If $(x) \neq p \in \mathrm{Spec}(k[x])$, then $x \notin p$, therefore $k[x]_p \cong (xk[x])_p$, and the stalk at p of the quotient is

$$\mathcal{O}_{X,p}/F_p = k[x]_p/xk[x]_p = 0.$$

If we take the stalk at (x), we find

$$k[x]_{(x)}/(xk[x])_{(x)} = k[x]_{(x)}/x(k[x]_{(x)}) = k.$$

Therefore the sheaf of modules $\mathrm{Coker}(m)$ is the skyscraper sheaf centered at 0 with value k. This is not a vector bundle because k is not a free $k[x]_{(x)}$-module.

However, observe that this example provides an instance of skyscraper sheaf of modules which is quasi-coherent, because quasi-coherent modules do form an abelian category.

Example 2.1.27. The category **Gr** of groups is not abelian. In fact, it's not even additive, since no natural structure of commutative operation can be given to the Hom-sets. However, a nice way to prove that is to consider a non abelian group G with a subgroup H which which is not normal, and hence the inclusion $H \to G$ cannot be a kernel, contradicting Remark 2.1.15.

Remark 2.1.28. The category of vector bundles over a ringed space $(X, \mathcal{O}_X)$ is additive, that is because we can look at it as the full subcategory of locally free $\mathcal{O}_X$-modules inside of the category of all $\mathcal{O}_X$-modules. This subcategory admits finite direct sums, which are products as well as coproducts, by the very definition of locally free $\mathcal{O}_X$-module, and clearly is induced the structure of abelian group on the Hom-sets by fullness.

Moreover, by the same argument, is additive also the category of finite rank vector bundles over a ringed space $(X, \mathcal{O}_X)$.

The non-example 2.1.26, as well as 2.1.20, shows that being abelian is a stronger requirement than being additive. The latter shows, moreover, that it is also stronger than being additive and having kernel and cokernel of any morphism.

2.2 The homotopy category of an abelian category

One of the most convincing arguments for the introduction of derived categories is that, as we are going to see, they lead to identify one object in some abelian category with its resolutions. In order to deal with a category where this makes sense, we need to enlarge the abelian category to the category of its complexes. The notion of homotopy category is not really necessary in order to define what the derived category is, but it's useful in order to describe its morphisms.

Definition 2.2.1. Let $\mathbf{A}$ be an abelian category. The *category of complexes* $\mathrm{Kom}(\mathbf{A})$ is the category whose objects $A^\cdot$ are chain complexes in $\mathbf{A}$, i.e. sequences of objects $(A^i)_{i\in\mathbb{Z}}$ and morphisms $d^i : A^i \to A^{i+1}$ with $d^i d^{i-1} = 0$ for all $i \in \mathbb{Z}$. A morphism $A^\cdot \to B^\cdot$ in $\mathrm{Kom}(\mathbf{A})$ is a family of morphisms $f^i : A^i \to B^i$ making the diagram

$$\begin{tikzcd} \cdots \arrow[r] & A^{i-1} \arrow[r] \arrow[d] & A^i \arrow[r] \arrow[d] & A^{i+1} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & B^{i-1} \arrow[r] & B^i \arrow[r] & B^{i+1} \arrow[r] & \cdots \end{tikzcd}$$

to commute.

Remark 2.2.2. The category of complexes of an abelian category $\mathbf{A}$ is abelian. (Co)kernels are pointwise the (co)kernels in $\mathbf{A}$.

Remark 2.2.3. Any abelian category $\mathbf{A}$ is a full subcategory of $\mathrm{Kom}(\mathbf{A})$ by mapping an object A to the complex which is A in degree 0 and zero everywhere else.

Remark 2.2.4. For any integer k is defined the cohomology functor

$$H^k : \mathrm{Kom}(\mathbf{A}) \to \mathbf{A}$$

by $A^\cdot \mapsto \mathrm{Coker}(\mathrm{Im}(d^{k-1}) \to \mathrm{Ker}(d^k))$. For a morphism $f : A^\cdot \to B^\cdot$, one has the pushout diagram

$$\begin{tikzcd} \mathrm{Im}(d_A^{k-1}) \arrow[r] \arrow[d] & \mathrm{Ker}(d_A^k) \arrow[r, "f^k"] \arrow[d] & \mathrm{Ker}(d_B^k) \arrow[r] & H^k(B^\cdot) \\ 0 \arrow[r] & H^k(A^\cdot) \arrow[urr, dashed, "H^k(f)"'] & & \end{tikzcd}$$

where by f^k we mean actually the restriction of f^k to $\mathrm{Ker}(d_A^k)$, which have in fact image contained in $\mathrm{Ker}(d_B^k)$ by definition of chain morphism. The first row composition gives in fact the zero morphism, because by definition $H^k(B^\cdot)$ is such that the morphism $\mathrm{Im}(d_B^{k-1}) \to \mathrm{Ker}(d_B^k) \to H^k(B^\cdot)$ is zero, and the composite $\mathrm{Im}(d_A^{k-1}) \to \mathrm{Ker}(d_A^k) \to \mathrm{Ker}(d_B^k)$ in the first row factors through $\mathrm{Im}(d_B^{k-1})$, again by commutativity in the definition of chain morphisms.

Lemma 2.2.5. *Let $\mathbf{A}$ be an abelian category. Then for any integer k the cohomology functor H^k is additive.*

Proof. Let $f, g : X \to Y$ be morphisms in $\mathrm{Kom}(\mathbf{A})$. By definition, $H^k(f+g)$ is the unique morphism $H^kX \to H^kY$ such that the diagram

$$\begin{tikzcd}
\mathrm{Im}(d_X^{k-1}) \arrow[r] \arrow[d] & \mathrm{Ker}(d_X^k) \arrow[r, "f^k+g^k"] \arrow[d, "\pi_X"] & \mathrm{Ker}(d_Y^k) \arrow[r, "\pi_Y"] & H^kY \\
0 \arrow[r] & H^kX \arrow[urr, dashed, "\exists!"'] & &
\end{tikzcd}$$

commutes. Thus it suffices to prove that $H^k(f)+H^k(g)$ works as well. That is by the easy calculation

$$(H^k f + H^k g)\pi_X = H^k f\pi_X + H^k g\pi_X = \pi_Y f^k + \pi_Y g^k = \pi_Y(f^k + g^k).$$

□

Definition 2.2.6. A complex $A^\cdot$ in $\mathrm{Kom}(\mathbf{A})$ is said to be *acyclic* whether its cohomology $H^k(A^\cdot)$ is zero for every integer k.

A fundamental characterization of exact functors is provided by the notion of acyclic complex in the category of complexes:

Proposition 2.2.7. *An additive functor $F : \boldsymbol{A} \to \boldsymbol{B}$ between abelian categories is exact if an only if its extension* $\mathrm{Kom}F : \mathrm{Kom}(\boldsymbol{A}) \to \mathrm{Kom}(\boldsymbol{B})$ *preserves acyclic objects.*

Proof. On one hand, any short exact sequence $0 \to A \to B \to C \to 0$ is surely acyclic, thus if F preserves acyclic, $0 \to FA \to FB \to FC \to 0$ will be exact. On the other hand consider a complex $A^\cdot$ in $\mathrm{Kom}(\mathbf{A})$, and let's focus on a degree k:

$$\longrightarrow A^{k-1} \xrightarrow{d^{k-1}} A^k \xrightarrow{d^k} A^{k+1} \longrightarrow$$

Consider then the commutative diagram

$$\begin{tikzcd}
 & & & & 0 \\
 & & & \mathrm{Im}(d^k) \arrow[ur] \arrow[dr, hook] & \\
A^{k-1} \arrow[rr, "d^{k-1}"] \arrow[dr, two heads] & & A^k \arrow[rr, "d^k"] \arrow[ur] & & A^{k+1} \\
 & \mathrm{Im}(d^{k-1}) = \mathrm{Ker}(d^k) \arrow[ur] & & & \\
0 \arrow[ur] & & & &
\end{tikzcd}$$

containing a short exact sequence, which remains exact if we apply F to the whole diagram. Thus, let's compute

$$\mathrm{Ker}(Fd^k) = \mathrm{Ker}(FA^k \to F\,\mathrm{Im}\,d^k \hookrightarrow FA^{k+1}) = \mathrm{Ker}(FA^k \to F\,\mathrm{Im}\,d^k)$$
$$= \mathrm{Im}(F\,\mathrm{Im}\,d^{k-1} \to FA^k) = \mathrm{Im}(FA^{k-1} \twoheadrightarrow F\,\mathrm{Im}\,d^{k-1} \to FA^k) = \mathrm{Im}(Fd^{k-1}).$$

$\square$

One of the main results on abelian categories, allowing to generate long exact cohomology sequences, is the following.

Theorem 2.2.8 (Snake Lemma)**.** *Let* $\boldsymbol{A}$ *be an abelian category and*

$$0 \longrightarrow A^\cdot \xrightarrow{f} B^\cdot \xrightarrow{g} C^\cdot \longrightarrow 0$$

be a short exact sequence of objects in $\mathrm{Kom}(\boldsymbol{A})$*. Then, there exists a family of morphisms* $\delta^k : H^k(C^\cdot) \to H^{k+1}(A^\cdot)$ *indexed by* $k \in \mathbb{Z}$ *such that*

$$\to H^k(A^\cdot) \stackrel{H^k(f)}{\to} H^k(B^\cdot) \stackrel{H^k(g)}{\to} H^k(C^\cdot) \stackrel{\delta^k}{\to} H^{k+1}(A^\cdot) \stackrel{H^{k+1}(f)}{\to} H^{k+1}(B^\cdot) \to$$

is a long exact sequence.

Proof. See any book in homological algebra, e.g. [Lan02] III.9. $\square$

Here it comes a particularly central notion. Roughly speaking, the derived category will be such that the morphisms that we are going to consider become isomorphism, and that happens in a universal way that will be later precised.

Definition 2.2.9. A morphism $f : A^\cdot \to B^\cdot$ in $\mathrm{Kom}(\mathbf{A})$ is said to be a *quasi-isomorphism* (or *qiso*, for short) if for every $i \in \mathbb{Z}$ the morphism $H^i(f) : H^i(A^\cdot) \to H^i(B^\cdot)$ is an isomorphism.

Proposition 2.2.10. *Let* $\boldsymbol{A}$ *be an abelian category and* A *an object. Any resolutions of an object* A*, that is any exact sequence*

$$\cdots \longrightarrow R_1 \longrightarrow R_0 \longrightarrow A \longrightarrow 0$$

is a quasi-isomorphism.

Proof. First observe that Remark 2.2.3 makes sense of the statement, because we actually have a morphism of complexes $R^\cdot \to A^\cdot$

$$\begin{tikzcd}
\cdots \arrow[r] & R_1 \arrow[r, "r"] \arrow[d] & R_0 \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & A \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

in which all cohomologies are 0 but the one in degree zero. Clearly the isomorphism on cohomology for $i \neq 0$ is induced by the only existing morphism in these degrees. In order to see that f induces isomorphism on cohomology at the zero degree, look at the canonical decomposition of f into

$$\begin{tikzcd}
R_1 \arrow[r] & R_0 \arrow[rr, "f"] \arrow[dr] & & A \arrow[r] & 0 \\
& & \operatorname{Coker}(\operatorname{Ker}(f) \to R_0) \cong \operatorname{Ker}(A \to 0) \arrow[ur, "\cong"'] & &
\end{tikzcd}$$

and observe that that by exactness $\operatorname{Coker}(\operatorname{Ker}(f) \to R_0) \cong \operatorname{Coker}(\operatorname{Im}(r) \to \operatorname{Ker}(R_0 \to 0)) = H^0(R^\cdot)$. This isomorphism, composed with the one provided by the decomposition, is an isomorphism between the 0-th cohomologies (by $\operatorname{Ker}(A \to 0) \cong A \cong H^0(A^\cdot)$), and fits into

$$\begin{tikzcd}
\operatorname{Im}(R_1 \to R_0) \arrow[r] \arrow[d] & R_0 \arrow[r, "f"] \arrow[d] & A \arrow[r, "\cong"] & H^0(A^\cdot) \\
0 \arrow[r] & H^0(R^\cdot) \arrow[urr, "\cong"'] & &
\end{tikzcd}$$

By the commutativity in the canonical decomposition of f, this last diagram commutes as well, and hence by uniqueness the isomorphism $H^0(R^\cdot) \to H^0(A^\cdot)$ is $H^0(f)$. $\square$

Example 2.2.11. Quasi-isomorphism are not invertible. In fact, in the category of abelian groups, between the complex

$$X = \cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$

and the complex

$$Y = \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \longrightarrow \cdots$$

there exists a quasi-morphism $X \to Y$ given by the unique non-trivial morphism $\mathbb{Z} \to \mathbb{Z}/2$, but no morphism at all $Y \to X$.

Remark 2.2.12. As we will see later on based on the previous example, it could happens that two complexes have the same cohomology in each degree but no quasi-isomorphism exists between them, neither way.

Definition 2.2.13. Two morphisms of complexes $f, g : X \to Y$ are called *homotopically equivalent*, in symbols $f \sim g$, if there exists a family of morphism $h^i : X^i \to Y^{i-1}$ for $i \in \mathbb{Z}$, such that

$$f^i - g^i = d_Y^{i-1} h^i + h^{i+1} d_X^i.$$

The *homotopy category* $\mathrm{K}(\mathbf{A})$ of an abelian category $\mathbf{A}$ is the category whose objects are the same as in $\mathrm{Kom}(\mathbf{A})$, and morphisms between two complexes X and Y are considered up to homotopical equivalence, that is $\operatorname{Hom}_{\mathrm{K}(\mathbf{A})}(X, Y) = \operatorname{Hom}_{\mathrm{Kom}(\mathbf{A})}(X, Y)/\sim$

Remark 2.2.14. The previous definition makes sense once proved that homotopical equivalence is, as the name suggests, an equivalence relation. Certainly $f \sim f$ because $h^i = 0$ for every i works. For $f \sim g$, by a family $(h^i)_i$ of morphisms, to consider $(-h^i)_i$ gives the desired family of morphisms showing $g \sim f$. Eventually, for $f_0 \sim f_1$ by $(h_0^i)_i$ and $f_1 \sim f_2$ by $(h_1^i)_i$, transitivity is proved taking $(h_0^i + h_2^i)$.

Remark 2.2.15. If $f : X \to Y$ and $g : Y \to X$ are morphisms in $\mathrm{Kom}(\mathbf{A})$ such that $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$ then f is *homotopical inverse* of g, as they are in fact inverses of each other in $\mathrm{K}(\mathbf{A})$.

Proposition 2.2.16. *Let $f, g : X \to Y$ be morphisms in $\mathrm{Kom}(\boldsymbol{A})$. If $f \sim g$, then $H^k(f) = H^k(g)$ for every $k \in \mathbb{Z}$.*

Proof. Let's prove $H^k(f - g) = 0$, the result follows since H^k is additive functor, Lemma 2.2.5. Recall that cohomology is defined on a morphisms f as the dashed pushout map in the diagram

$$\begin{tikzcd}
\mathrm{Im}(d_X^{k-1}) \arrow[r] \arrow[d] & \mathrm{Ker}(d_X^k) \arrow[r, "f^k"] \arrow[d] & \mathrm{Ker}(d_Y^k) \arrow[r] & H^k(Y) \\
0 \arrow[r] & H^k(X) \arrow[rru, dashed, "H^k(f)"'] & &
\end{tikzcd}$$

where as usual we mean by f^k the restriction

$$\mathrm{Ker}\, d_X^k \longrightarrow X^k \xrightarrow{f^k} Y^k$$

Thus, in order to prove that $H^k(f - g) = 0$ one can prove to be zero the map inducing it, that is

$$\mathrm{Ker}(d_X{}^k) \xrightarrow{f^k - g^k} \mathrm{Ker}(d_Y{}^k) \longrightarrow H^k(Y),$$

and since 0 will works, by uniqueness it has to be $H^k(f - g) = 0$.

Note that the composite $\mathrm{Ker}(d_X^k) \longrightarrow X^k \xrightarrow{f^k - g^k} Y^k$ is the same arrow as $\mathrm{Ker}(d_X{}^k) \longrightarrow X^k \xrightarrow{d^{k-1}h^k} Y^k$, and that is because $f^k - g^k = d^{k-1}h^k + h^{k+1}d^k$, but d^k is clearly zero on its kernel.

Thus the diagram

$$\begin{tikzcd}
 & & \mathrm{Ker}(d_Y^k) \arrow[rd] & \\
\mathrm{Ker}(d_X^k) \arrow[r] & X^k \arrow[ru, "f^k - g^k"] \arrow[rd, "h^k"'] & \mathrm{Im}(d_Y^{k-1}) \arrow[u, hook] \arrow[r, "0"] & H^k(Y) \\
 & & Y^{k-1} \arrow[u, "d^{k-1}"'] &
\end{tikzcd}$$

is commutative, and chasing it shows that

$$\operatorname{Ker}(d_X{}^k) \xrightarrow{f-g} \operatorname{Ker}(d_Y{}^k) \longrightarrow H^k(Y)$$

is in fact zero. $\square$

Corollary 2.2.17. *Let X and Y be complexes in* $\mathrm{Kom}(\boldsymbol{A})$ *and $f : X \to Y$, $g : Y \to X$ be homotopical inverses of each other. Then f and g are quasi-isomorphism. More precisely, for every $k \in \mathbb{Z}$ the inverse of $H^k(f)$ is $H^k(f)^{-1} = H^k(g)$.*

Proof. It's basically Lemma 2.2.16 and the fact that H^k is a functor:

$$H^k(g)H^k(f) = H^k(gf) = H^k(\mathrm{id}_X) = \mathrm{id}_{H^k X},$$

and similarly for the reverse composition. $\square$

2.3 Translations and Mapping cones

Let's introduce the objects that will give to the homotopy category, as well as to the derived category of an abelian category, the necessary structure in order to develop some homological algebra.

Definition 2.3.1. Let $\mathbf{A}$ be an abelian category, X an object in $\mathrm{Kom}(\mathbf{A})$ and n an integer number. The *translation by n* of X is a complex denoted by $X[n]$ and defined in degree i to be $X[n]^i = X^{n+i}$ for every integer i, and with differential $d^i_{X[n]}$ to be $(-1)^n d_X^{n+i}$.

More in general, there is a *translation functor* $[n]$, defined on a morphism $f : X \to Y$ to be $f[n]^i = f^{i+n}$.

Definition 2.3.2. Let $f : X \to Y$ be a morphism of complexes in $\mathrm{Kom}(\mathbf{A})$, for an abelian category $\mathbf{A}$. The *mapping cone* of f is the complex $C(f)$ defined in each degree to be $C^k(f) = X^{k+1} \oplus Y^k$ and with differentials $d^k : X^{k+1} \oplus Y^k \to X^{k+2} \oplus Y^{k+1}$ given by

$$d^k = \begin{pmatrix} -d_X^{k+1} & 0 \\ -f^{k+1} & d_Y^k \end{pmatrix}$$

acting to the left.

Remark 2.3.3. Many would agree that signs are troublesome. However, observe that one need to consider some minus signs defining differentials in the mapping cone of a morphism in order to turn it into an actual chain complex.

$$d^2 = \begin{pmatrix} -d & 0 \\ -f & 0 \end{pmatrix}^2 = \begin{pmatrix} d^2 & d^2 \\ fd - df & d^2 \end{pmatrix} = 0.$$

The following result, moreover, also explain the choice of $(-1)^n$ in the definition of the differentials of the translation $[n]$. In fact, we are going to see that there is, for any morphism of complexes $f : X \to Y$, a canonical projection morphism $C(f) \to X[1]$, which wouldn't make the squares commute if differentials in $X[1]$ weren't defined as such.

Lemma 2.3.4. *Let $\boldsymbol{A}$ be an abelian category, and $f : X \to Y$ a morphism in $\operatorname{Kom}(\boldsymbol{A})$. Then there is a short exact sequence of complexes*

$$0 \longrightarrow Y \longrightarrow C(f) \longrightarrow X[1] \longrightarrow 0$$

Proof. Non-trivial morphisms in each degree k are just the inclusion $Y^k \to X^{k+1} \oplus Y^k$ and the projection $X^{+1} \oplus Y^k \to X^{k+1}$. Let's prove them to define morphisms of chain complexes. Commutativity, for every k, of squares

$$\begin{tikzcd}
Y^{k-1} \arrow[r] \arrow[d, "d_Y^{k-1}"'] & X^k \oplus Y^{k-1} \arrow[r] \arrow[d, "d_{C(f)}^{k-1}"] & X^{k-1}[1] \arrow[d, "d_{X[1]}^{k-1} = -d_X^k"] \\
Y^k \arrow[r] & X^{k+1} \oplus Y^k \arrow[r] & X^k[1]
\end{tikzcd}$$

is given just checking that

$$d_{C(f)}^{k-1} \begin{pmatrix} 0 \\ \mathrm{id}_{Y^{k-1}} \end{pmatrix} = \begin{pmatrix} -d_X^k & 0 \\ -f^k & d_Y^{k-1} \end{pmatrix} \begin{pmatrix} 0 \\ \mathrm{id}_{Y^{k-1}} \end{pmatrix} = \begin{pmatrix} 0 \\ d_Y^{k-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{id}_{Y^k} \end{pmatrix} d_Y^{k-1}$$

and

$$\begin{pmatrix} \mathrm{id}_{X^{k+1}} & 0 \end{pmatrix} \begin{pmatrix} -d_X^k & 0 \\ -f^k & d_Y^{k-1} \end{pmatrix} = \begin{pmatrix} -d_X^k & 0 \end{pmatrix} = -d_X^k \begin{pmatrix} \mathrm{id}_{X^k} & 0 \end{pmatrix}.$$

□

Mapping cones, which will be needed in order to define the triangulated structure both of the homotopy and the derived category of an abelian category, are useful in order to characterize quasi-isomorphisms.

Proposition 2.3.5. *Let $\boldsymbol{A}$ be an abelian category. A morphism $f : X \to Y$ in $\operatorname{Kom}(\boldsymbol{A})$ is a quasi-isomorphism if and only if its mapping cone is an acyclic complex.*

Proof. Just consider, using the Snake Lemma, the cohomology long exact sequence associated to the short exact sequence

$$0 \longrightarrow Y \longrightarrow C(f) \longrightarrow X[1] \longrightarrow 0$$

of Lemma 2.3.4. That is

$$\to H^{k-1}C(f) \to H^{k-1}X[1] \to H^kY \to H^kC(f) \to H^kX[1] \to$$

where exactness tells us that $H^kX = H^{k-1}X[1] \to H^kY$ is isomorphism for every k if and only if $H^kC(f)$ is zero for every k. □

2.4 Axioms of triangulated categories

The notion of triangulated category is introduced abstracting the properties of the homotopy category of an abelian category. The homotopy category $\mathrm{K}(\mathbf{A})$ of an abelian category $\mathbf{A}$, as well as the derived category $\mathrm{D}(\mathbf{A})$, will happen in fact to be triangulated, and triangles are the data used as a replacement for exact sequences in order to develop the homological algebra of derived categories, which are almost never abelian.

Definition 2.4.1. Let $\mathbf{C}$ be a category with an endofunctor $T : \mathbf{C} \to \mathbf{C}$. A *triangle* in $\mathbf{C}$ is a tuple (X, Y, Z, u, v, w) of objects and morphisms in $\mathbf{C}$ giving a sequence

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

Often, for X and f an object and a morphism in $\mathbf{C}$, the object $T(X)$ and the morphism $T(f)$ are respectively denoted by $X[1]$ and $f[1]$, as well as $T^k(X)$ and $T^k(f)$ are denoted by $X[k]$ and $f[k]$.

Remark 2.4.2. There is a notion of morphism between triangles which is, for triangles $A \to B \to C \to A[1]$ and $A' \to B' \to C' \to A'[1]$, a triple of morphism (f, g, h) providing a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & A[1] \arrow[d, "f[1]"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]
\end{tikzcd}$$

A morphism between two triangles is isomorphism if all of the three f,g and h are in $\mathbf{C}$. We can take this as a definition or introduce in a obvious way a category whose objects are triangles of a triangulated category and check that this actually happens, the same way it happens for complexes.

Definition 2.4.3. A *triangulated category* is an additive category together with an endofunctor $T : \mathbf{C} \to \mathbf{C}$, which is called *shift functor* and required to be an additive equivalence, and a class of triangles called *distinguished triangles* satisfying the axioms **T1-T4** below.

Axiom **T1**:

(i) Any triangle of the form

$$A \xrightarrow{\mathrm{id}} A \longrightarrow 0 \longrightarrow TA$$

is distinguished

(ii) Any triangle isomorphic to a distinguished triangle is distinguished

(iii) For any morphism $f : A \to B$ in $\mathbf{C}$ there exist a distinguished triangle

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow TA$$

Axiom **T2**: Any triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$ is distinguished if and only if the rotated triangle

$$B \xrightarrow{v} C \xrightarrow{w} TA \xrightarrow{-Tu} TB$$

is distinguished.

Axiom **T3**: For any commutative pair of triangles $A \to B \to C \to TA$, $A' \to B' \to C' \to TA'$ and morphisms f and g making

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] & A[1] \arrow[d, "f[1]"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]
\end{tikzcd}$$

commutes, there exists a (not necessarily unique) morphism $h : C \to C'$ such that the completed diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & A[1] \arrow[d, "f[1]"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]
\end{tikzcd}$$

is commutative.

Axiom **T4**: Given three distinguished triangles of the form

$$A \xrightarrow{f} B \xrightarrow{h} D_1 \longrightarrow TA$$
$$B \xrightarrow{g} C \xrightarrow{k} D_2 \longrightarrow TB$$
$$A \xrightarrow{gf} C \xrightarrow{j} D_3 \longrightarrow TA,$$

there is a triangle

$$D_1 \longrightarrow D_3 \longrightarrow D_2 \longrightarrow TD_1$$

such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[rr, "gf"] \arrow[dr, "f"'] & & C \arrow[rr, "k"] \arrow[dr, "j"] & & D_2 \arrow[rr, dashed] \arrow[dr] & & TD_1 \\
& B \arrow[ur, "g"] \arrow[dr, "h"] & & D_3 \arrow[ur, dashed] \arrow[dr] & & TB \arrow[ur, "Th"'] & \\
& & D_1 \arrow[ur, dashed] \arrow[rr] & & TA \arrow[ur, "Tf"'] & &
\end{tikzcd}$$

Remark 2.4.4. Axioms **T1-T3** seems reasonable, they guarantee to have enough triangles to work with and ensure the class of triangles to be closed under isomorphism and "rotation": more precisely, axiom **T2** allows to think triangles as infinitely long helices

$$
\begin{tikzcd}
 & C \arrow[dd] & C[1] \arrow[dd] & C[2] & \\
 & & B \arrow[ul] & B[1] \arrow[ul] & B[2] \arrow[ul] \\
A \arrow[urr] & A[1] \arrow[urr] & A[2] \arrow[urr] & &
\end{tikzcd}
$$

such that any three consecutive vertices form a triangle. Axiom **T3** give a sufficient condition to the existence of morphisms between these helices.

However, the meaning of **T4** may look quite unclear. What goes on here is that, as we will see, a distinguished triangle $A \to B \to D \to TA$ of complexes in the derived category $\mathrm{D}(\mathbf{A})$ of an abelian category $\mathbf{A}$, arises exactly when we have a short exact sequence $0 \to A \to B \to D \to 0$ in $\mathrm{Kom}(\mathbf{A})$ (see Proposition 3.3.4). Hence, if we write D as the quotient of the previous ones, that is $D_1 \cong B/A$, $D_2 \cong C/B$ and $D_3 \cong C/A$, the requirement of axiom **T4** boils down to the statement that $C/B \cong (C/A)/(B/A)$. This is known is module theory as the Third Isomorphism Theorem.

Observe then, that this set of axiom is not independent. In fact, axiom **T3** is only stated because it's frequent to apply it on its own, but follows actually from the other ones.

Proposition 2.4.5. *Let* $\boldsymbol{C}$ *be an additive category equipped with a class of distinguished triangles and a shift functor satisfying axioms* ***T1*** *and* ***T4****, then it also satisfies axiom* ***T3****.*

Proof. Consider two distinguished triangles $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$ and $A' \xrightarrow{u'} B' \xrightarrow{v'} C' \xrightarrow{w'} TA'$ together with morphisms $f : A \to A'$ and $g : B \to B'$ making the following diagram to commute

$$
\begin{array}{ccccccc}
A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & TA \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle Tf} \\
A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & TA'
\end{array}
$$

Consider now, thanks to axiom **T1**, the completion to a distinguished triangle of the morphism $gu = u'f : A \to B'$, in order to get a triangle

$$A \overset{gu=u'f}{\longrightarrow} B' \xrightarrow{k} X \xrightarrow{j} TA$$

Also, complete the two morphisms f and g to a pair of distinguished triangles

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] & TA \arrow[d, "Tf"] \\
A' \arrow[r, "u'"] \arrow[d] & B' \arrow[r, "v'"] \arrow[d] & C' \arrow[r, "w'"] & TA' \\
D_1 \arrow[d] & D_2 \arrow[d] & & \\
TA & TB & &
\end{tikzcd}
$$

and use axiom **T4** in order to get triangles

$$
C \xrightarrow{s} X \xrightarrow{t} D_2 \longrightarrow TC
$$

$$
D_1 \xrightarrow{s'} X \xrightarrow{t'} C' \longrightarrow TD_1
$$

from the commutative diagrams

$$
\begin{tikzcd}
A \arrow[rr, "gu"] \arrow[dr, "u"'] & & B' \arrow[rr] \arrow[dr, "k"] & & D_2 \arrow[rr, dashed] \arrow[dr] & & TC \\
& B \arrow[ur, "g"'] \arrow[dr, "v"'] & & X \arrow[ur, dashed, "t"] \arrow[dr, "j"] & & TB \arrow[ur] & \\
& & C \arrow[ur, dashed, "s"] \arrow[rr, "w"'] & & TA \arrow[ur, "Tu"'] & &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
A \arrow[rr, "u'f"] \arrow[dr, "f"'] & & B' \arrow[rr, "v'"] \arrow[dr, "k"] & & C' \arrow[rr, dashed] \arrow[dr, "w'"] & & TD_1 \\
& A' \arrow[ur, "u'"'] \arrow[dr] & & X \arrow[ur, dashed, "t'"] \arrow[dr, "j"] & & TA' \arrow[ur] & \\
& & D_1 \arrow[ur, dashed, "s'"] \arrow[rr] & & TA \arrow[ur, "Tf"'] & &
\end{tikzcd}
$$

The two triangles so obtained suggest to consider $t's : C \to C'$ as morphism proving **T3**, and the two diagrams tell us that such a morphism works, because $t'sv = t'kg = v'g$, as well as $w't's = (Tf)js = (Tf)w$, that is, the diagram

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "t's"] & TA \arrow[d, "Tf"] \\
A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & TA'
\end{tikzcd}
$$

is commutative. $\square$

It is useful to define an environment in which to consider triangulated categories and morphisms between them. In fact, there is a category **TrCat** of triangulated categories as described below.

Definition 2.4.6. An additive functor $F : \mathbf{C} \to \mathbf{C}'$ between triangulated categories with shift functors T and T' is said to be a *triangulated functor* if it commutes with the shifts, i.e. there is a natural isomorphism $T' \circ F \cong F \circ T$, and map distinguished triangles to distinguished triangles, i.e. for any distinguished triangle $A \to B \to C \to TA$, the triangle

$$FA \to FB \to FC \to FTA \cong TFA$$

is a distinguished triangle.

TrCat is then the subcategory of **Cat** whose objects are triangulated categories and morphisms between them are triangulated functors.

Definition 2.4.7. A *triangulated subcategory* $\mathbf{B} \subseteq \mathbf{A}$ of a triangulated category $\mathbf{A}$ is a subcategory with a structure of triangulated category and such that the inclusion functor $i : \mathbf{B} \to \mathbf{A}$ is triangulated.

Let's now state and prove some of the main properties of triangulated categories, in order first of all to examine how triangles are related to the notion of exact sequence. We start with an easy observation.

Remark 2.4.8. In any triangle $A \to B \to C \to TA$, the composition $A \to C$ is zero. That is because **T1** provides the triangle $A \to A \to 0 \to TA$, and by **T3** there is a morphism $0 \to C$ such that the following diagram commute

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, dashed] & TA \arrow[d, "\mathrm{id}"] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & TA \end{tikzcd}$$

Hence $gf = 0$.

Lemma 2.4.9. *Let $A \to B \to C \to TA$ be a distinguished triangle in a triangulated category $\boldsymbol{A}$. Then for any object D in $\boldsymbol{A}$ the sequences obtained applying the* Hom *functors to $A \to B \to C$*

$$\mathrm{Hom}(C, D) \longrightarrow \mathrm{Hom}(B, D) \longrightarrow \mathrm{Hom}(A, D)$$
$$\mathrm{Hom}(D, A) \longrightarrow \mathrm{Hom}(D, B) \longrightarrow \mathrm{Hom}(D, C)$$

are exact.

Proof. The fact that the image of $\mathrm{Hom}(C, D) \to \mathrm{Hom}(B, D)$ is contained in the kernel of $\mathrm{Hom}(B, D) \to \mathrm{Hom}(A, D)$ is given by Remark 2.4.8. Conversely, for $f : B \to D$ to be in the kernel of $\mathrm{Hom}(B, D) \to \mathrm{Hom}(A, D)$ means that precomposed with $A \to B$ is the zero morphism. Then consider the triangle $D \to D \to 0 \to TD$ provided by **T1**, and its rotation, distinguished by **T2**, $0 \to D \to D \to 0$, where clearly $T0 = 0$ by additivity. The morphism f is hence such that the diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & TA \\ \downarrow & & \downarrow{\scriptstyle f} & & & & \downarrow \\ 0 & \longrightarrow & D & \xrightarrow{\mathrm{id}} & D & \longrightarrow & 0 \end{array}$$

commutes. Axiom **T3** provides then a morphism $C \to D$ such that precomposed with $B \to C$ gives f, that is f is in the image of $\mathrm{Hom}(C, D) \to \mathrm{Hom}(B, D)$.

The exactness of the other sequence is analogous: the first inclusion is still given by Remark 2.4.8, and conversely one consider a morphisms $f : D \to B$ which is zero ones composed with $B \to C$. Then there is a commutative diagram

$$\begin{array}{ccccccc} D & \xrightarrow{\mathrm{id}} & D & \longrightarrow & 0 & \longrightarrow & TD \\ & & \downarrow{\scriptstyle f} & & \downarrow & & \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & TA \end{array}$$

that gives, rotating by axiom **T2** and using **T3**, a morphism $D \to A$ lifting f to B, proving that f is in the image of $\mathrm{Hom}(D, A) \to \mathrm{Hom}(D, B)$. □

Remark 2.4.10. Thanks to axiom **T2**, if $A \to B \to C \to TA$ is a distinguished triangle and D any object, Lemma 2.4.9 provide actually long exact sequences induced by the Hom functors.

Proposition 2.4.11. *Let $A \xrightarrow{f} B \xrightarrow{g} C \to TA$ be a distinguished triangle in a triangulated category $\boldsymbol{A}$ such that g is a split monomorphism, then $f = 0$*

Proof. Consider, for any D in $\mathbf{A}$, the exact sequence

$$\mathrm{Hom}(D, A) \longrightarrow \mathrm{Hom}(D, B) \longrightarrow \mathrm{Hom}(D, C)$$

and observe that whenever $h_1, h_2 : D \to B$ are such that $gh_1 = gh_2$, we can take the left inverse g' of g and find that $h_1 = g'gh_1 = g'gh_2 = h_2$. Therefore, the morphism $\mathrm{Hom}(D, g) : \mathrm{Hom}(D, B) \to \mathrm{Hom}(D, C)$ is injective, and hence by exactness $\mathrm{Hom}(D, f) = 0$. Since D was any, we can take it to be A and observe that $f = f\mathrm{id}_A = 0$. □

The following result illustrates a phenomenon related to distinguished triangles which we will encounter later on.

Lemma 2.4.12. *Let the following be a morphism of distinguished triangles:*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & TA \arrow[d, "Tf"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & TA' \end{tikzcd}$$

Then, if two of the morphisms f, g and h are isomorphism, so is the third one.

Proof. Note, first of all, that by axiom **T2** one can prove the statement assuming a particular choice of which among the three are the isomorphisms. So, suppose f and g to be isomorphisms and consider, for any object D, the long exact sequence from Remark 2.4.10

$$\mathrm{Hom}(D,A) \to \mathrm{Hom}(D,B) \to \mathrm{Hom}(D,C) \to \mathrm{Hom}(D,TA) \to \mathrm{Hom}(D,TB)$$

Out of the terms of each degree but the mid one, are defined isomorphisms given by $\mathrm{Hom}(D,f)$, $\mathrm{Hom}(D,g)$ in the first two degrees, and $\mathrm{Hom}(D,Tf)$, $\mathrm{Hom}(D,Tg)$ in the last two. The mid morphism is $\mathrm{Hom}(D,h) : \mathrm{Hom}(D,B) \to \mathrm{Hom}(D,B')$, and the Five Lemma ensures that this also is an isomorphism. Now, this diagram is natural in D, hence we get an isomorphism of functors $\mathrm{Hom}(-,h) : \mathrm{Hom}(-,B) \to \mathrm{Hom}(-,B')$, and from the Yoneda Lemma, in particular Corollary 1.1.6, we get that $h : B \to B'$ is an isomorphism. $\square$

A similar argument is carried in order to prove the following result.

Proposition 2.4.13. *Let $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$ and $A' \xrightarrow{u'} B' \xrightarrow{v'} C' \xrightarrow{w'} TA'$ be two distinguished triangles. Then, their direct sum*

$$A \oplus A' \xrightarrow{u \oplus u'} B \oplus B' \xrightarrow{v \oplus v'} C \oplus C' \xrightarrow{w \oplus w'} TA \oplus TA' \cong T(A \oplus A')$$

is a distinguished triangle.

Proof. Consider the morphism $u \oplus u' = \begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix} : A \oplus A' \to B \oplus B'$ and complete it to a distinguished triangle

$$A \oplus A' \longrightarrow B \oplus B' \longrightarrow Z \longrightarrow T(A \oplus A')$$

Observe that by axiom **T3** there is a morphisms of distinguished triangles induced by inclusions

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & B \arrow[r] \arrow[d, hook] & C \arrow[r] \arrow[d, dashed, "\phi"] & TA \arrow[d, hook] \\ A \oplus A' \arrow[r] & B \oplus B' \arrow[r] & Z \arrow[r] & TA \oplus TA' \end{tikzcd}$$

as well as

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, hook] & B' \arrow[r] \arrow[d, hook] & C' \arrow[r] \arrow[d, dashed, "\phi'"] & TA' \arrow[d, hook] \\
A \oplus A' \arrow[r] & B \oplus B' \arrow[r] & Z \arrow[r] & TA \oplus TA'
\end{tikzcd}$$

Morphisms ϕ, ϕ' give rise to the morphism $\psi : C \oplus C' \to Z$, which we claim to be an isomorphism. In order to see that, it suffices to consider the long exact sequences induced by representable functors $\mathrm{Hom}(D, -)$, and observe that the exactness of both

$$\cdots \to \mathrm{Hom}(D, B) \to \mathrm{Hom}(D, C) \to \mathrm{Hom}(D, TA) \to \cdots$$

and

$$\cdots \to \mathrm{Hom}(D, B') \to \mathrm{Hom}(D, C') \to \mathrm{Hom}(D, TA') \to \cdots$$

implies, since both the Hom and cohomology functors preserve finite direct sums, the exactness of

$$\cdots \to \mathrm{Hom}(D, B \oplus B') \to \mathrm{Hom}(D, C \oplus C') \to \mathrm{Hom}(D, T(A \oplus A')) \to \cdots$$

Hence, the morphism induced by the identities and $\psi : C \oplus C' \to Z$ gives a diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}(D, B \oplus B') \arrow[r] \arrow[d] & \mathrm{Hom}(D, C \oplus C') \arrow[r] \arrow[d] & \mathrm{Hom}(D, T(A \oplus A')) \arrow[d] \\
\mathrm{Hom}(D, B \oplus B') \arrow[r] & \mathrm{Hom}(D, Z) \arrow[r] & \mathrm{Hom}(D, T(A \oplus A'))
\end{tikzcd}$$

which allows to conclude by the Five Lemma, that $\mathrm{Hom}(-, C \oplus C') \cong \mathrm{Hom}(-, Z)$, and therefore $C \oplus C' \cong Z$ via ψ by the Yoneda Lemma. This concludes because triangles isomorphic to distinguished triangles are themselves distinguished. $\square$

Lemma 2.4.14. *In any triangulated category, if a distinguished triangle* $A \xrightarrow{f} B \xrightarrow{g} C \to TA$ *is such that* $C \to TA$ *is zero, then* C *decomposes as* $C \cong A \oplus B$.

Proof. The long exact sequence obtained applying $\mathrm{Hom}(C, -)$ gives in fact a short exact sequence

$$0 \to \mathrm{Hom}(C, A) \xrightarrow{\bar{f}} \mathrm{Hom}(C, B) \xrightarrow{\bar{g}} \mathrm{Hom}(C, C) \to 0$$

since both the morphisms $\mathrm{Hom}(C, C[-1]) \to \mathrm{Hom}(C, A)$ and $\mathrm{Hom}(C, C) \to \mathrm{Hom}(C, A[1])$ are (precomposition with) zero by hypothesis. By exactness at $\mathrm{Hom}(C, C)$, the map $\bar{g} : k \mapsto gk$ is surjective, for any map $h : C \to C$ we find a map $k : B \to C$ such that $gk = h$, hence we get a right inverse $\bar{s} : \mathrm{Hom}(C, C) \to \mathrm{Hom}(C, B)$ mapping $h \mapsto k$. This implies by Splitting

Lemma that $\mathrm{Hom}(C, B) \cong \mathrm{Hom}(C, A) \oplus \mathrm{Hom}(C, C)$. Now set $s = \bar{s}(1_C) : C \to B$, then for every object E the exact sequence

$$0 \to \mathrm{Hom}(E, A) \to \mathrm{Hom}(E, B) \to \mathrm{Hom}(E, C) \to 0$$

also splits by the right inverse $\bar{s} : \mathrm{Hom}(E, C) \to \mathrm{Hom}(E, B)$ mapping $h \mapsto sh$. Moreover, everything is natural in E, and hence the isomorphism of functors

$$\mathrm{Hom}(-, A \oplus C) \cong \mathrm{Hom}(-, A) \oplus \mathrm{Hom}(-C) \cong \mathrm{Hom}(-, B)$$

gives, by Yoneda Lemma, the desired decomposition $A \oplus B \cong C$. $\square$

Remark 2.4.15. The zero object $0_{\mathbf{B}}$ in any triangulated subcategory $\mathbf{B} \subseteq \mathbf{A}$ is isomorphic to the zero object $0_{\mathbf{A}}$ of $\mathbf{A}$. Consider in fact, by axiom **T1**, for an object B in $\mathbf{B}$ the distinguished triangle $B \stackrel{\mathrm{id}}{\to} B \to 0_B \to TB$, then look at the distinguished triangle in $\mathbf{A}$ given by $i(B) \stackrel{\mathrm{id}}{\to} i(B) \to i(0_{\mathbf{B}}) \to i(TB) = Ti(B)$. The distinguished triangle in $\mathbf{A}$ given by axiom **T1** again, $i(B) \to i(B) \to 0_{\mathbf{A}} \to Ti(B)$, together with Lemma 2.4.12, gives then $i(0_{\mathbf{B}}) \cong 0_{\mathbf{A}}$.

The structure of triangulated category of a full triangulated subcategory $\mathbf{B} \subseteq \mathbf{A}$ is uniquely determined by the structure of triangulated category of $\mathbf{A}$. More precisely one has the following result, whose proof is quite trivial, but it's useful to keep it in mind.

Proposition 2.4.16. *Let $\boldsymbol{B} \subseteq \boldsymbol{A}$ be a full subcategory of a triangulated category $\boldsymbol{A}$. Then $\boldsymbol{B}$ is a triangulated subcategory if and only if $\boldsymbol{B}$ is invariant under the shift functor and any triangle $A \to B \to C \to TA$ in $\boldsymbol{A}$ such that A and B are objects in $\boldsymbol{B}$, is such that C is isomorphic to an object in $\boldsymbol{B}$.*

Proof. On one direction, assume the full subcategory to be triangulated with shift functor T'. First of all, given $A \to B \to C \to TA$ a triangle in $\mathbf{A}$ with A, B in $\mathbf{B}$, by fullness also $A \to B$ is a morphism in $\mathbf{B}$. Then use axiom **T1** for $\mathbf{B}$ in order to complete it to a triangle $A \to B \to C' \to T'A$, and apply to it the inclusion triangulated functor $i : \mathbf{B} \to \mathbf{A}$, finding again a distinguished triangle. Now, shifts commute with inclusion, that is $TA = Ti(A) = i(T'A) = T'A$, hence axiom **T3** for $\mathbf{A}$ with Lemma 2.4.12 provides the commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\mathrm{id}"] & B \arrow[r] \arrow[d, "\mathrm{id}"] & C \arrow[r] \arrow[d, dashed, "\cong"] & TA \arrow[d, "\mathrm{id}"] \\
A \arrow[r] & B \arrow[r] & C' \arrow[r] & TA
\end{tikzcd}$$

and the desired isomorphism $C \to C'$. Being invariant under the shift functor is implicit in the definition of triangulated subcategory: if B is an object in **B**, then $T'B$ certainly is, and as before, since inclusion functor is triangulated $T'B = TB$.

Conversely, the triangulated structure on the subcategory is defined inducing the shift functor, which requires to be invariant under its action, and declaring a triangle $A \to B \to C \to TA$ in **B** to be distinguished whether it is distinguished as a triangle in **A** after applying the inclusion functor. With this class of triangles, axioms of triangulated categories obviously hold true for the full subcategory **B**. $\square$

Remark 2.4.17. From now on, we will assume any triangulated subcategory $\mathbf{B} \subseteq \mathbf{A}$ to be *strictly full*, i.e. such that whenever an object A in **A** is isomorphic to an object B in **B**, then A is actually in **B**.

2.5 The homotopy category is triangulated

At this point, one can imagine that we will have triangulated categories whose objects are complexes, with shift functors the actual shifts of the complexes one step towards the left. Triangles will be given using the mapping cone construction.

Definition 2.5.1. Let **A** be an abelian category. For the category $\mathrm{K}(\mathbf{A})$ together with its shift functor $T = [1]$, a triangle

$$X^\cdot \longrightarrow Y^\cdot \longrightarrow Z^\cdot \longrightarrow X^\cdot[1]$$

is said to be *exact* whether it's isomorphic (through morphisms in $\mathrm{K}(\mathbf{A})$) to a triangle of the form

$$A^\cdot \xrightarrow{f} B^\cdot \longrightarrow C(f) \longrightarrow A^\cdot[1]$$

Example 2.5.2. This choice of triangles makes sense also for the category of complexes $\mathrm{Kom}(\mathbf{A})$, however, it's easily seen that they cannot match the requirements for the distinguished triangles of a triangulated category, because for any non-trivial complex X, the triangle

$$X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow TX$$

cannot be isomorphic to a triangle of the form

$$A \xrightarrow{f} B \longrightarrow C(f) \longrightarrow TA$$

because it should certainly be $A, B \ncong 0$, and the cone of any morphism $A \to B$ would not be isomorphic, in $\mathrm{Kom}(\mathbf{A})$, to zero.

More in general, one can prove (see [GM02], IV.1 Exercise 1), than any triangulated category which is also abelian, has to be *semisimple*, namely such that any short exact sequence splits, and this is not the case, e.g. for $\mathrm{Kom}(\mathbf{Ab})$.

Remark 2.5.3. Observe that for every X in $\mathrm{K}(\mathbf{A})$, the complex $C(\mathrm{id}_X)$ is homotopically equivalent to 0. In fact, there is an homotopy between the morphisms

$$0, \mathrm{id} : C(\mathrm{id}_X) \longrightarrow C(\mathrm{id}_X)$$

given by the morphisms (h^i) in

$$\begin{tikzcd}
\cdots \arrow[r] & C^{i-1}(f) \arrow[r, "d^{i-1}"] \arrow[d] & C^i(f) \arrow[r, "d^i"] \arrow[d, "\mathrm{id}"] \arrow[dl, "h^i"'] & C^{i+1}(f) \arrow[r] \arrow[d] \arrow[dl, "h^{i+1}"] & \cdots \\
\cdots \arrow[r] & C^{i-1}(f) \arrow[r, "d^{i-1}"'] & C^i(f) \arrow[r, "d^i"'] & C^{i+1}(f) \arrow[r] & \cdots
\end{tikzcd}$$

defined by

$$h^i = \begin{pmatrix} 0 & -\mathrm{id}_{X^i} \\ 0 & 0 \end{pmatrix}.$$

In fact, the morphism $d^{i-1}h^i + h^{i+1}d^i$ is computed as

$$\begin{pmatrix} -d_X^i & 0 \\ -\mathrm{id}_{X^i} & d_X^{i-1} \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{id}_{X^i} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\mathrm{id}_{X^{i+1}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -d_X^{i+1} & 0 \\ -\mathrm{id}_{X^{i+1}} & d_X^i \end{pmatrix} =$$
$$\begin{pmatrix} 0 & d_X^i \\ 0 & \mathrm{id}_{X^i} \end{pmatrix} + \begin{pmatrix} \mathrm{id}_{X^{i+1}} & -d_X^i \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{X^i} & 0 \\ 0 & \mathrm{id}_{X^{i+1}} \end{pmatrix} = \mathrm{id}_{C^i(f)}.$$

Also the rest of the axioms of triangulated categories can be proven for the homotopy category. More precisely, the following theorem holds true.

Theorem 2.5.4. *Let $\boldsymbol{A}$ be an abelian category. The category $\mathrm{K}(\boldsymbol{A})$ with its shift functor and its class of exact triangles as distinguished triangles is a triangulated category.*

Proof. See [GM02] IV.1 10-14. □

Remark 2.5.5. Observe the fact, implicitly proved yet in Lemma 2.3.4 and Proposition 2.3.5, that a distinguished triangle $A \to B \to C \to TA$ in $\mathrm{K}(\mathbf{A})$ gives rise, thanks to the Snake Lemma, to a long exact sequence in cohomology

$$\cdots \to H^{i-1}(C) \to H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \cdots$$

Chapter 3

Derived categories

The idea is the following: in order to define the derived category of an abelian category $\mathbf{A}$ we do not need the homotopy category $\mathrm{K}(\mathbf{A})$, and in a way that will be more precise using the notion of *localization*, that is because as a result of passing from the category of complexes to the homotopy category we get that homotopical inverses become isomorphisms, while the operation of constructing the derived category from the category of complexes consist of turn quasi-isomorphisms into isomorphisms. Corollary 2.2.17 tells us, however, that homotopically invertible morphisms are quasi-isomorphisms yet.

Why, then, we went through the previous discussion of homotopy categories? Because the "abstract" definition of derived category does not provide any tool aimed to compute and understand how the morphisms work in there, and the homotopy category is a natural context in order to do so.

3.1 Localization of morphisms

The following is a first, naïve but very general, definition of localization.

Definition 3.1.1. Let $\mathbf{C}$ be a category and S be a class of morphisms in $\mathbf{C}$, and introduce a variable symbol x_s for every s in S. The *localization at S of $\boldsymbol{C}$*, denoted by $\mathbf{C}[S^{-1}]$ is a category defined to have as objects the same objects as $\mathbf{C}$.

In order to define morphisms, consider an intermediate step in which we let, for every objects X and Y, an arrow $x_s : X \to Y$ for every $s : Y \to X$ in S, as well as all the compositions with these new morphisms. A morphism $X \to Y$ in $\mathbf{C}[S^{-1}]$ is then an equivalence class of morphisms $X \to Y$ in this intermediate category under the equivalence relation, for every A and B in $\mathbf{C}$,

$$x_s s \sim \mathrm{id}_A, \ s x_s \sim \mathrm{id}_B$$

for every $s : A \to B$ in S.

Remark 3.1.2. Any localization comes together with a *localization functor*

$$Q : \mathbf{A} \longrightarrow \mathbf{A}[S^{-1}]$$

mapping any object to itself and any morphism to its class.

This category defined as such, has the advantage to provide an easy proof of the existential result for derived categories, defined so in term of their universal property. However, it has the following main problem: denoting by s^{-1} the morphism x_s, we do not know how to compute composition of morphisms using desirable algebraic manipulations. As an example, for the composition $fs_1^{-1}gs_2^{-1}$, we're not able to find a common denominator s such that it becomes equal to $f'g's^{-1}$.

Theorem 3.1.3. *Let $\boldsymbol{C}$ be a category and S a class of morphisms in $\boldsymbol{C}$. If $F : \boldsymbol{C} \longrightarrow \boldsymbol{D}$ is a functor such that $F(s)$ is isomorphism in $\boldsymbol{D}$ for every s in S, then there exists a unique functor $F' : \boldsymbol{C}[S^{-1}] \longrightarrow \boldsymbol{D}$ such that $F'Q = F$*

$$\begin{tikzcd}
\boldsymbol{C} \arrow[r, "Q"] \arrow[dr, "F"'] & \boldsymbol{C}[S^{-1}] \arrow[d, "F'"] \\
 & \boldsymbol{D}
\end{tikzcd}$$

Proof. The functor F' is defined on objects to be $F'(X) = F(X)$ for every X in $\mathbf{C}$. If $[f]$ is a morphism of $\mathbf{C}[S^{-1}]$, first suppose it to be the class of a morphism which is actually in $\mathbf{C}$, which we still call f. In this case just set $F'([f]) = F(f)$. If $[f]$ is of the form $[x_s]$, then set $F'([f]) = F(s)^{-1}$ (this shows that the previous definition of F' on arrows in $\mathbf{C}$ is actually well defined on classes). This clearly suffices in order to define the functor F' on every morphism of $\mathbf{C}[S^{-1}]$, since these are composition of morphism from $\mathbf{C}$ and morphism of sort x_s, and hence F' applied to the class of this composition is the composition of F applied to the classes of the component.

Suppose $F'' : \mathbf{C}[S^{-1}] \to \mathbf{D}$ to be a functor with the same property of F', that is $F''Q = F$. Then F' and F'' clearly are the same on objects because $F'X = F'QX = FX = F''QX = F''X$. A morphism $[f]$, as before, is the class of a composition of morphisms $f_1 \cdots f_n$ which are either in $\mathbf{C}$ or of the form $f_i = x_{s_i}$, with s_i in S. Applying F'' will provide the composition $F'([f_1]) \cdots F'([f_n])$, and each of these components is either $F''Q(f_i) = F(f_i)$ if f_i is in $\mathbf{C}$, or $F''([x_{s_i}])$ if $f_i = x_{s_i}$. Observe, eventually, that the relation imposed on morphisms of $\mathbf{C}[S^{-1}]$ forces $F''([x_{s_i}])$ to be such that for all $s : A \to B$

$$F(s)F''([x_s]) = F''([s])F''([x_s]) = F''([sx_s]) = F''([\mathrm{id}_B]) = \mathrm{id}_{F''(B)}$$

as well as $F''([x_s])F([s]) = \mathrm{id}_{F''(A)}$. That is $F''([x_{s_i}]) = F(s)^{-1}$. So, the result is the same obtained applying F'. $\square$

Definition 3.1.4. Let $\mathbf{A}$ be an abelian category. The *derived category* $\mathrm{D}(\mathbf{A})$ of $\mathbf{A}$ is the localization $\mathrm{Kom}(\mathbf{A})[S^{-1}]$, where S is the class of all quasi-isomorphisms.

Definition 3.1.5. Define the full subcategories $\mathrm{Kom}^+(\mathbf{A}), \mathrm{Kom}^-(\mathbf{A}) \subseteq \mathrm{Kom}(\mathbf{A})$ whose objects are the bounded (i.e. eventually constant to 0), respectively below and above, chain complexes of objects in $\mathbf{A}$. Define as well their intersection $\mathrm{Kom}^b(\mathbf{A})$.

Remark 3.1.6. The same construction of the derived category can be made replacing $\mathrm{Kom}(\mathbf{A})$ with either $\mathrm{Kom}^+(\mathbf{A})$, $\mathrm{Kom}^-(\mathbf{A})$ and $\mathrm{Kom}^b(\mathbf{A})$. The resulting categories are denoted by $\mathrm{D}^+(\mathbf{A})$, $\mathrm{D}^-(\mathbf{A})$ and $\mathrm{D}^b(\mathbf{A})$.

Moreover, if the abelian category in question is a category of modules over a sheaf of rings $\mathcal{O}_X$, we abbreviate $\mathrm{Kom}(\mathbf{Mod}(\mathcal{O}_X))$ and $\mathrm{D}(\mathbf{Mod}(\mathcal{O}_X))$ with, respectively, $\mathrm{Kom}(\mathcal{O}_X)$ and $\mathrm{D}(\mathcal{O}_X)$. Analogously for a ring R.

Remark 3.1.7. Thanks to theorem 3.1.3, any functor $F : \mathrm{Kom}(\mathbf{A}) \to \mathbf{D}$ such that $F(i)$ is invertible for any quasi-isomorphism i, uniquely factors through $\mathrm{D}(\mathbf{A})$.

Example 3.1.8. Let $\mathbf{A} = \mathbf{Vec}_f(k)$ be the category of finite dimensional vector spaces over a field k. Then $\mathrm{D}(\mathbf{A})$ is equivalent to the product category $\prod_{i \in \mathbb{Z}} \mathbf{A}$. In order to prove that, we show that any object in $\mathrm{D}(\mathbf{A})$ is isomorphic to an object in its full subcategory of complexes with trivial differential (which is isomorphic to $\prod_{i \in \mathbb{Z}} \mathbf{A}$). More precisely, for any complex $V^\cdot$ there is a quasi-isomorphism

$$\begin{tikzcd}
\cdots \arrow[r] & V^{i-1} \arrow[r, "d^{i-1}"] \arrow[d] & V^i \arrow[r, "d^i"] \arrow[d] & V^{i+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^{i-1}(V^\cdot) \arrow[r, "0"] & H^i(V^\cdot) \arrow[r, "0"] & H^{i+1}(V^\cdot) \arrow[r] & \cdots
\end{tikzcd}$$

This really is because for any i one can find, in a highly non-canonical way, a subspace $W^i \subseteq V^i$ such that

$$V^i \cong \mathrm{Ker}(d^i) \oplus W^i \cong \mathrm{Im}(d^{i-1}) \oplus H^i(V^\cdot) \oplus W^i.$$

Thus, one have a quotient projection morphisms $f^i : V^i \to H^i$, and the resulting morphism $(f^i)_i$ between complexes certainly induces isomorphism on cohomology, hence is an isomorphism in $\mathrm{D}(\mathbf{A})$.

A fancier notion of localization is provided by the following definition, that will impose conditions on the class of morphisms at which we are allowed to localize, providing some useful computational tools, which are basically algebraic identities aimed to manipulate morphisms.

Definition 3.1.9. Let **C** be a category, and S be a class of morphisms in **C**. S is called a *multiplicative system* if it satisfies the following conditions:

(a) S is closed under composition, that is, for any X object in **C**, id_X is in S and for any pair of morphisms (f, g) of S such that the composition gf exists, then gf is in S

(b) Every diagram of the form

$$\begin{tikzcd} & Z \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \text{, respectively} \quad \begin{tikzcd} & Z \\ X & Y \arrow[l, "f"'] \arrow[u, "s"'] \end{tikzcd}$$

with s in S, may be completed to a commutative diagram

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "t"'] & Z \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \text{, respectively} \quad \begin{tikzcd} W & Z \arrow[l, "g"'] \\ X \arrow[u, "t"] & Y \arrow[l, "f"] \arrow[u, "s"'] \end{tikzcd}$$

where t is in S.

(c) Let f and g be morphisms $X \to Y$. There exists s in S with $sf = sg$ if and only if there exists t in S with $ft = gt$.

Remark 3.1.10. Multiplicative systems straightforward generalize the notion of *multiplicatively closed subset* of a commutative ring. In fact, given a commutative ring R, i.e. a pre-additive category with exactly one object X (and $R \cong \mathrm{Hom}(X, X)$) and commutative composition, S is a multiplicative system of this category if an only if $S \subseteq R$ is such that $1 \in S$ and $x, y \in S \Rightarrow xy \in S$. Certainly if S is a multiplicative system, then part (a) says exactly that it is a multiplicative set. Conversely, parts (b) and (c) are clear in the commutative case: if $X \xrightarrow{f} X \xleftarrow{s} X$, then the same morphisms give

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & X \arrow[d, "s"] \\ X \arrow[r, "f"] & X \end{tikzcd}$$

which is commutative by commutativity of R. The same works for the diagram with reversed arrows. Part (c) is given considering $s = t$.

Remark 3.1.11. Given a category $\mathbf{C}$ and a multiplicative system S, consider morphisms $X \xrightarrow{f} Y \xleftarrow{s} Z$ with s in S. There are, thanks to (b), morphisms t in S and g such that $ft = sg$. Now, in $\mathbf{C}[S^{-1}]$ there are morphisms $[x_s f]$ and $[g x_t]$, and these are the same since

$$ft = sg \Rightarrow [x_s f] = [x_s f t x_t] = [x_s s g x_t] = [g x_t].$$

Thus, for a morphism of the form (class of) $s_1^{-1} f_1 s_2^{-1} f_2$ there are t_1, t_2, g_1, g_2 such that it becomes $g_1 t_1^{-1} g_2 t_2^{-1}$, which again is of the form $g_1 g_0 t_0^{-1} t_2^{-1} = g_1 g_0 (t_2 t_0)^{-1}$. Similarly, the diagram with reversed arrows in (b) allows us to move the denominators to the left.

The point is that the class of quasi-isomorphism does not form in general a multiplicative system for $\mathrm{Kom}(\mathbf{A})$. However, the notion of quasi-isomorphism makes sense also in $\mathrm{K}(\mathbf{A})$, because the cohomologies of homotopically equivalent morphisms coincides (Proposition 2.2.16), and the following result holds true.

Proposition 3.1.12. *Let $\boldsymbol{A}$ be an abelian category. The class of quasi-isomorphisms in $\mathrm{K}(\boldsymbol{A})$ is a multiplicative system.*

Proof. See [GM02] III.4.4. ☐

Thus, the following Theorem will provide an operative definition of the derived category of an abelian category.

Theorem 3.1.13. *Let S be a multiplicative system in a category $\boldsymbol{C}$. Then the localization $\boldsymbol{C}[S^{-1}]$ can be described as the category whose objects are the same as in $\boldsymbol{C}$, while morphisms and compositions are described by*

(*i*) *A morphism $X \to Y$ in $\boldsymbol{C}[S^{-1}]$ is an equivalence class of diagrams, called "roofs", of the form*

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

denoted as a pair (s, f) with s in S and f a morphism in $\boldsymbol{C}$. Two such diagrams $(s, f) \sim (t, g)$ are equivalent if and only if there are r in S and h morphism in $\boldsymbol{C}$ forming a third common roof in the commutative diagram

$$\begin{tikzcd} & & X''' \arrow[dl, "r"'] \arrow[dr, "h"] & & \\ & X' \arrow[dl, "s"'] \arrow[drrr, "f" near end] & & X'' \arrow[dlll, "t"' near end] \arrow[dr, "g"] & \\ X & & & & Y \end{tikzcd}$$

Identity of X is the class of $(\mathrm{id}_X, \mathrm{id}_X)$

(*ii*) *The composition of two morphisms classes of diagrams* (s, f) *and* (t, g) *is given using the definition of multiplicative system in order to find morphisms* t' *in* S *and* f' *fitting the diagram*

$$\begin{tikzcd} & & X'' \arrow[dl, "t'"'] \arrow[dr, "f'"] & & \\ & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & & X \end{tikzcd}$$

That is, the composition is the class of (st', gf')

Proof. See [GM02] III.2.8. □

Proposition 3.1.14. *The derived category* $\mathrm{D}(\boldsymbol{A})$ *of an abelian category carries a natural structure of additive category.*

Proof. Given a pair of morphisms $X \to Y$ in $D(\mathbf{A})$, represent them using Theorem 3.1.13 as morphisms $X \overset{q_1}{\leftarrow} Z' \overset{f_1}{\to} Y$ and $X \overset{q_2}{\leftarrow} Z'' \overset{f_2}{\to} Y$ in the homotopy category, where q_1, q_2 are quasi-isomorphisms. Then consider the diagram $Z' \overset{q_1}{\to} X \overset{q_2}{\leftarrow} Z''$, and since quasi-isomorphisms are a multiplicative system for $\mathrm{K}(\mathbf{A})$ complete it to

$$\begin{tikzcd} Z \arrow[r, "p_1"] \arrow[d, "p_2"'] & Z' \arrow[d, "q_1"] \\ Z'' \arrow[r, "q_2"'] & X \end{tikzcd}$$

where p_2 (and hence also p_1) is qiso. Call q the morphism $q_1 p_1 = q_2 p_2 : Z \to X$, and just set $f_1 q_1^{-1} + f_2 q_2^{-1}$ to be the class of

$$X \overset{q}{\longleftarrow} Z \overset{f_1 p_1 + f_2 p_2}{\longrightarrow} Y$$

Inverse is given by $-(fq^{-1}) = (-f)q^{-1}$.

Moreover, finite products are the same as in $\mathrm{K}(\mathbf{A})$. If X and Y are complexes in $\mathrm{D}(\mathbf{A})$, the complex $X \times Y$ is such that whenever $P \to X$ and $P \to Y$ are two morphisms in $\mathrm{D}(\mathbf{A})$, represented as $P \leftarrow P' \to X$ and $P \leftarrow P'' \to Y$, one can complete $P'' \to P \leftarrow P'$ to

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & P' \arrow[d] \\ P'' \arrow[r] & P \end{tikzcd}$$

where, as before, all morphisms are quasi-isomorphisms, and consider the composition $q : Q \to P$ as well as the two morphisms $Q \to P' \to X$ and $Q \to P'' \to Y$, inducing a unique morphism $f : Q \to X \times Y$ commuting with projections in $\mathrm{K}(\mathbf{A})$. The morphism

$$P \xleftarrow{q} Q \xrightarrow{f} X \times Y$$

will be the unique proving that $\mathrm{D}(\mathbf{A})$ admits finite products. Hence, the category $\mathrm{D}(\mathbf{A})$ is an additive category. $\square$

So, by Theorem 3.1.13 we get an explicit description for the derived category of an abelian category, especially of its morphisms. One could think in first approximation, supported by Example 3.1.8, that studying the derived category of an abelian category amounts to look at the objects up to cohomology, but this is not right. There are in fact complexes whose cohomologies are isomorphic, but which aren't isomorphic objects in the derived category (see Example 3.4.8).

Moreover, it's not even true that between isomorphic objects in a derived category has to exists a quasi-isomorphism.

Example 3.1.15. If $\mathbf{A}$ is an abelian category and X and Y are isomorphic objects in $D(\mathbf{A})$, it may not exists a quasi-isomorphism between X and Y. In order to see that, start considering quasi isomorphic complexes $\tilde{X}$ and $\tilde{Y}$ such that there is no quasi-isomorphism $\tilde{Y} \to \tilde{X}$: for example in $\mathbf{A} = \mathbf{Ab}$

$$\tilde{X} = \cdots \to 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0 \to \cdots$$

and

$$\tilde{Y} = \cdots \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0 \cdots$$

The morphism $f : \tilde{X} \to \tilde{Y}$ given by the only non-trivial morphism $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ induces in fact isomorphism on cohomology. However, no nontrivial morphism at all exists from $\tilde{Y}$ to $\tilde{X}$, because $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$. This leads to consider two other complexes one build concatenating $\tilde{X}$ and $\tilde{Y}$, and the other concatenating $\tilde{Y}$ and $\tilde{X}$, in the respective positions, that is, in our example:

$$\begin{aligned} X &= \cdots 0 \to 0 \to \mathbb{Z}/2 \to 0 \to 0 \to \mathbb{Z} \longrightarrow \mathbb{Z} \to 0 \to \cdots \\ Y &= \cdots 0 \to \mathbb{Z} \longrightarrow \mathbb{Z} \to 0 \to 0 \to 0 \to \mathbb{Z}/2 \to 0 \to \cdots \end{aligned}$$

There is no nontrivial morphism neither from X to Y nor from Y to X, thus they aren't quasi-isomorphic, but for the complex X' given as the concatenation of two copies of $\tilde{X}$, clearly with the corresponding indexes:

$$X' = \cdots 0 \to \mathbb{Z} \longrightarrow \mathbb{Z} \to 0 \to 0 \to \mathbb{Z} \longrightarrow \mathbb{Z} \to 0 \to \cdots$$

there are quasi-isomorphisms $X \leftarrow X' \to Y$, mapping each copy of $\tilde{X}$ by f to the copies of $\tilde{Y}$, and by the identity on the copies of $\tilde{X}$.

Remark 3.1.16. For a general localization $Q : \mathbf{C} \to \mathbf{C}[S^{-1}]$, if a morphism $\eta : X \to Y$ is an isomorphism in $\mathbf{C}[S^{-1}]$, then any representing roof

$$X \xleftarrow{s} W \xrightarrow{f} Y$$

is such that Qf is isomorphism as well. Consider in $\mathbf{C}[S^{-1}]$ the diagram

$$\begin{tikzcd}
 & QW \arrow[dl, "Qs"'] \arrow[dr, "Qf"] & \\
QX \arrow[rr, "\eta"] & & QY
\end{tikzcd}$$

It is commutative since the composition ηQs is represented by the composition of roofs

$$\begin{tikzcd}
 & & QW \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & & \\
 & QW \arrow[dl, "\mathrm{id}"'] \arrow[dr, "s"] & & QW \arrow[dl, "s"'] \arrow[dr, "Qf"] & \\
QW & & QX & & QY
\end{tikzcd}$$

which gives exactly a roof representing Qf. Thus, Qf is invertible as composition of invertible morphisms.

In the particular case of the derived category, slightly more can be said, providing the following useful result. From now on, we use the notation $\sim$ above the arrows in order to indicate that it's a quasi-isomorphism.

Proposition 3.1.17. *If X and Y are isomorphic complexes in $D(\boldsymbol{A})$, then any representing morphisms of complexes $X \overset{\sim}{\leftarrow} A \to Y$ is such that also $A \to Y$ is a quasi-isomorphisms.*

Proof. Consider the compositions of the isomorphism $X \to Y$ with its inverse $Y \to X$. These are given as compositions of roofs

$$\begin{tikzcd}
 & & B_1 \arrow[dl, "\sim"] \arrow[dr, "g"] \arrow[ddll, bend right, "i"'] & & \\
 & A_1 \arrow[dl, "\sim"] \arrow[dr] & & A_2 \arrow[dl, "\sim"] \arrow[dr, "h"] & \\
X & & Y & & X
\end{tikzcd}$$

$$\begin{tikzcd}
 & & B_2 \arrow[dl, "\sim"] \arrow[dr, "g'"] \arrow[ddll, bend right, "j"'] & & \\
 & A_2 \arrow[dl, "\sim"] \arrow[dr, "h"] & & A_1 \arrow[dl, "\sim"] \arrow[dr, "h'"] & \\
Y & & X & & Y
\end{tikzcd}$$

and the fact that these are the identities respectively on X and Y in the derived category, means that there are commutative diagrams in the homotopy category of the form

$$\begin{tikzcd} & B_1 \arrow[dl, "\sim"'] \arrow[dr, "g"] & \\ A_1 \arrow[dr, "\sim"'] & C_1 \arrow[u] \arrow[d] & A_2 \arrow[dl, "h"] \\ & X & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & B_2 \arrow[dl, "\sim"'] \arrow[dr, "g'"] & \\ A_2 \arrow[dr, "\sim"'] & C_2 \arrow[u] \arrow[d] & A_1 \arrow[dl, "h'"] \\ & Y & \end{tikzcd}$$

Thus, in particular, hg and $h'g'$ are homotopically equivalent to the quasi-isomorphisms i and j respectively, and hence, by Proposition 2.2.16 they are quasi-isomorphisms as well. Therefore, for every integer k one has $H^k(g)H^k(h) = H^k(gh) = H^k(i)$, hence $H^k(h)$ has a right inverse given by $H^k(i)^{-1}H^k(g)$. On the other hand $H^k(g')$ has, in the same way, a left inverse which is $H^k(j)^{-1}H^k(h')$. Eventually, observe that from the commutativity of composition diagram given by $Y \to X \to Y$, the morphisms $H^k(g')$ and $H^k(h)$ are the same up to isomorphisms, thus the latter also has a right inverse. In conclusion h is a quasi-isomorphism, and hence $A_2 = A$ gives

$$Y \xleftarrow{\sim} A \xrightarrow{h} X$$

working as desired.

The fact that any representative is of this form easily follows because any other roof $Y \leftarrow A' \to X$ is such that there is a commutative diagram

$$\begin{tikzcd} & & C \arrow[dl, "\sim"'] \arrow[dr] & & \\ & A \arrow[dl, "\sim"'] \arrow[drrr, "\sim"] & & A' \arrow[dlll, "\sim"'] \arrow[dr] & \\ X & & & & Y \end{tikzcd}$$

But then, just being H^k a functor $\mathrm{Kom}(\mathbf{A}) \to \mathbf{Ab}$, shows that both $C \to A'$, and then also $A' \to Y$ are quasi-isomorphisms. $\square$

3.2 Verdier quotient

Another kind of structure that can be defined using localization of morphisms is the so called *Verdier quotient*, and involves triangulated categories. The structure of a triangulated category $\mathbf{K}$ with a full triangulated subcategory $\mathbf{J}$ allows in fact the construction of a new triangulated category $\mathbf{K}/\mathbf{J}$ enjoying the usual algebraic universal property of quotients. More precisely, its localization functor $Q : \mathbf{K} \to \mathbf{K}/\mathbf{J}$ is such that whenever a triangulated functor $G : \mathbf{K} \to \mathbf{H}$ maps every object in $\mathbf{J}$ to an object isomorphic to 0, then there exists a unique functor $\hat{G} : \mathbf{K}/\mathbf{J} \to \mathbf{H}$ such that the diagram

$$\begin{tikzcd} \mathbf{K} \arrow[r, "Q"] \arrow[dr, "G"'] & \mathbf{K}/\mathbf{J} \arrow[d, dashed, "\hat{G}"] \\ & \mathbf{H} \end{tikzcd}$$

commutes in **TrCat**.

In order to see that, let's first of all define it, and then describe its triangulated structure.

Definition 3.2.1. Let $\mathbf{K}$ be a triangulated category with shift T, and $\mathbf{J}$ a triangulated subcategory. Consider the class $S = S(\mathbf{J})$ of morphisms in $\mathbf{K}$ given by those $s : X \to Y$ such that the completion of s to a distinguished triangle $X \to Y \to Z \to TX$ is such that Z is in $\mathbf{J}$.

Then, set $\mathbf{K}/\mathbf{J} = \mathbf{K}[S(\mathbf{J})^{-1}]$

Remark 3.2.2. The little of ambiguity in the previous definition of the class S is justified by the fact that the completion to triangle of a morphism is unique up to isomorphism, thanks to Lemma 2.4.12, and the fact that any triangulated subcategory is assumed to be strictly full.

In order to being able to deal with morphisms in the quotient category, it's useful to prove that S is a multiplicative system, so that by Theorem 3.1.13, we will represent them through roofs.

Theorem 3.2.3. *Let $\boldsymbol{K}$ be a triangulated category and $\boldsymbol{J} \subseteq \boldsymbol{K}$ a triangulated subcategory. The class $S(\boldsymbol{J})$ is a multiplicative system for $\boldsymbol{K}$.*

Proof. Let's prove to hold conditions (a), (b) and (c) of Definition 3.1.9. Certainly 0 is an object of $\mathbf{J}$, therefore, for any object X in $\mathbf{K}$ the distinguished triangle $X \xrightarrow{\mathrm{id}} X \to 0 \to TX$ show that id_X is in S. Moreover, if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a composition of morphisms with f and g in S, one has distinguished triangles

$$X \xrightarrow{f} Y \longrightarrow D_1 \longrightarrow TX$$
$$Y \xrightarrow{g} Z \longrightarrow D_2 \longrightarrow TY$$
$$X \xrightarrow{gf} Z \longrightarrow D_3 \longrightarrow TX$$

with D_1 and D_2 in $\mathbf{J}$. We want to prove D_3 to be in $\mathbf{J}$, and that is by axiom **T4**, providing another distinguished triangle

$$D_1 \longrightarrow D_3 \longrightarrow D_2 \longrightarrow TD_1$$

in which two out of three vertices are in $\mathbf{J}$. Since $\mathbf{J}$ is a triangulated subcategory, D_3 also is in $\mathbf{J}$, by Proposition 2.4.16. This proves (a).

Claim: Let S be a class of morphisms satisfying (a) and such that

(i) if s is in S, so is each $T^n s$ for every integer n;

(ii) for every pair of distinguished triangles $X \to Y \to Z \to TX$, $X' \to Y' \to Z' \to TX'$ and morphisms $s' : X \to X'$, $s'' : Y \to Y'$ in S making

$$\begin{tikzcd} X \arrow[r] \arrow[d, "s'"] & Y \arrow[d, "s''"] \\ X' \arrow[r] & Y' \end{tikzcd}$$

to commute, there is a morphism $s : Z \to Z'$ in S turning

$$\begin{tikzcd} X \arrow[r] \arrow[d, "s"] & Y \arrow[r] \arrow[d, "s'"] & Z \arrow[r] \arrow[d, "s"] & TX \arrow[d, "Ts"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & TX' \end{tikzcd}$$

into a morphism of triangles.

Then, condition (b) holds.

In fact, consider a pair of morphisms (s, f) with s in S

$$\begin{tikzcd} & X' \arrow[d, "s"] \\ Z \arrow[r, "f"] & X \end{tikzcd}$$

and extend f to a triangle in order to get $Z \xrightarrow{f} X \xrightarrow{u} Y \to TZ$, and shifting,

$$X \xrightarrow{u} Y \xrightarrow{v} TZ \xrightarrow{Tf} TX$$

Extend then the morphism $us : X' \to Y$, and get the triangle

$$X' \xrightarrow{us} T \xrightarrow{v'} Z' \xrightarrow{w'} TX'$$

Certainly there is a commutative square

$$\begin{tikzcd} X' \arrow[r, "us"] \arrow[d, "s"'] & Y \arrow[d, "\mathrm{id}_Y"] \\ X \arrow[r, "u"] & Y \end{tikzcd}$$

with both vertical arrows in S. Thanks to condition (ii), there exists in S a morphism $t : Z' \to TZ$ such that

$$\begin{tikzcd} X' \arrow[r, "us"] \arrow[d, "s"'] & Y \arrow[r] \arrow[d, "\mathrm{id}_Y"] & Z' \arrow[r] \arrow[d, "t"] & TX' \arrow[d, "Ts"] \\ X \arrow[r, "u"] & Y \arrow[r] & TZ \arrow[r, "Tf"] & TX \end{tikzcd}$$

is commutative. In particular, a shift of the third square provides a morphism $T^{-1}t : T^{-1}Z' \to Z$, which still remain in S thanks to (i), and it completes the pair of morphisms (s, f), as required by (b) to a commutative square

$$\begin{tikzcd}
T^{-1}Z' \arrow[r] \arrow[d, "T^{-1}t"'] & X' \arrow[d, "s"] \\
Z \arrow[r, "f"] & X
\end{tikzcd}$$

So, let's prove (i) and (ii) to hold for S. Invariance under the shift functor follows from the same invariance of $\mathbf{J}$. If $s : X \to Y$ is such to fit a triangle

$$X \xrightarrow{s} Y \longrightarrow Z \longrightarrow TX$$

with Z in $\mathbf{J}$, then certainly $T^n s$ fits the shifted triangle

$$T^n X \xrightarrow{T^n s} T^n Y \longrightarrow T^n Z \longrightarrow T^{n+1} X$$

and since $T^n Z$ still happens to be in $\mathbf{J}$ by Proposition 2.4.16, being $\mathbf{J}$ a full triangulated subcategory. Then $T^n s$ is in S. Analogous argument proves also the dual part of (b).

Eventually, let's prove (c). Observe first of all that the pre-additive structure allows us to prove that for any pair of morphisms $f : X \to Y$, are equivalent the existence of a morphism s in S such that $sf = 0$ and the existence of a morphism t in S such that $ft = 0$.

Suppose then to have $s : Y \to Z$ such that $sf = 0$, and complete s to a distinguished triangle

$$Y \xrightarrow{s} Z \longrightarrow W \longrightarrow TY$$

where W is in $\mathbf{J}$ by our assumption. Then, since the composite $sf : X \to Y \to Z$ is zero, the diagram

$$\begin{tikzcd}
Y \arrow[r, "s"] & Z \arrow[r] & W \arrow[r] & TY \\
X \arrow[r] \arrow[u, "f"] & 0 \arrow[r] \arrow[u] & TX \arrow[r, "\mathrm{id}_{TX}"] & TX \arrow[u, "Tf"']
\end{tikzcd}$$

is commutative, and being both the rows distinguished triangles, can be completed to a morphism of distinguished triangles by an arrow $g : TX \to W$. Consider then a completion of this morphism to a distinguished triangle

$$TX \xrightarrow{g} W \longrightarrow V \longrightarrow T^2 X$$

and its shift

$$T^{-1}V \longrightarrow TX \xrightarrow{g} W \longrightarrow V$$

Call t the morphism $T^{-1}V \to TX$ and observe that it's in S because W is in **J** and, by Remark 2.4.8, that $gt = 0$. Hence $hgt = 0$, and this is $(Tf)t = 0$. This proves one implication up to replace t by $T^{-1}t$, giving $0 = T^{-1}0 = T^{-1}((Tf)t) = fT^{-1}t$. Certainly, $T^{-1}t$ still is in S by what proved above. The reverse part is given by a completely analogous proof. □

Now, we want to endow **K**/**J** with a triangulated structure turning the localization functor into an exact functor. The easy way to do that happens to work.

Definition 3.2.4. Let **K** be a triangulated category and $\mathbf{J} \subseteq \mathbf{K}$ a full triangulated subcategory. The shift functor $T_{\mathbf{K}/\mathbf{J}}$ is defined to be the same as T on objects. On a morphism $f : X \to Y$, which can be represented by a roof $X \leftarrow W \to Y$, Tf is defined to be the class of

$$TX \longleftarrow TW \longrightarrow TY$$

A triangle in **K**/**J** is said to be distinguished whether it is isomorphic to the image under the localization functor of a distinguished triangle in **K**.

In order to prove axioms of triangulated categories, especially axiom **T4**, to hold true for a Verdier quotient, we need the following lemma.

Lemma 3.2.5. *Let* $\boldsymbol{K}$ *be a triangulated category, and*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow TX$$

$$X' \longrightarrow Y' \longrightarrow Z' \longrightarrow TX'$$

two distinguished triangles in $\boldsymbol{K}$*. Suppose to have a pair of isomorphism in* $\boldsymbol{K}/\boldsymbol{J}$ *giving a commutative square*

$$\begin{array}{ccc} QX & \longrightarrow & QY \\ \downarrow & & \downarrow \\ QX' & \longrightarrow & QY' \end{array}$$

Then, there exists a third isomorphism $QZ \to QZ'$ *providing an isomorphism of triangles in* $\boldsymbol{K}/\boldsymbol{J}$*.*

Proof. See [Nee01] 2.1.38. □

Theorem 3.2.6. *The category* $\boldsymbol{K}/\boldsymbol{J}$ *with its shift functor and its class of distinguished triangles defines a triangulated category.*

Proof. Let's check the axioms. Recalling that axiom **T3** is in fact a theorem, it suffices to check **T1**, **T2** and **T4**.

Let A be an object in $\mathbf{K}/\mathbf{J}$. A is also an object in $\mathbf{K}$, hence consider the distinguished triangle in $\mathbf{K}$

$$A \xrightarrow{\mathrm{id}_A} A \longrightarrow 0 \longrightarrow TA$$

Since for any functor $Q(\mathrm{id}_A) = \mathrm{id}_{QA}$, and quite trivially

$$QT_{\mathbf{K}}A = T_{\mathbf{K}}A = T_{\mathbf{K}/\mathbf{J}}A = T_{\mathbf{K}/\mathbf{J}}QA,$$

the image under the localization functor gives the desired distinguished triangle in $\mathbf{K}/\mathbf{J}$, proving **T1**-(i). Moreover, if two triangles in $\mathbf{K}/\mathbf{J}$ are isomorphic, then composing the isomorphism leads to conclude that if the first one is isomorphic to a triangle of the form $QX \to QY \to QZ \to QTX \cong TQX$, so is the second one. This proves (ii) in axiom **T1**.

If $f : X \to Y$ is a morphism in $\mathbf{K}/\mathbf{J}$, represented in $\mathbf{K}$ by

$$us^{-1} : X \longleftarrow W \longrightarrow Y$$

complete in $\mathbf{K}$ the morphism $u : W \to Y$ to a distinguished triangle $W \to Y \to Z \to TW$, and apply Q to it in order to get a distinguished triangle in $\mathbf{K}/\mathbf{J}$

$$QW \longrightarrow QY \longrightarrow QZ \longrightarrow TQW$$

There is then an isomorphism

$$\begin{array}{ccccccc} QP & \xrightarrow{Qu} & QY & \xrightarrow{Qv} & QZ & \xrightarrow{Qw} & TQP \\ \downarrow{\scriptstyle Qs} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle TQs} \\ QX & \xrightarrow[f]{} & QY & \xrightarrow[Qv]{} & QZ & \xrightarrow[QTsQw]{} & TQX \end{array}$$

since Qs is isomorphism in $\mathbf{K}/\mathbf{J}$, as well as certainly are the identities. This gives **T1**-(iii)

Axiom **T2** is trivially true. The shift $Y \to Z \to TX \overset{-Tu}{\to} TY$ of a triangle $X \to Y \to Z \to TX$, isomorphic to the image of a triangle under the localization functor $QA \to QB \to QC \to TQA$ of a triangle in $\mathbf{K}$, is isomorphic to the image of the shift of the triangle in $\mathbf{K}$.

In order to check axiom **T4**, let's prove the following

Claim: Any commutative square in $\mathbf{K}/\mathbf{J}$ can be lifted by S to a commutative square in $\mathbf{K}$, that is, for any commutative square in the quotient

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

there exists a commutative square in $\mathbf{K}$

$$\begin{tikzcd}
W' \arrow[r] \arrow[d] & X' \arrow[d] \\
Y' \arrow[r] & Z'
\end{tikzcd}$$

together with morphisms $W' \to W$, $X' \to X$, $Y' \to Y$ and $Z' \to Z$ in $S(\mathbf{J})$ making the following to commute in $\mathbf{K}/\mathbf{J}$

$$\begin{tikzcd}
W \arrow[rrr] \arrow[ddd] & & & X \arrow[ddd] \\
& W' \arrow[ul] \arrow[r] \arrow[d] & X' \arrow[ur] \arrow[d] & \\
& Y' \arrow[dl] \arrow[r] & Z' \arrow[dr] & \\
Y \arrow[rrr] & & & Z
\end{tikzcd} \tag{3.1}$$

Start considering the morphism $W \to X \to Z$, represented in $\mathbf{K}$ by the composition of two roofs

$$\begin{tikzcd}
& & W_1 \arrow[dl] \arrow[dr] & & \\
& W_1' \arrow[dl] \arrow[dr] & & X' \arrow[dl] \arrow[dr] & \\
W & & X & & Z
\end{tikzcd}$$

where $X' \to X$ is in S. Let's write this roof avoiding explicit compositions as

$$\begin{tikzcd}
& W_1 \arrow[dl] \arrow[dr] & \\
W & & Z
\end{tikzcd}$$

with $W_1 \to W$ in S. The same morphism $W \to Y \to Z$ is represented also as

$$\begin{tikzcd}
& & W_2 \arrow[dl] \arrow[dr] & & \\
& W_2' \arrow[dl] \arrow[dr] & & Y' \arrow[dl] \arrow[dr] & \\
W & & Y & & Z
\end{tikzcd}$$

where $Y' \to Y$ is in S, and again we can write it as

$$\begin{tikzcd}
 & W_2 \arrow[dl] \arrow[dr] & \\
W & & Z
\end{tikzcd}$$

with $W_2 \to W$ in S.

Then, these two roofs represent the same morphism, hence they are in the same equivalence class, so that there exists an object W' and morphisms $W_1 \leftarrow W' \to W_2$ such that the following diagram is commutative

$$\begin{tikzcd}
 & & W' \arrow[dl] \arrow[dr] & & \\
 & W_1 \arrow[dl] \arrow[drrr] & & W_2 \arrow[dlll] \arrow[dr] & \\
W & & & & Z
\end{tikzcd}$$

where both $W' \to W_1$ and $W_1 \to W$ are in S, and so is their composition $W' \to W$.

Therefore, we have two commutative squares

$$\begin{tikzcd}
W' \arrow[r] \arrow[d] & W_1 \arrow[d] \\
W_2 \arrow[r] & Z
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
W' \arrow[r] \arrow[d] & W_1 \arrow[d] \\
W_2 \arrow[r] & W
\end{tikzcd}$$

Since the morphisms $W_i \to Z$, for $i = 1, 2$, are compositions $W_1 \to X' \to Z$ and $W_2 \to Y' \to Z$, the left square can be written as

$$\begin{tikzcd}
W' \arrow[r] \arrow[d] & W_1 \arrow[r] & X' \arrow[dd] \\
W_2 \arrow[d] & & \\
Y' \arrow[rr] & & Z
\end{tikzcd}$$

Eventually, one can even consider $Z' \to Z$ to be the identity. It remains to prove commutativity in (3.1), which is fairly obvious by construction. Let's prove for example to hold the commutativity on the left side of the diagram, that is

$$\begin{tikzcd}
W \arrow[d] & W' \arrow[l] \arrow[d] \\
Y & Y' \arrow[l]
\end{tikzcd}$$

The morphism $W' \to W \to Y$ is represented by

$$\begin{tikzcd}[column sep=small]
 & & W' \arrow[dl] \arrow[dr] & & \\
 & W' \arrow[dl] \arrow[dr] & & W_2' \arrow[dl] \arrow[dr] & \\
W' & & W & & Y
\end{tikzcd}$$

where, by definition, $W' \to W_2'$ factors as $W' \to W_2 \to W_2'$. The morphism $W' \to Y' \to Y$ is represented by

$$\begin{tikzcd}[column sep=small]
 & & W' \arrow[dl] \arrow[dr] & & \\
 & W' \arrow[dl] \arrow[dr] & & Y' \arrow[dl] \arrow[dr] & \\
W' & & Y' & & Y
\end{tikzcd}$$

where, by definition, $W' \to Y$ factors as $W' \to W_2 \to Y'$.

Thus, in order to see that these two roofs are in the same class, just take identities $W' \leftarrow W' \to W'$ as a common roof, and everything to check is the commutativity in $\mathbf{K}$ of

$$\begin{tikzcd}
W' \arrow[r] \arrow[d] & Y' \arrow[d] \\
W_2' \arrow[r] & Y
\end{tikzcd}$$

which holds thanks to the factorization just recalled and the commutativity, given by the definition of W_2, of

$$\begin{tikzcd}
W_2 \arrow[r] \arrow[d] & Y' \arrow[d] \\
W_2' \arrow[r] & Y
\end{tikzcd}$$

This proves the claim. Moreover, observe that if one of the morphisms between $X \to Z$ and $Y \to Z$ in the original square where an identity, let's say $Z = X$, then X' is defined to be X itself, with $X' \to X$ the identity.

Now, Consider three triangles in $\mathbf{K}/\mathbf{J}$

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow TX$$
$$Y \xrightarrow{g} Y' \longrightarrow Z'' \longrightarrow TY$$
$$X \xrightarrow{gf} Y' \longrightarrow Z' \longrightarrow TX$$

and consider the commutative square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & Y \arrow[d, "g"] \\
X \arrow[r, "gf"] & Y'
\end{tikzcd}$$

which by the claim can be lifted to a commutative square in $\mathbf{K}$, giving a commutative diagram

$$\begin{tikzcd}
X \arrow[rrr, "f"] \arrow[ddd, "\mathrm{id}"'] & & & Y \arrow[ddd, "g"] \\
& \overline{X} \arrow[r, "\overline{f}"] \arrow[d, "\mathrm{id}"'] \arrow[ul] & \overline{Y} \arrow[d, "\overline{g}"] \arrow[ur] & \\
& \overline{X} \arrow[r, "\overline{g}\overline{f}"] \arrow[dl] & \overline{Y'} \arrow[dr] & \\
X \arrow[rrr, "gf"] & & & Y'
\end{tikzcd}$$

with morphisms linking the two squares in S. Observe then, that sides

$$\begin{tikzcd}
\overline{X} \arrow[r, "\overline{f}"] \arrow[d, "s"'] & \overline{Y} \arrow[d, "t"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{Y} \arrow[r, "\overline{g}"] \arrow[d, "t"'] & \overline{Y'} \arrow[d, "t'"] \\
Y \arrow[r, "g"] & Y'
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{X} \arrow[r, "\overline{g}\overline{f}"] \arrow[d, "s"'] & \overline{Y'} \arrow[d, "t'"] \\
X \arrow[r, "gf"] & Y'
\end{tikzcd}$$

are such that vertical morphisms are isomorphism in $\mathbf{K}/\mathbf{J}$.

Complete $\overline{f}$, $\overline{g}$, $\overline{g}\overline{f}$ to distinguished triangles in $\mathbf{K}$

$$\overline{X} \xrightarrow{\overline{f}} \overline{Y} \longrightarrow \overline{Z} \longrightarrow T\overline{X}$$

$$\overline{Y} \xrightarrow{\overline{g}} \overline{Y'} \longrightarrow \overline{Z''} \longrightarrow T\overline{Y}$$

$$\overline{X} \xrightarrow{\overline{g}\overline{f}} \overline{Y'} \longrightarrow \overline{Z'} \longrightarrow T\overline{X}$$

Now, the given distinguished triangles in $\mathbf{K}/\mathbf{J}$ are by definition the image under Q of distinguished triangles in $\mathbf{K}$. Therefore, up to consider the distinguished triangles in $\mathbf{K}$ from which they come from, one has morphisms between distinguished triangles in $\mathbf{K}$

$$\begin{tikzcd}
\overline{X} \arrow[r, "\overline{f}"] \arrow[d, "s"'] & \overline{Y} \arrow[r] \arrow[d, "t"] & \overline{Z} \arrow[r] \arrow[d, dashed, "\phi"] & T\overline{X} \arrow[d, "Ts"] \\
X \arrow[r, "f"] & Y \arrow[r, "h"] & Z \arrow[r] & TX
\end{tikzcd}$$

$$\begin{tikzcd}
\overline{Y} \arrow[r, "\overline{g}"] \arrow[d, "t"'] & \overline{Y'} \arrow[r] \arrow[d, "t'"] & \overline{Z''} \arrow[r] \arrow[d, dashed, "\psi"] & T\overline{Y} \arrow[d, "Tt"] \\
Y \arrow[r, "g"] & Y' \arrow[r] & Z'' \arrow[r] & TY
\end{tikzcd}$$

$$\begin{tikzcd}
\overline{X} \arrow[r, "\overline{g}\overline{f}"] \arrow[d, "s"'] & \overline{Y'} \arrow[r] \arrow[d, "t'"] & \overline{Z'} \arrow[r] \arrow[d, dashed, "\lambda"] & T\overline{X} \arrow[d, "Ts"] \\
X \arrow[r, "gf"] & Y' \arrow[r, "k"] & Z' \arrow[r] & TX
\end{tikzcd}$$

where Lemma 3.2.5 provides the existence of the dashed isomorphisms of $\mathbf{K}/\mathbf{J}$. Let's use the fact that axiom **T4** holds for $\mathbf{K}$, hence deduce that there exists a triangle

$$\overline{Z} \xrightarrow{u} \overline{Z'} \xrightarrow{v} \overline{Z''} \xrightarrow{w} T\overline{Z}$$

filling the commutative diagram

$$\begin{tikzcd}
\overline{X} \arrow[rr, "\overline{g}\overline{f}"] \arrow[dr, "\overline{f}"'] & & \overline{Y'} \arrow[rr] \arrow[dr, "\overline{k}"] & & \overline{Z''} \arrow[rr, "w"] \arrow[dr] & & T\overline{Z} \\
& \overline{Y} \arrow[ur, "\overline{g}"'] \arrow[dr, "\overline{h}"'] & & \overline{Z'} \arrow[ur, "v"'] \arrow[dr] & & T\overline{Y} \arrow[ur] & \\
& & \overline{Z} \arrow[ur, "u"] \arrow[rr] & & T\overline{X} \arrow[ur] & &
\end{tikzcd}$$

Now we use this new triangle and previous isomorphisms to define the desired triangle $Z \to Z' \to Z'' \to TZ$.

Set

$$\hat{u} = \psi u \phi^{-1} : Z \to Z'$$

$$\hat{v} = \lambda v \psi^{-1} : Z' \to Z''$$

$$\hat{w} = (T\phi) w \lambda^{-1} : Z'' \to TZ$$

Let's check whether these morphism work, proving commutativity of (part of) the following diagram

$$\begin{tikzcd}
X \arrow[rr, "gf"] \arrow[dr, "f"'] & & Y' \arrow[rr] \arrow[dr, "k"] & & Z'' \arrow[rr, "\hat{w}"] \arrow[dr] & & TZ \\
& Y \arrow[ur, "g"'] \arrow[dr, "h"'] & & Z' \arrow[ur, "\hat{v}"'] \arrow[dr] & & TY \arrow[ur] & \\
& & Z \arrow[ur, "\hat{u}"] \arrow[rr] & & TX \arrow[ur] & &
\end{tikzcd}$$

namely of the square

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "h"'] & Y' \arrow[d, "k"] \\
Z \arrow[r, "\hat{u}"] & Z'
\end{tikzcd}$$

The rest is analogous diagram chasing. The morphism $\hat{u}h$ is by definition

$$\psi u\phi^{-1}h = \psi u\overline{h}t^{-1} = \psi\overline{k}\overline{g}t^{-1} = kt'\overline{g}t^{-1} = kg.$$

$\square$

Remark 3.2.7. It's now clear from how shifts and distinguished triangles are defined, that the localization functor $\mathbf{K} \to \mathbf{K}/\mathbf{J}$ is triangulated.

Therefore, one can prove the Verdier's Theorem stated as follows.

Theorem 3.2.8. *Given a triangulated category $\boldsymbol{K}$ with a triangulated subcategory $\boldsymbol{J} \subseteq \boldsymbol{K}$, the triangulated localization functor*

$$Q : \boldsymbol{K} \to \boldsymbol{K}/\boldsymbol{J}$$

is universal for the property of quotients. Namely, whenever a triangulated functor $G : \boldsymbol{K} \to \boldsymbol{H}$ is such that $G(J) \cong 0$ for any J in $\boldsymbol{J}$, then there exists a unique functor $\hat{G} : \boldsymbol{K}/\boldsymbol{J} \to \boldsymbol{H}$ such that the following commutes in ***TrCat***

$$\begin{tikzcd} \boldsymbol{K} \arrow[r, "Q"] \arrow[dr, "G"'] & \boldsymbol{K}/\boldsymbol{J} \arrow[d, dashed, "\hat{G}"] \\ & \boldsymbol{H} \end{tikzcd}$$

Proof. We are going to use the universal property of localizations in Theorem 3.1.3, so that it suffices to prove $G(s)$ to be isomorphism whenever $s : X \to Y$ is in $S(\mathbf{J})$. Just consider the triangle, with J in $\mathbf{J}$, witnessing s to be in S

$$X \xrightarrow{s} Y \longrightarrow J \longrightarrow TX$$

and apply to it the triangulated functor G, obtaining, up to isomorphism, a triangle

$$GX \xrightarrow{Gs} GY \longrightarrow 0 \longrightarrow TGX$$

which gives $Gs : GX \cong GY$. $\square$

3.3 The derived category is triangulated

The procedure illustrated in order to define a triangulated structure of the Verdier quotient (Theorem 3.2.6) enjoy a much greater generalization which applies to any localization by a multiplicative system.

Definition 3.3.1. Let $\mathbf{T}$ be a triangulated category and S a multiplicative system of morphisms in $\mathbf{T}$. Then, define a triangle in $\mathbf{T}[S^{-1}]$ to be distinguished if it's, up to isomorphism, the image of a distinguished triangle in $\mathbf{T}$ under the localization functor $Q : \mathbf{T} \to \mathbf{T}[S^{-1}]$.

The shift functor easily extend to $\mathbf{T}[S^{-1}]$ in the obvious way on objects, and for a morphism $X \to Y$ represented by $X \leftarrow W \to Y$, is defined to be the class of $TX \leftarrow TW \to TY$.

Theorem 3.3.2. *Let $\boldsymbol{T}$ be a triangulated category and S a multiplicative system of morphisms. The shift functor of $\boldsymbol{T}[S^{-1}]$ together with the class of distinguished triangles defines the structure of a triangulated category of the localization $\boldsymbol{T}[S^{-1}]$.*

Proof. See [GM02] IV.2 2-6. $\square$

In particular $\mathrm{D}(\mathbf{A})$ for any abelian category $\mathbf{A}$ has a natural structure of triangulated category, since $\mathrm{D}(\mathbf{A})$ is the localization of $\mathrm{K}(\mathbf{A})$ at the class of quasi-isomorphisms, which is a multiplicative system by Proposition 3.1.12.

Remark 3.3.3. It follows that distinguished triangles in $\mathrm{D}(\mathbf{A})$ are defined just as those triangles arising, up to isomorphism, as image under the localization functor of triangles in $\mathrm{K}(\mathbf{A})$. Therefore, they are exactly those triangles $X \overset{u}{\to} Y \overset{v}{\to} Z \overset{w}{\to} TX$ such that there are isomorphisms in $\mathrm{D}(\mathbf{A})$ providing an isomorphism of triangles

$$\begin{tikzcd} X^\cdot \arrow[r, "u"] \arrow[d] & Y^\cdot \arrow[r, "v"] \arrow[d] & Z^\cdot \arrow[r, "w"] \arrow[d] & TX^\cdot \arrow[d] \\ A^\cdot \arrow[r, "f"] & B^\cdot \arrow[r] & C(f) \arrow[r] & TA^\cdot \end{tikzcd}$$

Proposition 3.3.4. *If $0 \to A^\cdot \overset{f}{\to} B^\cdot \overset{g}{\to} C^\cdot \to 0$ is an exact sequence in $\mathrm{Kom}(\boldsymbol{A})$ for an abelian category $\boldsymbol{A}$, then there is a morphism $\phi : C^\cdot \to A^\cdot[1]$ in $\mathrm{D}(\boldsymbol{A})$ such that*

$$A^\cdot \xrightarrow{f} B^\cdot \xrightarrow{g} C^\cdot \xrightarrow{\phi} A^\cdot[1]$$

is a distinguished triangle in $\mathrm{D}(\boldsymbol{A})$.

Proof. Let's call f the given morphism $A^\cdot \to B^\cdot$ and let $f' : A^\cdot \to \mathrm{Im}(f)$ be the induced isomorphism. Therefore, we get that f' is in particular a quasi-isomorphism, and hence its mapping cone $C(f')$ is acyclic. Consider then the short exact sequence

$$0 \longrightarrow C(f') \longrightarrow C(f) \xrightarrow{\psi} C^\cdot \longrightarrow 0$$

where the morphism $C(f') \to C(f)$ is the obvious inclusion, while ψ is the composition of g with the projection $p : C(f) \to B^\cdot$. Observe that ψ composed to the inclusion $i : B^\cdot \to C(f)$ gives g. This short exact sequence induces a long exact cohomology sequence

$$\cdots \to H^k(C(f')) \to H^k(C(f)) \to H^k(C^\cdot) \to H^{k+1}(C(f')) \to \cdots$$

where, as observed, $H^k(C(f')) = 0$ for every k. It follows that ψ is a quasi-isomorphism, hence, we can consider its inverse ψ^{-1} as an arrow $C^\cdot \to C(f)$ in $\mathrm{D}(\mathbf{A})$, and the composition with the projection $q : C(f) \to A^\cdot[1]$ in order to get a morphism

$$\phi : C^\cdot \xrightarrow{\psi} C(f) \xrightarrow{q} A^\cdot[1]$$

This morphism is now such that there are isomorphisms

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & B \arrow[r, "i"] \arrow[d, "\mathrm{id}"] & C(f) \arrow[r, "q"] \arrow[d, "\psi"] & A[1] \arrow[d, "\mathrm{id}"] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r, "q\psi^{-1}"] & A[1]
\end{tikzcd}$$

proving that the bottom row is a distinguished triangle ad desired. □

3.4 Derived functors

Let's start observing that an additive functor $F : \mathbf{A} \to \mathbf{B}$ between abelian categories which is not exact doesn't preserves acyclic objects, by Proposition 2.2.7. That means, in particular, that applying it pointwise won't provide an extension to the derived category, at least as an additive functor, because an acyclic complex, which is the same as 0 in $\mathrm{D}(\mathbf{A})$, won't be mapped to 0.

The following starts from the observation that exact functors are quite rare, but a lot of interesting functors still carries some sort of exactness.

Definition 3.4.1. Let $0 \to A \to B \to C \to 0$ be a short exact sequence in some abelian category $\mathbf{A}$, and let $F : \mathbf{A} \to \mathbf{B}$ be an additive functor between abelian categories. Consider the sequence

$$0 \to FA \to FB \to FC \to 0$$

The functor F is said to be *left exact* if the sequence is exact except possibly at FC. Also, F is said to be *right exact* if the sequence is exact except possibly at FA.

Proposition 3.4.2. *An additive functor between abelian categories $F : \boldsymbol{A} \to \boldsymbol{B}$ is left exact if and only if for every exact sequence $0 \to A \to B \to C$, the sequence $0 \to FA \to FB \to FC$ is exact. Dually, F is right exact if and only if for every exact sequence $A \to B \to C \to 0$ the sequence $FA \to FB \to FC \to 0$ is exact.*

Proof. Suppose F to be left exact and observe first of all that F preserves monomorphisms. That is because if $f : A \to B$ is mono, one can consider the short exact sequence $0 \to A \to B \to \mathrm{Coker}(f) \to 0$, apply F and find that $0 \to FA \to FB$ if exact, that is $F(f)$ is monomorphism. So, let $0 \to A \xrightarrow{f} B \xrightarrow{g} C$, and consider the short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g'} \mathrm{Im}(g) \to 0$, where g' is clearly such that $ig' = g$ for $i : \mathrm{Im} f \to C$. Applying F gives the exact sequence

$$0 \longrightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg'} F(\mathrm{Im} g)$$

Now, since i is mono, so is Fi, hence $\mathrm{Im}(Ff) = \mathrm{Ker}(Fg') = \mathrm{Ker}(FiFg') = \mathrm{Ker}(Fg)$.

The reverse implication is obvious. The dual statement is analogous. □

Proposition 3.4.3. *An additive functor $F : \boldsymbol{A} \to \boldsymbol{B}$ between abelian categories is left exact if and only if it preserves finite limits. Dually, F is right exact if and only if it preserves finite colimits.*

Proof. On one hand if F preserves finite limits and $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence, then looking at

$$0 \longrightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC$$

one has $\mathrm{Ker}(Ff) = F(\mathrm{Ker}\, F) = F(0) = 0$, so $F(f)$ is mono as desired. Exactness at FB is now given since

$$\mathrm{Ker}(Fg) = F(\mathrm{Ker}\, g) = F(\mathrm{Im}\, f) = F(A) = \mathrm{Im}(Ff).$$

For the reverse implication one can use the fact that for a functor to preserve finite limits it suffices that it preserves finite products and equalizers. Being in the setting of abelian categories, the equalizer between f and g can be described as $\mathrm{Ker}(f-g)$. Now, the functor F is supposed to be additive, so it preserves finite products. It suffices then to prove that F preserves kernels. Let $f : A \to B$ be any morphism in $\mathbf{A}$, consider $0 \to \mathrm{Ker}(f) \to A \to B$. Being F left exact and thanks to Proposition 3.4.2, one has $0 \to F(\mathrm{Ker} f) \to FA \to FB$ exact, thus $\mathrm{Ker}(Ff) = F(\mathrm{Ker}\, f)$. □

Remark 3.4.4. By Proposition 3.4.3, one can define (left and right) exactness of functors between any finitely complete categories, i.e. categories admitting finite limits and colimits.

Remark 3.4.5. Let $\mathbf{A}$ and $\mathbf{B}$ be any finitely complete categories. If $F : \mathbf{A} \to \mathbf{B}$ is a left adjoint functor, then it preserves colimits, and in particular it is right exact. Dually, if F is a right adjoint functor, then it is left exact.

Remark 3.4.6. Recall that for any fixed R-module M there is an adjunction between functors $- \otimes_R M \dashv \mathrm{Hom}(M, -)$, that is, one has natural isomorphism, for any pair of R-modules N and K,

$$\mathrm{Hom}(N \otimes M, K) \cong \mathrm{Hom}(N, \mathrm{Hom}(M, K))$$

mapping $f : N \otimes M \to K$ to the map λ defined by $\lambda(n)(m) = f(n \otimes m)$.

Thus, $-\otimes M : \mathbf{Mod}(R) \to \mathbf{Mod}(R)$ is right exact, and dually $\mathrm{Hom}(M, -)$ is left exact.

Our purposes requires mainly to define the derived tensor product of chain complexes, which will provide more structure to the derived category, and consequently the internal Hom of a derived category.

Remark 3.4.7. In general, for a left exact functor $F : \mathbf{A} \to \mathbf{B}$, if the category $\mathbf{A}$ contains enough injective objects, there is a functor

$$\mathrm{R}F : \mathrm{D}^+(\mathbf{A}) \longrightarrow \mathrm{D}^+(\mathbf{B})$$

which is computed on an object X in $\mathrm{D}^+(\mathbf{A})$ by taking a bounded below complex of injective objects I quasi-isomorphic to X, applying to it the functor $\mathrm{K}(F)$, and looking at the result $\mathrm{K}F(I)$ in $\mathrm{D}(\mathbf{B})$ through the localization functor $\mathrm{K}^+(\mathbf{B}) \to \mathrm{D}^+(\mathbf{B})$.

Dually, if $\mathbf{A}$ has enough projective objects and $F : \mathbf{A} \to \mathbf{B}$ is a right exact functor, it can be induced a functor

$$\mathrm{L}F : \mathrm{D}^-(\mathbf{A}) \to \mathrm{D}^-(\mathbf{B})$$

computed on a complex X by taking a quasi-isomorphic bounded above complex of projective objects P, applying to it the functor $\mathrm{K}F$, and looking at the result $\mathrm{K}F(P)$ in $\mathrm{D}^-(\mathbf{B})$.

For more details and for the proof that this is a well posed definition see [Huy06] 2.2.

Now that we know the triangulated structure of derived categories and a bit of derived functors, we are able to show the following.

Example 3.4.8. Let's provide a concrete example of complexes of sheaves on a scheme X with same cohomology which aren't isomorphic in $\mathrm{D}(\mathcal{O}_X)$. Consider X to be a compact, simply connected complex surface with trivial bundle of differential 2-forms $\Omega_X \cong \mathcal{O}_X$, which is usually called K3 *surface*, and has sheaf cohomology $\Gamma(X, \Omega_X) = H^0(X, \Omega_X) \cong \mathbb{C}$.

It's also known, and is an extremely useful result to deal with derived categories, that for any abelian category $\mathbf{A}$ with enough injective objects, we can consider the Ext functors defined as

$$\mathrm{Ext}^i(A, -) = H^i(\mathrm{RHom}(A, -))$$

and it holds, for objects A and B in $\mathbf{A}$,

$$\mathrm{Ext}^i(A, B) \cong \mathrm{Hom}_{\mathrm{D}(\mathbf{A})}(A, B[i]).$$

The proof of this fact can be found in [Huy06] and is an easy consequence, at least for $\mathbf{A} = \mathbf{Mod}(\mathcal{O}_X)$, of Lemma 3.6.11 below.

Now, by Serre duality we get, for any coherent sheaf E, the isomorphism $\mathrm{Ext}^i(E, \Omega_X) \cong H^{2-i}(X, E) = H^{2-i}\mathrm{R}\Gamma(X, E)$. In particular, for $E \cong \mathcal{O}_X$ and $i = 2$, we find that

$$\mathrm{Ext}^2(\mathcal{O}_X, \mathcal{O}_X) \cong H^0(X, \Omega_X) \cong \mathbb{C}$$

and therefore $\mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(\mathcal{O}_X, \mathcal{O}_X[2]) \cong \mathbb{C}$. We can therefore consider a non-zero morphism $f : \mathcal{O}_X \to \mathcal{O}_X[2]$ and complete it to a distinguished triangle in $\mathrm{D}(\mathcal{O}_X)$

$$\mathcal{O}_X \xrightarrow{f} \mathcal{O}_X[2] \xrightarrow{g} K \xrightarrow{h} \mathcal{O}_X[1]$$

where neither f nor the inclusion g or the projection h are zero. In order to see this, observe that the distinguished triangle completing f is isomorphic in $\mathrm{D}(\mathcal{O}_X)$ to a distinguished triangle in $\mathrm{K}(\mathcal{O}_X)$

$$\mathcal{O}_X \xrightarrow{p} \mathrm{O}_X[2] \longrightarrow C(p) \longrightarrow \mathcal{O}_X[1]$$

Now, p is certainly not a qiso, and therefore $C(p)$ is not acyclic, i.e. not zero in $\mathrm{D}(\mathcal{O}_X)$, and hence are not zero the inclusion and the projection $\mathcal{O}_X[2] \to C(p)$ and $C(p) \to \mathcal{O}_X[1]$. It follows, since the two triangles are isomorphic in $\mathrm{D}(\mathcal{O}_X)$, that neither g nor h are zero. Therefore, the same hold true for the morphisms of the rotated triangle

$$\mathcal{O}_X \longrightarrow K[-2] \longrightarrow \mathcal{O}_X[-1] \longrightarrow \mathcal{O}_X[1]$$

and hence this triangle doesn't split, as a direct consequence of Proposition 2.4.11. That means, $K[-2]$ is not isomorphic to the complex with trivial differentials $\mathcal{O}_X \oplus \mathcal{O}_X[-1]$ in degrees 0 and 1

$$\longrightarrow 0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X \longrightarrow 0 \longrightarrow$$

However, the cohomology sheaves of $K[-2]$ are the same as those of $\mathcal{O}_X \oplus \mathcal{O}_X[-1]$. In order to see that consider, by the definition of the triangulated structure of $\mathrm{D}(\mathcal{O}_X)$, a distinguished triangle $A \to B \to C \to A[1]$ in $\mathrm{K}(\mathcal{O}_X)$, its image under the localization functor and an isomorphism of triangles in $\mathrm{D}(\mathcal{O}_X)$

$$\begin{array}{ccccccc}
A & \xrightarrow{g} & B & \longrightarrow & C & \longrightarrow & A[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{O}_X & \xrightarrow{f} & \mathcal{O}_X[2] & \longrightarrow & K & \longrightarrow & \mathcal{O}_X[1]
\end{array}$$

Inducing, in particular, isomorphism on the cohomology of each vertex. Thus, $H^i(A) \cong \mathcal{O}_X$ if $i = 0$, and is zero everywhere else, while $H^i(B) \cong \mathcal{O}_X$ if $i = -2$, and is zero everywhere else. Therefore, since $C \cong C(g)$, we can consider the long exact cohomology sequence

$$\to H^{-2}(B) \to H^{-2}(C) \to H^{-1}(A) \to H^{-1}(B) \to H^{-1}(C) \to H^0(A) \to$$

which reduces to

$$\cdots \to 0 \to \mathcal{O}_X \to H^{-2}(C) \to 0 \to 0 \to H^{-1}(C) \to \mathcal{O}_X \to 0 \to \cdots$$

Hence, $H^{-2}(K) \cong H^{-2}(C) \cong \mathcal{O}_X$ and $H^{-1}(K) \cong H^{-1}(C) \cong \mathcal{O}_X$, while $H^i(K) \cong 0$ everywhere else. This implies the cohomology of $K[-2]$ to be isomorphic to the cohomology of $\mathcal{O}_X \oplus \mathcal{O}_X[1]$.

3.5 Derived tensor product

Through this section we are going to define the tensor product of complexes of $\mathcal{O}_X$-modules and see how it induces a functor on the derived categories. This construction will in turn be a generalization of the derived tensor product obtained as in Remark 3.4.7 deriving the functor that takes the tensor product by a fixed $\mathcal{O}_X$-module.

Definition 3.5.1. Given a chain complex $K^\cdot$ of $\mathcal{O}_X$-modules, define the *tensor product of chain complexes* as the functor

$$- \otimes K^\cdot : \mathrm{Kom}(\mathcal{O}_X) \longrightarrow \mathrm{Kom}(\mathcal{O}_X)$$

mapping $N^\cdot \mapsto N^\cdot \otimes K^\cdot$ defined to be, in degree j,

$$\bigoplus_{p+q=j} N^p \otimes K^q$$

with differentials $d^j = d_N^j \otimes \mathrm{id} + (-1)^j \mathrm{id} \otimes d_K^j$. The signs as usual are needed in order to turn d into a differential.

On a morphism $f : N^\cdot \to L^\cdot$ the tensor product of chain complexes act diagonally: the j-th component of $f \otimes K^\cdot$ is $(f \otimes K^\cdot)^j : \bigoplus_{p+q=j} N^p \otimes K^q \to \bigoplus_{p+q=j} L^p \otimes K^q$ and is defined component-wise to be $f^p \otimes \mathrm{id}_{K^q}$.

Remark 3.5.2. When confusion could arise, we are going to indicate the complex $N^\cdot \otimes K^\cdot$ as $\mathrm{Tot}^\cdot(N^\cdot \otimes M^\cdot)$ as it is by definition the *total complex* of the double complex $N^\cdot \otimes K^\cdot$.

Recall now the following definition.

Definition 3.5.3. A module M over a (sheaf of) ring(s) is *flat* if the tensor product functor $- \otimes M$ is exact.

Fix now a complex $M^\cdot$ in $D^-(\mathcal{O}_X)$. What we are going to prove is that there is a bounded above resolution $K^\cdot$ of flat modules, i.e. a quasi-isomorphism $K^\cdot \to M^\cdot$ where K^j is flat for every integer j. Subsequently we will use this fact in order to define a functor inducing the tensor product by $M^\cdot$ at the level of the derived categories.

Lemma 3.5.4. *An $\mathcal{O}_X$-module F is flat if and only if for every $x \in X$ the $\mathcal{O}_{X,x}$-module F_x is flat.*

Proof. Follows by elementary property of the tensor product of $\mathcal{O}_X$-modules, and the fact that exactness may be checked on stalks. More precisely if one consider a short exact sequence on $\mathcal{O}_X$-modules

$$0 \longrightarrow K \longrightarrow L \longrightarrow N \longrightarrow 0$$

the resulting sequence

$$0 \longrightarrow K \otimes F \longrightarrow L \otimes F \longrightarrow N \otimes K \longrightarrow 0$$

is exact if an only if it is exact at each stalk, and each stalk, say $(K \otimes_{\mathcal{O}_X} F)_x$, is isomorphic to $K_x \otimes_{\mathcal{O}_{X,x}} F_x$. This proves F to be flat if each of F_x is flat. Conversely, let M be an $\mathcal{O}_{X,x}$-module and consider the skyscraper sheaf $\mathrm{Sk}_x(M)$ on x with value M. Since $(\mathrm{Sk}_x(M) \otimes F)_x = M \otimes F_x$, one finds that the functor $- \otimes F_x$ is the composition of $\mathrm{Sk}_x(-) \otimes F$ with taking stalks, and both are exact. $\square$

Lemma 3.5.5. *Let $\boldsymbol{A}$ be an abelian category and $\mathcal{P}$ a class of objects in $\boldsymbol{A}$ with 0 in $\mathcal{P}$ such that any object of $\boldsymbol{A}$ is a quotient of an object in $\mathcal{P}$. Let $m \in \mathbb{Z}$ and $K^\cdot$ a complex in $\mathrm{Kom}(\boldsymbol{A})$ bounded above by m, i.e. $\forall j > m$ $P^j = 0$, then there exists a complex $P^\cdot$ bounded above by m with P^i in $\mathcal{P}$ for every $i \in \mathbb{Z}$ and an epi quasi-isomorphism $P^\cdot \to K^\cdot$*

Proof. The proof is by induction on the integers. Let's start taking $n = m+1$, so that the following inductive hypothesis holds: there exist objects $\{P^j\}_{j \geq n}$ with morphisms $d^j : P^j \to P^{j+1}$ defining a chain complex $P^\cdot$ with $P^j = 0$ whenever $j < n$, and epimorphisms $\alpha^j : P^j \to K^j$ for every $j \geq n$, inducing isomorphism on cohomology for every $j > n$, and epimorphism $\mathrm{Ker}\, d^n \to \mathrm{Ker}\, d^n_K$:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P^n \arrow[r] \arrow[d] & P^{n+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & K^{n-1} \arrow[r] & K^n \arrow[r] & K^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

As said, the base step $n = m+1$ is easily true considering the zero complex. Let's now suppose the inductive hypothesis true and prove it for $n-1$, extending the complex $\cdots \to 0 \to P^n \to P^{n+1} \to \cdots$ one step further to a complex $\cdots \to 0 \to P^{n-1} \to P^n \to \cdots$. In order to define P^{n-1} and the morphisms d^{n-1} and α^{n-1} consider the pullback diagram

$$\begin{tikzcd}
K^{n-1} \times_{K^n} \mathrm{Ker}\, d^n \arrow[r] \arrow[d] & \mathrm{Ker}\, d^n \arrow[r, hook] \arrow[d] & P^n \arrow[dl, "\alpha^n"] \\
K^{n-1} \arrow[r, "d_K^{n-1}"] & K^n &
\end{tikzcd}$$

and observe that being $\mathrm{Im}(d_K^{n-1}) \subseteq \mathrm{Ker}(d_K^n)$, the pullback considered is $K^{n-1} \times_{K^n} \mathrm{Ker}\, d^n = K^{n-1} \times_{\mathrm{Ker}\, d_K^n} \mathrm{Ker}\, d^n$. Now since any object in $\mathbf{A}$ is quotient of an object in $\mathcal{P}$, find a projection $P^{n-1} \to K^{n-1} \times_{K^n} \mathrm{Ker}\, d^n = K^{n-1}$, and observe that the composite

$$\alpha^{n-1} : P^{n-1} \longrightarrow K^{n-1} \times_{K^n} \mathrm{Ker}\, d^n = K^{n-1} = K^{n-1} \times_{\mathrm{Ker}\, d_K^n} \mathrm{Ker}\, d^n \longrightarrow K^{n-1}$$

is an epimorphism because in any abelian category the pullback of an epimorphism is an epimorphism (Remark 2.1.18), and by assumption α^n induces epimorphism $\operatorname{Ker} d^n \to \operatorname{Ker} d^n_K$.

Then, define the new differential d^{n-1} to be the composite

$$P^{n-1} \longrightarrow K^{n-1} \times_{K^n} \operatorname{Ker} d^n \longrightarrow \operatorname{Ker} d^n \longrightarrow P^n$$

This morphism turn $P^\cdot$ into another chain complex, since $d^n d^{n-1} = 0$ from the fact that d^{n-1} factors though $\operatorname{Ker} d^n$. In order to see that this new complex gives isomorphism on cohomology for degrees $j > n-1$ one has to check only degree n.

The fact that the induced map on cohomology is epimorphism is given by the assumption that α^n is epimorphism on kernels, hence on cohomology. In order to show that in cohomology it becomes also a monomorphism, consider a cycle $[c] \in H^n(P^\cdot)$. To say that $[\alpha^n c] = 0$ means that there exists $k \in K^{n-1}$ such that $\alpha^n \xi = d^{n-1}_K(k)$, that is exactly to say that c sits in the image of d^{n-1}, i.e. that $[c] = 0$ in $H^n(P^\cdot)$. Eventually, the morphism α^{n-1} on kernels is epi, since any $k \in \operatorname{Ker} d^{n-1}_K$ comes from any preimage of $(k, 0) \in K^{n-1} \times_{K^n} \operatorname{Ker} d^n$ through the chosen projection $P^{n-1} \to K^{n-1} \times_{K^n} \operatorname{Ker} d^n$.

In conclusion, the inductive hypothesis holds true for $n-1$, as desired. □

Theorem 3.5.6. *Let $M^\cdot$ be any bounded above complex of $\mathcal{O}_X$-modules, and let m such that $M^j = 0$ for every $j > m$. There exists a bounded above complex of flat $\mathcal{O}_X$-modules $K^\cdot$, with $K^j = 0$ for every $j > m$, with an epi quasi-isomorphism $K^\cdot \to M^\cdot$.*

Proof. The proof immediately follows from Lemma 3.5.5, once observed that any $\mathcal{O}_X$-module is quotient of a flat $\mathcal{O}_X$-modules. In fact, if F is an $\mathcal{O}_X$-module, for any open subspace $j : U \subseteq X$ and any section $s \in F(U)$ consider the corresponding morphism $\mathcal{O}_U \to F|_U$ and its extension to zero $j_!\mathcal{O}_U \to F$, where $j_!\mathcal{O}_U$ is defined to be the sheafification of the presheaf

$$j_!\mathcal{O}_U(V) = \begin{cases} \mathcal{O}_U(V) & \text{if } V \subseteq U \\ 0 & \text{otherwise} \end{cases}$$

Now, all these morphism for each open U and each section in $F(U)$ define a morphism

$$\bigoplus_{\substack{U \subseteq X \\ s \in F(U)}} j_!\mathcal{O}_U \longrightarrow F$$

which is surjective on sections, and in particular is an epimorphism of sheaves.

Now, thanks to Lemma 3.5.4 and from the fact that direct sum of flat modules is flat since cohomology commutes with direct sums, one can conclude that the big module $\bigoplus_{\substack{U \subseteq X \\ s \in F(U)}} j_! \mathcal{O}_U$ is flat because each stalk at x of $j_! \mathcal{O}_U$ is either 0 or $\mathcal{O}_{X,x}$, which is free and hence flat. □

Eventually, let's prove that taking the tensor product with a bounded above chain complex of flat $\mathcal{O}_X$-modules preserves acyclic bounded above complexes. This will give, for a fixed module $M^\cdot$, a resolution $K^\cdot \to M^\cdot$ such that the functor $N^\cdot \mapsto N^\cdot \otimes K^\cdot$ maps quasi-isomorphism to quasi-isomorphism. Thus, it will induce the desired functor $- \otimes^{\mathrm{L}} M^\cdot : \mathrm{D}(\mathcal{O}_X) \to \mathrm{D}(\mathcal{O}_X)$.

Recall the following result about the spectral sequence associated to the total complex of a double complex.

Proposition 3.5.7. *Let $C^{\cdot,\cdot}$ be a double chain complex such that for every n there are finitely many (p,q) with $p+q=n$ such that $C^{p,q} \neq 0$. There is a spectral sequence*

$$E_2^{p,q} = H^p H^q(C^{\cdot,\cdot}) \Rightarrow H^{p+q}(\mathrm{Tot}^\cdot(C^{\cdot,\cdot}))$$

The first page of this spectral sequence is $E_1^{p,q} = H^q(C^{\cdot,p})$.

Proof. See [Huy06] Proposition 2.64. □

Theorem 3.5.8. *Let $K^\cdot$ be a bounded above complex of flat $\mathcal{O}_X$-modules, and let $A^\cdot$ be an acyclic complex. Then the tensor product of chain complex* $\mathrm{Tot}^\cdot(A^\cdot \otimes K^\cdot)$ *is acyclic.*

Proof. Fix $n \in \mathbb{Z}$. Thanks to boundedness of $K^\cdot$, any element in any section of the sheaf $H^n(\mathrm{Tot}^\cdot(A^\cdot \otimes K^\cdot))$ has components coming from the corresponding sections of modules in the following complex

$$\cdots \longrightarrow 0 \longrightarrow \operatorname{Ker} d^m \longrightarrow A^m \longrightarrow A^{m+1} \longrightarrow \cdots$$

for a fixed m depending on n. So, in order to prove that any such an element is zero, we can replace $A^\cdot$ with this new bounded below complex, which still remains acyclic.

The double complex is now such that each diagonal $p+q=n$ is eventually zero both sides, thus the hypothesis of Proposition 3.5.7 holds true, providing a spectral sequence $E_1^{p,q} = H^q(A^\cdot \otimes K^p) \Rightarrow H^{p+q}(\mathrm{Tot}^\cdot(A^\cdot \otimes K^\cdot))$. From the assumption of K^p being flat for every p and $A^\cdot$ being acyclic, we conclude that the n-th cohomology of the total complex is zero. Since n was any, the total complex is then acyclic. □

Lemma 3.5.9. *Let $K^\cdot$ be a bounded above complex of flat objects. Then the functor $- \otimes K^\cdot : \mathrm{Kom}(\mathcal{O}_X) \to \mathrm{Kom}(\mathcal{O}_X)$ maps quasi-isomorphisms to quasi-isomorphisms.*

Proof. Recall by Proposition 2.3.5 that $f : A^\cdot \to B^\cdot$ in $\mathrm{Kom}(\mathcal{O}_X)$ is qiso if and only if its mapping cone $C(f)$ is an acyclic complex. Thus, the result is true whenever the functor $- \otimes K^\cdot$ preserves acyclic complex, and that is the case thanks to Lemma 3.5.8. □

Remark 3.5.10. The functor $-\otimes K^\cdot$ composed with the localization functor $\mathrm{Kom}(\mathcal{O}_X) \to \mathrm{D}(\mathcal{O}_X)$ gives, for $K^\cdot$ bounded above complex of flat $\mathcal{O}_X$-modules, a functor mapping each quasi-isomorphism to an invertible map. One can apply then Theorem 3.1.3 in order to make sense of the following definition.

Definition 3.5.11. Fix a bounded above complex of $\mathcal{O}_X$-modules $M^\cdot$ in $\mathrm{D}^-(\mathcal{O}_X)$ and find (by Theorem 3.5.6) a resolution by a bounded above complex of flat modules $K^\cdot \to M^\cdot$. The *derived tensor product of complexes* is the functor

$$- \otimes^{\mathrm{L}} M^\cdot : \mathrm{D}(\mathcal{O}_X) \longrightarrow \mathrm{D}(\mathcal{O}_X)$$

induced by $N^\cdot \longmapsto N^\cdot \otimes K^\cdot$ composed with the localization functor.

In general, is defined the functor $- \otimes^{\mathrm{L}} - : \mathrm{D}^-(\mathcal{O}_X) \times \mathrm{D}^-(\mathcal{O}_X) \longrightarrow \mathrm{D}^-(\mathcal{O}_X)$.

Certainly, Definition 3.5.11 isn't automatically well posed. It is, however, thanks to the next result.

Proposition 3.5.12. *Let $f : P^\cdot \to Q^\cdot$ be a quasi-isomorphism between bounded above complexes of flat $\mathcal{O}_X$-modules. Then, for any bounded above complex $M^\cdot$, the morphism $M^\cdot \otimes f : M^\cdot \otimes P^\cdot \to M^\cdot \otimes Q^\cdot$ is a quasi-isomorphism.*

Proof. Thanks to Theorem 3.5.6, find a quasi-isomorphism $g : K^\cdot \to M^\cdot$ and consider the diagram

$$\begin{tikzcd} K^\cdot \otimes P^\cdot \arrow[r, "K^\cdot \otimes f"] \arrow[d, "g\otimes P^\cdot"'] & K^\cdot \otimes Q^\cdot \arrow[d, "g\otimes Q^\cdot"] \\ M^\cdot \otimes P^\cdot \arrow[r, "M^\cdot \otimes f"'] & M^\cdot \otimes Q^\cdot \end{tikzcd}$$

in which vertical arrows and the top horizontal one are quasi-isomorphisms because, by Lemma 3.5.9, taking tensor product by a bounded above complex of flat modules maps quasi-isomorphisms to quasi-isomorphisms. Thus, the bottom horizontal arrow also is a quasi-isomorphism. □

In particular, the derived tensor product is well defined because whenever one consider two resolutions by bounded complex of flat modules for a complex $M^\cdot$, say $K_1^\cdot \to M^\cdot$ and $K_2^\cdot \to M^\cdot$, then for any other complex $N^\cdot$ one has that both the complex $N^\cdot \otimes K_1^\cdot$ and $N^\cdot \otimes K_1^\cdot$ will be isomorphic to $N^\cdot \otimes M^\cdot$.

3.6 Internal Hom

In the following, we want to define the internal Hom in $\mathrm{D}(\mathcal{O}_X)$. Given two complexes $N^\cdot$, $M^\cdot$ of $\mathcal{O}_X$-modules, we want a third complex of $\mathcal{O}_X$-modules $R\mathcal{H}om(N^\cdot, M^\cdot)$ giving adjunction between the functor $- \otimes^{\mathrm{L}} N^\cdot$ and the functor $R\mathcal{H}om(N^\cdot, -)$. That is, for any $K^\cdot$ complex of $\mathcal{O}_X$-modules we are going to prove

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(K^\cdot, R\mathcal{H}om(N^\cdot, M^\cdot)) \cong \mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(K^\cdot \otimes^{\mathrm{L}} N^\cdot, M^\cdot)$$

In order to define such a functor, we let first of all the following definition.

Definition 3.6.1. Let $N^\cdot$, $M^\cdot$ be complexes of $\mathcal{O}_X$-modules. The *Hom complex* $\mathcal{H}om^\cdot(N^\cdot, M^\cdot)$ is defined by

$$\mathcal{H}om^n(N^\cdot, M^\cdot) = \prod_{p+q=n} \mathcal{H}om_{\mathcal{O}_X}(N^{-p}, M^q) = \prod_k \mathcal{H}om_{\mathcal{O}_X}(N^k, M^{k+n})$$

with differential d^n induced by morphisms, for every k,

$$\prod_{i\in\mathbb{Z}} \mathcal{H}om_{\mathcal{O}_X}(N^i, M^{i+n}) \to \mathcal{H}om_{\mathcal{O}_X}(N^k, M^{k+n+1})$$

defined on an open subset U to act on a morphism $(f^i) \in \prod_i \mathrm{Hom}_{\mathcal{O}_U}(N^k|_U, M^{n+k}|_U)$ as

$$(f^i) \mapsto d_{M|_U} f^k - (-1)^n f^k d_{N|_U}$$

For short, we are going to use the less explicit notation $df = d_M f - (-1)^n f d_N$.

Remark 3.6.2. The differential d^2 is actually 0 since it is on each open, where one has $(d_{M|_U} f - (-1)^{n+1} f d_{N|_U})(d_{M|_U} f - (-1)^n f d_{N|_U}) = 0$ because f commutes with differentials.

Proposition 3.6.3. *Let $N^\cdot$, $M^\cdot$ be complexes and let f be a morphism in $\mathcal{H}om^n(N^\cdot, M^\cdot)$, defined by a family of morphisms $(f^k)_{k\in\mathbb{Z}}$ with $f^k : N^k \to M^{k+n}$.*

- (i) *$df = 0$ if and only if f defines a morphism of chain complexes $N^\cdot \to M^\cdot[n]$*
- (ii) *$f = dg$ is differential of a morphism g in $\mathcal{H}om^{n-1}(N^\cdot, M^\cdot)$ if and only if f is homotopically equivalent to zero.*

Proof. In order to prove (i) just consider $0 = df = d_M f - (-1)^n f d_N$, and the fact that $d^k_{M[n]} = (-1)^n d^{k+n}_M$, saying exactly that for every k one has $d^{k+n}_M f^k = f^{k+1} d^k_N$.

For part (ii), observe first of all that by part (i) and Remark 3.6.2, f actually defines a morphism of complexes $N^\cdot \to M^\cdot[n]$. Now consider g as the family of morphisms $N^k \to M^{k+n-1}$ and the diagram

$$\begin{tikzcd}
{} \arrow[r] & N^{k-1} \arrow[r] & N^k \arrow[r, "d_N^k"] \arrow[dl, "g^k"'] \arrow[d, "f^k"] & N^{k+1} \arrow[r] \arrow[dl, "g^{k+1}"'] & {} \\
{} \arrow[r] & M^{k+n-1} \arrow[r, "d^{k-1}_{M[n]}"'] & M^{k+n} \arrow[r] & M^{k+n+1} \arrow[r] & {}
\end{tikzcd}$$

$f = dg$ means $f = d_M g - (-1)^{n-1} g d_N = d_M g + (-1)^n g d_N$, thus for every integer k

$$f^k = d_M^{k+n-1} g^k + (-1)^n g^{k+1} d_N^k$$

which means, if n is even,

$$f^k - 0 = f^k = g^{k+1} d_N^k + (-1)^n d_M^{k+n-1} g^k = g^{k+1} d_N^k + d_{M[n]}^{k-1} g^k,$$

while if n is odd just consider $(-1)^n(f^k - 0) = 0 - f^k$ giving the same result. Thus f is homotopically equivalent to zero. □

Corollary 3.6.4. *Let $N^\cdot$, $M^\cdot$ be complexes of $\mathcal{O}_X$-modules. For every open $U \subseteq X$ there is a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{O}_U)}(N^\cdot|_U, M^\cdot[n]|_U) \cong H^n(\mathcal{H}om^\cdot(N^\cdot, M^\cdot)(U))$$

Proof. It's just a reformulation of Proposition 3.6.3: Thanks to part (i) a morphism of complexes $N^\cdot|_U \to M^\cdot[n]|_U$ defines a cycle and vice versa, hence one get a surjective map between the two. Part (ii) says that this map is injective. □

Remark 3.6.5. $\mathcal{H}om^\cdot(N^\cdot, -)$ defines a functor mapping $f : M^\cdot \to L^\cdot$ to

$$\mathcal{H}om^\cdot(N^\cdot, f) : \mathcal{H}om^\cdot(N^\cdot, M^\cdot) \to \mathcal{H}om^\cdot(N^\cdot, L^\cdot)$$

defined in degree n to be

$$\mathcal{H}om^n(N^\cdot, f) : \prod_k \mathcal{H}om_{\mathcal{O}_X}(N^k, M^{k+n}) \to \prod_k \mathcal{H}om_{\mathcal{O}_X}(N^k, L^{k+n})$$

the diagonal morphism mapping, for a fixed k and open $U \subseteq X$, a morphism $g \in \mathcal{H}om_{\mathcal{O}_X}(N^k, M^{k+n})(U) = \mathrm{Hom}_{\mathcal{O}_U}(N^k|_U, M^{k+n}|_U)$ as

$$g \mapsto f^{k+n}|_U g \in \mathcal{H}om_{\mathcal{O}_X}(N^k, L^{k+n})(U)$$

So defined, the Hom complex functor is naturally right adjoint to the tensor product of complexes.

Proposition 3.6.6. *Let $K^\cdot$, $L^\cdot$, $M^\cdot$ be complexes of modules. There is a natural isomorphism*

$$\mathcal{H}om^\cdot(K^\cdot, \mathcal{H}om^\cdot(L^\cdot, M^\cdot)) \cong \mathcal{H}om^\cdot(\mathrm{Tot}^\cdot(K^\cdot \otimes L^\cdot), M^\cdot)$$

Proof. $\mathcal{H}om^\cdot(K^\cdot, \mathcal{H}om(L^\cdot, M^\cdot))$ has degree n given as

$$\prod_j \mathcal{H}om(K^j, \mathcal{H}om^{j+n}(L^\cdot, M^\cdot)) = \prod_j \mathcal{H}om(K^j, \mathcal{H}om(L^i, M^{i+j+n})) \cong$$
$$\prod_{i,j} \mathcal{H}om(K^j, \mathcal{H}om(L^i, M^{i+j+n}))$$

to which one can apply the usual adjunction in order to get

$$\prod_{i,j} \mathcal{H}om(K^j \otimes L^i, M^{i+j+n}) = \prod_{q, i+j=q-n} \mathcal{H}om(K^j \otimes L^i, M^q) \cong$$
$$\prod_q \mathcal{H}om(\bigoplus_{i+j=q-n} K^j \otimes L^i, M^q) = \prod_p \mathcal{H}om(\mathrm{Tot}^p(K^\cdot \otimes L^\cdot), M^{p+n})$$

which is exactly $\mathcal{H}om^n(\mathrm{Tot}^\cdot(K^\cdot \otimes L^\cdot), M^\cdot)$. $\square$

Remark 3.6.7. Recall that the category of $\mathcal{O}_X$-modules has enough injectives. That is, for any $\mathcal{O}_X$-module F there is an injective module I and a monomorphism $F \to I$.

Analogous to Lemma 3.5.6, but easier thanks to the existence of enough injective objects, is the following result.

Proposition 3.6.8. *Any bounded below complex of $\mathcal{O}_X$-modules $M^\cdot$ admits a resolution $M^\cdot \to I^\cdot$ by a bounded below complex of injective modules.*

Proof. Let's construct $I^\cdot$ by induction. Start with n small enough so that the zero complex $I^\cdot = 0$ works.

Suppose then to have defined a complex $I^\cdot$ with I^k injective for $k \leq n$ (and just set $I^k = 0$ for $k > n$), together with a morphism $f : M^\cdot \to I^\cdot$ such that $C(f)$ is acyclic in degrees $k < n$.

$$\begin{tikzcd}
\cdots \arrow[r] & M^{n-1} \arrow[r] \arrow[d] & M^n \arrow[r] \arrow[d, "f^n"] & M^{n+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & I^{n-1} \arrow[r] & I^n \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Let's define the injective object I^{n+1} together with the morphisms $d^n : I^n \to I^{n+1}$ and $f^{n+1} : M^{n+1} \to I^{n+1}$ redefining $I^\cdot$ and f, such that $C(f)$ is acyclic in degree n.

Consider $C = \mathrm{Coker}(d^{n-1}_{C(f)} : M^n \oplus I^{n-1} \to M^{n+1} \oplus I^n)$, and by Remark 3.6.7 take a monomorphism $C \to I^{n+1}$, for some injective object I^{n+1}. The morphism

$$h : M^{n+1} \oplus I^n \to C \to I^{n+1}$$

determines two morphisms which we call $h_1 = d^n : I^n \to I^{n+1}$ and $h_0 = -f^{n+1} : M^{n+1} \to I^{n+1}$.

One needs to check that these two morphisms define respectively the structures of a new chain complex $I^{\cdot}$ and of a morphism of chain complexes. This is given by the observation that the composition

$$hd^{n-1}_{C(f)} : M^n \oplus I^{n-1} \to M^{n+1} \oplus I^n \to I^{n+1}$$

is zero, since the second morphism factors through $\mathrm{Coker}(d^{n-1}_{C(f)})$, so that

$$(-f^{n+1}, d^n)\begin{pmatrix} -d^n_M & 0 \\ -f^n & d^{n-1} \end{pmatrix} = (f^{n+1}d^n_M - d^n f^n, d^n d^{n-1}) = 0$$

which means exactly that $\cdots \to I^{n-1} \to I^n \to I^{n+1} \to 0 \to \cdots$ defines a chain complex, as well as the desired commutativity in

$$\begin{tikzcd}
\cdots \arrow[r] & M^{n-1} \arrow[r] \arrow[d] & M^n \arrow[r, "d^n_M"] \arrow[d, "f^n"] & M^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\
\cdots \arrow[r] & I^{n-1} \arrow[r] & I^n \arrow[r, "d^n"] & I^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

The fact that $C(f)$ is acyclic in degree n follows by construction, because

$$\mathrm{Ker}(d^n_{C(f)}) = \mathrm{Ker}\begin{pmatrix} -d^{n+1}_M & 0 \\ -f^{n+1} & d^n \end{pmatrix} \subseteq \mathrm{Ker}(-f^{n+1}, d^n) = \mathrm{Ker}(h) =$$
$$\mathrm{Ker}(M^{n+1} \oplus I^n \to C) = \mathrm{Ker}(\mathrm{Coker}(d^{n-1}_{C(f)})) = \mathrm{Im}(d^{n-1}_{C(f)}).$$

□

Definition 3.6.9. Let $N^{\cdot}$ be an object in $D^-(\mathcal{O}_X)$. The functor

$$R\mathcal{H}om(N^{\cdot}, -) : \mathrm{D}^+(\mathcal{O}_X) \longrightarrow \mathrm{D}^+(\mathcal{O}_X)$$

is the functor induced by

$$\mathrm{K}^+(\mathcal{O}_X) \longrightarrow \mathrm{D}^+(\mathcal{O}_X)$$

defined on an object $M^{\cdot}$ by taking an injective resolution $M^{\cdot} \to I^{\cdot}$ and computing $\mathcal{H}om^{\cdot}(N^{\cdot}, I^{\cdot})$. On a morphism $f : M^{\cdot} \to L^{\cdot}$ it is

$$\mathcal{H}om^{\cdot}(N^{\cdot}, I^{\cdot}) \longrightarrow \mathcal{H}om^{\cdot}(N^{\cdot}, J^{\cdot})$$

for $\phi : M^{\cdot} \to I^{\cdot}$ and $\psi : N^{\cdot} \to J^{\cdot}$ injective resolutions, defined acting diagonally on the k-th component of a degree n section $g : N^k|_U \to I^{k+n}|_U$ as $\psi|_U f|_U \phi^{-1}|_U g : N^k|_U \to J^{k+n}|_U$.

Remark 3.6.10. Definition 3.6.9 gives a functor

$$R\mathcal{H}om(-,-) : \mathrm{D}^-(\mathcal{O}_X)^{\mathrm{op}} \times \mathrm{D}^+(\mathcal{O}_X) \longrightarrow \mathrm{D}^+(\mathcal{O}_X)$$

where the image is in fact bounded below since one has $N^\cdot$ bounded above, and given an injective resolution $M^\cdot \to I^\cdot$ with $I^\cdot$ in $\mathrm{D}^+(\mathcal{O}_X)$, one can take $n < \min\{j|I^j \neq 0\} - \max\{j|N^j \neq 0\}$, so that $R\mathcal{H}om(N^\cdot, M^\cdot) = \mathcal{H}om(N^\cdot, I^\cdot)$ has degree m given by

$$\prod_k \mathcal{H}om_{\mathcal{O}_X}(N^k, I^{k+m})$$

Now, by our choice of n, this product has each term zero whenever $m \leq n$. That is because if $N^k \neq 0$, then $k \leq \max\{j|N^j \neq 0\}$, hence

$$k + m \leq k + n < k + \min\{j|I^j \neq 0\} - \max\{j|N^j \neq 0\} \leq \min\{j|I^j \neq 0\}$$

Hence $I^{k+m} = 0$. Similarly, if $I^{k+m} \neq 0$, has to be $N^k = 0$.

Certainly, one has to prove this functor to be well defined. Precisely, one has to prove that the functor $\mathrm{K}^+(\mathcal{O}_X) \to \mathrm{D}^+(\mathcal{O}_X)$ actually induces one on the derived category, and also that it's independent on the choice of the injective resolution. Both these facts are provided by the following results, in particular Theorem 3.6.13, once observed that any pair of injective resolution $I_1 \leftarrow M \to I_2$ determines an isomorphism in $\mathrm{D}(\mathcal{O}_X)$.

Lemma 3.6.11. *Let $N^\cdot, I^\cdot$ be bounded below complexes of $\mathcal{O}_X$-modules, with I^j injective for every j. Then*

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{O}_X)}(N^\cdot, I^\cdot) \cong \mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(N^\cdot, I^\cdot)$$

Proof. Consider the natural morphism $\mathrm{Hom}_{\mathrm{K}(\mathcal{O}_X)}(N^\cdot, I^\cdot) \to \mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(N^\cdot, I^\cdot)$ mapping $f : N^\cdot \to I^\cdot$ to the class represented by $N^\cdot \xleftarrow{\mathrm{id}} N^\cdot \xrightarrow{f} I^\cdot$.

It suffices to prove that for any morphism $N^\cdot \to I^\cdot$ in $\mathrm{D}(\mathcal{O}_X)$, that is for any pair of morphisms in $\mathrm{K}(\mathcal{O}_X)$

$$N^\cdot \longleftarrow M^\cdot \longrightarrow I^\cdot$$

there exists a unique morphism $N^\cdot \to I^\cdot$ in $\mathrm{K}(\mathcal{O}_X)$ making the resulting triangle

$$\begin{tikzcd} & M^\cdot \arrow[dl, "\sim"'] \arrow[dr] & \\ N^\cdot \arrow[rr, dashed] & & I^\cdot \end{tikzcd}$$

to commute in $\mathrm{K}(\mathcal{O}_X)$. Such a morphism, if seen in the derived category as $N^\cdot \leftarrow N^\cdot \to I^\cdot$, will in fact have a common roof with the given $N^\cdot \leftarrow M^\cdot \to I^\cdot$ provided by $M^\cdot$ itself. Thus, the proof of the theorem reduces to the following.

Claim: Given a quasi-isomorphism $g : M^{\cdot} \to N^{\cdot}$ of bounded below complexes and a bounded below complex of injective objects $I^{\cdot}$, the induced morphism

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{O}_X)}(N^{\cdot}, I^{\cdot}) \to \mathrm{Hom}_{\mathrm{K}(\mathcal{O}_X)}(M^{\cdot}, I^{\cdot})$$

is a bijection.

In order to see that, use the fact that $\mathrm{K}(\mathcal{O}_X)$ is a triangulated category and consider the completion of g to a distinguished triangle

$$M^{\cdot} \longrightarrow N^{\cdot} \longrightarrow C^{\cdot} \longrightarrow N^{\cdot}[1]$$

by the mapping cone $C^{\cdot}$. Being g a quasi-isomorphism, its mapping cone $C^{\cdot}$ is acyclic by Corollary 2.3.5. Moreover, by Lemma 2.4.9 one can consider the long exact sequence of homomorphisms in $\mathrm{K}(\mathcal{O}_X)$

$$\mathrm{Hom}(C^{\cdot}, I^{\cdot}) \longrightarrow \mathrm{Hom}(N^{\cdot}, I^{\cdot}) \longrightarrow \mathrm{Hom}(M^{\cdot}, I^{\cdot}) \longrightarrow \mathrm{Hom}(C^{\cdot}[-1], I^{\cdot})$$

showing that in order to prove the claimed bijection, it suffices to prove that for any acyclic complex $A^{\cdot}$, any morphism $A^{\cdot} \to I^{\cdot}$ is homotopically equivalent to zero. This homotopy is constructed by induction using the boundedness of both $A^{\cdot}$ and $I^{\cdot}$, so that for any index less than a sufficiently small n, such that both A^n and I^n are zero, the homotopy is of course given by the zero morphism.

Now suppose to have homotopy $h^j : A^j \to I^j$ between g and 0 for every $j \leq n$. Let's build h^{n+1}:

$$\begin{tikzcd}
 & A^{n-1} \arrow[r] \arrow[dl, "h^{n-1}"'] \arrow[d, "g^{n-1}"] & A^n \arrow[r] \arrow[dl, "h^n"] \arrow[d, "g^n"] & A^{n+1} \arrow[dl, dashed] \\
I^{n-2} \arrow[r] & I^{n-1} \arrow[r] & I^n &
\end{tikzcd}$$

Consider the morphism

$$g^n - d_I^{n-1} h^n : A^n \longrightarrow I^n$$

and let's prove that it's well defined on the quotient $A^n / \operatorname{Im} d^{n-1}$, that is, $(g^n - d_I^{n-1} h^n) d^{n-1} = 0$. This is just the computation

$$\begin{aligned}
g^n d^{n-1} - d_I^{n-1} h^n d^{n-1} = d_I^{n-1} g^{n-1} - d_I^{n-1}(g^{n-1} - d_I^{n-2} h^{n-1}) = \\
d_I^{n-1}(g^{n-1} - g^{n-1} + d_I^{n-2} h^{n-1}) = d_I^{n-1} d_I^{n-2} h^{n-1} = 0
\end{aligned}$$

Thus, one can consider, using the isomorphisms induced by the differential $d : A^n \to A^{n+1}$

$$A^n / \operatorname{Im} d^{n-1} \cong A^n / \operatorname{Ker} d^n \cong \operatorname{Im} d^n$$

the morphism h^{n+1} induced by I^n being injective

$$\begin{tikzcd}
A^n/\operatorname{Im}(d^{n-1}) \arrow[r, "\cong"] \arrow[d, "g^n - d_I^{n-1}h^n"'] & \operatorname{Im}(d^n) \arrow[r, hook] & A^{n+1} \arrow[dll, dashed, "h^{n+1}"] \\
I^n & &
\end{tikzcd}$$

such that $h^{n+1}d^n = g^n - d_I^{n-1}h^n$. That is exactly what we want for h to be homotopy between g and 0. $\square$

Remark 3.6.12. Dually, one can prove that if $P^\cdot$ and $M^\cdot$ are bounded above complexes of $\mathcal{O}_X$-modules, with P^j projective for every integer j, then the natural morphism

$$\operatorname{Hom}_{\mathrm{K}(\mathbf{A})}(P^\cdot, M^\cdot) \longrightarrow \operatorname{Hom}_{\mathrm{D}(\mathbf{A})}(P^\cdot, M^\cdot)$$

is an isomorphism.

Theorem 3.6.13. *Let $I_1^\cdot$, $I_2^\cdot$ be bounded below complexes of injective $\mathcal{O}_X$-modules, and suppose $I_1^\cdot \cong I_2^\cdot$ in $\mathrm{D}(\mathcal{O}_X)$. Then, for every bounded below complex of $\mathcal{O}_X$-modules $N^\cdot$, one has*

$$\mathcal{H}om^\cdot(N^\cdot, I_1^\cdot) \cong \mathcal{H}om^\cdot(N^\cdot, I_2^\cdot)$$

in $\mathrm{D}(\mathcal{O}_X)$.

Proof. Let the isomorphism $I_1^\cdot \cong I_2^\cdot$ be represented, thanks to Proposition 3.1.17, by a pair of quasi-isomorphisms

$$I_1^\cdot \xleftarrow{i_1} M^\cdot \xrightarrow{i_2} I_2^\cdot$$

Thus, recall the claim proved in Lemma 3.6.11, giving that the morphism $M^\cdot \to I_1^\cdot$ induces isomorphism $\operatorname{Hom}_{\mathrm{K}(\mathcal{O}_X)}(I_1^\cdot, I_2^\cdot) \cong \operatorname{Hom}_{\mathrm{K}(\mathcal{O}_X)}(M^\cdot, I_2^\cdot)$ by precomposition with i_1. Consider then the morphism $f : I_1 \to I_2$ corresponding to i_2, and observe that the relation $fi_1 = i_2$, as well as the fact that H^k is a functor for every integer k, tells us that f also needs to be a qiso.

Let's prove then $\mathcal{H}om^\cdot(N^\cdot, f)$ to be a quasi-isomorphism. For any open $U \subseteq X$ consider the complex $\mathcal{H}om^\cdot(N^\cdot, I_1^\cdot)(U)$, whose n-th cohomology can be computed using Corollary 3.6.4 and Proposition 3.6.11 as

$$\begin{gathered}
H^n(\mathcal{H}om^\cdot(N^\cdot, I_1^\cdot)(U)) \cong \operatorname{Hom}_{\mathrm{K}(\mathcal{O}_X)}(N^\cdot|_U, I_1^\cdot[n]|_U) \cong \\
\operatorname{Hom}_{\mathrm{D}(\mathcal{O}_X)}(N^\cdot|_U, I_1^\cdot[n]|_U) \cong \operatorname{Hom}_{\mathrm{D}(\mathcal{O}_X)}(N^\cdot|_U, I_2^\cdot[n]|_U) \cong \\
\operatorname{Hom}_{\mathrm{K}(\mathcal{O}_X)}(N^\cdot|_U, I_2^\cdot[n]|_U) \cong H^n(\mathcal{H}om^\cdot(N^\cdot, I_2^\cdot)(U))
\end{gathered}$$

Therefore, is induced isomorphism on cohomology sheaves. $\square$

Observe that the same argument of the proof above proves the following.

Proposition 3.6.14. *Let $N^\cdot$ be a bounded above complex and $P^\cdot \to N^\cdot$ a quasi-isomorphism with $P^\cdot$ a bounded above complex of projective modules, and $M^\cdot$ a bounded below complex with $M^\cdot \to I^\cdot$ a quasi-isomorphism with $I^\cdot$ a bounded below complex of injective modules. Then, the induced morphisms*

$$\mathcal{H}om^\cdot(P^\cdot, M^\cdot) \longrightarrow \mathcal{H}om^\cdot(P^\cdot, I^\cdot) \longleftarrow \mathcal{H}om^\cdot(N^\cdot, I^\cdot)$$

are quasi-isomorphisms.

Proof. Take a generic integer n and an open $U \subseteq X$ and consider, thanks to the claim proved in Lemma 3.6.11, the isomorphism

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{O}_U)}(N^\cdot|_U, I^\cdot[n]|_U) \cong \mathrm{Hom}_{\mathrm{K}(\mathcal{O}_U)}(P^\cdot|_U, I^\cdot[n]|_U)$$

Thanks to Lemma 3.6.11, this isomorphism expands to an isomorphism

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{O}_U)}(N^\cdot|_U, I^\cdot[n]|_U) \cong \mathrm{Hom}_{\mathrm{D}(\mathcal{O}_U)}(P^\cdot|_U, I^\cdot[n]|_U)$$

Now, Corollary 3.6.4 gives isomorphism

$$H^n(\mathcal{H}om^\cdot(N^\cdot, I^\cdot))(U) \cong H^n(\mathcal{H}om^\cdot(P^\cdot, I^\cdot))(U)$$

proving that the morphism $\mathcal{H}om^\cdot(N^\cdot, I^\cdot) \to \mathcal{H}om^\cdot(P^\cdot, I^\cdot)$ induces isomorphism on cohomology sheaves.

The second part is proved using a dual version of the claim proved in Lemma 3.6.11, i.e. the fact that the natural morphism

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{O}_X)}(P^\cdot, M^\cdot) \longrightarrow \mathrm{Hom}_{\mathrm{K}(\mathcal{O}_X)}(P^\cdot, I^\cdot)$$

is isomorphism. Then, Remark 3.6.12 extends this to an isomorphism

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(P^\cdot, M^\cdot) \longrightarrow \mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(P^\cdot, I^\cdot)$$

and the argument is the same as above. □

Remark 3.6.15. The previous proposition allows to compute $\mathrm{R}\mathcal{H}om(N^\cdot, M^\cdot)$ by projective resolutions $P^\cdot \to N^\cdot$ as $\mathcal{H}om^\cdot(P^\cdot, M^\cdot)$.

Let's now observe that the adjunction holds at the level of the derived categories.

Theorem 3.6.16. *For complexes of $\mathcal{O}_X$-modules $K^\cdot$, $N^\cdot$ in $\mathrm{D}^-(\mathcal{O}_X)$, $M^\cdot$ in $\mathrm{D}^+(\mathcal{O}_X)$ it holds*

$$R\mathcal{H}om(K^\cdot, R\mathcal{H}om(N^\cdot, M^\cdot)) \cong R\mathcal{H}om(K^\cdot \otimes^{\mathrm{L}} N^\cdot, M^\cdot)$$

in $\mathrm{D}(\mathcal{O}_X)$.

Proof. Let's consider a resolution $M^\cdot \to I^\cdot$ by a bounded below complex of injective modules, and $F^\cdot \to N^\cdot$ a resolution by a bounded above complex of flat modules. Observe that $R\mathcal{H}om(N^\cdot, M^\cdot) \cong R\mathcal{H}om(F^\cdot, M^\cdot) = \mathcal{H}om^\cdot(F^\cdot, I^\cdot)$ is again a bounded below complex, because it suffices to consider $n < \min\{j | I^j \neq 0\} - \max\{k | F^k \neq 0\}$ in order to get that $\mathcal{H}om^n(F^\cdot, I^\cdot) = \prod\limits_k \mathcal{H}om(F^k, I^{k+n})$ has each factor equal to 0.

Moreover, let's prove that $\mathcal{H}om^\cdot(F^\cdot, I^\cdot)$ is a complex of injective modules yet, so that it will be $R\mathcal{H}om(K^\cdot, \mathcal{H}om^\cdot(F^\cdot, I^\cdot)) = \mathcal{H}om^\cdot(K^\cdot, \mathcal{H}om^\cdot(F^\cdot, I^\cdot))$. Recall that to be injective for an $\mathcal{O}_X$ module Q is the same as to say that the functor $\mathcal{H}om_{\mathcal{O}_X}(-, Q)$ is exact. Thus, consider for any pair of indexes k, j, the usual adjunction in order to get isomorphism of functors

$$\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{H}om_{\mathcal{O}_X}(F^k, I^j)) \cong \mathcal{H}om_{\mathcal{O}_X}(- \otimes_{\mathcal{O}_X} F^k, I^j)$$

and observe that the right hand side is exact, because it is composition of the functors $- \otimes_{\mathcal{O}_X} F^k$ and $\mathcal{H}om_{\mathcal{O}_X}(-, I^j)$ which both are exact for flatness of F^k and injectivity of I^j. Therefore the left hand side is also exact, that is, for any pair of indexes k, j the module $\mathcal{H}om_{\mathcal{O}_X}(F^k, I^j)$ is injective, and so is for every n the module $\mathcal{H}om^n(F^\cdot, I^\cdot)$, being product of injective modules.

Now, we reduced to prove the isomorphism in the non-derived case, so that we can use Proposition 3.6.6 in order to get

$$\begin{aligned} &R\mathcal{H}om(K^\cdot, \mathcal{H}om^\cdot(F^\cdot, I^\cdot)) = \mathcal{H}om^\cdot(K^\cdot, \mathcal{H}om^\cdot(F^\cdot, I^\cdot)) \cong \\ &\mathcal{H}om^\cdot(\mathrm{Tot}^\cdot(K^\cdot \otimes F^\cdot), I^\cdot) = R\mathcal{H}om(K^\cdot \otimes^{\mathrm{L}} N^\cdot, M^\cdot) \end{aligned}$$

□

Remark 3.6.17. As a special case, Theorem 3.6.16 provides what we claimed about $R\mathcal{H}om(N^\cdot, -)$ being right adjoint to $- \otimes^{\mathrm{L}} N^\cdot$, at least for the bounded case, where they actually restricts to functors in opposite directions.

Consider in fact the isomorphism

$$H^0(R\mathcal{H}om(K^\cdot, R\mathcal{H}om(N^\cdot, M^\cdot))(X)) \cong H^0(R\mathcal{H}om^\cdot(K^\cdot \otimes^{\mathrm{L}} N^\cdot, M^\cdot)(X))$$

which becomes, taking a bounded below resolution $M^\cdot \to I^\cdot$ by injective, and a bounded above resolution $F^\cdot \to N^\cdot$ by flats,

$$H^0(R\mathcal{H}om(K^\cdot, \mathcal{H}om^\cdot(F^\cdot, I^\cdot))(X)) \cong H^0(\mathcal{H}om(K^\cdot \otimes^{\mathrm{L}} N^\cdot, I^\cdot)(X))$$

Again, as in the proof of Theorem 3.6.16, one can observe that $\mathcal{H}om(F^\cdot, I^\cdot)$ need not to be resolved, since yet it is a bounded below complex of injective modules. Hence the isomorphism becomes

$$H^0(\mathcal{H}om^\cdot(K^\cdot, \mathcal{H}om^\cdot(F^\cdot, I^\cdot))(X)) \cong H^0(\mathcal{H}om^\cdot(K^\cdot \otimes^{\mathrm{L}} N^\cdot, I^\cdot)(X))$$

to which we can apply Corollary 3.6.4, in order to get

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(K^{\cdot}, \mathcal{H}om^{\cdot}(F^{\cdot}, I^{\cdot})) \cong \mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(K^{\cdot} \otimes^{\mathrm{L}} N^{\cdot}, I^{\cdot})$$

which clearly gives, since resolutions are isomorphisms in $\mathrm{D}(\mathcal{O}_X)$, the adjunction isomorphism

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(K^{\cdot}, R\mathcal{H}om(N^{\cdot}, M^{\cdot})) \cong \mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(K^{\cdot} \otimes^{\mathrm{L}} N^{\cdot}, M^{\cdot})$$

Remark 3.6.18. All the constructions given for the derived tensor product and the inner Hom can be generalized with a bit more of work to unbounded chain complexes. Our attention in the next chapters, however, will be drifted to a particularly nice subcategory of $\mathrm{D}(\mathcal{O}_X)$ which will in fact be equivalent to a subcategory of the bounded derived category $\mathrm{D}^b(\mathcal{O}_X)$.

Chapter 4

Perfect complexes

As we're going to see, the spectrum construction is a very general technique associating a locally ringed space to any (essentially small) triangulated category equipped with the structure given by a symmetric tensor product. Our aim is, however, to show how it works in a specific case, that will lead us to reconstruct a scheme X starting from the category of the so called *perfect complexes* on X. Perfect complexes are, roughly speaking, an enlargement of the concept of bounded complex of vector bundles, obtained refining the huge generalization consisting of the complexes of $\mathcal{O}_X$-modules.

Perfect complexes on a scheme X will form a triangulated subcategory of $D(\mathcal{O}_X)$ in which we are allowed to use general results about derived categories and derived functors (especially the derived tensor product $\otimes^{\mathrm{L}}_{\mathcal{O}_X}$) but still being able to compute them without much trouble, since perfect complexes will locally be, up to quasi-isomorphism, complexes of (locally) free, and hence projective, $\mathcal{O}_X$-modules.

4.1 Truncations and inductive construction

One of the main tool that we are going to use in order to deal with construction of complexes in the derived category is Lemma 4.1.9, which under rather technical hypothesis will provide, given a morphism of complexes, the existence of complexes inductively defined, quasi-isomorphic to the codomain and through which the given morphism factors injecting the domain in it. One reason why we care about this result is that the complex inductively defined will be a complex of objects in a general additive subcategory of an abelian category, if the domain is, while the codomain may be taken in a subcategory of cohomologically bounded above complexes of the ambient abelian category.

Definition 4.1.1. A morphism $f : A^{\cdot} \to B^{\cdot}$ in the category of complexes $\mathrm{Kom}(\mathbf{A})$ of an abelian category $\mathbf{A}$ is called *n-quasi-isomorphism* (or *n-*

qiso, for short) if the induced morphism $H^i(f) : H^i(A^{\cdot}) \to H^i(B^{\cdot})$ is an isomorphism for every integer $i > n$ and an epimorphism for $i = n$.

Remark 4.1.2. A morphism $f : A^{\cdot} \to B^{\cdot}$ in the category of complexes $\mathrm{Kom}(\mathbf{A})$ of an abelian category $\mathbf{A}$ is a quasi-isomorphism if an only if it is an n-quasi-isomorphism for every integer n.

Moreover, are defined the following operations on complexes.

Definition 4.1.3. Let $A^{\cdot}$ be a complex, with differential d, in $\mathrm{Kom}(\mathbf{A})$ for an abelian category $\mathbf{A}$. Define the *standard trucation* $\tau^n = \tau^{\geq n} A^{\cdot}$ to be the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathrm{Coker}(d^{n-1}) \longrightarrow A^{n+1} \longrightarrow \cdots$$

and similarly, $\tau^{\leq n} A^{\cdot}$ is the complex

$$\cdots \longrightarrow A^{n-1} \longrightarrow \mathrm{Ker}(d^n) \longrightarrow 0 \longrightarrow \cdots$$

Moreover, in a naïve way we get the *stupid truncations* $\sigma^n = \sigma^{\geq n} A^{\cdot}$ to be

$$\cdots \longrightarrow 0 \longrightarrow A^n \longrightarrow A^{n+1} \longrightarrow \cdots$$

and $\sigma^{\leq n} A^{\cdot} = A^{\cdot} / \sigma^{\geq n+1} A^{\cdot}$

$$\cdots \longrightarrow A^{n-1} \longrightarrow A^n \longrightarrow 0 \longrightarrow \cdots$$

Remark 4.1.4. In the definition of the standard truncation $\tau^n A^{\cdot}$, the morphisms $\mathrm{Coker}(d^{n-1}) \to A^{n-1}$ is defined by the pushout property for the map d^n:

$$\begin{tikzcd}
A^{n-1} \arrow[r] \arrow[d] & A^n \arrow[r] \arrow[d] & A^{n+1} \\
0 \arrow[r] & \mathrm{Coker}(d^{n-1}) \arrow[ur, dashed] &
\end{tikzcd}$$

as well as, dually, the map $A^{n-1} \to \mathrm{Ker}(d^n)$ defining $\tau^{\leq n} A^{\cdot}$ is given by the pullback property for the map d^{n-1}.

Remark 4.1.5. Both σ^n and $\tau^{\leq n}$ define filtrations of the complex. That is because there are monomorphisms

$$0 \longrightarrow \cdots \longrightarrow \sigma^n A^{\cdot} \longrightarrow \sigma^{n-1} A^{\cdot} \longrightarrow \cdots \longrightarrow A^{\cdot}$$

and monomorphisms

$$0 \longrightarrow \cdots \longrightarrow \tau^{\leq n} A^{\cdot} \longrightarrow \tau^{\leq n+1} A^{\cdot} \longrightarrow \cdots \longrightarrow A^{\cdot}$$

Hence, since these are all monomorphisms into $A^{\cdot}$, we can think of them as subobjects.

Proposition 4.1.6. *Truncations τ and σ define functors from the category of complexes of an abelian category to itself.*

Proof. Let's prove that if $f : A^\cdot \to B^\cdot$ one has $\tau^n f : \tau^n A^\cdot \to \tau^n B^\cdot$

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{Coker}(d_A^{n-1}) \arrow[r] \arrow[d] & A^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathrm{Coker}(d_B^{n-1}) \arrow[r] & B^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

The non obvious morphism is $\hat{f} : \mathrm{Coker}(d_A^{n-1}) \to \mathrm{Coker}(d_B^{n-1})$, which is given by the pushout property of $\mathrm{Coker}(d_A^{n-1})$ for the map $A^n \to B^n \to \mathrm{Coker}(d_B^{n-1}) = B^n / \mathrm{Im}(d_B^{n-1})$

$$\begin{tikzcd}
A^{n-1} \arrow[r] \arrow[d] & A^n \arrow[r, "f"] \arrow[d] & B^n \arrow[r] & \mathrm{Coker}(d_B^{n-1}) \\
0 \arrow[r] & \mathrm{Coker}(d_A^{n-1}) \arrow[urr, dashed, "\hat{f}"'] & &
\end{tikzcd}$$

where the first row composite is actually zero because $f^n d_A^{n-1}(A^{n-1}) = d_B^{n-1} f^{n-1}(A^{n-1}) \subseteq \mathrm{Im}(d_B^{n-1})$.

The fact that this defines a chain morphism is a consequence of the following computation. Call π_A, π_B the projections of respectively A and B on the cokernel of the respective differentials d^{n-1}, and call $r_A : \mathrm{Coker}(d_A^{n-1}) \to A^{n+1}$, $r_B : \mathrm{Coker}(d_B^{n-1}) \to B^{n+1}$ the morphisms in Remark 4.1.4. The situation is expressed by the diagram

$$\begin{tikzcd}
A^n \arrow[dr, "\pi_A"'] \arrow[drr, "d_A^n"] \arrow[ddd, "f^n"] & & \\
& \mathrm{Coker}(d_A^{n-1}) \arrow[r, "r_A"'] \arrow[d, "\hat{f}"'] & A^{n+1} \arrow[d, "f^{n+1}"] \\
& \mathrm{Coker}(d_B^{n-1}) \arrow[r, "r_B"] & B^{n+1} \\
B^n \arrow[ur, "\pi_B"] \arrow[urr, "d_B^n"'] & &
\end{tikzcd}$$

in which the commutativity of the central square is given by the computation

$$r_B \hat{f} \pi_A = r_B \pi_B f^n = d_B^n f^n = f^{n+1} d_A^n = f^{n+1} r_A \pi_A.$$

Eventually, since π_A is epimorphism, this gives the desired commutativity $r_B \hat{f} = f^{n+1} r_A$. $\square$

We now state and prove two categorical lemmas, modeling the situation that appear when we consider the category of $\mathcal{O}_X$-modules and its full subcategory of vector bundles.

Lemma 4.1.7. *Let* $\boldsymbol{A}$ *be an abelian category and* $\boldsymbol{B} \subseteq \boldsymbol{A}$ *a full additive subcategory. Suppose* $\boldsymbol{B}$ *to be closed under taking kernel of epimorphisms. Let* $C^\cdot$ *be a bounded above complex of objects in* $\boldsymbol{B}$ *and let* n *be integer such that* $H^k(C^\cdot) = 0$ *for* $k > n$*. Then, the object* $Z^n(C^\cdot) = \mathrm{Ker}(C^n \to C^{n+1})$ *is in* $\boldsymbol{B}$*, and the complex* $C^\cdot$ *is quasi-isomorphic to the truncated complex* $\tau^{\leq n} C^\cdot$ *given by*

$$\cdots \longrightarrow C^{n-2} \longrightarrow C^{n-1} \longrightarrow Z^n(C^\cdot) \longrightarrow 0 \longrightarrow \cdots$$

Proof. Let n be such that $H^k(C^\cdot)$ is zero for every $k > n$ and since $C^\cdot$ is bounded above, let N integer be such that $C^p = 0$ for every $p > N$. The proof is by induction on N by observing that the theorem is trivial for $N \leq n$: in this case in fact $C^\cdot$ is

$$\cdots \longrightarrow C^{N-1} \longrightarrow C^N \longrightarrow \overset{N+1}{0} \longrightarrow \cdots \longrightarrow \overset{n}{0} \longrightarrow \cdots$$

and $Z^n(C^\cdot) = \mathrm{Ker}(C^n \to C^{n+1}) = C^n$ is surely in $\mathbf{B}$. Clearly in this case the complex $C^\cdot$ is actually the same as its truncation.

Suppose then $N > n$ and the theorem to hold, by inductive hypothesis, for bounded complexes of objects in $\mathbf{B}$ starting vanishing before N. Since $N > n$, it holds $H^N(C^\cdot) = 0$. Hence, from the fact that $\mathrm{Ker}(C^N \to C^{N+1}) = C^N$, we deduce that the same is the image of $C^{N-1} \to C^N$, which is then an epimorphism. By our assumption we get that its kernel $Z^{N-1}(C^\cdot)$ is an object in $\mathbf{B}$ as well. This leads us to consider the shorter complex $C'^\cdot = \tau^{\leq N-1} C^\cdot$ which is again made of objects in $\mathbf{B}$:

$$\cdots \longrightarrow C^{N-2} \longrightarrow Z^{N-1}(C^\cdot) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

and is quasi-isomorphic to $C^\cdot$:

$$\cdots \longrightarrow C^{N-2} \longrightarrow C^{N-1} \longrightarrow C^N \longrightarrow 0 \longrightarrow \cdots$$

Thus $C'^\cdot$ satisfies the hypothesis of the theorem, and being shorter than $C^\cdot$ leads to conclude by induction that for the fixed n one has that $Z^n(C'^\cdot)$ is in $\mathbf{B}$. Moreover it's clear that $Z^k(C^\cdot) = Z^k(C'^\cdot)$ for every $k \leq N-1$, in particular for $k = n$. Together, we get that $Z^n(C^\cdot) = Z^n(C'^\cdot)$ is in $\mathbf{B}$.

Eventually, there are quasi-isomorphisms $C^\cdot \sim C'^\cdot \sim \tau^{\leq n} C'^\cdot \sim \tau^{\leq n} C^\cdot$ given respectively by what observed before, the inductive hypothesis, and a the general fact that truncations of quasi-isomorphic complexes are quasi-isomorphic. In fact, if $C^\cdot \to C'^\cdot$ is a morphism of complexes, one induces a quasi-isomorphism $\tau^{\leq n} C^\cdot \to \tau^{\leq n} C'^\cdot$ because $Z^n(C^\cdot)$ is actually mapped into $Z^n(C'^\cdot)$. $\square$

Now, observe that we can apply Lemma 4.1.7 when dealing with the category of locally free $\mathcal{O}_X$-modules.

Remark 4.1.8. Let $\phi : F \to G$ be an epimorphism of locally free $\mathcal{O}_X$-modules. Then, there is an exact sequence

$$0 \longrightarrow \mathrm{Ker}(\phi) \longrightarrow F \longrightarrow G \longrightarrow 0$$

and therefore, since the functor St_x preserves both colimits and finite limits, an exact sequence, for every $x \in X$,

$$0 \longrightarrow K = \mathrm{Ker}(\phi)_x \longrightarrow F_x \longrightarrow G_x \longrightarrow 0$$

Now, G_x is a free $\mathcal{O}_{X,x}$-module, and hence it is projective. Therefore, the exact sequence splits, and we get that K is a direct summand of the free module F_x. That is equivalent of K being projective. Now, since K is a projective module over a local ring, it is also free, as a consequence of the Nakayama Lemma.

Therefore, since any stalk is a free module, the sheaf of modules $\mathrm{Ker}(\phi)$ is locally free.

Lemma 4.1.9 (Inductive construction)**.** *Let $\boldsymbol{A}$ be an abelian category and $\boldsymbol{B}$ an additive subcategory. Let $\boldsymbol{C}$ be a full subcategory of* $\mathrm{Kom}(\boldsymbol{A})$ *and suppose it to be closed under taking quasi-isomorphic objects. Suppose that every complex in $\boldsymbol{C}$ is cohomologically bounded above. Suppose that any bounded complex $D^\cdot$ of objects in $\boldsymbol{B}$ is in $\boldsymbol{C}$, and that $\boldsymbol{C}$ contains, for any $D^\cdot$ bounded complex of objects in $\boldsymbol{B}$, the mapping cone of any $\boldsymbol{C}$-morphism $D^\cdot \to C^\cdot$, for any $C^\cdot$ in $\boldsymbol{C}$. Moreover, suppose the following condition to hold:*

- $(*)$ *for any integer n and any $C^\cdot$ in $\boldsymbol{C}$ such that $H^k(C^\cdot) = 0$ for $k \geq n$ and any epimorphism $A \to H^{n-1}(C^\cdot)$ in $\boldsymbol{A}$, there exists D in $\boldsymbol{B} \subseteq \boldsymbol{A}$ and a map $D \to A$ such that the composite $D \to H^{n-1}(C^\cdot)$ is epimorphism in $\boldsymbol{A}$.*

Then, for any $D^\cdot$ in $\mathrm{Kom}^-(\boldsymbol{B}) \cap \boldsymbol{C}$*, any $C^\cdot$ in $\boldsymbol{C}$ and any map $x : D^\cdot \to C^\cdot$ there exists a $D'^\cdot$ in* $\mathrm{Kom}^-(\boldsymbol{B}) \cap \boldsymbol{C}$*, a degree-wise split monomorphism $a : D^\cdot \to D'^\cdot$, and a quasi-isomorphism $x' : D'^\cdot \to C^\cdot$ such that $x = x'a$.*

Proof. The idea of the proof is to construct $D'^\cdot$ by induction. More precisely, we define for a sufficiently large n a bounded complex in $\mathbf{B}$ suggestively called $\sigma^n D'^\cdot$ (which will actually be a fortiori the n-th truncation of $D'^\cdot$) and maps of complexes called $\sigma^n a : \sigma^n D^\cdot \to \sigma^n D'^\cdot$ and $\sigma^n x' : \sigma^n D'^\cdot \to \sigma^n C^\cdot$ such that:

- (i) $\sigma^k a$ is a degree-wise split monomorphism for $k \geq n$
- (ii) $\sigma^n x = \sigma^n x' \sigma^n a$
- (iii) $\sigma^n x'$ composed with the inclusion $\sigma^n C^\cdot \to C^\cdot$ is a n-quasi-isomorphism $\sigma^n D'^\cdot \to C^\cdot$.

Then we will suppose for n to have constructed a bounded (below by n, above by whatever) complex called $\sigma^n D'^{\cdot}$ and maps $\sigma^n a$ and $\sigma^n x'$ satisfying conditions (i)-(iii), and the inductive step consists of defining the previous ones. That is, we will find a complex $\sigma^{n-1} D'^{\cdot}$ and morphisms $\sigma^{n-1} a : \sigma^{n-1} D^{\cdot} \to \sigma^{n-1} D'^{\cdot}$, $\sigma^{n-1} x' : \sigma^{n-1} D'^{\cdot} \to \sigma^{n-1} C^{\cdot}$ satisfying conditions (i)-(iii) above with n replaced by $n-1$.

It will eventually be clear that the n-th truncations of the object and morphisms constructed in the inductive step will be the assumed existing object and morphisms in the inductive hypothesis. This leads eventually to consider the diagram $(\mathbb{Z}, \geq) \to \mathrm{Kom}(\mathbf{A})$ given by the family $\{\sigma^n D'^{\cdot}\}_{n \in \mathbb{Z}}$ an call its colimit $D'^{\cdot}$. The analog has to be done with the morphisms $\sigma^n a$ and $\sigma^n x'$, and check that they works as desired.

In order to prove the base step, look at $x : D^{\cdot} \to C^{\cdot}$, defined with $D^{\cdot}$ bounded above complex of objects in $\mathbf{B}$ lying in $\mathbf{C}$, while $C^{\cdot}$ is in $\mathbf{C}$ and hence cohomologically bounded above. So, consider n such that both $D^{\cdot}$ and $C^{\cdot}$ are cohomologically 0 above n. For such a n set $\sigma^n D'^{\cdot} = \sigma^n D^{\cdot}$, $\sigma^n a : \sigma^n D^{\cdot} \to \sigma^n D'^{\cdot}$ the identity map and $\sigma^n x' = \sigma^n x$. Among conditions (i)-(iii) the unique not completely trivial is (iii) In fact, isomorphism on cohomology for $k > n$ $H^k(D^{\cdot}) \to H^k(C^{\cdot})$ gives isomorphism in cohomology for $k > n$ for the morphism $\sigma^n x$

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & D^n & \longrightarrow & D^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & C^n & \longrightarrow & C^{n+1} & \longrightarrow & \cdots
\end{array}$$

but this doesn't define isomorphism, and in general neither epimorphism, in degree n. However, condition (iii) holds, because in order to get an n-qiso it suffices to compose with the inclusion $\sigma^n C^{\cdot} \to C^{\cdot}$, since in degree n the composition will give on cohomology the epimorphism $\mathrm{Ker}(d^n) \to H^n(C^{\cdot}) \cong H^n(D^{\cdot}) = \mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1})$. This concludes the base step.

Now, suppose (i)-(iii) to hold for n. To fix notation, call d the differential in $D^{\cdot}$ and ∂^k for $k \geq n$ the differential in $\sigma^n D'^{\cdot}$. Let $M^{\cdot}$ be the mapping cone of the n-qiso $\sigma^n D'^{\cdot} \to C^{\cdot}$, that is

$$\longrightarrow C^{n-2} \longrightarrow (\sigma^n D')^n \oplus C^{n-1} \longrightarrow (\sigma^n D')^{n+1} \oplus C^n \longrightarrow$$

By our inductive assumption $\sigma^n D'^{\cdot}$ is a bounded (below by n, above by whatever) complex of objects in $\mathbf{B}$, hence, by assumptions of the theorem is in $\mathbf{C}$, as well as the mapping cone $M^{\cdot}$. Considering the long exact cohomology sequence

$$\begin{aligned}
H^n(\sigma^n D'^{\cdot}) \longrightarrow H^n(C^{\cdot}) \longrightarrow H^n(M^{\cdot}) \longrightarrow H^{n+1}(\sigma^n D'^{\cdot}) \\
\xrightarrow{\cong} H^{n+1}(C^{\cdot}) \longrightarrow H^{n+1}(M^{\cdot}) \longrightarrow H^{n+2}(\sigma^n D'^{\cdot}) \xrightarrow{\cong} H^{n+2}(C^{\cdot})
\end{aligned}$$

where the first arrow is epimorphism, provides $H^k(M^\cdot) = 0$ for every $k \geq n$. Now since $M^\cdot$ is in $\mathbf{C}$ and $H^k(M^\cdot) = 0$ for $k \geq n$, considering the epimorphism $Z^{n-1} \to H^{n-1}(M^\cdot)$ leads to apply the hypothesis $(*)$, so that there exists an object $\hat{D}$ in $\mathbf{B} \subseteq \mathbf{A}$ and a map $\hat{D} \to Z^{n-1}(M^\cdot)$ such that the composite $\hat{D} \to Z^{n-1}(M^\cdot) \twoheadrightarrow H^{n-1}(M^\cdot)$ is epimorphism in $\mathbf{A}$. Recall that the differential in $M^\cdot$ is given by

$$\partial_M^{n-1} : M^{n-1} = (\sigma^n D'^\cdot)^n \oplus C^{n-1} \longrightarrow M^n = (\sigma^n D'^\cdot)^{n+1} \oplus C^n$$

as the matrix $\begin{pmatrix} -\partial^n & 0 \\ -\sigma^n x' & d_C^{n-1}(c) \end{pmatrix}$.

Claim: $Z^{n-1}(M^\cdot)$ is the pullback of the diagram

$$\begin{tikzcd} & Z^n(\sigma^n(D'^\cdot)) \arrow[d, "\sigma^n x'"] \\ C^{n-1} \arrow[r, "d_C"] & Z^n(C^\cdot) \end{tikzcd}$$

In order to make sense of it, observe that $\sigma^n x' : \sigma^n(D'^\cdot) \to \sigma^n C^\cdot$ is a chain map, so actually maps kernel into kernel. Moreover, pullback arrows are the projections on the summands of the mapping cone restricted to the kernel of ∂^{n-1}. Note that over $Z^{n-1}(M^\cdot)$, the differential $\partial_M^{n-1}(a, c) = (-\partial^n(a), -\sigma^n x'(a) + d_C^{n-1}(c))$ is zero, so is the differential $\partial^n(a)$, as well as it holds $d_C^{n-1}(c) = \sigma^n x'(a)$. Hence is well defined and commutative the diagram

$$\begin{tikzcd} Z^{n-1}(M^\cdot) \arrow[r] \arrow[d] & Z^n(\sigma^n D'^\cdot) \arrow[d, "\sigma^n x'"] \\ C^{n-1} \arrow[r, "d_C"'] & C^n \end{tikzcd}$$

In order to see that it's a pullback, note that an object Q together with maps $f : Q \to Z^n(\sigma^n D'^\cdot)$ and $g : Q \to C^{n-1}$ making the relative square commute, forces $Q \to Z^{n-1}(M^\cdot)$ mapping $q \mapsto (f(q), g(q))$ to be the unique that composed with the projections provide f and g.

Thus, it is $Z^{n+1}(M^\cdot) = Z^n(\sigma^n D'^\cdot) \times_{Z^n(C^\cdot)} C^{n-1}$. This helps us in the following way: define the object $(\sigma^{n-1} D'^\cdot)^{n-1} := D^{n-1} \oplus \hat{D}$, and let it extend the complex $\sigma^n D'^\cdot$

$$\longrightarrow 0 \longrightarrow 0 \longrightarrow (\sigma^n D'^\cdot)^n \xrightarrow{\partial^n} (\sigma^n D'^\cdot)^{n+1} \xrightarrow{\partial^{n+1}}$$

one step below by

$$\longrightarrow 0 \longrightarrow (\sigma^{n-1} D'^\cdot)^{n-1} \xrightarrow{\partial^{n-1}} (\sigma^n D'^\cdot)^n \xrightarrow{\partial^n} (\sigma^n D'^\cdot)^{n+1} \xrightarrow{\partial^{n+1}}$$

The map ∂^{n-1} is defined on the first summand D^{n-1} as

$$(\sigma^n a)^n d^{n-1} : D^{n-1} \to D^n \to (\sigma^n D'^{\cdot})^n,$$

and on the second summand $\hat{D}$, thanks to the map $\hat{D} \to Z^{n-1}(M^{\cdot})$, as the composite

$$\hat{D} \longrightarrow Z^n(\sigma^n D'^{\cdot}) \times_{Z^n(C^{\cdot})} C^{n-1} \longrightarrow Z^n(\sigma^n D'^{\cdot}) \hookrightarrow (\sigma^n D'^{\cdot})^n$$

Call this new extending complex $\sigma^{n-1}D'^{\cdot}$. First of all, let's check that the map ∂ actually defines a complex, that is $\partial^n\partial^{n-1} = 0$. On the first summand D^{n-1}, the component of ∂^{n-1} composed with ∂^n is $\partial^n(\sigma^n a)^n d^{n-1} = (\sigma^n a)^{n+1} d^n d^{n-1} = 0$, while on the second summand $\hat{D}$, by definition ∂^{n-1} factors through $Z^n(\sigma^n D'^{\cdot})$, hence composing with ∂^n is zero again.

Let's define the arrows. Set $\sigma^{n-1}a : \sigma^{n-1}D^{\cdot} \to \sigma^{n-1}D'^{\cdot}$ to be $\sigma^n a$ for every degree $k \geq n$, and $(\sigma^{n-1}a)^{n-1} : D^{n-1} \to (\sigma^{n-1}D'^{\cdot})^{n-1} = D^{n-1} \oplus \hat{D}$ is just the split monomorphism given by the inclusion. The arrow $\sigma^{n-1}x'$ is as well defined to be $\sigma^n x'$ in degree greater than n, while

$$\sigma^{n-1}x' : D^{n-1} \oplus \hat{D} \to C^{n-1}$$

is defined to be $x^{n-1} : D^{n-1} \to C^{n-1}$ in the first component, and $\hat{D} \to Z^{n-1} \to C^{n-1}$ on the second component, where $Z^{n-1}(M^{\cdot}) \to C^{n-1}$ is the pullback map in the previous pullback diagram.

Now a bunch of things need to be checked, first of all that these two morphisms define chain maps. The only nontrivial part is the commutativity of the squares from degree $n-1$ to degree n. However, it's clear that the diagram

$$\begin{tikzcd}
{} \arrow[r] & D^{n-1} \arrow[r, "d"] \arrow[d, hook, "(\sigma^{n-1}a)^{n-1}"'] & D^n \arrow[r] \arrow[d, "(\sigma^n a)^n"] & {} \\
{} \arrow[r] & D^{n-1} \oplus \hat{D} \arrow[r, "\partial"] & (\sigma^{n-1}D'^{\cdot})^n \arrow[r] & {}
\end{tikzcd}$$

is commutative because ∂ is defined on the first component exactly as $(\sigma^n a)^n d^{n-1}$. For what concerns the morphism $\sigma^{n-1}x'$ one has the diagram

$$\begin{tikzcd}
{} \arrow[r] & D^{n-1} \oplus \hat{D} \arrow[r, "\partial"] \arrow[d, "(\sigma^{n-1}x')^{n-1}"'] & (\sigma^n D')^n \arrow[r] \arrow[d, "(\sigma^n x')^n"] & {} \\
{} \arrow[r] & C^{n-1} \arrow[r, "d_C"] & C^n \arrow[r] & {}
\end{tikzcd}$$

which is commutative since both squares

$$\begin{tikzcd}
D^{n-1} \arrow[r, "(\sigma^n a)^n d^{n-1}"] \arrow[d, "x^{n-1}"'] & (\sigma^n D'^{\cdot})^n \arrow[d, "(\sigma^n x')^n"] \\
C^{n-1} \arrow[r, "d_C"] & C^n
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\hat{D} \arrow[r] \arrow[d] & (\sigma^n D'^{\cdot})^n \arrow[d, "(\sigma^n x')^n"] \\
C^{n-1} \arrow[r, "d_C"] & C^n
\end{tikzcd}$$

commute. The left one because by inductive hypothesis holds true that $(\sigma^n x')^n(\sigma^n a)^n d^{n-1} = x^n d^{n-1}$, and this is $d_C^{n-1}x^{n-1}$ being x a chain map. The right one commutes because both the morphisms from $\hat{D}$ by definition factors through $Z^{n-1}(M^\cdot)$. In a diagram is

$$\begin{tikzcd}
\hat{D} \arrow[dr] \arrow[ddr] \arrow[drrr] & & & \\
& Z^{n-1}(M^\cdot) \arrow[r] \arrow[d] & Z^n(\sigma^n D'^\cdot) \arrow[r, hook] \arrow[d] & (\sigma^n D'^\cdot)^n \arrow[d, "(\sigma^n x')^n"] \\
& C^{n-1} \arrow[r, "d_C"] & Z(C^\cdot) \arrow[r, hook] & C^n
\end{tikzcd}$$

where each square and each triangle commute, and so does the exterior diagram.

It remains to prove (ii) and (iii). Being $(\sigma^{n-1}a)^{n-1}$ the inclusion $D^{n-1} \to D^{n-1} \oplus \hat{D}$, it's clear that $(\sigma^{n-1}x')^{n-1}(\sigma^{n-1}a)^{n-1}$ is just the first component of $(\sigma^{n-1}x')^{n-1}$, which by definition is $x^{n-1} = (\sigma^{n-1}x)^{n-1}$.

In order to prove that the composition

$$\sigma^{n-1}D'^\cdot \xrightarrow{\sigma^{n-1}x'} \sigma^{n-1}C^\cdot \hookrightarrow C^\cdot$$

is an $n-1$-qiso, consider its mapping cone $\tilde{M}^\cdot$

$$\longrightarrow C^{n-3} \longrightarrow D^{n-1} \oplus \hat{D} \oplus C^{n-2} \longrightarrow (\sigma^{n-1}D'^\cdot)^n \oplus C^{n-1} \longrightarrow$$

and recall that by $(*)$, existence of $\hat{D}$ comes with a morphism $f : \hat{D} \to Z^{n-1}(M^\cdot)$ such that

$$\hat{D} \to Z^{n-1}(M^\cdot) \to H^{n-1}(M^\cdot) = Z^{n-1}(M^\cdot)/B^{n-2}(M^\cdot)$$

is epimorphism, where we denote as usual with $B^k(M^\cdot)$ the image of $\partial_M^k : M^k \to M^{k+1}$. Then if we call s and t the pullback maps in

$$\begin{tikzcd}
Z^{n-1}(M^\cdot) \arrow[r, "s"] \arrow[d, "t"'] & Z^n(D'^\cdot) \subseteq D'^n \arrow[d, "x'^n"] \\
C^{n-1} \arrow[r, "d_C"] & Z^n(C^\cdot)
\end{tikzcd}$$

so that $\tilde{\partial}^{n-2} : D^{n-1} \oplus \hat{D} \oplus C^{n-2} \to (\sigma^{n-1}D'^\cdot)^n \oplus C^{n-1}$, which by definition is represented by the matrix $\begin{pmatrix} -\partial^{n-1} & 0 \\ -\sigma^{n-1}x' & d_C \end{pmatrix}$ can be written decomposing the arrows from $D^{n-1} \oplus \hat{D}$ as $\begin{pmatrix} -a^n & -sf & 0 \\ -x^{n-1} & -tf & d_C \end{pmatrix}$.

Claim: $\tilde{\partial}$ maps epimorphically onto Z^{n-1}. It suffices to prove that for the map restricted to the last two summands $\begin{pmatrix} -sf & 0 \\ -tf & d_C \end{pmatrix}$. Thanks to the epimorphism $\hat{D} \xrightarrow{f} Z^{n-1}(M^\cdot) \to H^{n-1}(M^\cdot)$ there is an epimorphism $\hat{D} \oplus B^{n-2}(M^\cdot) \to Z^{n-1}(M^\cdot)$ which is f in the first component and the inclusion in the second one. That means there's an epimorphism $(f, \partial_M) : \hat{D} \oplus M^{n-2} \to Z^{n-1}$. Now observe that $M^{n-2} = C^{n-2}$, and that $\partial_M = \begin{pmatrix} 0 \\ d_C \end{pmatrix}$. Hence the epimorphism (f, ∂_M) composed with $\begin{pmatrix} s \\ t \end{pmatrix} : Z^{n-1}(M^\cdot) \to D'^n \oplus C^{n-1}$ is, up to signs doesn't affecting the image, the restriction of $\tilde{\partial}$, showing that $\tilde{\partial}$ maps onto $Z^{n-1}(M^\cdot)$. This proves the claim.

Observe that $Z^{n-1}(M^\cdot)$ is the same as $Z^{n-1}(\tilde{M}^\cdot)$. In particular we get $H^{n-1}(\tilde{M}^\cdot) = 0$ as well as $H^k(\tilde{M}^\cdot) = H^k(M^\cdot) = 0$ for $k \geq n$. Thus looking at the long exact cohomology sequence for the mapping cone

$$\begin{aligned} \cdots \longrightarrow H^{n-1}(\sigma^{n-1}D'^\cdot) \longrightarrow H^{n-1}(C^\cdot) \longrightarrow H^{n-1}(\tilde{M}^\cdot) = 0 \\ \longrightarrow H^n(\sigma^{n-1}D'^\cdot) \xrightarrow{\cong} H^n(C^\cdot) \longrightarrow H^n(\tilde{M}^\cdot) = 0 \longrightarrow \cdots \end{aligned}$$

we get that $H^{n-1}(\sigma^{n-1}D'^\cdot) \to H^{n-1}(C^\cdot)$ is epimorphism and that the map $\sigma^{n-1}D'^\cdot \xrightarrow{\sigma^{n-1}} \sigma^{n-1}C^\cdot \hookrightarrow C^\cdot$ is $(n-1)$-qiso. This concludes the inductive step.

Eventually, we define $D'^\cdot$ to be the colimit of all the $\sigma^n D'^\cdot$ with inclusions between them. Observe now that in general, for a complex $X^\cdot$, truncations $\sigma^n X^\cdot$ for n running through integers come with inclusion monomorphisms $i_n : \sigma^n X^\cdot \subseteq \sigma^{n+1} X^\cdot$ as well as $\sigma^n X^\cdot \subseteq X^\cdot$, and this provides a limiting cocone structure for $X^\cdot$. In fact, given a family of morphisms $f_n : \sigma^n X^\cdot \to P^\cdot$, is clearly defined a unique chain map $g : X^\cdot \to P^\cdot$ making $g i_n = f_n$, that is $g^n = f^n_n$. That means we get also morphisms $a = \mathrm{colim}(\sigma^n a)$ and $x' = \mathrm{colim}(\sigma^n x')$, which clearly are as desired, because they are on each component. $\square$

Remark 4.1.10. A porism of theorem 4.1.9 is given observing that for a fixed integer m, if we assume the hypothesis to hold for any $n > m$, the inductive step can be applied until to construct a complex $\sigma^m D'^\cdot$ with the corresponding morphisms. More precisely, we get that given a bounded above complex $D^\cdot$ of objects in $\mathbf{B}$ and a map $x : D^\cdot \to C^\cdot$ such that the condition $(*)$ holds true for the $n > m$, then there are a complex $\sigma^m D'^\cdot$ and morphisms $\sigma^m a : \sigma^m D^\cdot \to \sigma^m D'^\cdot$ degree-wise split mono and $\sigma^m x' : \sigma^m D'^\cdot \to \sigma^m C^\cdot$ such that the composition $\sigma^m D'^\cdot \to \sigma^m C^\cdot \to C^\cdot$ is quasi-isomorphism, such that $\sigma^m x = \sigma^m x' \sigma^m a$.

Moreover, observe that we don't need to ask D^n to be in $\mathbf{B}$ for any n, but just for those $n \geq m$, so that any added piece of the truncation until the lower one $\sigma^m D'$ is build up from objects in $\mathbf{B}$.

Remark 4.1.11. Consider the assumption of Lemma 4.1.9 and assume, moreover, $x : D^\cdot \to C^\cdot$ to be a m-qiso for some integer m.

Claim: then a can be taken to be such that $a^n : D'^n \to D^n$ is isomorphism for $n \geq m$.

In the proof of lemma, observe that at the base step, for sufficiently large n, we take the map $\sigma^n a$ to be in fact the identity, and so $\sigma^n x = \sigma^n x'$. In the inductive step, this can't be done, because in general $\hat{D} \neq 0$. However, the assumption on x of being m-qiso and the new inductive hypothesis $\sigma^n x = \sigma^n x'$ leads to observe that, if $n \geq m$, the exact cohomology sequence for the mapping cone $M^\cdot$ of $\sigma^n x' = \sigma^n x$ becomes

$$\begin{aligned} &H^n(\sigma^n D^\cdot) \longrightarrow H^n(C^\cdot) \longrightarrow H^n(M^\cdot) \longrightarrow H^{n+1}(\sigma^n D^\cdot) \\ &\qquad \xrightarrow{\cong} H^{n+1}(C^\cdot) \longrightarrow H^{n+1}(M^\cdot) \longrightarrow H^{n+2}(\sigma^n D^\cdot) \xrightarrow{\cong} H^{n+2}(C^\cdot) \end{aligned}$$

with the first arrow epimorphism, that is $H^n(M^\cdot) = 0$ for every $n \geq m$. In particular the epimorphism $Z^n(M^\cdot) \to H^n(M^\cdot) = 0$ tells us that at any n-th step, the summand $\hat{D}$ of $D'^n = D^n \oplus \hat{D}$ can be taken to be 0. Thus, the inclusion $a^n : D^n \to D^n \oplus \hat{D}$ is isomorphism for every $n \geq m$.

4.2 Pseudocoherent complexes

Definition 4.2.1. Let X be a scheme and $m \in \mathbb{Z}$, a complex $E^\cdot$ of $\mathcal{O}_X$-modules is *strict m-pseudocoherent* if E^i is a locally free $\mathcal{O}_X$-module of finite rank for every $i \geq m$ and $E^i = 0$ for sufficiently large i.

$E^\cdot$ is *strict pseudocoherent* if it's strict m-pseudocoherent for every $m \in \mathbb{Z}$.

In other words, a strict pseudocoherent complex is a bounded above complex of vector bundles, while in a strict m-pseudocoherent we don't require conditions for the sheaves in degrees lower than m.

Definition 4.2.2. A complex $E^\cdot$ of $\mathcal{O}_X$-modules is *strictly perfect* if it is strictly pseudocoherent and bounded below. That is: $E^\cdot$ is a bounded complex of vector bundles.

Remark 4.2.3. Recall that the stalk of a quotient sheaf $(F/G)_x$ is the quotient of stalks F_x/G_x. That is true by the isomorphism between the stalk of any presheaf and the stalk of its sheafification, and by the fact that taking filtered colimits over the category of presheaves is an exact functor into the category of abelian groups. More precisely, it holds

$$(F/G)_x \cong \operatorname*{Colim}_{U \ni x}(F(U)/G(U)) \cong \operatorname*{Colim}_{U \ni x} F(U)/\operatorname*{Colim}_{U \ni x} G(U) = G_x/F_x$$

Moreover, the following will be needed below

Proposition 4.2.4. *Let X be a scheme and F, G be locally free $\mathcal{O}_X$-modules, with F of finite type. Then, a morphism $\phi : F_x \to G_x$ between* *stalks at $x \in X$ can be extended locally. That is, there exists an open neighborhood V of x and a morphism of $\mathcal{O}_V$-modules $F|_V \to G|_V$ inducing ϕ on its stalk at x.*

Proof. The morphism ϕ is given by $\underset{k}{\oplus}\phi_i : \bigoplus_k \mathcal{O}_{X,x} \to G_x$, where $\phi_i : \mathcal{O}_{X,x} \to G_x$. Thus, consider $\phi_i(1) \in G_x$ and extend it to a small neighborhood $W_i \ni x$, that is, let $t_i \in G(W_i)$ such that $(t_i)_x = \phi_i(1)$. Now, recall that there is a bijection between sections t in $G(W_i)$ and morphism $\hat{t} : \mathcal{O}_{W_i} \to G|_{W_i}$, hence the family $\{t_i|_{\bigcap_j W_j}\}$ (which we will still call $\{t_i\}$) determines a morphism $\underset{k}{\oplus}\hat{t_i} : \bigoplus_k \mathcal{O}_X|_{\bigcap_j W_j} \to G|_{\bigcap_j W_j}$. Let's call $V = \bigcap_j W_j$ and assume it, up to intersect with a trivializing neighborhood of x, to be trivializing. Therefore, the morphism $\underset{i}{\oplus}\hat{t_i} : \bigoplus_k \mathcal{O}_V \to G|_V$ is actually a morphism $F|_V \to G|_V$. Eventually, it's clear that it induces the original morphism on stalks. Its stalk is, on each component, $((\underset{i}{\oplus}\hat{t_i})_x)_i = (\hat{t_i})_x$ (since colimits commutes with colimits), which is the map $\mathcal{O}_{X,x} \ni a \mapsto a(t_i)_x = \phi_i(a) \in G_x$. $\square$

Lemma 4.2.5. *Let X be a scheme and $A^\cdot$ be a complex of $\mathcal{O}_X$-modules with cohomology $H^k(A^\cdot) = 0$ for every $k \geq m+1$. then $H^m(A^\cdot)$ is an $\mathcal{O}_X$-module of finite type if and only if for every $x \in X$ there exists an open neighborhood U of x and a quasi-isomorphism between $A^\cdot|_U$ and a strict m-pseudocoherent complex on U.*

Proof. $H^m(A^\cdot)$ being of finite type means that locally (for every $x \in X$ there's U neighborhood of x such that) there is an epimorphism

$$\bigoplus_k \mathcal{O}_X|_U \longrightarrow H^m(A^\cdot)|_U$$

Look at the induced epimorphism on stalks $\bigoplus_k \mathcal{O}_{X,x} \to H^m(A^\cdot)_x$. Now $\bigoplus_k \mathcal{O}_{X,x}$ is a free $\mathcal{O}_{X,x}$-module, hence is projective. Thus, the map on stalks lifts to the quotient projection $Z^m(A^\cdot)_x \twoheadrightarrow H^m(A^\cdot)_x \cong Z^m(A^\cdot)_x / B^{m-1}(A^\cdot)_x$ (see Remark 4.2.3), that is

$$\begin{tikzcd} & Z^m(A^\cdot)_x \arrow[d] \\ \bigoplus_k \mathcal{O}_{X,x} \arrow[r] \arrow[ur, dashed] & H^m(A^\cdot)_x \end{tikzcd}$$

Hence, by Proposition 4.2.4 we get a morphism $\bigoplus_k \mathcal{O}_X|_U \to Z^m(A^\cdot)|_U \subseteq A^m|_U$, for a suitable small neighborhood U of x. Since $H^m(A^\cdot)$ is of finite type, by shrinking U we can also assume that this open neighborhood of

x is such that composing with quotient projection gives the epimorphism $\bigoplus_k \mathcal{O}_X|_U \to H^m(A^\cdot)|_U$. Thus, we have a morphism, call it x, of complexes

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \bigoplus_k \mathcal{O}_X|_U & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & A^{m-1}|_U & \longrightarrow & A^m|_U & \longrightarrow & A^{m+1}|_U & \longrightarrow & \cdots \end{array}$$

inducing isomorphism in cohomology for $i \geq m+1$ (by hypothesis on $A^\cdot$) and epimorphism on m-th stage. This leads us to use Lemma 4.1.9 in order to deduce that there exists a complex $E^\cdot$ on U and a quasi-isomorphism $x' : E^\cdot \to A^\cdot|_U$. By Remark 4.1.11, since x is a m-qiso, this inductively constructed complex $E^\cdot$ is isomorphic, in degrees greater than m, to the complex $\cdots \to 0 \to \bigoplus_k \mathcal{O}_U \to 0 \to \cdots$, that is $E^m = \bigoplus_k \mathcal{O}_U$ ad $E^k = 0$ for $k \geq m$. Hence, $E^\cdot$ is strict m-pseudocoherent and quasi-isomorphic to $A^\cdot|_U$.

Conversely, suppose A to be locally quasi-isomorphic to a strict m-pseudocoherent complex, that is for any $x \in X$ there exists U open neighborhood of x and a strict m-pseudocoherent complex $A^\cdot_U$ quasi isomorphic to $A^\cdot|_U$. Being strict m-pseudocoherent, $A^\cdot_U$ is a bounded above complex, and if we consider its truncation $\sigma^m A^\cdot_U$ we get a bounded above (and below) complex of finite dimensional vector bundles such that for any $k \geq m+1$ it holds (see Remark 4.2.6 below)

$$H^k(\sigma^m A^\cdot_U) = H^k(A^\cdot_U) = H^k(A^\cdot|_U) = H^k(A^\cdot)|_U = 0,$$

by our assumption for such a k to have $H^k(A^\cdot) = 0$. The idea is to use Lemma 4.1.7 with $\mathbf{A} = \mathbf{Mod}(\mathcal{O}_X)$ and $\mathbf{B}$ its full subcategory of finite dimensional vector bundles, giving that $Z^m(\sigma^m A^\cdot_U) = Z^m(A^\cdot_U)$ is a finite dimensional vector bundle, hence of finite type. Then, $H^m(A^\cdot_U) = H^m(A^\cdot|_U)$ being a quotient of $Z^m(A^\cdot_U)$ is of finite type as well. That means $H^m(A^\cdot)|_U$ is of finite type, and so is $H^m(A^\cdot)$. $\square$

Remark 4.2.6. In the previous proof we made use of the fact that for a complex of sheaves $A^\cdot$ it holds $H^k(A^\cdot|_U) = H^k(A^\cdot)|_U$. That is because both taking kernel and cokernel commute in this sense with restrictions. While it is obvious for kernel, that $\mathrm{Ker}(A \to B)|_U(V) = \mathrm{Ker}(A(V) \to B(V)) = \mathrm{Ker}(A|_U \to B|_U)(V)$, this is less trivial for $\mathrm{Coker}(A \to B)$, defined to be the sheafification of the presheaf $U \mapsto \mathrm{Coker}(A(U) \to B(U))$. This is because, in general, the sheafification $\Gamma\Lambda_{F|_U}$ of the restriction of a presheaf F is isomorphic to the restriction of the sheafification $\Gamma\Lambda_F|_U$. In order to see that, consider the universal morphisms $\eta : F|_U \to \Gamma\Lambda_{F|_U}$ and $\phi : F \to \Gamma\Lambda_F$. Restricting ϕ to U provide a map $F|_U \to \Gamma\Lambda_F|_U$, and being $\Gamma\Lambda_F|_U$ a sheaf, we find by universal property a morphisms $\Gamma\Lambda_{F|_U} \to \Gamma\Lambda_F|_U$ making the diagram

$$\begin{tikzcd} F|_U \arrow[r, "\eta"] \arrow[dr, "\phi|_U"'] & \Gamma\Lambda_{F|_U} \arrow[d, dashed] \\ & \Gamma\Lambda_F|_U \end{tikzcd}$$

to commute. Now, the sheafification morphisms (and their restriction) are isomorphisms on stalks, thus, taking any stalk at x in the previous diagram shows that the map $(\Gamma\Lambda_{F|_U})_x \to (\Gamma\Lambda_F|_U)_x$ is also isomorphism, and so is the sheaf morphism $\Gamma\Lambda_{F|_U} \to \Gamma\Lambda_F|_U$.

Lemma 4.2.7. *Let U be a scheme and $x \in U$. Let $F^\cdot$, $G^\cdot$, $E^\cdot$ be complex of $\mathcal{O}_U$-modules with morphisms $F^\cdot \to G^\cdot \to E^\cdot$. Then, under the hypothesis in any of (a),(b) or (c) below, there are $V \subseteq U$ open neighborhood of x, a complex $E'^\cdot$ of $\mathcal{O}_V$-modules with maps $d : F^\cdot|_V \to E'^\cdot$ and $c : E'^\cdot \to G^\cdot|_V$ such that $cd = a|_V$*

$$\begin{tikzcd} & E'^\cdot \arrow[d, dashed, "c"] \\ F^\cdot \arrow[ur, dashed, "d"] \arrow[r, "a"'] & G^\cdot \arrow[d, "b"] \\ & E^\cdot \end{tikzcd}$$

and conclusion in respectively (a), (b) and (c) holds.

(a) *If both $E^\cdot$ and $F^\cdot$ are strict m-pseudocoherent and the truncation $\tau^m b : \tau^m G^\cdot \to \tau^m E^\cdot$ is qiso, then $E'^\cdot$ can be taken to be m-pseudocoherent and c to be qiso*

(b) *If $E^\cdot$ is m-pseudocoherent, $F^\cdot$ strictly perfect and the truncation $\tau^m b : \tau^m G^\cdot \to \tau^m E^\cdot$ is qiso, then $E'^\cdot$ can be taken to be strictly perfect and c to be m-qiso*

(c) *If both $E^\cdot$ and $F^\cdot$ are strictly perfect and $b : G^\cdot \to E^\cdot$ is qiso, then $E'^\cdot$ can be taken to be strictly perfect and c to be qiso.*

Proof. Let's start observing that in any of cases (a), (b) or (c), $F^\cdot$ and $E^\cdot$ are bounded above. So, since $\tau^m b$ is m-qiso, $G^\cdot$ is cohomologically bounded above by some integer k. That means we can, up to quasi-isomorphism, replace $G^\cdot$ with $\tau^{\leq k} G$. If we prove any of the thesis in (a),(b) or (c) for such a replaced $G^\cdot$, we can get the original statement composing c with the qiso $\tau^{\leq k} G^\cdot \to G^\cdot$.

Now fix m such that hypothesis holds. Let us pass to stalks and apply Lemma 4.1.9 with the following categories. Let:

A be the category of $\mathcal{O}_{U,x}$-modules,

B the category of finitely generated free $O_{U,x}$-modules,

$\mathbf{C}$ the category of complexes of objects in $\mathbf{A}$ having a map b into a strict m-pseudocoherent complex of $\mathcal{O}_{U,x}$-modules (that is: a bounded above complex of $\mathcal{O}_{U,x}$-modules free in degrees $k \geq m$) such that $\tau^m b$ is qiso.

These three categories satisfy the hypothesis of the Inductive Construction Lemma: any object in $\mathbf{C}$ is cohomologically bounded above as its τ^m truncation is quasi-isomorphic to a strict m-pseudocoherent complex, which is by definition bounded above, and if $D^{\cdot}$ is a bounded complex of objects in $\mathbf{B}$, then it is clearly in $\mathbf{C}$ being itself strictly m-pseudocoherent. In order to see that $\mathbf{C}$ contains mapping cones of morphisms $D^{\cdot} \to C^{\cdot}$, consider, as $C^{\cdot}$ in $\mathbf{C}$, a strict m-pseudocoherent $A^{\cdot}$ and a map $C^{\cdot} \to A^{\cdot}$ such that $\tau^m C^{\cdot} \to \tau^m A^{\cdot}$ is qiso. Its mapping cone is clearly mapped in each degree $D^{k+1} \oplus C^k$, with identity in the first component, to $D^{k+1} \oplus A^k$, which is again a strict m-pseudocoherent complex as $D^{\cdot}$ is perfect, and truncation is clearly still qiso. Eventually, condition $(*)$ holds true by considering the quasi-isomorphism $\tau^m b : \tau^m C^{\cdot} \to \tau^m A^{\cdot}$ with $A^{\cdot}$, and hence $\tau^m A^{\cdot}$, strict m-pseudocoherent, so that by Lemma 4.2.5 $H^m(\tau^m C^{\cdot}) = H^m(C^{\cdot})$ is finite type. Hence there exist k and an epimorphism from the free of finite type module $D = \bigoplus_k \mathcal{O}_{U,x} \to H^m(C^{\cdot})$. Thus, being D in particular projective module, any epimorphism onto $H^m(C^{\cdot})$ serves as factorization of $D \to H^m(C^{\cdot})$, and hence D in $\mathbf{B}$ uniformly shows $(*)$ to hold.

Now, $F^{\cdot}_x$ is a complex of objects which are in $\mathbf{B}$ in degrees $k \geq m$, since taking stalk of a locally free sheaf of finite type provides a finite type free module. Moreover $G^{\cdot}_x$ is clearly in $\mathbf{C}$. Thus Inductive Construction in it's form of Remark 4.1.10 applies to $F^{\cdot}_x \to G^{\cdot}_x$ providing a bounded above complex of $\mathcal{O}_{U,x}$-modules in $\mathbf{B}$ which we shall call $\sigma^m E'^{\cdot}_x$, together with a degree-wise split monomorphism which we call

$$\sigma^m d_x : \sigma^m F^{\cdot}_x \to \sigma^m E'^{\cdot}_x$$

and m-qiso

$$\sigma^m c_x : \sigma^m E'^{\cdot}_x \to \sigma^m G^{\cdot}_x$$

Since all of the three $\sigma^m E'^{\cdot}_x$, $\sigma^m F^{\cdot}_x$ and $\sigma^m G^{\cdot}_x$ are bounded complex of free modules of finite type, an argument analogous to the one in the proof of Proposition 4.2.4 tells us that we can degree-wise locally (on an open neighborhood V_i of x for each degree i) extend the complex given by the $\sigma^m E'^i_x = \bigoplus_{k_i} \mathcal{O}_{U,x}$ to a strict perfect complex having degrees $\sigma^m E'^i = \bigoplus_{k_i} \mathcal{O}_{V_i}$, as well as the morphisms $\sigma^m d_x$ and $\sigma^m c_x$ to morphisms of $\mathcal{O}_V$-modules $\sigma^m d$ and $\sigma^m c$ on a suitable small neighborhood $V = \bigcap_i V_i$ of x, where boundedness allows to do that, such that $b|_V \sigma^m c : \sigma^m : E'^{\cdot} \to E^{\cdot}|_V$ is a quasi-isomorphism. In order to see that the extended map $\sigma^m E'^{\cdot} \to G^{\cdot}|_V$ is an m-quasi-isomorphism, we use the fact that $\tau^m b$ is qiso, hence so is the extended $\sigma^m c : \sigma^m E'^{\cdot} \to G^{\cdot}|_V$.

Then, in order to build the rest of $E'^{\cdot}$, as well as of morphisms d and c, apply again Lemma 4.1.9 without any requirement on $\mathbf{B}$, that is $\mathbf{A} = \mathbf{B} = \mathcal{O}_V$-modules, and $\mathbf{C}$ the category of cohomologically bounded above complexes of $\mathcal{O}_V$-modules. The resulting $E'^{\cdot}$ is strict m-pseudocoherent since $\sigma^m E'^{\cdot}$ is strictly perfect. this concludes the proof of (a).

In order to prove (b), since we want $E'^{\cdot}$ to be strictly perfect, consider $\sigma^m E'^{\cdot}$ previously constructed over $V \subseteq U$ together with the map $\sigma^m F^{\cdot}|_V \to \sigma^m E'^{\cdot}$. Consider also the morphism $\sigma^m F^{\cdot} \to F^{\cdot}$, and call the pushout of these maps $E'^{\cdot}$. Consider then the pair consisting of $a : F^{\cdot} \to G^{\cdot}$ and the composition $\sigma^m E'^{\cdot} \to G^{\cdot}|_V$ given by the inclusion $\sigma^m G^{\cdot}|_V \to G^{\cdot}|_V$ after $\sigma^m c : E'^{\cdot} \to G^{\cdot}|_V$. This gives

$$\begin{tikzcd}
\sigma^m F^{\cdot}|_V \arrow[r] \arrow[d] & F^{\cdot}|_V \arrow[d] \arrow[ddr, "a|_V"] & \\
\sigma^m E'^{\cdot} \arrow[r, hook] \arrow[dr, "\sigma^m c"'] & E'^{\cdot} \arrow[dr, dashed] & \\
& \sigma^m G^{\cdot}|_V \arrow[r, hook] & G^{\cdot}|_V
\end{tikzcd}$$

Is induced a map which we shall call $c : E'^{\cdot} \to G^{\cdot}|_V$ extending $\sigma^m c$, and the commutativity of the lower part of the diagram says that it is an m-qiso since both the inclusions and $\sigma^m c$ are. In order to see that $E'^{\cdot}$ is strictly perfect, observe that it's bounded by the same bound of $F^{\cdot}$. Moreover, the pushout of free $\mathcal{O}_X$-modules by split monomorphisms is free. In fact, in for general degree-wise split monomorphisms of complexes of free modules, the pushout is given in each degree as

$$\begin{tikzcd}
\bigoplus_k \mathcal{O}_V \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & \bigoplus_h \mathcal{O}_V \arrow[d] \\
\bigoplus_l \mathcal{O}_V \arrow[r] & \bigoplus_h \mathcal{O}_V \oplus \bigoplus_l \mathcal{O}_V /\sim
\end{tikzcd}$$

where $\sim$ is the equivalence relation generated on each section over $W \subseteq V$ by $i(r) \sim j(r)$ for every $r \in \bigoplus_k \mathcal{O}_V(W)$. From i and j having both a left inverse we deduce that the pushout is isomorphic to $\bigoplus_{h+l-k} \mathcal{O}_V$, because the images of i and j are the same copy of $\bigoplus_k \mathcal{O}_V$ seen as submodule of $\bigoplus_h \mathcal{O}_V$ and of $\bigoplus_l \mathcal{O}_V$ respectively.

Let's eventually prove (c). Being both $F^{\cdot}$ and $E^{\cdot}$ strictly perfect, take m integer small enough so that $\forall k \leq m+1$ we have $E^k = F^k = 0$. For such an m, we can apply part (b) of the Lemma, which provides a strictly perfect complex of $\mathcal{O}_V$-modules $E'^{\cdot}$ with $\sigma^m E'^{\cdot} = E'^{\cdot}$ (since $\sigma^m F^{\cdot} = F^{\cdot}$) and a

m-qiso $c : E^\cdot \to G^\cdot|_V$. Our aim is to extend $E'^\cdot$ and c in degree $m-1$, in order to turn the latter into a quasi-isomorphism. Since b is by assumption a quasi-isomorphism, we get an m-qiso $b|_V c : E'^\cdot \to E^\cdot|_V$, whose mapping cone $M^\cdot = C(b|_V c)$ is

$$\longrightarrow E'^{m-1} \oplus E^{m-2} \longrightarrow E'^m \oplus E^{m-1} \longrightarrow E'^{m+1} \oplus E^m \longrightarrow$$

but by our assumption on m we get that this is just

$$\longrightarrow 0 \longrightarrow E'^m \longrightarrow E'^{m+1} \longrightarrow$$

where the morphism $E'^m \to E'^{m-1}$ is just $-d_{E'}$. Moreover, since $b|_V c$ is m-qiso, this complex is such that $H^k(M^\cdot) = 0$ for every $k \geq m$, and we just observed that $H^{m-1}(M^\cdot) = \operatorname{Ker}(E'^m \to E'^{m+1})$. Therefore, we can apply Proposition 4.1.7 with $\mathbf{B}$ the full subcategory of $\mathbf{Mod}(\mathcal{O}_U)$ of locally free $\mathcal{O}_U$-modules of finite rank, in order to get that

$$H^{m-1}(M^\cdot) = Z^{m-1}(M^\cdot) = Z^m(E'^\cdot)$$

is a locally free $\mathcal{O}_X$-module of finite rank.

Now, the morphism $c : E'^\cdot \to G^\cdot|_V$ is certainly well defined on kernels, in degree m as

$$c^m : Z^m(E'^\cdot) \longrightarrow Z^m(G^\cdot|_V)$$

Moreover, since $b|_V : G^\cdot|_V \to E^\cdot|_V$ is a quasi-isomorphism and we supposed m such that $E^m = 0$, then $0 = H^m(E^\cdot|_V) = H^m(G^\cdot|_V)$. Therefore, the map c^m is actually defined into $\operatorname{Im}(d^{m-1}_{G|_V}) = Z^m(G^\cdot|_V)$. Hence we have

$$\begin{tikzcd}
Z^m E'^\cdot \arrow[r, "c^m"] \arrow[dr, dashed, "c'^m"'] & \operatorname{Im}(d_G^{m-1}) \\
& G^{m-1}|_V \arrow[u, two heads, "d_G^{m-1}"']
\end{tikzcd}$$

where the morphism can be lifted because $Z^m E'^\cdot$ is a free $\mathcal{O}_U$-module of finite rank, and hence is projective in the category of coherent sheaves.

Now, our complex $E'^\cdot = \sigma^m E'^\cdot$ is ready to be extended in degree $m-1$ by the term $E'^{m-1} = Z^m E'^\cdot$, and boundary morphism the inclusion $d^{m-1}_{E'} : Z^m E'^\cdot \to E'^m$. This new complex $E'^\cdot$ is again strictly perfect, because $Z^m E'^\cdot$ is free of finite rank, and we can extend the morphism c in degree $m-1$ by the lifted morphism found above $c'^m : Z^m E'^\cdot \to G^{m-1}|_V$. Now the cohomology $H^m(E'^\cdot)$ is zero, just as $H^m(E^\cdot|_V) = H^m(G^\cdot|_V)$, hence the new morphism c, which is $c^m = c'^m$ in degree m, induces isomorphism in cohomology, as desired.

In degrees other than m there's not much to check. For greater ones we have the thesis from (b), for $m-1$ the kernel of the inclusion $d^{m-1}_{E'}$ is of course zero again, just as $H^{m-1}(G^\cdot|_V)$, and in lower degrees the terms themselves of $E'^\cdot$ are zeros. $\square$

Proposition 4.2.8. *On a scheme X, for a complex $E^\cdot$ of $\mathcal{O}_X$-modules, the following are equivalent*

(1) *$\forall x \in X$ there exist U neighborhood of x, a strict n-pseudocoherent $F^\cdot$ and a qiso $F^\cdot \to E^\cdot|_U$*

(2) *$\forall x \in X$ there exist U neighborhood of x, a strict perfect $F^\cdot$ and a n-qiso $F^\cdot \to E^\cdot|_U$*

(3) *$\forall x \in X$ there exist U neighborhood of x, a strict n-pseudocoherent $F^\cdot$ and an isomorphism between $F^\cdot$ and $E^\cdot|_U$ in the derived category $\mathrm{D}(\mathcal{O}_U)$*

(4) *$\forall x \in X$ there exist U neighborhood of x, a strict perfect $F^\cdot$ and a n-qiso $F^\cdot \to E^\cdot|_U$ in the derived category $\mathrm{D}(\mathcal{O}_U)$, i.e. there is a map $F^\cdot \to E^\cdot|_U$ in the derived category represented by a roof inducing on cohomology epimorphism in degree n and isomorphism for later degrees.*

Proof. (1) $\Rightarrow$ (2): Suppose to have a strict n-pseudocoherent $F^\cdot$ and a qiso $F^\cdot \to E^\cdot|_U$. Thus on the same open set, just consider $\sigma^n F^\cdot$ which is perfect and the induced restriction map $\sigma^n F^\cdot \to E^\cdot|_U$, being clearly isomorphism on cohomology for degrees $i > n$, and epimorphism $\mathrm{Ker}(F^n \to F^{n+1}) \to H^n(F^\cdot) \cong H^{n-1}(E^n|_U)$, with kernel $\mathrm{Im}(F^{n-1} \to F^n)$, in degree n.

(1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are obvious.

(3) $\Rightarrow$ (1): Any isomorphism $F \to E^\cdot|_U$ in $\mathrm{D}(\mathcal{O}_X)$ is represented by a roof of complex maps $F^\cdot \leftarrow G^\cdot \to E^\cdot|_U$, where both the arrows are quasi-isomorphisms by Proposition 3.1.17. Now, since $F^\cdot$ is strict n-pseudocoherent, we apply Lemma 4.2.7(a) to the qiso $b : G^\cdot \to F^\cdot$. It provides, up to shrink U to a smaller open neighborhood V of x, a strict n-pseudocoherent complex of $\mathcal{O}_V$-modules $F'^\cdot$ and a quasi-isomorphism $F'^\cdot \to G^\cdot|_V$. Hence the composite $F'^\cdot \to G^\cdot|_V \to E^\cdot|_V$ is the desired quasi-isomorphism.

(4) $\Rightarrow$ (1): It's similar to the previous implication. Let us represent the n-qiso $F^\cdot \to E^\cdot|_U$ by a pair of chain morphisms $F \leftarrow G \to E^\cdot|_U$, where the left one is as usual a qiso, while the right one is an n-qiso. Thus, being $F^\cdot$ perfect, by Lemma 4.2.7(c) we get, after shrinking U to a smaller neighborhood which we still call U, a perfect complex $F'^\cdot$ together with a qiso $F'^\cdot \to G^\cdot$. Hence the composition $F'^\cdot \to G^\cdot \to E^\cdot|_U$ is a n-qiso. Now $F'^\cdot$ is strict perfect, and this is more than (1) requires, however it only provides a n-qiso $F'^\cdot \to E^\cdot|_U$, while is required a quasi-isomorphism. It helps us Lemma 4.1.9 with $\mathbf{A} = \mathbf{B} = \mathbf{Mod}(\mathcal{O}_X)$ and $\mathbf{C}$ the category of cohomologically bounded above complexes of $\mathcal{O}_X$-modules. Thus, apply Inductive Construction Lemma starting from n, that is, construct inductively a complex $\hat{F}^\cdot$ using the base step n, which is true since we have yet an n-qiso, call it $x : G^\cdot \to E^\cdot|_U$, hence we can just consider $\sigma^n \hat{F}^\cdot = \sigma^n F'^\cdot$ (i.e. $\sigma^n a$ to be the identity) and $\sigma^n x' : \sigma^n \hat{F}^\cdot \to \sigma^n E^\cdot|_U$ to be just $\sigma^n x$. The composition

$\sigma^n F'^{\cdot} = \sigma^n \hat{F}^{\cdot} \to \sigma^n E^{\cdot}|_U \to E^{\cdot}|_U$ is then an n-qiso as required by inductive step, because $F'^{\cdot} \to E^{\cdot}|_U$ is, and this induces the same morphisms on cohomology for degrees greater than n, while in degree n we have surjectivity since $H^n(F'^{\cdot})$ is quotient of $H^n(\hat{F}^{\cdot})$. Thus we end up with a complex $\hat{F}^{\cdot}$ which is strict n-pseudocoherent because $\sigma^n \hat{F}^{\cdot} = \sigma^n F'^{\cdot}$ and $F'^{\cdot}$ is strict perfect, and a quasi isomorphism $\hat{F}^{\cdot} \to E^{\cdot}|_U$ as desired. □

Definition 4.2.9. A complex $E^{\cdot}$ of $\mathcal{O}_X$-modules on a scheme X is said to be *n-pseudocoherent* if it's locally quasi-isomorphic to a strict n-pseudocoherent complex. That is, if any of the equivalent conditions (1)-(4) in Proposition 4.2.8 holds. $E^{\cdot}$ is said to be *pseudocoherent* if it is n-pseudocoherent for every n.

Remark 4.2.10. In [TT90] 2.2.7, is argued that a pseudocoherent complex, being n-pseudocoherent for each integer n, admits locally a quasi-isomorphism with a strict n-pseudocoherent complex, but such a local neighborhood clearly could not be suitable for every n, giving that a priori a pseudocoherent complex couldn't be locally quasi-isomorphic to a strict pseudocoherent complex.

Moreover, see again [TT90] 2.2.7, it turns out that pseudocoherent complexes have quasi-coherent cohomology, and it happens to be true (see Proposition 4.3.13 and subsequent Remark) for complexes of quasi-coherent $\mathcal{O}_X$-modules that they are locally quasi-isomorphic to a strict pseudocoherent complex. Therefore [1], using a result by Bökstedt and Neeman [BN93] Corollary 5.5, we observe that any complex $E^{\cdot}$ with quasi-coherent cohomology is locally isomorphic in $\mathrm{D}(\mathcal{O}_X)$ to a complex of quasi-coherent sheaves on an open affine neighborhood U. Thanks to Proposition 4.2.8 such a complex $E^{\cdot}$ is locally quasi-isomorphic to a complex of quasi-coherent sheaves $F^{\cdot}$ on U via $F^{\cdot} \to E^{\cdot}|_U$. The complex $F^{\cdot}$ so found is again pseudocoherent, since for every n we can compose the isomorphisms in the derived category $E^{\cdot}|_{U'} \cong K^{\cdot}$, for $K^{\cdot}$ strict n-pseudocoherent on U', and $F^{\cdot} \cong E^{\cdot}|_U$, previously restricted to $U \cap U'$.

Summing up, this complex $F^{\cdot}$, being a pseudocoherent complex of quasi-coherent modules, is locally quasi-isomorphic to a strict pseudocoherent $G^{\cdot}$ on $V \subseteq U$, via $G^{\cdot} \to F^{\cdot}|_V$, and so is $E^{\cdot}$ by composing the quasi-isomorphisms

$$G^{\cdot} \longrightarrow F^{\cdot}|_V \longrightarrow E^{\cdot}|_V$$

The notion of pseudocoherent complex is well behaved with respect to the triangulated structure of the derived category $\mathrm{D}(\mathcal{O}_X)$, in the sense that they form a triangulated subcategory.

Proposition 4.2.11. *If $K \xrightarrow{f} L \xrightarrow{g} M \to K[1]$ is a distinguished triangle in $\mathrm{D}(\mathcal{O}_X)$ and K and L are respectively $m+1$-pseudocoherent and m-pseudocoherent, then M also is m-pseudocoherent.*

[1]Thanks to Alberto Canonaco for explaining this to me

Proof. The idea of the proof is to use condition (2) in Proposition 4.2.8 and lift the problem on the level of strict perfect complexes. So, for the m-pseudocoherent complex L, find an open cover $\{U_i\}$ and strict perfect complexes L_i with m-quasi-isomorphisms $\beta_i : L_i \to L|_{U_i}$. Analogously, up to refining the open cover, find strict perfect complexes K_i and $m+1$-quasi-isomorphisms $K_i \to K|_{U_i}$. We want to lift the morphism $f|_{U_i}$ to a morphism of perfect complexes $\gamma_i : K_i \to L_i$.

Consider then the complex $C(\beta_i)$ giving the diagram

$$\begin{tikzcd}
K_i \arrow[d, "\alpha_i"'] & L_i \arrow[d, "\beta_i"] & \\
K|_{U_i} \arrow[r, "f|_{U_i}"] & L|_{U_i} \arrow[r, "r_i"] & C(\beta_i)
\end{tikzcd}$$

and let's claim that the morphism $r_i f|_{U_i} \alpha_i : K_i \to C(\beta_i)$ is locally homotopically equivalent to 0. In fact, This can be proven by induction on the amplitude of the bounded complex K_i. Suppose $K_i = K_i^m[-m]$ to be non-zero only in degree m, then observe that the morphism $C^{m-1}(\beta_i) \to \mathrm{Ker}(d^m) \subseteq C^m(\beta_i)$ in

$$\begin{tikzcd}
{} \arrow[r] & 0 \arrow[r] \arrow[d] & K^m \arrow[r] \arrow[d, "h"] \arrow[dl, dashed] & 0 \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & C^{m-1}(\beta_i) \arrow[r, "d^{m-1}"'] & C^m(\beta_i) \arrow[r, "d^m"'] & C^{m+1}(\beta_i) \arrow[r] & {}
\end{tikzcd}$$

is epi, because $H^m(C(\beta_i)) = 0$ from the fact that β_i is a m-qiso (an easy generalization of Proposition 2.3.5). Therefore, since this is an epimorphism of sheaves, for every U and section $h(t) \in C^m(\beta_i)(U)$ there is an open cover $\{U_j\}$ of U such that $h(t)|_{U_i}$ is is the image of $d^{m-1}|_{U_i}$. This allows us to define the homotopy over each open.

If K_i has amplitude in the interval $[a, b]$, and we assume the result to hold for the degrees up to $b-1$, consider the exact sequence of complexes

$$0 \longrightarrow K_i^b[-b] \longrightarrow K_i \longrightarrow \sigma^{\leq b-1} K_i \longrightarrow 0$$

and the associated triangle in $\mathrm{K}(\mathcal{O}_X)$ induced by Theorem 3.3.4. Then, taking Hom in $\mathrm{K}(\mathcal{O}_X)$ induces a short exact sequence

$$\mathrm{Hom}(\sigma^{\leq b-1} K_i, C(\beta_i)) \longrightarrow \mathrm{Hom}(K_i, C(\beta_i)) \longrightarrow \mathrm{Hom}(K_i^b[-b], C(\beta_i))$$

and we can observe from the base step that the third group is zero. Therefore, by exactness, the morphism h is in the image of the first morphism, hence is the precomposition of the projection $K_i \to \sigma^{\leq m-1} K_i$ with a morphism, by inductive hypothesis, homotopically equivalent to 0, hence is itself homotopically equivalent to 0, by precomposition of the homotopy with the same projection.

This gives the existence of a pair of morphisms

$$\begin{tikzcd}
K_i \arrow[r, "\mathrm{id}"] \arrow[d, dashed] & K_i \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & K_i[1] \arrow[d, dashed] \\
L_i \arrow[r, "\beta_i"] & L|_{U_i} \arrow[r, "r_i"] & C(\beta_i) \arrow[r] & L_i[1]
\end{tikzcd}$$

which can be completed to a morphism of triangle by the desired morphism γ_i.

Eventually, consider the perfect complex $C(\gamma_i)$ and the morphism of triangles

$$\begin{tikzcd}
K_i \arrow[r, "\gamma_i"] \arrow[d] & L_i \arrow[r] \arrow[d] & C(\gamma_i) \arrow[r] \arrow[d, dashed] & K_i[1] \arrow[d] \\
K|_{U_i} \arrow[r, "f|_{U_i}"] & L|_{U_i} \arrow[r, "r_i"] & M|_{U_i} \arrow[r] & K|_{U_i}[1]
\end{tikzcd}$$

The induced morphism on the long exact cohomology sequences

$$\begin{tikzcd}
H^m L_i \arrow[r] \arrow[d, two heads] & H^m C(\gamma_i) \arrow[r] \arrow[d] & H^m K_i \arrow[r] \arrow[d, two heads] & H^{m+1} L_i \arrow[r] \arrow[d, "\cong"] & H^{m+1} C(\gamma_i) \arrow[d] \\
H^m L|_{U_i} \arrow[r] & H^m M|_{U_i} \arrow[r] & H^m K|_{U_i} \arrow[r] & H^{m+1} L|_{U_i} \arrow[r] & H^{m+1} M|_{U_i}
\end{tikzcd}$$

proves, by the Five Lemma, $C(\gamma_i) \to M|_{U_i}$ to be a isomorphism on cohomology for degree greater than m, and epimorphism in degree m. That means $M|_{U_i}$ is m-pseudocoherent. □

Lemma 4.2.12. *On a scheme X, for a complex $E^\cdot$ of $\mathcal{O}_X$-modules, are equivalent*

(1) $\forall x \in X$ exist U neighborhood of x, a strict perfect complex $F^\cdot$ on U and a quasi-isomorphism $F^\cdot \to E^\cdot|_U$

(2) $\forall x \in X$ exist U neighborhood of x, a strict perfect complex $F^\cdot$ on U and an isomorphism $F^\cdot \to E^\cdot|_U$ in $\mathrm{D}(\mathcal{O}_X)$

Proof. Clearly (1) $\Rightarrow$ (2). Conversely the isomorphism is given by a pair of quasi-isomorphisms $F^\cdot \leftarrow G^\cdot \to E^\cdot|_U$. Apply then Lemma 4.2.7 to the morphism $b : G^\cdot \to F^\cdot$ in order to produce, on a smaller neighborhood $V \subseteq U$, a strict perfect complex $F'^\cdot$ and a qiso $F'^\cdot \to G^\cdot|_V$, which gives a composed qiso $F'^\cdot \to G^\cdot|_V \to E|_V$. □

Definition 4.2.13. A complex $E^\cdot$ of $\mathcal{O}_X$-modules on a scheme X is said to be *perfect* if it is locally quasi-isomorphic to a strict perfect complex, i.e. if any of the two condition of Lemma 4.2.12 holds.

Remark 4.2.14. Observe that one could just say that a complex of $\mathcal{O}_X$-modules is perfect whether it's locally quasi-isomorphic to a bounded complex of free modules of finite type, not only *locally free*, as the definition of strict perfect complex requires.

If $E^{\cdot}$ is perfect, then for any x one can find an open neighborhood U such that $E^{\cdot}|_U$ is quasi-isomorphic though a quasi-isomorphism $F^{\cdot} \to E^{\cdot}|_U$ to a bounded complex $F^{\cdot}$ of locally free $\mathcal{O}_U$-modules of finite type. Consider then for the same x, and for any k such that $F^k \neq 0$, an open neighborhood U_k such that $F^k|_{U_k}$ is free $\mathcal{O}_{U_k}$-module. Since $F^{\cdot}$ is bounded, we are taking these k's from a finite set, hence on the finite intersection $V = \bigcap_k U_k$, one has that $F^{\cdot}|_V$ is a bounded complex of free $\mathcal{O}_V$-modules.

Perfect complexes are quite easy to handle because they have, locally, a resolution by free (and hence projective) modules. This makes them a convenient implement which we are able to deal with technically. On the other hand, we are going to see that they form a triangulated subcategory of $\mathrm{D}(\mathcal{O}_X)$ and that they also enjoy nice categorical properties.

4.3 Ample families of line bundles

Still following the work [TT90] by Thomason and Trobaugh, we're going to see in this section what is useful to assume in order to describe perfect complexes and pseudocoherent ones more explicitly, but still remaining under reasonable assumptions.

The possibility to describe globally a pseudocoherent or a perfect complex as quasi-isomorphic to a strict pseudocoherent or a strict perfect one, rely on a geometrical condition of the underlined scheme. While for pseudocoherent complexes we should require also the quasi-coherence of the modules, for perfect complexes the condition on the scheme is enough, and this provides a more explicit and clear picture of what perfect complexes look like, at least when the scheme is nice enough. To begin, assume every scheme in this section to be quasi-compact and quasi-separated.

Definition 4.3.1. A *line bundle* L on a scheme X is a locally free $\mathcal{O}_X$-module with rank 1.

Definition 4.3.2. A line bundle L on a scheme X is said to be *ample* if the set $\{X_f\}_f$, where

$$X_f = \{x \in X | f_x \notin m_x(L^{\otimes n})_x\}$$

and where f runs through all the global sections in the global sections modules $\{\Gamma(X, L^{\otimes n})\}_{n \geq 1}$, form a basis for the Zariski topology of X.

For sake of simplicity, having in mind what happens when the invertible sheaf in question is the sheaf rings of functions into a field, one usually abuses notation and indicates X_f as $\{x \in X | f(x) \neq 0\}$.

Remark 4.3.3. On any affine scheme $X = \text{Spec}R$ the structure sheaf $\mathcal{O}_X$ is ample, because a basis for the Zariski topology is given by distinguished open sets $D(f)$ for f running over $R = \Gamma(X, \mathcal{O}_X)$.

Usually (e.g. in [Har77]) ample line bundles are defined in an equivalent way that we are going to see.

Definition 4.3.4. An $\mathcal{O}_X$-module F on a scheme X is said to be *generated by global sections* if there is a set of global sections $\{s^i\}_{i\in I} \subseteq \Gamma(X, F)$ such that for any $x \in X$ the set of germs $\{(s^i)_x\}_i$ generates the $\mathcal{O}_{X,x}$-module F_x.

Remark 4.3.5. An $\mathcal{O}_X$-module is generated by global sections if and only if it is a quotient of a free $\mathcal{O}_X$-module. Both are in fact equivalent to the existence of a surjective morphism of sheaves

$$\bigoplus_I \mathcal{O}_X \longrightarrow F$$

Proposition 4.3.6. *Let $X = \text{Spec}(R)$ be an affine scheme, then any quasi-coherent $\mathcal{O}_X$-module is generated by global sections.*

Proof. Since F is quasi-coherent and X is affine, $F = \tilde{M}$ for an R-module M. Thus just consider a set of generators $\{m^i\}$ for $M = \Gamma(X, F)$. It works as desired because on any prime ideal $p \in X$ and any $t \in F_p = \tilde{M}_p = M_p$, will be $t = \frac{m}{s}$ with $m, s \in M$ and $s \notin p$. Thus if $m = \sum_{j\in J} a_j m^j$, then coefficients $b_j = \frac{a_j}{s} \in R_p$ provide $t = \sum_{j\in J} b_j m^j$. □

Proposition 4.3.7. *An locally free $\mathcal{O}_X$-module L is ample if and only if for every coherent $\mathcal{O}_X$-module F there exists $N \in \mathbb{N}$ such that for every $n > N$ the sheaf $F(n) = F \otimes L^{\otimes n}$ is generated by global sections*

Proof's idea. This can be done observing that if we denote by S the graded ring $\bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$ and if the set of $\{X_f\}$ with f homogeneous of positive degree covers X, then there is a canonical morphism

$$X \longrightarrow \text{Proj}(S)$$

Then, one can argue that both sides of the equivalence claimed boil down to this morphism being an open immersion. □

Remark 4.3.8. If $X = \text{Spec}R$ is affine, then not only the structure sheaf, but any line bundle is ample. That is because for any coherent sheaf F one clearly has that $F(n)$ is also coherent, and in general for quasi-coherent sheaves Proposition 4.3.6 applies.

Definition 4.3.9. A scheme is said to have an *ample family of line bundles* if it's quasi-compact, quasi-separated and there exists a set Λ and a family of line bundles $\{L_\alpha\}_{\alpha\in\Lambda}$ such that the set $\{X_f\}_f$, where

$$X_f = \{x \in X | f_x \notin m_x(L_\alpha^{\otimes n})_x\}$$

and where f runs through all the global sections in the global sections modules $\{\Gamma(X, L_\alpha^{\otimes n})\}_{\alpha\in\Lambda,n\in\mathbb{N}}$, is a basis for the Zariski topology of X.

Remark 4.3.10. Clearly, the existence of an ample line bundle $\mathcal{L}$ implies having an ample family with the one-point set Λ. In particular, any affine scheme $X = \mathrm{Spec}R$ has an ample family of line bundles.

Actually, a quite large class of schemes enjoy this property.

Proposition 4.3.11. *Suppose X to be any separated, regular and noetherian scheme, then X have an ample family of line bundles.*

Proof. See [BGI71] II 2.2.7.1. □

In the following we are going to state that for a scheme X with an ample family of line bundles and a perfect complex on X, there is a global isomorphism (in the derived category) with a strict perfect complex. The proof is not so hard but it's rather technical and we can avoid it without losing so much of the meaning of our construction. The proof uses this characterization for perfect complexes:

Theorem 4.3.12. *Let X be a scheme. The following are equivalent.*

(a) $E^\cdot$ is perfect.

(b) $E^\cdot$ is pseudocoherent and has locally finite Tor*-dimension, that is, X is covered by open subsets U over which the complex of $\mathcal{O}_U$-modules $E^\cdot|_U$ is such that for all $\mathcal{O}_U$-modules F, the module $H^k(E^\cdot|_U \otimes^L_{\mathcal{O}_U} F)$ is zero for every k out of a range of integer numbers $[a, b]$.*

Proof. See [TT90] 2.2.12. □

Proposition 4.3.13. *Suppose X to have an ample family of line bundles, then*

(a) Let $E^\cdot$ be a strict pseudocoherent complex, $F^\cdot$ a pseudocoherent complex of quasi-coherent $\mathcal{O}_X$-modules and $x : E^\cdot \to F^\cdot$ a chain morphism. Then, there exists a strict pseudocoherent complex $F'^\cdot$, a morphism $a : E^\cdot \to F'^\cdot$ and a quasi-isomorphism $x' : F'^\cdot \to F^\cdot$ such that $x'a = x$:

$$\begin{tikzcd} E^{\cdot} \arrow[r, "a"] \arrow[d, "x"'] & F'^{\cdot} \arrow[dl, "x'"] \\ F^{\cdot} & \end{tikzcd}$$

(b) *If $F^{\cdot}$ is any perfect complex of $\mathcal{O}_X$-modules (possibly not quasi-coherent), then there exists a strict perfect complex $E^{\cdot}$ and an isomorphism $F^{\cdot} \to E^{\cdot}$ in the derived category* $\mathrm{D}(\mathcal{O}_X)$

Proof. See [TT90] 2.3.1. □

Remark 4.3.14. In particular, when $E^{\cdot} = 0$, part (a) of the previous result tells us that a pseudocoherent complex of quasi-coherent $\mathcal{O}_X$-modules on any scheme X is locally quasi-isomorphic to a strict pseudocoherent complex, because this is true on any affine open subset since affine schemes certainly have an ample family of line bundles (by Remark 4.3.10). For what observed in Remark 4.2.10 we get that the quasi-coherence assumption can be dropped.

Moreover, thanks to (b) and using the really same proof of Lemma 4.2.12, we have that a perfect complex $F^{\cdot}$ on a scheme with an ample family of line bundles X is globally quasi-isomorphic to a strict perfect complex.

Proposition 4.3.15. *Let $f : X \to Y$ be a morphism of schemes and $E^{\cdot}$ a strict perfect complex in* $\mathrm{Kom}(\mathcal{O}_Y)$*. The pointwise pullback complex $f^*E^{\cdot}$ is a strict perfect complex in* $\mathrm{Kom}(\mathcal{O}_X)$*.*

Proof. Recall that $f^* : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Mod}(\mathcal{O}_X)$ is a left adjoint functor, so it preserves colimits.

It suffices to prove that if F is a locally free sheaf, then so is f^*F. for a fixed $y \in Y$ we can find an open neighborhood V such that $F \cong \bigoplus_n \mathcal{O}_X|_V$, thus

$$(f^*F)|_{f^{-1}(V)} = f^*(F|_V) = f^*\bigoplus_n \mathcal{O}_Y|_V = \bigoplus_n f^*(\mathcal{O}_Y|_V) = f^*\mathcal{O}_Y|_{f^{-1}(V)},$$

where f^* preserves direct sums, since they are colimits, and eventually it holds for any morphism of schemes that $f^*\mathcal{O}_Y = f^{-1}\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \cong \mathcal{O}_X$. □

Proposition 4.3.16. *For a morphism of schemes $f : X \to Y$, the left derived functor $\mathrm{L}f^* : \mathrm{D}^-(\mathcal{O}_Y) \to \mathrm{D}^-(\mathcal{O}_X)$ is well defined on the subcategories of perfect complexes.*

Proof. Let's assume first Y to be affine. Any complex $E^{\cdot}$ in $\mathrm{Pf}(Y) \subseteq \mathrm{D}^-(\mathcal{O}_Y)$ is, by Proposition 4.3.13, isomorphic to a strict perfect complex, which we still call $E^{\cdot}$. Such a complex is a complex of free and hence projective objects, and we can compute the derived functor $\mathrm{L}f^*(E^{\cdot})$ just as the pointwise functor $f^*(E^\cdot)$, providing a strict perfect complex by Proposition 4.3.15. For the general case, let's prove that $\mathrm{L}f^*(E^\cdot)$ is locally a strict perfect complex. Let $x \in X$, take $V \subseteq Y$ to be an affine open neighborhood of $f(x)$ and call $U = f^{-1}(V) \subseteq X$. Then $\mathrm{L}f^*(E^\cdot)|_U = \mathrm{L}f|_U^*(E^\cdot|_V)$ is strict perfect for the above case. $\square$

Definition 4.3.17. Let $f : A \to B$ be a ring morphism of finite presentation, i.e. f gives to B the structure of an A-module isomorphic to a quotient $A[x_1, \ldots, x_k]/(f_1, \ldots, f_m)$. A complex of B-modules $M^\cdot$ is said to be *n-pseudocoherent relative to A* if it is a n-pseudocoherent complex of $A[x_1, \ldots x_n]$-modules for some presentation (surjective map of A-modules) $A[x_1 \ldots, x_n] \to B$. It is *pseudocoherent relative to A* if it is n-pseudocoherent relative to A for every integer n.

A morphism of rings $f : A \to B$ is said to be *pseudocoherent* if B itself as a complex of B-modules is pseudocoherent relative to A.

Definition 4.3.18. A morphism of schemes $f : X \to Y$ is said to be *perfect* if for every affine subspaces $U \subseteq X$ and $V \subseteq Y$ such that $f(U) \subseteq V$ one has that the resulting morphism of rings

$$\mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(U)$$

is pseudocoherent and of finite Tor-dimension, where a morphism of rings $f : A \to B$ is said to have finite Tor-dimension if for every A-module M the module $\mathrm{Tor}^i_A(B, M) = 0$ for every i out of a bounded range.

Example 4.3.19. Let X and Y be schemes of finite type over a fixed scheme $S = \mathrm{Spec}(k)$ for a fixed field k. That certainly implies the morphisms $f : X \to S$, as well as $g : Y \to S$, to be flat, since each stalk

$$f^\# : k \longrightarrow \mathcal{O}_{X,x}$$

is a flat morphism, being $\mathcal{O}_{X,x}$ a vector space over k, and hence a necessarily flat k-module.

Let's now observe that the projection morphisms $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ in

$$\begin{tikzcd} X \times_S Y \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

are perfect. In order to do so, we use the fact that for a flat morphism to be perfect is equivalent to being of finite type, hence the morphism $f : X \to S$ is perfect because is of finite type by assumptions, as well as g. By taking open affine subschemes and by definition of perfect morphism of schemes, the problem can be reformulated by saying that if $A \to B$ and $A \to A'$ are perfect and flat ring morphisms (we just need the first one to be perfect and the second one to be flat), then their pushout

$$\begin{tikzcd} A' \otimes_A B & B \arrow[l, "i"'] \\ A' \arrow[u, "j"] & A \arrow[l] \arrow[u] \end{tikzcd}$$

gives a perfect morphism $j : A' \to A' \otimes_A B$. Therefore, we need to prove j to be of finite Tor-dimension and pseudocoherent. Being A' a flat A-module is obvious that for every A'-module M, which is also an A-module, the modules $\mathrm{Tor}^i_{A'}(A' \otimes_A B, M)$ and $\mathrm{Tor}^i_A(B, M)$ are isomorphic. Moreover, the morphism j is pseudocoherent because given, for each m, a presentation

$$A[x_1, \ldots, x_n] \longrightarrow B$$

such that B is a m-pseudocoherent $A[x_1, \ldots, x_n]$-module, we have an m-qiso $B' \to B$ with B' a bounded above complex of finite free $A[x_1, \ldots, x_n]$-modules. Therefore it suffices to consider the presentation given by taking the tensor product with the flat A-module A'

$$A'[x_1, \ldots, x_n] = A' \otimes_A A[x_1, \ldots, x_n] \longrightarrow A' \otimes_A B$$

which is certainly surjective because $A' \otimes -$ is always right exact. Since it's actually exact, the morphism $A' \otimes_A B' \to A' \otimes_A B$ is still an m-qiso, and $A' \otimes_A B'$ is a bounded above complex of free $A'[x_1, \ldots, x_n]$-modules since each term is the tensor product by A' of a free $A[x_1, \ldots, x_n]$-module.

Remark 4.3.20. The derived direct image $\mathrm{R}f_* : \mathrm{D}^+(\mathcal{O}_Y) \to \mathrm{D}^+(\mathcal{O}_X)$ also preserves perfect complexes, but we have to assume the morphism f to be perfect. More precisely, if $f : X \to Y$ is a perfect morphism and Y is locally noetherian, then any complex $E^\cdot$ in $\mathrm{Pf}(Y)$ is such that $\mathrm{R}f_*(E^\cdot)$ is a complex in $\mathrm{Pf}(X)$. See [TT90] 2.5.4.

The following will lead us to consider, for a sufficiently decent scheme X, the category of perfect complexes as a subcategory of the bounded derived category $\mathrm{D}^b(\mathcal{O}_X)$

Proposition 4.3.21. *Let X be a scheme which can be written as finite union of affine open subschemes (e.g. noetherian). Then, any perfect complex $F^\cdot$ on X is isomorphic in $\mathrm{D}(\mathcal{O}_X)$ to a bounded complex $E^\cdot$.*

Proof. Consider the affine cover $X = \bigcup_{i=1}^{n} U_i$ and assume moreover, thanks to Proposition 4.3.13, that these affine opens are such that for every i there exists a strict perfect complex $E_i^\cdot \cong F^\cdot|_{U_i}$ in $\mathrm{D}(\mathcal{O}_X)$. That means in particular $H^j(E_i) \cong H^j(F^\cdot|_U)$, so that one can set $k_i = \max\{j \in \mathbb{Z} | E_i^j \neq 0\}$ and $h_i = \min\{j \in \mathbb{Z} | E_i^j \neq 0\}$. Then, take $k = \max\{k_i\}_{i=1}^n$ and $h = \min\{h_i\}_{i=1}^n$, so that one has a cover $\{U_i\}_i$ of X such that for every integer i

$$H^j(F^\cdot|_{U_i}) \cong H^j(E_i) \cong 0$$

whenever $j \notin [h,k]$. Therefore, the sheaf $H^j(F^\cdot) \cong 0$ whenever $j \notin [h,k]$. Now, recall that truncations provide quasi-isomorphisms for cohomologically bounded complexes. Hence there are quasi-isomorphisms

$$\tau^h \tau^{\leq k} F^\cdot \longrightarrow \tau^{\leq k} F^\cdot \longrightarrow F^\cdot$$

which proves the theorem with $E^\cdot = \tau^h \tau^{\leq k} F^\cdot$ $\square$

The next step is to provide a description in terms of categorical properties of these objects when we look at them in the derived category $\mathrm{D}(\mathcal{O}_X)$. The full subcategory of $\mathrm{D}(\mathcal{O}_X)$ whose objects are perfect complexes is denoted by $\mathrm{Pf}(X)$.

Remark 4.3.22. Proposition 4.3.21 shows that up to equivalence $\mathrm{Pf}(X) \subseteq \mathrm{D}^b(X)$.

Theorem 4.3.23. $\mathrm{Pf}(X)$ *is a triangulated subcategory of* $\mathrm{D}(\mathcal{O}_X)$

Proof. To prove that $\mathrm{Pf}(X)$ is triangulated is equivalent, by Proposition 2.4.16, to prove that it is invariant under the shift functor and that the cone of a morphism of perfect complexes is again perfect. Being invariant under the shift functor is an obvious property of perfect complexes, since one can argue locally that strict perfect complexes are. In order to prove that the cone of a morphism of perfect complexes is again perfect, one consider a morphism $A^\cdot \to B^\cdot$ between perfect complexes, which is locally, on $U \subseteq X$, where ew can assume $A^\cdot|_U$ and $B^\cdot|_U$ to be strict perfect, represented by a pair o chain morphisms

$$A^\cdot|_U \longleftarrow G^\cdot \longrightarrow B^\cdot|_U$$

with $G^\cdot \to A^\cdot|_U$ qiso. Now apply to this morphism Lemma 4.2.7(c), in order to find a strict perfect complex of $\mathcal{O}_U$-modules $A'^\cdot$ and a quasi-isomorphism $A'^\cdot \to G^\cdot$. Since we're in $\mathrm{D}(\mathcal{O}_X)$, we can replace $A^\cdot|_U$ by the quasi-isomorphic $A'^\cdot$ and prove the theorem for the chain morphism of strict perfect complexes

$$A'^\cdot \longrightarrow G^\cdot \longrightarrow B^\cdot|_U$$

whose cone certainly is strict perfect and quasi-isomorphic to the (restriction at U) of the actual cone of $A^\cdot \to B^\cdot$, which is then perfect. $\square$

Another interesting tool applying to perfect complexes is the following isomorphism.

Proposition 4.3.24. *For a perfect complex $F^\cdot$ on X there is an isomorphism*

$$\mathrm{R}\mathcal{H}om(F^\cdot, \mathcal{O}_X) \otimes^{\mathrm{L}} F^\cdot \cong \mathrm{R}\mathcal{H}om(F^\cdot, F^\cdot)$$

Proof. Consider the basis formed by affine open subspaces $\{U_i\}$ and for every i, using Proposition 4.3.13 and Remark 4.2.14, take a complex of free $\mathcal{O}_{U_i}$-modules of finite type $E_i^\cdot$ with a qiso

$$E_i^\cdot \longrightarrow F^\cdot|_U$$

Being a free resolution, it's projective and flat, therefore by Remark 3.6.15 we can use it in order to compute, on an open U_i

$$\begin{aligned}(\mathrm{R}\mathcal{H}om(F^\cdot, \mathcal{O}_X) \otimes^{\mathrm{L}} F^\cdot)(U_i) &= (\mathcal{H}om^\cdot(E_i^\cdot, \mathcal{O}_X) \otimes^{\mathrm{L}} F^\cdot)(U_i) \\ &= \mathrm{Tot}^\cdot(\mathcal{H}om^\cdot(E_i^\cdot, \mathcal{O}_X) \otimes E_i^\cdot)(U_i)\end{aligned}$$

In each degree n, this complex of $\mathcal{O}_X(U_i)$-modules is

$$\bigoplus_{p+q=n} \mathcal{H}om^p(E_i^\cdot, \mathcal{O}_{U_i})(U_i) \otimes E_i^q(U_i)$$

that is, being $\mathcal{O}_{U_i}$ a complex centered in degree 0,

$$\bigoplus_{p+q=n} \mathrm{Hom}_{\mathcal{O}_{U_i}}(E_i^{-p}, \mathcal{O}_{U_i}) \otimes E_i^q(U_i)$$

Now recall the well known isomorphism for finite type free $\mathcal{O}_X$-modules A and B

$$\begin{aligned}\lambda : \mathrm{Hom}_{\mathcal{O}_X}(A, \mathcal{O}_X) \otimes B(X) &\cong \mathrm{Hom}_{\mathcal{O}_X}(A, B) \\ \phi \otimes s &\mapsto \tilde{\phi}\end{aligned}$$

defined by $\tilde{\phi}_U(t) = \phi_U(t)s|_U$ for every $U \subseteq X$. Surjectivity suffices to prove the isomorphism for (finite) dimensional reasons, and it's given taking basis $\{a_i\}$ and $\{b_i\}$ of the $\mathcal{O}_X(X)$-modules $A(X)$ and $B(X)$, and observing that the elements of the form $\lambda(a_i^* \otimes b_j)$, where ${a_i}_U^*(a_j|_U) = \delta_{ij} \in \mathcal{O}_X(U)$, give a basis for $\mathrm{Hom}_{\mathcal{O}_X}(A, B)$.

Therefore, what we found is isomorphic to

$$\bigoplus_{p+q=n} \mathrm{Hom}_{\mathcal{O}_{U_i}}(E_i^{-p}, E_i^q).$$

Now, the set of indexes is finite because $E^\cdot$ is bounded, hence by Proposition 2.1.6 it's isomorphic to

$$\prod_{p+q=n} \mathrm{Hom}_{\mathcal{O}_{U_i}}(E_i^{-p}, E_i^q) = \prod_{p+q=n} \mathcal{H}om_{\mathcal{O}_{U_i}}(E_i^{-p}, E_i^q)(U_i) = \mathcal{H}om^n(E_i^\cdot, E_i^\cdot)$$

that is the degree n module of $\mathrm{R}\mathcal{H}om(F^\cdot|_{U_i}, E_i^\cdot)(U_i) \cong \mathrm{R}\mathcal{H}om(F^\cdot, F^\cdot)(U_i)$. □

Recall now an important result relating geometric objects, here specifically finite dimensional vector bundles, with their algebraic counterpart, finitely generated projective modules, through the global section functor.

Theorem 4.3.25 (Serre-Swan duality)**.** *The category of finitely generated projective R-modules over a noetherian ring R is equivalent to the category of locally free $\mathcal{O}_X$-modules of finite rank over* $\operatorname{Spec} R$.

Proof. See [Ser55]. □

Proposition 4.3.26. *Let $X = \operatorname{Spec} R$ be an affine scheme over a noetherian ring R. Then the category $\mathrm{Pf}(X)$ is equivalent to the full subcategory of $\mathrm{D}(R) \simeq \mathrm{D}(\boldsymbol{Coh}(\mathcal{O}_X)) \subseteq \mathrm{D}(\mathcal{O}_X)$ consisting of bounded complexes of finitely generated projective R-modules.*

Proof. Since the scheme X has an ample family of line bundles, Proposition 4.3.13 tells us that $\mathrm{Pf}(X)$ is equivalent to the full subcategory of $\mathrm{D}(\mathcal{O}_X)$ whose objects are the strict perfect complexes, i.e. complexes of locally free $\mathcal{O}_X$-modules of finite rank. Serre-Swan duality then concludes the proof. □

Eventually, observe that we can deal with the derived tensor product on the category of perfect complexes.

Proposition 4.3.27. *The derived tensor product functor*

$$\otimes^{\mathrm{L}} : \mathrm{D}^-(\mathcal{O}_X) \times \mathrm{D}^-(\mathcal{O}_X) \longrightarrow \mathrm{D}^-(\mathcal{O}_X)$$

is a well defined functor on the subcategory $\mathrm{Pf}(X) \subseteq \mathrm{D}^b(\mathcal{O}_X)$. More precisely, if $E^\cdot$ and $F^\cdot$ are perfect complexes, the complex $E^\cdot \otimes^{\mathrm{L}} F^\cdot$ is again perfect

Proof. Fix $x \in X$ and open neighborhoods $V, W \subseteq X$ such that $E^\cdot|_V$ is quasi-isomorphic to a strict perfect complex $E'^\cdot$ on V, via $E'^\cdot \to E^\cdot|_V$, while $F^\cdot|_W$ is quasi-isomorphic to a strict perfect complex $F'^\cdot$ on W, via $F'^\cdot \to F^\cdot|_W$. Then, let's compute $E^\cdot \otimes F^\cdot$ by a bounded above resolution $K^\cdot \to F^\cdot$ of flat objects, so that

$$E^\cdot \otimes^{\mathrm{L}} F^\cdot = \mathrm{Tot}(E^\cdot \otimes K^\cdot).$$

Therefore, if we restrict $E^\cdot \otimes^{\mathrm{L}} F^\cdot$ to $U = V \cap W$, we get a chain of isomorphism in $\mathrm{D}(\mathcal{O}_X)$

$$E^\cdot \otimes^{\mathrm{L}} F^\cdot|_U = \mathrm{Tot}(E^\cdot \otimes K^\cdot)|_U \cong \mathrm{Tot}(E^\cdot|_U \otimes K^\cdot|_U) \cong$$
$$\mathrm{Tot}(E'^\cdot|_U \otimes K^\cdot|_U) \cong \mathrm{Tot}(E'^\cdot|_U \otimes F^\cdot|_U) \cong \mathrm{Tot}(E'^\cdot \otimes F'^\cdot|_U)$$

where Lemma 3.5.9 provides the quasi-isomorphism

$$\mathrm{Tot}(E'^\cdot|_U \otimes K^\cdot|_U) \longrightarrow \mathrm{Tot}(E^\cdot|_U \otimes K^\cdot|_U)$$

as well as the last two isomorphisms in $\mathrm{D}(\mathcal{O}_X)$, because locally free modules also are flat, since their stalks are free and flatness can be checked on stalks by Lemma 3.5.4. □

Chapter 5

Classification of $\otimes$-ideals

In this chapter we are going to study some triangulated subcategories of a triangulated category $\mathbf{T}$, and the objects that classifies them. At first it's described an important tool, the Grothendieck group associated to an essentially small triangulated category $\mathbf{T}$, encoding as subgroups a kind of strictly full triangulated subcategories called *dense*. Moreover, it will be introduced the crucial notion of *thick* subcategory of a triangulated category, and in the particular case of $\mathbf{T} = \mathrm{Pf}(X)$, for X a scheme with noetherian topological space, we are going to prove a correspondence between those thick subcategories acting as "ideals" for the derived tensor product and the set of those subsets of X with the topological property of being union of closed subspaces. This result will be crucial for the reconstruction of X from the category $\mathrm{Pf}(X)$.

5.1 Grothendieck groups

Definition 5.1.1. The *Grothendieck group* of an essentially small triangulated category $\mathbf{T}$ with shift functor T is the quotient group $K_0(\mathbf{T})$ of the free abelian group on the set of isomorphism classes $is(A)$ of objects A in $\mathbf{T}$ by the equivalence relation generated by

$$R = \{(is(B), is(A) + is(C)),\ \text{for } A \to B \to C \to TA \text{ triangle in } \mathbf{T}\}.$$

Remark 5.1.2. In the following we denote by $[A]$ the R-class of $is(A)$. In other words, the Grothendieck group is constructed by imposing the so called Euler relations $[B] = [A] + [C]$ whenever $A \to B \to C \to TA$ is a distinguished triangle.

Let's stress once and for all the importance of being essentially small, and hence the possibility to replace $\mathbf{T}$ with a small equivalent category. From now on, essential smallness is tacitly assumed whenever we consider Grothendieck groups.

Proposition 5.1.3. *In the Grothendieck group of a triangulated category $K_0(\boldsymbol{T})$ it holds $[A]+[B]=[A\oplus B]$ for every pair of objects A and B in $\boldsymbol{T}$.*

Proof. Consider the zero map $B \to TA$ and complete it to a distinguished triangle $B \xrightarrow{0} TA \to E \to TB$. Now apply axiom **T2** (i.e. rotate the triangle) twice in order to get a distinguished triangle

$$A \to T^{-1}E \to B \xrightarrow{0} TA$$

which, by Lemma 2.4.14, is isomorphic to $A \to A\oplus B \to B \to TA$, which is then a distinguished triangle by axiom **T1**, proving the desired equality. $\square$

Corollary 5.1.4. *In the Grothendieck group $K_0(\boldsymbol{T})$ of a triangulated category $\boldsymbol{T}$ it holds*

(i) $[0]=0$

(ii) $[TA]=-[A]$

Proof. Part (i) follows from Proposition 5.1.3 for $A\oplus 0\cong A$, that is $[A]+[0]=[A]$, hence $[0]=0\in K_0(\mathbf{T})$. Part (ii) comes from the distinguished triangle $A \xrightarrow{\text{id}} A \to 0 \to TA$, that rotated is $A\to 0\to TA\to TA$, giving by definition of $K_0(\mathbf{T})$ that $[A]+[TA]=[0]=0$. $\square$

Remark 5.1.5. The construction of K_0 gives a functor $\mathbf{TrCat}\to\mathbf{Ab}$ from the category of (small) triangulated categories and triangulated functors to the category of abelian groups. To any triangulated functor $F:\mathbf{S}\to\mathbf{T}$ we associate

$$K_0(F): K_0(\mathbf{S})\ni [A]\longmapsto [FA]\in K_0(\mathbf{T}).$$

Observe that K_0 is a group morphism since any additive functor preserves direct sums. Hence

$$\begin{aligned} K_0F([A]+[B]) &= K_0F([A\oplus B]) = [F(A\oplus B)] \\ &= [F(A)\oplus F(B)] = [F(A)]+[F(B)] = K_0F[A]+K_0F[B]. \end{aligned}$$

It's obvious that K_0 preserves identities and compositions.

The Grothendieck group of a triangulated category $\mathbf{T}$ enjoy the following universal property.

Proposition 5.1.6. *Let $\boldsymbol{T}$ be a triangulated category, G an abelian group, and $\delta: Ob(\boldsymbol{T})\to G$ a monoid morphism which is additive, i.e. if $A\to B\to C\to TA$ is a triangle in $\boldsymbol{T}$, then $\delta(B)=\delta(A)+\delta(C)$. Then, there exists a unique morphism of abelian groups $\hat{\delta}: K_0(\boldsymbol{T})\to G$ such that the diagram*

$$\begin{tikzcd} & K_0(\boldsymbol{T}) \arrow[d, dashed, "\exists!"] \\ Ob(\boldsymbol{T}) \arrow[ur] \arrow[r, "\delta"'] & G \end{tikzcd}$$

commutes. Here, the map $Ob(\boldsymbol{T}) \to K_0(\boldsymbol{T})$ is $D \mapsto [D]$, which is a commutative monoid morphism by Proposition 5.1.3.

Proof. The proof is quite trivial. The map $\hat{\delta}$ is obviously defined just as $\hat{\delta}([D]) = \delta(D)$ for D in $\mathbf{T}$ and extended by linearity. This gives uniqueness once proved that the map is well defined. First of all, $\hat{\delta}$ is defined on the isomorphism classes of objects: in fact, if $is(A) = is(B)$, consider the triangle $A \overset{\cong}{\to} B \to 0 \to TA$ which is isomorphic to $A \to A \to 0 \to TA$ and hence, as the latter, distinguished. In order to see that $\hat{\delta}$ is well defined on $K_0(\mathbf{T})$ consider two elements isomorphic in K_0, which are given by a triangle $A \to B \to C \to TA$ as elements $[B]$ and $[A]+[C]$. Using additivity of δ and linearity of $\hat{\delta}$ one has

$$\hat{\delta}([B]) = \delta(B) = \delta(A) + \delta(C) = \hat{\delta}([A]) + \hat{\delta}([C]) = \hat{\delta}([A]+[C]).$$

$\square$

The Grothendieck group in this pretty much general context will be useful in order to work with subcategories of the triangulated category.

Definition 5.1.7. A (strictly full) triangulated subcategory $\mathbf{A} \subseteq \mathbf{T}$ of an abelian category $\mathbf{T}$ is called *dense* if any object in $\mathbf{T}$ is a direct summand of an object in $\mathbf{A}$.

Theorem 5.1.8. *Let $\boldsymbol{T}$ be a triangulated category. There is a bijective correspondence between dense triangulated subcategories $\boldsymbol{A} \subseteq \boldsymbol{T}$ and subgroups H of the abelian group $K_0(\boldsymbol{T})$.*

Proof. By the fact that K_0 is a functor, for a dense triangulated subcategory $i : \mathbf{A} \subseteq \mathbf{T}$ there is a group morphism $K_0(i) : K_0(\mathbf{A}) \to K_0(\mathbf{T})$, giving the subgroup $\mathrm{Im}(K_0(\mathbf{A}) \to K_0(\mathbf{T}))$, which is just $K_0(\mathbf{A})$ because $K_0(i) : [A] \mapsto [iA] = [A]$ is clearly a monomorphism. Conversely, to a subgroup $H \subseteq K_0(\mathbf{T})$ one can associate the subcategory $\mathbf{A}_H$ of those elements D in $\mathbf{T}$ such that $[A] \in H$.

On one hand, it's cleat that $H = K_0(\mathbf{A}_H)$. In fact, $[A] \in K_0(\mathbf{A}_H)$ if and only if A is an object in $\mathbf{A}_H$, that is $[A] \in H$.

On the other hand, for a fixed subcategory $\mathbf{A} \subseteq \mathbf{T}$, we need to show $\mathbf{A}_{K_0(\mathbf{A})} = \mathbf{A}$. Let's restate that in the form of the following claim: for every object D in $\mathbf{T}$

$$D \text{ is object in } \mathbf{A} \iff [D] \in K_0(\mathbf{A}) \subseteq K_0(\mathbf{T}). \tag{5.1}$$

Equivalently, we can show that D is object in $\mathbf{A}$ if and only if $[D] = 0 \in K_0(\mathbf{T})/K_0(\mathbf{A})$.
Consider, thanks to essential smallness, the relation $\sim$ on the set of isomorphism classes of objects in $\mathbf{T}$ given by

$$[D] \sim [D'] \iff \exists A, A' \text{ in } \mathbf{A} \text{ s.t. } D \oplus A \cong D' \oplus A'.$$

It's obviously an equivalence relation. Let G be the quotient of the set of isomorphism classes by this relation, and denote $\langle D \rangle$ the $\sim$-class of $[D]$. Observe that an element D is in $\mathbf{A}$ if and only if $\langle D \rangle = \langle 0 \rangle$ in G. That is because if D is in $\mathbf{A}$, then for any A in $\mathbf{A}$ we can call $A' = D \oplus A$, and this shows that $\langle D \rangle = \langle 0 \rangle$. Conversely if $\langle D \rangle = \langle 0 \rangle$ there are by definition A and A' in $\mathbf{A}$ such that $D \oplus A \cong 0 \oplus A \cong A$, hence $D \oplus A$ is in $\mathbf{A}$, and the triangle

$$A \to D \oplus A \to D \to TA$$

tells us that D is in $\mathbf{A}$, since we can complete here the morphism $A \to D \oplus A$ to a triangle and the resulting object is in $\mathbf{A}$ and isomorphic to D.
Now, in order to see that (5.1) holds, it suffices to prove that there is a group isomorphism $K_0(\mathbf{T})/K_0(\mathbf{A}) \cong G$. So that will be D in $\mathbf{A}$ if and only if $\langle D \rangle = \langle 0 \rangle$ in G if and only if $[D] \in K_0(\mathbf{A})$.
We are going to prove that there is a group morphism

$$\begin{aligned} \pi : K_0(\mathbf{T}) &\xrightarrow{\cong} G \\ [D] &\longmapsto \langle D \rangle \end{aligned}$$

and that its kernel is $K_0(A) \subseteq K_0(\mathbf{T})$. First of all, observe that G is not just a commutative monoid with direct sum as operation $\langle D \rangle + \langle D' \rangle := \langle D \oplus D' \rangle$, but in fact an abelian group: by density of $\mathbf{A}$, given D in $\mathbf{T}$ and so $\langle D \rangle$ in G, we find another summand D' such that $D \oplus D'$ is in $\mathbf{A}$, but that means $\langle D \oplus D' \rangle = \langle 0 \rangle$. So $\langle D' \rangle$ is the desired inverse. Eventually one need to prove that π is a group morphism, and this is true in fact because π is the morphism given by the universal property in Proposition 5.1.6 once we prove that the map $D \mapsto \langle D \rangle$ is additive. Let $A \to B \to C \to TA$ be a triangle in $\mathbf{T}$, since $\mathbf{A}$ is dense, there are A' and C' in $\mathbf{T}$ such that $A \oplus A'$ and $C \oplus C'$ are in $\mathbf{A}$. Hence it holds $\langle A \oplus A' \rangle = \langle C \oplus C' \rangle = \langle 0 \rangle$, that is $\langle A' \rangle = -\langle A \rangle$ and $\langle C' \rangle = -\langle C \rangle$ in G. Now, observe that besides $A \to B \to C \to TA$ there are distinguished triangles

$$\begin{gathered} A' \to A' \to 0 \to TA' \\ 0 \to C' \to C' \to 0 \end{gathered}$$

and the direct sum of all of these three triangles gives, by Proposition 2.4.13, another distinguished triangle

$$A \oplus A \to B \oplus A' \oplus C' \to C \oplus C' \to T(A \oplus A')$$

with the two vertexes $A \oplus A'$ and $C \oplus C'$ sitting in $\mathbf{A}$, and hence so is $B \oplus A' \oplus C'$ because we can complete the triangle in $\mathbf{A}$, which is supposed to have all the objects isomorphic to some of its objects. Again, as showed before, this is equivalent to be 0 in G, that is

$$\langle 0 \rangle = \langle B \oplus A' \oplus C' \rangle = \langle B \rangle - \langle A \rangle - \langle C \rangle$$

This proves additivity, so that the map π is actually a group homomorphism by the universal property. Clearly π is surjective since is a quotient map. Eventually, since any element in $K_0(\mathbf{T})$ admits a representative in $\mathbf{T}$ such that it's of the form $[D]$, we can describe $\mathrm{Ker}(\pi)$ as the subgroup $\{[D] \in K_0(\mathbf{T}), \langle D \rangle = \langle 0 \rangle\}$, that is exactly the subgroup of those $[D] \in K_0(\mathbf{T})$ for which D is in $\mathbf{A}$, i.e. coming from $K_0(\mathbf{A})$. This concludes that $K_0(\mathbf{T})/K_0(\mathbf{A}) \cong G$. □

5.2 Triangulated subcategories of $\mathrm{Pf}(X)$

The following general definition indicates a first condition for subcategories of the category of perfect complexes. Subsequently, we are going to use the derived tensor structure of $\mathrm{Pf}(X)$ in order to refine the set of subcategories of our interest. These further notions will be naturally generalized to general tensor triangulated categories.

Definition 5.2.1. A full triangulated subcategory $\mathbf{A}$ of a triangulated category $\mathbf{T}$ is called *thick* if whenever X is an object in $\mathbf{A}$ splitting as $X \cong Y \oplus Z$, then both Y and Z are in $\mathbf{A}$.

Example 5.2.2. A first example is the whole category $\mathrm{Pf}(X) \subseteq \mathrm{D}(\mathcal{O}_X)$. Let $E^{\cdot} \oplus F^{\cdot}$ be a perfect complex. Then we know it from Theorem 4.3.12 to be pseudocoherent and of locally finite Tor-dimension. The fact that both $E^{\cdot}$ and $F^{\cdot}$ has locally finite Tor-dimension is clear since both the cohomology and the derived tensor product functors preserve direct sums.

If suffices then to prove that both $E^{\cdot}$ and $F^{\cdot}$ are pseudocoherent. Observe that there are distinguished triangles

$$E^{\cdot} \longrightarrow E^{\cdot} \longrightarrow 0 \longrightarrow E^{\cdot}[1]$$

and

$$F^{\cdot} \longrightarrow F^{\cdot} \longrightarrow F^{\cdot} \oplus F^{\cdot}[1] \longrightarrow F^{\cdot}[1]$$

Therefore, their direct sum

$$E^{\cdot} \oplus F^{\cdot} \longrightarrow E^{\cdot} \oplus F^{\cdot} \longrightarrow F^{\cdot} \oplus F^{\cdot}[1] \longrightarrow (E^{\cdot} \oplus F^{\cdot})[1]$$

is a distinguished triangle, and being the first two pseudocoherent by assumption, so is $F^{\cdot} \oplus F^{\cdot}[1]$ by Proposition 4.2.11. Surely, we have that all the complexes $F[n] \oplus F[n+1]$ are pseudocoherent.

For any fixed integer m, since $F^\cdot$ is (up to quasi-isomorphism) bounded above, one can take an $n > 0$ big enough so that $F[n]$ is $m+1$-pseudocoherent. Therefore, consider the triangle

$$F^\cdot[n] \longrightarrow F^\cdot[n] \oplus F^\cdot[n-1] \longrightarrow F^\cdot[n-1] \longrightarrow F^\cdot[n+1]$$

which gives, again by Proposition 4.2.11, that also $F^\cdot[n-1]$ is m-pseudocoherent. Since m was arbitrary, $F^\cdot[n-1]$ is pseudocoherent, and this iterates up to show that $F^\cdot$ is pseudocoherent. The same certainly holds true for $E^\cdot$.

Let's introduce an important notion describing some thick subcategories of a triangulated category. It will be crucial the notion of derived tensor product $\otimes^{\mathrm{L}}$ introduced in §3.5, defined on the derived category and well defined on perfect complexes by Proposition 4.3.27. From now on when we refer to complexes in $\mathrm{D}(\mathcal{O}_X)$, by the symbol $\otimes$ we mean the derived tensor product.

Definition 5.2.3. A full triangulated subcategory $\mathbf{A} \subseteq \mathrm{Pf}(X)$ is a $\otimes$-*ideal* ("tensor-ideal") if it is thick and such that whenever $F^\cdot$ is any object in $\mathbf{A}$ and $E^\cdot$ any object in $\mathrm{Pf}(X)$, then $F^\cdot \otimes E^\cdot$ is in $\mathbf{A}$.

Remark 5.2.4. The intersection of $\otimes$-ideals is again a $\otimes$-ideal. In particular, there exists the smallest $\otimes$-ideal containing a fixed object in $\mathbf{T}$.

Definition 5.2.5. A subspace $Y \subseteq X$ of a topological space X is said to be *specialization closed* if it is union of closed subspaces.

Remark 5.2.6. A subspace is specialization closed if and only if it is union of the closure of its points, that is $Y = \bigcup_{y \in Y} \overline{\{y\}}$. Clearly if Y is union of the closure of its points it is specialization closed. Conversely, if $Y = \bigcup_\alpha Y_\alpha$ is specialization closed and $y \in Y$, then find a closed Y_α such that $y \in Y_\alpha$ and it is $\overline{\{y\}} \subseteq Y_\alpha \subseteq Y$. Thus $\bigcup_{y \in Y} \overline{\{y\}} \subseteq Y$, and the reverse inclusion certainly holds true.

The main theorem of this section provides a correspondence between $\otimes$-ideal in $\mathrm{Pf}(X)$ and specialization closed subspaces of a noetherian scheme X. The constructions which will provide this correspondence are described in the following.

Definition 5.2.7. Let $E^\cdot$ be a perfect complex over a scheme X. The *cohomological support* of $E^\cdot$ is the subset of X given by

$$\mathrm{Supph}(E^\cdot) = \{x \in X | E^\cdot_x \text{ is not acyclic}\}$$

Remark 5.2.8. It holds clearly $\mathrm{Supph}(E^\cdot) = \bigcup_{k \in \mathbb{Z}} \mathrm{Supp}\, H^k(E^\cdot)$, where for a sheaf of modules F over X, its support $\mathrm{Supp}(F) \subseteq X$ denotes as usual the subset $\{x \in X | F_x \neq 0\}$.

A more convincing description of the cohomological support $\mathrm{Supph}(E^\cdot)$ may be as the subset of those $x \in X$ such that $E^\cdot_x \ncong 0$ in $\mathrm{D}(\mathcal{O}_{X,x})$.

Two of the main properties of the cohomological support, which will be axioms in a more general context (see §6.3) are the following.

Proposition 5.2.9. *Let X be a scheme and $E^\cdot$, $F^\cdot$ be complexes of finite type modules in $\mathrm{D}^-(\mathcal{O}_X)$ (e.g. perfect). Then it holds*

(a) $\mathrm{Supph}(E^\cdot \oplus F^\cdot) = \mathrm{Supph}(E^\cdot) \cup \mathrm{Supph}(F^\cdot)$

(b) $\mathrm{Supph}(E^\cdot \otimes F^\cdot) = \mathrm{Supph}(E^\cdot) \cap \mathrm{Supph}(F^\cdot)$*.*

Proof. Part (a) is easy. Thanks to the fact that colimits commutes with colimits, we get that $x \in \mathrm{Supph}(E^\cdot \oplus F^\cdot)$ if and only if there exists an integer k with

$$H^k((E^\cdot \oplus F^\cdot)_x) = H^k(E^\cdot_x \oplus F^\cdot_x) = H^k(E^\cdot_x) \oplus H^k(F^\cdot_x) \neq 0$$

which is true if and only if either one of the two summand is non-zero, that is $x \in \mathrm{Supph}(E^\cdot) \cup \mathrm{Supph}(F^\cdot)$.

Part (b) is a bit trickier, because dealing with cohomology of (derived) tensor products requires more advanced tools. On one hand it's clear that if $(E^\cdot \otimes F^\cdot)_x \neq 0$ in $\mathrm{D}(\mathcal{O}_{X,x})$, then the $\mathcal{O}_{X,x}$-module $E^\cdot_x \otimes_{\mathcal{O}_{X,x}} F^\cdot_x$ is non-zero, and so are both $E^\cdot_x$ and $F^\cdot_x$. This proves

$$\mathrm{Supph}(E^\cdot \otimes F^\cdot) \subseteq \mathrm{Supph}(E^\cdot) \cap \mathrm{Supph}(F^\cdot)$$

In order to prove the reverse inclusion, recall that for complexes of modules $K^\cdot$, $N^\cdot$, there is a spectral sequence

$$E_2^{p,q} = H^p(H^q(K^\cdot) \otimes^{\mathrm{L}} N^\cdot) \Longrightarrow H^{p+q}(K^\cdot \otimes^{\mathrm{L}} N^\cdot)$$

Let then $x \in \mathrm{Supph}(E^\cdot) \cap \mathrm{Supph}(F^\cdot)$, namely $E^\cdot_x$ and $F^\cdot_x$ are non-zero in $\mathrm{D}(\mathcal{O}_{X.x})$. Find then the maximal integers k_1 and k_2 such that $H^{k_1}(E^\cdot_x) \neq 0$ and $H^{k_2}(F^\cdot_x) \neq 0$.

The idea in order to prove $x \in \mathrm{Supph}(E^\cdot \otimes F^\cdot)$ is to prove

$$H^{k_1+k_2}((E^\cdot \otimes_{\mathcal{O}_X} F^\cdot)_x) = H^{k_1+k_2}(E^\cdot_x \otimes_{\mathcal{O}_{X,x}} F^\cdot_x) \neq 0,$$

and do so proving that the corresponding term in the second page of the spectral sequence $H^{k_1}(H^{k_2}(F^\cdot_x) \otimes_{\mathcal{O}_{X,x}} E^\cdot_x)$ is non-zero and is a vertex, i.e. all the terms in place (p, q) are zero whenever $p > k_1$ or $q > k_2$. Therefore, any subsequent page will keep the same non-zero term in place (k_1, k_2).

It helps a well know result from commutative algebra (see [AM71]) asserting that the tensor product of two finitely generated non-zero modules over a local ring is itself non-zero. Therefore, if we consider the derived tensor product

$$H^{k_2}(F^\cdot_x) \otimes_{\mathcal{O}_{X,x}} E^\cdot_x$$

it can be computed taking a bounded above resolution of flat $\mathcal{O}_{X,x}$ modules $M^\cdot \to E^\cdot_x$ as the complex having degree n given by

$$\bigoplus_{p+q=n} (H^{k_2}(F^\cdot_x))^p \otimes M^q = H^{k_2}(F^\cdot_x) \otimes M^{k_1}.$$

Observe that since the resolution is isomorphism on cohomology, the degree k_1 of $M^\cdot$ is non-zero, hence the n-th term just computed is non-zero because $H^{k_2}(F^\cdot)$ also is.

Now, in order to compute its cohomology H^{k_1}, observe that the differential morphisms of the total complex behave just like the differentials of $M^\cdot$, that is because $H^{k_2}(F^\cdot_x)$ is centered at degree 0. More precisely, the chain complex

$$H^{k_2}(F^\cdot_x) \otimes M^{k_1-1} \xrightarrow{\partial^{k_1-1}} H^{k_2}(F^\cdot_x) \otimes M^{k_1} \xrightarrow{\partial^{k_1}} H^{k_2}(F^\cdot_x) \otimes M^{k_1+1}$$

has $\partial = \mathrm{id} \otimes d_M$, up to a sign. Therefore,

$$H^{k_1}(H^{k_2}(F^\cdot_x) \otimes^{\mathrm{L}} E^\cdot_x) = H^{k_1}(H^{k_2}(F^\cdot_x) \otimes M^\cdot) = H^{k_1}(M^\cdot) = H^{k_1}(E^\cdot_x) \neq 0$$

The fact that $H^{k_1}(H^{k_2}(F^\cdot_x) \otimes E^\cdot_x)$ is a vertex is clear from the choice of k_1 and k_2. If $q > k_2$, is $H^q(F^\cdot_x)$ to be zero and hence certainly is $H^p(H^q(F^\cdot_x) \otimes^{\mathrm{L}} E^\cdot_x)$, while if $p > k_1$ the computation above gives

$$H^p(H^q(F^\cdot_x) \otimes^{\mathrm{L}} E^\cdot_x) = H^p(M^\cdot) = H^p(E^\cdot_x) = 0.$$

□

A useful property characterizing the cohomological support of a perfect complex is the following.

Proposition 5.2.10. *Let X be a noetherian scheme and $E^\cdot$ a perfect complex on X. For any $x \in X$ denote by $k(x)$ the residue field at x. Then $x \notin \mathrm{Supph}(E^\cdot)$ if and only if $E^\cdot_x \otimes^{\mathrm{L}}_{\mathcal{O}_{X,x}} k(x)$ (sometimes abbreviated as $E^\cdot \otimes k(x)$) is acyclic.*

Proof. Suppose $x \notin \mathrm{Supph}(E^\cdot)$, that is $E^\cdot_x$ acyclic, or equivalently $E^\cdot_x = 0$ in $\mathrm{D}(\mathcal{O}_{X,x})$. Therefore $E^\cdot_x \otimes_{\mathcal{O}_{X,x}} k(x)$ is certainly zero in $\mathrm{D}(\mathcal{O}_{X,x})$, namely acyclic, because the derived tensor product is an additive functor.

Conversely, suppose $E^\cdot_x \otimes_{\mathcal{O}_{X,x}} k(x) = 0$ in $\mathrm{D}(\mathcal{O}_{X,x})$. Observe that the skyscraper sheaf $\mathrm{Sk}_x(k(x))$ over x with value $k(x)$, is such that

$$(E^\cdot \otimes^{\mathrm{L}}_{\mathcal{O}_X} \mathrm{Sk}_x(k(x)))_x \cong E^\cdot_x \otimes^{\mathrm{L}}_{\mathcal{O}_{X,x}} k(x).$$

That is because everything works as in the non-derived case once we consider a flat resolution of $\mathrm{Sk}_x(k(x))$.

Therefore, $x \notin \operatorname{Supph}(E^\cdot \otimes_{\mathcal{O}_X} \operatorname{Sk}_x(k(x)))$ which is, by Proposition 5.2.9,

$$x \notin \operatorname{Supph}(E^\cdot) \cap \operatorname{Supph}(\operatorname{Sk}_x(k(x))).$$

Eventually, $\operatorname{Supph}(\operatorname{Sk}_x(k(x)))$ is the whole scheme, because the residue field is never zero, hence this proves $x \notin \operatorname{Supph}(E^\cdot)$. $\square$

Recall the following basic but crucial fact.

Proposition 5.2.11. *Let X be a scheme and F a coherent $\mathcal{O}_X$-module. Then* $\operatorname{Supp}(F)$ *is a closed subspace of X.*

Proof. Consider $x \notin \operatorname{Supp} F$ and by coherence an affine open neighborhood $V = \operatorname{Spec} R \ni x$ over which $F|_V \cong \tilde{M}$ for a finitely generated R-module M. Let x correspond to a prime ideal $p \subseteq R$, and consider a finite family $\{m_1, \dots m_n\}$ of generators for M. The stalk $\tilde{M}_x = M_p$ is zero, hence for every i the element $\frac{m_i}{1} \in M_p$ is zero, that means by definition of localization that there exists $f_i \in R \setminus p$ and an integer k_i such that $f_i^{k_i} m_i = 0$ in M. By taking k to be the maximum in $\{k_i\}$ and $f = (f_1 \cdots f_n)^k$, we've found that the open set $\{q \in \operatorname{Spec} A | f \notin q\} = D(f) \subseteq \operatorname{Spec} R$ is an open neighborhood of x, since $f_i \notin p$ for every i and so is f, and whenever y correspond to a prime ideal $q \in D(f)$ the stalk $\tilde{M}_y = M_q$ is zero, because $f^k m_i = 0$ with $f \notin q$. Therefore, $D(f)$ is the desired open neighborhood of x contained in $(\operatorname{Supp} F)^{\mathsf{c}}$. $\square$

Proposition 5.2.12. *Let $E^\cdot$ be a perfect complex on a noetherian scheme X, then* $\operatorname{Supph}(E^\cdot)$ *is a closed subspace.*

Proof. Consider locally a quasi-isomorphism $F^\cdot \to E^\cdot|_V$ with $F^\cdot$ strictly perfect, and since X is noetherian, find a finite cover of affine open subspaces $\{V_i\}$ and quasi-isomorphisms $F_i^\cdot \to E^\cdot|_{V_i}$. Thus we can compute the cohomological support of $E^\cdot$ as

$$\operatorname{Supph}(E^\cdot) = \bigcup_i \operatorname{Supph}(E^\cdot|_{V_i}) = \bigcup_k \bigcup_i \operatorname{Supp} H^k(E^\cdot|_{V_i}) = \bigcup_k \bigcup_i \operatorname{Supp} H^k F_i^\cdot$$

Observe than that in the last term both unions are finite, since supports of $H^k F^\cdot$ are empty when k is out of the finite range where $F^\cdot$ is bounded, so one has to prove $\operatorname{Supp} H^k F_i^\cdot$ to be closed. That is true because F^k's are free, hence coherent as well as their cohomology since $\mathbf{Coh}(X)$ is abelian, and eventually the support of coherent sheaves is closed by Proposition 5.2.11. $\square$

Definition 5.2.13. Let Y be any subspace of a scheme X. Denote by $\operatorname{Pf}_Y(X) \subseteq \operatorname{Pf}(X)$ the full subcategory whose objects $E^\cdot$ have $\operatorname{Supph}(E^\cdot) \subseteq Y$.

Lemma 5.2.14. *If $A^\cdot \to B^\cdot \to C^\cdot \to TA^\cdot$ is a distinguished triangle in* $\mathrm{D}(\mathcal{O}_X)$*, then*

$$\mathrm{Supph}(A^\cdot) \subseteq \mathrm{Supph}(B^\cdot) \cup \mathrm{Supph}(C^\cdot).$$

Proof. The distinguished triangle $A^\cdot \xrightarrow{f} B^\cdot \to C^\cdot(f) \to TA^\cdot$ gives a long exact cohomology sequence

$$H^{k-1}(B^\cdot) \longrightarrow H^{k-1}(C^\cdot) \longrightarrow H^k(A^\cdot) \longrightarrow H^k(B^\cdot) \longrightarrow H^k(C^\cdot)$$

Taking stalks, functor that certainly commutes with cohomology, gives the exact sequence

$$H^{k-1}(B^\cdot_x) \longrightarrow H^{k-1}(C^\cdot_x) \longrightarrow H^k(A^\cdot_x) \longrightarrow H^k(B^\cdot_x) \longrightarrow H^k(C^\cdot_x)$$

Therefore, if $x \in \mathrm{Supph}(A^\cdot)$, it exists a k such that $H^k(A^\cdot_x) \neq 0$, and hence at least one of the two $H^{k-1}(C^\cdot_x)$ and $H^k(B^\cdot_x)$ has to be non-zero, and therefore $x \in \mathrm{Supph}(B^\cdot) \cup \mathrm{Supph}(C^\cdot)$. □

Proposition 5.2.15. *Let Y be any subspace of a scheme X, then* $\mathrm{Pf}_Y(X) \subseteq \mathrm{Pf}(X)$ *is a $\otimes$-ideal.*

Proof. The subcategory $\mathrm{Pf}_Y(X)$ is triangulated because it's invariant under the shift functor and the mapping cone of two complexes supported in Y is again supported in Y. In fact, Lemma 5.2.14, gives that if in a triangle $A^\cdot \to B^\cdot \to C^\cdot \to TA^\cdot$ two objects, let's say $B^\cdot$ and $C^\cdot$ have cohomological support contained in Y, so does $A^\cdot$. The other cases are given by rotation of the triangle.

Thickness is given by part (a) of Proposition 5.2.9, since whenever $E^\cdot \cong F^\cdot \oplus G^\cdot$, has cohomological support contained in Y, then

$$\mathrm{Supph}(F^\cdot) \cup \mathrm{Supph}(G^\cdot) \subseteq Y$$

and hence both $F^\cdot$ and $G^\cdot$ are in $\mathrm{Pf}_Y(X)$.

In order to see that it's also a $\otimes$-ideal, we use part (b) of Proposition 5.2.9, which gives that if $E^\cdot$ is in $\mathrm{Pf}_Y(X)$ and $F^\cdot$ any other perfect complex, then

$$\mathrm{Supph}(E^\cdot \otimes F^\cdot) = \mathrm{Supph}(E^\cdot) \cap \mathrm{Supph}(F^\cdot) \subseteq Y.$$

□

The following results, whose proofs can be found in the huge paper by Thomason and Trobaugh, show how the Grothendieck group functor K_0 provides a criteria for the extension of a perfect complex defined on an compact open subset to the whole scheme, under suitable geometric assumptions.

Proposition 5.2.16. *Let X be a quasi-compact, quasi-separated scheme and $j : U \subseteq X$ a compact open subspace. For any two perfect complexes $E^{\cdot}$, $E'^{\cdot}$ on X and any morphism $b : j^*E^{\cdot} \to j^*E'^{\cdot}$ in $\mathrm{D}(\mathcal{O}_U)$, there exists a third perfect complex $E''^{\cdot}$ on X and morphisms $a : E''^{\cdot} \to E^{\cdot}$, $a' : E''^{\cdot} \to E'^{\cdot}$ in $\mathrm{D}(\mathcal{O}_X)$ such that j^*a is isomorphism in $\mathrm{D}(\mathcal{O}_U)$ and the diagram*

$$\begin{tikzcd} & j^*E''^{\cdot} \arrow[dl, "j^*a"'] \arrow[dr, "j^*a'"] & \\ j^*E^{\cdot} \arrow[rr, "b"] & & j^*E'^{\cdot} \end{tikzcd}$$

commutes in $\mathrm{D}(\mathcal{O}_U)$.

Proof. See [TT90] Proposition 5.2.3. □

Proposition 5.2.17. *Let X be a quasi-compact quasi-separated scheme and $j : U \subseteq X$ a compact open subspace. Let $E^{\cdot}$, $E'^{\cdot}$ be two perfect complexes on X, and suppose*

$$E^{\cdot} \underset{b}{\overset{a}{\rightrightarrows}} E'^{\cdot}$$

to be two morphisms in $\mathrm{D}(\mathcal{O}_X)$ such that they agree on U, that is $j^*a = j^*b$ in $\mathrm{D}(\mathcal{O}_U)$. Then, there exists a third perfect complex $E''^{\cdot}$ on X and a morphism

$$c : E''^{\cdot} \longrightarrow E^{\cdot}$$

in $\mathrm{D}(\mathcal{O}_X)$ equalizing a and b and such that j^*c is an isomorphism in $\mathrm{D}(\mathcal{O}_U)$.

Proof. See [TT90] Proposition 5.2.4. □

Proposition 5.2.18 (K_0 extension lemma)**.** *Let X be a quasi-compact quasi-separated scheme, $Y \subseteq X$ a closed subspace with quasi-compact complement and $U \subseteq X$ a quasi-compact open subscheme. Let $E^{\cdot}$ be a perfect complex in $\mathrm{Pf}_{Y\cap U}(U)$. Then, there exists in X a perfect complex $F^{\cdot}$ in $\mathrm{Pf}_Y(X)$ such that $F^{\cdot}|_U$ is quasi-isomorphic to $E^{\cdot}$ if and only if the class of $E^{\cdot}$ in $K_0(\mathrm{Pf}_{Y\cap U}(U))$ is in the image of the restriction homomorphism*

$$K_0(\mathrm{Pf}_Y(X)) \longrightarrow K_0(\mathrm{Pf}_{Y\cap U}(U)).$$

Proof. See [TT90] Proposition 5.2.2. □

Remark 5.2.19. In the following we will assume our scheme X to be noetherian, and not just quasi-compact and quasi-separated. Therefore, the results stated above, will apply on any open subspace of the scheme, since any open of a noetherian scheme is quasi-compact.

For the case of a noetherian scheme, let's observe that in the previous result it makes sense to consider Grothendieck groups of the category of perfect complexes and its subcategories.

Theorem 5.2.20. *Let X be a noetherian scheme. Then the category* $\mathrm{Pf}(X)$ *is essentially small.*

Proof. Let's assume $X = \mathrm{Spec}(R)$ to be affine. In this case we have proved in Proposition 4.3.26 that $\mathrm{Pf}(X)$ is equivalent to the full subcategory of $\mathrm{D}(R)$ whose objects are bounded complexes of finitely generated projective R-modules. In order to prove this category to be essentially small, it clearly suffices to prove that the subcategory $\mathbf{P} \subseteq \mathbf{Mod}(R)$ whose objects are finitely generated projective R-modules is essentially small. Once proved that, in fact, we will have a set of objects such that any finitely generated projective R-module is isomorphic to a module from this set, hence taking countably many copies of these modules with all the possible chain structure (recall that morphisms between two fixed modules surely form a set) will provide the desired set of complexes.

The fact that the $\mathbf{P}$ is essentially small follows from the fact that the whole subcategory of finitely generated R-modules is essentially small. Consider any finitely generated module M: by definition there is a surjection

$$\bigoplus_k R \longrightarrow M$$

with $k < \aleph_0$, showing that the cardinality of M is $|M| \leq |R|$ (in the case $|R| \geq \aleph_0$, otherwise one just have $|M| < \aleph_0$). Up to isomorphism, one can certainly take M to be hereditarily of cardinality less or equal to $|R|$: all of its elements in fact, can be chosen to be for example the ordinals up to $|R|$, because this doesn't affect at all the module structure. With this choice one can consider the subset

$$\{M | M \text{ has structure of } R\text{-module}\} \subseteq |R|^+$$

where $|R|^+$ denotes the successor cardinal (actually, it suffices to take the successor ordinal) of $|R|$. Therefore, we have proved that every finitely generated projective R-modules is isomorphic to a module that may be picked up from a fixed set.

Then let us assume X to be a separated noetherian scheme. For this case it suffices to prove that if $\mathrm{Pf}(U_1)$, $\mathrm{Pf}(U_2)$ and $\mathrm{Pf}(U_1 \cap U_2)$ are essentially small, so is $\mathrm{Pf}(U_1 \cup U_2)$.

In fact, by our further assumption and A.0.6 we can start the induction. More precisely, consider $X = \bigcup_i U_i$ be the finite union of its affine open subspaces, and let U_1, U_2 be affine subschemes. So the intersection $U_1 \cap U_2$ will also be affine, and by the previous part of the proof the categories of perfect complexes on these subschemes are essentially small. Suppose we have proved $\mathrm{Pf}(U_1 \cup U_2)$ to be essentially small, too. The next step is to consider another affine subspace U_3 and prove $\mathrm{Pf}(U_1 \cup U_2 \cup U_3)$ to be essentially small. That will be true because we will have $\mathrm{Pf}(U_1 \cup U_2)$, $\mathrm{Pf}(U_3)$

and $\mathrm{Pf}((U_1 \cup U_2) \cap U_3)$ essentially small, the first one for what will be proven, as well as the last one, since $(U_1 \cup U_2) \cap U_3 = (U_1 \cap U_3) \cup (U_2 \cap U_3)$ is union of two affine subschemes.

So, consider subspaces $U_1, U_2 \subseteq X$ and the commutative diagram with canonical inclusions

$$\begin{tikzcd} U_1 \cap U_2 \arrow[r, hook, "k'"] \arrow[d, hook, "j'"'] & U_1 \arrow[d, hook, "j"] \\ U_2 \arrow[r, hook, "k"'] & U_1 \cup U_2 \end{tikzcd}$$

and call $h = jk' = kj'$. Let S_1, S_2, S_{12} be sets in $\mathrm{Pf}(U_1), \mathrm{Pf}(U_2), \mathrm{Pf}(U_3)$ respectively, witnessing essential smallness. That is, any perfect complex is isomorphic to one of the respective set. Let $E^{\cdot}$ be a perfect complex on $U_1 \cup U_2$. Observe that thanks to the fact that open immersions are flat morphisms, i.e. their inverse image functors are exact, hence we can compute the derived inverse image pointwise as the non-derived functor. Therefore, there are perfect complexes $F_1^{\cdot}$, $F_2^{\cdot}$, $F_{12}^{\cdot}$ respectively in S_1, S_2, S_{12} and isomorphisms

$$\begin{aligned} \phi_1 &: j^* E^{\cdot} \longrightarrow F_1^{\cdot} \\ \phi_2 &: k^* E^{\cdot} \longrightarrow F_2^{\cdot} \\ \phi_{12} &: h^* E^{\cdot} \longrightarrow F_{12}^{\cdot} \end{aligned}$$

By the adjunction $f^* \dashv f_*$ for functors between the categories $\mathbf{Mod}(\mathcal{O}_X)$ and $\mathbf{Mod}(\mathcal{O}_Y)$ for any morphism of schemes $f : X \to Y$, we get also adjoint isomorphisms with the pointwise direct images

$$\begin{aligned} \tilde{\phi_1} &: E^{\cdot} \longrightarrow j_* F_1^{\cdot} \\ \tilde{\phi_2} &: E^{\cdot} \longrightarrow k_* F_2^{\cdot} \\ \tilde{\phi_{12}} &: E^{\cdot} \longrightarrow h_* F_{12}^{\cdot} \end{aligned}$$

Consider then morphisms $\tilde{\phi_{12}} \tilde{\phi_1}^{-1} : j_* F_1^{\cdot} \to h_* F_{12}^{\cdot}$ and $\tilde{\phi}_{12} \tilde{\phi_2}^{-1} : k_* F_2^{\cdot} \to h_* F_{12}^{\cdot}$. They define a morphisms

$$j_* F_1^{\cdot} \oplus k_* F_2^{\cdot} \to h_* F_{12}^{\cdot}$$

which we can complete to a triangle in $\mathrm{Pf}(X)$ given by

$$G^{\cdot} \longrightarrow j_* F_1^{\cdot} \oplus k_* F_2^{\cdot} \longrightarrow h_* F_{12}^{\cdot} \longrightarrow TG^{\cdot}$$

Now, by Mayer-Vietoris Theorem, there is a short exact sequence of complexes

$$0 \longrightarrow E^{\cdot} \longrightarrow j_* j^* E \oplus k_* k^* E^{\cdot} \longrightarrow h_* h^* E^{\cdot} \longrightarrow 0$$

inducing by Proposition 3.3.4 a distinguished triangle in the derived category

$$E^{\cdot} \longrightarrow j_*j^*E \oplus k_*k^*E^{\cdot} \longrightarrow h_*h^*E^{\cdot} \longrightarrow TE^{\cdot}$$

Observe then that there are isomorphism $j_*\phi_1 \oplus k_*\phi_2$ and $h_*\phi_{12}$ making the following diagram to commute, and hence there exists a third isomorphism $E^{\cdot} \to G^{\cdot}$ completing the morphism of triangles

$$\begin{tikzcd}
E^{\cdot} \arrow[r] \arrow[d, dashed] & j_*j^*E^{\cdot} \oplus k_*k^*E^{\cdot} \arrow[r] \arrow[d] & h_*h^*E^{\cdot} \arrow[r] \arrow[d] & TE^{\cdot} \arrow[d, dashed] \\
G^{\cdot} \arrow[r] & j_*F_1^{\cdot} \oplus k_*F_2^{\cdot} \arrow[r] & h_*F_{12}^{\cdot} \arrow[r] & TG^{\cdot}
\end{tikzcd}$$

This proves that a priori we could have fixed a set S of objects consisting of perfect complexes representing the isomorphism class of those complexes completing a triangle given by a morphism from a direct sum of direct images of complexes belonging to the sets S_1 and S_2 respectively, into the direct image of a complex belonging to the set S_{12}. What just proved is that any perfect complex on $U_1 \cup U_2$ is isomorphic to a perfect complex in this set, hence $\mathrm{Pf}(U_1 \cup U_2)$ is essentially small.

Eventually, let us drop the hypothesis of X being separated. Let us cover $X = U_1 \cup \dots \cup U_n$ by separated noetherian (e.g. affine) schemes. The proof is by induction on n, and the base case of $X = U_1$ is given by the proof under the assumption of X being separated. Now assume $X = U_1 \cup \dots U_n$, and $\mathrm{Pf}(U_1 \cup \dots \cup U_{n-1})$ to be essentially small, as well as $\mathrm{Pf}(U_n)$. We observe that by inductive hypothesis, the same is true for the category of perfect complexes on

$$(U_1 \cup \dots \cup U_{n-1}) \cap U_n = (U_1 \cap U_n) \cup \dots \cup (U_{n-1} \cap U_n)$$

since this also is union of $n-1$ separated subschemes. Therefore from what argued above we deduce that $\mathrm{Pf}(X)$ is essentially small. $\square$

Lemma 5.2.21. *Let X be a noetherian scheme and $Y \subseteq X$ a closed subspace. Then there exists a perfect complex $E^{\cdot}$ on X such that* $\mathrm{Supph}(E^{\cdot}) = Y$.

Proof. Since X is a noetherian schemes, it is a noetherian topological space, and so is its closed subscheme Y. That means Y has finitely many irreducible components $Y_1, \dots Y_k$ and each of them by Proposition A.0.7 has a unique generic point y_i, with $i \in \{1, \dots k\}$. If we prove that there exist perfect complexes $E_i^{\cdot}$ with $\mathrm{Supph}(E_i^{\cdot}) = \overline{\{y_i\}}$, then the complex $E^{\cdot} = \bigoplus_{i=1}^{k} E_i^{\cdot}$ has support the union $\bigcup \overline{\{y_i\}} = Y$.

So assume $Y = \overline{\{y\}}$ to be a single irreducible component. Let $U = \mathrm{Spec}(A) \subseteq X$ an affine open neighborhood of y. The subscheme $Y \cap U \subseteq U$ is irreducible, so it correspond to a prime ideal of A, which has a finite number of generator $\{f_1, ..., f_n\}$ since A is noetherian, and any affine subspace of a noetherian scheme is the spectrum of a noetherian ring. Consider the complexes given in degrees -1 and 0 by the multiplication $f_i : A \to A$, and let

$$K^\cdot = \bigotimes_{i=1}^{n} (\cdots \longrightarrow 0 \longrightarrow A \xrightarrow{f_i} A \longrightarrow 0 \longrightarrow \cdots)$$

be their derived tensor product.

Claim: $\mathrm{Supph}(K^\cdot) = Y \cap U$. On one hand,

$$\mathrm{Supph}(K^\cdot) = \bigcup_{i=1}^{n} \mathrm{Supp}\, H^i(K^\cdot) \supseteq \mathrm{Supp}\, H^0(K^\cdot).$$

Now the tensor product of these complexes is computed just as chain complexes, since they are free and hence flat. If $C_1 = (A \xrightarrow{f_1} A)$ and $C_2 = (A \xrightarrow{f_2} A)$, the tensor product $C_1 \otimes C_2$ is given as the complex having $(C_1 \otimes C_2)^k = \bigoplus_{i+j=k} C_1^i \otimes C_2^j$, and morphisms $\partial^k = d_{C_1} \otimes \mathrm{id} + (-1)^k \mathrm{id} \otimes d_{C_2}$, that is

$$A \longrightarrow A \oplus A \longrightarrow A$$

with morphisms $a \mapsto (f_1 a, -f_2 a)$ and $(a_1, a_2) \mapsto f_2 a_1 + f_1 a_2$. In general, so, we argue that $K^\cdot$ in degree 0 will be

$$\bigoplus_n A \longrightarrow A \longrightarrow 0$$

mapping $(a_1, \dots a_n) \mapsto f_1 a_1 + ... + f_n a_n$, so that $H^0(K^\cdot) = A/(f_1, \dots, f_n)$. Observe now that for any ideal $q \subseteq A$ one has $\mathrm{Supp}(A/q) = \overline{\{q\}} \subseteq \mathrm{Spec}(A)$, that is because for any ideal p one has $p \notin \overline{\{q\}} \iff p \not\supseteq q \iff \exists r \in q \backslash p \iff \forall a \in (A/q)_p$ it holds $a = \frac{1}{r} ra = 0$, since $r \in q$.

Hence

$$\mathrm{Supph}(K^\cdot) \supseteq \mathrm{Supp}(H^0 K^\cdot) = \mathrm{Supp}(A/(f_1, \dots, f_n)) = \overline{\{(f_1, \dots, f_n)\}} \cap \mathrm{Spec}\, A = Y \cap U.$$

One the other hand, observe that there is a resolution, and hence a quasi-isomorphism

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & R \arrow[r, "f_i"] \arrow[d] & R \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & R/(f_i) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

which implies that $K^{\cdot} = R/(f_1)\otimes^{\mathrm{L}}\cdots\otimes^{\mathrm{L}}R/(f_n)$, whose support is $\mathrm{Supph}(K^{\cdot}) = \bigcap_{i=1}^{n}\mathrm{Supph}(R/(f_i)) = \bigcap_{i=1}^{n}\overline{\{(f_i)\}} = \mathrm{Supp}(A/(f_1,\ldots,f_n)) = Y\cap U$.

Consider then the shift of the complex $K^{\cdot}$, namely $K^{\cdot}[1]$, and observe that both has the same cohomological support, as well as their sum $K^{\cdot}\oplus K^{\cdot}[1]$, whose cohomological support is the union of the two. Now in the group $K_0(\mathrm{Pf}_{Y\cap U}(U))$, one certainly has from Proposition 5.1.3 and Corollary 5.1.4 that $[K^{\cdot}\oplus K^{\cdot}[1]] = 0$ and so it is in the image of the restriction $K_0(\mathrm{Pf}_Y(X)) \to K_0(\mathrm{Pf}_{Y\cap U}(U))$. Hence, by Proposition 5.2.18, there is a perfect complex $E^{\cdot}$ on X acyclic off Y and such that $E^{\cdot}|_U$ is quasi-isomorphic to $K^{\cdot}\oplus K^{\cdot}[1]$. That means on one hand that $\mathrm{Supph}\,E^{\cdot}\subseteq Y$, and on the other hand $\mathrm{Supph}\,E^{\cdot}|_U = \mathrm{Supph}(K^{\cdot}\oplus K^{\cdot}[1]) = Y\cap U \ni y$, so that $y\in\mathrm{Supph}\,E^{\cdot}$ which is closed by Lemma 5.2.12, and hence $Y = \overline{\{y\}}\subseteq\mathrm{Supph}\,E^{\cdot}$. Thus $\mathrm{Supph}\,E^{\cdot} = Y$. □

The following computation will be needed in the subsequent Lemma.

Proposition 5.2.22. *Let $A \xrightarrow{f} B \longrightarrow C \longrightarrow TA$ be a distinguished triangle in $\mathrm{D}^-(\mathcal{O}_X)$ and D any object in $\mathrm{D}^-(\mathcal{O}_X)$. Then, the triangle*

$$D\otimes^{\mathrm{L}} A \xrightarrow{D\otimes^{\mathrm{L}} f} D\otimes^{\mathrm{L}} B \longrightarrow D\otimes^{\mathrm{L}} C \longrightarrow T(D\otimes^{\mathrm{L}} A)$$

is distinguished.

Proof. It clearly suffices to prove that $D\otimes^{\mathrm{L}}C(f)$ is isomorphic to $C(D\otimes^{\mathrm{L}} f)$. Start considering two flat resolutions by bounded above complexes of flat modules $K'^{\cdot}\to A^{\cdot}$ and $K''^{\cdot}\to B^{\cdot}$, and compute the i-th degree of $C(D\otimes^{\mathrm{L}} f)$ as

$$C(D\otimes^{\mathrm{L}} f)^i = (D\otimes^{\mathrm{L}} A)^{i+1}\oplus(D\otimes^{\mathrm{L}} B)^i = \bigoplus_{p+q=i+1} D^p\otimes K'^q \oplus \bigoplus_{p+q=i} D^p\otimes K''^q$$

Now, observe that the complex $K = K'^{\cdot}[1]\oplus K''^{\cdot}$ define, by diagonal mapping of the previous resolutions of $A^{\cdot}$ and $B^{\cdot}$, a bounded above resolution of flat objects for the complex $C(f)$. Therefore, we can compute

$$(D\otimes^{\mathrm{L}} C(f))^i = \bigoplus_{p+q=i} D^p\otimes K^q = \bigoplus_{p+q=i} D^p\otimes(K'^{q+1}\oplus K''^q) =$$
$$\bigoplus_{p+q=i}(D^p\otimes K'^{q+1})\oplus(D^p\otimes K''^{q+1}) = \bigoplus_{p+q=i+1} D^p\otimes K'^q \oplus \bigoplus_{p+q=i} D^p\otimes K''^q$$

So, they are the same, as well as the morphisms of these two complexes which are $\begin{pmatrix}-\mathrm{id}_D\otimes d_A & 0\\ -\mathrm{id}_D\otimes f & \mathrm{id}_D\otimes d_B\end{pmatrix} = \mathrm{id}_D\otimes\begin{pmatrix}-d_A & 0\\ -f & d_B\end{pmatrix}$. □

Let's now state a result known as Tensor Nilpotence Theorem, which will be needed in Lemma 5.2.24.

Theorem 5.2.23 (Tensor Nilpotence)**.** *Let X be a noetherian scheme, $E^\cdot$ and $G^\cdot$ perfect complexes on X, and $F^\cdot$ a complex with quasi-coherent cohomology. Let $f : E^\cdot \to F^\cdot$ and suppose for all $x \in \mathrm{Supph}(G^\cdot)$ that, denoting by $k(x)$ the residue field at x, one has $f \otimes k(x) = 0$ in $\mathrm{D}(k(x))$. Then there is a natural n such that $G^\cdot \otimes (\otimes^n f) : G^\cdot \otimes (\otimes^n E^\cdot) \to G^\cdot \otimes (\otimes^n F^\cdot)$ is zero in $\mathrm{D}(\mathcal{O}_X)$.*

Proof. See [Tho97] Theorem 3.8. $\square$

Lemma 5.2.24. *Let X be a noetherian scheme, and suppose $E^\cdot$, $F^\cdot$ to be perfect complexes in $\mathrm{Pf}(X)$ such that $\mathrm{Supph}(E^\cdot) \subseteq \mathrm{Supp}(F^\cdot)$. Then $E^\cdot$ is an object of the smallest $\otimes$-ideal of $\mathrm{Pf}(X)$ containing $F^\cdot$.*

Proof. Let us call $\mathbf{A} \subseteq \mathrm{Pf}(X)$ the smallest $\otimes$-ideal containing $F^\cdot$. Note that for a morphism $a : G^\cdot \to \mathcal{O}_X$ in $\mathrm{Pf}(X)$ we can set $C(a)$ to be the cone of a and get the triangle

$$G^\cdot \xrightarrow{a} \mathcal{O}_X \longrightarrow C(a) \longrightarrow TG^\cdot$$

The same way, for any natural $n \geq 1$, the map $\otimes^n a : \bigotimes_n G^\cdot \to \mathcal{O}_X$ gives us a triangle

$$\bigotimes_n G^\cdot \xrightarrow{\otimes^n a} \mathcal{O}_X \longrightarrow C(\otimes^n a) \longrightarrow T\bigotimes_n G^\cdot$$

By Proposition 5.2.22 there is a third distinguished triangle

$$G^\cdot \otimes (\bigotimes_n G^\cdot) \xrightarrow{\mathrm{id}_{G^\cdot} \otimes (\otimes^n a)} G^\cdot \longrightarrow G^\cdot \otimes C(\otimes^n a) \longrightarrow T(G^\cdot \otimes (\bigotimes_n G^\cdot))$$

Moreover, under the identification $\otimes^{n+1} a = a \circ \mathrm{id}_G \otimes (\otimes^n a)$ we are in the hypothesis of axiom **T4**, so that these three triangles in the three different colors gives a diagram of the form

$$\begin{tikzcd}
G^\cdot \otimes (\bigotimes_n G^\cdot) \arrow[r, "\otimes^{n+1} a"] \arrow[d, "\mathrm{id}_{G^\cdot} \otimes (\otimes^n a)"'] & \mathcal{O}_X \arrow[r] \arrow[d] & C(a) \arrow[r, dashed] \arrow[d] & T(G^\cdot \otimes C(\otimes^n a)) \\
G^\cdot \arrow[ur, "a"'] \arrow[d] & C(\otimes^{n+1} a) \arrow[ur, dashed] \arrow[d] & TG^\cdot \arrow[ur] & \\
G^\cdot \otimes C(\otimes^n a) \arrow[r] \arrow[ur, dashed] & T(G^\cdot \otimes (\bigotimes_n G^\cdot)) \arrow[ur] & &
\end{tikzcd}$$

and thus there exists a further distinguished triangle

$$G^{\cdot} \otimes C(\otimes^{n} a) \longrightarrow C(\otimes^{n+1} a) \longrightarrow C(a) \longrightarrow T(G^{\cdot} \otimes C(\otimes^{n} a))$$

Now, observe that for every natural $n \geq 1$ it holds

$$E^{\cdot} \otimes C(a) \text{ is in } \mathbf{A} \Longrightarrow E^{\cdot} \otimes C(\otimes^{n} a) \text{ also is in } \mathbf{A}.$$

This is easily proved by induction taking the tensor product (Proposition 5.2.22) by $E^{\cdot}$ with the new triangle obtained by axiom **T4**. The base step is in fact trivial, while if $E^{\cdot} \otimes C(a)$ and $E^{\cdot} \otimes C(\otimes^{n} a)$ are in $\mathbf{A}$, so is $E^{\cdot} \otimes G^{\cdot} \otimes C(\otimes^{n} a)$ being $\mathbf{A}$ a tensor ideal, hence two sides of a distinguished triangle in a triangulated subcategory are in the triangulated subcategory $\mathbf{A}$, and by Proposition 2.4.16 so is the third one, which is $E \otimes C(\otimes^{n+1} a)$.

Moreover the isomorphism of Proposition 4.3.24, for $F^{\cdot}$ perfect,

$$\mathrm{R}\mathcal{H}om(F^{\cdot}, \mathcal{O}_X) \otimes F^{\cdot} \cong \mathrm{R}\mathcal{H}om(F^{\cdot}, F^{\cdot})$$

tells us, since $F^{\cdot}$ is in the $\otimes$-ideal $\mathbf{A}$, that $\mathrm{R}\mathcal{H}om(F^{\cdot}, F^{\cdot})$ is in $\mathbf{A}$. Thus, consider $\mathrm{id}_{F^{\cdot}} \in \mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(F^{\cdot}, F^{\cdot})$ and its correspondent morphism, under the isomorphism

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(F^{\cdot}, F^{\cdot}) \cong \mathrm{Hom}_{\mathrm{D}(\mathcal{O}_X)}(\mathcal{O}_X, \mathrm{R}\mathcal{H}om(F^{\cdot}, F^{\cdot}))$$

given by the adjunction $- \otimes^{\mathrm{L}} F^{\cdot} \dashv \mathrm{R}\mathcal{H}om(F^{\cdot}, -)$,

$$f : \mathcal{O}_X \to \mathrm{R}\mathcal{H}om(F^{\cdot}, F^{\cdot}).$$

Let then $a : G^{\cdot} \to \mathcal{O}_X$ be its completion to a triangle, so that there is a triangle

$$G^{\cdot} \xrightarrow{a} \mathcal{O}_X \xrightarrow{f} \mathrm{R}\mathcal{H}om(F^{\cdot}, F^{\cdot}) \longrightarrow TG^{\cdot}$$

with $\mathrm{R}\mathcal{H}om(F^{\cdot}, F^{\cdot}) \cong C(a)$ an object in $\mathbf{A}$, as well as $E^{\cdot} \otimes C(a)$ and, for what proved above by induction, $E^{\cdot} \otimes C(\otimes^{n} a)$. What we are going to prove is that $E^{\cdot}$ is a direct summand of $E^{\cdot} \otimes C(\otimes^{n} a)$ for some n, and being the former in the $\otimes$-ideal $\mathbf{A}$, which is in particular thick, $E^{\cdot}$ will also be in $\mathbf{A}$.

Claim: There exists a natural $n \geq 1$ such that $E^{\cdot} \otimes (\otimes^{n} a)$ is the zero map in $\mathrm{Pf}(X)$. If we prove this last claim, for this particular n we have the distinguished triangle

$$E^{\cdot} \otimes (\bigotimes_{n} G^{\cdot}) \xrightarrow{0} E^{\cdot} \longrightarrow E \otimes C(\otimes^{n} a) \longrightarrow T(E^{\cdot} \otimes (\bigotimes_{n} G^{\cdot}))$$

whose shift

$$E^{\cdot} \longrightarrow E^{\cdot} \otimes C(\otimes^{n} a) \longrightarrow T(E^{\cdot} \otimes (\bigotimes_{n} G^{\cdot})) \xrightarrow{0} TE$$

shows by Lemma 2.4.14 that $E^{\cdot} \otimes C(\otimes^n a) \cong E^{\cdot} \oplus T(E^{\cdot} \otimes (\bigotimes_n G^{\cdot}))$, and eventually that $E^{\cdot}$ is in $\mathbf{A}$.

It remains then to prove the claim, and by Tensor Nilpotence Theorem 5.2.23 it suffices to prove that $\forall x \in \mathrm{Supph}(E^{\cdot})$ it holds $a \otimes k(x) = 0$ in $\mathrm{D}(k(x))$. Now we use the hypothesis, so far untouched, that $\mathrm{Supph}(E^{\cdot}) \subseteq \mathrm{Supph}(F^{\cdot})$. Thanks to Proposition 5.2.10 we have that $F^{\cdot} \otimes k(x) \ncong 0$ in $\mathrm{D}(k(x))$. Thus, under the derived $\otimes$-Hom adjunction (Theorem 3.6.16 and subsequent Remark) between the functors $F^{\cdot} \otimes -$ and $\mathrm{R}\mathcal{H}om(F, -)$

$$\mathrm{Hom}(F^{\cdot} \otimes k(x), F^{\cdot} \otimes k(x)) \cong \mathrm{Hom}(k(x), \mathrm{R}\mathcal{H}om(F^{\cdot}, F^{\cdot} \otimes k))$$

$\mathrm{id}_{F^{\cdot} \otimes k(x)}$ corresponds to a non-zero morphism $k(x) \to \mathrm{R}\mathcal{H}om(F^{\cdot}, F^{\cdot} \otimes k)$. As a non-zero monomorphism between chain complexes of vector spaces, it splits, and so does once composed with the isomorphism (for perfect complexes, Proposition 4.3.24)

$$\mathrm{R}\mathcal{H}om(F^{\cdot}, F^{\cdot} \otimes k(x)) \cong \mathrm{R}\mathcal{H}om(F^{\cdot}, F^{\cdot}) \otimes k(x)$$

which gives clearly $f \otimes k(x) : k(x) \to \mathrm{R}\mathcal{H}om(F^{\cdot}, F^{\cdot}) \otimes k(x)$.

Eventually, being $f \otimes k(x)$ split mono in the distinguished triangle

$$G \otimes k(x) \overset{a \otimes k(x)}{\longrightarrow} k(x) \overset{f \otimes k(x)}{\longrightarrow} \mathrm{R}\mathcal{H}om(F^{\cdot}, F^{\cdot}) \otimes k(x) \longrightarrow T(G \otimes k(x))$$

implies $a \otimes k(x) = 0$ by Proposition 2.4.11. $\square$

Theorem 5.2.25. *Suppose X to be a noetherian scheme. Let $\mathcal{T} = \{\boldsymbol{A} \subseteq \mathrm{Pf}(X) | \boldsymbol{A}$ is a $\otimes$-ideal$\}$ and $\mathcal{S} = \{Y \subseteq X | Y$ is specialization closed$\}$. There is a bijection between $\mathcal{T}$ and $\mathcal{S}$ provided by*

$$\begin{aligned} \mathcal{T} &\longleftrightarrow \mathcal{S} \\ \boldsymbol{A} &\mapsto \bigcup_{E^{\cdot} \in \mathrm{Ob}(\boldsymbol{A})} \mathrm{Supph}(E^{\cdot}) \\ \mathrm{Pf}_Y(X) &\leftarrow\!\shortmid Y \end{aligned}$$

Moreover, this mapping is also inclusion-preserving.

Proof. Let's give a name to these mappings $\phi : \mathcal{T} \to \mathcal{S}$ and $\psi : \mathcal{S} \to \mathcal{T}$. The fact that they are well-defined follows respectively from Propositions 5.2.12 and 5.2.15. Let's remark again the fact that $\mathbf{A}$ is a subcategory of a small category, so it has a set of objects $\mathrm{Ob}(\mathbf{A})$.

Now, on one hand it's clear that

$$\phi\psi(Y) = \bigcup_{E^{\cdot} \in \mathrm{Ob}(\mathrm{Pf}_Y(X))} \mathrm{Supph}(E^{\cdot}) \subseteq Y$$

since is union of subsets contained in Y by definition of $\mathrm{Pf}(Y)$. The reverse inclusion is given considering $x \in Y = \bigcup\limits_{\alpha} Y_\alpha$, with the Y_α's closed, so that there exists α with $x \in Y_\alpha$, and Lemma 5.2.21 provide a complex $E^\cdot_\alpha$ on X such that $\mathrm{Supph}(E^\cdot_\alpha) = Y_\alpha \subseteq Y$. Then $E^\cdot_\alpha \in \mathrm{Ob}(Pf_Y(X))$ and $x \in Y_\alpha = \mathrm{Supph}(E^\cdot_\alpha) \subseteq \phi\psi(Y)$

For the reverse composition, is clear that

$$\psi\phi(\mathbf{A}) = \mathrm{Pf}_{\bigcup \mathrm{Supph}(E^\cdot)}(X) \supseteq \mathbf{A},$$

since any object $E^\cdot$ in $\mathbf{A}$ has support contained in $\bigcup\limits_{E^\cdot \in \mathrm{Ob}(\mathbf{A})} \mathrm{Supph}(E^\cdot)$. Conversely, let $G^\cdot$ be any object in $\psi\phi(\mathbf{A})$, that means

$$\mathrm{Supph}(G^\cdot) \subseteq \bigcup_{E^\cdot \in \mathrm{Ob}(\mathbf{A})} \mathrm{Supph}(E^\cdot).$$

By Lemma A.0.8 let us find a finite set J indexing $\{E^\cdot_i\}_{i \in J} \subseteq \mathrm{Ob}(\mathbf{A})$ such that

$$\mathrm{Supph}(G^\cdot) \subseteq \bigcup_{i \in J} \mathrm{Supph}(E^\cdot_i) = \mathrm{Supph}(\bigoplus_{i \in J} E^\cdot_i).$$

Now, by Lemma 5.2.24, $G^\cdot$ is in the smallest $\otimes$-ideal containing $\bigoplus\limits_{i \in J} E^\cdot_i$, and since each of the $E^\cdot_i$ for $i \in J$ is in $\mathbf{A}$, so is their direct sum (any triangulated subcategory is closed under taking finite direct sum by additivity. Thus the smallest $\otimes$-ideal containing $\bigoplus\limits_{i \in J} E^\cdot_i$ is in particular contained in $\mathbf{A}$, hence $G^\cdot$ is in $\mathbf{A}$ as desired.

□

Chapter 6

The spectrum construction

In general, even for very well behaved schemes such as abelian varieties, the derived category of the category of sheaves of modules is not an invariant. A well known example is the one provided by Shigeru Mukai (see [Muk81]), where it's proved that for an abelian variety X and its dual $\hat{X}$ subsists an equivalence of categories $\mathrm{D}(\mathcal{O}_X) \simeq \mathrm{D}(\mathcal{O}_{\hat{X}})$. The equivalence is provided considering the product variety $X \times \hat{X}$ with its projection morphisms

$$\begin{tikzcd} & X \times \hat{X} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & \hat{X} \end{tikzcd}$$

Then, a particular line bundle $\mathcal{P}$ over $X \times \hat{X}$, provides an equivalence of categories

$$\begin{aligned} \Phi : \mathrm{D}(\mathcal{O}_X) &\longrightarrow \mathrm{D}(\mathcal{O}_{\hat{X}}) \\ E^{\cdot} &\longmapsto \mathrm{R}q_*(\mathrm{L}p^* E^{\cdot} \otimes^{\mathrm{L}} \mathcal{P}) \end{aligned}$$

We know by Example 4.3.19 that the projection q is a perfect morphism, hence we deduce from Proposition 4.3.16, Remark 4.3.20 and Proposition 4.3.27, that this equivalence passes to the full subcategories of perfect complexes.

The aim of the following chapter is to provide the structure needed to the category $\mathrm{Pf}(X)$ in order to characterize X. It happens to be sufficient to consider its derived tensor product structure, and this gives rise to the general theory of *tensor triangulated categories.*

6.1 Symmetric monoidal categories

A category is monoidal when is equipped with the structure of a "product" operation, satisfying certain associativity and existence of neutral element. Let's proceed by steps.

Definition 6.1.1. A *strict monoidal category* $(\mathbf{B}, \otimes, 1)$ is a category $\mathbf{B}$ together with a *tensor* functor $- \otimes - : B \times B \to B$ which is associative, that is one has identity of functors

$$\otimes(\otimes \times \mathrm{id}_{\mathbf{B}}) = \otimes(\mathrm{id}_B \times \otimes) : B \times B \times B \longrightarrow B$$

and such that 1 is neutral element for $\otimes$, that is

$$\otimes(1 \times \mathrm{id}_{\mathbf{B}}) = \mathrm{id}_{\mathbf{B}} = \otimes(\mathrm{id}_{\mathbf{B}} \times 1),$$

where clearly $1 \times \mathrm{id}_{\mathbf{B}} : B \mapsto B \times B$ is the functor $X \mapsto (1, X)$, and mapping a morphism $f : X \mapsto Y$ to $\mathrm{id}_1 \times f$

In a less cryptic way, the functor $\otimes$ maps each pair of objects (X, Y) to an object $X \otimes Y$, and we just required for any triple of objects (X, Y, Z) to holds $(X \otimes Y) \otimes Z = (X \otimes Y) \otimes Z$, as well as $1 \otimes X = X = X \otimes 1$.

Example 6.1.2. A monoid $(M, *)$, regarded a set of objects given by its elements and without non-identical morphisms, is a strict monoidal category with tensor structure given by $(a, b) \mapsto a * b$. Properties of strict monoidal category are the very definition of monoid.

Example 6.1.3. For a fixed category $\mathbf{X}$, the category $\mathrm{End}(\mathbf{X})$ whose objects are the functors $\mathbf{X} \to \mathbf{X}$ and whose morphisms are the natural transformations between them is a strict monoidal category. That is just because composition of natural transformations is associative and has a unit.

However, the category of sets endowed with its structure of product $\otimes = \times$ is not strictly monoidal, since given sets X, Y and Z, the sets $(A \times B) \times C$ and $A \times (B \times C)$ are not the same set. They are isomorphic in a natural way though.

Definition 6.1.4. A *monoidal category* $(\mathbf{B}, \otimes, 1, \alpha, \lambda, \rho)$ is a category $\mathbf{B}$ together with a functor $B \times B \to B$, an object 1 and natural isomorphism

$$\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$$

called *associator*, making the following pentagonal diagram to commute:

$$\begin{tikzcd}
X \otimes (Y \otimes (Z \otimes W)) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}\otimes\alpha"] & (X \otimes Y) \otimes (Z \otimes W) \arrow[r, "\alpha"] & ((X \otimes Y) \otimes Z) \otimes W \\
X \otimes ((Y \otimes Z) \otimes W) \arrow[rr, "\alpha"] & & (X \otimes (Y \otimes Z)) \otimes W \arrow[u, "\alpha\otimes\mathrm{id}"']
\end{tikzcd}$$

and left and right *unitors*

$$\lambda_X : 1 \otimes X \cong X \text{ and } \rho_X : X \otimes 1 \cong X$$

such that $\lambda_1 = \rho_1 : 1 \otimes 1 \to 1$ and making

$$
\begin{tikzcd}
X \otimes (1 \otimes Y) \arrow[rr, "\alpha"] \arrow[dr, "\mathrm{id}\otimes\lambda_Y"'] & & (X \otimes 1) \otimes Y \arrow[dl, "\rho_X\otimes\mathrm{id}"] \\
& X \otimes Y &
\end{tikzcd}
$$

to commute.

Example 6.1.5. Now, **Set** is certainly a monoidal category with product giving the tensor structure and the singleton terminal object giving the unit.

Remark 6.1.6. More in general, in any category with finite products the product gives a monoidal structure with unit the terminal object. Dually, any category with finite coproduct has a tensor structure given by the coproduct. The category of sets has both structures.

Example 6.1.7. The usual tensor product of modules over a fixed commutative ring R endows the category $\mathbf{Mod}(R)$ of the structure of a monoidal category. Defined through its universal property, in fact, the tensor product of two R-modules $M\otimes_R N$ comes with a bilinear morphism $M\times N \to M\otimes_R N$ mapping $(m,n) \mapsto m\otimes n$ which is universal for any other bilinear map from $M \times N$ into another R-module.

Thus, for modules N, M and L, the isomorphism $\alpha : M \otimes (N \otimes L) \to (M \otimes N) \otimes L$ is given by

$$
\begin{tikzcd}
M \times (N \otimes L) \arrow[r, "\otimes"] \arrow[dr, "f"'] & M \otimes (N \otimes L) \arrow[d, dashed, "\alpha"] \\
& (M \otimes N) \otimes L
\end{tikzcd}
$$

where $f : (m, n \otimes l) \mapsto (m \otimes n) \otimes l$, which is bilinear because the inner tensor product is. Its inverse is given by the same sort of diagram with $f : (M \otimes N) \times L \to M \otimes (N \otimes L)$ mapping $(m \otimes n, l) \mapsto m \otimes (n \otimes l)$. This gives a map α' such that $\alpha'((m \otimes n) \otimes l) = m \otimes (n \otimes l)$, which is clearly inverse to α.

The pentagon is easily seen to commute, and the unit is well known to be R, as well as the isomorphisms $R \otimes M \cong M \cong M \otimes R$.

Remark 6.1.8. The pentagon identity, although looks extremely natural, could seem sort of arbitrary. The reason beyond it is the so called Coherence Theorem (see [ML97] VII.2), which roughly speaking says that it is the minimal requirement in order to get any diagram involving only unitors and associators to be commutative.

A particular kind of monoidal categories are those where the tensor product is commutative. The following makes this precise.

Definition 6.1.9. A *symmetric monoidal category* $(\mathbf{B}, \otimes, 1, \alpha, \lambda, \rho)$ is a monoidal category equipped with isomorphisms, for every pair of objects X, Y,

$$\gamma_{X,Y} : X \otimes Y \cong Y \otimes X$$

natural in X and Y, such that $\gamma_{X,Y}\gamma_{Y,X} = \mathrm{id}$, $\rho_Y = \lambda_Y \gamma_{Y,1} : Y \otimes 1 \cong Y$ and making the following hexagon diagram to commute:

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}\otimes\gamma"] & (X \otimes Y) \otimes Z \arrow[r, "\gamma"] & Z \otimes (X \otimes Y) \arrow[d, "\alpha"] \\
X \otimes (Z \otimes Y) \arrow[r, "\alpha"] & (X \otimes Z) \otimes Y \arrow[r, "\gamma\otimes\mathrm{id}"] & (Z \otimes X) \otimes Y
\end{tikzcd}$$

Remark 6.1.10. There is a version of the Coherence Theorem (see Remark 6.1.8) also for symmetric monoidal categories, and that is the reason why we require the hexagon diagram to commute.

Example 6.1.11. The category of R modules is symmetric monoidal, since the morphism $M \otimes_R N \to N \otimes_R M$ mapping $m \otimes n \mapsto n \otimes m$ is an isomorphism. Analogous arguments and standard results for sheaves of modules prove also the category of sheaves of modules $\mathbf{Mod}(\mathcal{O}_X)$ on a scheme X to be symmetric monoidal under the tensor product of sheaves.

Remark 6.1.12. The derived category of bounded above complexes $\mathrm{D}^-(R)$ is a symmetric monoidal category with the monoidal structure given by the derived tensor product $\otimes^{\mathrm{L}}$. We avoid the proof, which follows by taking bounded above resolutions by complexes of flat objects and knowing the axioms of symmetric monoidal category to hold for the category of modules.

Moreover, in a completely analogous way, it can be proven that the category $\mathrm{D}^-(\mathcal{O}_X)$ and, since the derived tensor product is defined on perfect complexes, its full subcategory $\mathrm{Pf}(X)$ are symmetric monoidal.

6.2 Prime $\otimes$-ideals

Let's now go back to triangulated categories and introduce a way to combine their structure with the one of symmetric monoidal category.

Definition 6.2.1. A *tensor triangulated category* $(\mathbf{K}, \otimes, 1)$ is an essentially small triangulated category $(\mathbf{K}, T)$ endowed with the structure of a symmetric monoidal category given by a tensor product $\otimes$ such that

(*i*) $\otimes : \mathbf{K} \times \mathbf{K} \longrightarrow \mathbf{K}$ is a triangulated functor in each variable.

(*ii*) There is a natural isomorphism between $(T-) \otimes -$ and $T(- \otimes -)$

Definition 6.2.2. A *tensor triangulated functor* between tensor triangulated categories $F : \mathbf{K} \longrightarrow \mathbf{L}$ is a triangulated functor such that it respects monoidal structure and identity. That is, $F(A \otimes B) = F(A) \otimes F(B)$ and $F(1_{\mathbf{K}}) = 1_{\mathbf{L}}$.

Example 6.2.3. The category of perfect complexes on a scheme $(X, \mathcal{O}_X)$ has a structure of tensor triangulated category $(\mathrm{Pf}(X), \otimes^{\mathrm{L}}, \mathcal{O}_X)$.

In this situation, one can straightforward generalize Definition 5.2.3 of $\otimes$-ideal to an arbitrary tensor triangulated category:

Definition 6.2.4. A $\otimes$*-ideal* in a tensor triangulated category $\mathbf{K}$ is a triangulated thick subcategory $\mathbf{A} \subseteq \mathbf{K}$ such that whenever A is in $\mathbf{A}$, so is $A \otimes B$ for every B in $\mathbf{T}$

Remark 6.2.5. Since we are working with essentially small triangulated categories and $\otimes$-ideals are strictly full subcategories, there is a set of these subcategories, and each of them is equivalent to a set. Therefore, meaning to work up to equivalence, we are going to consider elements of these ideals.

Remark 6.2.6. The Verdier quotient $\mathbf{K}/\mathbf{A}$ of a tensor triangulated category $\mathbf{K}$ by a $\otimes$-ideal $\mathbf{A}$, carries the structure not only of triangulated category (Theorem 3.2.6), but also that of tensor triangulated category.

The tensor product functor $A \otimes -$ is trivially extended on objects in $\mathbf{K}/\mathbf{A}$, which are the same of $\mathbf{K}$. On a morphism $f : B \to C$, represented by a roof

$$B \overset{s}{\longleftarrow} W \longrightarrow C$$

with $C(s)$ in $\mathbf{A}$, the functor $A \otimes -$ is defined as the morphism represented by

$$A \otimes B \overset{A \otimes s}{\longleftarrow} A \otimes W \longrightarrow A \otimes C$$

and it's well defined because $C(A \otimes s) = A \otimes C(s)$ still is in $\mathbf{A}$, since A is.

A tensor triangulated category roughly speaking looks like a categorical version of a commutative ring. This can be made clearer using Grothendieck groups (§ 5.1), anyway the symmetric monoidal structure given by the tensor product correspond to the commutative monoid structure of the multiplication. Having this in mind makes us more comfortable with the next definition.

Definition 6.2.7. A proper $\otimes$-ideal $\mathbf{P} \subsetneq \mathbf{K}$ in a tensor triangulated category is said to be *prime* if whenever $A \otimes B \in \mathbf{P}$, then either A or B are in $\mathbf{P}$.

Now we are ready to define the spectrum at the level of sets.

Definition 6.2.8. The *spectrum* of a tensor triangulated category is the set

$$\mathcal{S}pec(\mathbf{K}) = \{\mathbf{P} \subseteq \mathbf{K} | \mathbf{P} \text{ is a prime } \otimes\text{-ideal}\}$$

Definition 6.2.9. For any set of objects $\mathbf{S} \subseteq \mathbf{K}$, define

$$Z(\mathbf{S}) = \{\mathbf{P} \in \mathcal{S}pec(\mathbf{K}) | \mathbf{P} \cap \mathbf{S} = \varnothing\}$$

Define also its complement $U(\mathbf{S}) = \{\mathbf{P} \in \mathcal{S}pec(\mathbf{K}) | \mathbf{P} \cap \mathbf{S} \neq \varnothing\}$

Proposition 6.2.10. *The set $\{Z(\boldsymbol{S}) | \boldsymbol{S} \subseteq \boldsymbol{K}\}$ defines a base of closed subsets for a topology on $\mathcal{S}pec(\boldsymbol{K})$.*

Proof. $\bigcap_{j \in J} Z(\mathbf{S}_j) = \{\mathbf{P} \in \mathcal{S}pec(\mathbf{K}) | \forall j \in J \ \mathbf{P} \cap \mathbf{S}_j = \varnothing\} = \{\mathbf{P} \in \mathcal{S}pec(\mathbf{K}) | \mathbf{P} \cap \bigcup_{j \in J} \mathbf{S}_j = \varnothing\} = Z(\bigcup_{j \in J} \mathbf{S}_j)$.

Let's prove that $Z(\mathbf{S}) \cup Z(\mathbf{T}) = Z(\mathbf{S} \oplus \mathbf{T})$, where $\mathbf{S} \oplus \mathbf{T} = \{A \oplus B \in \mathbf{K} | A \in \mathbf{S}, B \in \mathbf{T}\}$. Recall that as any triangulated subcategory, $\mathbf{P}$ is closed under taking direct sums, hence if $\mathbf{P} \notin Z(\mathbf{S}) \cup Z(\mathbf{T})$, by definition neither $\mathbf{P} \cap \mathbf{S}$ nor $\mathbf{P} \cap \mathbf{T}$ are empty, so there are A, B in these two sets showing that $A \oplus B \in \mathbf{P} \cap (\mathbf{S} \oplus \mathbf{T}) \neq \varnothing$, that is $\mathbf{P} \notin Z(\mathbf{S} \oplus \mathbf{T})$. Conversely, if $\mathbf{P} \notin Z(\mathbf{S} \oplus \mathbf{T})$, there are $A \in \mathbf{S}$ and $B \in \mathbf{T}$ with $A \oplus B \in \mathbf{P}$, and thickness of $\mathbf{P}$ says that both A and B are in $\mathbf{P}$, hence $P \notin Z(\mathbf{S}) \cup Z(\mathbf{T})$

Eventually, it's clear that $Z(\varnothing) = \mathcal{S}pec(\mathbf{K})$ and $Z(\mathbf{K}) = \varnothing$. □

The topology on $\mathcal{S}pec(\mathbf{K})$ generated by the closed subsets $\{Z(\mathbf{S}) | \mathbf{S} \subseteq \mathbf{K}\}$ is called *Zariski topology*. The structure of scheme will be later discussed.

Remark 6.2.11. Just as the operation V on the ideals of a ring R defining a base of closed subspaces of $\operatorname{Spec}(R)$ by $V(a) = \{p \in \operatorname{Spec}(R) | p \supseteq a\}$, the operation Z reverses inclusions. Although a lot of notions in this setting can be easily imported from standard affine algebraic geometry, one has to be careful with what goes on under these definitions, which doesn't match faithfully those from commutative algebra. A main difference is highlighted by the following phenomena of reversing inclusions with respect to what happens for spectral spaces. Closed points in $\mathcal{S}pec(\mathbf{K})$ are, in fact, *minimal* prime ideals.

Proposition 6.2.12. *For any point $\boldsymbol{P} \in \mathcal{S}pec(\boldsymbol{K})$, its closure is*

$$\overline{\{\boldsymbol{P}\}} = \{\boldsymbol{Q} \in \mathcal{S}pec(\boldsymbol{K}) | \boldsymbol{Q} \subseteq \boldsymbol{P}\}.$$

Proof. Let $\mathbf{S}_0 = \mathbf{K} \setminus \mathbf{P}$. Then certainly $\mathbf{P} \cap \mathbf{S}_0 = \varnothing$, that is $\mathbf{P} \in Z(\mathbf{S}_0)$. Moreover $\mathbf{S}_0$ is maximal with this property, $\mathbf{P} \in Z(\mathbf{S})$ if and only if $\mathbf{S} \subseteq \mathbf{S}_0$, which is equivalent to $Z(\mathbf{S}_0) \subseteq Z(\mathbf{S})$. In other words, we get that whenever $\mathbf{P}$ belongs to a closed basic subset, then $Z(\mathbf{S}_0)$ is contained in this subset. That means $Z(\mathbf{S}_0) = \{\mathbf{Q} \in \mathcal{S}pec | \mathbf{Q} \cap \mathbf{K} \setminus \mathbf{P} = \varnothing\} = \{\mathbf{Q} \in \mathcal{S}pec | \mathbf{Q} \subseteq \mathbf{P}\}$ is the closure of $\mathbf{P}$. □

Corollary 6.2.13. *Let $\boldsymbol{P}_1$, $\boldsymbol{P}_2 \in \mathcal{S}pec(\boldsymbol{K})$. If $\overline{\{\boldsymbol{P}_1\}} = \overline{\{\boldsymbol{P}_2\}}$, then $\boldsymbol{P}_1 = \boldsymbol{P}_2$*

Proof. Proposition 6.2.12 gives

$$\{\mathbf{Q} \in \mathcal{S}pec(\mathbf{K}) | \mathbf{Q} \subseteq \mathbf{P}_1\} = \{\mathbf{Q} \in \mathcal{S}pec(\mathbf{K}) | \mathbf{Q} \subseteq \mathbf{P}_2\},$$

and hence is obvious that $\mathbf{P}_1 = \mathbf{P}_2$. $\square$

6.3 Supported topological spaces

Inspired by the operator Supph, which is an assignment of a closed subspace of X to any perfect complex on X, we generalize to this setting the notion of support.

Definition 6.3.1. Let $(\mathbf{K}, \otimes, 1)$ be a fixed tensor triangulated category. A topological space X is said to be *supported* on $\mathbf{K}$ if it is endowed with an assignment

$$\sigma : \mathbf{K} \longrightarrow \{Y \subseteq X | Y \text{ is closed}\}$$

called *support* satisfying

(i) $\sigma(0) = \varnothing$ and $\sigma(1) = X$

(ii) $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$

(iii) $\sigma(TA) = \sigma(A)$

(iv) $\sigma(A) \subseteq \sigma(B) \cup \sigma(C)$ for any $A \to B \to C \to TA$ distinguished triangle

(v) $\sigma(A \otimes B) = \sigma(A) \cap \sigma(B)$

Definition 6.3.2. A morphism $f : (X, \sigma) \to (Y, \tau)$ of topological spaces supported on a tensor triangulated category $\mathbf{K}$ is a morphism of topological spaces $f : X \to Y$ such that

$$\sigma(A) = f^{-1}(\tau(A))$$

for any object A in $\mathbf{K}$.

Remark 6.3.3. The previous definitions provide a category $\mathbf{Top}_{\mathbf{K}}$, since composition of maps with the required property for being a morphism in $\mathbf{Top}_{\mathbf{K}}$, still has the property. Isomorphisms in this category are precisely the homeomorphism of topological spaces which are also morphism of supported topological spaces.

Remark 6.3.4. The cohomological support Supph gives to any scheme X the structure of a topological space supported on $\mathrm{Pf}(X)$. In Definition 6.3.1, conditions (iii) trivial for Supph, condition (i) is easy, since the zero complex has clearly empty cohomological support, while for any $x \in X$ the ring $\mathcal{O}_{X,x}$ cannot be zero. Conditions (ii) and (v) are provided by Proposition 5.2.9, and (iv) by Lemma 5.2.14.

The following, generalizes Definition 5.2.13.

Definition 6.3.5. Let (X, σ) be a topological space supported on $\mathbf{K}$ and $Y \subseteq X$ be any subspace. Define the full subcategory $\mathbf{K}_Y \subseteq \mathbf{K}$ to be the one whose objects are all those A in $\mathbf{K}$ with $\sigma(A) \subseteq Y$.

Lemma 6.3.6. *For any subspace $Y \subseteq X$ the category $\boldsymbol{K}_Y \subseteq \boldsymbol{K}$ is a $\otimes$-ideal.*

Proof. Observe first of all that support is invariant under the shift functor by (iii), and so is the full subcategory $\mathbf{K}_Y$. Moreover, if two objects in a triangle $A \to B \to C \to TA$ are in $\mathbf{K}_Y$, suppose up to shift that these two objects are B and C and use condition (iv) which gives $\sigma(A) \subseteq \sigma(B) \cup \sigma(C) \subseteq Y$. This proves $\mathbf{K}_Y$ to be a triangulated subcategory. Thickness is given by (ii), since $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B) \subseteq Y \Rightarrow \sigma(A), \sigma(B) \subseteq Y$. Eventually, condition (v) gives, for A in $\mathbf{K}_Y$, that $\sigma(A \otimes B) \subseteq \sigma(A) \subseteq Y$, that is $\mathbf{K}_Y$ is a $\otimes$-ideal. $\square$

A very special topological space supported on $\mathbf{K}$ is the spectrum $\mathcal{S}pec(\mathbf{K})$.

Definition 6.3.7. Given a tensor triangulated category $\mathbf{K}$, let the function

$$\mathrm{supp} : \mathbf{K} \longrightarrow \{Y \subseteq X | Y \text{ is closed}\}$$

be defined by $\mathrm{supp}(A) = Z(\{A\}) = \{\mathbf{P} \in \mathcal{S}pec(\mathbf{K}) | A \notin \mathbf{P}\}$.

Remark 6.3.8. Any basic closed subspace in $\mathcal{S}pec(\mathbf{K})$ is by definition of the form $Z(\mathbf{S})$ for $\mathbf{S} \subseteq \mathbf{K}$, and $Z(\mathbf{S}) = \bigcap_{A \in \mathbf{S}} \mathrm{supp}(A)$. Thus the family $\{\mathrm{supp}(A) | A \in \mathbf{K}\}$ is a base for the Zariski topology.

Proposition 6.3.9. *The space $(\mathcal{S}pec(\boldsymbol{K}), \mathrm{supp})$ is supported on $\boldsymbol{K}$*

Proof. Certainly if $\mathbf{P}$ is any $\otimes$-ideal, it's a triangulated subcategory, hence the zero object of $\mathbf{K}$ is isomorphic to the zero object of $\mathbf{P}$, and so is in $\mathbf{P}$. That means $\mathrm{supp}(0) = \{\mathbf{P} \in \mathcal{S}pec(\mathbf{K}) | 0 \notin \mathbf{P}\} = \varnothing$. It's also clear that 1 cannot belong to any prime, since otherwise for any object A one would have $A \otimes 1 \cong A$ in $\mathbf{P}$, which wouldn't be proper. This proves (i).

As observed yet, $\mathbf{P}$ being thick triangulated subcategory implies that

$$A \in \mathbf{P} \text{ and } B \in \mathbf{P} \Longleftrightarrow A \oplus B \in \mathbf{P}$$

. Thus $\mathrm{supp}(A \oplus B) = \{\mathbf{P} \in \mathcal{S}pec(\mathbf{K}) | A \oplus B \notin \mathbf{P}\} = \{\mathbf{P} \in \mathcal{S}pec(\mathbf{K}) | A \notin \mathbf{P} \text{ or } B \notin \mathbf{P}\} = \mathrm{supp}(A) \cup \mathrm{supp}(B)$. This proves (ii).

The fact that any triangulated subcategory $\mathbf{P}$ is invariant under the shift functor and the distinguished triangle $T^{-1}A \to 0 \to A \to A$ for any A in $\mathbf{P}$ show that A is in $\mathbf{P}$ if and only if TA is in $\mathbf{P}$. That means $\mathrm{supp}(A) = \mathrm{supp}(TA)$. This proves (iii).

If $A \to B \to C \to TA$ is a distinguished triangle and $P \notin \text{supp}(B) \cup \text{supp}(C)$, then both B and C are in $\mathbf{P}$, which is triangulated, hence also $A \in \mathbf{P}$, that is $\mathbf{P} \notin \text{supp}(A)$. This proves (iv).

Eventually, (v) actually needs $\mathbf{P}$ being prime $\otimes$-ideal, since in this case

$$A \otimes B \in \mathbf{P} \Longleftrightarrow A \in \mathbf{P} \text{ or } B \in \mathbf{P}$$

hence $A \otimes B \notin \mathbf{P} \Leftrightarrow A \notin \mathbf{P}$ and $B \notin \mathbf{P}$, that is $\text{supp}(A \otimes B) = \text{supp}(A) \cap \text{supp}(B)$. $\square$

Theorem 6.3.10. *Let $\boldsymbol{K}$ be a tensor triangulated category, then the object $(\mathcal{S}pec(\boldsymbol{K}), \text{supp})$ is final in the category $\boldsymbol{Top_K}$. Moreover, for any space (X, σ) supported on $\boldsymbol{B}$ the unique morphism $f : (X, \sigma) \longrightarrow (\mathcal{S}pec(\boldsymbol{K}), \text{supp})$ is explicitly given as*

$$f(x) = \{A \in \boldsymbol{K} | x \notin \sigma(A)\}.$$

Proof. One has to check that the morphism $(X, \sigma) \to (\mathcal{S}pec(\mathbf{K}), \text{supp})$ defined in the statement has actually range in the set of prime $\otimes$-ideal and that it defines a morphism of spaces supported on $\mathbf{K}$. For a fixed $x \in X$, Lemma 6.3.6 applied to the subspace $Y = X \setminus \{x\}$ shows that $f(x) = \{A \in \mathbf{K} | \sigma(A) \subseteq Y\} = \mathbf{K}_Y$ is a $\otimes$-ideal. In order to see that it's also prime, let $A \otimes B \in f(x)$, that means $x \notin \sigma(A \otimes B) = \sigma(A) \cap \sigma(B)$, thus either $x \notin \sigma(A)$ or $x \notin \sigma(B)$, i.e. either A or B are in $f(x)$.

The fact that f defines a morphism in $\mathbf{Top_K}$ is given by $x \in f^{-1}(\text{supp}(A))$ if and only if $f(x) \in \text{supp}(A)$, that is $A \notin f(x) = \{A \in \mathbf{K} | x \notin \sigma(A)\}$, equivalently $x \in \sigma(A)$ as desired.

Uniqueness is proved considering $f, g : (X, \sigma) \to (\mathcal{S}pec(\mathbf{K}), \text{supp})$ such that $f^{-1}(\text{supp}(A)) = \sigma(a) = g^{-1}(\text{supp}(A))$ for every object A in $\mathbf{K}$, and showing that then $f = g$. Let $x \in X$, then our assumption become $f(x) \in \text{supp}(A)$ if and only if $g(x) \in \text{supp}(A)$. For what noted in Remark 6.3.8, one can consider

$$\overline{\{f(x)\}} = \bigcap_{\{A \in \mathbf{K} | f(x) \in \text{supp}(A)\}} \text{supp}(A) = \bigcap_{\{A \in \mathbf{K} | g(x) \in \text{supp}(A)\}} \text{supp}(A) = \overline{\{g(x)\}}.$$

Corollary 6.2.13 provides then $f(x) = g(x)$. $\square$

Another notion that can be imported from commutative algebra is the following

Definition 6.3.11. Let $\mathbf{K}$ be a tensor-triangulated category, and $\mathbf{J} \subseteq \mathbf{K}$ a $\otimes$-ideal. The *radical* of $\mathbf{J}$ is defined to be

$$\sqrt{\mathbf{J}} = \{A \in \mathbf{K} | \exists n \geq 1 \bigotimes_n A \in \mathbf{J}\}.$$

The $\otimes$-ideal $\mathbf{J}$ is said to be *radical* whether $\mathbf{J} = \sqrt{\mathbf{J}}$.

Remark 6.3.12. For our purposes of studying the tensor-triangulated subcategory of perfect complex, it's irrelevant to introduce radical $\otimes$-ideals, because all $\otimes$-ideals are radical in $\mathrm{Pf}(X)$. This is given by the following argument.

Proposition 6.3.13. *Let $\boldsymbol{K}$ be a tensor triangulated category. Then, are equivalent:*

(a) Any $\otimes$-ideal is radical.

(b) Any object A in $\boldsymbol{K}$ is in the smallest $\otimes$-ideal containing $A \otimes A$.

Proof. Call $\mathbf{J}_A$ the smallest $\otimes$-ideal containing $A \otimes A$. Certainly (i) implies (ii), because if $\mathbf{J}_A$ is radical, then it contains A because it contains $A \otimes A$. Conversely, let $\mathbf{J}$ be any $\otimes$-ideal and suppose $\bigotimes\limits_n A$ to be in $\mathbf{J}$. Let's prove A to be in $\mathbf{J}$ by induction on n.

The base case $n = 1$ is trivial. Now, let n be an integer greater than 1. If n is even, consider $B = \bigotimes\limits_{\frac{n}{2}} A \in \mathbf{J}$, and by our assumption observe that $B \otimes B \in \mathbf{J}$ means $B \otimes B \in \mathbf{J}_B \subseteq \mathbf{J}$, hence $B \in \mathbf{J}_B \subseteq \mathbf{J}$. The theorem for B holds true, so that $A \in \mathbf{J}$.

If n is odd, use the fact that $\mathbf{J}$ is a $\otimes$-ideal, so that $\bigotimes\limits_{n+1} A = A \otimes \bigotimes\limits_n A$ is in $\mathbf{J}$. Use then the case n even, giving the result whenever $\frac{n+1}{2} < n$, i.e. whenever $n > 1$, which is the case. $\square$

Corollary 6.3.14. *Every $\otimes$-ideal in* $\mathrm{Pf}(X)$ *is radical.*

Proof. Recall that $\mathrm{Supph}(E^{\cdot} \otimes F^{\cdot}) = \mathrm{Supph}E^{\cdot} \cap \mathrm{Supph}F^{\cdot}$, so that always happens $\mathrm{Supp}E^{\cdot} \subseteq \mathrm{Supp}(E^{\cdot} \otimes E^{\cdot})$. Thus, Theorem 5.2.24 says that is satisfied the condition (ii) in Proposition 6.3.13, and so is the equivalent condition (i), saying that every $\otimes$-ideal is radical. $\square$

Remark 6.3.15. Let (X, σ) be a topological space supported on a tensor triangulated category $\mathbf{K}$, and $Y \subseteq X$ a subspace. Then the $\otimes$-ideal $\mathbf{K}_Y$ (see Lemma 6.3.6) is radical. That is because $\sigma(\bigotimes\limits_n A) = \bigcap\limits_n \sigma(A) = \sigma(A)$.

The next theorem has hypothesis mimicking the statement of Theorem 5.2.25, so that it will be applied subsequently.

Theorem 6.3.16. *Let $(X, \mathcal{O}_X)$ be a noetherian scheme with a support σ on a tensor triangulated category $\boldsymbol{K}$. Set $\mathcal{T} = \{\boldsymbol{A} \subseteq \boldsymbol{K} | \boldsymbol{A}$ is radical $\otimes$-ideal$\}$ and $\mathcal{S} = \{Y \subseteq X | Y$ is specialization closed$\}$ and suppose that there exists a bijection between $\mathcal{T}$ and $\mathcal{S}$ given by*

$$\begin{aligned} \phi : \mathcal{T} &\longleftrightarrow \mathcal{S} \\ \boldsymbol{A} &\mapsto \bigcup_{A \in \boldsymbol{A}} \sigma(A) \\ \boldsymbol{K}_Y &\mapsfrom Y \end{aligned}$$

Then the unique morphism $(X, \sigma) \to (\mathcal{S}pec(\boldsymbol{K}), \mathrm{supp})$ is a homeomorphism.

Proof. Let's first of all prove the following claim, which is an analog result of Lemma 5.2.21. The outline of the proof also is the same.
Claim: Whenever $Y \subseteq X$ is closed subspace, there exists an object A in $\mathbf{K}$ such that $Y = \sigma(A)$.

Being $Y \subseteq X$ closed subspace of noetherian scheme it is noetherian, and this gives a decomposition of Y as finite union of irreducible components, each of which has a unique generic point by Proposition A.0.7. Thus, if one supposes $Y = \overline{\{y\}}$ and prove the result, the general case follows considering the direct sum of the objects, whose support is the union of the single supports. For the supposed case, $Y = \overline{\{y\}} = \phi(\phi^{-1}(Y)) = \bigcup_{A \in \phi^{-1}(Y)} \sigma(A)$ gives the existence of an object $A \in \phi^{-1}(Y) = \{A \in \mathcal{S}pec(\mathbf{K}) | \sigma(A) \subseteq Y\}$ such that $y \in \sigma(A)$, but now $\sigma(A)$ is closed by definition, thus $\overline{\{y\}} \subseteq \sigma(A)$. That means both $\sigma(A) \subseteq Y$ and $Y \subseteq \sigma(A)$. This proves the claim.

Let's prove that f is bijective and closed. For a fixed $x \in X$ set

$$Y(x) = \{y \in X | x \notin \overline{\{y\}}\}$$

which is specialization closed since $Y(x) = \bigcup_{x \notin \overline{\{z\}}} \overline{\{z\}}$. The left to right inclusion is in fact obvious, and whenever there exists z such that $x \notin \overline{\{z\}}$ and $y \in \overline{\{z\}}$, then $x \notin \overline{\{z\}} \supseteq \overline{\{y\}}$, thus $x \notin \overline{\{y\}}$, i.e. $y \in Y(x)$.

Now let's see that for any object A in $\mathbf{K}$ it holds

$$\sigma(A) \subseteq Y(x) \Longleftrightarrow x \notin \sigma(A).$$

One one hand if $x \in \sigma(A)$, then one has found a point which is in $\sigma(A)$ but not in $Y(x)$, since clearly $x \in \overline{\{x\}}$. Conversely, if $x \notin \sigma(A)$, take any $y \in \sigma(A)$, which is closed and hence $x \notin \sigma(A) = \bigcup_{y \in \sigma(A)} \overline{\{y\}}$. So one has $x \notin \overline{\{y\}}$, that means $y \in Y(x)$. Thus $\sigma(A) \subseteq Y(x)$.

This provides a description of $f(x)$ as

$$f(x) = \{A \in \mathbf{K} | x \notin \sigma(A)\} = \{A \in \mathbf{K} | \sigma(A) \subseteq Y(x)\} = \phi^{-1}(Y(x)).$$

Thus if $f(x_1) = f(x_2)$ one has $\phi^{-1}(Y(x_1)) = \phi^{-1}(Y(x_2))$, and hence $Y(x_1) = Y(x_2)$. By definition that means $\{y \in X | x_1 \notin \overline{\{y\}}\} = \{y \in X | x_2 \notin \overline{\{y\}}\}$, which can be more clearly stated as $x_1 \in \overline{\{y\}} \Longleftrightarrow x_2 \in \overline{\{y\}}$. In particular one has for every y that $\overline{\{x_1\}} \subseteq \overline{\{y\}} \Longleftrightarrow \overline{\{x_2\}} \subseteq \overline{\{y\}}$, which shows by taking $y = x_1$ and $y = x_2$, that $\overline{\{x_1\}} = \overline{\{x_2\}}$. Eventually, being X a scheme, and hence T_0, that implies $x_1 = x_2$.

In order to prove f to be surjective, take any prime $\otimes$-ideal $\mathbf{P} \subseteq \mathbf{K}$, and look at the specialization closed subspace $Y = \phi(\mathbf{P})$. Being $\mathbf{P}$ a proper subcategory of $\mathbf{K}$ and since ϕ certainly maps $\mathbf{K} \mapsto X$ because $1 \in \mathbf{K}$, then

$Y \subsetneq X$. Consider thus $x, y \in X \setminus Y$ and find by the initial claim two objects A, B with supports respectively $\overline{\{x\}}$ and $\overline{\{y\}}$. Since $x, y \notin Y$, this gives $\sigma(A), \sigma(B) \nsubseteq Y$, which is by definition $A, B \notin \mathbf{K}_Y = \phi^{-1}(Y) = \mathbf{P}$. Being $\mathbf{P}$ prime, neither their tensor product $A \otimes B$ is in $\mathbf{K}_Y$, that is $\sigma(A \otimes B) \nsubseteq Y$. This leads us to find z again in $X \setminus Y$ with $z \in \sigma(A \otimes B)$, that is $z \in \sigma(A) \cap \sigma(B) = \overline{\{x\}} \cap \overline{\{y\}}$, implying $\overline{\{z\}} \subseteq \overline{\{x\}}, \overline{\{y\}}$.

In slightly more sophisticated terms, we just proved that the set partially ordered by inclusion

$$\mathcal{F} = \{\overline{\{x\}} \subseteq X | x \in X \setminus Y\}$$

is finitely complete, that is any finite family of objects admits a lower bound in $\mathcal{F}$. Moreover, since X is noetherian scheme, it has a noetherian topological space, hence any totally ordered subset of $\mathcal{F}$, being a chain of closed subspace, has a minimal element. These two facts implies that $\mathcal{F}$ has a minimum. That is because if it wouldn't have a minimum, for any $a_0 \in \mathcal{F}$, which is not a minimum one would find $a_1 \in \mathcal{F}$ with $a_1 \ngeq a_0$, that is either $a_1 < a_0$ or a_1 and a_0 are not comparable. In this case, find a lower bound in $\mathcal{F}$ for the two, and call it again a_1. This inductively defines a chain without minimal element.

So, there exists $x \in X \setminus Y$ such that whenever $y \in X \setminus Y$ it holds $\overline{\{x\}} \subseteq \overline{\{y\}}$, that means

$$X \setminus Y \subseteq \{y \in X | \overline{\{x\}} \subseteq \overline{\{y\}}\} = \{y \in X | x \in \overline{\{y\}}\}.$$

The reverse inclusion also hods true, because $x \notin Y = \bigcup_{y \in Y} \overline{\{y\}}$, that means for any $y \in Y$ one has $x \notin \overline{\{y\}}$. Whence for this minimal x one has $X \setminus Y = \{y \in X | x \in \overline{\{y\}}\}$, and thus

$$Y = \{y \in X | x \notin \overline{\{y\}}\} = Y(x).$$

Therefore, using the description of $f(x)$ as $\phi^{-1}(Y(x))$, one has the surjectivity proved since

$$\mathbf{P} = \phi^{-1}(Y) = \phi^{-1}(Y(x)) = f(x).$$

Eventually, given a closed subspace $Y \subseteq X$, let's find by the initial claim an object A such that $\sigma(A) = Y$. Being f a morphism of $\mathbf{Top_K}$ gives $f^{-1}(\mathrm{supp}(A)) = \sigma(A)$, and hence $f(Y) = f(\sigma(A)) = \mathrm{supp}(A)$, which is closed. $\square$

Corollary 6.3.17. *Let X be a noetherian scheme. The topological space* (X, Supph) *supported on* $\mathrm{Pf}(X)$ *is isomorphic to* $\mathcal{S}pec(\mathrm{Pf}((X))$ *through the map*

$$f(x) = \{A^\cdot \in \mathrm{Pf}(X) | A^\cdot_x \cong 0 \textit{ in } \mathrm{D}(\mathcal{O}_{X,x})\}.$$

Moreover, f maps $\mathrm{Supph}(A^\cdot)$ *to* $\mathrm{supp}(A^\cdot)$.

Proof. It's just the observation that Supph satisfies the hypothesis of Theorem 6.3.16, and that is thanks to Theorem 5.2.25 and Corollary 6.3.14. The map f is here defined to be $f(x) = \{A^\cdot \in \mathrm{Pf}(X) | x \notin \mathrm{Supph}(A^\cdot)\}$, that means $A^\cdot \in f(x)$ if and only if the stalk at x of $A^\cdot$ is not acyclic. The moreover part follows by the fact that this homeomorphism is a morphism of topological spaces supported on $\mathrm{Pf}(X)$. $\square$

6.4 Karoubi envelope

In this last part we are going to prove that the homeomorphism of Corollary 6.3.17 is part of an isomorphism of schemes, once will be given to $\mathcal{S}pec(\mathrm{Pf}(X))$ the structure of locally ringed space. In order to make this construction is needed the Verdier quotient, from § 3.2, and the so called Karoubi envelope.

Definition 6.4.1. Let $\mathbf{C}$ be a category and C an object in $\mathbf{C}$. A morphism $f : C \to C$ is said to be *idempotent* whether $f^2 = f$.

Definition 6.4.2. A morphism $f : C \to C$ in a category $\mathbf{C}$ is called *split idempotent* whether there exists an object D and morphisms

$$C \underset{i}{\overset{p}{\rightleftarrows}} D$$

such that $ip = f$ and $pi = \mathrm{id}_D$.

Remark 6.4.3. A split idempotent is certainly idempotent because

$$f^2 = (ip)(ip) = i(\mathrm{id}_D)p = ip = f$$

Definition 6.4.4. A category $\mathbf{C}$ is said to be *idempotent complete* if any idempotent is a split idempotent.

The following proposition explains the name "split".

Proposition 6.4.5. *Let $\boldsymbol{C}$ be an idempotent complete category. Then, any idempotent $e : X \to X$ has kernel and image, and there is a decomposition of X as*

$$X = \mathrm{Im}(e) \oplus \mathrm{Ker}(e).$$

Proof. Observe first of all that the morphism $\mathrm{id}_X - e$ is idempotent, since $(\mathrm{id}_X - e)^2 = \mathrm{id}_X - e - e + e = \mathrm{id}_X - e$. Therefore we can consider a split of e

$$X \underset{i}{\overset{r}{\rightleftarrows}} Y$$

with $ri = \mathrm{id}_Y$, $ir = e$, and also a split of $\mathrm{id}_X - e$

$$\begin{tikzcd} X \arrow[r, shift left, "r'"] & W \arrow[l, shift left, "i'"] \end{tikzcd}$$

with $r'i' = \mathrm{id}_W$, $i'r' = \mathrm{id}_X - e$. Let's prove that $W = \mathrm{Ker}(e)$ and $Y = \mathrm{Im}(e)$. The diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, "i'"'] & 0 \arrow[d] \\ X \arrow[r, "e"] & X \end{tikzcd}$$

is commutative since

$$ei' = -(\mathrm{id} - e - \mathrm{id})i' = -(\mathrm{id} - e)i' + i' = -i'r'i' + i' = -i' + i' = 0$$

For any morphism $j : Z \to X$ such that $ej = 0$, the morphism $r'j = Z \to W$ is such that $i'r'j = (\mathrm{id}_X - e)j = j - ej = j$. Uniqueness is immediate from the fact that i' is a left inverse of r', and hence a monomorphism.

Moreover, an analogous argument also proves that the morphism $r' : X \to W$ satisfies the universal property of the cokernel of e. Therefore, the image of e, if it exists, is the cokernel of r'. Let's show that Y works. The diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "i"'] & 0 \arrow[d] \\ X \arrow[r, "r'"] & W \end{tikzcd}$$

is commutative since

$$r'ir = re = -r'(\mathrm{id} - e - \mathrm{id}) = -r'(\mathrm{id} - e) + r' = -r'i'r' + r' = -r' + r' = 0.$$

If moreover a morphism $j : Z \to X$ is such that $r'j = 0$, the morphism $rj : Z \to Y$ is such that

$$ej = -(\mathrm{id}_X - e)j + j = -i'r'j + j = 0.$$

Uniqueness, is given again from i being mono.

Then, it remains to prove that the decomposition actually gives X, i.e. that X is the coproduct $W \oplus Y$. The coproduct morphisms are $i : Y \to X$ and $i' : W \to X$, and if another object Z comes with morphisms $j : Y \to Z$ and $j' : W \to Z$

$$\begin{tikzcd} & Z & \\ & X \arrow[u, dashed] & \\ Y \arrow[uur, "j"] \arrow[ur, "i"'] & & W \arrow[uul, "j'"'] \arrow[ul, "i'"] \end{tikzcd}$$

consider the morphism $j'r' + jr : X \to Z$ which is such that

$$(j'r' + jr)i' = j'r'i' + jri'$$

but $r'i' = \mathrm{id}_W$, while $ri' = 0$ because

$$ri'r' = r(\mathrm{id}_X - e) = r - re = r - rir = r - \mathrm{id}_Y r = 0$$

and r' is epi. Therefore $(j'r' + jr)i' = j'$. Analogously, $(j'r' + jr)i = j$.

Eventually, the morphism $j'r' + jr$ is unique, because if g and g' were two morphisms such that

$$gi' = j' = g'i' \text{ and } gi = j = g'i,$$

then $gi'r' = g'i'r'$, that is $g(\mathrm{id} - e) = g'(\mathrm{id} - e)$, giving $g - g(ir) = g' - g'(ir)$, and therefore

$$g - jr = g' - jr$$

which implies $g = g'$. □

Remark 6.4.6. There's certainly also a converse of the previous result. That is, if any idempotent $e : X \to X$ has kernel and image giving a decomposition of X as $\mathrm{Ker}(e) \oplus \mathrm{Im}(e)$, then the canonical projection $r : X \to \mathrm{Im}(e)$ and inclusion $i : \mathrm{Im}(e) \to X$ provide a split of e.

A first example of idempotent complete category is certainly any abelian category, where it's by definition required that kernel and image do exist for any morphism. However, also a lot of triangulated categories enjoy the property of splitting idempotent morphisms. A result by Bökstedt and Neeman asserts the following.

Theorem 6.4.7. *Any triangulated category with arbitrary direct sums is idempotent complete.*

Proof. See [BN93], Proposition 3.2. □

Remark 6.4.8. In particular, the derived category $\mathrm{D}(\mathcal{O}_X)$ is idempotent complete.

The following definition aims to construct the smallest idempotent complete enlargement of a category.

Definition 6.4.9. The *Karoubi envelope* of a category $\mathbf{C}$ is the full subcategory of $\bar{\mathbf{C}} \subseteq \hat{\mathbf{C}}$ consisting of objects which are retracts of representable presheaves. Namely, a presheaf P is in $\bar{\mathbf{C}}$ if and only if there exists an object C in $\mathbf{C}$ and morphisms in $\hat{\mathbf{C}}$

$$\mathrm{Hom}(-, C) \underset{i}{\overset{r}{\rightleftarrows}} P$$

such that $ri = \mathrm{id}_P$.

Remark 6.4.10. In the case of a retraction of $\mathrm{Hom}(-,C)$ to P, the morphism $ir : \mathrm{Hom}(-,C) \to \mathrm{Hom}(-,C)$ is idempotent, because $irir = i(\mathrm{id}_P)r = ir$.

Lemma 6.4.11. *Any cocomplete category $\boldsymbol{D}$ is idempotent complete.*

Proof. Let $f : D \to D$ be an idempotent. Since $\mathbf{D}$ is cocomplete, consider the coequalizer (E,p) of the morphisms $f, \mathrm{id}_D : D \to D$, and the unique morphism $i : E \to D$ induced by f in

$$\begin{tikzcd} D \arrow[r, shift left, "f"] \arrow[r, shift right, "\mathrm{id}_D"'] & D \arrow[r, "p"] \arrow[dr, "f"'] & E \arrow[d, dashed, "i"] \\ & & D \end{tikzcd}$$

such that $ip = f$. In order to see that also holds $pi = \mathrm{id}_E$, observe that being p equalizer one has $pf = p$, and therefore, from $f = ip$,

$$pip = p = \mathrm{id}_E p$$

and since p is equalizer and hence epi, we get eventually $pi = \mathrm{id}_E$. This proves f to split. $\square$

Proposition 6.4.12. *Let $\boldsymbol{C}$ be a category. Any idempotent $f : P \to P$ in $\bar{\boldsymbol{C}}$ is a split idempotent*

Proof. Since $\hat{\mathbf{C}}$ is cocomplete, consider a split of f in $\hat{\mathbf{C}}$

$$\begin{tikzcd} P \arrow[r, shift left, "p"] & H \arrow[l, shift left, "i"] \end{tikzcd}$$

with $pi = \mathrm{id}_H$ and $ip = f$.

Now, observe that by definition P is a retract of a representable presheaf, that is there exists an object C and morphisms

$$\begin{tikzcd} \mathrm{Hom}(-,C) \arrow[r, shift left, "r"] & P \arrow[l, shift left, "j"] \end{tikzcd}$$

such that $rj = \mathrm{id}_P$. But then assembling these morphisms one find that pr and ji defines a retraction of $\mathrm{Hom}(-,C)$ to H, because $prji = pi = \mathrm{id}_H$, and hence H is actually in $\bar{\mathbf{C}}$. $\square$

Remark 6.4.13. The Yoneda functor $y : \mathbf{C} \to \hat{\mathbf{C}}$, which is fully faithful by the Yoneda Lemma, obviously factors defining a fully faithful embedding $\mathbf{C} \hookrightarrow \bar{\mathbf{C}} \subseteq \hat{\mathbf{C}}$.

Theorem 6.4.14. *Let $\boldsymbol{C}$ be a category and $\boldsymbol{C} \hookrightarrow \bar{\boldsymbol{C}}$ its Karoubi envelope. For any idempotent complete category $\boldsymbol{D}$ and any functor $G : \boldsymbol{C} \to \boldsymbol{D}$ there exists a functor $\bar{G} : \bar{\boldsymbol{C}} \to \boldsymbol{D}$ lifting G, i.e. such that the triangle*

$$\begin{tikzcd} \boldsymbol{C} \arrow[r, hook, "y"] \arrow[dr, "G"'] & \bar{\boldsymbol{C}} \arrow[d, "\bar{G}"] \\ & \boldsymbol{D} \end{tikzcd}$$

commutes in ***Cat****.*

Proof. As proved in Lemma 6.4.11, the splitting of an idempotent $f : C \to C$ is just the same as the coequalizer of f and id_C. Therefore, the result follows in the same way as the construction of the Kan extension, in Theorem 1.1.10.

Explicitly, if P is in $\bar{\mathbf{C}}$ one has a retraction

$$y_C \underset{i}{\overset{r}{\rightleftarrows}} P$$

and $\bar{G}(P)$ is defined then to be the coequalizer of the parallel morphisms

$$GC \underset{\mathrm{id}}{\overset{G(ir)}{\rightrightarrows}} GC \xrightarrow{\ e\ } \bar{G}(P)$$

If $f : P \to P'$ is a morphism, P' comes analogously with a retraction given by $r' : P' \to y_{C'}$ and $i' : y_{C'} \to P$, and $\bar{G}(P')$ is the analogous equalizer, with morphism $e' : GC' \to \bar{G}P'$. Let's induce a morphism $\bar{G}P \to \bar{G}P'$.

Observe that the morphism $i'fr : y_C \to y_{C'}$ correspond by the Yoneda Lemma to a morphism $C \to C'$, and let us call it again $i'fr$. Then, the morphism

$$GC \overset{G(i'fr)}{\longrightarrow} GC' \xrightarrow{e'} \bar{G}(P')$$

defines a cocone for the same diagram defining the equalizer $(\bar{G}(P), e)$, in fact

$$e'G(i'fr)G(ir) = e'G(i'(\mathrm{id}_P)fr) = e'G(i'fr)\mathrm{id}_{FC}.$$

Therefore, it induces a unique morphism $u : \bar{G}(P) \to \bar{G}(P')$ such that $ue = e'F(i'fr)$. It's clear than this construction preserves identities, while for a composition

$$P \xrightarrow{f} P' \xrightarrow{g} P''$$

using analogous notations as above for the retraction of $y_{C''}$ to P'' and for the equalizer $\bar{G}(P'')$, one finds a morphism $u' : \bar{G}P' \to \bar{G}P''$ such that $u'e' = e''F(i''gr')$. It has to be proved that $u'u = \bar{G}(gf)$, i.e. that $u'ue = e''F(i''gfr)$, and that is because

$$u'ue = u'e'F(i'fr) = e''F(i''gr')F(i'fr) = e''F(i''g(\mathrm{id}_{P'})fr).$$

□

Remark 6.4.15. The idea behind the notion of the Karoubi envelope [1] is that we can think of it as if we formally (and iteratively) add for each

[1] Thanks to Eric Wofsey for explaining this to me.

idempotent $f : C \to C$ in $\mathbf{C}$ a new object D together with new morphisms making f to split

$$C \underset{i}{\overset{p}{\rightleftarrows}} D$$

and morphisms into D and out of D are those arising from the bijection induced by p and i.

More precisely, a morphism $g : E \to D$ will be a morphism arising from one of the form $h : E \to C$ with the property that $fh = h$ by composition with p. Compositions by p and i defines in fact, for every E, a bijection between the sets

$$\{g : E \to D\} \underset{p\circ}{\overset{i\circ}{\rightleftarrows}} \{h : E \to C | fh = h\}$$

because $iph = fh = h$ and $pig = \mathrm{id}_D g = g$. Conversely, morphisms out of D will be those $g : D \to E$ arising from precomposition by p of those $h : C \to E$ such that $hf = h$, under the analogous bijection

$$\{g : D \to E\} \underset{\circ i}{\overset{\circ p}{\rightleftarrows}} \{h : C \to E | hf = h\}$$

We have in fact the following result asserting that the one just described is the case.

Proposition 6.4.16. *Let $\boldsymbol{C}$ be a category and $f : C \to C$ an idempotent of $\boldsymbol{C}$. Let then D be an object in $\bar{\boldsymbol{C}}$ such that the induced idempotent $y(f)$: $\mathrm{Hom}(-, C) \to \mathrm{Hom}(-, C)$ splits as*

$$\mathrm{Hom}(-, C) \underset{i}{\overset{p}{\rightleftarrows}} D$$

Let $g : D \to E$ be a morphism of presheaves in $\bar{\boldsymbol{C}}$. Then $gpf = gp$ (i.e. g arise from a morphism $h = gp$ such that $hf = h$)

Proof. This is obvious since $gpf = gpip = g\mathrm{id}_D p = gp$. □

Remark 6.4.17. An analogous result describes the morphisms into D as a particular set of morphisms into C.

Lemma 6.4.18. *Let $\boldsymbol{C}$ be a category and $\boldsymbol{D}$ an idempotent complete category with a fully faithful functor*

$$G : \boldsymbol{C} \longrightarrow \boldsymbol{D}$$

Then, the induced functor $\bar{G} : \bar{\boldsymbol{C}} \to \boldsymbol{D}$ is fully faithful.

Proof. Let P be an object in $\bar{\mathbf{C}}$, which as seen arises as split of the idempotent $ir : y_C \to y_C$, where $i : P \to y_C$ and $r : y_C \to P$ define a retraction of y_C to P.

Let us confuse objects in $\mathbf{C}$ and the correspondent representable presheaves. Then, the morphism ir can be seen as an idempotent morphism $ir : C \to C$, and we can look at the idempotent $\bar{G}(ir) : GC \to GC$, which splits since $\mathbf{D}$ is idempotent complete as

$$GC \rightleftarrows D$$

Observe that $D = \bar{G}(P)$, directly from how we defined the extension $\bar{G}$ in Theorem 6.4.14. Therefore, if we consider any object E in $\mathbf{C}$, there is a natural bijection

$$\mathrm{Hom}(GE, \bar{G}P) = \{g : GE \to D\} \cong \{h : GE \to GC | \bar{G}(ir)h = h\}$$

where the last bijection is provided by compositions with $\bar{G}i$ and $\bar{G}r$, and is well defined because $\bar{G}(ir)\bar{G}(i)h = \bar{G}(iri)h = \bar{G}(i)h$. Now we know that the set $\{g : GE \to D\} \cong \{h : GE \to GC|\bar{G}(ir)h = h\}$ is in natural bijection, because G is fully faithful, with

$$\{h : E \to C|fh = h\}$$

which is isomorphic by Remark 6.4.15 to the set of morphisms $\mathrm{Hom}(E, D)$.

This proves the result $\mathrm{Hom}(E, P) \cong \mathrm{Hom}(\bar{G}E, \bar{G}P)$ when E is in $\mathbf{C} \subseteq \bar{\mathbf{C}}$. The general case is provided by a transfinite induction argument. □

Another result that will be needed below is the following.

Lemma 6.4.19. *Let $G : \boldsymbol{C} \to \boldsymbol{D}$ be a fully faithful functor into an idempotent complete category $\boldsymbol{D}$ with coproducts, and let F be an object in $\boldsymbol{D}$. If there exists an object W in $\boldsymbol{D}$ and an object E' in $\boldsymbol{C}$ such that $GE' \cong F \oplus W$, then there exists an object E in $\bar{\boldsymbol{C}}$ such that $\bar{G}E \cong F$.*

Proof. The direct sum decomposition $G(E') \cong F \oplus W$ provides natural inclusion an projection morphisms $i : F \to G(E')$ and $p : G(E') \to F$, giving a morphism $ip : G(E') \to G(E')$, idempotent because $pi = \mathrm{id}_F$, splitting as

$$G(E') \underset{i}{\overset{p}{\rightleftarrows}} F$$

Being G fully faithful, we find a morphism $f : E' \to E'$ with $Gf = ip$, which still is idempotent. That is because $G(f^2) = G(f)G(f) = ipip = ip$, so that f^2 corresponds under the natural bijection to ip, and hence $f^2 = f$.

Therefore, f splits in $\bar{\mathbf{C}}$, namely there exists an object E in $\bar{\mathbf{C}}$ with morphisms

$$E' \underset{j}{\overset{q}{\rightleftarrows}} E$$

with $jq = f$ and $qj = \mathrm{id}_E$. Applying $\bar{G}$ to this, provides a splitting of $\bar{G}(jq) = \bar{G}(f) = ip$. Summing up, we have two splitting of ip

$$G(E') \underset{i}{\overset{p}{\rightleftarrows}} F \qquad G(E') \underset{Gj}{\overset{Gq}{\rightleftarrows}} \bar{G}(E)$$

from which we get, thanks to Remark 6.4.15, that for any D in $\mathbf{D}$, the following natural bijections between set of morphisms holds:

$$\{g : D \to F\} \cong \{h : D \to G(E') | iph = h\} \cong \{g : D \to \bar{G}E\}.$$

Therefore, there's a natural isomorphism of functors

$$\mathrm{Hom}(-, F) \cong \mathrm{Hom}(-, \bar{G}(E))$$

which gives by the Yoneda Lemma the desired isomorphism $F \cong \bar{G}(E)$. $\square$

Remark 6.4.20. In the context of triangulated categories, one can prove that the Karoubi envelope of a ($\otimes$-)triangulated category $\mathbf{K}$ yields a natural structure of ($\otimes$-)triangulated category turning the inclusion $\mathbf{K} \to \bar{\mathbf{K}}$ into a triangulated functor. See [BS00].

6.5 The structure sheaf

Now, we're ready to define the structure sheaf of the spectrum.

Definition 6.5.1. Given a tensor triangulated category $\mathbf{K}$, the structure sheaf $\mathcal{O}_{\mathcal{S}pec(\mathbf{K})}$ on $\mathcal{S}pec(\mathbf{K})$ is defined to be the sheafification of the presheaf defined by

$$p\mathcal{O}_{\mathcal{S}pec(\mathbf{K})} : U \mapsto \mathrm{End}_{\mathbf{K}/\mathbf{K}_Z}(1_U)$$

where Z is the closed subspace $\mathcal{S}pec(\mathbf{K}) \setminus U$, and $1_U = Q_Z(1_{\mathbf{K}})$ is the image of the unit under the localization functor $Q_Z : \mathbf{K} \to \mathbf{K}/\mathbf{K}_Z$.

Remark 6.5.2. Restriction morphisms are given by the universal property of the localization. Observe, for $V \subseteq U$, that $Z = \mathcal{S}pec(\mathbf{K}) \setminus U \subseteq Y = \mathcal{S}pec(\mathbf{K}) \setminus V$, and hence $\mathbf{K}_Z \subseteq \mathbf{K}_Y$. A morphism $1_U \to 1_U$ induces then a morphism $1_V \to 1_V$ as image of the former under the functor

$$\mathbf{K}/\mathbf{K}_Z \longrightarrow \mathbf{K}/\mathbf{K}_Y$$

given by Verdier's Theorem in

$$\begin{tikzcd}
\mathbf{K} \arrow[r, "Q_Z"] \arrow[dr, "Q_Y"'] & \mathbf{K}/\mathbf{K}_Z \arrow[d, dashed] \\
& \mathbf{K}/\mathbf{K}_Y
\end{tikzcd}$$

since the localization functor Q_Y maps any object of $\mathbf{K}_Z$ to 0.

This sheaf turns the spectrum of any tensor triangulated category into a locally ringed space. In order to prove that, we characterize stalks of this sheaf as endomorphism rings.

Lemma 6.5.3. *Let $\boldsymbol{K}$ be a tensor triangulated category and $\boldsymbol{P} \in \mathcal{S}pec(\boldsymbol{K})$. Then, there's a natural isomorphism*

$$(\mathcal{O}_{\mathcal{S}pec(\boldsymbol{K})})_{\boldsymbol{P}} \cong \mathrm{End}_{\boldsymbol{K}/\boldsymbol{P}}(1)$$

Proof. Let U be an open neighborhood of $\mathbf{P}$ and Z its closed complement. A morphism in $\mathbf{K}/\mathbf{K}_Z$ is represented as a roof

$$1 \xleftarrow{s} A \longrightarrow 1$$

where, denoting as $C(s)$ the completion of s to a distinguished triangle, $\mathrm{supp}(C(s)) \subseteq Z$. It follows that a morphism in the colimit

$$(\mathcal{O}_{\mathcal{S}pec(\mathbf{K})})_{\mathbf{P}} = \underset{U \ni \mathbf{P}}{\mathrm{Colim}} \mathrm{End}_{\mathbf{K}/\mathbf{K}_Z}(1)$$

is a roof $1 \leftarrow A \rightarrow 1$ where $\mathrm{supp}(C(s)) \subseteq \mathcal{S}pec(\mathbf{K}) \setminus \{\mathbf{P}\}$, or equivalently

$$\mathbf{P} \notin \mathrm{supp}(C(s)) = \{\mathbf{Q} | C(s) \notin \mathbf{Q}\},$$

that is $C(s) \in \mathbf{P}$. Therefore, these are exactly the morphisms $1 \rightarrow 1$ in $\mathbf{K}/\mathbf{P}$. $\square$

Remark 6.5.4. Observe that when $\mathbf{P}$ is a prime $\otimes$-ideal, the category $\mathbf{K}/\mathbf{P}$ enjoy the usual property of quotient rings of being an integral domain. More precisely, if $A \otimes B \cong 0$ in $\mathbf{K}/\mathbf{P}$, then $A \otimes B$ is in $\mathbf{P}$, and hence either A or B is zero in $\mathbf{K}/\mathbf{P}$.

Proposition 6.5.5. *If $\boldsymbol{P} \subseteq \boldsymbol{K}$ is a prime $\otimes$-ideal, then the stalk $(\mathcal{O}_{\mathcal{S}pec(\boldsymbol{K})})_{\boldsymbol{P}}$ is a local ring.*

Proof. Thanks to Lemma 6.5.3, it suffices to prove that $\mathrm{End}_{\mathbf{K}/\mathbf{P}}(1)$ is a local ring. Let's prove then, that any morphism $f \in \mathrm{End}_{\mathbf{K}/\mathbf{P}}(1)$ is either invertible, or such that $\mathrm{id} - f$ is invertible.

In order to do that, we are going to prove that either $C(f)$, the completion of f to a distinguished triangle, or $C(\mathrm{id} - f)$, the completion of $\mathrm{id} - f$ to a distinguished triangle, is isomorphic to 0 in $\mathbf{K}/\mathbf{P}$. For such, being $\mathbf{P}$ prime, it suffices to prove $C(f) \otimes C(\mathrm{id} - f) \cong 0$.

Therefore, consider the distinguished triangles

$$1 \xrightarrow{\mathrm{id}-f} 1 \longrightarrow C(\mathrm{id}-f) \longrightarrow T1$$

and, being the tensor product a triangulated functor in each variable,

$$C(f) \xrightarrow{(\mathrm{id}-f)\otimes C(f)} C(f) \longrightarrow C(\mathrm{id}-f) \otimes C(f) \longrightarrow TC(f)$$

which tells us that it suffices to prove $(\mathrm{id} - f) \otimes C$ to be an isomorphism. Observe then that since the tensor product is a functor, from the morphism $\mathrm{id} - f : 1 \to 1$ and the distinguished triangle

$$1 \xrightarrow{f} 1 \xrightarrow{v} C(f) \xrightarrow{w} T1$$

we get a commutative diagram

$$\begin{tikzcd}
1\otimes 1 \arrow[r, "1\otimes f"] \arrow[d, "(\mathrm{id}-f)\otimes 1"] & 1\otimes 1 \arrow[r, "1\otimes v"] \arrow[d, "(\mathrm{id}-f)\otimes 1"] & 1\otimes C \arrow[r, "1\otimes w"] \arrow[d, "(\mathrm{id}-f)\otimes C"] & 1\otimes T1 \arrow[d, "(\mathrm{id}-f)\otimes T1"] \\
1\otimes 1 \arrow[r, "1\otimes f"'] & 1\otimes 1 \arrow[r, "1\otimes v"'] & 1\otimes C \arrow[r, "1\otimes w"'] & 1\otimes T1
\end{tikzcd}$$

which is, up to isomorphism

$$\begin{tikzcd}
1 \arrow[r, "f"] \arrow[d, "\mathrm{id}-f"] & 1 \arrow[r, "v"] \arrow[d, "\mathrm{id}-f"] & C \arrow[r, "w"] \arrow[d, "(\mathrm{id}-f)\otimes C"] & T1 \arrow[d, "T(\mathrm{id}-f)"] \\
1 \arrow[r, "f"'] & 1 \arrow[r, "v"'] & C \arrow[r, "w"'] & T1
\end{tikzcd}$$

Now consider another morphism between the same distinguished triangles, given by the commutative (because $vf = 0$) diagram

$$\begin{tikzcd}
1 \arrow[r, "f"] \arrow[d, "f"] & 1 \arrow[r, "v"] \arrow[d, "f"] & C \arrow[r, "w"] \arrow[d, "0"] & T1 \arrow[d, "Tf"] \\
1 \arrow[r, "f"'] & 1 \arrow[r, "v"'] & C \arrow[r, "w"'] & T1
\end{tikzcd}$$

The sum of these two morphisms of distinguished triangles gives another morphism of distinguished triangles

$$\begin{tikzcd}
1 \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & 1 \arrow[r, "v"] \arrow[d, "\mathrm{id}"] & C \arrow[r, "w"] \arrow[d, "(\mathrm{id}-f)\otimes C"] & T1 \arrow[d, "\mathrm{id}"] \\
1 \arrow[r, "f"'] & 1 \arrow[r, "v"'] & C \arrow[r, "w"'] & T1
\end{tikzcd}$$

Eventually, Proposition 2.4.12 tells us that $(\mathrm{id} - f) \otimes C$ is isomorphism. $\square$

Back to the case where $\mathbf{K} = \mathrm{Pf}(X)$ for a noetherian scheme X, as we are going to see, the presheaf described is actually a sheaf yet, and it leads to extend the homeomorphism

$$f : X \longrightarrow \mathcal{S}pec(\mathrm{Pf}(X))$$

of Corollary 6.3.17 to an isomorphism of schemes.

First, we need the following results.

Proposition 6.5.6. *Let X be a scheme. Then the category $\mathrm{Pf}(X)$ is idempotent complete.*

Proof. Let $e : X \to X$ be an idempotent in $\mathrm{Pf}(X)$. Since the category $\mathrm{D}(\mathcal{O}_X)$ is idempotent complete, take, thanks to Proposition 6.4.5, the decomposition

$$X \cong \mathrm{Ker}(e) \oplus \mathrm{Im}(e).$$

From Example 5.2.2 the subcategory $\mathrm{Pf}(X) \subseteq \mathrm{D}(\mathcal{O}_X)$ is thick, from which we get that both $\mathrm{Ker}(e)$ and $\mathrm{Im}(e)$ are perfect complexes. Therefore, by Remark 6.4.6, e splits in $\mathrm{Pf}(X)$. $\square$

Theorem 6.5.7. *Let X be a noetherian scheme and $U \subseteq X$ an open subspace, with complement $Z = X \backslash U$. There is an equivalence of $\otimes$-triangulated categories*

$$\overline{\mathrm{Pf}(X)/\mathrm{Pf}_Z(X)} \simeq \mathrm{Pf}(U)$$

Proof. In order to prove the equivalence, let's induce a full, faithful and essentially surjective functor

$$\overline{\mathrm{Pf}(X)/\mathrm{Pf}_Z(X)} \longrightarrow \mathrm{Pf}(U)$$

Consider the restriction functor $r_U : \mathrm{Pf}(X) \to \mathrm{Pf}(U)$ mapping $F^{\cdot}$ to $F^{\cdot}|_U$, and observe that any perfect complex $F^{\cdot}$ on X whose support is contained in Z, is such that the restriction $F^{\cdot}|_U$ has each stalk zero, and so is itself the zero complex in $\mathrm{Pf}(U)$. Therefore, it's induced a functor

$$\hat{r_U} : \mathrm{Pf}(X)/\mathrm{Pf}_Z(X) \longrightarrow \mathrm{Pf}(U)$$

such that $\hat{r_U} Q_Z = r_U$, where Q_Z is the localization functor.

Let's prove this functor to be full and faithful. Let $E^{\cdot}$, $E'^{\cdot}$ be perfect complexes on X and take a morphism in $\mathrm{Pf}(U)$

$$b : E^{\cdot}|_U \longrightarrow E'^{\cdot}|_U$$

By Proposition 5.2.16 there exists a third perfect complex $E''^{\cdot}$ on X and morphisms $a : E''^{\cdot} \to E^{\cdot}$, $a' : E''^{\cdot} \to E'$ in $\mathrm{Pf}(X)$ such that the diagram

$$
\begin{tikzcd}
& E''^{\cdot}|_U \arrow[dl, "a|_U"'] \arrow[dr, "a'|_U"] & \\
E^{\cdot}|_U \arrow[rr, "b"] & & E'^{\cdot}|_U
\end{tikzcd}
$$

commutes in $\mathrm{Pf}(U)$, with $a|_U$ isomorphism. Let's observe that the pair of morphisms (a, a') in $\mathrm{Pf}(X)$ defines a roof, and hence a morphism in the quotient category. That is true if a happens to be in $S(\mathrm{Pf}_Z(X))$, the class of morphisms whose cone is in $\mathrm{Pf}_Z(X)$. The complex $C(a)$, which is perfect because $\mathrm{Pf}(X) \subseteq \mathrm{D}(\mathcal{O}_X)$ is a triangulated subcategory, is in fact in $\mathrm{Pf}_Z(X)$, because $C(a)|_U = C(a|_U)$ is isomorphic to zero, being the cone of an isomorphism.

Therefore, we have a morphism in $\mathrm{Pf}(X)/\mathrm{Pf}_Z(X)$ whose restriction at U is the original morphism b. This proves $\hat{r_U}$ to be full.

In order to prove faithfulness, consider two perfect complexes $E^{\cdot}$, $E'^{\cdot}$ and two morphisms

$$
E^{\cdot} \overset{a}{\underset{b}{\rightrightarrows}} E'^{\cdot}
$$

in $\mathrm{Pf}(X)/\mathrm{Pf}_Z(X)$, and suppose $\hat{r_U}(a) = \hat{r_U}(b)$ in $\mathrm{Pf}(U)$. We can represent a and b as roofs in $\mathrm{Pf}(X)$, given as

$$
\begin{tikzcd}
& W_a^{\cdot} \arrow[dl, "s"'] \arrow[drr, "f"', near end] & W_b^{\cdot} \arrow[dll, "t", near end] \arrow[dr, "g"] & \\
E^{\cdot} & & & E'^{\cdot}
\end{tikzcd}
$$

Since $\hat{r_U} Q_Z = r_U$, the functor $\hat{r_U}$ applies on the roofs $E^{\cdot} \leftarrow W_a^{\cdot} \rightarrow E'^{\cdot}$, $E^{\cdot} \leftarrow W_b^{\cdot} \rightarrow E'^{\cdot}$, just as a restriction functor. Therefore, we have commutative triangles in $\mathrm{Pf}(U)$

$$
\begin{tikzcd}
& W_a^{\cdot}|_U \arrow[dl, "s|_U"'] \arrow[dr, "f|_U"] & \\
E^{\cdot}|_U \arrow[rr, "\hat{r_U}(a)"] & & E'^{\cdot}|_U
\end{tikzcd}
\qquad
\begin{tikzcd}
& W_b^{\cdot}|_U \arrow[dl, "t|_U"'] \arrow[dr, "g|_U"] & \\
E^{\cdot}|_U \arrow[rr, "\hat{r_U}(b)"] & & E'^{\cdot}|_U
\end{tikzcd}
$$

Observe then, that being $C(s)$ in $\mathrm{Pf}_Z(X)$, one has $C(s)|_U = C(s|_U) = 0$, and therefore $s|_U$ is isomorphism in $\mathrm{Pf}(U)$. The same is true for $t|_U$ of course, and thanks to these isomorphisms

$$
W_a^{\cdot}|_U \cong E^{\cdot}|_U \cong W_b^{\cdot}|_U
$$

we find morphisms $W_a^{\cdot}|_U \cong W_b^{\cdot}|_U \overset{g|_U}{\underset{f|_U}{\rightrightarrows}} E'^{\cdot}|_U$ to which we can apply Proposition 5.2.17 in order to find a perfect complex $W^{\cdot}$ on X and a morphism

$$
c : W^{\cdot} \longrightarrow E^{\cdot}
$$

with $gc = fc$ and such that $c|_U$ is isomorphism in $\mathrm{Pf}(U)$. It follows that the cone of c is in $\mathrm{Pf}_Z(X)$. Therefore, we have a commutative triangle in $\mathrm{Pf}(X)$

$$\begin{tikzcd} & W^{\cdot} \arrow[dl, "c"'] \arrow[dr, "ac=bc"] & \\ E^{\cdot} \arrow[rr, shift left, "a"] \arrow[rr, shift right, "b"'] & & E'^{\cdot} \end{tikzcd}$$

proving that a and b are the same morphism in $\mathrm{Pf}(X)/\mathrm{Pf}_Z(X)$. This gives faithfulness of $\hat{r_U}$.

Now Proposition 6.5.6 applied to U allows to lift the functor $\hat{r_U}$ to the idempotent completion, so that we get

$$\bar{r_U} : \mathrm{Pf}(X)/\bar{\mathrm{P}}\mathrm{f}_Z(X) \longrightarrow \mathrm{Pf}(U)$$

and Lemma 6.4.18 tells us that it remains full and faithful.

Then, let's see that it's also essentially surjective. Thanks to Lemma 6.4.19, it suffices to prove any perfect complex $F^{\cdot}$ on U to be direct summand of the restriction of a perfect complex on X. This is immediately given by the K_0 Extension Lemma, Proposition 5.2.18, applied with $Y = X$, once observed that the class $[F^{\cdot} \oplus TF^{\cdot}] = 0 \in \mathrm{K}_0(\mathrm{Pf}(U))$ is certainly in the image of the group homomorphism

$$\mathrm{K}_0(\mathrm{Pf}(X)) \longrightarrow \mathrm{K}_0(\mathrm{Pf}(U)).$$

□

Theorem 6.5.8. *Let X be a noetherian scheme. Then $(X, \mathcal{O}_X)$ is isomorphic to the locally ringed space $(\mathcal{S}pec(\mathrm{Pf}(X)), \mathcal{O}_{\mathcal{S}pec(\mathrm{Pf}(X))})$.*

Proof. Since we already have an homeomorphism $f : X \to \mathcal{S}pec(\mathrm{Pf}(X))$, it suffices to prove an isomorphism between the structure sheaves $\mathcal{O}_{\mathcal{S}pec}$ and $f_*\mathcal{O}_X$.

For any open $U \subseteq \mathcal{S}pec(\mathrm{Pf}(X))$, the sheaf $\mathcal{O}_{\mathcal{S}pec(\mathrm{Pf}(X))}(U)$ is the ring

$$\mathrm{End}_{\mathrm{Pf}(X)/(\mathrm{Pf}(X))_Z}(1_U),$$

where Z is the complement of U in $\mathcal{S}pec(\mathrm{Pf}(X))$. Observe then, that this endomorphism ring is equal to the endomorphism ring in the category $\mathrm{Pf}(X)/(\bar{\mathrm{P}}\mathrm{f}(X))_Z$, because the Yoneda functor injects faithfully any category in its Karoubi envelope.

Now, the category $(\mathrm{Pf}(X))_Z$ is given by the objects $E^{\cdot}$ in $\mathrm{Pf}(X)$ such that $\mathrm{supp}(E^{\cdot}) \subseteq Z$, or equivalently such that

$$f^{-1}(\mathrm{supp}(E^{\cdot})) = \mathrm{Supph}(E^{\cdot}) \subseteq f^{-1}(Z).$$

Therefore $(\mathrm{Pf}(X))_Z = \mathrm{Pf}_{f^{-1}(Z)}(X)$, and since 1 in $\mathrm{Pf}(X)$ (and hence 1_U in $\mathrm{Pf}(X)/\mathrm{Pf}(X)_Z$) is the complex $\mathcal{O}_X$ centered at degree zero, then

$$\mathcal{O}_{\mathcal{S}pec(\mathrm{Pf}(X))}(U) = \mathrm{End}_{\mathrm{Pf}(X)/\bar{\mathrm{Pf}}_{f^{-1}Z}(X)}(\mathcal{O}_X)$$

Now observe that the equivalence of Theorem 6.5.7, which is given as the functor induced by the restriction, provides isomorphism

$$\mathrm{End}_{\mathrm{Pf}(X)/\bar{\mathrm{Pf}}_{f^{-1}Z}(X)}(\mathcal{O}_X) \cong \mathrm{End}_{\mathrm{Pf}(f^{-1}U)}(\mathcal{O}_{f^{-1}U}).$$

Eventually, this is just $\mathrm{Hom}(\mathcal{O}_{f^{-1}U}, \mathcal{O}_{f^{-1}U})$, which is isomorphic to the ring of sections $\mathcal{O}_X(f^{-1}(U)) = f_*\mathcal{O}_X(U)$. □

Corollary 6.5.9. *Let X and Y be two noetherian schemes such that there exists a tensor triangulated equivalence of tensor triangulated categories*

$$\mathrm{Pf}(X) \simeq \mathrm{Pf}(Y).$$

Then there exists an isomorphism of schemes $X \cong Y$.

Proof. It certainly suffices to prove that equivalent tensor triangulated categories have the same spectrum. Let's call $F : \mathrm{Pf}(X) \to \mathrm{Pf}(Y)$ and $G : \mathrm{Pf}(Y) \to \mathrm{Pf}(X)$ the tensor triangulated functors witnessing the equivalence. They induce a bijection, which we still call F with a little of ambiguity,

$$\begin{aligned} F : \mathcal{S}pec(\mathrm{Pf}(X)) &\longrightarrow \mathcal{S}pec(\mathrm{Pf}(Y)) \\ \mathbf{P} &\longmapsto F(\mathbf{P}) \end{aligned}$$

with inverse $\mathbf{Q} \mapsto G(\mathbf{Q})$, where $F(\mathbf{P})$ and $G(\mathbf{Q})$ are respectively defined as the full subcategories of $\mathrm{Pf}(Y)$ and $\mathrm{Pf}(X)$ whose objects are, up to isomorphism, of the form $F(A)$ for $A \in \mathbf{P}$ and $G(B)$ for $B \in \mathbf{Q}$.

Everything that need to be checked is that these mappings are well defined, continuous and that it's induced an isomorphism of the level of structure sheaves. If FA is in $F(\mathbf{P})$, then $T(FA) \cong F(TA)$ also is in $F(\mathbf{P})$ since TA is in $\mathbf{P}$, while if $FA \to FB \to D \to TFA$ is a distinguished triangle with two objects, FA and FB, in $F(\mathbf{P})$, then there is a triangle

$$GFA \longrightarrow GFB \longrightarrow GD \longrightarrow TGFA$$

which is isomorphic to $A \to B \to GD \to TA$, which then is distinguished, too. Now since A and B are in $\mathbf{P}$, so is GD, and hence $D \cong FGD$ is in $F(\mathbf{P})$.

This shows that the full subcategory $F(\mathbf{P})$ is triangulated. In order to see that it's a prime $\otimes$-ideal, consider $F(A) \cong E \oplus D$, which gives $A \cong GFA \cong G(E \oplus D) \cong GE \oplus GD$, providing, since $\mathbf{P}$ is thick, that both GE and GD are in $\mathbf{P}$, and therefore both $E \cong FGE$ and $D \cong FGD$ are in $F(\mathbf{P})$. If FA is an object in $F(\mathbf{P})$ and E is any object in $\mathrm{Pf}(Y)$, then $G(FA \otimes E) \cong A \otimes GE$ which is in the ideal $\mathbf{P}$ since A is. Therefore $FA \otimes E \cong F(A \otimes GE)$ is in $F(\mathbf{P})$. This proves $\mathbf{P}$ to be a $\otimes$-ideal. Eventually, if $FA \cong E \otimes D$, then $A \cong GFA \cong G(E \otimes D) \cong GE \otimes GD$ implies that either GE or GD are in $\mathbf{P}$, and hence either $E \cong FGE$ or $D \cong FGD$ is in $F(\mathbf{P})$. So, $F(\mathbf{P})$ is a prime $\otimes$-ideal.

Continuity may be checked on the basis of closed of the form $\{\mathbf{Q}|A \notin \mathbf{Q}\} \subseteq \mathcal{S}pec(\mathrm{Pf}(Y))$, whose preimage is $\{G\mathbf{Q}|A \notin \mathbf{Q}\}$. Since $A \in \mathbf{Q}$ if and only if $GA \in G(\mathbf{Q})$, and G is essentially surjective, we get that the preimage is the basic closed set $\{\mathbf{P}|GA \notin \mathbf{P}\}$.

Eventually, observe that this homeomorphism comes together with an isomorphism

$$\mathcal{O}_{\mathcal{S}pec(\mathrm{Pf}(Y))} \longrightarrow F_*\mathcal{O}_{\mathcal{S}pec(\mathrm{Pf}(X))} \tag{6.1}$$

For each open $U \subseteq \mathcal{S}pec(\mathrm{Pf}(Y))$, the functor F restricts to an equivalence of subcategories $\mathrm{Pf}(Y)_Z \simeq \mathrm{Pf}(X)_{F^{-1}(Z)}$, where Z is the complement of U. This is because

$$\{FE|\operatorname{supp} E \subseteq F^{-1}(Z)\} = \{FE|F(\operatorname{supp} E) \subseteq Z\},$$

and since $F(\operatorname{supp} E) = F(\{\mathbf{P}|E \notin \mathbf{P}\}) = \{\mathbf{Q}|FE \notin \mathbf{Q}\} = \operatorname{supp} FE$, we get, from F being essentially surjective, that $F(\mathrm{Pf}(X)_{F^{-1}(Z)}) = \mathrm{Pf}(Y)_Z$. It follows that F passes to an equivalence between the quotients, and hence that there is an isomorphism

$$\mathrm{End}_{\mathrm{Pf}(Y)/\mathrm{Pf}(Y)_Z}(\mathcal{O}_Y) \cong \mathrm{End}_{\mathrm{Pf}(X)/\mathrm{Pf}(X)_{F^{-1}(Z)}}(F^{-1}(\mathcal{O}_Y))$$

which eventually, since F preserves the unit of the tensor structure $F^{-1}(\mathcal{O}_Y) = \mathcal{O}_X$, and $F^{-1}(Z) = F^{-1}(U)^{\mathsf{c}}$, is isomorphic to

$$\mathrm{End}_{\mathrm{Pf}(X)/\mathrm{Pf}_{F^{-1}(U)^{\mathsf{c}}}}(\mathcal{O}_X) = \mathcal{O}_{\mathcal{S}pec(\mathrm{Pf}(X))}(F^{-1}(U)).$$

This proves the desired isomorphism (6.1), giving an isomorphism of schemes. □

Appendix A

Some standard definitions and results on schemes

Proposition A.0.1. *(Isbell adjunction) There is an adjunction between the functors* $(\Gamma \dashv \mathrm{Spec})$ *between the categories of schemes and commutative rings.* $\Gamma : \boldsymbol{Sch} \to \boldsymbol{ComRing^{op}}$ *is the global section functor* $(X, \mathcal{O}_X) \mapsto \Gamma(X, \mathcal{O}_X)$*, while* $\mathrm{Spec} : \boldsymbol{ComRing^{op}} \to \boldsymbol{Sch}$ *is* $R \mapsto (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$*. More explicitly, for every scheme* X *and any commutative ring* R *there is a natural bijection*

$$\mathrm{Hom}_{\boldsymbol{ComRing}}(R, \Gamma(X, \mathcal{O}_X)) \cong \mathrm{Hom}_{\boldsymbol{Sch}}(X, \mathrm{Spec}\, R).$$

Proof. On one hand, a morphism $X \to \mathrm{Spec}\, R$ comes by definition with a morphism $f^\sharp : \mathcal{O}_{\mathrm{Spec}\, R} \to f_*\mathcal{O}_X$, which we can compute on global sections in order to get the desired morphism $R \to \Gamma(X, \mathcal{O}_X)$.

Conversely, let $\phi : R \to \Gamma(X, \mathcal{O}_X)$ be a ring homomorphism. The affine case is well known, since $\mathbf{AffSch} \cong \mathbf{ComRing^{op}}$. Thus cover $X = \bigcup\limits_i U_i$ by affine schemes $U_i = \mathrm{Spec}\, A_i$, and define for every index i a morphism $f_i : \mathrm{Spec}\, A_i \to \mathrm{Spec}\, R$ to be the spectrum of the morphism

$$R \longrightarrow \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(U_i, \mathcal{O}_X) \cong A_i.$$

In order to get a morphism $X \to \mathrm{Spec}\, R$ one has to prove that these morphisms f_i agrees on each intersection. Let $U_i \cap U_j$ be covered by affine subspaces $U_i \cap U_j = \bigcup\limits_k W_{ijk}$, it suffices then to prove that $f_i|_{W_{ijk}} \cong f_j|_{W_{ijk}}$. That is because for every i the morphism $f_i|_{W_{ijk}}$ is the spectrum of the map

$$R \longrightarrow \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(U_i, \mathcal{O}_X) \longrightarrow \Gamma(W_{ijk}, \mathcal{O}_X),$$

which doesn't really depend on U_i, since restrictions can compose.

It's clear that these operations are inverses to each other. $\square$

Corollary A.0.2. $\mathrm{Spec}(\mathbb{Z})$ *is the terminal object in the category of schemes.*

Proof. Since $\mathbb{Z}$ is initial object in the category of commutative rings, for any scheme X there is a unique arrow $X \to \mathrm{Spec}(\mathbb{Z})$ corresponding to the unique arrow $\mathbb{Z} \to \Gamma(X\mathcal{O}_X)$ under the Isbell adjunction, Proposition A.0.1. $\square$

Definition A.0.3. Let X be a scheme over S, that is a scheme together with a morphism $X \to S$. The *diagonal morphism* of X is the map $\Delta : X \to X \times_S X$ induced by the pullback property with the identity morphisms

$$\begin{tikzcd}
X \arrow[dr, dashed] \arrow[drr, "\mathrm{id}"] \arrow[ddr, "\mathrm{id}"'] & & \\
& X \times_S X \arrow[r] \arrow[d] & X \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

Remark A.0.4. The product in the category of schemes does not provide the same underlying set of the product in **Set**. In fact, being $\mathrm{Spec}(\mathbb{Z})$ the terminal object, it follows that in the category of schemes, the product $X \times Y$ is in fact the fibered product $X \times_{\mathrm{Spec}(\mathbb{Z})} Y$.

Definition A.0.5. Let X and Y be schemes. A morphism of schemes $f : X \to Y$ is said to be *quasi-separated* if the diagonal morphism is quasi-compact, i.e. if the preimage of any quasi-compact open subspace is again quasi-compact. A scheme X is said to be *quasi-separated* if the morphism $X \to \mathrm{Spec}\,\mathbb{Z}$ is quasi-separated.

The morphism $f : X \to Y$ is said to be *separated* if the diagonal morphism is a closed immersion. The scheme X is said to be *separated* if the morphism $X \to \mathrm{Spec}\,\mathbb{Z}$ is separated.

Proposition A.0.6. *Let X be a separated scheme and $A, B \subseteq X$ affine subschemes. Then $A \cap B$ is an affine subscheme.*

Proof. Declare the affine subspaces to be $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$, and recall that by definition of product in the category of schemes it holds $U \times_{\mathrm{Spec}\mathbb{Z}} V = \mathrm{Spec}(A \otimes_{\mathbb{Z}} B)$, which is then an affine open subscheme of $X \times_{\mathrm{Spec}\mathbb{Z}} X$.

Therefore, the result follows from the fact that closed immersions $f : Z \to Y$ into an affine scheme $Y = \mathrm{Spec}\,R$ correspond to ideals $I \subseteq R$ by taking the module associated to the quasi-coherent sheaf $\mathrm{Ker} f^{\#}$.

So, there's an ideal J of the $\mathbb{Z}$-algebra $A \otimes_{\mathbb{Z}} B$ such that the diagonal closed immersion, which clearly restricts as

$$\begin{tikzcd}
X \arrow[r, "\Delta"] & X \times_{\mathrm{Spec}\mathbb{Z}} X \\
U \cap V \arrow[r] \arrow[u, hook] & U \times_{\mathrm{Spec}\mathbb{Z}} V \arrow[u, hook]
\end{tikzcd}$$

provides $U \cap V = \mathrm{Spec}(A \otimes B/J)$. $\square$

Proposition A.0.7. *Any irreducible component of a scheme X has a unique generic point.*

Proof. Let $Y \subseteq X$ irreducible. For any affine subspace $U \subseteq X$, the subspace $Y \cap U$ of $U = \mathrm{Spec}(R)$ is given by $V(I) = \{p \in \mathrm{Spec}(R) | p \supseteq I\}$ for a unique radical ideal $I \subseteq R$. Now, $Y \cap U$ is either empty or irreducible itself, and we can certainly take at least one affine open U with $Y \cap U \neq \varnothing$. Thus for U such, $I = p$ is a prime ideal, i.e. a point in $p \in \mathrm{Spec}(R) \subseteq X$, and it's so clearly a generic point of its closure $V(p) = Y \cap U$. This holding for each U implies that $\overline{\{p\}} = Y$. Uniqueness holds since any scheme is T_0, and hence $\overline{\{x\}} = \overline{\{y\}} \Rightarrow x = y$. $\square$

Lemma A.0.8. *Let X be a noetherian scheme and consider a family of closed subspaces $\{Y_\alpha\}_{\alpha \in A}$, as well as a closed subspace $Y \subseteq \bigcup_{\alpha \in A} Y_\alpha$. Then there exists a finite set $E \subseteq A$ such that $Y \subseteq \bigcup_{\alpha \in E} Y_\alpha$.*

Proof. Decompose Y, which is closed in a noetherian scheme and hence noetherian, as the finite union of its irreducible components $Y = T_1 \cup \cdots \cup T_k$, each of which has a unique generic point y_i for $i = 1, \ldots, k$. Thus there are indexes $\alpha_1, \ldots, \alpha_k \in A$ such that for every $i \in \{1, \ldots, k\}$ $y_i \in Y_{\alpha_i}$, and being each Y_α closed, $T_i = \overline{\{y_i\}} \subseteq Y_{\alpha_i}$. That means $Y \subseteq \bigcup_{i=1}^{k} Y_{\alpha_i}$. $\square$

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