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Black Hole Thermodynamics and Boundary Terms in Teleparallel Gravity

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Abstract

Teleparallel equivalent of General Relativity (TEGR) is an alternative theory of gravity that describes gravitational interactions only in terms of spacetime torsion instead of curvature. This theory can also be nicely formulated as a gauge theory for the translation group. TEGR, as the name suggests, is believed to be equivalent to General Relativity (GR) since their corresponding actions are equal up to a boundary term B, which does not contribute to the equations of motion.

Even though TEGR and GR are dynamically equivalent, boundary terms do affect black hole thermodynamics. For this reason, one could expect to obtain different results, if compared to GR, concerning BH thermodynamics in TEGR. However, at least at the leading order, this turns out not to be the case. Indeed, in this work we compute entropy and energy of the Teleparallel equivalent of a Schwarzschild BH, and we find that these values agree with those obtained in GR. Also, we construct the TEGR analog of the Landau-Lifshitz energy-momentum pseudo-tensor, from which we obtain the total conserved energy of Teleparallel Schwarzschild BH. This allows us to confirm the previous results, reinforcing the equivalence of black hole thermodynamics in GR and TEGR at the leading order.

Upon quantization, however, higher-order terms are expected to show up and spoil this equivalence. Thus, we study as well one-loop corrections to the partition function. To this purpose, we employed heat kernel methods to calculate the one-loop divergences of the effective action of a scalar field minimally coupled to gravity in TEGR. We find that these divergences are the same as what one obtains in GR. Then, we give some hints on the analogous calculations for the gravitational sector of the theory. In particular, we find out the second-order differential operator relevant for the heat kernel method and we present a simple argument that seems to indicate that the one-loop divergences of quantum TEGR are the same as in quantum GR too. However, there is a possible shortcoming concerning 1-loop corrections, discussed in the conclusions, since the needed counterterms are not expected to have all the symmetries of TEGR.

"L'occhio non vede cose ma figure di cose che significano altre cose."

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Chapter 1

Introduction

Boundaries and boundary terms are very important in many areas of physics. In classical electrostatic, for example, a fundamental problem is to find the arrangement of the potential in a given disposition of conductors. The study of this problem is equivalent to find the solution of the Laplace equation in a space with boundaries, which are given by the conductors' surfaces. Another well-known example is the Casimir effect in quantum electrodynamics. In the simplest situation, the Casimir effect, which essentially is quantum, consists of an attractive force proportional to a^{-4}, where a is the distance between two conducting parallel plates of characteristic length L such that $a \ll L$.

In this work, we focus on a further area in which boundary terms play a dominant role, namely BH thermodynamics from path integrals [1–4]. Let us briefly discuss why this is the case. In the path integral approach, the partition function at the leading order is given by the classical action, which gives the semiclassical approximation to the path integral. Studying BH thermodynamics, however, it is found that the action is usually divergent. The most common strategy to cure this divergence has been developed by Gibbons and Hawking in [3]. It consists of cutting the spacetime at some fixed distance r from the BH, add a suitable counterterm, and send r back to infinity. Thus, we have to work with a spacetime with boundary. In presence of such a boundary, we need to add to the familiar Einstein-Hilbert action a boundary term, called Gibbons-Hawking-York (GHY) boundary term. Concisely, this is because the Einstein-Hilbert action contains second-order derivatives of the metric and the presence of the GHY term in the action ensures the existence of a well-defined variational principle, i.e., a variation of this total action yields the correct Einstein equations. Consider now a Schwarzschild BH, the system that we will focus on during the thesis. For a Schwarzschild BH, Einstein equations set the Einstein-Hilbert action to zero and, then, the action is completely determined by the GHY term. Therefore, BH thermodynamics provides as well a very good physical reason to introduce the GHY term. Indeed, without considering it, we would obtain that the partition function is identically zero.

In this thesis, we deal with the thermodynamics of a Teleparallel Schwarzschild BH, i.e., the Teleparallel equivalent of a Schwarzschild BH. Teleperallel Equivalent of General Relativity (TEGR) is a gauge theory for the translation group [5–9]. Because of the properties of this gauge group, TEGR will differ in many ways from standard gauge theories. The main difference is the presence of a tetrad field, which, among other things, ensures the presence of torsion, which turns out to be the field strength of the theory. Indeed, TEGR describes gravity in terms of torsion instead of curvature. We stress also that in many theories of gravity including torsion (e.g., Einstein-Cartan theory) curvature and torsion represent different gravitational degrees of freedom. In TEGR, instead, they are related to the same degrees of freedom. We notice that this fact, among the torsional modified theories of gravity, points in favor of TEGR. Indeed, so far, there is no experimental evidence for new physics associated with torsion. Furthermore, from the point of view of TEGR, torsion does not geometrize gravitational interactions. In this theory, test particles do not follow geodesics, but force equations in which torsion appears in the right-hand side of the equation of motion of a free particle, similarly to the Lorentz force in electromagnetism. Thus, although torsion has a precise geometrical meaning, it seems irrelevant for the teleparallel description of the gravitational interactions. TEGR offers also some advantages if compared to GR. For example, in TEGR it is possible to split gravity and inertia. Indeed, in this theory gravity is described by a gauge potential while inertial effects are described by the spin connection. For this reason, as we will see, it seems possible in this theory to construct a genuine energy-momentum tensor for the gravitational field. Besides, TEGR can be formulated without the weak equivalence principle [6]. Since the equivalence and the uncertainty principles seem to be in contradiction, TEGR could provide a better framework than GR to deal with the inconsistencies between Quantum Mechanics and gravity.

In the TEGR community, Teleparallel Gravity and GR are usually considered to be *fully* equivalent because their actions differ by a boundary term B. This implies that the two theories are dynamically equivalent, i.e., they have the same equations of motion. However, little attention is often paid to the TEGR boundary term B, which, though, should have an impact on BH thermodynamics in the context of Teleparallel Gravity. Moreover, it seems probable, at least in principle, that the boundary term B could lead to a modification of BH entropy and energy, spoiling the equivalence between the two theories. For these reasons, it is worth studying BH thermodynamics in TEGR. Although somewhat unexpected, we have found that, at the semiclassical level, BH thermodynamics in TEGR and GR are precisely the same. This turned out to be, as we will show later, because the TEGR boundary term B is found to be equal to the GHY term. After quantization, however, higher-order terms show up in the partition function, which in principle could spoil this equivalence. This leads us to consider one-loop corrections to the path integral, which, physically, can be thought of as the contribution of thermal graviton and matter quanta to the free energy.

Quantum gravity is one of the major open problems of modern theoretical physics.

Many attempts have been made to overcome this problem but it is out of our scope to give a detailed review of all these efforts. For this work we are interested in a quite conservative approach to quantum gravity, that is to apply effective field theory (EFT) methods to gravity using the standard definition of effective action. In this framework, however, we did not construct the effective action (EA) from first principles. Instead we just employed heat kernel methods to study the divergent part of the one-loop EA [10–14]. Indeed, this is much easier since the heat kernel method, combined with the background field formalism, provides a simple algorithm to compute such divergences starting only with the knowledge of the quadratic action in the quantum fields.

The thesis is organized as follows. In Chapter 2, we give the basics of TEGR. In particular, we provide some fundamental theoretical tools as the tetrad formulation, Lorentz connection, and the definitions of its curvature and torsion. Then, we construct TEGR gauging translations, presenting covariant derivatives, the field strength, the construction of the TEGR Lagrangian, and the field equations. Thereafter, we show some known results in the TEGR literature that will be useful in the following chapters, like the TEGR gravitational coupling and a proposal for a genuine gravitational energy-momentum tensor.

Chapter 3 is devoted to recalling some general results which we will employ later. In particular, we recall the introduction of the GHY term and the computations of energy and entropy from the path integral for a Schwarzschild BH in GR. Next, we describe the basics of the heat kernel theory presenting the algorithm for the one-loop divergences for a quite general class of second-order differential operators. More in detail, the most general operator that we will consider has the form $-g_{\mu\nu}\nabla^{\mu}\nabla^{\nu} + \mathbf{E}$, where the covariant derivative contains both the Levi-Civita and the gauge connection and $\mathbf{E}$ is known as "potential term". This class of operators, which contains second-order derivatives in the form of a D'Alambertian, are known in the literature as minimal operators.

In Chapter 4, we present our results about BH thermodynamics in TEGR. In particular, we start showing that the TEGR boundary term B is equal to the GHY. Next, we present the TEGR analog of the calculations of energy and entropy of a Teleparallel Schwarzschild BH. Moreove, we discuss an alternative way to regularize the action which makes use of the TEGR spin connection. Thereafter, we show a different approach to obtain the same values of energy and entropy. Fundamentally, we use an analogy between the Landau-Lifshitz energy-momentum pseudo-tensor and the TEGR equations of motion to obtain the energy E as a conserved charge of a suitable current. Within this approach, we obtain for the energy the same value as before. Using our result for the energy E, we naïvely construct the first law of BH mechanics in the form $TdS = dE + \sigma dA$, where, following reference [6], $\sigma = -\partial E/\partial A$ is a "surface" pressure defined in analogy with the ordinary pressure. From this equation, then, we obtain the entropy, which turns out to be given again by the same value as in the previous case. Thus, this approach strengthens the GR-TEGR equivalence in BH thermodynamics at the leading order. The last part of the Chapter is devoted to one-loop corrections to the partition function. In particular, we will start by considering the divergences at one-loop coming from the quantization of a scalar field on a classical TEGR background, that we explicitly compute using the aforementioned algorithm. Interestingly, the divergences that we find are the same as of a quantized scalar field on a GR (curved) background. In the conclusions, however, we discuss the probable existence of a shortcoming concerning the equivalence between TEGR and GR in BH thermodynamics at one-loop. The point is that the necessary counterterms to regularize these divergences, which are the same as in GR, are not expected to share all the symmetries of TEGR. Later, we give some hints on the evaluation of the divergent part of the effective action of TEGR itself. More in detail, we present a simple argument suggesting that also the divergent part of the TEGR effective action at one-loop is precisely the same as in GR. However, since TEGR and GR have deep conceptual differences, it is highly desirable to explicitly obtain the one-loop divergences. For this reason, we as well obtain the second-order differential operator relevant to the heat kernel method. Unfortunately, using the Lorentz gauge, we arrive at a so-called non-minimal operator from which is much more difficult to obtain one-loop divergences. Due to of lack of time we leave this computation for future work.

Chapter 2

Teleparallel Gravity: gravity and torsion

Gravity is a peculiar force among the four known universal forces. Indeed, it is well-known that gravity is equally felt by particles, irrespective of their masses or composition. Physicists usually refer to this universality as the *universality of free falling*. This is a peculiar aspect of the gravitational interaction when compared to the other fundamental forces. This universality is what allowed Einstein to build up General Relativity (GR), a theory describing gravity through the curvature of the spacetime and where the fundamental variable is the metric of the spacetime. In this case, the underlying spacetime is a pseudo-Riemaniann space. In the GR description of gravity, we stress a couple of interesting points for this work. The first one is the very well-known fact that gravity in GR does not act as a force, but particles follow geodesics lines. The second one is that in GR inertial effects are geometrized together with the gravitational forces: they are both contained in the usual Christoffel connection and in general cannot be separated from each other. In this thesis, we will work with a modified theory of gravity: the Teleparallel Equivalent of General Relativity (TEGR). TEGR is a theory equivalent to GR, at least at the level of the equations of motion, which describe gravity in terms of torsion instead of curvature. In fact, in this context one assumes curvature to be vanishing. Now the pseudo-Riemaniann space is replaced with a so-called Weitzenböck spacetime. A Weitzenböck spacetime is essentially a differential manifold equipped with a Weitzenböck connection (defined below) and so it is a flat but twisted manifold. Although equivalent, to some extent, TEGR is conceptually quite different from GR. For example, in this theory gravity is no longer geometrized but it acts as a force. As we will see, geodesics are replaced with force equations where the role of force is played by the contortion tensor. TEGR offers also some advantages if compared to GR. For example, in TEGR it turns out that it is possible to split gravitational and inertial effects by means of an appropriate spin connection. This intriguing possibility will offer us a possible expression of the gravitational energy-momentum tensor. The presence of the spin connection is also crucial for the local Lorentz invariance of the theory. Finally, TEGR admits a nice formulation as a gauge theory of the translation group, which is the strategy that we will follow to construct TEGR. In this approach, the torsion tensor naturally arises as field strength of what will be called translational covariant derivative. This gauge formulation of gravity is an interesting fact, for example, for quantization purposes.

In this chapter, we begin by introducing some concepts as tetrads fields, which are the very starting point of TEGR, and Lorentz connections. Then, we very briefly recall some crucial aspects of gauge theories with some references to gravitational theories. We then construct the Teleparallel action and the field equations. Next, we present the gravitational coupling and particle mechanics. Finally, we discuss the possibility of constructing the energy-momentum tensor in TEGR. For this Chapter our main references are [5–9], which are those reported in the introduction.

2.1 Preliminary concepts

Two very basic objects that will be fundamental in the construction of Teleparallel Gravity are linear frames and tetrads. Linear frames and tetrad fields are constitutive parts of a differential manifold and they are always present as soon as a manifold is assumed to be differential. As it is well-known, at each point of a general spacetime, which is a pseudo-Riemannian manifold M, there is a tangent space T_xM with the same dimensionality of the manifold. This tangent space is seen as a vector space and it is identified with a Minkowski spacetime. We recall here that the (disjoint) union of all tangent spaces forms the so-called tangent bundle $TM = \bigcup_{x\in M} T_xM$, which will be the geometrical setting for TEGR.

Now, the basic idea is to introduce at each point of the spacetime a set of four orthonormal vectors, constituting a basis for the tangent space, and which transform covariantly under local Lorentz transformations.

2.1.1 Tetrad formulation

As said, the main idea is to introduce at each point of the spacetime a basis for the tangent space. Thus, we take four orthonormal vectors $\{e_a\}$, one time-like and three space-like, called linear frames

$$g(e_a, e_b) = \eta_{ab}, \tag{2.1}$$

which are usually defined only in restricted domains. The whole set of these four vector fields constitutes the bundle of linear frames, the prototype of principal bundles. Of course, we can introduce the dual basis (co-frame) $\{e^a\}$ for 1-forms, defined as usual by the relation

$$e^a(e_b) = \delta^a_b. \tag{2.2}$$

As always, the coordinates basis satisfy as well a similar relation, $dx^\nu(\partial_\mu) = \delta^\nu_\mu$. Consider now that, in their common definition domain, frames and co-frames can be expanded in terms of the coordinate basis $\{\partial_\mu\}$ and the dual $\{dx^\mu\}$ as

$$e_a = e_a{}^\mu \partial_\mu, \qquad e^a = e^a{}_\mu dx^\mu. \tag{2.3}$$

Conversely, we of course also have

$$\partial_\mu = e^a{}_\mu e_a, \qquad dx^\mu = e_a{}^\mu e^a. \tag{2.4}$$

From these relations, we easily find that (2.1) can be written as

$$g_{\mu\nu}\, e_a{}^\mu\, e_b{}^\nu = \eta_{ab}, \tag{2.5}$$

and using $dx^\nu(\partial_\mu) = e_a{}^\nu e^b{}_\mu\, e^a(e_b) = e^a{}_\mu e_a{}^\nu = \delta^\nu_\mu$ we obtain that the inverse relation of (2.5) is

$$g_{\mu\nu} = \eta_{ab}\, e^a{}_\mu\, e^b{}_\nu. \tag{2.6}$$

In addition, a general linear basis $\{e_a\}$ satisfies the commutation relations

$$[e_a, e_b] = f^c{}_{ab}\, e_c, \tag{2.7}$$

where the anholonomy coefficients, using the expansion of the tetrads in terms of the coordinate basis, are easily found to be given by

$$f^c{}_{ab} = e_a{}^\mu e_b{}^\nu (\partial_\nu e^c{}_\mu - \partial_\mu e^c{}_\nu). \tag{2.8}$$

The "matrices" $e^a{}_\mu$ and $e_a{}^\mu$ just introduced are called tetrads fields. Since these different kinds of indices can be confusing, we highlight that our convention is to use Latin indices to denote the "internal" (or Lorentz) indices $a, b = 0, \ldots, 3$, i.e., living in the internal space where the gauge transformations will act, while we use Greek indices to denote "external" (or spacetime) indices $\mu, \nu = 0, .., 3$. Lorentz indices are raised and lowered with the Minkowski metric while spacetime indices are raised and lowered with the spacetime metric [15].

The inverse relation (2.6) will be of great importance later on. It allows us to find the tetrads fields once that the spacetime metric is known. However, it is important to notice that the spacetime metric has 10 independent degrees of freedom while the tetrads have 16 degrees of freedom. Equation (2.6), in fact, determine the tetrads fields up to local Lorentz transformations $e^a{}_\mu(x) \to \Lambda^a_b(x) e^b{}_\mu(x)$: this is the natural freedom of choosing an orthonormal basis in Minkowski spacetime.

Tetrads fields give us the possibility to introduce intuitively the concept of spin connection, also called Lorentz connection. To do this we notice that tetrads fields, carrying them two kinds of indices, allow to define tensors with Latin and/or mixed indices simply contracting a spacetime tensor $\mathcal{T}$ with the tetrads. For example,

$$\mathcal{T}^{a_1...a_n}{}_{b_1...b_n} = e^{a_1}{}_{\alpha_1}...e^{a_n}{}_{\alpha_n}\, \mathcal{T}^{\alpha_1...\alpha_n}{}_{\beta_1...\beta_n}\, e_{b_1}{}^{\beta_1}...e_{b_n}{}^{\beta_n}. \tag{2.9}$$

Once that these quantities with a new type of indices have been introduced, it is natural to extend the notion of parallel transport also to them. For this reason, one can introduce some new connection coefficients $A^a{}_{b\mu}$, which undergo the name of spin (or Lorentz) connection. We can now introduce "full" covariant derivatives of objects carrying both kinds of indices. For example, we can define

$$\tilde{\nabla}_\mu \mathcal{T}^{a\nu} = \partial_\mu \mathcal{T}^{a\nu} + \Gamma^\nu{}_{\rho\mu}\mathcal{T}^{a\rho} + A^a{}_{b\mu}\mathcal{T}^{b\nu}. \tag{2.10}$$

More formally, a Lorentz connection, a particular case of linear connection, can be defined as a 1-form taking values in the Lorentz algebra

$$A_\mu = \frac{1}{2}A^{ab}{}_\mu S_{ab}, \tag{2.11}$$

where S_{ab} are the Lorentz generators in some representation. For instance, for a scalar field ϕ they have the form $S_{ab} = 0$ while for a spinor field ψ, in terms of the Dirac matrices γ_a, they are $S_{ab} = \frac{i}{4}[\gamma_a, \gamma_b]$. Of course, the Lorentz connection $A^a{}_{b\mu}$ transforms as a good connection under local Lorentz transformation $\Lambda^a{}_b = \Lambda^a{}_b(x)$

$$A^a{}_{b\mu} \to \Lambda^a{}_c A^c{}_{d\mu}(\Lambda^{-1})_b{}^d + \Lambda^a{}_c\partial_\mu(\Lambda^{-1})_b{}^c, \tag{2.12}$$

in such a way that the covariant derivative built using this connection is Lorentz-covariant. This covariant derivatives is known as Fock-Ivanenko covariant derivative and is defined as $\mathscr{D}_\mu = \partial_\mu - \frac{i}{2}A^{ab}{}_\mu S_{ab}$ [6].

If we choose to consider that objects like $\mathcal{T}^{a\nu}$ and $\mathcal{T}^{\mu\nu}$ are the same invariant entity in different disguises, it is natural to require that the two notions of parallel transport defined by $A^a{}_{b\mu}$ and by $\Gamma^\nu{}_{\rho\mu}$ actually coincide. Formally, we can ask that the tetrads fields, which change the type of indices, commute with taking the covariant derivative of a tensor. A way to impose this condition is to require that the "full" covariant derivative of the tetrad vanishes

$$\partial_\mu e^a{}_\nu + A^a{}_{b\mu}e^b{}_\nu - \Gamma^\rho{}_{\nu\mu}e^a{}_\rho = 0, \tag{2.13}$$

which is also known as *tetrad postulate.* Equation (2.13) can be solved for the general linear connection as

$$\Gamma^\rho{}_{\nu\mu} = e_a{}^\rho\partial_\mu e^a{}_\nu + e_a{}^\rho A^a{}_{b\mu}e^b{}_\nu = e_a{}^\rho\mathscr{D}_\mu e^a{}_\nu, \tag{2.14}$$

where $\mathscr{D}_\mu$ is the Fock-Ivanenko covariant derivative for a Lorentz vector. Alternatively, it can be solved for the spin connection as

$$A^a{}_{b\mu} = e^a{}_\nu\partial_\mu e_b{}^\nu + e^a{}_\nu\Gamma^\nu{}_{\rho\mu}e_b{}^\rho = e^a{}_\nu\nabla_\mu e_b{}^\nu, \tag{2.15}$$

where ∇_μ is the standard covariant derivative in terms of $\Gamma^\nu{}_{\rho\mu}$. From equations (2.15) and (2.14), which will be very useful for our work, we see that, under the tetrad postulate, to each spin connection $A^a{}_{b\mu}$ there is an associated general linear connection and vice-versa.

We conclude this paragraph recalling that a connection $\Gamma^\nu{}_{\rho\mu}$ is said to be metric compatible if the following holds

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\rho{}_{\lambda\mu} g_{\rho\nu} - \Gamma^\rho{}_{\nu\lambda} g_{\mu\rho} = 0. \tag{2.16}$$

The right-hand side, if not zero, is a rank-three tensor known as nonmetricity tensor. In tetrad formalism, and using (2.13), equation (2.16) can be rewritten as

$$\partial_\lambda \eta_{ab} - A^d{}_{a\mu} \eta_{db} - A^d{}_{b\mu} \eta_{ad} = 0, \tag{2.17}$$

which can be expressed as

$$A_{ba\mu} = -A_{ab\mu}. \tag{2.18}$$

From the previous equation, we conclude that the metricity condition holds as long as we work with a Lorentz connection. TEGR will belong to the class of theories with zero nonmetricity.

2.1.2 Curvature and torsion

Curvature and torsion are tensorial properties of Lorentz connections. Physicists are used to speaking about curvature of spacetime. However, working with connections presenting different curvatures and torsions, it seems more convenient to follow the mathematicians and consider connections as additional structures over a manifold.

Formally, the curvature of a Lorentz connection is defined as a 2-form assuming values in the Lie algebra of the Lorentz group

$$\mathbf{R} = \frac{1}{4} R^a{}_{b\mu\nu}\, S_a{}^b \; dx^\mu \wedge dx^\nu, \tag{2.19}$$

where the components of the curvature are given by

$$R^a{}_{b\nu\mu} = \partial_\nu A^a{}_{b\mu} - \partial_\mu A^a{}_{b\nu} + A^a{}_{e\nu} A^e{}_{b\mu} - A^a{}_{e\mu} A^e{}_{b\nu}. \tag{2.20}$$

The torsion tensor is also a 2-form but assuming values in the Lie algebra of the translation group

$$\mathbf{T} = \frac{1}{2} T^a{}_{\nu\mu}\, P_a \; dx^\mu \wedge dx^\nu, \tag{2.21}$$

where the components of $\mathbf{T}$ are given by

$$T^a{}_{\nu\mu} = \partial_\nu e^a{}_\mu - \partial_\mu e^a{}_\nu + A^a{}_{e\nu} e^e{}_\mu - A^a{}_{e\mu} e^e{}_\nu. \tag{2.22}$$

Through a contraction with the tetrads and using equation (2.15), the spacetime-indexed quantities are found to be given by the usual expressions

$$R^{\rho}{}_{\lambda\nu\mu} = e_a{}^{\rho} e^b{}_{\lambda} R^a{}_{b\nu\mu} = \partial_\nu \Gamma^{\rho}{}_{\lambda\mu} - \partial_\mu \Gamma^{\rho}{}_{\lambda\nu} + \Gamma^{\rho}{}_{\gamma\nu}\Gamma^{\gamma}{}_{\lambda\mu} - \Gamma^{\rho}{}_{\gamma\mu}\Gamma^{\gamma}{}_{\lambda\nu}, \tag{2.23}$$

and

$$T^{\rho}{}_{\nu\mu} = e_a{}^{\rho} T^a{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu}. \tag{2.24}$$

Curvature and torsion tensors can be written also in terms of Lorentz indices only. Further contracting with the tetrad fields and recalling that $A^a{}_{bc} = A^a{}_{b\nu} e_c{}^{\nu}$, it can be verified that

$$R^a{}_{bcd} = e_c(A^a{}_{bd}) - e_d(A^a{}_{bc}) + A^a{}_{ec} A^e{}_{bd} - A^a{}_{ed} A^e{}_{bc} - f^e{}_{cd} A^e{}_{be}, \tag{2.25}$$

and

$$T^a{}_{bc} = A^a{}_{cb} - A^a{}_{bc} - f^a{}_{bc}, \tag{2.26}$$

where $f^a{}_{bc}$ are the anholonomy coefficients of the basis $\{e_a\}$, which can be expressed as in (2.3). Using equation (2.26) for three different combinations of indices, we can express the spin connection as

$$A^a{}_{bc} = \overset{\circ}{A}{}^a{}_{bc} + K^a{}_{bc}, \tag{2.27}$$

in which

$$\overset{\circ}{A}{}^a{}_{bc} = \frac{1}{2}(f_b{}^c{}_a + f_c{}^b{}_a - f^a{}_{bc}) \tag{2.28}$$

is the spin connection of General Relativity (all the quantities of GR will have the empty circle above), and

$$K^a{}_{bc} = \frac{1}{2}(T_b{}^c{}_a + T_c{}^b{}_a - T^a{}_{bc}), \tag{2.29}$$

is the contortion tensor. Equation (2.27) is actually the content of a theorem and will be of major importance afterwards. Expressing (2.27) through the corresponding general linear connection, we have the well-known result

$$\Gamma^{\rho}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu} + K^{\rho}{}_{\mu\nu}, \tag{2.30}$$

where $\overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu}$ is the familiar Levi-Civita connection and $K^{\rho}{}_{\mu\nu}$ is the spacetime-indexed contortion tensor, which reads

$$K^{\rho}{}_{\mu\nu} = \frac{1}{2}(T_\nu{}^{\rho}{}_{\mu} + T_\mu{}^{\rho}{}_{\nu} - T^{\rho}{}_{\mu\nu}). \tag{2.31}$$

2.1.3 Local Lorentz transformations

In addition to diffeomorphism invariance, TEGR is invariant under local Lorentz transformations, and it is translational gauge invariant. We will consider gauge invariance later, in section 2.2. We now take into account the issue of local Lorentz invariance. Essentially, a local Lorentz transformation is a point-dependent Lorentz transformation taking place in the tangent space. On tangent space coordinates it has form

$$x'^a = \Lambda^a{}_b(x)x^b. \tag{2.32}$$

Under local Lorentz transformations, frames and co-frames, of course, will transform as

$$e'^a = \Lambda^a{}_b(x)e^b \qquad \text{and} \qquad e'{}_a = \Lambda_a{}^b(x)e_b. \tag{2.33}$$

We have seen from equation (2.6) that the knowledge of the spacetime metric does not completely fix the tetrads, but we are left with the freedom to perform local Lorentz transformations in the tangent space indices. Consider in fact a locally Lorentz rotated frames $\{e'{}_a\}$. Then, equation (2.6) becames

$$g_{\mu\nu} = \eta_{cd}e'^c{}_\mu e'^d{}_\nu. \tag{2.34}$$

Contracting both side of the previous equation with $e_a{}^\mu e_b{}^\nu$, we obtain that

$$\eta_{ab} = \eta_{cd}(e'^c{}_\mu e_a{}^\mu)(e'^d{}_\nu e_b{}^\nu). \tag{2.35}$$

We see that, if we introduce the matrix

$$\Lambda^a{}_b(x) = e'^a{}_\mu e_b{}^\mu, \tag{2.36}$$

then, equation (2.35) became the condition for (2.36) to belong to the Lorentz group. Therefore, we can say that, in the previous equation, $\Lambda^a{}_b(x)$ is a Lorentz matrix. Next, inverting equation (2.36), we obtain the transformation law for the tetrads under local Lorentz transformations

$$e'^a{}_\mu = \Lambda^a{}_b(x)e^b{}_\mu. \tag{2.37}$$

We have already noticed in equation (2.12) that spin connections transform as good connections. For readability, we report here this transformation law

$$A'^a{}_{b\mu} = \Lambda^a{}_c A^c{}_{d\mu}(\Lambda^{-1})_b{}^d + \Lambda^a{}_c\partial_\mu(\Lambda^{-1})_b{}^c. \tag{2.38}$$

Then, from their definitions, one can verify in a similar way [6] that $R^a{}_{b\nu\mu}$ and $T^a{}_{\nu\mu}$ transform covariantly under local transformations

$$R'^a{}_{b\nu\mu} = \Lambda^a{}_c(x)\Lambda_a{}^d(x)R^c{}_{d\nu\mu}, \tag{2.39}$$

and

$$T'^{a}{}_{\nu\mu} = \Lambda^{a}{}_{b}(x) T^{b}{}_{\nu\mu}. \tag{2.40}$$

This tells us that the spacetime-indexed quantities $R^{\rho}{}_{\beta\nu\mu}$, $T^{\rho}{}_{\nu\mu}$ and $\Gamma^{\rho}{}_{\mu\nu}$, just as the spacetime metric, are (local) Lorentz invariants. This fact will guarantee that the TEGR action is invariant under local Lorentz transformations. Indeed, this action will be constructed through the so-called torsion scalar.

2.1.4 Trivial and nontrivial frames

Trivial frames, or trivial tetrads, are frames that are not related to gravitation. Physically, they represent observers in special relativity; while, mathematically, constitutes the general linear basis on Minkowski spacetime. Thus, they only exist when gravity is absent. This class of frames will be denoted as $\{E_a\}$ and $\{E^a\}$, and they are related by the usual dual relation $E^a(E_b) = \delta^a_b$. They can as well be expanded in terms of coordinates basis as in equations (2.3). Moreover they satisfy the commutation relations (2.7), that now can be written as

$$[E_a, E_b] = f^{c}{}_{ab}\, E_c, \tag{2.41}$$

where the anholonomy coefficients are

$$f^{c}{}_{ab} = E_{a}{}^{\mu} E_{b}{}^{\nu} \left(\partial_\nu E^{c}{}_{\mu} - \partial_\mu E^{c}{}_{\nu} \right). \tag{2.42}$$

A basic property of these frames is that we can identify among them the special class of inertial frames $\{E'_a\}$, in which the anholonomy coefficients vanish

$$f'^{c}{}_{ab} = 0. \tag{2.43}$$

This condition is valid everywhere for the class of inertial frames. Namely, it is not a local property. As usual, frames characterized by this condition are called holonomic, similarly to coordinate basis.

Another property of trivial tetrads, holonomic or not, is that they relate the tangent Minkowski space to a Minkowski spacetime, i.e., usually Minkowski spacetime in general coordinates. Consider the Minkowski metric. In Cartesian coordinates, it takes the familiar diagonal form

$$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1), \tag{2.44}$$

but, of course, in any other coordinate system, it will be a function of the spacetime coordinates. Trivial tetrads fields will relate the tangent Minkowski η_{ab} to this spacetime Minkowski metric $\eta_{\mu\nu}$. Thus, equation (2.5), can be written as

$$\eta_{\mu\nu}\, E_{a}{}^{\mu}\, E_{b}{}^{\nu} = \eta_{ab}, \tag{2.45}$$

as can be easily verified. The inverse relation is instead found to be

$$\eta_{\mu\nu} = \eta_{ab}\, e^a{}_\mu\, e^b{}_\nu, \tag{2.46}$$

repeating the same steps as before. Let us present now the concept of nontrivial frames.

Nontrivial frames are defined as those frames whose anholonomy coefficients are related to both gravitational and inertial effects. In this work, they will be denoted as

$$\{h_a\} \qquad \text{and} \qquad \{h^a\}. \tag{2.47}$$

They satisfy relations analogous to equations (2.3), and to the dual relations as $h^a(h_b) = \delta^a_b$. Of course, also the nontrivial tetrad satisfy the commutation relations

$$[h_a, h_b] = f^c{}_{ab}\, h_c, \tag{2.48}$$

with anholonomy coefficients given by

$$f^c{}_{ab} = h_a{}^\mu h_b{}^\nu \left(\partial_\nu h^c{}_\mu - \partial_\mu h^c{}_\nu\right). \tag{2.49}$$

However, $f^c{}_{ab}$ are now related to both inertia and gravitation. In particular, it is not possible to *globally* set the anholonomy coefficients to zero, although it is still possible to have $f^c{}_{ab} = 0$ *locally*. The condition $f^c{}_{ab} = 0$ locally, means that we are working in a local frame in which inertial effects compensate gravitational effects. Moreover, nontrivial tetrads relate the spacetime metric and the Minkowski tangent space metric via the usual relation

$$g_{\mu\nu}\, h_a{}^\mu\, h_b{}^\nu = \eta_{ab}, \tag{2.50}$$

which can be seen as the orthogonality relation $g(h_a, h_b) = \eta_{ab}$, in terms of the tetrads fields. From the dual relation $h^a{}_\mu h_a{}^\nu = \delta^\mu_\nu$ we obtain the inverse relation

$$g_{\mu\nu} = \eta_{ab}\, h^a{}_\mu\, h^b{}_\nu. \tag{2.51}$$

From this equation, we see as well that the metric determinant

$$g = \det(g_{\mu\nu}) \tag{2.52}$$

is related to the determinant of the tetrads by the equation

$$h = \det\left(h^a{}_\mu\right) = \sqrt{-g}. \tag{2.53}$$

We will frequently use this notation in the following sections.

2.1.5 TEGR Lorentz connection and Weitzenböck connection

As we will see more explicitly in the next section, in TEGR, since it is a gauge theory, the gravitational field is represented by a suitable gauge potential and not by a (spacetime or Lorentz) connection. In particular, this means that in TEGR the spin connection keeps its special relativistic role of describing inertial effects only. In special relativity, Lorentz connections represent inertial effects present in non-inertial frames and they are vanishing in inertial frames. We can see how a Lorentz connection appears in special relativity starting from relation (2.45), and performing a local Lorentz transformation. Let us denote as $\{E'_a\}$ the class of inertial frames, i.e., those for which the anholonomy coefficients $f^c{}_{ab}$ are zero. From the expansion in terms of the coordinate basis (2.3), and using the Cartan structure equation $dE'^c = -\frac{1}{2} f^c{}_{ab} E'^a \wedge E'^b \equiv 0$, we can see that the tetrads of this class of frames can be represented in general coordinates as

$$E'^a{}_\mu = \partial_\mu x'^a, \tag{2.54}$$

where $x'^a = x'^a(x)$ is a spacetime dependent Lorentz vector. The relation between the spacetime metric and the tangent space metric is as usual

$$\eta'_{\mu\nu} = \eta_{ab}\, E'^a{}_\mu\, E'^b{}_\nu. \tag{2.55}$$

Now, under a local transformation, x'^a transform as a Lorentz vector

$$x^a = \Lambda^a{}_b(x) x'^b \tag{2.56}$$

and the holonomic frame $E'^a{}_\mu$ as

$$E^a{}_\mu = \Lambda^a{}_b(x) E'^b{}_\mu. \tag{2.57}$$

Using equation (2.54) and (2.56) it is then easy to show that the locally Lorentz rotated tetrad (2.57) takes the form

$$E^a{}_\mu = \partial_\mu x^a + \overset{\bullet}{A}{}^a{}_{b\mu} x^b = \overset{\bullet}{\mathscr{D}}_\mu x^a, \tag{2.58}$$

in which

$$\overset{\bullet}{A}{}^a{}_{b\mu} = \Lambda^a{}_e(x) \partial_\mu \Lambda_b{}^e(x) \tag{2.59}$$

is a Lorentz connection depending solely on the Lorentz transformations, and consequently it only represents inertial effects in the rotated frame. As such, we will refer to (2.59) as the purely inertial spin connection, which will turn out to be the TEGR spin connection. The purely inertial spin connection is simply the spin connection obtained by local Lorentz transformation of the vanishing spin connection $\overset{\bullet}{A}{}'^a{}_{b\mu} = 0$. This can be easily seen from the transformation law (2.12)

$$\overset{\bullet}{A}{}^a{}_{b\mu} = \Lambda^a{}_c(x) \overset{\bullet}{A}{}'^c{}_{d\mu} \Lambda_b{}^d(x) + \Lambda^a{}_c(x) \partial_\mu \Lambda_b{}^c(x). \tag{2.60}$$

If we start in a frame in which $\overset{\bullet}{A'}{}^a{}_{b\mu} = 0$, that is an inertial frame, different classes of frames are reached through local Lorentz transformations, while within any class all the infinite reference frames are related by global (point independent) Lorentz transformation. Conversely, starting from a general frame, we can always impose the condition $\overset{\bullet}{A'}{}^a{}_{b\mu} = 0$, known as Weitzenböck gauge, through a Lorentz transformation. Of course, this choice breaks the Lorentz invariance of TEGR because we have chosen a specific class of frames. However, this is not a deficit since it is a sort of gauge gauge fixing procedure in gauge theories [20] and Lorentz invariance can be restored by using the appropriate spin connection.

We have seen in equation (2.14) that to each spin connection corresponds a general linear connection, because of the tetrad postulate. The spacetime connection corresponding to the purely inertial spin connection is called Weitzenböck connection and, for a generic tetrad, it is given by

$$\overset{\bullet}{\Gamma}{}^{\rho}{}_{\nu\mu} = e_a{}^{\rho}\partial_\mu e^a{}_\nu + e_a{}^{\rho}\overset{\bullet}{A}{}^a{}_{b\mu}e^b{}_\nu = e_a{}^{\rho}\overset{\bullet}{\mathscr{D}}_\mu e^a{}_\nu. \tag{2.61}$$

Some comments are in order. To begin with, we recall that the previous equation is equivalent to assume the validity of the tetrad postulate (2.13), that for a nontrivial tetrad takes the form

$$\partial_\mu h^a{}_\nu + \overset{\bullet}{A}{}^a{}_{b\mu}h^b{}_\nu - \overset{\bullet}{\Gamma}{}^{\rho}{}_{\nu\mu}h^a{}_\rho = 0. \tag{2.62}$$

In the class of reference frames, in which the purely inertial spin connection $\overset{\bullet}{A}{}^a{}_{b\mu}$ is zero, it simply becomes

$$\partial_\mu h^a{}_\nu - \overset{\bullet}{\Gamma}{}^{\rho}{}_{\nu\mu}h^a{}_\rho = 0, \tag{2.63}$$

which is the teleparallel condition (or distant parallelism condition), from where Teleparallel Gravity takes its name. Of course, this condition holds only in this particular class of reference frames but, for historical reasons, we shall keep call it in that way.
Next, we notice from equations (2.22) and (2.20) that the curvature of a spin connection depends only on the spin connection, while the torsion depends on tetrad fields too. Of course, as a purely inertial spin connection, the curvature

$$\overset{\bullet}{R}{}^a{}_{b\nu\mu} = \partial_\nu \overset{\bullet}{A}{}^a{}_{b\mu} - \partial_\mu \overset{\bullet}{A}{}^a{}_{b\nu} + \overset{\bullet}{A}{}^a{}_{e\nu}\overset{\bullet}{A}{}^e{}_{b\mu} - \overset{\bullet}{A}{}^a{}_{e\mu}\overset{\bullet}{A}{}^e{}_{b\nu} = 0 \tag{2.64}$$

of the connection (2.59) is vanishing and, for a trivial tetrad, its torsion too

$$\overset{\bullet}{T}{}^a{}_{\nu\mu} = \partial_\nu E^a{}_\mu - \partial_\mu E^a{}_\nu + \overset{\bullet}{A}{}^a{}_{e\nu}E^e{}_\mu - \overset{\bullet}{A}{}^a{}_{e\mu}E^e{}_\nu = 0 \tag{2.65}$$

is zero. However, we anticipate that in the next section we will obtain that nontrivial tetrads can be written as a trivial tetrad plus the gauge potential. Thus, we will see that for a *nontrivial* tetrad the Weitzenböck torsion is non-vanishing

$$\overset{\bullet}{T}{}^a{}_{\nu\mu} = \partial_\nu h^a{}_\mu - \partial_\mu h^a{}_\nu + \overset{\bullet}{A}{}^a{}_{e\nu}h^e{}_\mu - \overset{\bullet}{A}{}^a{}_{e\mu}h^e{}_\nu \neq 0, \tag{2.66}$$

while curvature, depending only on the spin connection, is still vanishing. In TEGR the spacetime is flat but twisted. For reference, we mention that from equation (2.24) we immediately obtain the spacetime indexed version of Weintzenböck torsion (2.66)

$$\overset{\bullet}{T}{}^{\rho}{}_{\nu\mu} = \overset{\bullet}{\Gamma}{}^{\rho}{}_{\mu\nu} - \overset{\bullet}{\Gamma}{}^{\rho}{}_{\nu\mu}, \tag{2.67}$$

where $\overset{\bullet}{\Gamma}{}^{\rho}{}_{\mu\nu}$ is the Weitzenböck connection (2.61) computed using a *nontrivial* tetrad. This situation is the opposite of what happen in GR, whose spin connection present non-vanishing curvature

$$\overset{\circ}{R}{}^{a}{}_{b\nu\mu} = \partial_\nu \overset{\circ}{A}{}^{a}{}_{b\mu} - \partial_\mu \overset{\circ}{A}{}^{a}{}_{b\nu} + \overset{\circ}{A}{}^{a}{}_{e\nu} \overset{\circ}{A}{}^{e}{}_{b\mu} - \overset{\circ}{A}{}^{a}{}_{e\mu} \overset{\circ}{A}{}^{e}{}_{b\nu} \neq 0, \tag{2.68}$$

but vanishing torsion

$$\overset{\circ}{T}{}^{a}{}_{\nu\mu} = \partial_\nu h^{a}{}_{\mu} - \partial_\mu h^{a}{}_{\nu} + \overset{\circ}{A}{}^{a}{}_{e\nu} h^{e}{}_{\mu} - \overset{\circ}{A}{}^{a}{}_{e\mu} h^{e}{}_{\nu} = 0. \tag{2.69}$$

2.2 Fundamentals of Teleparallel Gravity

We are now ready to present the Teleparallel Equivalent of General Relativity (TEGR) or, for short, Teleparallel Gravity. We will construct this theory as a gauge theory for the translation group. In this section, we start by recalling, very briefly, some basics facts about (classical) gauge theories and we make at the same time some comments on the most relevant differences between standard gauge theories and TEGR. Then, following the usual framework of gauge theories, we introduce gauge transformations, gauge potentials and covariant derivatives, Lagrangian, and field equations.

2.2.1 Gravity and Gauge Theories

Let us begin with a short and rough reminder about gauge theories. All the details can be found in many textbooks of Quantum Field Theory (QFT). For physicists, the prototypes of gauge theories are Yang-Mills' theories, which have successfully described three out of the four fundamental forces. Fiber bundles are composite manifolds that encode all the geometrical aspects of gauge theories. They are composite since they are obtained by combining, in such a way to have a differential manifold, a base-manifold, that for us will be the spacetime, and another space of interest, like the gauge group or any space carrying one of its representations. This bundle is such that in a given point p of the base-manifold it is locally the direct product of both involved spaces. For example, the gauge principal bundle is constructed by attaching to each spacetime point the gauge group G itself. Thus, for a principal bundle, each fiber is a group. Let us now present some important properties of gauge bundles. The fiber bundle is equipped with a

"projection" π that takes all the points of the fiber in p into the corresponding base-space point p. Conversely, there is as well a "section" σ taking points in the neighbourhood of p into the bundle manifold.
Along with principal bundles, we also have the so-called "associated" bundles. The associated bundles are obtained by substituting the group G with one of its linear representations. Source fields live in the vector spaces carrying such representations and then are objects carrying gauge indices, on which the gauge group acts, and depending on the base space coordinates. Thus, they are subject to gauge transformations

$$\Psi'^{i}(x) = U^{i}{}_{j}(x)\Psi^{j}(x), \tag{2.70}$$

where $U^{i}{}_{j}(x)$ are the entries of a gauge group element $U(x^{\mu})$, representing the action of the gauge group in the point p of coordinates x^{μ}. In this notation, we have $i, j = 1, 2, ..., d$, where d is the dimension of the representation of the group. The gauge group element can be expressed through the exponential representation as

$$U^{i}{}_{j}(x) = \left\{ \exp\left[\epsilon^{b}(x)T_{b}\right] \right\}^{i}{}_{j}, \tag{2.71}$$

where T_b are the group generators in a given representation, $\epsilon^{b}(x)$ are the group parameters, where $b = 1, 2, ..., n$ and n is the dimension of the group. The generators T_b satisfy the commutation relations

$$[T_a, T_b] = f^{c}{}_{ab}T_c, \tag{2.72}$$

where, of course, $f^{c}{}_{ab}$ are the structure constants of the gauge group. We recall that the adjoint representation is defined by the fact that the generators, denoted by J_a, are $n \times n$ matrices with entries given by the structure constants

$$(J_a)^{c}{}_{b} = f^{c}{}_{ab}. \tag{2.73}$$

We can also express the gauge transformation (2.70) dropping the matrix indices as

$$\Psi'(x) = \exp\left[\epsilon^{b}(x)T_{b}\right] \Psi(x). \tag{2.74}$$

The corresponding infinitesimal transformation is obtained for $|\epsilon^{b}(x)| << 1$, and it reads

$$\delta\Psi(x) = \Psi'(x) - \Psi(x) = \epsilon^{b}(x)T_{b}\Psi(x). \tag{2.75}$$

Then, one can introduce the gauge boson field, a 1-form taking values in the Lie algebra of the gauge group, as

$$A_{\mu} = T_{c}A^{c}{}_{\mu}dx^{\mu}, \tag{2.76}$$

from which, starting from the general definition of covariant derivative

$$D_{\mu}\Psi(x) = \partial_{\mu}\Psi(x) - A^{b}{}_{\mu}\frac{\delta\Psi(x)}{\delta\epsilon^{b}(x)}, \tag{2.77}$$

and using the infinitesimal transformation (2.75), we arrive at the conclusion that the covariant derivative acting on matter fields is

$$D_\mu \Psi(x) = \partial_\mu \Psi(x) - A^b{}_\mu T_b \Psi(x), \tag{2.78}$$

where now the gauge potential takes values in the appropriate representation, concerning the matter field Ψ, of the gauge group. After imposing, as usual, a suitable transformation law on the gauge field, the covariant derivative (2.78) truly transforms in the same way as the field Ψ on which it acts. Basically, if Ψ transforms as in equation (2.70), its covariant derivative transforms in the same way

$$D'_\mu \Psi'(x) = U(x) D_\mu \Psi(x). \tag{2.79}$$

Then, imposing this condition we find that the gauge potential has to transform as

$$A'_\mu = U(x) A_\mu U^{-1}(x) + U(x) \partial_\mu U^{-1}(x). \tag{2.80}$$

The gauge potential, which is a connection, belongs to the adjoint representation, and so the generators in (2.80) are the structure constants. Then, the infinitesimal version of (2.80) is

$$\delta A^a{}_\mu = A'^a{}_\mu - A^a{}_\mu = - \left[\partial_\mu \epsilon^a(x) - f^a_{\ bc} A^b{}_\mu \epsilon^c(x) \right] = -D_\mu \epsilon^a(x), \tag{2.81}$$

where D_μ is the gauge covariant derivative in the adjoint representation.
The field strength of the theory is defined as the commutator of covariant derivatives

$$F_{\mu\nu}(x) = F^a{}_{\mu\nu}(x) T_a = \left[D_\mu, D_\nu \right], \tag{2.82}$$

where $F^a{}_{\mu\nu}$ is given by

$$F^a{}_{\mu\nu}(x) = \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu + f^a_{\ bc} A^b{}_\mu A^c{}_\nu. \tag{2.83}$$

Under gauge transformation, the field strength transforms as

$$F'_{\mu\nu} = U(x) F_{\mu\nu} U^{-1}(x), \tag{2.84}$$

that infinitesimally is

$$\delta F^a{}_{\mu\nu} = F'^a{}_{\mu\nu} - F^a{}_{\mu\nu} = f^a_{\ bc} \epsilon^b F^c{}_{\mu\nu}. \tag{2.85}$$

To end this brief reminder about gauge theories we recall that the Yang-Mills Lagrangian, from which we derive the dynamical equations for the gauge potential, is defined as

$$\mathcal{L} = \mathcal{L}_s - \frac{1}{4} \gamma_{ab} F^a{}_{\mu\nu} F^{b\mu\nu}, \tag{2.86}$$

where

$$\gamma_{ab} = \mathrm{Tr}\,(J_a J_b) = f^c{}_{ad} f^d{}_{bc} \tag{2.87}$$

is the Cartan-Killing metric, which is a metric only for semi-simple groups (notice that for the group $U(1)$, for example, it is not defined), since for non semi-simple groups γ_{ab} is degenerate. $\mathcal{L}_s = \mathcal{L}_s(\Psi, D_\mu \Psi)$ is the matter Lagrangian obtained from the free Lagrangian using the minimal coupling prescription: $\partial_\mu \to D_\mu$.

Let us now make some additional comments about TEGR as a gauge theory. GR presents some problematics as a gauge theory (as well as TEGR actually). Concerning GR, some of the more relevant problems are

- the fundamental field of a gauge theory is the gauge potential, while, in GR, there is a connection (Levi-Civita connection) but it is not a fundamental field: it is uniquely defined by the metric which is the true fundamental field.
- Usually, gauge Lagrangians are quadratic in the curvature while the GR action is linear in the curvature.
- Gauge interactions are mediated by forces, in GR a gravitational force is absent.

We can ask if there is a way to describe gravitation in an alternative fashion more closely related to a gauge theory. Let us take Electromagnetism (EM) as example. EM is a gauge theory for the group $U(1)$. In this theory, the source of the electromagnetic field is the electric current, which is the conserved current associated, through Noether's theorem, to the invariance of the Dirac Lagrangian under global $U(1)$ transformations. Then, to keep the Dirac Lagrangian symmetric under local $U(1)$ transformations, we introduce a connection (the gauge potential) assuming values in the Lie algebra of $U(1)$ and transforming in a suitable way. Analogously, the gravitational field source is the energy-momentum tensor. According to Noether's theorem, the energy-momentum tensor is the conserved current associated with the invariance of the Lagrangian under spacetime translations. Therefore, the idea of relating TEGR, or more generally gravity, to translations is very natural. However, as we will see briefly below, extending that link to a gauge theory of the translations group could be more subtle.

2.2.2 Gauge transformations

Let us now start to build up TEGR gauging translations. For TEGR the geometrical setting is the tangent bundle. Tangent bundles are constructed taking a (pseudo)Riemannian spacetime $\mathbb{R}^{3,1}$ as base-space and as fiber the Minkowski tangent space $\mathbb{M} = T_p\mathbb{R}^{3,1}$, in which gauge transformations take place. For TEGR a gauge transformation is a point-dependent translation of the $T_p\mathbb{R}^{3,1}$ coordinates

$$x'^a = x^a + \epsilon^a(x^\mu). \tag{2.88}$$

The infinitesimal version of this gauge transformation is

$$\delta x^a = \epsilon^b P_b x^a, \tag{2.89}$$

where, of course, P_b are the generators of translations

$$P_a = \frac{\partial}{\partial x^a} = \partial_a, \tag{2.90}$$

which obviously satisfy the commutation relations

$$[P_a, P_b] = 0. \tag{2.91}$$

The tangent bundle is more closely related to the spacetime structure than the usual gauge principal bundle. The main difference, as already mentioned, is the presence of soldering. Soldering ensures the presence of tetrad fields, indeed it is using tetrad fields that internal indices can be transformed into spacetime indices. We can notice from equation (2.22) that, essentially, the torsion tensor is the Lorentz covariant derivative of the tetrad fields. Thus, in usual non-soldered gauge theories, torsion is simply not existent.

Source fields Ψ are usually defined as local sections of the fiber bundle

$$\Psi_V \ : \ V \to \pi^{-1}(V) \sim V \times F, \tag{2.92}$$

where π is the bundle projection from the fiber into the spacetime, V is an open set of the spacetime, F is a fiber and $\sim$ indicates that $\pi^{-1}(V)$ is diffeomorphic to $V \times F$, a general property of tangent bundles known as local triviality. This diffeomorphism is called local trivialization and can be indicated as

$$\phi_V \ : \ \pi^{-1}(V) \to V \times F. \tag{2.93}$$

Using the local trivialization ϕ, the local section Ψ_V can be equivalently expressed as the application

$$x^\mu \to \phi_V^{-1}\left(x^\mu, x^a\right), \tag{2.94}$$

where the coordinate set $\{x^\mu\}$ indicate a point in spacetime and $\{x^a(x^\mu)\}$ a point in the fiber attached in the point x^μ. Then, a source field, is an object that by definition depends on both x^μ and x^a (here $a = 1, 2, ...d$, with d the dimension of the fiber)

$$\Psi = \Psi\left(x^\mu, x^a\right) \equiv \Psi^a\left(x^\mu\right). \tag{2.95}$$

The last equality holds for usual Yang-Mills-like gauge theories, where the fiber are finite-dimensional vector spaces (carrying unitary finite-dimensional representations of the gauge group) and, then, the source fields are finite multiplets in which the internal coordinate are the components. For TEGR, a fiber is a copy of the entire Minkowski spacetime, i.e., it is not finite. Then, in this case, the continuum coordinates $x^a(x^\mu)$ (we continue to use the notation x^a to emphasize that the tangent Minkowski space is the fiber) takes the role of the components of the multiplet and then the source field can be written as

$$\Psi = \Psi\left(x^a\left(x^\mu\right)\right). \tag{2.96}$$

After that, it is immediate to find that, under a gauge translation of the kind (2.88), Ψ transforms as

$$\delta\Psi\left(x^a\left(x^\mu\right)\right) = \epsilon^a\left(x^\mu\right)\partial_a\Psi\left(x^a\left(x^\mu\right)\right), \tag{2.97}$$

or more easily

$$\delta\Psi = \epsilon^a\partial_a\Psi. \tag{2.98}$$

Equation (2.97) gives the variation of the matter field at fixed x^a and x^μ: the typical gauge transformations. We conclude this subsection by saying that, for quantum mechanical reasons, the representation of the gauge group must be unitary. It is, however, well-known that only compact groups have finite unitary representations; for non-compact groups as translations, unitarity is jeopardized.

2.2.3 Translational covariant derivative

We have just seen that matter fields $\Psi = \Psi\left(x^a\left(x^\mu\right)\right)$ transform under a gauge transformation (2.88) as

$$\delta\Psi = \epsilon^a\partial_a\Psi. \tag{2.99}$$

Then, as usual, under a local translation $\epsilon^a = \epsilon^a\left(x^\mu\right)$ the ordinary derivative $\partial_\mu\Psi$ does not transform covariantly

$$\delta(\partial_\mu\Psi) = \epsilon^a\partial_a(\partial_\mu\Psi) + \left(\partial_\mu\epsilon^a\right)\partial_a\Psi. \tag{2.100}$$

To introduce a derivative that transforms covariantly under a gauge transformation, we have to introduce a gauge potential B_μ taking values in the Lie algebra of the gauge group, i.e., translations

$$B_\mu = B^a{}_\mu P_a. \tag{2.101}$$

We can then introduce the translational gauge covariant derivative h_μ

$$h_\mu\Psi = \partial_\mu\Psi + B^a{}_\mu\partial_a\Psi. \tag{2.102}$$

This derivative transform covariantly under gauge transformations

$$\delta(h_\mu\Psi) = \epsilon^a\partial_a(h_\mu\Psi), \tag{2.103}$$

if the gauge field satisfies the transformation law

$$\delta B^a{}_\mu = -\partial_\mu\epsilon^a. \tag{2.104}$$

Then, the translational coupling prescription is obtained by substituting ordinary with covariant ones

$$\partial_\mu \Psi \to h_\mu \Psi. \tag{2.105}$$

We notice that, because of the soldering property of the bundle, equation (2.102) can be rewritten as

$$h_\mu \Psi = h^a{}_\mu \partial_a \Psi, \tag{2.106}$$

where $h^a{}_\mu$ is a nontrivial tetrad field of form

$$h^a{}_\mu = \partial_\mu x^a + B^a{}_\mu. \tag{2.107}$$

Of course, to be nontrivial it has to satisfy

$$B^a{}_\mu \neq \partial_\mu \epsilon^a, \tag{2.108}$$

because otherwise $B^a{}_\mu$ would be only a gauge transformation of the nontrivial tetrad. Next, from equation (2.107) we can see that we are working in the class of reference frames in which the spin connection vanishes. Indeed, we recognize in $\partial_\mu x^a$ the trivial tetrad (2.54). Thus, to obtain the translational covariant derivative in a general frame, we proceed as before and we apply a local Lorentz transformation

$$x^a \to \Lambda^a{}_b(x) x^b, \tag{2.109}$$

under which the matter field transforms in the usual way

$$\Psi \to U(\Lambda)\Psi, \tag{2.110}$$

where $U(\Lambda)$ is a Lorentz transformation in the appropriate representation for Ψ. Taking into account that the gauge potential is a Lorentz vector in the tangent space index, that is it transforms as

$$B^a{}_\mu \to \Lambda^a{}_b(x) B^b{}_\mu, \tag{2.111}$$

one obtains [6] that the translational covariant derivative

$$h_\mu \Psi = h^a{}_\mu \partial_a \Psi, \tag{2.112}$$

is expressed using the tetrad field

$$h^a{}_\mu = \underbrace{\partial_\mu x^a + \overset{\bullet}{A}{}^a{}_{b\mu} x^b}_{=\overset{\bullet}{\mathscr{D}}_\mu x^a \equiv E^a{}_\mu} + B^a{}_\mu, \tag{2.113}$$

and where $\overset{\bullet}{A}{}^a{}_{b\mu}$ is given by

$$\overset{\bullet}{A}{}^a{}_{b\mu} = \Lambda^a{}_e(x) \partial_\mu \Lambda_b{}^e(x). \tag{2.114}$$

In the previous equation, we recognize the purely inertial spin connection (2.59). Thus, we can write the tetrad field more compactly as

$$h^a{}_\mu = E^a{}_\mu + B^a{}_\mu, \tag{2.115}$$

where $E^a{}_\mu$ is the trivial (that is non gravitational) part of the tetrad. In the class of reference frames in which the spin connection is non-vanishing the gauge transformation of the gauge potential is

$$\delta B^a{}_\mu = -\overset{\bullet}{\mathscr{D}}_\mu \epsilon^a, \tag{2.116}$$

instead of (2.104). Now it can be easily verified that the tetrad (2.113) is invariant under the gauge transformations $\delta x^a = \epsilon^b P_b x^a$ and $\delta B^a{}_\mu = -\overset{\bullet}{\mathscr{D}}_\mu \epsilon^a$:

$$\delta h^a{}_\mu = 0. \tag{2.117}$$

This important result ensures the translational gauge invariance of the torsion tensor and consequently of the TEGR Lagrangian.

2.2.4 Translational field strength and fundamental fields

Finally, we are ready to obtain the translational field strength: the torsion tensor. The translational field strength can be obtained with the usual recipe of gauge theories, that is taking the commutator of gauge covariant derives (2.82). From the translational gauge covariant derivative (2.112) just presented, it is easy to obtain

$$[h_\mu, h_\nu] = \overset{\bullet}{T}{}^a{}_{\mu\nu}(x) P_a, \tag{2.118}$$

where the translational field strength is

$$\overset{\bullet}{T}{}^a{}_{\mu\nu} = \partial_\mu B^a{}_\nu - \partial_\nu B^a{}_\mu + \overset{\bullet}{A}{}^a{}_{e\mu} B^e{}_\nu - \overset{\bullet}{A}{}^a{}_{e\nu} B^e{}_\mu = \overset{\bullet}{\mathscr{D}}_\mu B^a{}_\nu - \overset{\bullet}{\mathscr{D}}_\nu B^a{}_\mu. \tag{2.119}$$

As the notation already indicates, it is easy to show that the translational field strength is nothing else but torsion. To reach this result, we add to the right-hand side of (2.119) the vanishing piece

$$\overset{\bullet}{\mathscr{D}}_\mu \left(\overset{\bullet}{\mathscr{D}}_\nu x^a \right) - \overset{\bullet}{\mathscr{D}}_\nu \left(\overset{\bullet}{\mathscr{D}}_\mu x^a \right) = \left[\overset{\bullet}{\mathscr{D}}_\mu, \overset{\bullet}{\mathscr{D}}_\nu \right] x^a = 0. \tag{2.120}$$

Then, we obtain

$$\overset{\bullet}{T}{}^a{}_{\mu\nu} = \overset{\bullet}{\mathscr{D}}_\mu \left(\overset{\bullet}{\mathscr{D}}_\nu x^a + B^a{}_\nu \right) - \overset{\bullet}{\mathscr{D}}_\nu \left(\overset{\bullet}{\mathscr{D}}_\mu x^a + B^a{}_\mu \right). \tag{2.121}$$

In the previous equation, we recognize the nontrivial tetrad (2.113)

$$h^a{}_\mu = \overset{\bullet}{\mathscr{D}}_\mu x^a + B^a{}_\mu. \tag{2.122}$$

We arrive at the conclusion that the translational field strength is just the torsion tensor (2.66)

$$\overset{\bullet}{T}{}^a{}_{\mu\nu} = \overset{\bullet}{\mathscr{D}}_\mu h^a{}_\nu - \overset{\bullet}{\mathscr{D}}_\nu h^a{}_\mu. \tag{2.123}$$

We conclude this paragraph by stating that, since the tetrad field is invariant under gauge transformations (2.117), then also the torsion tensor is invariant under gauge transformations

$$\overset{\bullet}{T}{}'^a{}_{\mu\nu} = \overset{\bullet}{T}{}^a{}_{\mu\nu}. \tag{2.124}$$

This is a somewhat expected result since the translation group, as the group $U(1)$ of electrodynamics, is abelian.

In TEGR the gravitational field is fully represented by the gauge potential $B^a{}_\mu$, a 1-form taking values in the Lie algebra of the translation group

$$\mathbf{B} = B^a{}_\mu P_a dx^\mu. \tag{2.125}$$

Indeed, the fundamental Lorentz connection of TEGR is the purely inertial spin connection, as can be seen in (2.113), which depends only on Lorentz transformations. As already said, this means that in TEGR the spin connection describes inertial effects only, as in special relativity. Now we can as well better understand why (2.66) holds. Consider a nontrivial tetrad

$$h^a{}_\mu = e^a{}_\mu + B^a{}_\mu \quad \text{with} \quad B^a{}_\mu \neq \overset{\bullet}{\mathscr{D}}_\mu \epsilon^a \tag{2.126}$$

thus, the torsion tensor for a nontrivial tetrad will be, in general, non-vanishing

$$\begin{aligned} \overset{\bullet}{T}{}^a{}_{\mu\nu} &= \overset{\bullet}{\mathscr{D}}_\mu h^a{}_\nu - \overset{\bullet}{\mathscr{D}}_\nu h^a{}_\mu \\ &= \partial_\mu B^a{}_\nu - \partial_\nu B^a{}_\mu + \overset{\bullet}{A}{}^a{}_{e\mu} B^e{}_\nu - \overset{\bullet}{A}{}^a{}_{e\nu} B^e{}_\mu \neq 0. \end{aligned} \tag{2.127}$$

As repeatedly said, the fact that the fundamental Lorentz connection of TEGR is the purely inertial spin connection means that its curvature is always vanishing. This also tells us that the fundamental linear connection of TEGR is the Weitzenböck connection (2.61), in terms of which we can express the spacetime indexed torsion as in (2.67).

2.3 Lagrangian, field equations and equivalence with GR

As a gauge theory for translations, the action of TEGR can be written in the general form

$$\overset{\bullet}{\mathcal{S}} = \frac{c^3}{16\pi G} \int \eta_{ab} \overset{\bullet}{T}{}^a \wedge \star \overset{\bullet}{T}{}^b. \tag{2.128}$$

In this equation, $\overset{\bullet}{T}{}^a$ is the torsion 2-form

$$\overset{\bullet}{T}{}^a = \frac{1}{2}\overset{\bullet}{T}{}^a{}_{\mu\nu} dx^\mu \wedge dx^\nu \tag{2.129}$$

and $\star\overset{\bullet}{T}{}^b$ is the associated dual form

$$\star\,\overset{\bullet}{T}{}^a = \frac{1}{2}\left(\star\overset{\bullet}{T}{}^a{}_{\mu\nu}\right) dx^\mu \wedge dx^\nu, \tag{2.130}$$

defined using the generalized dual form for soldered bundles [6]

$$\star\,\overset{\bullet}{T}{}^a{}_{\mu\nu} = \frac{h}{2}\epsilon_{\mu\nu\alpha\beta} S^{a\alpha\beta}. \tag{2.131}$$

Finally, we recall that $h = \det\left(h^a{}_\mu\right)$. Consequently, the TEGR action (2.128) can be written as

$$\overset{\bullet}{\mathcal{S}} = \frac{1}{8ck}\int \overset{\bullet}{T}{}_{a\mu\nu}\left(\star\overset{\bullet}{T}{}^a{}_{\rho\sigma}\right) dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma, \tag{2.132}$$

where $k = 8\pi G/c^4$. Using then the definition of the generalized dual for soldered bundles as well as the identity

$$dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = -\epsilon^{\mu\nu\rho\sigma} h\, d^4x, \tag{2.133}$$

we can arrive to write the action functional (2.128) as

$$\overset{\bullet}{\mathcal{S}} = \frac{1}{4ck}\int \overset{\bullet}{T}{}^a{}_{\mu\nu}\,\overset{\bullet}{S}{}_a{}^{\mu\nu}\; h\, d^4x. \tag{2.134}$$

In this equation

$$\overset{\bullet}{S}{}_a{}^{\mu\nu} = -\overset{\bullet}{S}{}_a{}^{\nu\mu} = h_a{}^\rho\left(\overset{\bullet}{K}{}^{\mu\nu}{}_\rho - \delta_\rho{}^\nu\,\overset{\bullet}{T}{}^{\sigma\mu}{}_\sigma + \delta_\rho{}^\mu\,\overset{\bullet}{T}{}^{\sigma\nu}{}_\sigma\right) \tag{2.135}$$

is called *superpotential* and

$$\overset{\bullet}{K}{}^{\mu\nu}{}_\rho = \frac{1}{2}\left(\overset{\bullet}{T}{}^{\nu\mu}{}_\rho + \overset{\bullet}{T}{}_\rho{}^{\mu\nu} - \overset{\bullet}{T}{}^{\mu\nu}{}_\rho\right) \tag{2.136}$$

is the contortion tensor of the Weitzenböck torsion (2.67). From the TEGR action (2.134), we read the Teleparallel Lagrangian density

$$\overset{\bullet}{\mathcal{L}} = \frac{c^4\, h}{32\pi G}\overset{\bullet}{T}{}^a{}_{\mu\nu}\,\overset{\bullet}{S}{}_a{}^{\mu\nu} \equiv \frac{c^4\, h}{32\pi G}\overset{\bullet}{T}{}_{\rho\mu\nu}\,\overset{\bullet}{S}{}^{\rho\mu\nu}, \tag{2.137}$$

where the last equality is obtained through obvious contraction with the tetrads. Using now the identity

$$\overset{\bullet}{K}^{\mu}{}_{\rho\mu} = \overset{\bullet}{T}^{\mu}{}_{\mu\rho}, \tag{2.138}$$

the TEGR lagrangian (2.137) can be rewritten in terms of the contortion only as

$$\overset{\bullet}{\mathcal{L}} = \frac{c^4\, h}{17\pi G} \left(\overset{\bullet}{K}^{\mu\nu\rho} \overset{\bullet}{K}_{\rho\nu\mu} - \overset{\bullet}{K}^{\mu\rho}{}_{\mu} \overset{\bullet}{K}^{\nu}{}_{\rho\nu} \right). \tag{2.139}$$

Using the definition of the contortion tensor (2.136), it can as well be written in terms of the torsion tensor only

$$\overset{\bullet}{\mathcal{L}} = \frac{h}{16\pi} \left(\frac{1}{4}\, \overset{\bullet}{T}^{\rho}{}_{\mu\nu}\, \overset{\bullet}{T}_{\rho}{}^{\mu\nu} + \frac{1}{2}\, \overset{\bullet}{T}^{\rho}{}_{\mu\nu}\, \overset{\bullet}{T}^{\nu\mu}{}_{\rho} - \overset{\bullet}{T}^{\rho}{}_{\mu\rho}\, \overset{\bullet}{T}^{\nu\mu}{}_{\nu} \right). \tag{2.140}$$

Some comments are in order. We see, from the previous equation, that the first term is the usual Lagrangian for the gauge fields. The presence of the other two terms can be understood from the soldering character of the tangent bundle: the tetrad field allows internal and external indices to be treated on the same footing, and, consequently, new kinds of contractions become possible. Contracting with the tetrads, the Lagrangian can also be written in terms of Lorentz indices only as

$$\overset{\bullet}{\mathcal{L}} = \frac{h}{16\pi} \left(\frac{1}{4}\, \overset{\bullet}{T}^{a}{}_{bc}\, \overset{\bullet}{T}_{a}{}^{bc} + \frac{1}{2}\, \overset{\bullet}{T}^{a}{}_{bc}\, \overset{\bullet}{T}^{cb}{}_{a} - \overset{\bullet}{T}^{a}{}_{ba}\, \overset{\bullet}{T}^{cb}{}_{c} \right). \tag{2.141}$$

From this equation, we see that, since the torsion tensor transforms covariantly under local Lorentz transformations, each term of the Lagrangian is a Lorentz invariant independently from the values of the coefficients.

Consider now the Lagrangian

$$\mathcal{L} = \overset{\bullet}{\mathcal{L}} + \mathcal{L}_m, \tag{2.142}$$

where $\overset{\bullet}{\mathcal{L}}$ is the TEGR Lagrangian and $\mathcal{L}_m$ is a general matter (or source) Lagrangian. Varying the full Lagrangian 2.142 with respect to the tetrads fields, or equivalently with respect to the gauge field, it can obtained, after a lengthy calculation, the TEGR equations of motion for the gravitational field [6]

$$\partial_\nu \left(h\ \overset{\bullet}{S}_{a}{}^{\mu\nu} \right) - kh \overset{\bullet}{J}_{a}{}^{\mu} = kh\Theta_{a}{}^{\mu}. \tag{2.143}$$

In the previous equation, we have

$$h\ \overset{\bullet}{S}_{a}{}^{\mu\nu} = -k \frac{\partial \overset{\bullet}{\mathcal{L}}}{\partial(\partial_\nu h^{a}{}_{\mu})}, \tag{2.144}$$

where $\overset{\bullet}{S}_a{}^{\mu\nu}$ is the superpotential defined in equation (2.135), whereas

$$h\overset{\bullet}{J}_a{}^{\mu} = -\frac{\partial\overset{\bullet}{\mathcal{L}}}{\partial h^a{}_\mu} = \frac{h}{k}\, h_a{}^{\rho}\ \overset{\bullet}{S}_c{}^{\sigma\mu}\ \overset{\bullet}{T}{}^c{}_{\sigma\rho} - h_a{}^{\mu}\overset{\bullet}{\mathcal{L}} + \frac{h}{k}\ \overset{\bullet}{A}{}^c{}_{a\sigma}\ \overset{\bullet}{S}_c{}^{\mu\sigma} \tag{2.145}$$

is the analogue of the gauge current. The remaining term, which reads

$$h\Theta_a{}^{\rho} = -\frac{\delta\mathcal{L}_s}{\delta h^a{}_\rho} \equiv -\left(\frac{\partial\mathcal{L}_s}{\partial h^a{}_\rho} - \partial_\mu\frac{\partial\mathcal{L}_s}{\partial_\mu\partial h^a}\right), \tag{2.146}$$

it is instead the energy-momentum tensor for the matter field. We can also recast the equation of motion (2.143) using spacetime-indexed quantities. We do this starting from (2.143) and expressing $\partial_\rho h_a{}^{\lambda}$ using the Weitzenbock connection without the spin connection. In this way it can be obtained that

$$E_\mu^{\ \rho} \equiv \partial_\sigma\left(h\ \overset{\bullet}{S}_\mu{}^{\rho\sigma}\right) + kh\overset{\bullet}{t}_\mu{}^{\rho} = kh\Theta_\mu{}^{\rho}. \tag{2.147}$$

where

$$h\overset{\bullet}{t}_\mu{}^{\rho} = \frac{1}{k}h\ \overset{\bullet}{\Gamma}{}^{\alpha}{}_{\sigma\mu}\ \overset{\bullet}{S}_\alpha{}^{\sigma\rho} + \delta_\mu^\rho\overset{\bullet}{\mathcal{L}} \tag{2.148}$$

is the spacetime-indexed gravitational energy-momentum pseudo tensor [8].

It is now quite easy to see the equivalence between TEGR and GR. To this purpose we recall that any connection satisfying the metricity condition (2.16) can be decomposed as the sum of the Levi-Civita connection plus the contortion torsion (2.30). This, in particular, holds for the Weitzenböck connection

$$\overset{\bullet}{\Gamma}{}^{\rho}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu} + \overset{\bullet}{K}{}^{\rho}{}_{\mu\nu}, \tag{2.149}$$

where $\overset{\bullet}{K}{}^{\rho}{}_{\mu\nu}$ is the contortion tensor (2.31). If we insert this decomposition in the general formula defining the Riemann tensor

$$\overset{\bullet}{R}{}^{\rho}{}_{\lambda\nu\mu} = \partial_\nu\overset{\bullet}{\Gamma}{}^{\rho}{}_{\lambda\mu} - \partial_\mu\overset{\bullet}{\Gamma}{}^{\rho}{}_{\lambda\nu} + \overset{\bullet}{\Gamma}{}^{\rho}{}_{\gamma\nu}\overset{\bullet}{\Gamma}{}^{\gamma}{}_{\lambda\mu} - \overset{\bullet}{\Gamma}{}^{\rho}{}_{\gamma\mu}\overset{\bullet}{\Gamma}{}^{\gamma}{}_{\lambda\nu}, \tag{2.150}$$

it is only a matter of calculations to see that the curvature undergoes a similar splitting

$$\overset{\bullet}{R}{}^{\rho}{}_{\lambda\nu\mu} = \overset{\circ}{R}{}^{\rho}{}_{\lambda\nu\mu} + \overset{\bullet}{Q}{}^{\rho}{}_{\lambda\nu\mu}, \tag{2.151}$$

where

$$\overset{\circ}{R}{}^{\rho}{}_{\lambda\nu\mu} = \partial_\nu\overset{\circ}{\Gamma}{}^{\rho}{}_{\lambda\mu} - \partial_\mu\overset{\circ}{\Gamma}{}^{\rho}{}_{\lambda\nu} + \overset{\circ}{\Gamma}{}^{\rho}{}_{\gamma\nu}\overset{\circ}{\Gamma}{}^{\gamma}{}_{\lambda\mu} - \overset{\circ}{\Gamma}{}^{\rho}{}_{\gamma\mu}\overset{\circ}{\Gamma}{}^{\gamma}{}_{\lambda\nu} \tag{2.152}$$

is the Levi-Civita curvature, and $\overset{\bullet}{Q}{}^{\rho}{}_{\lambda\nu\mu}$ is given by

$$\begin{aligned}\overset{\bullet}{Q}{}^{\rho}{}_{\lambda\nu\mu} =&\partial_\nu \overset{\bullet}{K}{}^{\rho}{}_{\lambda\mu} - \partial_\mu \overset{\bullet}{K}{}^{\rho}{}_{\lambda\nu} + \overset{\bullet}{\Gamma}{}^{\rho}{}_{\sigma\nu}\overset{\bullet}{K}{}^{\sigma}{}_{\lambda\mu} - \overset{\bullet}{\Gamma}{}^{\rho}{}_{\sigma\mu}\overset{\bullet}{K}{}^{\sigma}{}_{\lambda\nu} \\ &- \overset{\bullet}{\Gamma}{}^{\sigma}{}_{\lambda\nu}\overset{\bullet}{K}{}^{\rho}{}_{\sigma\mu} + \overset{\bullet}{\Gamma}{}^{\sigma}{}_{\lambda\mu}\overset{\bullet}{K}{}^{\rho}{}_{\sigma\nu} + \overset{\bullet}{K}{}^{\rho}{}_{\sigma\mu}\overset{\bullet}{K}{}^{\sigma}{}_{\lambda\nu} - \overset{\bullet}{K}{}^{\rho}{}_{\sigma\nu}\overset{\bullet}{K}{}^{\sigma}{}_{\lambda\mu},\end{aligned} \tag{2.153}$$

which is a tensor written in terms of the Weitzenböck only. We have said many times that the curvature of the Weitzenböck connection vanishes $\overset{\bullet}{R}{}^{\rho}{}_{\lambda\nu\mu} = 0$. Using this condition in equation (2.151), it becomes

$$\overset{\bullet}{Q}{}^{\rho}{}_{\lambda\nu\mu} = -\overset{\circ}{R}{}^{\rho}{}_{\lambda\nu\mu}. \tag{2.154}$$

Taking the usual contractions to obtain the Ricci scalar, this equation yields the aforementioned TEGR-GR equivalence

$$\overset{\bullet}{Q} = \left(\overset{\bullet}{K}{}^{\mu\nu\rho}\overset{\bullet}{K}{}_{\rho\nu\mu} - \overset{\bullet}{K}{}^{\mu\rho}{}_{\mu}\overset{\bullet}{K}{}^{\nu}{}_{\rho\nu}\right) + \frac{2}{h}\partial_\mu\left(h\ \overset{\bullet}{T}{}^{\nu\mu}{}_{\nu}\right) = -\overset{\circ}{R}, \tag{2.155}$$

where $\overset{\circ}{R}$ stands for the Ricci scalar. Thus, taking into account the expression of the TEGR Lagrangian in terms of the contortion tensor only (2.139), it can be verified that

$$\overset{\bullet}{\mathcal{L}} \equiv \overset{\circ}{\mathcal{L}} - \partial_\mu\left(\frac{h}{8\pi}\ \overset{\bullet}{T}{}^{\nu\mu}{}_{\nu}\right), \qquad \overset{\circ}{\mathcal{L}} = -\frac{1}{16\pi}\sqrt{-g}\overset{\circ}{R}. \tag{2.156}$$

So, as claimed, the TEGR Lagrangian is the same as the Einstein-Hilbert up to a boundary term depending only on torsion

$$B = \partial_\mu\left(\frac{h}{8\pi}\ \overset{\bullet}{T}{}^{\nu\mu}{}_{\nu}\right). \tag{2.157}$$

We will refer to this boundary term as the TEGR boundary term and later it will be of central importance.

2.4 Gravitational coupling prescription and particle mechanics

We now tackle the problem of the gravitational coupling prescription. In TEGR exists a coupling prescription completely equivalent to the gravitational coupling in GR [6]. The TEGR coupling is composed out of two parts:

- Translational coupling prescription. It is universal and it comes from the requirement of translational gauge invariance.

- Lorentz coupling prescription. It is non-universal, since it depends on the matter content, and it comes from the requirement of local Lorentz invariance.

Let us begin with the translational coupling prescription. The minimal coupling prescription in gauge theories amounts to substitute ordinary derivatives with gauge covariant derivatives. Considering a generic matter field Ψ, it amounts to substitute

$$\partial_\mu \Psi \rightarrow h_\mu \Psi. \tag{2.158}$$

Where h_μ is the translational covariant derivative. Rewriting this equation as

$$E^a{}_\mu \partial_a \Psi \rightarrow h^a{}_\mu \partial_a \Psi, \tag{2.159}$$

we see that the minimal translational coupling is equivalent to substitute a trivial tetrad with a nontrivial tetrad

$$E^a{}_\mu \rightarrow h^a{}_\mu. \tag{2.160}$$

Fundamentally, this substitution is equivalent to the replacement

$$\eta_{\mu\nu} \rightarrow g_{\mu\nu}, \tag{2.161}$$

since $E^a{}_\mu$ and $h^a{}_\mu$ satisfy the relations $\eta_{ab}\, E^a{}_\mu\, E^b{}_\nu = \eta_{\mu\nu}$ and $\eta_{ab}\, h^a{}_\mu\, h^b{}_\nu = g_{\mu\nu}$, respectively. Let us now consider the Lorentz part of the coupling. The requirement of local Lorentz covariance introduces, even although it is not a dynamical symmetry, an additional coupling that, loosely speaking, amount to substitute ordinary with Lorentz covariant derivatives. The Lorentz coupling can be obtained from the general covariance principle [6], and it amounts to perform the substitution

$$\partial_a \Psi \rightarrow \mathscr{D}_a \Psi = h_a \Psi - \frac{i}{2} \left(A^{bc}{}_a - K^{bc}{}_a \right) S_{bc} \Psi. \tag{2.162}$$

Combining now the translational coupling prescription (2.159) and the Lorentz coupling prescription (2.162), and using the TEGR spin connection $\overset{\bullet}{A}{}^a{}_{bc}$ and its contortion, we arrive to the *full gravitational coupling prescription*

$$E^a{}_\mu \partial_a \Psi \rightarrow h^a{}_\mu \mathscr{D}_a \Psi = h^a{}_\mu \left[h_a \Psi - \frac{i}{2} \left(\overset{\bullet}{A}{}^{bc}{}_a - \overset{\bullet}{K}{}^{bc}{}_a \right) S_{bc} \Psi \right]. \tag{2.163}$$

It is immediate to see that the full gravitational coupling can be written also as

$$\partial_\mu \Psi \rightarrow \overset{\bullet}{\mathscr{D}}_\mu \Psi = \partial_\mu \Psi - \frac{i}{2} \left(\overset{\bullet}{A}{}^{bc}{}_\mu - \overset{\bullet}{K}{}^{bc}{}_\mu \right) S_{bc} \Psi. \tag{2.164}$$

Recall that for a Lorentz vector ϕ^d the Lorentz generators are $(S_{bc})^a{}_d = i(\delta^a_b \eta_{cd} - \delta^a_b \eta_{bd})$. Thus we have $S_{bc}\Psi \equiv (S_{bc})^a{}_d \phi^d$. Then, for example, the full gravitational coupling prescription for $\Psi \equiv \phi^d$ takes the form

$$\partial_\mu \phi^d \to \overset{\bullet}{\mathscr{D}}_\mu \phi^d = \partial_\mu \phi^d + \left(\overset{\bullet}{A}{}^d{}_{c\mu} - \overset{\bullet}{K}{}^d{}_{c\mu} \right) \phi^c. \tag{2.165}$$

The corresponding spacetime version is obtained contracting with the tetrads [6], and it reads

$$\partial_\mu \phi^\rho \to \overset{\bullet}{\nabla}_\mu \phi^\rho = \partial_\mu \phi^\rho + \left(\overset{\bullet}{\Gamma}{}^\rho{}_{\lambda\mu} - \overset{\bullet}{K}{}^\rho{}_{\lambda\mu} \right) \phi^\lambda. \tag{2.166}$$

We notice that, because of the relation (2.27)

$$\overset{\circ}{A}{}^a{}_{bc} = \overset{\bullet}{A}{}^a{}_{bc} - \overset{\bullet}{K}{}^a{}_{bc}, \tag{2.167}$$

the TEGR gravitational coupling is equivalent to the GR gravitational coupling

$$\overset{\bullet}{\mathscr{D}}_\mu \Psi = \overset{\circ}{\mathscr{D}}_\mu \Psi. \tag{2.168}$$

From the gravitational coupling, we can derive the equation of motion of test particles using a variational principle. We begin with the equation of motion of free particles. Then, using the coupling prescription, we obtain the equation of motion of gravitationally coupled test particles. We present now the equivalence with the geodesic motion. We begin with Minkowski spacetime, whose quadratic spacetime interval reads

$$d\sigma^2 = \eta_{\mu\nu} dx^\mu dx^\nu. \tag{2.169}$$

Recalling that the four-velocity is defined as $u_\mu = \frac{dx_\mu}{d\sigma}$, we can express the spacetime interval as

$$d\sigma = u_\mu dx^\mu = \left(E^a{}_\mu u_a \right) \left(E_c{}^\mu E^c \right) = u_a E^a, \tag{2.170}$$

where we have used that $E^a{}_\mu E_c{}^\mu = \delta^a_c$. Then, the action functional for a free particle of mass m is given by

$$S = -mc \int d\sigma = -mc \int u_a E^a. \tag{2.171}$$

Remembering that a trivial frame can be expressed as (cf. (2.58))

$$E^a = dx^a + \overset{\bullet}{A}{}^a{}_{b\mu} x^b dx^\mu, \tag{2.172}$$

our action functional is now written as

$$S = -mc \int u_a \left(dx^a + \overset{\bullet}{A}{}^a{}_{b\mu} x^b dx^\mu \right). \tag{2.173}$$

Taking the variation under spacetime coordinates change $x^\mu \to x^\mu + \delta x^\mu$, and using

$$\delta x^a = \partial_\mu x^a \delta x^\mu, \qquad \delta \overset{\bullet}{A}{}^a{}_{b\mu} = \partial_\rho \overset{\bullet}{A}{}^a{}_{b\mu} \delta x^\rho, \tag{2.174}$$

one can arrive at the variation [6]

$$\delta S = mc \int \left[E^a{}_\mu \left(\frac{du_a}{d\sigma} - \overset{\bullet}{A}{}^b{}_{a\rho} u_b u^\rho \right) \right] d\sigma \delta x^\mu. \tag{2.175}$$

Imposing $\delta S = 0$ and using the arbitrariness of δx^μ, we obtain the equation of motion for a free particle

$$\frac{du_a}{d\sigma} - \overset{\bullet}{A}{}^b{}_{a\rho} u_b u^\rho = 0. \tag{2.176}$$

We can easily obtain the equation describing the motion of particles interacting with gravity following the gravitational coupling prescription. In equation (2.171) we easily substitute the trivial tetrad with a nontrivial tetrad (since there are no derivatives). One obtains the action functional for a gravitationally interacting particle of mass m in the form

$$S = -mc \int ds = -mc \int u_a h^a, \tag{2.177}$$

where ds is the spacetime interval

$$ds = u_\mu dx^\mu = u_a h^a, \tag{2.178}$$

obtained from $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ with the same steps as before. Taking now the variation of (2.177) with respect to spacetime coordinates and using

$$\delta x^a = \partial_\mu x^a \delta x^\mu, \qquad \delta \overset{\bullet}{A}{}^a{}_{b\mu} = \partial_\rho \overset{\bullet}{A}{}^a{}_{b\mu} \delta x^\rho, \qquad \delta B^a_\mu = \partial_\rho B^a_\mu \delta x^\rho \tag{2.179}$$

one obtains

$$\delta S = mc \int \left[h^a{}_\mu \left(\frac{du_a}{ds} - \overset{\bullet}{A}{}^b{}_{a\rho} u_b u^\rho \right) - \overset{\bullet}{T}{}^b{}_{\mu\rho} u_b u^\rho \right] ds\, \delta x^\mu. \tag{2.180}$$

Then, the equation of motion is

$$\frac{du_a}{ds} - \overset{\bullet}{A}{}^b{}_{a\rho} u_b u^\rho = \overset{\bullet}{T}{}^b{}_{a\rho} u_b u^\rho. \tag{2.181}$$

Making use of the identity

$$\overset{\bullet}{T}{}^b{}_{a\rho} u_b u^\rho = -\overset{\bullet}{K}{}^b{}_{a\rho} u_b u^\rho, \tag{2.182}$$

the equation of motion for the gravitationally interacting particle becames

$$\frac{du_a}{ds} - \overset{\bullet}{A}{}^b{}_{a\rho} u_b u^\rho = -\overset{\bullet}{K}{}^b{}_{a\rho} u_b u^\rho, \tag{2.183}$$

while its contravariant form is easily found to be

$$\frac{du^a}{ds} + \overset{\bullet}{A}{}^a{}_{b\rho} u^b u^\rho = \overset{\bullet}{K}{}^a{}_{b\rho} u^b u^\rho. \tag{2.184}$$

These equations are the teleparallel version of the equations of motion for a particle of mass m in a gravitational field. As already claimed, they are force equations in which the contortion (or torsion) tensor takes the role of gravitational force. Now, from equation (2.184) is easy to see the equivalence with the geodesic equation. Contracting with the tetrads and using the tetrad postulate (2.13), it becames

$$\frac{du^\mu}{ds} + \overset{\bullet}{\Gamma}{}^\mu{}_{\rho\nu} u^\rho u^\nu = \overset{\bullet}{K}{}^\mu{}_{\rho\nu} u^\rho u^\nu, \tag{2.185}$$

and through the equation

$$\overset{\bullet}{\Gamma}{}^\mu{}_{\rho\nu} u^\rho u^\nu - \overset{\bullet}{K}{}^\mu{}_{\rho\nu} u^\rho u^\nu = \overset{\circ}{\Gamma}{}^\mu{}_{\rho\nu} u^\rho u^\nu \tag{2.186}$$

we find that (2.185) is exactly the usual geodesic equation.

2.5 Energy and momentum of the gravitational field

We close our introduction to TEGR by presenting one last interesting result. That is the opportunity to obtain in TEGR a true tensorial expression for the energy-momentum tensor for gravity. This intriguing result is due to the possibility of separating gravitational and inertial effects by the means of an appropriate spin connection. Indeed, gravitation is represented by the translational gauge potential, while inertia by the purely inertial spin connection. Now, the sourceless gravitational field equation of TEGR is (2.143)

$$\partial_\nu \left(h \; \overset{\bullet}{S}{}_a{}^{\mu\nu} \right) - kh \overset{\bullet}{J}{}_a{}^\mu = 0, \tag{2.187}$$

where, we recall for readability, that the superpotential is (2.135)

$$\overset{\bullet}{S}{}_a{}^{\mu\nu} = -\overset{\bullet}{S}{}_a{}^{\nu\mu} = \overset{\bullet}{K}{}^{\mu\nu}{}_a - h_a{}^\nu \overset{\bullet}{T}{}^{\sigma\mu}{}_\sigma + h_a{}^\mu \overset{\bullet}{T}{}^{\sigma\nu}{}_\sigma, \tag{2.188}$$

while the energy-momentum current is given by

$$\overset{\bullet}{J}{}_a{}^\mu = -\frac{1}{h}\frac{\partial \overset{\bullet}{\mathcal{L}}}{\partial h^a{}_\mu} = \frac{1}{k} \; h_a{}^\rho \; \overset{\bullet}{S}{}_c{}^{\sigma\mu} \; \overset{\bullet}{T}{}^c{}_{\sigma\rho} - \frac{h_a{}^\mu}{h} \overset{\bullet}{\mathcal{L}} + \frac{1}{k} \; \overset{\bullet}{A}{}^c{}_{a\sigma} \; \overset{\bullet}{S}{}_c{}^{\mu\sigma}. \tag{2.189}$$

The central point is that the term of the field equation containing the derivative of the superpotential and the last term of the expression for $\overset{\bullet}{J}{}_a{}^\mu$ (the one proportional to the spin connection) make up a Lorentz covariant derivative

$$\partial_\nu \left(h \overset{\bullet}{S}{}_a{}^{\mu\nu} \right) - \; \overset{\bullet}{A}{}^c{}_{a\sigma} \left(h \; \overset{\bullet}{S}{}_c{}^{\mu\sigma} \right) = \overset{\bullet}{\mathscr{D}}_\nu \left(h \overset{\bullet}{S}{}_a{}^{\mu\nu} \right), \tag{2.190}$$

that acts on the algebraic indices only. Consequently, we can re-express the field equations as

$$\overset{\bullet}{\mathscr{D}}_\nu \left(h \; \overset{\bullet}{S}_a{}^{\mu\nu} \right) - kh\overset{\bullet}{t}_a{}^{\mu} = 0, \tag{2.191}$$

where $\overset{\bullet}{t}_a{}^{\mu}$, absorbing the last term of equation (2.189) in the Lorentz covariant derivative $\overset{\bullet}{\mathscr{D}}_\nu$, is given by

$$\overset{\bullet}{t}_a{}^{\mu} = \frac{1}{k} \; h_a{}^{\rho} \; \overset{\bullet}{S}_c{}^{\sigma\mu} \; \overset{\bullet}{T}{}^{c}{}_{\sigma\rho} - \frac{h_a{}^{\mu}}{h} \overset{\bullet}{\mathcal{L}}, \tag{2.192}$$

which is a tensorial current. Taking now into account that

$$\left[\overset{\bullet}{\mathscr{D}}_\mu, \overset{\bullet}{\mathscr{D}}_\nu \right] = 0, \tag{2.193}$$

since the TEGR spin connection has vanishing curvature (differently to GR), we arrive to conclude that

$$\overset{\bullet}{\mathscr{D}}_\mu \left(h\overset{\bullet}{t}_a{}^{\mu} \right) = 0, \tag{2.194}$$

because of the antisymmetry of the superpotential $\overset{\bullet}{S}_a{}^{\mu\nu} = -\overset{\bullet}{S}_a{}^{\nu\mu}$. Then, we see that we can interpret $\overset{\bullet}{t}_a{}^{\mu}$ as the energy-momentum density of the gravitational field only. Basically we have split the non-covariant energy-momentum current as

$$\overset{\bullet}{J}_a{}^{\mu} = \overset{\bullet}{t}_a{}^{\mu} + \overset{\bullet}{I}_a{}^{\mu}, \tag{2.195}$$

where

$$\overset{\bullet}{I}_a{}^{\mu} = \frac{1}{k} \; \overset{\bullet}{A}{}^{c}{}_{a\sigma} \; \overset{\bullet}{S}_c{}^{\mu\sigma} \tag{2.196}$$

can now be interpreted as the energy-momentum density of inertial effects. We recall that the covariant conservation law (2.194) does not lead to a conserved quantity, while instead we see, once again because of the antisymmetry of the superpotential, that the *total* non-covariant energy-momentum current satisfies a true conservation law

$$\partial_\mu \left(h\overset{\bullet}{J}_a{}^{\mu} \right) = 0. \tag{2.197}$$

Thus, we can say that the usual expressions for the gravitational energy-momentum density are pseudotensors because they contain, in addition to the energy-momentum density of gravitation, a contribution coming from inertial effects. We conclude with an important observation for the following sections. If one makes use of the identity

$$\partial_\rho h = h \; \overset{\circ}{\Gamma}{}^{\nu}{}_{\nu\rho} = h \left(\; \overset{\bullet}{\Gamma}{}^{\nu}{}_{\rho\nu} - \; \overset{\bullet}{K}{}^{\nu}{}_{\rho\nu} \right), \tag{2.198}$$

the conservation law $\partial_\rho(h\overset{\bullet}{J}_a{}^\rho) = 0$ can be rewritten in the covariant form

$$\overset{\bullet}{D}_\rho \overset{\bullet}{J}_a{}^\rho \equiv \partial_\rho \overset{\bullet}{J}_a{}^\rho + \left(\overset{\bullet}{\Gamma}{}^\rho{}_{\lambda\rho} - \overset{\bullet}{K}{}^\rho{}_{\lambda\rho} \right) \overset{\bullet}{J}_a{}^\lambda = 0, \tag{2.199}$$

where $\overset{\bullet}{D}_\rho$ is the so-called teleparallel covariant derivative. Next, we notice that $\overset{\bullet}{t}_\mu{}^\rho$, equation (2.148), is not merely the gauge current $\overset{\bullet}{J}_a{}^\rho$ with the internal index changed to a spacetime index. Indeed, from the derivative term in equation (2.143), we get an extra piece

$$\overset{\bullet}{t}_\lambda{}^\rho = h^a{}_\lambda \overset{\bullet}{J}_a{}^\rho + k \ \overset{\bullet}{\Gamma}{}^\mu{}_{\lambda\nu} \ \overset{\bullet}{S}_\mu{}^{\rho\nu}. \tag{2.200}$$

We can as well notice that, like the gauge current $(h\overset{\bullet}{J}_a{}^\rho)$, also the pseudo tensor $(h\overset{\bullet}{t}_\mu{}^\rho)$ is conserved because of the field equations

$$\partial_\rho(h\overset{\bullet}{t}_\mu{}^\rho) = 0. \tag{2.201}$$

However, because of the pseudotensorial character of $\overset{\bullet}{t}_\mu{}^\rho$, this conservation law cannot be expressed with the Teleparallel covariant derivative. Moreover, it can be shown that it corresponds exactly to the Möller energy-momentum pseudotensor. Due to these features, we can say that the gauge current, which is a true spacetime and gauge tensor, it is an improved version of the older Möller expression [8] and so we will use $\overset{\bullet}{J}_a{}^\rho$ to define an energy-momentum vector.

Chapter 3

Black Hole Thermodynamics and path integrals in General Relativity

In this chapter, we introduce the theoretical tools that we will use to study BH thermodynamics in Teleparallel Gravity. We begin by presenting the calculation of the Schwarzschild BH entropy and energy from the Euclidean Path integral, mainly following [1–3]. After that, we give a tool to study one-loop corrections to the partition function, the heat kernel method. For the heat kernel method we will mostly follow [10,12,14].

3.1 Black Holes Thermodynamics and the Euclidean action in GR

The basic idea of the path integral approach to BH thermodynamics is to exploit the well-known formal analogy between the canonical partition function and the Euclidean path integral with periodic boundary conditions. We briefly recall this analogy using a scalar field. The partition function of a canonical ensemble at temperature $T = 1/\beta$ is obtained from its density matrix $\rho = e^{-\beta H}$ as

$$Z(\beta) = tr\, e^{-\beta H} = \sum_{\phi_1} \langle \phi_1 | e^{-\beta H} | \phi_1 \rangle \,. \tag{3.1}$$

The path integral computes instead the amplitude

$$\langle \phi_2, t_2 | \phi_1, t_1 \rangle = \langle \phi_2 | \, e^{-iH(t_2 - t_1)} \, | \phi_1 \rangle = \int D\phi \; e^{iS[\phi]}. \tag{3.2}$$

Now, let's perform a Wick's rotation setting $t_2 - t_1 = -i\beta$. We can notice that this means that we are imposing that the field is periodic in imaginary time with period β.

Then, we impose periodic boundary conditions for the field $\phi_1 = \phi_2$ and, summing over ϕ_1, we have

$$Z(\beta) = \int D\phi \; e^{-S_E[\phi]}. \tag{3.3}$$

In the previous equation, S_E is the Euclidean action and the Euclidean path integral is computed over fields with periodic boundary conditions and which are periodic in imaginary time with period β. Then, the partition function for a d-dimensional theory on the sphere S^{d-1}, that is the relevant case for a Schwarzschild BH, is the path integral on $S^{d-1} \times S^1$, where S^1 represent the periodically identified Euclidean time [1]. Including the metric in our set of fields, the Euclidean path integral is

$$Z = \int DgD\phi \; e^{-S_E[g,\phi]}, \tag{3.4}$$

As in the previous case, its meaning depends on boundary conditions. Then, for a 4-dimensional theory, one defines the partition function $Z(\beta)$ as the Euclidean path integral evaluated on a Euclidean manifold given by $S^3 \times S^1$, so that its boundary is given by a 2-sphere of radius $r = r_0 = \mathit{const}$ times a circle of circumference β (representing again the periodically identified imaginary time axis). Topologically, the boundary of the spacetime is, in this case, simply given by

$$S^2 \times S^1,$$

and so is compact. We approximate now the expression for $Z(\beta)$ by Taylor expanding S_E around the background fields (a solution of the classical equation of motion obeying the correct boundary condition) $\bar{g}$ and $\bar{\phi}$

$$Z(\beta) = \exp\left(-S_E[\bar{g}, \bar{\phi}]\right) \int DgD\phi e^{-(S_E)_2[\delta g, \delta\phi]+\dots}. \tag{3.5}$$

where $g = \bar{g} + \delta g, \phi = \bar{\phi} + \delta\phi$ and $(S_E)_2$ is quadratic in the perturbations $\delta g, \delta\phi$. The first term (i.e., the classical action) in the semiclassical approximation to the path integral it is the leading term. The second term gives 1-loop corrections, which will be considered later. Thus, at the leading order the free energy is given by the Euclidean action

$$\ln Z(\beta) \approx -S_E[\bar{g}, \bar{\phi}]. \tag{3.6}$$

Then, recalling that $\ln Z(\beta) = S - \beta E$, we have

$$S = (1 - \beta\partial_\beta) \ln Z(\beta), \qquad E = -\partial_\beta \ln Z(\beta). \tag{3.7}$$

Since our goal is to compute the Euclidean action for a Schwarzschild BH, we begin by introducing the Euclidean Schwarzschild BH, obtained from a Wick's rotation.

3.1.1 Euclidean Schwarzschild Black Hole

The Euclidean Schwarzschild black hole is obtained by performing a Wick's rotation $\tau = -it$ of the usual Lorentzian solution. Using Kruskal-Szekeres coordinates $\{T, X, \theta, \phi\}$, we have instead to define the Euclidean Kruskal time $\mathcal{T} = iT$. We are going to see that this implies that Euclidean BH has no interior and that τ takes values in a circle of length $8\pi M$. To see this, consider the relation between the Schwarzschild time and X, T

$$\frac{X+T}{X-T} = e^{\frac{t}{2M}}. \tag{3.8}$$

Defining the imaginary time by $\tau = it$ from this equation, it follows that, to avoid conical singularity on the horizon, τ has to be periodic with period $8\pi M$ [3]. This result can as well be derived as follow. We start with the Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2^2. \tag{3.9}$$

Then, we set $\left(1 - \frac{2M}{r}\right) = f(r)$, and we make the coordinates transformation

$$f(r) \approx \epsilon^2, \quad \text{or} \quad r = 2M(1 + \epsilon^2). \tag{3.10}$$

Expanding (3.9) for small ϵ, we obtain

$$ds^2 = \underbrace{-\epsilon^2 dt^2 + 16M^2 d\epsilon^2}_{\text{Rindler}} + 4M^2 d\Omega_2^2 + ... \tag{3.11}$$

where the $\{t, \epsilon\}$ piece is the Rindler metric $ds^2 = -R^2 d\eta^2 + dR^2$, after appropriate rescaling. Rindler metric is related to polar coordinates $ds^2 = dR^2 + R^2 d\theta^2$ through $\eta = i\theta$, in which θ is a periodic variable: $\theta \approx \theta + 2\pi$. Therefore, we take η to be periodic in the imaginary direction

$$\eta \sim \eta + 2\pi i. \tag{3.12}$$

Looking now at equation (3.11), which relates the vicinity of the black hole horizon to Rindler space, we conclude that the Schwarzschild time has periodicity

$$t \sim t + 8i\pi M. \tag{3.13}$$

Recall now the relation between the Schwarzschild radius and X, T

$$\left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}} = X^2 - T^2. \tag{3.14}$$

If we now set $\mathcal{T} = iT$, this equation becames

$$\left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}} = X^2 + \mathcal{T}^2 > 0 \tag{3.15}$$

from which we see that the interior of the BH $r < 2M$ is not covered by the Euclidean coordinates $\{X, \mathcal{T}\}$. In other words, we can say that, in the Euclidean section, r must be equal or greater than $2M$. With these results at hand, we define the Euclidean Schwarzschild BH as

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2^2, \tag{3.16}$$

with the Euclidean time τ periodic with period $8\pi M$

$$\tau \sim \tau + 8\pi M. \tag{3.17}$$

Moreover, Euclidean BH has no interior or singularity. Therefore, we recover the well-known result $T_H = 8\pi M$ for the Hawing temperature of a Schwarzschild BH, which is usually obtained studying QFT on curved background [19]. We conclude this section by noticing that the Euclidean BH has just the correct boundary conditions to obtain the partition function from the path integral.

3.1.2 The Gibbons-Hawking-York term

Before arriving at the explicit calculation of the Euclidean action, we have to address one last problem. We have said that the Euclidean path integral will give the partition function if we compute it on a manifold with topology $S^{d-1} \times S^1$. To tackle divergences in the partition function for the Schwarzschild BH, the best strategy is to cut the spacetime at some large r, and at the end of the calculations send $r \to \infty$. Besides a needed regularization procedure developed in [3] that we will consider later, we have to recall that in presence of a spacetime boundary, which for us is $S^2 \times S^1$, we have to add another piece to the usual Einstein-Hilbert Lagrangian, called the Gibbons-Hawking-York term (GHY). The GHY is required to have a well-defined variational principle, i.e., to obtain the correct Einstein equations. Alongside this somewhat "formal" need of the GHY, we have as well a very good physical reason to add it, concerning BH thermodynamics from path integrals. Indeed, the Einstein-Hilbert action for a Schwarzschild BH is zero and, consequently, the full contribution to the classical action comes from GHY. Without adding it one would obtain a vanishing partition function. We now briefly recall in some more details how the GHY term is introduced. To do that, let's start by computing the variation of the GR action with respect to the metric. It is a standard computation [15] to show that

$$\begin{aligned}(16\pi)\, \delta S_{EH} &= \int_M d^4x\, \delta\left(\sqrt{-g}\; g^{\alpha\beta} \overset{\circ}{R}_{\alpha\beta}\right) \\ &= \int_M d^4x\, \overset{\circ}{G}_{\alpha\beta} \sqrt{-g} \delta g^{\alpha\beta} + \oint_{\partial M} \delta \overset{\circ}{A}{}^{\mu} n_\mu \sqrt{|h|} d^3y,\end{aligned} \tag{3.18}$$

where

$$\delta \overset{\circ}{A}^{\mu} = g^{\alpha\beta}\delta \overset{\circ}{\Gamma}{}^{\mu}{}_{\alpha\beta} - g^{\alpha\mu}\delta \overset{\circ}{\Gamma}{}^{\beta}{}_{\alpha\beta}, \tag{3.19}$$

in which $\overset{\circ}{\Gamma}{}^{\mu}{}_{\alpha\beta}$ is the Levi-Civita connection. We recall that, in our notation, quantities with the empty circle above are computed with respect to the Levi-Civita connection. To obtain this equation, we have used

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta} \tag{3.20}$$

$$\delta \overset{\circ}{R}_{\alpha\beta} = \overset{\circ}{\nabla}_{\mu}\left(\delta \overset{\circ}{\Gamma}{}^{\mu}{}_{\alpha\beta}\right) - \overset{\circ}{\nabla}_{\beta}\left(\delta \overset{\circ}{\Gamma}{}^{\mu}{}_{\alpha\mu}\right). \tag{3.21}$$

The second term in equation (3.18) is exactly the one which is cancelled by the GHY term. It can be checked by an explicit calculation that if we define the boundary term

$$S_{GHY} = \oint_{\partial M} K\sqrt{|h|}d^3y, \tag{3.22}$$

where ∂M is the spacetime boundary and $K = n^{\alpha}{}_{;\alpha}$ is the trace of the extrinsic curvature of the boudary, then the variation of equation (3.22) cancel out the boundary term in (3.18). As a result, we arrive to the following equation

$$\delta S_{tot} = \delta S_{EH} + \delta S_{GHY} = \frac{1}{16\pi}\int_M d^4x\, \overset{\circ}{G}_{\alpha\beta}\sqrt{-g}\delta g^{\alpha\beta}. \tag{3.23}$$

In the previous equation

$$S_{tot} = S_{EH} + S_{GHY} = \frac{1}{16\pi}\int_M d^4\sqrt{-g}\overset{\circ}{R} + \oint_{\partial M} K\sqrt{|h|}d^3y \tag{3.24}$$

is the total for the gravitational field in presence of a spacetime boundary.

3.1.3 BH Thermodynamics at the leading order in GR

Now we can finally evaluate the action. Of course, by Einstein equations, for a Schwarzschild BH we have $R = 0$. Because of this, the full contribution to the Euclidean version of (3.24) comes from the boundary term. The Schwarzschild Euclidean action can be written as [3]

$$S_E = -\frac{1}{16\pi}\int_M d^4x\sqrt{g}R - \frac{1}{8\pi}\int_{\partial M} d^3y\sqrt{h}K. \tag{3.25}$$

Evaluating now the GHY term on the 2-sphere $\times$ S^1, we obtain

$$\frac{1}{8\pi}\int_{\partial M} d^3y\sqrt{h}K = \frac{1}{8\pi}\left[8\pi\beta r_0 - 12\pi\beta M\right]. \tag{3.26}$$

To regularize this result (it clearly explode sending $r_0 \to \infty$), we consider a counter term that lead to the Euclidean action

$$S_E = -\frac{1}{16\pi}\int_M d^4x\sqrt{g}R - \frac{1}{8\pi}\int_{\partial M} d^3y\sqrt{h}K + \frac{1}{8\pi}\int_{\partial M} d^3y\sqrt{h}K_0, \tag{3.27}$$

where K_0 is given by the extrinsic curvature for a flat space-time with the same boundary ∂M, which for Schwarzschild is $S^2 \times S^1$. We notice that the counterterm depends only on the induced metric h on the boundary. To compute the counterterm, we repeat the previous calculation using the flat metric

$$ds^2_{subtraction} = \left(1 - \frac{2M}{r_0}\right) d\tau^2 + dr^2 + r^2 d\Omega_2^2. \tag{3.28}$$

This leads to the counterterm [1]

$$\frac{1}{8\pi}\int_{\partial M} d^3y\sqrt{h}K_0 = \frac{1}{8\pi}\left[8\pi\beta r_0 - 8\pi\beta M + \beta\, O(1/r_0)\right]. \tag{3.29}$$

Considering this counterterm, which regulates the divergence, we arrive at the following formula for the Euclidean action

$$S_E = \frac{\beta\, M}{2} = 4\pi M^2. \tag{3.30}$$

The partition function at leading order is then

$$Z(\beta) = \exp\left(-4\pi M^2\right) = \exp\left(-\frac{\beta^2}{16\pi}\right). \tag{3.31}$$

From this result, we easily find from standard thermodynamics that

$$\begin{aligned} S &= (1 - \beta\partial_\beta)\ln Z(\beta) = 4\pi M^2 \\ E &= -\partial_\beta \ln Z(\beta) = M. \end{aligned} \tag{3.32}$$

We notice that S is in agreement with the "area law" of BH thermodynamics, $S = \text{Area}/4$ and that, physically, the last result means that the energy of a Schwarzschild BH entirely comes from its rest mass.

3.2 One-loop Euclidean Effective Action and partition function

Now we see a general method to study one-loop corrections to the Euclidean path integral. We start recalling very briefly the general expression for the one-loop effective action [11].

To this purpose, we consider for simplicity a scalar field $\phi(x)$. In the Euclidean theory the generating functional is given by

$$Z[J] = \int D\phi \exp\left(-\frac{1}{\hbar}S[\phi] - \int d^4x J(x)\phi(x)\right) = \exp\left(-\frac{1}{\hbar}E[J]\right), \tag{3.33}$$

where E is the generating functional of the connected components of the n-points correlation functions. In particular, the mean field $\Phi(x)$ in presence of the source J is given by

$$\Phi(x) = -\frac{\delta E[J]}{\delta J(x)} = \frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J(x)}. \tag{3.34}$$

Assuming that this equation is invertible, we denote $J[\Phi]$ as the source which corresponds to the mean field configuration $\Phi(x)$. We use this expression to define the effective action $\Gamma[\Phi]$ as the Legendre transform of $E[J]$

$$\Gamma[\Phi] = E[J] - J \cdot \Phi, \tag{3.35}$$

where we have used the notation

$$J \cdot \Phi = \int d^4x J(x)\Phi(x). \tag{3.36}$$

The basic property of the effective action is that it is the generating functional of 1-particle-irreducible Green's functions. Differentiating with respect to the field Φ the definition of the effective action, we obtain

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi} = -J[\Phi](x). \tag{3.37}$$

A useful expression of the effective action can be obtained combining equations (3.33), (3.35) and equation (3.37). It reads

$$\exp\left[-\frac{1}{\hbar}\Gamma[\Phi]\right] = \int D\phi \exp\left[-\frac{1}{\hbar}\left(S - \int d^4x \frac{\delta\Gamma[\Phi]}{\delta\Phi}(\phi - \Phi)\right)\right]. \tag{3.38}$$

In addition, we recall that the effective action admit a formal expansion in powers of $\hbar$ [21]

$$\Gamma = \sum_{n=0}^{\infty} \hbar^n \Gamma_{(n)} = S + \hbar\Gamma_{(1)} + O\left(\hbar^2\right), \tag{3.39}$$

where $\Gamma_{(n)}$ is the n-loop contribution to Γ and so $S = \Gamma_{(0)}$ is the classical action and $\Gamma_{(1)}$ the one-loop contribution. Using the loop expansion (3.39) in equation (3.38) we can find an expression for the one-loop effective action. Consider indeed the splitting

$$\phi = \Phi + \sqrt{\hbar}\varphi. \tag{3.40}$$

Consequently, the classical action can be expanded as

$$S[\Phi + \sqrt{\hbar}\varphi] = S[\Phi] + \sum_{n=1}^{\infty} \frac{\hbar^{n/2}}{n!} \int dx_1 \ldots dx_n S_n(x_1 \ldots x_n; \Phi)\varphi(x_1)\ldots\varphi(x_n), \tag{3.41}$$

where

$$S_n(x_1 \ldots x_n; \Phi) = \left.\frac{\delta^n S}{\delta\phi(x_1)\ldots\delta\phi(x_n)}\right|_{\Phi}. \tag{3.42}$$

To further simplify the following expressions, we introduce the notations

$$\int dx_1 \ldots dx_n S_n(x_1 \ldots x_n; \Phi)\varphi(x_1)\ldots\varphi(x_n) \equiv S_n(\Phi)\varphi^n, \tag{3.43}$$

and

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi} \equiv \Gamma_1[\Phi] \qquad \int dx\varphi(x)\frac{\delta\Gamma[\Phi]}{\delta\Phi} \equiv \varphi\Gamma_1[\Phi]. \tag{3.44}$$

Considering the notations just introduced and using (3.41) and (3.40), equation (3.38) can be written as

$$\begin{aligned}\exp\left[-\frac{1}{\hbar}\Gamma[\Phi]\right] = \int D\varphi \exp\bigg[- \frac{1}{\hbar}\bigg(S[\Phi] + \frac{\hbar}{2}S_2\varphi^2 + \sum_{n=3}^{\infty}\frac{\hbar^{n/2}}{n!}S_n[\Phi]\varphi^n + \\ - \sqrt{\hbar}\varphi(\Gamma_1[\Phi] - S_1[\Phi])\bigg)\bigg].\end{aligned} \tag{3.45}$$

With some very simple passages, the previous equation can be written as

$$\begin{aligned}\exp\left[-\frac{1}{\hbar}(\Gamma[\Phi] - S[\Phi])\right] = \int D\varphi \exp\bigg[- \frac{1}{2}S_2[\Phi]\varphi^2 - \sum_{n=3}^{\infty}\frac{\hbar^{n/2-1}}{n!}S_n[\Phi]\varphi^n + \\ + \hbar^{-1/2}\varphi(\Gamma_1[\Phi] - S_1[\Phi])\bigg],\end{aligned} \tag{3.46}$$

where we see that the effective action appear only in the combination $\bar{\Gamma}[\Phi] \equiv \Gamma[\Phi] - S[\Phi]$. Thus, from the loop expansion (3.39), we immediately get $\bar{\Gamma}[\Phi] = \sum_{n=1}^{\infty} \hbar^n \Gamma_{(n)}$, which then represent the quantum corrections to the classical action. Inserting the expansion of $\bar{\Gamma}[\Phi]$ in equation (3.46), we can write it in the form

$$\begin{aligned}\exp\left[-\sum_{n=1}^{\infty}\hbar^{n-1}\Gamma_{(n)}\right] = \int D\varphi \exp\bigg[- \frac{1}{2}S_2[\Phi]\varphi^2 - \sum_{n=3}^{\infty}\frac{\hbar^{\frac{n}{2}-1}}{n!}S_n[\Phi]\varphi^n + \\ + \sum_{n=1}^{\infty}\hbar^{-\frac{1}{2}+n}\varphi(\Gamma_1)_{(n)}\bigg].\end{aligned} \tag{3.47}$$

This equation is the basis for the construction of a perturbation theory in powers of $\hbar$ known as loop expansion. Equation (3.47) in the one-loop approximation reads

$$\exp\left[-\Gamma_{(n)}[\Phi]\right] = \int D\varphi \exp\left[-\frac{1}{2}S_2[\Phi]\varphi^2\right]. \tag{3.48}$$

Recalling the rule for bosonic functional determinants

$$\int D\phi \exp\left[-\frac{1}{2}\int \phi\Delta\phi\right] = \frac{1}{(\det\Delta)^{1/2}}, \tag{3.49}$$

we arrive at the following expression for the one-loop corrections to the effective action

$$\Gamma_{(1)} = \frac{1}{2}\ln\det\Delta_\phi, \quad \text{where} \quad \Delta_\phi\delta(x,y) = \left.\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\right|_\Phi. \tag{3.50}$$

Dropping the subindex E, our free energy (3.5) at one-loop looks like

$$\ln Z(\beta) \approx -S[\bar{g},\bar{\phi}] + \ln\int D(\delta g)\exp\left(-S_2[\delta g]\right) + \ln\int D(\delta\phi)\exp\left(-S_2[\delta\phi]\right), \tag{3.51}$$

where the first term is the leading one and the remaining two are the one-loop corrections coming from the gravitational sector of the theory and from a scalar field minimally coupled to gravity. Physically, the 1-loop corrections can be thought of as the contribution from thermal gravitons and thermal scalar particles to the free energy. As we have seen, the one-loop terms are obtained by Taylor expanding the action in the path integral. Thus, roughly speaking, they are the Hessians matrices of the Euclidean GR action (its second functional derivative) and of the action of a scalar field minimally coupled to gravity. So, considering for simplicity only the scalar field, our one-loop corrections to the partition function are given by

$$\ln\int D\phi\exp\left(-\frac{1}{2}\int\left.\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\right|_{\bar{\phi}}\varphi(y)\varphi(x)d^4x d^4y\right) = -\frac{1}{2}\ln\det\Delta_\phi = -\Gamma_{(1)} \tag{3.52}$$

where the operator is given by

$$\Delta_\phi = \left.\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\right|_{\bar{\phi}}. \tag{3.53}$$

3.3 Heat kernel method

The heat kernel method is a very general and useful method to study the divergent part of the one-loop effective action. This section aims to introduce a general-enough algorithm for computing the divergent part of the effective action, which gives interesting pieces of information about the theory, such as renormalizability properties and the form of the necessary counterterms. As always throughout this essay, we work in the Euclidean formulation of the theory. For simplicity, we begin presenting the method for a Euclidean scalar field on a curved background, however, all the results can be obtained for a much more general theory, as we will see below. We begin recalling the Euclidean formulation of a scalar field minimally coupled to gravity. The action of a scalar field propagating on a d-dimensional curved manifold with metric $g_{\mu\nu}$ is

$$S[\phi, g] = \frac{1}{2}\int d^dx\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{3.54}$$

Performing then a Wick's rotation $t = -i\tau$

$$\begin{aligned} iS[\phi, g] &= i\frac{1}{2}\int dtd^{d-1}x\sqrt{-g}\,[g^{00}(\partial_t\phi)^2 - g^{ij}\partial_i\phi\partial_j\phi] \\ &= \frac{1}{2}\int d\tau d^{d-1}x\sqrt{-g}\,[-g^{00}(\partial_\tau\phi)^2 - g^{ij}\partial_i\phi\partial_j\phi] = -S_E[\phi, g_E], \end{aligned} \tag{3.55}$$

where

$$S_E[\phi, g_E] = \frac{1}{2}\int d^dx\sqrt{g_E}\,g_E^{\mu\nu}\partial_\mu\phi\partial_\nu\phi, \tag{3.56}$$

and

$$(g_E)_{\mu\nu} = \begin{pmatrix} g_{00} & 0 \\ 0 & g_{ij} \end{pmatrix} \tag{3.57}$$

is a Euclidean metric, i.e., positive definite. Since we will always work on the Euclidean manifold, we will drop the subindex E. Integrating by parts, and assuming that the boundary conditions on the fields are such that we can do that without having any boundary terms left out, we obtain

$$S[\phi, g] = \frac{1}{2}\int d^dx\sqrt{g}\,\phi\Delta\phi, \tag{3.58}$$

in which $\Delta = -g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the covariant Laplacian. The heat kernel $K_\Delta(x, y; t)$ is a solution of the heat equation

$$\frac{d\Psi}{dt} + \Delta\Psi = 0, \tag{3.59}$$

describing diffusion processes on a manifold M with metric g over an external "time" t, and satisfying boundary conditions

$$K_\Delta(x, y; 0) = \delta(x, y). \tag{3.60}$$

Notice that t has dimensions of length squared. From the definition of the heat kernel, we have that the field Ψ, at any time, can be expressed as

$$\Psi(x;t) = \int d^d y K_\Delta(x,y;t)\Psi(y;0). \tag{3.61}$$

We obtain then that the heat kernel satisfy an heat equation of the form

$$\frac{\partial K_\Delta(x,y;t)}{\partial t} + \Delta K_\Delta(x,y;t) = 0 \quad \text{with} \quad K_\Delta(x,y;0) = \delta(x,y). \tag{3.62}$$

Thus, formally, we can write the heat kernel as

$$K_\Delta(t) = e^{-t\Delta}. \tag{3.63}$$

Consider now the eigenvalue problem for the Laplacian

$$\Delta\phi_n = \lambda_n\phi_n, \tag{3.64}$$

where λ_n are the eigenvalues and ϕ_n are orthonormal eigenvectors with respect to the scalar product $(\phi_n,\phi_m) = \int_M d^d x\sqrt{g}\phi_n(x)\phi_m(x) = \delta_{nm}$. Then, from equation (3.63), we can obtain the heat kernel in terms of the eigenfunctions

$$K_\Delta(x,y;t) = \sum_n \phi_n(x)\phi_n(y)\, e^{-t\lambda_n}, \tag{3.65}$$

called spectral decomposition. We will be mainly interested in the trace of the heat kernel

$$\mathrm{Tr}K_\Delta(t) = \int d^d x\sqrt{g}K_\Delta(x,x;t) = \sum_n e^{-t\lambda_n}. \tag{3.66}$$

This is because we can formally express the effective action in terms of it, using the generalized ζ-function. Indeed, changing variable $\bar{t} = \lambda t$ in the definition of the gamma function $\Gamma(s) = \int_0^\infty d\bar{t}\,\bar{t}^{s-1}\exp\{-\bar{t}\}$, we have

$$\lambda^{-s} = \frac{1}{\Gamma(s)}\int dt\, t^{s-1}\exp\{-\lambda t\}. \tag{3.67}$$

Then the ζ-function $\zeta_\Delta(s) = \sum_n \lambda_n^{-s}$ can be written as

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\mathrm{Tr}K_\Delta(t). \tag{3.68}$$

It is well-know [17, 18] that we can express the one-loop effective action through the derivative of the ζ-function

$$\frac{1}{2}\ln\det\Delta_\phi = \frac{1}{2}\mathrm{Tr}\ln\Delta = -\frac{1}{2}\sum_n\frac{d}{ds}\lambda_n^{-s} = -\frac{d}{ds}\zeta_\Delta(s)_{|s=0}. \tag{3.69}$$

From this equation, using (3.68) and ignoring that the integrals are divergent, we have the formal expression for the one-loop effective action

$$\Gamma = -\frac{1}{2}\int_0^\infty dt\, t^{-1}\mathrm{Tr}K_\Delta(t). \tag{3.70}$$

So, we can express the effective action in terms of the trace of the heat kernel. This expression is formal in the sense that it is actually divergent. Indeed, without setting $s = 0$ in equation (3.69), we could write the "effective action" as a function of s as [14]

$$\begin{aligned}\Gamma_s &= -\frac{1}{2}\mu^{2s}\int_0^\infty dt\, t^{s-1}\mathrm{Tr}K_\Delta(t)\\ &= -\frac{1}{2}\mu^{2s}\Gamma(s)\zeta_\Delta(s),\end{aligned} \tag{3.71}$$

where μ is a mass parameter introduced to have the effective action with the proper dimension. Recall now that in $s = 0$ the gamma function has a simple pole

$$\Gamma(s) = \frac{1}{s} - \gamma_E + O(s), \tag{3.72}$$

in which γ_E is the Euler-Mascheroni constant. We see in this way that in $s = 0$ the effective action (3.71) also has a pole. We are about to see how, in general, we can obtain the form of this divergence. Notice that, since the parameter t has dimension of inverse squared length, the lower end of the integration range in (3.70) corresponds to the ultraviolet (UV) while the upper end corresponds to the infrared (IR) divergences. Then, to explicitly see the UV divergences, one introduces an UV cut-off Λ_{UV} and a finite reference mass $\mu < \Lambda_{UV}$ and then, finally, one can split the integral $\int_{1/\Lambda_{UV}^2}^\infty = \int_{1/\Lambda_{UV}^2}^{1/\mu^2} + \int_{1/\mu^2}^\infty$. Now, if Δ does not have negative or zero eigenvalues [12], the second piece is convergent. Instead, in the piece containing the UV divergences, we need an "early-time" expansion of the trace of the heat kernel, i.e., an expansion of $\mathrm{Tr}K_\Delta(t)$ valid for small t. This early-time expansion is well-known in the mathematical literature, however, without the pretension of being too formal, we can intuitively obtain such expansion. We consider that in flat spacetime the heat kernel equation can be solved through Fourier analysis. We denote the coordinates as d-dimensional vectors $\vec{x}$. In flat spacetime the heat equation (3.62) can be rewritten in terms of the Fourier transform $\widetilde{K}_\Delta(\vec{q},\vec{y};t)$ as

$$\frac{d\widetilde{K}_\Delta(\vec{q},\vec{y};t)}{\partial t} + q^2\widetilde{K}_\Delta(\vec{q},\vec{y};t) = 0 \quad \text{with} \quad \widetilde{K}_\Delta(\vec{q},\vec{y};0) = e^{-\vec{q}\cdot\vec{y}}. \tag{3.73}$$

The solution is then $\widetilde{K}_\Delta(\vec{q},\vec{y};t) = e^{-q^2t-\vec{q}\cdot\vec{y}}$. From the inverse Fourier transform then we obtain

$$K_\Delta(\vec{x},\vec{y};t) = \int \frac{d\vec{q}}{(2\pi)^d}e^{-q^2t-\vec{q}\cdot(\vec{x}-\vec{y})} = \frac{1}{(4\pi t)^{d/2}}\,e^{\frac{|\vec{x}-\vec{y}|^2}{4t}}. \tag{3.74}$$

And so the trace of the heat kernel (3.66) is

$$\text{Tr}K_\Delta(t) = \frac{V}{(4\pi t)^{d/2}}, \tag{3.75}$$

where V is the infinite volume of the manifold. Now comes the point. Since locally every manifold looks like a flat manifold, for $t \to 0$ the trace of the heat kernel has to reduce to equation (3.75). Because deviations from the flat space are represented by the curvature we expect that the corrections to (3.75) in curved spacetime are proportional to curvature invariants. The easiest form of the "corrected" trace of heat kernel for $t \to 0$ is then a power law of the form

$$\text{Tr}K_\Delta(t) \approx \frac{1}{(4\pi t)^{d/2}} \left[B_0(\Delta) + tB_2(\Delta) + t^2 B_4(\Delta) + ...\right], \tag{3.76}$$

where

$$B_n(\Delta) = \int d^4x \sqrt{g}\; b_n(\Delta), \tag{3.77}$$

and $b_n(\Delta)$ are constructed in terms of the curvature and its covariant derivatives. Now we can fix the dependence of $b_n(\Delta)$ on the curvature and its covariant derivatives by dimensional analysis. For dimensional reasons $b_n(\Delta)$ must contain n-derivatives of the metric tensor. Then $b_2(\Delta) \propto R$, while $b_4(\Delta) \propto R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}, R_{\mu\nu}R^{\mu\nu}, R^2, \nabla^2 R$ and so on. This line of reasoning left undetermined the numerical coefficients of the expansion, which however can be determined with iterative procedures [18]. The final result for the coefficients appearing in the early-time expansion of the trace of the heat kernel for the covariant Laplacian is

$$\begin{aligned} b_0(x,x) &= 1 \\ b_2(x,x) &= \frac{1}{6}R \\ b_4(x,x) &= \frac{1}{180}\left(R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - R_{\mu\nu}R^{\mu\nu} + \frac{5}{2}R^2 + 6\nabla_\mu\nabla^\mu R\right). \end{aligned} \tag{3.78}$$

These coefficients are known as HAMIDEW coefficients [16]. We can use them to obtain the divergent part of the effective action. We have previously introduced the decomposition $\int_{1/\Lambda_{UV}^2}^{\infty} = \int_{1/\Lambda_{UV}^2}^{1/\mu^2} + \int_{1/\mu^2}^{\infty}$ of the integral in equation (3.70). Inserting now the early-time expansion (4.82) in the first piece of this decomposition, we can write the effective action as

$$\begin{aligned} \Gamma_{div} &= -\frac{1}{2}\frac{1}{(4\pi)^{d/2}}\int d^dx\sqrt{g}\int_{1/\Lambda_{UV}^2}^{1/\mu^2} dt\left[t^{-\frac{d}{2}-1}b_0 + t^{-\frac{d}{2}}b_2 + \;...\; + t^{-1}b_d + ...\right] \\ &= -\frac{1}{2}\frac{1}{(4\pi)^{d/2}}\int d^dx\sqrt{g}\left[\frac{\Lambda_{UV}^d}{d/2}b_0 + \frac{\Lambda_{UV}^{d-2}}{\frac{d}{2}-1}b_2 + ... + \ln\frac{\Lambda_{UV}^2}{\mu^2}b_d + \text{finite terms}\right] \end{aligned} \tag{3.79}$$

where the coefficients b_n are listed in (3.78). Some comments are in order. Notice that the first diverging term is proportional to the volume of the spacetime and can absorbed in the definition of the cosmological constant, renormalizing it. The second term is proportional to the Einstein-Hilbert action and so can be used to renormalize the Newton's constant [22]. The most interesting it is the third terms, since it tells us that to renormalize the theory we have to add to the action terms which are second order in curvature and have essentially the form of b_4.

This construction can be greatly generalized substituting to Δ a generic differential operator $F(\nabla)$, where $\nabla = \nabla^R + \omega$ do not contain only the usual Levi-Civita connection but as well a gauge connection, and it contains, in addition, a so-called potential term (see below). All the equations obtained in this section, excluding of course the result for the HAMIDEW coefficient, are valid also in this much more general framework replacing $\Delta \to F(\nabla)$. In the next section, we will see some of these generalizations.

3.3.1 Generalizations and Master Formula

We have presented the heat kernel method for a scalar field on a curved background. However, it can be extended to compute the one-loop divergences in any QFT. In particular, we present now the result for the HAMIDEW coefficients for a much more general second-order differential operator, with form $-g^{\mu\nu}\nabla_\mu\nabla_\nu + \mathbf{E}$, where $\mathbf{E}$ is an endomorphism called potential term. For our work, we do not strictly need such generalizations, but we very briefly present it for the sake of completeness. Following reference [12], we consider covariant derivatives that, very schematically, reads $\nabla = \partial + A + \omega$; where A is the spin connection associated to the usual torsion-free Levi-Civita connection (see our discussion in section (1.1.1)), and ω is a gauge connection (gauge potential). Let us be now more precise. Let us consider a quantum field Ψ. This field has a given mass and spin, which means that it transforms in some well-defined representation σ of the Lorentz group $SO(1,3)$. The field Ψ transforms also under a gauge group G, which identifies a principal bundle P, in some representation ρ. The representation ρ identifies one of the associates of P, let us call it V. Thus, the field Ψ carries two kinds of indices $\Psi \equiv \Psi^i_A$. In this expression the index A is an index in the space S carrying the representation σ (it is a Lorentz index), while i is an index in the space V carrying the representation ρ (it is a gauge index). We can then introduce the "full" covariant derivative of Ψ^i_A as

$$\nabla_\mu \Psi^i_A = \partial_\mu \Psi^i_A + A^{ab}{}_\mu (\sigma_{ab})_A{}^B \Psi^i_B + \omega^m{}_\mu (\rho_m)^i{}_j \Psi^j_A. \tag{3.80}$$

where $A^{ab}{}_\mu$ is the GR spin connection and $\omega^m{}_\mu$ is a gauge potential. In addition, σ_{ab} are the generators of the Lorentz group in the representation σ. Instead, $(\rho_m)^i{}_j$, where m is an index of the Lie algebra of the gauge group G, are the generators of G in the representation ρ. We recall that this covariant derivatives can be equivalently written, using the tetrad postulate, through the usual Levi-Civita connection instead of a spin connection. In particular, there are no differences concerning the values of the HAMIDEW coefficients below. Let us define now a generalization of the Laplacian, given by

$$-g^{\mu\nu}\nabla_\mu\nabla_\nu, \tag{3.81}$$

where now ∇ is the covariant derivative (3.80). We also assume that the action can be written as

$$S(\Psi;A,\omega)=\frac{1}{2}\mathcal{G}(\Psi,\Delta\Psi) \quad \text{with} \quad \mathcal{G}(\Psi,\Psi')=\int dx\sqrt{g}G^{AB}_{ij}\Psi^i_A\Psi^j_B, \tag{3.82}$$

where G^{AB}_{ij} is a metric in the field space and Δ is a general second order differential operator with the form

$$\Delta=-g^{\mu\nu}\nabla_\mu\nabla_\nu+\mathbf{E}, \tag{3.83}$$

where $\mathbf{E}\equiv E_A{}^{Bi}{}_j$ is and endomorphism of $S\otimes V$, called potential term. Repeating step by step the calculations of the previous section we arrive to the following expression of the effective action, called master formula

$$\Gamma_{div}=-\frac{1}{2}\frac{1}{(4\pi)^{d/2}}\int d^dx\,\sqrt{g}\left[\frac{\Lambda^d_{UV}}{d/2}b_0+\frac{\Lambda^{d-2}_{UV}}{\frac{d}{2}-1}b_2+...+\ln\frac{\Lambda^2_{UV}}{\mu^2}b_d+\text{finite terms}\right]. \tag{3.84}$$

Where now the HAMIDEW coefficients for the operator (3.83) are given by [12]

$$\begin{aligned}
b_0(x,x)&=\mathrm{tr}\mathbf{1}\\
b_2(x,x)&=\frac{1}{6}R\mathbf{1}-\mathrm{tr}\mathbf{E}\\
b_4(x,x)&=\frac{1}{180}\left(R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}-R_{\mu\nu}R^{\mu\nu}+\frac{5}{2}R^2+6\nabla_\mu\nabla^\mu R\right)\mathrm{tr}\mathbf{1}\\
&\quad+\frac{1}{2}\mathrm{tr}\mathbf{E}^2-\frac{1}{6}R\mathbf{E}+\frac{1}{12}\mathrm{tr}\Omega_{\mu\nu}\Omega^{\mu\nu}-\frac{1}{6}\nabla^2\mathrm{tr}\mathbf{E}.
\end{aligned} \tag{3.85}$$

In the previous formulae $\Omega_{\mu\nu}$ is the so called gauge bundle curvature, defined by the commutation relations

$$[\nabla_\mu,\nabla_\nu]\Psi=\Omega_{\mu\nu}\Psi, \tag{3.86}$$

where ∇ is the covariant derivatives (3.80). So $\Omega_{\mu\nu}$, for a gauge theory on flat space, is just the field strength of the gauge field. The Riemann tensor is instead defined from the commutator of covariant derivatives (3.80), but acting on a spacetime vector V^μ

$$[\nabla_\rho,\nabla_\sigma]V^\mu=R^\mu_{\alpha\rho\sigma}V^\alpha. \tag{3.87}$$

Finally, $\mathbf{1}$ in equation (3.85) is the identity in $S\otimes V$. Thus, $\mathrm{tr}\mathbf{1}$ is equal to the product of the dimensions of the spaces carrying the representations σ and ρ. We conclude this chapter by mentioning that further generalizations of the heat kernel method exist. In particular, we have presented the heat kernel method for operators such as (3.83), whose principal symbol, defined as the higher derivative term with the derivatives replaced by a vector p_μ, is given by $p^2 = g^{\mu\nu} p_\mu p_\nu$. This type of operators are called *minimal operators*. However, exist a generalization [10] of the method for operators whose second derivatives do not form a d'Alembertian, the so-called *non-minimal operators*. The authors, in [10, 11], develop the method also for operators of any order, provided they satisfy some causality conditions. To conclude, we signal that another generalization, for Riemann-Cartan spacetimes, exists and it can be found in the intresting article [13]. However, since we will do not need this results, we do not explicitly report the values for the HAMIDEW coefficients.

Chapter 4

Black Hole Thermodynamics and path integrals in TEGR

This chapter aims to study the thermodynamics of the Teleparallel equivalent of a Schwarzschild black hole. In particular, we derive the thermodynamical entropy and energy from the path integral, following the lines of chapter 2. To begin with, we consider the partition function at the leading order and so our first job is to evaluate the classical TEGR action. One-loop corrections are considered at the end of the chapter.

4.1 Euclidean TEGR action for the path integral

To begin with, we find the Teleparallel description of the Euclidean Schwarzschild black hole. We recall that such a BH, which is static, can be obtained from the Lorentzian BH by means of a naïve Wick rotation of the time coordinate $t = -i\tau$. Consequently, the Euclidean Schwarzschild metric is given by

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2^2. \tag{4.1}$$

Essentially, we have to find the (Euclidean) tetrads corresponding to this metric. Instead of solving the field equations, we use equation (2.6). To this purpose, we notice that applying the Wick rotation $t = -i\tau$, the Minkowski metric in the tangent space get replaced by $(\eta_E)_{ab} = \delta_{ab} = \text{diag}(+1, +1, +1, +1)$ and, thus, equation (2.6) becames

$$(g_E)_{\mu\nu} = (\eta_E)_{ab}(h_E)^a{}_\mu(h_E)^b{}_\nu, \tag{4.2}$$

where $(g_E)_{\mu\nu}$ is the Euclidean metric and $(h_E)^a{}_\mu$ the corresponding Euclidean tetrad. Since we will always work in Euclidean signature, in the following we will usually refer to Euclidean quantities without any particular sign.

In the next subsections, we will see that the TEGR boundary term is equal to the GHY term, and, thus, the TEGR action (2.156) "behaves" under Wick rotation as the GR action in presence of a spacetime boundary, i.e., the Einstein-Hilbert action plus GHY. Indeed, we will consider as (classical) Euclidean action in TEGR the usual GR Euclidean action with the GHY term replaced by the TEGR boundary term, see equation (4.5) below. Further, we will see that for the actual calculations of S_E, we need only the r-component of the torsion vector $\overset{\bullet}{T}{}^{\nu\mu}{}_{\nu}$, which remains unchanged under Wick rotation. Using now equation (4.2) and the Euclidean Schwarzschild metric, it is immediate to find (up to local Lorentz transformations) the diagonal Euclidean nontrivial tetrads

$$\begin{aligned} h_a{}^{\mu} &= \mathrm{diag}(f(r)^{-\frac{1}{2}},\, f(r)^{\frac{1}{2}},\, r^{-1},\, r^{-1}\sin^{-1}\theta) \\ h^a{}_{\mu} &= \mathrm{diag}(f(r)^{\frac{1}{2}},\, f(r)^{-\frac{1}{2}},\, r,\, r\sin\theta), \end{aligned} \tag{4.3}$$

where $f(r) = \left(1 - \frac{2M}{r}\right)$. To compute the (classical) Euclidean action in a Teleparallel context, we start by considering the Lagrangian (2.156), which establish the equivalence between TEGR and GR up to a boundary term. Taking (2.156) and (3.27) into account, and performing a Wick's rotation, one could try to consider in TEGR an Euclidean action of the following form

$$\begin{aligned} S_E = &-\frac{1}{16\pi}\int_{\overset{\circ}{M}} d^4x\sqrt{g}\overset{\circ}{\mathcal{R}} - \frac{1}{8\pi}\int_{\partial\overset{\circ}{M}} d^3y\sqrt{h}K + \frac{1}{8\pi}\int_{\partial\overset{\circ}{M}} d^3y\sqrt{h}K_0 \\ &- \frac{1}{8\pi}\int_{\overset{\bullet}{M}} \partial_\mu\left(\sqrt{g}\ \overset{\bullet}{T}{}^{\nu\mu}{}_{\nu}\right)d^4x + \frac{1}{8\pi}\int_{\overset{\bullet}{M}} \partial_\mu\left(\sqrt{g}\ \overset{\bullet}{T}{}_0{}^{\nu\mu}{}_{\nu}\right)d^4x. \end{aligned} \tag{4.4}$$

In the previous equation, $\overset{\circ}{M}$ is the usual Euclidean Schwarzschild spacetime, therefore we have $\overset{\circ}{\mathcal{R}} = 0$. $\partial\overset{\circ}{M}$ is instead its boundary, which, topologically, as defined before, is $S^2 \times S^1$. So the first three terms are the same as in Euclidean GR. We recall that in our notation the empty circle just indicates quantities in GR. Indeed, the second term is the GHY term, and the third one is the counterterm (3.29). In the last two terms, instead, $\overset{\bullet}{M}$ is the Teleparallel spacetime equivalent to $\overset{\circ}{M}$, which correspond to a flat space endowed with the appropriate Weitzenböck connection, $\overset{\bullet}{M} = (\mathbb{R}^4,\ \overset{\bullet}{\Gamma}{}^{\rho}{}_{\mu\nu})$. The fourth term comes from the Lagrangian (2.156), whereas the last one is another needed counterterm defined as the same expression of the fourth term, but computed in absence of gravity, i.e., using a trivial tetrad field (see below). However, as we are about to show that the choice (4.4) is not the right one since adding the GHY term (and its regularization) actually is redundant. This is because the boundary term (2.156), due only to torsion, can already eliminate the boundary terms coming from the variation of the Einstein-Hilbert action in presence of a spacetime boundary. In other words, the TEGR boundary term of equation

(2.156) plays the role of the GHY, becoming pointless to add it in (4.4). Indeed, we are going to see that

$$S_E = -\frac{1}{16\pi}\int_{\overset{\circ}{M}} d^4x\sqrt{g}\overset{\circ}{\mathcal{R}} - \frac{1}{8\pi}\int_{\overset{\bullet}{M}} \partial_\mu\left(\sqrt{g}\ \overset{\bullet}{T}{}^{\nu\mu}{}_{\nu}\right) d^4x + \frac{1}{8\pi}\int_{\overset{\bullet}{M}} \partial_\mu\left(\sqrt{g}\ \overset{\bullet}{T}{}_0^{\nu\mu}{}_{\nu}\right) d^4x, \quad (4.5)$$

is the right choice for the Euclidean action. We present now some computations showing this claim.

4.1.1 Equivalence between the GHY term and the TEGR boundary term

One could suspect that GHY should not be taken into account by noticing that, ultimately, we need to add GHY in GR because the Lagrangian contains second-order derivatives of the metric. Indeed, a schematic variation of the Einstein-Hilbert action leads to [1]

$$\delta\int_M \sqrt{g}R \sim \int_M (eom)\delta g + \int_{\partial M}\left[A(g,\partial g)\delta g + B(g,\partial g)\partial\delta g\right], \quad (4.6)$$

where "eom" is, essentially, the left-hand side of the Einstein equations, and the second integral comes from integrating by parts. Thus, if one imposes, as usual, boundary conditions such that $\delta g_{|\partial M} = 0$ at $r = r_0$, we have that the first term in the second integral vanishes, but the second one does not. So, varying this action in presence of a boundary of the spacetime and imposing that $\delta g_{|\partial M} = 0$, one does not obtain the left-hand side of Einstein equations. GHY solves this problem since it is chosen so that $\delta\int_M \sqrt{g}R \sim \int_M (eom)\delta g$. Reconsider now the Teleparallel Lagrangian in terms of the torsion tensor only

$$\overset{\bullet}{\mathcal{L}} = \frac{h}{16\pi}\left(\frac{1}{4}\overset{\bullet}{T}{}^{\rho}{}_{\mu\nu}\ \overset{\bullet}{T}{}_{\rho}{}^{\mu\nu} + \frac{1}{2}\overset{\bullet}{T}{}^{\rho}{}_{\mu\nu}\ \overset{\bullet}{T}{}^{\nu\mu}{}_{\rho} - \overset{\bullet}{T}{}^{\rho}{}_{\mu\rho}\ \overset{\bullet}{T}{}^{\nu\mu}{}_{\nu}\right),$$

where, we recall, $h = \det\left(h^a_\mu\right) = \sqrt{-g}$. One immediately notices that the TEGR Lagrangian is only of first-order in the derivatives of the tetrads. Thus, for TEGR there is no need to add a boundary term in presence of a boundary of the spacetime, since varying $\overset{\bullet}{\mathcal{L}}$ one already obtains the correct equations of motion. We recall now equation (2.156) and that, for a Schwarzschild BH, $\overset{\circ}{R} = 0$. Thus, in this simple case, we obtain that computing the TEGR action it is equivalent to calculate the integral

$$\overset{\bullet}{\mathcal{S}} = \int \overset{\bullet}{\mathcal{L}}\ d^4x = -\int \partial_\mu\left(\frac{h}{8\pi}\ \overset{\bullet}{T}{}^{\nu\mu}{}_{\nu}\right) d^4x. \quad (4.7)$$

Following this argument we obtain, as will be explicitly derived in the following section, that for a Teleparallel Schwarzschild BH the Euclidean action is

$$S_E = \frac{\beta M}{2} = 4\pi M^2, \tag{4.8}$$

and so in TEGR, at least without loop contributions, we obtain from the TEGR boundary term the same thermodynamical entropy and energy as from GHY (cf. equation (3.32)). Thus, for the simple case of a Teleparallel Schwarzschild BH, we have no reasons to include GHY. Moreover, we notice that if we include both the boundary terms as in equation (4.4), we obtain a doubling of the values of the entropy and the energy of GR. This result leads to a physical interpretation problem that will be discussed at the end of the next section. However, this result can be avoided by proving, as previously said, that the GHY term should not be considered in TEGR. Let's now try to be a little bit more general considering the variation of $\overset{\bullet}{\mathcal{S}}$ without taking $\overset{\circ}{R} = 0$ from the beginning

$$\delta \int \overset{\bullet}{\mathcal{L}}\ d^4x = \delta \int \overset{\circ}{\mathcal{L}}\ d^4x - \delta \int \partial_\mu \left(\frac{h}{8\pi}\ \overset{\bullet}{T}{}^{\nu\mu}{}_\nu\ d^4x \right). \tag{4.9}$$

Where now the Ricci scalar can be thought of in terms of the tetrads. Looking at this equation, it seems that to safely exclude the GHY term from the action we have to study if the last integral in the previous equation is equal to the variation of GHY term in equation (3.23). So we ask: does the boundary term in equation (2.156) the same job as the GHY term? Now, let us verify if that is the case. Using that the equations of motion of TEGR and GR are the same, we can say that the variations of the bulk terms in the left and right-hand sides cancel each other out. Recalling now that $\overset{\bullet}{\mathcal{L}}$ contain only first derivatives, we see that from the variation of the TEGR action does not come out any boundary term. Thus, using Stokes theorem in the form

$$\begin{aligned} \int_M A^\mu_{;\mu} \sqrt{-g} d^4x = \int_M \left(\sqrt{-g} A^\mu\right)_{,\mu} d^4x &= \oint_{\partial M} A^\mu d\Sigma_\mu \\ &= \oint_{\partial M} A^\mu n_\mu \sqrt{|\sigma|} d^3y, \end{aligned} \tag{4.10}$$

where n_μ is the normalized normal vector to ∂M, and $\sigma = \det\{\sigma_{ab}\} = \det\{[g_{|\partial M}]_{ab}\}$. Taking care of the numerical factors, we can state that the following equation holds

$$\delta \left(2 \int \sqrt{\sigma}\ \overset{\bullet}{T}{}^{\nu\mu}{}_\nu n_\mu\ d^3y \right) = 2 \int \sqrt{\sigma}\ \delta\ \overset{\bullet}{T}{}^{\nu\mu}{}_\nu\ n_\mu\ d^3 = \int \delta \overset{\circ}{A}{}^\mu n_\mu \sqrt{\sigma} d^3y. \tag{4.11}$$

Where the first equality comes from the fact that if we take the boundary and the metric on the boundary fixed $\delta g_{|\partial M} = 0$, then $\delta \left(\sqrt{h}\ \overset{\bullet}{T}{}^{\nu\mu}{}_\nu n_\mu \right) = \sqrt{h} \left(\delta\ \overset{\bullet}{T}{}^{\nu\mu}{}_\nu \right) n_\mu$. In equation

(4.11) the last integral in the right-hand side is the boundary term coming from the variation of the Einstein-Hilbert action expressed in terms of the tetrads. Therefore, we can say that really the TEGR boundary term does the same job of the GHY, which thus can be excluded from the calculation of the Euclidean action.

4.2 BH Thermodynamics at the leading order in TEGR

We are now ready to present the calculations leading to the evaluation of the thermodynamical entropy (and energy) of a Teleparallel Schwarzschild BH. This is interesting because the divergence in equation (2.156) could lead to a different value for the entropy with respect to the GR case. Our goal is to compute the Euclidean action

$$S_E = -\frac{1}{8\pi}\int_{\overset{\bullet}{M}} \partial_\mu\left(\sqrt{g}\ \overset{\bullet}{T}{}^{\nu\mu}{}_\nu\right) d^4x + \frac{1}{8\pi}\int_{\overset{\bullet}{M}} \partial_\mu\left(\sqrt{g}\ \overset{\bullet}{T}{}_0^{\nu\mu}{}_\nu\right) d^4x, \tag{4.12}$$

which is the same action that one usually considers in GR but with GHY replaced by the TEGR boundary term. Let us begin with the first integral. We evaluate this term using again Stokes theorem (4.10)

$$\begin{aligned}\int_M A^\mu_{;\mu}\sqrt{-g}d^4x = \int_M \left(\sqrt{-g}A^\mu\right)_{,\mu} d^4x &= \oint_{\partial M} A^\mu d\Sigma_\mu \\ &= \oint_{\partial M} A^\mu n_\mu \sqrt{|\sigma|}d^3y.\end{aligned} \tag{4.13}$$

Hence, we have

$$\frac{1}{8\pi}\int_{\overset{\bullet}{M}} \partial_\mu\left(\sqrt{g}\ \overset{\bullet}{T}{}^{\nu\mu}{}_\nu\right) d^4x = \frac{1}{8\pi}\int_0^\beta d\tau \int_{S^2} \overset{\bullet}{T}{}^{\nu\mu}{}_\nu n_\mu \sqrt{\sigma} d^2y, \tag{4.14}$$

where we have used that $\partial\overset{\bullet}{M} = S^2 \times S^1$. The integral in the Euclidean time is done over S^1, which represents the periodically identified Euclidean time, that behaves, as stated before, as an angular-like coordinate $\tau \sim \tau + 8\pi M = \tau + \beta$. Now, the line element of the boundary is given by $ds^2 = f(r_0)d\tau^2 + r_0^2 d\Omega_2^2$, so that

$$\sqrt{\sigma} = f(r_0)^{\frac{1}{2}} r_0^2 \sin\theta. \tag{4.15}$$

We compute now the unitary normal vector to the surface $r = r_0$. The un-normalized normal is $\bar{n}_\mu = \partial_\mu r = (0, 1, 0, 0)$, so its module is given by

$$\epsilon = \bar{n}^\mu \bar{n}_\mu = g^{\mu\nu}\bar{n}_\nu\bar{n}_\mu = g^{rr} = f = 1 - \frac{2M}{r}.$$

Then, the properly normalized normal vector, is

$$n_\mu = \frac{\partial_\mu r}{\epsilon^{\frac{1}{2}}} = f^{-\frac{1}{2}}(0,\, 1,\, 0,\, 0) = (0,\, f^{-\frac{1}{2}},\, 0,\, 0). \tag{4.16}$$

We only have left to consider the torsion vector $\overset{\bullet}{T}{}^{\nu\mu}{}_\nu$. From equation (4.16), we see that we only need the r component of the torsion vector (which remains unmodified under Wick rotation). Let us now choose the class of reference frames in which the spin connection is zero. Using the definition of the Weitzenböck connection in such class $\overset{\bullet}{\Gamma}{}^\rho{}_{\mu\nu} = h_a{}^\rho \partial_\nu h^a{}_\mu$, the torsion tensor is written as

$$\overset{\bullet}{T}{}^\nu{}_{\rho\mu} = h_a{}^\nu \left(\partial_\rho h^a{}_\mu - \partial_\mu h^a{}_\rho\right). \tag{4.17}$$

From this equation, and using the tetrads (4.3), it is only a matter of algebra to find out the components of the torsion vector

$$\begin{aligned}
\overset{\bullet}{T}{}^\nu{}_{0\nu} &= 0 \\
\overset{\bullet}{T}{}^\nu{}_{1\nu} &= \frac{1}{2}\partial_r \ln f + \frac{2}{r} \\
\overset{\bullet}{T}{}^\nu{}_{2\nu} &= \cot\theta \\
\overset{\bullet}{T}{}^\nu{}_{3\nu} &= 0.
\end{aligned} \tag{4.18}$$

Looking at equation (4.16) and at (4.14), we see that we only need the component $\overset{\bullet}{T}{}^{\nu 1}{}_\nu$, that depends uniquely on the radial coordinate. Using (4.18), we easily have

$$\overset{\bullet}{T}{}^{\nu 1}{}_\nu = g^{1\rho}\ \overset{\bullet}{T}{}^\nu{}_{\rho\nu} = g^{11}\ \overset{\bullet}{T}{}^\nu{}_{1\nu} = f\left(\frac{f'}{2f} + \frac{2}{r}\right) = \frac{2}{r} - \frac{3M}{r^2}.$$

Then, taking into account the normal vector (4.16), we obtain

$$\overset{\bullet}{T}{}^{\nu\mu}{}_\nu n_\mu = \ \overset{\bullet}{T}{}^{\nu 1}{}_\nu f^{-\frac{1}{2}} = \left(\frac{2}{r} - \frac{3M}{r^2}\right) f^{-\frac{1}{2}}. \tag{4.19}$$

Collecting (4.19) and (4.15), the last integral in equation (4.14) gives

$$\begin{aligned}
\frac{1}{8\pi}\int_0^\beta d\tau \int_{r_0=const} \left(\frac{2}{r_0} - \frac{3M}{r_0^2}\right) r_0^2 \sin\theta\, d\theta\, d\phi &= \frac{1}{8\pi}\int (2r_0 - 3M)\sin\theta d\tau\, d\theta\, d\phi \\
&= \frac{1}{8\pi}\left(8\pi\beta r_0 - 12\pi\beta M\right)
\end{aligned} \tag{4.20}$$

Where obviously we have used that $\int \left(\int \sin\theta d\theta d\phi\right) d\tau = 4\pi\beta$. Let's notice that, interestingly, it is *exactly* the result obtained in GR (3.26), further confirming the result of the previous section. This integral needs to be regularized. To this purpose, we consider the counterterm

$$\frac{1}{8\pi}\int_{\overset{\bullet}{M}} \partial_\mu \left(\sqrt{g}\ \overset{\bullet}{T}_0{}^{\nu\mu}{}_\nu\right) d^4x = \frac{1}{8\pi}\int_0^\beta d\tau \int_{S^2} \overset{\bullet}{T}_0{}^{\nu\mu}{}_\nu n_\mu \sqrt{\sigma} d^2y, \tag{4.21}$$

where $\overset{\bullet}{T}_0{}^{\nu\mu}{}_\nu$ is the torsion corresponding to the (Euclidean) trivial tetrads associated with the flat metric (3.28) $ds^2_{subtraction} = \left(1 - \frac{2M}{r_0}\right) d\tau^2 + dr^2 + r^2 d\Omega_2^2$, so that again

$$\sqrt{\sigma} = f(r_0)^{\frac{1}{2}} r_0^2 \sin\theta.$$

In the present case, the normal unit vector is given by

$$n_\mu = \partial_\mu r = (0,\, 1,\, 0,\, 0), \tag{4.22}$$

since, comparing with the previous equations, $\epsilon = g^{rr} = 1$. Proceeding as before, the diagonal trivial tetrad corresponding to (3.28) are found to be

$$\begin{aligned} E_a{}^\mu &= \mathrm{diag}(f(r_0)^{-\frac{1}{2}},\, 1,\, r^{-1},\, r^{-1}\sin^{-1}\theta) \\ E^a{}_\mu &= \mathrm{diag}(f(r_0)^{\frac{1}{2}},\, 1,\, r,\, r\sin\theta). \end{aligned} \tag{4.23}$$

Assuming again vanishing spin connection, the Weitzenböck connection takes the form $\overset{\bullet}{\Gamma}{}^\rho{}_{\mu\nu} = E_a{}^\rho \partial_\nu E^a{}_\mu$ and the components of the torsion are now given by

$$\overset{\bullet}{T}{}^\nu{}_{\rho\mu} = E_a{}^\nu \left(\partial_\rho E^a{}_\mu - \partial_\mu E^a{}_\rho\right). \tag{4.24}$$

Looking at the last integral in (4.21) and considering (4.22), we see that again we need just the r-component of the torsion tensor $\overset{\bullet}{T}_0{}^{\nu\mu}{}_\nu n_\mu = \overset{\bullet}{T}_0{}^{\nu 1}{}_\nu$. Using (4.24), we find that $\overset{\bullet}{T}_0{}^\nu{}_{1\nu} = \frac{2}{r}$ and thus

$$\overset{\bullet}{T}_0{}^{\nu 1}{}_\nu = g^{1\rho}\ \overset{\bullet}{T}_0{}^\nu{}_{\rho\nu} = g^{11}\ \overset{\bullet}{T}_0{}^\nu{}_{1\nu} = \frac{2}{r}. \tag{4.25}$$

Using (4.25) and (4.23), the second integral in (4.21) is

$$\frac{1}{8\pi}\int_{r_0} \frac{2}{r} r^2 \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} \sin\theta d\tau\, d\theta\, d\phi = \frac{1}{8\pi}\left(8\pi\beta r_0 - 8\pi\beta M + \beta O\left(\frac{1}{r_0}\right)\right), \tag{4.26}$$

that again is the same result as (3.29), which gives another confirmation of what we have obtained in the previous section. The integral (4.26) regulates (4.20) and *change* *the finite term.* Considering (4.20) and (4.26), we finally obtain for the classical Euclidean action (4.12) the following value

$$S_E = \frac{\beta M}{2} = 4\pi M^2, \tag{4.27}$$

which is the precisely the result obtained in GR (cf. (3.30)). Thus, for the partition function of the Teleparallel Schwarzschild BH, we obtain

$$Z(\beta) = \exp\left(-4\pi M^2\right) = \exp\left(-\frac{\beta^2}{16\pi}\right), \tag{4.28}$$

which, obviously, is the same result as in (3.31). Therefore, also in TEGR, for entropy and energy of a Schwarzschild BH, we obtain the values (cf. (3.32))

$$\begin{aligned} S &= (1 - \beta\partial_\beta) \ln Z(\beta) = 4\pi M^2 \\ E &= -\partial_\beta \ln Z(\beta) = M. \end{aligned} \tag{4.29}$$

We notice that if we start with (4.4), instead of (4.5), then we would have obtained a doubling of the action. Consequently, we would have had as well a doubling of S and E. Indeed, using (3.30) and (4.27), one obtains. from the "wrong" Euclidean action (4.4), the result

$$S_E = \frac{\beta M}{2} + \frac{\beta M}{2} = \beta M = 8\pi M^2, \tag{4.30}$$

which is twice the result obtained in GR. Therefore, in this case, for the partition function we would obtain

$$Z(\beta) = \exp\left(-8\pi M^2\right) = \exp\left(-\frac{\beta^2}{8\pi}\right), \tag{4.31}$$

instead of (4.28). Thus, from the previous partition function, we would have entropy and energy equal to

$$\begin{aligned} S &= (1 - \beta\partial_\beta) \ln Z(\beta) = 8\pi M^2 \\ E &= -\partial_\beta \ln Z(\beta) = 2M. \end{aligned} \tag{4.32}$$

As stated before, we obtain twice the results in (4.29). Moreover, there would be a physical interpretation problem: where the additional factor M in (4.32) does come from? In GR, the result (4.29) for the energy, it is usually interpreted stating that the energy comes from the BH rest mass. One could then try to link the additional factor of M in (4.32) to the gravitational energy stored in the spacetime. Indeed, it seems quite natural to think that gravitational energy plays a role in the thermodynamical behaviour of the BH. However, as we will see below, this seems not to be the case. We will see, basically building up the Landau-Lifshitz energy-momentum pseudotensor [15], that the total conserved energy of a Teleparallel Schwarzchild BH seems to be M, in agreement with the result just obtained. Thus, using this different approach, we support further the equivalence between the TEGR boundary term and GHY.

Summarizing: we have obtained in (4.29) that in TEGR, although it introduces an (apparently) new boundary term, the BH thermodynamics is essentially the same. This unexpected result is due to the equivalence of the TEGR boundary term and the GHY term, a result which has been proved in section 4.1.1. Before presenting the construction of the Landau-Lifshitz energy-momentum pseudotensor in TEGR, we give an alternative way to regularize the TEGR Euclidean action.

4.3 A different way to regularize the action

We present now a different way to regularize the term (4.20). This approach, physically, is based on the idea that this divergence is due to inertial effects, which do not vanish at infinity. The idea is, then, to remove these inertial effects by considering a suitable, generally non-vanishing, purely inertial spin connection. This possibility can be seen in this way. We start by considering that a trivial tetrad (no gravity) can be formally written as

$$E^a{}_\mu = \partial_\mu x^a + \overset{\bullet}{A}{}^a{}_{b\mu} x^b, \tag{4.33}$$

where $\overset{\bullet}{A}{}^a{}_{b\mu}$ is the purely inertial spin connection (2.59) and therefore, as we have already observed, its torsion tensor vanishes identically

$$\overset{\bullet}{T}{}^a{}_{\mu\nu}(E^a{}_\mu, \overset{\bullet}{A}{}^a{}_{b\mu}) = 0, \tag{4.34}$$

while we have seen that it will be non-vanishing for a nontrivial tetrad. Despite this, we have explicitly seen in equation (4.26) that the action computed with a trivial tetrad and vanishing spin connection $\overset{\bullet}{S}_E(e^a{}_\mu, 0) = \int_{\overset{\bullet}{M}} \overset{\bullet}{\mathcal{L}}(e^a{}_\mu, 0)$ diverge. Instead, from equation (4.34), i.e., using the appropriate spin connection $\overset{\bullet}{A}{}^a{}_{b\mu}$, we find that

$$\overset{\bullet}{S}_E(E^a{}_\mu, \overset{\bullet}{A}{}^a{}_{b\mu}) = 0. \tag{4.35}$$

We see that by considering an appropriate spin connection we can remove all the inertial effect. This is indeed the action for inertial effects only since we are considering a trivial tetrad. Considering now that the spin connection of TEGR is purely inertial we can use the same spin connection of the previous equation as well in presence of a nontrivial tetrad. We should obtain a renormalized (finite) action (that takes into account only gravitation and without any inertial effect) computing

$$\overset{\bullet}{S}_E(h^a{}_\mu, \overset{\bullet}{A}{}^a{}_{b\mu}) = \int_{\overset{\bullet}{M}} \overset{\bullet}{\mathcal{L}}(h^a{}_\mu, \overset{\bullet}{A}{}^a{}_{b\mu}). \tag{4.36}$$

To check this we have to find the associated spin connection to a given tetrad and repeat the calculation that leads to equation (4.20) but considering a non vanishing spin connection.

4.3.1 Associated spin connection to a given tetrad

The basic idea is that we can compute the associated spin connection using a trivial tetrad $E^c_{\ \mu}$, obtained from the nontrivial tetrad (4.3) "switching off gravity". This is because $E^c_{\ \mu}$ and $h^c_{\ \mu}$ only differ for their "gravitational content". We can obtain an expression for the spin connection easily combining two well-established theoretical results that we have already presented. The first result is equation (2.27) for the purely inertial spin connection

$$\overset{\bullet}{A}{}^a{}_{bc} = \frac{1}{2}\left(f_b{}^a{}_c + \overset{\bullet}{T}{}_b{}^a{}_c + f_c{}^a{}_b + \overset{\bullet}{T}{}_c{}^a{}_b - f^a{}_{cb} - \overset{\bullet}{T}{}^a{}_{cb} \right). \tag{4.37}$$

This equation has been obtained taking three different combinations of the Lorentz indexed torsion tensor (2.26). We recall that $f^a_{\ bc}$ are the anholonomy coefficients of a nontrivial tetrad, which are given by equation (2.49)

$$f^c{}_{ab} = h_a{}^\mu h_b{}^\nu \left(\partial_\nu h^c{}_\mu - \partial_\mu h^c{}_\nu \right). \tag{4.38}$$

The second result is equation (4.34)

$$\overset{\bullet}{T}{}^a{}_{\mu\nu}(E^a{}_\mu, \overset{\bullet}{A}{}^a{}_{b\mu}) = 0, \tag{4.39}$$

which holds for a trivial tetrad. Using (4.34) in (4.37) we obtain the following expression for the spin connection

$$\overset{\bullet}{A}{}^a{}_{b\mu} = \frac{1}{2} E^c{}_\mu \left(f_b{}^a{}_c(E^c{}_\mu) + f_c{}^a{}_b(E^c{}_\mu) - f^a{}_{cb}(E^c{}_\mu) \right). \tag{4.40}$$

This equation gives the inertial spin connection associated to the tetrad $E^c_{\ \mu}$, that we recall is a trivial tetrad obtained from (4.3) "switching off gravity". However, we have said at the beginning of the paragraph that, since the nontrivial tetrad $h^c_{\ \mu}$ and $E^c_{\ \mu}$ differ only by their gravitational content, equation (4.40) represent the inertial spin connection naturally associated as well to $h^c_{\ \mu}$. Indeed, we will use (4.40) as the equation for the Lorentz connection associated to a nontrivial tetrad.

We would like to insert an additional comment. The main weakness of our procedure is how to "switch off" gravity. Indeed, in the simple case of a Schwarzschild BH is, at least, intuitive which flat metric we should use, but in a more general case is not clear which would be the form of the flat metric to be considered.

4.3.2 Using the spin connection to regularize the action

A natural choice for the trivial tetrads obtained from (4.3) "switching off gravity" is

$$\begin{aligned} E_a{}^\mu &= \mathrm{diag}(1,\, 1,\, r^{-1},\, r^{-1}\sin^{-1}\theta) \\ E^a{}_\mu &= \mathrm{diag}(1,\, 1,\, r,\, r\sin\theta) \end{aligned} \tag{4.41}$$

which are the tetrads corresponding to the flat metric $\delta_{\mu\nu}$ in spherical coordinates. Considering equations (4.41) and (4.40), the non-vanishing components of the associated spin connection are found to be

$$\overset{\bullet}{A}{}^{\hat{1}}{}_{\hat{2}\theta} = -\overset{\bullet}{A}{}^{\hat{2}}{}_{\hat{1}\theta} = -1,\;\; \overset{\bullet}{A}{}^{\hat{1}}{}_{\hat{3}\phi} = -\overset{\bullet}{A}{}^{\hat{3}}{}_{\hat{1}\phi} = -\sin\theta,\;\; \overset{\bullet}{A}{}^{\hat{2}}{}_{\hat{3}\phi} = -\overset{\bullet}{A}{}^{\hat{3}}{}_{\hat{2}\phi} = -\cos\theta. \tag{4.42}$$

We consider now the "full" Weitzenböck connection, that we recall is given by

$$\overset{\bullet}{\Gamma}{}^\rho{}_{\mu\nu} = h_a{}^\rho \partial_\nu h^a{}_\mu + h_a{}^\rho \overset{\bullet}{A}{}^a{}_{b\nu} h^b{}_\mu .$$

In the previous equation, of course, $h^c{}_\mu$ are the nontrivial tetrads (4.3)

$$\begin{aligned} h_a{}^\mu &= \mathrm{diag}(f(r)^{-\frac{1}{2}},\, f(r)^{\frac{1}{2}},\, r^{-1},\, r^{-1}\sin^{-1}\theta) \\ h^a{}_\mu &= \mathrm{diag}(f(r)^{\frac{1}{2}},\, f(r)^{-\frac{1}{2}},\, r,\, r\sin\theta), \end{aligned}$$

where $f(r) = \left(1 - \frac{2M}{r}\right)$. In this case, the components of the torsion tensor are given by

$$\overset{\bullet}{T}{}^\rho{}_{\nu\mu} = h_a{}^\rho \left(\partial_\nu h^a{}_\mu - \partial_\mu h^a{}_\nu\right) + h_a{}^\rho \left(\overset{\bullet}{A}{}^a{}_{b\nu} h^b{}_\mu - \overset{\bullet}{A}{}^a{}_{b\mu} h^b{}_\nu\right). \tag{4.43}$$

As before, we need only $\overset{\bullet}{T}{}^{\nu 1}{}_\nu$. From equations (4.43) and (4.42) we obtain

$$\begin{aligned} \overset{\bullet}{T}{}^{\nu 1}{}_\nu &= g^{1\rho}\overset{\bullet}{T}{}^\nu{}_{\rho\nu} = g^{11}\overset{\bullet}{T}{}^\nu{}_{1\nu} = f\left(\frac{1}{2}\frac{f'}{f} + \frac{2}{r} - \frac{2}{r}f^{-\frac{1}{2}}\right) \\ &= \frac{1}{2}f' + \frac{2}{r}f - \frac{2}{r}f^{\frac{1}{2}} = \frac{1}{2}\frac{2M}{r^2} + \frac{2}{r}\left(1 - \frac{2M}{r}\right) - \frac{2}{r}\left(1 - \frac{M}{r} + O\left(\frac{1}{r^2}\right)\right) \\ &= -\frac{M}{r^2} + O\left(\frac{1}{r^3}\right), \end{aligned} \tag{4.44}$$

and then

$$\overset{\bullet}{T}{}^{\nu\mu}{}_\nu n_\mu = \overset{\bullet}{T}{}^{\nu 1}{}_\nu f^{-\frac{1}{2}} = \left(-\frac{M}{r^2} + O\left(\frac{1}{r^3}\right)\right) f^{-\frac{1}{2}}, \tag{4.45}$$

to be compared to (4.19). Proceeding as in (4.20), the integral in (4.14) now become

$$\begin{aligned}&\frac{1}{8\pi}\int\left(-\frac{M}{r^2}+O\left(\frac{1}{r^3}\right)\right)\left(1+\frac{M}{r}+O\left(\frac{1}{r^2}\right)\right)r^2\sin\theta d\tau\, d\theta\, d\phi\\&\quad=\frac{1}{8\pi}(-4\pi\beta M)+O\left(\frac{1}{r_0}\right).\end{aligned}\tag{4.46}$$

Therefore, with this method, we obtain the same result as in (4.27)

$$S_E=\frac{\beta M}{2}=4\pi M^2.\tag{4.47}$$

and, of course, the same values for S and E as in (4.29).

4.4 Gravitational energy in TEGR

To begin with, we recall very briefly the construction of the Landau-Lifshitz energy-momentum tensor in GR. These introductory computations and all the details can be found in [16], but we report some steps here for reference. We do that because we will identify the Landau-Lifsits energy-momentum tensor in TEGR through an analogy between an equation satisfied by this object and the field equations of TEGR. The idea of this section is to study the link between the additional factor of M in the expression for E (4.32) to a possible expression of the energy of the gravitational field. We also include a "gravitational surface pressure" conjugate to the area $A = 4\pi r^2$, following the idea of reference [4].

In absence of gravity, the conservation law of energy and momentum of the matter is expressed as

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu}=0.$$

The generalization of this equation in presence of gravity is [15]

$$T^\nu_{\mu;\,\nu}=\frac{1}{\sqrt{-g}}\frac{\partial\left(T^\nu_\mu\sqrt{-g}\right)}{\partial x^\nu}-\frac{1}{2}\frac{\partial g_{\nu\rho}}{\partial x^\mu}T^{\nu\rho}=0,\tag{4.48}$$

which, however, does not express any conservation law, as usual. Let us now choose a coordinates system in such a way that all the first derivatives of metric tensor with respect to the coordinates are zero at a point. At this point, then, equation (4.48) becomes

$${T^{\mu\nu}}_{,\nu}=0.$$

The quantities that satisfy this equation can the be formally written as

$$T^{\mu\nu}=\frac{\partial\eta^{\mu\nu\rho}}{\partial x^\rho},$$

where the η's are antisymmetric with respect to the last two indices

$$\eta^{\mu\nu\rho} = -\eta^{\mu\rho\nu}.$$

Let us now write down $T^{\mu\nu}$ in this form. It can be derived from the Einstein field equations [15] that

$$T^{\mu\nu} = \frac{\partial}{\partial x^\rho}\left[\frac{1}{(-g)}\frac{\partial}{\partial x^\gamma}\left(\lambda^{\mu\nu\rho\gamma}\right)\right] = \frac{1}{(-g)}\frac{\partial}{\partial x^\rho}\left[\frac{\partial}{\partial x^\gamma}\left(\lambda^{\mu\nu\rho\gamma}\right)\right], \tag{4.49}$$

where we can bring out the factor $1/(-g)$, since we are working in a coordinate system in which $\partial g_{\mu\nu}/\partial x^\nu = 0$. In equation (4.49) we have set

$$\lambda^{\mu\nu\rho\gamma} = \frac{c^4}{16\pi k}(-g)(g^{\mu\nu}g^{\rho\gamma} - g^{\mu\rho}g^{\nu\gamma}).$$

Let's now set

$$h^{\mu\nu\rho} = \frac{\partial}{\partial x^\gamma}\lambda^{\mu\nu\rho\gamma}, \tag{4.50}$$

which is antisymmetric in the last two indices

$$h^{\mu\nu\rho} = -h^{\mu\rho\nu}.$$

We can then rewrite equation (4.49) as

$$\frac{\partial}{\partial x^\rho}h^{\mu\nu\rho} = (-g)T^{\mu\nu}.$$

As said, this equation holds if $\partial g_{\mu\nu}/\partial x^\nu = 0$. In an arbitrary coordinate system, the generically non-vanishing difference $\frac{\partial}{\partial x^\rho}h^{\mu\nu\rho} - (-g)T^{\mu\nu}$ can be called $(-g)t^{\mu\nu}$. Therefore, by definition, for a general coordinate system we can write

$$(-g)(T^{\mu\nu} + t^{\mu\nu}) = \frac{\partial}{\partial x^\rho}h^{\mu\nu\rho}. \tag{4.51}$$

The fundamental property of $t^{\mu\nu}$ is that it is not a tensor, as it is already clear from the fact that $\frac{\partial}{\partial x^\rho}h^{\mu\nu\rho}$ are ordinary derivatives and not covariant derivatives. $t^{\mu\nu}$ is called Landau-Lifshitz energy-momentum pseudotensor of the gravitational field. Now, from equation (4.51), we have that the following conservation law holds

$$\frac{\partial}{\partial x^\nu}[(-g)(T^{\mu\nu} + t^{\mu\nu})] = \frac{\partial^2}{\partial x^\nu \partial x^\rho}h^{\mu\nu\rho} = 0, \tag{4.52}$$

and so the quantities

$$P^\mu = \frac{1}{c}\int (-g)(T^{\mu\nu} + t^{\mu\nu})dS_\nu \tag{4.53}$$

are conserved. Substituting now equation (4.51) in equation (4.53), one finds out that the conserved quantities can be written as

$$P^\mu = \frac{1}{c}\int\left(\frac{\partial}{\partial x^\rho}\; h^{\mu\nu\rho}\right)dS_\nu.$$

Recalling now that, for antisymmetric tensors $A^{\mu\nu} = -A^{\nu\mu}$, we can transform this integral done on a 3d hypersurface to an integral on a 2d surface performing

$$\frac{1}{2}\int A^{\mu\nu} df^*_{\mu\nu} = \frac{1}{2}\int\left(dS_\mu\frac{\partial A^{\mu\nu}}{\partial x^\nu} - dS_\nu\frac{\partial A^{\mu\nu}}{\partial x^\mu}\right) = \int dS_\mu\frac{\partial A^{\mu\nu}}{\partial x^\nu},$$

where $df^*_{\mu\nu}$ is the dual of $df_{\mu\nu} = dx^\mu dx'^\nu - dx^\nu dx'^\mu$, and so $df^*_{\mu\nu} = \frac{1}{2}e_{\mu\nu\rho\gamma}df^{\rho\gamma}$. Taking this into account, we can finally rewrite the conserved quantities P^μ as

$$P^\mu = \frac{1}{2c}\oint h^{\mu\nu\rho}df^*_{\nu\rho}. \tag{4.54}$$

Take now $t = const$ as 3d hypersurface in equation (4.53). Then, the 2d surface in (4.54) is purely spatial, and so we have

$$P^\mu = \frac{1}{c}\oint h^{\mu 0 i}df_i, \tag{4.55}$$

where $df_i = df^*_{0i}$ are the components three-dimensional element of a 2d surface. We only need to consider df^*_{0i} because, in this case, all the other components vanish [15]. From the result (4.55), it is easy to guess a possible choice for the energy-momentum tensor in TEGR. We can notice in fact that field equation of TEGR really looks like equation (4.51). If we simply write them as

$$\partial_\sigma\left(h\;\overset{\bullet}{S}_a{}^{\rho\sigma}\right) = kh(\Theta_a{}^\rho - \overset{\bullet}{J}_a{}^\rho) = -kh\overset{\bullet}{J}_a{}^\rho,$$

the analogy with equation (4.51) is quite clear. In the last equality we have set the energy momentum tensor of the matter (source) to zero, as it is for the Schwarzschild spacetime. Recalling now that the super potential is antisymmetric in the last indices ($\rho \leftrightarrow \sigma$), one has the following conservation law

$$\partial_\rho(h\overset{\bullet}{J}_a{}^\rho) = -\frac{1}{k}\left[\partial_\rho\partial_\sigma\left(h\;\overset{\bullet}{S}_a{}^{\rho\sigma}\right)\right] = 0, \tag{4.56}$$

which comes directly from the field equations. Let us do a comment before proceeding. In chapter 1 we have noticed that the "gauge current" $\overset{\bullet}{J}_a{}^\rho$, which is a true spacetime and gauge tensor, can be considered as an improved version of spacetime-indexed pseudocurrent (2.148). For this reason, we use $\overset{\bullet}{J}_a{}^\rho$ to define an energy-momentum vector instead of the spacetime-indexed quantity $\overset{\bullet}{t}_\mu{}^\rho$. By analogy between equation (4.52) and equation (4.56), we identify

$$-\frac{1}{k} h \; \overset{\bullet}{S}{}^{a\rho\sigma} \sim h^{\mu\nu\rho}. \tag{4.57}$$

Consequently, we try to define the energy momentum vector in TEGR simply in analogy with equation (4.55) (and setting $c = 1$)

$$\begin{aligned} P^a &= -\frac{1}{k} \oint_{S_2} h \; \overset{\bullet}{S}{}^{a0i} df_i \\ &= -\frac{1}{k} \oint_{S_2} h \; \overset{\bullet}{S}{}^{a0i} \, n_i \, d\theta d\phi. \end{aligned} \tag{4.58}$$

This integral has to be done on a 2d sphere $r = const$ embedded in the 3d space $t = const$. In addition, n_i is the normal to S_2, and so it has only r-component n_1. To begin with, we try to compute the energy, that is given by

$$P^{\hat{0}} = -\frac{1}{k} \oint_{S_2} h \; \overset{\bullet}{S}{}^{\hat{0}01} \, n_1 \, d\theta d\phi. \tag{4.59}$$

The calculation proceed straightforwardly using the appropriate spin connection (4.42) and the diagonal tetrad (4.3). We have

$$h = r^2 \sin\theta$$

$$n_1 = \left(1 - \frac{2M}{r}\right)^{-1/2}$$

$$\overset{\bullet}{S}{}^{\hat{0}01} = -2 \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} \frac{M}{r^2}.$$

Let's present in some details only the calculation of the last equation. We start with the definition of the spacetime-indexed superpotential

$$\overset{\bullet}{S}{}^{\rho\mu\nu} = \; \overset{\bullet}{K}{}^{\mu\nu\rho} - g^{\rho\nu} \, \overset{\bullet}{T}{}^{\sigma\mu}{}_\sigma + g^{\rho\mu} \, \overset{\bullet}{T}{}^{\sigma\nu}{}_\sigma. \tag{4.60}$$

So we have

$$\overset{\bullet}{S}{}^{001} = \; \overset{\bullet}{K}{}^{010} - g^{01} \, \overset{\bullet}{T}{}^{\sigma 0}{}_\sigma + g^{00} \, \overset{\bullet}{T}{}^{\sigma 1}{}_\sigma = \; \overset{\bullet}{K}{}^{010} + g^{00} \, \overset{\bullet}{T}{}^{\sigma 1}{}_\sigma.$$

the last term have already been computed to find out equation (4.45) while, from the definition of the contortion tensor (2.136), one finds

$$\overset{\bullet}{K}{}^{010} = \frac{1}{2}\left(\overset{\bullet}{T}{}^{100} + \overset{\bullet}{T}{}^{001} - \overset{\bullet}{T}{}^{010} \right) = \overset{\bullet}{T}{}^{001}.$$

Now, using the definition (4.43) of the torsion tensor, which includes the spin connection (4.42) natural associated with the diagonal tetrads (4.3)

$$\overset{\bullet}{T}{}^{001} = g^{1\nu} g^{0\mu} \overset{\bullet}{T}{}^{0}{}_{\mu\nu} = \overset{\bullet}{T}{}^{0}{}_{01} = -f^{-1}\frac{M}{r^2},$$

and so we reach to the result $\overset{\bullet}{S}{}^{001} = -2\left[\left(1-\frac{2M}{r}\right)^{-1}\frac{M}{r^2}\right]$. Considering now that $\overset{\bullet}{S}{}^{a\mu\nu} = h^a{}_\rho \overset{\bullet}{S}{}^{\rho\mu\nu}$, we easily achive to the reported result. Thus, we can compute the integral (4.59), that reads

$$\begin{aligned} P^0 &= -\frac{1}{k}\oint_{S_2} r^2 \sin\theta \left(-f^{-\frac{1}{2}}\frac{2M}{r^2}\right) f^{-\frac{1}{2}} d\theta d\phi = M\left(1-\frac{2M}{r}\right)^{-1} \\ &= M\left(1+\frac{2M}{r}+O\left(\frac{1}{r^2}\right)\right) \xrightarrow[r\to\infty]{} M. \end{aligned} \tag{4.61}$$

We notice from this equation that, interestingly, at the real singularity $r = 0$ the gravitational energy is zero. However, we notice that our expression for the energy diverges at the horizon (this could be due to the coordinate singularty of the Schwarzschild BH) while inside the horizon it is negative. Therefore, we obtain that the energy of the Schwarzschild spacetime is

$$E = P^{\hat{0}} = M. \tag{4.62}$$

As anticipated we see that taking also into account a possible expression for the gravitational energy, the total thermodynamical energy seems to be M. We now try to write down an expression for the first law of BH mechanics in the form

$$TdS = dE + pdV.$$

Where the differentials should be understood as "variations" under a Penrose process. The idea is to derive the value of the entropy associated with our expression of gravitational energy. Following the reference [4], we use, instead of the usual pressure p, a "surface pressure" defined in analogy with the ordinary pressure as

$$\sigma = -\left(\frac{\partial E}{\partial A}\right)_S = \frac{1}{8\pi r}\frac{\partial E}{\partial r} = \frac{M^2}{r^3}\left(1-\frac{2M}{r}\right)^{-\frac{3}{2}}. \tag{4.63}$$

We do that since the spatial volume is not defined. Thus, we write the first law as

$$TdS = dE + \sigma dA. \tag{4.64}$$

From the expression of the energy, one finds that

$$dE = \frac{\partial E}{\partial M}\, dM + \frac{\partial E}{\partial r}\, dr = f^{-1}\left[1 + \frac{2M}{r} f^{-1}\right] dM - \frac{2M}{r^2} f^{-2} dr. \tag{4.65}$$

Using $A = 4\pi r^2$ and the expression for the surface pressure (4.63), we also find that

$$\sigma dA = \frac{M^2}{r^3}\left(1 - \frac{2M}{r}\right)^{-\frac{3}{2}} d\left(4\pi r^2\right) = \frac{8\pi}{r^2} M^2 f^{-\frac{3}{2}} dr. \tag{4.66}$$

Inserting then (4.66) and (4.65) into (4.64), one finds

$$TdS = dE + \rho dA = f^{-1}\left[1 + \frac{2M}{r} f^{-1}\right] dM + \frac{2M}{r^2} f^{-\frac{3}{2}}\left[4\pi M - f^{-\frac{1}{2}}\right] dr. \tag{4.67}$$

We recall now that asymptotically and at late time $T = (8\pi M)^{-1}$, and that $f \xrightarrow[r\to\infty]{} 1$. As a resul, we find from (4.67) that in the limit $r \to \infty$

$$dS = 8\pi M\, dM \tag{4.68}$$

and therefore that the entropy can be written as

$$S = 4\pi M^2. \tag{4.69}$$

This thermodynamical analysis again tells us that in TEGR, as anticipated, the thermodynamical entropy is the same as in GR.

4.5 One-loop corrections in TEGR

In this section, we consider one-loop corrections to the partition function. We take into account the effect of $(S_E)_2$, in the expression (3.5), on the free energy. Physically, one-loop corrections can be thought of as the contribution from thermal gravitons to the free energy. In GR, using translational invariance of path integral measure, one finds from (3.5) that the partition function takes the form (dropping the subindex E and substituting $\delta g \to h$ and $\delta\phi \to \phi$)

$$\ln Z(\beta) \approx -S[\bar{g}, \bar{\phi}] + \ln \int Dh \exp\left(-S_2[h]\right) + \ln \int D\phi \exp\left(-S_2[\phi]\right), \tag{4.70}$$

We have to study the second and third term of this equation in TEGR. Thus, roughly speaking, we have to evaluate the determinant of the hessian matrix of the Teleparallel action (its second functional derivative with respect to the gauge potential), and the hessian of the action of a scalar field in a Weitzenbock spacetime, i.e., a scalar field minimally coupled to gravity. To perform these calculations, it is convenient to start introducing some comments. To begin with, we assume to be in the class of reference frames in which the spin connection vanishes. Then, we perturb a nontrivial tetrad field around a nontrivial background

$$h^a{}_\mu = \bar{h}^a{}_\mu + B^a{}_\mu, \tag{4.71}$$

where $B^a{}_\mu$ is the translational gauge potential and $\bar{h}^a{}_\mu$ is a nontrivial background tetrad. Indeed, in approaching the quantization of gravity in TEGR, $B^a{}_\mu$ is the field that should be taken into account. In particular, the gauge potential should both enter in the path integral in functional quantization and be promoted to an operator in canonical formalism. So in TEGR, with no other field than the tetrads, the path integral should take the form

$$Z = \int DB^a{}_\mu \exp\left(-\overset{\bullet}{\mathcal{S}}\right), \tag{4.72}$$

where

$$\overset{\bullet}{\mathcal{S}} = \int \overset{\bullet}{\mathcal{L}}\; d^4x + \frac{1}{16\pi}\int \frac{h}{2\alpha}\chi^\mu c_{\mu\nu}\chi^\nu\; d^4x + (S_{gh}). \tag{4.73}$$

In the previous equation, $\overset{\bullet}{\mathcal{L}}$ is the (Euclidean) TEGR Lagrangian thought as a function of the gauge potential. The second term is instead a gauge-breaking action. In particular, $c_{\mu\nu}$ is a field independent matrix and $\chi^\mu = \chi^\mu(B^a{}_\mu)$ is a general gauge-averaging functional. The ghost action S_{gh} depends on the specific form of χ^μ. We will not explicitly consider S_{gh} in TEGR because we will use a gauge-fixing (Lorentz gauge) so that the ghost Lagrangian is the Lagrangian of a field minimally coupled to gravity, a problem that we will consider separately in the next section. For this reason, we put the ghost action between parenthesis and the ghost fields do not appear in the functional measure in equation (4.72). From the previous equation, we have that the partition function at one loop in TEGR has the form (including also a scalar field)

$$\ln Z(\beta) \approx -S[\bar{h},\bar{\phi}] + \ln\int DB^a{}_\mu \exp\left(-S_2[B]\right) + \ln\int D\phi \exp\left(-S_2[\phi]\right). \tag{4.74}$$

where $\bar{h}$ stands for the nontrivial background tetrad and it must not to be confused with the perturbations of the metric tensor. The first term is the leading one and has already been computed in the preceding sections. The other 2 terms are the one-loop corrections. If we neglect the presence of the ghost fields, the relevant differential operator, from which the 1-loop corrections are computed, schematically has the form

$$F_{ik} = \frac{\delta^2 S[B]}{\delta B^i\,\delta B^k} - \frac{\delta\chi^\mu}{\delta B^i}\,c_{\mu\nu}\,\frac{\delta\chi^\nu}{\delta B^k}, \tag{4.75}$$

where $B_i \equiv B_a{}^{\mu}(x)$. To obtain the form of this operator in the case of TEGR, the best strategy is to first consider the perturbation around a nontrivial background (4.71) and then expand the TEGR action (or the Lagrangian) up to second order in the gauge potential about the nontrivial tetrad configuration $\bar{h}^a{}_{\mu}$. For the scalar field, the situation is much easier. For example, we do not have to deal with gauge-breaking conditions and ghost fields, the action is already of second and we do not need the background field method. We begin now to consider this easier case and in the following section, we only present the computations to find out the relevant operator (4.75) which however turns out to be a non-minimal operator [11].

4.5.1 One-loop divergences of a scalar field in TEGR

As a simple starting point, we begin with taking into account one-loop corrections coming from a massless scalar field. The action of a scalar field minimally coupled to gravity can be written as

$$\mathcal{S}_{scalar} = \int d^4x \frac{h}{2} \left\{ g^{\mu\nu} \dot{\mathscr{D}}_{\mu}\phi \dot{\mathscr{D}}_{\nu}\phi \right\}. \tag{4.76}$$

This result is obtained from the full gravitational coupling prescription 2.164

$$\partial_{\mu}\psi \to \dot{\mathscr{D}}_{\mu}\psi = \partial_{\mu}\psi - \frac{i}{2}\left(\dot{A}^{ab}{}_{\mu} - \dot{K}^{ab}{}_{\mu} \right) S_{ab}\,\psi, \tag{4.77}$$

where ψ is a generic field carrying an arbitrary representation of the Lorentz group, and S_{ab} are the Lorentz generators in the ψ representation. This coupling is manifestly equivalent to the GR one and then also the Euclidean formulation will be the same as in GR. Now, for a scalar field, the Lorentz generators are $S_{ab} = 0$. Therefore, in our simple case, the full gravitational coupling takes the form

$$\partial_{\mu}\phi \to \dot{\mathscr{D}}_{\mu}\phi = \partial_{\mu}\phi. \tag{4.78}$$

Then, the action for the scalar field takes the simple form

$$\mathcal{S}_{scalar} = \int d^4x \frac{h}{2} \left\{ g^{\mu\nu} \partial_{\mu}\phi \partial_{\nu}\phi \right\}. \tag{4.79}$$

Therefore, in the Barvinsky language [11], we have that the operator of interest is

$$F(\nabla) = \Delta = -\dot{\nabla}_{\mu}\dot{\nabla}^{\mu}, \tag{4.80}$$

where $\dot{\nabla}_{\mu}$ is the operator (2.166), which schematically is given by

$$\dot{\nabla}_{\mu} = \partial_{\mu} + \dot{\Gamma}_{\mu} - \dot{K}_{\mu}. \tag{4.81}$$

Using equation (2.30), we can see that it contains exactly the Levi-Civita connection rephrased in terms of the Weintzenbock connection as $\overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu} = \overset{\bullet}{\Gamma}{}^{\rho}{}_{\mu\nu} - \overset{\bullet}{K}{}^{\rho}{}_{\mu\nu}$. Of course, then, the operator (4.80) is precisely the same as the GR covariant Laplacian and, thus, we will obtain the same one-loop correction as the usual scalar field on a curved background. We notice that this happens to be the case because the relevant connection, the one in term of which we write down the coefficients of the early-time expansion of the trace of the heat kernel, is the one that enters in (4.80). We recall that, for $t \to 0$, the trace of the heat kernel in d-dimensional spacetimes is expected to have an asymptotic expansion of the form

$$\mathrm{Tr}K_{\Delta}(t) \approx \frac{1}{(4\pi t)^{d/2}} \left[B_0(\Delta) + tB_2(\Delta) + t^2 B_4(\Delta) + ...\right], \tag{4.82}$$

where

$$B_n(\Delta) = \int d^4x\, h\; b_n(\Delta), \tag{4.83}$$

and $b_n(\Delta)$ are constructed in terms of the "generalized curvatures" entering in equation (3.85). However, since the operator (4.80) does not contain neither a gauge connection nor a potential term, we remain only with the "Riemann" curvature, which is obtained from the commutator of the covariant derivatives entering (4.80) and acting on a spacetime vector V^{α}

$$[\overset{\bullet}{\nabla}_{\mu}, \overset{\bullet}{\nabla}_{\nu}]V^{\alpha} = -\overset{\bullet}{Q}{}^{\alpha}{}_{\beta\mu\nu}V^{\beta} = \overset{\circ}{R}{}^{\alpha}{}_{\beta\mu\nu}V^{\beta}, \tag{4.84}$$

where we have used that $\overset{\bullet}{Q}{}^{\alpha}{}_{\beta\mu\nu} = -\overset{\circ}{R}{}^{\alpha}{}_{\beta\mu\nu}$, which holds because the curvature of the Weintzenbock connection vanishes. The tensor $\overset{\bullet}{Q}{}^{\alpha}{}_{\beta\mu\nu}$ is explicitly given by (cf. equation (2.153))

$$\begin{aligned}\overset{\bullet}{Q}{}^{\rho}{}_{\theta\mu\nu} =& \partial_{\mu}\overset{\bullet}{K}{}^{\rho}{}_{\theta\nu} - \partial_{\nu}\overset{\bullet}{K}{}^{\rho}{}_{\theta\mu} + \overset{\bullet}{\Gamma}{}^{\rho}{}_{\sigma\mu}\overset{\bullet}{K}{}^{\sigma}{}_{\theta\nu} - \overset{\bullet}{\Gamma}{}^{\rho}{}_{\sigma\nu}\overset{\bullet}{K}{}^{\sigma}{}_{\theta\mu} \\ &- \overset{\bullet}{\Gamma}{}^{\sigma}{}_{\theta\mu}\overset{\bullet}{K}{}^{\rho}{}_{\sigma\nu} + \overset{\bullet}{\Gamma}{}^{\sigma}{}_{\theta\nu}\overset{\bullet}{K}{}^{\rho}{}_{\sigma\mu} + \overset{\bullet}{K}{}^{\rho}{}_{\sigma\nu}\overset{\bullet}{K}{}^{\sigma}{}_{\theta\mu} - \overset{\bullet}{K}{}^{\rho}{}_{\sigma\mu}\overset{\bullet}{K}{}^{\sigma}{}_{\theta\nu}.\end{aligned} \tag{4.85}$$

Then, the first three HAMIDEW coefficients (all we need in d=4), in terms of which one writes down the divergent part of the effective action, are

$$\begin{aligned} b_0(x,x) &= 1 \\ b_2(x,x) &= \frac{1}{6}\overset{\bullet}{Q} \\ b_4(x,x) &= \frac{1}{180}\left(\overset{\bullet}{Q}_{\alpha\beta\mu\nu}\overset{\bullet}{Q}{}^{\alpha\beta\mu\nu} - \overset{\bullet}{Q}_{\mu\nu}\overset{\bullet}{Q}{}^{\mu\nu} + \frac{5}{2}\overset{\bullet}{Q}{}^2 - 6\overset{\bullet}{\nabla}_{\mu}\overset{\bullet}{\nabla}{}^{\mu}\overset{\bullet}{Q}\right). \end{aligned} \tag{4.86}$$

Consequently, we obtain from the master formula (3.84), the divergent part of the effective action

$$\Gamma_{div} = -\frac{1}{2}\frac{1}{(4\pi)^{d/2}}\int d^dx\, h\left[\frac{\Lambda_{UV}^d}{d/2}b_0 + \frac{\Lambda_{UV}^{d-2}}{\frac{d}{2}-1}b_2 + ... + \ln\frac{\Lambda_{UV}^2}{\mu^2}b_d + \text{finite terms}\right]$$
$$= (d=4) = -\frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\, h\left[\frac{\Lambda_{UV}^4}{2}b_0 + \Lambda_{UV}^2 b_2 + \ln\frac{\Lambda_{UV}^2}{\mu^2}b_4 + \text{finite terms}\right], \tag{4.87}$$

where the coefficients b_n are listed in (4.86).

As an aside, one could argue that the most natural choice for the gravitational coupling would be to just follow the gauge paradigm and to substitute the partial derivatives the gauge covariant derivatives as seen in a general Lorentz frame

$$\partial_\mu\psi \to h_\mu\psi = h_\mu{}^a\partial_a\psi, \tag{4.88}$$

where

$$h_\mu{}^a = \partial_\mu x^a + \overset{\bullet}{A}{}^a{}_{b\mu}x^b + B^a{}_\mu. \tag{4.89}$$

According to the coupling (4.88), one would obtain a completely different situation as compared to before. Indeed, instead of (4.86) we easily attain, from equation (3.85), the following HAMIDEW coefficients

$$\begin{aligned} b_0(x,x) &= 1 \\ b_2(x,x) &= 0 \\ b_4(x,x) &= \frac{1}{12}\text{Tr}\{F_{\mu\nu}F^{\mu\nu}\}, \end{aligned} \tag{4.90}$$

where $F_{\mu\nu}$ is the translational field strength, which is the torsion tensor $[h_\mu, h_\nu] = F_{\mu\nu} = \overset{\bullet}{T}{}^a{}_{\nu\mu}P_a$. In this case, we should have

$$\text{Tr}\{F_{\mu\nu}F^{\mu\nu}\} = \overset{\bullet}{T}{}^a{}_{\mu\nu}\;\overset{\bullet}{T}{}^{b\mu\nu}\underbrace{\text{tr}\,[P_bP_a]}_{\gamma_{ab}=\text{CK form}} = \overset{\bullet}{T}{}^a{}_{\mu\nu}\;\overset{\bullet}{T}{}^{b\mu\nu}\left[f^c_{ad}f^d_{bc}\right] = 0, \tag{4.91}$$

because translations are abelian. This result tells us that in (4.87) for $d=4$ only the quartic power divergence survive

$$\Gamma_{div} = -\frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\, h\left[\frac{\Lambda_{UV}^4}{2}b_0 + \text{finite terms}\right]. \tag{4.92}$$

The fact that b_4 vanishes suggests that, using the coupling (4.88), one would obtain a *renormalizable theory*. However, classically, it is very desirable to have a coupling that is equivalent to GR. Indeed, at the classical level, any modification to the coupling needs to be compared (and to be in agreement) with several experimental bounds before being used. Moreover, since the group manifold is essentially Minkowski, one should use the Minkowski metric η_{ab} instead of the degenerate Cartan-Killing bilinear form γ_{ab}. In this case, instead of a vanishing b_4, one would obtain the standard result for a gauge theory.

4.5.2 Towards one-loop divergences of TEGR

We now do some steps toward the computation of one-loop correction coming from the gravitational part of the Lagrangian. To begin with, we notice that the equation establishing the equivalence between TEGR and GR (2.156)

$$\overset{\bullet}{\mathcal{L}} = \overset{\circ}{\mathcal{L}}_{EH} - \partial_\mu \left(\frac{h}{8\pi} \overset{\bullet}{T}{}^{\nu\mu}{}_\nu \right) = \overset{\circ}{\mathcal{L}}_{EH} - \frac{h}{8\pi} \overset{\circ}{\nabla}_\mu \left(\overset{\bullet}{T}{}^{\nu\mu}{}_\nu \right), \tag{4.93}$$

suggest that one-loop corrections to the TEGR action will be the same as in GR. This statement is clear in absence of a boundary of the spacetime. In this case, we can in fact discard every boundary term and then the TEGR Lagrangian is exactly equal to the GR lagrangian. In presence of a boundary of the spacetime, the situation is more subtle. In this case, the HAMIDEW coefficients are modified by some boundary terms, below denoted with c. These coefficients are expressed in terms of geometric invariants evaluated at the boundary, which are built up using the "effective" connection entering the quadratic action. In particular, the early time asymptotic expansion (4.82) is modified into [11]

$$\begin{aligned} \mathrm{Tr}K(t) &\approx \frac{1}{(4\pi t)^{d/2}} \sum_{k=0}^{\infty} t^{k/2} B_k, \\ B_{2n} &= \int d^d x\, h\, b_n(x) + \int_{\partial M} d^{d-1} y\, \sigma^{1/2} c_{2n}(y), \\ B_{2n+1} &= \int_{\partial M} d^{d-1} y\, \sigma^{1/2} c_{2n+1}(y), \end{aligned} \tag{4.94}$$

where of course $\{y\}$ are coordinates on the boundary and σ is the determinant of the induced metric (or, equivalently, of the associated induced tetrad). The coefficients b_{2n} are the same as in the case of boundaryless spacetimes. The boundary terms in (4.94), essentially, depend on the boundary conditions (BC) on the fields. These boundary conditions must be chosen to have symmetric and elliptic differential operators [14]. Often, one choose BC on the fields such that the all the boundary terms vanish. Among the most used BC there are Dirichlet (D) and generalized Neumann (N) BC

$$\begin{aligned} \phi_{|\partial M} &= 0, & (D) \\ (\nabla_n - \hat{S})\phi_{|\partial M} &= 0, & (N) \end{aligned} \tag{4.95}$$

where ∇_n denotes derivatives normal to the boundary and $\hat{S}$ is a matrix valued function defined on the boundary. For these BC several HAMIDEW are known. Some of the lowest order c read

$$\begin{aligned} &c_0^{D,N} = 0 \\ &c_1^D = -\frac{\sqrt{\pi}}{2} tr\hat{1}, \;\; c_1^N = \frac{\sqrt{\pi}}{2} tr\hat{1} \\ &c_2^D = \frac{1}{3} K\, tr\hat{1}, \;\; c_2^N = tr\left(2\hat{S} + \frac{1}{3}K\hat{1}\right). \end{aligned} \tag{4.96}$$

We see that the boundary terms arising from the quadratic action will influence the choice of BC which, then, will modify the HAMIDEW coefficients. However, since we have shown that we can identify the TEGR boundary term B with the GHY term, we expect that no differences will arise in TEGR (with respect to GR of course), even in presence of a boundary of the spacetime. One should obtain different one-loop corrections in TEGR only in the case in which the TEGR boundary term would be different from GHY and the appropriate BC for GR (which seems to be still problematic [18]) are not "good" for TEGR.

Although this simple reasoning seems to indicate that in TEGR no differences with respect to GR will arise even at one-loop, it is very desirable to obtain such corrections directly from the TEGR action. Indeed, conceptually, TEGR and GR differ profoundly and then it seems worthy to, at least, explicitly check this equivalence. This, however, turns out to be a highly-nontrivial problem. The procedure to follow is quite clear: one should take a perturbation of a nontrivial background, obtain the action at second order in the perturbations of the tetrads fields and impose a gauge fixing condition, identifying in this way a second-order differential operator. On this operator, we then apply the heat kernel theory to obtain one-loop corrections, pretty much what we have done in the case of the scalar field. We will perform these computations applying the background field method.

To apply this method, one can start from the TEGR Lagrangian (2.140) written for convenience as

$$16\pi\overset{\bullet}{\mathcal{L}} = h\left(\frac{1}{4}\overset{\bullet}{T}{}^a{}_{\mu\nu}\overset{\bullet}{T}{}_a{}^{\mu\nu} + \frac{1}{2}\overset{\bullet}{T}{}^\rho{}_{\mu\nu}\overset{\bullet}{T}{}^{\nu\mu}{}_\rho - \overset{\bullet}{T}{}^\rho{}_{\mu\rho}\overset{\bullet}{T}{}^{\nu\mu}{}_\nu\right) \equiv h\mathbb{T}. \tag{4.97}$$

Then, basically, our second order Lagrangian is given by

$$16\pi\overset{\bullet}{\mathcal{L}}{}^{(2)} = h^{(0)}\mathbb{T}^{(2)} + h^{(1)}\mathbb{T}^{(1)} + h^{(2)}\mathbb{T}^{(0)}. \tag{4.98}$$

So our task is to expand at zero, first and second order in the gauge field (this is actually the meaning of the numbers in the apices), the inverse tetrad, the determinant and the torsion scalar. Let's present the formulas that we need. To begin with, we notice that the inverse tetrad, using the formula $(A+\delta A)^{-1} = A^{-1} - A^{-1}\delta A A^{-1} + A^{-1}(\delta A)A^{-1}(\delta A)A^{-1} + \ldots$, can be expanded as

$$h_a{}^\rho = \bar{h}_a{}^\rho - B_a{}^\rho + \bar{h}_c{}^\rho B_a{}^\delta B^c{}_\delta + \ldots \tag{4.99}$$

We further notice that we can rewrite the torsion tensor replacing the ordinary derivatives with covariant Levi-Civita derivative

$$\overset{\bullet}{T}{}^a{}_{\mu\nu} = \partial_\mu h^a{}_\nu - \partial_\nu h^a{}_\mu = \overset{\circ}{\nabla}_\mu h^a{}_\nu - \overset{\circ}{\nabla}_\nu h^a{}_\mu, \tag{4.100}$$

since the the Levi-Civita connection is symmetric. Of course, we as well have

$$\overset{\bullet}{T}{}^\rho{}_{\mu\nu} = h_a{}^\rho \left(\overset{\circ}{\nabla}_\mu h^a{}_\nu - \overset{\circ}{\nabla}_\nu h^a{}_\mu \right). \tag{4.101}$$

Doing this replacement, one has to notice that the perturbation in the tetrad field induces also a perturbation in the spacetime metric. Indeed, combining the equation $g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_\nu$ with the perturbations $g_\mu = \bar{g}_\mu + \delta g_{\mu\nu}$ and $h^a{}_\mu = \bar{h}^a{}_\mu + B^a{}_\mu$, one obtains, up to first order, $\delta g_{\mu\nu} = \bar{h}_{a\mu} B^a{}_\nu + \bar{h}_{a\nu} B^a{}_\mu$. Thus, in addition, we have to expand the covariant derivative as

$$\overset{\circ}{\nabla}_\mu = \overset{\circ}{\nabla}{}^{(0)}_\mu + \overset{\circ}{\Gamma}{}^{(1)}_\mu + \overset{\circ}{\Gamma}{}^{(2)}_\mu, \tag{4.102}$$

However we are about to see that only $\overset{\circ}{\nabla}{}^{(0)}_\mu$ will matter for the torsion perturbations. Now the apices indicate the power of the perturbation of the metric. It is now easy to obtain the following

$$\begin{cases} \overset{\bullet}{T}{}^{(0)a}{}_{\mu\nu} = \overset{\circ}{\nabla}{}^{(0)}_\mu \bar{h}^a{}_\nu - \overset{\circ}{\nabla}{}^{(0)}_\nu \bar{h}^a{}_\mu \\ \overset{\bullet}{T}{}^{(1)a}{}_{\mu\nu} = \overset{\circ}{\nabla}{}^{(0)}_\mu B^a{}_\nu - \overset{\circ}{\nabla}{}^{(0)}_\nu B^a{}_\mu \\ \overset{\bullet}{T}{}^{(2)a}{}_{\mu\nu} = 0 \end{cases} \qquad \begin{cases} \overset{\bullet}{T}{}^{(0)\rho}{}_{\mu\nu} = \bar{h}_a{}^\rho \left(\overset{\circ}{\nabla}{}^{(0)}_\mu \bar{h}^a{}_\nu - \overset{\circ}{\nabla}{}^{(0)}_\nu \bar{h}^a{}_\mu \right) \\ \overset{\bullet}{T}{}^{(1)\rho}{}_{\mu\nu} = \bar{h}_a{}^\rho\, \overset{\bullet}{T}{}^{(1)a}{}_{\mu\nu} - B_a{}^\rho\, \overset{\bullet}{T}{}^{(0)a}{}_{\mu\nu} \\ \overset{\bullet}{T}{}^{(2)\rho}{}_{\mu\nu} = \bar{h}_a{}^\rho\, \overset{\bullet}{T}{}^{(2)a}{}_{\mu\nu} - B_a{}^\rho\, \overset{\bullet}{T}{}^{(1)a}{}_{\mu\nu} + \\ \qquad + \bar{h}_c{}^\rho B_a{}^\delta B^c{}_\delta\, \overset{\bullet}{T}{}^{(0)a}{}_{\mu\nu}. \end{cases}$$

Finally, we recall that the determinant of the tetrad can be expanded as

$$h = \bar{h} \left[1 + \frac{1}{2} B + \left(\frac{1}{8} B^2 - \frac{1}{4} B^a{}_\mu B_a{}^\mu \right) + \ldots \right], \tag{4.103}$$

where $\bar{h} = \det\left(\bar{h}^a{}_\mu\right)$ and $B = B^a{}_\mu \bar{h}_a{}^\mu$. Now that we have all the ingredients, and after having inserted them into (4.98), it is just a matter of long calculations to obtain the following expression (omitting the apices (0) in the covariant derivatives)

$$16\pi\overset{\bullet}{\mathcal{L}}^{(2)} = h\,B^a{}_\nu\Bigg\{\left[-\frac{1}{2}\eta_{ab}g^{\mu\nu} + A^{(0)\mu\nu}_{ab}\right]\overset{\circ}{\nabla}_\rho\overset{\circ}{\nabla}^\rho - \frac{1}{2}\eta_{ab}\overset{\circ}{\nabla}^\mu\overset{\circ}{\nabla}^\nu + C^{(0)\,\nu}_{ab\rho}\overset{\circ}{\nabla}^\mu\overset{\circ}{\nabla}^\rho + g^{\mu\nu}D^{(0)}_{ab\alpha\rho}\overset{\circ}{\nabla}^\alpha\overset{\circ}{\nabla}^\rho +$$

$$\left[G^{(0)\,\nu\mu}_{ba\rho}\right]\overset{\circ}{\nabla}^\rho - \left[Q^{(0)\mu}_{ba}\right]\overset{\circ}{\nabla}^\nu + \left[L^{(0)\mu\nu}_{ba}\right]\Bigg\}B^b{}_\mu =$$

$$= h\,B^a{}_\nu\Bigg\{\left[-\frac{1}{2}\eta_{ab}g^{\mu\nu} + A^{(0)\mu\nu}_{ab}\right]\overset{\circ}{\nabla}_\rho\overset{\circ}{\nabla}^\rho - \frac{1}{2}\eta_{ab}\overset{\circ}{\nabla}^\nu\overset{\circ}{\nabla}^\mu + C^{(0)\,\nu}_{ab\rho}\overset{\circ}{\nabla}^\mu\overset{\circ}{\nabla}^\rho + g^{\mu\nu}D^{(0)}_{ab\alpha\rho}\overset{\circ}{\nabla}^\alpha\overset{\circ}{\nabla}^\rho +$$

$$\left[G^{(0)\,\nu\mu}_{ba\rho}\right]\overset{\circ}{\nabla}^\rho - \left[Q^{(0)\mu}_{ba}\right]\overset{\circ}{\nabla}^\nu + \left[L^{(0)\mu\nu}_{ba} - \frac{1}{2}\eta_{ab}\overset{\circ}{R}^{\mu\nu}\right]\Bigg\}B^b{}_\mu, \tag{4.104}$$

where the quantities with the apices (0) are various combinations of the background tetrad and torsion. The coefficients A, C, D, G, Q and L are reported in the concluding appendix. Now, imposing the gauge condition (Lorentz gauge)

$$\overset{\circ}{\nabla}^\mu B^b{}_\mu = 0, \tag{4.105}$$

our differential operator become

$$F^{\mu\nu}_{ab}(\nabla) = \Bigg\{\left[\left(-\frac{1}{2}\eta_{ab}g^{\mu\nu} + A^{(0)\mu\nu}_{ab}\right)g_{\alpha\rho} + C^{(0)\,\nu}_{ab\rho}\delta^\mu_\alpha + g^{\mu\nu}D^{(0)}_{ab\alpha\rho}\right]\overset{\circ}{\nabla}^\alpha\overset{\circ}{\nabla}^\rho +$$

$$+\left[\left(G^{(0)\,\nu\mu}_{ba\rho}\right) - \left(Q^{(0)\mu}_{ba}\delta^\nu_\rho\right)\right]\overset{\circ}{\nabla}^\rho + \left[L^{(0)\mu\nu}_{ba} - \frac{1}{2}\eta_{ab}\overset{\circ}{R}^{\mu\nu}\right]\Bigg\} \tag{4.106}$$

$$= \Bigg\{\left[N^{(0)\ \ \nu\mu}_{ab\alpha\rho}\right]\overset{\circ}{\nabla}^\alpha\overset{\circ}{\nabla}^\rho + \left[O^{(0)\,\nu\mu}_{ba\rho}\right]\overset{\circ}{\nabla}^\rho + \left[P^{(0)\mu\nu}_{ba}\right]\Bigg\}.$$

We arrive, using the Lorentz gauge, to a non-minimal second order operator from which is much more difficult to obtain the one-loop corrections. Moreover, it can be shown that the ghost operator associated to the Lorentz gauge is just the covariant Laplacian [16] considered in the previous section. Now, it is generally believed that a gauge choice for which an operator became minimal, where the computations are easier, always exists. This gauge choice is known as the minimal gauge [10]. Though, it is as well possible to obtain one-loop corrections in a non-minimal gauge. In this case, the basic quantity that are needed are the so called non-minimal propagators. We sketch now briefly the procedure in the more general case of non abelian gauge theories. Let's call $g^i = g^A(x)$ the set of our fields. The total action can then be written as

$$S[g] = S^G[g] + S^{GB}[g], \tag{4.107}$$

where $S^G[g]$ is a gauge invariant action and $S^{GB}[g]$ is the gauge breaking action with the generic form

$$S^{GB}[g] = -\frac{1}{2}\chi^\mu c_{\mu\nu}\chi^\nu. \tag{4.108}$$

In the previous equation $\chi^\mu = \chi^\mu(\lambda)$ is a certain linear gauge condition such that $\chi^\mu(0)$ is a minimal gauge and $c_{\mu\nu}$ is a local, invertible and field-independent matrix. Then, schematically, the operator of interest is

$$F_{ik} = \frac{\delta^2 S^G[g]}{\delta g^i \delta g^k} - \frac{\delta\chi^\mu}{\delta g^i} c_{\mu\nu} \frac{\delta\chi^\nu}{\delta g^k}. \tag{4.109}$$

It is shown in [10] that the effective action in the non.minimal gauge can be written in terms of the non-minimal propagators

$$F_{ik}G^{kn} = \delta_i^n, \qquad \nabla^i_\alpha \frac{\delta\chi^\mu}{\delta g^i} Q^\sigma_\mu = \delta^\sigma_\alpha. \tag{4.110}$$

of the gauge and ghost fields respectively. We could then follow this strategy but, however, because lack of time, we left these computations for future work. Anyway, we have presented as well a heuristic argument suggesting that TEGR one-loop corrections are same as in quantum GR.

Chapter 5

Conclusions

In the literature of TEGR, it is often stated that this theory is *fully* equivalent to GR because their actions are equal up to a boundary term B, which does not influence the equations of motion. Albeit boundary terms play an important role in physics, the effect of B is usually overlooked in the existent literature. For example, the boundary term B affects BH thermodynamics, which then is worth to be studied. Analyzing this effect has been the aim of this work. In particular, we have explicitly computed the classical TEGR action of a Schwarzschild BH, from which we have obtained that the values of entropy and energy at the leading order are the same as in GR. Therefore, we can say that, also in TEGR, the BH entropy obeys the well-known area law $S = \text{Area}/4$ and the BH energy is given by its rest mass. This unforeseen result is due to the equivalence of the TEGR boundary term and the GHY boundary term, as has been proved in subsection 4.1.1. Moreover, we have examined entropy and energy of a Schwarzschild BH in TEGR from a different perspective. Namely, we have constructed the TEGR analog of the Landau-Lifshitz energy-momentum pseudo-tensor of the gravitational field and computed the associated conserved charge. As we have seen, this approach leads to the same values of BH energy and entropy. Hence, these results provide further support to the aforementioned equivalence.

The classical action gives the leading correction in the semiclassical approximation to the path integral. Therefore, in section 4.5, we have studied if some discrepancies with GR arise upon quantization. In subsection 4.5.1, we have considered the one-loop effective action, which gives one-loop quantum corrections to the partition function, of a scalar field in a classical TEGR background. For this purpose, we have used the heat kernel method to find out the divergent part of the one-loop effective action. We have obtained that these divergences are precisely the same as in GR but rephrased in terms of the Weintzenböck connection. This result is due to the fact that the so-called full gravitational coupling prescription of TEGR is completely equivalent to the GR coupling prescription. Nevertheless, there could be a shortcoming. Indeed, let's consider the logarithmic divergence of the effective action (4.87). As in GR, this divergence is proportional to some terms that are not present in the TEGR Lagrangian $\overset{\bullet}{\mathcal{L}}$. Then, to renormalize the effective action, we should add some new counterterms to $\overset{\bullet}{\mathcal{L}}$ with the same structure of these divergences. However, any term added to the Lagrangian must be in agreement with its symmetries, that for the TEGR Lagrangian are diffeomorphism invariance, local Lorentz invariance, and translational gauge invariance. Since the divergent terms are the same as GR, the necessary counterterms are not expected to share all these symmetries; translational gauge invariance, in particular, seems unlikely to occur. We believe that this aspect deserves to be more deeply studied in the future, since, likely, it could lead to a discrepancy with GR, breaking thus the equivalence between the two theories. Finally, subsection 4.5.1 is concluded with an interesting observation: by choosing the usual gauge coupling prescription instead of the TEGR coupling prescription, we would obtain a renormalizable theory for a scalar field minimally coupled to gravity. Nonetheless, there are multiple problems with this conclusion that invalidate it. A first problem, of physical nature, is that any change of the coupling prescription, which is inequivalent to the GR coupling prescription, needs to pass several experimental tests before being safely used. Of course, this is not invalidating per se, indeed, it just means that we concretely must verify if this new coupling prescription is compatible with experimental bounds before drawing any conclusions from it. Another more serious issue is, instead, of mathematical nature. The point is that, in computing the trace of equation (4.91), we should admittedly use the Minkowski metric instead of the degenerate Cartan-Killing metric. For this reason, such conclusion is essentially unfounded. Thereafter, in subsection 4.5.2, we have given some hints on the effect of the quantization of the gravitational field. In particular, at the beginning of such subsection, we have presented a heuristic argument suggesting that the one-loop divergences of the TEGR effective action are the same as in quantum GR as well. However, because of the conceptual differences between TEGR and GR, it is desirable to explicitly evaluate the divergent part of the effective action. Following again the heat kernel method, we have found the relevant second-order differential operator. Unfortunately, using the Lorentz gauge, we arrive at a so-called non-minimal operator, from which is more difficult to obtain one-loop corrections. Because of lacking time and technical difficulties this computation is left for future work. We also point out a drawback of our approach. That is, for simplicity, we have chosen to deal only with the naïve definition of the effective action, which is known to depend off-shell on the gauge choice. We likewise want to stress that this gauge dependence takes place only off-shell, and then it does not appear in scattering amplitudes. Therefore, it would be interesting in the future to perform this analysis using the Vilkoviski unique effective action.

To conclude, our work seems to suggest that the equivalence between TEGR and GR can be extended to boundary terms and quantum (one-loop) corrections. Though, we want underline one more time that there are some subtle aspects about the latter. Indeed, the renormalization of the scalar field on a classical TEGR background will probably work out differently if compared to that of GR, since the necessary counterterms probably do not share the symmetries of TEGR. In our opinion, these features should to be deepened in the future.

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Appendix A

We report in this appendix the results for the coefficients appearing in the second order differential operator (4.106), obtained from the TEGR action expanded at second order in the gauge field. These coefficients can be expressed in terms of the background quantities only and so in the following expressions we omit the (0) and the bars above each term. The first three A, C and D have a simple form. The A coefficient reads

$$A^{(0)\mu\nu}_{ab} = \left(h_a{}^{\nu} h_b{}^{\mu} - \frac{1}{2} h_a^{\mu} h_b^{\nu}\right), \tag{A.1}$$

while C is

$$C^{(0)\,\nu}_{ab\rho} = \left(h_{a\rho} h_{b\rho} - 2 h_a{}^{\nu} h_{b\rho}\right), \tag{A.2}$$

and D is given by

$$D^{(0)}_{ab\alpha\rho} = \left(h_{a\alpha} h_{b\rho} - \frac{1}{2} h_{a\rho} h_{b\alpha}\right). \tag{A.3}$$

The last three coefficients have a more complicated form. The coefficient Q can be written as

$$Q^{(0)\,\mu}_{ba} = h_b{}^{\rho} \overset{\bullet}{T}_a{}^{\mu}{}_{\rho} + 2\delta_{ba} h_c{}^{\rho} \overset{\bullet}{T}{}^{c\mu}{}_{\rho}, \tag{A.4}$$

and L can be expressed in the following way

$$\begin{aligned} L^{(0)\,\nu\mu}_{ba} &= g^{\mu\nu} h_b{}^{\rho} h_c{}^{\beta} \left(\overset{\bullet}{T}_{a\gamma\beta} \overset{\bullet}{T}{}^{c\gamma}{}_{\rho} - 2\overset{\bullet}{T}_{a\gamma\rho} \overset{\bullet}{T}{}^{c\gamma}{}_{\beta}\right) + \left(h_a{}^{\nu} h_b{}^{\mu} - \frac{1}{4}\delta_{ab} g^{\mu\nu}\right) \mathbb{T} + \\ &+ \frac{1}{2} \overset{\bullet}{T}_{a\rho}{}^{\mu} \overset{\bullet}{T}_b{}^{\rho\nu} - \overset{\bullet}{T}_{a\rho}{}^{\nu} \overset{\bullet}{T}_b{}^{\rho\mu} + h_b{}^{\mu} h_c{}^{\rho} \left(\overset{\bullet}{T}_{a\gamma}{}^{\nu} \overset{\bullet}{T}{}^{c\gamma}{}_{\rho} - \frac{1}{2} \overset{\bullet}{T}_{a\gamma\rho} \overset{\bullet}{T}{}^{c\gamma\nu}\right). \end{aligned} \tag{A.5}$$

We have only the coefficient G left. Since its structure is rather complicated we split it as

$$G^{(0)\,\nu\mu}_{ab\rho} = E^{(0)\,\nu\mu}_{ab\rho} + g^{\mu\nu} F^{(0)}_{ab\rho} + H^{(0)\quad\mu)\nu}_{ab(\rho}, \tag{A.6}$$

where the second and the first terms contains derivatives of the tetrads field, and come from integration by parts of terms containing two derivatives. The round brackets are notation for $H^{(0)\ \ \mu)\nu}_{ab(\rho} = H^{(0)\,\mu\nu}_{ab\rho} - H^{(0)\,\mu\ \ \nu}_{ab\ \ \rho}$. The last term instead comes from terms containing one derivative. Explicitly they are

$$E^{(0)\,\nu\mu}_{ab\rho} = \overset{\circ}{\nabla}_\rho \left(h_a{}^\nu h_b{}^\mu\right) - \frac{1}{2}\overset{\circ}{\nabla}_\rho \left(h_a{}^\mu h_b{}^\nu\right) + \overset{\circ}{\nabla}{}^\mu \left(h_{a\rho} h_b{}^\nu\right) - 2\overset{\circ}{\nabla}{}^\mu \left(h_a{}^\nu h_{b\rho}\right), \tag{A.7}$$

while F is

$$F_{ab\rho} = \overset{\circ}{\nabla}_\alpha \left(h_a{}^\alpha h_{b\rho}\right) - \frac{1}{2}\overset{\circ}{\nabla}_\alpha \left(h_{a\rho} h_b{}^\alpha\right). \tag{A.8}$$

Finally, the tensor H is given by

$$H^{(0)\,\nu\mu}_{ab\rho} = 2h_b{}^\mu \overset{\bullet}{T}_{a\rho}{}^\nu - \delta_{ab} h_c{}^\mu \overset{\bullet}{T}{}^c{}_\rho{}^\nu + \frac{h_a{}^\nu}{2}\left[\frac{1}{2}\overset{\bullet}{T}_{b\rho}{}^\mu + \left(h_b{}^\gamma h_c{}^\nu - 2h_b{}^\nu h_c{}^\gamma\right)\overset{\bullet}{T}{}^c{}_{\rho\gamma}\right]. \tag{A.9}$$

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