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# Homeomorphic extension of Quasi-Isometries and Iteration Theory

Tesi di Laurea Magistrale in Analisi Complessa

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# Introduzione

A partire dal Teorema della Mappa di Riemann nasce l'interesse per biolomorfismi in una o più variabili. Poincaré mostrò che non esiste biolomorfismo tra polidisco e palla unitaria in $\mathbb{C}^2$. Questo fatto suggerisce che domini mappati biolomorficamente possano estendere la regolarità di questa mappa fino alle rispettive frontiere. Un approccio molto influente è stato affrontato da Fefferman in [18] (pubblicato nel 1974), provando che ogni biolomorfismo tra domini fortemente pseudoconvessi con frontiere lisce si estende a diffeomorfismo alle chiusure dei rispettivi domini. Nel suo articolo è utilizzato un risultato classico di invarianza della distanza/metrica di Bergmann tra domini biolomorfi e egli stesso ha notato un interessante comportamento delle geodetiche (rispetto a questa distanza) quando si avvicinano alla frontiera del dominio considerato ( [18], pagina 3).
La prima parte di questa tesi segue principalmente il lavoro di Abate in [3]. Nel primo capitolo sono introdotte due distanze (talvolta degeneri) su varietà complesse, inventate da Carathéodory e Kobayashi rispettivamente, invarianti per mappe biolomorfe. Nella seconda parte dell'Esempio 1.4 viene fornita la risposta a una domanda lasciata aperta in [ [2], pagina 52]. Per un dominio in più variabili complesse scopriremo che la pseudodistanza di Kobayashi $k$ può essere rappresentata come forma integrata di una precisa pseudo-metrica e questo risulterà molto utile quando necessiteremo di stime dal basso e dall'alto. Nell'ultima sezione del capitolo, spiegando l'articolo di Barth [5], cercheremo condizioni equivalenti tra la geometria di un dominio $D$ e lo spazio metrico $(D, k_D)$ che diventa completo o geodetico.

Nel Capitolo 2, sempre seguendo il lavoro di Abate in [2], proveremo l'estensione omeomorfa di un biolomorfismo tra domini fortemente pseudoconvessi con regolarità di frontiera $C^2$. L'obiettivo principale sarà trovare stime dall'alto e dal basso per la distanza di Kobayashi quando uno dei punti si avvicina alla frontiera.
La seconda parte della tesi presenta il lavoro di Bracci, Zimmer e Gaussier in [19]. Nel capitolo 3 introduciamo gli strumenti coinvolti come l'iperbolicità secondo Gromov e la Compattificazione di Gromov, il Teorema di Karlsson, Mappe commutative e 1-Lipschitz dello spazio in sè e infine la Compattificazione "Finale". Per ogni argomento saranno esposti diversi esempi per entrare nel profondo di queste definizioni e risultati. Un altro apporto cruciale per questo capitolo, dovuto a Ghys e De La Harpe in [23], è l'invarianza dell'iperbolicità secondo Gromov sotto l'azione di Quasi-Isometrie.
Nel Capitolo 4 presenteremo le dimostrazioni di Teoremi riguardanti l'estensione omeomorfa di Quasi-Isometrie e l'Iterazione di mappe olomorfe in sè per certi domini di $\mathbb{C}^d$. Per questo scopo necessitiamo di due fatti notevoli. Alla fine dell'anno 2000 è stato provato un risultato molto importante da Balogh e Bonk ( [31]); affermando che per domini limitati e fortemente pseudoconvessi la distanza di Kobayashi è Iperbolica secondo Gromov e la Frontiera di Gromov coincide con la Frontiera Euclidea. Inoltre, grazie al testo di Ghys e De La Harpe [23], si ha che spazi metrici con iperbolicità secondo Gromov, quasi-isometrici, presentano Compattificazioni di Gromov omeomorfe. Ricordiamo che ogni biolomorfismo tra domini è un'isometria, quando i domini sono dotati di distanza di Kobayashi. Si ha quindi che queste due conseguenze consentono una dimostrazione sull'estensione omeomorfa alle chiusure euclidee tra domini fortemente pseudoconvessi biolomorfi. Inoltre gli strumenti precedentemente illustrati permettono di modificare le nostre ipotesi sui due domini coinvolti e sul tipo di omeomorfismo ad essi relativo, che diventa una Quasi-Isometria, ma permettono comunque un'estensione omeomorfa alle Compattificazioni finali dei rispettivi domini (Teorema 4.2.1). Sebbene la conclusione sia più debole del risultato di Fefferman, questa vale

per una più estesa classe di mappe, ovvero quelle che sono Quasi-Isometrie rispetto alle distanze di Kobayashi dei due domini.
Ulteriori conseguenze del Capitolo 6 riguardano l'estensione del Teorema di Denjoy-Wolff 3.4.1 per domini di $\mathbb{C}^d$ e presentano una situazione tipica del Teorema di Denjoy Wolff per mappe commutative olomorfe senza punti fissi nel dominio (Corollari 4.3.1 e 4.3.2).
Nel capitolo 6 saranno esposte delle idee di base che possono concorrere per degli ulteriori sviluppi.

# Introduction

Starting from the Riemann Mapping Theorem it arises the interest for biholomorphisms over domains in one or several complex variables. Poincaré showed that there is no analytic isomorphism between the Polydisc and the unit ball already in $\mathbb{C}^2$. The previous fact may suggest that biholomorphic domains are a class of such well-behaved sets that could extend some regularity of the biholomorpshism until their respective boundaries. A very influent approach was faced by Fefferman in [18] (published in the year 1974), by proving that every biholomorphism between bounded strongly pseudoconvex domains with $C^\infty$ boundaries extends as a $C^\infty$ diffeomorphism to the closures of the domains. In this work is quoted a classical result that presents an isometry respect to the Bergman metric between biholomorphic domains and he noticed an interesting behaviour of geodesics when they are going to the boundary of a considered domain ( [18], page 3).

The first part of this thesis mainly follows Abate's work in [3]. In Chapter 1 there will be introduced two (sometimes degenerate) distances on complex manifolds, invented by Carathéodory and Kobayashi respectively, which are invariant under biholomorphic mappings. In the second part of Example 1.4 we answer to a question that was left unsolved in [ [2], page 52] about the Caratheodory distance of an annulus or a spherical shell in several variables. For a domain in several complex variables we will discover that the Kobayashi pseudodistance $k$ can be represented as the integrated form of a suitable pseudo-metric and this will turn very useful when we need some upper and lower bounds. In the last section of this chapter, by explaining

Barth's work [5], we will search for some equivalent conditions between the geometry of a domain $D$ and $(D, k_D)$ becoming a complete or geodesic metric space.

In Chapter 2 still following Abate's work [2] we are going to prove the homeomorphic extension of a biholomorphism between $C^2$-smooth strongly pseudoconvex domains. The main purpose will be to find some lower and upper bounds for the Kobayashi distance when one point is approaching to the boundary.

The second part of this thesis mainly follows the work of Bracci, Gaussier and Zimmer from [19]. In Chapter 3 we introduce the main tools involved such as Gromov Hyperbolicity and Gromov Compactification, Karlsson's Theorem, Commuting 1-Lipschitz selfmaps and the End Compactification. We present several examples for each topic to get deeper in these definitions and results. Another crucial intake for this chapter, due to the book of Ghys and De La Harpe ( [23]), is the Gromov Hyperbolicity invariance under Quasi-Isometries.

In Chapter 4 we provide the proofs of Theorems involving the Homeomorphic extension of Quasi-Isometries and Iteration of Holomorphic Selfmaps for certain domains in $\mathbb{C}^d$.

For this purpose we need two remarkable facts. At the end of the year 2000, an important result has come proved by Balogh and Bonk ( [31]), it claims that the Kobayashi distance on a bounded strongly pseudoconvex domain is Gromov hyperbolic and the Gromov boundary coincides with the Euclidean boundary. Moreover, thanks to Ghys and De La Harpe's work [23], for a Gromov hyperbolic metric space a homeomorphic quasi-isometry extend as homeomorphism between the Gromov compactifications of the metric spaces. Since every biholomorphism between two domains is an isometry when the domains are endowed with their Kobayashi distances, this provides a new proof that every biholomorphism between strongly pseudoconvex domains extends to a homeomorphism of the Euclidean closures. Moreover the previous illustrated tools allow us to modify our assumption about the two domains involved and homeomorphism related to them which then become a Quasi-Isometry, but still allow an homeomorphic extension to the End Compactifications of the respective domains (Theorem 4.2.1). Despite this conclusion is weaker than Fefferman's result, it holds for a much larger class of maps, the ones that are quasi-isometries relative to the Kobayashi distances.

Other consequences in this chapter are related to extend the Denjoy-Wolff Theorem 3.4.1 for domains in $\mathbb{C}^d$ and present the Denjoy-Wolff behaviour for commuting holomorphic selfmaps with no fixed point in the domain itself (Corollaries 4.3.1 and 4.3.2).

In Chapter 6 there will be exposed some basic ideas that may concur for a future work.

# Contents

# Chapter 1

# Invariant pseudodistances for a complex domain

## 1.1 The Carathéodory pseudodistance

In this and the next sections we are going to provide two attempts of giving a pseudodistance to a complex manifold.
There will be introduced some examples in this process that will lead us to understand the behaviour of holomorphic mappings from or to the unit disk by the geometry of the manifold itself.

**Definition 1.1.** A *pseudodistance* on a set $X$ is a function $d : X \times X \to \mathbb{R}^+$ so that:

(i) $d(x, x) = 0$ for every $x \in X$

(ii) $d(x, y) = d(y, x)$ for every $x, y \in X$

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for every $x, y, z \in X$

**Definition 1.2** (Poincaré distance)**.** In $\Delta$, the unit disk of $\mathbb{C}$, we define the Poincaré distance as:

$$\omega(z_1, z_2) = \frac{1}{2} \log \frac{1 + |\frac{z_1 - z_2}{1 - \overline{z_2} z_1}|}{1 - |\frac{z_1 - z_2}{1 - \overline{z_2} z_1}|} = \tanh^{-1} \big( \frac{z_1 - z_2}{1 - \overline{z_2} z_1} \big)$$

for every $z_1, z_2 \in \Delta$.

A list of properties that $\Delta$ gets when endowed with the distance $\omega$ can be found in Chapter 1.1 of [3].

**Definition 1.3** (Carathéodory pseudodistance)**.** Let $X$ be a complex connected manifold, the *Carathéodory pseudodistance* $c_X$ on X is defined by

$$\forall z, w \in X \;\; c_X = \sup\{\omega(h(z), h(w)) | h \in Hol(X, \Delta)\}.$$

**Proposition 1.1.1.** *$c_X(z, w)$ is finite for every $z, w$ in $X$.*

*Proof.* By contradiction suppose it is not finite.
This implies that for every $n$ it exists $(\varphi_n) \in Hol(X, \Delta)$ so that $\omega(\varphi_n(z), \varphi_n(w)) = +\infty$ for some $z, w \in X$ . By Montel's Theorem there is a $\varphi \in Hol(X, \Delta)$ uniform limit on compact subsets of $X$ for $(\varphi_n)_{n\in\mathbb{N}}$ up to its subsequences. Hence for the previous $z, w \in X$ it holds that $\omega(\varphi(z), \varphi(w)) = +\infty$. Then one between $\varphi(z)$ or $\varphi(w)$ belong to $\partial\Delta$, but we have that $\varphi$ gives values just inside the unit disk. □

*Remark* 1. With Proposition 1.1.1 it follows that $c_X$ is a pseudodistance on $X$.

*Remark* 2. We shall denote by $B_c(z, r) = \{w \in X | c_X(z, w) < r\}$, the open Carathéodory ball of center $z \in X$ and radius $r > 0$.

We will introduce some examples for computing the Carathéodory distance and its features in some particular complex manifolds.

**Example 1.1.** In case $X = \mathbb{C}$ as a consequence of Liouville's Theorem every entire function is constant and hence $c_X$ degenerates to the null function on all $\mathbb{C} \times \mathbb{C}$.

**Example 1.2.** Let $M$ be a compact and connected manifold. We can observe that every holomorphic function from this source space $M$ to the unit disk $\Delta$ is a constant function.
Indeed, we can argue that, said $f : M \to \mathbb{C}$ holomorphic and $z_0$ the point

of $M$ in which the maximum of $|f|$ is attained, $f \equiv f(z_0)$ on $M$. This is motivated by the fact that the set $N = \{z \in M \mid f(z) = f(z_0)\}$ is non-empty and closed, moreover the Maximum Modulus Principle assures us that if $w$ is in $N$ then it exists a whole neighbourhood $U$ of $w$ such that $f|_U \equiv f(z_0)$. Hence $N$ is also open, by connectedness $N = M$ and $f$ is constant.
In conclusion, as in the previous example, $c_M$ degenerates to the null function on the whole $M \times M$.

**Example 1.3.** Let $A$ be a proper subset of $\mathbb{C}^n$ $(n \geq 1)$ that is also an *analytic subset* i.e. for all $a \in A$ exists an open neighbourhood $U$ of $a$ and finitely many holomorphic functions $f_1, \cdots, f_p$ so that $A \cap U = \{z \in U | f_1(z) = \cdots = f_p(z) = 0\}$. Now set $M = \mathbb{C}^n \setminus A$ and by the Riemann's Continuation Theorem we have that all the holomorphic functions bounded on $M$ are holomorphic over all $\mathbb{C}^n$. Thus an holomorphic function $f : M \to \Delta$ has to be constant by the Liouville's Theorem.
We can conclude again that $c_M$ degenerates to the null function.

**Example 1.4.** Consider the spherical shell $M = \{z \in \mathbb{C}^n | r < |z| < R\}$ and the ball $B = \{z \in \mathbb{C}^n | |z| < R\}$ (with $n \geq 2$). Since $B$ is holomorphically convex we get that $B$ is a domain of holomorphy. We can see that the smallest holomorphically convex set containing $M$ is $B$, hence $B$ is the envelope of holomorphy of $M$. This means that every holomorphic function from $M$ to the unit disk $\Delta$ can be extended holomorphically to $B$. Moreover the Maximum Principle provides that the extension still gives values in the unit disk $\Delta$. Then $c_M \equiv c_B$.
**Case n=1** Consider the vertical strip $M' = \{z \in \mathbb{C} : \log r < \mathrm{Re} z < \log R\}$, we observe that $M'$ is simply connected and $M'$ is the universal cover of $M$. Moreover $M'$ is biholomorphic to $\Delta$ due to a composition with a suitable exponential function with target space the upper-half plane and then the Cayley Transform. In the next section we will discover that the Carathéodory distance is invariant under biholomorphisms (Corollary 1.2.2). This then shows that $c_\Delta < c_M$.

As we could see there were many cases in which the Carathéodory pseudodistance degenerated to 0, in order to improve the non-degeneracy of a pseudodistance we will introduce the next concept.

## 1.2 The Kobayashi pseudodistance

The "dual" concept is the function $\delta : X \times X \to [0, +\infty]$ defined by

$$\delta_X(z, w) = \inf\{\omega(\zeta, \eta) \mid \exists\, \varphi \in Hol(\Delta, X) : \varphi(\zeta) = z, \varphi(\eta) = w\} \tag{1.1}$$

for every $z, w \in X$.

Generally $\delta_X$ does not suffice the triangular inequality, we will explain it in the following

**Example 1.5.** Let $\Gamma_\varepsilon = \{(z, w) : |z| < 1, |w| < 1, |zw| < \varepsilon\}$, let $A = (0, \frac{1}{2})$ and $B = (\frac{1}{2}, 0)$.
We can observe, by using the holomorphic function $f : \Delta \to \Gamma_\varepsilon$ so that $f(z) = (0, z)$, that $\delta_{\Gamma_\varepsilon}((0, 0), A) \leq \omega(0, \frac{1}{2}) < +\infty$.
Similarly we can argue, with a $g : \Delta \to \Gamma_\varepsilon$ and $g(z) = (z, 0)$, that $\delta_{\Gamma_\varepsilon}(B, (0, 0)) \leq \omega(0, \frac{1}{2}) < +\infty$.
Now we claim that $\delta_{\Gamma_\varepsilon}(A, B) \to \infty$ when $\varepsilon \to 0$.
Let's find a contradiction by assuming this does not happen. This means that it exists a constant $R > 0$, a sequence $\varepsilon_j \to 0$ and a sequence of holomorphic functions $f_j = (g_j, h_j) : \Delta \to \Gamma_{\varepsilon_j}$ with $f_j(0) = A$ and $f_j(a_j) = B$, for some $a_j \in \Delta$, satisfying $\omega(0, a_j) \leq R$. By Montel's Theorem we have that $f_j$ admits a subsequence converging uniformly on compacts sets of $\Delta$. Let's call then such a limit $f = (g, h)$, holomorphic on the compact set $K$ . Let's modify again the subsequence of $f_j$ such that $a_j$ could converge to a point $a \in \Delta$ on the same compact $K$.
Now, $\sup_K |g_{j_k} h_{j_k}| < \varepsilon_j$, hence $g_j h_j$ converges to 0 local uniformly on compact sets and this implies that either g or h is the null function on the compact $K$. But we have for $g_j$ that $g_j(0) = \frac{1}{2}$ and then g cannot be identically 0; as well $h_j(a_j) = \frac{1}{2}$ and this gives $h(a) = \frac{1}{2}$, implying that $h$ cannot be 0.
In conclusion we have that for $\varepsilon$ small enough it holds

$$\delta_{\Gamma_\varepsilon}(A, B) > \delta_{\Gamma_\varepsilon}(A, 0) + \delta_{\Gamma_\varepsilon}(0, B)\,.$$

This example lead us to arrange the following definition

**Definition 1.4.** An *analytic chain* $\alpha = \{\zeta_0, \cdots, \zeta_m; \eta_0, \cdots, \eta_m; \varphi_0, \cdots, \varphi_m\}$ connecting two points, on a complex manifold $X$, denoted as $z_0$ and $w_0$ is a sequence of points $\zeta_0, \cdots, \zeta_m, \eta_0, \cdots, \eta_m \in \Delta$ and homomorphic functions $\varphi_0, \cdots, \varphi_m : \Delta \to X$ so that $\varphi_0(\zeta_0) = z_0, \varphi_j(\eta_j) = \varphi_{j+1}(\zeta_{j+1})$ for $j = 0, \cdots, m-1$ and $\varphi_m(\eta_m) = w_0$.
We define the length of the chain $\alpha$ to be

$$\omega(\alpha) = \sum_{j=0}^{m} \omega(\zeta_j, \eta_j) .$$

**Definition 1.5.** We define the *Kobayashi pseudodistance* $k_X$ on $X$ by

$$k_X(z, w) = \inf\{\omega(\alpha)\} \quad \forall z, w \in X.$$

Here the infimum is taken over all the the analytic chains connecting $z$ to $w$.

*Remark* 3. $X$ is a complex manifold and hence locally euclidean, if we add the connectedness condition on $X$ we gain that $X$ is path connected. This provide that $k_X$ is always finite
For this reason, from now on a manifold will be always meant as connected. Moreover, in this setting, $k_X$ respects the properties of a pseudodistance as in 1.1 . Indeed:

- $k_X(z, z) = 0 \; \forall z \in X$.
- $k_X(w, z) = k_X(z, w)$ by reversing the order of the elements in each analytic chain connecting $w$ to $z$ and using the symmetry property of the Poincaré distance.
- given three different points $z, w, y$ in $X$ and using the triangular inequality of the Poincaré distance, we get that $\omega(\alpha) \leq \omega(\beta) + \omega(\gamma)$ for each analytic chain $\alpha, \beta, \gamma$, linking respectively $z$ to $w$, $z$ to $y$ and $y$ to $w$. This implies that $k_X(z, w) \leq k_X(z, y) + k_X(y, w)$.

*Remark* 4. By applying the definition of $\delta_X$ in 1.1 to the analytic chains connecting two points of the manifold we get that:

$$k_X(z,w) = \inf\{\sum_{j=0}^{m} \delta_X(z_j, z_{j+1}) \mid z_0 = z, z_{m+1} = w, z_1, \cdots, z_m \in X, m \in \mathbb{N}\}. \tag{1.2}$$

A very important property of the Caratheodory and Kobayashi pseudodistances is that they are decreasing respect holomorphic functions.

**Proposition 1.2.1.** *Let $f : X \to Y$ be a holomorphic map between two complex manifolds.*
*Then for all $z, w \in X$*

$$c_Y(f(z), f(w)) \leq c_X(z,w)$$

*and*

$$k_Y(f(z), f(w)) \leq k_X(z,w).$$

*Proof.* Let's prove the first inequality.
If $f$ is a holomorphic map from $X$ to $Y$ and $\varphi : Y \to \Delta$, a holomorphic map of $Y$ into the unit disk $\Delta$, then $\varphi \circ f$ is a holomorphic map of $X$ into $\Delta$ and so

$$\omega\big(\varphi(f(z)), \varphi(f(w))\big) \leq c_X(z,w), \text{ for every pair of points } z, w \in X.$$

Taking the supremum over all such maps $\varphi$ we have the first thesis.

Now let's show the second inequality.
Using the analogous definition as in 1.2 we get that

$$k_Y(f(z), f(w)) =$$

$$\inf\{\sum_{j=0}^{m} \delta_Y(z'_j, z'_{j+1}) \mid z'_0 = f(z), z'_{m+1} = f(w), z'_1, \cdots, z'_m \in Y, m \in \mathbb{N}\} \leq$$

$$\inf\{\sum_{j=0}^{m} \delta_Y\big(f(z_j), f(z_{j+1})\big) \mid z'_0 = f(z), z'_{m+1} = f(w), z'_j = f(z_j)\ \forall j = 1, \cdots, m\}\ .$$

Now for each $j = 0, \cdots, m$ we can estimate the following:
$\delta_Y\big(f(z_j), f(z_{j+1})\big) \leq \omega(\zeta_j, \zeta_{j+1})$ where $\varphi : \Delta \to X$ is a holomorphic map so that $\varphi(\zeta_j) = z_j$ and $\varphi(\zeta_{j+1}) = z_{j+1}$. Taking the infimum over all such $\varphi$ we gain that $\delta_Y\big(f(z_j), f(z_{j+1})\big) \leq \delta_X\big(z_j, z_{j+1}\big)$. Hence this provides:

$$\inf\{\sum_{j=0}^{m} \delta_Y\big(f(z_j), f(z_{j+1})\big) |\ z'_0 = f(z), z'_{m+1} = f(w), z'_j = f(z_j)\ \forall j = 1, \cdots, m\} \leq$$

$$\inf\{\sum_{j=0}^{m} \delta_X(z_j, z_{j+1}) \mid z_0 = z, z_{m+1} = w, z_1, \cdots, z_m \in X, m \in \mathbb{N}\} = k_X(z, w)\ .$$

This inequality concludes the proof. □

The last statement may be interpreted as a generalization of the Schwarz–Ahlfors–Pick Theorem for the case of a complex manifold. Moreover the last Proposition gives two interesting results.

**Corollary 1.2.2.** *Let $X, Y$ be two complex manifolds, then every biholomorphic mapping $f : X \to Y$ is an isometry respect the two pseudodistances $c_X$ and $k_X$.*

*Proof.* The result follows by using the estimates from Proposition 1.2.1 for $f$ and $f^{-1}$. □

*Remark* 5. The last Corollary explains that the Carathéodory and the Kobayashi pseudodistances are invariants under biholomorphisms and this motivates why in different literatures one may see these concepts explained as invariant objects (see for instance [2] and [3] ).

**Corollary 1.2.3.** *If $Y$ is a submanifold of $X$ then for every $z, w \in Y$*

$$c_X(z, w) \leq c_Y(z, w) \text{ and } k_X(z, w) \leq k_Y(z, w).$$

*Proof.* The two results follow from the estimates of Proposition 1.2.1 applied to the holomorphic embedding $Y \hookrightarrow X$. □

**Proposition 1.2.4.** *Let $X$ be a complex manifold, and $d : X \times X \to \mathbb{R}^+$ a pseudodistance on $X$. Then* *(i) if* $d\big(\varphi(\zeta_1), \varphi(\zeta_2)\big) \leq \omega(\zeta_1, \zeta_2)$ *for all* $\zeta_1, \zeta_2 \in \Delta$ *and* $\varphi \in Hol(\Delta, X)$, *then* $d \leq k_X$*;*

*(ii) if* $d(z_1, z_2) \geq \omega\big(\varphi(z_1), \varphi(z_2)\big)$ *for all* $z_1, z_2 \in X$ *and* $\varphi \in Hol(X, \Delta)$, *then* $d \geq c_X$*.*

*Proof.* $(i)$ If $\alpha = \{\zeta_0, \cdots, \zeta_m; \eta_0, \cdots, \eta_m; \varphi_0, \cdots, \varphi_m\}$ is any analytic chain connecting two points $z, w \in X$ we have

$$d(z, w) \leq \sum_{j=0}^{m} d\big(\varphi(\zeta_j), \varphi_j(\eta_j)\big) \leq \sum_{j=0}^{m} \omega(\zeta_j, \eta_j) = \omega(\alpha) .$$

Indeed, the first inequality is just an iteration of the triangular inequality for the pseudodistance $d$ and the second inequality is motivated by the assumption. Taking the infimum over all the analytic chains we get that $d \leq k_X$.

The condition in $(ii)$ is granted by taking the supremum all over the $\varphi \in Hol(X, \Delta)$. □

**Corollary 1.2.5.** *For a complex manifold* $X$ *we have that* $c_X \leq k_X$*.*

*Proof.* From the definition of $k_X$ it follows that:

$$\omega\big(h(z), h(w)\big) \leq k_X(z, w) \;\; \forall z, w \in X \text{ and } \forall h \in Hol(X, \Delta) .$$

Taking the supremum, over the family $Hol(X, \Delta)$, we obtain

$$c_X(z, w) \leq k_X(z, w) \;\; \forall z, w \in X.$$

□

*Remark* 6. Thanks to Remark 3 and Corollary 1.2.5 we have another reason why $c_X$ is always finite.

Proposition 1.2.1 is already verified if the considered manifold is the unit disk $\Delta$, but we can also state more:

**Proposition 1.2.6.** $k_\Delta = \omega = c_\Delta$

*Proof.* We can first state that $k_\Delta \leq \omega$.
Indeed, considering the identity transformation of $\Delta$, we obtain the inequality

$$k_\Delta(z,w) \leq \omega(z,w)\ \forall z,w \in \Delta.$$

Moreover it holds for every pair of $z$ and $w$ in $\Delta$ that

$$\omega(z,w) = \omega\big(id(z), id(w)\big) \leq \sup\{\omega\big(\varphi(z),\varphi(w)\big) | \varphi \in Hol(\Delta,\Delta)\} = c_\Delta(z,w).$$

From Corollary 1.2.5 we can finally have the following chain of inequalities:

$$\omega \leq c_\Delta \leq k_\Delta \leq \omega\ .$$

□

The next purpose of this section is to find some estimates for the Carathéodory and Kobayashi pseudodistances for some precise kinds of complex manifolds.

**Proposition 1.2.7.** *Let* $\|\cdot\|_1 \colon \mathbb{C}^n \to \mathbb{R}^+$ *be a norm on* $\mathbb{C}^n$*, and* $B$ *the unit ball for this norm. Then for all* $z \in B$

$$c_B(0,z) = k_B(0,z) = \omega(0, \| z \|_1).$$

*Proof.* Consider $z \in B$ so that $z \neq 0$, we can define an holomorphic function $\varphi : \Delta \to B$ by $\varphi(\zeta) = \zeta \frac{z}{\|z\|_1}$ . Then, by applying Corollary 1.2.5 and Proposition 1.2.1, we obtain that:

$$c_B(0,z) \leq k_B(0,z) \leq \omega(0, \| z \|_1)\ .$$

On the other hand, a consequence of the Hahn-Banach Theorem applied on $Span(z)$ extended to the whole $\mathbb{C}^n$, assures that for every $z \in \mathbb{C}^n$ there exists a linear map $\lambda_z : \mathbb{C}^n \to \mathbb{C}$ such that $\lambda_z(z) = \| z \|_1$ and $\lambda_z(w) \leq \| w \|_1$ for all $w \in \mathbb{C}^n$.
Therefore the restriction of $\lambda_z$ over $B$ sends $B$ itself into $\Delta$ and, if $z \in B$

$$\omega(0, \| z \|_1) = c_\Delta(\lambda_z(0), \lambda_z(z)) \leq c_B(0,z)\ .$$

□

**Corollary 1.2.8.** *The Carathéodory and Kobayashi distances coincide on $B^n$.*

*Proof.* We start by introducing, for each fixed $z \in B^n$, an automorphism $\varphi_z : B^n \to B^n$ so that $\varphi_z(z) = 0$.
In order to do this we quote the following consequences of Theorem 2.2.2 at pages 26-27 from [4].
Fix $z \in B^n$, let $P_z$ be the orthogonal projection of $\mathbb{C}^n$ onto the subspace $Span(z)$,

$$\text{i.e. } P_z(w) = \frac{\langle z, w\rangle}{\langle z, z\rangle} z \text{ when } z \neq 0 \text{ and } P_0 = 0.$$

Let $Q_z = Id - P_z$ be the projection on the orthogonal complement of $Span(z)$.
Put $s_z = (1 - |z|^2)^{1/2}$ and define:

$$\varphi_z(w) = \frac{z - P_z(w) - s_z Q_z(w)}{1 - \langle w, z\rangle} \,. \tag{1.3}$$

It can be observed that:

- $\varphi_z$ is holomorphic on $B^n$ for every $z \in B^n$;
- $|\varphi_z(w)| < 1 \Leftrightarrow |w| < 1$ ;
- $\varphi_z \circ \varphi_z = Id$ .

Thus $\varphi_z$ is an automorphism of the unit ball $B^n$.
More over:

$$\varphi_z(z) = \frac{z - z - s_z \cdot 0}{1 - \langle z, z\rangle} = 0 \,.$$

By using Corollary 1.2.2 and Proposition 1.2.7, we have:

$$k_{B^n}(z, w) = k_{B^n}(\varphi_z(z), \varphi_z(w)) = k_{B^n}(0, \varphi_z(w)) = c_{B^n}(0, \varphi_z(w)) =$$

$$c_{B^n}(\varphi_z(z), \varphi_z(w)) = c_{B^n}(z, w)\ \forall z, w \in B^n.$$

□

Recalling that the unit polydisc $\Delta^n$ of $\mathbb{C}^n$ is the ball centered in 0 with unit radius respect the norm $\| (z_1, \cdots, z_n) \|_\infty = \max\{|z_1|, \cdots, |z_n|\}$.

**Corollary 1.2.9.** *In the unit polydisc* $\Delta^n$, *given* $\gamma_z(w) = \left(\frac{z_1-w_1}{1-\overline{z_1}w_1}, \cdots, \frac{z_n-w_n}{1-\overline{z_n}w_n}\right)$, *it holds that*

$$k_{\Delta^n}(z,w) = c_{\Delta^n}(z,w) = \omega(0, \| \gamma_z(w) \|_\infty) = \max_{j=1,\cdots,n} \omega(z_j, w_j).$$

*Proof.* We can notice that $\gamma_z$ is an automorphism of $\Delta^n$ where $\gamma_z(z) = 0$.
At this point, using Corollary 1.2.2 with $z, w \in \Delta^n$, we have

$$k_{\Delta^n}(z,w) = k_{\Delta^n}(\gamma_z(z), \gamma_z(w)) = k_{\Delta^n}(0, \gamma_z(w))$$

and

$$c_{\Delta^n}(z,w) = c_{\Delta^n}(\gamma_z(z), \gamma_z(w)) = c_{\Delta^n}(0, \gamma_z(w))\ .$$

Now let's apply Proposition 1.2.7 respect $\| \cdot \|_\infty$ and we get that:

$$c_{\Delta^n}(z,w) = c_{\Delta^n}(0, \gamma_z(w)) = \omega(0, \| \gamma_z(w) \|_\infty) = k_{\Delta^n}(0, \gamma_z(w)) = k_{\Delta^n}(z,w).$$

Finally with a straightforward computation it holds that:

$$\omega(0, \| \gamma_z(w) \|_\infty) = \omega\big(0, \max_{j=1,\cdots,n} \left|\frac{w_j - z_j}{1-\overline{z_j}w_j}\right|\big) = \max_{j=1,\cdots,n} \tanh^{-1}\big(\left|\frac{w_j - z_j}{1-\overline{z_j}w_j}\right|\big) =$$

$$\max_{j=1,\cdots,n} \omega(z_j, w_j).$$

□

**Proposition 1.2.10.** *Let* $X$ *and* $Y$ *be two complex manifolds,* $z_1, z_2 \in X$ *and* $w_1, w_2 \in Y$. *Then*

$$c_X(z_1,z_2)+c_Y(w_1,w_2) \geq c_{X\times Y}((z_1,w_1),(z_2,w_2)) \geq \max\{c_X(z_1,z_2), c_Y(w_1,w_2)\}$$

*and*

$$k_X(z_1,z_2)+k_Y(w_1,w_2) \geq k_{X\times Y}((z_1,w_1),(z_2,w_2)) \geq \max\{k_X(z_1,z_2), k_Y(w_1,w_2)\}.$$

*Proof.* Both of the right hand side inequalities descend from Proposition 1.2.1 respect the two holomorphic projections $(z,w) \mapsto z$ and $(z,w) \mapsto w$.
On the other hand, given $z_2$ in $X$ and $w_1$ in $Y$, we consider the holomorphic

maps $z \mapsto (z, w_1)$ and $w \mapsto (z_2, w)$. Then, by applying again Proposition 1.2.1 and the triangular inequality, we obtain:

$$c_X(z_1, z_2) + c_Y(w_1, w_2) \geq c_{X\times Y}((z_1, w_1), (z_2, w_1)) + c_{X\times Y}((z_2, w_1), (z_2, w_2)) \geq$$

$$c_{X\times Y}((z_1, w_1), (z_2, w_2))$$

and

$$k_X(z_1, z_2) + k_Y(w_1, w_2) \geq k_{X\times Y}((z_1, w_1), (z_2, w_1)) + k_{X\times Y}((z_2, w_1), (z_2, w_2)) \geq$$

$$k_{X\times Y}((z_1, w_1), (z_2, w_2)) .$$

□

## 1.3 The Kobayashi pseudometric

In this section we shall see that the Kobayashi pseudodistance is the integrated form of a precise pseudometric.
Having a new perspective for this pseudodistance will turn very useful in some computations for certain domains in $\mathbb{C}^n$.

**Definition 1.6.** Let $X$ be a domain in $\mathbb{C}^n$, its tangent space $T_pX$ is $\mathbb{C}^n$ itself for every choice of $p$ in $X$, then the Kobayashi pseudometric $\kappa_X : TX = X \times \mathbb{C}^n \to \mathbb{R}^+$is defined by:

$$\kappa_X(z; v) = \inf\{|\xi| \big| \exists \varphi \in Hol(\Delta, X) : \varphi(0) = z, d\varphi_0(\xi) = v\} .$$

In order to get closer into the comprehension of the last definition we introduce the following

**Example 1.6** (The Kobayashi Pseudometric of $\mathbb{C}^n, n \geq 1$)**.** We can first observe that if $\varphi \in Hol(\Delta, \mathbb{C}^n)$, such that $\varphi(0) = 0$ and $d\varphi_0(\xi) = v$, then also $m\varphi \in Hol(\Delta, \mathbb{C}^n)$ for every $m \in \mathbb{N}$.
We still have that:

$$d(m\varphi)_0(\frac{1}{m}\xi) = v \, \forall m \in \mathbb{N}.$$

Taking $m$ arbitrarily large we obtain: $\left|\frac{1}{m}\xi\right| \to 0$.
As a consequence we have that $\kappa_{\mathbb{C}^n}(0; v) = 0$.
At this point consider $\varphi \in Hol(\Delta, \mathbb{C}^n)$ so that $\varphi(0) = z$ and $d\varphi_0(\xi) = v$. Analogously, by using $\psi(\vartheta) = m(\varphi(\vartheta) - z) + z$, we can see that:

$$\psi(0) = z \text{ and } d\psi_0(\frac{1}{m}\xi) = v \, \forall m \in \mathbb{N} \,.$$

In conclusion $\frac{1}{m}\left|\xi\right| \to 0$ and this implies that $\kappa_{\mathbb{C}^n}(z; v) = 0$ for every $z, v$ in $\mathbb{C}^n$.

**Proposition 1.3.1.** *For every $z$ in $X$, $v$ in $\mathbb{C}^n$ and $\lambda$ in $\mathbb{C}$ it holds*

$$\kappa_X(z; \lambda v) = |\lambda| \kappa_X(z, v) \,.$$

*Proof.* Set $\xi \in \mathbb{C}$ and $\varphi \in Hol(\Delta, \mathbb{C}^n)$, such that $\varphi(0) = z$ and $d\varphi_0(\xi) = v$. From a straightforward computation we get

$$d\varphi_0(\lambda\xi) = \lambda v, \; \forall \lambda \in \mathbb{C} \,.$$

Thus

$$\{|\tilde{\xi}|\big|\exists\varphi : \varphi(0) = z, d\varphi_0(\tilde{\xi}) = \lambda v\} = \{|\lambda\xi|\big|\exists\varphi : \varphi(0) = z, d\varphi_0(\lambda\xi) = \lambda v\} =$$

$$|\lambda|\{|\xi|\big|\exists\varphi : \varphi(0) = z, d\varphi_0(\lambda\xi) = \lambda v\} \,.$$

By passing through an infimum argument on both sides of the equality we can finally conclude. □

*Remark* 7. This last Proposition can be interpreted as the homogeneity of degree one for $\kappa_X$.

At this point we can state some features of the Kobayashi pseudometric that remind the properties of the Kobayashi pseudodistance enumerated in the last section.

**Proposition 1.3.2.** *Let $X$ and $Y$ be two domains respectively in $\mathbb{C}^n$ and $\mathbb{C}^m$.*

*(i) If $f : X \to Y$ is an holomorphic map, then for all $z \in X$ and $v \in \mathbb{C}^n$*

$$\kappa_Y(f(z); df_z(v)) \le \kappa_X(z; v) .$$

*(ii) If $f \in Aut(X)$ then for all $z \in X$ and $v \in \mathbb{C}^n$*

$$\kappa_X(f(z); df_z(v)) = \kappa_X(z; v) .$$

*(iii) If $Y$ is a subset of $X$ then for all $z \in Y$ and $v \in \mathbb{C}^n$*

$$\kappa_X(z; v) \le \kappa_Y(z; v) .$$

*Proof.* $(i)$ Set $\xi \in \mathbb{C}$ and $\varphi : \Delta \to X$ such that $\varphi(0) = z$ and $d\varphi_0(\xi) = v$. Consider $\psi = f \circ \varphi : \Delta \to Y$, $\psi$ satisfies the following relations:

$$\psi(0) = f(z) \text{ and, by the chain rule, } d\psi_0(\xi) = df_{\varphi(0)} \cdot d\varphi_0(\xi) = df_z(v) .$$

Passing through an infimum argument we can conclude.
The proofs for $(ii)$ and $(iii)$ take the same path of Corollaries 1.2.2 and 1.2.3. □

**Proposition 1.3.3.** *$\kappa_\Delta$ coincides with the Poincaré metric.*

*Proof.* Given $(z; v)$ a point in the tangent bundle $\Delta \times \mathbb{C}$ of the Poincaré disk $\Delta$ we have that the Poincaré metric of $(z; v)$ is $\frac{|v|}{1-|z|^2}$.
Now, let $\varphi \in Hol(\Delta, \Delta)$ and $\xi \in \mathbb{C}$ so that $\varphi(0) = z$ and $\varphi'(0) \cdot \xi = v$. From the Schwarz-Pick Lemma yields that

$$\frac{|\varphi'(0)|}{1 - |\varphi(0)|^2} \le 1 \Leftrightarrow |\varphi'(0)| \le 1 - |z|^2 .$$

From $\varphi'(0) \cdot \xi = v$ we get $|\varphi'(0)||\xi| = |v|$ and this gives:

$$\frac{|v|}{1 - |z|^2} = \frac{|\varphi'(0)||\xi|}{1 - |z|^2} \le |\xi|.$$

By passing through the infimum for all the $\xi$ in $\mathbb{C}$ we obtain:

$$\frac{|v|}{1 - |z|^2} \le \kappa_\Delta(z; v).$$

For the other inequality we can consider the holomorphic function

$$\varphi(\vartheta) = \frac{z-\vartheta}{1-\overline{z}\vartheta}\ , \vartheta \in \Delta.$$

We can express $z \in \Delta$ and $\vartheta \in \partial\Delta$ as $z = \rho e^{i\eta}$ and $\vartheta = e^{i\lambda}$, for some suitable $\rho, \eta$ and $\lambda$. We have then that:

$$\left|\frac{\rho e^{i\eta} - e^{i\lambda}}{1-\rho e^{i(\lambda-\eta)}}\right| = |e^{i\lambda}|\left|\frac{\rho e^{i(\eta-\lambda)}-1}{1-\rho e^{i(\lambda-\eta)}}\right| = 1\ .$$

This means that $|\varphi(\vartheta)| = 1 \Leftrightarrow |\vartheta| = 1$ and the Maximum Modulus Principle assures that $\varphi \in Hol(\Delta, \Delta)$. Moreover, with a straightforward computation, we can see that $\varphi \circ \varphi = Id$.
Finally it holds that:

- $\varphi(0) = z$ ;
- $\varphi'(0) = \left[\frac{-(1-\overline{z}\vartheta)+\overline{z}(z-\vartheta)}{(1-\overline{z}\vartheta)^2}\right]_{|\vartheta=0} = -1+|z|^2$ .

In this way we get the inequality

$$\kappa_\Delta(z;v) \leq \frac{|v|}{1-|z|^2}\ ,$$

and it concludes the proof. □

**Proposition 1.3.4.** *Let* $\|\cdot\|_1 \colon \mathbb{C}^n \to \mathbb{R}^+$ *be a norm on* $\mathbb{C}^n$ *and* $B$ *the unit ball for this norm. Then for all* $v \in \mathbb{C}^n$ *we have*

$$\kappa_B(0;v) = \| v \|_1\ .$$

*Proof.* Analogously to 1.2.7, set $v \in B$ and define an holomorphic function $\varphi : \Delta \to B$ by $\varphi(\zeta) = \zeta\frac{v}{\|v\|_1}$. From Proposition 1.3.2 it descends that:

$$\kappa_B(0;v) \leq \kappa_\Delta(0, \| v \|_1) = \| v \|_1\ .$$

On the other side we can just follow the same path as described in 1.2.7, respect the linear form $\lambda_v : B \to \Delta$ such that $\lambda_v(v) = \| v \|_1$. □

We know that the distance associated to a Riemannian metric is obtained as infimum of length of curves. Following a similar path, the relation between the Kobayashi pseudodistance and the Kobayashi pseudometric needs primarily a meaning to the expression

$$\int_a^b \kappa_X(\sigma(t);\dot\sigma(t))dt\,, \tag{1.4}$$

where $\sigma:[a,b]\to X$ is a piecewise $C^1$ curve in $X$.
To obtain a meaning to such writing we need the following

**Lemma 1.3.5.** *Let $X$ be a domain in $\mathbb{C}^n$ and $\varphi \in Hol(\Delta, X)$ such that $\varphi'(0)\neq 0$. Then for every $r<1$ there exist a neighbourhood $U_r$ of $\overline{\Delta_r}\times\{0\}$ in $\Delta^n$ and a map $f_r\in Hol(U_r,X)$ such that $f_r|_{\overline{\Delta_r}\times\{0\}}=\varphi|_{\overline{\Delta_r}}$ and $f_r$ is a biholomorphism in a neighbourhood of $0$.*

*Proof.* Set $v_0=\varphi'(0)\neq 0$ and let $V$ denote the orthogonal complement of $v_0$ in $\mathbb{C}^n$. Define $g:\Delta\times V\to\mathbb{C}^n$ by

$$g(\zeta,w)=\varphi(\zeta)+w,\ \text{with } \zeta\in\Delta \text{ and } w\in V.$$

We can observe that $g$ is holomorphic, $g_{|\Delta\times\{0\}}\equiv\varphi$ and since $dg_{(0,0)}(\xi,w)=\xi v_0+w$, by Osgood's Theorem at pages 86-88 from [1], then $g$ is a biholomorphism in a suitable neighbourhood of the origin.
Now, since $\Delta_r\times\{0\}$ is compact and $g(\Delta_r\times\{0\})\subset\subset X$, there is a neighbourhood $U_r$ of $\Delta_r\times\{0\}$ in $\Delta^n$ such that $g(U_r)\subset X$.
Finally we can take $f_r=g_{|U_r}$ to conclude. □

**Theorem 1.3.6.** *Let $X$ be a domain in $\mathbb{C}^n$. Then the Kobayashi pseudometric is an upper semicontinuous function on $X\times\mathbb{C}^n$.*

*Proof.* Choose $z_0\in X$, $v_0\in\mathbb{C}^n$ and $\varepsilon>0$; we will show that there is a neighbourhood $\tilde V$ of $(z_0;v_0)$ in $X\times\mathbb{C}^n$ such that

$$\kappa_X(z;v)<\kappa_X(z_0;v_0)+\varepsilon\ ,\ \forall(z;v)\in\tilde V.$$

From the definition of $\kappa_X(z_0;v_0)$ as an infimum, it follows that there are $\varphi\in Hol(\Delta,X)$ and $\xi\in\mathbb{C}$ such that $\varphi(0)=z_0$, $d\varphi_0(\xi)=v_0$ and $|\xi|<$

$\kappa_X(z_0;v_0)+\varepsilon/2$. Pick $r_0<1$ such that $|\xi|/r_0$ is still less than $\kappa_X(z_0;v_0)+\varepsilon/2$, and let $U\subset\Delta^n$ and $f\in Hol(U,X)$ be given by Lemma 1.3.5 applied to $\varphi$ and $r_0$; we can consider $U=\Delta_{r_0}\times\Delta_\rho^{n-1}$ for a proper $\rho>0$.
Now, $f$ is a biholomorphism in a neighbourhood of 0, $f(0)=z_0$ and $df_0(\xi e_1)=v_0$, where $e_1=(1,0,\cdots,0)^t$.
Therefore we can find a neighbourhood $\tilde{U}$ of $(0;\xi e_1)$ in $TU=U\times\mathbb{C}^n$ such that the tangent map $(f;df):TU\to TV$ is a biholomorphism between $\tilde{U}$ and $\tilde{V}$. Since, by Proposition 1.3.4, $\kappa_U$ is a continuous function we can also assume that

$$\kappa_U(\zeta;v)\le\kappa_U(0;\xi e_1)+\varepsilon/2\ ,\ \forall(\zeta,\nu)\in\tilde{U}.$$

To conclude consider $(z;v)\in\tilde{V}$ and $(\zeta;\nu)\in\tilde{U}$ so that $z=f(\zeta)$ and $v=df_\zeta(\nu)$, then by Proposition 1.3.2 and the previous considerations it follows:

$$\kappa_X(z;v)\le\kappa_U(\zeta;\nu)\le\kappa_U(0;\xi e_1)+\varepsilon/2\le|\xi|/r_0+\varepsilon/2<\kappa_X(z_0;v_0)+\varepsilon\ .$$

□

So 1.4 is well-defined; at least, $\kappa_X$ is integrable.

*Remark* 8. We can state the same result for a bigger class of curves $\sigma$ in $X$, the ones which are absolutely continuous. This is granted by the Fundamental Theorem of Lebesgue Integral Calculus which assures us that $\sigma:[a,b]\to X$ has a derivative $\dot\sigma(t)$ almost everywhere, $\dot\sigma$ is Lebesgue integrable and it holds $\sigma(t)-\sigma(a)=\int_a^t\dot\sigma(\tau)d\tau$ for every $t\in[a,b]$.

We can also show that 1.4 is always finite:

**Lemma 1.3.7.** *Let $X$ be a domain of $\mathbb{C}^n$, then for every compact subset $K$ of $X$ there is a constant $c_K>0$ such that*

$$\forall z\in K\ \forall v\in\mathbb{C}^n\ \kappa_X(z;v)\le c_K\parallel v\parallel\ .$$

*Proof.* Since $K\subset\subset X$ we have that:

$$K\times\{v\in\mathbb{C}^n|\parallel v\parallel=1\}\subset\subset X\times\mathbb{C}^n.$$

Since $\kappa(z;v)$ is upper semicontinuous on $K \times \mathbb{C}^n$ then it attains a maximum on $K \times S^n$. We can name this maximum as $c_K$ and this gives:

$$\kappa(z;v) \leq c_K \,\forall (z;v) \in K \times S^n.$$

Now take $v \in \mathbb{C}^n$ so that $v \neq 0$ and for such a general $v$ it holds that:

$$\kappa(z; \frac{v}{\parallel v \parallel}) \leq c_K.$$

Due to 1.3.1 this is equivalent to:

$$\kappa(z;v) \leq c_K \parallel v \parallel .$$

Moreover we have that $\forall z \in K \; \kappa_K(z;0) = 0$, and this concludes our proof.

□

*Remark* 9. Given a domain $X$ in $\mathbb{C}^n$, due to Lemma 1.3.7 we have that for all the piecewise $C^1$(or absolutely continuous) curves $\sigma : [a,b] \to X$ it holds that:

$$\forall t \in [a,b] \;\; \kappa(\sigma(t), \dot{\sigma}(t)) \leq c_{[a,b]} \parallel \dot{\sigma}(t) \parallel .$$

Thus:

$$\int_a^b \kappa(\sigma(t), \dot{\sigma}(t))dt \leq c_{[a,b]} \int_a^b \parallel \dot{\sigma}(t) \parallel dt,$$

and the euclidean length of such curves $\sigma$ is finite.

An analogous fact to Lemma 1.3.7 can be proved for the Kobayashi pseudodistance:

**Proposition 1.3.8.** *Let $X$ be a domain of $\mathbb{C}^n$, and fix a point $z_0 \in X$, a neighbourhood $U$ of $z_0$ and a biholomorphism $\psi : U \to B^n$. Then for every compact subset $K$ of $U$ there is a constant $c'_K > 0$ such that*

$$\forall z, w \in K \; k_X(z,w) \leq c'_K \parallel \psi(z) - \psi(w) \parallel .$$

*Proof.* From Proposition 1.2.1 and Corollary 1.2.3 we have that for a compact set $K \subset X$ it holds:

$$k_X(z,w) \leq k_K(z,w) = k_{B^n}(\psi(z), \psi(w)).$$

At this point, for every $z \in K$, consider the automorphism $\varphi_{\psi(z)}$ of the unit ball as defined in 1.3.
Hence:

$$k_{B^n}(\psi(z), \psi(w)) = k_{B^n}(\varphi_{\psi(z)}(\psi(z)), \varphi_{\psi(z)}(\psi(w))) = k_{B^n}(0, \varphi_{\psi(z)}(\psi(w)))$$

Now, Proposition 1.2.7 gives

$$k_{B^n}(0, \varphi_{\psi(z)}(\psi(w))) = \omega(0, \| \varphi_{\psi(z)}(\psi(w)) \|)$$

and defining the continuous function $g : K \to \mathbb{R}^+$ by
$g(w) = \omega(0, \| \varphi_{\psi(z)}(\psi(w)) \|)$ we have that $\forall z \in K$ $g$ attains its maximum in $K$.
Summarizing up to here we have:

$$\forall z, w \in K \; k_{B^n}(\psi(z), \psi(w)) < +\infty.$$

Then it exists a constant $c'_K$ so that:

$$\forall z, w \in K \; k_{B^n}(\psi(z), \psi(w)) \leq c'_K \| \psi(z) - \psi(w) \| .$$

□

Now, let $\sigma : [a, b] \to X$ be a piecewise $C^1$ (or absolutely continuous) curve in a complex domain $X \subset \mathbb{C}^n$. Then the Kobayashi length $\ell_k(\sigma)$ of $\sigma$ is given by

$$\ell_k(\sigma) = \int_a^b \kappa_X(\sigma(t); \dot{\sigma}(t))dt \, .$$

By Theorem 1.3.6 and Lemma 1.3.7 we have that $\ell_k(\sigma)$ is well defined and always finite. Morover we can state that:

**Proposition 1.3.9.** *$\ell_k(\sigma)$ does not depend on the parametrization of $\sigma$.*

*Proof.* Consider $\sigma, \tau$ two equivalent parametrization for the same curve respect the real intervals $[a, b]$ and $[c, d]$.
This means that it exists a diffeomorphism $\varphi$ between $[a, b]$ and $[c, d]$ so that $\sigma = \tau \circ \varphi$ and from the chain rule it follows that:

$$\dot{\sigma}(t) = \dot{\tau}(\varphi(t))\varphi'(t).$$

Then, by Proposition 1.3.1, we have:

$$\int_a^b \kappa_X(\sigma(t);\dot\sigma(t))dt = \int_a^b \kappa_X(\sigma(t);\dot\tau(\varphi(t))\varphi'(t))dt =$$

$$\int_a^b |\varphi'(t)|\kappa_X(\tau(\varphi(t));\dot\tau(\varphi(t)))dt = \int_c^d \kappa_X(\tau(t);\dot\tau(t))dt.$$

□

At this point we can define a pseudodistance $k_X^i : X \times X \to \mathbb{R}^+$ on X, the *integrated form* of $\kappa_X$, by

$$\forall z, w \in X \quad k_X^i(z,w) = \inf \ell_k(\sigma),$$

where the infimum is taken with respect to the family of all piecewise $C^1$(or absolutely continuous) curves connecting $z$ to $w$.

Now we can finally state and prove the main result of this section which relates how $k_X^i$ is constructed starting from $\kappa_X$ exactly as the distance associated to a Riemannian metric with the Kobayashi pseudodistance.

**Theorem 1.3.10.** *Let $X$ be a complex domain of $\mathbb{C}^n$. Then $k_X$ is the integrated form of $\kappa_X$.*

*Proof.* Let's prove it by double inequality.
*First step: $k_X^i \leq k_X$.*
Given two points $z$ and $w$ in $X$, pick a finite sequence of points $\{z_j\}_{j=0}^{m+1}$ in $X$ so that $z_0 = z$ and $z_{m+1} = w$. Now consider the family of curves $\sigma_j$ that connect $z_j$ to $w_j$ and the family of curves $\sigma$ that connect $z$ to $w$. Clearly it follows that

$$k_X^i(z,w) = \inf\{\ell_k(\sigma)\} \leq \inf\{\sum_{j=0}^{m+1} \ell_k(\sigma_j)\}.$$

Following the definition of $k_X$ as in 1.2 it is not so hard to convince ourselves that $k_X^i \leq \delta_X$ suffices to prove $k_X^i \leq k_X$. In order to do this take $z_0, w_0 \in X$. If $\delta_X(z_0, w_0) = +\infty$, there is nothing more to prove; otherwise, fix $\varepsilon > 0$ and choose $\varphi \in Hol(\Delta, X)$ with $\varphi(0) = z_0$ and $\varphi(t_0) = w_0$ for a suitable $t_0 \in [0,1)$ such that the infimum construction of $\delta_X$ grants $\omega(0,t_0) < \delta_X(z_0,w_0) + \varepsilon$. Let $\sigma(t) = \varphi(t)$. Then, by applying some basic integral inequalities and Proposition 1.3.2 to $\varphi$, we have

$$k_X^i(z_0,w_0) \leq \int_0^{t_0} \kappa_X(\sigma(t);\dot{\sigma}(t))dt \leq \int_0^{t_0} \kappa_\Delta(t;1)dt = \omega(0,t_0) < \delta_X(z_0,w_0)+\varepsilon,$$

since $\varepsilon > 0$ is chosen arbitrarily we can affirm that $k_X^i \leq \delta_X$.

*Second step: $k_X \leq k_X^i$.*
In order to prove the second step we will show that for every piecewise $C^1$ (or absolutely continuous) curve $\sigma : [a,b] \to X$ connecting $z_0$ to $w_0$ it holds $k_X(z_0,w_0) \leq \ell_k(\sigma)$.
Let $f : [a,b] \to \mathbb{R}^+$ be defined by $f(t) = k_X(z_0, \sigma(t))$. By using Proposition 1.3.8 we have that for two given points $t, t'$ in $[a,b]$:

$$|f(t) - f(t')| = |k_X(z_0,\sigma(t)) - k_X(z_0,\sigma(t'))| \leq |k_X(\sigma(t),\sigma(t'))| \leq$$

$$c'_{[a,b]} \parallel \sigma(t) - \sigma(t') \parallel \leq c''_{[a,b]}|t-t'| \ ,$$

thus $f$ is locally Lipschitz. Since Lipschitz functions are absolutely continuous then $f$ is differentiable almost everywhere.
In particular,

$$k_X(z_0, w_0) = f(b) - f(a) \leq \int_a^b |f'(t)|dt \ ;$$

hence it suffices to prove that if $f$ is differentiable in $t_0 \in (a, b)$ then

$$|f'(t_0)| \leq \kappa_X(\sigma(t_0); \dot{\sigma}(t_0)) \ .$$

Fix $\varepsilon > 0$, and choose $\varphi \in Hol(\Delta, X)$ and $\xi \in \mathbb{C}$ such that $\varphi(0) = \sigma(t_0)$, $d\varphi_0(\xi) = \dot{\sigma}(t_0)$ and $|\xi| < \kappa_X(\sigma(t_0); \dot{\sigma}(t_0)) + \varepsilon$. Then if $h \in \mathbb{R}$ is small enough, applying some triangular inequalities, Proposition 1.3.2 and Proposition 1.3.3, we get:

$$|f(t_0+h)-f(t_0)| \leq k_X(\sigma(t_0+h), \sigma(t_0)) \leq k_X(\sigma(t_0+h), \varphi(h\xi))+k_X(\varphi(h\xi), \varphi(0)) \leq$$

$$k_X(\sigma(t_0 + h), \varphi(h\xi)) + \omega(0, h\xi).$$

Remembering that $\varphi(0) = \sigma(t_0)$, we see:

$$k_X(\sigma(t_0 + h), \varphi(h\xi)) \leq k_X(\sigma(t_0 + h), \sigma(t_0)) + k_X(\varphi(0), \varphi(h\xi)).$$

Hence by applying Proposition 1.3.8 we have that:

$$k_X(\sigma(t_0 + h), \varphi(h\xi)) = o(|h|).$$

Therefore we gained an estimate for:

$$|f'(t_0)| \leq \lim_{h\to 0} \frac{|f(t_0 + h) - f(t_0)|}{|h|} \leq \lim_{h\to 0} \frac{k_X(\sigma(t_0 + h), \varphi(h\xi)) + \omega(0, h\xi)}{|h|} =$$

$$\lim_{h\to 0} \frac{\omega(0, h\xi)}{|h|} = |\xi| < \kappa_X(\sigma(t_0); \dot{\sigma}(t_0)) + \varepsilon \ .$$

Since $\varepsilon > 0$ is chosen arbitrarely we can conclude the proof. □

**Example 1.7.** As a consequence of Theorem 1.3.10 and Example 1.6 we obtain that

$$k_{\mathbb{C}^n}(z, w) = 0 \ \forall z, w \in \mathbb{C}^n.$$

For deeper and further results about the Kobayashi pseudo-metric on complex manifolds one may check [41], especially Theorem 3.1.

## 1.4 The Kobayashi distance for $\mathbb{C}$-proper domains

The aim of this section is to qualify a complex domain that assures the non-degenerancy condition for the Kobayashi pseudodistance. In order to do this we will introduce the following

**Definition 1.7.** Let $D$ be a domain in $\mathbb{C}^d$, $D$ is called $\mathbb{C}$-proper if it does not contain any complex affine line.

**Example 1.8.** The domain

$$A := \{(z, w) \in \mathbb{C}^2 : |z| < 1 \text{ and } w \in C\},$$

$$\text{where } C = \{v \in \mathbb{C} : Im(v) > |Re(v)|\}$$

is $\mathbb{C}$-proper and unbounded.

*Remark* 10. If a domain is bounded then the property of being $\mathbb{C}$-proper follows automatically. One of the purposes of this definition is to generalize the concept of boundedness itself, in this way we have created a wider class of domains to deal with.

We would like to state something important about a space endowed with the pseudodistances we introduced in the last sections such as being a complete metric space. For this purpose Theodore J. Barth in [5] stated and proved the following

**Theorem 1.4.1.** *Let $X$ be a $\mathbb{C}$-proper and convex domain of $\mathbb{C}^n$. Then the Carathéodory pseudodistance $c_X$ is a distance and every closed Carathéodory ball is compact.*

*Proof. First step: $c_X$ is a distance.*
We know already that $c_X$ is symmetric and satisfies the triangular inequality. Then the only thing we have to check is the non degeneracy condition i.e. if $p$ and $q$ are distinct points in $X$ then $c_X(p, q) > 0$.

Consider $L$ as the complex affine line joining $p, q$. Seeing that $X$ is $\mathbb{C}$-proper we have that $L$ is not contained in $X$ and thus $L$ contains a boundary point $b$ of $X$.
Since $X$ is convex, by representing $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ we can take the real supporting hyperplane $V$ (of real dimension $2n-1$) to $X$ at the point $b$ (see pages 50-51 from [6]). This real supporting hyperplane $V$ splits $L$ into two open complex half lines, now let's call $H$ the complex half line which contains $L \cap X$. Moreover $V$ contains a unique complex affine hyperplane $P$ (with real dimension $2n-2$) that passes through $b$.
The holomorphic projection $\pi$ of $\mathbb{C}^n$ parallel to $P$ onto $L$ maps $X$ into the open half line $H$. The condition that $\pi(X) \subset H$ is granted because $\pi(X)$ is connected, hence it cannot be contained on both of the sides of $L$ which is splitted by the supporting hyperplane $V$; at the same time the Open Mapping Theorem grants that $\pi(X)$ is open hence $\pi(X)$ has no points in $V \cap L$.
Since $p, q \in L$, from Proposition 1.2.1 we have:

$$c_H(p, q) = c_H(\pi(p), \pi(q)) \leq c_X(p, q).$$

Now, for every half complex line we have a biholomorphism $\varphi$ with the unit disk $\Delta$ which is a composition of some suitable rotations, translations, homotheties and of the Cayley Transform.
Then, from Proposition 1.2.6 we have:

$$c_H(p, q) = \omega(\varphi(p), \varphi(q)) > 0.$$

*Second step: Closed Carathéodory balls are compact.*
Pick a sequence $\{q_j\}_{j\in\mathbb{N}}$ in the closed Carathéodory ball $\overline{B}_c(p, r)$ for some $p \in X$ and $r > 0$.
We are going to prove that it is possible to extract a subsequence converging to a point of $X$ and then, since the subsequnce takes values in the closed Carathéodory ball $\overline{B}_c(p, r)$, we get that $\overline{B}_c(p, r)$ is sequentially compact.
Without loss of generality we can assume that $p = 0$ and that $q_j \neq p$ for all

$j$. Consider as $\| \cdot \|$ the euclidean norm on $\mathbb{C}^n$. Since the unit sphere $S^n$ is compact, by taking a subsequence, we may assume that $v_j = q_j / \| q_j \| \to v$ with $\| v \| = 1$.
Let $L$ be the complex line joining $p = 0$ and $v$. Since $L$ is not contained in $X$, then $L$ contains a boundary point $b$ of $X$. Constructing the half complex line $H$ and the projection $\pi$ as in the *First step*, we obtain

$$r \geq c_X(p, q_j) \geq c_H(p, \pi(q_j)).$$

We know that the half line $H$ is biholomorphic to the unit disk $\Delta$ endowed with the Poincaré distance $\omega$, in which closed balls in respect of $\omega$ are compact. Hence the same compactness property is granted for the half line $H$.
As a consequence the sequence $\{\pi(q_j)\}_{j \in \mathbb{N}}$, that the takes values on a closed ball in $H$, converges, up to subsequences, to a point $q$ in $H$.
Noting that the mapping $\pi$ is linear, we obtain

$$\pi(q_j) = \| q_j \| \, \pi(v_j).$$

By construction of the projection $\pi$ we have $\pi(v) = v$. Then we can give an estimate to:

$$\| q_j \| = \frac{\| \pi(q_j) \|}{\| \pi(v_j) \|} \to \frac{\| q \|}{\| \pi(v) \|} = \frac{\| q \|}{\| v \|} = \| q \| \, .$$

Hence the expansion

$$q_j = \| q_j \| \, (v_j - \pi(v_j)) + \| q_j \| \, \pi(v_j) = \| q_j \| \, (v_j - \pi(v_j)) + \pi(q_j)$$

gives that $\{q_j\}_{j \in \mathbb{N}}$ converges as well to $q$, up to subsequences.
Finally we note that $q$ belongs to $X$. For otherwise we have two options $q \in \mathbb{C}^n$ or $q \in \partial X$. The first case is not possible because at some point the sequence $q_j$ would take points outside from $X$ and then outside from the closed Carathéodory ball contained in $X$. The second case means that we could have taken $b = q$ and considering the real supporting hyperplane $V$, which splits the complex line $L$ into two open complex half lines. Remembering that $H$ is the open complex half line which contains $X \cap L$, this gives $q \notin H$. $\square$

*Remark* 11. Corollary 1.2.5 assures us that the non degeneracy of $c_X$ grants also the non degeneracy of $k_X$.

*Remark* 12. The properties of $c_X$ involved in the second part of the last proof were contractivity under holomorphic maps and that $c_\Delta$ coincides with the Poincaré distance. The same properties hold for $k_X$ and this means that also closed Kobayashi balls are compact.

**Definition 1.8.** A metric space $(X, d)$ is said to be *proper* if closed balls in $X$ respect $d$ are compact.

*Remark* 13. Proper metric spaces are Cauchy complete metric spaces.

We know now that a convex and $\mathbb{C}$-proper domain $X$ of $\mathbb{C}^n$ is endowed with a norm for tangent vectors said $\kappa_X$. We can see that $\kappa_X$ is not given by an Hermitian product like in the case of a Riemannian Manifold. However $\kappa_X$ is still positively homogeneous of degree one and this property can lead us to introduce Finsler Manifolds as in [7]. Analogously with the case of a connected Riemannian Manifold we are able to state a result which assures that every pair of points is joined by a minimal geodesic (Chapter 6 from [7]). This fact is known as a consequence of the *Hopf-Rinow Theorem* for a connected Finsler manifold. We state the result in the following

**Theorem 1.4.2.** *Let $(M, F)$ be a connected Finsler manifold, where $F$ is an absolutely homogeneous metric of degree one. If the metric space $(M, d)$, endowed with the distance where $d$ is the integrated form of $F$, is Cauchy complete, then every pair of points in $M$ is joined by a minimizing geodesic.*

**Definition 1.9.** A metric space $(X, d)$ is said to be *geodesic* if every pair of points in $X$ can be joined by a geodesic segment.

*Remark* 14. We can observe that for a convex domain $\Omega$, for which $(\Omega, k_\Omega)$ is a proper geodesic metric space, it follows that is $\mathbb{C}$-proper. Indeed, if it would not be $\mathbb{C}$-proper we could map biholomorphically the affine complex line contained in $\Omega$ into $\mathbb{C}$ and, since $k_{\mathbb{C}} = 0$, then $k_\Omega$ would degenerate on this complex affine line.

In order to resume our path we can state the following

**Theorem 1.4.3.** *Suppose $\Omega$ is a convex domain of $\mathbb{C}^n$, then the following are equivalent:*

1. *$\Omega$ is $\mathbb{C}$-proper,*
2. *$k_\Omega$ is a non-degenerate distance on $\Omega$,*
3. *$(\Omega, k_\Omega)$ is a proper metric space,*
4. *$(\Omega, k_\Omega)$ is a proper geodesic metric space.*

We give also a result about the "dual" Carathéodory distance $\delta_D$ for a convex and bounded domain with the following

**Proposition 1.4.4.** *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain. Then $\delta_D$ is always finite and not degenerate.*

*Proof.* First of all we prove that $\delta_D(z, w) < +\infty$ for all $z, w$ in $D$. In order to do that we search for a map $\varphi : \Delta \to D$ so that for two points $\zeta, \eta \in \Delta$ we have $\varphi(\eta) = w$ and $\varphi(\zeta) = z$ , as in the definition provided in 1.1.
Indeed, consider

$$\mathbf{\Omega} = \{\lambda \in \mathbb{C} | (1 - \lambda)z + \lambda w \in D\}.$$

Since $D$ is bounded we can observe that $\mathbf{\Omega}$ is also bounded.
Moreover $\mathbf{\Omega}$ is convex: $\lambda_1, \lambda_2 \in \mathbf{\Omega}$ mean that $(1 - \lambda_1)z + \lambda_1 w \in D$ and $(1 - \lambda_2)z + \lambda_2 w \in D$, by convexity of $D$ for every $t \in [0, 1]$ we have

$$D \ni t((1 - \lambda_1)z + \lambda_1 w) + (1 - t)((1 - \lambda_2)z + \lambda_2 w) =$$

$$(1 - (t\lambda_1 + (1 - t)\lambda_2))z + (t\lambda_1 + (1 - t)\lambda_2)w$$

and this gives $t\lambda_1 + (1 - t)\lambda_2 \in D \; \forall t \in [0, 1]$.
Furthermore $\mathbf{\Omega}$ contains 0 and 1.
Thus $D$ is a simply connected set which contains 0 and it is a proper subset of $\mathbb{C}$. As a consequence of the Riemann Mapping Theorem and the transitive action of $\Delta$ under automorphisms, we can find a biholomorphism $\phi : \Delta \to \Omega$

so that $\phi(0) = 0$. In this way we can define an holomorphic map $\varphi : \Delta \to D$ to be

$$\varphi(\xi) = (1 - \phi(\xi))z + \phi(\xi)w$$

and it is such that $z, w \in \varphi(\Delta)$.
We can prove the non-degeneracy of $\delta_D$ in the following way: pick two different points $z, w \in D$, by definition of $k_D$ it follows that $\delta_D(z, w) \geq k_D(z, w)$, moreover Theorem 1.4.3 provides that $k_D(z, w) > 0$ and then we can conclude. □

# Chapter 2

# Boundary behaviour of the Kobayashi distance

The Kobayashi distance is still a mysterious tool to compute. For the applications, it becomes important to find a way of approximating it using something more explicit. In interior points of a complex domain Proposition 1.3.8 and Theorem 1.3.10 give good ideas for obtaining an upper bound.
In this chapter we will focus on presenting estimates for the Kobayashi distance, when it is not degenerate, near the boundary of a domain with certain properties.
One may check Corollary 2.1.14 and Proposition 2.3.14 from [3] to deduce that the Kobayashi pseudodistance is not degenerate in a strongly pseudoconvex domain.
Now we are going to quote some theory of strongly pseudoconvex domains with $C^2$ (or smooth) boundary from [8].

**Definition 2.1.** A domain $D \subset\subset \mathbb{C}^n$ is said to have $C^k$ (or smooth) boundary, $k \geq 1$, if there is a $k$ times continuously differentiable (or smooth) function $\rho : U \to \mathbb{R}$, defined on a neighborhood $U$ of the boundary, such that:

- $D \cap U = \{z \in U | \rho(z) < 0\}$;

- $\nabla\rho \neq 0$ on $\partial D$.

*Remark* 15. The previous setting for the defining function is given on a neighbourhood of the boundary of the domain. Anyways, by using a partition of unity, one can patch together the local defining functions and extend the definition above the whole $\mathbb{C}^n$.

**Proposition 2.0.1.** *A domain $D$ has a $C^k$ defining function, with $k \geq 1$, if and only if $\partial D$ is a $C^k$ manifold.*

*Proof.* For first we will prove that $\partial D$ is a $C^k$ manifold.
Given a $C^k$ defining function $\rho$ on $D$, we have that $\rho(\partial D) = 0$.
Pick a point $p = (p_1, \cdots, p_n) \in \partial D$ and consider a related neighborhood $N$ for $p$ in $\partial D$. On the other hand, from the assumptions, we know that $\nabla\rho \neq 0$; hence at least one of the partial derivatives of $\rho$ does not vanish and without loss of generality let's choose that $\frac{\partial\rho}{\partial z_n}(p) \neq 0$. In this way we can apply the Implicit Function Theorem and have an open set $N' \subset \mathbb{C}^{n-1}$, $(p_1, \cdots, p_{n-1}) \in N'$, a unique function $\varphi : N' \to \mathbb{C}$ that is continuously differentiable $k$-times so that $p_n = \varphi(p_1, \cdots, p_{n-1})$ and $\rho(p', \varphi(p')) = 0$ holds for $p' \in N'$. In conclusion we manage to express the points in a neighborhood of $\partial D$ as the graph of a $C^k$ function, hence $\partial D$ is a $C^k$ manifold.

For the converse proof one may check Proposition 5.43 at page 118 from [13], by adapting it to the $C^k$ case. □

**Definition 2.2.** Let $D \subset\subset \mathbb{C}^n$ be a domain with $C^2$ boundary. If $x \in \partial D$, then $x$ is a point of *Levi pseudoconvexity* if the Levi form $L_{\rho,x}$ is positive semi-definite on the space of $w \in T_x(D)$. Explicitly, $x \in \partial D$ is a point of *Levi pseudoconvexity* for $D = \{z \in \mathbb{C}^n | \rho(z) < 0\}$ if

$$L_{\rho,x}(w, w) = \sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial \overline{z_k}}(x) w_j \overline{w_k} \geq 0$$

for all $w \in \mathbb{C}^n$ that satisfy

$$\sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}(x) w_j = 0.$$

The point $x$ is a point of *strong pseudoconvexity* if the Levi form at $x$ is positive definite for some choice of defining function and $w \in T_x(D)$.
The domain $D$ is said to be *Levi pseudoconvex* (resp. *strongly pseudoconvex*) if every $x \in \partial D$ is a point of Levi pseudoconvexity (resp. strong pseudoconvexity).

*Remark* 16. These definitions are independent on the choice of defining functions.

**Definition 2.3.** A real valued function is $f \in C^2(D), D \subset \mathbb{C}^n$ domain, is *strictly plurisubharmonic* if

$$\sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \overline{z_k}}(z) w_j \overline{w_k} > 0$$

for every $z \in D$ and every $0 \neq w \in \mathbb{C}^n$.

**Proposition 2.0.2.** *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with $C^2$ boundary, then $D$ admits a defining function $\tilde{\rho}$ such that is strictly plurisubharmonic in a neighbourhood of $\partial D$.*

*Remark* 17. Denoted as $U$ such neighborhood of $\partial D$, it means that $L_{\rho,z}$ is positive definite for all $z \in U$.

Moreover we can also state the following

**Proposition 2.0.3.** *Since $\partial D$ is compact, there are $c_1, c_2 > 0$ such that for all $v \in \mathbb{C}^n$ and $x_0 \in \partial D$ we have*

$$c_1 \parallel v \parallel^2 \leq L_{\rho,x_0}(v,v) \leq c_2 \parallel v \parallel^2 .$$

**Definition 2.4.** The Levi polynomial of $\rho$ at $x \in \partial D$ is the expression

$$p_x(z) = \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(x)(z_j - x_j) + \frac{1}{2} \sum_{h,k=1}^{n} \frac{\partial^2 \rho}{\partial z_h \partial z_k}(x)(z_h - x_h)(z_k - x_k).$$

**Proposition 2.0.4.** *The expansion of $\rho$ about $x_0 \in \partial D$ can be written as*

$$\rho(z) = 2\mathrm{Re}(p_{x_0}(z)) + L_{\rho,x_0}(z - x_0, z - x_0) + o(\parallel z - x_0 \parallel^2).$$

*Proof.* Since $x_0 \in \partial D$ we have that: $\rho(x_0) = 0$. By a rotation and a translation of coordinates we may assume that $x_0 = 0$.
According to page 261 in Chapter IX from [9] we can express the Taylor series for a real valued $C^2$ function $\rho$ in a neighborhood of 0 in such a way:

$$\rho(z) = \rho(0) + \sum_{i=1}^{n}(a_i z_i + \overline{a}_i \overline{z}_i) + \frac{1}{2}\sum_{i,j=1}^{n}(a_{ij} z_i z_j + \overline{a}_{ij}\overline{z}_i\overline{z}_j) + \sum_{i,j=1}^{n} c_{ij} z_i \overline{z}_j + o(|z|^2),$$

where

$$a_i = \frac{\partial \rho}{\partial z_i}(0),\ a_{ij} = \frac{\partial^2 \rho}{\partial z_i \partial z_j}(0),\ c_{ij} = \frac{\partial^2 \rho}{\partial \overline{z}_j \partial z_i}(0).$$

Now we remember the relation:

$$\overline{\frac{\partial \rho}{\partial z_i}} = \frac{\partial \overline{\rho}}{\partial \overline{z}_i}.$$

In conclusion, we obtain:

$$\rho(z) = 2\mathrm{Re}(p_0(z)) + \sum_{j,k=1}^{n} c_{jk} z_j \overline{z_k} + o(|z|^2).$$

□

*Remark* 18. Given $U$ a suitable neighborhood of $\partial D$ as before, since $\rho(z) < 0$ in $D \cap U$ and $L_{\rho,x_0}$ is positive definite, there is a neighbourhood $V_{x_0}$ of $x_0$ such that $\mathrm{Re}(p_{x_0}) < 0$ in $V_{x_0} \cap D$. Moreover, since $\partial D$ is compact, we can assume that $V_{x_0}$ is of uniform size, that is that there is a fixed neighbourhood $V$ of the origin such that $V_{x_0} = x_0 + V$ for all $x_0 \in \partial D$.

**Proposition 2.0.5.** *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with $C^2$ boundary. If $\rho$ is a strictly plurisubharmonic defining function in a neighbourhood $U$ of $\partial D$ and $\psi$ is any $C^2$ real-valued function compactly supported in $U$, then for any $\varepsilon > 0$ sufficiently small the function $\rho - \varepsilon\psi$ is strictly plurisubharmonic in $U$ and*

$$\tilde{D} = \{z \in \mathbb{C}^n | (\rho - \varepsilon\psi)(z) < 0\}$$

*Remark* 19. This means that the notion of strongly pseudoconvex domain is stable under perturbation.
Furthermore this can prove that $\overline{D}$ has a fundamental system of neighbourhoods composed by strongly pseudoconvex $C^\infty$ domains.

By synthesizing and adapting to our purpose the main results obtained in [9] and [10], we can state the following

**Theorem 2.0.6.** *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with smooth boundary. Let $\eta$ be a $\overline{\partial}$-closed (i.e. $\overline{\partial}\eta = 0$) smooth $(0,1)$-form in $L^2_{(0,1)}(D)$. Then there is a unique smooth function $u = S\eta \in L^2(D)$ such that $\overline{\partial}u = \eta$ and $u$ is orthogonal in $L^2(D)$ to the holomorphic functions on $D$. Moreover, $S$ is a bounded linear operator, that is there exists $C > 0$ depending only on $D$ such that*

$$\| u \|_2 \leq C \| \eta \|_{L^2_{(0,1)}(D)} .$$

Two very important consequences of the last Theorem, proved in [11], are the next two following important results.

**Theorem 2.0.7.** *Let $M$ be a compact subset of $\mathbb{R}^N$ , and $D \subset\subset \mathbb{C}^n$ a strongly pseudoconvex domain with smooth boundary. Let $\eta : M \to L^\infty_{(0,1)}(D)$ be a continuous map such that $\eta_x = \eta(x)$ is smooth and $\overline{\partial}$-closed for every $x \in M$. Set $u_x = S\eta_x$. Then $u : M \times D \to \mathbb{C}$ given by $u(x,z) = u_x(z)$ is continuous on $M \times D$.*

*Remark* 20. The last statement can be interpreted as the continuous dependence on parameters of the solution to the $\overline{\partial}$-problem in strongly pseudoconvex domain.

**Definition 2.5.** Let $D$ be a domain of $\mathbb{C}^n$. A *peak function* for $D$ at a point $x \in \partial D$ is a holomorphic function $f$ defined in a neighbourhood of $\overline{D}$ such that $f(x) = 1$ and $|f(z)| < 1$ for all $z \in D \setminus \{x\}$.

There are several sufficient conditions that assure the existence of a *peak function* on certain domains of $\mathbb{C}^n$ and about this topic one may consult

Chapter 2.1 from [3].
By the way, the next result will focus on the dependence of *peak functions* on the boundary point $x_0$.

**Theorem 2.0.8.** *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex $C^2$ domain. Then there exist a neighbourhood $D'$ of $\overline{D}$ and a continuous function $\Psi : \partial D \times D' \to \mathbb{C}$ such that:*

*(i) $\Psi_{x_0} = \Psi(x_0, \cdot)$ is holomorphic in $D'$ for any $x_0 \in \partial D$;*

*(ii) $\Psi_{x_0}$ is a peak function for $D$ at $x_0$ for each $x_0 \in \partial D$.*

These definitions and results are all we need in order to investigate about the boundary behaviour of the Kobayashi distance.
If it is not explicitly specified, from now on we mean for a *strongly pseudoconvex* domain a bounded and strongly pseudoconvex domain with $C^2$ boundary.
Given a strongly pseudoconvex domain $D$ and $z \in D$, we denote with $d(z, \partial D)$ the euclidean distance of $z$ from the boundary.
The next step is to get results that are generalized versions of the following

**Lemma 2.0.9.** *Let $B_r$ be the euclidean ball of radius $r$ in $\mathbb{C}^n$ centered at the origin. Then, for every $z \in B_r$, we have:*

$$\frac{1}{2}\log r - \frac{1}{2}\log d(z, \partial B_r) \leq c_{B_r}(0, z) = k_{B_r}(0, z) \leq \frac{1}{2}\log 2r - \frac{1}{2}\log d(z, \partial B_r).$$

*Proof.* We start by applying Proposition 1.2.7 with $\| \cdot \|_1 = \frac{\|\cdot\|}{r}$. In this way we get

$$c_{B_r} = k_{B_r} = \omega(0, \frac{\| z \|}{r}).$$

Moreover, we explicit the euclidean distance from the boundary for a point $z \in B_r$ as

$$d(z, \partial B_r) = r - \| z \| .$$

Setting as $t = \frac{\|z\|}{r}$ and using the monotonicity of the logarithm, we then have:

$$\frac{1}{2}\log\frac{1}{1-t} \leq \frac{1}{2}\log\frac{1+t}{1-t} = \omega(0, t) \leq \frac{1}{2}\log\frac{2}{1-t}\,.$$

Thus, as a lower bound, we have:

$$\frac{1}{2}\log r - \frac{1}{2}\log(r - \parallel z \parallel) \leq \omega(0, \frac{\parallel z \parallel}{r}).$$

And, as an upper bound, we have:

$$\omega(0, \frac{\parallel z \parallel}{r}) \leq \frac{1}{2}\log 2r - \frac{1}{2}\log(r - \parallel z \parallel).$$

□

Now for the general case of a strongly pseudoconvex domain we need to deal with its boundary, that we know it is a $C^2$ manifold.
A well-known result assures that a compact and smooth hypersurface of $\mathbb{R}^N$ is orientable. (see Chapter 15 from [13])
In Section 3.3 from [14], one can understand that orientability has a homological characterization via torsion and cohomology is always torsion free.
Indeed, by using Alexander Duality, smoothness might not be required as assumption in order to get an orientation on a compact manifold. (see [15])
We remember also that for an orientable manifold there are only two choices for a unit normal vector field, denoted as **n** and −**n**.
In this way for a compact and $C^2$ hypersurface $M$ we consider the unit normal vector field **n**.
We shall say that such a manifold $M$ has a tubular neighbourhood $U_\varepsilon$ of radius $\varepsilon > 0$, if the segments $\{x + t\mathbf{n}_x | t \in (-\varepsilon, \varepsilon)\}$ are pairwise disjoint and we set

$$U_\varepsilon = \bigcup_{x \in M} \{x + t\mathbf{n}_x | t \in (-\varepsilon, \varepsilon)\}.$$

Note that if $M$ has a tubular neighbourhood of radius $\varepsilon$, then $d(x + t\mathbf{n}_x, M) = |t|$ for every $t \in (-\varepsilon, \varepsilon)$ and $x \in M$; in particular we can state the following

**Proposition 2.0.10.** *If $M$ is a compact and $C^2$ hypersurface that admits a tubular neighborhood $U_\varepsilon$, then*

$$U_\varepsilon = \bigcup_{x \in M} B(x, \varepsilon)\,.$$

*Proof.* Let's prove it by double inclusion.
Pick $y \in U_\varepsilon$, then there are $x^* \in M$ and $t^* \in (-\varepsilon, \varepsilon)$ so that $y = x^* + t\mathbf{n}_{x^*}$, thus $d(x^*, y) < \varepsilon$ and this gives $y \in \bigcup_{x \in M} B(x, \varepsilon)$.
On the other side, we pick $y \in \bigcup_{x \in M} B(x, \varepsilon)$, then it exists $x^* \in M$ so that $y \in B(x^*, \varepsilon)$. Since $M$ admits a tubular neighborhood there are $\hat{x}$ and $\hat{t} \in (-\varepsilon, \varepsilon)$ so that $y = \hat{x} + \hat{t}\mathbf{n}_{\hat{x}}$ and then $y \in U_\varepsilon$. □

A proof of the existence of a tubular neighbourhood of radius sufficiently small for any compact hypersurface of $\mathbb{R}^N$ can be found in Chapter 9 from [16].
At this point we can start to give some upper and lower bound for the Kobayashi distance, the upper estimate does not require the domain to be strongly pseudoconvex:

**Theorem 2.0.11.** *Let $D \subset\subset \mathbb{C}^n$ be a $C^2$ domain, and $z_0 \in D$. Then there is a constant $c_1 \in \mathbb{R}$ depending only on $D$ and $z_0$ such that for all $z \in D$ it holds*

$$c_D(z_0, z) \leq k_D(z_0, z) \leq c_1 - \frac{1}{2} \log d(z, \partial D) .$$

*Proof.* We observe that $D$ is a $C^2$ domain and the previous considerations with respect to $\partial D$ allow to admit tubular neighbourhoods $U_\varepsilon$, with a radius $\varepsilon < 1$ small enough. Denoting by $\operatorname{diam}(D)$ the euclidean diameter of $D$, we can define the constant:

$$c_1 = \{k_D(z_0, w) | w \in D \setminus U_{\varepsilon/4}\} + \max\{0, \frac{1}{2} \log \operatorname{diam}(D)\}.$$

As $c_1$ is defined we can see that it depends just on $D$.
Let's consider now two different cases.

If $z \in D \cap U_{\varepsilon/4}$, consider a boundary point $x \in \partial D$ that minimizes the distance from the boundary respect $z$, i.e. $\| z - x \| = d(z, \partial D)$. By construction of $U_{\varepsilon/2}$, as a tubular neighborhood for $\partial D$, it exists $t \in \mathbb{R}$ so that $w = t(x - z)$ stays inside $\partial U_{\varepsilon/2} \cap D$ and the euclidean ball $B(w, \varepsilon/2)$ is contained in $U_\varepsilon \cap D$ and $B(w, \varepsilon/2)$ is tangent to $\partial D$ at $x$. As a consequence of

Lemma 2.0.9 applied to $B(w, \varepsilon/2)$ we get the following upper bound:

$$k_{B(w,\varepsilon/2)}(w, z) \leq \frac{1}{2}\log\frac{\varepsilon}{2} - \frac{1}{2}\log d(z, \partial B(w, \varepsilon/2)) \leq \frac{1}{2}\log\varepsilon - \frac{1}{2}\log d(z, \partial B(w, \varepsilon/2)).$$

Moreover, since $\varepsilon < 1$ we have $\log\varepsilon < 0$ and by tangency condition of $B(w, \varepsilon/2)$ at $x$ we get $d(z, \partial B(w, \varepsilon/2)) = d(z, \partial D)$. Then the last considerations and the inclusion $B(w, \varepsilon/2) \subset D$ yield to:

$$c_D(z_0, z) \leq k_D(z_0, z) \leq k_D(z_0, w) + k_D(w, z) \leq k_D(z_0, w) + k_{B(w,\varepsilon/2)}(w, z) \leq$$

$$k_D(z_0, w) + \frac{1}{2}\log\varepsilon - \frac{1}{2}\log d(z, \partial B(w, \varepsilon/2)) \leq c_1 - \frac{1}{2}\log(z, \partial D).$$

Otherwise, if $z \in D \setminus U_{\varepsilon/4}$, by definition of $c_1$, we get:

$$c_D(z_0, z) \leq k_D(z_0, z) \leq c_1 - \frac{1}{2}\log\operatorname{diam}(D) \leq c_1 - \frac{1}{2}\log d(z, \partial D).$$

□

Now, we will take care of the lower estimate; in order to do this we will benefit of the existence of a *peak function*.

**Theorem 2.0.12.** *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain, and $z_0 \in D$. Then there is a constant $c_2 \in \mathbb{R}$ depending only on $D$ and $z_0$ such that for all $z \in D$*

$$c_2 - \frac{1}{2}\log d(z, \partial D) \leq c_D(z_0, z) \leq k_D(z_0, z)\ .$$

*Proof.* Since $D$ is compactly contained in $\mathbb{C}^n$, we can consider a set $D'$ so that $D \subset\subset D'$; in this way $D'$ works as a neighborhood for $\overline{D}$. Then consider $D'$ and $\Psi : \partial D \times D' \to \mathbb{C}$ as in the statement of Theorem 2.0.8.
We define now $\varphi : \partial D \times \Delta \to \mathbb{C}$ to be

$$\varphi(x, \zeta) = \frac{1 - \overline{\Psi(x, z_0)}}{1 - \Psi(x, z_0)} \cdot \frac{\zeta - \Psi(x, z_0)}{1 - \overline{\Psi(x, z_0)}\zeta}.$$

Then the function denoted as $\Phi(x, z) = \Phi_x(z) = \varphi(x, \Psi(x, z))$ is defined on a neighbourhood $\partial D \times D_0$ of $\partial D \times \overline{D}$ (with $D_0 \subset\subset D'$ ) and satisfies:

- $\Phi$ is continuous on $\partial D \times D_0$ since $\Psi$ is a peak function which guarantees that $\Phi$ has no poles on its source space;

- $\Phi_x$ is a holomorphic peak function for $D$ at $x$ for any $x \in \partial D$, indeed $\Phi_x(x) = \varphi(x, \Psi(x,x)) = \frac{1-\overline{\Psi(x,z_0)}}{1-\Psi(x,z_0)} \cdot \frac{1-\Psi(x,z_0)}{1-\overline{\Psi(x,z_0)}} = 1$ and the Maximum Modulus Principle assures that $|\Phi_x(z)| < 1 \forall z \in D$;

- for every $x \in \partial D$ we have that $\Phi_x(z_0) = 0$, indeed $\Phi_x(z_0) = \varphi(x, \Psi(x,z_0)) = \frac{1-\overline{\Psi(x,z_0)}}{1-\Psi(x,z_0)} \cdot \frac{\Psi(x,z_0)-\Psi(x,z_0)}{1-|\Psi(x,z_0)|^2} = 0$.

Given $P(x,\varepsilon)$ to be the polydisk of center $x$ and polyradius $(\varepsilon, \cdots, \varepsilon)$, we can define $U_\varepsilon = \bigcup_{x \in \partial D} P(x,\varepsilon)$. The family $\{U_\varepsilon\}_\varepsilon$ is a basis for the neighbourhoods of $\partial D$; hence there exists $\varepsilon > 0$ such that $U_\varepsilon \subset\subset D_0$ and $U_\varepsilon$ is contained in a tubular neighbourhood of $\partial D$. Then, for any $x \in \partial D$ and $z \in P(x, \varepsilon/2)$, the Fundamental Theorem of Calculus for Complex Analysis and the Cauchy estimates of the derivatives of first order of $\Phi$ yield to

$$|1 - \Phi_x(z)| = |\Phi_x(x) - \Phi_x(z)| \leq \| z - x \| \max_{w \in P(x,\varepsilon/2)} \| \frac{\partial \Phi_x}{\partial z}(w) \| =$$

$$\| z - x \| \max_{w \in P(x,\varepsilon/2)} \Big( \sum_{j=1}^{n} \big| \frac{\partial \Phi_x}{\partial z_j}(w) \big|^2 \Big)^{1/2} \leq \| z - x \| \frac{2}{\varepsilon}\sqrt{n} \max_{w \in P(x,\varepsilon/2)} |\Phi_x(w)|$$

$$\leq \| z - x \| \frac{2}{\varepsilon}\sqrt{n} \max_{(y,w) \in \partial D \times U_\varepsilon} |\Phi(y,w)| = M \| z - x \|,$$

where the constant $M$ depends only on $\partial D \times U_\varepsilon$ and not on precise values like $x$ or $z$.
We observe now that $\Phi_x(x) = 1$ and this grants

$$1 \leq \max_{(y,w) \in \partial D \times U_\varepsilon} |\Phi(y,w)| \, .$$

At this point we can define $c_2 = -\frac{1}{2} \log M$ and check that, if $1 \leq \max_{(y,w) \in \partial D \times U_\varepsilon} |\Phi(y,w)|$, then $c_2 \leq \frac{1}{2} \log(\varepsilon/2)$:

$$\frac{1}{\sqrt{n}} \leq 1 \leq \max_{(y,w) \in \partial D \times U_\varepsilon} |\Phi(y,w)| \Leftrightarrow \frac{2}{\varepsilon} \leq \frac{2}{\varepsilon}\sqrt{n} \max_{(y,w) \in \partial D \times U_\varepsilon} |\Phi(y,w)| \Leftrightarrow \frac{2}{\varepsilon} \leq M,$$

then, by monotonicity of the logarithm, we get

$$\log\frac{1}{M}\leq\log\frac{\varepsilon}{2}\Leftrightarrow -\frac{1}{2}\log M\leq\frac{1}{2}\log\frac{\varepsilon}{2}.$$

We are going to consider again two different cases.

If $z\in D\cap U_{\varepsilon/2}$, we take $x\in\partial D$ so that $\parallel x-z\parallel = d(z,\partial D)$ and since $\Phi_x(D)\subset\Delta$ and $\Phi_x(z_0)=0$, by applying Proposition 1.2.1, we get:

$$k_D(z_0,z)\geq c_D(z_0,z)\geq\omega(\Phi_x(z_0),\Phi_x(z))=\frac{1}{2}\log\frac{1+|\Phi_x(z)|}{1-|\Phi_x(z)|}\geq\frac{1}{2}\log\frac{1}{1-|\Phi_x(z)|}.$$

On the other hand:

$$1-|\Phi_x(z)|\leq|1-\Phi_x(z)|\leq M\parallel z-x\parallel = Md(z,\partial D)\,,$$

consequently

$$k_D(z_0,z)\geq c_D(z_0,z)\geq -\frac{1}{2}\log M-\frac{1}{2}\log d(z,\partial D)\geq c_2-\frac{1}{2}\log d(z,\partial D).$$

Otherwise, if $z\in D\setminus U_{\varepsilon/2}$, then $d(z,\partial D)\geq\varepsilon/2$ and thus:

$$k_D(z_0,z)\geq c_D(z_0,z)\geq 0\geq\frac{1}{2}\log(\varepsilon/2)-\frac{1}{2}\log d(z,\partial D)\geq c_2-\frac{1}{2}\log d(z,\partial D).$$

□

An interesting consequence is the following

**Corollary 2.0.13.** *In every strongly pseudoconvex domain of $\mathbb{C}^n$ closed balls are compact.*

*Proof.* Consider $D$ to be such a strongly pseudoconvex domain with $z_0\in D$ and, with $r>0$, let $z\in B_k(z_0,r)=\{w\in D|k_D(z_0,w)<r\}$. Now, by applying Theorem 2.0.12, we get:

$$c_2-\frac{1}{2}\log d(z,\partial D)\leq k_D(z_0,z)<r\Rightarrow\log d(z,\partial D)>2(c_2-r)\Rightarrow$$
$$d(z,\partial D)>\exp(2(c_2-r)).$$

Here we remember that $c_2$ is a constant which depends only on $z_0\in D$. Hence $B_k(z_0,r)\subset\subset D$ and this concludes the proof. □

Now we will refine the bounds for the Kobayashi distance of Theorems 2.0.11 and 2.0.12 in the particular case where the points get closer and closer to the boundary.

**Theorem 2.0.14.** *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain and $\delta > 0$. Then there exist $\varepsilon_1, \varepsilon_0 \in (0, \delta)$ with $\varepsilon_0 < \varepsilon_1$ and a constant $c \in \mathbb{R}$ such that for all $x_0 \in \partial D$ and $z \in D \cap B(x_0, \varepsilon_0)$ we have*

$$k_D(z, D \setminus B(x_0, 2\varepsilon_1)) \geq -\frac{1}{2} \log d(z, \partial D) + c.$$

*Proof.* Let $D \subset\subset D_0$ (neighborhood of $\overline{D}$), $\Psi : \partial D \times D_0 \to \mathbb{C}$ (peak function) be given by Theorem 2.0.8 and set again $U_\varepsilon = \bigcup_{x \in \partial D} P(x, \varepsilon)$.
Then pick $\varepsilon_1 \in (0, \delta)$ so that $U_{2\varepsilon_1}$ is contained in a tubular neighbourhood of $\partial D$ and moreover $U_{2\varepsilon_1} \subset\subset D_0$. Put

$$V_{\varepsilon_1} = \{(x, z_0) \in \partial D \times \overline{D} | \parallel z_0 - x \parallel \geq \varepsilon_1\};$$

since $V_{\varepsilon_1}$ is compact (bounded and close in $\partial D \times \overline{D}$) and $|\Psi(x, z_0)| < 1$ for all $(x, z_0) \in V_{\varepsilon_1}$, there is $\eta < 1$ such that $|\Psi(x, z_0)| < \eta < 1$ for all $(x, z_0) \in V_{\varepsilon_1}$. Define $\varphi : V_{\varepsilon_1} \times \Delta \to \mathbb{C}$ by

$$\varphi(x, z_0, \zeta) = \frac{1 - \overline{\Psi(x, z_0)}}{1 - \Psi(x, z_0)} \cdot \frac{\zeta - \Psi(x, z_0)}{1 - \overline{\Psi(x, z_0)}\zeta}$$

and fix $\gamma \in (\eta, 1)$. If we take a neighbourhood $D_0 \subset\subset D'$ of $\overline{D}$ such that $|\Psi(x, z)| < \gamma/\eta$ for all $x \in \partial D$ and $z \in \overline{D_0}$, then the map $\Phi(x, z_0, z) = \Phi_{x,z_0}(z) = \varphi(x, z_0, \Psi(x, z))$ is defined on $V_{\varepsilon_1} \times D_0$; moreover we can notice that, for a fixed $z_0$, this function shares the same properties of $\Phi$ that appear in the proof of Theorem 2.0.12. Furthermore we check that $\Phi$ is bounded on the source space, hence for every $(x, z_0, z) \in V_{\varepsilon_1} \times D_0$ we have

$$|\Phi(x, z_0, z)|^2 = \Phi(x, z_0, z)\overline{\Phi(x, z_0, z)} =$$

$$\frac{1 - \overline{\Psi(x, z_0)}}{1 - \Psi(x, z_0)} \cdot \frac{\Psi(x, z) - \Psi(x, z_0)}{1 - \overline{\Psi(x, z_0)}\Psi(x, z)} \cdot \frac{1 - \Psi(x, z_0)}{1 - \overline{\Psi(x, z_0)}} \cdot \frac{\overline{\Psi(x, z)} - \overline{\Psi(x, z_0)}}{1 - \Psi(x, z_0)\overline{\Psi(x, z)}} =$$

$$\frac{|\Psi(x, z)|^2 - 2\mathrm{Re}(\overline{\Psi(x, z)}\Psi(x, z_0)) + |\Psi(x, z_0)|^2}{1 - 2\mathrm{Re}(\overline{\Psi(x, z)}\Psi(x, z_0)) + |\Psi(x, z_0)|^2|\Psi(x, z)|^2} \leq$$

$$\frac{|\Psi(x,z)|^2+2|\Psi(x,z)||\Psi(x,z_0)|+|\Psi(x,z_0)|^2}{1-2|\Psi(x,z)||\Psi(x,z_0)|+|\Psi(x,z_0)|^2|\Psi(x,z)|^2}<\frac{(\gamma/\eta)^2+2\gamma+\eta^2}{1-2\gamma+\gamma^2}<+\infty.$$

Now choose $\varepsilon_0 \in (0, \varepsilon_1/2)$ so that $U_{2\varepsilon_0} \subset\subset D_0$. Then for every $(x, z_0) \in V_{\varepsilon_1}$ and $z \in B(x, \varepsilon_0) \subset P(x, \varepsilon_0)$ we have

$$|1-\Phi_{x,z_0}(z)| = |\Phi_{x,z_0}(x)-\Phi_{x,z_0}(z)| \leq \| z-x \| \max_{w\in P(x,\varepsilon_0)} \| \frac{\partial \Phi_{x,z_0}}{\partial z}(w) \| \leq$$

$$\| z-x \| \frac{\sqrt{n}}{\varepsilon_0} \max_{(y,w_0,w)\in V_{\varepsilon_1}\times D_0} |\Phi(y,w_0,w)| \,.$$

At this point we can define the constant

$$c = -\frac{1}{2}\log\Big(\frac{\sqrt{n}}{\varepsilon_0} \max_{(y,w_0,w)\in V_{\varepsilon_1}\times D_0} |\Phi(y,w_0,w)|\Big)$$

and observe that it does not depend on the choice of the points $x$, or $z_0$, or $z$; since it is a maximum attained on the whole $V_{\varepsilon_1} \times D_0$.
Now consider $x \in \partial D$, $z \in B(x, \varepsilon_0) \cap D$ and $z_0 \in D \setminus B(x, 2\varepsilon_1)$. Then there is $y \in B(x, 2\varepsilon_0) \cap \partial D$ such that $\| z - y \| = d(z, \partial D)$; furthermore we have

$$\| y - z_0 \| \geq \| x - z_0 \| - \| y - x \| > 2\varepsilon_1 - 2\varepsilon_0 \geq \varepsilon_1 \,,$$

and this means that $(y, z_0) \in V_{\varepsilon_1}$. Finally we can give the lower bound:

$$k_D(z,z_0) \geq c_D(z,z_0) \geq \omega(\Phi_{y,z_0}(z), \Phi_{y,z_0}(z_0)) = \omega(\Phi_{y,z_0}(z), 0) \geq$$

$$\frac{1}{2}\log\frac{1}{1-|\Phi_{y,z_0}(z)|} \geq c - \frac{1}{2}\log \| z-y \| = c - \frac{1}{2}\log d(z,\partial D).$$

□

And now a very crucial consequence of this last theorem

**Corollary 2.0.15.** *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain of $\mathbb{C}^n$, and choose two points $x_1, x_2 \in \partial D$ with $x_1 \neq x_2$. Then there exist $\varepsilon_0 > 0$ and $K \in \mathbb{R}$ such that for any $z_1 \in D \cap B(x_1, \varepsilon_0)$ and $z_2 \in D \cap B(x_2, \varepsilon_0)$ we have*

$$k_D(z_1, z_2) \geq -\frac{1}{2}\log d(z_1, \partial D) - \frac{1}{2}\log d(z_2, \partial D) + K.$$

*Proof.* Let $\varepsilon_0, \varepsilon_1 \in (0, \delta)$ be given through the proof of Theorem 2.0.14, where $\delta > 0$ is small enough to assure the condition $B(x_1, 2\delta) \cap B(x_2, 2\delta) = \varnothing$. Take $z_1 \in B(x_1, \varepsilon_0)$ and $z_2 \in B(x_2, \varepsilon_0)$. Now let $\sigma$ be any curve joining $z_1$ to $z_2$. Then part of the image of $\sigma$ should be outside both $B(x_1, 2\varepsilon_1)$ and $B(x_2, 2\varepsilon_1)$: this means that if we pick two points $z_1' \in \sigma \cap \partial B(x_1, 2\varepsilon_1)$ and $z_2' \in \sigma \cap \partial B(x_2, 2\varepsilon_1)$, they grant:

$$\ell_k(\sigma) \geq k_D(z_1, z_1') + k_(z_2, z_2') \geq k_D(z_1, D \setminus B(x_1, 2\varepsilon_1) + k_D(z_2, D \setminus B(x_2, 2\varepsilon_1).$$

Consequently the thesis of Theorem 2.0.14 leads to:

$$\ell_k(\sigma) \geq -\frac{1}{2} \log d(z_1, \partial D) + K_1 - \frac{1}{2} \log d(z_2, \partial D) + K_2.$$

We can notice now that $K = K_1 + K_2$ is a constant which does not depend on $z_1$ or $z_2$, thus by taking the infimum over all the curves joining $z_1$ to $z_2$ and then applying Theorem 1.3.10 we can write:

$$k_D(z_1, z_2) \geq -\frac{1}{2} \log d(z_1, \partial D) - \frac{1}{2} \log d(z_2, \partial D) + K.$$

□

At last we are going to describe what happens to the Kobayashi distance when the two points get closer to the same boundary point

**Theorem 2.0.16.** *Let $D \subset\subset \mathbb{C}^n$ be a $C^2$ domain and $x_0 \in \partial D$. Then there exist $\varepsilon > 0$ and $C \in R$ such that for all $z_1, z_2 \in D \cap B(x_0, \varepsilon)$ we have*

$$k_D(z_1, z_2) \leq -\frac{1}{2} \sum_{j=1}^{2} \log d(z_j, \partial D) + \frac{1}{2} \sum_{j=1}^{2} \log \left( d(z_j, \partial D) + \| z_1 - z_2 \| \right) + C.$$

*Proof.* For every $x \in \partial D$ denote by $\mathbf{n}_x$ the outer unit normal vector to $\partial D$ at $x$. Choose $\varepsilon > 0$ so small that $\partial D \cap B(x_0, 4\varepsilon)$ is connected and the following conditions are satisfied:

*(i)* since D is a $C^2$ domain it admits a unit normal smooth vector field which grants that $\| \mathbf{n}_x - \mathbf{n}_{x_0} \| < 1/8$ for all $x \in \partial D \cap B(x_0, \varepsilon)$;

*(ii)* for every $\delta \in [0, 2\varepsilon]$, $z \in D \cap B(x_0, \varepsilon)$ and $x \in \partial D \cap B(x_0, 4\varepsilon)$ we have $z - \delta \mathbf{n}_x \in D$ and

$$d(z - \delta \mathbf{n}_x, \partial D) > \frac{3}{4}\delta.$$

Set $U = B(x_0, \varepsilon)$. Let $z_1, z_2 \in U \cap D$, and choose $x_1, x_2 \in \partial D$ so that $\| z_j - x_j \| = d(z_j, \partial D)$ for $j = 1, 2$. Set $z'_j = z_j - \| z_1 - z_2 \| \mathbf{n}_{x_j}$; then the triangle inequality applied twice on the Kobayashi distance gives:

$$k_D(z_1, z_2) \leq k_D(z'_1, z'_2) + \sum_{j=1}^{2} k_D(z_j, z'_j).$$

Here it starts the way to bound from above the first of the last two terms, this bound will be plugged in the constant $C$.
Since $\| z_1 - z_2 \| < 2\varepsilon$, by $(ii)$ we have $d(z'_j, \partial D) > \frac{3}{4} \| z_1 - z_2 \|$. Moreover, by $(i)$ we have:

$$\| z'_1 - z'_2 \| = \| z_1 - z_2 + \| z_1 - z_2 \| \mathbf{n}_{x_1} - \| z_1 - z_2 \| \mathbf{n}_{x_2} \pm \| z_1 - z_2 \| \mathbf{n}_{x_0} \| < \frac{5}{4} \| z_1 - z_2 \| .$$

Now we define the open set $\Omega$ in $\mathbb{C}$ as

$$\Omega = \{\zeta \in \mathbb{C} | \min\{|\zeta|, |1 - \zeta|\} < \frac{3}{5}\}$$

and the holomorphic map $\varphi : \Omega \to \mathbb{C}^n$ by $\varphi(\zeta) = z'_1 + \zeta(z'_2 - z'_1)$.
Then $\varphi(\Omega) \subset B(z'_1, \frac{3}{4} \| z_1 - z_2 \|) \subset D$, $\varphi(0) = z'_1$ and $\varphi(1) = z'_2$; hence Proposition 1.2.1 gives:

$$k_D(z'_1, z'_2) \leq k_\Omega(0, 1).$$

In order to conclude the poof we must bound from above the term $k_D(z_j, z'_j)$.
Let $\varphi_j \in Hol(\mathbb{C}, \mathbb{C}^n)$ be defined as $\varphi_j(\zeta) = x_j - \zeta \mathbf{n}_{x_j}$; then $\varphi_j(0) = x_j$, $\varphi_j(d(z_j, \partial D)) = x_j - d(z_j, \partial D)\mathbf{n}_{x_j} = z_j$ and $\varphi_j(d(z_j, \partial D) + \| z_1 - z_2 \|) = x_j - \big(d(z_j, \partial D) + \| z_1 - z_2 \| \big)\mathbf{n}_{x_j} = z'_j$. For a fixed $\alpha > 0$ we define

$$\Omega_\alpha = \{\zeta = \xi + i\eta \in \mathbb{C} | |\zeta| < 4\varepsilon, \xi > \alpha|\eta|^2\}.$$

For $\alpha$ big enough then $\Omega_\alpha$ becomes thinner until it grants the condition $\varphi_j(\Omega_\alpha) = D \cap U$. Since $\partial\Omega_\alpha$ is obtained as the intersection between a circumference and a parabola, we will consider a subset $\Omega'_\alpha \subset \Omega_\alpha$ that is smooth in a neighbourhood of the angular points of $\partial\Omega_\alpha$. Again by Proposition 1.2.1 we have:

$$k_D(z_j, z'_j) \le k_{\Omega'_\alpha}\big(d(z_j, \partial D), d(z_j, \partial D) + \parallel z_1 - z_2 \parallel\big).$$

So it remains to show that if $a$ and $b$ are real numbers satisfying $0 < a < b < 3\varepsilon$, then

$$k_{\Omega'_\alpha}(a,b) \le \frac{1}{2}(\log b - \log a) + O(1).$$

Since $\Omega'_\alpha$ is simply connected and a proper subset of $\mathbb{C}$, the Riemann Mapping Theorem grants a biholomorphism with $\Delta$. Let $\tau$ be such biholomorphism of $\Omega'_\alpha$ with $\Delta$ so that $\tau(0) = 1$ and $\tau$ is real on the real axis. Since the construction of $\partial\Omega'_\alpha$ grants that is it smooth, Kellogg's Theorem (see Theorem 6 at page 426 in [17]) allows $\tau$ to extend as a diffeomorphism between $\overline{\Omega'_\alpha}$ and $\overline{\Delta}$. Therefore there are $K > 1$ and $\vartheta \in (-1, 1)$ such that for every $c \in (0, 3\varepsilon)$

$$\max\{\vartheta, 1 - Kc\} \le \tau(c) \le 1 - c/K.$$

Recalling that the geodesic respect the Poincaré metric of $\Delta$ for two points aligned with the origin is a straight line, thus

$$k_{\Omega'_\alpha}(a,b) = \omega(\tau(a), \tau(b)) = \omega(0, \tau(a)) - \omega(0, \tau(b)) = \frac{1}{2}\Big(\log\frac{1 + |\tau(a)|}{1 - |\tau(a)|} - \log\frac{1 + |\tau(b)|}{1 - |\tau(b)|}\Big) \le$$

$$\frac{1}{2}\Big(\log\frac{2}{a/K} - \log\frac{1+\vartheta}{Kb}\Big) \le \frac{1}{2}(\log b - \log a) + O(1).$$

□

And now an useful result concerning subharmonic functions that will be crucial to us later

**Lemma 2.0.17** (Hopf's Lemma)**.** *Let $U \subset \mathbb{R}^N$ be a $C^2$ domain. Let the function $f : \overline{U} \to \mathbb{R}$ be subharmonic in $U$, continuous in $\overline{U}$ and suppose that*

*$f$ has a local maximum at $x_0 \in \partial U$. Let $\mathbf{n} = \mathbf{n}_{x_0}$ be the outer unit normal to $\partial U$ at $x_0$; then*

$$\liminf_{t \to 0^+} \frac{f(x_0) - f(x_0 - t\mathbf{n})}{t} > 0.$$

*In particular, when it exists, it holds*

$$\frac{\partial f}{\partial \mathbf{n}}(x_0) > 0.$$

*Proof.* Since $U$ is open and $x_0$ is a local maximum we can consider $\varepsilon > 0$ to be such that there exists a ball $B$ of radius $\varepsilon$ internally tangent to $\partial U$ at $x_0$ so that $f(x_0) > f(x)$ for all $x \in B$. Up to a translation, we can assume that the center of $B$ is the origin. Respect this setting some straightforward computations lead to $\| x_0 \| = \varepsilon$ and $\langle x_0, \mathbf{n}_{x_0} \rangle = \varepsilon$. Let $B_1$ be a ball centered at $x_0$ of radius $\varepsilon_1 < \varepsilon$, and let $B' = B \cap B_1$. Then $\partial B'$ is the union of two hypersurfaces $S' = \partial B' \cap B$ and $S'_1 = \partial B' \cap B_1$. Define now $h : \mathbb{R}^N \to \mathbb{R}$ by

$$h(x) = e^{-\alpha \|x\|^2} - e^{-\alpha \varepsilon^2}$$

where $\alpha > 0$. For as it is defined we have $h > 0$ on $B' \subset B$ and

$$\nabla^2 h = \sum_{j=1}^{N} \frac{\partial^2 h}{\partial x_j^2} = \sum_{j=1}^{N} e^{-\alpha \|x\|^2} (4\alpha^2 x_j^2 - 2\alpha) = e^{-\alpha \|x\|^2} (4\alpha^2 \| x \|^2 - 2\alpha N).$$

In particular, if $\alpha$ is large enough then $\nabla^2 h > 0$ on $B'$. Now set

$$v(x) = f(x) + \delta h(x).$$

If $\delta$ is small enough then also $v(x) < f(x_0)$ on $S'_1$; moreover, since $h_{|S' \setminus \{x_0\}} \equiv 0$ we have $v(x) = f(x)$ on $S' \setminus \{x_0\}$. Since $v$ is subharmonic in $B'$, the Maximum Principle grants

$$\max_{x \in \overline{B'}} v(x) = f(x_0).$$

Therefore

$$\liminf_{t \to 0^+} \frac{v(x_0) - v(x_0 - t\mathbf{n}_{x_0})}{t} = \delta \frac{\partial h}{\partial \mathbf{n}_{x_0}}(x_0) + \liminf_{t \to 0^+} \frac{f(x_0) - f(x_0 - t\mathbf{n}_{x_0})}{t} \geq 0.$$

On the other hand we have that

$$\frac{\partial h}{\partial \mathbf{n}_{x_0}}(x_0) = \langle \nabla h(x_0), \mathbf{n}_{x_0} \rangle = -2\alpha e^{-\alpha \|x_0\|^2} \langle x_0, \mathbf{n}_{x_0} \rangle = -2\alpha \varepsilon e^{-\alpha \varepsilon^2} < 0$$

and this concludes our proof. □

**Theorem 2.0.18** (Fefferman's Theorem)**.** *Let $D, D' \subset\subset \mathbb{C}^n$ be strongly pseudoconvex domains and $f : D \to D'$ a biholomorphism. Then $f$ extends continuously to a homeomorphism of $\overline{D}$ with $\overline{D'}$.*

*Proof.* We start by observing that given $\rho$ to be a defining function for $D$, strictly plurisubharmonic in a neighbourhood $U$ of $\partial D$, it can be assumed that $U$ has $C^2$ boundary. In this way $U$ is contained in a suitable tubular neighbourhood of $\partial D$ and then $f(U \cap D)$ is contained in a tubular neighbourhood of $\partial D'$. Then we can apply Lemma 2.0.17 to the subharmonic function $\rho \circ f^{-1}$ defined on $f(U \cap D)$ which assumes maximum on $\partial D'$, obtaining that there exists $c > 0$ such that for all $x_0 \in \partial D'$

$$\liminf_{t \to 0^+} \frac{\rho \circ f^{-1}(x') - \rho \circ f^{-1}(x' - t\mathbf{n}_{x'})}{t} = \liminf_{t \to 0^+} \frac{\rho \circ f^{-1}(x' - t\mathbf{n}_{x'})}{-t} \geq c > 0\,,$$

where we set $\mathbf{n}_{x'}$ as the outer unit normal vector to $\partial D'$ at $x'$.
The last condition can be restated as there is $\varepsilon > 0$ such that

$$\rho \circ f^{-1}(x' - t\mathbf{n}_{x'}) \leq -ct$$

for all $t \in [0, \varepsilon]$ and $x' \in \partial D'$.
Moreover the tubular neighbourhood in $\partial D'$ which contains $f(U \cap D)$ grants that $t = d(x' - t\mathbf{n}_{x'}, \partial D')$. Then, eventually by taking an even smaller U, we infer

$$cd(f(z), \partial D') \leq -\rho(z)$$

for all $z \in U \cap D$.
On the other hand we have that $\nabla \rho$ does not vanish on $\partial D$ and the expansion of $\rho$, as in Proposition 2.0.4, then implies that $-\rho(z)$ is of the order of $d(z, \partial D)$ when approaching to $\partial D$. Thus there exists a different constant $K > 0$ such that

$$d(f(z), \partial D') \leq Kd(z, \partial D) \tag{2.1}$$

for all $z \in U \cap D$. Therefore, since $D \setminus U$ is compact, we can extend this bound to the whole $D$ up to adjusting properly the constant $K$.
Now we can show that $f$ extends continuously to $\partial D$. By contradiction

we suppose that if we pick a point $x_0 \in \partial D$ then there are two sequences $(z_\nu^1)_{\nu\in\mathbb{N}}, (z_\nu^1)_{\nu\in\mathbb{N}}$ of points in $D$ which converge to the same point $x_0$ and $f(z_\nu^1) \to y^1 \in \partial D'$,$f(z_\nu^2) \to y^2 \in \partial D'$ as $\nu \to \infty$ with $y^1 \neq y^2$.
In this way from Theorem 2.0.16 we get an upper bound for the Kobayashi distance when the two points are approaching the same boundary point in $\partial D$:

$$k_D(z_\nu^1, z_\nu^2) \leq -\frac{1}{2}\sum_{j=1}^{2} \log d(z_\nu^j, \partial D) + \frac{1}{2}\sum_{j=1}^{2} \log\big(d(z_\nu^j, \partial D) + \parallel z_\nu^1 - z_\nu^2 \parallel\big) + O(1). \tag{2.2}$$

On the other hand from Corollary 2.0.15 we get a lower bound of the Kobayashi distance when the two points are approaching two different boundary points of $\partial D'$ respectively:

$$k_{D'}\big(f(z_\nu^1), f(z_\nu^2)\big) \geq -\frac{1}{2}\sum_{j=1}^{2} \log(d(f(z_\nu^j), \partial D')) + O(1). \tag{2.3}$$

Now, thanks to Proposition 1.2.1 we gain $k_{D'}\big(f(z_\nu^1), f(z_\nu^2)\big) \leq k_D(z_\nu^1, z_\nu^2)$, thus we can plug together 2.2 and 2.3 obtaining:

$$\begin{aligned} -\frac{1}{2}\sum_{j=1}^{2} \log(d(f(z_\nu^j), \partial D')) \leq & -\frac{1}{2}\sum_{j=1}^{2} \log d(z_\nu^j, \partial D) + \\ & + \frac{1}{2}\sum_{j=1}^{2} \log\big(d(z_\nu^j, \partial D) + \parallel z_\nu^1 - z_\nu^2 \parallel\big) + O(1). \end{aligned} \tag{2.4}$$

At this point 2.1 gives the bound:

$$-\frac{1}{2}\sum_{j=1}^{2} \log(d(z_\nu^j, \partial D)) \leq -\frac{1}{2}\sum_{j=1}^{2} \log(d(f(z_\nu^j), \partial D')). \tag{2.5}$$

Hence by combining 2.4 and 2.5 we get:

$$-\frac{1}{2}\sum_{j=1}^{2} \log\big(d(z_\nu^j, \partial D) + \parallel z_\nu^1 - z_\nu^2 \parallel\big) \leq O(1). \tag{2.6}$$

Thus, by letting $j \to \infty$, both $d(z_\nu^j, \partial D)$ and $\| z_\nu^1 - z_\nu^2 \|$ become 0 and, combined with 2.6, this leads to a contradiction.
Since $f$ is a biholomorphism, the same argument works for proving that $f^{-1}$ extends with continuity from $\overline{D'}$ to $\overline{D}$.
Hence we can conclude that $f$ extends as an homeomorphism from $\overline{D}$ to $\overline{D'}$. □

*Remark* 21. The result proved by Fefferman in [18] has a stronger thesis since it gives a diffeomorphic estension to the boundaries, but it is stated for smooth strongly pseudoconvex domains.
The approach through his paper illustrates the boundary behaviour of geodesic respect the Bergman metric and it is not related into finding bounds for the Kobayashi distance.

*Remark* 22. Since a biholomorphism is an isometry respect the Kobayashi distance, in the literature, the last result is sometimes announced as the *Homeomorphic extension of an Isometry between Strongly Pseudoconvex Domains.*

# Chapter 3

# Main tools and examples

## 3.1 Gromov Hyperbolic metric spaces

Let $(X, d)$ be a metric space and let $I \subset \mathbb{R}$ be an interval, endowed with the Euclidean metric. An isometry $\gamma : I \to X$ is called a geodesic. If $I = [a, b]$, we call $\gamma$ a geodesic segment. If $I = \mathbb{R}_{\geq 0}$, we call $\gamma$ a geodesic ray. Finally, if $I = R$, we call $\gamma$ a geodesic line.
We also remember that $(X, d)$ is *geodesic* if every two points $x_1, x_2 \in X$ are joined by a geodesic segment.
If $(X, d)$ is a geodesic metric space, a *geodesic triangle* is the union of geodesic segments $\gamma_i : [a_i, b_i] \to X, i = 1, 2, 3$, such that $a_i < b_i$ for every $i = 1, 2, 3$ and $\gamma_1(b_1) = \gamma_2(a_2)$, $\gamma_2(b_2) = \gamma_3(a_3)$,$\gamma_3(b_3) = \gamma_1(a_1)$. The geodesic segments $\gamma_1$, $\gamma_2$ and $\gamma_3$ are called the sides of the triangle.

**Definition 3.1.** A geodesic metric space $X$ is *Gromov hyperbolic* or $\delta$-*hyperbolic*, if it exists $\delta \geq 0$ so that all geodesic triangles in $X$ are $\delta$-*thin*: i.e. every side is contained in the $\delta$-neighbourhood $N_\delta$ of the other two sides.

*Remark* 23. More precisely the $\delta$-thin condition for a geodesic triangle with vertices $a, b, c$ requires that the geodesic segments $[a, c] \subset N_\delta([a, b] \cup [b, c])$,$[b, c] \subset N_\delta([a, b] \cup [a, c])$ and $[a, b] \subset N_\delta([a, c] \cup [b, c])$.

**Example 3.1.**

- Any geodesic metric space $X$ of bounded diameter (respect the given distance of $X$) are diam$(X)$-*hyperbolic*.
- The real line $\mathbb{R}$ is 0-*hyperbolic* because every geodesic triangle in $\mathbb{R}$ is degenerate.

The following example needs some steps to be proved, precisely we followed the outline of Exercise 11.6 at page 196 from [22].

**Proposition 3.1.1.** *The hyperbolic plane* $\mathbb{H}^2 = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$ *is a* $\delta$*-hyperbolic metric space where the smallest* $\delta$ *that holds is* $\operatorname{arcsinh}(1)$.

*Proof.* First of all recall that all the geodesic lines in $\mathbb{H}^2$ are straight lines or circumferences both perpendicular to $\partial\mathbb{H}^2$.
Moreover every geodesic triangle is contained in a triangle with two vertices in $\partial\mathbb{H}^2$ and the third to be $\{\infty\}$.
Since the Poincaré distance is invariant under Möbius tranformations we can pick a geodesic triangle with vertices $A = 0, B = \infty$ and $C = 1$.
Let $p = iy \in \overline{AB}$, we need to compute the point $q \in \overline{BC}$ which realizes $d_{\mathbb{H}^2}(p, \overline{CB})$. Set $q = 1 + ia \in \overline{CB}$ with $a \geq 0$ and therefore

$$d_{\mathbb{H}^2}(iy, 1 + ia) = \operatorname{arccosh}(1 + \frac{1 + (y - a)^2}{2ay}) =: F(a) ,$$

thus by the sign of $F'(a)$ we deduce that the minimum satisfies $F'(a) = \frac{a^2 - y^2 - 1}{a\sqrt{(a^4 - 2a^2(y^2-1) + (y^2+1)^2))}} = 0$, hence $a = \sqrt{1 + y^2}$.
As a consequence $d_{\mathbb{H}^2}(p, \overline{CB}) = d_{\mathbb{H}^2}(iy, 1 + i\sqrt{1 + y^2}) = \operatorname{arcsinh}(1/y)$.
At this point we can define the Möbius tranformation

$$\varphi(z) = \frac{z - 1}{z}$$

that grants

$$\varphi(A) = \varphi(0) = \infty = B,\ \varphi(B) = \varphi(\infty) = 1 = C \text{ and } \varphi(C) = \varphi(1) = 0 = A.$$

Now we have:

$$d_{\mathbb{H}^2}(p, \overline{CA}) = d_{\mathbb{H}^2}(\varphi(p), \varphi(\overline{CA})) = d_{\mathbb{H}^2}(\varphi(p), \overline{AB}) = d_{\mathbb{H}^2}(1 + \frac{i}{y}, i\sqrt{1 + \frac{1}{y^2}}) = \operatorname{arcsinh}(y).$$

In conclusion we can compute the sharpest $\delta$ since:

$$d_{\mathbb{H}^2}(p, \overline{AC} \cup \overline{CB}) = \min_{z \in \overline{AC} \cup \overline{CB}} d_{\mathbb{H}^2}(p, z) = \min_{y \geq 0} (\max\{\operatorname{arcsinh}(y), \operatorname{arcsinh}(1/y)\}) = \operatorname{arcsinh}(1).$$

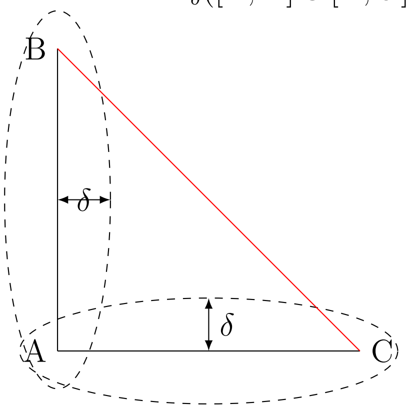
□

*Remark* 24. In [22], due to the characterisation of the automorphisms of $\mathbb{H}^n$, it is possible to understand that $\delta = \operatorname{arcsinh}(1)$ is the sharpest estimate in any $\mathbb{H}^n$.

And now we are going to present two **non**-examples

**Proposition 3.1.2.** *The Euclidean space* $\mathbb{R}^2$ *is not Gromov hyperbolic.*

*Proof.* For $\delta \in \mathbb{R}_{\geq 0}$, the Euclidean triangle with vertices $A = (0, 0), B = (0, 6\delta)$, and $C = (6\delta, 0)$ is not $\delta$-slim, the reason is that the side $[B, C]$ is not included in $N_\delta([A, B] \cup [A, C])$.




□

**Proposition 3.1.3.** *The bi-disc* $\Delta^2$ *endowed with the Kobayashi distance* $k_{\Delta^2}$ *is not Gromov hyperbolic.*

*Proof.* Thanks to Theorem 1.4.3 and Corollary 1.2.9 we have that $\Delta^2$ is a geodesic metric space endowed with the maximum distance. Consider the geodesic triangle with vertices:

$$O = (0, 0), \; p_m = \Big(1 - \frac{1}{m}, -1 + \frac{1}{m}\Big) \text{ and } q_m = \Big(-1 + \frac{1}{m}, -1 + \frac{1}{m}\Big).$$

Denote by $s_m^1$, respectively with $s_m^2$, the geodesics joining $O$ to $p_m$, respectively to $q_m$ .

Denote by $l_m$ the Kobayashi length of the unique geodesic joining $p_m$ to $q_m$. Then the unique point $z_m$ such that $k_{\Delta^2}(p_m, z_m) = \frac{l_m}{2}$ is at a Kobayashi distance $d_m$ from the two geodesics $s_m^1$ and $s_m^2$, with $\lim_{m\to\infty} d_m = +\infty$. This means that for every $\delta \geq 0$ it exists a $m_\delta$ so that a neighbourhood of $z_m$ is not contained in $N_\delta\big([O, p_m], [O, q_m]\big)$ for every $m \geq m_\delta$. Thus $\Delta^2$ is not Gromov hyperbolic. □

In the literature (see [2], section 2.3), the non-degeneracy of the Kobayashi distance is also referred as Kobayashi Hyperbolicity of the domain itself. Proposistion 3.1.3 proves then that $\Delta^2$ is an example of Kobayashi hyperbolicity which it is not Gromov hyperbolic.
Moreover one can find a generalization of last proof, holding in a product two complete non-compact geodesic metric spaces endowed with the maximum distance on [21], and more non-examples on [20].

## 3.2 Quasi-geodesics and quasi-isometries

**Definition 3.2.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and let $A \geq 1$, $B \geq 0$. If $I \subset \mathbb{R}$ is an interval, then a map $\gamma : I \to X$ is an $(A, B)$-*quasi-geodesic* if for all $s, t \in I$:

$$A^{-1}|t - s| - B \leq d_X(\gamma(s), \gamma(t)) \leq A|t - s| + B.$$

If $I = [a, b]$ (resp. $I = \mathbb{R} \geq 0$ or $I = \mathbb{R}$) we call $\gamma$ a quasi-geodesic segment (resp. quasi-geodesic ray or quasi-geodesic line).

**Definition 3.3.** Given $A \geq 1$, $B \geq 0$, a map $f : X \to Y$ is an $(A, B)$-*quasi-isometry* if for all $x_1, x_2 \in X$:

$$A^{-1}d_X(x_1, x_2) - B \leq d_Y(f(x_1), f(x_2)) \leq Ad_X(x_1, x_2) + B.$$

In order to get through the definition of a quasi-isometry we are going to show some examples and non-examples.

**Example 3.2.** The immersion $i : \mathbb{Z} \hookrightarrow \mathbb{R}$ is a $(1, 0)$-quasi-isometry.
On the other side, the projection $\pi : \mathbb{R} \to \mathbb{Z}$, defined by $x \to \lfloor x \rfloor$, is a $(1, 1)$-quasi-isometry.

**Example 3.3.** The function

$$\begin{aligned} \varphi : \mathbb{R} &\to \mathbb{R}^2 \\ x &\mapsto x(1, 0) \end{aligned}$$

is a $(1, 0)$-quasi-isometry.
More generally, consider $v \in \mathbb{R}^2 \setminus \{0\}$ and define

$$K = \max\{\frac{1}{\| v \|_2}, \| v \|_2\} ,$$

then

$$\begin{aligned} \varphi : \mathbb{R} &\to \mathbb{R}^2 \\ t &\mapsto tv \end{aligned}$$

is a $(K, 0)$-quasi-isometry.

**Example 3.4.** The map

$$\begin{aligned} \varphi_1 : \mathbb{R} &\to \mathbb{R} \\ t &\mapsto t^2 \end{aligned}$$

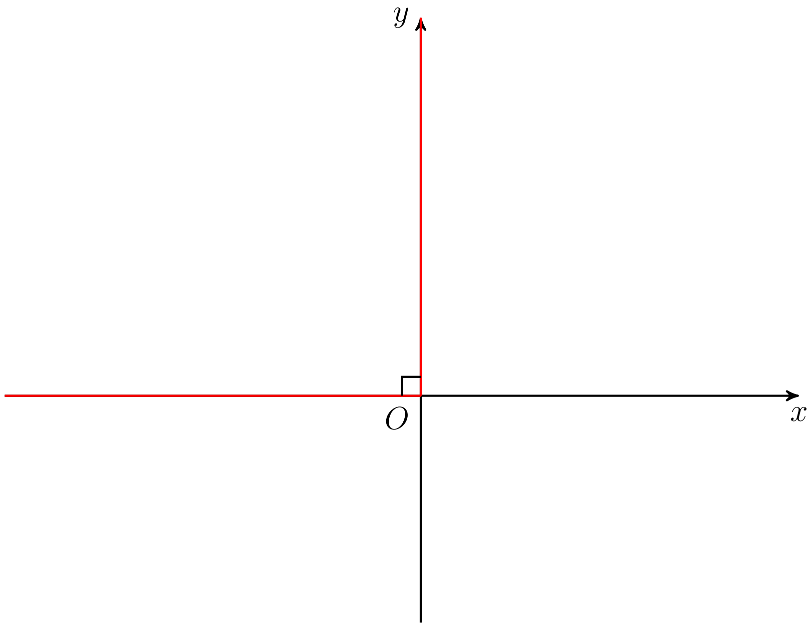
is **not** a quasi-isometry.
Indeed, the upper bound is not true since:
$d_{\mathbb{R}}(\varphi_1(0), \varphi_1(n)) = n^2 \not\leq An + B, \ \forall A \geq 1, \forall B \geq 0.$
Moreover, the map $\varphi_2 : \mathbb{R} \to \mathbb{R}$ defined by $\varphi_2(t) = \begin{cases} \sqrt{|t|}, \ t \geq 0 \\ -\sqrt{|t|}, \ t \leq 0 \end{cases}$ is **not** a quasi-isometry.
That is because:
$d_{\mathbb{R}}(\varphi_2(0), \varphi_2(n)) = \sqrt{n} \not\geq \frac{n}{A} - B, \ \forall A \geq 1, \forall B \geq 0.$

Anyway the image of a quasi-isometry can be very far from a line

**Example 3.5.** The function $f : \mathbb{R} \to \mathbb{R}^2$ defined by cases as

$$f(t) = \begin{cases} (t, 0), & t \leq 0 \\ (0, t), & t \geq 0 \end{cases}$$

is a $(\sqrt{2}, 0)$-quasi-isometry.



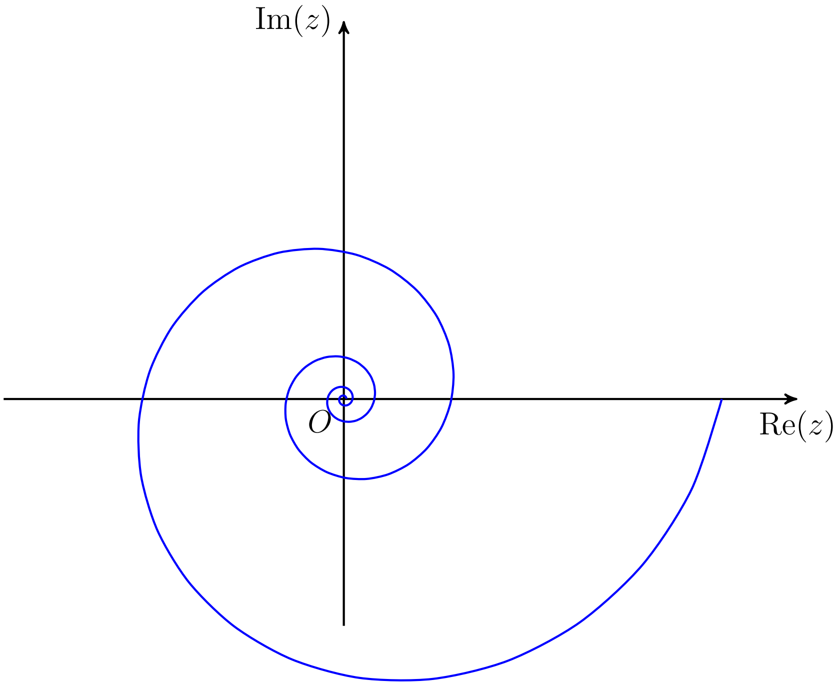

**Example 3.6.** The map

$$\begin{aligned} f : [0, \infty) &\to \mathbb{C} \\ t &\mapsto t e^{\pi i \log t} \end{aligned}$$

is a quasi-isometry.
Indeed:

$$|t - s| = |d_{\mathbb{C}}(f(0), f(t)) - d_{\mathbb{C}}(f(0), f(s))| \leq d_{\mathbb{C}}(f(t), f(s)) \leq |t - s| \max_{[t,s]} |f'| \,,$$

where $|f'(u)| = |e^{\pi i \log u} + \pi i e^{\pi i \log u}| \leq 1 + \pi$. Thus $f$ is a $(1 + \pi, 0)$-quasi-isometry.




Now we will illustrate some remarkable facts on quasi-geodesics and quasi-isometries

**Proposition 3.2.1.**

1. *Notice that an $(A, B)$-quasi-geodesic in $(X, d)$ is an $(A, B)$-quasi-isometry from $(I, |\cdot|)$, where $I$ is an interval of $\mathbb{R}$, to $(X, d)$.*

2. *When $f$ is a bijective $(A, B)$-quasi-isometry from $(X, d_X)$ to $(Y, d_Y)$, then $f^{-1}$is a $(A, AB)$-quasi-isometry.*

3. *If $f : (X, d_X) \to (Y, d_Y)$ and $g : (Y, d_Y) \to (Z, d_Z)$ are quasi-isometries then $g \circ f : (X, d_X) \to (Z, d_Z)$ is a quasi-isometry.*

**Proposition 3.2.2.** *Let $f : (X, d_X) \to (Y, d_Y)$ be a surjective mapping between two metric spaces where $X$ has bounded diameter. Then $f$ is a quasi-isometry if and only if $Y$ has bounded diameter.*

*Proof.* If $f$ is a surjective $(A, B)$-quasi-isometry:

$$\sup_{y_1,y_2\in Y} d_Y(y_1, y_2) = \sup_{x_1,x_2\in Y} d_Y(f(x_1), f(x_2)) \leq A \sup_{x_1,x_2\in X} d_X(x_1, x_2)+B < +\infty.$$

For the other implication we can use as additive constant $B = \mathrm{diam}(Y)$. □

Now we are going to prove that in a Gromov hyperbolic metric space a quasi-geodesic is always followed, like a shadow, by a geodesic with same starting and final points.

**Lemma 3.2.3** (Shadowing Lemma)**.** *Let $X$ be $\delta$-hyperbolic. For all $A \geq 1, B \geq 0$ there exists $K > 0$ so that any $(A,B)$-quasi geodesic segment $\rho$, with the same endpoints as a geodesic segment $\gamma \subset X$, satisfies $\rho \subset N_K(\gamma)$ and $\gamma \subset N_K(\rho)$.*

*Proof.* The proof is divided in two parts, for first a logarithmic bound and then it will follow an uniform bound for the quasi-geodesic $\rho$.
*First step:* We want to show that if $p \in X$ lays on the geodesic segment $[x, y]$ and $\alpha$ is any rectifiable path from $x$ to $y$, then we get

$$d(p, \alpha) \leq \delta \log_2(\ell(\alpha)) + 2.$$

Indeed, if $\ell(\alpha) \leq 2$ then $d(x, y) \leq 2$ and thus $d(p, \alpha) = \min\{d(p, q) | q \in \alpha\} \leq d(p, x) \leq 2$.
Otherwise, if $2 \leq \ell(\alpha) < +\infty$ we can pick a finite sequence of points $(q_i)_{i=1}^N \subset \alpha$ so that $N > 2$,$q_1 = x$,$q_N = y$ and $\ell([q_i, q_{i+1}]) = \ell([q_{i+1}, q_{i+2}]) = \frac{\ell(\alpha)}{N} \leq 2$ $\forall i \in \{1, \cdots, N-2\}$. Since the geodesic triangles $[x, y, q_i]$ are $\delta$-thin $\forall i \in \{1, \cdots, N\}$, we have $p \in N_\delta([x, q_i])$ $\forall i$ and this means

$$\forall i \exists p'_i \in [x, q_i] \text{ so that } d(p, p'_i) \leq \delta.$$

In this way it holds:

$$d(p, \alpha) \leq d(p, p'_i) + d(p'_i, [q_i, q_{i+1}]) \leq d(p, p'_i) + \delta \log_2 \big(\ell([q_i, q_{i+1}])\big) + 2 \leq$$

$$\delta(1 - \log_2 N) + \delta \log_2(\ell(\alpha)) + 2 \leq \delta \log_2(\ell(\alpha)) + 2.$$

As an application, if $\alpha = \rho$ is a $(A, B)$-quasi-geodesic segment, then $d(p, \rho) \leq \delta \log_2 \big(Ad(x, y) + B\big) + 2$.

*Second step:* We want to show $d(p, \rho) \leq m$ for some constant $m$ depending only on $A, B, \delta$.
Let $p \in [x, y]$ with $d(p, \rho) = \max\{d(q, \rho) | q \in [x, y]\} = m$. Now choose $x', y' \in [x, y]$ with $d(p, x') = d(p, y') = 2m$ (or $x' = x$ if $d(p, x) < 2m$, $y' = y$ if $d(p, y) < 2m$). Then pick $x'' \in \rho, y'' \in \rho$ with $d(x', x'') \leq m$ and $d(y', y'') \leq m$. We can consider the path $\beta$ that joints the segments $[x', x''], [x'', y''], [y'', y']$ and observe:

- $d(p, \beta) \geq d(p, \rho) = m$;
- $d(x'', y'') \leq d(x'', x') + d(x', y') + d(y', y'') \leq 6m$, then since $\beta \cap \rho$ is an $(A, B)$-quasi-geodesic it holds $\ell(\beta) \leq A6m + B + d(x', x'') + d(y', y'') \leq A6m + B + 2m$.

Hence the logarithmic bound allow us to infer:

$$m \leq d(p, \beta) \leq \delta \log_2(\ell(\beta)) + 2 \leq \delta \log_2 \big((6A + 2)m + B\big) + 2.$$

Thus $m$ has an upper bound with a constant depending only on $A, B, \delta$. Therefore $[x, y] \subset N_m(\rho)$.
We still need to show that $\rho \subset N_K([x, y])$ for some $K > 0$. (recall: we want $\gamma \subset N_K(\rho)$ and $\rho \subset N_K(\gamma)$.)
Let $q \in \rho$. If $d(q, [x, y]) \leq m$ we can conclude already.
For otherwise, consider $\rho_1, \rho_2$ to be subpaths of $\rho$ meeting at $q$, where $x \in \rho_1$ and $y \in \rho_2$. Since $[x, y] \subset N_m(\rho)$ this grants that at some point $p \in [x, y]$ we have:

$$d(p, \rho_1) \leq m \text{ and } d(p, \rho_2) \leq m\,.$$

Then consider the subpath $\alpha \subset \rho$, so that it contains $q$. Now name its endpoints $q_1$, the one closer to $x$, and $q_2$, the one closer to $y$. Since $q_1 \in \rho_1$ and $q_2 \in \rho_2$ then we have:

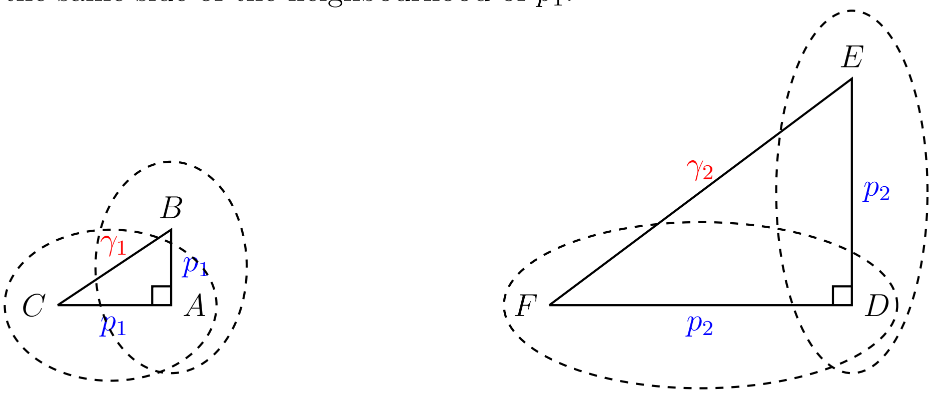
$$d(p, q_1) \leq m \text{ and } d(p, q_2) \leq m\,.$$

Since $d(q_1, q_2) \leq d(q_1, p) + d(p, q_2) \leq 2m$ and $\alpha$ is an $(A, B)$-quasi-geodesic we get:

$$d(q, [x, y]) \leq d(q, p) \leq d(q, q_1) + d(q_1, p) \leq \ell(\alpha) + m \leq 2mA + B + m = K\,.$$

□

*Remark* 25. The *Shadowing Lemma* provides another reason to the fact that $\mathbb{R}^2$ is not Gromov hyperbolic. Indeed, Example 3.5, grants that $p_1$ and $p_2$ are quasi-geodesics and $\gamma_2$ is not contained on the neighbourhood of $p_2$ with the same side of the neighbourhood of $p_1$.




**Proposition 3.2.4.** *If $f : (X, d_X) \to (Y, d_Y)$ is a quasi-isometry and a bijection between proper geodesic metric spaces, then $(X, d_X)$ is Gromov hyperbolic if and only if $(Y, d_Y)$ is Gromov hyperbolic.*

*Proof.* Let $f : X \to Y$ be a $(A, B)$-quasi-isometry; suppose $Y$ is $\delta$-hyperbolic. Let $\triangle = [a, b, c] \subset X$ be a geodesic triangle and take $p \in [a, c]$. Then Lemma 3.2.3 grants that $f(p) \in N_K([f(a), f(c)])$. Then the $\delta$-hyperbolicity of $Y$ grants $f(p) \in N_{K+\delta}([f(a), f(b)])$ (or also $f(p) \in N_{K+\delta}([f(b), f(c)])$), and again Lemma 3.2.3 gives $f(p) \in N_{2K+\delta}(f[a, b])$.
Now, let $q \in [a, b]$ with $d_Y(f(p), f(q)) \leq 2K + \delta$, then the $(A, B)$-quasi-isometric property of $f$ grants :

$$d_X(p, q) \leq A(2K + \delta) + AB.$$

In conclusion, by defining $\delta' = A(2K + \delta) + AB$, $\triangle$ is $\delta'$-thin. □

**Example 3.7.** The Poincaré disc is biholomorphic to the hyperbolic plane $\mathbb{H}^2$ by the Cayley tranform, thus Corollary 1.2.2 grants the isometry with respect to the Kobayashi distance and hence the Poincaré disc is Gromov hyperbolic.

## 3.3 The Gromov Compactification and its properties

We assume for the rest of this section that $(X, d)$ is a proper geodesic Gromov hyperbolic metric space.
Consider $x_0 \in X$ and then let $\mathcal{G}_{x_0}$ denote the space of geodesic rays $\gamma : [0, +\infty) \to X$ such that $\gamma(0) = x_0$, endowed with the topology of uniform convergence on compact subsets of $[0, +\infty)$. We construct on $\mathcal{G}_{x_0}$ the equivalence relation $\sim$ defined by

$$\gamma \sim \lambda \Leftrightarrow \sup_{t \geq 0} d(\gamma(t), \lambda(t)) < +\infty.$$

Indeed, reflexivity and symmetry are an immediate consequences of degeneracy and symmetry of the distance $d$ and the triangle inequality grants the transitivity. At this point we can endow $\mathcal{G}_{x_0}/\sim$ with the quotient topology. Now we are going to prove that a certain homeomorphism arises naturally from the construction of this quotient.

**Proposition 3.3.1.** *Given $x_0, x_1$ points in $X$ then $\mathcal{G}_{x_0}/\sim$ is homeomorphic to $\mathcal{G}_{x_1}/\sim$.*

*Proof.* We can define a map $J : \mathcal{G}_{x_0}/\sim \to \mathcal{G}_{x_1}/\sim$ as follows. Let $[\gamma] \in \mathcal{G}_{x_0}/\sim$, where $\gamma : [0, +\infty) \to X$ is a geodesic ray such that $\gamma(0) = x_0$. For $n \in \mathbb{N}$, let $\eta_n : [0, R_n] \to X$ be a geodesic segment such that $\eta_n(0) = x_1$ and $\eta_n(R_n) = \gamma(n)$. Via Arzelà-Ascoli's Theorem we can argue that the sequence $(\eta_n)_{n \in \mathbb{N}}$ converges, up to subsequences, locally uniformly to a geodesic ray $\eta : [0, +\infty) \to X$ such that $\eta(0) = x_1$. We then let $J([\gamma]) = [\eta]$.
We can notice that for a fixed $t \geq 0$ the geodesic triangle $\triangle = [\eta(t), \gamma(t), x_0]$ is $\delta$-thin and then each point of $\eta$ is contained in a neighborhood of $\gamma$. The consequence is that $J([\gamma])$ does not depend on the choice of the representative $\gamma$: indeed, if we pick $\gamma_1, \gamma_2$ as $\gamma$ so that $\gamma_1 \sim \gamma_2$ then their respective images $J([\gamma_1]) = [\eta_1], J([\gamma_2]) = [\eta_2]$ and are contained respectively in neighborhoods

of $\gamma_1, \gamma_2$ and hence $J([\gamma_1]) = [\eta_1] = [\eta_2] = J([\gamma_2])$. Similarly we can argue that $J$ is injective, surjective and continuous.
In the same manner we can define $J^{-1} : \mathcal{G}_{x_1}/\sim \to \mathcal{G}_{x_0}/\sim$ and thus $J$ is an homeomorphism. □

*Remark* 26. This last result can be interpreted as the fact that the choice of the base point $x_0$ does not matter particularly in $\mathcal{G}_{x_0}/\sim$.

Everything is set up to give the two following

**Definition 3.4.** The *Gromov boundary* $\partial_G X$ of $X$ is defined to be the quotient space $\mathcal{G}_{x_0}/\sim$.

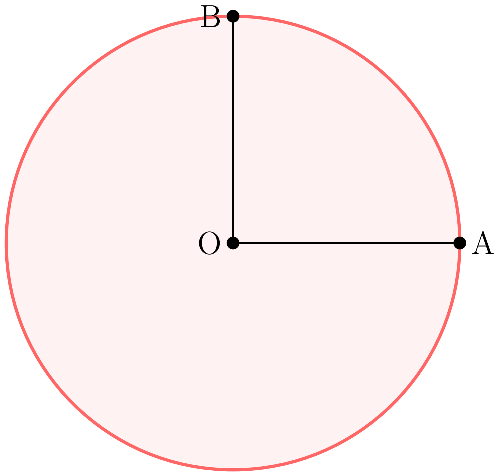
**Definition 3.5.** The *Gromov closure* of $X$ is $\overline{X}^G := X \cup \partial_G X$.

From [25], Chapter III.H.3, pages 427-432 one can see such interesting properties, for the Gromov closure of $X$, as:

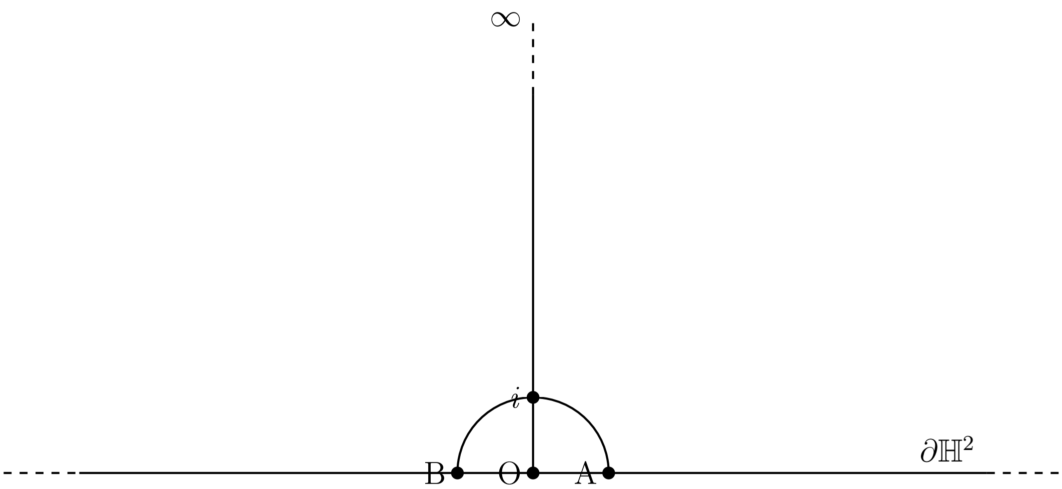
- $\overline{X}^G$ is a compactification of $X$;
- $\overline{X}^G$ is first countable and Hausdorff.

We can then show an example of how the Gromov Boundary works for the Poincaré Disc $\Delta$ and for the Hyperbolic plane $\mathbb{H}^2$, since for both of them we know they are proper geodesic Gromov hyperbolic metric spaces.

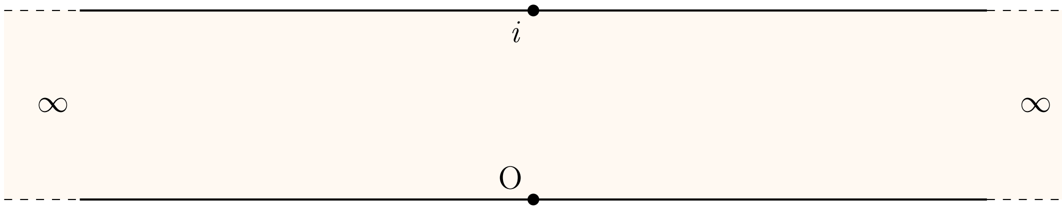
**Example 3.8.** Due to Proposition 3.3.1 we can consider geodesic rays with base point 0. In this way we have straight lines that are not related respect $\sim$ if and only if they intersect different points in the euclidean boundary of $\Delta$, since points in $\partial\Delta$ are infinitely far respect the Poincaré distance. Thus $\partial_G \Delta = \partial\Delta$.




**Example 3.9.** With the same approach as before we can pick $i \in \mathbb{H}^2$ and then the geodesic rays will be vertical lines or circular paths with the endpoint orthogonal to $\partial\mathbb{H}^2$. In this way we can deduce that $\partial_G(\mathbb{H}^2) = \partial(\mathbb{H}^2) \cup \{\infty\} = \{z \in \mathbb{C} | \mathrm{Im}(z) = 0\} \cup \{\infty\}$.




**Example 3.10.** Consider the complex stripe $S = \{z \in \mathbb{C} | 0 < \mathrm{Im} z < 1\}$. We can endow $S$ with a Gromov hyperbolic structure via the biholomorphism $f : \Delta \to S$ so that $f(z) = \frac{e^{\pi z} - i}{e^{\pi z} + i}$ which it extends continuously to the respective boundaries. We can deduce, sending by $f$ the geodesic rays of $\Delta$, that $\partial_G(S) = \partial S \cup \{\infty\} = \{z \in \mathbb{C} | \mathrm{Im} z = 0\} \cup \{z \in \mathbb{C} | \mathrm{Im} z = 1\} \cup \{\infty\}$.




At this point, in order to get a better understanding for the topology of $\overline{X}^G$, we introduce some additional notation.

**Definition 3.6.** For a geodesic ray $\sigma \in \mathcal{G}_{x_0}$ we define $End(\sigma)$ to be the equivalence class of $\sigma$.
Moreover, for a geodesic segment $\sigma : [0, R] \to X$ such that $\sigma(0) = x_0$, we define $End(\sigma) = \sigma(R)$.
Then $\xi_n \to \xi$ in $\overline{X}^G$ if and only if for every choice of geodesics $\sigma_n$ with $\sigma_n(0) = x_0$ and $End(\sigma_n) = \xi_n$ every subsequence of $\{\sigma_n\}_{n\in\mathbb{N}}$ has a subsequence which converges locally uniformly to a geodesic $\sigma$ with $End(\sigma) = \xi$.

*Remark* 27. The last definition describes the topology of $\overline{X}^G$: the closed subsets $B \subset \overline{X}^G$ are those which satisfy the condition

$$\xi_n \in B,\ \forall n \in \mathbb{N} \textit{ and } \xi_n \to \xi \textit{ then } \xi \in B.$$

**Lemma 3.3.2.** *Suppose that $(X, d)$ is a proper geodesic Gromov hyperbolic metric space. If $\sigma : \mathbb{R} \mapsto X$ is a geodesic, then the limits*

$$\lim_{t\to-\infty} \sigma(t) \textit{ and } \lim_{t\to+\infty} \sigma(t)$$

*both exist in $\overline{X}_G$ and are distinct.*

*Proof.* Recalling that $\mathcal{G}_{x_0}/\sim$ is the set of the equivalence class of the geodesics rays starting from a point $x_0 = \sigma(0) \in X$ with the relation

$$\gamma \sim \sigma \Leftrightarrow \sup_{t\geq 0} d(\gamma(t), \sigma(t)) < +\infty.$$

Since $\overline{X}^G$ is a compactification of $X$, then $\lim_{t\to+\infty} \sigma(t) = \xi \in \overline{X}^G$.
Moreover if $\tilde{\sigma} : [0, +\infty) \to X$ is defined as $t \mapsto \sigma(-t)$ then it follows that

$\lim_{t\to-\infty}\sigma(t)=\eta\in\overline{X}^G$.
Finally:

$$d(\xi,\eta)=lim_{t\to+\infty}d(\sigma(t),\tilde{\sigma}(t))=lim_{t\to+\infty}2t=+\infty.$$

Hence $\xi$ and $\eta$ are two different points in $\overline{X}^G$. □

The following result in [24], chapter denoted as *Lecture 1*, ninth consequence from the Lemma at page 54, illustrates a property that the geodesics joining two points on the boundary "bend" into the space itself.

**Theorem 3.3.3.** *Let $(X,d)$ be a proper geodesic Gromov hyperbolic metric space. If $\xi,\eta\in\partial_G X$ and $V_\xi,V_\eta$ are neighborhoods of $\xi,\eta$ in $\overline{X}^G$ so that $V_\xi\bigcap V_\eta=\varnothing$, then there exist a compact set $K\subset X$ with the following property: if $\sigma:[a,b]\to X$ is a geodesic with $\sigma(a)\in V_\xi$ and $\sigma(b)\in V_\eta$, then $\sigma\cap K\neq\varnothing$.*

We underline that such compact subset $K\subset X$ works for all the geodesic segments connecting neighbourhoods of two different points in the Gromov boundary.
It comes now an interesting consequence

**Corollary 3.3.4.** *Let $(X,d)$ be a proper geodesic Gromov hyperbolic metric space and let $x_0\in X$. If $\xi,\eta\in\partial_G X$ and $V_\xi,V_\eta$ are neighbourhoods of $\xi,\eta$ in $\overline{X}^G$ so that $\overline{V_\xi}\cap\overline{V_\eta}=\varnothing$, then there exists some $A\geq 0$ such that*

$$d(x,y)\leq d(x,x_0)+d(x_0,y)\leq d(x,y)+A$$

*for all $x\in V_\xi$ and $y\in V_\eta$.*

*Proof.* Let $K$ be the compact set from the previous Theorem. Now define

$$A=2\max_{k\in K}d(x_0,k).$$

Then suppose $x\in V_\xi$, $y\in V_\eta$ are joined by a geodesic segment $\sigma:[a,b]\to X$ so that $\sigma(a)=x$ and $\sigma(b)=y$. Thus there exists some $t\in[a,b]$ such that $\sigma(t)\in K$. By minimizing distance property of a geodesic we have:

$$d(x,y)=d(x,\sigma(t))+d(\sigma(t),y)$$

and applying twice the triangle inequality we gain:

$$d(x,\sigma(t)) + d(\sigma(t),y) \geq [d(x,x_0) - d(\sigma(t),x_0)] + [d(x_0,y) - d(x_0,\sigma(t))] \geq$$

$$d(x,x_0) + d(x_0,y) - 2\max_{k\in K} d(x_0,k) = d(x,x_0) + d(x_0,y) - A$$

□

The last Corollary leads to the following

**Definition 3.7.** Let $(X,d)$ be a metric space and let $x,y,z \in X$. Then the *Gromov product* of $y$ and $z$ at $x$, denoted $(y,z)_x$, is defined by

$$(y,z)_x = \frac{1}{2}(d(x,y) + d(x,z) - d(y,z)) .$$

*Remark* 28. In the settings of Corollary 3.3.4 we have that for every $x_0 \in X$ and $\xi, \eta \in \partial_G X$ with disjoint neighbourhoods $V_\xi, V_\eta$, it exists $K \subset X$ compact set so that it holds

$$(x,y)_{x_0} \leq d(x_0,K) \text{ for all } x \in V_\xi \text{ and } y \in V_\eta.$$

Now we are going to restate the Shadowing Lemma 3.2.3 with two quasi-geodesics that have the same endpoints

**Theorem 3.3.5.** *Suppose that $(X,d)$ is a proper geodesic Gromov hyperbolic metric space. For any $A \geq 1$ and $B \geq 0$ there exists $R > 0$ such that: if $\gamma_1 : [a_1,b_1] \to X$ and $\gamma_2 : [a_2,b_2] \to X$ are two $(A,B)$-quasi-geodesic segments with $\gamma_1(a_1) = \gamma_2(a_2)$ and $\gamma_1(b_1) = \gamma_2(b_2)$, then the Hausdorff distance between the two quasi-geodesic segments satisfies:*

$$\mathcal{H}(\gamma_1,\gamma_2) := \max\{\max_{t\in[a_1,b_1]} d(\gamma_1(t),\gamma_2([a_2,b_2])), \max_{t\in[a_2,b_2]} d(\gamma_1([a_1,b_1]),\gamma_2(t))\} \leq R.$$

**Lemma 3.3.6.** *If $\sigma : \mathbb{R} \mapsto X$ is an $(A,B)$-quasi-geodesic, then the limits*

$$\lim_{t\to-\infty} \sigma(t) \text{ and } \lim_{t\to+\infty} \sigma(t)$$

*both exist in $\overline{X}_G$ and are distinct.*

*Proof.* With the notion of quasi-geodesic rays we mean the whole $(A, B)$-quasi-geodesic rays with any $A \geq 1$ and $B \geq 0$. Let's define a new equivalence relation for quasi-geodesic rays. Given two quasi-geodesic rays $\gamma, \sigma$ starting from a point $x_0 = \sigma(0) \in X$ we introduce the equivalence relation

$$\gamma \sim \sigma \Leftrightarrow \sup_{t \geq 0} d(\gamma(t), \sigma(t)) < +\infty.$$

Now we denote $\partial_{qG}(X)$ to be the set of equivalence classes of quasi-geodesic rays.
The claim is that there is a natural bijection between $\partial_G X$ and $\partial_{qG} X$.
Since geodesic rays are obviously quasi-geodesic rays we have the immersion $\partial_G X \hookrightarrow \partial_{qG} X$. At this point we define an inverse in the following way: consider $p \in X$ and a quasi-geodesic ray $\sigma : [0, \infty) \to X$, let $\sigma_n$ be a geodesic segment with $\sigma_n(0) = p$ that joins $p$ to $\sigma(n)$. Since $X$ is proper, due to the Arzelà-Ascoli Theorem, a subsequence of $\sigma_n$ converges to a geodesic ray $\sigma_\infty : [0, \infty) \to X$. Thus the Shadowing Lemma 3.2.3 provides a constant $R$ so that the Hausdorff distance between $\sigma([0, n])$ and the image of $\sigma_n$ is less than $R$; therefore we obtain a bound on the Hausdorff distance between $\sigma$ and $\sigma_\infty$. As a consequence the inverse is $\partial_{qG}(X) \ni \sigma \mapsto \sigma_\infty \in \partial_G(X)$.
Hence Lemma 3.3.2 completes our proof. □

## 3.4 The Denjoy-Wolff Theorem and Karlsson's Theorem

We are going to deal with iteration theory of holomorphic self-maps in the unit disk $\Delta$. The main result is the Denjoy-Wolff Theorem and then we will present a generalization for proper geodesic Gromov hyperbolic metric spaces.
To introduce the Denjoy-Wolff Theorem we can start thinking that thanks to the Schwarz-Pick Lemma the unit disk endowed with the Poincaré distance becomes a complete metric space, and the holomorphic functions from the

disk to itself decrease the distance between the points in the Poincaré metric, the decrease of distance is not enough to apply the Banach fixed point Theorem for contractions but we can still say something.
The following result comes from [29], in Chapter IV.3, at pages 79/80.

**Theorem 3.4.1.** *[Denjoy-Wolff Theorem] Let $f : \Delta \to \Delta$ be a holomorphic map. Then either:*

1. *$f$ has a fixed point in $\Delta$; or*
2. *there exists a point $\xi \in \partial\Delta$ so that $lim_{n\to\infty} f_n(x) = \xi$ for any $x \in \Delta$, this convergence is meant uniform on compact subsets of $\Delta$.*

**Example 3.11.** Some of the holomorphic selfmaps which behave as described in the first point are:

- Contractions $f(z) = az$, $|a| < 1$ and $z \in \Delta$, the fixed point is 0;
- Rotations $f(z) = e^{i\vartheta}z$, $\forall z \in \Delta$ and $\vartheta \in \mathbb{R}$, the fixed point is still 0.

For an example of an holomorphic selfmap that behaves as described in the second point consider $\phi \circ f \circ \phi^{-1}$, where $f : \mathbb{H}^2 \to \mathbb{H}^2$ is defined as $f(z) = z+i$ and $\phi : \mathbb{H}^2 \to \Delta$ is the Cayley Transform.
Indeed,

$$\lim_{n\to\infty} f^n(z) = \lim_{n\to\infty} z + ni = \infty \in \partial_G \mathbb{H}^2.$$

Then, for $n \to \infty$,

$$\phi \circ f^n \circ \phi^{-1}(z) = \phi\Big(\frac{z+1}{i(z-1)} + ni\Big) = \frac{\frac{z+1}{i(z-1)} + (n-1)i}{\frac{z+1}{i(z-1)} + (n+1)i} \to 1 \in \partial_G(\Delta).$$

**Theorem 3.4.2** (Karlsson's Theorem)**.** *Let $(X, d)$ be a proper geodesic Gromov Hyperbolic metric space and let $f : X \to X$ be a 1-Lipschitz selfmap. Then either:*

1. *for every $x \in X$, the orbit $\{f^n(x) : n \in \mathbb{N}\}$ is bounded in $(X, d)$, or*
2. *there exists a unique $\xi \in \partial_G X$ so that for all $x \in X$ $\lim_{n\to\infty} f^n(x) = \xi$, in the Gromov compactification.*

*Proof.* Suppose we have $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$, then Banach fixed-point Theorem grants that it exists $p \in X$, fixed point for $f$, so that $\lim_{n\to\infty} f^n(p) = p$. This means that the orbit $\{f^n(p) : n \in \mathbb{N}\}$ is bounded. Now pick $x \in X$, then $d(p, x) < \infty$. In this way, for a fixed $m \in \mathbb{N}$ there exists a constant $K > 0$ so that $d(f^m(p), f^n(x)) \leq d(p, x) + K < \infty$ for all $n \in \mathbb{N}$. Therefore, for each point $x \in X$ the orbit $\{f^n(x) : n \in \mathbb{N}\}$ is bounded in $(X, d)$.

Otherwise, if there is no point so that its orbit is bounded then consider $y \in X$, where $a_n = d(y, f^n(y)) \to \infty$ for $n \to \infty$. Now, since $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers which is unbounded from above, there are infinitely many $n$ such that $a_k < a_n$ for all $k \leq n$. Thus for $k \leq n_i$, it holds

$$(f^{n_i}(y), f^k(y))_y = \frac{1}{2}\big(d(y, f^{n_i}(y)) + d(y, f^k(y)) - d(f^{n_i}(y), f^k(y))\big) =$$

$$\frac{1}{2}\big(a_{n_i} + a_k - d(f^{n_i}(y), f^k(y)\big) \geq \frac{1}{2}\big(a_{n_i} + a_k - a_{n_i-k}\big) > \frac{1}{2}a_k.$$

Since by assumption $a_k \to \infty$, we get from the above inequality, with $k = n_j$,

$$(f^{n_i}(y), f^{n_j}(y))_y \to \infty \text{ for } i, j \to \infty.$$

At this point consider the geodesic triangle $\triangle = [y, f^{n_i}(y), f^{n_j}(y)]$ and, since $\triangle$ is $\delta$-thin, we have that:

$$d(y, [f^{n_i}(y), f^{n_j}(y)]) \leq \delta + (f^{n_i}(y), f^{n_j}(y))_y$$

and also

$$\frac{1}{2}\big(d(y, f^{n_i}(y)) + d(y, f^k(y)) - d(f^{n_i}(y), f^k(y))\big) \leq \frac{1}{2}(\delta+\delta-\delta) \leq \delta + d(y, [f^{n_i}(y), f^{n_j}(y)]).$$

Then this implies

$$|d(y, [f^{n_i}(y), f^{n_j}(y)]) - (f^{n_i}(y), f^{n_j}(y))_y| \leq \delta.$$

As a consequence of $(f^{n_i}(y), f^{n_j}(y))_y \to \infty$, we have that $d(y, [f^{n_i}(y), f^{n_j}(y)]) \to \infty$ for $i, j \to \infty$. Thus, by means of convergence in the Gromov compactification as in Definition 3.6, we can construct a sequence of geodesic segments

$\{\sigma_n\}_{n\in\mathbb{N}}$ with base point $\sigma_n(0) = y$ and $End(\sigma_n) = f^n(y)$. Since $X$ is proper, Arzelà-Ascoli Theorem grants that we can extract a subsequence which converges locally uniformly to a geodesic $\sigma$ with $End(\sigma) =: \xi \in \overline{X}^G$. Since the condition $d(y, [f^{n_i}(y), f^{n_j}(y)]) \to \infty$ for $i, j \to \infty$ takes neighbourhood in $\partial_G X$, then we have that $\xi \in \partial_G X$.
Now it remains to show that such $\xi$ is unique for all $x \in X$. Indeed, suppose there exist $\xi_1, \xi_2 \in \partial_G X$ so that for $x_1, x_2 \in X$, with $x_1 \neq x_2$ and $f^m(x_1) \to \xi_1, f^n(x_2) \to \xi_2$ for $m, n \to \infty$. Since $f$ is 1-Lipschitz there exists a constant $C > 0$ so that

$$\sup_{n\in\mathbb{N}} d(f^n(x_1), f^n(x_2)) \leq C + d(x_1, x_2) < \infty$$

and, as a consequence, $\xi_1 = \xi_2 \in \partial_G X$. □

**Example 3.12.** Consider the unit ball $B^d$ ($d$ positive integer) endowed with the Kobayashi distance, Theorem 1.4.3 grants that it is a proper geodesic metric space. Moreover, since $B^d$ is a strongly pseudoconvex domain, from Theorem 1.4. in [31], $(B^d, k_{B^d})$ becomes a Gromov hyperbolic metric and its Gromov boundary coincides with the euclidean one.
For a result of the first case of Karlsson's Theorem 3.4.2 consider the biholomorphism $f : B^d \to B^d$ defined as $f(z) = Uz$, where $U$ is a unitary matrix. Indeed, for every $z \in B^d$, it holds:

$$k_{B^d}(0, U^n z) \leq k_{B^d}(0, Uz) \leq k_{B^d}(0, z) = \omega(0, \| z \|) < \infty.$$

For a result of the second case we can find a generalized Cayley Transform at pages 31-32 from [4] and follow a similar path as in Example 3.11. Indeed, consider the "Siegel upper-half plane" of $\mathbb{C}^d$ defined as

$$\Omega = \{(w_1, w') \in \mathbb{C}^d \big| \operatorname{Im}(w_1) > |w'|^2\}$$

where $w' = (w_2, \cdots, w_d)$ and $|w'|^2 = |w_2|^2 + \cdots + |w_d|^2$. The generalized Cayley transform is then the map $\phi : B^d \to \Omega$ defined as $\phi(z) = i\frac{e_1+z}{1-z_1}$, where $e_1 = (1, 0')$. At this point consider the holomorphic selfmap $F : \Omega \to \Omega$

defined as $F(z) = z + (i, 0')$. Then, by composing $\phi^{-1} \circ F \circ \phi : B^d \to B^d$ we get an holomorphic selfmap of $B^d$ which is 1-Lipschitz due to Proposition 1.2.1 and it holds:

$$\lim_{n\to\infty} \phi^{-1} \circ F^n \circ \phi(z) = \lim_{n\to\infty} \phi^{-1}\big(i\frac{e_1 + z}{1 - z_1} + n(i, 0')\big) = \lim_{n\to\infty} \phi^{-1}\Big(i\big(\frac{1}{z_1 - 1} + z_1 + n\big)e_1 + (0, z')\Big)$$

$$= \lim_{n\to\infty} \frac{2\big(\frac{i(e_1+z)}{1-z_1} + n(i, 0')\big)}{i\big(1 + \frac{1}{1-z_1} + z_1 + n\big)} - e_1 = \frac{2(i, 0')}{i} - e_1 = e_1 \in \partial_G B^d.$$

## 3.5 Commuting 1-Lipschitz self maps of a Gromov hyperbolic metric space

In this section we are going to deal with commuting 1-Lipschitz maps for a proper geodesic Gromov hyperbolic metric space into itself.
From now on in this section, we consider $(X, d)$ such metric space.
Further, suppose that $f, g : X \to X$ are commuting 1-Lipschitz maps and that there exist $\xi_f, \xi_g \in \partial_G X$ so that for all $x \in X$, it holds

$$f^n(x) \to \xi_f \text{ and } g^n(x) \to \xi_g \,. \tag{3.1}$$

**Proposition 3.5.1.** *With the notation above, suppose that $\xi_g \neq \xi_f$ . Then there exists a compact set $K \subset X$ such that: for every $m \geq 0$ there exists $n = n(m) \geq 0$ with*

$$K \cap f^m g^n(K) \neq \varnothing \,.$$

*Proof.* Fix some $x_0 \in X$. Since $\xi_f \neq \xi_g$, Theorem 3.3.3 implies that there exists some $r > 0$, related to a compact set $K' \subset X$, such that: if $m, n \geq 0$ and $\gamma : [a, b] \to X$ is a geodesic segment with $\gamma(a) = f^m(x_0)$ and $\gamma(b) = g^n(x_0)$, then there exists some $t \in [a, b]$ such that $\gamma \cap K' \neq \varnothing$ and therefore

$$d(\gamma(t), x_0) \leq \mathcal{H}(\gamma, K') + d(K', x_0) := r \,.$$

Then, in the setting of Corollary 3.3.4 with $m, n \geq 0$ it holds:

$$d(f^m(x_0), g^n(x_0)) \geq d(f^m(x_0), x_0) + d(x_0, g^n(x_0)) - 2r. \tag{3.2}$$

By Theorem 3.3.5 there exists $R \geq 0$ so that: if $\gamma_1 : [a_1, b_1] \to X$ and $\gamma_2 : [a_2, b_2] \to X$ are $(1, 2r)$-quasi-geodesic segments with $\gamma_1(a_1) = \gamma_2(a_2)$ and $\gamma_1(b_1) = \gamma_2(b_2)$, then $\mathcal{H}(\gamma_1, \gamma_2) \leq R$.
Define $C = d(x_0, g(x_0))$ and fix $m \geq 0$. The main claim is that there exists $n = n(m) > 0$ so that

$$d(f^m g^n(x_0), x_0) \leq 4r + 2R + C/2.$$

With this claim every closed ball $\overline{B}(x_0, \rho)$, with $\rho \geq 4r + 2R + C/2$ is a compact set so that $f^m g^n(x_0) \in \overline{B}(x_0, \rho) \cap f^m g^n(\overline{B}(x_0, \rho))$.
We can deduce that the sequence $\{d(g^j(x_0), x_0)\}_{j \in \mathbb{N}}$ has points far at most $C$ and diverges: indeed for every $n > 0$

$$|d(g^n(x_0), x_0) - d(g^{n+1}(x_0), x_0)| \leq d(g^n(x0), g^{n+1}(x_0)) \leq d(x_0, g(x_0)) = C$$

and

$$\lim_{j \to \infty} d(g^j(x_0), x_0) = \infty.$$

Thus there exists some $n \geq 0$ so that the amount $d(f^m(x_0), x_0)$ is between some points of the sequence $\{d(g^j(x_0), x_0)\}_{j \in \mathbb{N}}$, more precisely

$$|d(f^m(x_0), x_0) - d(g^n(x_0), x_0)| \leq C/2.$$

We remark that in order to get the last inequality now $m$ depends on $n$.
Let $\gamma_1 : [0, T_1] \to X$ be a geodesic segment with $\gamma_1(0) = f^m g^n(x_0)$ and $\gamma_1(T_1) = f^m(x_0)$. Also let $\gamma_2 : [0, T_2] \to X$ be a geodesic segment with $\gamma_2(0) = f^m g^n(x_0)$ and $\gamma_2(T_2) = g^n(x_0)$. Finally define the curve $\gamma : [-T_1, T_2] \to X$ by

$$\gamma(t) = \begin{cases} \gamma_1(-t), & \text{if } t \leq 0 \\ \gamma_2(t), & \text{if } t \geq 0. \end{cases}$$

**Claim 1.** $\gamma : [-T_1, T_2] \to X$ is a $(1, 2r)$-quasi-geodesic.
*Proof of Claim 1.* Since $\gamma_1$ and $\gamma_2$ are geodesics, when $s$ and $t$ both belong to $[-T_1, 0]$ or $[0, T_2]$ it holds

$$d(\gamma(t), \gamma(s)) = |t - s|\ ;$$

moreover, when $s \in [-T_1, 0]$ and $t \in [0, T_2]$ it holds

$$d(\gamma(t),\gamma(s)) \leq d(\gamma(s),\gamma(0))+d(\gamma(0),\gamma(t)) = d(\gamma_1(-s),\gamma_1(0))+d(\gamma_2(0),\gamma_2(t)) =$$

$$t-s \leq |t-s|.$$

Therefore we obtain $d(\gamma(t),\gamma(s)) \leq |t-s|$ for all $s,t \in [-T_1, T_2]$, that is even better that the upper bound for a $(1, 2r)$-quasi-geodesic.
Similarly, since $\gamma_1$ and $\gamma_2$ are geodesics, when $s$ and $t$ both belong to $[-T_1, 0]$ or $[0, T_2]$ it holds

$$|t-s| - 2r \leq |t-s| = d(\gamma(t),\gamma(s)) \,.$$

Thus it remains to show that

$$(t-s) - 2r \leq d(\gamma(s),\gamma(t)),$$

for all $-T_1 \leq s \leq 0 \leq t \leq T_2$.
In this case, remembering that $f$ and $g$ commute, we have

$$\begin{aligned}
d(\gamma(s),\gamma(t)) &= d(\gamma_1(-s),\gamma_2(t)) \\
&\geq d(\gamma_1(T_1),\gamma_2(T_2)) - d(\gamma_1(-s),\gamma_1(T_1)) - d(\gamma_2(T_2),\gamma_2(t)) \\
&= d(\gamma_1(T_1),\gamma_2(T_2)) - (T_1+s) - (T_2-t) \\
&= (t-s) + d(\gamma_1(T_1),\gamma_2(T_2)) - T_1 - T_2 \\
&= (t-s) + d(\gamma_1(T_1),\gamma_2(T_2)) - d(\gamma_1(0),\gamma_1(T_1)) - d(\gamma_2(0),\gamma_2(T_2)) \\
&= (t-s) + d(f^m(x_0), g^n(x_0)) - d(f^m(x_0), f^m g^n(x_0)) - d(g^n(x_0), f^m g^n(x_0)) \\
&\geq (t-s) + d(f^m(x_0), g^n(x_0)) - d(x_0, g^n(x_0)) - d(x_0, f^m(x_0)).
\end{aligned}$$

So by equation 3.2, we have $d(\gamma(s),\gamma(t)) \geq (t-s) - 2r$.

**Claim 2.** $T_2 \leq d(x_0, f^m(x_0)) \leq T_2 + 2r$ and $T_1 \leq d(x_0, g^n(x_0)) \leq T_1 + 2r$.
*Proof of Claim 2.* Since $f$ and $g$ are commuting 1-Lipschitz maps we have

$$T_1 = d(\gamma(T_1), 0) = d(f^m(x_0), f^m g^n(x_0)) \leq d(x_0, g^n(x_0))$$

and

$$T_2 = d(\gamma(T_2), 0) = d(g^n(x_0), f^m g^n(x_0)) \leq d(x_0, f^m(x_0)).$$

For the upper bounds, remembering 3.2, we have:

$$\begin{aligned} T_1 + T_2 &= d(f^m(x_0), f^m g^n(x_0)) + d(g^n(x_0), f^m g^n(x_0)) \\ &\geq d(f^m(x_0), g^n(x_0)) \geq d(f^m(x_0), x_0) + d(x_0, g^n(x_0) - 2r. \end{aligned} \tag{3.3}$$

Then, we can plug $T_1 \leq d(x_0, g^n(x_0))$ into 3.3 and obtain

$$T_2 + 2r \geq d(f^m(x_0), x_0).$$

Similarly, by using $T_2 \leq d(x_0, f^m(x_0))$ in 3.3, we get

$$T_1 + 2r \geq d(g^n(x_0), x_0).$$

With the bounds obtained in **Claim 2** we estimate $T_1 - T_2$ and $T_2 - T_1$, therefore

$$|T_1 - T_2| \leq 2r + |d(x_0, f^m(x_0)) - d(x_0, g^n(x_0))| \leq 2r + C/2.$$

At this point, let $\sigma : [0, T] \to X$ be a geodesic segment with $\sigma(0) = f^m(x_0)$ and $\sigma(T) = g^n(x_0)$.
Then by the way to choose $R$, we have for all $t \in [0, T]$

$$d(\sigma(t), \gamma) \leq R.$$

By the definition of $r$, there exists some $t_0 \in [-T_1, T_2]$ so that for the point $\gamma(t_0)$ it holds

$$d(\gamma(t_0), x_0) \leq r + R. \tag{3.4}$$

In this last step we are going to prove that

$$d(x_0, f^m g^n(x_0)) \leq 4r + 2R + C/2.$$

For first let's observe that 3.4 and $\gamma(-T_1) = f^m(x_0)$ grants

$$t_0 + T_1 = d(\gamma(t_0), \gamma(-T_1)) \geq d(x_0, \gamma(-T_1)) - d(x_0, \gamma(t_0))$$

$$\geq d(x_0, f^m(x_0)) - r - R \geq T_2 - r - R.$$

The last inequality implies then

$$t_0 \geq T_2 - T_1 - r - R. \tag{3.5}$$

In a similar manner, remembering 3.4 and $\gamma(T_2) = g^n(x_0)$, it also holds

$$T_2 - t_0 = d(\gamma(t_0), \gamma(T_2)) \geq T_1 - r - R.$$

Then this gives

$$t_0 \leq T_2 - T_1 + r + R. \tag{3.6}$$

With the bounds from 3.5 and 3.6 we can estimate both $t_0$ and $-t_0$, therefore if holds:

$$|t_0| \leq |T_1 - T_2| + r + R \leq 2r + C/2 + r + R \leq 3r + R + C/2. \tag{3.7}$$

In conclusion, from 3.4 and 3.7, we have

$$\begin{aligned} d(x_0, f^m g^n(x_0)) &\leq d(x_0, \gamma(t_0)) + d(\gamma(t_0), f^m g^n(x_0)) \\ &= d(x_0, \gamma(t_0)) + d(\gamma(t_0), \gamma(0)) \\ &\leq r + R + |t_0| \\ &\leq 4r + 2R + C/2. \end{aligned}$$

□

*Remark* 29. Since $(X, d)$ is a proper geodesic Gromov hyperbolic metric space, due to Karlsson's Theorem 3.4.2, the existence of a point $x_0 \in X$ such that $f^n(x_0)$ converges to $\xi_f$ is equivalent to the convergence of $f^n(x)$ to $\xi_f$, for all $x \in X$. Therefore we found a condition to imply 3.1 and obtain the last Proposition.

**Definition 3.8.** A subset $M$ of the metric space $(X, d)$ is called totally geodesic if any geodesic on $M$, with its induced distance $d_{|M}$, is also a geodesic on the metric space $(X, d)$.

The next result updates its assumptions thanks to the last Remark and it reveals more informations about the convergence in the Gromov compactification.

**Theorem 3.5.2.** *Let $(X, d)$ be a proper geodesic Gromov hyperbolic metric space. Let $f, g : X \to X$ be commuting 1-Lipschitz maps. Suppose there exist $\xi_f \neq \xi_g \in \partial_G X$ and $x_0 \in X$ such that:*

$$\lim_{n\to\infty} f^n(x_0) = \xi_f \text{ and } \lim_{n\to\infty} g^n(x_0) = \xi_g \, ,$$

*in the Gromov compactification. Then there exist a totally geodesic closed subset $M \subset X$ and a 1-Lipschitz map $\rho : X \to M$ such that:*

1. $\rho \circ \rho = \rho$
2. *$f(M) = g(M) = M$ and $f_{|M}$ and $g_{|M}$ are isometries of $(M, d_{|M})$.*

*Proof.* We start by observing that the family $\{f^m \circ g^n\}_{m,n\in\mathbb{N}}$ is equicontinuous and, as a consequence of Proposition 3.5.1, we have that for every $m \geq 0$ there exists $n(m) \geq 0$ so that for every $x \in X$ the set $\{f^m \circ g^{n(m)}(x)\}$ is relatively compact in $X$. Then it follows from the Arzelà-Ascoli Theorem that there exist sequences of natural numbers, namely $\{m_k\}, \{n_k\} \subset \mathbb{N}$ so that $f^{m_k} \circ g^{n_k}$ converges uniformly on compact sets of $X$ to a 1-Lipschitz map $h : X \to X$. Moreover we assume the following

$$p_k := m_{k+1} - m_k \to \infty, \; p'_k := n_{k+1} - n_k \to \infty$$

and

$$q_k := p_k - m_k \to \infty, \; q'_k := p'_k - n_k \to \infty.$$

Since, commutativity of $f$ and $g$ and uniform convergence of $f^{m_k} \circ g^{n_k}$ imply

$$(f^{p_k} \circ g^{p'_k})(f^{m_k}(g^{n_k}(z))) = (f^{m_{k+1}} \circ g^{n_{k+1}})(z) \to h(z), \tag{3.8}$$

we get that $f^{p_k} \circ g^{p'_k}$ converges uniformly on compact sets of $X$ to a 1-Lipschitz self-map denoted as $\rho : X \to X$. Moreover, 3.8 combined with commutativity between $f$ and $g$ imply:

$$h \circ \rho = \lim_{k\to\infty} (f^{m_k} \circ g^{n_k}) \circ (f^{p_k} \circ g^{p'_k}) = \lim_{k\to\infty} (f^{p_k} \circ g^{p'_k}) \circ (f^{m_k} \circ g^{n_k}) = \rho \circ h = h.$$

On the other hand we have:

$$(f^{q_k} \circ g^{q'_k})(f^{m_k}(g^{n_k}(z))) = (f^{p_k} \circ g^{p'_k})(z) \to \rho(z),$$

therefore, following the same previous path, by passing to a subsequence if necessary, $f^{q_k} \circ g^{q'_k}$ converges uniformly on compact sets to a 1-Lipschitz map $\chi : X \to X$ such that

$$\chi \circ h = h \circ \chi = \rho.$$

In this way we have that

$$\rho \circ \rho = (\chi \circ h) \circ \rho = \chi \circ (h \circ \rho) = \chi \circ h = \rho.$$

Now we denote $M = \rho(X)$ and it must be a closed subset of $X$. Therefore $\rho : X \to M$ and, since commutativity between $f$ and $g$ implies

$$f \circ \rho = \rho \circ f \text{ and } g \circ \rho = \rho \circ g \ ,$$

we also have that $f(M) \subset M$ and $g(M) \subset M$.
Further, since

$$(f^{(p_k-1)} \circ g^{(p'_k-1)}) \circ (f \circ g) = f^{p_k} \circ g^{p'_k} \to \rho,$$

by passing eventually to a subsequence, converges uniformly on compact sets to a 1-Lipschitz selfmap $\psi : X \to X$. Since the target space of $\rho$ is $M$ we then have $\psi(M) \subset M$. Hence, for $z \in M$,

$$(\psi \circ (f \circ g))(z) = z.$$

Therefore, $f \circ g$ is a 1-bi-Lipschitz selfmap of $M$, an isometry for $(M, d_{|M})$. Since $f \circ g = g \circ f$, it follows that both $f$ and $g$ are bijective and then isometries of $(M, d_{|M})$ and due to this we deduce also that $M$ is totally geodesic. □

## 3.6 The End compactification

The notion of an "end" of a space firstly appeared in [26], which is H.Freudental PhD thesis, submitted for publication in March 1930.
For introducing the basic idea we will only consider the case when $X$ is a manifold. As a consequence there exists an increasing sequence $K_0 \subset K_1 \subset$

$K_2 \subset \cdots$ of compact subsets with $X = \cup_{n \geq 0} K_n$. By compactness, each $X \setminus K_n$ has finitely many components. An *end* of $X$ is a decreasing sequence $U_0 \supset U_1 \supset U_2 \supset \cdots$ of open sets where each $U_n$ is a connected component of $X \setminus K_n$. Let $E[X]$ denote the set of ends. The set $X \cup E[X]$ has a natural topology making it a compactification of X where each end $(U_j)_{j \geq 0} \in E[X]$ has a neighborhood basis

$$U_k \cup \{(V_j)_{j \geq 0} \in E[X] : V_j = U_j \text{ for } j \leq k\}, \; k \geq 0.$$

The main results from this theory, also explained in [27], can be resumed as

**Proposition 3.6.1.** *For the end compactification the following facts hold:*

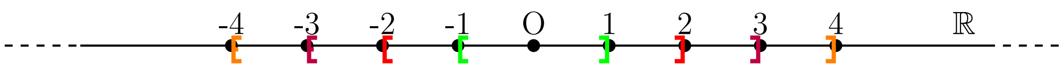
1. $X \cup E[X]$ *is compact and Hausdorff;*

2. $E[X]$ *is closed and totally disconnected with respect to the topology just defined;*

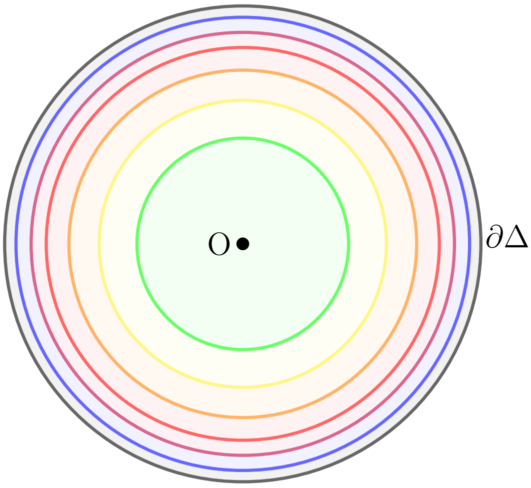
3. $X \cup E[X]$ *does not depend on the choice of compact sets* $K_0 \subset K_1 \subset K_2 \subset \cdots$.

**Example 3.13.** If $K$ is a compact manifold we clearly have that $E[K] = \varnothing$.

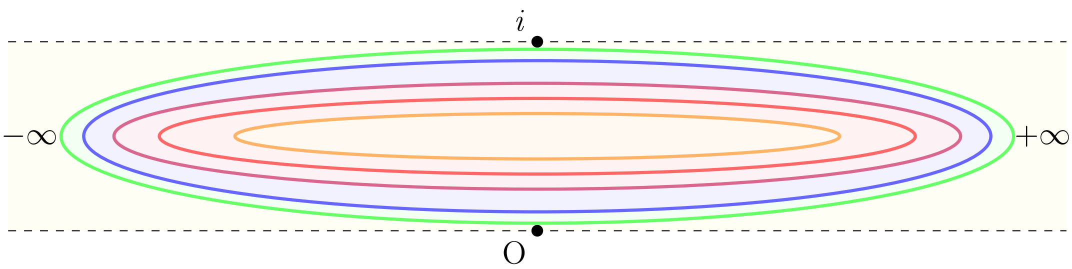
**Example 3.14.** The real line $\mathbb{R}$ has two ends, i.e. $E[\mathbb{R}] = \{-\infty, +\infty\}$. Indeed we can consider $\{K_n\}_{n \in \mathbb{N}}$, where $K_n = [-n, n]$.




**Example 3.15.** For the Poincaré Disk $\Delta$, we can consider the euclidean topology since the Poincaré distance $\omega$ generates equivalent open sets. Therefore a sequence of compact sets $\{K_n\}_{n \in \mathbb{N}}$, defined as $K_n = \overline{B}(0, r_n)$ with $r_n \to 1$, invades $\Delta$ and then for each fixed $m \in \mathbb{N}$ there is only one connected component in $\Delta \setminus \cup_{n=1}^{m} K_n$. This gives $E[\Delta] = \partial\Delta$.

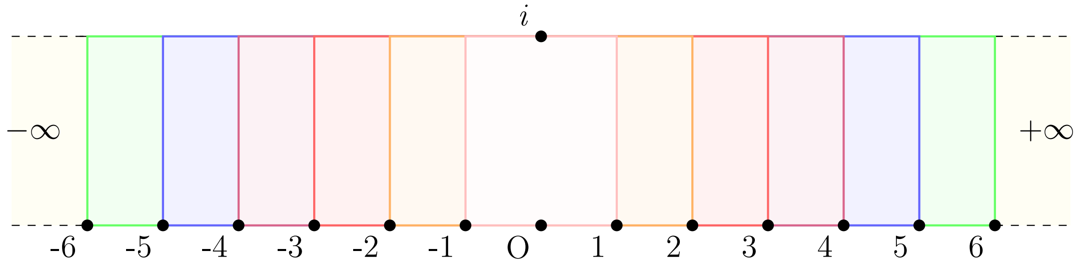



**Example 3.16.**
Consider the open stripe of $\mathbb{C}$ defined as $S = \{z \in \mathbb{C} | 0 < \mathrm{Im} z < 1\}$, one of the possibles invading sequences of compact sets is represented by ellipses with foci laying on the line $S = \{z \in \mathbb{C} | \mathrm{Im} z = \frac{1}{2}\}$.



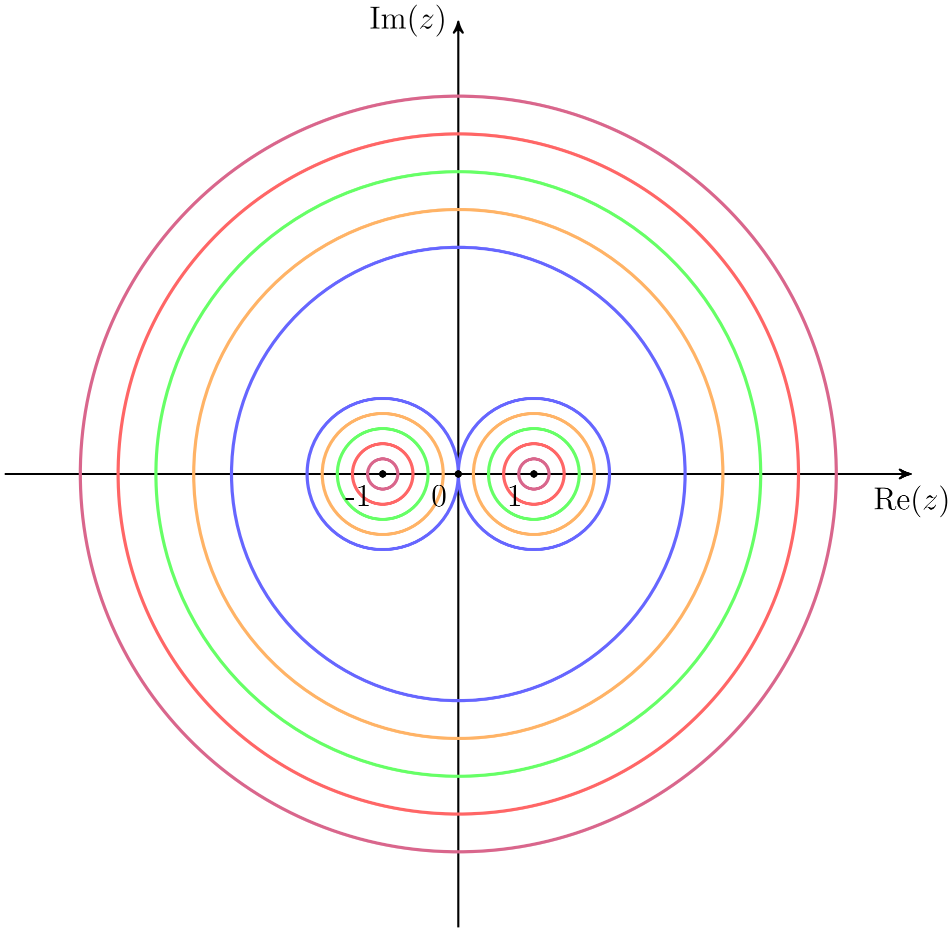

From the picture we can deduce that there is only one connected component in the complement, thus one end and we can improperly write it as $E[S] = \partial S \cup \{-\infty\} \cup \{+\infty\}$.

**Example 3.17.**
Consider the closed stripe of $\mathbb{C}$ defined as $\overline{S} = \{z \in \mathbb{C} | 0 \leq \mathrm{Im} z \leq 1\}$, one of the possibles invading sequences of compact sets is represented by rectangles with two sides on the lines of $\partial S = \{z \in \mathbb{C} | \mathrm{Im} z = 0 \vee \mathrm{Im} z = 1\}$.
For instance, we take the sequence $\{K_n\}_{n \in \mathbb{N}}$, where $K_n = \{z \in \mathbb{C} |\ 0 \leq \mathrm{Im} z \leq 1 \text{ and } |\mathrm{Re} z| \leq n\}$.




From the picture we deduce that for a fixed $m$ there are always two connected components in the complement $\overline{S} \setminus \cup_{n=1}^{m} K_n$, therefore two ends. We can then write $E[\overline{S}] = \{-\infty, +\infty\}$.

**Example 3.18.** Consider $X = \mathbb{C} \setminus \{-1, 1\}$ and take the invading sequence of compacts $\{K_n\}_{n \in \mathbb{N}}$, where $K_n = \overline{B}(0, n) \setminus B(1, 1/n) \setminus B(-1, 1/n)$. This explains that there are three ends, more precisely $E[X] = \{-1, 1, \infty\}$.



### 3.6.1 Unbounded convex domains

In the setting of unbounded convex domains, in $\mathbb{R}^l$ and then in $\mathbb{C}^d$, we are going to present a characterisation of their ends.

For $x \in \mathbb{R}^l$ and $R > 0$ we define $B(x, R) := \{w \in \mathbb{R}^l : \| w - x \| < R\}$.

**Definition 3.9.** Let $D \subset \mathbb{R}^l$ be an unbounded convex domain. A vector $v \in \mathbb{R}^l$, $\| v \| = 1$, is called a direction at $\infty$ for $D$ if there exists $x \in D$ such that $x + tv \in D$ for all $t \geq 0$. Then let $S_\infty(D) \subset \mathbb{R}^l$ be the set of directions at infinity for $D$.

**Proposition 3.6.2.** *If $v$ is a direction at $\infty$ for $D$ convex, then for every $z \in D$ and all $t \geq 0$ it holds $z + tv \in D$.*

*Proof.* By definition of $v$ it exists a $z' \in D$ so that $z' + tv \in D$ for all $t \geq 0$. Since $z$ and $z' + tv$ for all $t \geq 0$ both belong to the convex set $D$ we can write the convex combination with $\lambda \in [0, 1]$

$$\lambda(z' + tv) + (1 - \lambda)z = z + \lambda(z' - z + tv) = z + \lambda t\Big(\frac{z' - z}{t} + v\Big) \in D \; \forall t > 0.$$

Thus the direction $\frac{z'-z}{t} + v \to v$ for $t \to \infty$ and then $v$ is a direction at $\infty$ also respect to $z$. □

**Lemma 3.6.3.** *Let $D \subset \mathbb{R}^l$ be an unbounded convex domain. Then there exists at least one direction $v$ at $\infty$ for $D$. Moreover:*

1. *either $D \setminus \overline{B(0, R)}$ has only one unbounded connected component for all $R > 0$ or*
2. *there exists $R_0 > 0$ such that $D \setminus \overline{B(0, R)}$ has two unbounded connected components for all $R \geq R_0$. This is the case if and only if the only directions at $\infty$ for $D$ are $v$ and $-v$.*

*Proof.* Since $D$ is unbounded, there exists a sequence $\{p_k\}_{k \in \mathbb{N}} \subset D$ such that $\lim_{k \to \infty} \| p_k \| = \infty$. Up to extracting subsequences, we can assume that

$\lim_{k\to\infty} p_k/ \parallel p_k \parallel = v$. Since $D$ is convex, for a fixed $z$ in $D$, the real segment $[z, p_k]$ is contained in $D$ for all $k$. Hence, since $\{z+tv, t \geq 0\}$ is the limit, for the local Hausdorff convergence, of the segments $[z, p_k]$, then $\{z+tv, t \geq 0\}$ is contained in $\overline{D}$. Finally, since $z \in D$, by convexity of $D$ it follows that $z + tv \in D$ for all $t \geq 0$ and therefore $v$ is a direction at $\infty$.
Next, assume that there exists $R > 0$ such that $D \setminus \overline{B(0, R)}$ is not connected. We claim that $D \setminus \overline{B(0, R)}$ has at most two unbounded components. Indeed, if $U$ is an unbounded connected component of $D \setminus \overline{B(0, R)}$ and $\{p_k\}_{k\in\mathbb{N}} \subset U$ converges in norm to $\infty$ and $\lim_{k\to\infty} p_k/ \parallel p_k \parallel = v$. Then, $U$ is clearly convex and for every $z \in U$ such that $\parallel z \parallel > R$ and for every $t \geq 0$, due to Proposition 3.6.2 it holds $z+tv \in U$. Hence, for every unbounded connected component of $D \setminus \overline{B(0, R)}$ there exists $v \in \mathbb{R}^l$, $\parallel v \parallel = 1$, such that $z+tv$ belongs to such a component for every $t \geq 0$ and some $z \in D$. If the unbounded components were more than two, there would exist two components $U$ and $U'$ and two directions $v$ and $w$ at $\infty$ for $D$, which are $\mathbb{R}$-linearly independent and such that $z_0 + tv \in U$ for all $t \geq 0$ and some $z_0 \in D$, and $z_1 + tw \in U'$ for all $t \geq 0$ and some $z_1 \in D$. But then, since $v$ and $w$ are $\mathbb{R}$-linearly independent, for $a, b$ sufficiently large the intersection $[z_0 + av, z_1 + bw] \cap \overline{B(0, R)}$ is empty. On the other side $D \cap \overline{B(0, R)}$ is convex and therefore contains all the segments joining $U$ with $U'$, subsets connected in $D$ and this presents a contradiction.
We have then proved that, if $D \setminus \overline{B(0, R)}$ is not connected, then it has at most two unbounded connected components. If $D \setminus \overline{B(0, R)}$ contains two unbounded connected components, then it follows automatically that for every $R' > R$, also $D \setminus \overline{B(0, R)}$ contains two unbounded connected components.
Moreover, we proved also that if there are two $\mathbb{R}$-linearly independent directions at $\infty$, then for every $R > 0, D \setminus \overline{B(0, R)}$ has only one unbounded connected component.
Therefore, if $D \setminus \overline{B(0, R)}$ has two unbounded connected components, then there are only two directions at $\infty$ for $D$. Hence, for some $v \in \mathbb{R}^l, \parallel v \parallel = 1$, we denote such directions at $\infty$ as $v$ and $-v$ .

Conversely, assume $v, -v$ are the only directions at $\infty$ for $D$. Suppose by contradiction that for every $R > 0$ the open set $D \setminus \overline{B(0,R)}$ had only one unbounded connected component. Let $z_0 \in D$, and $R > \| z_0 \|$, by definition of direction at $\infty$ then there exists $t_R \in (0, \infty)$ such that $z_0 + tv, z_0 - tv \in D \setminus \overline{B(0,R)}$ for all $t \geq t_R$. Since $D \setminus \overline{B(0,R)}$ has only one unbounded connected component, the points $z_0 + t_R v$ and $z_0 - t_R v$ can be joined by a continuous path $\gamma_R$ laying in the unique unbounded connected component of $D \setminus \overline{B(0,R)}$. Let $H$ be the real affine hyperplane through $z_0$ orthogonal to $v$. Then, by construction, the continuous path $\gamma_R$ allways goes through $H$ in order to join $z_0 + t_R v$ to $z_0 - t_R v$, for all $R \geq \| z_0 \|$. Hence $H \cap \gamma_R \neq \varnothing$ for all $R \geq \| z_0 \|$. Therefore, there exists a sequence $\{p_k\}_{k \in \mathbb{N}} \subset H \cap D$ converging to $\infty$ such that $w := \lim_{k \to \infty} p_k / \| p_k \|$ is a direction at $\infty$ for $D$. We observe that $w$ belongs to $H$, thus it is $\mathbb{R}$-linearly independent with $v$ and this presents a contradiction. □

**Lemma 3.6.4.** *Let $D \subset \mathbb{R}^l$ be an unbounded convex domain and $S_\infty(D) = \{v, -v\}$ for some $v \in \mathbb{R}^l$. Let $H$ be the real orthogonal complement of $\mathbb{R}v$ in $\mathbb{R}^l$. Then there exists a bounded convex domain $\Omega \subset H$ such that $D = \Omega + \mathbb{R}v$.*

*Proof.* Let's define $\Omega := D \cap H$. The set $\Omega$ is an open convex set in $H$, and, due to Proposition 3.6.2 every direction at $\infty$ for $\Omega$ is also a direction at $\infty$ for $H$, the set $\Omega$ must be bounded.
Consider $p \in D$, Proposition 3.6.2 grants that $p + tv \in D$ for all $t \in \mathbb{R}$ and then there exists $t_0 \in \mathbb{R}$ such that $p' := p - t_0 v \in H$. Hence, $p = p' + t_0 v$. Since $p \in D$ was arbitrary, we have $D = \Omega + \mathbb{R}v$. □

Furthermore, when $D$ is an unbounded convex domain of $\mathbb{R}^l$, Lemma 3.6.3 and Lemma 3.6.4 also describe the behaviour of $E[\overline{D}]$. To sum up we can state the following

**Corollary 3.6.5.** *Let $D \subset \mathbb{R}^l$ be an unbounded convex domain. Then $\overline{D}$ has either one or two ends. Moreover,*

1. *$\overline{D}$ has one end if and only if for every $R > 0$ the open set $D \setminus \overline{B(0,R)}$ has only one unbounded connected component,*

*2.* $\overline{D}$ *has two ends if and only if* $S_\infty(D) = \{v, -v\}$ *for some* $v \in \mathbb{R}^l$*.*

### 3.6.2 The Gromov boundary and ends in $\mathbb{C}^d$

Assume now that $D \subset \mathbb{C}^d$ is an unbounded $\mathbb{C}$-proper convex domain such that $(D, k_D)$ is Gromov hyperbolic. As usual, we let $\overline{D} \subset \mathbb{C}^d$ denote the closure of $D$ in $\mathbb{C}^d$, $\partial D = \overline{D} \setminus D$ and we define $\overline{D}^\star = \overline{D} \cup E[\overline{D}]$.
To distinguish between the Gromov compactification and the End compactification, we will write $\xi_n \overset{\text{Gromov}}{\longrightarrow} \xi$ when $\xi_n \in \overline{D}^G$ is a sequence converging to $\xi \in \overline{D}^G$.
Now we are going to prove a chain of Lemmas aiming to describe the ends for $D$ and their relation with the Gromov boundary.

**Lemma 3.6.6.** *For any* $v \in S_\infty(D)$*, there exists a point* $\zeta_v \in \partial_G D$ *such that if* $p_n \in D$ *is a sequence with* $\|p_n\| \to \infty$ *and* $p_n/\|p_n\| \to v$*, then* $p_n \overset{\text{Gromov}}{\longrightarrow} \zeta_v$*.*

*Proof.* Consider the 1-Lipschitz selfmap of $(D, k_D)$ defined as $D \ni z \mapsto z + v \in D$. Then by Karlsson's Theorem 3.4.2 there exists some $\zeta_v \in \partial_G D$ such that

$$z + nv \overset{\text{Gromov}}{\longrightarrow} \zeta_v$$

for all $z \in D$.
Now fix a sequence $p_n \in D$ with $\|p_n\| \to \infty$ and, up to subsequences, $p_n/\|p_n\| \to v$. Assume for a contradiction that $p_n$ does not converge to $\zeta_v$ in $\overline{D}^G$. Then by passing to a subsequence we can suppose that $p_n \overset{\text{Gromov}}{\longrightarrow} \xi \in \partial_G D$, with $\xi \neq \zeta_v$.
Consider, for $n \geq 0$ and a fixed point $z_0$ in $D$, the 1-Lipschitz function $b_n : (D, k_D) \to (\mathbb{R}, |\cdot|)$ defined by

$$b_n(z) = k_D(z, p_n) - k_D(p_n, z_0)\,.$$

Indeed $|b_n(z) - b_n(z')| = |k_D(z, p_n) - k_D(z', p_n)| \leq k_D(z, z')$ for all $z, z' \in D$ and $n \geq 0$. Moreover, since $b_n(z_0) = 0$ for every $n \geq 0$, we have that

$$|b_n(z)| = |b_n(z) - b_n(z_0)| \leq k_D(z, z_0).$$

Therefore Arzelà-Ascoli's Theorem grants that, by passing to a subsequence, $b_n$ converges uniformly on compact sets to some function $b$.
**Claim.** For each $n$, the set $b_n^{-1}((-\infty, 0])$ is convex.
*Proof of Claim.* For every $z \in D$ and every $r > 0$, we define the closed ball with center $z$ and radius $r$ respect the Kobayashi distance to be

$$\overline{B_D^k(z,r)} := \{w \in D : k_D(z,w) \leq r\}.$$

We observe that $\overline{B_D^k(z,r)}$ is convex. Indeed when $D$ is bounded one can see Proposition 2.3.56 from [3] at page 182. For the unbounded case, let $D_R$ be the intersection of $D$ with an Euclidean ball of center the origin and radius $R > 0$. Consider $B_D^k(z_0, \varepsilon)$, then the convex sets $B_{D_R}^k(z_0,\varepsilon) \subset B_{D_{R+\delta}}^k(z_0,\varepsilon) \subset B_D^k(z_0,\varepsilon)$ for all $R >> 1, \delta > 0$, and since $\lim_{R\to\infty} k_{D_R} = k_D$ (see Proposition 2.3.34 from [3] at page 173) then we have the increasing union of convex set $\cup_{R\geq 0} B_{D_R}^k(z_0,\varepsilon) = B_D^k(z_0,\varepsilon)$ to be convex.
In particular we have $b_n^{-1}((-\infty,0]) = \overline{B_D^k(p_n, k_D(p_n,z_0))}$, and therefore it is convex.

As a consequence, for every $n \geq 0$, the set $b_n^{-1}((-\infty, 0])$ contains the line segment

$$[z_0, p_n] = \{tz_0 + (1-t)p_n : 0 \leq t \leq 1\}.$$

Since $\lim_{n\to\infty}(p_n/\|p_n\|) = v$, then the set $b_n^{-1}((-\infty,0])$ contains the real line $\Omega := z_0 + R_{\geq 0} \cdot v$.
Let's define, for every $m \geq 0$, the sequence $z_m := z_0 + mv$. We consider $\xi \neq \zeta_v$ and, up to extracting a suitable subsequence of $\{z_m\}_{m\in\mathbb{N}}$ in order to satisfy the assumptions of Corollary 3.3.4, it follows:

$$k_\Omega(z_m, p_n) \geq k_\Omega(z_m, z_0) + k_\Omega(z_0, p_n) - M,$$

for every $n \geq 0$ and $m \geq 0$.
In this way we have, for every $m \geq 0$:

$$0 \geq b(z_m) = \lim_{n\to\infty} \big(k_\Omega(z_m,p_n) - k_\Omega(p_n,z_0)\big) \geq k_\Omega(z_m,z_0) - M.$$

Thus $k_\Omega(z_m, z_0) \leq M$ for every $m \geq 0$ and this contradicts $k_\Omega(z_0, z_m) \to \infty$. □

**Lemma 3.6.7.** *Suppose that $\overline{D}$ has one end. Then $\zeta_v = \zeta_w$ for all $v, w \in S_\infty(D)$.*

*Proof.* We start by considering the case where $v, w \in S_\infty(D)$ are linearly independent over $\mathbb{R}$.

Suppose for a contradiction that $\zeta_v \neq \zeta_w$. Consider the 1-Lipschitz selfmaps $f, g : (D, k_D) \to (D, k_D)$ defined by

$$f(z) = z + v \text{ and } g(z) = z + w.$$

Then $f^m(z) \overset{\text{Gromov}}{\longrightarrow} \zeta_v$ and $g^n(z) \overset{\text{Gromov}}{\longrightarrow} \zeta_w$ for all $z \in D$ by Lemma 3.6.6. So by Proposition 3.5.1 there exist $m_k, n_k \to \infty$ such that

$$\lim_{k\to\infty} f^{m_k} g^{n_k}(z_1) = z_2$$

for some $z_1, z_2$ contained in a compact set of $D$.

On the other side we have

$$f^{m_k} g^{n_k}(z_1) = z_1 + m_k v + n_k w \, .$$

Since $v$ and $w$ are linearly independent, there is a contradiction comparing the last two equalities. This means that $\zeta_w = \zeta_v$.

Now if $v, w \in S_\infty(D)$ are linearly dependent over $\mathbb{R}$ and distinct, then $w = -v$. Since, by assumption, $\overline{D}$ has one end then there exists some $u \in S_\infty(D)$ such that $u, v$ are linearly independent over $\mathbb{R}$. Then $\zeta_v = \zeta_u = \zeta_w$. □

**Lemma 3.6.8.** *Let $C$ be a convex domain of $\mathbb{C}^d$.*
*For $z \in C$ and $w \in \mathbb{C}^d$, let*

$$\delta_C(z; w) := \inf\{\| z - u \| : u \in \partial C \cap (z + \mathbb{C} \cdot w)\} \, .$$

*Then the Kobayashi metric $\kappa_C(z; w)$ is bounded from above and below as*

$$\frac{\| w \|}{2\delta_C(z; v)} \leq \kappa_C(z; w) \leq \frac{\| w \|}{\delta_C(z; v)}$$

*for all $z \in C$ and $w \in \mathbb{C}^d$.*

*Proof.* We start by proving the upper bound. Recall that $\kappa_C(z;w) = \inf\{|\xi| : \exists\varphi \in Hol(\Delta, C) \text{ so that } \varphi(0) = z,\ d\varphi_0(\xi) = w\}$. Then let's define a suitable $\varphi$ to get required inequality.

Consider $\varphi : \Delta \to C$ as $\varphi(\vartheta) = z + \vartheta w \delta_C(z,w)\frac{1}{\|w\|}$, we have that $\varphi(0) = z$ and, for a given $\xi \in \mathbb{C}^d$ so that $w = d\varphi_0(\xi) = \delta_C(z,w)\frac{1}{\|w\|}\xi w$, it follows

$$|\xi| = \frac{\|w\|}{\delta_C(z,w)}.$$

For the lower bound we start proving

$$\|w\| = \frac{\inf\{\|z-u\| : u \in \partial C \cap (z + \mathbb{C}\cdot w))\}}{\sup\{r > 0 : z + \Delta(0,r)w \subset C\}}.$$

Indeed let $\lambda \in \mathbb{C}$ so that $u = z + \lambda w$ with $u \in \partial C$ and let $\lambda' \in \mathbb{C}$ so that $z + \lambda' w \in C$, we can then consider

$$\frac{\inf\{\|z-u\| : u \in \partial C \cap (z + \mathbb{C}\cdot w))\}}{\sup\{r > 0 : z + \Delta(0,r)w \subset C\}} = \frac{\inf\{\|z-u\| : u = z + \lambda w, u \in \partial C\}}{\sup\{|\lambda'| : z + \lambda' w \in C\}}$$

$$= \|w\|\frac{\inf\{|\lambda| : z + \lambda w \in \partial C\}}{\sup\{|\lambda'| : z + \lambda' w \in C\}} = \|w\|.$$

Then we can apply Lemma 11.3.7 and Corollary 11.3.8 from [28] at pages 382-383 and conclude. However, in order to be consistent with the suggested literature we will prove the following

**Borel–Carathéodory Lemma.** Let $\psi : \Delta \to \mathbb{C}$ be a holomorphic map such that $\psi(0) = 0$ and $\mathrm{Re}\,\psi(z) \le 1$, $\lambda \in \Delta$. Then $|\psi(\lambda)| \le \frac{2|\lambda|}{1-|\lambda|}$, $\lambda \in \Delta$.

*Proof of the Lemma.* We observe that: $\mathrm{Re}\,\omega \le 1 \Leftrightarrow \left|\frac{\omega}{\omega-2}\right| \le 1$, indeed:

$$\mathrm{Re}\,\omega \le 1 \Leftrightarrow 0 \le -4\mathrm{Re}\,\omega + 4 \Leftrightarrow (\mathrm{Re}\,\omega)^2 + (\mathrm{Im}\,\omega)^2 \le (\mathrm{Re}\,\omega)^2 - 4\mathrm{Re}\,\omega + 4 + (\mathrm{Im}\,\omega)^2$$

$$\Leftrightarrow \left|\frac{\omega}{\omega-2}\right|^2 \le 1 \Leftrightarrow \left|\frac{\omega}{\omega-2}\right| \le 1\,.$$

Set $S = \{\omega \in \mathbb{C} : \mathrm{Re}\,\omega \le 1\}$, we can define the biholomorphism $\phi : S \to \overline{\Delta}$ by $\phi(\omega) = \frac{\omega}{\omega-2}$. Thus $\phi \circ \psi : \Delta \to \overline{\Delta}$ is an holomorphic map so that $\phi \circ \psi(0) = 0$ and, by Schwarz's Lemma, it holds for each $\lambda$ in $\Delta$:

$$\left|\frac{\psi(\lambda)}{\psi(\lambda)-2}\right| \le |\lambda| \Rightarrow |\psi(\lambda)| \le (|\psi(\lambda)| + 2)|\lambda| \Rightarrow |\psi(\lambda)|(1 - |\lambda|) \le 2|\lambda|$$

$$\Rightarrow |\psi(\lambda)| \leq \frac{2|\lambda|}{1-|\lambda|}\,.$$

□

**Lemma 3.6.9.** *Suppose that $\overline{D}$ has two ends, that is $S_\infty(D) = \{v, -v\}$ for some $v \in \mathbb{C}^d$. Then $\zeta_v \neq \zeta_{-v}$.*

*Proof.* To start we need the following

**Claim.** Let $z_0 \in D$. Then there exist $A > 1$ such that the curve $\sigma : \mathbb{R} \to D$ given by $\sigma(t) = z_0 + tv$ is an $(A, 0)$-quasi-geodesic.

*Proof of the Claim.* Since $S_\infty(D) = \{v, -v\}$, by Proposition 3.6.2, it holds for every $z \in D$ and $w \in \mathbb{C}^d$

$$u \in \partial D \cap (z + \mathbb{C} \cdot w) \Rightarrow u \in \partial D \cap (z + tv + \mathbb{C} \cdot w)\ \forall t \in \mathbb{R}$$

and

$$\|z - u\| = \|z + tv - (u + tv)\|\,.$$

Therefore we have:

$$\delta_D(z; w) = \delta_D(z + tv; w)$$

for all $z \in D$, $w \in \mathbb{C}^d$ and $t \in \mathbb{R}$. This implies that

$$\delta_D(\sigma(t); \sigma'(t)) = \delta_D(z_0 + tv; v) = \delta_D(z_0; v)$$

for all $t \in \mathbb{R}$.

We observe now that $D \subset \mathbb{C}^d \simeq \mathbb{R}^{2d}$, and the notions of convexity and $S_\infty(D)$ are preserved, therefore we can apply Lemma 3.6.4 and get $D = \Omega + \mathbb{R} \cdot v$, where $\Omega$ is a bounded convex domain laying on the real orthogonal complement of $\mathbb{R}v$. In this way $\partial D = \partial(\Omega + \mathbb{R}v)$ and then there exists $\alpha > 0$ so that

$$\delta_D(z; w) \leq \alpha$$

for all $z \in D$ and $w \in \mathbb{C}^d$.

Now fix $a \leq b$. Then,

$$k_D(\sigma(a), \sigma(b)) \leq \int_a^b \kappa_D(\sigma(t); \sigma'(t))dt \leq \int_a^b \frac{\|\sigma'(t)\|}{\delta_D(\sigma(t); \sigma'(t))}dt = \int_a^b \frac{\|\sigma'(t)\|}{\delta_D(\sigma(t); v)}dt$$

$$= \frac{1}{\delta_D(z_0; v)} \int_a^b \|\sigma'(t)\| dt = \frac{1}{\delta_D(z_0; v)} (b - a) \, .$$

Now consider any absolutely continuous curve $\gamma : [0,1] \to D$ so that $\gamma(0) = \sigma(a) = z_0 + av$ and $\gamma(1) = \sigma(b) = z + bv$. Then

$$\ell_k(\gamma) = \int_0^1 \kappa_D(\gamma(t); \gamma'(t)) dt \geq \int_0^1 \frac{\|\gamma'(t)\|}{2\delta_D(\gamma(t); \gamma'(t))} dt \geq \frac{1}{2\alpha} \int_0^1 \|\gamma'(t)\| dt$$

$$\geq \frac{1}{2\alpha} \|\gamma(1) - \gamma(0)\| = \frac{1}{2\alpha}(b - a).$$

Since $\gamma$ is an arbitrary absolutely continuous curve joining $\sigma(a)$ to $\sigma(b)$ we have that

$$k_D(\sigma(a), \sigma(b)) \geq \frac{1}{2\alpha}(b - a) \, .$$

In this way we proved that $\sigma$ is a $(A, 0)$-quasi-geodesic for some $A > 1$.

Then, Lemma 3.3.6 implies that the limits (in the Gromov meaning of convergence)

$$\lim_{t \to \infty} \sigma(t) \text{ and } \lim_{t \to -\infty} \sigma(t)$$

both exist in $\partial_G D$ and are distinct. Therefore we have $\zeta_v \neq \zeta_{-v}$. □

Thanks to the Lemmas 3.6.6, 3.6.7 and 3.6.9, we can now summarize the behaviour of the ends of $D$ with the following

**Proposition 3.6.10.** *Let $D \subset \mathbb{C}^d$ be an unbounded $\mathbb{C}$-proper convex domain so that $(D, k_D)$ is Gromov hyperbolic.*
*Suppose $x$ is an end of $\overline{D}$. Then there exists $\zeta_x \in \partial_G D$ such that: if $z_n \in D$ converges to $x$ in $\overline{D}^\star$, then*

$$z_n \overset{Gromov}{\longrightarrow} \zeta_x \, .$$

*Moreover, if $\overline{D}$ has two ends $x, y$, then $\zeta_x \neq \zeta_y$.*

# Chapter 4

# Homeomorphic Extension of Quasi-Isometries and Iteration Theory

## 4.1 Homeomorphic extension of the identity map for $\mathbb{C}$-proper and convex domains

We begin with the proof of a result known as the homeomorphic extension of the identity map for a $\mathbb{C}$-proper convex domain of $\mathbb{C}^d$ where $(D, k_D)$ is Gromov hyperbolic.
The first thing to do is to introduce some definitions

**Definition 4.1.** An *analytic disc* in $\mathbb{C}^d$ is a non-constant holomorphic map $\phi : \Delta \to \mathbb{C}^d$. We shall improperly consider with the same notation both the embedding and its image. If $\phi$ extends continuosly to $\overline{\Delta}$ then we call $\phi(\overline{\Delta})$ a closed analytic disc and $\phi(\partial\Delta)$ the boundary of the analytic disc.

The basic properties and examples for analytic discs can be found in [8], Chapter 3, pages 136/137.

**Definition 4.2.** A domain or compact subset $E$ in $\mathbb{C}^d$ is said to be $\mathbb{C}$-convex if for any complex line $l \subset \mathbb{C}^d$ the intersection $E \cap l$ is both connected and simply connected.

*Remark* 30. A convex set of $\mathbb{C}^d$ is clearly a $\mathbb{C}$-convex set.

**Example 4.1.** Consider as a subset of $\mathbb{C}^2$

$$E = \{z \in \mathbb{C} : (\mathrm{Re}z + 1)^2(\mathrm{Re}z - 2)^2 < \mathrm{Im}z\} \times \Delta \,.$$

$E$ is a $\mathbb{C}$-convex unbounded $\mathbb{C}$-proper domain which is not convex.

By establishing some basic properties of geodesics in $D$, we are going to present a chain of Lemmas with the aim of proving Theorem 4.1.7.
Throughout the chapter, for $z, w \in \mathbb{C}^d$ let

$$[z, w] = \{tz + (1-t)w : 0 \le t \le 1\}$$

denote the Euclidean line segment joining them.

**Lemma 4.1.1.** *Let $D \subset \mathbb{C}^d$ be an unbounded $\mathbb{C}$-proper convex domain. If $(z_n)_{n\in\mathbb{N}}, (w_n)_{n\in\mathbb{N}} \subset D$ are sequences with $\lim_{n\to\infty} \|z_n\| = \infty$ and $\lim_{n\to\infty} w_n = \xi \in \partial D$, then $\lim_{n\to\infty} k_D(z_n, w_n) = \infty$.*

*Proof.* According to Theorem 7.6 at page 200 in [33] there is a complex affine isomorphism $A : \mathbb{C}^d \to \mathbb{C}^d$ such that $A(D) \subset \tilde{\mathbb{H}}^d$ , where $\tilde{\mathbb{H}} := \{z \in \mathbb{C} : \mathrm{Re}z < 0\}$. Moreover $\tilde{\mathbb{H}}^d$ is biholomorphic to $\Delta^d$ via Möbius transformations that act component-wise denoted as $\phi_i$ for $i = 1, \cdots, d$.
Then:

$$k_D(z_n, w_n) = k_{A(D)}(Az_n, Aw_n) \ge k_{\tilde{\mathbb{H}}^d}(Az_n, Aw_n) = \max_{i=1,\cdots,d} \omega(\phi_i(Az_n), \phi_i(Aw_n)).$$

We can observe that

$$z_n \not\to \xi \in \partial D \text{ as } n \to \infty$$

and, for every $i = 1, \cdots, d$, it holds

$$\lim_{n\to\infty} \phi_i(Aw_n) \in \partial\Delta.$$

Therefore $k_D(z_n, w_n) \to \infty$ as $n \to \infty$. □

*Remark* 31. The same result holds also when $\lim_{n\to\infty} z_n \in D$ since $\phi_i(Az_n) \in \Delta$ for every $i = 1, \cdots, d$.

**Lemma 4.1.2.** *Let $C \subset \mathbb{C}^d$ be a bounded convex domain and $z$ in $C$. If $(w_n)_{n\in\mathbb{N}} \subset D$ is a sequence with $\lim_{n\to\infty} w_n = \xi \in \partial D$, then $\lim_{n\to\infty} k_D(z, w_n) = \infty$.*

*Proof.* Since $C$ is bounded then there exists $r > 0$ so that $C \subset P(0, r)$, where $P(0, r)$ denotes the polydisc of radius $r$ in $\mathbb{C}^d$. Clearly $P(0, r)$ is biholomorphic to $\Delta^d$ and with the same argument of Lemma 4.1.1 we can conclude the proof. □

**Lemma 4.1.3.** *Let $D$ be a $\mathbb{C}$-proper convex domain in $\mathbb{C}^d$ and suppose that $(D, k_D)$ is Gromov hyperbolic. If $z_n, w_n \in D$ are sequences with $lim_{n\to\infty} z_n = \xi \in \partial D$ and*

$$\sup_{n\in\mathbb{N}} k_D(z_n, w_n) < +\infty\ ,$$

*then $w_n \to \xi$.*

*Proof.* Since $\overline{D}^{\star}$ is compact we can assume, up to subsequences, that $w_n \to \eta$ for some $\eta \in \overline{D}^{\star}$. As a consequence of Lemmas 4.1.1 and 4.1.2 we must have that $\eta \in \partial D$. Suppose for a contradiction that $\xi \neq \eta$.
Since every convex domain is also $\mathbb{C}$-convex and

$$\sup_{n\in\mathbb{N}} k_D(z_n, w_n) < +\infty\ ,$$

we can apply Proposition 3.5 at page 8 from [36]. If $L$ is the complex line containing $\xi$ and $\eta$, we then have that the interior of $\overline{D} \cap L$ in $L$ contains $\xi$ and $\eta$. In this way, due to $\mathbb{C}$-convexity, $\mathbb{C}$-properness and the Riemann Mapping Theorem, $\xi$ and $\eta$ are contained into an analytic disc inside $\partial D$.
On the other hand Theorem 3.1 at page 10 from [35] proves that no analytic disc can be contained in $\partial D$. This contradiction concludes the proof. □

**Lemma 4.1.4.** *Let $D$ be a $\mathbb{C}$-proper convex domain in $\mathbb{C}^d$ and suppose that $(D, k_D)$ is Gromov hyperbolic. If $\sigma : [0, +\infty) \to D$ is a geodesic ray, then $lim_{t\to\infty}\sigma(t)$ exists in $\overline{D}^{\star}$.*

*Proof.* Let $L \subset \overline{D}^\star$ denote the set of points $x \in \overline{D}^\star$ where there exists $t_n \to \infty$ such that $\sigma(t_n) \to x$. Suppose for a contradiction that $L$ is not a single point. By definition, it follows that $L$ is path connected and therefore connected. If we claimed that $L \cap \partial D$ had no points, since $E[\overline{D}]$ is totally disconnected, it would imply $L$ to be not connected.
As consequence $L$ contains at least one point in $\partial D$. Then, with the same connectedness argument, $L$ must contain at least two points in $\partial D$.
Then, by definition of $L$, we can find two sequences $a_n, b_n \to \infty$ and two distinct points $\xi, \eta \in \partial D$ such that $\sigma(a_n) \to \xi$ and $\sigma(b_n) \to \eta$.
However, by the definition of the Gromov boundary, if $t_n \to \infty$, then

$$\sigma(t_n) \overset{\text{Gromov}}{\longrightarrow} [\sigma] \, .$$

Now fix some $z_0 \in D$, by Lemma 3.2 at page 10 from [35], there exists some $A \geq 1$ such that the line segments $[z_0, \sigma(b_n)]$ are $(A, 0)$-quasi-geodesics.
Then by Lemma 3.2.3, there exists some $R > 0$ and $z_n \in [z_0, \sigma(b_n)]$ such that

$$k_D(z_n, \sigma(a_n)) \leq R < +\infty$$

for all $n$.
Thus

$$\sup_{n\in\mathbb{N}} k_D(z_n, \sigma(a_n)) \leq R \tag{4.1}$$

and, since $\sigma(a_n) \to \xi$, by Lemma 4.1.3 we have that $z_n \to \xi$.
On the other hand we have $\sigma(b_n) \to \eta$, $z_n \in [z_0, \sigma(b_n)]$, then the triangle inequality, 4.1 and Lemma 4.1.2 imply

$$\lim_{n\to\infty} k_D(z_n, z_0) \geq \lim_{n\to\infty} k_D(\sigma(a_n), \sigma(0)) - k_D(\sigma(0), z_0) - R = \infty$$

and this means that $z_n \to \eta$. Since $\eta \neq \xi$ we have a contradiction with the uniqueness of the limit for $z_n$. □

**Lemma 4.1.5.** *Let $D$ be a $\mathbb{C}$-proper convex domain in $\mathbb{C}^d$ and suppose that $(D, k_D)$ is Gromov hyperbolic. If $T_n \in (0, +\infty]$, $\sigma_n : [0, T_n) \to D$ is a sequence of geodesics, and $\sigma_n$ converges locally uniformly to a geodesic $\sigma : [0, +\infty) \to D$, then*

$$\lim_{t\to\infty} \sigma(t) = \lim_{n\to\infty} \lim_{t\to T_n} \sigma_n(t) \, .$$

*Proof.* The proof follows the same basic ideas we used for proving the previous Lemma. Since $\overline{D}^\star$ is compact, it is enough to consider the case when

$$\lim_{n\to\infty}\lim_{t\to T_n}\sigma_n(t)$$

exists in $\overline{D}^\star$. Let $\xi = \lim_{t\to\infty}\sigma(t) \in \overline{D}^\star$, $\xi_n = \lim_{t\to T_n}\sigma_n(t) \in \overline{D}^\star$ and

$$\xi_\infty = \lim_{n\to\infty}\lim_{t\to T_n}\sigma_n(t) \in \overline{D}^\star.$$

Suppose for a contradiction that $\xi \neq \xi_\infty$.
Since $\lim_{n\to\infty}\sigma_n(t) = \sigma(t)$ for every $t$, we can pick $a_n \to \infty$ such that $\sigma_n(a_n) \to \xi$. We can also pick $b_n \to \infty$ such that $\sigma_n(b_n) \to \xi_\infty$.
**Claim.** After possibly passing to a subsequence, there exists $c_n$, $a_n \leq c_n \leq b_n$ such that $\sigma_n(c_n)$ converges to $\eta \in \partial D$ and $\eta \neq \xi$.
*Proof of the Claim.*We define a distance $d$ on $\overline{D}^\star$ as follows:

$$d(x,y) = \begin{cases} 0\ ,\ \text{if } x = y \\ \|x-y\|\ ,\ \text{if } x,y \in \mathbb{C}^d \\ \infty\ ,\ \text{if } x \neq y \text{ and at least one of } x,y \text{ is an end.} \end{cases}$$

Since $\xi \neq \xi_\infty$, we can pick $c_n$, $a_n \leq c_n \leq b_n$ such that

$$\liminf_{n\to\infty} d(\sigma_n(a_n),\sigma_n(c_n)) > 0$$

$$\liminf_{n\to\infty} d(\sigma_n(b_n),\sigma_n(c_n)) > 0 \text{ and}$$

$$\limsup_{n\to\infty} \|c_n\| < \infty\ .$$

Such a sequence $c_n$ exists when at least one between $\xi, \xi_\infty$ is not an end. On the other side when $\xi$ and $\xi_\infty$ are two different ends for $D$, unbounded convex domain, thanks to Lemma 3.6.4 we have that all the conditions are satisfied as well.
Then we can pass to a subsequence such that $\sigma_n(c_n)$ converges to $\eta \in \partial D$ and $\eta \neq \xi$.

Now fix some $z_0 \in D$, by Lemma 3.2 at page 10 from [35], there exists some $A \geq 1$ such that the line segments $[z_0, \sigma(b_n)]$ are $(A,0)$-quasi-geodesics. Then by Lemma 3.2.3, there exists some $R > 0$ and $z_n \in [z_0, \sigma(b_n)]$ such that

$$k_D(z_n, \sigma(a_n)) \leq R < +\infty$$

for all $n$.

Thus

$$\sup_{n\in\mathbb{N}} k_D(z_n, \sigma(a_n)) \leq R \tag{4.2}$$

and, since $\sigma(a_n) \to \xi$, by Lemma 4.1.3 we have that $z_n \to \xi$.

On the other hand we have $\sigma(b_n) \to \eta$, $z_n \in [z_0, \sigma(b_n)]$, then the triangle inequality, 4.2 and Lemma 4.1.2 imply

$$\lim_{n\to\infty} k_D(z_n, z_0) \geq \lim_{n\to\infty} k_D(\sigma(a_n), \sigma(0)) - k_D(\sigma(0), z_0) - R = \infty$$

and this means that $z_n \to \eta$. Since $\eta \neq \xi$ we have a contradiction with the uniqueness of the limit for $z_n$. □

**Lemma 4.1.6.** *Let $D$ be a $\mathbb{C}$-proper convex domain in $\mathbb{C}^d$ and suppose that $(D, k_D)$ is Gromov hyperbolic. If $\sigma_1, \sigma_2 : [0,+\infty) \to D$ are geodesics, then*

$$\lim_{t\to\infty} \sigma_1(t) = \lim_{t\to\infty} \sigma_2(t)$$

*in $\overline{D}^{\star}$ if and only if $[\sigma_1] = [\sigma_2]$.*

*Proof.* First we prove the sufficient condition. Suppose that $[\sigma_1] = [\sigma_2]$ and let $\xi_j = \lim_{t\to\infty} \sigma_j(t)$ in $\overline{D}^{\star}$ for $j = 1, 2$. Since $\sigma_1$ and $\sigma_2$ are equivalent geodesic rays we have

$$\sup_{t\geq 0} k_D(\sigma_1(t), \sigma_2(t)) < +\infty. \tag{4.3}$$

Therefore Lemma 4.1.1 implies that $\xi_1 \in \mathbb{C}^d$ is equivalent to $\xi_2 \in \mathbb{C}^d$. Then, if $\xi_1 \notin \mathbb{C}^d$, Proposition 3.6.10 combined with 4.3 grants that $\xi_1 = \xi_2$. Otherwise, if $\xi_1 \in \mathbb{C}^d$, then Lemma 4.1.3 implies that $\xi_1 = \xi_2$. Thus in either case

$$\lim_{t\to\infty} \sigma_1(t) = \lim_{t\to\infty} \sigma_2(t)\ .$$

Conversely, suppose that

$$\lim_{t\to\infty}\sigma_1(t)=\lim_{t\to\infty}\sigma_2(t)=\xi\in\overline{D}^{\star}\,.$$

If $\xi\notin\mathbb{C}^d$, then Proposition 3.6.10 grants $[\sigma_1]=[\sigma_2]$.
Therefore we may assume that $\xi\in\mathbb{C}^d$. Fix $T>0$. We are going to bound $k_D(\sigma_1(T),\sigma_2(T))$ from above. Then fix some $z_0\in D$, by [ [35], page 10,Lemma 3.2] there exists some $A\geq 1$ such that the line segments $[z_0,\sigma_j(t)]$ are $(A,0)$-quasigeodesics for $j=1,2$. Then Shadowing Lemma 3.2.3 implies that there exists some $R>0$ such that: for every $t\geq T$, there exists $z_t\in[z_0,\sigma_1(t)]$ with

$$k_D(z_t,\sigma_1(T))\leq R. \tag{4.4}$$

Further, since

$$\lim_{t\to\infty}\sigma_1(t)=\lim_{t\to\infty}\sigma_2(t)$$

and, by 4.4,

$$k_D(z_t,\sigma_1(0))\leq k_D(z_t,\sigma_1(T))+k_D(\sigma_1(T),\sigma_1(0))\leq T+R$$

we then have that there exists $w_t\in[z_0,\sigma_2(t)]$ so that

$$\lim_{t\to\infty}k_D(w_t,z_t)=0\,.$$

At this point fix $t$ big enough such that

$$k_D(w_t,z_t)\leq 1\,. \tag{4.5}$$

Again, by Shadowing Lemma 3.2.3 there exists $S\in[0,t]$ so that

$$k_D(\sigma_2(S),w_t)\leq R\,. \tag{4.6}$$

Then, by 4.4,4.6 and 4.5 it holds

$$k_D(\sigma_1(T),\sigma_2(S))\leq k_D(,\sigma_2(S),w_t)+k_D(w_t,z_t)+k_D(z_t,\sigma_1(T))\leq 2R+1\,.$$

Since, by last inequality, square inequality and $\sigma_1,\sigma_2$ geodesics it holds

$$2R+1\geq k_D(\sigma_1(T),\sigma_2(S))\geq|k_D(\sigma_1(T),\sigma_1(0))-k_D(\sigma_2(0),\sigma_2(S))|-k_D(\sigma_1(0),\sigma_2(0))$$

$$= |T - S| - k_D(\sigma_1(0), \sigma_2(0)),$$

we have then

$$k_D(\sigma_1(T), \sigma_2(T)) \leq k_D(\sigma_1(T), \sigma_2(S)) + k_D(\sigma_2(S), \sigma_2(T))$$

$$= k_D(\sigma_1(T), \sigma_2(S)) + |T - S| = \leq (2R + 1) + \big(2R + 1 + k_D(\sigma_1(0), \sigma_2(0))\big).$$

Since $T > 0$ was chosen arbitrarely, we have

$$\sup_{t \geq 0} k_D(\sigma_1(t), \sigma_2(t)) < +\infty$$

which implies $[\sigma_1] = [\sigma_2]$, concluding the proof. □

**Theorem 4.1.7.** *Let $D$ be a $\mathbb{C}$-proper convex domain in $\mathbb{C}^d$.*
*If $(D, k_D)$ is Gromov hyperbolic, then the identity map $\boldsymbol{id} : D \to D$ extends to a homeomorphism $\overline{id} : \overline{D}^* \to \overline{D}^G$.*

*Proof.* Define $\Phi : \overline{D}^G \to \overline{D}^\star$ by $\Phi(z) = z$ when $z \in D$ and

$$\Phi([\sigma]) = \lim_{t \to \infty} \sigma(t)$$

when $[\sigma] \in \partial_G D$.
By Lemma 4.1.6, $\Phi$ is well defined and injective.
By Lemma 4.1.5 is continuous.
Since $D \subset \overline{D}^G$, we clearly have $\Phi(D) \subset \Phi(\overline{D}^G)$. By construction $\Phi(D) = D$, thus $D \subset \Phi(\overline{D}^G)$. Further $\Phi$ is continuous, $\overline{D}^G$ is compact and Hausdorff, then $\Phi(\overline{D}^G)$ is closed. On the other hand $D$ is dense in $\overline{D}^\star$, hence it follows that $\overline{D}^\star \subset \Phi(\overline{D}^G)$ and therefore $\Phi$ is surjective.
The map $\Phi$ being continuous, injective and surjective, between compact Hausdorff spaces, it is a homeomorphism. To conclude we define $\overline{id} : \overline{D}^\star \to \overline{D}^G$ as $\overline{id} = \Phi^{-1}$. □

## 4.2 Homeomorphic extension of Quasi-Isometries

We begin this section with the following

**Example 4.2.** Consider, as in [8] Example at page 133, the "solid torus" in $\mathbb{C}^2$ defined as

$$D_r = \{(z_1, z_2) \in \mathbb{C}^2 : \rho_r(z_1, z_2) = (|z_1| - 1)^2 + |z_2|^2 \leq r^2\}$$

which is strongly pseudoconvex for $r$ so that $0 < r < 1/2$.
Further, $D_r$ can be topologically contracted to $D_0 = \{(z_1, 0) \in \mathbb{C}^2 : |z_1| = 1\}$ via

$$(z_1, z_2) \mapsto \left(\frac{z_1}{|z_1|}(1-t) + tz_1, t^N z_2\right)$$

with $t \in [0, 1]$ and for a suitable $N \in \mathbb{N}$. Since $D_r$ is path connected, this shows that its fundamental group is $\mathbb{Z}$, and on the other hand the fundamental group of any convex set, which is contractible, is trivial.
Therefore $D_r$ is not even homeomorphic to any convex domain.

As showed in last example, convex and strongly pseudoconvex domains are not invariant under biholomorphisms, however when they are biholomorphic we can extend some regularities with the following

**Theorem 4.2.1.** *Let $D$ and $\Omega$ be domains in $C^d$. We assume:*

1. *$D$ is either a bounded, $C^2$-smooth strongly pseudoconvex domain, or a convex $\mathbb{C}$-proper domain, such that $(D, k_D)$ is Gromov Hyperbolic,*

2. *$\Omega$ is convex.*

*Then every quasi-isometric homeomorphism $F : (D, k_D) \to (\Omega, k_\Omega)$ extends as a homeomorphism $F : \overline{D}^\star \to \overline{\Omega}^\star$. In particular, every biholomorphism $F : D \to \Omega$ extends as a homeomorphism $F : \overline{D}^\star \to \overline{\Omega}^\star$.*

*Proof.* Since $F$ is a homeomorphism, it must be a proper map. Thus, since $(D, k_D)$ is a proper metric space, we see that $(\Omega, k_\Omega)$ is a proper metric space. So $\Omega$ is $\mathbb{C}$-proper by Theorem 1.4.3.
According to [31], if $D$ is bounded and $C^2$-smooth strongly pseudoconvex, then $(D, k_D)$ is Gromov hyperbolic. Then Theorem 4.1.7 implies that the identity map $id_D : D \to D$ extends to a homeomorphism $\overline{id} : \overline{D} \to \overline{D}^G$ (since

$\overline{D} = \overline{D}^{\star}$).

On the other hand, if $D$ is convex, then Theorem 4.1.7 implies that the identity map $id_D : D \to D$ extends to a homeomorphism $\overline{id}_D : \overline{D}^{\star} \to \overline{D}^G$.

Since $F : (D, k_D) \to (\Omega, k_\Omega)$ is a quasi-isometry, due to Proposition 3.2.4 $(\Omega, k_\Omega)$ is also Gromov hyperbolic. Then Theorem 4.1.7 implies that the identity map $id_\Omega : \Omega \to \Omega$ extends to a homeomorphism $\overline{id}_\Omega : \overline{\Omega}^{\star} \to \overline{\Omega}^G$.

Via this argument, we gained an homeomorphic extension for both cases in the first assumption on $D$.

Finally, since $F : (D, k_D) \to (\Omega, k_\Omega)$ is a quasi-isometry and $F$ is a homeomorphism, $F$ extends to a homeomorphism $\overline{F} : \overline{D}^G \to \overline{\Omega}^G$. (see [23], Proposition 14, page 128). Hence, F extends to the homeomorphism $(\overline{id}_\Omega)^{-1} \circ \overline{F} \circ (\overline{id}_D) : \overline{D}^{\star} \to \overline{\Omega}^{\star}$. □

Now we are going to see that the Theorem 4.2.1 does not hold with weaker assumptions.

**Example 4.3.** Let $D := \Delta \times \mathbb{C}$. Note that $D$ is convex, unbounded but not $\mathbb{C}$-proper. Consider the automorphism of $D$ given by $F(z, w) = (z, w + g(z))$, where $g : D \to \mathbb{C}$ is a holomorphic map which is continuous at no points of $\partial D$. Therefore $F$ does not extend continuously at any point of $\partial D$.

**Example 4.4.** Let $D = \Delta^2$. Note that $D$ is convex, $\mathbb{C}$-proper but $(D, k_D)$ is not Gromov hyperbolic due to Proposition 3.1.3. Pick points $z_n, w_n \in D$ with $z_n \to (1, 0)$, $w_n \to (1, 1/2)$. Note that, due to Corollary 1.2.9, it holds $k_D(z_n, w_n) \to \omega(1/2, 0)$ for $n \to \infty$ and therefore

$$R := \sup_{n \in \mathbb{N}} k_D(z_n, w_n) < \infty.$$

Then for each integer $n$, pick a small tubular neighbourhood $U_n$ of a geodesic joining $z_n$ to $w_n$. By shrinking each $U_n$ and passing to a subsequence we can assume that $\overline{U}_1, \overline{U}_2, \cdots$ are all disjoint and the $k_D$-diameter of each $U_n$ is less than $2R$. Now for each $n$ construct a homeomorphism $f_n : U_n \to U_n$ with $f_{|\partial U_n} = id$ where $f_n(z_n) = w_n$ and $f_n(w_n) = z_n$ if $n$ is odd and $f_n(z_n) = z_n$ and $f_n(w_n) = w_n$ if $n$ is even. Let $U = \cup_{n \geq 1} U_n$ and construct a map $f : D \to D$

where $f_{|D\setminus U} = id$ and $f_{|U_n} = f_n$. Then $f$ is a $(1, 2R)$-quasi-isometry; indeed by construction the $k_D$-diameter of each $U_n$ is bounded by $2R$, with $(U_n)_{n\in\mathbb{N}}$ invading $U$ and $f$ on $D \setminus U$, being the identity, is an isometry.
On the other hand by construction $f$ does not extend continuously to $\partial D$ since both $f(z_n)$ and $f(w_n)$ do not converge for $n \to \infty$.

**Example 4.5.** According to Theorem 1.8 at page 4 in [36] the convex domain

$$D = \{(z_0, z) \in \mathbb{C} \times \mathbb{C}^d : \mathrm{Im}(z_0) > \|z\|\}$$

is Gromov hyperbolic with respect to the Kobayashi distance $k_D$. By Subsection 1.3 in [36] the map $f : \mathbb{C}^{d+1} \setminus \{(-i, w') \in \mathbb{C}^{d+1} : w' \in \mathbb{C}^d\} \to \mathbb{C}^{d+1}$ defined as

$$f(z_0, \cdots, z_d) = \Big(\frac{1}{z_0 + i}, \frac{z_1}{z_0 + i}, \cdots, \frac{z_d}{z_0 + i}\Big).$$

induces a biholomorphism of $D$ onto a bounded $\mathbb{C}$-convex domain $\Omega := f(D)$. Indeed, $f$ is holomorphic, injective and therefore surjective on $f(D)$, by Osgood's Theorem $f^{-1}$ is also holomorphic. Moreover $f$ is the restriction of a projective automorphism $F : \mathbb{P}(\mathbb{C}^{d+2}) \to \mathbb{P}(\mathbb{C}^{d+2})$ and thus affine complex lines are sent into affine complex lines. Since $D$ is convex we have that $D$ is $\mathbb{C}$-convex, then let $\ell$ be any complex affine line and therefore it exists $\ell'$ so that $\ell = F(\ell')$. We recall that the continuous image of a connected set is connected and, since $f$ is injective, then $f(D \cap \ell')$ is also simply connected. In this way, by injectivity of $f$, we have $\Omega \cap \ell = f(D) \cap F(\ell') = f(D \cap \ell')$ and due to this $\Omega$ is connected and simply connected.
However $\Omega$ is not convex.
Since $f$ is a biholomorphism, by Proposition 3.2.4 we than have that $(\Omega, k_\Omega)$ is Gromov hyperbolic.
Further, $\Omega$ is bounded in each component where the set $\{0\} \times \{(z_1, \cdots, z_d) : \sum_i |z_i|^2 < 1\}$ is contained in $\partial\Omega$ and $f^{-1}$ maps this whole set to $\{\infty\}$ (in the end compactification of D). Hence, $f$ does not extend continuously to $\overline{D}^{\star}$.

## 4.3 Iteration theory for $\mathbb{C}$-convex and proper domains

In this last section we are going to show some important consequences for Theorem 4.1.7 and Theorem 4.2.1.
As a direct corollary to Karlsson's Theorem 3.4.2 and Theorem 4.1.7 we have the following:

**Corollary 4.3.1.** *Let $D \subset \mathbb{C}^d$ be a $\mathbb{C}$-proper convex domain such that $(D, k_D)$ is Gromov hyperbolic. If $f : D \to D$ is holomorphic, then either:*

1. *$f$ has a fixed point in $D$; or*
2. *there exists a point $\xi \in \overline{D}^{\star} \setminus D$, called the Denjoy-Wolff point of $f$, so that*
$$\lim_{n\to\infty} f^n(x) = \xi$$
*for any $x \in D$. Moreover this convergence is uniform on compact subsets of $D$. In particular, either $\xi \in \partial D$ and $\lim_{n\to\infty} f^n(x) = \xi$ or $\lim_{n\to\infty} \|f^n(x)\| = \infty$ for all $x \in D$.*

This result extends what Abate proved in [ [37], page 5, Theorem 0.5] for bounded $C^2$-smooth strongly pseudoconvex domains and it extends what Abate and Raissy proved in [ [38], pages 8-9 , Corollary 3.2 and Corollary 3.12] for bounded $C^2$-smooth strictly $\mathbb{C}$-linearly convex domains.

Now we will introduce the concept of retract with addition of a holomorphic structure with the following

**Definition 4.3.** Let $X$ be a domain of $\mathbb{C}^d$ and consider $Y \subset X$. We define $Y$ to be a *holomorphic retract* if it exists a holomorphic retraction $r : X \to Y$, i.e. a holomorphic map so that $r_{|Y} \equiv id_Y$.

This definition means that the identity function on $Y$ can be extended holomorphically to $X$. Further a comprehensive dissertation on holomorphic retracts of the unit polydisc can be found in [39].

In the end, as a direct consequence of Corollary 4.3.1 and Theorem 3.5.2, we have

**Corollary 4.3.2.** *Let $D$ be a $\mathbb{C}$-proper convex domain in $\mathbb{C}^d$ such that $(D, k_D)$ is Gromov hyperbolic and let $f, g$ be commuting holomorphic selfmaps for $D$. Suppose that $f$ and $g$ have no fixed points in $D$ and let $p_f \in \overline{D}^\star \setminus D$ (resp. $p_g \in \overline{D}^\star \setminus D$) be the Denjoy-Wolff point of $f$ (resp. of $g$). Then, either $p_f = p_g$ or there exists a holomorphic retract $M$ of $D$, of complex dimension $1 \leq k \leq d$, such that $p_f, p_g \in \overline{D}^\star \setminus D, f(M) = g(M) = M$ and $f_{|M}, g_{|M} \in Aut(M)$.*

# Chapter 5

# Conclusions and outline for further work

As we could see many results, such as Theorem 4.1.7 and Theorem 4.2.1, present a very strong assumption that lead us to the following question.
For which domains $D \subset \mathbb{C}^d$ is the Kobayashi distance $k_D$ a complete Gromov hyperbolic distance?
In the literature is well-known a characterisation for the group of biholomorphisms with the unit ball (see [4]). A good start could be to classify the quasi-isometries with the unit ball/polydisc or Reinhardt domains which are not biholomorphisms and then apply Proposition 3.2.4.
On the other hand we could find other Gromov hyperbolic metric spaces [see [40] , section 4, page 11] and then apply Karlsson's Theorem 3.4.2, or Commuting 1-Lipschitz selfmaps Theorems 3.5.1 and 3.5.2 to get new conclusions for Iteration Theory in these metric spaces.

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